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BOOK REVIEW

H. Isozaki: "Maxwell Equation—Inverse Scattering in Electromagnetism"

World Scientific, 2018

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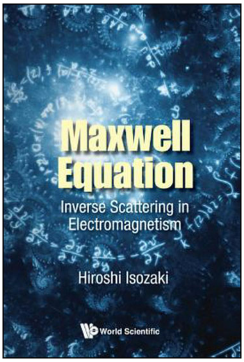

In the year 1873 James Clark Maxwell proposed a system of partial differential equations, which seemed to be appropriate for a complete theoretical description of electromagnetic phenomena. It turned out that this was the case with an at his time incalculable impact to science and technology. For instance, the equations allow for solutions in the specific case of vacuum. About twenty years later the predicted existence of such electromagnetic waves was confirmed experimentally by R. Hertz. But not only from a physical point of view Maxwell's equations do constitute a historical breakthrough. Also mathematical questions concerned with Maxwell's equations have been and are still an extremly fruitful starting point for new and profound developments.

A class of challenging questions are corresponding inverse problems. Thus, we ask for information on electromagnetic media which can be gained from scattered waves measured at some distance from the object. Most inverse scattering problems considered in mathematics are ill-posed, i.e., existence, uniqueness and stability of solutions cannot be ensured in suitable function spaces. Through his monograph on Maxwell's equations Hiroshi Isozaki introduces the readers to the mathematical background of scattering theory for electromagnetic waves and recent results on uniqueness of some specific inverse scattering poblems.

As mentioned by the author the book focuses on an overview for graduate students and young researchers. It is divided into three parts. Chapters 1–3 are devoted to basic

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knowledge on vector calculus and potential theory. Subsequently, the chapters 4–8 introduce into advanced mathematical techniques for electromagnetic waves. Finally, the last two chapters illuminate the inverse scattering theory.

The book starts with some elementary facts from vector calculus. The first chapter focuses on C^∞-smooth surfaces and submanifolds. Additionally, the notation of differential forms is introduced. Some more specific formulae and concepts from calculus are shifted to an appendix. The fundamental solutions of the Laplace operator, some knowledge on harmonic functions, and Rellich's Lemma for solutions of the Helmholtz equation are presented next. A short collection of solution formulae for the wave equation like the d'Alembert-, Poisson- or Duhamel-formulae finishes the second chapter.

Single and double layer potentials for the Laplace equation are introduced next. There, jump conditions at interfaces and the mapping properties of the corresponding boundary integral operators give the functional analytic background which allow for an application of Fredholm's alternative. This so called integral equation method for solving elliptic boundary value problems is explained in some detail in case of the exterior Dirichlet problem for the Laplace equation and brings to an end the basic part of the book.

After this preparation the reader is introduced to Maxwell's equations, that is Gauß' electric and magnetic laws

$$\operatorname{div}_x D = \rho, \quad \operatorname{div}_x B = 0,$$

Ampere's law

$$\partial_t D = \operatorname{curl}_x H - J,$$

and Faraday's law

$$\partial_t B = -\operatorname{curl}_x E$$

which describe the relationships of the electric field E, the dielectric displacement D, the magnetic field H and the magnetic flux density B, if the fields are rationalized by the MKS-system. As a first observation the system implies the equation of continuity

$$\partial_t \rho = -\operatorname{div}_x J$$

for the charge density ρ and the current density J of the medium.

Throughout the book isotropic media are considered, i.e., we have the constitutive equations

$$D = \varepsilon E, \qquad B = \mu H$$

with permittivity and permeability $\varepsilon, \mu : \mathbb{R}^3 \to \mathbb{R}$. In the fourth chapter some specific aspects of Maxwell's equations are presented in case of vacuum, i.e., ε and μ are positiv constants. Using Kirchhoff's formula solutions of the corresponding initial value problem in the whole space are given. Additionally, the choice of potential representations under the Coulomb and the Lorentz gauge is explained. Some short remarks are added with respect to the behavior of solutions of Maxwell's equations at interfaces of different media. An existence result for solutions to the Dirichlet boundary value problem in electrostatics finishes the chapter.

A main ingredient in the mathematical theory of Maxwell’s equations is the Helmholtz decomposition or even more specific the Hodge decomposition of differentiable vector fields based on the observation that any smooth vector field in a domain can be decomposed by $V = \nabla\varphi + \mathrm{curl} A$. This is explained in the next chapter which focuses mainly on topological aspects of the Hodge theory. The first Betti number of a surface and the first order cohomology group are introduced. Finally, this leads to the isomorphism between the cohomology group and the space of Neumann harmonic vector fields.

The next topic is the initial value problem for the free Maxwell equations in vacuum. Some remarks on the Fourier transformation, on distributions, and on Sobolev spaces prepare for the algebraic view on the time evolution. In detail it is shown how the Fourier transformation in space applied to Maxwell’s equations leads to a representation of the solution of the initial value problem by matrix exponentials. Additionally, the extension of the approach in case of smooth isotropic media is explained.

Chapter 7 is devoted to the initial boundary value problem for Maxwell’s equations in the exterior of a smooth domain with homogeneous Maxwell boundary condition. The approach requires spectral properties of the free Maxwell operator as well as for the Maxwell operator of the scattering problem. These are presented in detail including an introduction to the Sobolev spaces $H(\mathrm{curl}, \Omega)$ and $H(\mathrm{div}, \Omega)$ and traces in these spaces to establish the domains of definition of the operators. The corresponding stationary problem is considered in the next chapter. By the limiting absorption principle ingoing and outgoing waves are distinguished. Thus, by establishing the Sommerfeld radiation condition and the Silver-Müller radiation condition the asymptotic behaviour of scattered waves is analyzed. Properties of the scattering amplitude, the S-matrix as well as a corresponding far field operator are gathered, which for instance lead to a uniqueness result for the stationary scattering problem assuming the Silver-Müller radiation condition.

In preparation of the investigation on inverse scattering problems in the next chapter layer potentials, jump conditions and the corresponding boundary integral operators are considered. The extension of these operators to suitable trace spaces and mapping properties are collected. Finally it is shown that the S-matrix determines the so called E-M map and vice versa. Thus, by this result the inverse scattering problems under consideration can be reduced to problems on bounded artificial domains containing the scattering obstacles.

In the last chapter the challenging question of uniqueness of inverse scattering problems is discussed. Three different types of inverse problems are emphasized: recovery of the electric parameters, the support of perturbations of these parameters, and the Betti number of scattering obstacles. An approach, which is based on Faddeev’s Green function and so called complex geometric optics solutions, is explained to some extent and leads to uniqueness results. In 1987 the approach was used by J. Sylvester and G. Uhlmann to show uniqueness in electric impedance tomography. The presentation here concentrates on its modification in showing that the S-matrix operator in case of a fixed frequency uniquely determines an isotropic medium. Then, in the sense of a brief overview some remarks are added on known results on boundary identification, the linear sampling method, the relation to transmission eigenvalues for Maxwell's equation, and the determination of Betti number from scattering data.

In this book, the author certainly succeeded in presenting a useful collection of basics in electromagnetic scattering theory. Graduate students as well as researchers, who like to understand the challenging uniqueness results in inverse scattering theory based on complex geometric optics solutions, will find the necessary material developed in the text or cited by valuable references. Additionally, the book may also serve as a useful source for courses or seminars on the mathematics of electromagnetic waves. Since the book focuses on uniqueness questions in inverse electromagnetic scattering, the reader may not expect a complete overview on inverse scattering theory. For instance learning about the ill-posed nature of such problems or numerical aspects requires further material. However, the book introduces many mathematical aspects of electromagnetic waves and leads to a deep understanding of uniqueness questions in inverse scattering theory.

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