Prediction and Reconstruction of Distributed Dynamic Phenomena Characterized by Linear Partial Differential Equations

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***Abstract*— A primary challenge for the reconstruction of continuous-time, continuous-amplitude distributed parameter systems is the inclusion of recent discrete-time, discrete-amplitude, spatially discrete measurements. Hence, a systematic method for data processing is required that also handles incomplete and noisy data, e.g. data from a sensor network. This article presents two approaches to the reconstruction of distributed parameter systems that can be described by linear partial differential equations (PDEs) and involve one or several discrete measurement points. In both approaches, the linear PDE is first converted into a bank of linear lumped systems by means of modal analysis. In addition, a measurement equation relating state and (sensor) data is derived. In the second step, a Kalman filter (KF) is used to dynamically estimate the state of the lumped systems, which provides an approximation of the solution of the underlying PDE. The first approach uses Fourier Analysis. The second approach uses Fourier Analysis and the collocation method. The approaches are both demonstrated for a simple linear inhomogeneous PDE, the one-dimensional heat equation.**

I. Introduction

Centralized and decentralized sensor networks are becoming increasingly popular for monitoring distributed phenomena [1], [2]. Monitoring applications with sensor networks can be used for example in industrial, medical, urban and many other environments. The distributed sensor data has to be fused and evaluated systematically, in order to obtain meaningful data [3]. Many data fusion methods for sensor network applications have been investigated, i.e. the decentralized Kalman filter [4], value fusion [5], decision fusion [5], the Fuzzy approach [6], the Bayesian method [6] and the Dempster-Shafer method [6]. Other fusion methods are cited e.g. in [4], [7] and [8].

In this paper, we present a new method of systematic data processing for distributed physical phenomena, which can be monitored by centralized sensor networks, and which can be described by means of linear homogeneous and inhomogeneous PDEs (i.e., the temperature in solid bodies, the vibration of a membrane). Examples for such PDEs used in practice are: Computing the dispersion of acoustic waves and generating approximated sounds by motion of solid objects [9], simulating the motion of a piano string [10], controlling the fluid-flow in a thermal reactor [11], or bio-potential computing of electrical activity in the human body [12].

In order to predict and reconstruct the state of a continuous linear distributed phenomenon, we will derive a dynamic state space model. The estimate of the state is improved via a set of n-dimensional discrete sensor measurements. Of course, the measurements are noisy and can be incomplete. We introduce two new approaches for designing such a state space model. The two approaches are based on the preparatory work of [13], which did not consider the noise processes in a state space model. In order to use the model-based approach we have to solve the linear PDE. It can be solved with numerical methods such as the finite element method or the finite difference method. Both methods have expensive computation costs, if they are used to calculate an accurate solution for a PDE. Therefore, they are unpractical for real-time applications. In order to realize a real-time application we use the modal analysis to calculate the solution of the PDE and to derive the state space model.

For the systematic data processing method we propose two new approaches (in the following, approach 1 and approach 2). They are each divided into a two-step process. In the first step of both approaches, the chosen PDE has to be solved with respect to the boundary conditions. We solve the linear PDE by means of modal analysis and set up an equation, which has to be satisfied both in every spatial point and at every time. In approach 1 a single spatial point is chosen and inserted into this equation. As with the collocation method many collocation points are chosen in approach 2 and then inserted into this equation, so that a system of equations is established. After this the components of the resulting equation or the resulting equation system are split into a bank of lumped systems. In order to perform the split we use the Fourier Analysis. In the second step, we use the Kalman filter in both approaches to estimate the state of the lumped systems, i.e., the state of the dynamic system model. In addition, the measurement equation, which maps the system state to the sensor data, is derived. This two-step process provides an approximation of the unique solution of the underlying linear PDE. The theoret-

ical considerations are demonstrated by a practical application with a simple one-dimensional linear inhomogeneous PDE, the heat equation. The novelty in designing the state space model compared to [13] is, that we can handle noise processes of the system model and of the measurement data. In addition we are using a collocation method, which opens up the possibility of reconstructing the PDE solution in case the control function of linear or non-linear PDEs is unknown.

The remainder of this paper is organized as follows. In chapter II, the problem of building dynamic models of continuous distributed phenomena is formulated in detail. Chapter III and IV discuss the two approaches with the above mentioned two-step processes. Finally, in chapter V we show the simulation results of the two approaches for the prediction and reconstruction with the heat equation.

II. Problem Formulation

Our goal is to design a dynamic system model for the purpose of estimating the state of a distributed phenomenon, which is monitored by a centralized sensor network. We first discuss the required system theoretical equations of such a model. After this we explain the role and the solution procedure of the PDEs.

For the system model, which estimates and updates the state of the true system (= considered distributed phenomenon), a state equation and a measurement equation have to be defined. A discrete-time state equation for prediction purposes can be linear

$$\underline{x}_{k+1} = \mathbf{A}\underline{x}_k + \mathbf{B}(\underline{u}_k + \underline{w}_k) \tag{1}$$

or non-linear

$$\underline{x}_{k+1} = a_k(\underline{x}_k, \underline{u}_k, \underline{w}_k). \tag{2}$$

In addition, a measurement equation is noted as linear

$$\underline{y}_k = \mathbf{H}_k\underline{x}_k + \underline{v}_k, \tag{3}$$

or non-linear

$$\underline{y}_k = h_k(\underline{x}_k) + \underline{v}_k. \tag{4}$$

In the above formulas (1) to (4) $\underline{x}_k$ is the system state, matrix $\mathbf{A}$ describes the state change from $\underline{x}_k$ to $\underline{x}_{k+1}$, $\underline{u}_k$ is a known system control input, $\underline{w}_k$ is the process noise, matrix $\mathbf{B}$ relates the control input $\underline{u}_k$ to the state $\underline{x}_{k+1}$. In (3), (4) $\underline{y}_k$ is the measurement with the measurement uncertainty $\underline{v}_k$. $\mathbf{H}_k$ is a matrix, that maps the measurement $\underline{y}_k$ to the actual state $\underline{x}_k$. There are various filter methods to reconstruct the system state from noisy measurements. In the case of a linear system and a linear measurement equation the Kalman Filter is used. In the case of non-linear equations non-linear filters, such as the Extended Kalman Filter or the Progressive Bayes algorithm [14] have to be used.

In this paper we only consider dynamically distributed phenomena, which can be described by linear PDEs. Linear PDEs can correctly describe a class of dynamic behaviour in linear physical phenomena. We exploit the PDEs for systematic data processing of sensor data. Dynamic linear PDEs have the general form

$$F(p_1, ..., p_n, t; u; u_{p_1}, ..., u_{p_n}, u_t; ...; u_{p_1 ... p_1}, ..., u_{t...t}) = f(p_1, ..., p_n, t) \text{ for } n \geq 1, \tag{5}$$

where the unknown function u and its partial derivatives (e.g. u_{p_1}) occur only in linear form. The linear PDE is homogeneous if the control function $f(p_1, ..., p_n, t) = 0$, otherwise it is inhomogeneous. In addition, side conditions are defined for a PDE, i.e., initial and/or boundary conditions. The opposite of a distributed system is a lumped system. In contrast, a lumped system describes a mathematical model, where time is the only independent variable. Examples of lumped systems are chemical reactions or motion and vibrations in mechanical components. They have the general form

$$F(t; u; \dot{u}, \ddot{u}, ..., u^{(n)}) = f(t). \tag{6}$$

In our processing approach we use lumped systems to solve the linear PDE.

In order to work with a linear PDE, the PDE has to be converted into the system theoretical equations (1), (3). Therefore, we propose two approaches, each with a two–step process, which are presented in the next two chapters. In the first step (ch. III-A, ch. IV-A) the linear PDE is converted into a bank of linear lumped systems. This step is split into 4 substeps. Substeps 1, 2 and 3 are the same in both approaches. In the first substep, the homogeneous linear PDE is converted into a system of ordinary differential equations with modal analysis and the particular solution is calculated (ch. III-A.1). In substep 2, we adapt the particular solution to the boundary conditions (ch. III-A.2). In the third substep, we transform the solution into a normalized form and approximate the control function $f(p_1, ...p_n, t)$ of the inhomogeneous PDE by means of a Fourier series (ch. III-A.3). In the fourth substep of approach 1, the equation, which has to be satisfied, is derived and is decomposed into a bank of lumped systems (ch. III-A.4). In substep 4 of approach 2 (ch. IV-A.1) collocation points are additionally chosen. The second step (ch. III-B, ch. IV-B) of our two-step process uses the KF. For this purpose, we specify the time state, the state equation and the measurement equation by means of the results from substep 4 of step 1 (ch. III-B.1, ch. IV-B.1). In chapter III-B.2 the update and filter equations are specified. In addition, a reconstruction formula for non-measurement points is given (ch. III-B.3). At the end of each substep the theoretical considerations are demonstrated by means of two examples. *Example 1* is the one-dimensional heat equation of a bar of length l

$$u_t = Zu_{pp} + \frac{f(p,t)}{\gamma\varrho}, \quad \text{for } 0 < p < l \tag{7}$$

with homogeneous boundary conditions ($u(0,t) = 0$, $u(l,t) = 0$). The variables Z, γ, ϱ are material coefficients. *Example 2* is the two-dimensional wave equation of a rectangular vibrating

membrane with height a and width b

$$u_{tt} = G(u_{p_1p_1} + u_{p_2p_2}) + \frac{f(p_1,p_2,t)}{c}, \quad \text{for } 0 < p_1 < a, 0 < p_2 < b, \tag{8}$$

where G, c are material coefficients. The homogeneous boundary conditions are $u(p_1,0,t) = 0$, $u(0,p_2,t) = 0$, $u(p_1,b,t) = 0$, $u(a,p_2,t) = 0$.

III. The Two-step Process of Approach 1

A. Step1 - Conversion of the PDE to a Bank of Lumped Systems

In substeps 1 and 2 we consider only the homogeneous part of the linear PDE. In substep 3 the procedure is extended to the inhomogeneous PDE.

1) Substep 1 – Calculation of the Particular Solution by Means of Modal Analysis: The solution $u_H(p_1,....,p_n,t)$ of the homogeneous PDE is decomposed into independent parts φ_j according to

$$u_H(p_1,p_2,...,p_n,t) = \varphi_1(p_1) \cdot ... \cdot \varphi_n(p_n) \cdot \varphi_{n+1}(t). \tag{9}$$

Formula (9) is then inserted into the homogeneous PDE. By means of the separation method we separate the existing variables from (9) and obtain $n+1$ ordinary differential equations. For each ordinary differential equation, a general solution is computed. The product of these general solutions is the particular solution of the linear homogeneous PDE.

Example 1 and 2: The calculation of this supstep can be found in [15].

2) Substep 2 – Adaptation of the Particular Solution to Boundary Conditions: In order to adapt the particular solution to the boundary conditions, we insert the boundary values into the particular solution and calculate the desired coefficients. In general, we obtain different solutions $u_i(p_1,...,p_n,t) = \psi_i(p_1,..,p_n)\alpha_i(t)$ consisting of spatially dependent eigenfunctions $\psi_i(p_1,..,p_n)$ and time dependent eigenfunctions $\alpha_i(t)$. Thanks to the linearity of the PDE, the superposition of the different solutions $u_i(p_1,...,p_n,t)$ is also a solution of the homogeneous PDE. In this paper we approximate the solution

$$u_H(p_1,..,p_n,t) = \sum_{i=1}^{\infty} u_i(p_1,..,p_n,t)$$

$$\approx \sum_{i=1}^{N} \psi_i(p_1,..,p_n)\alpha_i(t) = \underline{\psi}(p_1,..,p_n)^T \underline{\alpha}(t) \tag{10}$$

with N eigenfunctions. The particular solution does not need to be adapted to the initial conditions. We will include the required variables in the time dependent part. In our system model this part is the unknown variable, which has to be calculated (see III-B).

Example 1 and 2: The calculation of this substep can be found in [15]. The approximated solution of the one-dimensional heat equation is

$$u(p,t) = \sum_{i=1}^{N} \underbrace{\sin\frac{i\pi p}{l}}_{\psi_i(p)} \underbrace{F_i e^{-\frac{-i^2\pi^2 Z t}{l^2}}}_{\alpha_i(t)}, \tag{11}$$

and the approximated solution of the two-dimensional wave equation, with $N = N1 \cdot N2$, is

$$u(p_1,p_2,t) = \sum_{m=1}^{N1}\sum_{n=1}^{N2} \underbrace{\sin\frac{m\pi p_1}{a}\sin\frac{n\pi p_2}{b}}_{\psi_i(p_1,p_2)} \underbrace{(A_{mn}\cos(\lambda_{mn}t) + B_{mn}\sin(\lambda_{mn}t))}_{\alpha_i(t)} \tag{12}$$

where $\psi_i(p)$ and $\psi_i(p_1,p_2)$ respectively are the spatially dependent parts and $\alpha_i(t)$ are the time dependent parts.

3) Substep 3 – Transformation of the Particular Solution into a normalized Form and Approximation of the Control Function: Now the solution (10) is transformed into the normalized form, in such a way that the spatially dependent eigenfunctions $\psi_i(p_1,...,p_n)$ of our solutions are orthonormal (i.e., orthogonal and with norm equal 1). With the normalized eigenfunctions $\psi_i(p_1,...,p_n)$ we can derive a Fourier series which converges to the solution $u_H(p_1,...,p_n,t)$. The Fourier series has the form

$$u_H(p_1,...,p_n,t) = \sum_{i=1}^{\infty} \psi_i(p_1,...,p_n) \cdot \alpha_i(t), \tag{13}$$

where $\alpha_i(t)$ are Fourier coefficients of the form

$$\alpha_i(t) = \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} u_H(p_1,..,p_n,t)\psi_i(p_1,..,p_n)\mathrm{d}p_1..\mathrm{d}p_n. \tag{14}$$

As in formula (10), we approximate the solution u_H in (13) with N normalized eigenfunctions.

The Fourier series (13) converges not only to the solution u_H, but also to the solution of the inhomogeneous PDE u_I, if the Fourier coefficients $\alpha_i(t)$ are adapted respectively. Consequently, in formula (13) the function u_H can be replaced by u_I.

After the transformation of the solution into normalized form we consider the control function $f(p_1,...,p_n,t)$ of the inhomogeneous PDE. In analogy to the homogeneous case mentioned above, we describe the control function as a Fourier series and approximate it by means of N orthonormal eigenfunctions $\psi_i(p_1,...,p_n)$, as

$$f(p_1,...,p_n,t) \approx \sum_{i=1}^{N} \psi_i(p_1,...,p_n) \cdot f_i(t), \tag{15}$$

where the Fourier coefficients $f_i(t)$ are of the form

$$f_i(t) = \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(p_1,...,p_n,t)\psi_i(p_1,...,p_n)\mathrm{d}p_1...\mathrm{d}p_n. \tag{16}$$

By decomposing the control function by means of the approximated Fourier series into N terms we gain an advantage. In the following substep, we can decompose the PDE system into N lumped systems.

Example 1: The normalized spatial eigenfunctions $\psi_i(p)$ have the form

$$\psi_i(p) = \sqrt{\frac{2}{l}} \sin \frac{i\pi p}{l} \text{ for } i > 0. \tag{17}$$

Example 2: The normalized spatial eigenfunctions $\psi_i(p_1, p_2)$ have the form

$$\psi_i(p_1, p_2) = \sqrt{\frac{4}{ab}} \sin \frac{m\pi p_1}{l} \sin \frac{n\pi p_2}{l} \text{ for } m, n, i > 0. \tag{18}$$

4) Substep 4 – Calculating the Equation to be Satisfied and the Decomposition into a Bank of Lumped Systems: In this step the approximation results of the previous steps are used to derive a differential equation, that has to be satisfied in every point and at a fixed time.

The approximation formula (15) and the approximated solution u_H or u_I (13) along with the calculated spatially dependent part $\underline{\psi}$ and time dependent part $\underline{\alpha}$ are inserted into the inhomogeneous linear PDE. The resulting equation should hold true at a fixed time and at an arbitrary point. In the following, we choose an arbitrary point. The equation of the distributed phenomenon is split into a bank of N linear lumped systems. For this, we consider one solution of the superposition (13) that consists of a spatially dependent and of a time dependent eigenfunction. With the corresponding spatially dependent eigenfunction in the control function, we can derive for each solution a linear ordinary differential equation, which only depends on time, a so-called lumped system.

The N differential equations are summarized and reduced to a first-order differential equation of the form

$$\dot{\underline{\beta}}(t) = \mathbf{A}\underline{\beta}(t) + \mathbf{B}\underline{u}(t). \tag{19}$$

The dimension of vector $\underline{\beta}$ can be oN and depends on the highest order o of the time derivatives in the PDE and the number of used eigenfunctions N. For example, the vector can have the form $\underline{\beta}(t) = \underline{\alpha}(t)$ if $o = 1$, $\underline{\beta}(t) = \begin{pmatrix} \underline{\alpha}(t) \\ \dot{\underline{\alpha}}(t) \end{pmatrix}$ if $o = 2$.

Example 1: The equation that has to be satisfied in every point has the form

$$\sum_{i=1}^{N} \psi_i(p)\dot{\alpha}_i(t) = Z \sum_{i=1}^{N} \psi_i^{''}(p)\alpha_i(t) + \frac{1}{\gamma\varrho} \sum_{i=1}^{N} \psi_i(p) f_i(t). \tag{20}$$

In order to split the equation into a lumped system we consider one solution $u_i(p,t) = \psi_i(p)\alpha_i(t)$

$$\psi_i(p)\dot{\alpha}_i(t) = Z\psi_i^{''}(p)\alpha_i(t) + \frac{1}{\gamma\varrho} f_i(t)\psi_i(p). \tag{21}$$

If the equation is divided by $\psi_i(p)$ and $\lambda_i = \frac{i\pi}{l}$ we obtain

$$\dot{\alpha}_i(t) = -Z\lambda_i^2 \alpha_i(t) + \frac{1}{\gamma\varrho} f_i(t). \tag{22}$$

In (22) we derived a corresponding lumped system for each solution u_i. The N lumped systems can be summarized in form of (19) with $\underline{\beta}(t) = \underline{\alpha}(t)$,

$$\mathbf{A} = \begin{pmatrix} -Z\lambda_1^2 & 0 & \cdots & 0 \\ 0 & -Z\lambda_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -Z\lambda_N^2 \end{pmatrix} \tag{23}$$

and

$$\mathbf{B}\underline{u}(t) = \begin{pmatrix} \frac{1}{\gamma\varrho} & 0 & \cdots & 0 \\ 0 & \frac{1}{\gamma\varrho} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{\gamma\varrho} \end{pmatrix} \begin{pmatrix} f_1(t) \\ \vdots \\ f_N(t) \end{pmatrix}. \tag{24}$$

Example 2: The lumped systems can be summarized to form (19) with

$$\underline{\beta}(t) = \begin{pmatrix} \underline{\alpha}(t) \\ \dot{\underline{\alpha}}(t) \end{pmatrix}, \quad \lambda_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2, \tag{25}$$

$$\mathbf{A} = \left(\begin{array}{ccc|c} & \mathbf{0}_{N\times N} & & \mathbf{I}_{N\times N} \\ \hline -G\lambda_{11}^2 & 0 & 0 & \\ 0 & \ddots & 0 & \mathbf{0}_{N\times N} \\ 0 & 0 & -G\lambda_{N1N2}^2 & \end{array}\right) \tag{26}$$

and

$$\mathbf{B}\underline{u}(t) = \left(\begin{array}{c|c} \mathbf{0}_{N\times N} & \frac{1}{c}\mathbf{I}_{N\times N} \end{array}\right) \left(\begin{array}{c} \mathbf{0}_{N\times 1} \\ \hline f_1(t) \\ \vdots \\ f_N(t) \end{array}\right). \tag{27}$$

B. Step 2 – Derivation of the State Equation and the Measurement Equation, Kalman Filter Equations and Reconstruction formula

1) State and Measurement Equation: After the completion of equation (19) with the process noise $\mathbf{B}\underline{w}(t)$ the discretization follows, and the linear state equation (1) for the system model is received. Our state $\underline{x}_k$ is equal to the time dependent vector $\underline{\beta}(t_k)$ and has the dimension oN. This $\underline{x}_k$ is an unknown variable in our system model and has to be estimated. A linear measurement $\underline{y}_k$ (3) includes oL measurement values of L measurement points, thus $\underline{y}_k \in \mathbb{R}^{oL}$. Varying from case to case the type of a measurement depends on the order o of the time derivations of the PDE. We have to distinguish several cases. If $o = 1$, the measurement $\underline{y}_k$ consists of the approximated solution u_H or u_I for the specified measurement points (=$\mathbf{H}_k\underline{x}_k$) plus an additive white noise term $\underline{v}_k$. Otherwise if $o > 1$ the measurement $\underline{y}_k$ consists of approximated PDE time derivatives (e.g. u_t, u_{tt}) and of the approximated solution u_H or u_I for the specified measurement points (=$\mathbf{H}_k\underline{x}_k$) plus an additive white noise term $\underline{v}_k$. Matrix $\mathbf{H}_k$ contains the spatially dependent parts of the eigenfunctions $\psi_i(p_1, ..., p_n)$ for any measurement point. The number of rows

of matrix $\mathbf{H}_k$ depends on oL; the number of columns depends on the number of eigenfunctions N. Thus, $\underline{y}_k$ does not depend upon the dimension of $\underline{x}_k$.

Example 1: Because $o = 1$ the linear measurement equation is

$$\underline{y}_k = \underbrace{\begin{pmatrix} \psi_1(p_1) & \cdots & \psi_N(p_1) \\ \vdots & \vdots & \vdots \\ \psi_1(p_L) & \cdots & \psi_N(p_L) \end{pmatrix}}_{\psi^H} \underline{\alpha}(t_k) + \underline{v}_k. \tag{28}$$

Example 2: Because $o = 2$ the linear measurement equation is

$$\underline{y}_k = \begin{pmatrix} \psi^H & \mathbf{0}_{L\times N} \\ \mathbf{0}_{L\times N} & \psi^H \end{pmatrix} \begin{pmatrix} \underline{\alpha}(t_k) \\ \dot{\underline{\alpha}}(t_k) \end{pmatrix} + \underline{v}_k, \tag{29}$$

where ψ^H is as in (28) with $\psi(p_1, p_2)$ instead of $\psi(p_1)$

2) Kalman Filter Equations: Now we specify the update and filter equations for the KF. Because the state equation and the measurement equation are linear, we will use the linear KF. The time update at each time step $t_k = kT_s$ consists of the linear system equation (1) and the error covariance equation of the estimated state

$$\mathbf{C}_{k+1} = \mathbf{A}\mathbf{C}_k\mathbf{A}^T + \mathbf{B}\mathbf{Q}_k\mathbf{B}^T. \tag{30}$$

In (30), the matrices $\mathbf{A}$ and $\mathbf{B}$ are defined as in III-B.1. The process noise $\underline{w}_k$ has the time-variant error covariance $\mathbf{Q}_k$. If the system model receives, at an arbitrary time t_c, a measurement of the true system, then the filtering step is started. Equation (3) with $\mathbf{H}_k$ as defined in III-B.1 describes the measurement context. The measurment noise $\underline{v}_k$ and the covariance matrix $\mathbf{R}_k$ describe the measurement error, e.g. of the used temperature sensors or velocity sensors. We assume that $\underline{w}_k$ and $\underline{v}_k$ are uncorrelated and have zero mean. The current predicted state $\underline{x}_c$ is corrected with the Kalman gain

$$\mathbf{K}_k = \mathbf{C}_c\mathbf{H}_k^T(\mathbf{H}_k\mathbf{C}_c\mathbf{H}_k^T + \mathbf{R}_k)^{-1} \tag{31}$$

and the result is a newly estimated state of the true system, given by

$$\underline{x}_k = \underline{x}_c + \mathbf{K}_k(\underline{y}_c - \mathbf{H}_k\underline{x}_c). \tag{32}$$

Also the error covariance is updated according to

$$\mathbf{C}_k = \mathbf{C}_c - \mathbf{K}_k\mathbf{H}_k\mathbf{C}_c. \tag{33}$$

3) Reconstruction Formula: With the current estimated state $\underline{x}_c$ we can calculate the solution u for every arbitrary point $(p_1, ..., p_n)$ in the true system. The approximate reconstruction formula is

$$u(p_1, .., p_n, t_c) \approx \psi(p_1, ..., p_n)^T\underline{x}_c. \tag{34}$$

In addition, we suggest including the approximation errors (10), (15) in the update and filter equations, e.g. with an additive noise term $\underline{j}_k$ in (1), (32), (34), and an additive covariance matrix $\mathbf{J}_k$ in (30), (31), (33). The estimate of the approximation error will be considered in a subsequent paper.

This completes our description of the two-step process of approach 1. In the next chapter we explain approach 2.

IV. The Two-step process of Approach 2

A. Step1 - Conversion of the PDE into a Bank of Lumped Systems

The substeps 1, 2 and 3 are the same as those in approach 1 in chapter III.

1) Substep 4 – Calculating the Equation to be Satisfied, Choosing the Collocation Points and Decomposition into a Bank of Lumped Systems: In this step the approximation results of the previous steps are used to derive a differential equation that has to be satisfied in every point. After this the choosing of collocation points follows.

The approximation formula (15) and the approximated solution u_H or u_I (13), with the calculated spatially dependent part $\underline{\psi}$ and time dependent part $\underline{\alpha}$, are inserted into the inhomogeneous linear PDE. The resulting equation should hold at a fixed time at arbitrary collocation points. We choose arbitrary collocation points. The selection criteria for the collocation points has not been determined until now. Also the amount M of collocation points has not been specified. We will discuss the selection criteria in a subsequent paper.

With the inserted collocation points we obtain many differential equations, which are summarized and reduced to the form

$$\mathbf{E}\dot{\underline{\beta}}(t) = \mathbf{A}\underline{\beta}(t) + \mathbf{B}\underline{u}(t). \tag{35}$$

$\mathbf{E}$, $\mathbf{A}$ and $\mathbf{B}$ are of size $M \times oN$. The matrices can be rectangular. The differential equation (35) with the inserted collocation points only depends on time and represents a bank of oM linear lumped systems.

Example 1: After inserting the approximation formula (15), the approximated solution (13) and the M collocation points in (7), we receive M differential equations. These differential equations are summarized in form of (35) as

$$\begin{aligned} &\underbrace{\begin{pmatrix} \psi_1(p_1) & \cdots & \psi_N(p_1) \\ \vdots & \vdots & \vdots \\ \psi_1(p_M) & \cdots & \psi_N(p_M) \end{pmatrix}}_{\mathbf{E}} \dot{\underline{\alpha}}(t) \\ &= \underbrace{Z\begin{pmatrix} \psi_1''(p_1) & \cdots & \psi_N''(p_1) \\ \vdots & \vdots & \vdots \\ \psi_1''(p_M) & \cdots & \psi_N''(p_M) \end{pmatrix}}_{\mathbf{A}} \underline{\alpha}(t) \\ &+ \underbrace{\frac{1}{\gamma\varrho}\begin{pmatrix} \psi_1(p_1) & \cdots & \psi_N(p_1) \\ \vdots & \vdots & \vdots \\ \psi_1(p_M) & \cdots & \psi_N(p_M) \end{pmatrix}}_{\mathbf{B}} \underline{f}(t). \end{aligned} \tag{36}$$

Example 2: Because $o = 2$ we receive oM ($M = M1 \cdot M2$) differential equations, which are summarized in form of (35)

as

$$\underbrace{\begin{pmatrix} \mathbf{I}_{M\times N} & \mathbf{0}_{M\times N} \\ \mathbf{0}_{M\times N} & \psi \end{pmatrix}}_{\mathbf{E}} \begin{pmatrix} \dot{\underline{\alpha}}(t) \\ \ddot{\underline{\alpha}}(t) \end{pmatrix}$$
$$= \underbrace{G \begin{pmatrix} \mathbf{0}_{M\times N} & \mathbf{I}_{M\times N} \\ \psi^G & \mathbf{0}_{M\times N} \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} \underline{\alpha}(t) \\ \dot{\underline{\alpha}}(t) \end{pmatrix}$$
$$+ \underbrace{\frac{1}{c} \begin{pmatrix} \mathbf{0}_{M\times N} \mid \psi \end{pmatrix}}_{\mathbf{B}} \begin{pmatrix} \mathbf{0}_{N\times 1} \\ \underline{f}(t) \end{pmatrix}, \tag{37}$$

with

$$\psi = \begin{pmatrix} \psi_1(p_{11}, p_{12}) & \cdots & \psi_N(p_{11}, p_{12}) \\ \vdots & \vdots & \vdots \\ \psi_1(p_{M1}, p_{M2}) & \cdots & \psi_N(p_{M1}, p_{M2}) \end{pmatrix} \tag{38}$$

and

$$\psi^G = \begin{pmatrix} \psi_1''(p_{11}, p_{12}) & \cdots & \psi_N''(p_{11}, p_{12}) \\ \vdots & \vdots & \vdots \\ \psi_1''(p_{M1}, p_{M2}) & \cdots & \psi_N''(p_{M1}, p_{M2}) \end{pmatrix}. \tag{39}$$

B. Step 2 – Derivation of the State Equation and the Measurement Equation, Kalman Filter Equations and Reconstruction formula

1) State and Measurement Equation: In order to derive the state equation we complete (35) with the white noise process $\mathbf{B}\underline{w}(t)$ and discretize it. If $\mathbf{E}$ in (35) is invertible we can convert it to (1). But if $\mathbf{E}$ is not regular then we can solve this problem using the theory of descriptor systems. For the discrete-time case, equation (35) can be converted from a non regular system to a regular system by means of the extended shuffle algorithm [16]. Other approaches in the case of rectangular matrices are discussed for the linear discrete-time stochastic system case in [17], [18].

The above descriptions of the time state $\underline{x}_k$ and the measurement equation in chapter (III-B.1) are also valid for this approach.

Example 1: If $\mathbf{E}$ in (36) is invertible, we receive without discretization the linear system equation

$$\dot{\underline{\alpha}} = \mathbf{E}^{-1}\mathbf{A}\underline{\alpha}(t) + \mathbf{E}^{-1}\mathbf{B}\underline{f}(t) + \mathbf{E}^{-1}\underline{w}(t). \tag{40}$$

The measurement equation is equal to (28).

Example 2: If $\mathbf{E}$ in (37) is invertible, we receive without discretization the linear system equation

$$\begin{pmatrix} \dot{\underline{\alpha}}(t) \\ \ddot{\underline{\alpha}}(t) \end{pmatrix} = \mathbf{E}^{-1}\mathbf{A} \begin{pmatrix} \underline{\alpha}(t) \\ \dot{\underline{\alpha}}(t) \end{pmatrix} + \mathbf{E}^{-1}\mathbf{B}\left(\begin{pmatrix} \mathbf{0}_{N\times 1} \\ \underline{f}(t) \end{pmatrix} + \underline{w}(t) \right). \tag{41}$$

The measurement equation is equal to (29).

2) Kalman Filter Equations and Reconstruction Formula: The descriptions from chapter III-B.2 and chapter III-B.3 are also valid for this approach.

With the use of the collocation method we would like to reconstruct an unknown distributed control function. The idea is to find the solution based on the collocation points. The reconstruction problem is still under investigation and will be presented in a later paper. Because we do not precisely know how many collocation points we need, we have to implement a descriptor system, so that we can test it.

V. Simulation with Approach 1 and Approach 2

In this section we simulate the one-dimensional heat equation (7) of a bar of length $l = 10$ length units (LU) with both approaches and discuss the results. The simulation examples are considered under the same conditions, i.e. the same true function which has to be approximated, the same initial function $\phi(p) = \sin(\frac{\pi p}{l})$, the same diffusion coefficient Z, the same white process noise $\underline{w}_k$, the same white measurement noise $\underline{v}_k$, the same amount of eigenfunctions N and the same measurement points.

In order to generate the true function and the measurement values $\underline{y}_k$ we use the numerical PDE solution of MATLAB (see figure 1).

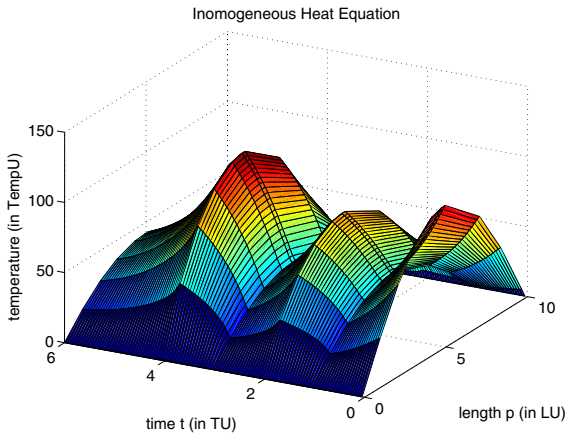

Fig. 1. The numerical solution

The simulations are performed in the time interval t $\in [0, 6]$ time units (TU). The control function $f(p,t)$, which affects the behaviour of (7), is defined as

$$f(p,t) = \begin{cases} 2 & \text{for} \quad t < 1 \\ 20 & \text{for} \quad t \in [1,2) \\ 2 & \text{for} \quad t \in [2,3) \\ 30 & \text{for} \quad t \in [3,4) \\ 5 & \text{for} \quad t > 4. \end{cases} \tag{42}$$

In our simulation we use the KF equations from chapter III-B and chapter IV-B with $N = 10$ normalized eigenfunctions, $\mathbf{R} = 2\mathbf{I}_{L\times L}$, $\mathbf{Q} = \mathbf{I}_{N\times N}$ in approach 1 and $\mathbf{Q} = \mathbf{I}_{M\times M}$ in approach 2. The measurement rate for the update is 4/TU.

First, we present the simulation with approach 1 and then with approach 2. The only measurement point for the simulations is located at $p = 4.3$. Figures 2, 3, 4, 5 show the results, where the noisy true function is marked gray and the predicted function is marked black.

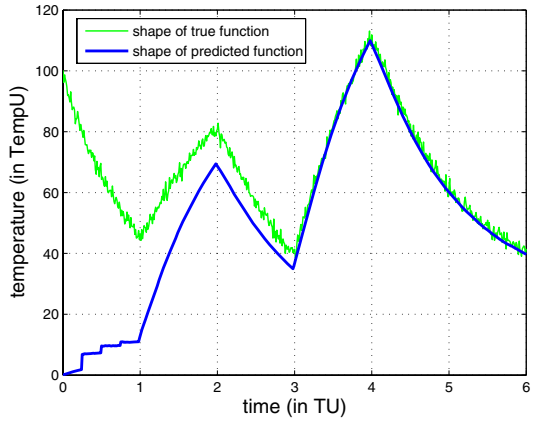

Fig. 2. Results at measurement point p=4.3

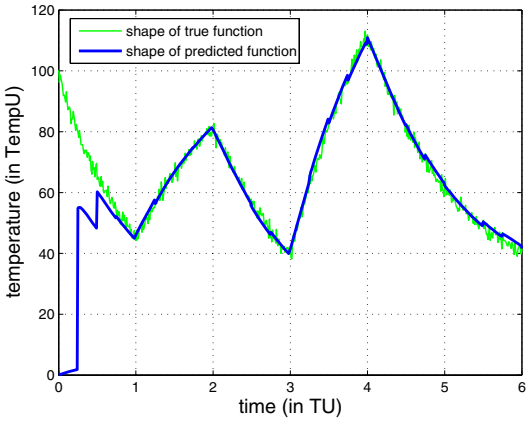

Fig. 4. Results at measurement point p=4.3

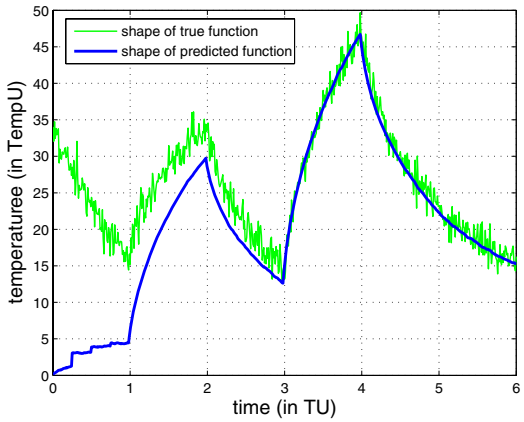

Fig. 3. Results at non-measurement point p=1.1

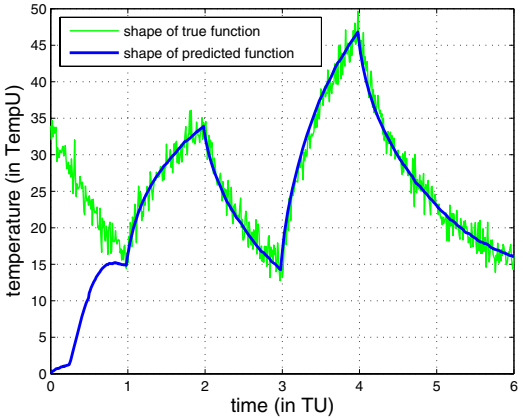

Fig. 5. Results at non-measurement point p=1.1

A. *Simulation with approach 1*

In figure 2, the shapes of the noisy true function and the predicted function at the measurement point are shown. In figure 3 the shapes of the functions at the non-measurement point $p = 1.1$ are shown. The shape of the predicted function at the non-measurement point is reconstructed with the time-variant calculated states $\underline{x}_k$ and the reconstruction formula (34). Now we discuss the shape of the functions. At time $t = 0$, the prediction begins with $\underline{x}(0) = [0...0]^T$. The time dependent part of the true system is not the same as the initial state of our true model, hence the predicted function has smaller values than the true function. At time $t = 0.25$ the first measurement update is received and the shape of the predicted function is increasing a little bit toward the true function. The estimation for the time dependent eigenfunction of the true function is still not good enough. The estimation is improved only after a few measurements are processed. From the time $t = 3.25$ when the state estimation is good enough, the predicted function is very similar to the true function. Hence, the prediction results are now acceptable. This estimation behaviour is also shown for a non-measurement point in figure 3.

B. *Simulation with approach 2*

In this simulation we use $M = 10$ collocation points. In figure 4, the shapes of the noisy true function and the predicted function at the measurement point are shown. In figure 5 the shapes of the functions at the non-measurement point $p = 1.1$ are shown. The shapes of the functions are similar to the simulation of approach 1, except that the estimation of the time dependent parts improve earlier. In this case the state of estimation for prediction is already promising from time $t = 1$ on.

The above simulation confirms the theoretical considerations for calculating the time dependent eigenfunctions. The simulation shows good prediction results after only few mea-

surements, based on a single measurement point! If multiple measurement points are used, the results are similar to those above. Consequently, we conclude from the above simulations that the only difference between the use of approach 1 and of approach 2 is, that in approach 2 a promising state estimation for the prediction is reached more quickly than for approach 1.

VI. Conclusion

We have developed two new state space models for prediction and reconstruction of distributed phenomena, which can be described by means of linear homogeneous and inhomogeneous PDE. The state estimate is improved by one or multiple noisy measurements of sensor data from a possibly distributed sensor network. In both approaches we convert the linear PDE by means of modal analysis and Fourier analysis into a bank of lumped systems. In addition, collocation points are used in the lumped systems of approach 2. Based on the lumped systems, the system equation and the measurement equation are derived for use with the Kalman filter. The Kalman filter estimates the time dependent variables of the lumped systems, which can be used for approximating the PDE solution. A great advantage of these approaches is that unknown initial conditions of the PDE do not have to be specified. Only the boundary conditions have to be specified. Simulation of the approaches for the one-dimensional example PDE show very good prediction and reconstruction results, based on a single measurement point. Both approaches are practically usable, because they have high precision and low computation costs.

Besides the consideration of noise processes a novelty is the use of the collocation method. In the lumped systems of approach 1, spatial variables are not considered. However, in the lumped systems of approach 2 spatially dependent collocation points were chosen and inserted. With approach 2 we will reconstruct the unknown control functions of the PDE in order to satisfy the system equations at the collocation points and to approximate the solution function of the PDE. Also an extension to the nonlinear case is conceivably.

Our future projects are the extension of the one-dimensional case to a higher dimensional case and the examination of real environments (i.e., real sensor data from temperature sensors). In order to use many collocation points we have to implement a descriptor system and to establish the rule for selecting collocation points. Furthermore, the system model has to adapt to the available situation, i.e., it should be a trainable system model, which can estimate unknown system parameters, for example the coefficient Z of the heat equation. Incorporating continuous-time descriptor systems [19] for estimating unknown parameters will also be investigated.

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