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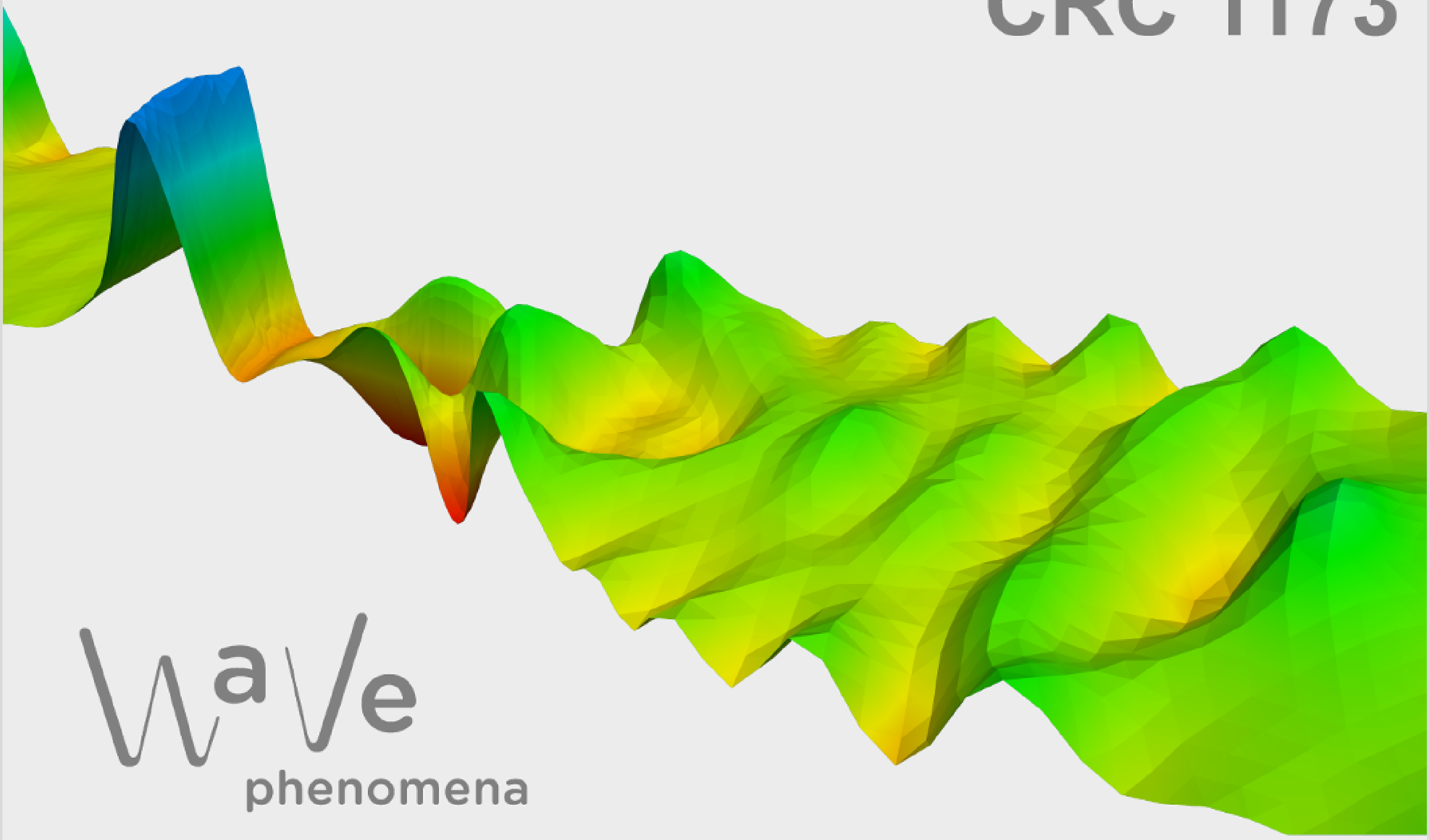

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LOW REGULARITY WELL-POSEDNESS FOR GENERALIZED BENJAMIN-ONO EQUATIONS ON THE CIRCLE

KIHYUN KIM AND ROBERT SCHIPPA

Abstract. New low regularity well-posedness results for the generalized Benjamin-Ono equations with quartic or higher nonlinearity and periodic boundary conditions are shown. We use the short-time Fourier transform restriction method and modified energies to overcome the derivative loss. Previously, Molinet–Ribaud established local well-posedness in $H^1(\mathbb{T},\mathbb{R})$ via gauge transforms. We show local existence and a priori estimates in $H^s(\mathbb{T},\mathbb{R})$, $s > 1/2$, and local well-posedness in $H^s(\mathbb{T},\mathbb{R})$, $s \geq 3/4$ without using gauge transforms. In case of quartic nonlinearity we prove global existence of solutions conditional upon small initial data.

1. Introduction

In this article we improve the well-posedness theory for the *k-generalized periodic Benjamin-Ono equation* in L^2-based Sobolev spaces

$$\begin{cases} \partial_t u + \mathcal{H}\partial_{xx} u &= \mp \partial_x (u^k) \quad (t,x) \in \mathbb{R}\times\mathbb{T}, \\ u(0) &= u_0 \in H^s(\mathbb{T},\mathbb{R}), \end{cases} \tag{1.1}$$

where $k \geq 4$ and $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Throughout this article, $\mathcal{H}$ denotes the *Hilbert transform*, i.e.,

$$\mathcal{H} : L^2(\mathbb{T}) \to L^2(\mathbb{T}), \quad (\mathcal{H}f)^\wedge(\xi) = -i\mathrm{sgn}(\xi)\hat{f}(\xi).$$

Note that real-valued initial data give rise to real-valued solutions. We shall implicitly consider real-valued initial data in the following, unless stated otherwise.

By *local well-posedness* we refer to the following: the data-to-solution mapping $S_T^\infty : H^\infty(\mathbb{T}) \to C([0,T], H^\infty(\mathbb{T}))$ assigning smooth initial data to smooth solutions admits a continuous extension $S_T^s : H^s \to C_T H^s$ with $T = T(\|u_0\|_{H^s})$, which can be chosen continuously on $\|u_0\|_{H^s}$. Existence and continuity of $S_T^s : H^s \to C_T H^s$ for $s > 3/2$ follows from the classical energy method (cf. [3, 1]). Solutions to (1.1) on the real line admit the scaling symmetry

$$u(t,x) \to \lambda^{-\frac{1}{k-1}} u(\lambda^{-2}t, \lambda^{-1}x).$$

This leads to the scaling critical space $\dot{H}^{s_c}(\mathbb{R})$, $s_c(k) = \frac{1}{2} - \frac{1}{k-1}$, which is the largest L^2-Sobolev space for which local well-posedness can be expected.

Conserved quantities of solutions to (1.1) are the *mass*, i.e., the L^2-norm,

$$M(u_0) = \int_{\mathbb{T}} u_0^2 dx,$$

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and the energy, related with the $H^{1/2}$-norm,

$$E(u_0) = \int_{\mathbb{T}} \frac{u_0 \mathcal{H} \partial_x u_0}{2} \pm \frac{u_0^{k+1}}{k+1} dx.$$

The $\pm$ signs correspond to (1.1).

When k is even, there is no difference between the dynamics of (1.1) with $\pm$ signs in front of the nonlinearity, because if u is a solution to (1.1) with $+$ sign, then $-u$ is a solution to (1.1) with $-$ sign, and vice versa. However, when k is odd, there is a big difference between the dynamics. (1.1) with a minus sign is referred to as *defocusing* equation and with a plus sign as *focusing* equation. The energy is positive definite, and a local well-posedness result in $H^{1/2}$ can be extended globally in the defocusing case. On the contrary, in the focusing case, Martel–Pilod [32] recently proved the existence of minimal blow-up solutions in the energy space for $k = 3$ on the real line (see also [25]). This indicates blow-up in the periodic case for focusing nonlinearities.

Equations (1.1) have mostly been studied on the real line, where the dispersive effects are stronger and the solutions are easier to handle. We digress for a moment to review the results on the real line to highlight key-points of the local well-posedness on the real line. Some transpire to the periodic case. We shall refer to the most recent results and the references therein.

The *Benjamin-Ono equation* ($k = 2$) is completely integrable and has been studied extensively. We first note that the High$\times$Low$\to$High-interaction

$$\partial_x (P_N u P_K u), \tag{1.2}$$

with P_L, $L \in 2^{\mathbb{N}_0}$, localizing to frequencies of size about L, leads to derivative loss. This makes it impossible to solve the Benjamin-Ono equation via Picard iteration (cf. [33, 26]). Via gauge transform (introduced by Tao in [49]), Ionescu–Kenig proved global well-posedness in $L^2(\mathbb{R})$ in [24] making use of Fourier restriction spaces; see also [36]. Ifrim–Tataru significantly simplified the proof by normal form transformations and relying only on Strichartz spaces in [22]. Recently, Talbut proved a priori estimates up to the scaling critical regularity in [47] via complete integrability as well on the real line as on the circle.

For the *modified Benjamin-Ono equation* ($k = 3$), Guo showed global well-posedness for complex-valued initial data in the energy space in [14]. He used smoothing effects on the real line instead of the gauge transform to overcome the derivative loss. Moreover, he proved a priori estimates up to $s > 1/4$ using short-time Fourier transform restriction. In this work becomes clear that for $k \geq 3$ the High$\times$High$\times$High$\to$High-interaction is also problematic below $H^{1/2}$:

$$P_N \partial_x (P_{N_1} u P_{N_2} u P_{N_3} u),$$

where $N \sim N_1 \sim N_2 \sim N_3$. In this case the resonance function (see Section 5) can become arbitrarily small. Furthermore, in [14] was shown how smoothing effects on the real line can replace the gauge transform for $k = 3$. For $k = 4$, Vento [51] proved local well-posedness in $H^{1/3}$, which turned out to be the limit of fixed point arguments, and reached the scaling critical regularity for $k \geq 5$; see also [2, 38, 37, 6].

There are fewer results for (1.1) with periodic boundary conditions. Molinet [34] adapted the gauge transform to the periodic Benjamin-Ono equation to prove global well-posedness in $L^2(\mathbb{T})$. Herr [21] showed that the Benjamin-Ono equation with periodic boundary conditions cannot be solved via Picard iteration directly. By complete integrability, Gérard–Kappeler–Topalov [12] proved global well-posedness up to the scaling critical regularity $s_c = -1/2$; see also [13]. For $k = 3$, Guo–Lin–Molinet [15] showed global well-posedness in the energy space by adapting the gauge transform and using Fourier restriction spaces. For $k = 3$, the second author proved existence and a priori estimates for $s > 1/4$ using short-time Fourier restriction, but not relying on gauge transforms, in [42]. For $k \geq 4$, Molinet–Ribaud [39] proved local well-posedness in $H^1(\mathbb{T})$ via gauge transforms and Strichartz estimates.

At last, we address ill-posedness issues. Firstly, we remark that Christ's argument [7], originally applied to the quadratic derivative nonlinear Schrödinger equation

$$i\partial_t u + \partial_{xx} u = iu\partial_x u, \quad (t,x) \in \mathbb{R} \times \mathbb{T},$$

shows norm inflation for complex-valued initial data at any Sobolev regularity.
Secondly, the High×Low-interaction (1.2) leads to the failure of the multilinear $X^{s,b}$-estimate[1]

$$\|\partial_x(u_1 \dots u_k)\|_{X^{s,-1/2}} \lesssim \prod_{i=1}^{k} \|u_i\|_{X^{s,1/2}},$$

also after removing trivial resonances.
This contrasts with the generalized KdV-equations on the circle

$$\partial_t u + \partial_x^3 u = \partial_x(u^k),$$

where Colliander *et al.* [11] showed the crucial multilinear $X^{s,b}$-estimate for $s = 1/2$ after renormalizing the nonlinearity.

We now state our main results. Our first result shows the local existence and a priori estimates for $s > 1/2$.

Theorem 1.1 (Local existence and a priori estimates). *Let $k \geq 4$ and $s > 1/2$. Then, for any $u_0 \in H^s(\mathbb{T})$ with $\|u_0\|_{H^s} \leq R$, there is $T = T(R) > 0$ such that a solution $u \in C_T H^s$ to (1.1) exists in the sense of distributions and the a priori estimate*

$$\sup_{t \in [0,T]} \|u(t)\|_{H^s} \lesssim_R \|u_0\|_{H^s} \tag{1.3}$$

holds true.

The result can be globalized in case of quartic nonlinearity.

Theorem 1.2 (Global existence for quartic nonlinearity). *In the quartic case $k = 4$, the statement of Theorem 1.1 holds under the assumption $\|u_0\|_{H^{1/2}} \leq R$ instead of $\|u_0\|_{H^s} \leq R$. If in addition R is sufficiently small, then we can find a global solution u.*

Our second result shows local well-posedness for $s \geq 3/4$. Since the difference equation satisfies less symmetries than the original equation, we can only prove the following weaker result for continuous dependence.

Theorem 1.3 (Local well-posedness). *Let $k \geq 4$ and $s \geq 3/4$. Then, we find (1.1) to be locally well-posed.*

[1] We set $b = 1/2$ for simplicity. Note that in this limiting case one actually has to consider a smaller function space.

Comments on Theorems 1.1, 1.2, and 1.3.

1. The restriction $s > 1/2$ in Theorem 1.1. In [42, Theorem 1.1], the second author proved the analog of Theorem 1.1 in case $k = 3$ with improved range $s > 1/4$. This improvement relies on the resonance relation. In $k = 3$, the zero set of the resonance function is nontrivial, but the (symmetrized) multiplier used in the energy estimates simultaneously vanishes. However, this does not necessarily hold for $k \geq 4$ and our restriction $s > 1/2$ comes from these resonant interactions. In the quartic case, we can show a priori estimates in the short-time function space $F^{1/2}$. In the quintic and higher cases, our argument merely gives a priori estimates in the Besov refinements $F_1^{1/2}$. As these miss the energy space, we cannot extend them globally. See Section 5 for more details.

2. The restriction $s \geq 3/4$ of Theorem 1.3. To prove local well-posedness, we need to consider the difference equation. When deriving energy estimates, the lack of symmetry does not allow for the same favorable cancellations, as for solutions. The restriction $s \geq 3/4$ again comes from the resonant interactions. More precisely, it comes from High×High×High×High interactions. See Section 6.3 for details.

3. Gauge transforms. Our method does not make use of gauge transforms in contrast to the works [51, 39]. Here we mean by gauge transforms the usual ones used in the previous literatures.[2]

The main reason is that the gauge transforms in our case do not behave as well as on the real line or for $k \in \{2, 3\}$. By a gauge transform, it is possible to delete Low×High interactions of the nonlinearity. However, we have to deal with error terms generated by the gauge transforms (e.g. when ∂_t falls onto the gauge transforms). On the real line, these can be estimated[3] using better linear estimates than on the torus. When $k \in \{2, 3\}$, these errors have better structure than those of $k \geq 4$; compare [15] of $k = 3$ and [39] of $k \geq 4$. When $k \geq 4$, ∂_t acting on the gauge transforms leads to problematic High×High→Low interactions, so we choose to avoid using gauge transforms. Hence, we have to deal with Low×High interactions in the original nonlinearity, and we choose to work with short-time Fourier restriction spaces to recover the derivative loss (cf. (1.2)).

Avoiding the use of gauge transforms, our method can be adopted to other models where the gauge transforms become very involved, if available at all. Examples include the dispersion-generalized models (cf. [44]). Moreover, for quadratic nonlinearities, an improvement of the energy method was proposed by Molinet–Vento [40]. This makes use of a precise comprehension of the resonance function and avoids gauge transforms, too.

4. Extension of the methods for $k \in \{2, 3\}$. We remark that our method can be easily modified to yield the same results (i.e., $s > \frac{1}{4}$ in Theorem 1.1, and $s > 1/2$ for $k = 2$ and $s \geq \frac{3}{4}$ for $k = 3$ in Theorem 1.3) for the cases $k \in \{2, 3\}$. However, there are stronger results for $k \in \{2, 3\}$ as mentioned above.

To prove Theorems 1.1-1.3, we use short-time Fourier restriction spaces as in [42]. We extend the approach of [42] to $k \geq 4$ in the present paper. In the following we elaborate on short–time Fourier restriction and the proof of the theorems. Since

[2]Any transform of the kind $u \mapsto e^{i\chi}u$ is a gauge transforms. For the current discussion we focus on the gauge transform introduced by Tao [49] and variants thereof.

[3]Or, one can use a fixed-time gauge transform as in Vento [51].

the body of literature on short-time Fourier restriction is already huge, we do not aim for an exhaustive review of references. We also refer to the references within the discussed literature and the PhD thesis of the second author [43].

The first key ingredient of our method is the use of short-time Fourier restriction spaces. This relies on the observation that the frequency dependent time localization allows to prove low regularity results for quasilinear dispersive equations. By *quasilinear* we mean that the equations cannot be solved by fixed point argument in L^2-based Sobolev spaces. In Euclidean space, early works on short-time Fourier restrictions are due to Koch–Tataru [27], Christ *et al.* [8], and Ionescu *et al.* [23]. Guo *et al.* observed in [17] that the frequency dependent time localization $T = T(N) = N^{-1}$ allows to overcome the derivative loss for High $\times$ Low $\to$ High–interaction (1.2) on the real line. This showed how to avoid the gauge transform and proved that inviscid limits recover solutions to the Benjamin-Ono equation.

The second author observed [42, 43] that this extends to periodic solutions. Although dispersive effects on tori are weaker, for time intervals of length $T = T(N) = N^{-1}$ Schrödinger wave packets cannot distinguish between Euclidean space and compact manifolds. Hence, Strichartz estimates on frequency dependent time intervals remain valid on compact manifolds. This was observed for linear estimates by Staffilani–Tataru [46] and Burq *et al.* [4] and for bilinear estimates by Moyua–Vega [41] and Hani [20].

The second key ingredient is to use cancellation effects, which allows to control low Sobolev norms, in a similar spirit with the *I-method* (cf. [10]). For differences of solutions, due to less symmetries, this is known as *normal form transformations* or *modified energies* as used by Kwon [31] and Kwak [29, 30]. We are not aware of previous instances of short-time Fourier restriction analysis combined with modified energies for quartic or higher nonlinearities. We hope that the arguments of the present work can be applied more generally. For instance, the model

$$\partial_t u + \mathcal{H}\partial_{xx} u = \partial_x (e^u)$$

seems to be in the scope of the methods of the paper.

We end the introduction by explaining key steps of the short-time analysis. Further details of the proofs are provided in Section 3. For solutions, the above program leads to the following set of estimates for solutions u in the short-time function space $F^s(T)$, $s > 1/2$, $\varepsilon = \varepsilon(s) > 0$:

$$\left\{ \begin{array}{rl} \|u\|_{F^s(T)} & \lesssim \|u\|_{E^s(T)} + \|\partial_x (u^k)\|_{N^s(T)} \\ \|\partial_x (u^k)\|_{N^s(T)} & \lesssim \|u\|^k_{F^s(T)} \\ \|u\|^2_{E^s(T)} & \lesssim \|u_0\|^2_{H^s} + \|u\|^{k+1}_{F^{s-\varepsilon}(T)} + \|u\|^{2k}_{F^{s-\varepsilon}(T)}. \end{array} \right. \tag{1.4}$$

This gives a priori estimates and existence of solutions by standard bootstrap and compactness arguments (cf. [16]) for *small* initial data u_0. To deal with large initial data, we rescale the torus yielding small initial data on tori with large period λ. Molinet introduced this argument in the context of short-time Fourier restriction in [35]; see also [42]. We omit the standard arguments and refer to the literature.

For the proof of Theorem 1.3 we firstly show Lipschitz continuity in the weaker norm $H^{-1/4}$ by the set of estimates for $v = u_1 - u_2$, u_i solutions to (1.1) in $F^s(T)$:

$$
\left\{
\begin{array}{ll}
\|v\|_{F^{-1/4}(T)} & \lesssim \|\partial_x(vu^{k-1})\|_{N^{-1/4}(T)} + \|v\|_{E^{-1/4}(T)} \\
\|\partial_x(vu^{k-1})\|_{N^{-1/4}(T)} & \lesssim \|v\|_{F^{-1/4}(T)}(\|u_1\|_{F^{3/4}(T)} + \|u_2\|_{F^{3/4}(T)})^{k-1} \\
\|v\|^2_{E^{-1/4}(T)} & \lesssim \|v(0)\|^2_{H^{-1/4}} \\
 & \quad +\|v\|^2_{F^{-1/4}(T)}(\|u_1\|_{F^{3/4}(T)} + \|u_2\|_{F^{3/4}(T)})^{k-1} \\
 & \quad +\|v\|^2_{F^{-1/4}(T)}(\|u_1\|_{F^{3/4}(T)} + \|u_2\|_{F^{3/4}(T)})^{2k-2}.
\end{array}
\right. \tag{1.5}
$$

In the above display u^m denotes a linear combination of $u_1^i u_2^{m-i}$, $i = 0, \dots, m$. This set of estimates yields Lipschitz continuous dependence in $H^{-1/4}$ for *small* initial data in $H^{3/4}$. We extend this to large initial data by rescaling the torus as above. To prove continuous dependence in $H^{3/4}$, we prove in addition the following set of estimates:

$$
\left\{
\begin{array}{ll}
\|v\|_{F^{3/4}(T)} & \lesssim \|\partial_x(vu^{k-1})\|_{F^{3/4}(T)} + \|v\|_{E^{3/4}(T)} \\
\|\partial_x(vu^{k-1})\|_{N^{3/4}(T)} & \lesssim \|v\|_{F^{3/4}(T)}(\|u_1\|_{F^{3/4}(T)} + \|u_2\|_{F^{3/4}(T)})^{k-1} \\
\|v\|^2_{E^{3/4}(T)} & \lesssim \|v(0)\|^2_{H^{3/4}} \\
 & \quad +\|v\|^2_{F^{3/4}(T)}(\|v\|_{F^{3/4}(T)} + \|u_2\|_{F^{3/4}(T)})^{k-1} \\
 & \quad +\|v\|_{F^{-1/4}(T)}\|v\|_{F^{3/4}(T)}\|u_2\|_{F^{7/4}(T)} \\
 & \qquad (\|v\|_{F^{3/4}(T)} + \|u_2\|_{F^{3/4}(T)})^{k-2} \\
 & \quad +\|v\|^2_{F^{3/4}(T)}(\|v\|_{F^{3/4}(T)} + \|u_2\|_{F^{3/4}(T)})^{2k-2} \\
 & \quad +\|v\|_{F^{-1/4}(T)}\|v\|_{F^{3/4}(T)}\|u_2\|_{F^{7/4}(T)} \\
 & \qquad (\|v\|_{F^{3/4}(T)} + \|u_2\|_{F^{3/4}(T)})^{2k-3}.
\end{array}
\right. \tag{1.6}
$$

We finish the proof of Theorem 1.3 for *small* initial data by a variant of the Bona–Smith method (cf. [3, 23, 44]).

The case of large initial data additionally requires rescaling to small initial data on tori with large periods as above. For this, we need to modify the Sobolev weights for the frequencies less than 1 (2.1).

Outline of the paper. In Section 2 we introduce notations, function spaces, and recall short–time (bilinear) Strichartz estimates. In Section 3 we conclude the proofs of the main results with the crucial short-time nonlinear and energy estimates at hand. In Section 4 we propagate the nonlinear interaction. In Section 5 we bound the energy norm for solutions and in Section 6 the energy norm for differences of solutions.

In the following we assume that k, the power of the nonlinearity, satisfies $k \geq 4$. Moreover, we suppress dependence on k for the implicit constants. The parameter λ (see Section 2.1) is always assumed to be $\lambda \in 2^{\mathbb{N}_0}$ and $\lambda \geq 1$. The dyadic frequencies range from $2^{\mathbb{Z}} \cap [\lambda^{-1}, \infty)$.

2. Function spaces, and linear and bilinear short-time estimates

2.1. Fourier analysis on $\lambda\mathbb{T}$. As mentioned above, the (local-in-time) large-data-theory is reduced to the small-data-theory via a scaling argument on circles. For this purpose, we need to develop our arguments working uniformly for functions with large periods. Set $\lambda\mathbb{T} = \mathbb{R}/(2\pi\lambda\mathbb{Z})$ with $\lambda \geq 1$. The Fourier transform of a function on $\lambda\mathbb{T}$ will have the domain $\mathbb{Z}/\lambda$.

Throughout this article, we assume

$$\lambda \in 2^{\mathbb{N}_0} \quad \text{so that} \quad \lambda \geq 1.$$

We will also assume that the dyadic frequencies always range in $[\lambda^{-1},\infty)$, e.g.

$$N, N_i, M, M_i, K, K_i, \dots \in 2^{\mathbb{Z}} \cap [\lambda^{-1},\infty).$$

We define the Lebesgue spaces $L^p(\lambda\mathbb{T})$ on $\lambda\mathbb{T}$ through the norm

$$\|f\|_{L^p_\lambda} = \left(\int_{\lambda\mathbb{T}} |f(x)|^p dx \right)^{1/p}, \quad 1 \leq p < \infty,$$

with the usual modification for $p=\infty$.

We turn to the Fourier transform on $\lambda\mathbb{T}$. As guideline for the conventions from below, we require that Plancherel's theorem remains valid; see [10]. The Fourier coefficients of $f \in L^1(\lambda\mathbb{T})$ are given by

$$\hat{f}(\xi) = \int_{\lambda\mathbb{T}} f(x) e^{-i\xi x} dx, \quad \xi \in \mathbb{Z}/\lambda$$

such that we have the Fourier inversion formula

$$f(x) = \frac{1}{\lambda} \sum_{\xi \in \mathbb{Z}/\lambda} \hat{f}(\xi) e^{ix\xi}$$

and Plancherel's theorem:

$$\|f\|^2_{L^2_\lambda} \sim \frac{1}{\lambda} \sum_{\xi \in \mathbb{Z}/\lambda} |\hat{f}(\xi)|^2.$$

The Littlewood-Paley projectors are defined as follows. Let $\chi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a smooth, compactly supported, radially decreasing function with $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. For dyadic[4] μ, set $\chi_\mu(x) = \chi_{\lambda^{-1}}(x) = \chi(\lambda x)$ if $\mu = \lambda^{-1}$ and $\chi_\mu(x) = \chi(x/2\mu) - \chi(x/\mu)$ otherwise. For the sequence of functions $\{\chi_{\lambda^{-1}}, \chi_{2\lambda^{-1}}, \chi_{4\lambda^{-1}}, \dots\}$, we denote the corresponding Fourier multipliers by $P_{\lambda^{-1}}$, $P_{2\lambda^{-1}}$, $\dots$. We refer to these as Littlewood-Paley projectors. We note that Bernstein's inequality holds as on $\mathbb{T}$ and $\mathbb{R}$, uniformly in λ.

The L^2-based Sobolev spaces $H^s(\lambda\mathbb{T})$ on $\lambda\mathbb{T}$ are defined through the norm

$$\|f\|^2_{H^s_\lambda} = \frac{1}{\lambda} \sum_{\xi \in \mathbb{Z}/\lambda} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2, \quad s \in \mathbb{R},$$

where we denoted $\langle \xi \rangle = (\xi^2+1)^{1/2}$. In view of Plancherel's theorem, $H^0(\lambda\mathbb{T}) = L^2(\lambda\mathbb{T})$. In terms of Littlewood-Paley projectors, we have

$$\|f\|^2_{H^s_\lambda} \approx \sum_{N<1} \|P_N f\|^2_{L^2_\lambda} + \sum_{N \geq 1} N^{2s} \|P_N f\|^2_{L^2_\lambda}.$$

However, as (1.1) for $k \geq 4$ is L^2-supercritical, the usual Sobolev spaces H^s does not work well with the scaling argument. The remedy is to consider a norm with a different weight on low frequencies. Set for $\overline{s} = (s_1, s_2) \in \mathbb{R}^2$:

$$\|f\|^2_{H^{\overline{s}}_\lambda} = \|f\|^2_{H^{s_1,s_2}_\lambda} = \frac{1}{\lambda} \sum_{\substack{\xi \in \mathbb{Z}/\lambda, \\ |\xi|<1}} (\lambda^{-1} + |\xi|)^{2s_1} |\hat{f}(\xi)|^2 + \frac{1}{\lambda} \sum_{\substack{\xi \in \mathbb{Z}/\lambda, \\ |\xi| \geq 1}} |\xi|^{2s_2} |\hat{f}(\xi)|^2. \tag{2.1}$$

[4] Recall that we assume the dyadic numbers to range in $2^{\mathbb{Z}} \cap [\lambda^{-1},\infty)$ throughout this article.

Note that $H^s(\lambda\mathbb{T}) = H^{0,s}(\lambda\mathbb{T})$. It is convenient to introduce the notations

$$N^{\overline{s}} = N^{s_1,s_2} = \begin{cases} N^{s_1} & \text{if } N < 1, \\ N^{s_2} & \text{if } N \geq 1, \end{cases}$$

and

$$\overline{s} + s' = (s_1, s_2 + s'), \qquad s' \in \mathbb{R},$$
$$cs' = (cs_1, cs_2), \qquad c \in \mathbb{R}.$$

so that

$$\|f\|^2_{H^{\overline{s}}_\lambda} \approx \sum_{N<1} N^{2s_1} \|P_N f\|^2_{L^2_\lambda} + \sum_{N\geq 1} N^{2s_2} \|P_N f\|^2_{L^2_\lambda} \approx \sum_N N^{2\overline{s}} \|P_N f\|^2_{L^2}.$$

For the scaling argument, we will use $\overline{s} = (s_1, s_2)$ such that $s_c < s_1 \leq s_2$, i.e., the *subcritical* regularities. Let $s_2 = s$ as in our main theorems. Any large data $u_0 \in H^s(\mathbb{T})$ is reduced to a small data by

$$\lambda^{-\frac{1}{k-1}} \|u_0(\lambda^{-1}\cdot)\|_{H^{\overline{s}}_\lambda} \to 0 \text{ as } \lambda \to \infty$$

More precisely, we have

$$\lambda^{s_c - s_2} \|u_0\|_{H^s} \lesssim \|\lambda^{-\frac{1}{k-1}} u_0(\lambda^{-1}\cdot)\|_{H^{\overline{s}}_\lambda} \lesssim \lambda^{s_c - s_1} \|u_0\|_{H^s}. \tag{2.2}$$

2.2. **U^p-/V^p-function spaces.** We consider short-time U^p-/V^p-function spaces as in [45]. Adapted U^p-/V^p-function spaces to treat nonlinear dispersive equations were introduced in the work [18, 19]. There are several reasons for this choice. Firstly, (bi-)linear estimates for linear solutions transfer well to these spaces. Secondly, the duality estimates are also available. Lastly, these spaces behave nicely with sharp time localizations. We shall be brief and for details refer to [45].

For a time interval I, we set

$$\|u\|_{U^p_{BO}(I)_\lambda} = \|e^{-t\mathcal{H}\partial_{xx}} u\|_{U^p(I;L^2_\lambda)},$$
$$\|v\|_{V^p_{BO}(I)_\lambda} = \|e^{-t\mathcal{H}\partial_{xx}} v\|_{V^p(I;L^2_\lambda)},$$
$$\|f\|_{DU^p_{BO}(I)_\lambda} = \|e^{-t\mathcal{H}\partial_{xx}} v\|_{DU^p(I;L^2_\lambda)}.$$

Let $T \in (0,1]$ and $\overline{s} = (s_1, s_2)$ be given. We define the $F^{\overline{s}}_\lambda(T)$-norm (for solutions) and the $N^{\overline{s}}_\lambda(T)$-norm (for nonlinearities) by

$$\|u\|^2_{F^{\overline{s}}_\lambda(T)} = \sum_{\substack{\lambda^{-1}\leq N\leq 1, \\ N\in 2^{\mathbb{Z}}}} N^{2s_1} \|P_N u\|^2_{U^2_{BO}([0,T])_\lambda} + \sum_{N\in 2^{\mathbb{N}_0}} N^{2s_2} \sup_{\substack{I\subseteq[0,T], \\ |I|=N^{-1}}} \|P_N u\|^2_{U^2_{BO}(I)_\lambda},$$

$$\|v\|^2_{N^{\overline{s}}_\lambda(T)} = \sum_{\substack{\lambda^{-1}\leq N\leq 1, \\ N\in 2^{\mathbb{Z}}}} N^{2s_1} \|P_N u\|^2_{DU^2_{BO}([0,T])_\lambda} + \sum_{N\in 2^{\mathbb{N}_0}} N^{2s_2} \sup_{\substack{I\subseteq[0,T], \\ |I|=N^{-1}}} \|P_N u\|^2_{DU^2_{BO}(I)_\lambda}.$$

As in [45], if $T \leq N^{-1}$, read (likewise for V^p and DU^p):

$$\sup_{\substack{I\subseteq[0,T], \\ |I|=N^{-1}}} \|f\|_{U^p_{BO}(I)} = \|f\|_{U^p_{BO}([0,T])},$$

We define the energy norm $E^{\overline{s}}_\lambda(T)$ (for solutions) by

$$\|u\|^2_{E^{\overline{s}}_\lambda(T)} = \|P_{\leq 1} u(0)\|^2_{H^{\overline{s}}_\lambda} + \sum_{N>1} N^{2s_2} \sup_{t\in[0,T]} \|P_N u(t)\|^2_{L^2_\lambda}.$$

A consequence of the definition of the function spaces is the following linear estimate; see [9] for the proof in a different context.

Lemma 2.1 (Linear estimate)**.** *Let $T \in (0,1]$, $\overline{s} = (s_1, s_2)$, and let u be a smooth solution to*

$$\partial_t u + \mathcal{H}\partial_{xx} u = v \text{ on } (-T,T) \times \lambda\mathbb{T}.$$

Then, we find the following estimate to hold:

$$\|u\|_{F^{\overline{s}}_\lambda(T)} \lesssim \|u\|_{E^{\overline{s}}_\lambda(T)} + \|v\|_{N^{\overline{s}}_\lambda(T)}.$$

2.3. **Short-time linear and bilinear Strichartz estimates.** Here we record short-time linear and bilinear Strichartz estimates.

Proposition 2.2 (Short-time linear Strichartz estimates)**.** *Let $p, q \in [2,\infty]$ be such that $\frac{2}{q} + \frac{1}{p} = \frac{1}{2}$. Then, we find the following estimate to hold:*[5]

$$\|P_N e^{-t\mathcal{H}\partial_{xx}} f\|_{L^q_t([0,\lambda N^{-1}], L^p_\lambda)} \lesssim \|P_N f\|_{L^2_\lambda}.$$

The analogous linear estimates for the Schrödinger propagator $e^{\pm it\partial_{xx}}$ were proved in [5, 46]. Proposition 2.2 for $\lambda = 1$ follows after projecting to positive and negative frequencies due to $P_\pm e^{-t\mathcal{H}\partial_{xx}} = e^{\pm it\partial_{xx}} P_\pm$, where $P_\pm$ is the Fourier multiplier operator $\mathbf{1}_{\pm|\xi|\geq 0}$. The general case $\lambda \geq 1$ follows from the scaling argument. We omit the proof.

Proposition 2.3 (Short-time bilinear Strichartz estimates)**.** *Let $r \geq \lambda^{-1}$ and $N \geq 1$. Let $\eta_1, \eta_2 \in \mathbb{R}$ be such that $||\xi_1| - |\xi_2|| \gtrsim N > 0$ for $\xi_i \in B(\eta_i, r)$. Then, we find the following estimate to hold for $f_i \in L^2(\lambda\mathbb{T})$ with supp$(\hat{f}_i) \subseteq B(\eta_i, r)$:*

$$\|e^{-t\mathcal{H}\partial_{xx}} f_1 e^{-t\mathcal{H}\partial_{xx}} f_2\|_{L^2_t([0,\lambda N^{-1}], L^2_\lambda)} \lesssim N^{-1/2} \|f_1\|_{L^2_\lambda} \|f_2\|_{L^2_\lambda}. \tag{2.3}$$

Remark 2.4. The frequency separation in its *magnitude* is required.

For the Schrödinger case, it is proved in [45] that Proposition 2.3 holds under the (weaker) assumption $|\xi_1 - \xi_2| \gtrsim N > 0$ instead of $||\xi_1| - |\xi_2|| \gtrsim N > 0$. This is the transversality assumption $|\partial_\xi h(\xi_1) - \partial_\xi h(\xi_2)| \gtrsim N$, where $h(\xi) = |\xi|^2$ is the dispersion relation for the linear Schrödinger flow. For the Benjamin-Ono case, this becomes $||\xi_1| - |\xi_2|| \gtrsim N$ as the dispersion relation is $h(\xi) = |\xi|\xi$. We omit the proof of Proposition 2.3 and refer to [45].

For Schrödinger equations on compact manifolds, such short-time bilinear estimates for dyadically separated frequencies were discussed in [41, 20].

In most cases, we apply Proposition 2.3 for dyadic frequency interactions. Moreover, we will restrict the time interval $[0, \lambda N^{-1}]$ to $[0, N^{-1}]$. One consequence of this restriction is the gain when a low frequency wave interacts.

Corollary 2.5 (Bilinear Strichartz for dyadic frequency interactions)**.** *Let $N_1 \gtrsim 1$. Let $u_i(t,x) = [e^{-t\mathcal{H}\partial_{xx}} f_i](x)$ for $f_i \in L^2_\lambda$, where $i \in \{1, \ldots, 4\}$.*

- *(High×Low) Assume $N_1 \gg N_2$. Then, we find the following estimate to hold:*

$$\|P_{N_1} u_1 P_{N_2} u_2\|_{L^2_t([0,N_1^{-1}], L^2_\lambda)} \lesssim N_1^{-1/2} N_2^{1/2,0} \|f_1\|_{L^2_\lambda} \|f_2\|_{L^2_\lambda}.$$

[5]Note that λN^{-1} is the maximal time scale, on which a wave packet with frequency N cannot distinguish the domains $\lambda\mathbb{T}$ and $\mathbb{R}$.

- *(High×High×High→Low) Assume $N_1 \sim N_2 \sim N_3 \gg N_4$. Then, we find the following estimate to hold:*
$$\int_0^{N_1^{-1}} \int_{\lambda \mathbb{T}} \prod_{i=1}^4 P_{N_i} u_i(s,x)\, dxds \lesssim N_1^{-1} N_4^{1/2,0} \prod_{i=1}^4 \|f_i\|_{L^2_\lambda}.$$

Proof. The first estimate for $N_2 \gtrsim 1$ follows from Proposition 2.3 with almost orthogonality. For $N_2 \ll 1$, we combine Hölder's and Bernstein's inequality to find:
$$\begin{aligned} &\|P_{N_1} u_1 P_{N_2} u_2\|_{L^2_t([0,N_1^{-1}],L^2_\lambda)} \\ &\lesssim N_1^{-1/2} \|P_{N_1} u_1 P_{N_2} u_2\|_{L^\infty_t([0,N_1^{-1}],L^2_\lambda)} \\ &\lesssim N_1^{-1/2} \|P_{N_1} u_1\|_{L^\infty_t([0,N_1^{-1}],L^2_\lambda)} \|P_{N_2} u_2\|_{L^\infty_t([0,N_1^{-1}],L^\infty_\lambda)} \\ &\lesssim N_1^{-1/2} N_2^{1/2} \|f_1\|_{L^2_\lambda} \|f_2\|_{L^2_\lambda}. \end{aligned}$$

We turn to the second estimate. Let c denote a small constant (depending on implicit constants involved in $N_1 \sim N_2 \sim N_3 \gg N_4$). Let I_j denote consecutive intervals of length cN_1 and write $P_{N_i} = \sum_{j=1}^M P_{I_j} P_{N_i}$, where $M \lesssim 1$ and P_{I_j} is a (smooth) Littlewood-Paley projector onto the frequency interval I_j. Due to $M \lesssim 1$, it is enough to show
$$\begin{aligned} &\int_0^{N_1^{-1}} \int_{\lambda \mathbb{T}} P_{I_1} u_1(s,x) P_{I_2} u_2(s,x) P_{I_3} u_3(s,x) P_{N_4} u_4(s,x)\, dxds \\ &\lesssim N_1^{-1} N_4^{1/2,0} \prod_{i=1}^4 \|f_i\|_{L^2_\lambda} \end{aligned} \tag{2.4}$$
because the claim follows from crudely summing over the I_i.

We claim that for any nontrivial interaction in (2.4), there are $i,j \in \{1,2,3\}$, $i \neq j$, such that $||\xi_i| - |\xi_j|| \gtrsim N$ for $\xi_i \in I_i$ and $\xi_j \in I_j$. At first, if (I_1 and I_2) or (I_1 and $-I_2$) are not neighboring intervals, we find $||\xi_1| - |\xi_2|| \gtrsim N$ for $\xi_i \in I_i$. Now suppose that I_1 and I_2 are neighboring intervals. Due to otherwise impossible frequency interaction, we find $||\xi_2| - |\xi_3|| \gtrsim N$ for $\xi_2 \in I_2$ and $\xi_3 \in I_3$. The case when I_1 and $-I_2$ are neighboring intervals is impossible: it would imply $|\xi_1 + \xi_2| \leq 2cN$, contradicting $|\xi_3 + \xi_4| \gtrsim N$, if c was chosen sufficiently small.

Having settled the claim, we finish the proof. Say $(i,j) = (1,2)$ is a pair of the indices as in the claim. We apply a bilinear Strichartz estimates (Proposition 2.3) and the first estimate to find
$$\begin{aligned} &\left| \int_0^{N_1^{-1}} \int_{\mathbb{T}} P_{I_1} u_1 P_{I_2} u_2 P_{I_3} u_3 P_{N_4} u_4 dxds \right| \\ &\leq \|P_{I_1} u_1 P_{I_2} u_2\|_{L^2_t([0,N_1^{-1}],L^2_\lambda)} \|P_{I_3} u_3 P_{N_4} u_4\|_{L^2_t([0,N_1^{-1}],L^2_\lambda)} \\ &\lesssim N_1^{-1/2} \|P_{I_1} f_1\|_{L^2_\lambda} \|P_{I_2} f_2\|_{L^2_\lambda} N_1^{-1/2} N_4^{1/2,0} \|P_{I_3} f_3\|_{L^2_\lambda} \|P_{N_4} f_4\|_{L^2_\lambda} \\ &\lesssim N_1^{-1} N_4^{1/2,0} \prod_{i=1}^4 \|P_{N_i} f_i\|_{L^2_\lambda}. \end{aligned}$$
This completes the proof. □

We will not treat the case $N_1 \lesssim 1$ with short-time bilinear Strichartz estimates because Hölder's and Bernstein's inequality suffice.

We record the corresponding estimates for U^p_{BO}-/V^p_{BO}-functions.

Proposition 2.6. *Let $N_1 \gtrsim 1$, $N_1 \gg N_2$ and $|I| = N_1^{-1}$. Then, we find the following estimates to hold:*

- *(Short-time U^2_{BO}-estimate)*

$$\begin{aligned} &\|P_{N_1}u_1 P_{N_2}u_2\|_{L^2_t(I,L^2_\lambda)} \\ &\lesssim N_1^{-1/2} N_2^{1/2,0} \|P_{N_1}u_1\|_{U^2_{BO}(I)_\lambda} \|P_{N_2}u_2\|_{U^2_{BO}(I)_\lambda}. \end{aligned} \tag{2.5}$$

- *(Short-time V^2_{BO}-estimate)*

$$\begin{aligned} &\|P_{N_1}u_1 P_{N_2}u_2\|_{L^2_t(I,L^2_\lambda)} \\ &\lesssim \log\langle \frac{N_1}{N_2}\rangle N_1^{-1/2} N_2^{1/2,0} \|P_{N_1}u_1\|_{V^2_{BO}(I)_\lambda} \|P_{N_2}u_2\|_{V^2_{BO}(I)_\lambda}. \end{aligned} \tag{2.6}$$

- *(Linear $L^6_{t,x}$ and $L^8_t L^4_x$ estimates)*

$$\|P_{N_2}u\|_{L^6_t(I,L^6_\lambda)} + \|P_{N_2}u\|_{L^8_t(I,L^4_\lambda)} \lesssim \|P_{N_2}u\|_{V^2_{BO}(I)_\lambda}. \tag{2.7}$$

Proof. (2.5) is an immediate consequence of the atomic representation of U^p-- functions; see [18, Section 2] for details. In a similar spirit, (2.7) follows from $\|P_{N_2}u\|_{L^6_t(I,L^6_\lambda)} \lesssim \|P_{N_2}u\|_{U^6_{BO}(I)_\lambda} \lesssim \|P_{N_2}u\|_{V^2_{BO}(I)_\lambda}$ and likewise for the $L^8_t L^4_x$-norm. For the proof of (2.6) we note that

$$\begin{aligned} \|P_{N_1}u_1 P_{N_2}u_2\|_{L^2_t(I,L^2_\lambda)} &\leq \|P_{N_1}u_1\|_{L^4_t(I,L^4_\lambda)} \|P_{N_2}u_2\|_{L^4_t(I,L^4_\lambda)} \\ &\lesssim N_1^{-1/4} \|P_{N_1}u_1\|_{L^8_t(I,L^4_\lambda)} \|P_{N_2}u_2\|_{L^8_t(I,L^4_\lambda)} \\ &\lesssim N_1^{-1/4} \|P_{N_1}u_1\|_{U^8_{BO}(I)_\lambda} \|P_{N_2}u_2\|_{U^8_{BO}(I)_\lambda} \\ &\lesssim N_1^{-1/4} \|P_{N_1}u_1\|_{U^8_{BO}(I)_\lambda} \|P_{N_2}u_2\|_{U^8_{BO}(I)_\lambda}, \end{aligned}$$

by Hölder's inequality, Proposition 2.3, and the transfer principle. Interpolating (cf. [18, Proposition 2.20, p. 930]) with (2.5) yields (2.6). ☐

3. Proof of main results

In this section we show how to conclude Theorems 1.1 - 1.3 with short-time nonlinear and energy estimates at hand. Many of the below arguments are standard in the literature, where short-time Fourier restriction is used (cf. [23]).

We note that existence of solutions for (rough) $H^s(\mathbb{T})$-data follows from the smoothing effect in the energy estimate for smooth solutions and a compactness argument. We refer to the literature [16] for details, and focus on a priori estimates for smooth solutions subsequently.

Local-in-time a priori estimates and existence for small initial data in $H^s(\mathbb{T})$, $s > 1/2$:

Let $u \in C^\infty_{t,x}([0,T_+),\mathbb{T})$ be a smooth solution to (1.1) with maximal forward-in-time lifespan $[0,T_+)$ and $\|u_0\|_{H^s} \leq \epsilon$ for sufficiently small $\epsilon > 0$. Note that the existence and uniqueness of smooth solutions are ensured by the classical energy method (cf. [1]). Gathering the linear short-time energy estimate (Lemma 2.1), the nonlinear short-time estimate (Proposition 4.1), and the energy estimate for solutions (Proposition 5.1) for $\lambda = 1$, we find the set of estimates (1.4) for $T \in (0,1] \cap (0,T_+)$.

Set $X(T) = \|\partial_x(u^k)\|_{N^s(T)} + \|u\|_{E^s(T)}$ for $T \in (0,1] \cap (0,T_+)$. Thus (1.4) reads

$$X(T) \lesssim \|u_0\|_{H^s} + X(T)^{\frac{k+1}{2}} + X(T)^k.$$

Since the function $T \mapsto X(T)$ is continuous, non-decreasing, and satisfies

$$\lim_{T\to 0} X(T) \lesssim \|u_0\|_{H^s}$$

(cf. [27, Section 1]), a continuity argument gives

$$X(T) \lesssim \|u_0\|_{H^s} \lesssim \epsilon$$

uniformly for $T \in (0,1] \cap (0,T_+)$, provided that ϵ is sufficiently small.

Note the following persistence property: For $1/2 < s \leq s'$ we find the following estimates in addition: (see for instance Proposition 5.1)

$$(3.1)\quad \begin{cases} \|u\|_{F^{s'}(T)} & \lesssim \|u\|_{E^{s'}(T)} + \|\partial_x(u^k)\|_{N^{s'}(T)}, \\ \|\partial_x(u^k)\|_{N^{s'}(T)} & \lesssim \|u\|_{F^{s'}(T)} \|u\|_{F^s(T)}^{k-1}, \\ \|u\|_{E^{s'}(T)}^2 & \lesssim \|u_0\|_{H^{s'}}^2 + \|u\|_{F^{s'}(T)}^2 \|u\|_{F^s(T)}^{k-1} + \|u\|_{F^{s'}(T)}^2 \|u\|_{F^s(T)}^{2k-2}. \end{cases}$$

This gives

$$\|u\|_{F^{s'}(T)} \lesssim \|u_0\|_{H^{s'}} + \|u\|_{F^{s'}(T)} \|u\|_{F^s(T)}^{\frac{k-1}{2}} + \|u\|_{F^{s'}(T)} \|u\|_{F^s(T)}^{k-1}.$$

Since we know that $\|u\|_{F^s(T)} \lesssim X(T) \lesssim \varepsilon$ is small, we find

$$\|u\|_{F^{s'}(T)} \lesssim \|u_0\|_{H^{s'}},$$

uniformly for $T \in (0,1] \cap (0,T_+)$. Recall the blow-up alternative (cf. [1]): either $\lim_{t\uparrow T_+} \|u(t)\|_{H^3} = \infty$, or u can be extended beyond T_+. This points out that $T_+ > 1$ so we can take $T = 1$ to get $\|u\|_{F^s(1)} \lesssim \|u_0\|_{H^s}$.

Local-in-time a priori estimates and existence of solutions for arbitrary initial data in $H^s(\mathbb{T})$, $s > 1/2$:

To extend the previous claims to large data $u_0 \in H^s(\mathbb{T})$, we rescale: $u_{0,\lambda}(x) := \lambda^{-\frac{1}{k-1}} u_0(\lambda^{-1}x)$. Let $s_c < s_1 < 1/2$ and set $\overline{s} = (s_1, s)$. For sufficiently small $\epsilon > 0$, we choose $\lambda = \lambda(\|u_0\|_{H^s}, \epsilon) \geq 1$ such that $\|u_{0,\lambda}\|_{H^{\overline{s}}_\lambda} \leq \varepsilon$ by (2.2). For the corresponding solutions u_λ on $\lambda\mathbb{T}$, we find

$$(3.2)\quad \begin{cases} \|u_\lambda\|_{F^{\overline{s}}_\lambda(T)} & \lesssim \|u_\lambda\|_{E^{\overline{s}}_\lambda(T)} + \|\partial_x(u_\lambda^k)\|_{N^{\overline{s}}_\lambda(T)}, \\ \|\partial_x(u_\lambda^k)\|_{N^{\overline{s}}_\lambda(T)} & \lesssim \|u_\lambda\|_{F^{\overline{s}}_\lambda(T)}^k, \\ \|u\|_{E^{\overline{s}}_\lambda(T)}^2 & \lesssim \|u_0\|_{H^{\overline{s}}_\lambda}^2 + \|u\|_{F^{\overline{s}}_\lambda(T)}^{k+1} + \|u\|_{F^{\overline{s}}_\lambda(T)}^{2k}. \end{cases}$$

We recall that the implicit constants can be chosen independently of λ. The reason is that the underlying Strichartz estimates and pointwise estimates are independent of λ. This yields like above a priori estimates for u_λ on $[0,1]$

$$\sup_{t\in[0,1]} \|u_\lambda(t)\|_{H^{\overline{s}}_\lambda} \lesssim \|u_{0,\lambda}\|_{H^{\overline{s}}_\lambda}$$

with persistence of regularity. Scaling back using (2.2) gives

$$\sup_{t\in[0,\lambda^{-2}]} \|u(t)\|_{H^s} \lesssim \lambda^{s-s_1} \|u_0\|_{H^s}.$$

This proves Theorem 1.1.

Improved local-in-time a priori estimates and existence of solutions in the quartic case:

Here, it suffices to show that we can choose T depending only on $\|u_0\|_{H^{1/2}}$. Let $s_c < s_1 < 1/2$ and set $\overline{s} = (s_1, 1/2)$. Choose $\lambda = \lambda(\|u_0\|_{H^{1/2}}, \epsilon) \geq 1$ such that $\|u_{0,\lambda}\|_{H^{\overline{s}}_\lambda} \leq \epsilon$. For the quartic case, we have (3.2) uniformly in λ, even for $\overline{s} = (s_1, 1/2)$. Together with persistence of regularity and the blow-up alternative, we have a priori estimates for u_λ on $[0,1]$,

$$\sup_{t \in [0,1]} \|u_\lambda(t)\|_{H^{s_1,s}_\lambda} \lesssim \|u_{0,\lambda}\|_{H^{s_1,s}_\lambda}$$

for any $s > 1/2$. Scaling back, we have

$$\sup_{t \in [0,\lambda^{-2}]} \|u(t)\|_{H^s} \lesssim \lambda^{s-s_1} \|u_0\|_{H^s}.$$

Since λ only depends on $\|u_0\|_{H^{1/2}}$, the desired local-in-time a priori estimate is proved.

Global existence for $H^s(\mathbb{T})$-solutions, $s > 1/2$, with small $H^{1/2}(\mathbb{T})$ initial data, in the quartic case:

Again, we focus on a priori estimates for smooth solutions $u \in C^\infty_{t,x}([0,T_+), \mathbb{T})$ with $\|u_0\|_{H^{1/2}} \leq \epsilon$ for sufficiently small $\epsilon > 0$. Recall that the energy and mass

$$E[u] = \int_{\mathbb{T}} \frac{u \mathcal{H} \partial_x u}{2} dx + \int_{\mathbb{T}} \frac{u^5}{5} dx, \quad M[u] = \int_{\mathbb{T}} u^2 dx$$

are conserved quantities. The potential energy is small relative to the sum of kinetic energy and mass:

$$\Big| \int_{\mathbb{T}} \frac{u^5}{5} dx \Big| \lesssim \|u\|^5_{H^{1/2}} \ll \|u\|^2_{H^{1/2}} \sim \int_{\mathbb{T}} \frac{u \mathcal{H} \partial_x u}{2} dx + \int u^2 dx,$$

provided that $\|u\|_{H^{1/2}}$ is small. Therefore, the conservation of mass and energy yields $\sup_{t \in [0,T_+)} \|u(t)\|_{H^{1/2}} \lesssim \epsilon$, provided that ϵ is sufficiently small. Together with persistence of regularity and the blow-up alternative, we find $T_+ > 1$ and $\sup_{t \in [0,1]} \|u(t)\|_{H^{s'}} \lesssim \|u_0\|_{H^{s'}}$ for any $s' > 1/2$. However, the $H^{1/2}$-norm of $u(t)$ is uniformly bounded, the argument can be iterated. Thus $T_+ = \infty$ and iterating $\|u(n+1)\|_{H^{s'}} \lesssim \|u(n)\|_{H^{s'}}$ gives

$$\|u(t)\|_{H^{s'}} \lesssim e^{Ct} \|u_0\|_{H^{s'}}$$

for any $s' > 1/2$. This yields global existence in the quartic case. This concludes the proof of Theorem 1.2.

We turn to the proof of Theorem 1.3. We restrict to the lowest regularity $s = 3/4$.

Lipschitz continuous dependence in $H^{-1/4}$:

For small initial data in $H^{3/4}$, say $\|u_i(0)\|_{H^{3/4}} \leq \varepsilon \ll 1$ for $i = 1, 2$, we can proceed as follows. Let $v = u_1 - u_2$ denote the difference of solutions u_i to (1.1) in $F^{3/4}(1)$. v is governed by (cf. (6.3))

$$\partial_t v + \mathcal{H} \partial_{xx} v = \partial_x (v(u_1^{k-1} + u_1^{k-2} u_2 + \ldots + u_2^{k-1})).$$

Lemma 2.1, Proposition 4.1, and Proposition 6.1 yield the set of estimates (1.5). Due to smallness of $\|u_i\|_{F^{3/4}(1)} \lesssim \epsilon$ (by the a priori estimates for solutions), the set of estimates (1.5) yields

$$\|v\|_{F^{-1/4}(1)} \lesssim \|v(0)\|_{H^{-1/4}}.$$

The previous argument can be extended to the large data case paralleling the arguments for the a priori estimates. Let $s_c < s_1 < 1/2$ and set $\overline{s} = (s_1, 3/4)$.

Choose λ sufficiently large such that $\|u_{i,\lambda}(0)\|_{H^{\overline{s}}_\lambda} \leq \epsilon \ll 1$. We find the following set of estimates independent of λ:[6]

$$\begin{cases} \|v_\lambda\|_{F^{\overline{s}-1}_\lambda} & \lesssim \|v_\lambda\|_{E^{\overline{s}-1}_\lambda} + \|\partial_x(v_\lambda u_\lambda^{k-1})\|_{N^{\overline{s}-1}_\lambda} \\ \|\partial_x(v_\lambda u_\lambda^{k-1})\|_{N^{\overline{s}-1}_\lambda} & \lesssim \|v_\lambda\|_{F^{\overline{s}-1}_\lambda}(\|u_{1,\lambda}\|_{F^{\overline{s}}_\lambda} + \|u_{2,\lambda}\|_{F^{\overline{s}}_\lambda})^{k-1}, \\ \|v_\lambda\|^2_{E^{\overline{s}-1}_\lambda} & \lesssim \|v_\lambda(0)\|^2_{H^{\overline{s}-1}_\lambda} + \|v_\lambda\|^2_{F^{\overline{s}-1}_\lambda}(\|u_{1,\lambda}\|_{F^{\overline{s}}_\lambda} + \|u_{2,\lambda}\|_{F^{\overline{s}}_\lambda})^{2k-2}. \end{cases}$$

By the a priori estimates for solutions, $\|u_{i,\lambda}\|_{F^{\overline{s}}_\lambda(1)} \lesssim \epsilon$. This smallness yields

$$\sup_{t\in[0,1]} \|v_\lambda(t)\|_{H^{\overline{s}-1}_\lambda} \lesssim \|v_\lambda\|_{F^{\overline{s}-1}_\lambda(1)} \lesssim \|v_\lambda(0)\|_{H^{\overline{s}-1}_\lambda}.$$

Scaling back yields

$$\sup_{t\in[0,\lambda^{-2}]} \|v(t)\|_{H^{-1/4}} \lesssim \lambda^{s_1+\frac{1}{4}} \|v(0)\|_{H^{-1/4}}.$$

Since λ only depends on the $H^{3/4}$-norms of initial data, this yields Lipschitz continuous dependence in $H^{-1/4}$.

Continuous dependence in $H^{3/4}$:

Lastly, we prove continuous dependence with the Bona–Smith approximation. Let $s_c < s_1 < 1/2$ and set $\overline{s} = (s_1, 3/4)$. Lemma 2.1, Proposition 4.1, and Proposition 6.1 read

$$\begin{cases} \|v\|_{F^{\overline{s}}_\lambda} & \lesssim \|\partial_x(vu^{k-1})\|_{N^{\overline{s}}_\lambda} + \|v\|_{E^{\overline{s}}_\lambda}, \\ \|\partial_x(vu^{k-1})\|_{N^{\overline{s}}_\lambda} & \lesssim \|v\|_{F^{\overline{s}}_\lambda}(\|u_1\|_{F^{\overline{s}}_\lambda} + \|u_2\|_{F^{\overline{s}}_\lambda})^{k-1}, \\ \|v\|^2_{E^{\overline{s}}_\lambda} & \lesssim \|v(0)\|^2_{H^{\overline{s}}_\lambda} + \|v\|^2_{F^{\overline{s}}_\lambda}(\|v\|_{F^{\overline{s}}_\lambda} + \|u_2\|_{F^{\overline{s}}_\lambda})^{k-1} \\ & \quad + \|v\|_{F^{\overline{s}}_\lambda}\|v\|_{F^{\overline{s}-1}_\lambda}\|u_2\|_{F^{\overline{s}+1}_\lambda}(\|v\|_{F^{\overline{s}}_\lambda} + \|u_2\|_{F^{\overline{s}}_\lambda})^{k-2} \\ & \quad + \|v\|^2_{F^{\overline{s}}_\lambda}(\|v\|_{F^{\overline{s}}_\lambda} + \|u_2\|_{F^{\overline{s}}_\lambda})^{2k-2} \\ & \quad + \|v\|_{F^{\overline{s}}_\lambda}\|v\|_{F^{\overline{s}-1}_\lambda}\|u_2\|_{F^{\overline{s}+1}_\lambda}(\|v\|_{F^{\overline{s}}_\lambda} + \|u_2\|_{F^{\overline{s}}_\lambda})^{2k-3}. \end{cases}$$

Consider a $H^{\overline{s}}_\lambda$-Cauchy sequence of smooth initial data $u_0^n \in C^\infty(\lambda\mathbb{T})$ with $\|u_0^n\|_{H^{\overline{s}}_\lambda} < \epsilon$. Note that this smallness can be assumed thanks to the scaling argument. Due to the smallness of $H^{\overline{s}}_\lambda$-norm, we can define $S_1^\infty(u_0^n)$ by the classical solution to (1.1) with initial data u_0^n, on the time interval $[0,1]$. By the density argument, it suffices to show that $S_1^\infty(u_0^n)$ is a Cauchy sequence in $C([0,1], H^{\overline{s}}_\lambda)$.

To show this, we start from writing

$$\begin{aligned} S_1^\infty(u_0^n) - S_1^\infty(u_0^m) = (S_1^\infty(u_0^n) - S_1^\infty(u^n_{0,\leq N})) + (S_1^\infty(u^n_{0,\leq N}) - S_1^\infty(u^m_{0,\leq N})) \\ + (S_1^\infty(u^m_{0,\leq N}) - S_1^\infty(u_0^m)), \end{aligned} \tag{3.3}$$

where $f_{\leq N}$ denotes $P_{\leq N} f$ and $N > 1$ will be chosen large. For the first (and third) term of (3.3), temporarily denoting $v = S_1^\infty(u_0^n) - S_1^\infty(u^n_{0,\leq N})$, we use the above set of estimates to get

$$\|v\|^2_{F^{\overline{s}}_\lambda(1)} \lesssim \|u^n_{0,>N}\|^2_{H^{\overline{s}}_\lambda} + \|v\|^2_{F^{\overline{s}}_\lambda(1)}\epsilon^{k-1} + \|v\|_{F^{\overline{s}}_\lambda(1)}\|v\|_{F^{\overline{s}-1}_\lambda(1)} N\epsilon^{k-1},$$

where the factor N comes from the persistence estimate

$$\|S_1^\infty(u^n_{0,\leq N})\|_{F^{\overline{s}+1}_\lambda(1)} \lesssim \|u^n_{0,\leq N}\|_{H^{\overline{s}+1}_\lambda} \lesssim N\|u^n_{0,\leq N}\|_{H^{\overline{s}}_\lambda} \lesssim N\epsilon.$$

The above loss of N is balanced by the Lipschitz dependence in $H^{\overline{s}-1}_\lambda$:

$$\|v\|_{F^{\overline{s}-1}_\lambda(1)} \lesssim \|u^n_{0,>N}\|_{H^{\overline{s}-1}_\lambda} \lesssim N^{-1}\|u^n_{0,>N}\|_{H^s_\lambda}.$$

[6] $T = 1$ is omitted in the estimate for brevity.

Therefore, smallness of ϵ implies

$$\|S_1^\infty(u_0^n) - S_1^\infty(u_{0,\leq N}^n)\|_{F_\lambda^{\overline{s}}(1)} = \|v\|_{F_\lambda^{\overline{s}}(1)} \lesssim \|u_{0,>N}^n\|_{H_\lambda^{\overline{s}}},$$

uniformly in N and n. As u_0^n is a Cauchy sequence in $H_\lambda^{\overline{s}}$, we have

$$\sup_n \|S_1^\infty(u_{0,\leq N}^n) - S_1^\infty(u_0^n)\|_{C([0,1],H_\lambda^{\overline{s}})} \to 0 \text{ as } N \to \infty.$$

The second term in (3.3) is bounded by the standard energy estimate[7]

$$\begin{aligned}
\|S_1^\infty(u_{0,\leq N}^n) - S_1^\infty(u_{0,\leq N}^m)\|_{C([0,1],H_\lambda^{\overline{s}})} &\lesssim \|S_1^\infty(u_{0,\leq N}^n) - S_1^\infty(u_{0,\leq N}^m)\|_{C([0,1],H_\lambda^1)} \\
&\lesssim N^C \|P_{\leq N}(u_0^n - u_0^m)\|_{H_\lambda^1} \\
&\lesssim N^C \lambda^{s_1} \|u_0^n - u_0^m\|_{H_\lambda^{\overline{s}}}.
\end{aligned}$$

Thus, for any N, this term converges to zero. This concludes that $S_1^\infty(u_0^n)$ is a Cauchy sequence in $C([0,1],H_\lambda^{\overline{s}})$. The proof of Theorem 1.3 is completed.

4. Nonlinear estimates

The main purpose of this section is to propagate the nonlinearity in short-time function spaces:

Proposition 4.1 (Nonlinear estimates)**.** *Let $T \in (0,1]$. Then, there is $s_k < 1/2$ such that we find the following estimate to hold:*

$$\|\partial_x(\prod_{i=1}^k u_i)\|_{N_\lambda^{\overline{s}}(T)} \lesssim \prod_{i=1}^k \|u_i\|_{F_\lambda^{\overline{s}}(T)}, \tag{4.1}$$

provided that $0 < s_1 < 1/2$ and $s_2 > s_k$, and we denoted $\overline{s} = (s_1, s_2)$.

In Section 4.1, we prove Proposition 4.1. In Section 4.2, we estimate the nonlinearity in the smaller space $L_t^1 L_x^2$; see Lemma 4.3. This aids in simplifying the energy estimates in Sections 5 and 6.

4.1. Short-time DU_{BO}^2-estimate.

In this subsection, we prove Proposition 4.1. After localizing the frequencies of the u_i and the output frequency, we reduce the proof of Proposition 4.1 to estimates of the following kind:

$$\begin{aligned}
\|P_N \partial_x(P_{N_1}u_1 \dots P_{N_k}u_k)\|_{DU_{BO}^2(I)_\lambda} &\lesssim C(N, N_1, \dots, N_k) \\
&\qquad \prod_{i=1}^k \sup_{\substack{I_i \subseteq [0,T], \\ |I_i| = N_i^{-1}}} \|P_{N_i}u_i\|_{U_{BO}^2(I_i)_\lambda},
\end{aligned} \tag{4.2}$$

where $I \subseteq [0,T]$, $|I| = N^{-1}$. $C(N, N_1, \dots, N_k)$ is a constant allowing for dyadic summation after adding frequency weights giving (4.1).

By symmetry and otherwise impossible frequency interaction, we can suppose that $N_1 \geq N_2 \geq \dots \geq N_k$ and $N_1 \sim N_2$. We give an overview of the arising frequency interactions.

- Firstly, suppose that $N \sim N_1$ with $N_1 \gtrsim 1$.
  - If $N_2 \ll N_1$, we apply Hölder in time and a bilinear Strichartz estimate to ameliorate the derivative loss. The remaining factors are estimated with pointwise bounds. In the following cases, we do not mention the application of pointwise bounds to the remaining factors.

[7] In the following estimate, $C = C(k)$ varies line by line.

  - If $N_1 \sim N_2 \gg N_3$, we can apply two bilinear Strichartz estimates after dualization.
  - If $N_1 \sim N_2 \sim N_3 \gtrsim N_4$, we use Hölder in time and linear Strichartz estimates on the high frequencies to estimate the interaction.
- Secondly, suppose that $N \ll N_1$ with $N_1 \gtrsim 1$. Due to otherwise impossible frequency interaction, we can suppose that $N_1 \sim N_2$.
  - If $N_1 \sim N_2 \gg N_3$, we apply two bilinear Strichartz estimates after making use of duality.
  - If $N_1 \sim N_2 \sim N_3 \gg N_4$, we still apply two bilinear Strichartz estimates as observed in Corollary 2.5.
  - If $N_1 \sim N_2 \sim N_3 \sim N_4$, we use linear Strichartz estimates and Hölder's inequality to estimate the interaction.
- Finally, suppose that $N_1 \ll 1$. We merely use Hölder and Bernstein's inequalities.

Lemma 4.2 (Short-time DU^2_{BO}-estimate)**.** *Let $T \in (0,1]$ and $N_1 \gtrsim \ldots \gtrsim N_k$. The constant $C(N, N_1, \ldots, N_k)$ can be chosen as follows:*

- Case A: $N \sim N_1 \gtrsim 1$.

$$C(N, N_1, \ldots, N_k) \lesssim \begin{cases} N_2^{1/2,0} \prod_{i=3}^k N_i^{1/2} & \text{if } N_1 \gg N_2, \\ \log\langle N \rangle N_3^{1/2,0} \prod_{i=4}^k N_i^{1/2} & \text{if } N_1 \sim N_2 \gg N_3, \\ \prod_{i=3}^k N_i^{1/2} & \text{if } N_1 \sim N_2 \sim N_3. \end{cases}$$

- Case B: $N \ll N_1 \sim N_2$ *with* $N_1 \gtrsim 1$.

$$C(N, N_1, \ldots, N_k) \lesssim N^{3/2,0} \begin{cases} \log\langle N_1 \rangle N_3^{1/2,0} \prod_{i=4}^k N_i^{1/2} & \text{if } N_1 \gg N_3, \\ \log\langle N_1 \rangle \prod_{i=4}^k N_i^{1/2} & \text{if } N_1 \sim N_3 \gg N_4, \\ N^{0,1/2} \prod_{i=4}^k N_i^{1/2} & \text{if } N_1 \sim N_4. \end{cases}$$

- Case C: $N_1 \lesssim 1$.

$$C(N, N_1, \ldots, N_k) \lesssim N^{1/2} \prod_{i=1}^k N_i^{1/2}.$$

Proof. **Case A:** $N \sim N_1 \gtrsim 1$.

Subcase I: $N_1 \gg N_2$. We claim that (4.2) holds with $C(N, N_1, \ldots, N_k) = N_2^{1/2,0} \prod_{i=3}^k N_i^{1/2}$. We use the embedding $L^1(I, L^2_\lambda) \hookrightarrow DU^2_{BO}(I)_\lambda$, Hölder in time, bilinear Strichartz (2.5), and pointwise bounds to find

$$\begin{aligned} &\|P_N \partial_x (P_{N_1} u_1 \ldots P_{N_k} u_k)\|_{DU^2_{BO}(I)_\lambda} \\ &\lesssim N \|P_N (P_{N_1} u_1 \ldots P_{N_k} u_k)\|_{L^1(I, L^2_\lambda)} \\ &\lesssim N^{1/2} \|P_{N_1} u_1 \ldots P_{N_k} u_k\|_{L^2_t(I, L^2_\lambda)} \\ &\lesssim N^{1/2} \|P_{N_1} u_1 P_{N_2} u_2\|_{L^2_t(I, L^2_\lambda)} \prod_{i=3}^k \|P_{N_i} u_i\|_{L^\infty_t(I, L^\infty_\lambda)} \\ &\lesssim N_2^{1/2,0} \Big(\prod_{i=3}^k N_i^{1/2} \Big) \prod_{i=1}^k \sup_{\substack{I_i \subseteq [0,T], \\ |I_i| = N_i^{-1}}} \|P_N u_i\|_{U^2_{BO}(I_i)_\lambda}. \end{aligned}$$

Subcase II: $N_1 \sim N_2 \gg N_3$. We claim that (4.2) holds with $C(N, N_1, \ldots, N_k) = \log\langle N\rangle N_3^{1/2,0} \prod_{i=4}^k N_i^{1/2}$. We use duality (cf. [18, Proposition 2.10]) to write

$$\begin{aligned} &\|\partial_x(\prod_{i=1}^k P_{N_i} u_i)\|_{DU^2_{BO}(I)_\lambda} \\ &= \sup_{\|v\|_{V^2_{BO}(I)_\lambda}=1} \left| \int_I \int_{\lambda\mathbb{T}} (\partial_x P_N v(s,x)) P_{N_1} u_1(s,x) \ldots P_{N_k} u_k(s,x) dx ds \right|. \end{aligned} \tag{4.3}$$

As in Corollary 2.5, either $P_N v P_{N_1} u_1$ or $P_N v P_{N_2} u_2$ are amenable to a bilinear Strichartz estimate (after breaking the frequency support into intervals of length cN for some $c \ll 1$). Say we can use the bilinear Strichartz estimate for $P_N v P_{N_1} u_1$. Then we find using (2.5) and pointwise bounds:

$$\begin{aligned} (4.3) &\lesssim N \sup_{\|v\|_{V^2_{BO}}=1} \|P_N v P_{N_1} u_1\|_{L^2_t(I, L^2_\lambda)} \|P_{N_2} u_2 P_{N_3} u_3\|_{L^2_t(I,L^2_\lambda)} \\ &\quad \prod_{i=4}^k \|P_{N_i} u_i\|_{L^\infty_t(I, L^\infty_\lambda)} \\ &\lesssim \log\langle N\rangle N_3^{1/2,0} \left(\prod_{i=4}^k N_i^{1/2}\right) \prod_{i=1}^k \sup_{\substack{I\subseteq[0,T],\\ |I|=N_i^{-1}}} \|P_N u_i\|_{U^2_{BO}(I)_\lambda}. \end{aligned}$$

Subcase III: $N_1 \sim N_2 \sim N_3$. We claim that (4.2) holds with $C(N, N_1, \ldots, N_k) = \prod_{i=3}^k N_i^{1/2}$. Indeed, we use the embedding $L^1(I, L^2_\lambda) \hookrightarrow DU^2_{BO}(I)_\lambda$, Hölder in time, and $L^6_{t,x}$-Strichartz (2.7) to find

$$\begin{aligned} &\|P_N \partial_x (P_{N_1} u_1 \ldots P_{N_k} u_k)\|_{DU^2_{BO}(I)_\lambda} \\ &\lesssim N \|P_{N_1} u_1 \ldots P_{N_k} u_k\|_{L^1_t(I, L^2_\lambda)} \\ &\lesssim N^{1/2} \|P_{N_1} u_1 \ldots P_{N_k} u_k\|_{L^2_t(I,L^2_\lambda)} \\ &\lesssim N^{1/2} \prod_{i=1}^k \|P_{N_i} u_i\|_{L^6_t(I,L^6_\lambda)} \prod_{i=4}^k \|P_{N_i} u_i\|_{L^\infty_t(I,L^\infty_\lambda)} \\ &\lesssim \left(\prod_{i=3}^k N_i^{1/2}\right) \prod_{i=1}^k \sup_{\substack{I_i\subseteq[0,T],\\ |I_i|=N_i^{-1}}} \|P_{N_i} u_i\|_{U^2_{BO}(I_i)_\lambda}. \end{aligned}$$

Case B: $N_1 \sim N_2 \gg N$, $N_1 \gtrsim 1$.

In this case, we have to increase the time localization matching the highest frequency.

Subcase I: $N_1 \gg N_3$. We claim that (4.2) holds with $C(N, N_1, \ldots, N_k) = N^{3/2,0} \log\langle N_1\rangle N_3^{1/2,0} \prod_{i=4}^k N_i^{1/2}$. Indeed, we use duality to write

$$\begin{aligned} &\|P_N \partial_x (P_{N_1} u_1 \ldots P_{N_k} u_k)\|_{DU^2_{BO}(I)_\lambda} \\ &= \sup_{\|v\|_{V^2_{BO}(I)_\lambda}=1} \left| \int_I \int_{\lambda\mathbb{T}} (\partial_x P_N v)(s,x) P_{N_1} u_1(s,x) \ldots P_{N_k} u_k(s,x) dx ds \right| \\ &\leq \sup_{\|v\|_{V^2_{BO}(I)_\lambda}=1} \sum_{\substack{J\subseteq I,\\ |J|=N_1^{-1}}} \left| \int_J \int_{\lambda\mathbb{T}} (\partial_x P_N v)(s,x) P_{N_1} u_1(s,x) \ldots P_{N_k} u_k(s,x) dx ds \right|. \end{aligned}$$

For an interval $J \subseteq [0,T]$ with $|J| = N_1^{-1}$, we estimate the above integral by

$$\begin{aligned}
&\|(\partial_x P_N v) P_{N_1} u_1 \dots P_{N_k} u_k\|_{L^1_t(J,L^2_\lambda)} \\
&\lesssim \|(\partial_x P_N v) P_{N_1} u_1\|_{L^2_t(J,L^2_\lambda)} \|P_{N_2} u_2 P_{N_3} u_3\|_{L^2_t(J,L^2_\lambda)} \prod_{i=4}^k \|P_{N_i} u_i\|_{L^\infty_t(J,L^\infty_\lambda)} \\
&\lesssim N_1^{-1} \log\langle N_1 \rangle N^{1/2,0} N_3^{1/2,0} \|\partial_x P_N v\|_{V^2_{BO}(J)_\lambda} \prod_{i=1}^3 \|P_{N_i} u_i\|_{U^2_{BO}(J)_\lambda} \\
&\quad \prod_{i=4}^k N_i^{1/2} \|P_{N_i} u_i\|_{U^2_{BO}(J)_\lambda}.
\end{aligned}$$

The claim follows from replacing ∂_x by N and summing over $J \subseteq I$, $|J| = N_1^{-1}$, which loses $N^{0,-1} N_1$.

Subcase II: $N_1 \sim N_3 \gg N_4$. We claim that (4.2) holds with $C(N, N_1, \dots, N_k) = N^{3/2,0} \log\langle N_1 \rangle \prod_{i=4}^k N_i^{1/2}$. Indeed, we notice that as in Corollary 2.5, (after breaking the frequency support into intervals of length cN for some $c \ll 1$) there are two frequencies among N_1, N_2, N_3 for which the bilinear Strichartz estimate applies. Thus the same proof as in Subcase B.I works.

Subcase III: $N_1 \sim N_4$. We claim that (4.2) holds with $C(N, N_1, \dots, N_k) = N^{3/2,1/2} \prod_{i=4}^k N_i^{1/2}$. We use the embedding $L^1(I, L^2_\lambda) \hookrightarrow DU^2_{BO}(I)_\lambda$, Hölder in time, and Bernstein to find:

$$\begin{aligned}
&\|P_N \partial_x (P_{N_1} u_1 \dots P_{N_k} u_k)\|_{DU^2_{BO}(I)_\lambda} \\
&\lesssim N \|P_N (P_{N_1} u_1 \dots P_{N_k} u_k)\|_{L^1_t(I,L^2_\lambda)} \\
&\lesssim N_1 N^{1,0} \sup_{\substack{J \subseteq I, \\ |J| = N_1^{-1}}} \|P_N (P_{N_1} u_1 \dots P_{N_k} u_k)\|_{L^1_t(J,L^2_\lambda)} \\
&\lesssim N_1^{1/2} N^{3/2,1/2} \sup_{\substack{J \subseteq I, \\ |J| = N_1^{-1}}} \|P_{N_1} u_1 \dots P_{N_k} u_k\|_{L^2_t(J,L^1_\lambda)}.
\end{aligned}$$

For an interval $J \subseteq [0,T]$ with $|J| = N_1^{-1}$, we estimate the above $L^2_J L^1_\lambda$-norm using $L^8_t L^4_x$-Strichartz (2.7) and pointwise bounds to find

$$\begin{aligned}
&\|P_{N_1} u_1 \dots P_{N_k} u_k\|_{L^2_t(J,L^1_\lambda)} \\
&\lesssim \prod_{i=1}^4 \sup_{\substack{I_i \subseteq [0,T], \\ |I_i| = N_i^{-1}}} \|P_{N_i} u_i\|_{L^8_t(I_i,L^4_\lambda)} \prod_{i=5}^k \sup_{\substack{I_i \subseteq [0,T], \\ |I_i| = N_i^{-1}}} \|P_{N_i} u_i\|_{L^\infty_t(I_i,L^\infty_\lambda)} \\
&\lesssim \Big(\prod_{i=5}^k N_i^{1/2} \Big) \Big(\prod_{i=1}^k \sup_{\substack{I_i \subseteq [0,T], \\ |I_i| = N_i^{-1}}} \|P_{N_i} u_i\|_{U^2_{BO}(I_i)_\lambda} \Big).
\end{aligned}$$

The claim follows from substituting this bound into the above $DU^2_{BO}(I)_\lambda$ bound with $N_1 \sim N_4$.

Case C: $N_1 \lesssim 1$.

Note that $N \lesssim N_1 \lesssim 1$. Thus we may assume that $I = [0,T]$. We use the embedding $DU^2_{BO}(I)_\lambda \hookrightarrow L^1_t(I, L^2_\lambda)$, Hölder and Bernstein's inequalities to find

$$\begin{aligned}
&\|P_N \partial_x (P_{N_1} u_1 \dots P_{N_k} u_k)\|_{DU^2_{BO}(I)_\lambda} \\
&\lesssim N \|P_{N_1} u_1 \dots P_{N_k} u_k\|_{L^1_t(I, L^2_\lambda)} \\
&\lesssim N \|P_{N_1} u_1\|_{L^\infty_t(I, L^2_\lambda)} \prod_{i=2}^k \|P_{N_i} u_i\|_{L^\infty_t(I, L^\infty_\lambda)} \\
&\lesssim N^{1/2} \prod_{i=1}^k N_i^{1/2} \|P_{N_i} u_i\|_{U^2_{BO}(I)_\lambda}.
\end{aligned}$$

This completes the proof. $\square$

Proposition 4.1 follows from Lemma 4.2 by summing over the Littlewood-Paley pieces. We remark that one should use the gain (e.g. $N_2^{1,0}$ in Case A.I.) for low frequencies $\lesssim 1$ under the condition $s_1 < 1/2$. We omit the proof.

4.2. $L^1_t L^2_x$**-estimates.** In this section, we estimate the nonlinearity in $L^1_t L^2_x$. Due to the embedding $L^1_t L^2_x \hookrightarrow DU^2_{BO}$, the $L^1_t L^2_x$-norm is never smaller than the DU^2_{BO}-norm. Indeed, if one uses Lemma 4.3 (see below) instead of Lemma 4.2, then the short-time nonlinear estimate follows only for $s_2 \geq \frac{1}{2}$. However, $L^1_t L^2_x$ is well-suited for the product estimate (e.g. $\|fg\|_{L^1_t L^2_x} \lesssim \|f\|_{L^1_t L^2_x} \|g\|_{L^\infty_{t,x}}$). This helps us to deal with the spacetime error terms in Section 6.

We start with the short-time $L^1_t L^2_x$-estimate of each Littlewood-Paley piece.

Lemma 4.3 (Short-time $L^1_t L^2_x$-estimate)**.** *Let $T \in (0,1]$ and $N_1 \gtrsim N_2 \gtrsim \dots \gtrsim N_k$. We find*

$$\begin{aligned}
&\sup_{I \subseteq [0,T],\, |I| = N^{-1}} \|P_N \partial_x (P_{N_1} u_1 \dots P_{N_k} u_k)\|_{L^1_t(I, L^2_\lambda)} \\
&\lesssim \begin{cases} N^{1,0} N_2^{1/2,0} \prod_{i=3}^k N_i^{1/2} \cdot \prod_{i=1}^k \|P_{N_i} u_i\|_{F^0_\lambda}, & \text{if } N \sim N_1 \gg N_2, \\ N^{1/2} N_1^{1,0} N_3^{1/2,0} \prod_{i=4}^k N_i^{1/2} \cdot \prod_{i=1}^k \|P_{N_i} u_i\|_{F^0_\lambda} & \text{if } N_1 \sim N_2 \gtrsim N. \end{cases}
\end{aligned}$$

Proof. By the proof of Lemma 4.2, it suffices to deal with Cases A.II, B.I, and B.II.

<u>Case A.II:</u> $N \sim N_1 \sim N_2 \gg N_3$ with $N \gtrsim 1$. We use Hölder in time, bilinear Strichartz, and pointwise bounds to find

$$\begin{aligned}
&\|P_N \partial_x (P_{N_1} u_1 \dots P_{N_k} u_k)\|_{L^1_t(I, L^2_\lambda)} \\
&\lesssim N^{1/2} \|P_{N_1} u_1\|_{L^\infty_t(I, L^\infty_\lambda)} \|P_{N_2} u_2 P_{N_3} u_3\|_{L^2_t(I, L^2_\lambda)} \prod_{i=4}^k \|P_{N_i} u_i\|_{L^\infty_t(I, L^\infty_\lambda)} \\
&\lesssim N^{1/2} N_3^{1/2,0} \prod_{i=4}^k N_i^{1/2} \cdot \prod_{i=1}^k \|P_{N_i} u_i\|_{F^0_\lambda}.
\end{aligned}$$

<u>Case B.I:</u> We consider the case $N_1 \sim N_2 \gg \max(N, N_3)$ with $N_1 \gtrsim 1$. Here we need to increase time localization. Decompose I into $O(N^{0,-1} N_1)$-many intervals

J of length N_1^{-1}. We apply Bernstein and bilinear Strichartz to find

$$\begin{aligned}
&\|P_N \partial_x (P_{N_1} u_1 \dots P_{N_k} u_k)\|_{L_t^1(J,L_\lambda^2)} \\
&\lesssim N^{3/2} \|P_{N_1} u_1 \dots P_{N_k} u_k\|_{L_t^1(J,L_\lambda^1)} \\
&\lesssim N^{3/2} N_1^{-1} \|P_{N_1} u_1 P_{N_3} u_3\|_{L_t^2(J,L_\lambda^2)} \|P_{N_2} u_2 P_{N_4} u_3\|_{L_t^2(J,L_\lambda^2)} \prod_{i=5}^{k} \|P_{N_i} u_i\|_{L_t^\infty(I,L_\lambda^\infty)} \\
&\lesssim \Big(N^{3/2} N_1^{-1} N_3^{1/2,0} N_4^{1/2,0} \prod_{i=5}^{k} N_i^{1/2} \Big) \prod_{i=1}^{k} \|P_{N_i} u_i\|_{F_\lambda^0}.
\end{aligned}$$

The claim follows from summing over $J \subseteq I$, $|J| = N_1^{-1}$, which loses $N^{0,-1} N_1$.

Case B.II: We consider the case $N_1 \sim N_2 \sim N_3 \gg \max(N, N_4)$ with $N_1 \gtrsim 1$. As in the proof of Lemma 4.2, there is a pair of frequencies among N_1, N_2, N_3 amenable to the bilinear Strichartz estimate. Say N_1, N_3 is such a pair. One then proceeds as in Case B.I. This completes the proof. $\square$

We turn to the $L_T^1 L_x^2$-estimate of the nonlinearity. For later purposes, we estimate each P_N-portion of the nonlinearity, i.e., $P_N \partial_x (u^k)$. Due to High×High→Low interaction, we cannot say that $P_N \partial_x (u^k) = (P_N \partial_x u) u^{k-1}$. However we want to ensure $P_N \partial_x (u^k) \lesssim c_N u^{k-1}$, by introducing some ℓ_N^2-sequence c_N that mimics $\|P_{\sim N} u\|_{F^0}$ and also incorporates frequencies other than N. This is related with the frequency envelopes in [48].

Let $\overline{s} = (s_1, s_2)$, $0 < s_1 < 1/2 < s_2$, $\delta := \min(\frac{1}{2} - s_1, s_2 - \frac{1}{2}, \frac{1}{100})$, and $s' \in \{-1\} \cup [0, \infty)$. Define for dyadic N

$$c_N^{(v,\overline{s},s')} := \|P_{\sim N} v\|_{F_\lambda^0} + N^{\delta,-\delta} \cdot N^{-(\overline{s}+s')} \|v\|_{F_\lambda^{\overline{s}+s'}} \tag{4.4}$$

We use short-hand notation

$$c_N^{(u,\overline{s})} = c_N^{(u,\overline{s},0)}.$$

Set

$$d_N^{(v,\overline{s},s')} = N^{\overline{s}+s'} c_N^{(v,\overline{s},s')}. \tag{4.5}$$

Due to $\delta > 0$, we have a ℓ_N^2-summation property

$$\|d_N^{(v,\overline{s},s')}\|_{\ell_N^2} \lesssim \|v\|_{F_\lambda^{\overline{s}+s'}}. \tag{4.6}$$

Lemma 4.4 ($L_T^1 L_x^2$-estimate)**.** *Let $T \in (0,1]$; let $\overline{s} = (s_1, s_2)$, $0 < s_1 < 1/2 < s_2$, $\delta := \min(\frac{1}{2} - s_1, s_2 - \frac{1}{2}, \frac{1}{100})$, and $s' \in \{-1\} \cup [0, \infty)$. Then we have the $L_T^1 L_x^2$-nonlinear estimates*

$$\begin{aligned}
&\|P_N \partial_x (v u^{k-1})\|_{L_t^1([0,T],L_\lambda^2)} \\
&\lesssim N^{1/2,0} \|u\|_{F_\lambda^{\overline{s}}}^{k-2} \cdot \begin{cases} \|u\|_{F_\lambda^{\overline{s}}} c_N^{(v,\overline{s},-1)} & \text{if } s' = -1, \\ c_N^{(v,\overline{s},s')} \|u\|_{F_\lambda^{\overline{s}}} + c_N^{(u,\overline{s},s')} \|v\|_{F_\lambda^{\overline{s}}} & \text{if } s' \geq 0. \end{cases}
\end{aligned}$$

Remark 4.5. Lemma 4.4 can be viewed as

$$P_N \partial_x (v u^{k-1}) \approx (P_N \partial_x v) u^{k-1} + (P_N \partial_x u) v u^{k-2},$$

provided that $v \in F_\lambda^{\overline{s}-1}$ and $u \in F_\lambda^{\overline{s}}$ for $s' = -1$, or $v, u \in F_\lambda^{\overline{s}+s'}$ for $s' \geq 0$. We need to choose a function space to make sense of $\approx$. The crude choice $L_t^\infty L_x^2$ rather than $L_t^1 L_x^2$ would work for $s_2 > \frac{3}{4}$, but it seems difficult to reach $s_2 > \frac{1}{2}$ due to problematic High×High→Low interaction of vu. In order to reach $s_2 > \frac{1}{2}$, we choose a slightly larger space $L_t^1 L_x^2$.

Remark 4.6. These estimates will be used in the energy estimates for the differences of solutions. The case $s' = -1$ will be used for the Lipschitz estimates in a weaker topology; see (6.7). The case $s' \in \{0, 1\}$ will be used in establishing continuity estimates; see (6.8) and (6.9).

Proof of Lemma 4.4. For technical convenience, let us give a proof only for $1/2 < s_2 < 1$. The case $s_2 \geq 1$ is in fact easier, because we have higher regularity.

We decompose $[0, T]$ into $O(N^{0,1})$-many intervals of length $\sim \min(N^{-1}, T)$. It suffices to show the estimates

$$\|P_N \partial_x (v u^{k-1})\|_{L_t^1(I, L_\lambda^2)} \lesssim N^{1/2,0} \|u\|_{F_\lambda^{\bar{s}}}^{k-1} c_N^{(v,\bar{s},-1)}, \tag{4.7}$$

$$\|P_N \partial_x (v u^{k-1})\|_{L_t^1(I, L_\lambda^2)} \lesssim N^{1/2,0} \|u\|_{F_\lambda^{\bar{s}}}^{k-2} (c_N^{(v,\bar{s},s')} \|u\|_{F_\lambda^s} + c_N^{(u,\bar{s},s')} \|v\|_{F_\lambda^{\bar{s}}}) \tag{4.8}$$

uniformly for $I \subseteq [0, T]$ with $|I| = \min(N^{-1}, T)$. Fix such I. We will estimate each piece

$$\|P_N \partial_x (P_K v P_{K_1} u \dots P_{K_{k-1}} u)\|_{L_t^1(I, L_\lambda^2)} \tag{4.9}$$

and sum over $K, K_1, \dots, K_{k-1}$. Denote by $M_1, \dots, M_k$ the decreasing rearrangement of $K, K_1, \dots, K_{k-1}$. Due to otherwise impossible frequency interactions, we consider two cases: (Case A) $N \sim M_1 \gg M_2$ and (Case B) $M_1 \sim M_2 \gtrsim N$.

Case A: $N \sim M_1 \gg M_2$. Recall from Lemma 4.3 that

$$(4.9) \lesssim N^{1,0} M_2^{1/2,0} \Big(\prod_{i=3}^{k} M_i^{1/2} \Big) \|P_K v\|_{F_\lambda^0} \prod_{i=1}^{k-1} \|P_{K_i} u\|_{F_\lambda^0}.$$

<u>Subcase I:</u> $K = M_1$. We find

$$(4.9) \lesssim N^{1,0} \|P_K v\|_{F_\lambda^0} \prod_{i=1}^{k-1} \|P_{K_i} u\|_{F_\lambda^{1/2}}.$$

Summing over $K_1, \dots, K_{k-1}$ and $K \sim N$ yields both (4.7) and (4.8). (Recall $s_1 < \frac{1}{2} < s_2$.)

<u>Subcase II:</u> $K \in \{M_2, \dots, M_k\}$. Note that $K_1 = M_1 \sim N$. We find

$$(4.9) \lesssim N^{1,0} \|P_{K_1} u\|_{F_\lambda^0} K^{\frac{1}{2} - s_1, -(s_2 + s')} \|P_K v\|_{F_\lambda^{\bar{s}+s'}} \prod_{i=2}^{k-1} \|P_{K_i} u\|_{F_\lambda^{1/2}}.$$

We sum over $K, K_2, \dots, K_{k-1}$ and then use $K_1 \sim N$. If $s' = -1$, then this sums to $N^{\frac{3}{2} - s_1, -2s_2 + 1} \|v\|_{F_\lambda^{\bar{s}-1}} \|u\|_{F_\lambda^{\bar{s}}}^{k-1}$, yielding (4.7). If $s' \geq 0$, then this sums to $N^{\frac{3}{2} - s_1, 0} \|P_{\sim N} u\|_{F_\lambda^0} \|v\|_{F_\lambda^{\bar{s}}} \|u\|_{F_\lambda^{\bar{s}}}^{k-2}$, yielding (4.8).

Case B: $M_1 \sim M_2 \gtrsim N$. Recall from Lemma 4.3 that

$$(4.9) \lesssim \Big(N^{1/2} M_1^{1,0} M_3^{1/2,0} \prod_{i=4}^{k} M_i^{1/2} \Big) \|P_K v\|_{F_\lambda^0} \prod_{i=1}^{k-1} \|P_{K_i} u\|_{F_\lambda^0}.$$

<u>Subcase I:</u> $K \in \{M_1, M_2\}$. Note that $\{K, K_1\} = \{M_1, M_2\}$. We find

$$(4.9) \lesssim N^{\frac{1}{2}} M_1^{-2s_1+1, -2s_2-s'} \|P_K v\|_{F_\lambda^{\bar{s}+s'}} \|P_{K_1} u\|_{F_\lambda^{\bar{s}}} \prod_{i=2}^{k-1} \|P_{K_i} u\|_{F_\lambda^{1/2}}.$$

Since $s_2 > \frac{1}{2}$ and $s' \geq -1$, we have $-2s_2 - s' < 0$. Separately considering the cases $N \leq 1$ and $N \geq 1$, this sums to $N^{\frac{1}{2},-2s_2-s'+\frac{1}{2}}\|v\|_{F_\lambda^{\overline{s}+s'}}\|u\|_{F_\lambda^{\overline{s}}}^{k-1}$, yielding both (4.7) and (4.8).

Subcase II: $K \in \{M_3, \ldots, M_k\}$. Note that $\{K_1, K_2\} = \{M_1, M_2\}$. We find

$$\begin{aligned}(4.9) \lesssim N^{1/2} M_1^{1-2s_1,-2s_2-\ell} &\|P_{M_1}u\|_{F_\lambda^{\overline{s}+\ell}}\|P_{M_2}u\|_{F_\lambda^{\overline{s}}} \\ &\times K^{\frac{1}{2}-s_1,-(s_2+j)}\|P_K v\|_{F_\lambda^{\overline{s}+j}}\prod_{i=3}^{k-1}\|P_{K_i}u\|_{F_\lambda^{1/2}},\end{aligned}$$

where $j, \ell \in \mathbb{R}$. Choosing $j = -1$ and $\ell = 0$, the above sums to

$$N^{\frac{1}{2},-3s_2+\frac{3}{2}}\|v\|_{F_\lambda^{\overline{s}-1}}\|u\|_{F_\lambda^{\overline{s}}}^{k-1},$$

yielding (4.7). Choosing $j = 0$ and $\ell = s' \geq 0$, the above sums to

$$N^{\frac{1}{2},-2s_2+\frac{1}{2}-s'}\|v\|_{F_\lambda^{\overline{s}}}\|u\|_{F_\lambda^{\overline{s}+s'}}\|u\|_{F_\lambda^{\overline{s}}}^{k-2},$$

yielding (4.8). This finishes the proof. □

5. Energy estimates

The purpose of this section is to propagate the energy norm.

Proposition 5.1 (Energy estimates for solutions)**.** *Let $0 < s_1 < 1/2 \leq s_2$ and set $\overline{s} = (s_1, s_2)$. Let $s' \geq 0$. Then there exists $\epsilon = \epsilon(\overline{s}) > 0$ with the following properties: (denote $\overline{s}(\epsilon) = (s_1 + \epsilon, s_2 - \epsilon)$)*

- *If $s_2 > \frac{1}{2}$ and u is a smooth solution to (1.1) on $[0,T] \subseteq [0,1]$, then we have*

$$\|u\|_{E_\lambda^{\overline{s}+s'}(T)}^2 \lesssim \|u_0\|_{H_\lambda^{\overline{s}+s'}}^2 + \|u\|_{F_\lambda^{\overline{s}(\epsilon)+s'}(T)}^2\big(\|u\|_{F_\lambda^{\overline{s}(\epsilon)}(T)}^{k-1} + \|u\|_{F_\lambda^{s_1+\varepsilon,\frac{1}{2}-\varepsilon}(T)}^{2k-2}\big). \tag{5.1}$$

- *If $s_2 = \frac{1}{2}$ and u is a smooth solution to (1.1) with $k = 4$ on $[0,T] \subseteq [0,1]$, then we have*

$$\|u\|_{E_\lambda^{\overline{s}+s'}(T)}^2 \lesssim \|u_0\|_{H_\lambda^{\overline{s}+s'}}^2 + \|u\|_{F_\lambda^{\overline{s}+s'}(T)}^2\big(\|u\|_{F_\lambda^{\overline{s}}(T)}^{k-1} + \|u\|_{F_\lambda^{\overline{s}(\epsilon)}(T)}^{2k-2}\big). \tag{5.2}$$

Remark 5.2. The case $s' > 0$ corresponds to the persistence of regularity estimates.

Remark 5.3 (Besov version for $k \geq 5$ at $s_2 = 1/2$). If u is a smooth solution to (1.1) with $k \geq 5$, the argument of the proof of (5.2) gives the estimate

$$\|u\|_{E_{1,\lambda}^{s_1,1/2}(T)}^2 \lesssim \|u_0\|_{B_{1,\lambda}^{s_1,1/2}(T)}^2 + \|u\|_{F_{1,\lambda}^{s_1+\varepsilon,1/2}(T)}^{k+1} + \|u\|_{F_{1,\lambda}^{s_1+\varepsilon,1/2}(T)}^{2k}.$$

In the above display $B_{1,\lambda}^{s_1,1/2}$, $E_{1,\lambda}^{s_1,1/2}(T)$, and $F_{1,\lambda}^{s_1+\varepsilon,1/2}(T)$ denote 1-Besov variants, i.e.,

$$\|u\|_{E_{1,\lambda}^{s_1,1/2}(T)} = \|P_{\leq 1}u(0)\|_{H_\lambda^{s_1,1/2}} + \sum_{N\geq 1} N^{1/2}\sup_{t\in[0,T]}\|P_N u(t)\|_{L_\lambda^2},$$

and likewise for $B_{1,\lambda}^{s_1,1/2}$ and $F_{1,\lambda}^{s_1,1/2}(T)$.

To prove Proposition 5.1, we have to estimate the $E_\lambda^{\overline{s}+s'}$-norm

$$\|u\|_{E_\lambda^{\overline{s}+s'}(T)}^2 = \|P_{\leq 1}u(0)\|_{H_\lambda^{\overline{s}+s'}}^2 + \sum_{N>1} N^{2(s_2+s')}\|P_N u\|_{L_t^\infty([0,T],L_\lambda^2)}^2.$$

Since the low frequency part (≤ 1) is only written in terms of the initial data, it suffices to consider the high frequency part (> 1). Instead of estimating each $N^{2(s_2+s')}\|P_N u\|^2_{L^\infty_T L^2_\lambda}$ (having the symbol $N^{2(s_2+s')}\chi_N^2$) directly, it is more convenient to estimate a variant of $N^{2(s_2+s')}\|P_N u\|^2_{L^\infty_T L^2_\lambda}$ with slowly varying symbols, which are smoother in a sense.

More concretely, we consider the following set of symbols:

$$\begin{aligned} S^{s_2+s'} &= \{a_N^{s_2+s'} : N \gtrsim 1\}, \\ a_N^{s_2+s'}(\xi) &= N^{2(s_2+s')}(1-\chi(4\xi/N)). \end{aligned}$$

In what follows, we use short-hand notations $a_N = a_N^{s_2+s'}$ or $a = a_N$. These symbols satisfy the following properties (uniformly in $N \gtrsim 1$):

- (Vanishing) $a_N(\xi) = 0$ if $|\xi| \leq N/4$.
- (Comparison with Sobolev weights)

$$\begin{aligned} a_N(\xi) &= N^{2(s_2+s')} \text{ for } |\xi| \geq N/2, \\ \sum_{N>1} a_N(\xi) &\lesssim |\xi|^{2(s_2+s')}. \end{aligned} \tag{5.3}$$

- (Regularity) $|\partial^\alpha a_N(\xi)| \lesssim_\alpha \max(N^{2(s_2+s')}, a_N(\xi))\langle\xi\rangle^{-\alpha}$ for $|\xi| \geq N/4$.

For $a \in S^{s_2+s'}$, we define the generalized Sobolev (semi-)norm

$$\|u(t)\|^2_{H^a} = \frac{1}{\lambda}\sum_{\xi\in\mathbb{Z}/\lambda} a(\xi)|\hat{u}(t,\xi)|^2 = \int_{\Gamma^2_\lambda} a(\xi_1)\hat{u}(t,\xi_1)\hat{u}(t,\xi_2)d\Gamma^2_\lambda,$$

where the real-valuedness $\overline{u} = u$ is used in the last equality. The measure Γ^k_λ on $\{(\xi_1,\dots,\xi_k) \in \times_{i=1}^k \mathbb{Z}/\lambda \mid \sum_{i=1}^k \xi_i = 0\}$ is defined via pullback:

$$\int_{\Gamma^k_\lambda} f(\xi_1,\dots,\xi_k)d\Gamma^k_\lambda = \frac{1}{\lambda^{k-1}}\sum_{\xi_1,\dots,\xi_{k-1}\in\mathbb{Z}/\lambda} f(\xi_1,\dots,\xi_{k-1},-\xi_1-\dots-\xi_{k-1}).$$

Following [28, Section 6] we compute by the fundamental theorem of calculus and *symmetrization*

$$\begin{aligned} &\|u(t)\|^2_{H^a} \\ &= \|u_0\|^2_{H^a} + C\int_0^t \int_{\Gamma^{k+1}_\lambda} A_k(\xi_1,\dots,\xi_{k+1})\hat{u}(s,\xi_1)\dots\hat{u}(s,\xi_{k+1})d\Gamma^{k+1}_\lambda ds \\ &=: \|u_0\|^2_{H^a} + CR_k(u,t), \end{aligned}$$

where

$$A_k(\xi_1,\dots,\xi_{k+1}) = \sum_{i=1}^{k+1} a(\xi_i)\xi_i.$$

This symmetrization can be viewed as a multilinear version of the integration by parts in space. As the symbol $\xi \mapsto a(\xi)\xi$ is odd, $A_k(\xi_1,\dots,\xi_{k+1})$ enjoys better bounds than $a(\xi_i)\xi_i$ on the diagonal set Γ^{k+1}_λ. For instance, when $(\xi_1,\dots,\xi_{k+1}) \in$

Γ_λ^{k+1} satisfies $|\xi_1| \sim |\xi_2| \gtrsim |\xi_3| \gtrsim \cdots \gtrsim |\xi_{k+1}|$, we have

$$\begin{aligned} A_k(\xi_1,\dots,\xi_{k+1}) &= a(\xi_1)\xi_1 + a(\xi_2)\xi_2 + O(a(\xi_3)|\xi_3|) \\ &= a(\xi_1)(\xi_1+\xi_2) + (a(\xi_2) - a(-\xi_1))\xi_2 + O(a(\xi_3)|\xi_3|) \\ &= O(a(\xi_1)|\xi_3|), \end{aligned} \tag{5.4}$$

which is better than the crude bound $a(\xi_1)|\xi_1|$.

The symmetrization by itself does not make use of dispersion. To observe further smoothing effects from the dispersion, we change to the interaction picture via $\hat{v}(t,\xi) = e^{it\xi|\xi|}\hat{u}(t,\xi)$ to find

$$\begin{aligned} &R_k(u,t) \\ &= \int_0^t \int_{\Gamma_\lambda^{k+1}} e^{-is\Omega_k(\xi_1,\dots,\xi_{k+1})} A_k(\xi_1,\dots,\xi_{k+1})\hat{v}(s,\xi_1)\dots\hat{v}(s,\xi_{k+1}) d\Gamma_\lambda^{k+1} ds, \end{aligned} \tag{5.5}$$

where Ω_k is the *k-resonance function*

$$\Omega_k(\xi_1,\dots,\xi_{k+1}) = \sum_{i=1}^{k+1} \xi_i|\xi_i|.$$

For frequency interactions such that $\Omega_k(\xi_1,\dots,\xi_{k+1})$ is large, i.e., the *nonresonant interactions*, we can use that $e^{is\Omega_k}$ rapidly oscillates by writing $e^{-is\Omega_k} = (-i\Omega_k)^{-1}\partial_s e^{-is\Omega_k}$ and integrate by parts in time. See for instance Lemma 5.6.

In previous works on the Benjamin-Ono and the modified Benjamin-Ono equation (see [42, 44] for details), we have

$$|\Omega_2(\xi_1,\xi_2,\xi_3)| \sim |\xi_2\xi_3|,$$
$$\left|\frac{A_k(\xi_1,\dots,\xi_{k+1})}{\Omega_k(\xi_1,\dots,\xi_{k+1})}\right| \lesssim \frac{a(\xi_1)}{|\xi_1|} \qquad \forall k \in \{2,3\}.$$

In particular, in case of $k=2$, frequency interactions among mean zero functions are always nonresonant. In case of $k=3$, there are nontrivial resonant interactions, but the multiplier A_k also vanishes simultaneously.

In case of $k \geq 4$, the multiplier A_k does not necessarily vanish for resonant interactions. However, we show that A_k specialized to resonant interaction satisfies better bounds than (5.4). It turns out (Lemma 5.5) that we have a decomposition

$$A_k = b\Omega_k + c,$$

such that

$$|b| \lesssim a(\xi_1)|\xi_1|^{-1} \quad \text{and} \quad |c| \lesssim \big(a(\xi_1)|\xi_1|^{-1}|\xi_3| + a(\xi_3)\big)|\xi_5|.$$

Thanks to (5.3), Proposition 5.1 follows from summing over N the following.

Lemma 5.4 (Estimate of each $\|u\|^2_{L_T^\infty H_\lambda^{a_N}}$). *Assume the hypotheses of Proposition 5.1. Let $a_N \in S^{s_2+s'}$ for $N \gtrsim 1$.*

- *If $s_2 > 1/2$, there exists $\epsilon = \epsilon(\overline{s}) > 0$ such that we have the estimates*
$$\sup_{t\in[0,T]} |R_k(u,t)| \lesssim N^{-\epsilon}(\|u\|^{k+1}_{F_\lambda^{\overline{s}(\epsilon)}(T)} + \|u\|^{2k}_{F_\lambda^{s_1+\epsilon,1/2-\epsilon}(T)}).$$
- *If $k=4$ and $s_2 = 1/2$, there exists $\epsilon > 0$ such that we have the estimates*
$$\sup_{t\in[0,T]} |R_4(u,t)| \lesssim \|u\|^2_{F_\lambda^{a_N}(T)}\|u\|^3_{F_\lambda^{\overline{s}}(T)} + N^{-\varepsilon}\|u\|^2_{F_\lambda^{\overline{s}(\epsilon)+s'}(T)}\|u\|^6_{F_\lambda^{\overline{s}(\epsilon)}(T)}.$$

Here, we recall that $\overline{s} = (s_1, s_2)$ *and* $\overline{s}(\epsilon) = (s_1 + \epsilon, s_2 - \epsilon)$, *and used the notation*

$$\|u\|^2_{F^{a_N}_\lambda(T)} = N^{2(s_2+s')} \sum_{N' \geq N} \sup_{\substack{I \subseteq [0,T], \\ |I| = (N')^{-1}}} \|P_{N'} u\|^2_{U^2_{BO}(I;L^2_\lambda)}. \tag{5.6}$$

Henceforth, we focus on showing Lemma 5.4. In Section 5.1, we give details of the above decomposition of A_k. In Section 5.2, the contribution of $b\Omega_k$ in $R_k(u,t)$ is handled in a favorable way through integration by parts. In Section 5.3, the contribution of c in $R_k(u,t)$ is estimated. We do not integrate by parts in time, but we can use the bound of c in (5.8), which is better than the integration by parts in space bound (5.4).

5.1. **Decomposition of A_k.** The aim of this subsection is to detail the decomposition of A_k: Lemma 5.5.

Let us introduce some notations. Let $N_1, \ldots, N_{k+1} \in 2^{\mathbb{Z}} \cap [\lambda^{-1}, \infty)$. For $\pm_1, \ldots, \pm_{k+1} \in \{+, -\}$, define

$$\begin{aligned} D_{\pm_1 N_1, \ldots, \pm_{k+1} N_{k+1}} \coloneqq \{&(\xi_1, \ldots, \xi_{k+1}) \in \mathbb{Z}^{k+1}/\lambda : \\ &\pm_i \xi_i \geq 0,\ |\xi_i| \sim N_i, i \in \{1, \ldots, k+1\}\} \end{aligned}$$

with the exception to take $|\xi_i| \lesssim N_i$ for $N_i = \lambda^{-1}$. We then define the set $D_{N_1,\ldots,N_{k+1}}$ relevant to the frequency interactions:

$$\begin{aligned} D_{N_1,\ldots,N_{k+1}} \coloneqq \bigcup \Big\{ D_{\pm_1 N_1, \ldots, \pm_{k+1} N_k} : D_{\pm_1 N_1, \ldots, \pm_{k+1} N_k} \cap \Gamma^{k+1}_\lambda \neq \emptyset, \\ \pm_1, \ldots, \pm_{k+1} \in \{+, -\} \Big\}. \end{aligned}$$

In other words, $D_{N_1,\ldots,N_{k+1}}$ is the union of rectangles on which nontrivial frequency interactions can occur.

Lemma 5.5 (Decomposition of A_k). *Fix* $a = a_N \in S^{s_2+s'}$ *for* $N \gtrsim 1$. *Let* $N_1 \sim N_2 \gtrsim \cdots \gtrsim N_{k+1}$. *Then, there exist* b *and* c *defined on* $D_{N_1,\ldots,N_{k+1}}$ *satisfying the following:*

(1) *(Decomposition of* A_k *on* Γ^{k+1}_λ*) We have*

$$A_k = b\Omega_k + c \quad \text{on } \Gamma^{k+1}_\lambda \cap D_{N_1,\ldots,N_{k+1}}.$$

(2) *(Symbol regularity) For* $\alpha_1, \ldots, \alpha_{k+1} \in \mathbb{N}_0$, *the symbols* b *and* c *satisfy*[8]

$$|\partial^{\alpha_1}_{\xi_1} \ldots \partial^{\alpha_{k+1}}_{\xi_{k+1}} b(\xi_1, \ldots, \xi_{k+1})| \lesssim_{\alpha_1,\ldots,\alpha_{k+1}} a(N_1) N_1^{-1} N^{-\alpha}, \tag{5.7}$$

$$|\partial^{\alpha_1}_{\xi_1} \ldots \partial^{\alpha_{k+1}}_{\xi_{k+1}} c(\xi_1, \ldots, \xi_{k+1})| \lesssim_{\alpha_1,\ldots,\alpha_{k+1}} (a(N_1) N_1^{-1} N_3 + a(N_3)) N_5 N^{-\alpha}, \tag{5.8}$$

where we denoted $N^{-\alpha} = N_1^{-\alpha_1} \ldots N_{k+1}^{-\alpha_{k+1}}$.

(3) *(Support property of* c*)* c *is supported on the region where* $N_3 \sim N_4$.

Proof. For the symbol regularity, we only show the pointwise bounds of b and c on $\Gamma^{k+1}_\lambda \cap D_{N_1,\ldots,N_{k+1}}$ for simplicity. The symbol regularity estimates on the larger set $D_{N_1,\ldots,N_{k+1}}$ follow from deriving the explicit representation formulas of b and c; see for instance [28, Proposition 6.3].

We can suppose that $N_1 \gtrsim 1$ as for $N_1 \ll 1$ due to vanishing property of a on low frequencies $A_k \equiv 0$, and we can set $b = c \equiv 0$.

---

[8] In fact, when $a = a_N \in S^{s_2}$ and $N_1 \sim N$, then one should replace $a(N_1)$ by N^{2s_2} because $a(N_1)$ itself may vanish. The same remark applies for $a(N_3)$, and other estimates having upper bounds in terms of $a(N_1)$ with $N_1 \sim N$.

We proceed by Case-by-Case analysis.

Case A: $N_3 \gg N_4$.

In this case, there are only nonresonant interactions; we claim that

$$|\Omega_k(\xi_1, \dots, \xi_{k+1})| \sim N_1 N_3 \sim |\Omega_3(\xi_1, \xi_2, \xi_3)|. \tag{5.9}$$

By otherwise impossible interaction and symmetry, we can suppose that $\xi_1 > 0$, $\xi_2 < 0$, $\xi_3 < 0$. We compute using $\xi_1 = -(\xi_2 + \xi_3) + O(N_4)$

$$\begin{aligned} \Omega_k(\xi_1, \dots, \xi_{k+1}) &= \xi_1^2 - \xi_2^2 - \xi_3^2 + O(N_4^2) \\ &= 2\xi_2\xi_3 + 2(\xi_2 + \xi_3) \cdot O(N_4) + O(N_4^2) \\ &= 2\xi_2\xi_3 + O(N_2 N_4). \end{aligned}$$

Since $N_3 \gg N_4$ and $N_1 \sim N_2$, we have

$$|\Omega_k(\xi_1, \dots, \xi_{k+1})| \sim N_1 N_3 \sim |\Omega_3(\xi_1, \xi_2, \xi_3)|.$$

Having settled the claim, we set b and c by

$$b := \frac{A_k}{\Omega_k} \quad \text{and} \quad c := 0.$$

The estimate of b follows from (5.4) and (5.9).

Case B: $N_1 \sim N_2 \gg N_3 \sim N_4$.

Due to otherwise impossible interaction and symmetry, we can suppose that $\xi_1 > 0$, $\xi_2 < 0$. Choose $\ell \in \{4, \dots, k+1\}$ such that $N_1 \sim N_2 \gg N_3 \sim N_\ell \gg N_{\ell+1}$ (set $N_{k+2} = 0$ when $\ell = k+1$).

Subcase I: $\xi_3 \dots, \xi_\ell$ have the same sign. By $\xi_1 > 0$ and $\xi_2 < 0$, this means $\xi_3, \dots, \xi_l < 0$. Then, the following computation shows that this case is nonresonant:

$$\begin{aligned} \Omega_k &= \xi_1^2 - \xi_2^2 + O(N_3^2) \\ &= (\xi_1 - \xi_2)(-\xi_3 - \dots - \xi_\ell + O(N_{\ell+1})) + O(N_3^2) \sim N_1 N_3, \end{aligned}$$

where in the last inequality we used the fact that $\xi_3, \dots, \xi_\ell$ have same sign. Therefore, we set

$$b := \frac{A_k}{\Omega_k} \quad \text{and} \quad c := 0.$$

The estimate of b follows from (5.4) and $\Omega_k \sim N_1 N_3$.

Subcase II: $\xi_3, \dots, \xi_\ell$ have both positive and negative signs. This case is similar to [28, Proposition 6.3]. By symmetry, we may assume $\xi_3 > 0$ and $\xi_4 < 0$. We first observe that

$$\begin{aligned} \Omega_k &= \xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 + O(N_5^2) \\ &= (\xi_1 - \xi_2)(\xi_1 + \xi_2) + (\xi_3 - \xi_4)(-\xi_1 - \xi_2 + O(N_5)) + O(N_5^2) \\ &= (\xi_1 - \xi_2 - \xi_3 + \xi_4)(\xi_1 + \xi_2) + O(N_3 N_5) \\ &= -2(\xi_1 + \xi_4)(\xi_1 + \xi_2) + O(N_3 N_5). \end{aligned}$$

We define a smooth function (because a is even and smooth)

$$q(\xi, \eta) := \frac{a(\xi)\xi + a(\eta)\eta}{\xi + \eta} \text{ such that } |q(\xi, \eta)| \lesssim a(\max\{|\xi|, |\eta|\})$$

and observe

$$a(\xi_1)\xi_1 + a(\xi_2)\xi_2 = q(\xi_1,\xi_2)(\xi_1+\xi_2) = -\frac{q(\xi_1,\xi_2)}{2(\xi_1+\xi_4)}(\Omega_k + O(N_3N_5)),$$

$$a(\xi_3)\xi_3 + a(\xi_4)\xi_4 = -q(\xi_3,\xi_4)(\xi_1+\xi_2+O(N_5)) = \frac{q(\xi_3,\xi_4)}{2(\xi_1+\xi_4)}(\Omega_k + O(N_1N_5)).$$

Thus

$$\begin{aligned} A_k &= a(\xi_1)\xi_1 + \cdots + a(\xi_4)\xi_4 + O(a(N_5)N_5) \\ &= \frac{q(\xi_3,\xi_4)-q(\xi_1,\xi_2)}{2(\xi_1+\xi_4)}\Omega_k + O(a(N_1)N_1^{-1}N_3N_5 + a(N_3)N_5). \end{aligned}$$

Therefore, we set

$$b \coloneqq \frac{q(\xi_3,\xi_4)-q(\xi_1,\xi_2)}{2(\xi_1+\xi_4)} \quad \text{and} \quad c \coloneqq A_k - b\Omega_k.$$

The estimate $|b| \lesssim a(N_1)N_1^{-1}$ follows from $|q(\xi_3,\xi_4)|, |q(\xi_1,\xi_2)| \lesssim a(N_1)$. The estimate for c is immediate.

Case C: $N_1 \sim N_2 \sim N_3 \sim N_4$.

Subcase I: $N_4 \gg N_5$. Due to otherwise impossible interactions and symmetry, it suffices to consider the following two cases: the four highest frequencies contain (i) three positive and one negative frequencies (say $\xi_1,\xi_2,\xi_3 > 0$ and $\xi_4 < 0$); (ii) two positive and two negative frequencies (say $\xi_1,\xi_3 > 0$ and $\xi_2,\xi_4 < 0$).

The case (i) is nonresonant:

$$\begin{aligned} \Omega_k &= \xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_4^2 + O(N_5^2) \\ &= \xi_1^2 + \xi_2^2 + \xi_3^2 - (\xi_1+\xi_2+\xi_3+O(N_5))^2 + O(N_5^2) \\ &= -2(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1) + O(N_1N_5) \sim N_1^2, \end{aligned}$$

where in the last inequality we used the fact that ξ_1,ξ_2,ξ_3 have same sign. Therefore, we set

$$b \coloneqq \frac{A_k}{\Omega_k} \quad \text{and} \quad c \coloneqq 0.$$

The estimate $|b| \lesssim a(N_1)N_1^{-1}$ follows from $|A_k| \lesssim a(N_1)N_1$ and $\Omega_k \sim N_1^2$.

The case (ii) is similar to [28, Proposition 6.3]. Proceeding as in Case B.II, we have

$$\begin{aligned} \Omega_k &= -(\xi_1-\xi_2-\xi_3+\xi_4)(\xi_1+\xi_2) + O(N_3N_5), \\ a(\xi_1)\xi_1 + \cdots + a(\xi_4)\xi_4 &= (q(\xi_1,\xi_2)-q(\xi_3,\xi_4))(\xi_1+\xi_2), \\ a(\xi_5)\xi_5 + \cdots + a(\xi_{k+1})\xi_{k+1} &= O(a(N_5)N_5). \end{aligned}$$

If we define a smooth function (because $\partial_\xi q(\xi,\eta) = -\partial_\eta q(\xi,\eta)$)

$$r(\xi_1,\xi_2,\xi_3,\xi_4) \coloneqq \frac{q(\xi_1,\xi_2)-q(\xi_3,\xi_4)}{\xi_1-\xi_2-\xi_3+\xi_4} \text{ such that } |r| \lesssim a(N_1)N_1^{-1},$$

then

$$A_k(\xi_1,\ldots,\xi_{k+1}) = r(\xi_1,\xi_2,\xi_3,\xi_4)\Omega_k(\xi_1,\ldots,\xi_{k+1}) + O(a(N_1)N_5).$$

Therefore, we set

$$b(\xi_1,\ldots,\xi_{k+1}) \coloneqq r(\xi_1,\xi_2,\xi_3,\xi_4) \quad \text{and} \quad c \coloneqq A_k - b\Omega_k.$$

Subcase II: $N_4 \sim N_5$. In this case, we set

$$b := 0 \quad \text{and} \qquad c := A_k.$$

The estimate of c follows from the crude estimate $|A_k| \lesssim a(N_1)N_1$.

This completes the proof. $\square$

5.2. **Contribution of $b\Omega_k$.** The goal of this subsection is to estimate the contribution of $b\Omega_k$ in $R_k(u,t)$: Lemma 5.6. We integrate by parts in time. The case $k=3$ is detailed in [42]. Here we deal with general k.

Lemma 5.6 (Contribution of $b\Omega_k$)**.** *Let $T \in (0,1]$ and $0 < s_1 < 1/2 \le s_2$. For sufficiently small $\epsilon > 0$, we find*

$$\begin{aligned}(5.10) \qquad & \sup_{t\in[0,T]} \sum_{N_1\sim N_2 \gtrsim \cdots \gtrsim N_{k+1}} \Big| \int_0^t \int_{\Gamma_\lambda^{k+1}} b(\xi_1,\dots,\xi_{k+1})\Omega_k(\xi_1,\dots,\xi_{k+1}) \\ & \times \prod_{i=1}^{k+1} \chi_{N_i}(\xi_i)\hat{u}(s,\xi_i)\, d\Gamma_\lambda^{k+1} ds \Big| \lesssim N^{-\epsilon} \|u\|^2_{F_\lambda^{\overline{s}(\epsilon)+s'}} (\|u\|^{k-1}_{F_\lambda^{s_1+\epsilon,1/2-\epsilon}} + \|u\|^{2k-2}_{F_\lambda^{s_1+\epsilon,1/2-\epsilon}}),\end{aligned}$$

where we recall that $\overline{s} = (s_1,s_2)$ and $\overline{s}(\epsilon) = (s_1+\epsilon, s_2-\epsilon)$.

Remark 5.7. We remark that $s_2 = 1/2$ is included in Lemma 5.6. Moreover, it is possible to lower the threshold for s_2 slightly. Thus the contribution of $b\Omega_k$ is *not* the source for the restriction $s_2 > 1/2$ in our energy estimate (Proposition 5.1).

We first estimate each Littlewood-Paley piece.

Lemma 5.8 (Integration by parts in time)**.** *Let $T \in (0,1]$. We have the following integration by parts in time estimate:*

$$\begin{aligned} & \sup_{t\in[0,T]} \Big| \int_0^t \int_{\Gamma_\lambda^{k+1}} b(\xi_1,\dots,\xi_{k+1})\Omega_k(\xi_1,\dots,\xi_{k+1}) \prod_{i=1}^{k+1} \chi_{N_i}(\xi_i)\hat{u}(s,\xi_i)\, d\Gamma_{k+1} ds \Big| \\ & \lesssim a(N_1)N_1^{-1} \Big(\prod_{i=3}^{k+1} N_i^{1/2} \Big) \Big(\prod_{i=1}^{k+1} \|P_{N_i}u\|_{L_t^\infty([0,T],L_\lambda^2)} \Big) \\ & \quad + a(N_1)N_1^{-1}N_3 \sup_{t\in[0,T]} \big| \int_0^t \int_{\lambda\mathbb{T}} P_{N_j}(u^k) \prod_{i\neq j} P_{N_i} u dx ds \big|. \end{aligned}$$

Remark 5.9 (Separation of variables)*.* In what follows, we need to estimate the expression of the form $\int_{\Gamma_\lambda^{k+1}} b(\xi_1,\dots,\xi_{k+1}) \prod_{i=1}^{k+1} \hat{f}_i(\xi_i) d\Gamma_\lambda^{k+1}$. We will simply estimate this by $\|b\|_{L^\infty} \int_{\lambda\mathbb{T}} \prod_{i=1}^{k+1} f_i(x)dx$. By the symbol regularity (5.7), $b(\xi_1,\dots,\xi_{k+1})$ has a rapidly converging Fourier series, say $\sum b_{\lambda_1,\dots,\lambda_{k+1}} e^{i(\lambda_1\xi_1+\cdots+\lambda_{k+1}\xi_{k+1})}$ with rapidly decaying coefficients $b_{\lambda_1,\dots,\lambda_{k+1}}$, where the summation ranges over λ_i's in $N_i^{-1}\mathbb{Z}$. As the modulation on the Fourier space corresponds to the translation on the physical space, estimating $\int_{\Gamma_{k+1}} b(\xi_1,\dots,\xi_{k+1}) \prod_{i=1}^{k+1} \hat{f}_i(\xi_i) d\Gamma_\lambda^{k+1}$ essentially reduces to estimating $\|b\|_{L^\infty} \int_{\lambda\mathbb{T}} \prod_{i=1}^{k+1} f_i(x)dx$, as long as our estimates are translation invariant. This can be justified in this paper because we rely on frequency interactions, and our arguments do not depend on how functions are distributed in physical space. For a detailed discussion, see [50] and [9] in the context of dispersive equations.

Proof of Lemma 5.8. Changing to the interaction picture by introducing

$$\hat{v}(s,\xi) := e^{is|\xi|\xi}\hat{u}(s,\xi),$$

we need to estimate

(5.11)

$$\int_0^t \int_{\Gamma_\lambda^{k+1}} e^{-is\Omega_k(\xi_1,\dots,\xi_{k+1})} b(\xi_1,\dots,\xi_{k+1})\Omega_k(\xi_1,\dots,\xi_{k+1}) \prod_{i=1}^{k+1} \chi_{N_i}(\xi_i)\hat{v}(s,\xi_i)\, d\Gamma_\lambda^{k+1} ds.$$

Integrating by parts in time using

$$\begin{aligned} \partial_s e^{-is\Omega_k} &= -i\Omega_k e^{is\Omega_k}, \\ \partial_s \hat{v}(s,\xi) &= (i\xi) e^{is|\xi|\xi} (u^k(s,\cdot))^\wedge(\xi), \end{aligned}$$

we find

$$\begin{aligned} (5.11) = &\int_{\Gamma_\lambda^{k+1}} b(\xi_1,\dots,\xi_{k+1}) \prod_{i=1}^{k+1} \chi_{N_i}(\xi_i)\hat{u}(s,\xi_i) d\Gamma_\lambda^{k+1} \Big|_0^t \\ &+ \sum_{j=1}^{k+1} \int_0^t \int_{\Gamma_\lambda^{k+1}} b(\xi_1,\dots,\xi_{k+1}) \prod_{i=1}^{k+1} \chi_{N_i}(\xi_i) \cdot \xi_j (u^k)^\wedge(s,\xi_j) \\ &\quad \prod_{i\neq j} \hat{u}(s,\xi_i) d\Gamma_\lambda^{k+1} ds. \end{aligned}$$

We call the first term the *boundary term* and the second term the *spacetime term*. We turn to the estimates.

We firstly estimate the boundary term. By Remark 5.9, we may replace b by $a(N_1)N_1^{-1}$ and return to the physical space representation:

$$a(N_1)N_1^{-1} \int_{\lambda\mathbb{T}} \prod_{i=1}^{k+1} P_{N_i} u \, dx \Big|_0^t .$$

Applying L_x^2 to the highest two frequencies and pointwise bounds to the remaining ones, the boundary term is estimated by

$$a(N_1)N_1^{-1} \Big(\prod_{i=3}^{k+1} N_i^{1/2} \Big) \prod_{i=1}^{k+1} \| P_{N_i} u \|_{L_t^\infty([0,T],L_\lambda^2)} .$$

Next, we estimate the spacetime term. We focus on the $\sum_{j=1}^2$-part of the spacetime term. We rewrite the $\sum_{j=1}^2$-part as

$$\begin{aligned} \int_0^t \int_{\Gamma_\lambda^{2k}} \Big(& b(\xi_{sum},\xi_1,\xi_2,\dots,\xi_k)\xi_{sum}\chi_{N_1}(\xi_{sum})\chi_{N_2}(\xi_1) \\ &+ b(\xi_1,\xi_{sum},\xi_2,\dots,\xi_k)\xi_{sum}\chi_{N_1}(\xi_1)\chi_{N_2}(\xi_{sum}) \Big) \Big(\prod_{i=3}^{k+1} \chi_{N_i}(\xi_{i-1}) \Big) \prod_{i=1}^{2k} \hat{u}(s,\xi_i) ds, \end{aligned}$$

where we denote $\xi_{sum} = \xi_{k+1} + \cdots + \xi_{2k}$. We then *integrate by parts in space* as in (5.4); the following holds after taking $\int_{\Gamma_{2k}} (\cdot) \prod_{i=3}^{k+1} \chi_{N_i}(\xi_{i-1}) \prod_{i=1}^{2k} \hat{u}(s,\xi_i) d\Gamma_{2k}$:

$$\begin{aligned} &b(\xi_{sum},\xi_1,\xi_2,\dots,\xi_k)\xi_{sum}\chi_{N_1}(\xi_{sum})\chi_{N_2}(\xi_1) \\ &\quad + b(\xi_1,\xi_{sum},\xi_2,\dots,\xi_k)\xi_{sum}\chi_{N_1}(\xi_1)\chi_{N_2}(\xi_{sum}) \\ &= b(\xi_{sum},\xi_1,\xi_2,\dots,\xi_k)(\xi_1+\xi_{sum})\chi_{N_1}(\xi_{sum})\chi_{N_2}(\xi_1) \\ &= -b(\xi_{sum},\xi_1,\xi_2,\dots,\xi_k)(\xi_2+\cdots+\xi_k)\chi_{N_1}(\xi_{sum})\chi_{N_2}(\xi_1) \\ &= O(N_3)\cdot b(\xi_{sum},\xi_1,\xi_2,\dots,\xi_k))\chi_{N_1}(\xi_{sum})\chi_{N_2}(\xi_1), \end{aligned}$$

where we changed the variables $\xi_1 \leftrightarrow \xi_{sum}$ and used frequency localization $|\xi_{i-1}| \sim N_i$ for $3 \leq i \leq k+1$. For the $\sum_{j=3}^{k+1}$-part, we simply bound ξ_j by N_3.

As a result, the spacetime term is estimated by

$$a(N_1)N_1^{-1}N_3 \Big| \int_0^t \int_{\lambda\mathbb{T}} P_{N_j}(u^k) \prod_{i\neq j} P_{N_i} u \, dxds \Big|$$

by Remark 5.9. The proof is completed. $\square$

For $s_2 > 1/2$, we can estimate the spacetime term with L_x^2 in the highest frequencies and pointwise bounds for the remaining ones. We then give L_t^1 to $P_{N_j}(u^k)$ and apply Lemma 4.4 with $s' = 0$. Then, the spacetime term is estimated by

$$\|u\|_{F_\lambda^{s_1+\varepsilon,s_2-\varepsilon}}^{k-1} a(N_1)N_1^{-1}N_3 \Big(\prod_{i=3}^{k+1} N_i^{1/2} \Big) \prod_{i=1}^{k+1} c_{N_i}^{(u,(s_1+\varepsilon,s_2-\varepsilon))},$$

with $\epsilon(\overline{s}) > 0$ sufficiently small. For $s_2 = 1/2$, this argument fails as it does not allow for the use of multilinear estimates involving factors from $P_{N_j}(u^k)$ and $P_{N_i}u$, $i \neq j$. This shortcoming is remedied in the following:

Lemma 5.10 (Estimates of spacetime term)**.** *Let $j \in \{1,\dots k+1\}$ and*

$$R_{2k,j}(N_1,\dots,N_{k+1},M_1,\dots,M_k) = \Big| \int_0^t \int_{\lambda\mathbb{T}} P_{N_j}\big(\prod_{i=1}^k P_{M_i} u\big) \prod_{i\neq j} P_{N_i} u dxds \Big|.$$

Let $K_1,\dots,K_{2k}$ denote a decreasing rearrangement of $M_1,\dots,M_k,N_i \, (i\neq j)$ with multiplicity. Then, for $K_1 \sim K_2 \gtrsim 1$, we find the following estimate to hold:

$$R_{2k,j}(N_1,\dots,N_{k+1},M_1,\dots,M_k) \lesssim C(K_1,\dots,K_{2k}) \prod_{i=1}^{2k} \|P_{K_i} u\|_{F_\lambda^0},$$

where

$$C(K_1,\dots,K_{2k}) = \begin{cases} K_3^{1/2,0} K_4^{1/2,0} \prod_{i=5}^{2k} K_i^{1/2} & \text{if } K_2 \gg K_3 \text{ or } K_1 \sim K_3 \gg K_4, \\ K_1^{1/2} K_5^{1/2,0} \prod_{i=6}^{2k} K_i^{1/2} & \text{if } K_1 \sim K_4 \gg K_6 \text{ or } K_1 \sim K_5 \gg K_6, \\ K_1 \prod_{i=7}^{2k} K_i^{1/2} & \text{if } K_1 \sim K_6. \end{cases} \tag{5.12}$$

Proof. As explained in Remark 5.9, we can dispose of P_{N_j} by expanding the Fourier multiplier into a rapidly converging Fourier series. The resulting expression

$$\int_0^t \int_{\lambda\mathbb{T}} P_{K_1} u \dots P_{K_{2k}} u dxds$$

is estimated by linear and bilinear Strichartz estimates after partitioning $[0,t]$ into $O(K_1^{-1})$ time intervals of length K_1^{-1}:

(1) If $K_1 \sim K_2 \gg K_3$ or $K_1 \sim K_3 \gg K_4$, following the proof of Corollary 2.5 we can apply two bilinear Strichartz estimates involving $K_1, \dots, K_4$. This gives the first bound.
(2) If $K_1 \sim K_4 \gg K_5$ or $K_1 \sim K_5 \gg K_6$, we can apply one bilinear Strichartz estimate and three $L^6_{t,x}$-Strichartz estimates involving $K_1, \dots, K_5$. Pointwise bounds for the lower frequencies give the second estimate.
(3) If $K_1 \sim K_6$, the claim follows from six linear $L^6_{t,x}$-Strichartz estimates.

□

Proof of Lemma 5.6. By Lemma 5.8, it suffices to estimate the summation of boundary terms and spacetime terms. For the boundary terms, we use

$$a(N_1)N_1^{-1}\Big(\prod_{i=3}^{k+1} N_i^{1/2}\Big) \lesssim N^{-\epsilon} N_1^{\frac{1}{2},s_2+s'-2\epsilon} N_2^{\frac{1}{2},s_2+s'-2\epsilon} \prod_{i=3}^{k+1} N_i^{\frac{1}{2},\frac{1}{2}-2\varepsilon}$$

to find

$$\begin{aligned}&\sum_{N_1\sim N_2\gtrsim\dots\gtrsim N_{k+1}} a(N_1)N_1^{-1}\Big(\prod_{i=3}^{k+1} N_i^{1/2}\Big)\Big(\prod_{i=1}^{k+1} \|P_{N_i}u\|_{L_t^\infty([0,T],L^2_\lambda)}\Big)\\ &\lesssim N^{-\epsilon}\|u\|^2_{F_\lambda^{\overline{s}(\epsilon)+s'}}\|u\|^{k-1}_{F_\lambda^{s_1+\epsilon,1/2-\epsilon}}.\end{aligned}$$

For the spacetime terms, by Lemma 5.10, it suffices to estimate

$$\sum_{\substack{K_1\sim K_2\gtrsim\dots\gtrsim K_{2k}\\ K_1\sim K_2\gtrsim 1}} N^{2(s_2+s')-1}K_3 C(K_1,\dots,K_{2k})\prod_{i=1}^{2k}\|P_{K_i}u\|_{F^0_\lambda}, \tag{5.13}$$

where $C(K_1,\dots,K_{2k})$ is given in (5.12), and we used $a(N_1)N_1^{-1} \lesssim N^{2s_2-1}$ and $K_1 \sim K_2 \gtrsim N \gtrsim 1$. Due to (5.12), we have

$$N^{2(s_2+s')-1}K_3 C(K_1,\dots,K_{2k}) \lesssim N^{-\epsilon}K_1^{s_2+s'-2\epsilon}K_2^{s_2+s'-2\epsilon}\prod_{i=3}^{2k}K_i^{\frac{1}{2},\frac{1}{2}-2\epsilon},$$

provided that ϵ is sufficiently small. Therefore,

$$(5.13) \lesssim N^{-\epsilon}\|u\|^2_{F_\lambda^{\overline{s}(\epsilon)+s'}}\|u\|^{2k-2}_{F_\lambda^{s_1+\epsilon,1/2-\epsilon}}.$$

This completes the proof. □

5.3. **Contribution of c.**

Lemma 5.11 (Contribution of c)**.** *Let $T \in (0,1]$ and $0 < s_1 < 1/2$.*

- *If $s_2 > 1/2$, then there exists $\epsilon = \epsilon(\overline{s}) > 0$ such that we find*

$$\begin{aligned}&\sum_{\substack{N_1\sim N_2\gtrsim\dots\gtrsim N_{k+1}\\ N_1\sim N_2\gtrsim N}}\Big|\int_0^T\int_{\Gamma_\lambda^{k+1}} c(\xi_1,\dots,\xi_{k+1})\prod_{i=1}^{k+1}\chi_{N_i}(\xi_i)\hat{u}(s,\xi_i)\,d\Gamma_\lambda^{k+1}ds\Big|\\ &\lesssim N^{-\epsilon}\|u\|^2_{F_\lambda^{\overline{s}(\epsilon)+s'}}\|u\|^{k-1}_{F_\lambda^{\overline{s}(\epsilon)}}.\end{aligned}$$

- *If $k=4$ and $s_2=1/2$, we find*

$$\sum_{\substack{N_1\sim N_2\gtrsim\cdots\gtrsim N_{k+1}\\ N_1\sim N_2\gtrsim N}}\Big|\int_0^T\int_{\Gamma_\lambda^5} c(\xi_1,\dots,\xi_{k+1})\prod_{i=1}^5\chi_{N_i}(\xi_i)\hat{u}(s,\xi_i)\,d\Gamma_\lambda^5 ds\Big|$$
$$\lesssim \|u\|^2_{F_\lambda^{a_N}}\|u\|^3_{F_\lambda^{\bar{s}}}.$$

Here, we recall that the $F_\lambda^{a_N}$-norm is defined in (5.6).

Proof. By Lemma 5.5, c is nonzero only when $N_3\sim N_4$. By Remark 5.9, we may replace c by $(a(N_1)N_1^{-1}N_3+a(N_3))N_5$ and change back to the physical space representation. $a=a_N$ enables us to restrict to $N_1\sim N_2\gtrsim N\gtrsim 1$. Thus it suffices to estimate

$$\sum_{\substack{N_1\sim N_2\gtrsim N_3\sim N_4,\\ N_5\gtrsim\cdots\gtrsim N_{k+1},\\ N_1\sim N_2\gtrsim 1}}(a(N_1)N_1^{-1}N_3+a(N_3))N_5\Big|\int_0^T\int_{\lambda\mathbb{T}}\prod_{i=1}^{k+1}P_{N_i}u\,dxds\Big|.$$

We also note that the proof of Lemma 5.10 implies

$$\Big|\int_0^T\int_{\lambda\mathbb{T}}\prod_{i=1}^{k+1}P_{N_i}u\,dxds\Big|\lesssim\prod_{i=1}^4\|P_{N_i}u\|_{F_\lambda^0}\prod_{i=5}^{k+1}N_i^{1/2}\|P_{N_i}u\|_{F_\lambda^0}.$$

Therefore, it suffices to estimate

$$\sum_{\substack{N_1\sim N_2\gtrsim N_3\sim N_4,\\ N_5\gtrsim\cdots\gtrsim N_{k+1},\\ N_1\sim N_2\gtrsim 1}}(a(N_1)N_1^{-1}N_3+a(N_3))N_5\prod_{i=5}^{k+1}N_i^{1/2}\prod_{i=1}^{k+1}\|P_{N_i}u\|_{F_\lambda^0}. \tag{5.14}$$

If $s_2>1/2$, then we find

$$\begin{aligned}&(a(N_1)N_1^{-1}N_3+a(N_3))N_5\prod_{i=5}^{k+1}N_i^{1/2}\\ &\lesssim N^{-\epsilon}N_1^{1/2,s_2+s'-2\varepsilon}N_2^{1/2,s_2+s'-2\varepsilon}\prod_{i=3}^{k+1}N_i^{1/2,s_2-2\varepsilon}.\end{aligned}$$

Therefore,

$$(5.14)\lesssim N^{-\epsilon}\|u\|^2_{F_\lambda^{\bar{s}(\epsilon)+s'}}\|u\|^{k-1}_{F_\lambda^{\bar{s}(\epsilon)}}.$$

If $k=4$ and $s_2=1/2$, then we find

$$(a(N_1)N_1^{-1}N_3+a(N_3))N_5^{3/2}\lesssim a(N_1)N_5^{3/2}\lesssim N^{2s'+1}N_5^{3/2}.$$

Therefore,

$$\begin{aligned}(5.14)&\lesssim\sum_{N_1\sim N_2\gtrsim N_3\sim N_4\gtrsim N_5}N^{2s'+1}N_5^{3/2}\prod_{i=1}^5\|P_{N_i}u\|_{F_\lambda^0}\\ &\lesssim\|u\|_{F_\lambda^{\bar{s}}}\sum_{N_1\sim N_2\gtrsim N_3\sim N_4}N^{2s'+1}N_4\prod_{i=1}^4\|P_{N_i}u\|_{F_\lambda^0}\lesssim\|u\|^2_{F_\lambda^{\bar{s}+s'}}\|u\|^3_{F_\lambda^{\bar{s}}}\end{aligned}$$

$\square$

The proof of Lemma 5.4 (and hence that of Proposition 5.1) is now completed by Lemmas 5.6 and 5.11.

6. Estimates for differences of solutions

The goal of this section is to estimate the differences of solutions: Proposition 6.1. We firstly prove the Lipschitz continuity in a weaker topology $H^{\overline{s}-1}_{\lambda}$ for $H^{\overline{s}}_{\lambda}$-solutions (6.1). Furthermore, we bound the $H^{\overline{s}}_{\lambda}$-difference of solutions by the $H^{\overline{s}-1}_{\lambda}$-difference estimate and the $H^{\overline{s}+1}_{\lambda}$-a priori bound (6.2), as usual for the Bona-Smith approximation (cf. [23, Section 4]).

Proposition 6.1 (Energy estimates for differences of solutions)**.** *Let $\overline{s} = (s_1, s_2)$ with $0 < s_1 < 1/2$ and $s_2 \geq 3/4$. Let u_1 and u_2 be smooth solutions to (1.1) defined on $[0,T] \subseteq [0,1]$. Set $v = u_1 - u_2$. Then, we find the following estimates to hold:*

(1) *Lipschitz continuity in $H^{\overline{s}-1}_{\lambda}$ for $H^{\overline{s}}_{\lambda}$ solutions:*

$$\begin{aligned} \|v\|^2_{E^{\overline{s}-1}_{\lambda}(T)} \lesssim \|v(0)\|^2_{H^{\overline{s}-1}_{\lambda}} &+ \|v\|^2_{F^{\overline{s}-1}_{\lambda}}(\|u_1\|_{F^{\overline{s}}_{\lambda}} + \|u_2\|_{F^{\overline{s}}_{\lambda}})^{k-1} \\ &+ \|v\|^2_{F^{\overline{s}-1}_{\lambda}}(\|u_1\|_{F^{\overline{s}}_{\lambda}} + \|u_2\|_{F^{\overline{s}}_{\lambda}})^{2k-2}. \end{aligned} \tag{6.1}$$

(2) *Continuity in $H^{\overline{s}}_{\lambda}$:*

$$\begin{aligned} \|v\|^2_{E^{\overline{s}}_{\lambda}(T)} \lesssim \|v(0)\|^2_{H^{\overline{s}}_{\lambda}} &+ \|v\|^2_{F^{\overline{s}}_{\lambda}}(\|v\|_{F^{\overline{s}}_{\lambda}} + \|u_2\|_{F^{\overline{s}}_{\lambda}})^{k-1} \\ &+ \|v\|_{F^{\overline{s}}_{\lambda}}\|v\|_{F^{\overline{s}-1}_{\lambda}}\|u_2\|_{F^{\overline{s}+1}_{\lambda}}(\|v\|_{F^{\overline{s}}_{\lambda}} + \|u_2\|_{F^{\overline{s}}_{\lambda}})^{k-2} \\ &+ \|v\|^2_{F^{\overline{s}}_{\lambda}}(\|v\|_{F^{\overline{s}}_{\lambda}} + \|u_2\|_{F^{\overline{s}}_{\lambda}})^{2k-2} \\ &+ \|v\|_{F^{\overline{s}}_{\lambda}}\|v\|_{F^{\overline{s}-1}_{\lambda}}\|u_2\|_{F^{\overline{s}+1}_{\lambda}}(\|v\|_{F^{\overline{s}}_{\lambda}} + \|u_2\|_{F^{\overline{s}}_{\lambda}})^{2k-3}. \end{aligned} \tag{6.2}$$

We start with the equation for v:

$$\partial_t v + \mathcal{H}\partial_{xx} v = \partial_x(u_1^k - u_2^k).$$

We write $\partial_x(u_1^k - u_2^k)$ in two ways. A standard way of writing $\partial_x(u_1^k - u_2^k)$ is

$$\partial_x(u_1^k - u_2^k) = \partial_x(v(u_1^{k-1} + u_1^{k-2}u_2 + \cdots + u_2^{k-1})).$$

For simplicity of notations, let us express this as

$$\partial_x(u_1^k - u_2^k) = \partial_x(vu^{k-1}), \tag{6.3}$$

where u^{k-1} means a linear combination of $u_1^{k-1}, u_1^{k-2}u_2, \ldots, u_2^{k-1}$. We use (6.3) to show (6.1). However, when we show (6.2), we express $\partial_x(u_1^k - u_2^k)$ in another way. It is straight-forward that there exist integers $c_0, \ldots, c_{k-2}$ and $d_0, \ldots, d_{k-3}$ such that

$$\partial_x(u_1^k - u_2^k) = (\partial_x v)(\sum_{i=0}^{k-2} c_i u_1^{k-1-i}u_2^i) + v(\partial_x u_2)(\sum_{i=0}^{k-3} d_i u_1^{k-2-i}u_2^i).$$

We compactly write this as

$$\partial_x(u_1^k - u_2^k) = (\partial_x v)u^{k-1} + vw^{k-1}, \tag{6.4}$$

where u^{k-1} means a linear combination of $u_1^{k-1}, u_1^{k-2}u_2, \ldots, u_2^{k-1}$ as above, but w^{k-1} means a linear combination of $u_1^{k-2}\partial_x u_2, u_1^{k-3}u_2\partial_x u_2, \ldots, u_2^{k-2}\partial_x u_2$. An advantage of using (6.4) is that ∂_x is not applied to u_1 so that we can avoid $F^{7/4}$-norm for u_1, as stated in (6.2).

For (6.1), we take P_N to (6.3) for $N \geq 1$ (the case $N < 1$ does not require an estimate by the definition of the E^s_λ-norm), multiply with $N^{s_2-1} P_N v$, and then integrate on $[0,t] \times \lambda\mathbb{T}$ to get

$$N^{2s_2-2} \|P_N v(t)\|^2_{L^2_\lambda} = N^{2s_2-2} \|P_N v(0)\|^2_{L^2_\lambda} + N^{2s_2-2} \int_0^t \int_{\lambda\mathbb{T}} P_N v \partial_x P_N (v u^{k-1}) dx ds$$

for any $t \in [0,T]$ and N. Thus,

$$\|v\|^2_{E^{\bar{s}-1}_\lambda} \leq \|v(0)\|^2_{H^{\bar{s}-1}_\lambda} + \sum_{N \geq 1} N^{2s_2-2} \sup_{t \in [0,T]} \Big| \int_0^t \int_{\lambda\mathbb{T}} P_N v \partial_x P_N (v u^{k-1}) dx ds \Big|.$$

We split the integrand into the sum of Littlewood-Paley pieces:

$$\begin{aligned} (6.5) \quad & \sum_{N \geq 1} N^{2s_2-2} \sup_{t \in [0,T]} \Big| \int_0^t \int_{\lambda\mathbb{T}} P_N v \partial_x P_N (v u^{k-1}) dx ds \Big| \\ & \leq \sum_{\substack{K, K_1, \ldots, K_{k-1}, \\ N \geq 1}} N^{2s_2-2} \sup_{t \in [0,T]} \Big| \int_0^t \int_{\lambda\mathbb{T}} P_N v \partial_x P_N (P_K v P_{K_1} u \ldots P_{K_{k-1}} u) dx ds \Big|. \end{aligned}$$

Here, $N \geq 1$ and $P_{K_i} u$ can be either $P_{K_i} u_1$ or $P_{K_i} u_2$.

For (6.2), we similarly find

$$\|v\|^2_{E^{\bar{s}}_\lambda} \leq \|v(0)\|^2_{H^{\bar{s}}_\lambda} + \sum_{N \geq 1} N^{2s_2} \sup_{t \in [0,T]} \Big| \int_0^t \int_{\lambda\mathbb{T}} P_N v \partial_x P_N (v u^{k-1}) dx ds \Big|,$$

and further using the expression (6.4),

$$\begin{aligned} (6.6) \quad & \sum_{N \geq 1} N^{2s_2} \sup_{t \in [0,T]} \Big| \int_0^t \int_{\lambda\mathbb{T}} P_N v \, P_N ((\partial_x v) u^{k-1} + v w^{k-1}) dx ds \Big| \\ & \leq \sum_{\substack{K, K_1, \ldots, K_{k-1}, \\ N \geq 1}} N^{2s_2} \Big(\sup_{t \in [0,T]} \Big| \int_0^t \int_{\lambda\mathbb{T}} P_N v P_N ((\partial_x P_K v) P_{K_1} u \ldots P_{K_{k-1}} u) dx ds \Big| \\ & \qquad + \sup_{t \in [0,T]} \Big| \int_0^t \int_{\lambda\mathbb{T}} P_N v P_N (P_K v P_{K_1} w \ldots P_{K_{k-1}} w) dx ds \Big| \Big). \end{aligned}$$

Here, $P_{K_i} u$ can be either $P_{K_i} u_1$ or $P_{K_i} u_2$; $P_{K_i} w$ can be either $P_{K_i} u_1$, $P_{K_i} u_2$, or $P_{K_i} \partial_x u_2$, but $P_{K_i} \partial_x u_2$ should appear *exactly once* among $P_{K_1} w, \ldots, P_{K_{k-1}} w$.

We may assume $K_1 \geq \cdots \geq K_{k-1}$. Let $M_1, \ldots, M_{k+1}$ be the decreasing rearrangement of $N, K, K_1, \ldots, K_{k-1}$. In particular, $M_1 \sim M_2 \gtrsim 1$. We distinguish three cases:

- $M_1 \sim M_2 \gtrsim M_3 \gg M_4$: We treat this case in Section 6.1. Here, we have seen in (5.9) that $|\Omega_k| \sim M_1 M_3$ and integrate by parts in time. The estimates allow for $s_2 > \frac{1}{2}$.
- $M_1 \sim M_2 \gg M_3 \sim M_4$: We treat this case in Section 6.2. We do not integrate by parts in time, but apply two bilinear Strichartz estimates to the four highest frequencies. The estimates allow for $s_2 > \frac{1}{2}$.
- $M_1 \sim M_2 \sim M_3 \sim M_4$: We treat this case in Section 6.3. We do not integrate by parts in time. We merely apply the linear Strichartz estimates ($L^8_t L^4_x$) to the highest four frequencies. The estimates allow for $s_2 \geq \frac{3}{4}$. This is why we set $s_2 = \frac{3}{4}$ in Proposition 6.1.

6.1. **Case** $M_1 \sim M_2 \gtrsim M_3 \gg M_4$**.** Recall from (5.9) that

$$|\Omega_k| \sim M_1 M_3.$$

We handle these interactions via integration by parts in time (only for $M_3 \gtrsim 1$). The following lemma systematically treats the error terms arising from integration by parts in time.

Lemma 6.2 (Integration by parts in time)**.** *Let $T \in (0,1]$; let $\overline{s} = (s_1, s_2)$ be such that $0 < s_1 < 1/2 < s_2$. Assume $M_1 \sim M_2 \gtrsim M_3 \gg M_4$ and $M_1 \gtrsim 1$.*

- *For the $H^{\overline{s}-1}_\lambda$-estimate, we find*

$$\sup_{t \in [0,T]} \Big| \int_0^t \int_{\lambda \mathbb{T}} P_N v P_K v P_{K_1} u \dots P_{K_{k-1}} u \, dx ds \Big|$$
$$\lesssim (M_1^{-1} + \|u\|^{k-1}_{F^{\overline{s}}_\lambda}) M_3^{0,-1} \Big(\prod_{i=3}^{k+1} M_i^{1/2} \Big) c_N^{(v,\overline{s},-1)} c_K^{(v,\overline{s},-1)} \prod_{i=1}^{k-1} c_{K_i}^{(u,\overline{s})}. \tag{6.7}$$

- *For the $H^{\overline{s}}_\lambda$-estimate, we find*

$$\sup_{t \in [0,T]} \Big| \int_0^t \int_{\lambda \mathbb{T}} P_N v \partial_x P_K v P_{K_1} u \dots P_{K_{k-1}} u \, dx ds \Big|$$
$$\lesssim K M_3^{0,-1} \Big(\prod_{i=3}^{k+1} M_i^{1/2} \Big) \Big(\prod_{i=1}^{k-1} c_{K_i}^{(u,\overline{s})} \Big) \tag{6.8}$$
$$\times \Big\{ (M_1^{-1} + \|u\|^{k-1}_{F^{\overline{s}}_\lambda}) c_N^{(v,\overline{s})} c_K^{(v,\overline{s})} + \|u\|^{k-2}_{F^{\overline{s}}_\lambda} \|v\|_{F^{\overline{s}}_\lambda} (c_N^{(v,\overline{s})} c_K^{(u,\overline{s})} + c_N^{(u,\overline{s})} c_K^{(v,\overline{s})}) \Big\}$$

and

$$\sup_{t \in [0,T]} \Big| \int_0^t \int_{\mathbb{T}} P_N v P_K v P_{K_1} w \dots P_{K_{k-1}} w \, dx ds \Big|$$
$$\lesssim M_3^{0,-1} \Big(\prod_{i=3}^{k+1} M_i^{1/2} \Big) \Big(\prod_{i=1}^{k-1} c_{K_i}^{(w,\overline{s})} \Big) \tag{6.9}$$
$$\times \Big\{ (M_1^{-1} + \|u\|^{k-1}_{F^{\overline{s}}_\lambda}) c_N^{(v,\overline{s})} + \|u\|^{k-2}_{F^{\overline{s}}_\lambda} \|v\|_{F^{\overline{s}}_\lambda} c_N^{(u,\overline{s})} \Big\} c_K^{(v,\overline{s},-1)},$$

where $c_{K_i}^{(w,\overline{s})} = K_i c_{K_i}^{(u_2,\overline{s},1)}$ when $w = \partial_x u_2$.

Here, we recall that the quantities $c_N^{(v,\overline{s},s')}$ are defined in (4.4).

Proof. When $M_3 \ll 1$, the estimates follow from applying two short-time bilinear Strichartz estimates to $M_1, \dots, M_4$ after localizing to intervals of length M_1^{-1}.

From now on, we assume $M_3 \gtrsim 1$. Here we focus on (6.7). The proofs of (6.8) and (6.9) will be briefly sketched at the end of the proof. Fix $t \in [0,T]$. The following estimates will be uniform for $t \in [0,T]$. Since $|\Omega_k| \sim M_1 M_3$, we integrate

$$\int_0^t \int_{\lambda \mathbb{T}} P_N v P_K v P_{K_1} u \dots P_{K_{k-1}} u \, dx ds$$

by parts in time to get the boundary term

$$M_1^{-1} M_3^{-1} \int_{\lambda \mathbb{T}} P_N v P_K v P_{K_1} u \dots P_{K_{k-1}} u \, dx \Big|_0^t$$

and the spacetime term

$$\begin{aligned} M_1^{-1}M_3^{-1}\Big(&\int_0^t \int_{\lambda\mathbb{T}} P_N\partial_x(vu^{k-1})P_K v P_{K_1}u\dots P_{K_{k-1}}u\,dxds \\ &+ \int_0^t \int_{\lambda\mathbb{T}} P_N v P_K\partial_x(vu^{k-1})P_{K_1}u\dots P_{K_{k-1}}u\,dxds \\ &+ \sum_{j=1}^{k-1}\int_0^t \int_{\lambda\mathbb{T}} P_N v P_K v P_{K_j}\partial_x(u^{k-1})\prod_{i\neq j} P_{K_i}u\,dxds\Big). \end{aligned}$$

The boundary term is estimated by applying L_x^2 to the two highest frequencies, applying pointwise bounds for the remaining factors:

$$\Big(M_1^{-1}M_3^{-1}\prod_{i=3}^{k+1} M_i^{1/2}\Big)\|P_N v\|_{L_t^\infty([0,T],L_\lambda^2)}\|P_K v\|_{L_t^\infty([0,T],L_\lambda^2)}\prod_{i=1}^{k-1}\|P_{K_i}u\|_{L_t^\infty([0,T],L_\lambda^2)}.$$

The spacetime term is estimated by applying L_x^2 to the two highest frequencies, pointwise bounds to the remaining factors, and L_t^1 to $\partial_x(vu^{k-1})$ or $\partial_x(u^k)$. We find by Lemma 4.4 ($s'=-1$ for $\partial_x(vu^{k-1})$ and $s'=0$ for $\partial_x(u^k)$):

$$\|u\|_{F_\lambda^{\overline{s}}}^{k-1}\Big(M_3^{-1}\prod_{i=3}^{k+1}M_i^{1/2}\Big)c_N^{(v,\overline{s},-1)}c_K^{(v,\overline{s},-1)}\prod_{i=1}^{k-1}c_{K_i}^{(u,\overline{s})}.$$

This completes the proof of (6.7).

For (6.8), we replace ∂_x by K and integrate by parts in time. We apply Lemma 4.4 with $s'=0$ for both $\partial_x(vu^{k-1})$ and $\partial_x(u^k)$. For (6.9), we apply Lemma 4.4 with $s'=-1$ for $P_K\partial_x(vu^{k-1})$, $s'=0$ for $P_N\partial_x(vu^{k-1})$ and $P_{K_i}\partial_x(u^k)$, and $s'=1$ for $P_{K_i}\partial_{xx}(u_2^k)$. For the last one $P_{K_i}\partial_{xx}(u_2^k)$, we use

$$\|P_{K_i}\partial_{xx}(u_2^k)\|_{L_t^1([0,T],L_\lambda^2)} \lesssim K_i\|P_{K_i}\partial_x(u_2^k)\|_{L_t^1([0,T],L_\lambda^2)} \lesssim K_i^{1/2,1}\|u_2\|_{F_\lambda^{\overline{s}}}^{k-1}K_i c_{K_i}^{(u_2,\overline{s},1)}.$$

This finishes the proof. $\square$

From now on, we estimate (6.5) and (6.6) using the above integration by parts in time lemma. Recall again that $M_1 \sim M_2 \gtrsim N \gtrsim 1$.

Case A: $N\in\{M_1,M_2\}$.

<u>Subcase I:</u> $K\in\{M_1,M_2\}$. We start with the $H_\lambda^{\overline{s}-1}$-estimate. In this case, we can integrate by parts in (6.5) to move the derivative ∂_x to $K_1,\dots,K_{k-1}$. We may replace ∂_x by K_1 and let $K=N$ (due to $M_1\sim M_2$). Then (6.5) is of the form

$$\sum_{N\gtrsim 1}\sum_{K_1,\dots,K_{k-1}\lesssim N} N^{2s_2-2}K_1\sup_{t\in[0,T]}\Big|\int_0^t\int_{\lambda\mathbb{T}}(P_N v)^2 P_{K_1}u\dots P_{K_{k-1}}u\,dxds\Big|.$$

By (6.7), each summand is estimated by

$$(N^{-1}+\|u\|_{F_\lambda^{\overline{s}}}^{k-1})(d_N^{(v,\overline{s},-1)})^2\prod_{i=1}^{k-1}K_i^{1/2}c_{K_i}^{(u,\overline{s})},$$

where we recall the definition $d_N^{(v,\overline{s},-1)}$ of (4.5). Using (4.6), this sums up to (6.1).

For the $H_\lambda^{\overline{s}}$-estimate, we consider (6.6). As before, we can move the derivative ∂_x on v to $K_1,\dots,K_{k-1}$. We may replace ∂_x by K_1 and let $K=N$. Then (6.6) is of the form

$$\sum_{N\gtrsim 1}\sum_{K_1,\dots,K_{k-1}\lesssim N} N^{2s_2}K_1\sup_{t\in[0,T]}\left|\int_0^t\int_{\lambda\mathbb{T}}(P_Nv)^2P_{K_1}u\dots P_{K_{k-1}}udxds\right|.$$

By a variant of (6.8) (deleting K), each summand is estimated by

$$\Big(\prod_{i=1}^{k-1}K_i^{1/2}c_{K_i}^{(u,\overline{s})}\Big)\Big\{(N^{-1}+\|u\|_{F_\lambda^{\overline{s}}}^{k-1})(d_N^{(v,\overline{s})})^2+\|u\|_{F_\lambda^{\overline{s}}}^{k-2}\|v\|_{F_\lambda^{\overline{s}}}d_N^{(v,\overline{s})}d_N^{(u,\overline{s})}\Big\}.$$

Using (4.6), this sums up to (6.2).

<u>Subcase II:</u> $K\in\{M_3,\dots,M_{k+1}\}$. We may assume $K_1=N$. We start with the $H_\lambda^{\overline{s}-1}$-estimate. Replacing ∂_xP_N by N, we read (6.5) as

$$\sum_{N\gtrsim 1}\sum_{K,K_2,\dots,K_{k-1}\lesssim N} N^{2s_2-1}\sup_{t\in[0,T]}\left|\int_0^t\int_{\lambda\mathbb{T}}P_NvP_KvP_Nu\dots P_{K_{k-1}}udxds\right|.$$

By (6.7), each summand is estimated by (use $M_3^{0,-1}\lesssim K^{0,-1}$ and $K^{\frac12-s_1,-s_2+\frac12}\lesssim K^{0+,0-}$)

$$\Big(\prod_{i=2}^{k-1}K_i^{1/2}c_{K_i}^{(u,\overline{s})}\Big)(N^{-1}+\|u\|_{F_\lambda^{\overline{s}}}^{k-1})d_N^{(v,\overline{s},-1)}K^{0+,0-}d_K^{(v,\overline{s},-1)}d_N^{(u,\overline{s})}.$$

This sums up to (6.1).

For the H_λ^s-estimate, we apply (6.8) and (6.9) to each summand of the expression (6.6) by the sum of (use again $M_3^{0,-1}\lesssim K^{0,-1}$ and $K^{\frac12-s_1,-s_2+\frac12}\lesssim K^{0+,0-}$)

$$d_N^{(u,\overline{s})}\Big(\prod_{i=2}^{k-1}K_i^{1/2}c_{K_i}^{(u,\overline{s})}\Big)K^{0+,0-}\Big\{(N^{-1}+\|u\|_{F_\lambda^{\overline{s}}}^{k-1})d_N^{(v,\overline{s})}d_K^{(v,\overline{s})}$$
$$+\|u\|_{F_\lambda^{\overline{s}}}^{k-2}\|v\|_{F_\lambda^{\overline{s}}}(d_N^{(v,\overline{s})}d_K^{(u,\overline{s})}+d_N^{(u,\overline{s})}d_K^{(v,\overline{s})})\Big\}$$

and

$$d_N^{(w,\overline{s})}\Big(\prod_{i=2}^{k-1}K_i^{1/2}c_{K_i}^{(w,\overline{s})}\Big)K^{0+,0-}\Big\{(N^{-1}+\|u\|_{F_\lambda^{\overline{s}}}^{k-1})d_N^{(v,\overline{s})}+\|u\|_{F_\lambda^{\overline{s}}}^{k-2}\|v\|_{F_\lambda^{\overline{s}}}d_N^{(u,\overline{s})}\Big\}d_K^{(v,\overline{s},-1)}.$$

Using $\|w\|_{F_\lambda^{\overline{s}}}^{k-1}\lesssim\|u_2\|_{F_\lambda^{\overline{s}+1}}\|u\|_{F_\lambda^{\overline{s}}}^{k-2}$, these sum up to (6.2).

Case B: $N\in\{M_3,\dots,M_{k+1}\}$.

<u>Subcase I:</u> $K\in\{M_1,M_2\}$. Note that $K_1\sim K$. We start with the $H_\lambda^{\overline{s}-1}$-estimate. Replacing ∂_xP_N by N, we read (6.5) as

$$\sum_{N\gtrsim 1}\sum_{\substack{K\sim K_1\gtrsim N\\ K_2,\dots,K_{k-1}\lesssim K_1}} N^{2s-1}\sup_{t\in[0,T]}\left|\int_0^t\int_{\lambda\mathbb{T}}P_NvP_KvP_{K_1}u\dots P_{K_{k-1}}udxds\right|.$$

By (6.7), each summand is estimated by (use $M_3^{0,-1}\le N^{-1}$)

$$d_{K_1}^{(u,\overline{s})}\Big(\prod_{i=2}^{k-1}K_i^{1/2}c_{K_i}^{(u,\overline{s})}\Big)(K^{-1}+\|u\|_{F_\lambda^{\overline{s}}}^{k-1})N^{s_2-\frac12}K^{-2s_2+1}d_N^{(v,\overline{s},-1)}d_K^{(v,\overline{s},-1)}.$$

Using $s_2>\frac12$, this easily sums up to (6.1).

For the H^s_λ-estimate, we apply (6.8) and (6.9) to each summand of the expression (6.6) by the sum of (use again $M_3^{0,-1} \le N^{-1}$)

$$d_{K_1}^{(u,\bar{s})}\Big(\prod_{i=2}^{k-1} K_i^{1/2} c_{K_i}^{(u,\bar{s})}\Big) N^{s_2-\frac{1}{2}} K^{-2s_2+1}\Big\{(K^{-1} + \|u\|_{F_\lambda^{\bar{s}}}^{k-1}) d_N^{(v,\bar{s})} d_K^{(v,\bar{s})} + \|u\|_{F_\lambda^{\bar{s}}}^{k-2}\|v\|_{F_\lambda^{\bar{s}}}(d_N^{(v,\bar{s})} d_K^{(u,\bar{s})} + d_N^{(u,\bar{s})} d_K^{(v,\bar{s})})\Big\}$$

and

$$d_{K_1}^{(w,\bar{s})}\Big(\prod_{i=2}^{k-1} K_i^{1/2} c_{K_i}^{(w,\bar{s})}\Big) N^{s_2-\frac{1}{2}} K^{-2s_2+1} \times \Big\{(N^{-1} + \|u\|_{F_\lambda^{\bar{s}}}^{k-1}) d_N^{(v,\bar{s})} + \|u\|_{F_\lambda^{\bar{s}}}^{k-2}\|v\|_{F_\lambda^{\bar{s}}} d_N^{(u,\bar{s})}\Big\} d_K^{(v,\bar{s},-1)}.$$

Using $s_2 > \frac{1}{2}$ and $\|w\|_{F_\lambda^{\bar{s}}}^{k-1} \lesssim \|u_2\|_{F_\lambda^{\bar{s}+1}} \|u\|_{F_\lambda^{\bar{s}}}^{k-2}$, these sum up to (6.2).

Subcase II: $K \in \{M_3, \ldots, M_{k+1}\}$. Note that $K_1 = M_1$ and $K_2 = M_2$. We start with the $H_\lambda^{\bar{s}-1}$-estimate. We replace $\partial_x P_N$ by N to rewrite (6.5) as

$$\sum_{N\ge 1} \sum_{\substack{K_1\sim K_2 \gtrsim N \\ K,K_3,\ldots,K_{k-1}\lesssim K_2}} N^{2s_2-1} \sup_{t\in[0,T]} \Big| \int_0^t \int_{\lambda\mathbb{T}} P_N v P_K v P_{K_1} u \ldots P_{K_{k-1}} u dx ds\Big|.$$

By (6.7), each summand is estimated by (use $M_3^{0,-1} \lesssim K^{0,-1}$ and $K^{\frac{1}{2}-s_1,-s_2+\frac{1}{2}} \lesssim K^{0+,0-}$)

$$(K_1^{-1} + \|u\|_{F_\lambda^{\bar{s}}}^{k-1}) N^{s_2+\frac{1}{2}} K^{0+,0-} K_1^{-2s_2} d_{K_1}^{(u,\bar{s})} d_{K_2}^{(u,\bar{s})} \Big(\prod_{i=3}^{k-1} K_i^{1/2} c_{K_i}^{(u,\bar{s})}\Big) d_N^{(v,\bar{s},-1)} d_K^{(v,\bar{s},-1)}.$$

Using $s_2 > \frac{1}{2}$, this sums up to (6.1).

For the H^s_λ-estimate, we apply (6.8) and (6.9) to each summand of the expression (6.6) by the sum of (use again $M_3^{0,-1} \lesssim K^{0,-1}$ and $K^{\frac{1}{2}-s_1,-s_2+\frac{1}{2}} \lesssim K^{0+,0-}$)

$$N^{s_2+\frac{1}{2}} K^{0+,0-} K_1^{-2s_2} d_{K_1}^{(u,\bar{s})} d_{K_2}^{(u,\bar{s})} \Big(\prod_{i=3}^{k-1} K_i^{1/2} c_{K_i}^{(u,\bar{s})}\Big) \times \Big\{(K_1^{-1} + \|u\|_{F_\lambda^{\bar{s}}}^{k-1}) d_N^{(v,\bar{s})} d_K^{(v,\bar{s})} + \|u\|_{F_\lambda^{\bar{s}}}^{k-1}\|v\|_{F_\lambda^{\bar{s}}}(d_N^{(v,\bar{s})} d_K^{(u,\bar{s})} + d_N^{(u,\bar{s})} d_K^{(v,\bar{s})})\Big\}$$

and

$$N^{s_2+\frac{1}{2}} K^{0+,0-} K_1^{-2s_2} d_{K_1}^{(w,\bar{s})} d_{K_2}^{(w,\bar{s})} \Big(\prod_{i=3}^{k-1} K_i^{1/2} c_{K_i}^{(w,\bar{s})}\Big) \times \Big\{(K_1^{-1} + \|u\|_{F_\lambda^{\bar{s}}}^{k-1}) d_N^{(v,\bar{s})} + \|u\|_{F_\lambda^{\bar{s}}}^{k-2}\|v\|_{F_\lambda^{\bar{s}}} d_N^{(u,\bar{s})}\Big\} d_K^{(v,\bar{s},-1)}.$$

Using $s_2 > \frac{1}{2}$ and $\|w\|_{F_\lambda^{\bar{s}}}^{k-1} \lesssim \|u_2\|_{F_\lambda^{\bar{s}+1}} \|u\|_{F_\lambda^{\bar{s}}}^{k-2}$, these easily sum up to (6.2).

Therefore, the proofs of (6.1) and (6.2) are completed in case of $M_1 \sim M_2 \gtrsim M_3 \gg M_4$.

6.2. **Case** $M_1 \sim M_2 \gg M_3 \sim M_4$**.** We directly estimate (6.5) and (6.6) in this case. We do not integrate by parts in time. We can use two bilinear Strichartz estimates in the form

$$\begin{aligned} &\sup_{t\in[0,T]} \left| \int_0^t \int_{\lambda\mathbb{T}} P_{M_1}u_1 \dots P_{M_m}u_m dxds \right| \\ &\lesssim M_3^{1/2,0} M_4^{1/2,0} \Big(\prod_{i=5}^m M_i^{1/2} \Big) \prod_{i=1}^m \|P_{M_i}u_i\|_{F_\lambda^0}. \end{aligned} \tag{6.10}$$

Case A: $N \in \{M_1, M_2\}$.

Subcase I: $K \in \{M_1, M_2\}$. We start with the $H_\lambda^{\bar{s}-1}$-estimate. In this case, we can perform integration by parts in space to the expression (6.5) to move the derivative ∂_x to $K_1, \dots, K_{k-1}$. Replacing ∂_x by K_1, we need to estimate

$$\sum_{N\geq 1} \sum_{\substack{K_1\sim K_2 \ll N \\ K_3,\dots,K_{k-1}\lesssim K_2}} N^{2s_2-2}K_1 \sup_{t\in[0,T]} \left| \int_0^t \int_{\lambda\mathbb{T}} (P_N v)^2 P_{K_1}u \dots P_{K_{k-1}}u \, dxds \right|.$$

By (6.10) and $K_1 \sim K_2$, each summand is estimated by

$$\|P_N v\|_{F_\lambda^{\bar{s}-1}}^2 \prod_{i=1}^{k-1} \|P_{K_i}u\|_{F_\lambda^{1/2}}.$$

This sums up to (6.1).

For the $H_\lambda^{\bar{s}}$-estimate, we also integrate by parts in space to the expression (6.6) to assume that ∂_x is applied to $K_1, \dots, K_{k-1}$. Replacing ∂_x by K_1, it suffices to estimate

$$\sum_{N\geq 1} \sum_{\substack{K_1\sim K_2 \ll N \\ K_3,\dots,K_{k-1}\lesssim K_2}} N^{2s_2}K_1 \sup_{t\in[0,T]} \left| \int_0^t \int_{\lambda\mathbb{T}} (P_N v)^2 P_{K_1}u \dots P_{K_{k-1}}u \, dxds \right|.$$

By (6.10) and $K_1 \sim K_2$, each summand is estimated by

$$\|P_N v\|_{F_\lambda^{\bar{s}}}^2 \prod_{i=1}^{k-1} \|P_{K_i}u\|_{F_\lambda^{1/2}}.$$

This sums up to (6.2).

Subcase II: $K \in \{M_3, \dots, M_{k+1}\}$. We may assume $K_1 = N$. For the $H_\lambda^{\bar{s}-1}$-estimate, it suffices to estimate

$$\sum_{N\geq 1} \sum_{\substack{M_3\sim M_4 \ll N \\ M_5,\dots,M_{k+1}\lesssim M_3}} N^{2s_2-1} \sup_{t\in[0,T]} \left| \int_0^t \int_{\lambda\mathbb{T}} P_N v P_K v P_N u \dots P_{K_{k-1}}u \, dxds \right|.$$

Applying (6.10) and $M_3^{1/2,0}M_4^{1/2,0} \lesssim K^{1/2,-1/2}K_2^{1/2}$ (due to $M_3 \sim M_4 \sim K_2 \gtrsim K$), each summand is estimated by

$$\begin{aligned} &N^{2s_2-1}K^{1/2,-1/2}\|P_N v\|_{F_\lambda^0}\|P_K v\|_{F_\lambda^0}\|P_N u\|_{F_\lambda^0} \prod_{i=2}^{k-1} \|P_{K_i}u\|_{F_\lambda^{1/2}} \\ &\lesssim \|P_N v\|_{F_\lambda^{\bar{s}-1}} \|P_N u\|_{F_\lambda^{\bar{s}}} K^{0+,0-} \|P_K v\|_{F_\lambda^{\bar{s}-1}} \prod_{i=2}^{k-1} \|P_{K_i}u\|_{F_\lambda^{1/2}}. \end{aligned}$$

This sums up to (6.1).

For the $H^{\overline{s}}_\lambda$-estimate, we apply (6.10) to (6.6) to estimate each summand by

$$\|P_N v\|_{F^{\overline{s}}_\lambda}\Big(K^{0+,0-}\|P_K v\|_{F^{\overline{s}}_\lambda}\|P_N u\|_{F^{\overline{s}}_\lambda}\prod_{i=2}^{k-1}\|P_{K_i}u\|_{F^{1/2}_\lambda} + K^{0+,0-}\|P_K v\|_{F^{\overline{s}-1}_\lambda}\|P_N w\|_{F^{\overline{s}}_\lambda}\prod_{i=2}^{k-1}\|P_{K_i}w\|_{F^{1/2}_\lambda}\Big).$$

This sums up to (6.2).

Case B: $N \in \{M_3,\dots,M_{k+1}\}$.

Subcase I: $K \in \{M_1, M_2\}$. Note that $K_1 \sim K$ and $M_3 \sim M_4 \gtrsim N \gtrsim 1$. For the $H^{\overline{s}-1}_\lambda$-estimate, it suffices to estimate

$$\sum_{\substack{M_1\sim M_2\gg M_3\sim M_4\gtrsim 1\\ M_5,\dots,M_{k+1}\lesssim M_4}} N^{2s_2-1}\sup_{t\in[0,T]}\Big|\int_0^t\int_{\lambda\mathbb{T}} P_N v P_K v P_{K_1}u\dots P_{K_{k-1}}u\,dxds\Big|.$$

Using (6.10), $K_1 \sim K$, and $N^{1/2} \lesssim K_2^{1/2}$ (due to $K_2 \in \{M_3, M_4\}$), each summand is estimated by

$$\begin{aligned}&N^{2s_2-1}\Big(\prod_{i=5}^{k+1}M_i^{1/2}\Big)\|P_N v\|_{F^0_\lambda}\|P_K v\|_{F^0_\lambda}\prod_{i=1}^{k-1}\|P_{K_i}u\|_{F^0_\lambda}\\ &\lesssim N^{s_2-\frac12}K^{-2s_2+1}\|P_N v\|_{F^{\overline{s}-1}_\lambda}\|P_K v\|_{F^{\overline{s}-1}_\lambda}\|P_{K_1}u\|_{F^{\overline{s}}_\lambda}\prod_{i=2}^{k-1}\|P_{K_i}u\|_{F^{1/2}_\lambda}.\end{aligned}$$

Using $s_2 > \frac12$ to guarantee $-s_2 + \frac12 < 0$, this sums up to (6.1).

For the $H^{\overline{s}}_\lambda$-estimate, we apply (6.10) to (6.6) to estimate each summand by

$$N^{s_2-\frac12}K^{-2s_2+1}\|P_N v\|_{F^{\overline{s}}_\lambda}\Big(\|P_K v\|_{F^{\overline{s}}_\lambda}\|P_{K_1}u\|_{F^{\overline{s}}_\lambda}\prod_{i=2}^{k-1}\|P_{K_i}u\|_{F^{1/2}_\lambda} + \|P_K v\|_{F^{\overline{s}-1}_\lambda}\|P_{K_1}w\|_{F^{\overline{s}}_\lambda}\prod_{i=2}^{k-1}\|P_{K_i}w\|_{F^{1/2}_\lambda}\Big).$$

Using $s_2 > \frac12$ to guarantee $-s_2 + \frac12 < 0$, this sums up to (6.2).

Subcase II: $K \in \{M_3,\dots,M_{k+1}\}$. We still have $M_3 \sim M_4 \gtrsim N \gtrsim 1$. For the $H^{\overline{s}-1}_\lambda$-estimate, it suffices to estimate

$$\sum_{\substack{M_1\sim M_2\gg M_3\sim M_4\gtrsim 1\\ M_5,\dots,M_{k+1}\lesssim M_4}} N^{2s_2-1}\sup_{t\in[0,T]}\Big|\int_0^t\int_{\lambda\mathbb{T}} P_N v P_K v P_{K_1}u\dots P_{K_{k-1}}u\,dxds\Big|.$$

Using (6.10) and $\{K_1, K_2\} = \{M_1, M_2\}$, each summand is estimated by

$$
\begin{aligned}
&N^{2s_2-1} M_3^{1/2,0} M_4^{1/2,0} \Big(\prod_{i=5}^{k+1} M_i^{1/2} \Big) \|P_N v\|_{F^0_\lambda} \|P_K v\|_{F^0_\lambda} \prod_{i=1}^{k-1} \|P_{K_i} u\|_{F^0_\lambda} \\
&\lesssim N^{s_2} K^{\frac{1}{2}-s_1,-s_2+1} K_1^{-2s_2} \|P_N v\|_{F^{\overline{s}-1}_\lambda} \|P_K v\|_{F^{\overline{s}-1}_\lambda} \\
&\quad \times \|P_{K_1} u\|_{F^{\overline{s}}_\lambda} \|P_{K_2} u\|_{F^{\overline{s}}_\lambda} \prod_{i=3}^{k-1} \|P_{K_i} u\|_{F^{1/2}_\lambda}.
\end{aligned}
$$

Using $s_2 > \frac{1}{2}$ to guarantee $-2s_2 + 1 < 0$, this sums up to (6.1).

For the $H^{\overline{s}}_\lambda$-estimate, we apply (6.10) to (6.6) to estimate each summand by

$$
\begin{aligned}
N^{s_2} K^{\frac{1}{2}-s_1,-s_2+1} K_1^{-2s_2} \|P_N v\|_{F^{\overline{s}}_\lambda} \Big(&\|P_K v\|_{F^{\overline{s}}_\lambda} \|P_{K_1} u\|_{F^{\overline{s}}_\lambda} \|P_{K_2} u\|_{F^{\overline{s}}_\lambda} \prod_{i=3}^{k-1} \|P_{K_i} u\|_{F^{1/2}_\lambda} \\
&+ \|P_K v\|_{F^{\overline{s}-1}_\lambda} \|P_{K_1} w\|_{F^{\overline{s}}_\lambda} \|P_{K_2} w\|_{F^{\overline{s}}_\lambda} \prod_{i=3}^{k-1} \|P_{K_i} w\|_{F^{1/2}_\lambda} \Big).
\end{aligned}
$$

Using $s_2 > \frac{1}{2}$ to guarantee $-2s_2 + 1 < 0$, this sums up to (6.2).

Therefore, the proofs of (6.1) and (6.2) are completed in case of $M_1 \sim M_2 \gg M_3 \sim M_4$.

6.3. **Case $M_1 \sim M_2 \sim M_3 \sim M_4$.** We estimate (6.5) and (6.6) via linear Strichartz estimates:

$$
\sup_{t \in [0,T]} \Big| \int_0^t \int_{\lambda \mathbb{T}} P_{M_1} u_1 \dots P_{M_m} u_m dx ds \Big| \lesssim M_1^{1/2} \Big(\prod_{i=5}^{m} M_i^{1/2} \Big) \prod_{i=1}^{m} \|P_{M_i} u_i\|_{F^0_\lambda}. \tag{6.11}
$$

Here, our estimates only allow for $s_2 \geq 3/4$ due to the loss of $M_1^{1/2}$ compared to the previous case.

Case A: $N \in \{M_1, \dots, M_4\}$.

<u>Subcase I:</u> $K \in \{M_1, \dots, M_4\}$. Note that $N \sim K \sim K_1 \sim K_2$ so we may assume $K = K_1 = K_2 = N$. For the $H^{\overline{s}-1}_\lambda$-estimate, we use (6.11) to estimate each summand of (6.5) by

$$
N^{2s_2 - \frac{1}{2}} \|P_N v\|^2_{F^0_\lambda} \|P_N u\|^2_{F^0_\lambda} \prod_{i=3}^{k-1} \|P_{K_i} u\|_{F^{1/2}_\lambda}.
$$

We use $s_2 \geq \frac{3}{4}$ to estimate the above by

$$
\|P_N v\|^2_{F^{\overline{s}-1}_\lambda} \|P_N u\|^2_{F^{\overline{s}}_\lambda} \prod_{i=3}^{k-1} \|P_{K_i} u\|_{F^{1/2}_\lambda}.
$$

This sums up to (6.1).

For the $H^{\overline{s}}_\lambda$-estimate, we replace ∂_x of (6.6) (even for w when $w = \partial_x u_2$) by N and use (6.11) to estimate each summand by

$$
N^{2s_2 + \frac{3}{2}} \|P_N v\|^2_{F^0_\lambda} \|P_N u\|^2_{F^0_\lambda} \prod_{i=3}^{k-1} \|P_{K_i} u\|_{F^{1/2}_\lambda}.
$$

We use $s_2 \geq \frac{3}{4}$ to estimate the above by

$$\|P_N v\|^2_{F^{\overline{s}}_\lambda} \|P_N u\|^2_{F^{\overline{s}}_\lambda} \prod_{i=3}^{k-1} \|P_{K_i} u\|_{F^{1/2}_\lambda}.$$

This sums up to (6.2).

Subcase II: $K \in \{M_5, \ldots, M_{k+1}\}$. Note that $N \sim K_1 \sim K_2 \sim K_3$, so we may assume $K_1 = K_2 = K_3 = N$. For the $H^{\overline{s}-1}_\lambda$-estimate, we use (6.11) to estimate each summand of (6.5) by

$$N^{-2s_2+\frac{1}{2}} K^{\frac{1}{2}-s_1,-s_2+\frac{3}{2}} \|P_N v\|_{F^{\overline{s}-1}_\lambda} \|P_K v\|_{F^{\overline{s}-1}_\lambda} \|P_N u\|^3_{F^{\overline{s}}_\lambda} \prod_{i=4}^{k-1} \|P_{K_i} u\|_{F^{1/2}_\lambda}.$$

This sums up to (6.1) provided that $s_2 \geq \frac{2}{3}$.

For the $H^{\overline{s}}_\lambda$-estimate, we find similarly

$$\begin{aligned} N^{-2s_2+\frac{1}{2}} K^{\frac{1}{2}-s_1,-s_2+\frac{3}{2}} \|P_N v\|_{F^{\overline{s}}_\lambda} \Big(&\|P_K v\|_{F^{\overline{s}}_\lambda} \|P_N u\|^3_{F^{\overline{s}}_\lambda} \prod_{i=4}^{k-1} \|P_{K_i} u\|_{F^{1/2}_\lambda} \\ &+ \|P_K v\|_{F^{\overline{s}-1}_\lambda} \|P_N w\|^3_{F^{\overline{s}}_\lambda} \prod_{i=4}^{k-1} \|P_{K_i} w\|_{F^{1/2}_\lambda} \Big). \end{aligned}$$

This sums up to (6.2) provided that $s_2 \geq \frac{2}{3}$.

Case B: $N \in \{M_5, \ldots, M_{k+1}\}$.

Subcase I: $K \in \{M_1, \ldots, M_4\}$. Note that $K \sim K_1 \sim K_2 \sim K_3$, so we may assume $K = K_1 = K_2 = K_3$. For the $H^{\overline{s}-1}_\lambda$-estimate, we use (6.11) to estimate each summand of (6.5) by

$$\begin{aligned} &N^{2s_2-\frac{1}{2}} K^{1/2} \|P_N v\|_{F^0_\lambda} \|P_K v\|_{F^0_\lambda} \|P_K u\|^3_{F^0_\lambda} \prod_{i=4}^{k-1} \|P_{K_i} u\|_{F^{1/2}_\lambda} \\ &\lesssim N^{s_2+\frac{1}{2}} K^{-4s_2+\frac{3}{2}} \|P_N v\|_{F^{\overline{s}-1}_\lambda} \|P_K v\|_{F^{\overline{s}-1}_\lambda} \|P_K u\|^3_{F^{\overline{s}}_\lambda} \prod_{i=4}^{k-1} \|P_{K_i} u\|_{F^{1/2}_\lambda}. \end{aligned}$$

This sums up to (6.1) provided that $s_2 \geq \frac{2}{3}$.

For the $H^{\overline{s}}_\lambda$-estimate, we find similarly

$$\begin{aligned} N^{s_2+\frac{1}{2}} K^{-4s_2+\frac{3}{2}} \|P_N v\|_{F^{\overline{s}}_\lambda} \Big(&\|P_K v\|_{F^{\overline{s}}_\lambda} \|P_N u\|^3_{F^{\overline{s}}_\lambda} \prod_{i=4}^{k-1} \|P_{K_i} u\|_{F^{1/2}_\lambda} \\ &+ \|P_K v\|_{F^{\overline{s}-1}_\lambda} \|P_N w\|^3_{F^{\overline{s}}_\lambda} \prod_{i=4}^{k-1} \|P_{K_i} w\|_{F^{1/2}_\lambda} \Big). \end{aligned}$$

This sums up to (6.2) provided that $s_2 \geq \frac{2}{3}$.

Subcase II: $K \in \{M_5, \ldots, M_{k+1}\}$. Note that $K_1 \sim K_2 \sim K_3 \sim K_4$, so we may assume $K_1 = K_2 = K_3 = K_4$ are the largest four frequencies. Using (6.11), each summand of (6.5) is estimated by

$$N^{2s_2-\frac{1}{2}}K^{1/2}K_1^{1/2}\|P_Nv\|_{F_\lambda^0}\|P_Kv\|_{F_\lambda^0}\|P_{K_1}u\|^4_{F_\lambda^0}\prod_{i=5}^{k-1}\|P_{K_i}u\|_{F_\lambda^{1/2}}$$

$$\lesssim N^{s_2+\frac{1}{2}}K^{\frac{1}{2}-s_1,-s_2+\frac{3}{2}}K_1^{-4s_2+\frac{1}{2}}\|P_Nv\|_{F_\lambda^{\overline{s}-1}}\|P_Kv\|_{F_\lambda^{\overline{s}-1}}\|P_{K_1}u\|^4_{F_\lambda^{\overline{s}}}\prod_{i=5}^{k-1}\|P_{K_i}u\|_{F_\lambda^{1/2}}.$$

This sums up to (6.1) provided that $s \geq \frac{5}{8}$.

For the $H_\lambda^{\overline{s}}$-estimate, we find similarly for each summand of (6.6)

$$N^{s_2+\frac{1}{2}}K^{\frac{1}{2}-s_1,-s_2+\frac{3}{2}}K_1^{-4s_2+\frac{1}{2}}\|P_Nv\|_{F_\lambda^{\overline{s}}}\Big(\|P_Kv\|_{F_\lambda^{\overline{s}}}\|P_{K_1}u\|^4_{F_\lambda^{\overline{s}}}\prod_{i=5}^{k-1}\|P_{K_i}u\|_{F_\lambda^{1/2}}$$
$$+\|P_Kv\|_{F_\lambda^{\overline{s}-1}}\|P_{K_1}w\|^4_{F_\lambda^{\overline{s}}}\prod_{i=5}^{k-1}\|P_{K_i}w\|_{F_\lambda^{1/2}}\Big).$$

This sums up to (6.2) provided that $s_2 \geq \frac{5}{8}$.

Therefore, the proofs of (6.1) and (6.2) are completed in case of $M_1 \sim M_2 \sim M_3 \sim M_4$.

The proof of Proposition 6.1 is finished. □

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