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Cohomogeneity one Alexandrov spaces in low dimensions

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Abstract

Alexandrov spaces are complete length spaces with a lower curvature bound in the triangle comparison sense. When they are equipped with an effective isometric action of a compact Lie group with one-dimensional orbit space, they are said to be of cohomogeneity one. Well-known examples include cohomogeneity-one Riemannian manifolds with a uniform lower sectional curvature bound; such spaces are of interest in the context of non-negative and positive sectional curvature. In the present article we classify closed, simply connected cohomogeneity-one Alexandrov spaces in dimensions 5, 6 and 7. This yields, in combination with previous results for manifolds and Alexandrov spaces, a complete classification of closed, simply connected cohomogeneity-one Alexandrov spaces in dimensions at most 7.

Keywords Cohomogeneity one · Alexandrov space · Orbifold

Mathematics Subject Classification 53C23 · 57S10

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1 Introduction

Alexandrov spaces (with curvature bounded from below) are complete length spaces with a lower curvature bound in the triangle comparison sense; they generalize Riemannian manifolds with a uniform lower sectional curvature bound. Instances of Alexandrov spaces include Riemannian orbifolds (with a lower sectional curvature bound), orbit spaces of isometric actions of compact Lie groups on Riemannian manifolds with sectional curvature bounded below, or Gromov–Hausdorff limits of sequences of n-dimensional Riemannian manifolds with a uniform lower bound on the sectional curvature.

The classification of spaces with compact Lie group actions is a central problem in the theory of transformation groups. In this context, a space with an effective action of a compact Lie group is of *cohomogeneity one* if its orbit space is one-dimensional. In the topological and smooth categories, the geometry and topology of cohomogeneity-one manifolds have been studied extensively (see, for example, [16, 36] and references therein) and closed, simply connected cohomogeneity-one manifolds of dimension at most 7 have been classified (see [16, 22, 26–28, 30]).

Our main theorem is a complete equivariant classification of closed, simply connected Alexandrov spaces of cohomogeneity one in dimensions 5, 6 and 7. In combination with classification results for Alexandrov spaces of cohomogeneity one in dimensions 2, 3 and 4 (see [15]) and the manifold classification results cited above, our result yields a complete equivariant classification of closed, simply connected cohomogeneity-one Alexandrov spaces in dimensions at most 7.

Theorem A *Let X be a closed, simply connected Alexandrov space of dimension* 5, 6 *or* 7 *with an (almost) effective cohomogeneity one isometric action of a compact connected Lie group. If the action is not equivalent to a smooth action on a smooth manifold, then it is given by one of the diagrams in Table* 1 *if* $\dim X = 5$, Table 2 *if* $\dim X = 6$*, or Tables* 3, 4, 5, 6*, and* 7 *if* $\dim X = 7$.

Let us discuss the context for Theorem A in more detail. In dimensions two and three, the basic topological properties of Alexandrov spaces are fairly well-understood. Indeed, two-dimensional Alexandrov spaces are topological two-manifolds, possibly with boundary (see [8, Corollary 10.10.3]); closed (i.e., compact and without boundary) three-dimensional Alexandrov spaces are either topological three-manifolds or are homeomorphic to

Table 1 Group diagrams in dimension 5

Diagram	Space	Orbifold
$(S^3 \times S^1, \Gamma \times \mathbb{Z}_k, \Gamma \times S^1, S^3 \times \mathbb{Z}_k)$	$(S^3/\Gamma) * \mathbb{S}^1$	Yes
$(S^3 \times S^1, \{(e^{\frac{2\pi i}{k}\frac{lk+mps}{m}}, e^{\frac{2\pi i}{k}qs})\}, (\mathbb{Z}_m \times 1)K_0^-, S^3 \times \mathbb{Z}_{k/q})$, $K_0^- = \{(e^{ip\theta}, e^{iq\theta}\}, q\|(m,k), (p,k) = 1$	$\mathbb{S}^5/\mathbb{Z}_m$	Yes
$(S^3 \times S^3, N_{S^3}(S^1) \times S^1, N_{S^3}(S^1) \times S^1, S^3 \times S^1)$	$\mathbb{R}P^2 * \mathbb{S}^2$	Yes
$(S^3 \times S^3, N_{S^3}(S^1) \times S^1, S^3 \times S^1, S^3 \times S^1)$	$\mathrm{Susp}\,(\mathbb{R}P^2) \times \mathbb{S}^2$	Yes
$(S^3 \times S^3, N_{S^3}(S^1) \times N_{S^3}(S^1), N_{S^3}(S^1) \times S^3, S^3 \times N_{S^3}(S^1))$	$\mathbb{R}P^2 * \mathbb{R}P^2$	Yes
$(\mathrm{SU}(3), \mathrm{U}(2), \mathrm{SU}(3), \mathrm{SU}(3))$	$\mathrm{Susp}\,(\mathbb{C}P^2)$	No
$(\mathrm{Spin}(5), N_{\mathrm{Spin}(5)}(\mathrm{Spin}(4)), \mathrm{Spin}(5), \mathrm{Spin}(5))$	$\mathrm{Susp}\,(\mathbb{R}P^4)$	Yes

Table 2 Group diagrams in dimension 6

Diagram	Space	Orbifold
$(S^3\times S^3, N_{S^3}(S^1)\times 1, N_{S^3}(S^1)\times S^3, S^3\times 1)$	$\mathbb{R}P^2 * \mathbb{S}^3$	Yes
$(S^3\times S^3, N_{S^3}(S^1)\times \mathbb{Z}_k, N_{S^3}(S^1)\times S^1, S^3\times \mathbb{Z}_k)$	$(\mathbb{R}P^2 * \mathbb{S}^1)$-bundle over $\mathbb{S}^2$	Yes
$(S^3\times S^3, N_{S^3}(S^1)\times 1, S^3\times 1, S^3\times 1)$	Susp $(\mathbb{R}P^2)\times \mathbb{S}^3$	Yes
$(S^3\times S^3, N_{S^3}(S^1)\times \Gamma, N_{S^3}(S^1)\times S^3, S^3\times \Gamma)$	$\mathbb{R}P^2 * (S^3/\Gamma)$	Yes
$(S^3\times S^3, \Delta S^1\cup (j,j)\Delta S^1, S^3\times N_{S^3}(S^1), \Delta S^3)$	$\mathbb{R}P^2 * \mathbb{S}^3$	Yes
$(S^3\times S^3, \Delta S^1\cup (j,j)\Delta S^1, T^2\cup (j,j)T^2, \Delta S^3)$	$(\mathrm{SO}(5)/(\mathrm{SO}(2)\mathrm{SO}(3)))/\mathbb{Z}_2$	Yes
$(S^3\times S^3, \pm\Delta S^1\cup (j,\pm j)\Delta S^1, T^2\cup (j,j)T^2, \pm\Delta S^3)$	$\mathbb{C}P^3/\mathbb{Z}_2$	Yes
$(S^3\times S^3, \pm\Delta S^1\cup (j,\pm j)\Delta S^1, S^3\times N_{S^3}(S^1), \pm\Delta S^3)$	$\mathbb{R}P^2 * \mathbb{R}P^3$	Yes
$(S^3\times S^3, \Delta S^1\cup (j,j)\Delta S^1, \Delta S^3, \Delta S^3)$	Susp $(\mathbb{R}P^2)\times \mathbb{S}^3$	Yes
$(S^3\times S^3, \mathrm{D}^*_{2m}\times S^1, N_{S^3}(S^1)\times S^1, S^3\times S^1)$	$\mathbb{S}^2\times (\mathbb{C}P^2/\mathrm{D}^*_{2m})$	Yes
$(S^3\times S^3, \{(e^{ip\theta}\lambda, e^{i\theta})\mid \theta\in\mathbb{R}, \lambda\in\mathbb{Z}_k\}, T^2, S^3\times S^1)$	$(\mathbb{C}P^2/\mathbb{Z}_k)$-bundle over $\mathbb{S}^2$	Yes
$(S^3\times S^3, \Gamma\times S^1, \Gamma\times S^3, S^3\times S^1)$	$(S^3/\Gamma) * \mathbb{S}^2$	Yes
$(S^3\times S^3, \pm\Delta S^1\cup (j,\pm j)\Delta S^1, S^3\times N_{S^3}(S^1), \pm\Delta S^3)$	$\mathbb{R}P^2 * \mathbb{R}P^3$	Yes
$(S^3\times S^3, \mathbb{Z}_k\times S^1, \mathbb{Z}_k\times S^3, S^3\times S^1)$	$(S^3/\mathbb{Z}_k) * \mathbb{S}^2$	Yes
$(S^3\times S^3, \pm\Delta S^1, \pm\Delta S^3, S^3\times S^1)$	$\mathbb{R}P^3 * \mathbb{S}^2$	Yes
$(S^3\times S^3, \Gamma\times S^1, S^3\times S^1, S^3\times S^1)$	Susp $(S^3/\Gamma)\times \mathbb{S}^2$	Yes
$(S^3\times S^3, \{(e^{ip\theta}\lambda, e^{i\theta})\}, S^3\times S^1, S^3\times S^1)$	$(\mathbb{S}^4/\mathbb{Z}_k)$-bundle over $\mathbb{S}^2$	Yes
$(S^3\times S^3, \{(e^{i\theta}\lambda, e^{i\theta})\}, S^1\times S^3, S^3\times S^1)$	$\mathbb{C}P^3/\mathbb{Z}_k$	Yes
$(S^3\times S^3\times S^1, N_{S^3}(S^1)\times S^1\times \mathbb{Z}_k, N_{S^3}(S^1)\times S^1\times S^1, S^3\times S^1\times \mathbb{Z}_k)$	$(\mathbb{R}P^2 * \mathbb{S}^1)\times \mathbb{S}^2$	Yes
$(\mathrm{SU}(3), \mathrm{S}(\mathrm{U}(2)\mathbb{Z}_k), \mathrm{U}(2), \mathrm{SU}(3))$	$\mathbb{C}P^3/\mathbb{Z}_k$	Yes
$(\mathrm{SU}(3), \mathrm{S}(\mathrm{U}(2)\mathbb{Z}_k), \mathrm{SU}(3), \mathrm{SU}(3))$	Susp $(\mathbb{S}^5/\mathbb{Z}_k)$	Yes
$(\mathrm{SU}(3)\times S^1, \mathrm{U}(2)\times \mathbb{Z}_k, \mathrm{U}(2)\times S^1, \mathrm{SU}(3)\times \mathbb{Z}_k)$	$\mathbb{C}P^2 * \mathbb{S}^1$	No
$(\mathrm{Sp}(2)\times S^1, N_{\mathrm{Sp}(2)}(\mathrm{Sp}(1)\mathrm{Sp}(1))\times \mathbb{Z}_k, N_{\mathrm{Sp}(2)}(\mathrm{Sp}(1)\mathrm{Sp}(1))\times S^1, \mathrm{Sp}(2)\times \mathbb{Z}_k)$	$\mathbb{R}P^4 * \mathbb{S}^1$	Yes
$(\mathrm{Spin}(6), N_{\mathrm{Spin}(6)}(\mathrm{Spin}(5)), \mathrm{Spin}(6), \mathrm{Spin}(6))$	Susp $(\mathbb{R}P^5)$	Yes

quotients of smooth three-manifolds by orientation reversing involutions with isolated fixed points, and closed four-dimensional Alexandrov spaces are locally homeomorphic to orbifolds (see [14, Corollary 2.3]). In higher dimensions, however, similar general results are lacking and considering spaces with large isometry groups provides a systematic way of studying Alexandrov spaces. This yields manageable families of spaces with a reasonably simple structure but flexible enough to generate interesting examples on which to test conjectures or carry out geometric constructions. This framework has been successfully used in the smooth category to construct, for instance, Riemannian manifolds satisfying given geometric conditions, such as positive Ricci or sectional curvature (see [10, 18–20]).

One of the measures for the size of an isometric action of a compact Lie group G on an Alexandrov space X is its *cohomogeneity*, defined as the dimension of the orbit space X/G. This quotient space, when equipped with the orbital distance metric, is itself an Alexandrov space with the same lower curvature bound as X. From the point of view of cohomogeneity, transitive actions are the largest one can have. These actions preclude any topological or metric singularities: by the work of Berestovskiĭ [5], homogeneous Alexandrov spaces are isometric to Riemannian manifolds. The next simplest case to consider is when the orbit space is one-dimensional, i.e., when the action is of *cohomogeneity one*. Alexandrov spaces of cohomogeneity one were first studied in [15],

Table 3 Group diagrams for $S^3 \times S^3$ in dimension 7

Diagram	Space	Orbifold
$(S^3 \times S^3, \Gamma \times \mathbb{Z}_k, \Gamma \times S^1, S^3 \times \mathbb{Z}_k)$	$((S^3/\Gamma) * \mathbb{S}^1)$-bundle over $\mathbb{S}^2$	Yes
$(S^3 \times S^3, \mathbb{Z}_k \times 1, S^1 \times 1, S^3 \times 1)$	$(\mathbb{C}P^2/\mathbb{Z}_k) \times \mathbb{S}^3$	Yes
$(S^3 \times S^3, \mathrm{D}^*_{2m} \times 1, N_{S^3}(S^1) \times 1, S^3 \times 1)$	$(\mathbb{C}P^2/\mathbb{Z}_m) \times \mathbb{S}^3$	Yes
$(S^3 \times S^3, \left\{\left(e^{\frac{lk+mps}{km}2\pi i}, e^{\frac{2\pi qsi}{k}}\right)\right\}, (\mathbb{Z}_m \times 1)K_0^-, S^3 \times \mathbb{Z}_{k/(k,q)})$	$((\mathbb{S}^5/\mathbb{Z}_q)/\mathbb{Z}_m\mathbb{Z}_k)$-bundle over $\mathbb{S}^2$	Yes
$(S^3 \times S^3, \Gamma \times 1, \Gamma \times S^3, S^3 \times 1)$	$S^3 * (S^3/\Gamma)$	Yes
$(S^3 \times S^3, \mathbb{Z}_2 \times 1, \pm\Delta S^3, S^3 \times 1)$	$\mathbb{R}P^3 * \mathbb{S}^3$	Yes
$(S^3 \times S^3, \Gamma \times 1, S^3 \times 1, S^3 \times 1)$	$\mathrm{Susp}\,(S^3/\Gamma) \times S^3$	Yes
$(S^3 \times S^3, \Gamma \times \Lambda, \Gamma \times S^3, S^3 \times \Lambda)$	$(S^3/\Gamma) * (S^3/\Lambda)$	Yes
$(S^3 \times S^3, \pm\Delta\Lambda, \pm\Delta S^3, S^3 \times \Lambda)$	$(\mathbb{R}P^3) * (S^3/\Lambda)$	Yes
$(S^3 \times S^3, \Delta\mathbb{Z}_k, S^1 \times \mathbb{Z}_k, \Delta S^3)$	–	Yes
$(S^3 \times S^3, \Delta\mathrm{D}^*_{2m}, (S^1 \times 1)\Delta\mathrm{D}^*_{2m}, \Delta S^3)$	–	Yes
$(S^3 \times S^3, \Delta\mathbb{Z}_k, \{(e^{ip\theta}, e^{iq\theta})\}, \Delta S^3)$ where $k\|(p-q)$ and if k is even p, q are odd	–	Yes
$(S^3 \times S^3, \Delta\mathbb{Z}_k, \{(e^{ip\theta}, e^{-iq\theta})\}, \Delta S^3)$ where $k\|(p+q)$ and if k is even p, q are odd.	–	Yes
$(S^3 \times S^3, \Delta\mathrm{D}^*_{2m}, \{(e^{ip\theta}, e^{iq\theta})\} \cup \{(je^{ip\theta}, je^{iq\theta})\}, \Delta S^3)$ where $m\|(p-q)$ and if m is even p, q are odd	–	Yes
$(S^3 \times S^3, \Delta\mathrm{D}^*_{2k}, \{(e^{ip\theta}, e^{-iq\theta})\} \cup \{(je^{ip\theta}, je^{-iq\theta})\}, \Delta S^3)$ where $k\|(p+q)$ and if k is even p, q are odd	–	Yes
$(S^3 \times S^3, \pm\Delta\mathbb{Z}_k, S^1 \times \mathbb{Z}_k, \pm\Delta S^3)$	–	Yes
$(S^3 \times S^3, \pm\Delta\mathrm{D}^*_{2m}, (S^1 \times 1)\Delta\mathrm{D}^*_{2m}, \pm\Delta S^3)$	–	Yes
$(S^3 \times S^3, \pm\Delta\mathbb{Z}_k, \{(e^{ip\theta}, e^{iq\theta})\}, \pm\Delta S^3)$	–	Yes
$(S^3 \times S^3, \pm\Delta\mathbb{Z}_k, \{(e^{ip\theta}, e^{-iq\theta})\}, \pm\Delta S^3)$	–	Yes
$(S^3 \times S^3, \pm\Delta\mathrm{D}^*_{2k}, \{(e^{ip\theta}, e^{iq\theta})\} \cup \{(je^{ip\theta}, je^{iq\theta})\}, \pm\Delta S^3)$	–	Yes
$(S^3 \times S^3, \pm\Delta\mathrm{D}^*_{2k}, \{(e^{ip\theta}, e^{-iq\theta})\} \cup \{(je^{ip\theta}, je^{-iq\theta})\}, \pm\Delta S^3)$	–	Yes
$(S^3 \times S^3, \Delta\Gamma, S^3 \times \Gamma, \Delta S^3)$	$\mathbb{S}^7/\Gamma$	Yes
$(S^3 \times S^3, \Delta\Gamma, \Delta S^3, \Delta S^3)$	$\mathrm{Susp}\,(S^3/\Gamma) \times \mathbb{S}^3$	Yes
$(S^3 \times S^3, \pm\Delta\mathbb{Z}_{2k}, \pm\Delta_j S^3, \pm\Delta S^3)$	–	Yes
$(S^3 \times S^3, \pm\Delta\mathrm{D}^*_{2k}, \pm\Delta_j S^3, \pm\Delta S^3)$	–	Yes
$(S^3 \times S^3, \pm\Delta\mathrm{D}^*_{2k}, \pm\Delta_i S^3, \pm\Delta S^3)$	–	Yes

where the authors obtained a structure result and classified these spaces (up to equivariant homeomorphism) in dimensions 4 and below. Simple instances of these spaces are, for example, spherical suspensions of homogeneous spaces X with sectional curvature bounded below by 1, equipped with the canonical suspension action of the transitive action on X.

It was shown in [15, Proposition 5] that the orbit space of an isometric cohomogeneity one G-action on a closed, simply connected Alexandrov space X is homeomorphic to a closed interval $[-1, 1]$ and there exist compact Lie subgroups H and $K^{\pm}$ of G such that $H \subseteq K^{\pm} \subseteq G$ and $K^{\pm}/H$ are positively curved homogeneous spaces. The group H is the principal isotropy group of the action, and the groups $K^{\pm}$ are isotropy groups of points in

Table 4 Group diagrams for $S^3 \times S^3 \times S^1$ and $S^3 \times S^3 \times S^3$ in dimension 7

Diagram	Space	Orbifold
$(S^3 \times S^3 \times S^1, N(S^1) \times \mathbb{Z}_k, N(S^1) \times T^1, S^3 \times \mathbb{Z}_k)$	$(\mathbb{R}P^2 * \mathbb{S}^1) \times \mathbb{S}^3$	Yes
$(S^3 \times S^3 \times S^1, N_{\Delta S^3}(\Delta S^1)\mathbb{Z}_k, T^2 \cup (j,j,1)T^2, \Delta S^3 \mathbb{Z}_k)$	$B^7_d/\mathbb{Z}_2$	Yes
$(S^3 \times S^3 \times S^1, \Gamma \times S^1 \times \mathbb{Z}_k, \Gamma \times T^2, S^3 \times S^1 \times \mathbb{Z}_k)$	$(S^3/\Gamma * \mathbb{S}^1) \times \mathbb{S}^2$	Yes
$(S^3 \times S^3 \times S^3, N_{S^3}(S^1) \times T^2, N_{S^3}(S^1) \times S^3 \times S^1, S^3 \times T^2)$	$(\mathbb{R}P^2 * \mathbb{S}^2) \times \mathbb{S}^2$	Yes
$(S^3 \times S^3 \times S^3, N_{S^3}(S^1) \times T^2, S^3 \times T^2, S^3 \times T^2)$	$\mathrm{Susp}\,(\mathbb{R}P^2) \times (\mathbb{S}^2 \times \mathbb{S}^2)$	Yes
$(S^3 \times S^3 \times S^3, N_{S^3}(S^1) \times N_{S^3}(S^1) \times S^1, N_{S^3}(S^1) \times S^3 \times S^1, S^3 \times N_{S^3}(S^1) \times S^1)$	$(\mathbb{R}P^2 * \mathbb{R}P^2) \times \mathbb{S}^2$	Yes

the orbits corresponding to the boundary points ± 1 of the orbit space. The groups $K^\pm$ are called *non-principal isotropy groups*, and the orbits $G/K^\pm$ are called *non-principal orbits*. We collect these groups in the quadruple (G, H, K^-, K^+), called the *group diagram* of the action. The space X is the union of two bundles whose fibers are cones over the positively curved homogeneous spaces $K^\pm/H$. Conversely, any diagram (G, H, K^-, K^+), with $K^\pm/H$ positively curved homogeneous spaces, gives rise to a cohomogeneity one Alexandrov space. In the present article we complete the classification of these spaces in dimensions 5, 6 and 7 (assuming simply connectedness) and identify which of these spaces are smooth orbifolds.

Closed, smooth manifolds of cohomogeneity one have been classified by Mostert [26, 27] and Neumann [28] in dimensions 2 and 3, and by Parker [30] in dimension 4, without assuming any restrictions on the fundamental group. In dimensions 5, 6 and 7, Hoelscher [22] obtained the equivariant classification of closed smooth cohomogeneity one manifolds assuming simply connectedness. It is well-known that these manifolds admit invariant Riemannian metrics and are therefore Alexandrov spaces of cohomogeneity one. In the topological category, the corresponding classification results in dimensions at most 7 follow from combining the smooth classification with the classification of closed, simply connected cohomogeneity one topological manifolds with a non-smooth cohomogeneity one action in dimensions at most 7, obtained in [16]. It was also shown in [16] that closed, simply connected cohomogeneity one topological manifolds decompose as double cone bundles whose fibers are cones over spheres or the Poincaré homology sphere, and hence they admit invariant Alexandrov metrics. Our main result completes the equivariant classification of closed, simply connected Alexandrov spaces in dimensions 5, 6 and 7. Along the way, we obtain topological characterizations for most of the spaces in the classification.

We point out that the diagrams (G, H, K^-, K^+) in Tables 1, 2, 3, 4, 5, 6, and 7 contain, as particular cases, the diagrams of non-smoothable cohomogeneity one actions on closed, simply connected topological manifolds in [16]; in this special situation the positively curved homogeneous spaces $K^\pm/H$ are either spheres or the Poincaré homology sphere. Compared to the smooth and topological cases, the number of closed, simply connected cohomogeneity one Alexandrov spaces that are not manifolds increases substantially, due to the fact that at least one of the positively curved homogeneous spaces $K^\pm/H$ is no longer a sphere or the Poincaré homology sphere. In many cases, we can identify the spaces in Theorem A as joins, suspensions, products or bundles of familiar spaces. Moreover, many of the spaces in Theorem A are equivariantly homeomorphic to smooth cohomogeneity one orbifolds. Indeed, they admit a double cone bundle decomposition, where the cones are taken over spherical homogeneous spaces; this structure characterizes closed, smooth orbifolds of cohomogeneity one whose orbit space is a closed interval (see [17]).

Table 5 Group diagrams for SU(3) and SU(3) × S^i, $i = 1, 3$, in dimension 7

Diagram	Space	Orbifold
$(\mathrm{SU}(3), T^2, \mathrm{SU}(3), \mathrm{SU}(3))$	Susp (W^6)	No
$(\mathrm{SU}(3), T^2\mathbb{Z}_2, \mathrm{SU}(3), \mathrm{SU}(3))$	Susp $(\mathrm{W}^6/\mathbb{Z}_2)$	No
$(\mathrm{SU}(3), T^2, \mathrm{U}(2), \mathrm{SU}(3))$	–	No
$(\mathrm{SU}(3), T^2\mathbb{Z}_2, \mathrm{U}(2), \mathrm{SU}(3))$	–	No
$(\mathrm{SU}(3), T^2\mathbb{Z}_2, \mathrm{U}(2), \mathrm{U}(2))$	Susp $(\mathbb{R}P^2)$-bundle over $\mathbb{C}P^2$	Yes
$(\mathrm{SU}(3) \times S^1, S(U(2)\mathbb{Z}_k) \times \mathbb{Z}_l, S(\mathrm{U}(2)\mathbb{Z}_k) \times S^1, \mathrm{SU}^3 \times \mathbb{Z}_l)$	$(\mathbb{S}^5/\mathbb{Z}_k) * \mathbb{S}^1$	Yes
$(\mathrm{SU}(3) \times S^3, \mathrm{U}(2) \times S^1, \mathrm{SU}(3) \times S^1, \mathrm{SU}(3) \times S^1)$	Susp $(\mathbb{C}P^2) \times \mathbb{S}^2$	No
$(\mathrm{SU}(3) \times S^3, \mathrm{U}(2) \times S^1, \mathrm{SU}(3) \times S^1, \mathrm{U}(2) \times S^3)$	$\mathbb{C}P^2 * \mathbb{S}^2$	No
$(\mathrm{SU}(3) \times S^3, \mathrm{U}(2) \times N_{S^3}(S^1), \mathrm{SU}(3) \times N_{S^3}(S^1), \mathrm{U}(2) \times S^3)$	$\mathbb{C}P^2 * \mathbb{R}P^2$	No
$(\mathrm{SU}(3) \times S^3, \mathrm{U}(2) \times N_{S^3}(S^1), \mathrm{U}(2) \times S^3, \mathrm{U}(2) \times S^3)$	Susp $(\mathbb{R}P^2) \times \mathbb{C}P^2$	Yes

Table 6 Group diagrams for Sp(2) and Sp(2) × S^3 in dimension 7

Diagram	Space	Orbifold
$(\mathrm{Sp}(2), \mathrm{Sp}(1)\mathrm{SO}(2), \mathrm{Sp}(2), \mathrm{Sp}(2))$	Susp $(\mathbb{C}P^3)$	No
$(\mathrm{Sp}(2), \mathrm{Sp}(1)\mathrm{SO}(2)\mathbb{Z}_2, \mathrm{Sp}(2), \mathrm{Sp}(2))$	Susp $(\mathbb{C}P^3/\mathbb{Z}_2)$	No
$(\mathrm{Sp}(2), \mathrm{Sp}(1)\mathrm{SO}(2), \mathrm{Sp}(1)\mathrm{Sp}(1), \mathrm{Sp}(2))$	–	No
$(\mathrm{Sp}(2), \mathrm{Sp}(1)\mathrm{SO}(2)\mathbb{Z}_2, \mathrm{Sp}(1)\mathrm{Sp}(1), \mathrm{Sp}(2))$	–	No
$(\mathrm{Sp}(2), \mathrm{Sp}(1)\mathrm{SO}(2)\mathbb{Z}_2, \mathrm{Sp}(1)\mathrm{Sp}(1), \mathrm{Sp}(1)\mathrm{Sp}(1))$	Susp $(\mathbb{R}P^2)$-bundle over $\mathbb{S}^4$	Yes
$(\mathrm{Sp}(2) \times S^3, \mathrm{Sp}(1)\mathrm{Sp}(1) \times N_{S^3}(S^1), \mathrm{Sp}(1)\mathrm{Sp}(1) \times S^3, \mathrm{Sp}(1)\mathrm{Sp}(1) \times S^3)$	Susp $(\mathbb{R}P^2) \times \mathbb{S}^4$	Yes
$(\mathrm{Sp}(2) \times S^3, \mathrm{Sp}(1)\mathrm{Sp}(1) \times N_{S^3}(S^1), \mathrm{Sp}(1)\mathrm{Sp}(1) \times S^3, \mathrm{Sp}(2) \times N_{S^3}(S^1))$	$\mathbb{S}^4 * \mathbb{R}P^2$	Yes
$(\mathrm{Sp}(2) \times S^3, \mathrm{Sp}(1)\mathrm{Sp}(1)\mathbb{Z}_2 \times S^1, \mathrm{Sp}(1)\mathrm{Sp}(1)\mathbb{Z}_2 \times S^3, \mathrm{Sp}(2) \times S^1)$	$\mathbb{S}^2 * \mathbb{R}P^4$	Yes
$(\mathrm{Sp}(2) \times S^3, \mathrm{Sp}(1)\mathrm{Sp}(1)\mathbb{Z}_2 \times N_{S^3}(S^1), \mathrm{Sp}(1)\mathrm{Sp}(1)\mathbb{Z}_2 \times S^3, \mathrm{Sp}(2) \times N_{S^3}(S^1))$	$\mathbb{R}P^4 * \mathbb{R}P^2$	Yes
$(\mathrm{Sp}(2) \times S^3, \mathrm{Sp}(1)\mathrm{Sp}(1)\mathbb{Z}_2 \times S^1, \mathrm{Sp}(2) \times S^1, \mathrm{Sp}(2) \times S^1)$	Susp $(\mathbb{R}P^4) \times \mathbb{S}^2$	Yes

Table 7 Group diagrams for G_2, SU(4), SU(4) × S^1 and Spin(7) in dimension 7

Diagram	Space	Orbifold
$(\mathrm{G}_2, N_{\mathrm{G}_2}(\mathrm{SU}(3)), \mathrm{G}_2, \mathrm{G}_2)$	Susp $(\mathbb{R}P^6)$	Yes
$(\mathrm{SU}(4), \mathrm{U}(3), \mathrm{SU}(4), \mathrm{SU}(4))$	Susp $(\mathbb{C}P^3)$	No
$(\mathrm{SU}(4) \times S^1, \mathrm{Sp}(2)\mathbb{Z}_2 \times \mathbb{Z}_k, \mathrm{Sp}(2)\mathbb{Z}_2 \times S^1, \mathrm{SU}(4) \times \mathbb{Z}_k)$	$\mathbb{R}P^5 * \mathbb{S}^1$	Yes
$(\mathrm{Spin}(7), N_{\mathrm{Spin}(7)}(\mathrm{Spin}(6)), \mathrm{Spin}(7), \mathrm{Spin}(7))$	Susp $(\mathbb{R}P^6)$	Yes

As in the smooth and topological cases, the proof of Theorem A follows from a case-by-case analysis of the possible group actions. Using dimension restrictions, one first determines the possible groups that can act. One then considers each group action individually, taking into account the fact that the groups must satisfy restrictions imposed by the fact that the homogeneous spaces $K^{\pm}/H$ are positively curved. In this way, one obtains all the possible diagrams

(G, H, K^{-}, K^{+}), which determine the equivariant type of the Alexandrov space. Recognition results for specific types of actions help us identify the topological type of the space.

Our article is divided as follows. In Sect. 2 we collect background material on cohomogeneity one Alexandrov spaces and prove some results we will use in the proof of Theorem A. The proof of this theorem is contained in Sect. 3.

2 Preliminaries

In this section, we collect some background material which we will use in the proof of Theorem A.

2.1 Group actions

Let X be a topological space, and let x be a point in X. Given a topological (left) action $G \times X \to X$ of a Lie group G, we let $G(x) = \{ gx \mid g \in G \}$ be the *orbit* of x under the action of G. The *isotropy group* of x is the subgroup $G_x = \{ g \in G \mid gx = x \}$. Observe that $G(x) \approx G/G_x$, where the symbol "$\approx$" denotes homeomorphism between topological spaces. We will denote the orbit space of the action by X/G, and let $\pi : X \to X/G$ be the orbit projection map. The *(ineffective) kernel* of the action is the subgroup $K = \bigcap_{x \in X} G_x$. The action is *effective* if K is the trivial subgroup $\{e\}$ of G; the action is *almost effective* if K is finite.

We will say that two G-spaces are *equivalent* if they are equivariantly homeomorphic. From now on, we will suppose that G is compact and connected, and assume that the reader is familiar with the basic notions of compact transformation groups (see, for example, Bredon [7]). We will assume all spaces to be connected, unless stated otherwise. We will denote the identity component of a Lie group H by H_0.

2.2 Alexandrov spaces

A finite (Hausdorff)-dimensional length space (X, d) has curvature bounded below by k, denoted by $\mathrm{curv}\,(X) \geq k$, if every point $x \in X$ has a neighborhood U such that, for any collection of four different points (x_0, x_1, x_2, x_3) in U, the following condition holds:

$$\angle_k x_1 x_0 x_2 + \angle_k x_2 x_0 x_3 + \angle_k x_3 x_0 x_1 \leq 2\pi.$$

Here, $\angle_k x_i x_0 x_j$, called the *comparison angle*, is the angle at $x_0(k)$ in the geodesic triangle in M^2_k, the simply connected Riemannian 2-manifold with constant curvature k, with vertices $(x_0(k), x_i(k), x_j(k))$, which are the isometric images of (x_0, x_i, x_j). An *Alexandrov space* is a complete length space with finite Hausdorff dimension and curvature bounded below by k for some $k \in \mathbb{R}$. Recall that the Hausdorff dimension of an Alexandrov space is an integer and is equal to its topological dimension. The *space of directions* of a general Alexandrov space X^n of dimension n at a point x is, by definition, the completion of the space of geodesic directions at x. We will denote it by $\Sigma_x X^n$. It is a compact Alexandrov space of dimension $n-1$ with curvature bounded below by 1. We refer the reader to [8, 9] for the basic results on Alexandrov geometry. We will say that an Alexandrov space is *closed* if it is compact and has no boundary.

We now recall the definitions of a spherical suspension and of a spherical join (cf. [9, Sections 4.3.1 and 4.4.4]). These spaces will play an important role in the homeomorphism classification.

Definition 2.1 Let (X, d_X) be an Alexandrov space with $\operatorname{curv}(X) \geq 1$. The *(topological) suspension* of X is the space

$$\operatorname{Susp}(X) = (X \times [0, \pi])/\sim,$$

where $(x_1, 0) \sim (x_2, 0)$ and $(x_1, \pi) \sim (x_2, \pi)$ for all $x_1, x_2 \in X$. We endow $\operatorname{Susp}(X)$ with a metric given by

$$\cos(d([x_1, t_1], [x_2, t_2])) = \cos t_1 \cos t_2 + \sin t_1 \sin t_2 \cos d_X(x_1, x_2).$$

With this metric, $(\operatorname{Susp}(X), d)$ is an Alexandrov space with curv ≥ 1 and is called the *spherical suspension* of (X, d).

Definition 2.2 Let (X, d_X), (Y, d_Y) be two Alexandrov spaces with curvature bounded below by 1. The *(topological) join* of X and Y is the space

$$X * Y = (X \times Y \times [0, \pi/2])/\sim,$$

where $(x_1, y_1, t_1) \sim (x_2, y_2, t_2)$, if and only if $t_1 = t_2 = 0$ and $x_1 = x_2$ or $t_1 = t_2 = \pi/2$ and $y_1 = y_2$. We endow $X * Y$ with a metric defined by

$$\cos(d([x_1, y_1, t_1], [x_2, y_2, t_2])) = \cos t_1 \cos t_2 \cos d_X(x_1, x_2) + \sin t_1 \sin t_2 \cos d_Y(y_1, y_2).$$

Note that, since X and Y have curvature bounded below by 1, their diameter is bounded above by π (see [8, Theorem 10.4.1]). Hence the metric on $X * Y$ given by the preceding equation is well defined. The space $(X * Y, d)$ is the *spherical join* of (X, d_X) and (X, d_Y) and is an Alexandrov space with curv ≥ 1.

2.3 Group actions on Alexandrov spaces

Let X be an n-dimensional Alexandrov space. Fukaya and Yamaguchi proved in [12, Theorem 1.1] that $\operatorname{Isom}(X)$, the isometry group of X, is a Lie group. Moreover, $\operatorname{Isom}(X)$ is compact, if X is compact and connected (see [31, p. 370, Satz I] or [24, Corollary 4.10 and its proof in pp. 46–50]). As in the Riemannian case, the maximal dimension of $\operatorname{Isom}(X)$ is $n(n+1)/2$ and, if equality holds, X must be isometric to a Riemannian manifold (see [13, Theorems 3.1 and 4.1]).

As for locally smooth actions (see [7, Ch. IV, Section 3]), for an isometric action of a compact Lie group G on an Alexandrov space X there also exists a maximal orbit type G/H (see [13, Theorem 2.2]). This orbit type is the *principal orbit type*, and orbits of this type are called *principal orbits*. A non-principal orbit is *exceptional* if it has the same dimension as a principal orbit.

The structure of the space of directions in the presence of an isometric action is given by the following proposition.

Proposition 2.3 ([15, Proposition 4]) *Let X be an Alexandrov space with an isometric G-action and fix $x \in X$ with $\dim(G/G_x) > 0$. Let $S_x \subseteq \Sigma_x X$ be the unit tangent space to the orbit $G(x) \simeq G/G_x$, and let $S_x^{\perp} = \{v \in \Sigma_x X : \angle(v,w) = \pi/2 \text{ for all } w \in S_x\}$ be the set of normal directions to S_x. Then the following hold:*

(1) *The set $S_x^{\perp}$ is a compact, totally geodesic Alexandrov subspace of $\Sigma_x X$ with curvature bounded below by 1, and the space of directions $\Sigma_x X$ is isometric to the join $S_x * S_x^{\perp}$ with the standard join metric.*
(2) *Either $S_x^{\perp}$ is connected or it contains exactly two points at distance π.*

2.4 Alexandrov spaces of cohomogeneity one

In this subsection we collect basic facts on cohomogeneity one Alexandrov spaces and prove some preliminary results that we will use in the proof of Theorem A. For cohomogeneity one actions on smooth or topological manifolds, we refer the reader to [16, 22], respectively.

Definition 2.4 Let X be a connected n-dimensional Alexandrov space with an isometric action of a compact connected Lie group G. The action is of *cohomogeneity one* if the orbit space is one-dimensional or, equivalently, if there exists an orbit of dimension $n-1$. A connected Alexandrov space with an isometric action of cohomogeneity one is a *cohomogeneity one Alexandrov space*.

Cohomogeneity one Alexandrov spaces were first studied in [15]. Recall that the orbit space X/G of an Alexandrov space X by an isometric action of a group G with closed orbits is again an Alexandrov space (see [8, Proposition 10.2.4]). Since one-dimensional Alexandrov spaces are topological manifolds, the orbit space of a cohomogeneity one Alexandrov space is homeomorphic to a connected 1-manifold (possibly with boundary). When the orbit space is homeomorphic to $[-1, 1]$, we denote the isotropy groups corresponding to a point in the orbit mapped to ± 1 by $K^{\pm}$. By the Isotropy lemma (see [13, Lemma 2.1]) and the fact that principal orbits are open and dense, the orbits that project to the interior $(-1, 1)$ of the orbit space all have the same isotropy group H (up to conjugacy) and H is a subgroup of $K^{\pm}$. The subgroup H is the principal isotropy group of the action, and the corresponding orbits are the principal orbits. Let us now show that H is a proper subgroup of $K^{\pm}$. It suffices to show that if $\dim K^{\pm} = \dim H$, then $K^{\pm} \neq H$. Observe first that, in this case, $S^{\perp} = \mathbb{S}^0$ with a transitive action of $K^{\pm}$ with isotropy H. Hence $K^{\pm}/H = \mathbb{S}^0$, which shows that $K^{\pm} \neq H$. We call the orbits mapped to ± 1 *non-principal* orbits.

Let X be a closed cohomogeneity one Alexandrov G-space. Since the orbit space X/G must be a compact one-manifold, it must be either a circle or a closed interval. When X/G is a circle, X is equivariantly homeomorphic to a fiber bundle over $\mathbb{S}^1$ with fiber a principal orbit G/H. In particular, X is a smooth manifold (see [15, Theorem A]). Since we are interested in non-manifold Alexandrov spaces, we will focus our attention on the case where X/G is a compact interval.

A cohomogeneity one G-action on a closed Alexandrov space whose orbit space is an interval determines a group diagram

$$(G, H, K^{-}, K^{+}),$$

where $K^{\pm}$ are isotropy subgroups at the non-principal orbits corresponding to the endpoints of the interval, and H is the principal isotropy group of the action. The following theorem determines the structure of closed cohomogeneity-one Alexandrov spaces with orbit space an interval.

Theorem 2.5 ([15, Theorem A]) *Let X be a closed Alexandrov space with an effective isometric G-action of cohomogeneity one with principal isotropy H and orbit space homeomorphic to* $[-1, 1]$. *Then X is the union of two fiber bundles over the two singular orbits whose fibers are cones over positively curved homogeneous spaces, that is,*

$$X = G \times_{K^-} C(K^-/H) \bigcup_{G/H} G \times_{K^+} C(K^+/H).$$

The group diagram of the action is given by (G, H, K^-, K^+), *where* $K^{\pm}/H$ *are positively curved homogeneous spaces. Conversely, a group diagram* (G, H, K^-, K^+), *where* $K^{\pm}/H$ *are positively curved homogeneous spaces, determines a cohomogeneity one Alexandrov space.*

We will use the following proposition to identify equivalent actions.

Proposition 2.6 ([15, Proposition 9]) *If a cohomogeneity one Alexandrov space is given by a group diagram* (G, H, K^-, K^+), *then any of the following operations on the group diagram will result in an equivalent Alexandrov space:*

(1) *Switching* K^- and K^+,
(2) *Conjugating each group in the diagram by the same element of* G,
(3) *Replacing* K^- *with* gK^-g^{-1} *for* $g \in N(H)_0$, *the identity component of* $N(H)$.

Conversely, the group diagrams for two equivalent cohomogeneity one, closed Alexandrov space must be mapped to each other by some combination of these three operations.

Let G be a compact connected Lie group acting on a closed Alexandrov space X with cohomogeneity one, and let $\pi : X \to X/G = [0, 1]$ be the projection map. A minimizing geodesic $\gamma : [0, d] \to X$ between non-principal orbits has the following properties (see [13, Lemma 2.1]):

- it goes through all principal orbits,
- for all $t \in (0, d)$, $H = G_{c(t)} \subset G_{c(0)}, G_{c(d)}$, and
- the direction of γ is horizontal.

We set $K^- = G_{c(0)}$ and $K^+ = G_{c(d)}$. We call such a geodesic a *normal geodesic* (cf. [1, Section 4]).

Definition 2.7 We say that the cohomogeneity one Alexandrov space X is *non-primitive* if it has some group diagram representation (G, H, K^-, K^+) for which there is a proper connected closed subgroup $L \subset G$ with $K^{\pm} \subset L$. It then follows that (L, H, K^-, K^+) is a group diagram which determines some cohomogeneity one Alexandrov space Y.

Proposition 2.8 ([15, p. 96]) *Take a non-primitive cohomogeneity one Alexandrov space* X with L *and* Y as *in Definition* 2.7*. Then* X*is equivalent to* $(G \times Y)/L$*, where* L*acts on* $G \times Y$ *by* $l \cdot (g, y) = (gl^{-1}, ly)$*. Hence, there is a fiber bundle*

$$Y \to X \to G/L.$$

Definition 2.9 A cohomogeneity one action of a compact Lie group G on an Alexandrov space X is called *reducible* if there is a proper closed normal subgroup of G that acts on X with the same orbits.

We now recall the following results which describe the reduction or extension of certain cohomogeneity one actions (cf. [22, Section 1.11] and [15, Section 2]).

Proposition 2.10 ([15, Proposition 11]) *Let* X *be the cohomogeneity one Alexandrov space given by the group diagram* (G, H, K^-, K^+) *and suppose that* $G = G_1 \times G_2$ *with* $\mathrm{Proj}_2(H) = G_2$*. Then the subaction of* $G_1 \times 1$ *on* X *is also of cohomogeneity one, has the same orbits as the action of* G*, and has isotropy groups* $K_1^{\pm} = K^{\pm} \cap (G_1 \times 1)$ *and* $H_1 = H \cap (G_1 \times 1)$*.*

For the next proposition we will need the concept of a *normal extension*, which we now recall.

Definition 2.11 Let X be a cohomogeneity one Alexandrov space with group diagram (G_1, H_1, K^-, K^+), and let L be a compact, connected subgroup of $N(H_1) \cap N(K^-) \cap N(K^+)$. Observe that the subgroup $L \cap H_1$ is normal in L and define $G_2 := L/(L \cap H_1)$. We can then define an action of $G_1 \times G_2$ on X orbitwise by letting

$$(\hat{g}_1, [l]) \cdot g_1(G_1)_x = \hat{g}_1 g_1 l^{-1} (G_1)_x$$

on each orbit $G_1/(G_1)_x$ for $(G_1)_x = H_1$ or $K^{\pm}$. Such an extension is called a *normal extension* of G_1.

Proposition 2.12 ([15, Proposition 12]) *A normal extension of* G_1 *describes a cohomogeneity one action of* $G := G_1 \times G_2$ *on* X *with the same orbits as* G_1*and with group diagram*

$$(G_1 \times G_2, (H_1 \times 1)\Delta L, (K^- \times 1)\Delta L, (K^+ \times 1)\Delta L),$$

where $\Delta L = \{(l, [l]) : l \in L\}$*.*

Proposition 2.13 ([15, Proposition 13]) *For* X *as in Proposition* 2.10*, the action by* $G = G_1 \times G_2$ *occurs as the normal extension of the reduced action of* $G_1 \times 1$ *on* X*.*

Recall that every compact Lie group has a finite cover of the form $G_1 \times \cdots \times G_l \times T^n$, where the G_i are simple Lie groups. Therefore, every cohomogeneity one action can be written as an action of $G = G_1 \times \cdots \times G_l \times T^n$ if one allows for a finite ineffective kernel. In this case, as pointed out in [22, Section 1.11] in the manifold case, the action is reducible if and only if the principal isotropy group H projects onto some factor of G. Propositions 2.10, 2.12 and 2.13 show that the classification of cohomogeneity one Alexandrov spaces can be reduced to the classification of those with non-reducible actions. Thus we will assume from now on that all our cohomogeneity one actions are non-reducible.

2.5 Further tools

The following proposition, whose proof is as in [22, Proposition 1.25], yields bounds on the dimension of a Lie group acting by cohomogeneity one in terms of the dimension of a principal isotropy subgroup.

Proposition 2.14 *Let X be a closed Alexandrov space of cohomogeneity one with group diagram (G, H, K^-, K^+). Suppose that G acts non-reducibly on X and that G is the product of groups*

$$G = \mathrm{SU}(4)^i \times (G_2)^j \times \mathrm{Sp}(2)^k \times \mathrm{SU}(3)^l \times (S^3)^m \times (S^1)^n.$$

Then

$$\dim(H) \leq 10i + 8j + 6k + 4l + m.$$

We now state some useful results on the fundamental group of cohomogeneity one Alexandrov spaces. Their proofs follow as in the manifold case (see [22, Section 1.6] and [16, Section 4]).

Proposition 2.15 (Corollary to the van Kampen Theorem [22, Proposition 1.8]) *Let X be the closed cohomogeneity one Alexandrov space given by the group diagram (G, H, K^-, K^+) with $\dim(K^\pm/H) \geq 1$. Then*

$$\pi_1(X) \cong \pi_1(G/H)/N^-N^+,$$

where

$$N^\pm = \ker\{\pi_1(G/H) \to \pi_1(G/K^\pm)\} = \mathrm{Im}\{\pi_1(K^\pm/H) \to \pi_1(G/H)\}.$$

Corollary 2.16 ([16, Corollary 4.4]) *Let X be the closed simply connected cohomogeneity one Alexandrov space given by the group diagram (G, H, K^-, K^+), with $\dim(K^\pm/H) \geq 1$, and $K^-/H = \mathbb{S}^l$, for $l \geq 2$. Then G/K^+ is simply connected and, if G is connected, then K^+ is also connected.*

Lemma 2.17 ([22, Lemma 1.10]) *Let X be the closed cohomogeneity one Alexandrov space given by the group diagram (G, H, K^-, K^+) , and let $K_0^\pm$ and H_0 be the identity components of $K^\pm$ and H, respectively. Denote $H^\pm = H \cap K_0^\pm$, and let $\alpha_\pm^i : [0,1] \to K_0^\pm$ be curves that generate $\pi_1(K^\pm/H)$, with $\alpha_\pm^i(0) = 1 \in G$. The space X is simply connected if and only if*

(1) *H is generated as a subgroup by H^- and H^+, and*
(2) *α_-^i and α_+^i generate $\pi_1(G/H_0)$.*

We will use the following results on transitive actions.

Lemma 2.18 (cf. [16, Lemma 4.11]) *Let G_1 be a compact, connected, simply connected, simple Lie group of dimension n. Assume that G_1 is, up to a finite cover, the only Lie group that acts transitively and (almost) effectively on a manifold M with isotropy group H. Let G_2 be a compact, connected Lie group of dimension at most $n-1$. If $G_1 \times G_2$ acts transitively on M, then the following hold:*

(1) *The G_2 factor acts trivially on M and*
(2) *The isotropy group K of the $(G_1 \times G_2)$-action is $H \times G_2$.*

Proof Let $L \subseteq G_1 \times G_2$ be the kernel of the action of $G_1 \times G_2$ on M. Then $(G_1 \times G_2)/L$ is isomorphic to G_1. Hence, $\dim G_2 = \dim L$. Since L is a normal and connected subgroup of $G_1 \times G_2$, $\mathrm{Proj}_1(L)$ is a normal connected subgroup of G_1. Thus $\mathrm{Proj}_1(L)$ is trivial, since $\dim G_2 \leq n-1$. As a result, $L = 1 \times G_2$ and $K = H \times G_2$. □

Proposition 2.19 ([29, Ch. 1, §5 Proposition 7]) *Let a Lie group G act transitively on a manifold M. Then G_0acts transitively on any connected component of M. In particular, if $M$is connected, then G_0 acts transitively on M, and $G = G_0 G_x$, for all $x \in M$.*

The following two results give restrictions on the groups that may act by cohomogeneity one on a closed Alexandrov space. The next proposition can be found in [22, Proposition 1.19] for smooth actions. It was proven in [16] in the slightly more general case of topological actions on topological manifolds. The proof for Alexandrov spaces follows as in the topological case [16, Proposition 4.7], taking into account that, by the principal orbit theorem for Alexandrov spaces [13, Theorem 2.2], all principal isotropy groups are conjugate to each other and conjugate to a subgroup of non-principal isotropy groups.

Proposition 2.20 (cf. [16, Proposition 4.7]) *If a compact connected Lie group G acts (almost) effectively on an Alexandrov space with principal orbits of dimension k, then $k \leq \dim G \leq k(k+1)/2$.*

An argument as in the proof of [22, Proposition 1.18] yields the following lemma:

Lemma 2.21 *Let X be a closed, simply connected Alexandrov space with an (almost) effective cohomogeneity one action of a compact Lie group G. Suppose that the following conditions hold:*

- *$G = G_1 \times T^m$ and G_1 is semisimple;*
- *G acts non-reducibly;*
- *at least one of the homogeneous spaces $K^{\pm}/H$ is other than standard spheres.*

Then, $G_1 \neq 1$ and $m \leq 1$. Moreover, if $m = 1$, then one of the homogeneous spaces $K^{\pm}/H$, say K^-/H, is a circle and $K_0^- = H_0 \cdot S^-$, where S^- is a circle group with $\mathrm{Proj}_2(S^-) = T^1$ and $K_0^+ \subset G_1 \times 1$.

2.6 Special actions and recognition results

In this subsection we list some special types of cohomogeneity one actions and prove some recognition results that will allow us to identify such actions (cf. [22, 1.21]).

Definition 2.22 (*Product action*) Let G_1 and G_2 be Lie groups such that G_1 acts on an Alexandrov space X with cohomogeneity one and G_2 acts on a homogeneous space G_2/L transitively. We call the natural action of $G_1 \times G_2$ on $X \times G_2/L$ given by

$$(g_1, g_2) \cdot (x, gL) = (g_1 x, g_2 gL)$$

the *product action* of $G_1 \times G_2$.

Proposition 2.23 *Suppose that G_1 acts on an Alexandrov space X with cohomogeneity one and with group diagram (G_1, H, K^-, K^+), and G_2 acts transitively on the homogeneous space G_2/L. Then the product action of $G_1 \times G_2$on $X \times G_2/L$ is of cohomogeneity one with group diagram*

$$(G_1 \times G_2, H \times L, K^- \times L, K^+ \times L). \tag{2.1}$$

Conversely, a cohomogeneity one action of $G_1 \times G_2$ with the above group diagram, and $G_1/K^\pm$ positively curved homogeneous spaces, is equivalent to a product action of $G_1 \times G_2$ on $X \times G_2/L$, where Xis the cohomogeneity one Alexandrov space determined by the diagram (G_1, H, K^-, K^+).

Proof It is clear that the product action of $G_1 \times G_2$ on $X \times G_2/L$ is of cohomogeneity one. Now we prove that its group diagram is as in (2.1). Let γ be a normal geodesic between the non-principal orbits $G_1/K^\pm$ in X giving the group diagram (G_1, H, K^-, K^+). If we fix a G_2 -invariant metric on G_2/L, then, in the product metric on $X \times G_2/L$, the curve $\tilde{\gamma} = (\gamma, 1)$ is a shortest geodesic between non-principal orbits. The resulting diagram is

$$(G_1 \times G_2, H \times L, K^- \times L, K^+ \times L),$$

as claimed. The converse follows from Proposition 2.6. □

Definition 2.24 (*Join action*) Let G_1 and G_2 be two Lie groups which act on Alexandrov spaces X_1 and X_2, respectively. The action of $G_1 \times G_2$ on $X_1 * X_2$ is called *join action*, if $G_1 \times G_2$ acts on $X_1 * X_2$ naturally, i.e.

$$(g_1, g_2) \cdot [(x, y, t)] = [(g_1 x, g_2 y, t)].$$

Proposition 2.25 *If two Lie groups $G_1$and G_2 act transitively on positively curved homogeneous spaces M_1 and M_2 with isotropy groups H_1 and H_2, respectively, then the join action of $G = G_1 \times G_2$ on $M_1 * M_2$ is of cohomogeneity one with the following diagram:*

$$(G_1 \times G_2, H_1 \times H_2, G_1 \times H_2, H_1 \times G_2).$$

Conversely, a cohomogeneity one action of $G_1 \times G_2$ with the above group diagram, and G_i/H_i positively curved homogeneous spaces, for $i = 1, 2$, is equivalent to the join action of G on $(G_1/H_1) * (G_2/H_2)$.

Proof Let $x \in M_1$ and $y \in M_2$ be such that $H_1 = (G_1)_x$ and $H_2 = (G_2)_y$. The curve

$$\gamma : [0, \pi/2] \to X_1 * X_2$$
$$t \mapsto [x, y, t]$$

is a shortest geodesic between $[x, y, 0]$ and $[x, y, \pi/2]$ which goes through all orbits. Furthermore, $G_{\gamma(0)} = H_1 \times G_2$, $G_{\gamma(\pi/2)} = G_1 \times H_2$, and $t \in (0, \pi/2)$, $G_{\gamma(t)} = H_1 \times H_2$. Therefore, the action is of cohomogeneity one with the given diagram. By Proposition 2.6, the converse is immediate. □

Definition 2.26 (*Suspension action*) Let G be a Lie group which acts on an Alexandrov space X. The action of G on $\mathrm{Susp}\,(X)$ is called *suspension action*, if G acts on $\mathrm{Susp}\,(X)$ as follows:

$$g \cdot [(x, t)] = [(gx, t)].$$

Proposition 2.27 *Let G act transitively on a positively curved homogeneous space M with isotropy group H. Then the suspension action of G on* $\mathrm{Susp}\,(M)$ *is of cohomogeneity one with diagram (G, H, G, G). Conversely, a cohomogeneity one action of G with the above group diagram, and G/H a positively curved homogeneous space, is equivalent to the suspension action of G on* $\mathrm{Susp}\,(G/H)$.

Proof Let $x \in M$ be such that $H = G_x$. The curve

$$\gamma : [0, \pi] \to \mathrm{Susp}\,(M)$$
$$t \mapsto [x, t]$$

is a shortest geodesic between $[x, 0]$ and $[x, \pi]$ which goes through all orbits. Furthermore, $G_{\gamma(0)} = G$, $G_{\gamma(\pi)} = G$, and for $t \in (0, \pi)$, $G_{\gamma(t)} = H$. Therefore, the action is of cohomogeneity one with given diagram. By Proposition 2.6, the inverse is clear. □

Proposition 2.28 (*Spin action*) *Let G be a compact, simply connected Lie group which acts almost effectively and by cohomogeneity one on a closed, simply connected Alexandrov space X^n with group diagram (G, H, K^-, K^+). If X^n is not a manifold and* $\dim G = n(n-1)/2$, *then G is isomorphic to* $\mathrm{Spin}(n)$ *and the action is equivalent to the cohomogeneity one action of* $\mathrm{Spin}(n)$ *on* $\mathrm{Susp}\,(\mathbb{R}P^{n-1})$, *which is the suspension of the transitive action of* $\mathrm{Spin}(n)$ *on* $\mathbb{R}P^{n-1}$.

Proof The proof of this proposition is analogous to Hoelscher's proof in [22, Proposition 1.20] for the manifold case, with slight changes. Namely, since in our case $K^{\pm}$ is not a sphere, $H_0 \neq H$. As the only proper subgroup of $\mathrm{Spin}(n)$ containing $\mathrm{Spin}(n-1)$ is $N_{\mathrm{Spin}(n)}(\mathrm{Spin}(n-1))$, we have $H = N_{\mathrm{Spin}(n)}(\mathrm{Spin}(n-1))$ and $K^{\pm}/H = \mathbb{R}P^{n-1}$. □

2.7 Transitive actions on spheres

We conclude this section by recalling the well-known classification of almost effective transitive actions on spheres (see [3] and the references therein). We will use this classification throughout our work. First, we briefly explain the notation used for the representations appearing in the classification.

The symbols ρ_n, μ_n and ν_n denote, respectively, the standard representations of $\mathrm{SO}(n)$ in $\mathbb{R}^n$, of $\mathrm{SU}(n)$ or $\mathrm{U}(n)$ in $\mathbb{C}^n$, and of $\mathrm{Sp}(n)$ in $\mathbb{H}^n$. If $\pi = \mu_n$ or $\pi = \nu_n$, we denote by $\pi_{\mathbb{R}}$ the underlying real representation on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ or $\mathbb{H}^n \simeq \mathbb{R}^{4n}$, respectively, where we consider $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ or $\mathbb{H}^n \simeq \mathbb{R}^{4n}$ as real vector spaces. Moreover, if L^* denotes the adjoint of a linear map L between vector spaces, then we define $\pi^*(g) := (\pi(g^{-1}))^*$. Finally, ϕ_2 stands for the standard representation of G_2 in SO(7) and Δ_n is the spin representation (see, for example, [25, Section 3.7.5]).

Theorem 2.29 ([3, Section 2.1]) *Suppose that a compact, connected Lie group G acts almost effectively and transitively on the sphere $\mathbb{S}^{n-1} (n \geq 2)$. Then the G-action on $\mathbb{S}^{n-1}$ is equivalent to the following linear action of G on $\mathbb{S}^{n-1}$ via the representation $\iota : G \to \mathrm{SO}(n)$ with an isotropy subgroup H.*

(i) *If n is odd, then $G$is simple and (G, n, ι, H) are*

$$(\mathrm{SO}(n), n, \rho_n, \mathrm{SO}(n-1)), \tag{2.2}$$

$$(\mathrm{G}_2, 7, \phi_2, \mathrm{SU}(3)). \tag{2.3}$$

(ii) *If n is even, then G contains a simple normal subgroup G' such that the restricted G' -action on $\mathbb{S}^{n-1}$ is transitive and $G/G'$is of rank at most 1, and (G, n, ι, H) is*

$$(\mathrm{SO}(n), n, \rho_n, \mathrm{SO}(n-1)), \quad (n \neq 4), \tag{2.4}$$

$$(\mathrm{Spin}(7), 8, \Delta_7, \mathrm{G}_2), \tag{2.5}$$

$$(\mathrm{U}(k), 2k, (\mu_k)_{\mathbb{R}}, \mathrm{U}(k-1)), \tag{2.6}$$

$$(\mathrm{Sp}(k), 4k, (\nu_k)_{\mathbb{R}}, \mathrm{Sp}(k-1)), \tag{2.7}$$

$$(\mathrm{Sp}(k) \times S^1, 4k, (\nu_k \otimes \mu_1^*)_{\mathbb{R}}, \mathrm{Sp}(k-1) \times S^1), \tag{2.8}$$

$$(\mathrm{Sp}(k) \times S^3, 4k, (\nu_k \otimes \nu_1^*)_{\mathbb{R}}, \mathrm{Sp}(k-1) \times S^3), \tag{2.9}$$

$$(\mathrm{Spin}(9), 16, \Delta_9, \mathrm{Spin}(7)), \tag{2.10}$$

$$(\mathrm{SU}(k), 2k, (\mu_k)_{\mathbb{R}}, \mathrm{SU}(k-1)). \tag{2.11}$$

3 Proof of Theorem A

3.1 Possible groups

We first list the Lie groups that can act (almost) effectively and by cohomogeneity one on an Alexandrov space of dimension 5, 6 or 7. This list is obtained as in the manifold case, and we refer the reader to [22, Section 1.24] for more details.

Let G be a compact connected Lie group acting (almost) effectively and by cohomogeneity one on an n-dimensional Alexandrov space X^n. It is well-known that every compact

Table 8 Compact, connected, simply connected simple Lie groups in dimensions 21 and less

Group	Dimension	Rank
$S^3 \cong \mathrm{SU}(2) \cong \mathrm{Sp}(1) \cong \mathrm{Spin}(3)$	3	1
SU(3)	8	2
$\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$	10	2
G_2	14	2
$\mathrm{SU}(4) \cong \mathrm{Spin}(6)$	15	3
Sp(3)	21	3
Spin(7)	21	3

Table 9 Compact, connected, proper subgroups of dimension k, up to conjugation

Group	Dimensions	Subgroups
T^2	$k \geq 1$	$\{(e^{ip\theta}, e^{iq\theta})\}$
S^3	$k \geq 1$	$\{e^{x\theta} = \cos\theta + x\sin\theta\}$, where $x \in \mathrm{Im}(\mathbb{H}) \cap S^3 \subseteq \mathrm{Im}(\mathbb{H})$.
SU(3)	$k \geq 1$	$S^1 \subset T^2$, T^2, SO(3), SU(2) and U(2)
Sp(2)	$k \geq 4$	U(2), Sp(1)SO(2) and Sp(1)Sp(1)
G_2	$k \geq 8$	SU(3)
SU(4)	$k \geq 9$	U(3) and Sp(2)

and connected Lie group has a finite cover of the form $G_{ss} \times T^k$, where G_{ss} is semisimple and simply connected, and T^k is a torus. The classification of simply connected simple Lie groups is well-known, and all the possibilities are listed in Table 8 for dimensions 21 and less.

If an arbitrary compact connected Lie group G acts on an Alexandrov space X, then every cover $\widetilde{G}$ of G still acts on X, although less effectively. Hence, allowing for a finite ineffective kernel, and because G will always have dimension 21 or less, we can assume that G is a product of groups from Table 8 with a torus T^k.

In Table 9 we list the proper, connected, non-trivial closed subgroups of the groups in Table 8, in dimensions at most 15, and of T^2; these are the dimensions that will be relevant in our case. These subgroups are well-known (see, for example, [11] or [21, Tables 2.2.1 and 2.2.2]). Note that for the group Sp(2), we only list the subgroups in dimensions at least 4; for G_2, the subgroups in dimension at least 8; and for SU(4), the subgroups in dimension at least 9. The subgroups of lower dimensions can be distinguished by the information from the subgroups of the previous groups. The explicit embeddings of each subgroup depend on the way these groups are acting and are described in the course of the classification.

3.2 Possible normal spaces of directions

As stated in Theorem 2.5, for a cohomogeneity one action with group diagram (G, H, K^-, K^+), the homogeneous spaces $K^\pm/H$ are positively curved. The classification of simply connected positively curved homogeneous spaces has been carried out by Berger [6], Wallach [32], Aloff and Wallach [2], Berard-Bergery [4] and Wilking [33] (for

Table 10 Positively curved homogeneous spaces in dimensions at most 6

Dimension	Space
0	$\mathbb{S}^0$
1	$\mathbb{S}^1$
2	$\mathbb{S}^2, \mathbb{R}P^2$
3	3-Dimensional spherical space forms
4	$\mathbb{S}^4, \mathbb{R}P^4, \mathbb{C}P^2$
5	5-Dimensional spherical space forms
6	$\mathbb{S}^6, \mathbb{R}P^6, \mathbb{C}P^3, \mathbb{C}P^3/\mathbb{Z}_2, \mathrm{W}^6, \mathrm{W}^6/\mathbb{Z}_2$.

a complete exposition of the classification, correcting some oversights in the literature, see the article by Wilking and Ziller [34]). Combining this with the classification of homogeneous space forms due to Wolf [35], and the fact that in even dimensions there can be at most $\mathbb{Z}_2$ quotients, by Synge's theorem, it follows that the positively curved homogeneous spaces in dimensions 5 and below are (diffeomorphic to) $\mathbb{S}^0$, $\mathbb{S}^1$, $\mathbb{S}^2$, $\mathbb{R}P^2$, the three-dimensional spherical space forms S^3/Γ, where Γ is a finite subgroup of S^3 as in [35, Corollary 2.7.2], $\mathbb{S}^4$, $\mathbb{R}P^4$, $\mathbb{C}P^2$ (noting that $\mathbb{C}P^2$ admits no $\mathbb{Z}_2$ quotient) and, in dimension 5, the five-dimensional spherical space forms. In dimension 6, there appear $\mathbb{S}^6$, $\mathbb{R}P^6$, $\mathbb{C}P^3$, $\mathbb{C}P^3/\mathbb{Z}_2$ and, finally, the Wallach manifold $\mathrm{W}^6 = \mathrm{SU}(3)/T^2$ and its $\mathbb{Z}_2$ quotient. We collect this information in Table 10.

Let X be a closed Alexandrov space of cohomogeneity one. If both $K^\pm/H$ are spheres, then X is equivalent to a smooth manifold. These manifolds and their actions have been classified by Mostert [27] and Neumann [28] in dimensions 2 and 3, Parker [30] in dimension 4 and Hoelscher [22] in dimensions 5, 6 and 7 (assuming X is simply connected). If both $K^\pm/H$ are integral homology spheres, then X is equivalent to a topological manifold and $K^\pm/H$ must be either a sphere or the Poincaré homology sphere $\mathbf{P}^3$ (see [16]). These manifolds and their actions have been classified in [16] up to dimension 7, assuming, as in the manifold case, simply connectedness in dimensions 5, 6 and 7. From now on we will assume that at least one of the homogeneous spaces $K^\pm/H$ is not a sphere, i.e., that the action is not equivalent to a smooth action on a smooth manifold. We remind the reader that we assume all cohomogeneity one actions to be non-reducible.

3.3 Classification in dimension 5

To find the group diagrams of cohomogeneity one actions on closed, simply connected Alexandrov spaces in dimension 5, we first determine the acting groups. By Proposition 2.20, $4 \leq \dim G \leq 10$. Hence, in Table 8, G has the form $(S^3)^m \times T^n$, $\mathrm{SU}(3) \times T^n$ or Spin(5). From Proposition 2.21, we have $n \leq 1$. Since $\dim H = \dim G - 4$, Proposition 2.14 gives the possible groups. These are, up to a finite cover:

$$S^3 \times S^1, S^3 \times S^3, \mathrm{SU}(3), \text{ or } \mathrm{Spin}(5).$$

Now we examine the action of each group case by case.

$\mathbf{G = S^3 \times S^1}$. In this case, $\dim H = 0$, so $H_0 = \{1\}$. By Proposition 2.21, and without loss of generality, we can assume that $K^-/H = \mathbb{S}^1$. Therefore, $K_0^- = \{(e^{xp\theta}, e^{iq\theta}) \mid \theta \in \mathbb{R}\} \subseteq S^3 \times S^1$, with $x \in \operatorname{Im}(\mathbb{H})$, $q \neq 0$ and $(p, q) = 1$. Now we want to determine K^+/H. Since we have assumed that the action is non-smoothable, K^+/H is not a sphere. Hence, the possible dimensions for K^+/H are 2, 3 or 4. Since, by Proposition 2.19, K_0^+ acts transitively on K^+/H , it cannot be 1-dimensional. Further, by Proposition 2.21, $K_0^+ \subseteq S^3 \times 1$. Therefore, $K_0^+ = S^3 \times 1$ and $K^+/H = S^3/\Gamma$, with $\Gamma \neq \{1\}$. Consequently, by Proposition 2.19, we have that $H^+ = K_0^+ \cap H = \Gamma \times 1$.

Let $p = 0$. Then $H^- = K_0^- \cap H = 1 \times \mathbb{Z}_k$. Thus by Lemma 2.17, $H = \langle H^+, H^- \rangle = \Gamma \times \mathbb{Z}_k$. Therefore, by Proposition 2.19, $K^- = K_0^- H = \Gamma \times S^1$ and $K^+ = K_0^+ H = S^3 \times \mathbb{Z}_k$, and we obtain the diagram

$$(S^3 \times S^1, \Gamma \times \mathbb{Z}_k, \Gamma \times S^1, S^3 \times \mathbb{Z}_k). \tag{3.1}$$

By Proposition 2.25, this action is a join action and therefore X is equivariantly homeomorphic to $(S^3/\Gamma) * \mathbb{S}^1$ with the join action of $S^3 \times S^1$.

Now let $p \neq 0$. After conjugation, we may assume that $K_0^- = \{(e^{ip\theta}, e^{iq\theta}) \mid \theta \in \mathbb{R}\}$. Since $\Gamma \times 1 \subseteq H \subseteq K^- \subseteq N_G(K_0^-) = S^1 \times S^1$, we have that $\Gamma = \mathbb{Z}_m$, for $m \geq 2$. Moreover,

$$H^- = H \cap K_0^- = \mathbb{Z}_k := \left\langle \left(e^{\frac{2\pi i}{k} p}, e^{\frac{2\pi i}{k} q} \right) \right\rangle.$$

Then, by Lemma 2.17,

$$H = \langle H^-, H^+ \rangle = \left\{ \left(e^{\frac{2\pi i}{k} \frac{lk+mps}{m}}, e^{\frac{2\pi i}{k} qs} \right) \mid 1 \leq s \leq k, 1 \leq l \leq m \right\}.$$

By Proposition 2.19, we have then that

$$K^+ = K_0^+ H = S^3 \times \mathbb{Z}_{k/(k,q)}$$

and

$$K^- = K_0^- H = (\mathbb{Z}_m \times 1) K_0^-.$$

We now look for conditions on the parameters p, q, m, k. By Proposition 2.15, and the long exact sequences of homotopy groups of the fiber bundles

$$K^\pm/H \to G/H \to G/K^\pm,$$
$$K^- \to G \to G/K^-,$$

one can see that $\pi_0(K^-) = \mathbb{Z}_m/\mathbb{Z}_q$. Thus $q|m$. In addition, since

$$H^- \cap H^+ = \left\{ \left(e^{\frac{2\pi i}{k} \frac{k}{(k,q)} ps}, 1 \right) \mid 1 \leq s \leq (k,q) \right\},$$

we have $(k, q) = q$, i.e., $q|k$. We can also assume that $H \cap (1 \times S^1) = 1$ to have a more effective action. This condition gives, in particular, that $(p, k) = 1$. Therefore, the diagram is given by

$$\left(S^3 \times S^1, \left\{ \left(e^{\frac{2\pi i}{k} \frac{lk+mps}{m}}, e^{\frac{2\pi i}{k} qs} \right) \mid 1 \leq s \leq k, 1 \leq l \leq m \right\}, (\mathbb{Z}_m \times 1).K_0^-, S^3 \times \mathbb{Z}_{k/q} \right), \tag{3.2}$$

where $(p, k) = 1$ and $q|(m, k)$.

$\mathbf{G = S^3 \times S^3}$. We have $\dim H = 2$. Since the only connected 2-dimensional subgroup of G is its maximal torus, we have that $H_0 = T^2$. Therefore, $K_0^{\pm}$, which contain T^2, must be $S^3 \times S^1$ or $S^1 \times S^3$. In particular, $K^{\pm}/H$ is 2-dimensional. Since at least one of the positively curved homogeneous spaces $K^{\pm}/H$ is not a sphere, we may assume, without loss of generality, that $K^+/H = \mathbb{R}P^2$. The other homogeneous space K^-/H can be $\mathbb{S}^2$ or $\mathbb{R}P^2$.

First assume that $K^-/H = \mathbb{S}^2$. Then by Proposition 2.15, K^+ is connected. Let $K^+ = S^3 \times S^1$. Recall that S^3 is, up to a finite cover, the only Lie group that acts (almost) effectively and transitively on $\mathbb{R}P^2$. Then by Proposition 2.18, $H = N_{S^3}(S^1) \times S^1$. Consequently, K^- has to be $N_{S^3}(S^1) \times S^3$ since K^- contains H and $K^-/H = \mathbb{S}^2$. Therefore we have the diagram

$$(S^3 \times S^3, N_{S^3}(S^1) \times S^1, N_{S^3}(S^1) \times S^3, S^3 \times S^1), \tag{3.3}$$

which corresponds to a join action. By Proposition 2.25 X is equivariantly homeomorphic to $\mathbb{R}P^2 * \mathbb{S}^2$ with the join action of $S^3 \times S^3$.

Now let $K^-/H = \mathbb{R}P^2$. Assume that $K_0^+ = S^3 \times S^1$ and $K_0^- = S^3 \times S^1$. First notice that since $T^2 \subseteq K_0^{\pm}$, the circles in the second component of $K_0^{\pm}$ are the same, so $K_0^- = K_0^+$. Since $K^{\pm}$ acts transitively on $\mathbb{R}P^2$, so does $K_0^{\pm}$. Furthermore, by Theorem 2.29, $S^3 \times S^1$ does not act almost effectively on $\mathbb{R}P^2$. Thus, by Lemma 2.18, the second factor acts trivially and $H \cap K_0^{\pm} = N_{S^3}(S^1) \times S^1$. Since X is simply connected, by Lemma 2.17, $H = \langle H^+, H^- \rangle = N_{S^3}(S^1) \times S^1$. Therefore, $K^{\pm}$ are both connected and we obtain the diagram

$$(S^3 \times S^3, N_{S^3}(S^1) \times S^1, S^3 \times S^1, S^3 \times S^1). \tag{3.4}$$

This action is non-primitive with $L = S^3 \times S^1$ as in Definition 2.7. Thus

$$(S^3 \times S^1, N_{S^3}(S^1) \times S^1, S^3 \times S^1, S^3 \times S^1) \tag{3.5}$$

is a group diagram of a cohomogeneity one action on a 3-dimensional Alexandrov space Y. By Proposition 2.8, X is then equivariantly homeomorphic to $G \times_L Y$, which is a the total space of a Y-bundle over G/L. The action on $G \times_L Y$ is given by $g.[a, y] = [ga, y]$. Therefore, to obtain the homeomorphism type of X, we need to figure out what Y could be.

Since $\mathrm{proj}_2(H) = S^1$, by Proposition 2.10, the subaction of $S^3 \times 1$ on Y is also by cohomogeneity one with group diagram

$$(S^3 \times 1, N_{S^3}(S^1) \times 1, S^3 \times 1, S^3 \times 1),$$

which is the group diagram of the suspension action of S^3 on $\mathrm{Susp}\,(\mathbb{R}P^2)$. By Proposition 2.13, the group diagram (3.5) is just the normal extension of the suspension action of $S^3 \times 1$ on $\mathrm{Susp}\,(\mathbb{R}P^2)$. Whence Y is equivariantly homeomorphic to $\mathrm{Susp}\,(\mathbb{R}P^2)$, and X is equivariantly homeomorphic to the total space of a $\mathrm{Susp}\,(\mathbb{R}P^2)$-bundle over $\mathbb{S}^2$.

Assume now that $K_0^+ = S^3 \times S^1$ and $K_0^- = S^1 \times S^3$. Thus $H^+ = N_{S^3}(S^1) \times S^1$ and $H^- = S^1 \times N_{S^3}(S^1)$. As before, the assumption that X is simply connected implies, by Lemma 2.17, that $H = \langle H^+, H^- \rangle = N_{S^3}(S^1) \times N_{S^3}(S^1)$. Therefore we get the following diagram:

$$(S^3 \times S^3, N_{S^3}(S^1) \times N_{S^3}(S^1), N_{S^3}(S^1) \times S^3, S^3 \times N_{S^3}(S^1)). \tag{3.6}$$

By Proposition 2.25, this action is a join action and X is equivariantly homeomorphic to $\mathbb{R}P^2 * \mathbb{R}P^2$ with the join action of $S^3 \times S^3$.

G = SU(3). In this case $\dim H = 4$. In Table 9, one can see that the only 4-dimensional subgroup of SU(3) is U(2). Therefore, $H = H_0 = \mathrm{U}(2)$, as U(2) is a maximal subgroup of SU(3). Since X is simply connected, the action does not have any exceptional orbits. Hence, $K^{\pm}$ must be SU(3). Thus the diagram is

$$(\mathrm{SU}(3), \mathrm{U}(2), \mathrm{SU}(3), \mathrm{SU}(3)). \tag{3.7}$$

By Proposition 2.27, this action is a suspension action of SU(3) on Susp $(\mathbb{C}P^2)$ and thus X is equivalent to Susp $(\mathbb{C}P^2)$.

G = Spin(5). Since $\dim G = 10$, by Proposition 2.28, the group diagram is

$$(\mathrm{Spin}(5), N_{\mathrm{Spin}(5)}(\mathrm{Spin}(4)), \mathrm{Spin}(5), \mathrm{Spin}(5)), \tag{3.8}$$

and, by Proposition 2.28, X is equivariantly homeomorphic to Susp $(\mathbb{R}P^4)$ with the spin action of Spin(5).

3.4 Classification in dimension 6

Proceeding as in dimension 5, we see that $5 \leq \dim G \leq 15$ and $\dim H = \dim G - 5$. It follows from Propositions 2.14 and 2.21 that G is one of the following Lie groups:

$$S^3 \times S^3,\ S^3 \times S^3 \times S^1,\ \mathrm{SU}(3),\ \mathrm{SU}(3) \times S^1,\ \mathrm{Sp}(2),\ \mathrm{Sp}(2) \times S^1 \text{ or } \mathrm{Spin}(6).$$

If $G = \mathrm{Sp}(2)$, then dim $H = 5$. Since Sp(2) does not have a subgroup of dimension 5, we can rule it out. We now carry out the classification for the remaining groups in the list.

Notational convention. The binary dihedral group D^*_{2m} of order $4m$, $m \geq 2$, is a finite subgroup of S^3 (see [35], Section 2.6). Throughout the rest of the paper, we consider it as the following subgroup:

$$D^*_{2m} = \langle e^{\pi/mi}, j\rangle \subseteq S^3. \tag{3.9}$$

If, in the right-hand side of (3.9), we assume that $m = 1$, then $\langle e^{\pi/mi}, j\rangle = \mathbb{Z}_4$. Therefore, we use the notation D^*_{2m} for $m \geq 2$ (the binary dihedral group as in [35]), and, when $m = 1$, D^*_{2m} will correspond to the cyclic subgroup $\langle j\rangle$ of S^3 generated by j.

$\mathbf{G = S^3 \times S^3}$. In this case the principal isotropy group H is 1-dimensional. Thus $H_0 = T^1 \subseteq S^3 \times S^3$. After conjugation, we can assume that $H_0 = \{(e^{ip\theta}, e^{iq\theta}) \mid \theta \in \mathbb{R}\}$ with $(p, q) = 1$. Exploring the subgroups of G and the homogeneous spaces with positive curvature, we see that the normal space of directions to the singular orbits has to be a sphere, a real projective plane or $\mathbb{S}^3/\Gamma$ with $\Gamma \neq \{1\}$.

First, suppose that $K^+/H = \mathbb{R}P^2$. Therefore, K^+_0 is one of the subgroups $S^3 \times 1, 1 \times S^3$ or ΔS^3. Let $K^+_0 = S^3 \times 1$. Then $q = 0$ and $H^+ = H \cap K^+_0 = N_{S^3}(S^1) \times 1$. We now consider the different possibilities for K^-/H, namely, $\mathbb{S}^l$, $l \geq 1$, $\mathbb{R}P^2$, and $\mathbb{S}^3/\Gamma$ with $\Gamma \neq \{1\}$.

Let $K^-/H = \mathbb{S}^l$, $l \geq 2$. Then K^+ is connected and $H = N_{S^3}(S^1) \times 1$. Further, the only subgroup K^- of G containing H and satisfying $K^-/H = \mathbb{S}^l$ is $N_{S^3}(S^1) \times S^3$. Hence we have the following diagram:

$$(S^3 \times S^3, N_{S^3}(S^1) \times 1, N_{S^3}(S^1) \times S^3, S^3 \times 1). \tag{3.10}$$

By Proposition 2.25, this action is a join action and X is equivariantly homeomorphic to $\mathbb{R}P^2 * \mathbb{S}^3$ with the join action of $S^3 \times S^3$.

Let $K^-/H = \mathbb{S}^1$. Then $K_0^- = T^2$. Since $S^1 \times 1 \subseteq H \cap T^2 \subseteq S^1 \times S^1$, and $H^- = H \cap T^2$ is a finite extension of $S^1 \times 1$, we have $H \cap T^2 = S^1 \times \mathbb{Z}_k$. Consequently $H = \langle H^-, H^+ \rangle = N_{S^3}(S^1) \times \mathbb{Z}_k$. Thus we obtain the diagram

$$(S^3 \times S^3, N_{S^3}(S^1) \times \mathbb{Z}_k, N_{S^3}(S^1) \times S^1, S^3 \times \mathbb{Z}_k). \tag{3.11}$$

This action is non-primitive with $L = S^3 \times S^1$. Therefore, by Proposition 2.8, X is equivariantly homeomorphic to the total space of a Y-bundle over $\mathbb{S}^2$, where Y is the cohomogeneity one Alexandrov space given by the group diagram

$$(S^3 \times S^1, N_{S^3}(S^1) \times \mathbb{Z}_k, N_{S^3}(S^1) \times S^1, S^3 \times \mathbb{Z}_k). \tag{3.12}$$

By Proposition 2.25, it is the join action of $S^3 \times S^1$ on $\mathbb{R}P^2 * \mathbb{S}^1$. Therefore, Y is equivariantly homeomorphic to $\mathbb{R}P^2 * \mathbb{S}^1$.

Let $K^-/H = \mathbb{R}P^2$. Hence, K^- is a 3-dimensional subgroup of G, namely $S^3 \times 1$, $1 \times S^3$, or ΔS^3. However, since $S^1 \times 1 = H_0 \subseteq K_0^-$, the group K_0^- must be $S^3 \times 1$, and $H^- = K_0^- \cap H = N_{S^3}(S^1) \times 1$. Therefore $H = \langle H^-, H^+ \rangle = N_{S^3}(S^1) \times 1$, and we get the following diagram:

$$(S^3 \times S^3, N_{S^3}(S^1) \times 1, S^3 \times 1, S^3 \times 1). \tag{3.13}$$

This action is equivalent to the following action on $\operatorname{Susp}(\mathbb{R}P^2) \times \mathbb{S}^3$:

$$\begin{aligned}(S^3 \times S^3) \times (\operatorname{Susp}(\mathbb{R}P^2) \times \mathbb{S}^3) &\to (\operatorname{Susp}(\mathbb{R}P^2) \times \mathbb{S}^3)\\ ((g,h),([x,t],y)) &\mapsto ([gxg^{-1},t],hy).\end{aligned}$$

Let $K^-/H = S^3/\Gamma$ with $\Gamma \neq \{1\}$. Therefore, $K_0^- = S^3 \times S^1$, or $K_0^- = S^1 \times S^3$. Assume $\Gamma \neq \mathbb{Z}_k$. In this case, since S^3 is, up to a finite cover, the only Lie group which acts transitively and almost effectively on S^3/Γ, by Lemma 2.18, $H \cap K_0^-$ is $\Gamma \times S^1$ and $S^1 \times \Gamma$, respectively. As $H \cap K_0^- \subseteq K^+ = S^3 \times \Gamma_1$, where Γ_1 is a finite subgroup of S^3, we must have $H \cap K_0^- = S^1 \times \Gamma$. The diagram is then given by

$$(S^3 \times S^3, N_{S^3}(S^1) \times \Gamma, N_{S^3}(S^1) \times S^3, S^3 \times \Gamma). \tag{3.14}$$

By Proposition 2.25, the action is a join action and X is equivariantly homeomorphic to $\mathbb{R}P^2 * (S^3/\Gamma)$ with the join action of $S^3 \times S^3$.

Now let $\Gamma = \mathbb{Z}_k$. According to Theorem 2.29, $S^1 \times S^3$ acts on $S^3/\mathbb{Z}_k$ in the following way:

$$\begin{aligned}(S^1 \times S^3) \times S^3/\mathbb{Z}_k &\to S^3/\mathbb{Z}_k\\ ((z,v),[x]) &\mapsto [vx\bar{z}^p].\end{aligned}$$

Thus $H \cap K_0^- = \{(z, \lambda z^p) \mid z \in S^1, \lambda \in \mathbb{Z}_k\}$. However, $H \cap K_0^-$ is a subset of $K^+ = S^3 \times \Gamma_1$, which yields $p = 0$. Therefore we have the following diagram:

$$(S^3 \times S^3, N_{S^3}(S^1) \times \mathbb{Z}_k, N_{S^3}(S^1) \times S^3, S^3 \times \mathbb{Z}_k). \tag{3.15}$$

By Proposition 2.25, this action is a join action and X is equivariantly homeomorphic to $\mathbb{R}P^2 * (S^3/\mathbb{Z}_k)$ with the join action of $S^3 \times S^3$. For $K_0^- = S^3 \times S^1$, we have $H \cap K_0^- = \{(\lambda z^p, z) \mid z \in S^1, \lambda \in \mathbb{Z}_k\}$, which is not a subset of $K^+ = S^3 \times \Gamma_1$. Therefore, this case does not occur.

We now repeat the above procedure for $K_0^+ = \Delta S^3$. In this case $H_0 = \Delta S^1$ and

$$H^+ = H \cap K_0^+ = \Delta S^1 \cup (j,j)\Delta S^1.$$

We consider the different possibilities for K^-/H, namely, $\mathbb{S}^l$, $l \geq 1$, $\mathbb{R}P^2$, and $\mathbb{S}^3/\Gamma$ with $\Gamma \neq \{1\}$.

If $K^-/H = \mathbb{S}^l$, $l \geq 1$, then, as before, $l = 1, 3$ only. First, suppose that $K^-/H = \mathbb{S}^3$. Therefore, K^+ is connected and $H = \Delta S^1 \cup (j,j)\Delta S^1$. Since K_0^- is 4-dimensional, after exchanging the factors of G if necessary, we can assume that $K_0^- = S^3 \times S^1$. Hence $K^- = K_0^- H = S^3 \times N_{S^3}(S^1)$, and the following diagram is obtained

$$(S^3 \times S^3, \Delta S^1 \cup (j,j)\Delta S^1, S^3 \times N_{S^3}(S^1), \Delta S^3). \tag{3.16}$$

This action is equivalent to the following action:

$$\begin{aligned}(S^3 \times S^3) \times (S^3 * \mathbb{R}P^2) &\to (S^3 * \mathbb{R}P^2)\\ ((g,h),[x,[y]]) &\mapsto [gxh^{-1},[hyh^{-1}]].\end{aligned}$$

That is, X is equivariantly homeomorphic to $S^3 * \mathbb{R}P^2$.

Now let $K^-/H = \mathbb{S}^1$. Then $K_0^- = T^2$. Since

$$\begin{aligned}H \subseteq N(K^-) \cap N(K^+) &= \pm\Delta S^3 \cap (N(S^1) \times N(S^1))\\ &= \pm\Delta S^1 \cup (j, \pm j)\Delta S^1,\end{aligned}$$

where $\pm\Delta S^1 = \{(g,g)\} \cup \{(g,-g)\}$, we have two cases: $H = \Delta S^1 \cup (j,j)\Delta S^1$ or $H = \pm\Delta S^1 \cup (j, \pm j)\Delta S^1$. Thus we have the following diagrams:

$$(S^3 \times S^3, \Delta S^1 \cup (j,j)\Delta S^1, T^2 \cup (j,j)T^2, \Delta S^3) \tag{3.17}$$

and

$$(S^3 \times S^3, \pm\Delta S^1 \cup (j, \pm j)\Delta S^1, T^2 \cup (j,j)T^2, \pm\Delta S^3). \tag{3.18}$$

Now assume that $K^-/H = \mathbb{R}P^2$. Thus K^- is a 3-dimensional subspace containing $\Delta S^1 \cup (j,j)\Delta S^1$, which gives in particular that K_0^- must be ΔS^3, and $H^- = H \cap K_0^- = \Delta S^1 \cup (j,j)\Delta S^1$. Thus $H = \langle H^-, H^+ \rangle = \Delta S^1 \cup (j,j)\Delta S^1$ and the following diagram is obtained:

$$(S^3 \times S^3, \Delta S^1 \cup (j,j)\Delta S^1, \Delta S^3, \Delta S^3). \tag{3.19}$$

Note that this action is equivalent to the following action:

$$\begin{aligned}(S^3 \times S^3) \times (\operatorname{Susp}(\mathbb{R}P^2) \times \mathbb{S}^3) &\to (\operatorname{Susp}(\mathbb{R}P^2) \times \mathbb{S}^3)\\ ((g,h),([x,t],y)) &\mapsto ([gxg^{-1},t], hyg^{-1}).\end{aligned}$$

Thus X is equivariantly homeomorphic to $\operatorname{Susp}(\mathbb{R}P^2) \times \mathbb{S}^3$.

Now, let $K^-/H = S^3/\Gamma$ with $\Gamma \neq \{1\}$. Then $K_0^- = S^3 \times S^1$ or $K_0^- = S^1 \times S^3$. After exchanging the factors of G, if necessary, we can assume that $K_0^- = S^3 \times S^1$. If $\Gamma \neq \mathbb{Z}_k$, then $H \cap K_0^- = \Gamma \times S^1$. Since $\Delta S^1 = H_0 \subseteq H \cap K_0^-$, this cannot happen. Therefore $\Gamma = \mathbb{Z}_k$, and the action of $S^3 \times S^1$ on $S^3/\mathbb{Z}_k$ is given by:

$$\begin{aligned}(S^3 \times S^1) \times S^3/\mathbb{Z}_k &\to S^3/\mathbb{Z}_k\\ ((v,z),[x]) &\mapsto [vx\bar{z}^p].\end{aligned}$$

Thus $H \cap K_0^- = \{(\lambda z^p, z) \mid z \in S^1, \lambda \in \mathbb{Z}_k\}$. As $\Delta S^1 \subseteq H \cap K_0^-$, we have that $p = 1$. Further, $H \cap K_0^- \subseteq N_{S^3 \times S^3}(\Delta S^3) = \pm\Delta S^3$, which implies $\mathbb{Z}_k = \mathbb{Z}_2$. Therefore $H = \pm\Delta S^1 \cup (j, \pm j)\Delta S^1$, and the following diagram is obtained

$$(S^3 \times S^3, \pm\Delta S^1 \cup (j, \pm j)\Delta S^1, S^3 \times N_{S^3}(S^1), \pm\Delta S^3). \tag{3.20}$$

This action is equivalent to the action given by

$$\begin{aligned}(S^3 \times S^3) \times (\mathbb{R}P^2 * \mathbb{R}P^3) &\to \mathbb{R}P^2 * \mathbb{R}P^3 \\ ((g, h), [x, y, t]) &\mapsto [gxg^{-1}, hy, t].\end{aligned}$$

Thus X is equivariantly homeomorphic to $\mathbb{R}P^2 * \mathbb{R}P^3$.

Now assume that $K^+/H = S^3/\Gamma$ with $\Gamma \neq \{1\}$. Thus dim $K^+ = 4$. Since the connected 4-dimensional subgroups of $S^3 \times S^3$ are $S^3 \times S^1$ and $S^1 \times S^3$, we can assume, without loss of generality, that $K_0^+ = S^3 \times S^1$. The possibilities for K^-/H are $\mathbb{S}^l$, $l \geq 1$, $\mathbb{R}P^2$, and S^3/Λ, where Λ is a non-trivial finite subgroup of S^3. The case where $K^-/H = \mathbb{R}P^2$ has been treated above, so we only examine the cases where K^-/H is a sphere or a 3-dimensional spherical space form.

First assume that $K^-/H = \mathbb{S}^1$. Then $K_0^- = T^2$. Since $H_0 = \{(e^{pi\theta}, e^{qi\theta})\} \subseteq T^2$, and $H_0 = \{(e^{pi\theta}, e^{qi\theta})\} \subseteq S^3 \times S^1$, the circles in the second component of K_0^- and K_0^+ are the same, that is $S^1 = \{e^{i\theta}\}$. This implies that $K_0^- \subseteq K_0^+$. Therefore, $H = \langle H^-, H^+ \rangle = H^+ = H \cap K_0^+$. Let $\Gamma \neq \mathbb{Z}_k \subseteq \{e^{i\theta}\} \subseteq S^3$. Then by Lemma 2.18, $H \cap K_0^+ = \Gamma \times S^1$, since by Theorem 2.29, S^3 is the only compact connected Lie group which acts almost effectively on S^3/Γ. Further, $H \subseteq K^- \subseteq N_{S^3 \times S^3}(T^2) = N_{S^3}(S^1) \times N_{S^3}(S^1)$. Among the finite subgroups of S^3, only $\mathbb{Z}_k \subseteq \{e^{i\theta}\}$ and D^*_{2m} are contained in $N_{S^3}(S^1) = S^1 \cup jS^1$. Thus $\Gamma = \mathrm{D}^*_{2m}$, which implies that $K^- = T^2 \cup (j, 1)T^2$. Therefore, we have the following diagram:

$$(S^3 \times S^3, \mathrm{D}^*_{2m} \times S^1, N_{S^3}(S^1) \times S^1, S^3 \times S^1). \tag{3.21}$$

Now, suppose $\Gamma = \mathbb{Z}_k \subseteq \{e^{i\theta}\}$. The transitive action of $S^3 \times S^1$ on $S^3/\mathbb{Z}_k$ gives

$$H^+ = K_0^+ \cap H = \{(e^{ip\theta}\lambda, e^{i\theta}) \mid \theta \in \mathbb{R}, \lambda \in \mathbb{Z}_k\}.$$

Thus we have the following diagram:

$$(S^3 \times S^3, \{(e^{ip\theta}\lambda, e^{i\theta}) \mid \theta \in \mathbb{R}, \lambda \in \mathbb{Z}_k\}, T^2, S^3 \times S^1). \tag{3.22}$$

Assume now that $K^-/H = \mathbb{S}^l$, $l \geq 2$. Hence by Corollary 2.16, K^+ is connected. First assume that $\Gamma \neq \mathbb{Z}_k$. As a result $H = \Gamma \times S^1$. For $l = 2$, the only possibility for K_0^- is $1 \times S^3$. Then we obtain the following diagram:

$$(S^3 \times S^3, \Gamma \times S^1, \Gamma \times S^3, S^3 \times S^1). \tag{3.23}$$

By Proposition 2.25, this action is a join action and therefore X is equivariantly homeomorphic to $(S^3/\Gamma) * S^2$ with the join action of $S^3 \times S^3$.

For $l \geq 3$, there are no subgroups of G such that $K^-/H = \mathbb{S}^l$; thus, we need not consider these cases.

Now let $\Gamma = \mathbb{Z}_k$. Then the isotropy subgroup of the transitive action of $S^3 \times S^1$ on $S^3/\mathbb{Z}_k$ would be $\{(e^{ip\theta}\lambda, e^{i\theta}) \mid \theta \in \mathbb{R}, \lambda \in \mathbb{Z}_k\}$. Assume that $l = 2$. Then $K_0^- = 1 \times S^3$ or $K_0^- = \Delta S^3$. Therefore, $p = 0$, or $p = 1$, respectively. If $p = 1$, then $\mathbb{Z}_k = \mathbb{Z}_2$ since $H \subseteq N(\Delta S^3) = \pm\Delta S^3$. Thus we have the following diagrams corresponding to $p = 0$ and $p = 1$, respectively:

$$(S^3 \times S^3, \mathbb{Z}_k \times S^1, \mathbb{Z}_k \times S^3, S^3 \times S^1), \tag{3.24}$$

and

$$(S^3 \times S^3, \pm\Delta S^1, \pm\Delta S^3, S^3 \times S^1). \tag{3.25}$$

By Proposition 2.25, the first action is equivalent to the join action of $S^3 \times S^3$ on $(S^3/\mathbb{Z}_k) * \mathbb{S}^2$. The second one is the action of $S^3 \times S^3$ on $\mathbb{R}P^3 * S^2$ given by

$$(S^3 \times S^3) \times (\mathbb{R}P^3 * \mathbb{S}^2) \to \mathbb{R}P^3 * \mathbb{S}^2$$
$$(g,h) \cdot [x,y,t] = [gxh^{-1}, hyh^{-1}, t].$$

For $l \geq 3$, there are no subgroups of G such that $K^-/H = \mathbb{S}^l$.

Assume now that $K^-/H = S^3/\Lambda$ with Λ a non-trivial subgroup of S^3. Therefore, $K_0^- = S^3 \times S^1$ or $K_0^- = S^1 \times S^3$. First assume that $K_0^- = S^3 \times S^1$. Note that according to the classification of the transitive actions on 3-dimensional space forms, $q \neq 0$, which gives that the circles in the second component of $K_0^\pm$ are the same, so $K_0^- = K_0^+$, and $H = K_0^+ \cap H = K_0^- \cap H$. Thus $\Gamma = \Lambda$. Consequently, for $\Gamma \neq \mathbb{Z}_k$, we have the following diagram:

$$(S^3 \times S^3, \Gamma \times S^1, S^3 \times S^1, S^3 \times S^1). \tag{3.26}$$

By Proposition 2.23, this action is equivalent to the product action of $S^3 \times S^3$ on $\mathrm{Susp}\,(S^3/\Gamma) \times \mathbb{S}^2$. If $\Gamma = \mathbb{Z}_k$, the following diagram is obtained:

$$(S^3 \times S^3, \{(e^{ip\theta}\lambda, e^{i\theta}) \mid \theta \in \mathbb{R}, \lambda \in \mathbb{Z}_k\}, S^3 \times S^1, S^3 \times S^1). \tag{3.27}$$

By Proposition 2.23, for $p = 0$, this action is equivalent to the product action of $S^3 \times S^3$ on $\mathrm{Susp}\,(S^3/\mathbb{Z}_k) \times \mathbb{S}^2$. For $p \neq 0$, the action is non-primitive. In particular, in the preceding diagram, if $\mathbb{Z}_k = \mathbb{Z}_2$ and $p = 1$, then the action is as follows:

$$(S^3 \times S^3) \times (\mathrm{Susp}\,(\mathbb{R}P^3) \times \mathbb{S}^2) \to \mathrm{Susp}\,(\mathbb{R}P^3) \times \mathbb{S}^2$$
$$(g,h) \cdot ([x,t], y) = ([gxh^{-1}, t], hyh^{-1}).$$

Now let $K_0^- = S^1 \times S^3$. Hence $H_0 = \Delta S^1$, and both Γ and Λ are cyclic subgroups of S^3, say $\mathbb{Z}_k$ and Z_l, respectively. Then we have $H^+ = \{(e^{i\theta}\lambda, e^{i\theta}) \mid \theta \in \mathbb{R}, \lambda \in \mathbb{Z}_k\}$ and $H^- = \{(e^{i\theta}\lambda, e^{i\theta}) \mid \theta \in \mathbb{R}, \lambda \in \mathbb{Z}_l\}$. Hence H^+ and H^- are subgroups of both K_0^+ and K_0^-, which gives that $H = \langle H^+, H^- \rangle \subseteq K_0^\pm$. It follows then from Proposition 2.19 that $K^+ = K_0^+ H = K_0^+$ and $K^- = K_0^- H = K_0^-$. Thus $H^- = H = H^+$ and, in particular, $\Gamma = \Lambda$. The diagram is then given by

$$(S^3 \times S^3, \{(e^{i\theta}\lambda, e^{i\theta}) \mid \theta \in \mathbb{R}, \lambda \in \mathbb{Z}_k\}, S^1 \times S^3, S^3 \times S^1). \tag{3.28}$$

$\mathbf{G = S^3 \times S^3 \times S^1}$. In this case, $\dim H = 2$ and $H_0 \subseteq S^3 \times S^3 \times 1$, since the action is non-reducible. As the maximal torus of $S^3 \times S^3$ is the only 2-dimensional subgroup of $S^3 \times S^3$, we have $H_0 = T^2$. Further, by Proposition 2.21, $K^-/H = \mathbb{S}^1$, and $K_0^+ \subseteq S^3 \times S^3 \times 1$. As a result, $K_0^- = T^3$. Since $T^2 = H_0 \subseteq K_0^+ \subseteq S^3 \times S^3 \times 1$, we have $K_0^+ = S^3 \times S^1$, $K_0^+ = S^1 \times S^3$, or $K_0^+ = S^3 \times S^3$. However, $S^3 \times S^3$ does not act transitively on a 4-dimensional homogeneous space with positive curvature (see [34]). Therefore, $K_0^+ = S^3 \times S^1$ or $K_0^+ = S^1 \times S^3$. Without loss of generality, we can assume that $K_0^+ = S^3 \times S^1$. Thus $K^+/H = \mathbb{R}P^2$. By the classification of the transitive actions on spheres, S^3, up to a finite cover, is theonly Lie group which acts transitively and almost effectively on $\mathbb{R}P^2$. Therefore, $K_0^+ \cap H = N_{S^3}(S^1) \times S^1 \times 1$, and we obtain the following diagram:

$$(S^3 \times S^3 \times S^1, N_{S^3}(S^1) \times S^1 \times \mathbb{Z}_k, N_{S^3}(S^1) \times S^1 \times S^1, S^3 \times S^1 \times \mathbb{Z}_k). \tag{3.29}$$

By Proposition 2.23, this action is equivalent to the product action of $S^3 \times S^3 \times S^1$ on $(\mathbb{R}P^2 * \mathbb{S}^1) \times \mathbb{S}^2$.

$\mathbf{G = SU(3)}$. In this case, $\dim H = 3$. Thus the only possibilities for H_0 are SO(3) and SU(2). If $H_0 = \mathrm{SO}(3)$, then $K^\pm = \mathrm{SU}(3)$ since SO(3) is a maximal connected subgroup and there are no exceptional orbits. This cannot happen since there are no homogeneous spaces with positive curvature with an SU(3)-action and SO(3) as the isotropy group (see [34]). Hence $H_0 = \mathrm{SU}(2)$. The subgroups of G which contain H_0 properly are U(2) and SU(3). As $\mathrm{U}(2)/H = \mathbb{S}^1$, at least one of the singular isotropy groups, say K^+, is equal to SU(3). Therefore, $\dim\ K^+/H = 5$, which gives that $K^+/H = S^5/\mathbb{Z}_k$. The classification of the transitive actions on spheres then shows that $H = \mathrm{S}(\mathrm{U}(2)\mathbb{Z}_k)$. Depending on whether $K^- = \mathrm{U}(2)$ or SU(3), we have the following two diagrams:

$$(\mathrm{SU}(3), \mathrm{S}(\mathrm{U}(2)\mathbb{Z}_k), \mathrm{U}(2), \mathrm{SU}(3)), \tag{3.30}$$

$$(\mathrm{SU}(3), \mathrm{S}(\mathrm{U}(2)\mathbb{Z}_k), \mathrm{SU}(3), \mathrm{SU}(3)). \tag{3.31}$$

By Proposition 2.27, the space determined by diagram (3.31) is equivalent to Susp $(\mathbb{S}^5/\mathbb{Z}_k)$ with the suspension action of SU(3).

$\mathbf{G = SU(3) \times S^1}$. In this case, $\dim H = 4$. By Proposition 2.21, $H_0, K_0^+ \subseteq \mathrm{SU}(3) \times 1$ and $K^-/H = \mathbb{S}^1$. Therefore $H_0 = \mathrm{U}(2) \times 1$, $K_0^+ = \mathrm{SU}(3) \times 1$, and $K_0^- = \mathrm{U}(2) \times S^1$. Hence the following diagram is obtained:

$$(\mathrm{SU}(3) \times S^1, \mathrm{U}(2) \times \mathbb{Z}_k, \mathrm{U}(2) \times S^1, \mathrm{SU}(3) \times \mathbb{Z}_k). \tag{3.32}$$

By Proposition 2.25, the space determined by this diagram is equivalent to $\mathbb{C}P^2 * \mathbb{S}^1$ with the join action of $\mathrm{SU}(3) \times S^1$.

$\mathbf{G = Sp(2) \times S^1}$. In this case, $\dim H = 6$. As above, by Proposition 2.21, $H_0, K_0^+ \subseteq \mathrm{Sp}(2) \times 1$, and $K^-/H = \mathbb{S}^1$. Therefore $H_0 = \mathrm{Sp}(1)\mathrm{Sp}(1) \times 1$, and $K_0^+ = \mathrm{Sp}(2) \times 1$, for Sp(1)Sp(1) is a maximal connected subgroup of Sp(2). Thus $\dim\ K^+/H = 4$, and therefore $K^+/H = \mathbb{R}P^4$ (note that the other positively curved homogeneous space in dimension 4 is $\mathbb{C}P^2$, which does not admit an Sp(2)-transitive action (see [34, Table B])). Hence we get the following diagram

$$(\mathrm{Sp}(2) \times S^1, N_{\mathrm{Sp}(2)}(\mathrm{Sp}(1)\mathrm{Sp}(1)) \times \mathbb{Z}_k, N_{\mathrm{Sp}(2)}(\mathrm{Sp}(1)\mathrm{Sp}(1)) \times S^1, \mathrm{Sp}(2) \times \mathbb{Z}_k). \tag{3.33}$$

By Proposition 2.25, X is equivalent to $\mathbb{R}P^4 * \mathbb{S}^1$ with the join action of $\mathrm{Sp}(2) \times S^1$.

$\mathbf{G = Spin(6)}$. In this case, since $\dim\ G = 15 = 6(6-1)/2$, by Proposition 2.28, we obtain the diagram

$$(\mathrm{Spin}(6), N_{\mathrm{Spin}(6)}(\mathrm{Spin}(5)), \mathrm{Spin}(6), \mathrm{Spin}(6)). \tag{3.34}$$

and X is equivariantly homeomorphic to Susp $(\mathbb{R}P^5)$ with the spin action of Spin(6).

3.5 Classification in dimension 7

By Proposition 2.20, we have $6 \leq \dim G \leq 21$ and $\dim H = \dim G - 6$. As before, Propositions 2.14 and 2.21give us the possible acting groups:

$$S^3 \times S^3,\ S^3 \times S^3 \times S^1,\ \mathrm{SU}(3),\ S^3 \times S^3 \times S^3,\ \mathrm{SU}(3) \times S^1,\ \mathrm{Sp}(2),$$
$$\mathrm{SU}(3) \times S^3,\ \mathrm{Sp}(2) \times S^3,\ G_2,\ \mathrm{SU}(4),\ \mathrm{SU}(4) \times S^1,\ \mathrm{Spin}(7).$$

Now we examine each group case by case.

$\mathbf{G = S^3 \times S^3}$. In this case, $\dim H = 0$. Having looked at the classification of homogeneous spaces with positive curvature, and the subgroups of $S^3 \times S^3$, one can see that the only homogeneous spaces with positive curvature that can happen as the normal space of directions of singular orbits are 3-dimensional spherical space forms.

Assume that $K^+/H = S^3/\Gamma$, with Γ a nontrivial finite subgroup of S^3. Then K^+ is 3-dimensional and, as a result, K_0^+ can be $S^3 \times 1$, $1 \times S^3$ or $\Delta_{g_0} S^3 = \{(g, g_0 g g_0^{-1}) \mid g \in S^3\}$, for some fixed $g_0 \in S^3$.

Suppose first that $K_0^+ = S^3 \times 1$. Then $H \cap K_0^+ = \Gamma \times 1$. Furthermore, K^-/H is one of the spaces $\mathbb{S}^1$, $\mathbb{S}^3$, or S^3/Λ with Λ a nontrivial finite subgroup of S^3.

First assume that $K^-/H = \mathbb{S}^1$. Thus $K_0^- = \{(e^{xp\theta}, e^{yq\theta}) \mid \theta \in \mathbb{R}, x, y \in \mathrm{Im}(\mathbb{H}) \cap S^3\}$. If $p = 0$, then we have the diagram

$$(S^3 \times S^3, \Gamma \times \mathbb{Z}_k, \Gamma \times S^1, S^3 \times \mathbb{Z}_k). \tag{3.35}$$

This action is a non-primitive action with $L = S^3 \times S^1$. Therefore, by Proposition 2.8, X is equivariantly homeomorphic to the total space of an $((S^3/\Gamma) * \mathbb{S}^1)$-bundle over $\mathbb{S}^2$.

If $q = 0$, then

$$\Gamma \times 1 \subseteq H \subseteq N_{S^3 \times S^3}(S^1 \times 1) = N_{S^3}(S^1) \times S^3,$$

which implies that $\Gamma = \mathbb{Z}_k$ or $\Gamma = \mathrm{D}^*_{2m}$. For $\Gamma = \mathbb{Z}_k$, $H^+ \subseteq K_0^-$. Therefore, $H = \langle H^+, H^- \rangle \subseteq K_0^-$, which gives, by Proposition 2.19, that $K^- = K_0^-$. Thus we get the following diagram:

$$(S^3 \times S^3, \mathbb{Z}_k \times 1, S^1 \times 1, S^3 \times 1). \tag{3.36}$$

By Proposition 2.23, this action is equivalent to the product action of $S^3 \times S^3$ on $X^4 \times \mathbb{S}^3$, where X^4 is the 4-dimensional Alexandrov space with the following diagram (see [15]):

$$(S^3, \mathbb{Z}_k, S^1, S^3).$$

Indeed, X^4 is equivariantly homeomorphic to $\mathbb{C}P^2/\mathbb{Z}_k$ with an S^3-action induced by a cohomogeneity one action of S^3 on $\mathbb{C}P^2$.

For $\Gamma = \mathrm{D}^*_{2m}$, we have $K^- = N_{S^3}(S^1) \times 1$, and we obtain the following diagram:

$$(S^3 \times S^3, \mathrm{D}^*_{2m} \times 1, N_{S^3}(S^1) \times 1, S^3 \times 1). \tag{3.37}$$

Similarly, this action is equivalent to the product action of $S^3 \times S^3$ on $X^4 \times \mathbb{S}^3$, where X^4 is given by

$$(S^3, \mathrm{D}^*_{2m}, N_{S^3}(S^1), S^3).$$

Again, X^4 is equivariantly homeomorphic to $\mathbb{C}P^2/\mathbb{Z}_m$ with an S^3-action induced by a cohomogeneity one action of S^3 on $\mathbb{C}P^2$.

If $pq \neq 0$, then, since

$$\Gamma \times 1 \subseteq H \subseteq N_{S^3\times S^3}(\{(e^{xp\theta}, e^{yq\theta})\}) = \{(e^{x\theta}, e^{y\phi})\} \cup (z, w)\{(e^{x\theta}, e^{y\phi})\},$$

where $z \in x^{\perp} \cap \mathrm{Im}\,(\mathbb{H}) \cap S^3$ and $w \in y^{\perp} \cap \mathrm{Im}\,(\mathbb{H}) \cap S^3$, we have $\Gamma = \mathbb{Z}_k$. Also, without loss of generality, we may assume that $K_0^- = \{(e^{ip\theta}, e^{iq\theta})\}$. Therefore, we get the following diagram:

$$\left(S^3 \times S^3, \left\{ \left(e^{\frac{lk+mps}{km}2\pi i}, e^{\frac{2\pi psi}{k}}\right) \mid 1 \leq s \leq k, 1 \leq l \leq m\right\}, (\mathbb{Z}_m \times 1)K_0^-, S^3 \times \mathbb{Z}_{k/(k,q)}\right), \tag{3.38}$$

where $(k, q) = (q, m)$.

Now, assume that $K^-/H = \mathbb{S}^3$. As a result, K^+ is connected and $H = \Gamma \times 1$. On the other hand, K_0^- is a 3-dimensional subgroup containing $\Gamma \times 1$. Therefore, there are two possibilities: $K_0^- = 1 \times S^3$, and $K_0^- = \Delta_{g_0}S^3$. If $K_0^- = 1 \times S^3$, then we obtain the following diagram:

$$(S^3 \times S^3, \Gamma \times 1, \Gamma \times S^3, S^3 \times 1). \tag{3.39}$$

By Proposition 2.25, X is equivariantly homeomorphic to $S^3 * (S^3/\Gamma)$ with the join action of $S^3 \times S^3$.

Now let $K_0^- = \Delta_{g_0}S^3$. Since $1 \times S^3 \subseteq N(H)_0$, by Proposition 2.6 we can conjugate K^- by $(1, g_0^{-1})$ without changing the spaces. Moreover, $K^- \subseteq N(\Delta_{g_0}S^3) = \pm\Delta_{g_0}S^3$, so we can assume that $g_0 = 1$. Now, since K^-/H is simply connected, the number of connected components of K^- and H are the same. Since $H \neq 1$, and K^- has at most two components, we conclude that $\Gamma = \mathbb{Z}_2$. Thus, we get the following diagram:

$$(S^3 \times S^3, \mathbb{Z}_2 \times 1, \pm\Delta S^3, S^3 \times 1). \tag{3.40}$$

This action is equivalent to the following action on $\mathbb{R}P^3 * \mathbb{S}^3$:

$$(S^3 \times S^3) \times (\mathbb{R}P^3 * \mathbb{S}^3) \to \mathbb{R}P^3 * \mathbb{S}^3$$
$$(g, h) \cdot [x, y, t] = [gxh^{-1}, hy, t].$$

If $K^-/H = S^3/\Lambda$, then K_0^- is equal to one of the subgroups $S^3 \times 1$, $1 \times S^3$ or $\Delta_{g_0}S^3$. First assume that $K_0^- = S^3 \times 1$. Then

$$\Gamma \times 1 = K_0^+ \cap H = K_0^- \cap H = \Lambda \times 1.$$

Therefore, $\Gamma = \Lambda$ and by Lemma 2.17, $H = \langle H^+, H^-\rangle = \Gamma \times 1$ and we obtain the diagram

$$(S^3 \times S^3, \Gamma \times 1, S^3 \times 1, S^3 \times 1). \tag{3.41}$$

This action is equivalent to the product action of $S^3 \times S^3$ on $\mathrm{Susp}\,(S^3/\Gamma) \times S^3$.

Now let $K_0^- = 1 \times S^3$. In this case, $\Gamma \times 1 = K_0^+ \cap H$, and $K_0^- \cap H = \Lambda \times 1$, so by Lemma 2.17, $H = \Gamma \times \Lambda$. Hence, we get the following diagram:

$$(S^3 \times S^3, \Gamma \times \Lambda, \Gamma \times S^3, S^3 \times \Lambda). \tag{3.42}$$

Proposition 2.25 implies that X is equivariantly homeomorphic to $(S^3/\Gamma) * (S^3/\Lambda)$ with the join action of $S^3 \times S^3$.

Finally, suppose that $K_0^- = \Delta_{g_0} S^3$. Since $\Gamma \times 1 \subseteq K^- \subseteq N(\Delta_{g_0} S^3) = \pm\Delta_{g_0} S^3$, and $\Gamma \neq 1$, then K^- has to be $\pm\Delta_{g_0} S^3$, and $\Gamma = \mathbb{Z}_2$. Also, the classification of transitive actions on spheres gives us that $K_0^- \cap H = \Delta_{g_0}\Lambda$. Therefore $H = \pm\Delta_{g_0}\Lambda$, and the following diagram is obtained:

$$(S^3 \times S^3, \pm\Delta_{g_0}\Lambda, \pm\Delta_{g_0} S^3, S^3 \times g_0\Lambda g_0^{-1}). \tag{3.43}$$

According to Proposition 2.6 and Equation (3.43), we can assume that $g_0 = 1$. This action is equivalent to the following action, and X is then equivariantly homeomorphic to $\mathbb{R}P^3 * (S^3/\Lambda)$:

$$(S^3 \times S^3) \times (\mathbb{R}P^3 * S^3/\Lambda) \to \mathbb{R}P^3 * (S^3/\Lambda)$$
$$(g,h)\cdot[x,y,t] = [gxh^{-1}, hy, t].$$

Now assume that $K_0^+ = \Delta_{g_0} S^3$. Thus $H \cap K_0^+ = \Delta_{g_0}\Gamma$. As before, K^-/H can be a circle, a 3-sphere or a 3-dimensional spherical space form.

First suppose that $K^-/H = \mathbb{S}^1$. Therefore, $K_0^- = \{(e^{xp\theta}, e^{yq\theta})\}$ and, after conjugation, we can assume that it is equal to $K_0^- = \{(e^{ip\theta}, e^{iq\theta})\}$. Since $K^+ \subseteq N(K_0^+) = \pm\Delta_{g_0}(S^3)$, there are two possibilities for K^+: either $K^+ = \Delta_{g_0} S^3$ or $K^+ = \pm\Delta_{g_0} S^3$.

Assume that $K^+ = \Delta_{g_0} S^3$. Therefore $H = \Delta_{g_0}\Gamma$. Let $q = 0$. Then $K_0^- = S^1 \times 1$ and

$$K^- = K_0^- H = \{(za, g_0 a g_0^{-1}) \mid a \in \Gamma,\ z \in S^1\}.$$

Since $\mathrm{Proj}_1(K^-) \subseteq S^1 \cup jS^1$, we have $\Gamma = \mathbb{Z}_k$, or $\Gamma = \mathrm{D}^*_{2m}$. Thus K^- is equal to $S^1 \times g_0\mathbb{Z}_k g_0^{-1}$ or $(S^1 \times 1)\Delta_{g_0}\mathrm{D}^*_{2m}$, respectively. By conjugating the subgroups by $(1, g_0^{-1})$, we have the following diagrams:

$$(S^3 \times S^3, \Delta\mathbb{Z}_k, S^1 \times \mathbb{Z}_k, \Delta S^3), \tag{3.44}$$

$$(S^3 \times S^3, \Delta\mathrm{D}^*_{2m}, (S^1 \times 1)\Delta\mathrm{D}^*_{2m}, \Delta S^3). \tag{3.45}$$

If $p = 0$, we have, similarly,

$$(S^3 \times S^3, \Delta\mathbb{Z}_k, \mathbb{Z}_k \times S^1, \Delta S^3), \tag{3.46}$$

$$(S^3 \times S^3, \Delta\mathrm{D}^*_{2m}, (1 \times S^1)\Delta\mathrm{D}^*_{2m}, \Delta S^3). \tag{3.47}$$

Observe that these two diagrams are the same as Diagrams (3.44) and (3.45) up to exchanging the factors of G.

Now assume that $pq \neq 0$. Then $N(K_0^-) = \{(e^{i\theta}, e^{i\phi})\} \cup \{(je^{i\theta}, je^{i\phi})\}$. Thus $K^- = \{(e^{ip\theta}, e^{iq\theta})\}$ or $K^- = \{(e^{ip\theta}, e^{iq\theta})\} \cup \{(je^{ip\theta}, je^{iq\theta})\}$. If $K^- = \{(e^{ip\theta}, e^{iq\theta})\}$, then $\Gamma = \mathbb{Z}_k$. We have

$$\{(a, g_0 a g_0^{-1}) \mid a \in \mathbb{Z}_k\} = \Delta_{g_0}\mathbb{Z}_k = \left\{\left(e^{\frac{2\pi i}{k}p}, e^{\frac{2\pi i}{k}q}\right)\right\}.$$

Therefore,

$$g_0 e^{\frac{2\pi i}{k}p} g_0^{-1} = e^{\frac{2\pi i}{k}q}.$$

Since $e^{\frac{2\pi i}{k}p}$ and $e^{\frac{2\pi i}{k}q}$ are elements in the maximal torus $S^1 = \{e^{\theta i} \mid \theta \in \mathbb{R}\} \subseteq S^3$, by [23, Proposition 4.53], they are conjugate in the Weyl group $W(S^3, S^1) = \{S^1, jS^1\}$. Thus one of the following cases occurs:

- $g_0 \in \{e^{i\theta}\}$ and $e^{\frac{2\pi i}{k}p} = e^{\frac{2\pi i}{k}q}$. Consequently $k|(p-q)$, and if k is even, then p, q are odd. By conjugating the isotropy groups by $(1, g_0^{-1})$, we have $K^+ = \Delta S^3$, $K^- = \{(e^{ip\theta}, e^{iq\theta})\}$ and $H = \Delta\mathbb{Z}_k$.
- $g_0 \in \{je^{i\theta}\}$ and $e^{\frac{2\pi i}{k}p} = e^{-\frac{2\pi i}{k}q}$ which gives $k|(p+q)$, and if k is even, then p, q are odd. Again, by conjugating the isotropy groups by $(1, g_0^{-1})$, we have $K^+ = \Delta S^3$, $K^- = \{(e^{ip\theta}, e^{-iq\theta})\}$ and $H = \Delta\mathbb{Z}_k$.

Summing up, we have the following diagrams:

$$(S^3 \times S^3, \Delta\mathbb{Z}_k, \{(e^{ip\theta}, e^{iq\theta})\}, \Delta S^3), \tag{3.48}$$

where $k|(p-q)$, and if k is even, then p, q are odd, and we get

$$(S^3 \times S^3, \Delta\mathbb{Z}_k, \{(e^{ip\theta}, e^{-iq\theta})\}, \Delta S^3), \tag{3.49}$$

where $k|(p+q)$, and if k is even, then p, q are odd.

If $K^- = \{(e^{ip\theta}, e^{iq\theta})\} \cup \{(je^{ip\theta}, je^{iq\theta})\}$, similar arguments as above give rise to the following diagrams with the same conditions, respectively:

$$(S^3 \times S^3, \Delta\mathrm{D}^*_{2m}, \{(e^{ip\theta}, e^{iq\theta})\} \cup \{(je^{ip\theta}, je^{iq\theta})\}, \Delta S^3), \tag{3.50}$$

$$(S^3 \times S^3, \Delta\mathrm{D}^*_{2m}, \{(e^{ip\theta}, e^{-iq\theta})\} \cup \{(je^{ip\theta}, je^{-iq\theta})\}, \Delta S^3). \tag{3.51}$$

If $K^+ = \pm\Delta_{g_0} S^3$, then $H = \pm\Delta_{g_0}\mathbb{Z}_k$ or $H = \pm\Delta_{g_0}\mathrm{D}^*_{2m}$. By the same argument, we obtain the following diagrams:

$$(S^3 \times S^3, \pm\Delta\mathbb{Z}_k, S^1 \times \mathbb{Z}_k, \pm\Delta S^3), \tag{3.52}$$

$$(S^3 \times S^3, \pm\Delta\mathrm{D}^*_{2m}, (S^1 \times 1)\Delta\mathrm{D}^*_{2m}, \pm\Delta S^3), \tag{3.53}$$

$$(S^3 \times S^3, \pm\Delta\mathbb{Z}_k, \mathbb{Z}_k \times S^1, \pm\Delta S^3), \tag{3.54}$$

$$(S^3 \times S^3, \pm\Delta\mathrm{D}^*_{2m}, (1 \times S^1)\Delta\mathrm{D}^*_{2m}, \pm\Delta S^3). \tag{3.55}$$

Observe that the last two diagrams are the same as Diagrams (3.52) and (3.53) up to exchanging the factors of G.

In the following diagrams, p is odd and q is even, so k has to be odd:

$$(S^3 \times S^3, \pm\Delta\mathbb{Z}_k, \{(e^{ip\theta}, e^{iq\theta})\}, \pm\Delta S^3), \tag{3.56}$$

$$(S^3 \times S^3, \pm\Delta\mathbb{Z}_k, \{(e^{ip\theta}, e^{-iq\theta})\}, \pm\Delta S^3), \tag{3.57}$$

$$(S^3 \times S^3, \pm\Delta\mathrm{D}^*_{2m}, \{(e^{ip\theta}, e^{iq\theta})\} \cup \{(je^{ip\theta}, je^{iq\theta})\}, \pm\Delta S^3), \tag{3.58}$$

$$(S^3 \times S^3, \pm\Delta \mathrm{D}^*_{2m}, \{(e^{ip\theta}, e^{-iq\theta})\} \cup \{(je^{ip\theta}, je^{-iq\theta})\}, \pm\Delta S^3). \tag{3.59}$$

Now assume that $K^-/H = \mathbb{S}^3$. Therefore, $K^+ = K_0^+ = \Delta_{g_0} S^3$ and $H = \Delta_{g_0}\Gamma$. Further, K_0^- is equal to $S^3 \times 1, 1 \times S^3$ or $\Delta_{g_1} S^3$. For $K_0^- = S^3 \times 1, 1 \times S^3$, we have

$$(S^3 \times S^3, \Delta\Gamma, S^3 \times \Gamma, \Delta S^3), \tag{3.60}$$

$$(S^3 \times S^3, \Delta\Gamma, \Gamma \times S^3, \Delta S^3). \tag{3.61}$$

If $K_0^- = \Delta_{g_1} S^3$, since K^-/H is simply connected, $\pi_0(K^-) = \pi_0(H)$. The number of connected components of K^- is at most 2, for $K^- \subseteq N(K_0^-) = \pm\Delta_{g_1} S^3$. Thus $H = \langle(-1, 1)\rangle$. But then H is not a subgroup of K^+. Therefore, this case cannot occur.

Now, let $K^-/H = \mathbb{S}^3/\Lambda$ with Λ a non-trivial subgroup of S^3. Again, we have three possibilities for K_0^-: $S^3 \times 1$, $1 \times S^3$, $\Delta_{g_1} S^3$. For $K_0^- = S^3 \times 1$, $1 \times S^3$, we obtain a diagram equivalent to diagram (3.43) by Proposition 2.6. Hence, suppose that $K_0^- = \Delta_{g_1} S^3$. If $K^+ = \Delta_{g_0} S^3$, then $H = \Delta_{g_0}\Gamma$, so $K_0^- \cap H = \Delta_{g_1}\Lambda \subseteq \Delta_{g_0}\Gamma$ which implies $g_0^{-1} g_1 \in C_{S^3}(\Lambda)$, where

$$C_{S^3}(\Lambda) = \begin{cases} \{\pm 1\} & \text{if } \Lambda \neq \mathbb{Z}_k, \\ \{e^{i\theta} \mid \theta \in \mathbb{R}\} & \text{if } \Lambda = \mathbb{Z}_k. \end{cases} \tag{3.62}$$

Moreover, $\{(-a, g_1 a g_1^{-1}) \mid a \in \Lambda\} \nsubseteq \Delta_{g_0}\Gamma$. Therefore, $K^- = \Delta_{g_0 z} S^3$, $\Delta_{g_1}\Lambda = \Delta_{g_0}\Gamma$, where $z \in C(\Gamma)$. If $\Gamma \neq \mathbb{Z}_k$, then by (3.62), $z = \pm 1$ and in particular $K^- = \Delta_{g_0} S^3$. Hence, after conjugating all subgroups by $(1, g_0^{-1})$, we obtain an equivalent diagram given by

$$(S^3 \times S^3, \Delta\Gamma, \Delta S^3, \Delta S^3). \tag{3.63}$$

If $\Gamma = \mathbb{Z}_k$, then we first conjugate all subgroups by $(1, g_0^{-1})$. Then, since $(1, z) \in N(H)_0$ by (3.62), we can conjugate K^- by $(1, z)$ to obtain diagram (3.63).

This action is equivalent to the following action on $\mathrm{Susp}\,(S^3/\Gamma) \times \mathbb{S}^3$:

$$(S^3 \times S^3) \times (\,\mathrm{Susp}\,(S^3/\Gamma) \times \mathbb{S}^3) \to \mathrm{Susp}\,(S^3/\Gamma) \times \mathbb{S}^3$$
$$(g, h) \cdot ([x, t], y) = ([gx, t], gyh^{-1}).$$

Now assume that $K^+ = \pm\Delta_{g_0} S^3$. Then, $K^- = \pm\Delta_{g_1} S^3$ and $\pm\Delta_{g_0}\Gamma = H = \pm\Delta_{g_1}\Lambda$. Thus, we have $\Gamma = \Lambda$ and $g_1 = g_0 z$, for some $z \in N(\Gamma)$. We claim that if $z \in C_{S^3}(\Gamma)$, then this case cannot happen, since we have assumed that X is simply connected. Indeed, if $z \in C_{S^3}(\Gamma)$ then $\Delta_{g_0}\Gamma = \Delta_{g_0 z}\Gamma = \Delta_{g_1}\Gamma$, and therefore, $H^- = H^+$. Since X is simply connected, by Proposition 2.15, $H = \langle H^-, H^+\rangle = \Delta_{g_0}\Gamma$, which is a contradiction. Assume now that $z \notin C_{S^3}(\Gamma)$. A direct computation shows that either $\Gamma = \mathrm{D}^*_{2m}$, with $z = i$ or $z = j$, where in the latter case $2|m$, or $\Gamma = \mathbb{Z}_k$, with $z = j$ and $4|k$. As a result, we have the following diagram:

$$(S^3 \times S^3, \pm\Delta\Gamma, \pm\Delta_z S^3, \pm\Delta S^3), \tag{3.64}$$

where Γ and z are as above, respectively.

$\mathbf{G = S^3 \times S^3 \times S^1}$. In this case, $\dim H = 1$. By Proposition 2.21, $K^-/H = \mathbb{S}^1$ and both K_0^+ and H_0 are subgroups of $S^3 \times S^3 \times 1$. Thus we can assume that $H_0 = \{(e^{ip\theta}, e^{iq\theta})\} \times 1$ and $\mathrm{Proj}_3(K_0^-) = S^1$. Since $K^+ \subseteq S^3 \times S^3 \times 1$, an examination of the subgroups of $S^3 \times S^3$ shows that the only possibilities for K^+/H are $\mathbb{R}P^2$ and S^3/Γ with Γ finite and non-trivial.

First assume that $K^+/H = \mathbb{R}P^2$. Therefore, $\dim K^+ = 3$. The possibilities for K_0^+ are $S^3 \times 1 \times 1, 1 \times S^3 \times 1$ and $\Delta_{g_0} S^3 \times 1$.

Let $K_0^+ = S^3 \times 1 \times 1$. Then $H_0 = S^1 \times 1 \times 1$, $H \cap K_0^+ = N(S^1) \times 1 \times 1$, and $K_0^- = S^1 \times T^1$, where $T^1 \subseteq S^3 \times S^1$ is given by $T^1 = \{(e^{ip\theta}, e^{i\theta}) \mid \theta \in \mathbb{R}\}$, for some non-negative integer p. Since $N(S^1)$ is a maximal subgroup of S^3, $H \cap K_0^- = S^1 \times \mathbb{Z}_k$, where $\mathbb{Z}_k \subseteq T^1$. Therefore, we obtain the following diagram:

$$(S^3 \times S^3 \times S^1, N(S^1) \times \mathbb{Z}_k, N(S^1) \times T^1, S^3 \times \mathbb{Z}_k). \tag{3.65}$$

We can define an action of $S^3 \times S^3 \times S^1$ on $(\mathbb{R}P^2 * \mathbb{S}^1) \times \mathbb{S}^3$ with group diagram (3.65) as follows. Let $\mathbb{H}$ denote the quaternions, and consider the action

$$(S^3 \times S^3 \times S^1) \times \mathbb{S}^4 \times \mathbb{S}^3 (\subseteq (\mathrm{Im}\mathbb{H} \times \mathbb{C}) \times \mathbb{H}) \to \mathbb{S}^4 \times \mathbb{S}^3$$
$$(((g_1, g_2), z), (x, w, y)) \mapsto (g_1 x g_1^{-1}, z^k w, g_2 y \bar{z}^p).$$

By considering the $\mathbb{S}^4$ factor as the join $\mathbb{S}^2 * \mathbb{S}^1$, the antipodal map on the $\mathbb{S}^2$ factor commutes with the action of $S^3 \times S^3 \times S^1$ on $(\mathbb{S}^2 * \mathbb{S}^1) \times \mathbb{S}^3$ above and induces an action of $S^3 \times S^3 \times S^1$ on $(\mathbb{R}P^2 * \mathbb{S}^1) \times \mathbb{S}^3$.

Let $K_0^+ = 1 \times S^3 \times 1$. This case only differs from the previous one by an isomorphism of G which exchanges the factors. Therefore the analysis is analogous to the one in the preceding case.

Now let $K_0^+ = \Delta S^3 \times 1$. As a result, $H_0 = \Delta S^1 \times 1$, $K_0^+ \cap H = N_{\Delta S^3}(\Delta S^1) \times 1$ and

$$\begin{aligned} K_0^- &= (\Delta S^1 \times 1)\{(e^{ia\theta}, e^{ib\theta}, e^{ic\theta})\} \\ &= \{(e^{i(\phi + a\theta)}, e^{i(\phi + b\theta)}, e^{ic\theta})\} \\ &= \{(e^{i\phi}, e^{i\phi} e^{ip\theta}, e^{ic\theta})\}. \end{aligned}$$

Since the action is non-reducible, $\mathrm{Proj}_3(H \cap K_0^-)$ is a proper subgroup of S^1, namely $\mathrm{Proj}_3(H \cap K_0^-) = \mathbb{Z}_k$. Therefore, $H \cap K_0^- = \{(e^{i\phi}, e^{i\phi} e^{\frac{2\pi pli}{k}}, e^{\frac{2\pi cli}{k}}) \mid 1 \le l \le k\}$. The long exact sequence of homotopy groups corresponding to the fiber bundle

$$K^- \to G \to G/K^-$$

shows that $\pi_0(K^-) = \pi_1(G/K^-)/\mathbb{Z}_c$ (note that $\pi_0(K^-)$ is not trivial since $K_0^+ \cap H = N_{\Delta S^3}(\Delta S^1) \times 1 \subseteq K^-$). By Proposition 2.15, $\pi_1(G/K^-) = \mathbb{Z}_2$ as X is simply connected. Thus $c = 1$. On the other hand, $\mathbb{Z}_k \subseteq N(K_0^+)$, which gives $k|2p$, and since we can assume $H \cap (1 \times 1 \times S^1) = 1$ to have a more effective action, $k = 1, 2$. Therefore, we obtain the following diagram:

$$(S^3 \times S^3 \times S^1, N_{\Delta S^3}(\Delta S^1)\mathbb{Z}_k, T^2 \cup (j, j, 1)T^2, \Delta S^3 \mathbb{Z}_k), \tag{3.66}$$

where $T^2 = \{(e^{i\phi}, e^{i\phi} e^{ip\theta}, e^{i\theta}) \mid \phi, \theta \in \mathbb{R}\}$, and $k = 1, 2$.

Now let $K^+/H = S^3/\Gamma$ with Γ finite and non-trivial. Therefore, $K_0^+ = S^3 \times S^1 \times 1$ or $K_0^+ = S^1 \times S^3 \times 1$.

Suppose that $K_0^+ = S^3 \times S^1 \times 1$ and $\Gamma = \mathbb{Z}_k$. Then $H \cap K_0^+ = \{(e^{\frac{2\pi li}{k}} e^{p\theta i}, e^{\theta i}, 1) \mid 1 \le l \le k\}$. Since X is simply connected, by Proposition 2.15, and the exact sequences of homotopy groups related to the fiber bundles

$$K^{\pm}/H \to G/H \to G/K^{\pm},$$
$$K^- \to G \to G/K^-,$$

one can see that $\pi_0(K^-) = \mathbb{Z}_k/\mathbb{Z}_c$. Therefore, $c|k$ and the following diagram is obtained:

$$\left(S^3 \times S^3 \times S^1, \left\{ \left(e^{\frac{2\pi ari}{m}} e^{\frac{2\pi li}{k}} e^{p\theta i}, e^{\theta i}, e^{\frac{2\pi cri}{m}} \right) \mid 1 \le r \le m, 1 \le l \le k \right\}, \right.$$
$$\left. \left\{ \left(e^{\frac{2\pi li}{k}} e^{ip\theta} e^{ia\phi}, e^{i\phi}, e^{ic\theta} \right), 1 \le l \le k \right\}, S^3 \times S^1 \times \mathbb{Z}_{\frac{k}{c}} \right). \tag{3.67}$$

Now let $\Gamma \neq \mathbb{Z}_k$. By Lemma 2.18, $K_0^+ \cap H = \Gamma \times S^1 \times 1$. Therefore, $H_0 = 1 \times S^1 \times 1$ and, by Lemma 2.21,

$$\begin{aligned} K_0^- &= H_0\{(e^{ia\theta}, e^{ib\theta}, e^{ic\theta})\} \\ &= \{(e^{ia'\theta}, e^{i\phi}, e^{ic'\theta})\}, \end{aligned}$$

for some integers a', c'. If $a' = 0$, then we have the following diagram:

$$(S^3 \times S^3 \times S^1, \Gamma \times S^1 \times \mathbb{Z}_k, \Gamma \times T^2, S^3 \times S^1 \times \mathbb{Z}_k). \tag{3.68}$$

By Proposition 2.23, this action is a product action and X is equivariantly homeomorphic to $(S^3/\Gamma * \mathbb{S}^1) \times \mathbb{S}^2$. If $a' \neq 0$, then $N(K^-) = \{(e^{ia'\theta}, e^{i\phi}, e^{ic'\theta})\} \cup \{(e^{ia'\theta}, je^{i\phi}, e^{ic'\theta})\}$ which implies that $\Gamma = \mathbb{Z}_k$. Thus, this case cannot happen.

$\mathbf{G = S^3 \times S^3 \times S^3}$. In this case, $\dim H = 3$. Since the action is non-reducible, $\mathrm{Proj}_i(H) \subsetneq S^3$, $i = 1, 2, 3$. Therefore, H_0 must be a maximal torus of G. Further, by considering the subgroups of G containing H, we only have $\mathbb{R}P^2$ and $\mathbb{S}^2$ as the normal spaces of directions of singular orbits.

Assume, without loss of generality, that $K^+/H = \mathbb{R}P^2$. Then there are two possibilities for K^-/H, namely, $K^-/H = \mathbb{S}^2$ or $K^-/H = \mathbb{R}P^2$.

Let $K^-/H = \mathbb{S}^2$. Therefore, K^+ is connected and, after exchanging the factors of G if necessary, we can assume that $K^+ = K_0^+ = S^3 \times T^2$. Thus $H = H \cap K_0^+ = N_{S^3}(S^1) \times T^2$. Also, since K^-/H is simply connected, $\pi_0(K^-) = \pi_0(H) = \mathbb{Z}_2$, and their components intersect each other. Hence $K^- = N_{S^3}(S^1) \times S^3 \times S^1$, and we get the following diagram:

$$(S^3 \times S^3 \times S^3, N_{S^3}(S^1) \times T^2, N_{S^3}(S^1) \times S^3 \times S^1, S^3 \times T^2). \tag{3.69}$$

By Proposition 2.23, this action is a product action and X is equivariantly homeomorphic to $(\mathbb{R}P^2 * \mathbb{S}^2) \times \mathbb{S}^2$.

Now let $K^-/H = \mathbb{R}P^2$. Thus K_0^- is equal to $S^3 \times T^2$ or $S^1 \times S^3 \times S^1$. If $K_0^- = S^3 \times T^2$, then $H = H \cap K_0^- = H \cap K_0^+ = N_{S^3}(S^1) \times T^2$, and the following diagram is obtained:

$$(S^3 \times S^3 \times S^3, N_{S^3}(S^1) \times T^2, S^3 \times T^2, S^3 \times T^2). \tag{3.70}$$

By Proposition 2.23, this action is a product action and X is equivariantly homeomorphic to $\mathrm{Susp}(\mathbb{R}P^2) \times (\mathbb{S}^2 \times \mathbb{S}^2)$. If $K_0^- = S^1 \times S^3 \times S^1$, then $H \cap K_0^- = S^1 \times N_{S^3}(S^1) \times S^1$, and so $H = N_{S^3}(S^1) \times N_{S^3}(S^1) \times S^1$. Hence we have the diagram

$$(S^3 \times S^3 \times S^3, N_{S^3}(S^1) \times N_{S^3}(S^1) \times S^1, N_{S^3}(S^1) \times S^3 \times S^1, S^3 \times N_{S^3}(S^1) \times S^1). \tag{3.71}$$

By Proposition 2.23, this action is a product action and X is equivariantly homeomorphic to $(\mathbb{R}P^2 * \mathbb{R}P^2) \times \mathbb{S}^2$.

$\mathbf{G = SU(3)}$. In this case, $\dim H = 2$, and so $H_0 = T^2$. The subgroups containing T^2 are $\mathrm{U}(2)$ and $\mathrm{SU}(3)$.

Assume first that $K^\pm = \mathrm{SU}(3)$. Then the two following diagrams occur:

$$(\mathrm{SU}(3), T^2, \mathrm{SU}(3), \mathrm{SU}(3)), \tag{3.72}$$

$$(\mathrm{SU}(3), T^2\mathbb{Z}_2, \mathrm{SU}(3), \mathrm{SU}(3)), \tag{3.73}$$

By Proposition 2.27, the space X is equivariantly homeomorphic to $\mathrm{Susp}\,(\mathrm{W}^6)$ or $\mathrm{Susp}\,(\mathrm{W}^6/\mathbb{Z}_2)$, respectively, where $\mathrm{W}^6 = \mathrm{SU}(3)/T^2$ is the Wallach flag manifold, and X is equipped with the respective suspension action of SU(3).

Suppose now that $K^+ = \mathrm{SU}(3)$ and $K_0^- = \mathrm{U}(2)$. Since U(2) is a maximal subgroup of SU(3), $K^- = K_0^- = \mathrm{U}(2)$. We also have $N_{\mathrm{U}(2)}(T^2)/T^2 = \mathbb{Z}_2$, which gives $H = T^2$ or $T^2\mathbb{Z}_2$. As a result, the two following diagrams are obtained:

$$(\mathrm{SU}(3), T^2, \mathrm{U}(2), \mathrm{SU}(3)), \tag{3.74}$$

$$(\mathrm{SU}(3), T^2\mathbb{Z}_2, \mathrm{U}(2), \mathrm{SU}(3)). \tag{3.75}$$

Finally, assume that $K^\pm = U(2)$ (up to conjugation in G). Let $T^2 = \mathrm{diag}\,(\mathrm{SU}(3))$. If $K^\pm$ contains this T^2, then it must be a conjugate of U(2) by an element of the group $N(T^2)/T^2$. Therefore, there are two possibilities for the pair K^+, K^- up to conjugacy of G: S(U(1)U(2)), and S(U(2)U(1)) (see [22, Case 4_7]). On the other hand, since $\mathrm{U}(2)/H_0 = \mathbb{S}^2$, H must be $T^2\mathbb{Z}_2$. However, $\mathrm{S(U(1)U(2))} \cap \mathrm{S(U(2)U(1))} = T^2$, so K^-, K^+ should be the same. Thus we obtain the following diagram:

$$(\mathrm{SU}(3), T^2\mathbb{Z}_2, \mathrm{U}(2), \mathrm{U}(2)). \tag{3.76}$$

This action is a non-primitive action, and X is equivariantly homeomorphic to the total space of a $\mathrm{Susp}\,(\mathbb{R}P^2)$-bundle over $\mathbb{C}P^2$.

$\mathbf{G = SU(3) \times S^1}$. In this case, $\dim H = 3$. By Proposition 2.21, H, and $K_0^+ \subseteq \mathrm{SU}(3) \times 1$, $K^-/H = \mathbb{S}^1$. Since H is 3-dimensional, H_0 must be $\mathrm{SO}(3) \times 1$ or $\mathrm{SU}(2) \times 1$. If $H_0 = \mathrm{SO}(3) \times 1$, then K^+ has to be $\mathrm{SU}(3) \times 1$ since there is no exceptional orbit. However, the classification of positively curved homogeneous spaces shows that this cannot occur. Therefore, $H_0 = \mathrm{SU}(2) \times 1$. Since K^+/H is not a sphere, $K^+ = \mathrm{SU}(3) \times 1$, and $K^+/H = \mathbb{S}^5/\mathbb{Z}_k$. Thus $K_0^+ \cap H = \mathrm{S}(\mathrm{U}(2)\mathbb{Z}_k)$. On the other hand, K^- is a 4-dimensional subgroup of G containing $\mathrm{S}(\mathrm{U}(2)\mathbb{Z}_k)$ whose projection to S^1 is S^1. Hence it has to be $\mathrm{S}(\mathrm{U}(2)\mathbb{Z}_k) \times S^1$. As a result, we get the following diagram:

$$(\mathrm{SU}(3) \times S^1, \mathrm{S}(\mathrm{U}(2)\mathbb{Z}_k) \times \mathbb{Z}_l, \mathrm{S}(\mathrm{U}(2)\mathbb{Z}_k) \times S^1, \mathrm{SU}(3) \times \mathbb{Z}_l). \tag{3.77}$$

By Proposition 2.25, X is equivariantly homeomorphic to $(\mathbb{S}^5/\mathbb{Z}_k) * \mathbb{S}^1$ with the join action of $\mathrm{SU}(3) \times S^1$.

$\mathbf{G = SU(3) \times S^3}$. In this case, $\dim H = 5$. Since $\mathrm{Proj}_2(H_0) \subsetneq S^3$, we have $H_0 = \mathrm{U}(2) \times S^1$. Thus $\mathrm{U}(2) \subseteq \mathrm{Proj}_1(K_0^\pm) \subseteq \mathrm{SU}(3)$ and $S^1 \subseteq \mathrm{Proj}_2(K_0^\pm) \subseteq S^3$. Since U(2) is a maximal subgroup of SU(3), and S^1 is a maximal connected subgroup of S^3, $\mathrm{Proj}_1(K_0^\pm) = \mathrm{U}(2)$ or SU(3), and $\mathrm{Proj}_2(K_0^\pm) = S^1$ or S^3. But G/H is not homeomorphic to a positively curved homogeneous space, so $K^\pm$ are proper subgroups of G. Moreover, $\dim K^\pm > \dim H$, for X does not have an exceptional orbit. Therefore, we have the following cases: $K_0^\pm = \mathrm{SU}(3) \times S^1$, $K_0^+ = \mathrm{SU}(3) \times S^1$ and $K_0^- = \mathrm{U}(2) \times S^3$, and $K_0^\pm = \mathrm{U}(2) \times S^3$.

Assume first that $K_0^\pm = \mathrm{SU}(3) \times S^1$. Since U(2) is maximal, $H = H_0 = \mathrm{U}(2) \times S^1$, and we obtain the following diagram:

$$(\mathrm{SU}(3) \times S^3, \mathrm{U}(2) \times S^1, \mathrm{SU}(3) \times S^1, \mathrm{SU}(3) \times S^1). \tag{3.78}$$

By Proposition 2.23, this action is a product action and X is equivariantly homeomorphic to $\mathrm{Susp}\,(\mathbb{C}P^2) \times \mathbb{S}^2$.

Suppose now that $K_0^+ = \mathrm{SU}(3) \times S^1$ and $K_0^- = \mathrm{U}(2) \times S^3$. In this case, K^-/H can be either $\mathbb{S}^2$ or $\mathbb{R}P^2$. Therefore, we have the following diagrams, respectively,

$$(\mathrm{SU}(3) \times S^3, \mathrm{U}(2) \times S^1, \mathrm{SU}(3) \times S^1, \mathrm{U}(2) \times S^3), \tag{3.79}$$

$$(\mathrm{SU}(3) \times S^3, \mathrm{U}(2) \times N_{S^3}(S^1), \mathrm{SU}(3) \times N_{S^3}(S^1), \mathrm{U}(2) \times S^3). \tag{3.80}$$

By Proposition 2.25, X is equivariantly homeomorphic to $\mathbb{C}P^2 * \mathbb{S}^2$ or $\mathbb{C}P^2 * \mathbb{R}P^2$, respectively, equipped with the respective join action of $\mathrm{SU}(3) \times S^3$.

Finally, suppose that $K_0^\pm = \mathrm{U}(2) \times S^3$. Since $K_0^\pm = \mathrm{U}(2) \times S^3$ is a maximal subgroup of $\mathrm{SU}(3) \times S^3$, $K^\pm = K_0^\pm$. Thus $K^\pm/H$ has to be $\mathbb{R}P^2$, and consequently, we get the following diagram:

$$(\mathrm{SU}(3) \times S^3, \mathrm{U}(2) \times N_{S^3}(S^1), \mathrm{U}(2) \times S^3, \mathrm{U}(2) \times S^3). \tag{3.81}$$

By Proposition 2.23, this action is a product action and X is equivariantly homeomorphic to $\mathrm{Susp}\,(\mathbb{R}P^2) \times \mathbb{C}P^2$.

$\mathbf{G = Sp(2)}$. In this case, $\dim H = 4$. Hence $H_0 = \mathrm{U}(2)_{\max}$ (the maximal subgroup of $\mathrm{Sp}(2)$) or $H_0 = \mathrm{Sp}(1)\mathrm{SO}(2)$. If $H_0 = \mathrm{U}(2)_{\max}$, then $K^\pm$ have to be $\mathrm{Sp}(2)$, which is impossible, for $\mathrm{Sp}(2)/\mathrm{U}(2)_{\max}$ does not admit a positively curved metric (see [34]). Thus $H_0 = \mathrm{Sp}(1)\mathrm{SO}(2)$. Since the only proper subgroup of G containing H_0 is $\mathrm{Sp}(1)\mathrm{Sp}(1)$, we have the following cases: $K_0^\pm = \mathrm{Sp}(2)$, $K^+ = \mathrm{Sp}(2)$ and $K_0^- = \mathrm{Sp}(1)\mathrm{Sp}(1)$, and $K_0^\pm = \mathrm{Sp}(1)\mathrm{Sp}(1)$.

Assume first that $K_0^\pm = \mathrm{Sp}(2)$. Therefore we have the following diagrams:

$$(\mathrm{Sp}(2), \mathrm{Sp}(1)\mathrm{SO}(2), \mathrm{Sp}(2), \mathrm{Sp}(2)), \tag{3.82}$$

$$(\mathrm{Sp}(2), \mathrm{Sp}(1)\mathrm{SO}(2)\mathbb{Z}_2, \mathrm{Sp}(2), \mathrm{Sp}(2)). \tag{3.83}$$

By Proposition 2.27, X is equivariantly homeomorphic to $\mathrm{Susp}\,(\mathbb{C}P^3)$ or to $\mathrm{Susp}\,(\mathbb{C}P^3/\mathbb{Z}_2)$, respectively, equipped with the corresponding suspension action of $\mathrm{Sp}(2)$.

Suppose now that $K^+ = \mathrm{Sp}(2)$ and $K_0^- = \mathrm{Sp}(1)\mathrm{Sp}(1)$. The space K^+/H is equal to either $\mathbb{C}P^3$ or $\mathbb{C}P^3/\mathbb{Z}_2$, so $H = \mathrm{Sp}(1)\mathrm{SO}(2)$ or $H = \mathrm{Sp}(1)\mathrm{SO}(2)\mathbb{Z}_2$, respectively. Therefore, we obtain the following diagrams:

$$(\mathrm{Sp}(2), \mathrm{Sp}(1)\mathrm{SO}(2), \mathrm{Sp}(1)\mathrm{Sp}(1), \mathrm{Sp}(2)), \tag{3.84}$$

$$(\mathrm{Sp}(2), \mathrm{Sp}(1)\mathrm{SO}(2)\mathbb{Z}_2, \mathrm{Sp}(1)\mathrm{Sp}(1), \mathrm{Sp}(2)). \tag{3.85}$$

Finally, assume that $K_0^\pm = \mathrm{Sp}(1)\mathrm{Sp}(1)$ (up to a conjugation). Since $K_0^\pm$ both contain $H_0 = \mathrm{Sp}(1)\mathrm{SO}(2)$, they should be equal, so $H \cap K_0^+ = H \cap K_0^-$, which in turn implies that $K^\pm$ is connected. On the other hand, $K^\pm/H$ should be a positively curved homogeneous space not homeomorphic to a sphere. Therefore $H = \mathrm{Sp}(1)\mathrm{SO}(2)\mathbb{Z}_2$, and we get the following diagram:

$$(\mathrm{Sp}(2), \mathrm{Sp}(1)\mathrm{SO}(2)\mathbb{Z}_2, \mathrm{Sp}(1)\mathrm{Sp}(1), \mathrm{Sp}(1)\mathrm{Sp}(1)). \tag{3.86}$$

This action is a non-primitive action with $L = \mathrm{Sp}(1)\mathrm{Sp}(1)$, and X is equivariantly homeomorphic to the total space of a $\mathrm{Susp}\,(\mathbb{R}P^2)$-bundle over $\mathbb{S}^4$.

$\mathbf{G = Sp(2) \times S^3}$. Since the action is non-reducible, $\mathrm{Proj}_2(H_0) \subsetneq S^3$. Since $\dim H = 7$, and the highest dimension of a proper subgroup of Sp(2) is 6, $\mathrm{Proj}_2(H_0)$ must be equal to S^1. Thus $H_0 = \mathrm{Sp}(1)\mathrm{Sp}(1) \times S^1$. Maximality of S^1 in S^3, and of Sp(1)Sp(1) in Sp(2), gives rise to the following cases: $K_0^{\pm} = \mathrm{Sp}(1)\mathrm{Sp}(1) \times S^3$, $K_0^{+} = \mathrm{Sp}(1)\mathrm{Sp}(1) \times S^3$ and $K_0^{-} = \mathrm{Sp}(2) \times S^1$, and $K_0^{\pm} = \mathrm{Sp}(2) \times S^1$.

Assume first that $K_0^{\pm} = \mathrm{Sp}(1)\mathrm{Sp}(1) \times S^3$. Since $H \cap K_0^- = H \cap K_0^+$, we have $H = H \cap K_0^- = H \cap K_0^+ \subseteq K_0^{\pm}$. Therefore, $K^{\pm}$ are connected, and we get the following diagram:

$$(\mathrm{Sp}(2) \times S^3, \mathrm{Sp}(1)\mathrm{Sp}(1) \times N_{S^3}(S^1), \mathrm{Sp}(1)\mathrm{Sp}(1) \times S^3, \mathrm{Sp}(1)\mathrm{Sp}(1) \times S^3). \tag{3.87}$$

By Proposition 2.23, this action is a product action and X is equivariantly homeomorphic to $\mathrm{Susp}\,(\mathbb{R}P^2) \times \mathbb{S}^4$ with the product action of $\mathrm{Sp}(2) \times S^3$.

Suppose now that $K_0^+ = \mathrm{Sp}(1)\mathrm{Sp}(1) \times S^3$ and $K_0^- = \mathrm{Sp}(2) \times S^1$. We have the following diagrams:

$$(\mathrm{Sp}(2) \times S^3, \mathrm{Sp}(1)\mathrm{Sp}(1) \times N_{S^3}(S^1), \mathrm{Sp}(1)\mathrm{Sp}(1) \times S^3, \mathrm{Sp}(2) \times N_{S^3}(S^1)), \tag{3.88}$$

corresponding to the join action of $\mathrm{Sp}(2) \times S^3$ on $\mathbb{S}^4 * \mathbb{R}P^2$;

$$(\mathrm{Sp}(2) \times S^3, \mathrm{Sp}(1)\mathrm{Sp}(1)\mathbb{Z}_2 \times S^1, \mathrm{Sp}(1)\mathrm{Sp}(1)\mathbb{Z}_2 \times S^3, \mathrm{Sp}(2) \times S^1), \tag{3.89}$$

corresponding to the join action of $\mathrm{Sp}(2) \times S^3$ on $\mathbb{S}^2 * \mathbb{R}P^4$; and

$$(\mathrm{Sp}(2) \times S^3, \mathrm{Sp}(1)\mathrm{Sp}(1)\mathbb{Z}_2 \times N_{S^3}(S^1), \mathrm{Sp}(1)\mathrm{Sp}(1)\mathbb{Z}_2 \times S^3, \mathrm{Sp}(2) \times N_{S^3}(S^1)). \tag{3.90}$$

which corresponds to the join action of $\mathrm{Sp}(2) \times S^3$ on $\mathbb{R}P^4 * \mathbb{R}P^2$.

Finally, assume that $K_0^{\pm} = \mathrm{Sp}(2) \times S^1$. In this case $K^{\pm}$ are connected and $H = \mathrm{Sp}(1)\mathrm{Sp}(1)\mathbb{Z}_2 \times S^1$. Thus we obtain the following diagram:

$$(\mathrm{Sp}(2) \times S^3, \mathrm{Sp}(1)\mathrm{Sp}(1)\mathbb{Z}_2 \times S^1, \mathrm{Sp}(2) \times S^1, \mathrm{Sp}(2) \times S^1). \tag{3.91}$$

By Proposition 2.23, this action is a product action and X is equivariantly homeomorphic to $\mathrm{Susp}\,(\mathbb{R}P^4) \times \mathbb{S}^2$ with the product action of $\mathrm{Sp}(2) \times S^3$.

$\mathbf{G = G_2}$. Since $\dim H$ has to be 8, we have $H = \mathrm{SU}(3)$. Thus $K^{\pm} = \mathrm{G}_2$, for SU(3) is a maximal connected subgroup of G_2 and there are no exceptional orbits. As a result, we have the following diagram:

$$(\mathrm{G}_2, N_{\mathrm{G}_2}(\mathrm{SU}(3)), \mathrm{G}_2, \mathrm{G}_2). \tag{3.92}$$

By Proposition 2.27, X is equivariantly homeomorphic to $\mathrm{Susp}\,(\mathbb{R}P^6)$ with the suspension action of G_2.

$\mathbf{G = SU(4)}$. In this case, $\dim H = 9$, so $H_0 = \mathrm{U}(3)$. Because U(3) is a maximal subgroup of G, we have $K^{\pm} = \mathrm{SU}(4)$, and the following diagram is obtained:

$$(\mathrm{SU}(4), \mathrm{U}(3), \mathrm{SU}(4), \mathrm{SU}(4)). \tag{3.93}$$

By Proposition 2.27, this space is equivariantly homeomorphic to $\mathrm{Susp}\,(\mathbb{C}P^3)$ with the suspension action of SU(4).

$\mathbf{G = SU(4) \times S^1}$. In this case, $\dim H = 10$. We have $K_0^+, H_0 \subseteq \mathrm{SU}(4) \times 1$ and $K^-/H = \mathbb{S}^1$ by Proposition 2.21. Therefore, $K_0^+ = \mathrm{SU}(4) \times 1$, $H_0 = \mathrm{Sp}(2) \times 1$, $K_0^- = \mathrm{Sp}(2) \times S^1$, and we get the following diagram:

$$(\mathrm{SU}(4) \times S^1, (\mathrm{Sp}(2)\mathbb{Z}_2) \times \mathbb{Z}_k, (\mathrm{Sp}(2)\mathbb{Z}_2) \times S^1, \mathrm{SU}(4) \times \mathbb{Z}_k). \tag{3.94}$$

By Proposition 2.25, this action is equivalent to the join action of $\mathrm{SU}(4) \times S^1$ on $\mathbb{R}P^5 * \mathbb{S}^1$.

$\mathbf{G} = \mathbf{Spin(7)}$. In this case, since $\dim G = 21 = (6 \times 7)/2$, by Proposition 2.28, we have

$$(\mathrm{Spin}(7), N_{\mathrm{Spin}(7)}(\mathrm{Spin}(6)), \mathrm{Spin}(7), \mathrm{Spin}(7)), \tag{3.95}$$

and X is equivariantly homeomorphic to Susp $(\mathbb{R}P^6)$ with the spin action of Spin(7).

This concludes the proof of Theorem A. □

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