The L^2-Cheeger-Müller Theorem for Representations of Hyperbolic Lattices

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der KIT-Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

DISSERTATION

von

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Tag der mündlichen Prüfung:	30. Juli 2020
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Chapter 1

Introduction

The study of torsion invariants, which this thesis is devoted to, has been a driving force in the advance of modern mathematics in the last 100 years. To provide the necessary context, we aim to give a comprehensive, yet by no means exhaustive, summary of important foundational works and key results in that field. Being a prime motivation behind the development and main results of this thesis, the summary will have a strong focus on the close connection between torsion invariants and L^2-invariants, as well as the relationship between their respective *cellular*, *topological* and *analytical* versions.

At the beginning of the last century, the *primary invariants* of topological spaces, such as the fundamental group, the (co-)homology groups, as well as derived quantities such as the Betti-numbers and the Euler characteristic, had already been well-known and put to great use. A commonality of all these objects (and simultaneously a reason for their naming) is that they are *homotopy invariants*, i.e. they agree on homotopy equivalent spaces, and thus they can be used to distinguish between non-homotopy equivalent spaces. As such, they alone are however insufficient in classifying spaces up to more rigid transformations, such as homeomorphisms, diffeomorphisms or isometries. In fact, little was known at that time on elaborate methods that could distinguish between homotopy equivalent, yet non-homeomorphic spaces.

1.1 Brief history on classical torsion invariants

The first prominent result in this vein was established by Kurt Reidemeister [80] in 1935. Given a finite simplicial complex K with $\widetilde{K}$ its universal cover and a unitary representation $\rho : \pi_1(K) \to U(n)$, one can form the twisted cellular cochain complex

$$\begin{aligned} C^*(K,\rho) &\coloneqq C^*(\widetilde{K}) \otimes_\rho \mathbb{C}^n \\ \sigma.\gamma \otimes v &= \sigma \otimes \rho(\gamma)v. \end{aligned} \tag{1.1.1}$$

Together with a choice of homology bases $\mu \subseteq H^*(K,\rho)$, one can define a positive real number

$$T(K,\rho,\mu) \in \mathbb{R}_{>0}$$

nowadays called *Reidermeister-Torsion* (shortly R-torsion) in honor of its inventor. Provided that ρ is additionally *acyclic*, i.e. the twisted cohomology $H^*(K,\rho)$ vanishes (so no choice of μ is necessary), it can be computed as follows: Choosing a representative lift for a given oriented basis of cocells in $C^*(K)$, we obtain linearly independent elements in $C^*(\widetilde{K})$. Together with the canonical basis of $\mathbb{C}^n$, this set of representatives tensors up to become a complex, finite basis for the twisted complex $C^*(K,\rho)$. Choosing the unique inner product on $C^*(K,\rho)$ with respect to which this basis is orthonormal, we can further construct (combinatorial) Laplacians $\Delta_p^c : C^p(K,\rho) \to C^p(K,\rho)$ for each degree $0 \leq p \leq \dim(M)$. Since ρ was assumed to be acyclic, each Δ_p^c is a positive, invertible endomorphism, and we get an equality

$$\log(T(K,\rho)) = \frac{1}{2} \sum_{p=0}^{\dim(M)} (-1)^p p \log(\det(\Delta_p^c)). \tag{1.1.2}$$

Using R-torsion, Reidemeister was able to give a complete *combinatorial* classification of 3-dimensional lens spaces. While he defined the torsion only for complexes on lens spaces, Franz [37] generalized the definition of R-torsion to arbitrary finite simplicial complexes. In the same process, he generalized Reidemeister's classification result onto higher-dimensional lens spaces. Both authors implicitly used the important fact that the R-torsion associated to acyclic representations is a *combinatorial* invariant: Two simplicial structures of the same space admitting isomorphic subdivisions have the same acyclic R-torsion, a fact that was later proven rigorously in a more general fashion by Whitehead in a series of papers [100], [99], [98], [97]. Namely, he proved that R-torsion is invariant under a special class of homotopy equivalences, so-called *simple homotopy equivalences.* In the same body of work, he also gave the homotopy classification of lens spaces and provided examples of homotopy equivalent lens spaces with different R-torsions, thus showing that not every homotopy equivalence is simple. All of this he achieved by introducing a new torsion invariant, called the *Whitehead torsion*, which provided an effective new tool in analyzing two homotopy equivalent spaces. Lastly, he also raised the question whether all homeomorphims are simple homotopy equivalences. Combined with his previous result, this would imply that R-torsion for acyclic representations is a homeomorphism invariant.

Arguably the decade in which Whitehead torsion was used most beneficially, the 1960's began with a satisfying completion of the earlier mentioned classification results by Reidemeister, Franz and Whitehead, obtained by Brody [17] in 1960. Namely, he proved that the simple homotopy class, the combinatorial class and the homeomorphism class of lens spaces all agree. Shortly afterwards in 1961, Milnor [67] prominently applied a relative version of R-torsion to disprove the "Hauptvermutung", one of the famous topological conjectures of its time which stated that two homeomorphic finite-dimensional simplical complexes always would always have isomorphic subdivisions. He was also the one to define another torsion invariant, called the *Milnor torsion*, which would generalize Whitehead torsion even further. Thirdly (and perhaps most famously), Whitehead torsion played an important part in the proof of the *s-cobordism theorem*, shown independently by Mazur [65], Stallings [93], and Barden [5] for the category of piecewise-linear and smooth manifolds, later extended to the category of topological manifolds by Kirby and Siebenmann [49, Essay II]. It says that for $n \geq 5$, given a topological/PL/smooth inclusion $M^n \hookrightarrow N^{n+1}$ of a closed n-manifold into a compact $(n+1)$-manifold that is a homotopy equivalence, one has $N \cong M \times [0,1]$ (here, $\cong$ stands for the isomorphism in the respective category) if and only if the inclusion is a simple homotopy equivalence. Most notably, the s-cobordism theorem has as an (almost) immediate consequence the generalized Poincaré conjecture for dimensions $n \geq 5$: *Any n-manifold homotopy equivalent to the n-sphere S^n must in fact be homeomorphic to S^n.* To close off the decade, it was shown by Kirby and Siebenmann [50] in 1969 for simplicial structures on manifolds, later by Chapman [24] in 1975 for arbitrary simplicial complexes, that all homeomorphisms are simple homotopy equivalences, thereby confirming Whitehead's old conjecture that R-torsion for acyclic representations indeed is a homeomorphism invariant.

Among the most influential mathematical achievements that were brought forth in the 60's also ranks the *Atiyah-Singer index theorem*. Roughly stated, this deep result relates the index of certain elliptic differential operators defined over a manifold M to various of its topological properties. In the aftermath of the proof, more and more people became interested in discovering new sophisticated analytic quantities on manifolds that could possibly shed light on equally sophisticated topological properties.

One of the quantities discovered this way was the *analytic torsion*, first appearing in a paper by Ray and Singer [78] in 1971, where they rigorously defined it and proved some of its basic properties (therefore, it is sometimes called *Ray-Singer torsion*):

Given a closed Riemannian manifold (M, g), and a unitary representation $\rho : \pi_1(M) \to U(n)$, one obtains the flat, complex vector bundle $E^\rho \downarrow M$ over M associated to ρ, with E^ρ the total space of the bundle and M the base space (this notation will be used throughout the thesis). $E^\rho \downarrow M$ comes equipped with a canonical flat bundle metric h_ρ, since ρ is unitary. If $\Omega^\bullet(M, E^\rho)$ denotes the de Rham complex of E^ρ-valued differential forms over M, then the pair of metrics (g, h_ρ), together with the differential $d_\rho \colon \Omega^\bullet(M, E^\rho) \to \Omega^{\bullet+1}(M, E^\rho)$ give rise for each $0 \leq p \leq \dim(M)$ to the p-th *Hodge-Laplacian* $\Delta_p := (d_\rho^p)^* d_\rho^p + d_\rho^{p-1}(d_\rho^{p-1})^* \colon \Omega^p(M, E^\rho) \to \Omega^p(M, E^\rho)$. In that situation, it is due to the classic de Rham theorem that $\ker(\Delta_p)$, the space of *harmonic* p-forms, is a finite-dimensional complex vector space isomorphic to the twisted singular cohomology $H^p(M, \rho)$. Moreover, Δ_p is a non-negative, elliptic operator with discrete spectrum $\mathrm{spec}(\Delta_p)$. One can therefore at least formally define the *ζ-function* of Δ_p as the series

$$\zeta_{\Delta_p}(s) := \sum_{0 \neq \lambda \in \mathrm{spec}(\Delta_p)} \lambda^{-s}$$

for varying $s \in \mathbb{C}$. In analogy with the classical Riemannian ζ-function, this expression determines a convergent series for $\Re(s) >> 0$ sufficiently large that extends to a meromorphic function on all of $\mathbb{C}$ with 0 being a regular point. The extension is also denoted by $\zeta_{\Delta_p}(s)$. As such, the *ζ-regularized determinant* of Δ_p can now be defined as

$$\det{}^\zeta(\Delta_p) := e^{-\zeta'_{\Delta_p}(0)}$$

and the *analytic/Ray-Singer torsion* $T^{An}(M, \rho, g) \in \mathbb{R}_{>0}$ as

$$\log(T^{An}(M, \rho, g)) = \frac{1}{2} \sum_{p=0}^{\dim(M)} (-1)^p p \log(\det{}^\zeta(\Delta_p)).$$

Of course, the naming "determinant" is not without reason: It is easy to check that, replacing Δ_p by an invertible $n \times n$-matrix A, one recovers the classical determinant of A in the process previously described. Inspired by the de Rham theorem and the Atiyah-Singer-index theorem, Ray and Singer believed that for unitary representations $\rho : \pi_1(M) \to U(n)$, the resulting analytic torsion $T^{An}(M, \rho, g)$ must have a topological interpretation. Indeed, further substantiating their conjecture, they showed that, if ρ is unitary, $T^{An}(M, \rho, g)$ is always independent of the choice of g and for even-dimensional M always equal to 1. The precise statement of their conjecture is as follows: If K is a smooth simplicial structure on M, then the *integration over simplices map* determines a cochain map $I : \Omega^*(M, E^\rho) \to C^*(K, \rho)$, so that its restriction to harmonic forms, post-composed with the projection map, induces by Hodge-de Rham's theorem for each $0 \leq p \leq m$ an isomorphism $\Phi : \ker(\Delta_p) \to H^p(K, \rho)$. Choosing a basis $\mu \subseteq \ker(\Delta_*)$ of harmonic forms that is orthonormal with respect to the inner product on $\Omega^*(M, E^\rho)$ defined by g and h_ρ, we obtain by push-forward a basis $\Phi(\mu) \subseteq H^*(K, \rho)$ for twisted simplicial homology. Then Ray and Singer conjectured that

$$T^{An}(M, \rho, g) = T(K, \rho, \Phi(\mu)). \tag{1.1.3}$$

In particular, provided that ρ is acyclic, it would follow that $T^{An}(M,\rho,g)$ is a homeomorphism invariant. The conjecture was proven independently by Müller in 1978 [27], and by Cheeger in 1979 [27], and is nowadays consequently referred to as the *Cheeger-Müller theorem*.
Since then, the result has been extended and generalized in multiple different directions – there are versions for manifolds with singularities as well as for families of representations, all of which are often referred to as *Cheeger-Müller (type) theorems*. For the purpose of this thesis, we restrict our attention on the versions for manifolds with boundary and unimodular representations. The table below collects all relevant results in this vein.

Cheeger-Müller	ρ *unitary*	ρ *arbitrary & unimodular*
M *closed*	Cheeger, 1979 [27] Müller, 1978 [74]	Bismut/Zhang, 1992 [12] Müller, 1993 [73]
M *compact with boundary*	Vishik, 1987 [96] Lück, 1993 [58]	Brüning/Ma, 2013 [18]
M *non-compact*		Müller/Rochon, 2019 [68]

Observe that in the construction of both the Reidemeister torsion and the Ray-Singer torsion of a pair (M,ρ), the deck group action of $\pi_1(M)$ on the universal cover $\widetilde{M}$ is already a key ingredient. For the former torsion element, it was used to construct the relevant cellular cochain complex $C^*(M,\rho)$, while for the latter, it allowed us to construct the flat bundle $E^\rho \downarrow M$ associated to ρ. In the construction of an L^2*-torsion*, which is the subject of the next section and generalizes the ordinary torsion invariants we have introduced, the deck group action will also play an important role, although in a slightly different manner.

1.2 L^2-torsion on finite CW-complexes and compact manifolds

Let K be a finite, d-dimensional connected CW-complex, let $p : \widetilde{K} \to K$ be the universal cover and let $\Gamma \coloneqq \mathrm{deck}(p) \cong \pi_1(K)$ be the corresponding deck group. The associated cellular cochain complex $C^*(\widetilde{K})$ with integer coefficients then has the structure of a finitely generated, free $\mathbb{Z}[\Gamma]$-module cochain complex. A preferred *admissible* $\mathbb{Z}[\Gamma]$-basis $E \subseteq C^*(\widetilde{K},\rho)$ is given by a choice of lifts of oriented cocells of K, one for each Γ-orbit of cocells.
Now assume that we are given a complex, finite-dimensional representation $\rho : \Gamma \to \mathrm{GL}(V)$. This allows us to form the twisted cochain complex $C^*(\widetilde{K},\rho) \coloneqq C^*(\widetilde{K}) \otimes_{\mathbb{Z}} V$. The diagonal action of Γ on elementary tensors, given by $\gamma.(\omega \otimes v) \coloneqq (\gamma.\omega) \otimes (\rho(\gamma)\cdot v)$ intertwines with the natural $\mathbb{C}$-multiplication on the right factor, endowing $C^*(\widetilde{K},\rho)$ with the structure of a finitely-generated free $\mathbb{C}[\Gamma]$-module.
Observe that $C^*(\widetilde{K},\rho)$ differs from the cochain complex $C^*(K,\rho)$ that was defined in Equation 1.1.1 (in case that Γ is infinite, $C^*(\widetilde{K},\rho)$ is not a finite-dimensional complex vector space, unlike $C^*(K,\rho)$).
The tensor product of a basis $B \subset V$ with an admissible basis of $C^*(\widetilde{K})$ produces a $\mathbb{C}[\Gamma]$-basis $E \otimes B$ of $C^*(\widetilde{K},\rho)$, so that the Γ-orbit $\Gamma.(E \otimes B)$ forms a $\mathbb{C}$-basis for $C^*(\widetilde{K},\rho)$ (infinite whenever Γ is infinite). Equipping $C^*(\widetilde{K},\rho)$ with the unique inner product with respect to which the $\mathbb{C}$-basis $\Gamma.(E \otimes B)$ is orthonormal and taking the L^2-completion, we obtain a cochain complex of Hilbert spaces $C^*_{(2)}(\widetilde{K},\rho)$, which is in fact a *finite Hilbert* $\mathcal{N}(\Gamma)$*-module cochain complex* (cf. Definition 4.1.28). As such, the resulting boundary operators $\partial^p_\rho \colon C^p_{(2)}(\widetilde{K},\rho) \to C^{p+1}_{(2)}(\widetilde{K},\rho)$ can further be regarded as bounded morphisms of finite-dimensional Hilbert $\mathcal{N}(\Gamma)$-modules, thus admitting a *Fuglede-Kadison determinant* $\det_\Gamma(\partial^p_\rho) \in \mathbb{R}_{\geq 0}$ (see

Definition 4.1.9). Now assume that

- the representation ρ is *unimodular*, i.e. one has $|\det(\rho(\gamma))| = 1$ for each $\gamma \in \Gamma$,
- the pair (M, ρ) is *combinatorially L^2-acyclic* (shortly: c-L^2-acyclic), which means that $\ker(\partial_\rho^{p+1}) = \overline{\mathrm{im}(\partial_\rho^p)}$, and
- the pair (M, ρ) is of *combinatorial determinant class* (shortly: c-determinant class), which means that $\det_\Gamma(\partial_\rho^p) > 0$ for each $0 \leq p \leq d$.

In this case, one can define the topological L^2-torsion element $T_{(2)}^{Top}(K, \rho) \in \mathbb{R}_{>0}$ as

$$\log(T_{(2)}^{Top}(K, \rho)) = \sum_{p=0}^{\infty} (-1)^{p+1} \log(\det_\Gamma(\partial_\rho^p)). \tag{1.2.1}$$

It does not depend on the explicit choices of bases E and B that we have made (Corollary 5.2.10). Most importantly, however, it is a *homeomorphism* invariant of the pair (K, ρ): If $f : L \to K$ is a (not necessarily cellular) homeomorphism between finite CW-complexes and $\rho : \pi_1(K) \to \mathrm{GL}(V)$ is a representation as above, then we obtain a pullback representation $\rho \circ f_* : \pi_1(L) \to \mathrm{GL}(V)$ (unique up to conjugation), so that $T_{(2)}^{Top}(L, \rho \circ f_*) = T_{(2)}^{Top}(K, \rho)$. In fact, provided that the Whitehead group $\mathrm{Wh}(\Gamma)$ of Γ vanishes, $T_{(2)}^{Top}(K, \rho)$ is even a *homotopy* invariant of the pair (K, ρ). Crucially, the latter observation allows us to define a topological L^2-torsion $T_{(2)}^{Top}(X, \rho)$ for any space X that is not necessarily compact, but modeled on some finite CW-complex and satisfying $\mathrm{Wh}(\pi_1(X)) = 0$. The proofs of these statements are carried out in Sections 5.1 – 5.2.

Any smooth, compact d-dimensional manifold M admits a finite CW-structure. Therefore, given some CW-structure K on M and a unimodular, L^2-acyclic representation $\rho : \pi_1(K) \cong \pi_1(M) \to \mathrm{GL}(V)$ of determinant class, we can define the topological L^2-torsion $T_{(2)}^{Top}(M, \rho) = T_{(2)}^{Top}(K, \rho) \in \mathbb{R}_{>0}$. By what we have said before, this depends only on M and the representation ρ itself, but not on the specific choice of K. This suggests that there must be a way to compute the quantity $T_{(2)}^{Top}(M, \rho)$ without employing any CW-structures at all:

Indeed, assume that $\rho : \pi_1(M) \to \mathrm{GL}(V)$ is an arbitrary finite-dimensional, complex representation (not necessarily c-L^2-acyclic or unimodular). Then we can always form the Γ-equivariant vector bundle $\widetilde{E^\rho} := \widetilde{M} \times V \downarrow \widetilde{M}$ over the universal cover $\widetilde{M}$ of M, on the total space of which Γ acts diagonally via $\gamma.(v, x) := (\rho(\gamma)v, \gamma.x)$. Note that $\widetilde{E^\rho} \downarrow \widetilde{M}$ is precisely the pullback under the covering map $p : \widetilde{M} \to M$ of the flat, canonical bundle $E^\rho \downarrow M$ over M associated to ρ.

By acting fiberwise, this Γ-action extends in natural fashion to a $\mathbb{C}[\Gamma]$-action on the associated de Rham complex of $\widetilde{E^\rho}$-valued forms $\Omega^*(\widetilde{M}, \widetilde{E_\rho})$. Picking a Γ-invariant Riemannian metric g on M, as well as a Γ-invariant Hermitian form h on $\widetilde{E^\rho}$ further gives rise to a natural inner product structure on the subcomplex $\Omega_c^*(\widetilde{M}, \widetilde{E_\rho}) \subseteq \Omega^*(\widetilde{M}, \widetilde{E_\rho})$ of *compactly supported forms*. The choice of g and h guarantees that the complex $\Omega_{(2)}^*(\widetilde{M}, \widetilde{E_\rho})$ obtained by L^2-completion is a Hilbert space with isometric Γ-action, so that the closures of the exterior differentials $d^p : \Omega_{(2)}^p(\widetilde{M}, \widetilde{E_\rho}) \to \Omega_{(2)}^{p+1}(\widetilde{M}, \widetilde{E_\rho})$ are Γ-equivariant, densely defined operators. In fact, more is true:

1. $\Omega_{(2)}^*(\widetilde{M}, \widetilde{E_\rho})$ is a Hilbert $\mathcal{N}(\Gamma)$-module. In fact, if $\mathcal{F}$ is an arbitrary fundamental domain for the Γ-action of $\widetilde{M}$, then $\Omega_{(2)}^*(\widetilde{M}, \widetilde{E_\rho})$ is Γ-equivariantly isomorphic to the Hilbert space tensor product $L^2(\Gamma) \hat{\otimes} \Omega_{(2)}^*(\mathcal{F}, \widetilde{E^\rho}|_{\mathcal{F}})$.

2. The *Hodge-Laplacian* $\Delta_p := (d^p)^* d^p + d^{p-1}(d^{p-1})^*$ is an elliptic differential operator of order 2 (with certain boundary conditions if $\partial M \neq \emptyset$), as well as a positive, self-adjoint unbounded morphism of Hilbert $\mathcal{N}(\Gamma)$-modules.

3. For each $t > 0$, the *heat operator* $e^{-t\Delta_p} : \Omega^p_{(2)}(\widetilde{M}, \widetilde{E_\rho}) \to \Omega^p_{(2)}(\widetilde{M}, \widetilde{E_\rho})$ defined via Borel functional calculus on Δ_p is a positive, *bounded* morphism of Hilbert $\mathcal{N}(\Gamma)$-modules. Moreover, for fixed $p \in \mathbb{N}$, the assignment $t \mapsto \operatorname{tr}_\Gamma(e^{-t\Delta_p})$ determines a smooth, real-valued non-negative function in t.

Here, as everywhere else, tr_Γ denotes the *von Neumann trace*. It can be viewed as a generalization of the usual trace of positive bounded operators onto the realm of positive, bounded Γ-equivariant operators. We refer to Section 4.1 for a precise definition, as well as a list of its most important features. All this permits us to define the p-th *analytic L^2-Betti number* $b_p^{(2)} := \lim_{t\to\infty} \operatorname{tr}_\Gamma(e^{-t\Delta_p}) \in \mathbb{R}_{\geq 0}$. We say that the pair (M, ρ) is *analytically L^2-acyclic* (shortly: a-L^2-acyclic) if $b_p^{(2)} = 0$ for each $0 \leq p \leq n$.
We wish to define the L^2-Torsion of the complex $\Omega^*_{(2)}(\widetilde{M}, \widetilde{E_\rho})$ using suitable *determinants* of the Γ-equivariant Laplacians Δ_p. However, since $\widetilde{M}$ is in general non-compact, $\operatorname{spec}(\Delta_p)$ is in general not discrete, which is why a different approach than in the construction of ordinary analytic torsion is needed. Remedy comes in form of the von Neumann trace $\operatorname{tr}_\Gamma(e^{-t\Delta_p})$. Namely, with the aid of $\operatorname{tr}_\Gamma(e^{-t\Delta_p})$, we will essentially mimic the regularization procedure used in the previous section to define the ζ-regularized determinant. First, under the assumptions made on g and h, the small-time asymptotics of the function $\operatorname{tr}_\Gamma(e^{-t\Delta_p})$ are sufficiently well understood. This is why there exists $\epsilon > 0$ small, so that the formal expression $\zeta_p(s) := \int_0^\epsilon t^{s-1}(\operatorname{tr}_\Gamma(e^{-t\Delta_p}) - b_p^{(2)})dt$ determines a holomorphic function for $\Re(s) >> 0$ large that extends to a meromorphic function on all of $\mathbb{C}$ with 0 being a regular point.
On the other hand, the large-time asymptotics can potentially be complicated. We say that (M, ρ) *is of analytic determinant class* (short: a-determinant class) if the expression $\operatorname{tr}_\Gamma(e^{-t\Delta_p}) - b_p^{(2)}$ *decays sufficiently fast* as $t \to \infty$, in the sense that $\int_\epsilon^\infty t^{-1}(\operatorname{tr}_\Gamma(e^{-t\Delta_p}) - b_p^{(2)})dt < \infty$ for each $0 \leq p \leq n$. Just as the L^2-Betti numbers, the determinant class property is independent of the particular choices of g and h.
Provided that $(\widetilde{M}, \widetilde{E_\rho})$ is of a-determinant class, we can now define the L^2-analytic torsion $T^{An}_{(2)}(M, \rho, g, h) \in \mathbb{R}_{>0}$ as

$$\log(T^{An}_{(2)}(M, \rho, g, h)) := \frac{1}{2} \sum_{p=0}^{d} (-1)^{p+1} p \left(\frac{d}{ds} \zeta_p(s)|_{s=0} + \int_\epsilon^\infty t^{-1} \operatorname{tr}_\Gamma(e^{-t\Delta_p} - b_p^{(2)})dt \right). \tag{1.2.2}$$

As outlined before, the right-hand side of the equation should be understood as an alternating weighted sum of (logarithms of) ζ-regularized determinants of the Laplacians. Further, as already indicated in the notation, $T^{An}_{(2)}(M, \rho, g, h)$ *depends in general on g and h*, even if M is compact. However, in case that M is compact and odd-dimensional, the *metric anomaly*, i.e. the difference $\log(T^{An}_{(2)}(M, \rho, g, h)) - \log(T^{An}_{(2)}(M, \rho, g', h'))$ equals a sum of integral expressions over only the boundary ∂M. Assume additionally that $\chi(M) = \chi(\partial M) = 0$ (which is true whenever M is odd-dimensional with empty or toroidal boundary, for example) and that the representation ρ and the metrics h, h' are unimodular (cf. Section 5.3.1). In this case, the anomaly further reduces to an integral expression depending only on $\dim(\rho)$ and the metrics g, vanishing whenever both g and g' are product near ∂M, as shown by Ma and Zhang [60]. Its relative rigidity under metric transformations raises the suspicion that $T^{An}_{(2)}(M, \rho, g, h)$ must in fact be a topological quantity.
Many partial results in that spirit have been obtained in the past. The case when ρ is a *unitary* representation and h is the flat canonical metric associated to ρ was famously covered by Burghelea, Friedlander, Kappeller and MacDonald in [22] and [21]. The case when ρ is unimodular and M is closed was dealt

with by Zhang [102]. All of these results, however, have in common that they do not relate $T^{An}_{(2)}$ to its natural counterpart, $T^{Top}_{(2)}$. Instead, they employ the L^2-*Morse-Smale* torsion $T^{MS}_{(2)}$, a variant of the topological torsion which we will define in Section 5.3. Although probably known to the authors, they do not explicitly establish a relation between $T^{MS}_{(2)}$ and $T^{Top}_{(2)}$. This, along with generalizing the techniques of [21] to our setting, is what goes into the proof of two main theorems of this thesis, to be proven in Chapter 6:

Theorem A (Theorem 6.3.5)**.** *Let M be a compact manifold and $\rho : \pi_1(M) \to \mathrm{GL}(V)$ a complex, finite-dimensional representation, and let K be a CW-structure on M. Then there is an L^2-chain homotopy equivalence of Hilbert $\mathcal{N}(\Gamma)$-module cochain complexes $\Omega^*_{(2)}(\widetilde{M}, \widetilde{E^\rho}) \simeq C^*_{(2)}(\widetilde{K}, \rho)$. In particular*

1. *(M, ρ) is c-L^2-acyclic if and only if (M, ρ) is a-L^2-acyclic.*
2. *(M, ρ) is of c-determinant class if and only if (M, ρ) is of a-determinant class.*

Theorem B (Theorem 6.1.8)**.** *Let (M, g) be a compact, odd-dimensional, oriented Riemannian manifold with $\mathrm{Wh}(\pi_1(M)) = \{0\}$. There exists a form $B(g) \in \Omega^{n-1}(\partial M)$ depending only on the restriction of g to a neighborhood of ∂M and vanishing whenever g is a product near ∂M, so that the following holds: Let $\rho : \pi_1(M) \to \mathrm{GL}(V)$ be any complex, finite-dimensional, unimodular representation so that*

- *ρ is L^2-acyclic and of determinant class,*
- *the pair $(\partial M, \rho|_{\pi_1(\partial M)})$ is of determinant class.*

Then, for a choice of Γ-invariant, unimodular metric h on $\widetilde{E^\rho} \downarrow M$, one has

$$\log\left(\frac{T^{An}_{(2)}(M, \rho, g, h)}{T^{Top}_{(2)}(M, \rho)}\right) = \frac{1}{2}\dim(\rho)\int_{\partial M} B(g). \tag{1.2.3}$$

Remark 1.2.1. Via methods different from the ones we will employ, Theorem B has recently also been proven by Guangxiang Su in a currently unpublished paper.

Recall that under certain circumstances, there is a reasonable topological L^2-torsion for non-compact spaces. On the other hand, in our definition of analytic L^2-torsion, as well as all results mentioned so far, we have restricted our attention exclusively to compact manifolds. In what follows, we will introduce a realm of spaces, many of them non-compact, in which an analytic L^2-torsion is *always* defined: *Locally symmetric spaces.*

1.3 L^2-torsion on locally symmetric spaces

Throughout this section, we assume that we are given a **linear algebraic group G** defined over $\mathbb{Q}$, by which we mean a subgroup of $\mathrm{GL}(n, \mathbb{C})$ for some $n \in \mathbb{N}$ that is the zero locus of a set of polynomials in the n^2 variables with coefficients in $\mathbb{Q}$. We set G to be the identity component of $\mathbf{G}(\mathbb{R}) = \mathbf{G} \cap \mathrm{GL}(n, \mathbb{R})$. Then G is a real Lie group, which we assume from now on to be semi-simple without compact factors. For $K \subseteq G$ a maximal compact subgroup, the quotient space $X \coloneqq G/K$ then has the natural structure

of a non-positively curved globally symmetric space: As a smooth manifold, X is diffeomorphic to $\mathbb{R}^d$ for appropriate $d \in \mathbb{N}$ and there exists a canonical Riemannian metric g on X of non-positive sectional curvature, unique up to a positive scalar, turning the transitive action of G on X into an action by *isometries*. In the sequel, we will often employ the **fundamental rank** $\delta(G)$ of G, defined as the non-negative integer $\delta(G) \coloneqq \operatorname{rank}_{\mathbb{C}}(G) - \operatorname{rank}_{\mathbb{C}}(K) \in \mathbb{N}_0$.
Let $\rho\colon G \to \mathrm{GL}(V)$ be a complex, finite-dimensional irreducible representation. Consider the G-equivariant bundle $E^\rho \coloneqq X \times V \downarrow X$, on the total space of which G acts diagonally via $\gamma.(x,v) \coloneqq (\gamma.x, \rho(\gamma)v)$. Due to a result by Matsushima and Murakami [64, Lemma 3.1], E^ρ can be equipped with a canonical G-equivariant Hermitian metric h_ρ, unique up to a positive scalar. For each degree $0 \le p \le d$, the pair of metrics (g, h_ρ) induce on the associated E^ρ-valued de Rham complex $\Omega^*(X, E^\rho)$ Hodge-Laplacians $\Delta_p : \Omega^p(X, E^\rho) \to \Omega^p(X, E^\rho)$. Crucially, for each $t > 0$, Δ_p possesses a smooth heat kernel $e^{-t\Delta_p}(x,y) : X \times X \to \mathrm{End}(V)$. Due to G-equivariance of the pair (g, h_ρ), one has $e^{-t\Delta_p}(x,y) = e^{-t\Delta_p}(\gamma.x, \gamma.y)$ for each $\gamma \in G$. It follows that there exists a smooth, non-negative, monotonically decreasing function $H^p(\rho, t)$ in $t > 0$ which satisfies $\operatorname{tr}(e^{-t\Delta_p}(x,x)) \equiv H^p(\rho,t)$. Therefore, we can define for each $0 \le p \le d$ the non-negative real number

$$b^p(\rho) \coloneqq \lim_{t\to\infty} H^p(\rho, t). \tag{1.3.1}$$

It vanishes precisely when there are no harmonic, L^2-integrable p-forms in $\Omega^p(X, E^\rho)$. We say that ρ is *L^2-acyclic* if and only if $b^p(\rho) = 0$ for each $0 \le p \le d$.

Remark 1.3.1. With the aid of some uniform lattice $\Gamma < G$, it is easily verified that there exists a constant $\chi = \chi(G) \in \mathbb{Z}$, such that for all representations $\rho : G \to \mathrm{GL}(V)$ under consideration, one has

$$\sum_{p=0}^{d} (-1)^p b^p(\rho) = \dim(\rho) \cdot \chi. \tag{1.3.2}$$

Furthermore, in case that $\rho = \mathbb{1}_{\mathbb{C}}$ is the trivial complex representation, it was shown by Olbrich [76] that ρ is not L^2-acyclic if and only if both $\delta(G) = 0$ and d is even, in which case $b^p(\rho) > 0$ precisely when $p = d/2$. The main idea of his proof was to establish a correspondence between L^2-harmonic forms in $\Omega^*(X, E_{\mathbb{1}_{\mathbb{C}}})$ and *discrete series representations* of the ambient Lie group G. It is a classic result that there exist such representations if and only if $\delta(G) = 0$.
Together with Equation 1.3.2, we now conclude that there exists a positive constant $c > 0$, such that for any representation $\rho : G \to \mathrm{GL}(V)$, we have

$$\sum_{p=0}^{d} (-1)^p b^p(\rho) = \begin{cases} 0 & \text{if } \delta(G) \neq 0 \text{ or } d \text{ odd}, \\ (-1)^{d/2} \cdot \dim(\rho) \cdot c & \text{else}. \end{cases} \tag{1.3.3}$$

Consequently, we can conclude that a representation $\rho : G \to \mathrm{GL}(V)$ can be L^2-acyclic *only if* either $\delta(G) \neq 0$ or d is odd. In fact, the same methods applied in [76] should still be applicable in order to prove the equivalence of the two conditions for any arbitrary irreducible representation ρ, namely that ρ is L^2-acyclic *if and only if* either $\delta(G) \neq 0$ or $d =$ odd. To the author's knowledge, this is not explicitly written down anywhere in its full generality. However, partial results in that direction, which are also relevant for the results of this thesis, can be found in the literature (cf. [30]).

As we briefly explain here, analogous to the construction of the analytic L^2-torsion as in the previous paragraph, one can construct from the collection of functions $H^p(\rho, t)$ an L^2-torsion element $\tau_{(2)}(\rho) \in \mathbb{R}_{>0}$.

For this, one first needs to consider to small-time asymptotics of $H^p(\rho,t)$, which are sufficiently well-understood. Namely, it was shown by Bergeron and Venkatesh [9, Lemma 3.8] that for $t\to 0$, one has $H^p(\rho,t)\in\mathcal{O}(t^{-\dim(X)/2})$. It follows that for $s\in\mathbb{C}$ with $\Re(s)>>0$, there exists $\epsilon>0$ such the expression $\zeta^p_\rho(s):=\Gamma(s)^{-1}\int_0^\epsilon t^{s-1}(H^p(\rho,t)-b^p(\rho))dt$ determines a holomorphic function, that extends to a meromorphic function on all of $\mathbb{C}$ which is regular at 0.
However, to define an L^2-torsion element, one also needs information about the large-time asymptotics of $H^p(\rho,t)$. Assuming that these are sufficiently well-behaved, so that $\int_\epsilon^\infty t^{-1}(H^p(\rho,t)-b^p(\rho))dt<\infty$ for each $0\le p\le d$, we can define the L^2-torsion element $\tau_{(2)}(\rho)\in\mathbb{R}$ of ρ as

$$\tau_{(2)}(\rho):=\frac{1}{2}\sum_{p=0}^{\dim(X)}(-1)^{p+1}p\left(\frac{d}{ds}\zeta_\rho(s)|_{s=0}+\int_\epsilon^\infty t^{-1}(H^p(\rho,t)-b^p(\rho))dt\right),$$

and the analytic L^2-torsion of a given lattice $\Gamma<G$ as

$$T^{An}_{(2)}(\Gamma,\rho):=\exp\left(\mathrm{Vol}(\Gamma)\cdot\tau_{(2)}(\rho)\right).$$

Here, $\mathrm{Vol}(\Gamma)$ denotes the (finite) Riemannian volume of a fundamental domain $\mathcal{F}\subseteq X$ for the Γ-action on the symmetric space X. It was proven [9, Proposition 5.2] that $\tau_{(2)}(\rho)$ (and therefore also $T^{An}_{(2)}(\Gamma,\rho)$ for each $\Gamma<G$) is indeed always well-defined. In fact, using representation theory, the authors were able to provide a very explicit description of a positive number $c(\rho)>0$ depending on ρ, so that

$$\tau_{(2)}(\rho)=\begin{cases}0 & \text{if } \delta(G)\neq 1,\\ (-1)^{\frac{d-1}{2}}c(\rho) & \text{if } \delta(G)=1.\end{cases}\tag{1.3.4}$$

We now sketch how these torsion elements relate to the L^2-analytic torsion of the corresponding locally symmetric quotient spaces. For this, first observe that for each torsion-free lattice $\Gamma<G$, the bundle $E^\rho\downarrow X$ descends to a flat bundle $\Gamma\backslash E^\rho\downarrow\Gamma\backslash X$, which is precisely the flat bundle associated to the restricted representation $\rho|_\Gamma$. Furthermore, since both g and h_ρ are G-equivariant, they descend to metrics g^Γ and h^Γ_ρ on $\Gamma\backslash X$ and $\Gamma\backslash E^\rho$, respectively. It is now easily verified from the definitions that the flat bundle $(\Gamma\backslash E^\rho\downarrow\Gamma\backslash X)$ is det-L^2-acyclic if and only if ρ is det-L^2-acyclic, in which case the equality

$$T^{An}_{(2)}(\Gamma\backslash X,\rho|_\Gamma,g^\Gamma,h^\Gamma_\rho)=T^{An}_{(2)}(\Gamma,\rho)\tag{1.3.5}$$

holds.

Remark 1.3.2. The importance of $\tau_{(2)}(\rho)$ becomes apparent in the far-reaching conjecture by Bergeron and Venkatesh: Namely, given a *congruence lattice* $\Gamma<G$, one always finds a basis $B\subseteq V$, such that the free abelian group $A:=\mathbb{Z}.B\subset V$ generated by B is an *arithmetic* Γ-module, meaning that the Γ-action on V induced by ρ leaves A invariant. Now let $(\Gamma_N)_{N\in\mathbb{N}}$ be a nested sequence of finite-index congruence subgroups of Γ with trivial intersection. Since A is an arithmetic Γ-module, it follows that for each $n\in\mathbb{N}$, A is a $\mathbb{Z}[\Gamma_N]$-module. Thus, we can consider for each $0\le p\le d$ the singular homology group $H_p(\Gamma_N\backslash X,A):=H_p(C_*(X)\otimes_{\mathbb{Z}[\Gamma_N]}A)$ with coefficients in A. Observe that $H_p(\Gamma_N\backslash X,A)$ is a finitely generated abelian group, and thus splits as a direct sum of its *free* and its *torsion* part. The conjecture by Bergeron and Venkatesh [9, Conjecture 1.3] now predicts a growth estimate of the (finite) torsion part, with the asymptotic limit being the twisted L^2-torsion

$$\lim_{N\to\infty}\frac{\log|H_p(\Gamma_N\backslash X,A)_{tors}|}{[\Gamma:\Gamma_N]}=\begin{cases}0 & \text{if } 2p\neq d-1,\\ \log(T^{An}_{(2)}(\Gamma,\rho)) & \text{if } 2p=d-1.\end{cases}$$

For a detailed discussion on this conjecture, as well as proofs on partial results, we refer to the original paper [9]. For an overview on its various possible ramifications, we refer to the survery article [10] instead.

The torsion element $\tau_{(2)}(\rho)$ does not depend on the normalization constant of the Hermitian form h_ρ, and changes by the factor C^{-d} when scaling the Riemannian metric g by the factor $C > 0$. In particular, $T^{An}_{(2)}(\Gamma, \rho)$ does not depend on the normalization constants of g and h_ρ, but only on Γ and the given representation ρ. Just like in the previous instances, this suggests that there must be a topological counterpart to $T^{An}_{(2)}(\Gamma, \rho)$. Indeed, if G is a connected semi-simple Lie group with finite center and no compact factors and $\Gamma < G$ is a torsion-free lattice, the quotient manifold $\Gamma\backslash X$, although not necessarily compact, is always a CW-model for the classifying space $B\Gamma$. That is because $X \cong \mathbb{R}^d$ is contractible. Notably, however, it is not finite CW-model whenever Γ is not uniform. Regardless, it is known, cf. [4, Theorem 13.1], that $\Gamma\backslash X$ is always the interior of a compact manifold with boundary, which we denote by $\overline{\Gamma\backslash X}$. As such, a given CW-structure on $\overline{\Gamma\backslash X}$ always serves as a *finite* CW-model for $B\Gamma$.
Identifying Γ with the fundamental group of $\overline{\Gamma\backslash X}$ under the homotopy equivalent inclusion $\Gamma\backslash X \hookrightarrow \overline{\Gamma\backslash X}$, choosing a finite CW-structure on $\overline{\Gamma\backslash X}$ and some basis on the representation space V, we can form the L^2-cochain complex $C^*_{(2)}(Y, \rho)$. In this instance, Y denotes a preferred universal cover of $\overline{\Gamma\backslash X}$, equipped with the Γ-CW structure induced by the chosen CW-structure on $\overline{\Gamma\backslash X}$. Since Y is a finite Γ-CW complex, it follows that $C^*_{(2)}(Y, \rho)$ is a finite cochain complex of Hilbert $\mathcal{N}(\Gamma)$-modules. We say that the pair (Γ, ρ) is *det-L^2-acyclic* if the combinatorial complex $C^*_{(2)}(Y, \rho)$ is det-L^2-acyclic. This property neither depends on choice of basis on V nor the specific CW-structure on $\overline{\Gamma\backslash X}$. Crucially, semi-simplicity of G implies that the representation ρ must in fact be *unimodular* (see for example [73, Lemma 4.3]). As such, if (Γ, ρ) is det-L^2-acyclic, we can define the topological L^2-torsion

$$T^{Top}_{(2)}(\Gamma, \rho) = T^{Top}_{(2)}(\overline{\Gamma\backslash X}, \rho) \in \mathbb{R}_{>0}\,. \tag{1.3.6}$$

The choice of $\overline{\Gamma\backslash X}$ as the finite model of $\Gamma\backslash X$ might seem arbitrary. However, it was proven by Farrell and Jones [36, Proposition 0.10] that the *Whitehead group* $\mathrm{Wh}(\Gamma)$ (cf. Equations 5.3.5,5.3.18) of Γ vanishes. From this, it follows in fact that $T^{Top}_{(2)}(\Gamma, \rho)$ is a *homotopy invariant*, in the sense that we may have chosen *any* finite CW-model of $B\Gamma$ to define the same number $T^{Top}_{(2)}(\Gamma, \rho)$ in the above fashion (this will be explained in Definition 5.3.14).
Since we speculate the equality between the two torsion invariants $T^{An}_{(2)}$ and $T^{Top}_{(2)}$, let us first of all specify the realm of Lie groups and representations, in which both torsion invariants could be reasonably defined. For this, recall that the L^2-analytic torsion $T^{An}_{(2)}(\Gamma, \rho)$ is always defined, while the L^2-topological torsion $T^{An}_{(2)}(\Gamma, \rho)$ is only defined if the pair (Γ, ρ) is det-L^2-acyclic.
In view of Remark 1.3.1, it is therefore reasonable to restrict our attention to the case that either $\delta(G) \neq 0$ or d odd. In fact, the next result, to be proven in Section 6.6, is an almost immediate consequence of the preceding two Theorems A and B:

Corollary C. *In the above situation, suppose that ρ is L^2-acyclic and that $\Gamma < G$ is a* ***uniform*** *lattice. Then the pair (Γ, ρ) is det-L^2-acyclic. Moreover, if d is odd, we have an equality of L^2-torsion elements*

$$T^{Top}_{(2)}(\Gamma, \rho) = T^{An}_{(2)}(\Gamma, \rho). \tag{1.3.7}$$

However, if Γ is *not* uniform, no such relation is known to hold in full generality. All in all, this motivates the next conjecture:

Conjecture D. *Let G be a connected, semi-simple Lie group with no compact factors, let $K \subseteq G$ be a maximal compact subgroup and let $d := \dim(G/K)$. Suppose that either $\delta(G) \neq 0$ or that d is odd. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible, finite-dimensional, complex representation and $\Gamma < G$ a torsion-free lattice. Then the following holds:*

1. *The pair (Γ, ρ) is det-L^2-acyclic.*

2. *One has*
$$T^{An}_{(2)}(\Gamma, \rho) = T^{Top}_{(2)}(\Gamma, \rho). \tag{1.3.8}$$

Example 1.3.3. Suppose that $\rho = \mathbb{1}_{\mathbb{C}} : G \to \mathbb{C}^\times$ is the trivial representation. Then the first assertion is well-known to hold in full generality for all lattices, while the second assertion holds true for uniform lattices by the first assertion and the celebrated result in [22]. Namely, it was shown by Olbrich [76] that L^2-acyclicity of the pair $(\Gamma, \mathbb{1}_{\mathbb{C}})$ always holds whenever $\delta(G) \neq 0$ and $\Gamma < G$ is uniform. By Gaboriau's proportionality principle [38, Corollary 0.2], this results extends to arbitrary lattices in G.
Moreover, it is under the correct basis identification that the boundary operators of the corresponding L^2-cochain complex $C^*_{(2)}(\overline{\Gamma\backslash X}, \rho|_\Gamma)$ are elements of $\mathrm{Mat}(k, l, \mathbb{Z}[\Gamma])$ for appropriate integers $k, l \in \mathbb{N}$. Since Γ is *residually finite* as a finitely generated linear group by the classic result of Malcev [62], it follows from [54, Lemmas 13.6, 13.11] that any matrix in $\mathrm{Mat}(k, l, \mathbb{Z}[\Gamma])$ has Fuglede-Kadison determinant ≥ 1. In particular, the complex $C^*_{(2)}(\overline{\Gamma\backslash X}, \rho|_\Gamma)$, thus also the pair (Γ, ρ), is of determinant class as stated. We remark that the *det* ≥ 1*-property* we have employed here is much stronger than the determinant class conjecture from above. In fact, it cannot be used to prove the conjecture for arbitrary representations, since the corresponding boundary operators are in general elements of $\mathrm{Mat}(k, l, \mathbb{C}[\Gamma])$, and we find elements inside $\mathrm{Mat}(k, l, \mathbb{C}) \subseteq \mathrm{Mat}(k, l, \mathbb{C}[\Gamma])$ with arbitrarily small non-zero Fuglede-Kadison determinant.

Example 1.3.4. For $\rho = \mathbb{1}_{\mathbb{C}}$ and $G = SO(n, 1)$ with n odd, the conjecture is known to hold for *all* lattices $\Gamma < G$ by the main result of [55], established by Lück and Schick.

The last main theorem of this thesis, proven in Section 6.6 and generalizing the main result of [55], can now be stated as follows:

Theorem E. *Conjecture D holds true in full generality for $G = SO_0(n, 1)$ with $n \in \mathbb{N}$ odd.*

Remark 1.3.5. An analogous comparison result for the ordinary (i.e. non-L^2) versions of analytic and topological torsions for hyperbolic lattices was achieved very recently by Müller and Rochon in [68].

We emphasize that the proof of the above theorem strongly relies on the well-understood end structure of complete, finite-volume hyperbolic manifolds (see Section 2.3). In fact, this is also what allowed Lück and Schick to achieve their original result in [55], and a good part of this thesis will be devoted to generalizing their methods. However, aside from the geometry itself, another key ingredient of the proof is derived from the fact that the fundamental group of each such end is finitely generated free abelian. This allows us to take advantage of two recent results by Lück on finite-dimensional representations factoring over $\mathbb{Z}^d$ [52, 53], both of which play an essential part in the proof that (Γ, ρ) is always c-det-L^2-acyclic, even if Γ is not uniform.
To the author's knowledge, none of these ingredients do admit straightforward generalizations onto non-hyperbolic lattices.

Putting the results of this thesis into perspective, the table below collects all versions of the L^2-Cheeger-Müller theorem known to date.

L^2-Cheeger-Müller	ρ *unitary*	ρ *arbitrary & unimodular*
M *closed*	Burghelea et al., 1996 [22]	Burghelea et al., 2001 [20] Zhang, 2005 [102]
M *compact with boundary*	Burghelea et al., 1999 [21]	Su, 2019 (unpublished) Theorem B
M *non-compact*	Lück/Schick, 1999 [55]	Theorem E

Acknowledgements

First and foremost, I am indebted to my advisor Roman Sauer. His belief in my mathematical abilities encouraged me to start this work, while his constant assistance and patience throughout my research made it possible for me to shape and complete this work.
From the bottom of my heart, I thank Gertrud, Niklas, Ruth, Olga, Lucas, Atti, as well as my close friends for their incessant support, love and encouragement, which turned out to be invaluable in the difficult times of my studies. In no particular order, I owe my gratitude to Nicolas Bergeron, Holger, Lizhen Qin, Thomas Schick and Weiping Zhang, who have all aided me in tackling and solving some of the more intricate problems of my thesis. I thank my colleagues Caterina and Michael for proofreading parts of my thesis. I would also like to thank Alessandro, Annette, David, Georg, Leonid, Martin, Philipp and Sebastian for their many helpful comments and suggestions regarding the first version of my defense talk.
Last, but not least, I am grateful for the financial support I received from the *Deutsche Forschungsgemeinschaft* through their funding of the project "Asymptotic Invariants and Limits of Groups and Spaces", which I was a member of during my main research period undertaken at the *Karlsruhe Institute of Technology.* I also gratefully acknowledge the research grant from the *Consejo Superior de Investigaciones Cientficas*, which I received during my time abroad in Madrid at the *Instituto de Ciencias Mathemticas.*

Chapter 2

Equivariant Hermitian bundles

In this chapter, we will carefully introduce the fundamental complex, with aid of which all analytic L^2-invariants to be studied in this thesis are defined: The *twisted De Rham* complex associated to an *equivariant Hermitian bundle*. This will take up Sections 2.1 – 2.2. In Section 2.3, we introduce for $G = \mathrm{Isom}^+(\mathbb{H}^n)$, the group of orientation-preserving hyperbolic isometries, and for a given torsion-free lattice $\Gamma < G$ an exhaustion $\{M_R\}_{R\in\mathbb{R}}$ of $\mathbb{H}^n$ of complete manifolds-with-boundary. The isometric action of Γ on $\mathbb{H}^n$ leaves each M_R invariant and restricts on each M_R to a cocompact action. We will then apply the theory from the previous two sections to the finite-volume, hyperbolic manifolds $\Gamma\backslash M_R$ and $\Gamma\backslash\mathbb{H}^n$ and certain types of finite-dimensional, complex representations $\rho : \Gamma \to \mathrm{GL}(V)$ of their (common) fundamental group. As a consequence of results on uniform ellipticity of associated Hodge-Laplacians, shown in Chapter 3, we can define *analytic L^2-torsions* $T^{An}_{(2)}(\Gamma\backslash M_R, \rho)$, $T^{An}_{(2)}(\Gamma\backslash\mathbb{H}^n)$ and finally state two of the theorems, which will be proven in later chapters.

2.1 The twisted De Rham complex

We will commence by introducing some of the fundamental frameworks this thesis builds upon. Detailed introductions, discussions, and proofs of the well-established theory about to be presented can be found, for example, in [48].

Let (M, g) be an n-dimensional complete oriented Riemannian manifold, possibly with boundary. Let $E \downarrow M$ be a m-dimensional complex vector bundle over M and denote by $\Gamma(E)$ the space of smooth sections into E.

For $0 \le k \le n$, we define the space of *E-valued k-forms* over M as

$$\Omega^k(M, E) \coloneqq \Omega^k(M) \otimes_{C^\infty(M,\mathbb{R})} \Gamma(E) = \Gamma(\Lambda^k T^*M \otimes_{\mathbb{R}} E),$$

and the *twisted de Rham complex of E-valued differential forms as*

$$\Omega^\bullet(M, E) \coloneqq \bigoplus_{k=0}^{n} \Omega^k(M, E).$$

Since $\Lambda^k T^*M \otimes_{\mathbb{R}} E$ has the natural structure of a complex vector space induced by $\mathbb{C}$-multiplication on the left factor, it follows that $\Omega^\bullet(M, E)$ has the natural structure of a *graded* $C^\infty(M, \mathbb{C})$-module with the

obvious grading coming from the individual summands $\Omega^k(M,E), 0 \le k \le n$.
The wedge product on ordinary differential forms gives naturally rise to a $C^\infty(M,\mathbb{C})$-bilinear pairing

$$\wedge : \Omega^k(M,E) \times \Omega^l(M) \to \Omega^{k+l}(M,E), \qquad 0 \le k,l \le n.$$

Moreover, by intertwining the ordinary wedge product with the evaluation pairing with E and its *dual bundle* E^*, we also obtain a natural $C^\infty(M,\mathbb{C})$-bilinear pairing

$$\wedge : \Omega^k(M,E) \times \Omega^l(M,E^*) \to \Omega^{k+l}(M), \qquad 0 \le k,l \le n.$$

Observe that, if either the first or the second factor in the pairing is compactly supported, the resulting form is also compactly supported.
A **connection** on E is a $\mathbb{C}$-linear map $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes_{\mathbb{R}} E)$, satisfying the *Leibnitz-rule:*

$$\nabla(f \cdot \omega) = df \otimes \omega + f \cdot \nabla\omega, \quad \forall f \in C^\infty(M,\mathbb{R}) \text{ and } \forall \omega \in \Gamma(E).$$

Here, $df \in \Omega^1(M)$ denotes the ordinary exterior derivative of the smooth function f. A choice of connection ∇ on $E \downarrow M$ gives rise to a $\mathbb{C}$-linear operator $d_\nabla \colon \Omega^\bullet(M,E) \to \Omega^{\bullet+1}(M,E)$, that is uniquely determined by its behavior on elementary tensors: For $\sigma \in \Omega^k(M)$ and $\omega \in \Gamma(E)$, we have

$$d_\nabla(\sigma \otimes \omega) \coloneqq d\sigma \otimes \omega + (-1)^k \sigma \wedge \nabla\omega \in \Omega^{k+1}(M,E). \tag{2.1.1}$$

d_∇ is called the *covariant exterior derivative* induced by ∇.

Definition 2.1.1. ∇ is called a **flat** connection if $d_\nabla^2 = 0$, i.e. if d_∇ takes the form of a differential on $\Omega^\bullet(M,E)$. A complex vector bundle $E \downarrow M$, equipped with a fixed flat connection is called a **flat bundle**.

Since it will always be made clear from the context which derivative is used, we simplify the notation and drop the subscript ∇ from the covariant exterior derivative d_∇, from now on.
The differential d canonically induces a *dual differential* $d^* \colon \Omega^\bullet(M,E^*) \to \Omega^{\bullet+1}(M,E^*)$, which is completely determined by the following *Leibnitz-identity:*

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge d^*\sigma, \qquad \forall \omega \in \Omega^k(M,E) \text{ and } \forall \sigma \in \Omega^l(M,E^*). \tag{2.1.2}$$

In this instance, $d(\omega \wedge \sigma)$ denotes the ordinary exterior derivative of the $\mathbb{C}$-valued differential form $\omega \wedge \sigma$.
If $f : N \to M$ is a smooth embedding between smooth manifolds, and if $f^*(E) \downarrow N$ denotes the (smooth) pullback bundle of $E \downarrow M$ over N, then any connection ∇ on $E \downarrow M$ pulls back to a connection $f^*\nabla$ on the pullback bundle $f^*(E) \downarrow N$. This connection is uniquely determined by the identity

$$(f^*\nabla)_X (f^*s) = f^*\left(\nabla_{df(X)} s\right), \qquad \forall s \in \Gamma^\infty(E). \tag{2.1.3}$$

Here, $f^*(s) \in \Gamma(f^*(E))$ denotes the smooth pullback section induced by a section $s \in \Gamma(E)$, $X \in \Gamma(TN) = \Omega^0(N)$ is a vector field over N and $df : TN \to TM$ denotes the differential induced by f. This determines a connection over $f^*(E)$, since *every* section in $\Gamma(f^*(E))$ is the pullback under f of some section in $\Gamma(E)$. Assuming that ∇ is flat, the above equation implies that the pullback connection $f^*\nabla$ gives rise to a degree-1 differential $d_{f^*\nabla}$ on the complex $\Omega^*(M, f^*(E))$, so that f^* extends to a $\mathbb{C}$-linear map between *cochain complexes*

$$\begin{aligned} &f^* \colon (\Omega^\bullet(M,E), d_\nabla) \to (\Omega^\bullet(N, f^*(E)), d_{f^*\nabla}), \\ &f^*(d_\nabla \omega) = d_{f^*\nabla} f^*(\omega) \quad \forall \omega \in \Omega^\bullet(M,E). \end{aligned} \tag{2.1.4}$$

Now let g be a Riemannian metric on M. Together with the chosen orientation on M, it gives rise to the classical Hodge $*$-operator $*_g$, a bundle isomorphism $*_g : \Lambda^k T^*M \to \Lambda^{n-k}T^*M$ for each $0 \leq k \leq n$ uniquely determined by the identity

$$v \wedge *_g w = \langle v, w \rangle_{g(x)} \cdot d\mu_g(x), \quad v, w \in (\Lambda^k T^*M)_x$$

for each $x \in M$. $\langle \ , \ \rangle_{g(x)}$ denotes the inner product on the fiber $(\Lambda^\bullet T^*M)_x$ induced by the metric $g(x)$ on TM, while $d\mu_g \in \Omega^n(M)$ denotes the volume form on M induced by g and the fixed orientation on M.

By letting it act on the first factor, $*_g$ extends for each $k \in \mathbb{N}$ to two smooth bundle isomorphisms $\Lambda^k T^*M \otimes E \cong \Lambda^{m-k}T^*M \otimes E$ and $\Lambda^k T^*M \otimes \overline{E^*} \cong \Lambda^{m-k}T^*M \otimes \overline{E^*}$, and therefore to two $C^\infty(M, \mathbb{C})$-linear isomorphisms $*_g : \Omega^\bullet(M, E) \to \Omega^{m-\bullet}(M, E)$ and $*_g : \Omega^\bullet(M, \overline{E^*}) \to \Omega^{m-\bullet}(M, \overline{E^*})$ (in both cases, we use by slight abuse of notation the same symbol). Here, as everywhere else in the paper, $\overline{E^*}$ denotes the *conjugate dual bundle* of E.

Our intermediate goal is to define a *formal adjoint* of the differential operator d, which can be established, in the case that E is the trivial complex line bundle over M, purely by means of the previously defined *Hodge $*$-operator*. For general flat complex vector bundles, however, we will also need a way to *canonically* identify E-valued forms with $\overline{E^*}$-valued forms. This is done via means of a fixed *Hermitian form* $h \in \Gamma(\mathrm{GL}_\mathbb{C}(E, \overline{E^*}))$. Recall that a section $h \in \Gamma(\mathrm{GL}_\mathbb{C}(E, \overline{E^*}))$ is called an Hermitian form (or Hermitian metric), if for each $x \in M$, the isomorphism of complex vector spaces

$$h_x \colon E_x \overset{\cong}{\to} \overline{E^*_x} \tag{2.1.5}$$

is conjugate-symmetric, i.e. $h_x(v)(w) = \overline{h_x(w)(v)}$ for any pair $v, w \in E_x$, and non-degererate, i.e. $h_x(v)(v) \neq 0$ for $0 \neq v \in E_x$. Equipped with h, we call $(E, h) \downarrow M$ an *Hermitian bundle* (whenever clear from the context, h will be left out from the notation and we simply write $E \downarrow M$). In the obvious manner, the isomorphism described in 2.1.5 induces a degree-0 $C^\infty(M, \mathbb{C})$-linear isomorphism between graded modules

$$\#_h : \Omega^\bullet(M, E) \to \Omega^\bullet(M, \overline{E^*}).$$

Note that, however, since h is in general not parallel with respect to the chosen connection ∇, $\#_h$ is in general **not** a map between cochain complexes, i.e. one generally does **not** have $d^* \circ \#_h = \#_h \circ d$. Nevertheless, we obtain a bundle isomorphism that further gives rise to a $C^\infty(M, \mathbb{C})$-linear isomorphism

$$\# \colon = *_g \circ \#_h = \#_h \circ *_g : \Omega^\bullet(M, E) \to \Omega^{m-\bullet}(M, \overline{E^*}),$$

called the *Hermitian Hodge $*$-operator* on $(E, h) \downarrow (M, g)$. The underlying bundle isomorphism is uniquely determined by the identity

$$v \wedge \# w = \langle v, w \rangle_{h(\pi(v))} \cdot d\mu_g(x), \quad v, w \in (\Lambda^k T^*M \otimes_\mathbb{R} E)_x \tag{2.1.6}$$

for each $x \in M$. Here, as everywhere else in this paper, $\langle \ , \ \rangle_{h(x)}$ denotes the inner product on the fiber $(E \otimes \Lambda^\bullet T^*M)_x$ induced by $g(x)$ and $h(x)$, where g is implicit and left out from the notation.

Finally, we can define for each $0 \leq k \leq n$ a $\mathbb{C}$-valued pairing on compactly supported forms

$$\langle \ , \ \rangle : \Omega^k_c(M, E) \times \Omega^k_c(M, E) \to \mathbb{C}, \tag{2.1.7}$$

$$\langle \nu, \sigma \rangle := \int_M \nu \wedge \# \sigma = \int_M \langle \nu(x), \sigma(x) \rangle_{h(x)} d\mu_g(x). \tag{2.1.8}$$

Just like for the trivial bundle, one verifies that this pairing is $\mathbb{C}$-linear in the first argument, conjugate-symmetric and positive definite. Therefore, it defines an inner product on each module of *compactly supported k-forms*

$$\Omega_c^k(M,E) := \{\omega \in \Omega^k(M,E) : \overline{\operatorname{supp}(\omega)} \text{ compact}\}.$$

Endowing the subcomplex of compactly supported forms $\Omega_c^\bullet(M,E) := \bigoplus_{k=0}^n \Omega_c^k(M,E)$ with the direct sum inner product structure, we obtain a cochain complex of pre-Hilbert spaces. This further allows us to define the Hilbert space

$$\Omega_{(2)}^\bullet(M,E) = \bigoplus_{k=0}^n \Omega_{(2)}^k(M,E), \tag{2.1.9}$$

obtained by L^2-completion of $\Omega_c^\bullet(M,E)$ with respect to the inner product previously defined. In order to set it apart from the other summands, we denote the *space of L^2-sections* into E by

$$L^2(E) := \Omega_{(2)}^0(M,E).$$

Also, if there two or more pairs of metrics (g,h) at play, we will denote the corresponding L^2-de-Rham complex by $\Omega_{(2)}^\bullet(M,E,g,h)$.
As announced, the Hodge $*$-operator induced from g and h is also used to construct a *formal adjoint* δ to d, defined as the following degree -1 differential:

$$\delta := (-1)^{\bullet+1}\#^{-1} \circ d^* \circ \# \colon \Omega^\bullet(M,E) \to \Omega^{\bullet-1}(M,E).$$

Since M might have non-empty boundary, we do **not** have $\langle d\omega, \sigma\rangle = \langle \omega, \delta\sigma\rangle$ for arbitrary compactly supported forms that do not vanish at the boundary. To deal with that issue, we start by considering the boundary inclusion map $i : \partial M \to M$ and define for *any* bundle $E \downarrow M$ the *restriction bundle* $E|_{\partial M} \downarrow \partial M$ simply to be the pullback $i^*E \downarrow \partial M$. Note that a metric h on E pulls back to a metric on $i^*E = E|_{\partial M}$, which we fittingly denote by $h|_{\partial M}$. As in 2.1.4, there is a flat connection on $E|_{\partial M} \downarrow M$, whose induced differential on $\Omega^*(\partial M, E|_{\partial M})$ is also denoted by d, so that the inclusion map i naturally gives rise, as in 2.1.4, to a degree-0 $\mathbb{C}$-linear map of cochain complexes $i^* : \Omega^\bullet(M,E) \to \Omega^\bullet(\partial M, E|_{\partial M})$. This means that

$$d \circ i^* = i^* \circ d. \tag{2.1.10}$$

i^* is called the *tangential boundary projection*. It is a pseudo-differential operator of order 1/2.
Lastly, observe that the pair of restricted metrics $g|_{\partial M}$ and $h|_{\partial M}$ gives rise to a Hodge -operator $\hat{\#}$: $\Omega^\bullet(\partial M, E|_{\partial M}) \to \Omega^{n-1-\bullet}(\partial M, \overline{E^*}|_{\partial M})$ and to an inner product $\langle\ .\ ,\ .\ \rangle$ on $\Omega^*(\partial M, E|_{\partial M})$. Using Stokes' theorem, one can now easily verify the following:

Lemma 2.1.2. *Let $0 \le k \le n$ and suppose that $\omega \in \Omega^{k-1}(M,E)$ and $\sigma \in \Omega^k(M,E)$ are forms, so that either ω or σ is compactly supported. Then*

$$\langle d\omega, \sigma\rangle = \langle \omega, \delta\sigma\rangle + \langle i^*\omega, \hat{\#}^{-1} i^* \# \sigma\rangle.$$

Proof. Stokes' theorem says that for any compactly supported $n-1$-form $\omega \in \Omega_c^{n-1}(M)$, one has

$$\int_M d\omega = \int_{\partial M} i^*\omega. \tag{2.1.11}$$

It is also easily verified that the tangential boundary projection respects wedge products, i.e. we have for any bundle $E \downarrow M$ and any pair of differential forms $\omega \in \Omega^k(M,E), \sigma \in \Omega^l(M, \overline{E^*})$ the equality

$$i^*(\omega \wedge \sigma) = i^*\omega \wedge i^*\sigma \in \Omega^{k+l}(\partial M). \tag{2.1.12}$$

Using 2.1.2, 2.1.11 and 2.1.12, one now computes

$$\begin{aligned}\langle d\omega,\sigma\rangle &= \int_M d\omega\wedge\#\sigma = \int_M d(\omega\wedge\#\sigma) + (-1)^k\int_M \omega\wedge d^*\#\sigma\\ &= \int_{\partial M} i^*\omega\wedge i^*\#\sigma + \int_M \omega\wedge\#\,\overbrace{(-1)^k\#^{-1}d^*\#}^{=\delta}\,\sigma\\ &= \int_{\partial M} i^*\omega\wedge\hat{\#}\hat{\#}^{-1}i^*\#\sigma + \langle\omega,\delta\sigma\rangle = \langle i^*\omega,\hat{\#}^{-1}i^*\#\sigma\rangle + \langle\omega,\delta\sigma\rangle.\end{aligned}$$

$\square$

From now on, we fix two smooth submanifolds $\partial_1 M$ and $\partial_2 M$ of ∂M (either of which could be empty), so that ∂M decomposes as the topological disjoint union $\partial M = \partial_1 M\dot{\cup}\partial_2 M$. With respect to the corresponding differentials induced by the flat connection on $E\downarrow M$, this yields a natural isomorphism *of cochain complexes*

$$\Omega^\bullet(\partial M, E|_{\partial M}) \cong \Omega^\bullet(\partial_1 M, E|_{\partial_1 M}) \oplus \Omega^\bullet(\partial_2 M, E|_{\partial_2 M}).$$

Here, the right-hand direct sum is orthogonal with respect to the induced pair of metrics $g|_{\partial M}$ and $h|_{\partial M}$. Throughout the rest of the thesis, $\partial_1 M$ constitutes the part of the boundary on which *relative (Dirichlet)* boundary conditions are imposed, and $\partial_2 M$ the part on which we impose *absolute (Neumann)* boundary conditions. For $j = 1,2$, denote by $i_j : \partial_i M \to M$ the respective smooth inclusion maps and by $i_j^* : \Omega^\bullet(M,E)\to\Omega^\bullet(\partial_i M, E|_{\partial_i M})$ the tangential boundary projections. We define subspaces of compactly supported forms satisfying certain *boundary conditions*

$$\begin{aligned}&\Omega^\bullet(M,\partial_1 M,E) \coloneqq \{\omega\in\Omega_c^\bullet(M,E)\colon i_1^*\omega = 0\},\\ &\Omega^\bullet(M,\partial_2 M,E) \coloneqq \{\omega\in\Omega_c^\bullet(M,E)\colon i_2^*\#\omega = 0\},\\ &\Omega^\bullet(M,\partial M,E) \coloneqq \{\omega\in\Omega_c^\bullet(M,E)\colon i_1^*\omega = i_1^*\delta\omega = i_2^*\#\omega = i_2^*\#d\omega = 0\}\end{aligned}$$

and (graded) linear maps

$$d_1 = d_1^\bullet \coloneqq d|_{\Omega^\bullet(M,\partial_1 M,E)}\colon \Omega^\bullet(M,\partial_1 M,E)\to\Omega_c^{\bullet+1}(M,E), \tag{2.1.13}$$

$$\delta_1 = \delta_1^\bullet \coloneqq \delta|_{\Omega^\bullet(M,\partial_2 M,E)}\colon \Omega^\bullet(M,\partial_2 M,E)\to\Omega_c^{\bullet-1}(M,E), \tag{2.1.14}$$

$$\Delta = \Delta_\bullet \coloneqq \delta_1 d_1 + d_1\delta_1 : \Omega^\bullet(M,\partial M,E)\to\Omega_c^\bullet(M,E). \tag{2.1.15}$$

Observe that 2.1.10 shows that, in fact, we have both $\mathrm{im}(d_1)\subseteq\Omega^\bullet(M,\partial_1 M,E)$ and $\mathrm{im}(\delta_1)\subseteq\Omega^\bullet(M,\partial_2 M,E)$. We regard the operators 2.1.13–2.1.15 as unbounded, densely defined operators over the L^2-completion $\Omega_{(2)}^\bullet(M,E)$.

Lemma 2.1.3. *The unbounded, densely defined operators $d_1,\delta_1,\Delta\colon\Omega_{(2)}^\bullet(M,E)\to\Omega_{(2)}^\bullet(M,E)$ are closable. Moreover, the closure of the operator Δ is symmetric.*

Proof. Since the proof is completely analogous for all operators under consideration, we will only show that d_1 is closable. For that purpose, let $\omega_n\in\Omega_c^k(M,\partial_1 M,E)$ be a sequence satisfying both $\lim_{n\to\infty}\omega_n = 0$ and $\lim_{n\to\infty} d_1\omega_n = \sigma\in\Omega_{(2)}^{k+1}(M,E)$. To show that d_1 is closable, we must verify that $\sigma = 0$. For that purpose, let $x\in\Omega_c^{k+1}(M,E)$ be a $k+1$-form with $x\equiv 0$ on a neighborhood $U\supset\partial M$. Then, applying Lemma 2.1.2 and L^2-continuity of the inner product, it follows that

$$\langle\sigma,x\rangle = \lim_{n\to\infty}\langle d_1\omega_n,x\rangle = \lim_{n\to\infty}\langle\omega_n,\delta x\rangle = 0.$$

Since the subset of compactly supported forms vanishing on a neighborhood of the boundary is L^2-dense in $\Omega^\bullet_{(2)}(M,E)$, it follows that $\langle \sigma, y\rangle = 0$ for all $y \in \Omega^\bullet_{(2)}(M,E)$, hence $\sigma = 0$ as desired.
Using Lemma 2.1.2, it is also easily verified that Δ is symmetric. Since closures of symmetric operators are always again symmetric, the second assertion follows. $\square$

Remark 2.1.4. The (closure of) the operator Δ is called the **Bochner-Laplace operator** on $\Omega^\bullet(M,E)$. In case that $\partial M \neq \emptyset$, we say that Δ satisfies **relative**, respectively **absolute boundary conditions** if $\partial M = \partial_1 M$, respectively if $\partial M = \partial_2 M$. Often, we will employ the symbol E of the total space in the notation and write $\Delta[E]$ instead of Δ. This will be helpful in order to distinguish between two or more Laplacians arising from distinct Hermitian bundles.

From now on, as a consequence of the previous lemma, we can, and will, identify each of the respective unbounded operators with their *minimal closure* inside $\Omega^\bullet_{(2)}(M,E)$.
We want to retrieve meaningful numerical quantities from these operators. First, equipped with (the minimal closure of) the differential d_1, we call $\Omega^\bullet_{(2)}(M,E)$ the **L^2-de-Rham cochain complex** of $E \downarrow M$. Although not apparent in the notation, note that $\Omega^\bullet_{(2)}(M,E)$ depends on the choice of Riemannian metric g on M, the choice of Hermitian form h on $E \downarrow M$, as well as on the decomposition $\partial M = \partial_1 M \dot\cup \partial_2 M$ of the boundary. For notational convenience, these quantities are mostly left out from the notation, but will be included whenever it becomes necessary. Also, observe that since $d^2 = 0$, we have $\mathrm{im}(d_1) \subseteq \ker(d_1)$. Moreover, since d_1 is a closed operator, $\ker(d_1^k) \subseteq \Omega^k_{(2)}(M,E)$ is a closed subspace for each $0 \leq k \leq n$, so that we have

$$\overline{\mathrm{im}(d_1^{k-1})} \subseteq \ker(d_1^k). \tag{2.1.16}$$

This permits the next definition:

Definition 2.1.5 (L^2-de-Rham cohomology)**.** For $0 \leq k \leq n$, the k-th **L^2-de-Rham cohomology** is defined as the quotient Hilbert space

$$H^k_{(2)}(M,E) = \ker(d_1^k)/\overline{\mathrm{im}(d_1^{k-1})}. \tag{2.1.17}$$

As for the L^2-de-Rham complex, if there are more than one pair of metrics (g,h) at play, we will denote the corresponding L^2-de-Rham cohomology by $H^k_{(2)}(M,E,g,h)$. In general, i.e. if M is non-compact, $H^k_{(2)}(M,E)$ need not be a finite-dimensional complex vector space. However, the main focus of this thesis will be bundles with the property that $H^k_{(2)}(M,E)$ has finite *von Neumann dimension* (see 4.1.3).
To derive this, a key concept we will take advantage of are so-called *smoothing* operators:
If $E \downarrow M$ is a Hermitian bundle, then its induced homomorphism bundle is defined as the complex vector bundle

$$\hom\left(\pi_2^*(E), \pi_1^*(E)\right) \downarrow M \times M,$$

where $\pi_i : M \times M \to M$ denotes the projection onto the i-th factor for $i = 1,2$. The fiber $\hom(\pi_2^*(E), \pi_1^*(E))_{(x,y)}$ at a point $(x,y) \in M \times M$ is precisely the space $\hom(E_y, E_x)$ of $\mathbb{C}$-linear homomorphisms from E_y to E_x. Observe that any Hermitian form h on $E \downarrow M$ extends naturally to an Hermitian form on $\hom(\pi_2^*(E), \pi_1^*(E)) \downarrow M \times M$. With respect to this form, we can and will regard $\hom(\pi_2^*(E), \pi_1^*(E)) \downarrow M \times M$ also as an Hermitian bundle.

Definition 2.1.6 (Smoothing operator)**.** Let $E \downarrow M$ be a Hermitian bundle. A bounded operator $A : L^2(E) \to L^2(E)$ is called **smoothing** if $A(\Gamma_c(E)) \subseteq \Gamma^\infty(E) \cap L^2(E)$ and if there exists a smooth, integrable section

$$A(x, y) \in \Gamma^\infty(\hom(\pi_2^*(E), \pi_1^*(E))) \cap L^2(\hom(\pi_2^*(E), \pi_1^*(E)),$$

such that, for each $\phi \in \Gamma_c(E)$ and each $x \in M$, we have

$$A\phi(x) = \int_M A(x, y)\phi(y)dy.$$

The section $A(x, y)$ is called the (integral) **kernel** of the operator A.

In the next chapter, we will introduce the notion of a bundle *of bounded geometry*. The class of such bundles is quite large, including for example all Hermitian bundles $E \downarrow M$ over compact Riemannian manifolds M and all *lifts* $\widetilde{E} \downarrow \widetilde{M}$ thereof (here, $\widetilde{M}$ is an arbitrary covering space of the compact manifold M). As two main results, we will obtain the following:

Theorem 2.1.7. *Let $E \downarrow M$ be a flat Hermitian bundle that is the lift of a flat Hermitian bundle $\hat{E} \downarrow \hat{M}$ over a compact normal quotient $\hat{M}$ of M (under the corresponding covering map and with lifted Hermitian metric). Then, in the notation established as above, we have for each $0 \leq k \leq n$, that*

1. *The (closed) unbounded operator $\Delta_k \colon \Omega^k_{(2)}(M, E) \to \Omega^k_{(2)}(M, E)$ is positive and self-adjoint.*
2. *Let $f \in \mathcal{B}(\mathbb{R}^+)$ be a rapidly-decreasing Borel function, that is, we have for all $j \in \mathbb{N}_0$, that*

$$\sup_{\lambda \in \mathbb{R}^+} |\lambda^j \cdot f(\lambda)| \leq C_j$$

for some constant $C_j \geq 0$. Furthermore, let

$$f(\Delta_k) \colon \Omega^\bullet_{(2)}(M, E) \to \Omega^\bullet_{(2)}(M, E)$$

be the bounded operator defined via Borel functional calculus of the self-adjoint Δ_k. Then $f(\Delta_k)$ is a smoothing operator.

3. *One has $d_1^* = \delta_1$ and $\delta_1^* = d_1$, i.e. the Hilbert-space adjoint of the exterior derivative d_1 is precisely the (closure of) the formal adjoint δ_1, and vice versa. In particular, since both d_1 and δ_1 are closed and densely defined, one has*

$$\ker(d_1)^\perp = \overline{\operatorname{im}(\delta_1)}, \tag{2.1.18}$$

$$\ker(\delta_1)^\perp = \overline{\operatorname{im}(d_1)}. \tag{2.1.19}$$

4. *For each $0 \leq k \leq n$, we have the orthogonal **Hodge-decomposition***

$$\Omega^k_{(2)}(M, E) = \ker(\Delta_k) \oplus \overline{\operatorname{im}(d_1^{k-1})} \oplus \overline{\operatorname{im}(\delta_1^k)} \overset{3}{=} \ker(\Delta_k) \oplus \overline{\operatorname{im}(d_1^{k-1})} \oplus \ker(d_1^k)^\perp. \tag{2.1.20}$$

5. *With respect to the above decomposition, the Laplacian Δ_k decomposes as the orthogonal direct sum (of unbounded operators)*

$$\Delta_k = 0 \oplus d_1^{k-1}\delta_1^{k-1} \oplus \delta_1^k d_1^k. \tag{2.1.21}$$

Moreover, assertions 1 *and* 2 *are also true for the operator* $(\Delta_k)^\perp$*, which is defined as the restriction of* Δ_k *onto* $\ker(\Delta_k)^\perp$.

Proof. Assertions 1–4 are proven in Theorem 3.4.1, Proposition 3.4.2 and Proposition 3.4.6. To see that assertions 1 and 2 hold also for $(\Delta_k)^\perp$, we only need to observe that

$$(\Delta_k)^\perp = \Delta_k - \chi_{\{0\}}(\Delta_k), \tag{2.1.22}$$

and that by assertion 2, $\chi_{\{0\}}(\Delta_k)$ is a bounded, self-adjoint, smoothing operator. □

Corollary 2.1.8. *Let* $E \downarrow M$ *be a flat bundle satisfying the assumptions of Theorem 2.1.7. Then, for each* $0 \leq k \leq n$*, we have* $\ker(\Delta_k) \subseteq \ker(d_1^k)$*. Moreover, the restriction of the canonical projection* $\pi : \ker(d_1^k) \to H^k_{(2)}(M, E)$ *onto* $\ker(\Delta_k)$ *is an isometry of Hilbert spaces.*

2.2 Flat bundle isometries

Until the end of this section, we will use the results from Theorem 2.1.7. They allow us to effectively compare the Laplace operators of two flat bundles of bounded geometry.
For that purpose, let (M, g) and (M', g') be two Riemannian manifolds and let $(E, h) \downarrow M$ and $(E', h') \downarrow M'$ be Hermitian bundles. A bundle map

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \big\downarrow{\scriptstyle \pi_E} & & \big\downarrow{\scriptstyle \pi_{E'}} \\ M & \xrightarrow{f} & M' \end{array}$$

is called a **bundle isomorphism**, if

1. $f : M \to M'$ is a diffeomorphism, and
2. for any $x \in M$, $F_x \coloneqq F|_{E_x}$ is a linear isomorphism between the vector spaces E_x and $E'_{f(x)}$.

To simplify and streamline notation, we will from now on identify $M \subseteq E$ with the zero section and simply write $E \downarrow M \xrightarrow{\text{F}} E' \downarrow M$ for a bundle isomorphism. Each such bundle isomorphism naturally induces an isomorphism $F^* \colon \Omega^k(M', E') \to \Omega^k(M, E)$ for each $0 \leq k \leq n$, defined on elementary tensors $\omega \otimes s \in \Omega^k(M') \otimes_{\mathbb{C}^\infty(M', \mathbb{C})} \Gamma(M', E') = \Omega^k(M', E')$ via

$$F^*(\omega \otimes s)(x) \coloneqq Df^*(\omega)(x) \otimes F_x^{-1} \cdot s(F(x)).$$

Observe that F^* identifies $\Omega^k_c(M, \partial M; E)$ with $\Omega^k_c(M', \partial M'; E')$.
If $f = F|_M$ is additionally a Riemannian isometry between M and M', and $F_x : (E_x, h_x) \to (E'_{F(x)}.h'_{F(x)})$ is a linear isometry for each $x \in M$, then $E \downarrow M \xrightarrow{\text{F}} E' \downarrow M'$ is called a **bundle isometry**. Note that, if F is a bundle isometry, the induced map F^* satisfies for each $\sigma \in \Omega^k_c(M', \partial M'; E')$

$$\int_M ||F^*(\sigma(x))||_{h_x} d\mu_g(x) = \int_{M'} ||\sigma(x)||_{h'_x} d\mu_{g'}(x).$$

Hence, F^* extends to an isometry between L^2-completions $F^*\colon \Omega^k_{(2)}(M',E') \to \Omega^k_{(2)}(M,E)$.
Lastly, suppose that both $E \downarrow M$ and $E' \downarrow M'$ come equipped with flat connections, giving rise to degree-1 differentials $d_E : \Omega^\bullet(M,E) \to \Omega^{\bullet+1}(M,E)$ and $d_{E'}\colon \Omega^\bullet(M',E') \to \Omega^{\bullet+1}(M',E')$ as explained above. A bundle isomorphism $E \downarrow M \xrightarrow{\mathrm{F}} E' \downarrow M'$ is called **flat** if

$$d_E \circ F^* = F^* \circ d_{E'},$$

i.e. if F^* extends to a morphism of complexes $F^*\colon \Omega^\bullet(M',E') \to \Omega^\bullet(M,E)$. Equivalently, F is flat if and only if

$$\nabla_E(F^*(\omega)) = F^*(\nabla_{E'}\omega)$$

for any $\omega \in \Gamma(E')$, where $\nabla_E, \nabla_{E'}$ are the connections inducing d_E and $d_{E'}$ (as in 2.1.1).
Let $(E,h) \downarrow M \xrightarrow{\mathrm{F}} (E',h') \downarrow M$ be a flat bundle isometry between flat Hermitian bundles. For $0 \le k \le n$, denote by $\Delta_k[E]$, respectively $\Delta_k[E']$, the p-th Bochner Laplace operator on $\Omega^k(M,\partial M;E)$, respectively $\Omega^k(M',\partial M';E')$. Furthermore, let $\phi : \mathbb{R}^+ \to \mathbb{C}$ be a rapidly decreasing Borel function. Then the respective operators $\phi(\Delta_k[E])$ and $\phi(\Delta_k[E'])$, defined via Borel functional calculus, have integral kernels $\phi(\Delta_k[E'])(x,y)$ and $\phi(\Delta_k[E])(x,y)$ by Theorem 2.1.72

Proposition 2.2.1. *In the above situation, we have for any pair $x,y \in M$ the equality*

$$\phi(\Delta_k[E'])(F(x),F(y)) = F_x \cdot \phi(\Delta_k[E])(x,y) \cdot F_y^{-1}.$$

In particular, we have

$$\operatorname{tr}(\phi(\Delta_k[E])(x,x)) = \operatorname{tr}(\phi(\Delta_k[E'])(F(x),F(x))),$$
$$||\, \phi(\Delta_k[E])(x,y)\, || = ||\, \phi(\Delta_k[E'])(F(x),F(y))\, ||.$$

Here, as everywhere else in this paper, tr *denotes the complex trace of finite-dimensional endomorphisms and $||\ .\ ||$ denotes the norm on the bundles* $\hom(\pi_2^*(E),\pi_1^*(E)) \downarrow M \times M$*, respectively* $\hom(\pi_2^*(E'),\pi_1^*(E')) \downarrow M' \times M'$*, induced by the pair of Hermitian forms h and h'.*

Proof. We prove the result only for $k = 0$, the methods employed here can easily be extended to higher degrees. Denote by d_E, respectively $d_{E'}$, the differential on $\Omega^\bullet(M,E)$, respectively $\Omega^\bullet(M',E')$, induced by the flat connections. Similarly, denote by $\#_E : \Omega^\bullet(M,E) \to \Omega^{m-\bullet}(M,\overline{E^*})$, respectively by $\#_{E'}\colon \Omega^\bullet(M',E') \to \Omega^{m-\bullet}(M',(E')^*)$, the isomorphism as defined in Equation 2.1, induced by the corresponding Riemannian metric and Hermitian form.
By assumption, F is a flat isomorphism, therefore $F^* \circ d_{E'} = d_E \circ F^*$. Secondly, the fact that F is a bundle isometry implies that $\#_E \circ F^* = F^* \circ \#_{E'}$. Taken together, we obtain that $\Delta_0[E] \circ F^* = F^* \circ \Delta_0[E']$, which, by the spectral theorem, further implies that

$$\phi(\Delta_0[E]) \circ F^* = F^* \circ \phi(\Delta_0[E']).$$

Now let $\omega \in \Omega^0(M',\partial M';E')$ be arbitrary. Then, for any $x \in M$, we compute

$$\begin{aligned}&\int_{M'} \phi(\Delta_0[E])(x,F^{-1}(z)) \cdot F^{-1}_{F^{-1}(z)} \cdot \omega(z)dz = \int_M \phi(\Delta_0[E])(x,y) \cdot F_y^{-1} \cdot \omega(F(y))dy \\ &= (\phi(\Delta_0[E]) \circ F^*\omega)(x) = (F^* \circ \phi(\Delta_0[E'])\omega)(x) = \int_{M'} F_x^{-1} \cdot \phi(\Delta_0[E'])(F(x),z) \cdot \omega(z)dz.\end{aligned}$$

Fixing $y \in M$ and a vector $v \in E'_{F(y)}$, we can choose a sequence of smooth functions $(\omega_m)_{m\in\mathbb{N}} \subseteq \Omega_0(M', \partial M'; E')$ that converge as distributions to $\delta_{F(y)} \cdot v$, where $\delta_{F(y)}$ is the Dirac delta function, centered at $F(y)$. Then, using the Transformation formula, the previous equation implies that for any $x \in M$, we have

$$F_x^{-1} \cdot \phi(\Delta_0[E'])(F(x), F(y)) \cdot v = \lim_{m\to\infty} \int_{M'} F_x^{-1} \cdot \phi(\Delta_0[E'])(F(x), z) \cdot \omega_m(z) dz$$
$$\lim_{m\to\infty} \int_{M'} \phi(\Delta_0[E])(x, F^{-1}(z)) \cdot F_{F^{-1}(z)}^{-1} \cdot \omega_m(z) dz = \phi(\Delta_0[E])(x, y) \cdot F_y^{-1} \cdot v.$$

Since v and y were chosen arbitrarily, the result follows. $\square$

Definition 2.2.2. Let (M, g) be a Riemannian manifold, let $(E, h) \downarrow M$ be a flat Hermitian bundle over M and let $G \subseteq \mathrm{Isom}(M, g)$ be a subgroup. If $(E, h) \downarrow M$ is equipped with an G-action by flat bundle isometries that extends the isometric G-action on M, we say that the bundle $(E, h) \downarrow M$ is **G-equivariant** and the group G is **compatible with** $E \downarrow M$.

In the following, we will fix a subgroup $G < \mathrm{Isom}(M, g)$ of isometries on (M, g).

Definition 2.2.3. Let (M, g) be a Riemannian manifold and let $(E, h) \downarrow M$ a flat Hermitian bundle. We say that $(E, h) \downarrow M$ is of **trace class** if

$$\sup_{x\in M} || \operatorname{tr}(e^{-t\Delta_k[E]}(x, x)|| < \infty \tag{2.2.1}$$

for each $t > 0$ and each $0 \leq k \leq n$. If $(E, h) \downarrow M$ is additionally G-equivariant for some subgroup $G < \mathrm{Isom}(M, g)$, we can define for any lattice $\Gamma < G < \mathrm{Isom}(M, g)$ the **Γ-regularized trace** of $e^{-t\Delta_k[E]}$ as

$$\operatorname{tr}_\Gamma(e^{-t\Delta_k[E]}) := \int_{\mathcal{F}} \operatorname{tr}(e^{-t\Delta_k[E]}(x, x)) d\mu_g(x),$$

where $\mathcal{F}$ is an arbitrary fundamental domain for the Γ-action on M and $d\mu_g$ is the volume element on M induced by g.

Observe that if G contains a *uniform* lattice Γ, then $(E, h) \downarrow M$ is automatically of trace class. Namely, we can choose in that case a compact Γ-fundamental domain $\mathcal{F} \subset M$ and obtain from Proposition 2.2.1, that $\sup_{x\in M} || \operatorname{tr}(e^{-t\Delta_k[E]}(x, x))|| = \sup_{x\in\mathcal{F}} || \operatorname{tr}(e^{-t\Delta_k[E]}(x, x))|| < \infty$. In particular, any flat Hermitian bundle $(E, h) \downarrow M$ over a compact manifold M is always of trace class. Furthermore, Proposition 2.2.1 also shows that the above definition of $\operatorname{tr}_\Gamma(e^{-t\Delta_k[E]})$ makes sense, i.e. does not depend on the choice of Γ-fundamental domain $\mathcal{F}$ on M.

Remark 2.2.4. We will employ the same notation tr_Γ for the *von Neumann trace*, to be introduced in Section 4.1. This will be no cause of confusion, since these two traces coincide in all instances relevant for this thesis (see Proposition 4.2.2).

2.3 The flat, canonical ρ-bundle over $\mathbb{H}^n$

For $n \in \mathbb{N}$ odd, we set $G := SO_0(n, 1)$ be and let $K := SO(n) \subseteq G$. Then K is a maximal compact subgroup of G and we can identify the quotient G/K with the n-dimensional hyperbolic space $\mathbb{H}^n$.

Conversely, we can identify G with $\mathrm{Isom}_0(\mathbb{H}^n)$, the identity component of the hyperbolic isometry group. Let $\Gamma \subseteq G$ be a non-uniform lattice. Here, as everywhere else in this paper, lattices are always assumed to be *torsion-free* (this way, the induced quotient map $\mathbb{H}^n \to \Gamma\backslash\mathbb{H}^n$ is an honest covering projection). It is well-known (see for example [8, Chapter 4] or [45] embedded within a more general context) that, associated to Γ, we then find a totally ordered set

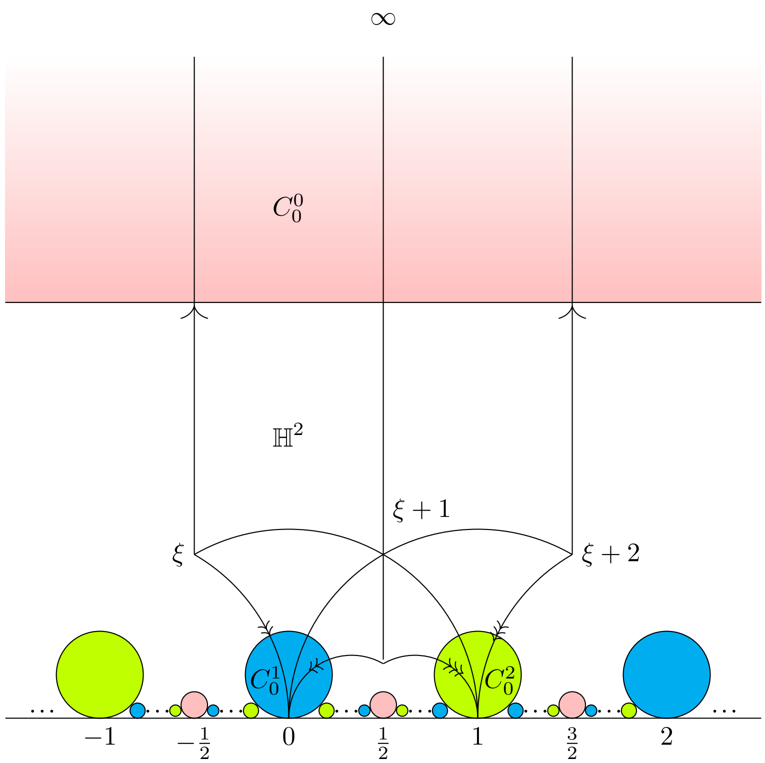
$$\{M_R \subseteq \mathbb{H}^n : R \in [0,\infty)\} \tag{2.3.1}$$

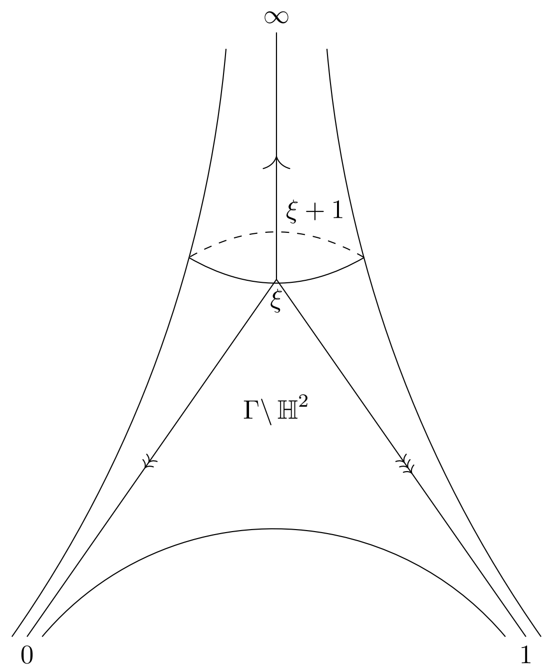
of complete Γ-invariant submanifolds of $\mathbb{H}^n$ (with $M_R \subset M_{R'}$ if $R < R'$), such that, additionally,

1. $\mathbb{H}^n = \bigcup_{R>0} M_R$,
2. Γ acts cocompactly on each M_R,
3. the complete submanifold
$$C_R \coloneqq \mathrm{clos}(\mathbb{H}^n \setminus M_R) \tag{2.3.2}$$
is also Γ-invariant. Moreover, there exists an integer $k \in \mathbb{N}$ and, for each $1 \leq j \leq k$, complete, connected submanifolds C_0^j of C_0 with $C_R^j \coloneqq C_R \cap C_0^j$ complete, connected submanifolds of C_R for each $R \geq 0$, such that the following holds:
   (a) $C_0^j \cong [0,\infty) \times \mathbb{R}^{n-1}$ under a diffeomorphism that identifies C_R^j with $[R,\infty) \times \mathbb{R}^{n-1}$. Furthermore, under the aforementioned identification, the hyperbolic metric restricted to C_0^j is of the form
   $$dt^2 + e^{-2t}dx^2, \tag{2.3.3}$$
   where dt^2 is the Euclidean metric on $[0,\infty)$ and dx^2 the Euclidean metric on $\mathbb{R}^{n-1}$.
   (b) For each $R \geq 0$, we have an equality of stabilizer subgroups $\Gamma_0^j \coloneqq \Gamma_{C_0^j} = \Gamma_{C_R^j} < \Gamma$. The action of Γ_0^j on $C_0^j \cong [0,\infty) \times \mathbb{R}^{n-1}$ is the product of the trivial action on the first factor $[0,\infty)$ and a cocompact, free, properly discontinuous action by Euclidean isometries on the second factor $\mathbb{R}^{n-1}$ of C_0^j. In particular, Γ_0^j is isomorphic to $\mathbb{Z}^{n-1}$.
   (c) For each $R \geq 0$, we have an isometric diffeomorphism of principal Γ-bundles
   $$C_R \cong \coprod_{j=1}^{k} \Gamma \times_{\Gamma_0^j} C_R^j. \tag{2.3.4}$$

Remark 2.3.1. In fact, the above decomposition of $\mathbb{H}^n$ into Γ-invariant parts still holds true if Γ is uniform (i.e. Γ acts cocompactly on $\mathbb{H}^n$) for trivial reasons. Namely, in this instance, we can simply define $M_R \coloneqq \mathbb{H}^n$ for all $R \geq 0$.

Example 2.3.2. Below left, we have sketched the decomposition of $\mathbb{H}^2$ as defined above, along with a fundamental domain for the lattice $\Gamma < \mathrm{Isom}^+(\mathbb{H}^2)$ (freely) generated by the matrices $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (the action on $\mathbb{H}^2$ is by Moebius transformations). In this instance, we have $k = 3$. The colors indicate which horoballs are identified in the quotient space $\Gamma\backslash\mathbb{H}^2$, sketched below right, which is homeomorphic to a three-holed sphere.



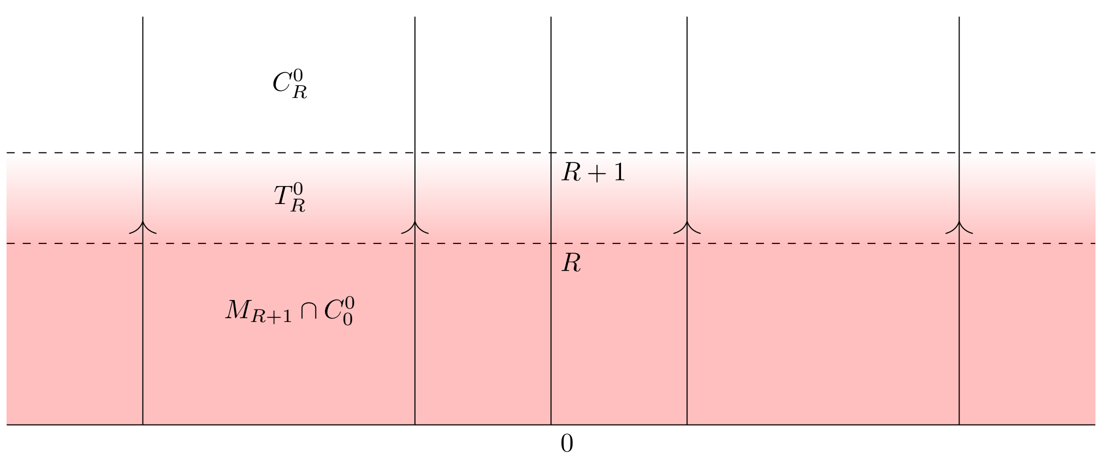


For each $R \geq 0$ and each $1 \leq j \leq k$, we further define the complete submanifolds

$$T_R \coloneqq C_R \cap M_{R+1}, \tag{2.3.5}$$

$$T_R^j \coloneqq C_R^j \cap T_R. \tag{2.3.6}$$

From the above, it follows that each T_R is Γ-invariant, and that the stabilizer of T_R^j inside Γ equals Γ_0^j. Moreover, we can identify T_R^j with $[R, R+1] \times \mathbb{R}^{n-1}$ and the hyperbolic metric correspondingly with $dt^2 + e^{-2t}dx^2$. Finally, it follows that also T_R is a principal Γ-bundle, isometrically diffeomorphic to $\coprod_{j=1}^k \Gamma \times_{\Gamma_0^j} T_R^j$.



Consider an irreducible representation $\rho : G_{\mathbb{C}} \to GL(V)$ of the complexification $G_{\mathbb{C}}$ of G on some complex, finite-dimensional vector space V. Observe that ρ gives rise to a diagonal action of G on the product $\mathbb{H}^n \times V$. Evidently, this determines an action on the vector bundle $\mathbb{H}^n \times V \downarrow \mathbb{H}^n$ by flat bundle isomorphisms, so that the projection map becomes G-equivariant (with respect to the G-actions on the base space and the total space). Here, we choose as flat connection ∇ the trivial one on $\mathbb{H}^n \times V$, defined by

$$\nabla f := \sum_{k=0}^{n} \frac{\partial f}{\partial x_i} \otimes dx_i \in \Omega^1(\mathbb{H}^n, V) \tag{2.3.7}$$

for any function $f \in C^\infty(V) = \Gamma(\mathbb{H}^n \times V)$.

Our next result is concerned with the existence of a special *Hermitian metric* h_ρ on that bundle.

Lemma 2.3.3. *There exists a distinguished Hermitian metric $h_\rho : \mathbb{H}^n \to GL(V, V^*)$, so that the resulting Hermitian bundle $(\mathbb{H}^n \times V, h_\rho) \downarrow M$ is G-equivariant (in the sense of Definition 2.2.2). The Hermitian bundle $(\mathbb{H}^n \times V, h_\rho) \downarrow \mathbb{H}^n$ is called the* ***flat, canonical*** ρ***-bundle*** *over $\mathbb{H}^n$ and is denoted by $E^\rho \downarrow \mathbb{H}^n$.*

Proof. First, consider the trivial vector bundle $G \times V \downarrow G$ and define both a left G-action and a right K-action of bundle isomorphisms on it via

$$\gamma.(g, v) := (\gamma g, v) \quad \gamma \in G,$$
$$(g, v).k := (gk^{-1}, \rho(k)v) \quad k \in K.$$

Clearly, any one action commutes with the other one. It follows that the G-action descends onto an action of bundle isomorphisms on the homogeneous quotient bundle $G \times_K V \downarrow G/K = \mathbb{H}^n$.

Moreover, since K is compact and the representation $\rho : G_{\mathbb{C}} \to GL(V)$ is assumed to be irreducible, there exists by [64, Lemma 3.1] a *canonical K-**invariant*** inner product $\langle\ ,\ \rangle$ on V, i.e. we have

$$\langle \rho(k)v, \rho(k)w \rangle = \langle v, w \rangle,$$

for any $k \in K$ and any two $v, w \in V$. Consequently, we obtain a *canonical G-**equivariant*** bundle metric $\langle\ ,\ \rangle$ on the quotient bundle $G \times_K V$ over $\mathbb{H}^n$, i.e. we have for any $p \in \mathbb{H}^n$, any $\gamma \in G$ and any pair of vectors $v, w \in (G \times_K V)_p$, that

$$\langle v, w \rangle_p = \langle \gamma \cdot v, \gamma \cdot w \rangle_{\gamma.p}.$$

Next, observe that (the trivial) bundle $\mathbb{H}^n \times V \downarrow \mathbb{H}^n$ is an obvious quotient bundle of $G \times V \downarrow G$ obtained by dividing out the K-action on the first factor. Moreover, it is easy to see that the bundle automorphism $(g, v) \mapsto (g, \rho(g)v)$ of $G \times V \downarrow G$ descends to a G-equivariant bundle isomorphism from $G \times_K V \downarrow \mathbb{H}^n$ to $E^\rho \downarrow \mathbb{H}^n$. Under this isomorphism, the canonical G-equivariant Hermitian metric on $G \times_K V \to \mathbb{H}^n$, as constructed above, pushes forward to a Hermitian bundle metric on $E^\rho \downarrow \mathbb{H}^n$, which we denote by $h_\rho : \mathbb{H}^n \to GL(V, V^*)$ and which satisfies for any $p \in \mathbb{H}^n$, any pair of vectors $v, w \in V$ and any $\gamma \in G$ the desired equality

$$\langle v, w \rangle_{h_\rho(p)} = \langle \rho(\gamma) \cdot v, \rho(\gamma) \cdot w \rangle_{h_\rho(\gamma.p)}.$$

$\square$

In other words, the action of G on E^ρ is by *flat bundle isometries.* This fact will be of central importance throughout this paper.

For $X \subseteq \mathbb{H}^n$ a complete, codimension 0 hyperbolic submanifold, we let $E^\rho_X \downarrow X$ be the Hermitian *restriction bundle* of E^ρ over X, obtained by pulling back the Hermitian bundle E^ρ through the inclusion $X \hookrightarrow \mathbb{H}^n$.

Let $\Omega^\bullet(X, E^\rho_X)$ be the de Rham complex of E^ρ_X-valued differential forms over X (with pulled-back differential and inner product). Also, we set $G_X := \{\gamma \in G : \gamma.X = X\} < G$ to be the subgroup of G leaving X invariant. We will show the following:

Lemma 2.3.4. *If X is connected, then the Hermitian bundle $E_X^\rho \downarrow X$ is G_X-equivariant.*

This is an immediate consequence of Corollary 2.3.6, which is in turn the immediate consequence of the next result:

Lemma 2.3.5. *Let $X, Y \subseteq \mathbb{H}^n$ be connected Riemannian codimension-0 submanifolds and let $f : X \to Y$ be an orientation-preserving isometry. Then there exists a global isometry $\gamma \in \mathrm{Isom}^+(\mathbb{H}^n)$, such that*

$$f = \gamma|_X.$$

Proof. Let $p \in \mathring{X}$. Since X has codimension 0, we find an open subset $U \ni p$ contained in X, which is diffeomorphic to an open subset $V \subset T_pX = T_p\,\mathbb{H}^m$ via the Riemannian exponential map $\exp_p^X = \exp_p^{\mathbb{H}^n} : T_pX \to \mathbb{H}^n$. Let $f_p^* : T_p\,\mathbb{H}^m \to T_{f(p)}\,\mathbb{H}^n$ be the differential of f at p. Since $\exp_q^{\mathbb{H}^n} : T_q\,\mathbb{H}^m \to \mathbb{H}^n$ is a diffeomorphism for any $q \in \mathbb{H}^n$, we can define a global isometry $\gamma : \mathbb{H}^n \to \mathbb{H}^n$ as

$$\gamma \coloneqq \exp_{f(p)}^{\mathbb{H}^n} \circ f_p^* \circ (\exp_p^{\mathbb{H}^n})^{-1}.$$

One now easily verifies that the subset $\{q \in X : f(q) = \gamma(q)\} \subseteq X$ is non-empty, open and closed in X. Since X is assumed to be connected, the result now follows. ☐

Corollary 2.3.6. *Let $X, Y \subseteq \mathbb{H}^n$ be two connected, codimesion 0 Riemannian submaifolds of $\mathbb{H}^n$ and let $f : X \to Y$ be an isometry. Then f extends to a flat bundle isometry $F : E_X^\rho \to E_Y^\rho$. Namely, there exists a unique element $\gamma_f \in G$, such that $f = \gamma_f|_X$ and that for any pair $(x, v) \in X \times V = E_X^\rho$, we have $F(x, v) = (\gamma_f.x, \rho(\gamma_f) \cdot v)$.*

For any $R \geq 0$, we introduce the following notational conventions

$$E_{R^-}^\rho \coloneqq E_{M_R}^\rho, \tag{2.3.8}$$

$$E_{R^+}^\rho \coloneqq E_{C_R}^\rho, \tag{2.3.9}$$

$$E_R^\rho \coloneqq E_{T_R}^\rho = E_{R^+}^\rho \cap E_{(R+1)^-}^\rho. \tag{2.3.10}$$

Here, M_R, C_R and T_R are the complete submanifolds of $\mathbb{H}^n$ as defined in Equations 2.3.1,2.3.2 and 2.3.5. As before, we let, for each $1 \leq j \leq k$, $C_0^j \cong [0, \infty) \times \mathbb{R}^{m-1}$ be a connected component of C_0, so that for any $R \geq 0$, $C_R^j \coloneqq C_0^j \cap C_R$ and $T_R^j \coloneqq T_R \cap C_0^j$ are connected components of C_R, respectively T_R.

Lemma 2.3.7. *For each $1 \leq j \leq k$ and any $R \geq 0$, the collection of hyperbolic isometries*

$$f_R^j : C_R^j \cong [R, \infty) \times \mathbb{R}^{m-1} \to C_0^j \cong [0, \infty) \times \mathbb{R}^{m-1},$$
$$(t, x) \mapsto (t - R, e^{-R}x)$$

extend to a flat bundle isometry

$$F_R : E_{R^+}^\rho \downarrow C_R \to E_{0^+}^\rho \downarrow C_0, \tag{2.3.11}$$

which induces by restriction a flat bundle isometry

$$F_R|_{E_R^\rho} : E_R^\rho \downarrow T_R \to E_0^\rho \downarrow T_0. \tag{2.3.12}$$

Proof. By Corollary 2.3.6, there exists a unique hyperbolic isometry $\gamma_R^j \in G$ extending f_R^j and a flat bundle isometry $F_R^j : E^\rho_{C_R^j} \downarrow C_R^j \to E^\rho_{C_0^j} \downarrow C_0^j$ of the form $F_R^j((t,x),v) = (\gamma_R^j.(t,x), \rho(\gamma_R^j)\cdot v)$. Notice that we have obvious identifications $\coprod_{j=1}^k \Gamma \times_{\Gamma_0^j} E^\rho_{C_0^j} \cong E^\rho_{0+}$ and likewise $\coprod_{j=1}^k \Gamma \times_{\Gamma_0^j} E^\rho_{C_R^j} \cong E^\rho_{R+}$ (as bundles over C_0, respectively C_R). Lastly, observe that the diffeomorphism

$$\begin{aligned} F_R : \coprod_{j=1}^k \Gamma \times E^\rho_{C_R^j} &\to \coprod_{j=1}^k \Gamma \times E^\rho_{C_0^j}, \\ \coprod_j (\gamma,(t,x),v) &\mapsto \coprod_j \left(\gamma(\gamma_R^j)^{-1}, \gamma_R^j.(t,x), \rho(\gamma_R^j)\cdot v\right) \end{aligned}$$

descends to a flat bundle isometry $F_R : \coprod_{j=1}^k \Gamma \times_{\Gamma_0^j} E^\rho_{C_R^j} \xrightarrow{\cong} \coprod_{j=1}^k \Gamma \times_{\Gamma_0^j} E^\rho_{C_0^j}$. The result follows. □

Let us now take advantage of these geometric results within the framework developed in the previous section. For that purpose, we consider the four L^2-cochain complexes $\Omega^\bullet_{(2)}(M_R, E^\rho_{R-})$, $\Omega^\bullet_{(2)}(C_R, E^\rho_{R+})$, $\Omega^\bullet_{(2)}(T_R, E^\rho_R)$ and $\Omega_{(2)}(\mathbb{H}^n, E^\rho)$ as defined in Section 2.1, with inner product induced by the hyperbolic metric g and the Hermitian form h_ρ constructed above, all with *absolute boundary conditions* (that is, we set $\partial M = \partial_2 M$ for $M = M_R, T_R, C_R$). Let

$$\Delta[E^\rho_{R-}] \colon \Omega^\bullet_{(2)}(M_R, E^\rho_{R-}) \to \Omega^\bullet_{(2)}(M_R, E^\rho_{R-}), \tag{2.3.13}$$

$$\Delta[E^\rho] \colon \Omega^\bullet_{(2)}(\mathbb{H}^n, E^\rho) \to \Omega^\bullet_{(2)}(\mathbb{H}^n, E^\rho). \tag{2.3.14}$$

be the respective Bochner-Laplace operators. Due to Theorem 2.1.7, it then follows that both operators are self-adjoint. In particular, for any rapidly decreasing Borel function $f \in \mathcal{B}(\mathbb{R}^+)$, the bounded operators $f(\Delta_k[E^\rho_{R-}])$ and $f(\Delta[E^\rho])$ are well-defined via Borel functional calculus and have well-defined smooth integral kernels. Combining Lemma 2.3.3, Lemma 2.3.4 and Proposition 2.2.1, we also obtain the following:

Lemma 2.3.8. *The bundle $E^\rho \downarrow \mathbb{H}^n$ is G-equivariant, so that for all $0 \le k \le n$ and any rapidly decreasing Borel function $f \in \mathcal{B}(\mathbb{R}^+)$, we have*

$$\operatorname{tr}(f(\Delta_k[E^\rho])(x,x)) = \operatorname{tr}(f(\Delta_k[E^\rho])(\gamma.x,\gamma.x)), \quad \forall x \in \mathbb{H}^n \text{ and } \forall \gamma \in G. \tag{2.3.15}$$

Moreover, for all $R > 0$, the bundle $E^\rho_{R-} \downarrow M_R$ is Γ-equivariant, so that for all $0 \le k \le n$ and all $t > 0$ we have

$$\operatorname{tr}(f(\Delta_k[E^\rho_{R-}])(x,x)) = \operatorname{tr}(f(\Delta_k[E^\rho_{R-}])(\gamma.x,\gamma.x)), \quad \forall x \in M_R \text{ and } \forall \gamma \in \Gamma. \tag{2.3.16}$$

The main takeaway from Lemma 2.3.8 is that for each rapidly-decreasing Borel function $f \in \mathcal{B}(\mathbb{R}^+)$, the bounded operators $f(\Delta_k[E^\rho_{R-}])$ and $f(\Delta_k[E^\rho])$ are both of *trace class*, as according to Definition 2.2.3. Consequently, if $\mathcal{F} \subseteq \mathbb{H}^n$ is a fundamental domain for the Γ-action on $\mathbb{H}^n$ and, for each $R \ge 0$, $\mathcal{F}_R \subseteq M_R$ a fundamental domain for the Γ-action on M_R, the Γ-*regularized heat traces*

$$\operatorname{tr}_\Gamma(e^{-t\Delta_k[E^\rho]^\perp}) = \int_{\mathcal{F}} \operatorname{tr}(e^{-t\Delta_k[E^\rho]^\perp}(x,x))\, d\mu_g(x), \tag{2.3.17}$$

$$\operatorname{tr}_\Gamma(e^{-t\Delta_k[E^\rho_{R-}]^\perp}) = \int_{\mathcal{F}_R} \operatorname{tr}(e^{-t\Delta_k[E^\rho_{R-}]^\perp}(x,x))\, d\mu_g(x) \tag{2.3.18}$$

are convergent integrals for each $t > 0$, whose respective values do not depend on the explicit choice of $\mathcal{F}$, respectively $\mathcal{F}_R$.

Observe that, since G acts transitively on $\mathbb{H}^n$ (i.e. $\mathbb{H}^n$ is a homogeneous space), the first result of Lemma 2.3.8 in fact implies the existence of a smooth function $H_\rho(t) : \mathbb{R}_{>0} \to \mathbb{R}$, satisfying

$$\mathrm{tr}(e^{-t\Delta_k[E^\rho]^\perp}(x,x)) \equiv H_\rho(t). \tag{2.3.19}$$

Using the *Plancherel Formula*, $H_\rho(t)$ can actually be explicitly computed, as done in [72, Section 9]. The conclusion to be drawn from this observation that is relevant for our purposes is the following:

Corollary 2.3.9. *Let $\Lambda, \Gamma < \mathrm{Isom}^+(\mathbb{H}^n, g)$ be two hyperbolic lattices. Then, for all $t > 0$, we have*

$$\frac{\mathrm{tr}_\Gamma(e^{-t\Delta_k[E^\rho]^\perp})}{\mathrm{tr}_\Lambda(e^{-t\Delta_k[E^\rho]^\perp})} = \frac{\mathrm{Vol}(\Gamma)}{\mathrm{Vol}(\Lambda)}. \tag{2.3.20}$$

Here, as everywhere else, $\mathrm{Vol}(\Gamma)$ *denotes the hyperbolic volume of the quotient* $\Gamma \backslash \mathbb{H}^n$.

An important result of this paper, proven in Corollary 4.3.4, can now be stated:

Theorem 2.3.10. *For each $0 \leq k \leq n$ and $s \in \mathbb{C}$ with $Re(s) >> 0$ the integral expressions*

$$\zeta_k(s) \coloneqq \Gamma(s)^{-1} \int_0^1 t^{s-1}\, \mathrm{tr}_\Gamma(e^{-t\Delta_k[E^\rho]^\perp})\, dt, \tag{2.3.21}$$

$$\zeta_k^R(s) \coloneqq \Gamma(s)^{-1} \int_0^1 t^{s-1}\, \mathrm{tr}_\Gamma(e^{-t\Delta_k[E^\rho_{R^-}]^\perp})\, dt, \quad R \geq 0 \tag{2.3.22}$$

determine holomorphic functions, each admitting meromorphic extensions on all of $\mathbb{C}$ which are regular at 0.

Remark 2.3.11. The meromorphic extensions will also be denoted by $\zeta_k(s)$ and $\zeta_k^R(s)$, respectively.

Another key result of this paper, obtained from Proposition 4.2.11 and Corollary 4.2.18, is as follows:

Theorem 2.3.12. *For each $0 \leq k \neq n$, we have*

$$\int_1^\infty t^{-1}\, \mathrm{tr}_\Gamma(e^{-t\Delta_k[E^\rho]^\perp}) dt < \infty.$$

Similarly, for all $R > 0$, we have

$$\int_1^\infty t^{-1}\, \mathrm{tr}_\Gamma(e^{-t\Delta_k[E^\rho_{R^-}]^\perp}) dt < \infty.$$

We will follow the strategy developed in [55] for the case of the trivial bundle (i.e. the bundle $E_\rho \downarrow \mathbb{H}^n$ associated to the trivial representation $\rho : G \to \mathbb{C}$) and show that it extends to the general case that we are concerned with here. The integrals from Theorem 2.3.10 will be investigated in Section 4.3, while the integrals from Theorem 2.3.12 will be the main focus of Section 4.2. In each of the previously mentioned sections, the key results will be extracted from a thorough inspection of the asymptotic behavior of $\mathrm{tr}_\Gamma(e^{-t\Delta_k[E^\rho]^\perp})$ and $\mathrm{tr}_\Gamma(e^{-t\Delta_k[E^\rho_{R^-}]^\perp})$ for *small time* $t \to 0$ (Section 4.3), respectively for *large time* $t \to \infty$ (Section 4.2).
The respective methods involved in the inspection will actually be quite distinct, since the small time asymptotics depend only on the *local geometry* of $\mathbb{H}^n$, while for the large time asymptotics, the large scale geometry of the quotients $\Gamma \backslash M_R$, $\Gamma \backslash \mathbb{H}^n$ comes into play.

As a consequence of Theorems 2.3.10 and 2.3.12, we can finally define the **analytic L^2-torsion** $T^{An}_{(2)}(\Gamma\backslash M_R,\rho)$ and $T^{An}_{(2)}(\Gamma\backslash\mathbb{H}^n,\rho)$ of the Hermitian bundles $E^\rho_{R^-}\downarrow M_R$ and $E^\rho\downarrow\mathbb{H}^n$ as

$$\log\left(T^{An}_{(2)}(\Gamma\backslash\mathbb{H}^n,\rho)\right) \coloneqq \sum_{k=0}^{n}\frac{k}{2}(-1)^{k+1}\left(\frac{d}{ds}\zeta_k(s)|_{s=0}+\int_1^\infty t^{-1}\operatorname{tr}_\Gamma(e^{-t\Delta_k[E^\rho]})dt\right) \tag{2.3.23}$$

$$\log\left(T^{An}_{(2)}(\Gamma\backslash M_R,\rho)\right) \coloneqq \sum_{k=0}^{n}\frac{k}{2}(-1)^{k+1}\left(\frac{d}{ds}\zeta^R_k(s)|_{s=0}+\int_1^\infty t^{-1}\operatorname{tr}_\Gamma(e^{-t\Delta_k[E^\rho_{R^-}]})dt\right). \tag{2.3.24}$$

Observe that from 2.3.19, it actually follows that there exists a number $\tau(\rho)\in\mathbb{R}$ depending only on the representation ρ, such that for any lattice $\Gamma<\operatorname{Isom}^+(M,g)$, one has

$$\log\left(T^{An}_{(2)}(\Gamma\backslash\mathbb{H}^n,\rho)\right)=\operatorname{Vol}(\Gamma)\cdot\tau(\rho).$$

For a detailed description of the element $\tau(\rho)$, we refer again to [72, Section 9]. The two main results of Chapter 4, Theorems 4.2.21 and 4.3.7, can now be summarized:

Theorem 2.3.13. *For each $0\le k\le n$, one has*

$$\lim_{R\to\infty}\frac{d}{ds}\zeta^R_k(s)|_{s=0}=\frac{d}{ds}\zeta_k(s)|_{s=0}, \tag{2.3.25}$$

$$\lim_{R\to\infty}\int_1^\infty t^{-1}\operatorname{tr}_\Gamma(e^{-t\Delta_k[E^\rho_{R^-}]})dt=\int_1^\infty t^{-1}\operatorname{tr}_\Gamma(e^{-t\Delta_k[E^\rho]})dt. \tag{2.3.26}$$

In particular

$$\lim_{R\to\infty}T^{An}_{(2)}(\Gamma\backslash M_R,\rho)=T^{An}_{(2)}(\Gamma\backslash\mathbb{H}^n,\rho). \tag{2.3.27}$$

Remark 2.3.14. The quantities $T^{An}_{(2)}(\Gamma\backslash M_R,\rho)$ and $T^{An}_{(2)}(\Gamma\backslash\mathbb{H}^n,\rho)$ both depend on the pair of metrics (g,h_ρ), which is why we will often write $T^{An}_{(2)}(\Gamma\backslash\mathbb{H}^n,\rho,g,h_\rho)$, respectively $T^{An}_{(2)}(\Gamma\backslash\mathbb{H}^n,\rho,g,h_\rho)$, to emphasize this dependency.

Chapter 3

Analysis on bundles of bounded geometry

Throughout this chapter, we will investigate in Sobolev spaces of sections over certain Riemannian manifolds M and metric bundles $E \downarrow M$ over them, so-called manifolds/bundles *of bounded geometry.* These generalize bundles over compact manifolds. On the basis of the classic theory around uniformly elliptic differential operators over compact manifolds, one derives that the Hodge-Laplacians Δ_* defined over space of differential forms $\Omega^*(M, E)$ with values in certain flat bundles $E \downarrow M$ are essentially self-adjoint, even if M is not necessarily compact. This works even if M has boundary – one then has to add certain boundary conditions to the domain space of Δ_*, coming from either the Dirichlet, Neumann, or from mixed boundary conditions on the complex $\Omega^*(M, E)$ itself. Firstly, this allows us to give a coordinate-free description of the associated Sobolev spaces. Secondly, and perhaps most importantly, we can apply spectral theory to Δ_* and define for each $t > 0$ the *heat operator* $e^{-t\Delta_*}$. It is an L^2-bounded operator, defined over the L^2-completion $\Omega^*_{(2)}(M, E)$ and taking values in smooth, L^2-integrable sections. Interpreting an input function $f \in \Omega^*_{(2)}(M, E)$ as an initial assignment of heat along the closed system M, we can further interpret $e^{-t\Delta} f$ as the heat distribution on M after time t has passed. Crucially, $e^{-t\Delta}$ admits an integral kernel $e^{-t\Delta}(x, y)$, the so-called *heat kernel.* Given a manifold M and a complete, codimension 0 submanifold $N \subseteq M$, we will also compare the heat kernels coming from a flat bundle $E \downarrow M$ with the ones coming from the restriction $E|_N \downarrow N$ of E to N and derive pointwise estimates. These comparison results will be fundamental for the convergence results of the next chapter.

3.1 Bundles of bounded geometry

For the sequel, we will denote for $m \in \mathbb{N}$ by $\mathbb{R}^n_{\geq 0} := \{(x_1, \ldots, x_m) \in \mathbb{R}^n : x_m \geq 0\}$ the **upper half-space** of $\mathbb{R}^n$.

Definition 3.1.1. [Normal coordinates on a manifold with boundary] Let (M, g) be a complete Riemannian manifold. For $r > 0$ and $x_0 \in M$, we say that x_0 **admits r-normal coordinates** if either

1. $x_0 \in M \setminus \partial M$ and the Riemannian exponential map $\exp^M_{x_0} : \mathbb{R}^n \cong T_{x_0}M \to M$ maps *the Euclidean ball* $B_r(0) \subseteq \mathbb{R}^n$ diffeomorphically onto its image, denoted by $N(r, x_0)$, or

2. $x_0 \in \partial M$ and the boundary exponential map $\partial \exp^M_{x_0} : T_{x_0}\partial M \times [0, r) \to M$, defined via

$$\partial \exp^M_{x_0}(v,t) := \exp^M_{\exp^{\partial M}_{x_0}(v)}(t\mu(\exp^{\partial M}_{x_0}(v))). \tag{3.1.1}$$

maps *the Euclidean cylinder* $B_r(0) \times [0, r) \subseteq \mathbb{R}^n_{\geq 0}$ (with $B_r(0) \subset \mathbb{R}^{n-1}$ the $m-1$-dimensional Euclidean ball) diffeomorphically onto its image, also denoted by $N(r, x_0)$. Here, $\exp^{\partial M}$ is the exponential map of the Riemannian submanifold $(\partial M, g|_{\partial M})$ and μ is the inward unit normal field.

From now on, we will denote such a normal chart at a point $x_0 \in M$ simply by $\kappa_{x_0} : \mathbb{R}^n_{\geq 0} \supset U_{x_0} \to N(x_0, r_{x_0})$.

Throughout, we will use the letter $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n_0$ for a general **multi-index** (of size n) and set its *length* to be

$$|\alpha| := \sum_{j=1}^{n} \alpha_i \in \mathbb{N}_0.$$

Further, for an open subset $O \subseteq \mathbb{R}^n_{\geq 0}$ and an integer $K \in \mathbb{N}$, we set $C^K(O, \mathbb{F}^m)$ to be the space of all K-times continuously differentiable vector fields over O. For a function $f \in C^K(O, \mathbb{F}^m)$ and a multi-index α with $|\alpha| \leq K$, we define

$$\partial^\alpha f := \partial^{\alpha_1}_{x_1} \ldots \partial^{\alpha_m}_{x_m} f \ \in C^{K-|\alpha|}(O, \mathbb{F}^m). \tag{3.1.2}$$

Also, for $f \in C^0(O, \mathbb{F}^m)$, we define its ∞-norm

$$||f||_\infty := \sup_{x \in O} |f(x)|. \tag{3.1.3}$$

In this instance, $|f(x)|$ denotes the norm of the vector $f(x) \in \mathbb{F}^m$ induced by the standard orthonormal basis of $\mathbb{F}^m$.

Definition 3.1.2 (Uniformly bounded sets of functions)**.** Let $n, m \in \mathbb{N}$ and let I be an index set. For each $i \in I$, let $O_i \subseteq \mathbb{R}^n_{\geq 0}$ be an open subset and $f_i \in C^0(O_i, \mathbb{F}^m)$ be a continuous function. For $K \in \mathbb{N}_0$, we say that the *set of functions* $U := \{f_i : i \in I\}$ is K**-uniformly bounded** if $f_i \in C^K(O_i, \mathbb{F}^m)$ and if there exist a universal upper bound for the ∞-norm of all f_i and all their partial derivatives up to order K. More explicitly, this means that there exists a constant $C_K > 0$, such that for each non-negative integer $k \leq K$, one has

$$\sup_{i \in I} \sup_{|\alpha| = k} ||\partial^\alpha f_i||_\infty < C_K. \tag{3.1.4}$$

U is ∞**-uniformly bounded** if it is K-uniformly bounded for each $K \in \mathbb{N}$.

Definition 3.1.3 (Manifold of bounded geometry)**.** A Riemannian manifold (M, g) is said to be **of bounded geometry** if there exists constants $R_I, R_C > 0$, such that each of the following conditions hold:

(1) The injectivity radius of ∂M is bounded from below by R_C.

(2) The *geodesic collar*

$$\begin{aligned} \partial M \times [0, R_C) &\to M, \\ (x,t) &\mapsto \exp_x^M(t\mu(x)) \end{aligned}$$

is a diffeomorphism onto its image, denoted by N. As before, μ denotes the inward unit normal field on ∂M. For $0 < q \leq 1$, we set $N_q \subseteq N$ to be the image of $\partial M \times [0, q \cdot R_C)$ under the geodesic collar map.

(3) Each $p \in M \setminus N_{2/3}$ admits R_I-normal coordinates.

(4) The set of functions consisting of all Riemannian metric tensors and their inverses induced by g, pulled back via sufficiently small normal normal charts, is ∞-uniformly bounded.

$$\begin{tikzcd} U_p \arrow[r, "\kappa_p"] \arrow[dd, dashed] & N(p,r) \arrow[d, "g"] \\ & TM \otimes TM^*|_{N(p,r)} \arrow[d, "\tilde{\kappa}_p \otimes \tilde{\kappa}_p^*"] \\ \mathbb{R}^{n^2} & N(p,r) \times \mathbb{R}^{n^2} \arrow[l, "pr_2"'] \end{tikzcd}$$

Here, $\tilde{\kappa}_p : TM|_{N(p,r)} \to N(p,r) \times \mathbb{R}^n$ denotes the local trivialization of the tangent bundle TM over $N(p,r)$ that is naturally induced by the smooth normal chart κ_p, and $r \leq \min\{2R_C, R_I\}$.

Although not necessary for this particular paper, we remark that there is an equivalent, coordinate-free characterization for a Riemannian manifold (M, g) to be of bounded geometry. Namely, one may replace condition (4) by uniform bounds on the covariant derivative of the curvature tensor and the second fundamental form on M with respect to the Levi-Civita connection induced by g, compare for example with [25, Page 33] or [84, Definition 3.1] (the equivalence to the above definition was shown in [85, Proposition 3.7, Appendix A]). For the sequel, we introduce the letter $\mathbb{F}$, which will stand for both the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.

Definition 3.1.4 (Bundle of bounded geometry)**.** Let $(E,h){\downarrow}M$ be a m-dimensional metric $\mathbb{F}$-vector bundle over a Riemannian manifold (M, g) of bounded geometry, with h either a Riemannian metric, for $\mathbb{F} = \mathbb{R}$, or an Hermitian metric, for $\mathbb{F} = \mathbb{C}$. Let $\pi : E \to M$ denote the projection map, and let I be an index set. Further, let $R > 0$ and assume that $\{x_i\}_{i \in I}$ is a set of points, each of which admits R-normal coordinates. A set $\mathcal{P} \coloneqq \{t_i : N(x_i, R) \times \mathbb{F}^m \to \pi^{-1}(N(x_i, R)) : i \in I\}$ of local trivializations, covering all of $E \downarrow M$, is called *bounded*, if both of the following two properties are satisfied:

1. The corresponding set of all transition functions between overlapping trivializations, regarded in normal coordinates, is ∞-uniformly bounded.

$$\begin{tikzcd} U_{x_i} \times \mathbb{F}^m \supseteq \kappa_{x_i}^{-1}(N(x_i,R) \cap N(x_j,R)) \times \mathbb{F}^m \arrow[r, "\kappa_{x_i} \times \mathbb{1}"] \arrow[dd, dashed] & N(x_i,R) \cap N(x_j,R) \times \mathbb{F}^m \arrow[d, "t_i^{-1}"] \\ & E|_{N(x_i,R) \cap N(x_j,R)} \arrow[d, "t_j"] \\ U_{x_j} \times \mathbb{F}^m \supseteq \kappa_{x_j}^{-1}(N(x_i,R) \cap N(x_j,R)) \times \mathbb{F}^m & N(x_i,R) \cap N(x_j,R) \times \mathbb{F}^m \arrow[l, "\kappa_{x_j}^{-1} \times \mathbb{1}"'] \end{tikzcd}$$

2. The corresponding set of all metric tensors and their inverses induced by h, pulled back via the trivializations and regarded in normal coordinates, is ∞-uniformly bounded.

$$\begin{tikzcd}
U_{x_i} \arrow[r, "\kappa_{x_i}"] \arrow[dd, dashed] & N(x_i, R) \arrow[d, "h"] \\
& E \otimes \overline{E^*}|_{N(x_i,R)} \arrow[d, "t_i \otimes {t_i}^*"] \\
\mathbb{F}^{m^2} & N(x_i, R) \times \mathbb{F}^{m^2} \arrow[l, "pr_2"']
\end{tikzcd}$$

If $E \downarrow M$ comes also equipped with a flat connection, we say that the covering $\mathcal{P}$ is *flat*, if

1. for any $i \in I$, the pullback $t_i^* \nabla$ of the flat connection ∇ on $E \downarrow M$ is the trivial connection on $N(x_i, R) \times \mathbb{F}^m \downarrow N(x_i, R)$, and

2. every transition function between two overlapping trivializations of $\mathcal{P}$ is locally constant.

$(E, h) \downarrow M$ is a **(flat) bundle of bounded geometry** if it admits a bounded (and flat) set $\mathcal{P}$ of trivializations.

Example 3.1.5. 1. Every compact Riemannian manifold M and every (flat) metric $\mathbb{F}$-bundle $E \downarrow M$ over it are of bounded geometry. This is easily verified, taking a finite cover $\{U_i\}$ of M, so that $E|_{U_i}$ is trivial (and flat), and (with the aid of Lebesgue's lemma) choosing an appropriate $R > 0$, so that each $x \in M$ admits R-normal coordinates and each normal chart of size R lies in some U_i.

2. From this, it also follows that, if (M, g) is a non-compact Riemannian manifold admitting a uniform (i.e. cocompact) lattice $\Gamma < \mathrm{Isom}(M, g)$, then (M, g) is of bounded geometry. If, moreover, $E \downarrow M$ is a (flat) metric $\mathbb{F}$-bundle over M that is Γ-equivariant, then $E \downarrow M$ is a (flat) metric bundle of bounded geometry.

3. If (M, g) is a Riemannian manifold of bounded geometry, then for any $m \in \mathbb{N}$, the trivial $\mathbb{F}$-bundle $M \times \mathbb{F}^m \downarrow M$ with trivial flat connection and constant (canonical) metric is a flat bundle of bounded geometry over M.

4. If (M, g) is a Riemannian manifold of bounded geometry, then its tangent bundle $TM \downarrow M$, with obvious metric and trivializations given by normal charts, is a bundle of bounded geometry.

5. The class of bundles of bounded geometry over a fixed Riemannian manifold (M, g) is closed under all common algebraic operations, including (but not limited to) taking duals, Whitney sums, tensor products and exterior powers.

3.2 Sobolev spaces

Our goal for this subsection is to define Sobolev spaces on bundles $E \downarrow M$ of bounded geometry. Naively, we would like to take the same local-to-global approach as it is available for flat Hermitian bundles over compact manifolds: Choose *some* boundedly flat trivialization $\mathcal{P}$ of $E \downarrow M$ of bounded geometry and

define the Sobolev norm on compactly supported (or measurable) sections simply in the usual fashion, via passing to Euclidean charts, taking the standard Sobolev norm there, and patching everything together using an appropriate partition of unity (cf. [51, Chapter III, 2]).
However, unlike in the case for compact manifolds, a different choice of trivialization $\mathcal{P}'$ might result in a non-equivalent norm. Fortunately, we do not have to steer too far away from this naive approach, since there is always a class so-called *admissible* trivializations on a bundle of bounded geometry that all give rise to equivalent norms:

Lemma 3.2.1 (Admissible triple)**.** *Let $E \downarrow M$ be a (flat) bundle of bounded geometry. Then there exists a constant $R_E > 0$, such that for any $r \in (0, R_E]$, we find a countable subset $\{b_i\}_{i\in\mathbb{Z}} \subset M$, along with*

1. *a bounded (and flat) set $\{t_i : N(b_i, r) \times \mathbb{F}^m \to \pi^{-1}(N(b_i, r)) \colon i \in \mathbb{Z}\}$ of trivializations,*
2. *associated normal coordinate charts $\{\kappa_i : \mathbb{R}^n_{\geq 0} \supset U_i \overset{\cong}{\to} N(b_i, r) \colon i \in \mathbb{Z}\}$, and*
3. *a smooth partition of unity $\{\psi_i \in C^\infty(M) \colon i \in \mathbb{Z}\}$ with $\operatorname{supp}(\psi_i) \subseteq N(b_i, r/2)$,*

such that all of the following additional properties are satisfied:

- *For any $s \in [r/2, r]$, the trivialization $\{t_i : N(b_i, s) \times \mathbb{F}^m \to \pi^{-1}(N(b_i, s)) \colon i \in \mathbb{Z}\}$ still covers all of $E \downarrow M$.*
- *$b_i \in \partial M$ for $i < 0$, while $N(b_i, r/2) \cap \partial M = \emptyset$ for $i \geq 0$.*
- *The underlying covering of M is uniformly locally finite: There exists some $D_E > 0$, such that for any $b \in M$, the index set $\{i \in \mathbb{Z} : \exists s < R_E \text{ with } N(b, s) \cap N(b_i, r) \neq \emptyset\}$ has cardinality at most D_E.*
- *The set of real-valued functions $\{\psi_i \circ \kappa_j : N(b_j, r/2) \cap N(b_i, r/2) \neq \emptyset\}$ is ∞-uniformly bounded.*

*The resulting sequence of triples $(t_i, \kappa_i, \psi_i)_{i\in\mathbb{Z}}$ will be called an **admissible triple** for the bundle $E \downarrow M$. The underlying, boundedly flat bundle trivialization $(t_i)_{i\in\mathbb{Z}}$ is called an **admissible trivialization**.*

Proof. See [85, Lemma 3.22] for the elementary, but technical proof. ☐

Suppose that an admissible triple $(t_i, \kappa_i, \psi_i)_{i\in\mathbb{Z}}$ for a bundle $E \downarrow M$ of bounded geometry is fixed. Then, for any smooth section $f \in \Gamma(E)$ and any $i \in \mathbb{Z}$, we obtain a smooth vector field

$$\begin{aligned} &f_i \in \Gamma(\mathbb{R}^n_{\geq 0}, \mathbb{F}^m), \\ &f_i(x) = \begin{cases} pr_2 \circ t_i^{-1} \circ (\psi_i \cdot f) \circ \kappa_i(x) & x \in U_i \\ 0 & x \notin U_i \end{cases}. \end{aligned} \tag{3.2.1}$$

Below is the schematic commutative diagram, highlighting the general situation:

$$\begin{tikzcd} \mathbb{F}^m & N(b_i, r) \times \mathbb{F}^m \arrow[l, "pr_2"'] & \pi^{-1}(N(b_i, r)) \arrow[l, "t_i^{-1}"'] \\ \mathbb{R}^n_{\geq 0} \supset U_i \arrow[u, "f_i"] \arrow[rr, "\kappa_i"] & & N(b_i, r) \arrow[u, "\psi_i \cdot f"'] \end{tikzcd}$$

Having established this notation, we can now finally define the natural Sobolev Spaces on bundles of bounded geometry:

Definition 3.2.2 (Sobolev Spaces)**.** Let $E \downarrow M$ be a bundle of bounded geometry. Further, let $s \geq 0$ and let $(t_i, \kappa_i, \psi_i)_{i\in\mathbb{Z}}$ be an admissible triple for $E \downarrow M$. Let $L(E)$ be the space of equivalence classes of measurable sections of E, where as usual, two sections are identified if they agree almost everywhere. For $s \geq 0$, we define the *Sobolev space of sections* as the subspace

$$\mathcal{W}_s(E) \coloneqq \{f \in L(E) : f_i \in \mathcal{W}_s(\mathbb{R}^n_{\geq 0}, \mathbb{F}^m)\ \forall i \in \mathbb{Z}\ \wedge \sum_{i\in\mathbb{Z}} ||f_i||^2_{\mathcal{W}_s(\mathbb{R}^n_{\geq 0},\mathbb{F}^m)} < \infty\}. \tag{3.2.2}$$

Here, $\mathcal{W}_s(\mathbb{R}^n_{\geq 0}, \mathbb{F}^m)$ denotes the standard Sobolev norm of vector fields on the upper half-plane, i.e. we have

$$||f_i||^2_{\mathcal{W}_s(\mathbb{R}^n_{\geq 0},\mathbb{F}^m)} = \int_{\mathbb{R}^n_{\geq 0}} (1 + |\xi|^2)^s \cdot |\widehat{f_i(\xi)}|^2\, d\xi,$$

where $\widehat{f_i(\xi)}$ denotes the Fourier transform of the vector field f_i. On $\mathcal{W}_s(E)$, we define an inner product $\langle\ ,\ \rangle_s$ via

$$\langle f, g\rangle_s \coloneqq \sum_{i\in\mathbb{Z}} \langle f_i, g_i\rangle_{\mathcal{W}_s(\mathbb{R}^n_{\geq 0},\mathbb{F}^m)}. \tag{3.2.3}$$

The induced norm will be denoted by $||\ .\ ||_s$. Sometimes, we will use the abbreviation $L^2(E) = \mathcal{W}_0(E)$. Lastly, we define the **s-th local Sobolev** space

$$\mathcal{W}_{s,loc}(E) \coloneqq \bigcap_K \mathcal{W}_s(E|_K), \tag{3.2.4}$$

where K ranges over all compact, codimension-0 Riemannian submanifolds $K \subseteq M$.

One can show that the definition of $\mathcal{W}_s(E)$, as well as the equivalence class of norm $||\ .\ ||_s$, does not depend on the particular choice of admissible triple $(t_i, \kappa_i, \psi_i)_{i\in\mathbb{Z}}$, see [85, Lemma 3.24]. Equipped with the inner product $\langle\ ,\ \rangle_s$, $\mathcal{W}_s(E)$ becomes a Hilbert space. Moreover, completeness of M ensures that $\Gamma^\infty_c(E) \subseteq \mathcal{W}_s(E)$ is a dense subspace. Lastly, note that if $E \downarrow M$ is a bundle of bounded geometry, so is $\Lambda^p T^*M \otimes E \downarrow M$ for any $0 \leq p \leq m$ (with respect to the natural induced (intertwined) metric). Therefore, we can define the s-th Sobolev space of differential p-forms as

$$\mathcal{W}^p_s(E) \coloneqq \mathcal{W}_s(\Lambda^p T^*M \otimes E), \tag{3.2.5}$$
$$\mathcal{W}^p_{s,loc}(E) \coloneqq \bigcap_{K\subseteq M} \mathcal{W}^p_s(E|_K). \tag{3.2.6}$$

From the definition, it becomes obvious that for any $\omega \in \Omega^p_c(M, E)$ and any pair $s > k$, we have $||\omega||_s \geq ||\omega||_k$. Furthermore, it is clear to see that on $\Omega_c(M, E)$, the norm $||\quad||_0$ is equivalent to the norm $||\quad||$, as defined in 2.1.7 via means of the wedge product and the Hermitian Hodge $\#$-operator. These two observations allow us to think of Sobolev spaces of differential forms as naturally sitting boundedly nested inside $\Omega^\bullet_{(2)}(M, E)$, i.e. we have in particular

$$\cdots \subseteq \mathcal{W}^\bullet_2(E) \subseteq \mathcal{W}^\bullet_1(E) \subseteq \mathcal{W}^\bullet_0(E) = \Omega^\bullet_{(2)}(M, E), \tag{3.2.7}$$

and each inclusion is a continuous embedding of Hilbert spaces with dense image. Lastly, this further allows us to define the *infinite order Sobolev space* of differential forms as

$$\mathcal{W}^\bullet_\infty(E) \coloneqq \bigcap_{k=1}^\infty \mathcal{W}^\bullet_k(E) \subset \Omega^\bullet_{(2)}(M, E). \tag{3.2.8}$$

Definition 3.2.3 (Bounded differential operator)**.** Let $E \downarrow M$ and $E' \downarrow M$ be two bundles of bounded geometry over a Riemannian manifold M. A differential operator $A : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E')$ is called **bounded** if the set of all complex valued coefficient functions of A, obtained via an arbitrary admissible trivialization, is ∞-uniformly bounded. A *bounded boundary differential operator* $B : \Omega^\bullet(M, E) \to \Omega^\bullet(\partial M, E'|_{\partial M})$ is the composition of a bounded differential operator $A : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E')$ with the tangential boundary projection $i^* : \Omega^\bullet(M, E') \to \Omega^\bullet(\partial M, E'|_{\partial M})$.

Lemma 3.2.4. *Let $E \downarrow M$ be a flat bundle of bounded geometry. Then all of the operators $\mathbb{1}, \#, d, \delta$ and $\Delta = d\delta + \delta d$ are bounded differential operators.*

Proof. Boundedness for $\mathbb{1}$ is on the nose, while boundedness for d is also obvious, since any admissible trivialization of $E \downarrow M$ is also always a flat trivialization by requirement. $\#$ is bounded, since its coefficient functions in an admissible trivialization involve only the Riemannian and Hermitian metric tensors, which are ∞-uniformly bounded by assumption. Finally, both δ and Δ are by definition sums and compositions of the bounded differential operators $\#$ and d, and therefore also bounded themselves. □

Definition 3.2.5. Let $E \downarrow M$ be a bundle of bounded geometry, let $(t_i, \kappa_i, \psi_i)_{i \in \mathbb{Z}}$ be an admissible triple and let $K \in \mathbb{N}$. We define the normed vector space

$$\Gamma_b^K(E) := \{f \in \Gamma^K(E) \colon \{f_i, i \in \mathbb{Z}\} \text{ is } K\text{-uniformly bounded}\}$$

with norm

$$|f|_K := \sup_{i \in \mathbb{Z}} \sup_{|\alpha| \leq K} ||\partial^\alpha f_i||_\infty.$$

The next proposition is a collection of all the results on Sobolev spaces of bundles of bounded geometry that, using an admissible trivialization, easily extend from well-known classic results in the Euclidean setting. The proofs, or at least clear guidelines of such, can be found in [85, Proposition 3.23].

Proposition 3.2.6 (Elementary properties of Sobolev spaces)**.** *Let $E \downarrow M$ and $E' \downarrow M$ be two bundles of bounded geometry over a Riemannian manifold M, $A : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E')$ a bounded differential operator of order μ, and $s, k \in \mathbb{R}_{\geq 0}$. Then,*

1. *for any $K \in \mathbb{N}_0$ and any $s \in \mathbb{R}$ with $s > m/2 + K$, we have a bounded embedding of normed vector spaces $\mathcal{W}_s(E) \hookrightarrow \Gamma_b^K(E)$. In particular, we have $\mathcal{W}^\bullet_\infty(E) \subseteq \Omega^\bullet(M, E) \cap \Omega^\bullet_{(2)}(M, E)$.*
2. *If $s \geq \mu$, then A extends to a bounded operator $A : \mathcal{W}^\bullet_s(E) \to \mathcal{W}^\bullet_{s-\mu}(E')$.*
3. *For any $0 \leq t \leq R_C$, identify $\partial M \times \{t\}$ with its image in M under the geodesic collar map, see Definition 3.1.3. Further, let $i^*_{(t)} : \Omega^\bullet(E') \to \Omega^\bullet(E'|_{\partial M \times \{t\}})$ be the pullback-map induced by the inclusion. Then if $s > \mu + 1/2$, the operator $i^*_{(t)} A : \Omega^\bullet(E) \to \Omega^\bullet(E'|_{\partial M \times \{t\}})$ extends to a bounded operator $i^*_{(t)} A : \mathcal{W}^\bullet_s(E) \to \mathcal{W}^\bullet_{s-\mu-1/2}(E'|_{\partial M \times \{t\}})$ (the corresponding norm depends continuously on t).*

Together with Lemma 2.1.2, we obtain a generalized version of Stokes' theorem for Sobolev 1-forms:

Corollary 3.2.7. *Let $E \downarrow M$ be a flat bundle of bounded geometry, $1 \le k \le n$, $\omega \in \mathcal{W}_1^{k-1}(E)$ and $\sigma \in \mathcal{W}_1^k(E)$. Then $d\omega \in \mathcal{W}_0^k(E)$, $\delta\sigma \in \mathcal{W}_0^{k-1}(E)$, $i^*\omega, \hat{\#}^{-1}i^*\#\sigma \in \mathcal{W}_{1/2}^{k-1}(E|_{\partial M})$, and*

$$\langle d\omega, \sigma\rangle = \langle \omega, \delta\sigma\rangle + \langle i^*\omega, \hat{\#}^{-1}i^*\#\sigma\rangle.$$

For the next technical lemma, we define for an admissible triple (t_i, κ_i, ϕ_i), any $f \in \mathcal{W}_s(E)$ and any measurable subset $N \subseteq M$ the *restricted Sobolev norm*

$$||f|_N||_s^2 \coloneqq \sum_{i\in\mathbb{Z}} \int_{\kappa_i^{-1}(N)} (1+|\xi|^2)^s \cdot |\widehat{f_i(\xi)}|^2 \, d\xi \tag{3.2.9}$$

Evidently, if N is a set of measure zero, then we have $||f|_N||_s^2 = 0$. In particular, if $N = \partial M$, the restricted Sobolev norm should **not** be confused with the Sobolev norm of the pull-back section i^*f on ∂M, where $i : \partial M \to M$ denotes the boundary inclusion.

Lemma 3.2.8. *Suppose that $N_1 \supseteq N_2 \supseteq N_3 \supseteq \dots$ is a nested sequence of measurable subsets of M with $N \coloneqq \bigcap_{k=0}^\infty N_k$. Let $s \in \mathbb{R}$ and $f \in \mathcal{W}_s(E)$. Then we have*

$$\lim_{k\to\infty} ||f|_{N_k}||_s^2 = ||f|_N||_s^2$$

Proof. For $i \in \mathbb{Z}$ and $k \in \mathbb{K}$, we define the positive real numbers

$$\begin{aligned} a(i,k) &\coloneqq \int_{\mathbb{R}^n_{\ge 0}} \chi(\kappa_i^{-1}(N_k)) \cdot (1+|\xi|^2)^s \cdot |\widehat{f_i(\xi)}|^2 \, d\xi, \\ a(i) &\coloneqq \int_{\mathbb{R}^n_{\ge 0}} \chi(\kappa_i^{-1}(N)) \cdot (1+|\xi|^2)^s \cdot |\widehat{f_i(\xi)}|^2 \, d\xi, \\ b(i) &\coloneqq \int_{\mathbb{R}^n_{\ge 0}} (1+|\xi|^2)^s \cdot |\widehat{f_i(\xi)}|^2 \, d\xi. \end{aligned}$$

Then $||f|_{N_k}||_s^2 = \sum_{i\in\mathbb{Z}} a(i,k)$, $||f|_N||_s^2 = \sum_{i\in\mathbb{Z}} a(i)$ and $||f||_s^2 = \sum_{i\in\mathbb{Z}} b(i)$. Since $a(i,k) \le b(i) < \infty$ for all i and all k, and since $g_i(k) \coloneqq \chi(\kappa_i^{-1}(N_k))$ converges point-wise to $g_i \coloneqq \chi(\kappa_i^{-1}(N))$ for each i, we can apply the dominated convergence theorem to obtain that $\lim_{k\to\infty} a(i,k) = a(i)$ for each i. Since moreover $\sum_{i\in\mathbb{Z}} b(i) < \infty$, we can apply the same theorem a second time to obtain that

$$\lim_{k\to\infty} ||f|_{N_k}||_s^2 = \lim_{k\to\infty} \sum_{i\in\mathbb{Z}} a(i,k) = \sum_{i\in\mathbb{Z}} \lim_{k\to\infty} a(i,k) = \sum_{i\in\mathbb{Z}} a(i) = ||f|_N||_s^2.$$

The result follows. □

Lemma 3.2.9. *Let $E \downarrow M$ be a bundle of bounded geometry, let $N \cong \partial M \times [0,1]$ be a collar neighborhood of ∂M and let $0 \le p \le m$. For $0 \le t \le 1$, let $i_{(t)}^* : \mathcal{W}_1^p(E) \to \mathcal{W}_0^p(E_{\partial M \times \{t\}})$ be the (continuous) tangential boundary projection induced by the smooth inclusion $i_{(t)} : \partial M \times \{t\} \hookrightarrow M$. Then there exists constants $C, \epsilon > 0$, such that for all $\omega \in \mathcal{W}_1^p(E)$ and all $0 < t < \epsilon$, it holds that*

$$||i_{(t)}^*\omega||_0^2 \le C\left(||i_{(0)}^*\omega||_0^2 + t||\omega|_{\partial M \times [0,t]}||_1^2\right). \tag{3.2.10}$$

Proof. Because of the $\mathcal{W}_1$-continuity of all operators involved (Proposition 3.2.6) it suffices to prove these statements for elements of $\Omega_c^p(M,E)$. Choose an admissible triple $(t_i, \kappa_i, \psi_i)_{i\in\mathbb{Z}}$ for $E \downarrow M$. Recall that, by definition, the indexing is chosen in such a way that $i \le 0 \Leftrightarrow b_i \in \partial M$. We can choose $0 < \epsilon < 1$ small enough such that for any $i > 0$, we have $\kappa_i^{-1}(N_\epsilon) = 0$, where $N_\epsilon \cong \partial M \times [0,\epsilon)$ is the collar neighborhood of ∂M as in Definition 3.1.3.

Choose some $i \geq 0$ and let ω_i be the vector field on $B_r(0) \times [0, r)$ derived from ω with the aid of the triple (t_i, κ_i, ϕ_i) as explained in Equation 3.2.1 . In this way, we have a decomposition

$$\omega_i(x,t) = \omega_{i,1}(x,t) + \omega_{i,2}(x,t) \wedge dt \tag{3.2.11}$$

with $\omega_{i,1}(x,t)$ a *tangential form* with image in $\mathbb{F}^m \otimes \Lambda^p \mathbb{R}^{n-1}$ and $\omega_{i,2}(x,t)$ a *normal form* with image in $\mathbb{F}^m \otimes \Lambda^{p-1} \mathbb{R}^{n-1}$. Now applying the fundamental theorem of Calculus, we have for any $0 \leq t \leq r$ and any $x \in B(0,r)$, that

$$\omega_{i,1}(x,t) = \omega_{i,1}(x,0) + \int_0^t \frac{d}{du}\omega_{i,1}(x,u)du. \tag{3.2.12}$$

Using the triangle inequality, the fact that $2ab \leq a^2 + b^2$ for any two real numbers a, b, along with Hölder's inequality, we therefore obtain

$$|\omega_{i,1}(x,t)|^2 \leq 2\left(|\omega_{i,1}(x,0)|^2 + t\int_0^t |\frac{d}{du}\omega_{i,1}(x,u)|^2 du\right). \tag{3.2.13}$$

Integrating over $B_r(0)$ then yields

$$\begin{aligned}
&\int_{B_r(0)} |\omega_{i,1}(x,t)|^2 dx \leq 2\left(\int_{B_r(0)} |\omega_{i,1}(x,0)|^2 dx + t\int_{B_r(0)}\int_0^t |\frac{d}{du}\omega_{i,1}(x,u)|^2 du\right)\\
&\leq 2\left(\int_{B_r(0)} |\omega_{i,1}(x,0)|^2 dx + t\int_{B_r(0)}\int_0^t |D\omega_i(x,u)|^2 du dx\right)\\
&\leq C\cdot\left(\int_{B_r(0)} |\omega_{i,1}(x,0)|^2 dx + t\int_{B_r(0)\times[0,t]} (1+|\xi|^2)\cdot|\widehat{\omega_i}(\xi)|^2\, d\xi\right)
\end{aligned}$$

for a constant $C > 0$ that depends only on the H^1-norm on $\mathbb{R}^n_{\geq 0}$, but *not* on i. For any $t < \epsilon$, we then have

$$\begin{aligned}
||i^*_{(t)}\omega||_0^2 &= \sum_{i=-\infty}^{0} \int_{B_r(0)} |\omega_{i,1}(x,t)|^2 dx\\
&\leq C\left(\sum_{i=-\infty}^{0}\int_{B_r(0)} |\omega_{i,1}(x,0)|^2 + t\int_{B_r(0)\times[0,t]} (1+|\xi|^2)\cdot|\widehat{\omega_i}(\xi)|^2\, d\xi\right)\\
&= C\left(||i^*_{(0)}\omega||_0^2 + t||\omega|_{\partial M\times[0,t]}||_1^2\right).
\end{aligned}$$

□

3.3 Uniformly elliptic boundary value problems

Definition 3.3.1 (Elliptic boundary value problem)**.** Let M be a manifold of bounded geometry and let $E \downarrow M$, $F \downarrow M$ and, for each $i = 0, \ldots, n$, $X_i \downarrow M$ be bundles over M of bounded geometry. A system of operators $\mathcal{A} := (A, p_0, \ldots, p_n) : \Gamma(E) \to \Gamma(F) \oplus \bigoplus_{i=0}^n \Gamma(X_i|_{\partial M})$ is called an **elliptic boundary value problem** of order μ, if

1. A is a bounded differential operator of order $\mu \geq n + 1/2$,

2. p_i are bounded boundary differential operators of order at most i,

3. $\mathcal{A}$ is elliptic in the sense of Schwartz [88, Definition 1.6.1].

In [84, Definition 4.3], systems of operators satisfying properties (1) and (2) are defined to be boundary value problems, while boundary values problems that additionally satisfy condition (3) are called *elliptic* in slight distinction. However, for the purpose of this work, we will exclusively look at elliptic boundary value problems, which is why we include this particular property already in the basic definition. Note that we have refrained from explicitly writing down the complicated definition of ellipticity for a boundary value problem, which involves delving deep into local coordinates at the boundary. Instead, we will focus on one of the main applications of ellipticity, the one that motivates the extensive study behind such problems: *Elliptic regularity.* To begin with, observe that, because of assertion (1) and (2) in the above definition and Proposition 3.2.6, we have:

Corollary 3.3.2. *Any elliptic boundary value problem*

$$\mathcal{A} = (A, p_0, \ldots, p_n) : \Gamma(E) \to \Gamma(F) \oplus \bigoplus_{i=0}^{n} \Gamma(X_i|_{\partial M})$$

of order μ extends for each $s \geq 0$ to a bounded operator

$$\mathcal{A} : \mathcal{W}_{s+\mu}(E) \to \mathcal{W}_s(F) \oplus \bigoplus_{i=0}^{n} \mathcal{W}_{s+\mu-i-\frac{1}{2}}(X_i|_{\partial M}). \tag{3.3.1}$$

Definition 3.3.3 (Formally self-adjoint boundary value problem)**.** In the situation as in the previous definition, an elliptic boundary value problem $\mathcal{A} := (A, p_0, \ldots, p_n) : \Gamma(E) \to \Gamma(E) \oplus \bigoplus_{i=0}^{n} \Gamma(X_i|_{\partial M})$ is called **formally self-adjoint** if there exists a system $(\vec{q}_1, \ldots, \vec{q}_n) : \Gamma(E) \to \bigoplus_{i=0}^{n} \Gamma(X_i|_{\partial M})$ of bounded boundary differential operators, such that, for any pair $\omega, \sigma \in \Gamma(E)$ with either ω or σ compactly supported, we have

$$\langle A\omega, \sigma\rangle - \langle \omega, A\sigma\rangle = \sum_{i=0}^{n} \langle \vec{p}_i\omega, \vec{q}_i\sigma\rangle - \langle \vec{q}_i\omega, \vec{p}_i\sigma\rangle. \tag{3.3.2}$$

An important feature of formally self-adjoint boundary value problems is that they give rise to a *Hodge-Decomposition* as follows:

Theorem 3.3.4. [84, Corollary 4.20] *Let M be a manifold of bounded geometry and let $E \downarrow M$, and, for each $i = 0, \ldots, n$, $X_i \downarrow M$ be bundles over M of bounded geometry. Further, let $\mathcal{A} := (A, p_0, \ldots, p_n) : \Gamma(E) \to \Gamma(E) \oplus \bigoplus_{i=0}^{n} \Gamma(X_i|_{\partial M})$ be an elliptic, formally self-adjoint boundary value problem. Consider the subspace $\Gamma(E, \vec{t}) := \{f \in \Gamma_c(E) : p_0 f = \cdots = p_n f = 0\}$ of compactly supported functions satisfying certain boundary conditions. Then we get an orthogonal decomposition*

$$L^2(E) = \{f \in \Gamma(E) \cap L^2(E) : Af = p_0 f = \ldots p_n f = 0\} \oplus \overline{A\Gamma(E, \vec{t})}. \tag{3.3.3}$$

A classical problem in the field of partial differential equations, translated into the language of linear operators, asks whether such an $\mathcal{A}$ has a (bounded) inverse, a so-called *solution.* As a matter of fact, ellipticity ensures that this is always *locally* the case. More precisely, a classic result by Hörmander [44, 7.3.1, 10.4.1] states that, for any elliptic boundary value problem $\mathcal{A} = (A, p_0, \ldots, p_n)$ as above, one has the following properties:

1. For each $x \in M$, there exists constants $D_x > 0$, and $0 < \epsilon_x < R_C$ depending continuously on x and a bounded linear operator

$$S_x : \mathcal{W}_0(E_{N(x,\epsilon_x)}) \oplus \sum_{l=0}^{n} \mathcal{W}_{\mu-l-1/2}(E_{N(x,\epsilon_x)\cap\partial M}) \to \mathcal{W}_\mu(E_{N(x,\epsilon_x)}), \tag{3.3.4}$$

called a *local fundamental solution* of $\mathcal{A}$, satisfying

   (a) $N(x, \epsilon_x) \cap \partial M = \emptyset$, if $x \notin \partial M$, in which case we set $\sum_{l=0}^{n} \mathcal{W}_{\mu-l-1/2}(E_{N(x,\epsilon_x)\cap\partial M}) := 0$.

   (b) One has $||S_x|| \leq D_x$.

   (c) The restriction $\mathcal{A}|_{\mathcal{W}_\mu(E_{N(b_x,\epsilon_x)})}$ of $\mathcal{A}$ to $\mathcal{W}_\mu(E_{N(b_x,\epsilon_x)})$ is a left inverse for S_x, i.e. $\mathcal{A}|_{\mathcal{W}_\mu(E_{N(b_x,\epsilon_x)})} \circ S_x = \mathbb{1}_{\mathcal{W}_0(E_{N(x,\epsilon_x)} \oplus \sum_{l=0}^{n} \mathcal{W}_{\mu-l-1/2}(E_{N(x,\epsilon_x)\cap\partial M})}$.

   (d) For any $f \in \mathcal{W}_\mu(E)$ that is compactly supported inside $N(x, \epsilon_x)$, it also holds that $S_x \circ \mathcal{A}(f) = f$.

Definition 3.3.5 (Uniformly elliptic boundary value problem)**.** In the above situation, an elliptic boundary value problem $\mathcal{A} = (A, p_0, \ldots, p_n)$ is called *uniformly elliptic* if

1. there exists a global constant $C_\mathcal{A} > 0$, bounding D_x from above for all $x \in \partial M$ and, and
2. there exists a global constant $r_\mathcal{A} > 0$, bounding ϵ_x from below for all $x \in M$.
3. The differential operator A is uniformly elliptic, that is

   (a) the matrix $a(x, \xi)$, representing the principal symbol of A at x in admissible normal coordinates of E and F, is invertible [ellipticity].

   (b) There exists a constant $C > 0$ independent of x or ξ, such that

$$|a^{-1}(x, \xi)| \cdot |\xi|^\mu \leq C,$$

   where $\mu \in \mathbb{N}_0$ is the order of A. Here, the (matrix-)norms used are the ones induced by the chosen admissible trivialization.

Remark 3.3.6. As shown in [85, 4.10, 4.11] a consequence of uniform ellipticity of a boundary value problem is that the regularity of local fundamental solutions scales proportionally with the regularity of the initial problem. Precisely, this means that for each $s \in \mathbb{R}$ and each $x \in M$, the bounded map S_x as in 3.3.4 can be chosen with domain space $\mathcal{W}_s(E_{N(x,\epsilon_x)}) \oplus \sum_{l=0}^{n} \mathcal{W}_{s+\mu-l-1/2}(E_{N(x,\epsilon_x)\cap\partial M})$ and target space $\mathcal{W}_{\mu+s}(E_{N(x,\epsilon_x)})$.

Example 3.3.7. Any elliptic boundary value problem $\mathcal{A} := (A, p_0, \ldots, p_n)$ over a compact manifold M is automatically uniformly elliptic. The existence of the global constants $C_\mathcal{A} > 0, r_\mathcal{A} > 0$ are direct consequences of compactness and Lebesgue's lemma.
What might be less obvious is the existence of the constant C bounding $|a^{-1}(x, \xi)| \cdot |\xi|^\mu$ from above for all relevant pairs (x, ξ). Note that, because of compactness of M, it suffices to prove the existence of such a constant C_x for any fixed $x \in M$. To this effect, first observe that ellipticity of A implies that one finds a constant $c_x > 0$ such that $\sup_{|\xi|=1} |a(x, \xi)| > c_x$. Now observe that for fixed x, $a(x, \xi)$ is a homogeneous (matrix-valued) polynomial in ξ of degree μ, which is why we obtain for any $\xi \in T_x^*M$ that $|a(x, \xi)| = |\xi|^\mu |a(x, \xi/|\xi|)| > |\xi|^\mu c_x$. Setting $C_x := c_x^{-1}$, the previous arguments imply that $|a(x, \xi)^{-1}| < |\xi|^{-\mu} C_x$ for any $0 \neq |\xi|$ as desired.

Example 3.3.8. More generally, if $\widetilde{M}$ is a normal covering of a compact manifold M and if $\mathcal{A}$ is a system of differential operators on M, we can lift $\mathcal{A}$ to a system $\widetilde{\mathcal{A}}$ of differential operators over $\widetilde{M}$, defined on sections of corresponding lifted bundles. Then, as the lift of a uniformly elliptic boundary value problem, it is evident that $\widetilde{\mathcal{A}}$ itself must be uniformly elliptic.

Example 3.3.9. For an elliptic boundary value problem $\mathcal{A} = (A, p_0, \dots, p_n)$ over a Riemannian manifold (M, g) of bounded geometry to be uniformly elliptic, it is not necessarily required that $\mathcal{A}$ by itself is the lift of a boundary value problem over a compact manifold. Indeed, suppose that there exists Riemannian manifolds X, Y of bounded geometry and the same dimension as M, such that

1. both M and Y are complete Riemannian submanifolds of X,
2. there exists a uniformly elliptic boundary value problem $\mathcal{A}_X = (A_X, \dots)$ over X with the property that $A = A_X|_M$, i.e. the elliptic operator A is the restriction of the uniformly elliptic operator A_X to M,
3. ∂M is a connected component of ∂Y. Moreover, there exists a neighborhood $U \subseteq M \cap Y$ of ∂M inside both M and Y and a uniformly elliptic boundary value problem $\mathcal{A}_Y$ over Y so that $\mathcal{A}|_U = \mathcal{A}_Y|_U$.

Then it is easily verified that $\mathcal{A}$ itself must be uniformly elliptic.

We will apply three essential results on general uniformly elliptic boundary value problems, whose proofs can be found in [84, Theorem 4.14, Theorem 4.23, Theorem 4.26] (in that order):

Proposition 3.3.10 (Elliptic regularity)**.** *Let $E \downarrow M$ be a flat bundle of bounded geometry and $\mathcal{A} = (A, p_0, \dots, p_l)$ a uniformly elliptic boundary value problem of order μ. Then, for any $s \in \mathbb{R}$, there exists a constant $C(s, \mu) > 0$, such that if $\omega \in \mathcal{W}_s(E) \cap \Gamma(E)$ and $\mathcal{A}\omega \in \mathcal{W}_s(E) \oplus \bigoplus_{j=0}^{l} \mathcal{W}_{s+\mu-j-1/2}(X_i) \coloneqq \mathcal{M}_{s,\mu}$, then $\omega \in \mathcal{W}_{s+\mu}(E)$ and*

$$||\omega||^2_{s+\mu} \leq C(s, \mu) \cdot \left(||\mathcal{A}\omega||^2_{\mathcal{M}_{s,\mu}} + ||\omega||^2_s \right).$$

Proposition 3.3.11 (Self-adjoint closures)**.** *Let $E \downarrow M$ be a bundle of bounded geometry and let $(A, p_0, \dots, p_n) : \Gamma(E) \to \Gamma(E) \bigoplus_{k=1}^{n} \Gamma(X_i|_{\partial M})$ be a uniformly elliptic, formally self-adjoint boundary value problem of order μ. Consider the subspace $\Gamma(E, \vec{t}) \coloneqq \{f \in \Gamma_c(E) : p_0 f = \dots = p_n f = 0\}$ of compactly supported functions with boundary conditions and define the unbounded operator*

$$A_0 \coloneqq A|_{\Gamma(E,\vec{t})} : \mathcal{W}_0(E) \to \mathcal{W}_0(E). \tag{3.3.5}$$

Then A_0 is essentially self-adjoint, i.e. A_0 admits a minimal closure with the property that $\overline{A_0} = A_0^*$. Moreover,

$$\begin{aligned} \operatorname{dom}(\overline{A_0}) &= \{\omega \in \mathcal{W}_\mu(E) : p_0\omega = \dots = p_n\omega = 0\}, \\ \ker(\overline{A_0}) &= \{\omega \in \mathcal{W}_\infty(E) : A_0\omega = p_0\omega = \dots p_n\omega = 0\}. \end{aligned}$$

Last but not least, if $\mathcal{A} = (A, p_0, \dots, p_n)$ and $\mathcal{B} = (B, q_0, \dots, q_m)$ are two elliptic boundary value problems, and if the individual composite operators $A \circ B$, as well as $q_i \circ A$ $(1 \leq i \leq m)$ are well-defined, we can consider the composite system of operators

$$\mathcal{A} \circ \mathcal{B} \coloneqq (A \circ B, q_1 \circ A, \dots, q_m \circ A, p_1, \dots, p_n). \tag{3.3.6}$$

Observe that we may rearrange the boundary differential operators in the system $\mathcal{A} \circ \mathcal{B}$, such that (after possibly filling up with some zero operators) the i-th boundary operator has order at most i. In this way, it becomes natural to ask whether the composition of two elliptic boundary value problems remains elliptic. Schick gives a partial positive answer which is sufficient for our purposes.

Proposition 3.3.12. [84, Proposition 4.15] *Let $\mathcal{A}$ and $\mathcal{B}$ be uniformly elliptic boundary problems of order μ, respectively ν. Then, if it is well-defined, $\mathcal{A} \circ \mathcal{B}$ is a uniformly elliptic boundary value problem of order $\mu + \nu$.*

3.3.1 The standard boundary value problem on flat bundles

Throughout this section, we will fix an n-dimensional manifold (M, g) of bounded geometry and a *flat* Hermitian bundle $(E, h) \downarrow M$ of bounded geometry over M. Just like in Section 2.1, we fix a (possibly empty) decomposition $\partial M = \partial_1 M \dot\cup \partial_2 M$ of ∂M into its *Dirichlet boundary* $\partial_1 M$ and *Neumann boundary* $\partial_2 M$. Denote by $i_1 : \partial_1 M \to M$ and $i_2 : \partial_2 M \to M$ the respective smooth inclusion maps. Our goal of this section is to define elliptic boundary value problems over M of varying degrees that depend only on the flat bundle structure, the choice of bounded metrics, and the boundary decomposition of ∂M.
As before, we let $\Omega^\bullet(M, E) = \Gamma(M, \bigoplus_{k=0}^n \Lambda^k T^* M \otimes_{\mathbb{R}} E) \cong \bigoplus_{k=0}^n \Omega^k(M, E)$ and define bounded boundary differential operators

$$t_0, n_0, t_1, n_1 : \Omega^\bullet(M, E) \to \Omega^\bullet(\partial M, E|_{\partial M}), \tag{3.3.7}$$

$$t_0(\omega) \coloneqq i_1^* \omega + \widehat{\#}^{-1} i_2^* \# \omega, \qquad n_0(\omega) \coloneqq \widehat{\#}^{-1} i_1^* \# d\omega - i_2^* \delta\omega, \tag{3.3.8}$$

$$t_1(\omega) \coloneqq i_1^* \delta\omega + \widehat{\#}^{-1} i_2^* \# d\omega, \qquad n_1(\omega) \coloneqq \widehat{\#}^{-1} i_1^* \# \omega - i_2^* \omega, . \tag{3.3.9}$$

As usual, $\#$ denotes the Hermitian Hodge $*$-operator on $E \downarrow M$ and $\widehat{\#}$ the (invertible) Hermitian Hodge $*$-operator on the restriction bundle $E \downarrow \partial M$. Note that, as boundary differential operators, t_0 and n_1 both have order 0, while t_1 and n_0 both have order 1. Combining them produces further boundary differential operators

$$\vec{t}, \vec{n} : \Omega^\bullet(M, E) \to \Omega^\bullet(\partial M, E|_{\partial M})^2, \tag{3.3.10}$$

$$\vec{t}(\omega) \coloneqq t_0 \omega \oplus t_1 \omega, \qquad \vec{n}(\omega) \coloneqq n_0 \omega \oplus n_1 \omega. \tag{3.3.11}$$

Lastly, for fixed $0 \le p \le n$, we let $\vec{t}^p$ be the restriction of $\vec{t}$ onto p-forms and define a (sub)bundle $\operatorname{im}(\vec{t}^p)$ over ∂M of bounded geometry via

$$\operatorname{im}(\vec{t}^p) \coloneqq (\Lambda^p T^* \partial M \oplus \Lambda^{p-1} T^* \partial M) \otimes_{\mathbb{R}} E.$$

We consider the first-order differential operator

$$d + \delta : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E),$$

along with, for $0 \le p \le n$, the second-order differential operator

$$\Delta_p \colon \delta^{p+1} d^p + d^p \delta^{p-1} = (d + \delta)^2|_{\Omega^p(M,E)} \colon \Omega^p(M, E) \to \Omega^p(M, E).$$

Taken together, we can define systems of differential operators on M that depend only on the flat bundle $E \downarrow M$ and the choice of boundary decomposition $\partial M = \partial_1 M \dot\cup \partial_2 M$:

Definition 3.3.13 (Standard boundary value problem)**.** Let $E \downarrow M$ be a bundle of bounded geometry and let $\partial M = \partial_1 M \dot{\cup} \partial_2 M$ be a decomposition of ∂M. We set

$$\mathcal{A} \coloneqq (d + \delta, t_0) : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E) \oplus \Omega^\bullet(\partial M, E|_{\partial M}).$$

Further, for fixed $0 \leq p \leq n$, we define a system of differential operators $\mathcal{B}^{\mathrm{p}}_{\mathrm{k}}$ for any $k \in \mathbb{N}$ as follows:

$$\mathcal{B}^p_1 \coloneqq \left(\Delta_p, \vec{t}\right) : \Omega^p(M, E) \to \Omega^p(M, E) \oplus \Gamma(\mathrm{im}(\vec{t}^p)),$$
$$\mathcal{B}^{\mathrm{p}}_{\mathrm{k}} \coloneqq \underbrace{\mathcal{B}^p_1 \circ \mathcal{B}^p_1 \circ \cdots \circ \mathcal{B}^p_1}_{k \text{ times}} \quad k \in \mathbb{N}.$$

Explicitly, this means that for $k \in \mathbb{N}$, we have

$$\mathcal{B}^{\mathrm{p}}_{\mathrm{k}} \coloneqq \left(\Delta^k_p, \vec{t}, \ldots, \vec{t}\Delta^{k-1}_p\right) : \Omega^p(M, E) \to \Omega^p(M, E) \oplus \Gamma(\mathrm{im}(\vec{t}))^k.$$

$\mathcal{A}$ and $\mathcal{B}^{\mathrm{p}}_{\mathrm{k}}$ are called the *standard boundary value problems* over M, associated to E and the decomposition $\partial_1 M \dot{\cup} \partial_2 M$.

Lemma 3.3.14. *The system $\mathcal{A}$, as well as the systems $\mathcal{B}^{\mathrm{p}}_{\mathrm{k}}$ for each $0 \leq p \leq n$ and each $k \in \mathbb{N}$ are elliptic, formally self-adjoint boundary value problems of order 1, respectively $2k$. Whenever $\omega, \sigma \in \Omega^\bullet(M, E)$ are forms (of the correct degree), so that either ω or σ is also compactly supported, we get*

$$\langle (d + \delta)\omega, \sigma\rangle - \langle \omega, (d + \delta)\sigma\rangle = \langle p_0\omega, n_1\sigma\rangle - \langle n_1\omega, p_0\sigma\rangle, \tag{3.3.12}$$

$$\langle \Delta^k_p\omega, \sigma\rangle - \langle \omega, \Delta^k_p\sigma\rangle = \sum_{i=0}^{k-1} \langle \vec{t}\Delta^i_p\omega, \vec{n}\Delta^{k-1-i}_p\sigma\rangle - \langle \vec{n}\Delta^{k-1-i}\omega, \vec{t}\Delta^i\sigma\rangle. \tag{3.3.13}$$

Proof. Equations 3.3.12 and 3.3.13 both follow from an iterative application of Lemma 2.1.2. It is a well-known classic result that the systems $\mathcal{A}$ and $\mathcal{B}^p_1$ are elliptic boundary value problems, see for example [88, Lemma 1.6.5] or [51, Page 169]. Since compositions of elliptic boundary value problems, whenever they can be defined, remain elliptic boundary value problems [85, Proposition 4.16], the result follows for $\mathcal{B}^{\mathrm{p}}_{\mathrm{k}}$ with $k \geq 2$. □

Together with the arguments laid out in Examples 3.3.7–3.3.9, we obtain the two following important results:

Corollary 3.3.15. *Let $E \downarrow M$ be a flat, Hermitian bundle of bounded geometry over a manifold of bounded geometry. Further, suppose that there exists manifolds X, Y of bounded geometry and of the same dimension as M, as well as flat bundles $E_X \downarrow X$, $E_Y \downarrow Y$ of bounded geometry, such that*

1. *both M and Y are Riemannian submanifolds of X and both E and E_Y are the restrictions of E_X to M, respectively Y,*
2. *E_X is Γ-equivariant with respect to some uniform lattice $\Gamma < \mathrm{Isom}^+(X)$,*
3. *E_Y is Λ-equivariant with respect to some uniform lattice $\Lambda < \mathrm{Isom}^+(Y)$,*
4. *the intersection $M \cap Y$ is a codimension 0-submanifold containing $\partial M \subset \partial Y \cup \partial M$.*

Then the the standard elliptic boundary value problems $\mathcal{A}$ and $\mathcal{B}^{\mathrm{p}}_{\mathrm{k}}$ derived from $E \downarrow M$, constructed as above with respect to a decomposition $\partial M = \partial_1 M \dot{\cup} \partial_2 M$, are uniformly elliptic.

Remark 3.3.16. Note that bundles $E \downarrow M$ that are themselves Γ-equivariant for some uniform lattice $\Gamma < \mathrm{Isom}^+(M)$, the most common case appearing throughout this thesis, trivially satisfy the assumptions of the above corollary (apparent from choosing $X = Y = M$).

3.4 Applications to the De Rham complex

From now on, until the end of this chapter, we will now make the following global assumption: **All appearing (metric) bundles $E \downarrow M$ satisfy the assumptions of Corollary 3.3.15**.
This is because all the examples presented in forthcoming chapters are exclusively of this shape and the derived standard boundary value problems $\mathcal{A}$ and $\mathcal{B}^{\mathrm{p}}_{\mathrm{k}}$ are then uniformly elliptic. Together with Theorem 3.3.11, we first obtain:

Theorem 3.4.1. *For each $0 \leq p \leq n$ and each $k \in \mathbb{N}$, the operator $\Delta^k_p[E]$, defined as the unbounded operator Δ^k_p on $\mathcal{W}^p_0(E)$ with domain $\{\omega \in \Omega^p_c(M,E) : \vec{t}\omega = \cdots = \vec{t}\Delta^{k-1}\omega\}$ is essentially self-adjoint. Its minimal, self-adjoint closure, also denoted by $\Delta^k_p[E]$, has domain $\{\omega \in \mathcal{W}^p_{2k}(E) : \vec{t}\omega = \cdots = \vec{t}\Delta^{l-1}\omega = 0\}$.*

Among many other things, Theorem 3.4.1 now guarantees the existence of several smoothing smoothing operators constructed via the spectral theory of the self-adjoint $\Delta_p[E]$:

Proposition 3.4.2 (Kernel)**.** *Let $E \downarrow M$ be a bundle of bounded geometry and let $0 \leq p \leq n$. For $f \in \mathbb{R}^+ \to \mathbb{R}^+$ a rapidly-decreasing, positive Borel function, let $f(\Delta_p[E]) : L^2(E) \to L^2(E)$ be the bounded, self-adjoint operator defined via Borel functional calculus of $\Delta_p[E]$. Then $f(\Delta_p[E])$ is a smoothing operator.*

Proof. We claim that, for each $k \in \mathbb{N}$, $f(\Delta_p[E])$ has image in $\mathrm{dom}(\Delta^k_p[E]) \subseteq \mathcal{W}^p_{2k}(E)$. To see how the result follows from the claim, we get as an immediate consequence that $f(\Delta_p[E])$ has image in $\mathcal{W}^p_\infty(E)$. By Proposition 3.2.6, it follows that $f(\Delta_p[E])$ has image in $\Gamma_b(E)$. Since $f(\Delta_p[E])$ is self-adjoint, the existence of a smooth integral Kernel, hence the result, then follows from [85, Lemma 13.6].
It therefore remains to prove the claim. For $n \in \mathbb{N}$, let $\chi_{[0,n]}$ be the characteristic function of the interval $[0,n] \subseteq \mathbb{R}^+$ and let $x \cdot \chi_{[0,n]} : \mathbb{R}^+ \to \mathbb{R}^+ \in B(\mathbb{R})$ be the bounded, positive Borel function, defined for $n \in \mathbb{N}$. Moreover, let $\phi : B(\mathbb{R}) \to \mathcal{B}(\Omega^p_{(2)}(M,E))$ be the Borel functional calculus of the self-adjoint $\Delta_p[E]$. In particular, we have $f(\Delta_p[E]) = \phi(f)$ in that notation. Since $\Delta^k_p[E]$ is simply the k-th power of the self-adjoint operator $\Delta_p[E]$, we get from the spectral theorem for all $\omega \in \Omega^p_{(2)}(E)$, that

$$\omega \in \mathrm{dom}(\Delta^k_p[E]) \Leftrightarrow \lim_{n\to\infty} \phi(x^k \cdot \chi_{[0,n]})\omega \text{ exists.} \tag{3.4.1}$$

Since $f \in B(\mathbb{R})$ is rapidly decreasing, we get both that $x^k \cdot f \in B(\mathbb{R})$ and that $x^{k+1} \cdot f \in B(\mathbb{R})$. In particular, we find for any $\epsilon > 0$ an index $N \in \mathbb{N}$, such that for all $n \geq N$, we get $|x^k \cdot f \cdot (1 - \chi_{[0,n]})|^2_\infty < \epsilon$. From the spectral theorem, we therefore obtain that

$$\lim_{n\to\infty} \phi(x^k \cdot \chi_{[0,n]})\phi(f)\omega = \phi(x^k \cdot f)\omega, \tag{3.4.2}$$

implying that $\phi(f)\omega \in \mathrm{dom}(\Delta^k_p[E])$ for any $\omega \in \Omega^p_{(2)}(M,E)$. $\square$

Applying [85, Theorem 4.26] to the uniformly elliptic boundary value problem $\mathcal{B}^p_1$, we further obtain:

Theorem 3.4.3 (Hodge decomposition)**.** *For $0 \leq p \leq m$, define the* ***space of harmonic integrable p-forms with boundary conditions***

$$\mathcal{H}^p(M, \partial M, E) \coloneqq \{\omega \in \Omega^p(M, E) \cap \Omega^p_{(2)}(M, E) : \Delta\omega = 0, i_1^*\omega = i_2^*(\#\omega) = 0\}.$$

Then $\mathcal{H}^p(M, \partial M, E) = \ker(\Delta_p[E])$. Moreover, for each $k \in \mathbb{N}$ we obtain the following orthogonal decomposition of the Sobolev space $W_0^p(E)$ called ***Hodge decomposition***

$$W_0^p(E) = \mathcal{H}^p(M, \partial M, E) \oplus \overline{d^{p-1}\Omega^{p-1}(M, \partial_1 M, E)} \oplus \overline{\delta^p\Omega^{p+1}(M, \partial_2 M, E)}. \tag{3.4.3}$$

Theorem 3.4.4 (Elliptic regularity)**.** *Let $E \downarrow M$ be a bundle of bounded geometry. We consider the elliptic boundary value problem $\mathcal{A}$ as an unbounded operator between Hilbert spaces*

$$\mathcal{A} : \mathcal{W}_0^\bullet(E) \to \mathcal{W}_0^\bullet(E) \oplus \mathcal{W}_{1/2}^\bullet(E|_{\partial M}) \tag{3.4.4}$$

with initial domain $\Omega_c^\bullet(M, E)$. Moreover, for each $k \in \mathbb{N}$, we consider the elliptic boundary value problem $\mathcal{B}_{\mathrm{k}}^{\mathrm{p}} = \mathcal{B}_{\mathrm{k}}^{\mathrm{p}}$ as an unbounded operator between Hilbert spaces

$$\mathcal{B}_{\mathrm{k}}^{\mathrm{p}} : \mathcal{W}_0^*(E) \to \mathcal{W}_0^*(E) \oplus \bigoplus_{j=0}^{k-1} \mathcal{W}^*_{2k-2j-1/2}(E|_{\partial M}) \oplus \mathcal{W}^*_{2k-2j-3/2}(E_{\partial M}) \tag{3.4.5}$$

with initial domain $\Omega_c^p(M, E)$. Then

1. *both $\mathcal{A}$ and $\mathcal{B}_{\mathrm{k}}^{\mathrm{p}}$ are closable. We set* $\mathrm{dom}(\mathcal{A})$*, respectively* $\mathrm{dom}\,(\mathcal{B}_{\mathrm{k}}^{\mathrm{p}})$*, to be the domain of its minimal closed extension.*

2. *The bilinear forms $\widetilde{\langle\omega, \sigma\rangle}_1 \coloneqq \langle(\mathbb{1} + \mathcal{A})\omega, (\mathbb{1} + \mathcal{A})\sigma\rangle$, respectively $\widetilde{\langle\omega, \sigma\rangle}_{2k} \coloneqq \langle(\mathbb{1} + \mathcal{B}_{\mathrm{k}}^{\mathrm{p}})\omega, (\mathbb{1} + \mathcal{B}_{\mathrm{k}}^{\mathrm{p}})\sigma\rangle$ for $k \in \mathbb{N}$, define complete inner products on* $\mathrm{dom}(\mathcal{A})$*, respectively* $\mathrm{dom}\,(\mathcal{B}_{\mathrm{k}}^{\mathrm{p}})$*. In fact, the identity map $\mathbb{1} : \Omega_c^p(M, E) \to \Omega_c^p(M, E)$ extends to isomorphisms of Hilbert spaces*

$$\mathrm{dom}(\mathcal{A}) \cong \mathcal{W}_1^\bullet(E), \tag{3.4.6}$$
$$\mathrm{dom}\,(\mathcal{B}_{\mathrm{k}}^{\mathrm{p}}) \cong \mathcal{W}_{2k}^p(E). \tag{3.4.7}$$

Proof. 1: For the sake of brevity, we will prove this statement explicitly only for $\mathcal{A}$, the remaining cases follow by analogous arguments: Let $x_n \in \mathcal{W}_0^p(E)$ be a sequence with $\lim^0_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} \mathcal{A}x_n = y$ for some $y = (y_1, y_2) \in \mathcal{W}_0^p(E) \oplus \mathcal{W}_{1/2}^p(E)$. Therefore, we have both $\lim_{n\to\infty}(d^p + \delta^{p-1})x_n = y_1$ and $\lim_{n\to\infty} p_0 x_n = y_2$. We must show that both y_1 and y_2 vanish, starting with the former: For every $\sigma \in \Omega_c^\bullet(M, E)$ with $\sigma \equiv 0$ on a neighborhood of ∂M, we get by Equation 3.3.12, that

$$\langle y_1, \sigma\rangle = \lim_{n\to\infty} \langle(d^p + \delta^{p-1})x_n, \sigma\rangle = \lim_{n\to\infty} \langle x_n, (d^{p-1} + \delta^p)\sigma\rangle = 0. \tag{3.4.8}$$

Since the subspace of all such σ forms an L^2-dense subspace of $\Omega^\bullet_{(2)}(M, E)$, we must have $y_1 = 0$. Now $y_2 = 0$ in $\mathcal{W}_{1/2}^p(E|_{\partial M})$ if and only if $y_2 = 0$ in $\mathcal{W}_0^p(E|_{\partial M})$. Hence, we can use same trick of testing y_2 against an appropriate, L^2-dense subspace of $\mathcal{W}_0^p(E|_{\partial M})$. Note that we have $p_0 x_n = i_1^* x_n + \widehat{\#}^{-1} i_2^* \# x_n$. We show separately that both $\lim_{n\to\infty} i_1^* x_n = 0$ and $\lim_{n\to\infty} \widehat{\#}^{-1} i_2^* \# x_n = 0$, which then implies that $y_2 = 0$. For the first equality, it suffices to show that $\lim_{n\to\infty}\langle i_1^* x_n, z\rangle = 0$ for any $z \in \Omega_c^p(\partial_1 M, E|_{\partial_1 M})$. Therefore, we may also assume that $z|_{\partial_2 M} = 0$. We can construct a $p+1$-form $\omega \in \Omega_c^1(M, E)$, whose normal component equals z near $\partial_1 M$, vanishes near $\partial_2 M$, and whose tangential component vanishes on all of ∂M. Explicitly, this means that $p_0\omega = 0 = i_2^*\omega$ and $n_1\omega = \hat{\#}^{-1}i_1^*\#\omega = z$. Using Equation 3.3.12, we therefore have

$$\begin{aligned}
&\lim_{n\to\infty}\langle i_1^* x_n, z\rangle = \lim_{n\to\infty}\langle i_1^* x_n, \hat{\#}^{-1}i_1^*\#\omega\rangle \overset{i_2^*\omega=0}{=} \lim_{n\to\infty}\langle p_0 x_n, n_1\omega\rangle \\
&= \lim_{n\to\infty}(\langle n_1 x_n, \underbrace{p_0\omega}_{=0}\rangle + \langle (d^p + \delta^{p-1})x_n, \omega\rangle - \langle x_n, (d^{p-1}+\delta^p)\omega\rangle\big) = 0.
\end{aligned}$$

The identity $\lim_{n\to\infty}\widehat{\#}^{-1}i_2\#x_n = 0$ can be proven similarly, finally showing that $y_2 = 0$ and, hence, that $\mathcal{A}^p$ is closable.

2: From Corollary 3.3.1, we obtain that both $\mathcal{A}$ and $\mathcal{B}_\mathrm{k}^\mathrm{p}$ are uniformly elliptic. Proposition 3.3.10 then provides us with constants $C, C_k > 0$, such that

$$\begin{aligned}
C^{-1}\widetilde{||\omega||}_1 &\le ||\omega||_1 \le C\widetilde{||\omega||}_1, \\
C_k^{-1}\widetilde{||\omega||}_{2k} &\le ||\omega||_{2k} \le C_k\widetilde{||\omega||}_{2k}
\end{aligned}$$

for any $\omega \in \Omega_c^p(M, E)$, from which the result immediately follows. ☐

Beginning with the proof of the next lemma, we will make use of the the following notational conventions for subsets $A \subseteq \mathcal{W}_s^\bullet(E)$ and subspaces $V \subseteq \mathcal{W}_s^\bullet(E)$:

$$\begin{aligned}
\overline{A}^s &\coloneqq \mathcal{W}_s\text{-closure of } A \text{ inside } \mathcal{W}_s^\bullet(E), \quad (\text{with } \overline{A} \coloneqq \overline{A}^0), \\
V^{\perp_s} &\coloneqq \mathcal{W}_s\text{-orthogonal complement of } V \text{ inside } \mathcal{W}_s^\bullet(E) \quad (\text{with } V^\perp \coloneqq V^{\perp_0}).
\end{aligned}$$

We say that a subspace $A \subseteq \mathcal{W}_0^\bullet(E)$ is *s-closed* if $A = \overline{A}^s$. Recall also that, for each $p \in \mathbb{N}$, we have previously (cf. Section 2.1) defined the subspaces of forms satisfying certain boundary conditions, as well as closed, densley defined operators over their respective L^2-completions:

$$\Omega^p(M, \partial M_1, E) = \{\omega \in \Omega_c^p(M, E) : i_1^*\omega = 0\} \tag{3.4.9}$$

$$\Omega^p(M, \partial M_2, E) = \{\omega \in \Omega_c^p(M, E) : i_2^*\#\omega = 0\}, \tag{3.4.10}$$

$$d_1^p \coloneqq \overline{d|_{\Omega^p(M,\partial_1 M,E)}}\colon W_0^p(E) \to W_0^{p+1}(E), \tag{3.4.11}$$

$$\partial_1^p \coloneqq \overline{\partial|_{\Omega^{p+1}(M,\partial_2 M,E)}}\colon W_0^{p+1}(E) \to W_0^p(E). \tag{3.4.12}$$

With aid of the results on elliptic regularity and Hodge decomposition, our intermediate goal is now to show that the two operators d_1^p and ∂_1^{p+1} are mutually adjoint for each $0 \le p \le n-1$. In order to do so, we need the following auxiliary lemma:

Lemma 3.4.5 (Sobolev functions with boundary conditions)**.** *Let $E \downarrow (M, g)$ be a bundle of bounded geometry. For $p \ge 0$, define the subspaces*

$$\begin{aligned}
\mathcal{W}_1^p(E, \partial_1 M) &\coloneqq \{\omega \in \mathcal{W}_1^p(E) : i_1^*\omega = 0\} \subseteq \mathcal{W}_1^p(E), \\
\mathcal{W}_1^p(E, \partial_2 M) &\coloneqq \{\omega \in \mathcal{W}_1^p(E) : i_2^*\#\omega = 0\} \subseteq \mathcal{W}_1^p(E), \\
\mathcal{W}_2^p(E, \partial M) &\coloneqq \{\omega \in \mathcal{W}_2^p(E) : i_1^*\omega = i_1^*\delta\omega = i_2^*\#\omega = i_2^*\#d\omega = 0\}.
\end{aligned}$$

Then

1. *$\mathcal{W}_1^p(E, \partial_1 M)$ and $\mathcal{W}_1^p(E, \partial_2 M)$ are 1-closed subspaces of $\mathcal{W}_1^p(E)$,*

2. $\mathcal{W}_1^p(E,\partial_1 M) = \overline{\Omega^p(M,\partial_1 M,E)}^1 \subseteq \mathrm{dom}(d_1^p)$,

3. $\mathcal{W}_1^p(E,\partial_2 M) = \overline{\Omega^p(M,\partial_2 M,E)}^1 \subseteq \mathrm{dom}(\delta_1^p)$,

4. $\mathrm{dom}(\Delta_p[E]) = \mathcal{W}_2^p(E,\partial M)$.

Proof. 1 : follows from Theorem 3.4.4.
2 : First, we prove the inclusion $\overline{\Omega^p(M,\partial_1 M,E)}^1 \subseteq \mathrm{dom}(d_1^p)$. Therefore, let $\omega \in \overline{\Omega^p(M,\partial_1 M,E)}^1$ and choose a sequence $\omega_n \in \Omega^p(M,\partial_1 M,E)$ with $\lim_{n\to\infty} ||\omega - \omega_n||_1^2 = 0$. By Proposition 3.2.6, we get both $0 = \lim_{n\to\infty} ||\omega - \omega_n||_0^2$ and $0 = \lim_{n\to\infty} ||d^p\omega - d^p\omega_n||_0^2 = \lim_{n\to\infty} ||d^p\omega - d_1^p\omega_n||_0^2$. Therefore, $\omega \in \mathrm{dom}(d_1^p)$ (and $d_1^p\omega = d^p\omega$). The inclusion $\overline{\Omega^p(M,\partial_1 M,E)}^1 \subseteq \mathcal{W}_1^p(E,\partial_1 M)$, now follows directly from 1. The non-trivial part is to show the inclusions $\mathcal{W}_1^p(E,\partial_1 M) \subseteq \overline{\Omega^p(M,\partial_1 M,E)}^1$.
For that purpose, let $N \cong \partial M \times [0,1]$ be a regular neighborhood of ∂M. For any form $\omega \in \Omega^p(M,E)$, we can write

$$\omega|_U(x,t) = \omega_1(x,t) + \omega_2(x,t) \wedge dt \tag{3.4.13}$$

for a *tangential* form $\omega_1 \in \Omega_c^p(M,E)$ and a *normal* form $\omega_2 \in \Omega_c^{p-1}(M,E)$ that both contain no dt-factor. Since $||\omega_i||_r^2 \leq ||\omega||_r^2$ for $i = 1,2$ and any $r \geq 0$, such a decomposition into tangential and boundary parts still exists for forms in $\mathcal{W}_r^p(E)$ and varies continuously within $\mathcal{W}_r^p(E)$ (in $\mathcal{W}_r$-norm). We can write $N = N_1 \dot\cup N_2$, where for $i = 1,2$, N_i is a regular neighborhood of $\partial_i M$. For the course of the proof, we will define for $i = 1,2$ the subspaces

$$\Omega_0^p(M,\partial_i M,E) \coloneqq \{\omega \in \Omega_c^p(M,E) : \omega_i = 0 \text{ in a neighborhood of } \partial M_i\}, \tag{3.4.14}$$

$$\mathcal{W}_{1,0}^p(E,\partial_i M) \coloneqq \overline{\Omega_0^p(M,\partial_i M,E)}^1 \subseteq \mathcal{W}_1^p(E,\partial_i M). \tag{3.4.15}$$

Since $\Omega_0^p(M,\partial_i M,E) \subset \Omega^p(M,\partial_i M,E)$, 3 and 4 will be a consequence of the inclusions $\mathcal{W}_1^p(E,\partial_1 M) \subseteq \mathcal{W}_{1,0}^p(E,\partial_1 M)$ and $\mathcal{W}_1^p(E,\partial_2 M) \subseteq \mathcal{W}_{1,0}^p(E,\partial_2 M)$.
Let $\phi : \mathbb{R} \to [0,1]$ be a smooth function with $\phi \equiv 0$ on $[0,1]$, $\phi \equiv 1$ on $(2,\infty)$ and $||\phi'|| \leq 1$. For each $n \in \mathbb{N}$, we set $\phi_n(t) \coloneqq \phi(2nt)$ and define a linear map

$$F_n : \Omega_c^p(M,E) \to \Omega_c^p(M,E)$$

$$F_n(\omega) = \begin{cases} \omega & \text{on } M \setminus N, \\ \phi_n(t) \cdot \omega_1(x,t) + \omega_2(x,t)dt & \text{on } N. \end{cases}$$

Clearly, it holds that $F_n(\Omega_c^p(M,E)) \subseteq \Omega_0^p(M,\partial_1 M,E)$. Moreover, it is also clear that there exists a constant $C_n > 0$, such that $||F_n(\omega)||_1 \leq C_n||\omega||_1$. Therefore, F_n extends to a continuous map from $\mathcal{W}_1^p(E)$ to $\mathcal{W}_{1,0}^p(E,\partial_1 M)$. We claim that $\lim_{n\to\infty} ||\omega - F_n(\omega)||_1 = 0$ for any $\omega \in \mathcal{W}_1^p(E,\partial_1 M)$. This implies in particular the inclusion $\mathcal{W}_1^p(E,\partial_1 M) \subseteq \mathcal{W}_{1,0}^p(E,\partial_1 M)$.
For $0 < t \leq 1$, set $N_1^t \coloneqq \partial_1 M \times [0,t]$. Then, for any smooth form $\omega \in \Omega_c^p(M,E)$, we easily see that

$$||\omega - F_n(\omega)||_0^2 \leq ||\omega|_{N_1^n}||_0^2. \tag{3.4.16}$$

In order to properly estimate the remaining terms of $||\omega - F_n(\omega)||_1^2$, we temporarily assume that $g|_{N_1}$ is of the form $dt^2 + g_{\partial_1 M}$, i.e. g is a a product near the boundary component $\partial_1 M$. First, we establish an upper bound on $||d\phi_n \wedge \omega_1||_0^2$. For $0 < t < 1$, let $i_{1,(t)} : \partial_1 M \times \{t\} \hookrightarrow M$ be the smooth boundary inclusion at level t. Then, using Lemma 3.2.9, we can further compute

$$
\begin{aligned}
||d\phi_n \wedge \omega_1||_0^2 &= \int_0^{\frac{1}{n}} \int_{\partial_1 M} |d\phi_n \wedge \omega_1(x,t)|^2 dxdt \leq 4n^2 \int_0^{\frac{1}{n}} \int_{\partial_1 M} |\omega_1(x,t)|^2 dxdt \\
&= 4n^2 \int_0^{\frac{1}{n}} ||i_{1,(t)}^* \omega||_0^2 dt \leq 4Cn^2 \int_0^{\frac{1}{n}} ||i_1^*\omega||_0^2 + t||\omega|_{N_1^n}||_1^2 dt = 4C \left(n||i_1^*\omega||_0^2 + ||\omega|_{N_1^n}||_1^2 \right)
\end{aligned}
$$

for some constant $C > 0$ independent of n and ω. This implies that

$$
\begin{aligned}
&||d(\omega - F_n(\omega))||_0^2 = ||d(\omega - F_n(\omega))|_N||_0^2 \leq ||(1-\phi_n) \cdot d\omega||_0^2 + ||d\phi_n \wedge \omega_1||_0^2 \\
&\leq ||d\omega|_{N_1^n}||_0^2 + 4C \left(n||i_1^*\omega||_0^2 + ||\omega|_{N_1^n}||_1^2 \right). \qquad (3.4.17)
\end{aligned}
$$

Moreover, if $\widehat{\#}$ denotes the Hodge $*$-operator on the bundle $E|_{\partial_1 M} \downarrow \partial_1 M$, we find that

$$
\#\omega|_{N_1} = \widehat{\#}\omega_2 + \widehat{\#}\omega_1 dt.
$$

Using this, one computes further

$$
||\delta(\omega - F_n(\omega))||_0^2 \leq ||d\widehat{\#}\omega_1|_{N_1^n}||_0^2. \qquad (3.4.18)
$$

By continuity of all maps involved, inequalities 3.4.16, 3.4.17 and 3.4.18 also hold true for any form $\omega \in \mathcal{W}_1^p(E)$. Furthermore, since any metric g on M of bounded geometry contains in its bounded conformal class a metric that is a product near the boundary (see [84, Proposition 7.3]), these same inequalities hold true for any metric of bounded geometry (up to some constant factor).
Summarizing, we therefore obtain the following limits

$$
\lim_{n\to\infty} ||\omega - F_n(\omega)||_0^2 \leq \lim_{n\to\infty} ||\omega|_{N_1^n}||_0^2 = 0, \qquad (3.4.19)
$$

$$
\lim_{n\to\infty} ||\partial\left(\omega - F_n(\omega)\right)||_0 \leq \lim_{n\to\infty} ||d\widehat{\#}\omega_1|_{N_1^n}||_0^2 \leq \lim_{n\to\infty} ||\omega|_{N_1^n}||_1^2 = 0. \qquad (3.4.20)
$$

Moreover, if $\omega \in \mathcal{W}_1^p(E, \partial_1 M)$, we have $i_1^*\omega = 0$, hence

$$
\begin{aligned}
&\lim_{n\to\infty} ||d(\omega - F_n(\omega))||_0^2 \leq \lim_{n\to\infty} \left(||d\omega|_{N_1^n}||_0^2 + 4C||\omega|_{N_1^n}||_1^2 \right) \\
&\leq \lim_{n\to\infty} (1+4C) \cdot ||\omega|_{N_1^n}||_1^2 = 0.
\end{aligned}
$$

The equality $\lim_{n\to\infty} ||\omega - F_n(\omega)||_1^2 = 0$ follows now from Corollary 3.4.4, from which we finally obtain the inclusion $\mathcal{W}_1^p(E, \partial_1 M) \subseteq \mathcal{W}_{1,0}^p(E, \partial_1 M)$.
3 : is proven in the same way as 2.
4 : follows directly from Theorem 3.4.1 □

Proposition 3.4.6. *Let $E \downarrow M$ be a bundle of bounded geometry and $0 \leq p \leq m$. Then the following holds true:*

1. *One has $(d_1^p)^* = \delta_1^p$ and $(\delta_1^p)^* = d_1^p$.*

2. *One has $\langle \Delta_p[E]\omega, \omega\rangle = ||d_1^p\omega||^2 + ||\delta_1^{p-1}\omega||^2$ for any $\omega \in \mathrm{dom}(\Delta_p[E])$. In particular, $\Delta_p[E]$ is a positive operator.*

3. *With respect to the Hodge-decomposition 3.4.3, we can write the orthogonal complement $\Delta_p[E]^\perp$ as the direct sum of self-adjoint operators $\Delta_p[E]^\perp = ((d_1^p)^* \circ d_1^p)^\perp \oplus (d_1^{p-1} \circ (d_1^{p-1})^*)^\perp$.*

Proof. 1 : Let $\omega \in \operatorname{dom}(\delta_1^p)$. By definition, there exists then a sequence $\omega_n \in \Omega^{p+1}(M, \partial_2 M, E)$ with L^2-limits

$$\lim_{n\to\infty} \omega_n = \omega,$$
$$\lim_{n\to\infty} \delta^p \omega_n = \delta_1^p \omega.$$

Therefore, we have for any $\sigma \in \Omega^p(M, \partial_1 M, E)$, that

$$\langle d_1^p \sigma, \omega\rangle = \langle d^p \sigma, \omega\rangle = \lim_{n\to\infty} \langle d^p\sigma, \omega_n\rangle = \lim_{n\to\infty} \langle \sigma, \delta^p \omega_n\rangle + \langle \underbrace{i_1^*\sigma}_{=0}, \widehat{\#}^{-1} i_1^* \#\omega_n\rangle$$
$$+ \langle i_2^*\sigma, \widehat{\#}^{-1} \underbrace{i_2^*\#\omega_n}_{=0}\rangle = \langle \sigma, \delta_1^p\omega\rangle.$$

By definition of the adjoint, we obtain that $\omega \in (d_1^p)^*$ with $(d_1^p)^*\omega = \delta_1^p\omega$, showing that $\delta_1^p < (d_1^p)^*$ (in the sense of unbounded operators).
The inequality $(d_1^p)^* < \delta_1^p$ is considerably more difficult to show. We will proceed as in [55, Lemma 5.16]. Throughout, we will use the Hodge decomposition

$$\Omega^*_{(2)}(M, E) = \mathcal{H}^*(M, \partial M, E) \oplus \overline{d_1\Omega^{*-1}(M, \partial_1 M, E)} \oplus \overline{\delta_1 \Omega^{*+1}(M, \partial_2 M, E)}.$$

Since d_1^p is closed and densely defined, it follows that $\operatorname{im}(d_1^p)^\perp = \ker((d_1^p)^*)$. From this and the fact that $\operatorname{im}(d_1^p) \subseteq \overline{d_1^p\Omega^p(M, \partial_1 M, E)}$, it now follows that

$$\mathcal{H}^p(M, \partial M, E) \oplus \overline{\delta_1^{p+1}\Omega^{p+2}(M, \partial_2 M, E)} \subseteq \ker((d_1^p)^*).$$

Next, observe that $\delta^{p+1}\Omega^{p+2}(M, \partial_2 M, E) \subseteq \Omega^{p+1}(M, \partial_2 M, E) \cap \ker(\delta^p)$, from which immediately follows that $\overline{\delta^{p+1}\Omega^{p+2}(M, \partial_2 M, E)} \subseteq \ker(\delta_1^p)$ (here, we have used Equation 2.1.10 and that $\ker(\delta_1)$ is L^2-closed in $\Omega^*_{(2)}(M, E)$). By Theorem 3.4.1 and the previous lemma, we obtain that

$$\mathcal{H}^{p+1}(M, \partial M, E) = \ker(\Delta_{p+1}[E]) \subseteq \overline{\Omega^{p+1}(M, \partial_2 M, E)}^1 \cap \ker(\delta^p) \subseteq \ker(\delta_1^p).$$

Summarizing, we have

$$\mathcal{H}^{p+1}(M, \partial M, E) \oplus \overline{\delta^{p+1}\Omega^{p+2}(M, \partial_2 M, E)} \subseteq \ker(\delta_1^p) \cap \ker((d_1^p)^*). \tag{3.4.21}$$

Therefore, to show that $(d_1^p)^* < \delta_1^p$, it now remains to show that

$$(d_1^p)^*|_{\overline{d_1\Omega^{*-1}(M, \partial_1 M, E)}} < \delta_1^p|_{\overline{d_1\Omega^{*-1}(M, \partial_1 M, E)}}. \tag{3.4.22}$$

For this, let $\omega \in \overline{d_1\Omega^{*-1}(M, \partial_1 M, E)} \cap \operatorname{dom}((d_1^p)^*)$ and let $\sigma \in \Omega^*_{p,0}(M, E) = \{\omega \in \Omega_c^*(M, E) : i_1^*\omega = i_2^*\#\omega = 0\}$ be arbitrary. We decompose σ into its harmonic, exact and coexact parts, according to the Hodge decomposition

$$\sigma = \sigma_\Delta + \sigma_d + \sigma_\delta. \tag{3.4.23}$$

Observe that $\sigma \in \operatorname{dom}(d_1) \cap \operatorname{dom}(\delta_1) = \operatorname{dom}(d_1 + \delta_1)$. Since $\sigma_\Delta + \sigma_d \in \ker(d_1)$ and $\sigma_\Delta + \sigma_\delta \in \ker(\delta_1)$, we obtain both $\sigma_d \in \operatorname{dom}(\delta_1)$ and $\sigma_\delta \in \operatorname{dom}(d_1)$. Therefore, we compute

$$\langle (d_1^p)^*\omega, \sigma\rangle = \langle \omega, d_1\sigma\rangle = \langle \omega, d_1\sigma_\Delta\rangle + \langle \omega, d_1\sigma_d\rangle + \langle \omega, d_1\sigma_\delta\rangle$$
$$= \langle \omega, \underbrace{(d_1 + \delta_1)\sigma_\Delta}_{=0}\rangle + \langle \omega, \underbrace{(d_1 + \delta_1)\sigma_d}_{\perp\omega}\rangle + \langle \omega, \underbrace{(d_1 + \delta_1)\sigma_\delta}_{=0}\rangle = \langle \omega, (d + \delta)\sigma\rangle.$$

From [84, Theorem 4.18], we obtain that ω is *locally* integrable with a local weak derivative, i.e. $\omega \in \mathcal{W}^{p+1}_{1,loc}(E)$, that $d^{p+1}\omega, \delta^p\omega \in \mathcal{W}^*_{0,loc}(E)$ and both $i^*\#\omega, i^*\omega \in \mathcal{W}^*_{0,loc}(E|_{\partial M})$, and finally, that $(d^p_1)^*\omega = (d^{p+1} + \delta^p)\omega \in \Omega^p_{(2)}(M, E)$. Furthermore, for any $x \in \Omega^p_c(M, E)$, the equality

$$\langle x, \delta^p\omega\rangle = \langle d^p x, \omega\rangle + \langle i^*x, \widehat{\#}^{-1} i^*\#\omega\rangle \tag{3.4.24}$$

holds. We claim that $\omega \in \mathcal{W}^p_1(E, \partial_2 M)$ and that $(d^p_1)^*\omega = \delta^p\omega$. By the previous lemma, this implies that $\omega \in \mathrm{dom}(\delta^p_1)$, that $\delta^p\omega = \delta^p_1\omega$ and hence the desired equality of operators $\delta^p_1 = (d^p_1)^*$.
For this, we need to show by elliptic regularity that both $d^{p+1}\omega, \delta^p\omega \in \Omega^*_{(2)}(M, E)$, $i^*_1\omega \in \Omega^*_{(2)}(M, E)$ and $i^*_2\#\omega = 0$. Now,

$$\begin{aligned}
&\omega \in \mathrm{im}(d^p) \subseteq \ker(d^{p+1}) \implies d^{p+1}\omega = 0,\\
&\delta^p\omega = (d^{p+1} + \delta^p)\omega = (d^p_1)^*\omega \in \Omega^p_{(2)}(M, E),\\
&\omega \in \mathrm{im}(d^p_1) \implies i^*_1\omega = 0.
\end{aligned}$$

Therefore, it remains to show that $i^*_2\#\omega = 0$. For this, let $\sigma \in \Omega^{p+1}_0(M, \partial_1 M, E)$ (so that $\sigma = 0$ in a neighborhood of $\partial_1 M$). Then $\delta^p\omega \in \mathrm{dom}(d^p_1)$ and

$$\langle d^p_1\delta^p\sigma, \omega\rangle = \langle \delta^p\sigma, (d^p_1)^*\omega\rangle = \langle \delta^p\sigma, \delta^p\omega\rangle = \langle d^p_1\delta^p\sigma, \omega\rangle + \langle i^*_2\delta^p\sigma, \widehat{\#}^{-1} i^*_2\#\omega\rangle.$$

Since $i^*_2\delta^p(\Omega^{p+1}_0(M, \partial_1 M, E))$ is L^2-dense in $\Omega^p_{(2)}(\partial_2 M, E|_{\partial_2 M})$, we finally obtain that $i^*_2\#\omega = 0$ as desired.
Now since d^p_1 is closed and densely defined, and $\delta^p_1 = (d^p_1)^*$, we also get

$$(\delta^p_1)^* = ((d^p_1)^*)^* = d^p_1. \tag{3.4.25}$$

This finally proves 1.
2 : By the previous lemma, we have $\mathrm{dom}(\Delta_p[E]) = \mathcal{W}^p_2(M, \partial M)$, $\mathcal{W}^p_1(M, \partial_1 M) \subseteq \mathrm{dom}(d^p_1)$ and $\mathcal{W}^{p+1}_1(M, \partial_2 M) \subseteq \mathrm{dom}(\delta^p_1)$. In particular, we get from 1 that

$$\begin{aligned}
\mathrm{dom}(\Delta_p[E]) &\subseteq \mathrm{dom}(d^{p-1}_1\delta^{p-1}_1) \cap \mathrm{dom}(\delta^p_1 d^p_1) \cap \mathrm{dom}(d^p_1) \cap (\delta^{p-1}_1)\\
&= \mathrm{dom}(d^{p-1}_1\delta^{p-1}_1) \cap \mathrm{dom}(\delta^p_1 d^p_1) \cap \mathrm{dom}((\delta^p_1)^*) \cap \mathrm{dom}((d^{p-1}_1)^*).
\end{aligned}$$

Therefore, we can write

$$\begin{aligned}
&\langle \Delta_p[E]\omega, \omega\rangle = \langle (d^{p-1}_1\delta^{p-1}_1 + \delta^p_1 d^p_1)\omega, \omega\rangle = \langle d^{p-1}_1\delta^{p-1}_1\omega, \omega\rangle + \langle \delta^p_1 d^p_1\omega, \omega\rangle\\
&\langle \delta^{p-1}_1\omega, \delta^{p-1}_1\omega\rangle + \langle d^p_1\omega, d^p_1\omega\rangle = ||\delta^{p-1}_1\omega||^2_0 + ||d^p_1\omega||^2_0.
\end{aligned}$$

3 : We have

$$\Delta_p[E] = \delta^p_1 d^p_1 + d^{p-1}_1\delta^{p-1}_1 \stackrel{1}{=} (d^p_1)^* d^p_1 + d^{p-1}_1(d^{p-1}_1)^*.$$

From this, it is clear that $\ker(d^p_1) \cap \ker(\delta^{p-1}_1) \subseteq \ker(\Delta_p[E])$. The reverse inclusion is an immediate consequence of 2. Therefore $\ker(\Delta_p[E]) = \ker(d^p_1) \cap \ker(\delta^{p-1}_1)$ and the result follows from the Hodge decomposition 3.4.3. $\square$

3.5 Heat kernel estimates

In this section, we will generalize the important Sobolev estimates and heat kernel comparison results from [55, Theorems 2.4, 2.26] onto differential forms with values in a general flat bundle $E \downarrow M$ of bounded geometry that satisfy the assumptions of Corollary 3.3.15. To this effect, we will closely follow the methods employed there.

Throughout, we fix a bundle $E \downarrow M$ of bounded geometry and a constant $R > 0$ such that for any $0 < r \leq R$ and *any* $x \in M$, we find normal coordinates $N(2r, x)$ around x and a bundle trivialization $N(2r, x) \times \mathbb{F}^m \cong E|_{N(2r,x)}$ around x that is contained in an admissible trivialization of the whole bundle $E \downarrow M$. For the remainder of this section, let us fix some notation: For numbers $a, b, c \in \mathbb{R}$, we will write

$$a \overset{c}{\leq} b, \tag{3.5.1}$$

if $a \leq c \cdot b$ holds and

$$a \overset{c}{\simeq} b, \tag{3.5.2}$$

if both $a \overset{c}{\leq} b$ and $b \overset{c}{\leq} a$ hold.

Lemma 3.5.1. *Let $E \downarrow M$ be a flat bundle of bounded geometry and let $0 < r \leq R_E$, where R_E is the constant from Lemma 3.2.1. Furthermore, let $\phi, \psi : M \to [0,1]$ be two smooth functions with* $\operatorname{supp}(\psi) \subseteq \operatorname{supp}(\phi) \subset N(r, x_0)$ *and $\phi \equiv 1$ on* $\operatorname{supp}(\psi)$. *Via an admissible normal trivialization around x_0, we identify $E|_{N(r,x_0)}$ with $U_r(x_0) \times \mathbb{F}^m$, where $U_r(x_0) \subseteq \mathbb{R}^n_+$ is some open, relatively compact subset. Let $k, l \in \mathbb{N}$. Then there exists a constant $C_r(k,l) > 0$, depending only on k, l, and the partial derivatives of ϕ in $U_r(x_0)$ up to order $2l$, such that for each $\omega \in \Omega^\bullet(M,E) \cap \operatorname{dom}(\Delta^l[E])$, we have*

$$||\phi \cdot \omega||^2_{H^{k+2l}(\mathbb{R}^n_{\geq 0}, \mathbb{F}^m)} \overset{C_r(k,l)}{\leq} ||\phi \cdot \Delta^l \omega||^2_{H^k(\mathbb{R}^n_{\geq 0}, \mathbb{F}^m)} + ||\phi \cdot \omega||^2_{H^k(\mathbb{R}^n_{\geq 0}, \mathbb{F}^m)} + ||\psi \cdot \omega||^2_{H^{k+2l-1}(\mathbb{R}^n_{\geq 0}, \mathbb{F}^m)}. \tag{3.5.3}$$

Proof. Without loss of generality, we may assume that for each $s \in \mathbb{R}$, the Sobolev norm $||\ .\ ||_s$ defined on $E \downarrow M$ satisfies

$$||\sigma||_s \overset{C}{\simeq} ||\sigma||_{H^s(\mathbb{R}^n_+, \mathbb{F}^m)} \tag{3.5.4}$$

for an appropriate constant $C > 0$ depending only on the geometry of the bundle and for any form $\sigma \in \Omega(M,E)$ with $\operatorname{supp}(\sigma) \subseteq N(r, x_0) \cong U_r(x_0)$. This follows because there is only one equivalence class of Sobolev norms on $E \downarrow M$ (and by extension on $E \otimes \Lambda^* T^* M \downarrow M$) induced by admissible trivialization, and by assumption on r, we can find via Lemma 3.2.1 an admissible trivialization of $E \downarrow M$, whose induced Sobolev norm satisfies the above equality. Therefore, the assertion of the lemma will follow once we show that

$$||\phi \cdot \omega||_{k+2l} \overset{C_r}{\leq} (||\phi \cdot \Delta^l \omega||^2_k + ||\phi \cdot \omega||^2_k + ||\psi \cdot \omega||^2_{k+2l-1}). \tag{3.5.5}$$

By Proposition 3.3.10, we find a constant $C_1 = C_1(k,l)$, depending only on k and l, such that

$$\begin{aligned} ||\phi\omega||^2_{k+2l} &\overset{C_1}{\leq} ||\Delta^l(\phi\omega)||^2_k + ||\phi\omega||^2_k + \sum_{j=0}^{l-1} ||(i_1^* \Delta^j + i_2^* \# \Delta^j)\phi\omega||^2_{k+2(l-j)-1/2} \\ &+ |(i_1^* \delta \Delta^j + i_2^* \# d \Delta^j)\phi\omega||^2_{k+2(l-j)-3/2}. \end{aligned}$$

Since $\omega \in \operatorname{dom}(\Delta^l[E])$, we have

$$i_1^* \Delta^j \omega = i_2^* \# \Delta^j \omega = i_1^* \delta \Delta^j \omega = i_2^* \# d \Delta^j \omega = 0$$

for all $0 \leq j \leq l-1$. Therefore, the right-hand side of the above inequality can be further estimated from above by the term

$$\begin{aligned} &||\phi \Delta^l \omega|^2_k + ||[\Delta^l, \phi]\omega||^2_k + ||\phi\omega||^2_k + \sum_{j=0}^{l-1} ||(i_1^*[\Delta^j, \phi] + i_2^*[\#\Delta^j, \phi])\omega||^2_{k+2(l-j)-1/2} \\ &+ ||(i_1^*[\partial\Delta^j, \phi] + i_2^*[\# d\Delta^j, \phi])\omega||^2_{k+2(l-j)-3/2}. \end{aligned} \tag{3.5.6}$$

For $0 \leq j \leq l$, the commutators $[\Delta^j, \phi]$ and $[\#\Delta^j, \phi]$, respectively $[\partial\Delta^j, \phi]$ and $[\#d\Delta^j, \phi]$, are bounded differential operators of order $2j-1$, respectively $2j$, whose norms depend only on the partial derivatives of ϕ, as well as on the partial derivatives of the metric and Hermitian tensors, pulled back from M through the admissible normal trivialization. Since $E \downarrow M$ is a bundle of bounded geometry, the latter two terms can be estimated from above by a uniform constant, depending only on ϕ and the degree of differentiation (but *neither* on x_0 **nor** r). Also, since $\phi \equiv 1$ on $\mathrm{supp}(\psi)$, it follows that ω by $\psi\omega$ have the same image under any of the aforementioned four differential operators. Using Proposition 3.2.6, we can therefore find a constant $C_2(r) > 0$, depending only on the partial derivatives of the pullback of ϕ onto $U_r(x_0)$ (up to some finite order), such that Term 3.5.6 is bounded from above by $C_2(r)$-times:

$$||\phi\Delta^l\omega||_k^2 + ||\psi\omega||_{k+2l-1}^2 + ||\phi\omega||_k^2. \tag{3.5.7}$$

The result now follows. $\square$

Lemma 3.5.2. *In the setting of the previous lemma, with $\sigma_r := \sigma|_{N(r,x_0)}$ for any $\sigma \in \Omega^*_{(2)}(M,E)$, there exists a constant $C_r > 0$, depending only on k and the partial derivatives of ϕ and ψ on $U_r(x_0)$, such that we have for any $k \geq 1$ and any $\omega \in \Omega^\bullet(M,E) \cap \mathrm{dom}(\Delta^l[E])$.*

$$||\phi\omega||_{H^{2k}(\mathbb{R}^n_{\geq 0},\mathbb{F}^m)} \tag{3.5.8}$$

$$\overset{C_r(k)}{\leq} \begin{cases} ||(\Delta^k\omega)_r||_0^2 + ||(\Delta^{k-1}\omega)_r||_0^2 + ||(\Delta\omega)_r||_0^2 + ||\omega_r||_0^2 + ||\psi\omega||_{H^{2k-2}(\mathbb{R}^n_{\geq 0},\mathbb{F}^m)} & k > 1 \\ ||(\Delta^k\omega)_r||_0^2 + ||(\Delta^{k-1}\omega)_r||_0^2 + ||(\Delta\omega)_r||_0^2 + ||\omega_r||_0^2 + ||\psi\omega||_0 & k = 1 \end{cases} \tag{3.5.9}$$

Proof. Firstly, one verifies via elementary computations (cf. [55, Lemma 2.5]) that

$$||d(f\cdot\sigma)||_0^2 + ||\delta(f\cdot\sigma)||_0^2 = \langle\Delta\sigma, f^2\sigma\rangle + ||df\wedge\sigma||_0^2 + ||df\wedge\#\sigma||_0^2 \tag{3.5.10}$$

for any $\sigma \in \Omega^\bullet(M,E) \cap \mathrm{dom}(\Delta[E])$ and any $f \in C^\infty(M,\mathbb{C})$. Choose an *intermediate* smooth cut-off function $\widehat{\phi} : M \to [0,1]$ with $\phi \equiv 1$ on $\mathrm{supp}(\widehat{\phi})$, $\widehat{\phi} \equiv 1$ on $\mathrm{supp}(\psi)$ and such that the partial derivatives of $\widehat{\phi}$ can be estimated from above and below by the partial derivatives of ϕ and ψ (on $U_r(x_0)$). Using the previous lemma twice, we obtain

$$\begin{aligned} ||\phi\omega||^2_{H^{2k}(\mathbb{R}^n_{\geq 0},\mathbb{F}^m)} &\overset{C(0,k)}{\leq} ||\phi\omega||_0^2 + ||\phi\Delta^k\omega||_0^2 + ||\widehat{\phi}\omega||^2_{H^{2k-1}(\mathbb{R}^n_{\geq 0},\mathbb{F}^m)} \\ &\overset{C(1,k-1)}{\leq} ||\omega_r||_0^2 + ||(\Delta^k\omega)_r||_0^2 + ||\widehat{\phi}\omega||_1^2 + ||\widehat{\phi}\Delta^{k-1}\omega||_1^2 + ||\psi\omega||^2_{H^{2k-2}(\mathbb{R}^n_{\geq 0},\mathbb{F}^m)} \end{aligned}$$

for constants $C_r(0,k)$ and $C_r(1,k-1)$ depending only on the partial derivatives of ϕ and ψ on $U_r(x_0)$. Furthermore, we compute

$$\begin{aligned} ||\widehat{\phi}\omega||_1^2 &\overset{D}{\leq} ||\widehat{\phi}\omega||_0^2 + ||d(\widehat{\phi}\omega)||_0^2 + ||\delta(\widehat{\phi}\omega)||_0^2 \\ &\leq ||\omega_r||_0^2 + \langle\Delta\omega, \widehat{\phi}^2\omega\rangle + ||d\widehat{\phi}\wedge\omega|||_0^2 + ||d\widehat{\phi}\wedge\#\omega||_0^2 \\ &\overset{1+2\cdot\sup|\widehat{\phi}|_1}{\leq} ||\omega_r||_0^2 + ||(\Delta\omega)_r||_0 \cdot ||\widehat{\phi}^2\omega||_0 \leq 2||\omega_r||_0^2 + ||(\Delta\omega)_r||_0^2. \end{aligned}$$

Here, we have used (in order) Proposition 3.3.10 for the constant $D > 0$ that depends only on the geometry of the bundle $E \downarrow M$, Equation 3.5.10, the Cauchy-Schwarz inequality and the fact that $ab \leq a^2 + b^2$ for any two real numbers a, b. The resulting constant C_r' therefore depends only on first partial derivatives of ϕ and ψ. Analogously, one obtains an estimate

$$||\widehat{\phi}\Delta^{k-1}\omega||_1^2 \overset{C_r''}{\leq} ||(\Delta^{k-1}\omega)_r||_0^2 + ||(\Delta^k\omega)_r||_0^2.$$

Putting all of the inequalities together yields the desired result. $\square$

Lemma 3.5.3. *Let $m, k \in \mathbb{N}$ be fixed integers. Then, for each $1 \leq l \leq k$, there exists a smooth 1-parameter family of bump functions $\phi_l[r] \in C^\infty(\mathbb{R}^n, [0,1])$, varying smoothly in $r \in \mathbb{R}^+$ and satisfying for each $r > 0$:*

1. *If $1 \leq l \leq k-1$, we have $\operatorname{supp}(\phi_{l+1}[r]) \subseteq \operatorname{supp}(\phi_l[r])$ and $\phi_l[r] \equiv 1$ on $\operatorname{supp}(\phi_{l+1}[r])$.*
2. *$\phi_k[r](0) = 1$, and*
3. *$\operatorname{supp}(\phi_1[r]) \subseteq B_r(0)$.*

Proof. Denote by $| \ . \ |$ the standard Euclidean norm on $\mathbb{R}^n$. For fixed $k \in \mathbb{N}$, each $r > 0$ and each $1 \leq l \leq k$, define

$$\phi_l[r](x) := \begin{cases} 1 & |x| \leq \frac{r}{l+1}, \\ 1 - \left(1 + \exp(-\frac{l^2}{l^2|x|^2 - r^2} + \frac{(l+1)^2}{(l+1)^2|x|^2 - r^2})\right)^{-1} & \frac{r}{l+1} < |x| < \frac{r}{l}, \\ 0 & |x| \geq \frac{r}{l}. \end{cases} \tag{3.5.11}$$

It is clear that the family $(\phi_l[r])_{1 \leq l \leq k}^{r \in \mathbb{R}^+}$ has the required properties. □

Theorem 3.5.4 (Sobolev estimates)**.** *Let $(E,h) \downarrow (M,g)$ be a flat bundle of bounded geometry. Then there exists a smooth, monotonically decreasing function $C : \mathbb{R}^+ \to \mathbb{R}^+$, such that for each $r > 0$, each $x_0 \in M$ with the property that x_0 admits r-normal coordinates, and each form $\omega \in \Omega^\bullet(M,E)$ satisfying relative boundary conditions, we have*

$$|\omega(x_0)|^2_{h(x_0)} \overset{C(r)}{\leq} \sum_{i=0}^{m} ||(\Delta^i \omega)|_{N(x_0,r)}||_0^2. \tag{3.5.12}$$

Proof. Again, we will use the abbreviation $\omega_r := \omega_{N(x_0,r)}$. Also, denote by $| \ . \ |$ the standard norm on $\mathbb{F}^m$. Since $E \downarrow M$ is of bounded geometry, there exists a universal constant $C_h > 0$, such that for any $x_0 \in M$, any admissible normal trivialization of the bundle $E \downarrow M$ around x_0 and any form $\omega \in \Omega(E)$, we have

$$|\omega(x_0)|^2_{h(x_0)} \leq C_h |\omega(x_0)|^2, \tag{3.5.13}$$

where ω is regarded as a $\mathbb{F}^m$-valued form via the aforementioned trivialization. Identifying $N(r, x_0)$ with $B(r, 0) \subseteq \mathbb{R}^n$, let $\phi_1[r], \ldots, \phi_{2m}[r]$ be the family of smooth bump functions from the previous lemma, each supported on a neighborhood of x_0, so that $\operatorname{supp}(\phi_1) \subseteq \operatorname{supp}(\phi_2) \subseteq \cdots \subseteq \operatorname{supp}(\phi_{2m}) \subseteq N(r, x_0)$, satisfying $\phi_{i+1} \equiv 1$ on $\operatorname{supp}(t_i)$ for each $0 \leq i \leq 2m$. Then we first obtain be the Sobolev lemma on Euclidean space the following inequality

$$|\omega(x_0)|^2 = |\phi_{2m}\omega(x_0)|^2 \overset{C_1}{\leq} ||\phi_{2m}\omega||^2_{H^{2m}(\mathbb{R}^n, \mathbb{F}^m)} \tag{3.5.14}$$

with a constant $C_1 > 0$ depending only on the dimension m of M. Now, by an inductive application of the previous corollary to ω, ϕ_{i+1} and ϕ_i, we obtain

$$||\phi_{2m}\omega||^2_{H^{2m}(\mathbb{R}^n, \mathbb{F}^m)} \overset{C_2[r]}{\leq} \sum_{i=0}^{m} ||(\Delta^i \omega)_r||_0^2, \tag{3.5.15}$$

where the constant $C_2[r] > 0$ depends only on the ∞-norm of the $\phi_i[r]'s$ and their respective partial derivatives. Now since $\phi_i[r]$ varies smoothly in r (for fixed i), it is apparent that the constant $C_2[r]$ also varies smoothly (and monotonically decreasing) in r. The result now follows. □

Theorem 3.5.5 (Properties of the solution to the wave equation)**.** *Let (M, g) be a Riemannian manifold of bounded geometry and let $(E, h) \downarrow M$ be a flat Hermitian bundle of bounded geometry over M. Further, let $u \in \Omega^p(M; E)$ be a smooth p-form, compactly supported in $\mathring{M}$. Then, for any $s \in \mathbb{R}_{\geq 0}$, the p-form $\cos(s\sqrt{\Delta_p[E]})u \in \Omega^p_{(2)}(M; E)$, defined via the spectral theorem, is sufficiently smooth in $\mathbb{R}_{\geq 0} \times M$ and the* ***unique*** *(sufficiently smooth) solution to the wave equation*

$$\frac{\partial^2}{\partial s^2}v + \Delta_p[E]v = 0, \tag{3.5.16}$$

$$v(0, x) = u(x), \tag{3.5.17}$$

$$\frac{\partial}{\partial s}v(0, x) = 0. \tag{3.5.18}$$

Moreover, the support of the solution propagates at unit speed in time from $\mathrm{supp}(u)$*. Explicitly, this means that we have*

$$\mathrm{supp}(\cos(s\Delta_p[E])u) \subseteq B_s(\mathrm{supp}(u)). \tag{3.5.19}$$

Proof. It is well-known that $v(s, x) \coloneqq \cos(s\sqrt{\Delta_p[E]})u$ is the unique sufficiently smooth solution for the wave equation, see for example [25], with the unit propagation speed being also well-established in the case that u is a *function*, see [95, Theorem 6.1]. What remains to be shown is that the unit-propagation speed property of a solution $v(s, x)$ which we assume to be twice continuously differentiable in s, also holds for forms. For notational convenience, we will abbreviate $v_s \coloneqq \frac{\partial}{\partial s}v$. In this notation, the solution v satisfies

$$v_{ss} + \Delta_p v \equiv 0, \tag{3.5.20}$$

$$v(0, x) = u(x), \tag{3.5.21}$$

$$v_s(0, x) = 0 \tag{3.5.22}$$

for all $(s, x) \in \mathbb{R}_{\geq 0} \times M$. Let $x_0 \in \mathring{M}$ and let $R_M > 0$ denote the positive injectivity radius of the complete manifold (M, g). Further, let $0 < r < R_M$, so that $\exp_x : T_xM \supset B_r(0) \to B(x_0, r)$ is a diffeomorphism onto the geodesic ball $B(x_0, r)$ of radius r around x_0.

Claim 1: Let $0 < r < R_M$ and suppose that $v(t_0, y) = 0$ for all $y \in B(x_0, r)$ and some $t_0 \in \mathbb{R}_{\geq 0}$. Then $v \equiv 0$ on the cone $C(x_0, r, t_0) \coloneqq \{(s, y) \in \mathbb{R}_{\geq 0} \times M : t_0 \leq s \leq t_0 + r$ and $y \in U(x_0, r + t_0 - s)\}$.

Proof of Claim 1: We may assume without loss of generality that $t_0 = 0$. Define $0 \leq s \leq r$ the *energy functional*

$$E(s) \coloneqq \frac{1}{2}\int_{B(x_0, r-s)} ||v_s(s, y)||^2_{h(y)} + ||dv(s, y)||^2_{h(y)} + ||\delta v(s, y)||^2_{h(y)}\, dvol_g.$$

Then $E \in C^2([0, r], \mathbb{R})$. Moreover, note that s-differentiation commutes with both d and δ, i.e. we have both

$$(dv)_s = d(v_s) =: dv_s,$$

$$(\delta v)_s = \delta(v_s) =: \delta v_s.$$

For notational simplicity, we will set $B_s \coloneqq B(x_0, r - s)$, $\partial B_s \coloneqq \partial B(x_0, r - s)$ and denote by $dVol_{\partial B_s}$ the Riemannian volume form on the submanifold ∂B_s induced by the restriction of g. Using Green's theorem and Equation 3.5.20, one now computes that

$$\frac{\partial}{\partial s}E(s) = \Re(\int_{B_s} \langle v_s, v_{ss}\rangle + \langle dv_s, dv\rangle + \langle \delta v_s, \delta v\rangle \, dVol_M$$
$$- \frac{1}{2}\int_{\partial B_s} ||v_s||^2 + ||dv||^2 + ||\delta v||^2 dVol_{\partial B_s}) = \Re(\int_{B_s} \langle v_s, v_{ss} + \Delta_p v\rangle dVol_M$$
$$+ \int_{\partial B(x_0, r-s)} i_s^* v_s \wedge \widehat{\#}^{-1} i_s^* \# dv - \widehat{\#}^{-1} i_s^* \# v_s \wedge i_s^* \delta v - \frac{1}{2}\left(||v_s||^2 + ||dv||^2 + ||\delta v||^2\right) \, dVol_{\partial B_s})$$
$$= \Re\left(\int_{\partial B_s} i_s^* v_s \wedge \widehat{\#}^{-1} i_s^* \# dv - \widehat{\#}^{-1} i^* \# v_s \wedge i_s^* \delta v - \frac{1}{2}\left(||v_s||^2 + ||dv||^2 + ||\delta v||^2\right) \, dVol_{\partial B_s}\right).$$

Here, as before, $\# : \Omega^\bullet(M, E) \to \Omega^{n-\bullet}(M, E^*)$ and $\widehat{\#} : \Omega^\bullet(\partial B_s, E_{\partial B_s}) \to \Omega^{n-1-\bullet}(\partial B_s, E^*_{\partial B_s})$ denote the isometric Hodge-$*$ operators on the respective twisted de Rham complexes induced by the Riemannian metric g and the Hermitian metric h of E. Furthermore, $i_s^* : \Omega^\bullet(M, E) \to \Omega^\bullet(\partial B_s, E_{\partial B_s})$ denotes the tangential boundary projection induced by the smooth inclusion $\partial B_s \subset M$. By Gauss's lemma, one has for any $\omega \in \Omega^\bullet(M, E)$, that

$$||\omega||^2 = ||i_s^* \omega||^2 + ||i_s^* \# \omega||^2 \tag{3.5.23}$$

on all of ∂B_s. Next, note that for any two differential forms ω, σ of complementary dimensions, one has $||\omega \wedge \sigma|| = |\langle \omega, \#^{-1}\sigma\rangle| \leq ||\omega|| \cdot ||\#^{-1}\sigma|| = ||\omega|| \cdot ||\sigma||$. Along with the triangle inequality and the elementary fact that $ab \leq \frac{1}{2}(a^2 + b^2)$ for any two real numbers $a, b \in \mathbb{R}$, one obtains that

$$||i_s^* v_s \wedge \widehat{\#}^{-1} i_s^* \# dv - \widehat{\#}^{-1} i_s^* \# v_s \wedge i_s^* \delta v|| \leq ||i_s^* v_s|| \cdot ||\widehat{\#}^{-1} i_s^* \# dv|| + ||\widehat{\#}^{-1} i_s^* \# v_s|| \cdot ||i_s^* \delta v||$$
$$= ||i_s^* v_s|| \cdot ||i_s^* \# dv|| + ||i_s^* \# v_s|| \cdot ||i_s^* \delta v|| \leq \frac{1}{2}\left(||i_s^* v_s||^2 + ||i_s^* \# v_s||^2 + ||i_s^* \# dv||^2 + ||i_s^* \delta v||^2\right)$$
$$\leq \frac{1}{2}\left(||v_s||^2 + ||dv||^2 + ||\delta v||^2\right).$$

Therefore, $\frac{\partial}{\partial s}E(s) \leq 0$ for all $0 \leq s \leq r$. Since $v_s(0, y) = 0 = v(0, y)$ for all $y \in B(x_0, r)$, we obtain that $E(0) = 0$, and therefore also $E(s) = 0$ for all $0 \leq s \leq r$. This in return implies that $v_s(s, y) = 0$ for all $y \in B(x_0, r - s)$, from which then also follows that $v(s, y) = 0$. This finally proves Claim 1.

Now let $B(x_0, r)$ denote a general geodesic ball of some positive radius $r \geq R_M$ around some point x_0.

Claim 2: Let $r \geq R_M$ and suppose that $v(0, y) = 0$ for all $y \in B(x_0, r)$. Then $v \equiv 0$ on the cone $C(x_0, r, 0) := \{(s, y) \in \mathbb{R}_{\geq 0} \times M : 0 \leq s \leq r \,\textbf{and}\, y \in B(x_0, r - s)\}$. Observe that Claim 2 obviously immediately implies the unit propagation speed of the solution.

Proof of Claim 2: First, observe that for any $0 \leq s < r$, we have equality of sets

$$B(x_0, r) = \bigcup_{x \in B(x_0, r-s)} B(x, s). \tag{3.5.24}$$

The inclusion $B(x_0, r) \subseteq \bigcup_{x \in B(x_0, r-s)} B(x, s)$ follows from the fact that, due to completeness of M, any $y \in B(x_0, r)$ can be connected to x_0 by a minimizing geodesic γ. By the intermediate value theorem, we therefore find some $x \in \gamma$ with $d(y, x) < s$ and $d(x, x_0) < r - s$. The reverse inclusion immediately follows from the triangle inequality.

Now choose some fixed $0 < \tau < R_M \leq r$ and an integer $k \in \mathbb{N}$ so that $0 < r - k\tau < R_M \leq r - (k - 1)\tau$. For $1 \leq l \leq k$, set $U_l := \bigcup_{x \in B(x_0, r - l\tau)} C(x, \tau, (l - 1)\tau)$. Equation 3.5.24 now implies all of the following equalities of sets

$$\begin{aligned}
C(x_0, r, 0) &= \bigcup_{l=1}^{k} U_l \cup C(x_0, r - k\tau, k\tau),\\
U_{l+1} \cap U_l &= \bigcup_{x \in B(x_0, r-(l+1)\tau)} \{l\tau\} \times B(x, \tau),\\
U_k \cap C(x_0, r - k\tau, k\tau) &= \{k\tau\} \times B(x_0, r - k\tau).
\end{aligned}$$

Therefore, using the assumption of the claim and starting with U_1, an iterative application of Claim 1 can be used to show that $u \equiv 0$ on all U_l and $C(x_0, r - k\tau, k\tau)$, and therefore also on $C(x_0, r, 0)$.

$\square$

Theorem 3.5.6 (Heat kernel estimates)**.** *Let (M, g) be a Riemannian manifold of bounded geometry and $E{\downarrow}M$ a* ***flat*** *trivial bundle of bounded geometry over M. Further, let $N \subseteq M$ be a (topologically) closed submanifold and let $\tilde{E}{\downarrow}N$ be the flat bundle over N, obtained by restriction of E to N. For $p \geq 0$, let $\Delta_p[E_M]$ and $\Delta_p[E_N]$ be the corresponding Bochner-Laplace operators on twisted p-forms on the restriction bundles E_M, respectively E_N. For $t > 0$, $k \in \mathbb{N}_0$ and $x, y \in N$, denote by*

$$\Delta_p^k[E_M]e^{-t\Delta_p[E_M]}(x, y) : E_x \to E_y, \tag{3.5.25}$$

$$\Delta_p^k[E_N]e^{-t\Delta_p[E_N]}(x, y) : E_x \to E_y \tag{3.5.26}$$

the respective smooth heat kernels.
Then the following two results hold true:

1. *There exists a constant $\kappa > 0$ depending only on the dimension of M, and, for each $k \in \mathbb{N}$ and any $D > 0$, a constant $C_k(D) > 0$, depending only on the bundle geometry of $(E, h) \downarrow M$ (but* ***not*** *on N), such that for any pair $x_0, y_0 \in N$ with $d_N(x_0) := d(x_0, M \setminus N) \geq D$ and $d_N(y_0) \geq D$, we have the inequality*

$$||\Delta_p[E_M]^k e^{-t\Delta[M]}(x_0, y_0) - \Delta_p[E_N]^k e^{-t\Delta[N]}(x_0, y_0)|| \leq C_k(D)e^{-\frac{d_M(x_0)+d_M(y_0)+2d(x_0,y_0)}{\kappa t}}. \tag{3.5.27}$$

2. *For any $t_0 > 0$, there exists a constant $c(t_0)$, such that for all $t \geq t_0$, we have*

$$||e^{-t\Delta_p[E_M]}(x, y)|| \leq c(t_0), \tag{3.5.28}$$

$$||e^{-t\Delta_p[E_N]}(x, y)|| \leq c(t_0). \tag{3.5.29}$$

Proof. 1: We will proceed as in the proof of [55, Theorem 2.26]. An elementary, yet essential observation we will take advantage of is the fact that, for E-valued p-forms ω, compactly supported *inside* $\mathring{N}$ (so that, in particular, ω lies in the domain of both $\Delta_p[E_N]$ and $\Delta_p[E_M]$), we have

$$\Delta_p[E_N]\omega = \Delta_p[E_M]\omega \qquad \text{in } \mathring{N}. \tag{3.5.30}$$

Throughout this proof, the symbol Δ stands for both $\Delta_p[E_M]$ and $\Delta_p[E_N]$. By well-known properties of the Fourier Transform of complex-valued functions and the spectral theorem, we have for any triple m, l, k of non-negative integers and any $t > 0$ the following equality:

$$\Delta^m \Delta^l \Delta^k e^{-t\Delta} = \frac{(-1)^{m+l+k}}{\sqrt{\pi t}} \cdot \int_0^\infty \frac{d^{2(m+l+k)}}{ds^{2(m+l+k)}} e^{-s^2/4t} \cos(s\Delta) ds. \tag{3.5.31}$$

Now let x_0 and y_0 be two points satisfying the assumptions of the theorem. Then the geodesic ball $B_{D/4}(y_0)$ of radius $D/4$ around y_0 is the same for M and N, and we have $B_{D/4}(y_0) \subseteq \mathring{N}$. Choose a p-form u with compact support inside $B_{D/4}(y_0)$, so that u lies in the domain of both $\Delta_p[E_N]$ and $\Delta_p[E_M]$. Then, by the previous equation, there exists for any triple m, l, k of integers a universal polynomial $P : \mathbb{R}^2 \to \mathbb{R}$, independent of x_0, y_0 and N, such that for

$$f := (\Delta[N]^k e^{-t\Delta[N]} - \Delta[M]^k e^{-t\Delta[N]})u, \tag{3.5.32}$$

we have

$$\Delta^m \Delta^l f = \int_0^\infty t^{-2(m+l+k)-1/2} P(\sqrt{t}, s) e^{-s^2/4t} (\cos(s\sqrt{\Delta_p[E_M]}) - \cos(s\sqrt{\Delta_p[E_N]}))u \, ds. \tag{3.5.33}$$

Next, we will need to find a good upper bound for $||\Delta^m \Delta^l f||_{B_{D/4}(x_0)}$. For that precise purpose, we first prove the following:

Claim 1: For any $0 \le s < L(x_0, y_0) := \max\{d(x_0, y_0)/2, d_N(y_0)/2\}$, we have

$$(\cos(s\sqrt{\Delta_p[E_M]}) - \cos(s\sqrt{\Delta_p[E_N]}))u = 0 \qquad \text{on } B_{D/4}(x_0). \tag{3.5.34}$$

Proof of Claim 1: By definition, we have that for any $s < d_N(y_0)/2$ the geodesic ball $B_s(\mathrm{supp}(u))$ satisfies $B_s(\mathrm{supp}(u)) \subseteq B_{s+D/4}(y_0) \subseteq B_{\frac{3}{4}d_N(y_0)}(y_0) \subseteq \mathring{N}$ since $D \le d_N(y_0)$ by assumption. From the unit propagation speed property of Theorem 3.5.5 and Equation 3.5.30, it follows that

$$\Delta_p[E_M] \cos(s\sqrt{\Delta_p[E_M]})u = \Delta_p[E_N] \cos(s\sqrt{\Delta_p[E_M]})u \quad 0 \le s < d_N(y_0)/2. \tag{3.5.35}$$

This implies that $\cos(s\sqrt{\Delta_p[E_M]})u$ satisfies the wave equation on $[0, d_N(y_0)/2] \times N$, given by Theorem 3.5.5 with $\Delta_p = \Delta_p[E_N]$, which, by the first part of that theorem, is *uniquely* solved on that domain by $\cos(s\sqrt{\Delta_p[E_N]})u$ (and vice versa). It follows that $\cos(s\sqrt{\Delta_p[E_M]})u = \cos(s\sqrt{\Delta_p[E_N]})u$ for $0 \le s < d_N(y_0)/2$. If $d(x_0, y_0) \ge d_N(y_0) \ge D$, we have $B_{D/4}(x_0) \cap B_s(\mathrm{supp}(u)) = \emptyset$ for any $s < d(x_0, y_0)/2$, so by the unit propagation speed property of Theorem 3.5.5, applied to both $\cos(s\sqrt{\Delta_p[E_M]})u$ and $\cos(s\sqrt{\Delta_p[E_N]})u$, we have $\cos(s\sqrt{\Delta_p[E_M]})u = \cos(s\sqrt{\Delta_p[E_N]})u \equiv 0$ on $B_{D/4}(x_0)$, finally proving Claim 1.

Together with the observation that $|\cos(r)| \le 1$ for all $r \in \mathbb{R}$ and the spectral theorem, we can now compute

$$\begin{aligned} ||\Delta^m \Delta^l f||_{B_{D/4}(x_0)} &\le 2\left(\int_{L(x_0,y_0)}^\infty t^{-2(m+l+k)-1/2} P(\sqrt{t}, s) e^{-s^2/4t} ds\right) \cdot ||u||_0 \\ &\le 2C_{m,l,k} e^{-L(x_0,y_0)^2/4t} \cdot ||u|| \end{aligned} \tag{3.5.36}$$

for an appropriate constant $C_{m,l,k} > 0$ independent of x_0, y_0 or N. We further obtain a pointwise estimate

$$|\Delta^l f(x_0)| \le C'_{l,k}(D) e^{-L(x_0,y_0)^2/4t} \cdot ||u||_0, \tag{3.5.37}$$

for an appropriate constant $C'_{l,k}(D)$ independent of x_0, y_0 or N. This follows from the previous estimate, along with the Sobolev estimates from Theorem 3.5.4 and the assumption that (M, g) is of bounded geometry. Now, the definition of f, together with the equality $\Delta^n e^{-t\Delta} = e^{-t\Delta}\Delta^n$ for any $n \in \mathbb{N}$, and an iterative application of Stokes' theorem imply that

$$\Delta^l f(x_0) = \int_M \Delta_y^l (\Delta[N]^k e^{-t\Delta[N]}(x_0, y) - \Delta[M]^k e^{-t\Delta[N]}(x_0, y)) u(y) dy, \tag{3.5.38}$$

where Δ_y^l denotes the appropriate Laplacian in y-coordinates. Choosing a sequence of smooth functions $(u_n)_{n\in\mathbb{N}} \subset L^2(B_{D/4}(y_0))$, with compact support inside $B_{D/4}(y_0)$ and converging in L^2-norm to the restriction of $\Delta^l(\Delta[N]^k e^{-t\Delta[N]}(x_0,y) - \Delta[M]^k e^{-t\Delta[N]}(x_0,y)) \in L^2(B_{D/4}(y_0))$, we obtain from 3.5.37, that

$$||\Delta^l(\Delta[N]^k e^{-t\Delta[N]}(x_0,y) - \Delta[M]^k e^{-t\Delta[N]}(x_0,y))||_{L^2(B_{D/4}(y_0))} \leq C'_{l,k}(D)e^{-L(x_0,y_0)^2/4t}.$$

In the very same way as above, we use Theorem 3.5.4 and the assumption that (M,g) is of bounded geometry (and therefore, also the associated bundle $\pi_1^*E^* \otimes \pi_2^*E \downarrow M \times M$) to pass from the above L^2-estimate to a point-wise estimate

$$|(\Delta[N]^k e^{-t\Delta[N]}(x_0,y_0) - \Delta[M]^k e^{-t\Delta[N]}(x_0,y_0))| \leq C_k(D)e^{L(x_0,y_0)^2/4t} \tag{3.5.39}$$

for a constant $C_k(D)$ as in the original assertion of the theorem. Analogously, swapping the roles of x_0 and y_0 and using the fact that the heat kernel is adjoint-symmetric, i.e $\Delta^k e^{-t\Delta}(x,y) = (\Delta^k e^{-t\Delta}(y,x))^*$ holds for any appropriate pair x,y, we obtain

$$|(\Delta[N]^k e^{-t\Delta[N]}(x_0,y_0) - \Delta[M]^k e^{-t\Delta[N]}(x_0,y_0))| \tag{3.5.40}$$

$$= |(\Delta[N]^k e^{-t\Delta[N]}(y_0,x_0) - \Delta[M]^k e^{-t\Delta[N]}(y_0,x_0))^*| \tag{3.5.41}$$

$$= |(\Delta[N]^k e^{-t\Delta[N]}(y_0,x_0) - \Delta[M]^k e^{-t\Delta[N]}(y_0,x_0))| \leq C_k(D)e^{L(y_0,x_0)^2/4t}, \tag{3.5.42}$$

where $L(y_0,x_0) \coloneqq \max\{d(x_0,y_0)/2, d_N(x_0)/2\}$. Since
$\max\{L(x_0,y_0), L(y_0,x_0)\} \geq \frac{1}{8}(d_N(x_0) + d_N(y_0) + 2d(x_0,y_0))$, the result follows.
2 : is proven using the same methods as in 1, cf. [55, Theorem 2.35]. □

Chapter 4

Analytic torsion

Applying the technical results from the previous chapter, we can finally define in general the analytic L^2-invariants associated to a (not necessarily compact) given manifold-with-boundary M and a representation $\rho : \pi_1(M) \to \mathrm{GL}(V)$ of its fundamental group, cf. Definition 4.2.3. As the title of this chapter indicates, of particular interest is the *analytic L^2-torsion* $T^{An}_{(2)}(M, \rho) \in \mathbb{R}_{>0}$, which can only be defined if the pair (M, ρ) satisfies the technical *det-L^2-acyclicity* condition.
Regarding the main results of this thesis, we will then focus on a torsion-free lattice $\Gamma < G := \mathrm{Isom}^+(\mathbb{H}^n)$ of orientation-preserving isometries on odd-dimensional hyperbolic n-space and an irreducible representation $\rho : G \to \mathrm{GL}(V)$. Recall from Section 2.3 the associated exhaustion $(M_R)_{R\in\mathbb{R}_{\geq 0}}$ of $\mathbb{H}^n$ by Γ-invariant submanifolds. From Section 4.2.3 onwards, we will show that the pair $(\Gamma\backslash \mathbb{H}^n, \rho)$ as well as each pair of the family $\{(\Gamma\backslash M_R, \rho) : R \in \mathbb{R}_{\geq 0}\}$ meets the det-L^2-acyclicity condition. Thus, we obtain L^2-torsion elements $T^{An}_{(2)}(\Gamma\backslash \mathbb{H}^n, \rho)$ and $T^{An}_{(2)}(\Gamma\backslash M_R, \rho)$. The two main results of this chapter, Theorem 4.2.21 and Theorem 4.3.7, then establish the *large-time convergence* and the *small-time convergence* of the respective summands in the function $T^{An}_{(2)}(\Gamma\backslash M_R, \rho)$ as $R \to \infty$. Taken together, these then imply the fundamental convergence result $\lim_{R\to\infty} T^{An}_{(2)}(\Gamma\backslash M_R, \rho) = T^{An}_{(2)}(\Gamma\backslash \mathbb{H}^n, \rho)$ from Theorem 2.3.13.
First of all, however, we will introduce in Section 4.1 the algebraic foundation of general L^2-invariants: *Hilbert $\mathcal{N}(\Gamma)$-Modules and Hilbert $\mathcal{N}(\Gamma)$-cochain complexes.* This way, we also provide the framework, with the aid of which the combinatorial L^2-invariants of the next chapter are defined.

4.1 Hilbert $\mathcal{N}(\Gamma)$-modules

Throughout, we fix a countable group Γ. We denote by $L^2(\Gamma)$ the complex Hilbert space with orthonormal basis the set Γ. It comes equipped with a natural left, linear Γ-action by isometries, arising as the extension of the left multiplication by Γ on itself.

Definition 4.1.1. A **Hilbert $\mathcal{N}(\Gamma)$-module** is a complex Hilbert space $\mathcal{H}$, equipped with a left, linear Γ-action by isometries, such that there exists a separable Hilbert space H and an isometric, Γ-equivariant embedding $\mathcal{H} \hookrightarrow L^2(\Gamma)\hat{\otimes}H$. Here, $\hat{\otimes}$ denotes the *tensor product of complex Hilbert spaces* and the Γ-action on $L^2(\Gamma)\hat{\otimes}H$ is the natural one induced by the canonical isometric Γ-action on the left factor (as described above). $\mathcal{H}$ is called *finitely generated* if we can chose $H = \mathbb{C}^n$ in the above identification.

A (not necessarily bounded) linear operator $f : \mathcal{H} \to \mathcal{H}'$ between two Hilbert $\mathcal{N}(\Gamma)$-modules is called a **morphism** of Hilbert $\mathcal{N}(\Gamma)$ modules if it is closed, densely defined and Γ-equivariant. Here, Γ-equivariant means that $\Gamma.\operatorname{dom}(f) = \operatorname{dom}(f)$ and $g.f(x) = f(g.x)$ for any $x \in \operatorname{dom}(f)$ and $g \in \Gamma$.
If f is, additionally, bounded, we say that f is a *bounded morphism (of Hilbert $\mathcal{N}(\Gamma)$-modules)*. Observe that by the *closed graph theorem*, any bounded morphism is automatically everywhere defined, from which immediately follows that the composition of two bounded morphisms is again a bounded morphism.
Since the property of being closed and densely defined is preserved under taking adjoints, the Hilbert space adjoint $f^* : \mathcal{H}' \to \mathcal{H}$ of a morphism $f : \mathcal{H} \to \mathcal{H}'$ is also again a morphism. Less trivial, but still true, is the following important result:

Theorem 4.1.2. *Let $f : \mathcal{H} \to \mathcal{H}$ be a morphism between Hilbert $\mathcal{N}(\Gamma)$-modules. Then the composition $f^*f : \mathcal{H} \to \mathcal{H}$ with $\operatorname{dom}(f^*f) := f^{-1}(\operatorname{dom}(f^*))$ is a positive, self-adjoint morphism.*

Proof. It is easily verified that f^*f is Γ-equivariant, positive and symmetric. The proof of the fact that f^*f is self-adjoint and still densely defined can be found, for example, in [47, Page 275, Theorem 3.24]. □

For the purpose of this paper, the most important consequence of this theorem is that for any morphism $f : \mathcal{H} \to \mathcal{H}'$ of Hilbert $\mathcal{N}(\Gamma)$-modules, the induced operator f^*f has a functional calculus, and that, since f^*f is a morphism of Hilbert $\mathcal{N}(\Gamma)$-modules, the same is true for any operator constructed via functional calculus of f^*f.
We denote by $\mathfrak{B}_\Gamma(\mathcal{H})$ to be the space of all bounded endomorphisms of $\mathcal{H}$ and by $\mathfrak{P}_\Gamma(\mathcal{H})$ the sub-monoid of *positive* bounded endomorphisms. Note that $\mathfrak{P}_\Gamma(\mathcal{H})$ includes in particular all projections onto closed, Γ-invariants subspaces of $\mathcal{H}$: Perhaps **the** essential feature of Hilbert $\mathcal{N}(\Gamma)$-modules is the existence of a particular positive function $\operatorname{tr}_\Gamma : \mathfrak{P}_\Gamma(\mathcal{H}) \to [0, \infty]$, the so-called *von Neumann trace*. As indicated by its properties below, tr_Γ can be viewed as a generalization of the standard trace of finite-dimensional endomorphisms:

1. $\operatorname{tr}_\Gamma(A) = 0 \Leftrightarrow A = 0$ [Faithfulness],
2. $\operatorname{tr}_\Gamma(A^*A) = \operatorname{tr}_\Gamma(AA^*)$ for all $A \in \mathfrak{B}_\Gamma(\mathcal{H})$ [Adjoint Symmetry].
3. If $A, B \in \mathfrak{P}_\Gamma(\mathcal{H})$ with $A \leq B$ (as positive operators), then $\operatorname{tr}_\Gamma(A) \leq \operatorname{tr}_\Gamma(B)$. [Monotonicity].
4. For any $\lambda \geq 0$ and all $A, B \in \mathfrak{P}_\Gamma(\mathcal{H})$, we have $\operatorname{tr}_\Gamma(A + \lambda B) = \operatorname{tr}_\Gamma(A) + \lambda \operatorname{tr}_\Gamma(B)$ [Linearity].
5. tr_Γ is ultra-weakly continuous.
6. If $\mathcal{H}$ is a finitely generated Hilbert $\mathcal{N}(\Gamma)$-module, then $\operatorname{tr}_\Gamma(A) < \infty$ for any $A \in \mathfrak{P}_\Gamma(\mathcal{H})$.

In fact, as shown [54, Definition 1.8], the von Neumann trace tr_Γ has the following explicit description: Let $e \in L^2(\Gamma)$ be the unit element. Further, let $\Psi : \mathcal{H} \hookrightarrow L^2(\Gamma)\hat{\otimes} H$ be *some* Γ-equivariant, isometric embedding and $\{x_i\}_{i \in I} \subseteq H$ *some* (countable) orthonormal basis of H. Then, for any $A \in \mathfrak{P}_\Gamma(\mathcal{H})$, one has

$$\operatorname{tr}_\Gamma(A) = \sum_{i \in I} \langle A\Psi^*(e \otimes x_i), \Psi^*(e \otimes x_i)\rangle \in [0, \infty]. \tag{4.1.1}$$

Here, as everywhere else, $e \in \Gamma$ denotes the unit of Γ.

Definition 4.1.3. A bounded morphism $f \in \mathfrak{P}_\Gamma(\mathcal{H})$ is said to be of *trace class* if

$$\mathrm{tr}_\Gamma(|f|) < \infty, \tag{4.1.2}$$

where $|f| := \sqrt{f^*f}$ is the positive, self-adjoint *square root* of f defined via Borel functional calculus of f^*f. We denote by $\mathfrak{B}_\Gamma(\mathcal{H})_1 \subseteq \mathfrak{B}_\Gamma(\mathcal{H})$ the subset of trace class operators.

One can show that $\mathfrak{B}_\Gamma(\mathcal{H})_1$ is, in fact, a subspace of $\mathfrak{B}_\Gamma(\mathcal{H})$ and that tr_Γ extends to a linear functional $\mathrm{tr}_\Gamma : \mathfrak{B}_\Gamma(\mathcal{H})_1 \to \mathbb{C}$ that still satisfies Identity 4.1.1. The well-known proof in the case $\Gamma = \{0\}$, see for example [79, Section VI.6], can be adapted to the case of general countable groups Γ without difficulty. Observe that $\mathfrak{B}_\Gamma(\mathcal{H}) = \mathfrak{B}_\Gamma(\mathcal{H})_1$ whenever $\mathcal{H}$ is finitely generated.
Finally, we can define the *von Neumann Dimension* of a Hilbert $\mathcal{N}(\Gamma)$-module as

$$\dim_{\mathcal{N}(\Gamma)}(\mathcal{H}) = \mathrm{tr}_\Gamma(\mathbb{1}_\mathcal{H}) \in [0,\infty]. \tag{4.1.3}$$

The von Neumann dimension is an isomorphism invariant of finitely-generated Hilbert $\mathcal{N}(\Gamma)$-modules, see [54, Theorem 1.12]. We also remark that for a generic infinite group Γ, any real Number can occur as the von Neumann dimension of an appropriate Hilbert $\mathcal{N}(\Gamma)$-module, and that there exists infinitely-generated Hilbert $\mathcal{N}(\Gamma)$-modules that are still have finite von Neumann-dimension, see [54, Examples 1.11,1.14].
Lemma 4.1.2 now permits the next definition:

Definition 4.1.4. The **spectral density function** of a morphism $f : \mathcal{H} \to \mathcal{H}'$ between two Hilbert $\mathcal{N}(\Gamma)$-modules is defined as

$$F(f,\lambda) := \mathrm{tr}_\Gamma(\chi_{[0,\lambda]}(|f|)) \in [0,\infty], \tag{4.1.4}$$

where $\chi_{[0,\lambda]}$ is the indicator function of the corresponding closed set in $\mathbb{R}$ and $\chi_{[0,\lambda]}(|f|)$ is the associated positive, bounded morphism defined via Borel functional calculus of $|f|$. If $F(f,0) < \infty$, we also define

$$\hat{F}(f,\lambda) = F(f,\lambda) - F(f,0) \in [0,\infty]. \tag{4.1.5}$$

For a closed, Γ-invariant subspace $L \subseteq \mathcal{H}$, we denote by $p_L : \mathcal{H} \to \mathcal{H}$ the orthogonal projection onto L. It is easy to see that p_L is a bounded, positive morphism of Hilbert $\mathcal{N}(\Gamma)$-modules (with $||p_L|| = 1$).

Lemma 4.1.5. [54, Section 2.1] *Let $f : \mathcal{H} \to \mathcal{H}'$ and $g : \mathcal{H}'' \to \mathcal{H}'$ be two morphisms of Hilbert $\mathcal{N}(\Gamma)$-modules. Then, for any $\lambda \geq 0$, the following holds true:*

1. *We have $F(f^*f, \sqrt{\lambda}) = F(f,\lambda) = F(|f|,\lambda)$.*
2. *We have $F(f \oplus g, \lambda) = F(f,\lambda) + F(g,\lambda)$.*
3. *We have $F(f,\lambda) = \sup\{\mathrm{tr}_\Gamma(p_L) : L \in \mathcal{L}(f,\lambda)\}$, where*
   $\mathcal{L}(f,\lambda) := \{L \subseteq \mathcal{H} \text{ closed, } \Gamma\text{-invariant subspace} : ||f(x)|| \leq \lambda||x|| \; \forall x \in L\}$.

Definition 4.1.6. A morphism $f : \mathcal{H} \to \mathcal{H}'$ between Hilbert $\mathcal{N}(\Gamma)$-modules is said to be **Fredholm** if

$$F(f,\lambda) < \infty \qquad \forall \lambda < ||f||. \tag{4.1.6}$$

$f : \mathcal{H} \to \mathcal{H}'$ is called *left Fredholm* if the weaker condition $F(f,\lambda) < \infty$ for some $\lambda > 0$ holds. Here, we use the convention $||f|| = \infty$ whenever f is unbounded.

Using Lemma 4.1.5, we find:

Corollary 4.1.7. *Let $f : \mathcal{H} \to \mathcal{H}'$ and $g : \mathcal{H}'' \to \mathcal{H}$ be two morphisms of Hilbert $\mathcal{N}(\Gamma)$-modules. Then*

1. *f is (left) Fredholm $\Leftrightarrow$ f^*f is (left) Fredholm $\Leftrightarrow$ $|f|$ is (left) Fredholm.*
2. *f and g are (left) Fredholm $\Leftrightarrow$ $f \oplus g$ is (left) Fredholm.*

Note that *any morphism $f : \mathcal{H} \to \mathcal{H}'$ between Hilbert $\mathcal{N}(\Gamma)$ is automatically Fredholm whenever $\mathcal{H}$ has finite von Neumann dimension.* This follows since $\chi_{[0,\lambda]}(|f|) \leq \mathbb{1}_{\mathcal{H}}$ (as positive operators), and hence also $F(f, \lambda) = \mathrm{tr}_\Gamma(\chi_{[0,\lambda]}(|f|)) \leq \mathrm{tr}_\Gamma(\mathbb{1}_{\mathcal{H}}) = \dim_{\mathcal{N}(\Gamma)}(\mathcal{H}) < \infty$.
If f is Fredholm, it follows from the monotonicity of the trace and Lemma 4.1.5,(3) that $F(f, \ .\ )$ is a non-decreasing, right-continuous function. Therefore, it defines a Borel measure F_f on $\mathbb{R}_{\geq 0}$ that is uniquely determined by the identity

$$F_f((a, b]) \coloneqq F(f, b) - F(f, a) \tag{4.1.7}$$

for any half-open interval $(a, b] \subseteq \mathbb{R}_{>0}$. From its definition, it is evident that the support of the measure F_f equals the spectrum of f^*f, i.e.

$$\mathrm{supp}(F_f) = \sigma(|f|).$$

Moreover, under the convention that $||f|| \coloneqq \infty$ whenever f is unbounded, we have $\sigma(|f|) \subseteq [0, ||f||]$. In particular, if f is a bounded morphism, F_f is a compactly supported measure.

Definition 4.1.8 (Determinant class)**.** Let $f : \mathcal{H} \to \mathcal{H}'$ be a Fredholm morphism. Then f is **of determinant class** if $\int_{0+}^{1} \log(\lambda) dF_f(\lambda) > -\infty$.

Definition 4.1.9 (Fuglede-Kadison determinant)**.** Let $f : \mathcal{H} \to \mathcal{H}'$ be a bounded morphism with $\mathcal{H}$ finite-dimensional. Define the **Fuglede-Kadison determinant** $\det_\Gamma(f) \in \mathbb{R}_{\geq 0}$ of f as

$$\det_\Gamma(f) = \begin{cases} \exp(\int_{0+}^{\infty} \log(\lambda) dF_f(\lambda)) \in \mathbb{R}_{>0} & \text{if f is of determinant class,} \\ 0 & \text{else.} \end{cases} \tag{4.1.8}$$

Remark 4.1.10. As stated above, any bounded morphism f over a finite-dimensional Hilbert $\mathcal{N}(\Gamma)$-module is automatically Fredholm and has compactly supported measure F_f. That is why we have an equality $\int_1^\infty \log(\lambda) dF_f(\lambda) = \int_1^{||f||} \log(\lambda) dF_f(\lambda)$, i.e. $\int_1^\infty \log(\lambda) dF_f(\lambda)$ is always a convergent integral. Provided that f is also of determinant class, the integral $\int_{0+}^\infty \log(\lambda) dF_f(\lambda)$ therefore also always converges.

Example 4.1.11. Assume that $\Gamma = \{0\}$. In this instance, any finitely generated Hilbert $\mathcal{N}(\Gamma)$-module $\mathcal{H}$ is simply a finite-dimensional complex inner product space, and tr_Γ becomes the usual trace for linear endomorphisms of finite-dimensional spaces. This means that for any $f \in \mathfrak{B}_\Gamma(\mathcal{H})$, $\mathrm{tr}_\Gamma(f)$ is just the sum of all eigenvalues of f, counted with multiplicity. This implies that the spectral density function of f is a step function taking the form

$$F(f, x) = \sum_{\lambda \in E(|f|) \leq x^2} j_\lambda,$$

where $E(|f|)$ the set of all (positive) eigenvalues of $|f|$ and j_λ is the geometric multiplicity of λ. In particular, the measure F_f is simply the finite sum of Dirac measures, which is why f is of determinant class and

$$\log \det_\Gamma(f) = \sum_{0 \neq \lambda \in E(|f|)} j_\lambda \log(\lambda). \tag{4.1.9}$$

Since for any invertible matrix, the eigenvalues of $|f|$ are just the absolute values of the eigenvalues of f (with same geometric multiplicities), we deduce from this example, that:

Lemma 4.1.12. *Let $\Gamma = \{0\}$ be the trivial group and let $f \in \mathrm{GL}_n(\mathbb{C})$. Then*

$$\det_\Gamma(f) = |\det(f)|, \tag{4.1.10}$$

where det *denotes the usual algebraic determinant.*

Note that unlike the algebraic determinant, the appearance of the adjoint in the definition of $\det_\Gamma$ suggest that for general (non-invertible) operators $f \in \mathfrak{B}_\Gamma(\mathcal{H}, \mathcal{H}')$, the function $\det_\Gamma \in \mathfrak{B}_\Gamma(\mathcal{H}, \mathcal{H}')$ *does* depend of the choice of inner products on $\mathcal{H}$ and $\mathcal{H}'$.

Example 4.1.13. That this is indeed the case can already be witnessed in the basic case $\Gamma = \{0\}$: For $i = 1, 2$, let $\mathcal{H}_i := (\mathbb{C}^2, \langle \cdot\,,\, \cdot \rangle_i)$, where $\langle \cdot\,,\, \cdot \rangle_i$ is the inner product induced by the positive matrix A_i with

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

Then the endomorphism

$$B := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

when regarded as an element of $\mathfrak{B}_\Gamma(\mathcal{H}_1)$ satisfies $\det_\Gamma(B) = 1$, since

$$B^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in this case, so that the matrix B^*B has 1 as its unique non-zero eigenvalue. However, when regarded as an element of $\mathfrak{B}_\Gamma(\mathcal{H}_2)$, we compute that

$$B^* = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix},$$

which is why we obtain $\det_\Gamma(B) = 2$ in that case.

Conversely, while the algebraic determinant is only well-defined for finite-dimensional *endomorphisms*, the Fuglede-Kadison determinant has the added advantage of being defined for *all* bounded morphisms between *any* two arbitrary finitely-generated Hilbert $\mathcal{N}(\Gamma)$-modules. Moreover, it has various natural and useful properties, many of which are (slightly modified) generalizations of properties of the algebraic determinant.

Proposition 4.1.14. [54, Theorem 3.14, Lemma 3.15] *Let $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{H}''$ and $\mathcal{H}'''$ be finitely generated Hilbert $\mathcal{N}(\Gamma)$-module. Further, let $f : \mathcal{H} \to \mathcal{H}'$, $g : \mathcal{H}'' \to \mathcal{H}'''$ and $h : \mathcal{H}'' \to \mathcal{H}'$ be bounded morphisms. Then:*

1. *One has $\det_\Gamma(\lambda \cdot f) = |\lambda| \cdot \det_\Gamma(f)$ for any $\lambda \in \mathbb{C}^\times$.*
2. *If $f : \mathcal{H} \to \mathcal{H}'$ is a partial isometry, then $\det_\Gamma(f) = 1$.*
3. *If $f : \mathcal{H} \to \mathcal{H}'$ has dense image and $g : \mathcal{H}' \to \mathcal{H}''$ is injective, then $\det_\Gamma(g \circ f) = \det_\Gamma(g) \cdot \det_\Gamma(f)$.*

4. Let $\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} : \mathcal{H} \oplus \mathcal{H}' \to \mathcal{H}'' \oplus \mathcal{H}'''$ *be a morphism, such that* $f : \mathcal{H} \to \mathcal{H}''$ *has dense image and* $g : \mathcal{H}' \to \mathcal{H}'''$ *is injective. Then* $\det_\Gamma \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} = \det_\Gamma(f) \cdot \det_\Gamma(g)$.

Any morphism $f : \mathcal{H} \to \mathcal{H}'$ admits an *orthogonal complement*

$$f^\perp := f|_{\ker(f)^\perp} : \ker(f)^\perp \to \overline{\operatorname{im}(f)}. \tag{4.1.11}$$

It is an injective morphism with dense image that satisfies $(f^\perp)^* = (f^*)^\perp$ and $(f^\perp)^* f^\perp = (f^* f)^\perp$. From this, it follows that $f^\perp$ is Fredholm whenever f is Fredholm. The converse direction need not necessarily hold, since $f^\perp$ doesn't "see" the kernel of f, which could be of infinite von Neumann dimension. In general, if f has finite-dimensional kernel, we have the correspondence

$$F(f^\perp, \lambda) = \hat{F}(f, \lambda) = F(f, \lambda) - F(f, 0) \tag{4.1.12}$$

for any $\lambda > 0$, showing that, in this case, f is Fredholm if and only if $f^\perp$ is Fredholm. In this case, it follows that $F_f(\lambda) = F_{f^\perp}(\lambda)$ for all $\lambda > 0$, which implies that $\det_\Gamma(f) = \det_\Gamma(f^\perp)$, if $f : \mathcal{H} \to \mathcal{H}'$ is bounded and $\mathcal{H}$ finite-dimensional.

Lemma 4.1.15. *Let* H *be a finite-dimensional complex vector space and let* $\mathcal{H} := L^2(\Gamma) \otimes_{\mathbb{C}} H$ *be the Hilbert* $\mathcal{N}(\Gamma)$*-module with its obvious (left)* Γ*-action. Then, for any bounded morphism* $f \in \mathcal{B}_\Gamma(L^2(\Gamma))$ *that is injective with dense image and any invertible endomorphism* $A \in \mathrm{GL}(H)$*, the tensor product* $f \otimes A$ *lies in* $\mathcal{B}_\Gamma(L^2(\Gamma) \otimes_{\mathbb{C}} H)$ *and satisfies*

$$\det_\Gamma(f \otimes A) = \det_\Gamma(f)^{\dim_{\mathbb{C}}(H)} |\det(A)|. \tag{4.1.13}$$

Proof. We can write $(f \otimes A) = (f \otimes \mathbb{1}_H) \circ (\mathbb{1}_{L^2(\Gamma)} \otimes A) = (\mathbb{1}_{L^2(\Gamma)} \otimes A) \circ (f \otimes \mathbb{1}_H)$. Since A is invertible and f is injective with dense image, we obtain from Proposition 4.1.14,(2), that

$$\det_\Gamma(f \otimes A) = \det_\Gamma(f \otimes \mathbb{1}_H) \det_\Gamma(\mathbb{1}_{L^2(\Gamma)} \otimes A). \tag{4.1.14}$$

To compute $\det_\Gamma(f \otimes \mathbb{1}_H)$, we observe that under a linear isometry $H \cong \mathbb{C}^n$ with $n = \dim_{\mathbb{C}}(H)$, we can identify $(f \otimes \mathbb{1}_H)$ with a diagonal matrix over $L^2(\Gamma)^n$ with diagonal entries f. Consequently, by Proposition 4.1.14,(2)–(4), we get

$$\det_\Gamma(f \otimes \mathbb{1}_H) = \det_\Gamma(f)^{\dim_{\mathbb{C}}(H)}. \tag{4.1.15}$$

From [54, Theorem 3.14,(6)] and Lemma 4.1.12, we also get

$$\det_\Gamma(\mathbb{1}_{l^2(\Gamma)} \otimes A) = \det_{\{0\}}(A) = |\det(A)|. \tag{4.1.16}$$

The result now follows from 4.1.14–4.1.16. □

We wish to extend the concept of the Fuglede-Kadison determinant onto unbounded Fredholm operators over infinite-dimensional Hilbert $\mathcal{N}(\Gamma)$-modules. However, although the notion of determinant class, i.e. the question of convergence of the integral $\int_{0+}^1 \log(\lambda) dF_f(\lambda)$ still makes sense in this scenario, the measure F_f of an unbounded operator is not compactly supported anymore. Therefore, if f is unbounded, the integral $\int_1^\infty \log(\lambda) dF_f$ will never converge, which is why a different method is needed in order to define a reasonable determinant of f.

For an arbitrary Borel measure μ on $\mathbb{R}_{\geq 0}$, we introduce the space of *essentially bounded Borel functions*

$$\mathcal{B}(\mathbb{R}_{\geq 0}, \mu) := \{\phi : \mathbb{R}_{\geq 0} \to \mathbb{C} \text{ Borel} : ||\phi||_\infty := \sup_{x \in \operatorname{supp}(\mu)} |\phi(x)| < \infty\}.$$

Lemma 4.1.16. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of Borel measures on $\mathbb{R}_{\geq 0}$. Further, assume that there exists a Borel measure μ on $\mathbb{R}_{\geq 0}$ with the property that, for every measurable subset $O \subseteq \mathbb{R}^+$*

1. *$\mu_n(O) \leq \mu(O)$, and*
2. *if $\mu(O) < \infty$, we have $\lim_{n\to\infty} \mu_n(O) = \mu(O)$.*

Then, for any positive function $\phi \in \mathcal{B}(\mathbb{R}_{\geq 0}, \mu)$, we have an equivalence

$$\phi \in L^1(\mathbb{R}_{\geq 0}, \mu) \Longleftrightarrow \phi \in \bigcap_{n\in\mathbb{N}} L^1(\mathbb{R}, \mu_n) \text{ and } \liminf_{n\to\infty} \int_{\mathbb{R}_{\geq 0}} \phi \, d\mu_n < \infty. \tag{4.1.17}$$

Finally, if one of the two equivalent conditions hold, then

$$\lim_{n\to\infty} \int_{\mathbb{R}_{\geq 0}} \phi \, d\mu_n = \int_{\mathbb{R}_{\geq 0}} \phi \, d\mu. \tag{4.1.18}$$

Proof. First, we show that $\phi \in L^1(\mathbb{R}_{\geq 0}, \mu)$ implies both $\phi \in \bigcap_{n\in\mathbb{N}} L^1(\mathbb{R}, \mu_n)$ and $\lim_{n\to\infty} \int_{\mathbb{R}_{\geq 0}} \phi \, d\mu_n = \int_{\mathbb{R}_{\geq 0}} \phi \, d\mu$. The weaker property $\liminf_{n\to\infty} \int_{\mathbb{R}_{\geq 0}} \phi \, d\mu_n < \infty$ then clearly follows. Since $\phi \in B(\mathbb{R}_{\geq 0}, \mu) \cap L^1(\mathbb{R}_{\geq 0}, \mu)$ by assumption, the containment $\phi \in B(\mathbb{R}_{\geq 0}, \mu_n) \cap L^1(\mathbb{R}, \mu_n)$ for each $n \in \mathbb{N}$ follows immediately from assertion 1. Now let $\epsilon > 0$. As $\phi \in L^1(\mathbb{R}, \mu)$, there exists some large $K >> 0$, so that $|\int_K^\infty \phi \, d\mu| < \epsilon/4$. From assertion 1, it follows that also $|\int_K^\infty \phi \, d\mu_n| < \epsilon/4$ for each $n \in \mathbb{N}$. Moreover, by assertion 2, we find $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|\mu([0, K)) - \mu_n([0, K))| < \frac{1}{2}\epsilon/||\phi||_\infty$. Therefore, for all $n \geq N$, we get

$$\begin{aligned} |\int_{\mathbb{R}_{\geq 0}} \phi \, d\mu - \int_{\mathbb{R}_{\geq 0}} \phi \, d\mu_n| &\leq |\int_{[0,K)} \phi \, d\mu - \int_{[0,K)} \phi \, d\mu_n| + |\int_K^\infty \phi \, d\mu| + |\int_K^\infty \phi \, d\mu_n| \\ &\leq ||\phi||_\infty |\mu([0, K)) - \mu_n([0, K))| + \epsilon/2 \leq \epsilon. \end{aligned}$$

Letting $\epsilon > 0$, the result follows.

Conversely, if $\phi \in \bigcap_{n\in\mathbb{N}} L^1(\mathbb{R}, \mu_n)$ and $\liminf_{n\to\infty} \int_{\mathbb{R}_{\geq 0}} \phi d\mu_n < \infty$, then the containment $\phi \in L^1(\mathbb{R}_{\geq 0}, \mu)$ follows from assertion 2 and Fatou's lemma, since

$$\int_{\mathbb{R}_{\geq 0}} \phi \, d\mu \leq \liminf_{n\to\infty} \int_{\mathbb{R}_{\geq 0}} \phi \, d\mu_n < \infty,$$

completing the proof. □

Lemma 4.1.17. *Let $f : \mathcal{H} \to \mathcal{H}'$ be a Fredholm morphism, and let $\phi \in \mathcal{B}(\mathbb{R}_{\geq 0}, F_f)$ be positive. Then the bounded morphism $\phi(|f|)$ is of trace class if and only if $\phi \in L^1(\mathbb{R}_{\geq 0}, F_f)$, in which case we get*

$$\operatorname{tr}_\Gamma(\phi(|f|)) = \int_0^\infty \phi(\lambda) \, dF_f(\lambda). \tag{4.1.19}$$

In particular, $\phi(|f|)$ is of trace class if either

1. *ϕ or F_f is compactly supported, or*

2. $\phi \in C^1(\mathbb{R})$, $\phi'(\lambda) \cdot F(f,\lambda) \in L^1(\mathbb{R})$ *and* $\lim_{\lambda \to +\infty} \phi(\lambda) \cdot F(f,\lambda) = 0$.

Proof. From spectral theory, it follows that $\phi(|f|)$ is bounded with norm estimate $||\phi(|f|)|| \leq ||\phi||_\infty < \infty$. Recall from 4.1.1 that we find a countable subset $\{e_i\}_{i=1}^\infty \subseteq \mathcal{H}$ (with $e_j := 0$ for all $j > \dim_{\mathcal{N}(\Gamma)}(\mathcal{H})$), such that the von Neumann trace $\mathrm{tr}_\Gamma(g)$ for any positive, bounded morphism $g \in \mathcal{P}(\mathcal{H})$ can be written as

$$\mathrm{tr}_\Gamma(g) = \sum_{i=1}^{\infty} \langle g e_i, e_i \rangle \in [0, \infty].$$

In particular, we get

$$F_f((a,b]) = \mathrm{tr}_\Gamma(\chi_{(a,b]}(f)) = \sum_{i=1}^{\infty} \langle \chi_{(a,b]}(|f|) e_i, e_i \rangle < \infty.$$

This observation allows us to define for any $n \in \mathbb{N}$ the right-continuous Borel measure $F_{f,n}$ on $\mathbb{R}_+$ via

$$F_{f,n}((a,b]) := \sum_{i=1}^{n} \langle \chi_{(a,b]}(|f|) e_i, e_i \rangle$$

for any half-open interval $(a,b] \in \mathbb{R}^+$. By spectral theory, any bounded Borel function $\phi \in B(\mathbb{R}_{\geq 0}, F_f)$ lies in $L^1(\mathbb{R}_{\geq 0}, F_{f,n})$ for any $n \in \mathbb{N}$ and satisfies

$$\int_0^\infty \phi \cdot dF_{f,n} = \sum_{i=1}^{n} \langle \phi(|f|) e_i, e_i \rangle.$$

Now observe that the pair $((F_{f,n})_{n\in\mathbb{N}}, F_f)$ satisfies the assumptions of Lemma 4.1.16. Using the results of the same lemma, we have the following chain of equivalences

$$\begin{aligned} &\mathrm{tr}_\Gamma(\phi(|f|)) = \lim_{n\to\infty} \int_0^\infty \phi(\lambda) \cdot dF_{f,n}(\lambda) < \infty \Leftrightarrow \phi \in L^1(\mathbb{R}_{\geq 0}, dF_f) \\ \text{and } &\int_0^\infty \phi(\lambda) dF_f(\lambda) = \lim_{n\to\infty} \int_0^\infty \phi(\lambda) \cdot dF_{f,n}(\lambda). \end{aligned}$$

Next, under the additional assumptions made on ϕ, we show that the integral $\int_0^\infty \phi(\lambda) \cdot dF_f(\lambda)$ converges. This is immediately obvious if ϕ or F_f is compactly supported, since then, one finds a compact subset $K \subseteq \mathbb{R}_{\geq 0}$, so that $\int_0^\infty \phi(\lambda) \cdot dF_f(\lambda) \leq ||\phi||_\infty F_f(K) < \infty$. Under the the second assumption, we have

$$\int_0^\infty \phi(\lambda) \cdot dF_f(\lambda) = \lim_{k\to\infty} \left(-\int_0^k \phi'(\lambda) \cdot F(f,\lambda) d\lambda + \phi(k) \cdot F(f,k) - \phi(0) \cdot F_f(0) \right) \in \mathbb{R}\,. \tag{4.1.20}$$

We refer to [57, Lemma 4.1] for a proof of the above identity. In other words, we also have $\phi \in L^1(\mathbb{R}, dF_f)$. □

Suppose that f is a Fredholm homomorphism with the property that for each $t > 0$, the *heat evolution operator* $e^{-t|f|^\perp}$ is of trace class. At least formally, we can now consider the (truncated/small-time) *zeta-function* of f as

$$\zeta_f(s) := \Gamma(s)^{-1} \int_0^1 t^{s-1} \, \mathrm{tr}_\Gamma(e^{-t|f|^\perp}) dt, \tag{4.1.21}$$

for any $s \in \mathbb{C}$. Here, $\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt$ denotes the complex *Gamma function.*

Definition 4.1.18 (ζ-regular)**.** A Fredholm morphism $f : \mathcal{H} \to \mathcal{H}'$ is said to be ζ-*regular* if the following two conditions hold:

1. For each $t > 0$, the operator $e^{-t|f|^\perp}$ is of trace class.

2. There exists some constant $C > 0$, such that $\zeta_{f,sm}(s)$ is a holomorphic function on $\{s \in \mathbb{C} : \Re(s) > C\}$, extending to a meromorphic function on all of $\mathbb{C}$ that is regular (i.e. holomorphic) at $s = 0$.

Remark 4.1.19. Observe that for any morphism $f : \mathcal{H} \to \mathcal{H}'$ with the property that $\mathrm{tr}_\Gamma(e^{-t|f|^\perp})$ is of trace class for each $t > 0$, it actually follows that $f^\perp$ is Fredholm. Indeed, notice that $\chi_{[0,\lambda]}(|f|^\perp) = \chi_{[0,\lambda]}(|f|^\perp) \cdot e^{|f|^\perp} \cdot \chi_{[0,\lambda]}(|f|^\perp) \cdot e^{-|f|^\perp}$ for each $\lambda > 0$. Together with the fact that Γ-trace class morphisms form an ideal inside the algebra of bounded endomorphisms over a Hilbert $\mathcal{N}(\Gamma)$-module, it follows that $\chi_{[0,\lambda]}(|f|^\perp)$ is of Γ-trace class, i.e. that $f^\perp$ is Fredholm. Provided that $F(f, 0) = \dim_{\mathcal{N}(\Gamma)}(\ker(f)) < \infty$, it also now follows that $F(f, \lambda) = F(f^\perp, \lambda) + F(f, 0) < \infty$ for each $\lambda \geq 0$, i.e. that f is Fredholm.

For a ζ-regular morphism $f : \mathcal{H} \to \mathcal{H}'$, the (complex) derivative $\frac{d}{ds}\zeta_f(s)|_{s=0} \in \mathbb{C}$ of ζ_f at 0 is well-defined. In fact, we must have $\frac{d}{ds}\zeta_f(s)|_{s=0} \in \mathbb{R}$. This follows since $\zeta_f(s)$ is of the integral shape as in 4.1.21 for any $s \in \mathbb{C}$ with sufficiently large real part, from which one deduces that $\overline{\zeta_f(\overline{s})} = \zeta_f(s)$ for all sufficiently large s. Therefore, the same equality holds true for the meromorphic extension, which is why in particular $\zeta_f(s) \in \mathbb{R}$ for all $s \in \mathbb{R}$ near 0, proving that $\frac{d}{ds}\zeta_f(s)|_{s=0} \in \mathbb{R}$.

Proposition 4.1.20. [54, Lemma 3.139] *Let $f : \mathcal{H} \to \mathcal{H}'$ be a Fredholm morphism, such that* $\mathrm{tr}_\Gamma(e^{-t|f|^\perp}) < \infty$ *for all $t > 0$. Then f is of determinant class if and only if*

$$\int_1^\infty t^{-1}\, \mathrm{tr}_\Gamma(e^{-t|f|^\perp})dt < \infty. \tag{4.1.22}$$

Definition 4.1.21 (ζ-regularized Determinant)**.** Let $f : \mathcal{H} \to \mathcal{H}'$ be a Fredholm morphism that is ζ-regular. Then its **ζ-regularized determinant** $\det^\zeta_\Gamma(f) \in \mathbb{R}_{\geq 0}$ is defined as

$$\det{}^\zeta_\Gamma(f) := \begin{cases} \exp\left(-\zeta_f'(0) - \int_1^\infty t^{-1}\, \mathrm{tr}_\Gamma(e^{-t|f|^\perp})dt\right) \in \mathbb{R}_{>0} & \text{if } f \text{ is of determinant class,} \\ 0 & \text{else.} \end{cases} \tag{4.1.23}$$

Remark 4.1.22. Observe that by definition, one has $\det^\zeta_\Gamma(f) = \det^\zeta_\Gamma(f^\perp)$.

Before we unwind this complicated definition and relate it to the Fuglede-Kadison determinant, let us first address one of the main problems that we will frequently encounter, namely the question of whether or not a given Fredholm morphism $f : \mathcal{H} \to \mathcal{H}'$ is of determinant class. The quantity we will introduce next will play an important part in that.

Definition 4.1.23 (Novikov-Shubin invariant)**.** Let f be a morphism that is left Fredholm. The **Novikov-Shubin invariant** $\alpha(f) \in [0, \infty]\dot{\cup}\{\infty^+\}$ of f is defined via

$$\alpha(f) := \begin{cases} \sup\{\alpha \in [0, \infty] : \hat{F}(f, \lambda) \in \mathcal{O}(\lambda^\alpha) \text{ for } \lambda \to 0\} & \text{if } \hat{F}(f, \lambda) > 0 \ \ \forall \lambda > 0, \\ \infty^+, & \text{else.} \end{cases} \tag{4.1.24}$$

Observe that $\alpha(f)$ will attain the value of the formal symbol ∞^+ precisely when $f^\perp$ has a spectral gap at zero, i.e. when $f^\perp$ is has a bounded inverse (with norm exactly the size of the spectral gap). Using [54, Lemma 3.139,(6)], it can be checked that for morphism satisfying $\mathrm{tr}_\Gamma(e^{-t|f|^\perp}) < \infty$ for all $t > 0$, one has

$$\alpha(f) = \begin{cases} \sup\{\alpha \in [0, \infty] : \mathrm{tr}_\Gamma(e^{-t|f|^\perp}) \in \mathcal{O}(t^{-\alpha}) \text{ for } t \to \infty\} & \text{if } \hat{F}(f, \lambda) > 0 \ \ \forall \lambda > 0 \\ \infty^+ & \text{else.} \end{cases} \tag{4.1.25}$$

Proposition 4.1.24. [54, Theorem 3.14,(4)] *Let $f : \mathcal{H} \to \mathcal{H}'$ be a Fredholm morphism, such that $\mathrm{tr}_\Gamma(e^{-t|f|^\perp}) < \infty$ for all $t > 0$ and $\alpha(f) \neq 0$. Then f is of determinant class.*

For the next proposition that relates the Fuglede-Kadison determinant to the ζ-regularized determinant, we introduce the *complete zeta-function* and the *large time zeta-function* of f as the formal expressions

$$\zeta_{f,cp}(s) := \Gamma(s)^{-1} \int_0^\infty t^{s-1}\, \mathrm{tr}_\Gamma(e^{-t|f|^\perp}) dt, \tag{4.1.26}$$

$$\zeta_{f,la}(s) := \Gamma(s)^{-1} \int_1^\infty t^{s-1}\, \mathrm{tr}_\Gamma(e^{-t|f|^\perp}) dt. \tag{4.1.27}$$

Proposition 4.1.25. *Let $f : \mathcal{H}' \to \mathcal{H}$ be a bounded morphism on a finite-dimensional Hilbert $\mathcal{N}(\Gamma)$-module $\mathcal{H}'$, such that $\alpha(f) = \infty^+$. Then f is ζ-regular. In fact, the integral expression for $\zeta_{f,cp}(s)$ determines a holomorphic function on the half-plane $\{\Re(s) > 0\}$, which extends to an entire function on all of $\mathbb{C}$. Moreover, we have*

$$\det{}_\Gamma(f) = \exp(-\frac{d}{ds}\zeta_{f,cp}(s)|_{s=0}) = \det{}^\zeta_\Gamma(f), \tag{4.1.28}$$

i.e. the Fuglede-Kadison determinant agrees with the ζ-regularized determinant of f.

Proof. First, recall that $\alpha(f) = \infty^+$ is just equivalent to the statement that $f^\perp$ has a spectral gap at 0. In other words, there exists an $\epsilon > 0$, so that $\sigma(|f|^\perp) \subseteq [\epsilon, ||f||]$. Together with Lemma 4.1.17, we obtain that

$$\mathrm{tr}_\Gamma(e^{-t|f|^\perp}) = \int_\epsilon^{||f||} e^{-t\lambda} dF_f(\lambda) \leq \dim_{\mathcal{N}(\Gamma)}(\mathcal{H}) e^{-t\epsilon}.$$

We use the well-known identity

$$\int_0^\infty t^{s-1} e^{-tz} dt = \Gamma(s) z^{-s} = \Gamma(s) \cdot e^{-s\log(z)}$$

for any pair of complex numbers $z, s \in \mathbb{C}$ with $z \in \mathbb{C}\backslash(-\infty, 0]$ and $\Re(s) > 0$. Observe also that for $t > 0$, one has $|t^{s-1}| = t^{\Re(s)-1}$.
This way, we obtain for any $s \in \mathbb{C}$ with $\Re(s) > 0$ that

$$\begin{aligned}\int_0^\infty t^{\Re(s)-1}\, \mathrm{tr}_\Gamma(e^{-t|f|^\perp})\, dt &\leq \dim_{\mathcal{N}(\Gamma)}(\mathcal{H}) \int_0^\infty t^{\Re(s)-1} e^{-t\epsilon} dt \\ &= \dim_{\mathcal{N}(\Gamma)}(\mathcal{H}) \Gamma(\Re(s)) \epsilon^{\Re(s)} < \infty.\end{aligned}$$

We may therefore apply the Fubini-Tonelli theorem and obtain for $\Re(s) > 0$ that

$$\begin{aligned}\zeta_{f,cp}(s) &= \Gamma(s)^{-1} \int_0^\infty t^{s-1}\, \mathrm{tr}_\Gamma(e^{-t|f|^\perp})\, dt = \int_\epsilon^{||f||} \Gamma(s)^{-1} \int_0^\infty t^{s-1} e^{-t\lambda}\, dt\, dF_f(\lambda) \\ &= \int_\epsilon^{||f||} \lambda^{-s} dF_f(\lambda).\end{aligned} \tag{4.1.29}$$

It is well-known that for fixed $\lambda > 0$, the function λ^{-s} is an entire functions on $s \in \mathbb{C}$. Conversely, fixing $s \in \mathbb{C}$ and varying λ, the function λ^{-s}, as well as its s-derivative $-\log(\lambda)\lambda^{-s}$, is uniformly bounded on the interval $[\epsilon, ||f||]$, which is why both are integrable over $[\epsilon, ||f||]$ with respect to the spectral measure F_f.
From the dominated convergence theorem, we now conclude that the integral expression of $\zeta_{f,cp}(s)$

determines a holomorphic function on the half-plane $\{s \in \mathbb{C} : \Re(s) > 0\}$ and extends via the formula 4.1.29 to an entire function on all of $\mathbb{C}$, such that

$$\frac{d}{ds}\zeta_{f,cp}(s)|_{s=s_0} = -\int_{\epsilon}^{||f||} \log(\lambda)\lambda^{-s_0} dF_f. \tag{4.1.30}$$

In particular, one has

$$\log \det_\Gamma(f) = -\frac{d}{ds}\zeta_{f,cp}(s)|_{s=0}. \tag{4.1.31}$$

Furthermore, as the integral $\int_1^\infty t^{s-1}e^{-tz}$ converges absolutely for *any* $s \in \mathbb{C}$ and any $z \in \mathbb{C}$ with $\Re(z) > 0$, the same holds true for $\int_1^\infty t^{s-1}\operatorname{tr}_\Gamma(e^{-t|f|^\perp})dt$, since

$$\int_1^\infty |t^{s-1}\operatorname{tr}_\Gamma(e^{-t|f|^\perp})|dt = \int_1^\infty t^{\Re(s)-1}\operatorname{tr}_\Gamma(e^{-t|f|^\perp})dt \leq \dim_{\mathcal{N}(\Gamma)}(\mathcal{H})\int_1^\infty t^{\Re(s)-1}e^{-t\epsilon}dt < \infty.$$

Therefore, the expressions $\int_1^\infty t^{s-1}\operatorname{tr}_\Gamma(e^{-t|f|^\perp})dt$ and $\zeta_{f,la}(s)$ actually are entire functions for $s \in \mathbb{C}$ (no regularization is required here).
Together with the above, we conclude that the expression $\zeta_{f,sm}(s) = \zeta_{f,cp}(s) - \zeta_{f,la}(s)$ determines a holomorphic function on the half-plane $\{s \in \mathbb{C} : \Re(s) > 0\}$ and extends via the expression $\int_\epsilon^{||f||}\lambda^{-s}dF_f(\lambda) - \zeta_{f,la}(s)$ to an entire function on all of $\mathbb{C}$. In particular, f is ζ-regular and we get

$$\frac{d}{ds}\zeta_{f,cp}(s)|_{s=0} = \frac{d}{ds}\zeta_{f,sm}(s)|_{s=0} + \frac{d}{ds}\zeta_{f,la}(s)|_{s=0}. \tag{4.1.32}$$

Lastly, it is well-known that for any complex-valued function $h(s)$ holomorphic at 0, it holds that $\frac{d}{ds}(\Gamma(s)^{-1}h(s))|_{s=0} = h(0)$. Applied to $h(s) = \int_1^\infty t^{s-1}\operatorname{tr}_\Gamma(e^{-t|f|^\perp})$, we therefore obtain

$$\frac{d}{ds}\zeta_{f,la}(s)|_{s=0} = \int_1^\infty t^{-1}\operatorname{tr}_\Gamma(e^{-t|f|^\perp}). \tag{4.1.33}$$

The equality $\exp(-\frac{d}{ds}\zeta_{f,cp}(s)|_{s=0}) = \det_\Gamma^\zeta(f)$ now follows from 4.1.31–4.1.33. $\square$

Remark 4.1.26. In order to justify the introduction of the ζ-regularized determinant, one has to provide examples of unbounded operators (for which $\det_\Gamma$ *cannot* be defined, as already mentioned) that are still ζ-regular and of determinant class. In fact, many such examples exist and will be thoroughly studied in the next chapters.

Although the previous lemma shows that they agree in some instances, $\det_\Gamma^\zeta$ is in general *not* as well-behaved as $\det_\Gamma$. For example, a multiplicativity formula as in Lemma 4.1.14 does in general *not* hold for $\det_\Gamma^\zeta$. We refer to [33] for more details on this phenomenon. In spite of this, the crucial *additivity* property from $\det_\Gamma$ still extends to $\det_\Gamma^\zeta$.

Lemma 4.1.27. *Let $f : \mathcal{H} \to \mathcal{H}'$ and $g : \mathcal{H}'' \to \mathcal{H}'''$ be morphisms of Hilbert $\mathcal{N}(\Gamma)$-modules. Then the direct sum $f \oplus g : \mathcal{H} \oplus \mathcal{H}'' \to \mathcal{H}' \oplus \mathcal{H}'''$ is a Fredholm morphism if and only if f and g are Fredholm morphisms. Moreover, for each $t > 0$, $e^{-t|f\oplus g|^\perp}$ is of trace class if and only if both $e^{-t|f|^\perp}$ and $e^{-t|g|^\perp}$ are of trace class, in which case we get*

$$\operatorname{tr}_\Gamma(e^{-t|f\oplus g|^\perp}) = \operatorname{tr}_\Gamma(e^{-t|f|^\perp}) + \operatorname{tr}_\Gamma(e^{-t|g|^\perp}). \tag{4.1.34}$$

In particular, if both f and g are ζ-regular, so is $f \oplus g$, in which case we get

$$\det_\Gamma^\zeta(f \oplus g) = \det_\Gamma^\zeta(f)\det_\Gamma^\zeta(g) \in \mathbb{R}_{\geq 0}\,. \tag{4.1.35}$$

Proof. First, observe first that $(f \oplus g)^\perp = f^\perp \oplus g^\perp$. Together with Lemma 4.1.5, we deduce that

$$\forall \lambda \in \mathbb{R}_{\geq 0} : F((f \oplus g)^\perp, \lambda) = F(f^\perp, \lambda) + F(g^\perp, \lambda) \Longrightarrow F_{(f \oplus g)^\perp} = F_{f^\perp} + F_{g^\perp}. \tag{4.1.36}$$

Applying Proposition 4.1.17, it follows that for any $t > 0$, $e^{-t|f \oplus g|^\perp}$ is of trace class if and only if both $\operatorname{tr}_\Gamma(e^{-t|f|^\perp})$ and $\operatorname{tr}_\Gamma(e^{-t|g|^\perp})$ are of trace class

$$\begin{aligned} \operatorname{tr}_\Gamma(e^{-t|f \oplus g|^\perp}) &= \int_0^\infty e^{-t\lambda} dF_{(f \oplus g)^\perp}(\lambda) = \int_0^\infty e^{-t\lambda} \left(dF_{f^\perp}(\lambda) + dF_{g^\perp}(\lambda) \right) \\ &= \operatorname{tr}_\Gamma(e^{-t|f|^\perp}) + \operatorname{tr}_\Gamma(e^{-t|g|^\perp}). \end{aligned}$$

□

4.1.1 Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

Definition 4.1.28. A **Hilbert $\mathcal{N}(\Gamma)$-cochain complex** (C_*, c_*) is a sequence

$$(C_*, c_*) : 0 \to C_0 \xrightarrow{c_0} C_1 \xrightarrow{c_1} C_2 \xrightarrow{c_2} \ldots, \tag{4.1.37}$$

where C_p is a Hilbert $\mathcal{N}(\Gamma)$-module, each c_p is a morphism between Hilbert $\mathcal{N}(\Gamma)$-modules, satisfying $c_{p+1} \circ c_p = 0$. (C_*, c_*) is called **finite** if additionally, $C_p = 0$ for all but finitely many p's, each c_p is bounded, and each C_p is a finitely-generated Hilbert $\mathcal{N}(\Gamma)$-module.

A **morphism** $f_* : (C_*, c_*) \to (D_*, d_*)$ of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes is a family $f_p : C_p \to D_p$ of *bounded* morphisms, satisfying for each $p \in \mathbb{N}_0$

- $f_p(\operatorname{dom}(c_p)) \subseteq \operatorname{dom}(d_p)$,
- $f_{p+1} c_p = d_p f_p$ on $\operatorname{dom}(c_p)$.

We say that two morphisms $f_*, g_* : C_* \to D_*$ between Hilbert $\mathcal{N}(\Gamma)$-cochain complexes are **L^2-chain homotopic** (written $f \simeq g$) if there exists a collection of bounded morphisms $K_* : C_* \to D_{*-1}$, satisfying for each $p \in \mathbb{N}_0$

- $K_p(\operatorname{dom}(c_p)) \subseteq \operatorname{dom}(d_{p-1})$,
- $f_p - g_p = K_{p+1} c_p + d_{p-1} K_p$ on $\operatorname{dom}(c_p)$.

K_* is called an **chain homotopy** between f_* and g_*. Two Hilbert $\mathcal{N}(\Gamma)$-cochain complexes (C_*, c_*) and (D_*, d) are called **chain homotopy equivalent** (written $C_* \sim D_*$) if there exists morphisms $f_* : C_* \to D_*$ and $g_* : D_* \to C_*$ such that $f_* g_* \simeq \mathbb{1}_{D_*}$ and $g_* f_* \simeq \mathbb{1}_{C_*}$. f_* is called a **chain homotopy equivalence** between C_* and D_* with *chain homotopy inverse* g_*.
A cochain complex (C_*, c_*) is said to be *contractible* if it is chain homotopy equivalent to the trivial complex $(\{0\}_*, 0_*) = \{0\} \to \{0\} \to \{0\} \to \ldots$. Equivalently, (C_*, c_*) is contractible if the identity map $\mathbb{1}_{C_*} : C_* \to C_*$ and the trivial map $0_{C_*} : C_* \to C_*$ are chain homotopic.
Since c_p is by requirement closed and Γ-equivariant, it follows that for each $p \in \mathbb{N}_0$, $\ker(c_p) \subseteq C_p$ is a closed, Γ-equivariant subspace. Since $\operatorname{im}(c_{p-1}) \subseteq \ker(c_p)$, we must therefore also have $\overline{\operatorname{im}(c_{p-1})} \subseteq \ker(c_p)$, so that the next definition makes sense.

Definition 4.1.29 (L^2-cohomology)**.** Let (C_*, c_*) be a cochain complex of Hilbert $\mathcal{N}(\Gamma)$-modules. For $p \in \mathbb{N}_0$, the p-th L^2**-cohomology** of (C_*, c_*) is defined to be the Hilbert $\mathcal{N}(\Gamma)$-module (with induced inner product and Γ-action)

$$H_p(C) := \ker(c_p)/\overline{\operatorname{im}(c_{p-1})}. \tag{4.1.38}$$

Similarly, the p-th (unreduced, ordinary) **cohomology** of (C_*, c_*) is defined to be the (not necessarily complete) quotient space

$$\widehat{H}_p(C) := \ker(c_p)/\operatorname{im}(c_{p-1}). \tag{4.1.39}$$

Definition 4.1.30. The p-th L^2**-Betti-number** of a Hilbert $\mathcal{N}(\Gamma)$-cochain complex (C_*, c_*) is defined as

$$b_p^{(2)}(C_*) := \dim_{\mathcal{N}(\Gamma)}(H_p(C_*)) \in [0, \infty]. \tag{4.1.40}$$

we say that (C_*, c_*) is (weakly) **acyclic** if $b_p^{(2)}(C_*) = 0$ (equivalently, if $H_p(C_*) = \{0\}$) for all $p \in \mathbb{N}_0$.

Using the definitions, the next lemma can be proven straightforwardly:

Lemma 4.1.31. *Let $(C_*, c*)$ and (D_*, d_*) be two Hilbert $\mathcal{N}(\Gamma)$-cochain complexes. Then any morphism $f_* : C_* \to D_*$ induces for each $p \in \mathbb{N}_0$ a bounded morphism of Hilbert $\mathcal{N}(\Gamma)$ modules $H_*(f_*) : H_*(C) \to H_*(D)$ that is* ***natural*** *in the sense that*

1. *if $g_* : D_* \to E_*$ is a another morphism of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes, then $H_*(f_* g_*) = H_*(f_*)H_*(g_*)$,*
2. *$H_*(1_{C_*}) = 1_{H_*(C)}$ for $1_{C_*} : C_* \to C_*$ the identity, and*
3. *if $f \simeq g$, then $H_*(f_*) = H_*(g_*)$.*

Immediately, we obtain

Corollary 4.1.32. *If $C_* \sim D_*$ via the chain homotopy equivalences $f_* : C* \to D_*$ and $g_* : D_* \to C_*$, then $H_*(f_*) : H_*(C) \to H_*(D)$ is an isomorphism of Hilbert $\mathcal{N}(\Gamma)$-modules with inverse $H_*(g_*)$. In particular, we get $b_p^{(2)}(C_*) = b_p^{(2)}(D_*)$ for any $p \in \mathbb{N}_0$.*

Given a chain complex (C_*, c_*) and $p \in \mathbb{N}_0$, we define its p-th *Laplacian*

$$\Delta_p := c_{p-1}c_{p-1}^* + c_p^* c_p \tag{4.1.41}$$

simply as the sum of two self-adjoint operators. Since both $\operatorname{dom}(c_{p-1}c_{p-1}^*)$ and $\operatorname{dom}(c_p^* c_p)$ are Γ-equivariant by Theorem 4.1.2, it follows that Δ_p is a Γ-equivariant operator. That Δ_p is, in fact, a self-adjoint morphism will follow from the next proposition:

Proposition 4.1.33. *For each $p \in \mathbb{N}_0$, we have*

$$\ker(\Delta_p) = \ker(c_p) \cap \ker(c_{p-1}^*) \tag{4.1.42}$$

and an orthogonal ***Hodge*** *decomposition*

$$C_p = \ker(\Delta_p) \oplus \ker(c_{p-1}^*)^\perp \oplus \overline{\operatorname{im}(c_p^*)} = \ker(\Delta_p) \oplus \overline{\operatorname{im}(c_{p-1})} \oplus \ker(c_p)^\perp. \tag{4.1.43}$$

In particular, we have a natural isomorphism of Hilbert $\mathcal{N}(\Gamma)$-modules

$$\ker(\Delta_p) \cong H_p(C_*).$$

Moreover, with regards to the above Hodge decomposition, Δ_p decomposes as the diagonal operator

$$\Delta_p := 0 \oplus c_{p-1}^{\perp}(c_{p-1}^{\perp})^* \oplus (c_p^{\perp})^* c_p^{\perp} \tag{4.1.44}$$

and is therefore self-adjoint (as the direct sum of self-adjoint operators).

Proof. The inclusion $\ker(c_p) \cap \ker(c_{p-1}^*) \subseteq \ker(\Delta_p)$ follows directly from the definition. For the opposite inclusion, observe first that, by the definition of sum and composition of unbounded operators, we have $\mathrm{dom}(\Delta_p) = \mathrm{dom}(c_{p-1}c_{p-1}^*) \cap \mathrm{dom}(c_p^* c_p) = (c_{p-1}^*)^{-1}\,\mathrm{dom}(c_{p-1}) \cap c_p^{-1}\,\mathrm{dom}(c_p^*) \subseteq \mathrm{dom}(c_{p-1}^*) \cap \mathrm{dom}(c_p) = \mathrm{dom}(c_{p-1}^*) \cap \mathrm{dom}((c_p^*)^*)$. Therefore, if $x \in \ker(\Delta_p)$, we can make the following algebraic simplifications

$$0 = \langle \Delta_p x, x\rangle = \langle c_{p-1}c_{p-1}^* x, x\rangle + \langle c_p^* c_p x, x\rangle = ||c_{p-1}^* x||^2 + ||c_p x||^2, \tag{4.1.45}$$

implying that $x \in \ker(c_{p-1}^*) \cap \ker(c_p)$, therefore 4.1.42 as desired.
Using the inclusion $\overline{\mathrm{im}(c_{p-1})} \subseteq \ker(c_p)$ and the equality $\mathrm{im}(c_{p-1})^{\perp} = \ker(c_{p-1}^*)$, we can now compute

$$\begin{aligned} C_p &= \ker(c_p) \oplus \ker(c_p)^{\perp} = \big(\ker(c_p) \cap \mathrm{im}(c_{p-1})^{\perp}\big) \oplus \overline{\mathrm{im}(c_{p-1})} \oplus \ker(c_p)^{\perp} \\ &= \big(\ker(c_p) \cap \ker(c_{p-1}^*)\big) \oplus \overline{\mathrm{im}(c_{p-1})} \oplus \ker(c_p)^{\perp} = \ker(\Delta_p) \oplus \overline{\mathrm{im}(c_{p-1})} \oplus \ker(c_p)^{\perp}. \end{aligned}$$

□

For a Hilbert $\mathcal{N}(\Gamma)$-cochain complex (C_*, c_*) and $p \in \mathbb{N}_0$, we set

$$F_p(C_*, \lambda) := F(c_p|_{\mathrm{im}(c_{p-1})^{\perp}}, \lambda) \in [0, \infty]. \tag{4.1.46}$$

If $F_p(C_*, 0) < \infty$, we can further define

$$\hat{F}_p(C_*, \lambda) := F_p(C_*, \lambda) - F_p(C_*, 0). \tag{4.1.47}$$

Such a complex (C_*, c_*) is called **Fredholm at** p if the restricted morphism $c_p|_{\mathrm{im}(c_{p-1})^{\perp}}$ is Fredholm. (C_*, c_*) is called a *Fredholm complex* if it is Fredholm at p all $p \in \mathbb{N}_0$, i.e. if all of its restricted boundary maps are Fredholm operators. Observe that any finite Hilbert $\mathcal{N}(\Gamma)$-cochain complex is automatically Fredholm.
For $p \in \mathbb{N}_0$, define the p-th **Novikov-Shubin invariant** of (C_*, c_*) that is Fredholm at p via

$$\alpha_p(C_*) := \alpha(c_p|_{\mathrm{im}(c_{p-1})^{\perp}}). \tag{4.1.48}$$

Observe that the inclusion $\mathrm{im}(c_{p-1}) \subseteq \overline{\mathrm{im}(c_{p-1})}$ induces a canonical embedding of vector spaces $H_p(C) \subseteq \hat{H}_p(C)$. Therefore, $\hat{H}_p(C) = 0$ certainly also implies $H_p(C) = 0$. A more precise relation is formulated next:

Lemma 4.1.34. [54, Lemma 2.18] *Let (C_*, c_*) be a cochain complex of Hilbert $\mathcal{N}(\Gamma)$-modules. Then the following properties are equivalent:*

1. *(C_*, c_*) is strongly acyclic, that is $\widehat{H}_p(C_*) = 0$ for all $p \in \mathbb{N}_0$,*

2. Δ_p *is boundedly invertible for all* $p \in \mathbb{N}_0$,

3. (C_*, c_*) *is Fredholm, acyclic and satisfies* $\alpha_p(C_*) = \infty^+$ *for all* $p \in \mathbb{N}_0$.

Given a sequence of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

$$0 \to C_* \xrightarrow{f_*} D_* \xrightarrow{g_*} E_* \to 0 \tag{4.1.49}$$

that is *exact* (i.e. $f_p : C_p \to D_p$ is injective, $g_p : D_p \to E_p$ is surjective and $\ker(g_p) = \operatorname{im}(f_p)$ for each $p \in \mathbb{N}$), we obtain a sequence of morphisms in reduced cohomology

$$\cdots \to H_p(C_*) \xrightarrow{H_p(f_*)} H_p(D_*) \xrightarrow{H_p(g_*)} H_p(E_*) \xrightarrow{\partial_p} H_{p+1}(C_*) \to \ldots, \tag{4.1.50}$$

which is weakly exact whenever all three complexes C_*, D_*, E_* are Fredholm, see [54, Theorem 1.21]. $\partial_p : H_p(E_*) \to H_{p+1}(C_*)$ is the *connecting homomorphim* induced be the above short exact sequence. A central result that relates the spectral density functions of cochain complexes in a given short exact sequence is the following:

Proposition 4.1.35. [55, Theorem 4.11] *Let* (C_*, c_*), (D_*, d_*) *and* (E_*, e_*) *be three Hilbert* $\mathcal{N}(\Gamma)$*-cochain complexes, and*

$$0 \to C_* \xrightarrow{f_*} D_* \xrightarrow{g_*} E_* \to 0 \tag{4.1.51}$$

a short exact sequence of morphisms between them. Suppose that C_* *and* E_* *are both Fredholm at* p. *Then* D_* *is also Fredholm at* p *and the connecting homomorphism* $\partial_p : H_p(E) \to H_{p+1}(C)$, *induced by the above short exact sequence, is left Fredholm. Moreover, we have*

$$\hat{F}_p(D_*, \lambda) \leq \hat{F}_p(C_*, c_C \cdot \lambda^{1/2}) + \hat{F}(\partial_p, c_\partial \cdot \lambda^{1/4}) + \hat{F}_p(E_*, c_E \cdot \lambda^{1/2}) \ \ \textit{for} \ \ 0 \leq \lambda < c_1, \tag{4.1.52}$$

where

$$c_C \coloneqq ||f_{p+1}^{-1}||^{1/2} \cdot ||f_p||, \tag{4.1.53}$$

$$c_E \coloneqq (4 + 2||d_p||)||g_{p+1}|| \cdot ||g_p^{-1}||, \tag{4.1.54}$$

$$c_\partial \coloneqq ||f_{p+1}^{-1}||^{1/2}(4 + 2||f_{p+1}^{-1}|| \cdot ||d_p||) \cdot ||g_p^{-1}||, \tag{4.1.55}$$

$$c_1 \coloneqq (4 + 2||d_p||)^{-1/2}. \tag{4.1.56}$$

Definition 4.1.36. A Hilbert $\mathcal{N}(\Gamma)$-cochain complex (C_*, c_*) is *of determinant class* if all of its boundary morphisms c_p are of determinant class.

The next central result, due to Gromov and Shubin, states that near 0, the spectral densities of homotopy equivalent complexes decay at essentially the same speed. More precisely, the spectral densities are said to be *dilatationally equivalent.*

Theorem 4.1.37. [42, Proposition 4.1] *Let* (C_*, c_*) *and* (D_*, d_*) *be two Hilbert* $\mathcal{N}(\Gamma)$*-cochain complexes,* $f_* : C_* \to D_*$ *a chain homotopy equivalence with chain homotopy inverse* $g_* : D_* \to C_*$, *and* $K_* : C_* \to C_{*-1}$ *a chain homotopy between* $g_* f_*$ *and* $\mathbb{1}_{C_*}$. *Then, for each* $p \in \mathbb{N}_0$, *we have*

$$F_p(C_*, \lambda) \leq F_p(D_*, ||f_{p+1}||^2 ||g_p||^2 \lambda) \qquad \textit{for}\, 0 \leq \lambda < ||2K_{p+1}||^{-2}. \tag{4.1.57}$$

The most important consequence that will be frequently used throughout this thesis is the following:

Corollary 4.1.38. *Let (C_*, c_*) and (D_*, d_*) be two Hilbert $\mathcal{N}(\Gamma)$-cochain complexes with $C_* \sim D_*$. Then*

1. *For all $p \in \mathbb{N}_0$, C_* is left Fredholm at p if and only if D_* is left Fredholm at p. In this case, we have $\alpha_p(C_*) = \alpha_p(D_*)$.*

2. *In case that both complexes are Fredholm, C_* is of determinant class if and only if D_* is of determinant class.*

Definition 4.1.39 (L^2-torsion of finite-type chain complexes)**.** Let (C_*, c_*) be a Hilbert $\mathcal{N}(\Gamma)$-module cochain complex of finite type and of determinant class. We define the **L^2-torsion** $T(C_*) \in \mathbb{R}_+$ **of** C_* by

$$\log T(C_*) := \sum_{p=0}^{\infty} (-1)^p \log(\det_\Gamma(c_p)). \tag{4.1.58}$$

The L^2-torsion of finite-type chain complexes behaves nicely with respect to chain homotopy equivalences, as shown in the next proposition:

Proposition 4.1.40. [54, Lemma 3.44] *Let (C_*, c_*) and (D_*, d_*) be two finite-type cochain complexes of Hilbert $\mathcal{N}(\Gamma)$-modules and let $f_* : C_* \to D_*$ be a chain isomorphism. Then, for any $p \in \mathbb{N}_0$, $H_p(f_*) : H_p(C_*) \to H_p(D_*)$ is an isomorphism of Hilbert $\mathcal{N}(\Gamma)$-modules. Moreover, (C_*, c_*) is of determinant class if and only if (D_*, d_*) is of determinant class, in which case we get*

$$\log(T(C_*)) - \log(T(D_*)) = \sum_{p=0}^{\infty} (-1)^p \log(\det_\Gamma(f_p)) - \sum_{p=0}^{\infty} (-1)^p \log\left(\det_\Gamma(H_p(f_p))\right). \tag{4.1.59}$$

Using the ζ-regularized determinant, we wish to extend the notion of L^2-torsion onto Hilbert $\mathcal{N}(\Gamma)$-cochain complexes (C_*, c_*) that are not necessarily of finite type, but still of finite length (that is, $C_n = 0$ for all but finitely many $n \in \mathbb{N}$). To motivate the definition, we remark that one can easily verify directly with aid of Proposition 4.1.33 that a Hilbert $\mathcal{N}(\Gamma)$-cochain complex is Fredholm and of determinant class if and only if all of its Laplacians Δ_p are Fredholm and of determinant class. Moreover, it is shown in [54, Lemma 3.30] that on a finite type Hilbert $\mathcal{N}(\Gamma)$-cochain complex (C_*, c_*) of determinant class, one has

$$\log T(C_*) = \sum_{p=1}^{\infty} \frac{1}{2}\, p\, (-1)^{p+1} \log(\det_\Gamma(\Delta_p)).$$

For practical purposes yet to be established, it will be convenient in many instances to consider the determinant of the Laplacians instead of the determinant of the boundary operators.

Definition 4.1.41 (Regularized L^2-torsion for finite-length chain complexes)**.** Let (C_*, c_*) be a Hilbert $\mathcal{N}(\Gamma)$-cochain complex of finite length that is Fredholm and of determinant class. We say that C_* is **ζ-regular** if all of its Laplacians Δ_p are ζ-regular. In this case, we define the **regularized L^2-torsion** $T_\zeta(C_*) \in \mathbb{R}_+$ of C_* as

$$\log T_\zeta(C_*) := \sum_{p=1}^{\infty} \frac{1}{2}\, p\, (-1)^{p+1} \log(\det{}_\Gamma^\zeta(\Delta_p)). \tag{4.1.60}$$

As already mentioned, the ζ-regularized determinant is not as well behaved as the Fuglede-Kadison determinant, which is among the reasons why we cannot expect a similar formula as in 4.1.40 to hold for the ζ-regularized torsion in general. In spite of this, using Proposition 4.1.14 and Lemma 4.1.27, we still get sum formulas for *both* torsion elements.

Proposition 4.1.42 (Sum formula)**.** *Let (C_*, c_*) and (D_*, d_*) be two Hilbert $\mathcal{N}(\Gamma)$-cochain complexes, and let $(C_* \oplus D_*, c_* \oplus d_*)$ be the direct sum complex. Then,*

1. *if (C_*, c_*) and (D_*, d_*) are of finite type, then so is $(C_* \oplus D_*, c_* \oplus d_*)$. Moreover, (C_*, c_*) and (D_*, d_*) are of determinant class if and only if $(C_* \oplus D_*, c_* \oplus d_*)$ is of determinant class, in which case we get*
$$\log T(C_* \oplus D_*) = \log T(C_*) + \log T(D_*). \tag{4.1.61}$$

2. *If (C_*, c_*) and (D_*, d_*) are of finite length, ζ-regular and of determinant class, then the same properties hold for $(C_* \oplus D_*, c_* \oplus d_*)$, in which case we get*
$$\log T_\zeta(C_* \oplus D_*) = \log T_\zeta(C_*) + \log T_\zeta(D_*). \tag{4.1.62}$$

4.2 The De Rham and Sobolev complexes

Throughout the whole section, we assume that $(E, h) \downarrow (M, g)$ is a flat Hermitian bundle of bounded geometry over a complete Riemannian manifold (M, g) of dimension n and of bounded geometry. We also assume *absolute boundary conditions*, that is $\partial M = \partial_2 M$ throughout. This allows us to draw algebraic conclusions out of the next geometric constructions.

Definition 4.2.1 (The de Rham and Sobolev complexes)**.** Let $E \downarrow M$ be as above and let $0 \leq p \leq n$.

1. The **Sobolev chain complex at level** p, denoted by $D_p[E]$ is the cochain complex of Hilbert spaces, defined as
$$\cdots \to 0 \to \mathcal{W}_2^{p-1}(M, E) \xrightarrow{d} \mathcal{W}_1^p(M, E) \xrightarrow{d} \mathcal{W}_0^{p+1}(M, E) = \Omega_{(2)}^{p+1}(M, E) \to 0 \to \ldots. \tag{4.2.1}$$

2. The **absolute Sobolev chain complex at level** p, denoted by $D_{p,abs}[E]$ is the cochain complex of Hilbert spaces, defined as
$$\cdots \to 0 \to \mathcal{W}_{2,abs}^{p-1}(M, E) \xrightarrow{d} \mathcal{W}_{1,abs}^p(M, E) \xrightarrow{d} \mathcal{W}_0^{p+1}(M, E) \to 0 \to \ldots, \tag{4.2.2}$$
where
$$\mathcal{W}_{2,abs}^{p-1}(M, E) \coloneqq \{\omega \in \mathcal{W}_2^{p-1}(M, E) : i^*(\#\omega) = 0 = i^*(\#d\omega)\}, \tag{4.2.3}$$
$$\mathcal{W}_{1,abs}^p(M, E) \coloneqq \{\omega \in \mathcal{W}_1^p(M, E) : i^*(\#\omega) = 0\}. \tag{4.2.4}$$
and the inner product on each space is the one induced by $\mathcal{W}_2^{p-1}(M, E)$, respectively $\mathcal{W}_1^p(M, E)$.

3. The **de Rham complex at level** p, denoted by $L_p[E]$ is the cochain complex of Hilbert spaces, defined as
$$\cdots \to 0 \to \Omega_{(2)}^{p-1}(M, E) \xrightarrow{d} \Omega_{(2)}^p(M, E) \xrightarrow{d} \Omega_{(2)}^{p+1}(M, E) \to 0 \to \ldots. \tag{4.2.5}$$

Observe that by Theorem 3.4.4, we have an inclusion of cochain complexes $D_{p,abs}[E] \subseteq D_p[E] \subset L_p[E] \subseteq \Omega_{(2)}^*(M, E)$ for any $0 \leq p \leq n$, so that that $D_{p,abs}[E] \subseteq D_p[E]$ is a closed subcomplex. Note also by Proposition 3.2.6 that the differentials on $D_p[E]$ are bounded and everywhere defined as linear operators. Using the machinery developed in the previous section, we get:

Proposition 4.2.2. *Let $(E,h) \downarrow (M,g)$ be as above. Suppose that $\Gamma \subseteq \mathrm{Isom}^+(M)$ is a lattice (not necessarily uniform) compatible with $(E,h) \downarrow (M,g)$. Then the following holds:*

1. *For each $0 \leq p \leq n$, there exists a linear Γ-action on $\Omega^p(M,E)$, so that, with respect to this action, the differential $d : \Omega^p(M,E) \to \Omega^{p+1}(M,E)$ is Γ-equivariant. Moreover, the action restricts to an action on $\Omega^p_c(M,E)$ by bundle isometries.*
2. *$\Omega^*_{(2)}(M,E)$ and $L_p[E]$ are Hilbert $\mathcal{N}(\Gamma)$-cochain complexes,*
3. *$D_p[E]$ and $D_{p,abs}[E]$ are bounded Hilbert $\mathcal{N}(\Gamma)$-cochain complexes.*
4. *Assume also that that there exists some **uniform** lattice $\Lambda < \mathrm{Isom}^+(M,g)$ compatible with $(E,h) \downarrow M$. Then:*
   - *(a) The Laplacian $\Delta_p[E] : \Omega^p_{(2)}(M,E) \to \Omega^p_{(2)}(M,E)$, defined as the minimal closure of the operator $\delta^p d^p + d^{p-1}\delta^{p-1}$ with initial domain $\{\omega \in \Omega^p_c(M,E) : i^*(\#\omega) = i^*(\#d\omega) = 0\}$ is a self-adjoint morphism of Hilbert $\mathcal{N}(\Gamma)$-modules.*
   - *(b) Let $f : [0,\infty] \to \mathbb{R}_{\geq 0}$ be a rapidly decreasing Borel function, let $f(\Delta_p[E]) : \Omega^p_{(2)}(M,E) \to \Omega^p_{(2)}(M,E)$ be the positive, bounded operator defined via the spectral theorem applied to $\Delta_p[E]$. Then $f(\Delta_p[E])$ is a positive, bounded, trace class morphism of Hilbert $\mathcal{N}(\Gamma)$-modules, that is moreoever an integral operator: Let $f(\Delta_p[E])(x,y)$ be its smooth Schwartz kernel and let $\mathcal{F} \subseteq M$ be a fundamental domain for the Γ-action on M. Then the von Neumann trace of $f(\Delta_p[E])$, as introduced at the beginning of Section 4.1, coincides with its Γ-regularized trace from Definition 2.2.3, i.e.*

     $$\mathrm{tr}_\Gamma(f(\Delta_p[E])) = \int_{\mathcal{F}} \mathrm{tr}(f(\Delta_p[E])(x,x))dx < \infty \tag{4.2.6}$$

     holds. In particular, both $\Omega^*_{(2)}(M,E)$ and $L_p[E]$ are Fredholm complexes of Hilbert $\mathcal{N}(\Gamma)$-modules.

Proof. 1: Since Γ acts on $E \downarrow M$ by bundle isomorphisms, we obtain a $\mathbb{C}$-linear Γ-action on the space of sections $\Gamma(E) = \Omega^0(M,E)$, defined for $f \in \Gamma(E)$, $\gamma \in \Gamma$ and $x \in M$ via

$$(\gamma.f)(x) := \gamma \cdot f(\gamma^{-1}(x)) \in E_x. \tag{4.2.7}$$

Similarly, for $p \geq 1$ and $\Omega^p(M,E) = \Omega^p(M) \otimes_{C^\infty(M)} \Gamma(E)$, we define the linear Γ-action to be the induced tensor product action (on $\Omega^p(M)$, Γ acts by the standard pullback). Flatness of the Γ-action on $E \downarrow M$ implies that the action thus defined commutes with the differential $d : \Omega^p(M,E) \to \Omega^{p-1}(M,E)$. It is clear that, with respect to this action, the subspace $\Omega^p_c(M,E)$ is Γ-invariant. We will prove that Γ acts isometrically on $\Omega^0_c(M,E) = \Gamma_c(E)$, for $p \geq 1$, the proof works completely analogous. Let h be the Hermitian form on $E \downarrow M$. Using the transformation formula and the fact that Γ acts by bundles isometries on $E \downarrow M$, we compute for any $f \in \Gamma_c(E)$ and any $\gamma \in \Gamma$, that

$$\begin{aligned} ||\gamma.f||_0^2 &= \int_M ||\gamma.f(x)||^2_{h(x)}dx = \int_M ||\gamma \cdot f(\gamma^{-1}(x))||^2_{h(x)}dx \\ &= \int_M ||f(\gamma^{-1}(x))||^2_{h(\gamma^{-1}(x))}dx = \int_M ||f(x)||^2_{h(x)}dx = ||f||_0^2. \end{aligned}$$

2: The Γ-action, defined in (1), extends to an action on $\Omega^*(M,E)$ and $L_p[E]$ by isometries. We claim that for each $0 \leq p \leq m$ and for $\mathcal{F} \subseteq M$ a closed fundamental domain for the Γ-action on M, there exists a Γ-equivariant isometry of Hilbert spaces $\Omega^p_{(2)}(M,E) \cong L^2(\Gamma)\hat{\otimes}\Omega^p_{(2)}(\mathcal{F}, E|_{\mathcal{F}})$. (2) then follows from the claim. To prove the claim, we only need to show that the $\mathbb{C}$-linear map

$$\begin{aligned} G : \Omega^p_c(M,E) &\to \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \Omega^p_c(\mathcal{F}, E|_{\mathcal{F}}), \\ f &\mapsto \sum_{\gamma\in\Gamma} \gamma \otimes (\gamma^{-1}.f)|_{\mathcal{F}} \end{aligned}$$

of inner product spaces is Γ-equivariant, isometric and has dense image. Observe that left-hand sum is finite, since f is compactly supported, so G is well-defined. Since

$$\begin{aligned} G(\gamma'.f) &= \sum_{\gamma\in\Gamma} \gamma \otimes ((\gamma^{-1}\cdot\gamma').f)|_{\mathcal{F}} \overset{\hat{\gamma}:=\gamma'^{-1}\cdot\gamma}{=} \sum_{\hat{\gamma}\in\Gamma} \gamma'\cdot\hat{\gamma} \otimes (\hat{\gamma}^{-1}.f)|_{\mathcal{F}} \\ &= \gamma'.\sum_{\gamma\in\Gamma} \gamma \otimes (\gamma^{-1}.f)|_{\mathcal{F}} = \gamma'.G(f), \end{aligned}$$

G is Γ-equivariant. Moreover, considering the image of forms having compact support in $\gamma.\mathcal{F}$ with γ ranging over all of Γ, one sees that, with respect to the inner product defined on $\mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\Omega^p_c(\mathcal{F},E|_{\mathcal{F}})$, $\Im(G)$ is also dense. Finally, we calculate

$$\begin{aligned} ||f||_0^2 &= \int_M ||f(x)||^2_{h(x)}dx = \sum_{\gamma\in\Gamma}\int_{\gamma.\mathcal{F}} ||f(x)||^2_{h(x)}dx = \sum_{\gamma\in\Gamma}\int_{\mathcal{F}} ||f(\gamma.x)||^2_{h(\gamma.x)}dx \\ &= \sum_{\gamma\in\Gamma}\int_{\mathcal{F}} ||(\gamma.f)(x)||^2_{h(x)}dx = ||G(f)||^2. \end{aligned}$$

3: The Γ-action, defined in (1), extends to an action on $D_p[E]$ by isometries. Here, one additionally needs the fact that the differential, the Hodge-# operator and the restriction to the boundary ∂M commute with the Γ-action. Now for any $r > 0$, one establishes a Γ-equivariant, isometric isomorphism $\mathcal{W}^p_r(M,E) = L^2(\Gamma)\hat{\otimes}\,\mathcal{W}^p_r(\mathcal{F},E|_{\mathcal{F}})$ just like in assertion (2). The assertion for $D_p[E]$ then follows. The assertion for $D_{p,abs}[E]$ follows, since it is a closed, Γ-invariant subspace of $D_p[E]$.

(4a): By our initial assumption, there exists *some* uniform lattice Λ that is compatible with $E \downarrow M$. Therefore, $E \downarrow M$ is the lift of a flat bundle over the compact quotient M/Λ. By Theorem 3.4.1, it follows that $\Delta_p[E]$ is self-adjoint. Recall that $\Delta_p[E]$ is the minimal closure of the symmetric Hodge-Laplacian with initial domain $\Omega^p(M,\partial M,E) = \{\omega \in \Omega^p_c(M,E) : i^*\#\omega = i^*\#d\omega = 0\}$, where $i : \partial M \to M$ denotes the smooth inclusion map. Since $\Omega^p(M,\partial M,E)$ is obviously Γ-invariant, the same holds true for $\mathrm{dom}(\Delta_p[E])$. We have already mentioned that Γ commutes with the differential and the Hodge-# operator. Therefore, $\gamma\cdot\Delta_p[E] = \Delta_p[E]\cdot\gamma$ and the assertion follows.

(4b): That $f(\Delta_p[E])$ is a morphism of Hilbert $\mathcal{N}(\Gamma)$-modules follows from (4) and the spectral theorem, while the result that $f(\Delta_p[E])$ posseses a smooth integral kernel follows from Proposition 3.4.2. Finally, the equality $\mathrm{tr}_\Gamma(f(\Delta_p[E])) = \int_{\mathcal{F}} \mathrm{tr}(f(\Delta_p[E](x,x))dx$ is shown in [1, Proposition 4.16, pp. 63-65]. □

Definition 4.2.3. Let $E \downarrow M$ be a flat bundle over a Riemannian manifold (M,g) associated to a finite-dimensional, complex representation $\rho : \pi_1(M) =: \Gamma \to \mathrm{GL}(V)$ and let $\widetilde{E} \downarrow \widetilde{M}$ be the Γ-equivariant lift of $E \downarrow M$ onto the universal cover $\widetilde{M}$ of M. Further, let g be a Riemannian metric on M, h a Hermitian form on E and $\widetilde{g}$,$\widetilde{h}$ their respective lifts. We say that $(E,h) \downarrow (M,g)$ is **L^2-acyclic/Fredholm/ζ-regular/of analytic determinant class**, if the corresponding L^2-cochain complex $\Omega_{(2)}(\widetilde{M},\widetilde{E}) = \Omega_{(2)}(\widetilde{M},\widetilde{E},\widetilde{g},\widetilde{h})$ of Hilbert $\mathcal{N}(\Gamma)$-modules with absolute boundary conditions is L^2-acyclic/Fredholm/ζ-regular/of determinant class.

For $0 \leq k \leq \dim(M)$, we define the **k-th analytic L^2-Betti Number** $b^{An}_{(2),k}(M,\rho) \in [0,\infty]$ as

$$b^{An}_{(2),k}(M,\rho) \coloneqq b^{(2)}_k(\Omega_{(2)}(\widetilde{M},\widetilde{E})). \tag{4.2.8}$$

In case that $(E,h) \downarrow (M,g)$ is Fredholm, we define for $k \in \mathbb{N}$ the **k-th analytic Novikov Shubin invariant** $\alpha^{An}_k(M,\rho) \in [0,\infty] \cup \{\infty^+\}$ as

$$\alpha^{An}_k(M,\rho) \coloneqq \alpha_k(\Omega^\bullet_{(2)}(\widetilde{M},\widetilde{E})). \tag{4.2.9}$$

If $(E,h) \downarrow (M,g)$ is also ζ-regular and of determinant class, we define the **analytic L^2-torsion** $T^{An}_{(2)}(M,\rho) \in \mathbb{R}$ as

$$\log(T^{An}_{(2)}(M,\rho)) \coloneqq \log(T_\zeta(\Omega^\bullet_{(2)}(\widetilde{M},\widetilde{E})) = \sum_{p=1}^{\infty} \frac{1}{2}\, p\, (-1)^{p+1} \log(\det{}^\zeta_\Gamma(\Delta_p)). \tag{4.2.10}$$

Remark 4.2.4. Suppose that M is compact. Then, if g' is another metric on M and h' another metric on E, the identity map $\mathbb{1} : \Omega^\bullet_{(2)}(\widetilde{M},\widetilde{E},\widetilde{g},\widetilde{h}) \to \Omega^\bullet_{(2)}(\widetilde{M},\widetilde{E},\widetilde{g'},\widetilde{h'})$ is an isomorphism of Hilbert $\mathcal{N}(\Gamma)$-modules. Together with Proposition 4.1.35, as well as Corollaries 4.1.32 and 4.1.38, it follows that the Fredholm and determinant class properties of $E \downarrow M$, as well as the L^2-Betti numbers $b^{An}_{(2),k}(M,\rho)$ and Novikov-Shubin invariants $\alpha^{An}_k(M,\rho)$, don't actually depend on the specific choice of metrics, and can thus be seen as invariants and properties of the tuple (M,ρ), as indicated in the notation. On the other hand, the term $T^{An}_{(2)}(M,\rho)$ *does* depend on the pair of metrics (g,h), which is why we will sometimes write $T^{An}_{(2)}(M,\rho,g,h)$, respectively $T^{An}_{(2)}(M,E,g,h)$ in order to highlight this dependency.

Remark 4.2.5. In the notation, if the group $\Gamma \coloneqq \pi_1(M)$ is clear from the context, we will also often replace M by its universal cover $\widetilde{M}$ and write $b^{An}_{(2),k}(\widetilde{M},\rho), \alpha^{An}_k(\widetilde{M},\rho)$ and $T^{An}_{(2)}(\widetilde{M},\rho)$ instead.

4.2.1 Estimates for the spectral density function

Throughout this subsection, we fix a lattice $\Gamma \subseteq \mathrm{Isom}^+(M)$ compatible with $E \downarrow M$ and also assume that there exists *some* uniform lattice $\Lambda \subseteq \mathrm{Isom}^+(M)$ compatible with $E \downarrow M$. Proposition 4.2.2 allows us to consider the spectral density functions of the Hilbert $\mathcal{N}(\Gamma)$-cochain complexes $\Omega^*_{(2)}(M,E), L_p[E], D_p[E]$ and $D_{p,abs}[E]$. This section is devoted to comparing their respective behavior near 0. Our first result in that vein is the following:

Proposition 4.2.6. *Let $E \downarrow M$ be as before and $0 \leq p \leq m$, let $\Delta^\perp_p[E] : \Omega^p_{(2)}(M,E) \to \Omega^p_{(2)}(M,E)$ the orthogonal Laplacian. Suppose that $\Omega^p_{(2)}(M,E)$ has trivial L^2-cohomology. Then we have*

$$F(\Delta_p[E],\sqrt{\lambda}) = F(\Delta^\perp_p[E],\sqrt{\lambda}) = F_p(L_p[E],\lambda) + F_{p-1}(L_{p-1}[E],\lambda) \tag{4.2.11}$$

for all $\lambda \geq 0$.

Proof. Since there is no L^2-cohomology, we have both $\Delta_p[E] = \Delta^\perp_p[E]$, as well as $F_p(L_p[E],\lambda) = F((d^p_{abs})^\perp,\lambda)$ for each p. Using Proposition 3.4.6 and Lemma 4.1.5, we now compute:

$$\begin{aligned}
&F(\Delta^\perp_p[E],\sqrt{\lambda}) = F\left(((d^p_{abs})^* d^p_{abs})^\perp \oplus (d^{p-1}_{abs}(d^{p-1}_{abs})^*)^\perp, \sqrt{\lambda}\right) \\
&= F(((d^p_{abs})^* d^p_{abs})^\perp,\sqrt{\lambda}) + F_p((d^{p-1}_{abs}(d^{p-1}_{abs})^*)^\perp,\sqrt{\lambda}) = F((d^p_{abs})^\perp,\lambda) + F((d^{p-1}_{abs})^\perp,\lambda) \\
&= F_p(L_p[E],\lambda) + F_{p-1}(L_{p-1}[E],\lambda).
\end{aligned}$$

□

For the next result, we assume that the underlying vector bundle of $E \downarrow M$ is *trivial*, i.e, of the form $M \times \mathbb{C}^n$ for some $n \in \mathbb{N}$ (however, we do *not* assume that the Hermitian form is constant on the fibers) and that $\Gamma \subseteq \mathrm{Isom}^+(M)$ is *uniform*. Triviality of $E \downarrow M$ allows us to identify smooth sections of E with smooth maps from M to $\mathbb{C}^n$, which further leads to an identification of $C^\infty(M, \mathbb{C})$-modules

$$\Omega^p(M, E) \cong \Omega^p(M)^n, \tag{4.2.12}$$

$$\Omega^p(\partial M, E) \cong \Omega^p(\partial M)^n. \tag{4.2.13}$$

Assume that $\partial M \neq \emptyset$. For appropriate $w > 0$, let $\partial M_w \cong [0, w) \times \partial M$ be the geodesic collar around ∂M of width w. Under this identification, we therefore find for any $\omega \in \Omega^p_c(M, E)$ appropriate smooth 1-parameter families of forms $\omega_1(t) \subset \Omega^p_c(\partial M)^n$ and $\omega_2(t) \subset \Omega^{p-1}_c(\partial M)^n$, such that

$$\omega(t, x) = \omega_1(t)(x) + dt \wedge \omega_2(t)(x), \qquad \forall (t, x) \in [0, w) \times \partial M \cong \partial M_w. \tag{4.2.14}$$

Denote by $\hat{\Delta}_p[E]$ the p-th Laplacian on the flat Hermitian restriction bundle $E \downarrow \partial M$. Let $\phi : [0, w] \to \mathbb{R}^+$ be a smooth map identically 1 near 0 and identically 0 for all $t > w/2$. With this data in mind, define

$$K^p : \Omega^p_c(M, E) \to \Omega^{p-1}(M, E), \tag{4.2.15}$$

$$K^p \omega := \begin{cases} \phi(u) \cdot \int_0^u e^{-te^{1+\hat{\Delta}_p[E]}} \omega_2(t)(\ .\ ) dt & \text{on } \partial M_w, \\ 0 & \text{elsewhere.} \end{cases} \tag{4.2.16}$$

An immediate, but important consequence is that $K^p \omega$ depends only on the restriction $\omega|_{\partial M_w}$ and that the support of $K^p \omega$ lies in ∂M_w. Just as in the case with for the trivial bundle, one can now proceed line by line as in [56, Lemma 5.5, Proposition 5.6] to show the following:

Proposition 4.2.7. *Let $p \in \mathbb{N}$. Then, for $r = 0, 1, 2$, the map K^p extends to a bounded operator.*

$$K^p_r : \mathcal{W}^p_r(M, E) \to \mathcal{W}^{p-1}_{r+1}(M, E). \tag{4.2.17}$$

The norm $||K^p_r||$ depends only on r and flat isometry class of the restriction bundle $E|_{\partial M_w} \downarrow \partial M_w$. Setting $K^p_{-1} := 0$ for all $p \in \mathbb{N}$, we furthermore obtain

1. *For $* = 0, 1, 2$ and each $p \in \mathbb{N}$, the map*

$$j^{p+1-*}_p = 1 - dK^{p+1-*}_* + K^{p+2-*}_{*-1} d : \mathcal{W}^{p+1-*}_*(M, E) \to \mathcal{W}^{p+1-*}_*(M, E) \tag{4.2.18}$$

   has image in $\mathcal{W}^{p+1-*}_{*,abs}(M, E)$ and extends to a morphism of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes $j_p : D_p[E] \to D_{p,abs}[E]$.

2. *The reduced complexes*

$$\overline{D}_p[E] := \ldots 0 \to \mathcal{W}^p_1(E)/\overline{\mathrm{im}(d)} \to \mathcal{W}^{p+1}_0(E) \to 0 \ldots,$$
$$\overline{D}_{p,abs}[E] := \ldots 0 \to \mathcal{W}^p_{1,abs}(E)/\overline{\mathrm{im}(d)} \to \mathcal{W}^{p+1}_0(E) \to 0 \ldots$$

   are chain homotopy equivalent. More precisely, the map j_p descends to a map $\overline{j}_p : \overline{D}_p[E] \to \overline{D}_{p,abs}[E]$, that is the chain homotopy inverse to the induced inclusion $\overline{i}_p : \overline{D}_{p,abs}[E] \to \overline{D}_p[E]$. The respective null-homotopies are induced from $i_* \circ K^*_*$ and $K^*_* \circ i_*$.

From this, we now obtain the following important, intermediate result:

Proposition 4.2.8. *Let $E \downarrow M$ be a bundle of bounded geometry over M and Γ a lattice, compatible with $E \downarrow M$, so that, if $\partial M \neq \emptyset$, we additionally assume that Γ is uniform and that $E \downarrow M$ is trivial. Then we find constants $C_1, C_2 > 0$, depending only on the flat isometry class of the restriction bundle $E|_{\partial M_w} \downarrow \partial M_w$, (with $\partial M_w := \emptyset$ if $\partial M = \emptyset$) such that all of the following hold:*

1. *We have*
$$F_p(D_{p,abs}[E], C_1^{-1}\lambda) \leq F_p(D_p[E], \lambda) \leq F_p(D_{p,abs}[E], C_1\lambda)$$
for all $\lambda \leq C_2$.

2. *We have*
$$F_p(L_p[E], \lambda) \leq F_p(D_{p,abs}[E], \lambda) \leq F_p(L_p[E], \sqrt{2}\lambda)$$
for all $\lambda \leq \frac{1}{\sqrt{2}}$.

3. *We have*
$$F_p(L_p[E], C_1^{-1}\lambda) \leq F_p(D_p[E], \lambda) \leq F_p(L_p[E], C_1\sqrt{2}\lambda).$$
for all $\lambda \leq \min\{C_2, \frac{1}{C_2\sqrt{2}}\}$.

Proof. (1): Follows from Proposition 4.2.7 and Theorem 4.1.37.
(2): Throughout the proof, we will use the equalities

$$\ker(d^p|_{\mathcal{W}^{1,p}_{abs}(E)})^{\perp_1} = \mathcal{D}(d^p) \cap \ker(d^p)^{\perp} = \mathcal{W}^{1,p}_{abs}(E) \cap \overline{\delta^p \Omega^{p+1}_{abs}(M, E)}, \tag{4.2.19}$$

as established in the proof of Proposition 3.4.6, as well as the identity

$$F(f, \lambda) = \sup\{\mathrm{tr}_\Gamma(p_L) : L \subseteq \mathcal{H} \text{ closed, } \Gamma\text{-invariant subspace } : ||f(x)|| \leq \lambda||x|| \; \forall x \in L\}, \tag{4.2.20}$$

as established in Lemma 4.1.5. Recall that for a subspace $A \subseteq \mathcal{W}^{1,*}(E)$, $A^{\perp_1} \subseteq \mathcal{W}^{1,*}(E)$ denotes the orthogonal complement of A with respect to the Sobolev 1-inner product.
We will first show that

$$F_p(L_p[E], \lambda) \leq F_p(D_{p,abs}[E], \lambda).$$

For this, we let $L \subseteq \mathcal{D}(d^p) \cap \ker(d^p)^{\perp}$ be a 0-closed, Γ-invariant subspace, such that each $v \in L$ satisfies $||dv||_0 \leq \lambda||v||_0$. By the left-hand equality of 4.2.19, and the fact that the 1-topology is stronger than the 0-topology, it follows that $L \subseteq W^{1,p}_{abs}(E)$ is a 1-closed, Γ-invariant subspace. Since $||v||_0 \leq ||v||_1$, we also have $||dv||_0 \leq \lambda||v||_1$. All in all, this implies that $F_p(L_p[E], \lambda) \leq F_p(D_{p,abs}[E], \lambda)$.
In order to show that

$$F_p(D_{p,abs}[E], \lambda) \leq F_p(L_p[E], \sqrt{2}\lambda),$$

we let $L \subseteq \ker(d^p|_{\mathcal{W}^{1,p}_{abs}(E)})^{\perp_1}$ be a 1-closed, Γ-invariant subspace, satisfying $||dv||_0 \leq \lambda||v||_1$ for any $v \in L$. Now observe that by the equality of the first and the last term in Equation 4.2.19, we also have $||v||_1^2 = ||v||_0^2 + ||dv||_0^2$ for any $v \in L$, which implies that

$$||v||_0 \leq ||v||_1 \leq \sqrt{1 - \lambda^2}||v||_0, \tag{4.2.21}$$

for any $v \in L$. Therefore, L is also 0-closed inside $\mathcal{D}(d^p) \cap \ker(d^p)^\perp$. Moreover,

$$\begin{aligned} ||dv||_0^2 &\leq \lambda^2 ||v||_1^2 \leq \lambda^2 ||v||_0^2 + \frac{1}{2}||dv||_0 \\ \implies ||dv||_0 &\leq \sqrt{2}\lambda ||v||_0, \end{aligned}$$

from which the inequality $F_p(D_{p,abs}[E], \lambda) \leq F_p(L_p[E], \sqrt{2}\lambda)$, and therefore the desired claim, follows.
(3): This follows immediately from (1) and (2). □

4.2.2 Comparison with the combinatorial complex

For the purpose of the next important result, we introduce the notion of a Γ-CW-cochain, further details will be developed in Chapter 5: Let X be a CW-complex, on which a countable group Γ acts freely, cellularly and co-compactly with a finite number of cell orbits. Such X is called a *free, finite* Γ-CW complex. The associated cellular cochain complex $C^*(X, \mathbb{C})$ with complex coefficients then has the structure of a complex of free and finitely generated $\mathbb{C}[\Gamma]$-bimodules, with $\mathbb{C}[\Gamma]$-bases given by explicit representatives of orbits $\Gamma.e$ for each cocell e of X.
Now assume that there exists a complex, finite dimensional representation $\rho : \Gamma \to \mathrm{GL}(V)$. Such a representation naturally endows V with the structure of a *left* $\mathbb{C}[\Gamma]$-module. We therefore can form the cochain complex

$$C^*(X, \rho) := C^*(X, \mathbb{C}) \otimes_{\mathbb{C}} V \tag{4.2.22}$$

with (left) diagonal Γ-action given by $\gamma.(e \otimes v) := \gamma.e \otimes \rho(\gamma)v$ on elementary tensors. It is still a complex of free, finitely generated left $\mathbb{C}[\Gamma]$-modules. Let $E \subseteq C^*(X, \mathbb{C})$ the set of all cocells, each equipped with some orientation. Additionally fixing a $\mathbb{C}$-basis $B \subseteq V$, the Γ-orbit $\Gamma.(E \otimes B)$ of the subset $E \otimes B := \{e \otimes b : e \in E, b \in B\} \subseteq C^*(X, \rho)$ is a $\mathbb{C}$-basis for $C^*(X, \rho)$, which is infinite whenever Γ is infinite. Equipping $C^*(X, \rho)$ with the unique inner product, with respect to which the set $\Gamma.(E \otimes B)$ is orthogonal, and forming the corresponding L^2-completion, we obtain a cochain complex of Hilbert spaces $C^*_{(2)}(X, \rho)$. As a matter of fact, it is a *finite* Hilbert $\mathcal{N}(\Gamma)$-cochain complex.
Let (M, g) is a Riemannian manifold and let $\Gamma \subset \mathrm{Isom}(M, g)$ be a uniform lattice. Choose on the quotient manifold M/Γ a (finite) CW structure $\bar{X}$, so that, additionally, the restriction to the boundary $\partial\bar{X} = \bar{X} \cap \partial(M/\Gamma)$ is a CW-structure on $\partial(M/\Gamma)$. The *lift* X of $\bar{X}$ onto M is then a free, finite Γ-CW structure on M, so that $X \cap \partial M$ is a free, finite Γ-CW structure on ∂M. Any CW-structure X on M obtained this way is called an *admissible* Γ-*CW structure* on M. The next important theorem is shown in Chapter 6:

Theorem 6.3.5 *Let (M, g) be a simply-connected Riemannian manifold and let $\Gamma \subset \mathrm{Isom}^+(M, g)$ be a uniform lattice. Further, let $\rho : \Gamma \to \mathrm{GL}(V)$ be some finite-dimensional, complex representation and let $E := M \times V \downarrow M$ be the associated flat, Γ-equivariant bundle over M. Choose some (finite) admissible Γ-CW structure X on M and let $\Omega^*_{(2)}(M, E)$ be the L^2-cochain complex with absolute boundary conditions (constructed with respect to some choice of Γ-equivariant Hermitian form h). Then there is a L^2-chain homotopy equivalence of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes*

$$\Omega^*_{(2)}(M, E) \simeq C^*_{(2)}(X, \rho). \tag{4.2.23}$$

Remark 4.2.9. In Section 5.1, we will define the notion of combinatorial determinant class (short: c-determinant class). Namely, a pair (X, ρ), where X is a finite Γ-CW-complex and $\rho : \Gamma \to \mathrm{GL}(V)$

is a finite-dimensional representation is said to be *of c-determinant class* if the corresponding cellular L^2-cochain complex of Hilbert $\mathcal{N}(\Gamma)$-modules $C^*_{(2)}(X,\rho)$ is of determinant class. In case that X is the Γ-CW structure of a smooth Riemannian manifold (M,g) and $\Gamma \subseteq \mathrm{Isom}^+(M,g)$ is a uniform lattice, then the above theorem, along with Proposition 4.2.2 and Corollary 4.1.38, shows that the pair (M,ρ) is of analytic determinant class (short: a-determinant class) if and only if (X,ρ) is of c-determinant class. Throughout this thesis, both notions will be used interchangeably.

4.2.3 Applications to the representation bundle $E^\rho \downarrow \mathbb{H}^n$

Recall the following notions and facts from Section 2.3: For $\mathbb{H}^n$ the hyperbolic n-space with n odd, set $G := \mathrm{Isom}^+(\mathbb{H}^n)$ and let $E^\rho \downarrow \mathbb{H}^n$ be the flat, canonical, Hermitian bundle associated to an irreducible, complex, finite-dimensional representation $\rho : G_{\mathbb{C}} \to \mathrm{GL}(V)$. For a fixed non-uniform lattice $\Gamma \subset G$ and $R \geq 0$, let M_R, C_R and T_R be the complete, Γ-invariant submanifolds associated to it. Equip E^ρ with the canocial bundle metric h^ρ. We've seen that Γ is compatible with $E^\rho \downarrow \mathbb{H}^n$, in the sense that the action of Γ on M extends to an action on $E^\rho \downarrow \mathbb{H}^n$ by bundle isometries. Hence, Γ is also compatible with the restriction of E^ρ over each M_R, C_R and T_R, which we have denoted respectively by E^ρ_{R-}, E^ρ_{R+} and E^ρ_R. Because of Proposition 4.2.2, the associated four L^2-de Rham cochain complexes are Hilbert $\mathcal{N}(\Gamma)$-module cochain complexes. Moreover, since Γ acts cocompactly on M_R and T_R and is compatible with the overlying bundles, the bundles $E^\rho_{R-} \downarrow M_R$ and $E^\rho_R \downarrow T_R$ are Fredholm by Proposition 4.2.2. The same result holds true for the cochain complex induced by $E^\rho \downarrow \mathbb{H}^n$, using that E^ρ is G-equivariant. The ultimate goal of this section is to find a uniform polynomial upper bound for the spectral density functions of the complex $\Omega^*(M_R, E_R)$, **independent of** R, which will then easily imply the desired large-time convergence.

Proposition 4.2.10. *There exists constants $C_1, C_2 > 0$ independent of $R \geq 1$, such that for all $p = 0, \ldots, m$ and all $\lambda \leq \min\{C_2, \frac{1}{C_2\sqrt{2}}\}$, we have*

$$F_p(L_p[E^\rho], C_1^{-1}\lambda) \leq F_p(D_p[E^\rho], \lambda) \leq F_p(L_p[E^\rho], C_1\sqrt{2}\lambda),$$
$$F_p(L_p[E^\rho_{R-}], C_1^{-1}\lambda) \leq F_p(D_p[E^\rho_{R-}], \lambda) \leq F_p(L_p[E^\rho_{R-}], C_1\sqrt{2}\lambda),$$
$$F_p(L_p[E^\rho_R], C_1^{-1}\lambda) \leq F_p(D_p[E^\rho_R], \lambda) \leq F_p(L_p[E^\rho_R], C_1\sqrt{2}\lambda).$$

Proof. Since $E^\rho \downarrow \mathbb{H}^n$ is a bundle of bounded geometry and Γ acts cocompactly on both M_R and T_R, we get the above inequalities from Proposition 4.2.8, choosing $w = 1/3$ (the width of the geodesic collar around the boundary), with constants $C_1(R), C_2(R)$, depending *a priori* on $R > 1$, but only on the flat isometry classes of the restrictions $E^\rho|_{\partial(M_R)_{1/3}} \downarrow \partial(M_R)_{1/3}$ and $E^\rho|_{\partial(T_R)_{1/3}} \downarrow \partial(T_R)_{1/3}$. Let $\partial(M_R)^0_{1/3}$ be a connected component of $\partial(M_R)_{1/3}$ and let $\partial(T_R)^0_{1/3}$ be the intersection of $\partial(T_R)_{1/3}$ with a connected component of T_R. Then, from the explicit end structure laid out in Section 2.3, there are isometric diffeomorphisms

$$\partial(M_R)^0_{1/3} \cong [R - 1/3, R] \times \mathbb{R}^{n-1},$$
$$\partial(T_R)^0_{1/3} \cong ([R, R + 1/3] \dot\cup [R + 2/3, R + 1]) \times \mathbb{R}^{n-1},$$

each sending the hyperbolic metric to the warped product metric $dt^2 + e^{-2t}dx^2$. Using the very same local isometries and following the same arguments as in Lemma 2.3.7, we obtain two bundle isometries

$$E^\rho|_{\partial(M_R)_{1/3}} \downarrow \partial(M_R)_{1/3} \cong E^\rho|_{\partial(M_1)_{1/3}} \downarrow \partial(M_1)_{1/3},$$
$$E^\rho|_{\partial(T_R)_{1/3}} \downarrow \partial(T_R)_{1/3} \cong E^\rho|_{\partial(T_1)_{1/3}} \downarrow \partial(T_1)_{1/3}.$$

Consequently, we get $C_1(R) = C_1(1) =: C_1$ and $C_2(R) =: C_2(1) =: C_2$ for any $R > 1$ and the result follows. ☐

Proposition 4.2.11. [30, Theorem 1.1, Proposition 1.2] *The Hilbert $\mathcal{N}(\Gamma)$-cochain complex $\Omega^*_{(2)}(\mathbb{H}^n, E^\rho)$ has trivial cohomology and positive Novikov-Shubin invariants.*

We want to show that the same holds true for the Hilbert $\mathcal{N}(\Gamma)$-cochain complex $\Omega^*_{(2)}(T_1, E_1^\rho)$. For that purpose, we need some additional preparation:
Let Γ be a countable group, equipped with some finite-dimensional representation $\rho : \Gamma \to \mathrm{GL}(V)$. Further, let $\Gamma_0 < \Gamma$ be a subgroup and denote by ρ_0 the restriction of ρ to Γ_0. Choose X be a finite Γ_0-CW complex and let $C^*(X, \rho_0)$ be the associated free, finite Hilbert $\mathcal{N}(\Gamma_0)$ cochain complex, whose construction was laid out in the previous section. We now explain how $C^*(X, \rho_0)$ gives rise to a free, finite Hilbert $\mathcal{N}(\Gamma)$-cochain complex via the *principle of induction*: Since $\Gamma_0 < \Gamma$, we can naturally regard the group ring $\mathbb{C}[\Gamma]$ as a *right* $\mathbb{C}[\Gamma_0]$-module. Hence, we can define the following cochain complex of free, finitely generated left $\mathbb{C}[\Gamma]$-modules

$$C^*(X, \rho_0, \Gamma) \coloneqq \mathbb{C}[\Gamma] \otimes_{\mathbb{C}[\Gamma_0]} C^*(X, \rho_0). \tag{4.2.24}$$

Similarly as before, the canonical inner product on the group ring $\mathbb{C}[\Gamma]$ turns $C^*(X, \rho_0, \Gamma)$ into a complex of inner product spaces, whose L^2-completion we denote by $C^*_{(2)}(X, \rho_0, \Gamma)$. It is a free, finite Hilbert $\mathcal{N}(\Gamma)$-cochain complex. It follows from [54, Lemma 1.24] that, for each $p \in \mathbb{N}$, we have

$$b^p_{(2)}(C^*_{(2)}(X, \rho_0, \Gamma), \Gamma) = b^p_{(2)}(C^*_{(2)}(X, \rho), \Gamma_0), \tag{4.2.25}$$
$$\alpha_p(C^*_{(2)}(X, \rho_0, \Gamma), \Gamma) = \alpha_p(C^*_{(2)}(X, \rho_0), \Gamma_0). \tag{4.2.26}$$

Now consider the principal Γ-bundle $Y \coloneqq \Gamma \times_{\Gamma_0} X$. The CW-structure on X then extends to a free, finite Γ-CW structure on Y. Just as above, form the twisted L^2-cochain complex $C^*_{(2)}(Y, \rho)$. It is proven in [52, Lemma 1.1, Theorem 6.7,(5)] that there is an isomorphism of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

$$C^*_{(2)}(Y, \rho) \cong C^*_{(2)}(X, \rho_0, \Gamma). \tag{4.2.27}$$

Using Corollary 4.1.38, along with equations 4.2.25 and 4.2.26, we thus obtain for each $p \in \mathbb{N}$ the Equalities

$$b^p_{(2)}(C^*_{(2)}(Y, \rho), \Gamma) = b^p_{(2)}(C^*_{(2)}(X, \rho_0), \Gamma_0), \tag{4.2.28}$$
$$\alpha_p(C^*_{(2)}(Y, \rho), \Gamma) = \alpha_p(C^*_{(2)}(X, \rho_0), \Gamma_0). \tag{4.2.29}$$

Proposition 4.2.12. *The Hilbert $\mathcal{N}(\Gamma)$-cochain complex $\Omega^*_{(2)}(T_1, E_1^\rho)$ has trivial cohomology and positive Novikov-Shubin invariants.*

Proof. For each $1 \leq j \leq k$, let T_1^j be the complete submanifolds of T_1 with $\Gamma_0^j \coloneqq \{\gamma \in \Gamma : \gamma.T_1^j = T_1^j\}$ the stabilizer of T_1^j, so that we have a decomposition

$$T_1 \cong \coprod_{j=1}^k \Gamma \times_{\Gamma_0^j} T_1^j,$$

as detailed in Section 2.3. In the same section, we have shown that $\Gamma_0^j \subset \mathrm{Isom}^+(T_1^j)$ is a uniform lattice, isomorphic to $\mathbb{Z}^{d-1}$. Therefore, we can choose a finite Γ_0^j-CW-structure on T_1^j, giving rise to the free, finite Hilbert $\mathcal{N}(\Gamma_0^j)$-complex $C^*_{(2)}(T_1^j, \rho_0^j)$, where ρ_0^j denotes the restriction of ρ to Γ_0^j. Moreover, since we have an identification $T_1 \cong \coprod_{j=1}^k \Gamma \times_{\Gamma_0^j} T_1^j$, the free, finite Γ_0^j-CW structure on each T_1^j naturally extends to a free, finite Γ-CW structure on T_1. In particular, we can form the associated free, finite Hilbert $\mathcal{N}(\Gamma)$-cochain complex $C^*_{(2)}(T_1, \rho)$, so that we have an orthogonal direct sum decomposition

$$C^*_{(2)}(T_1, \rho) = \bigoplus_{j=1}^{k} C^*_{(2)}(\Gamma \times_{\Gamma_0^j} T_1^j, \rho). \tag{4.2.30}$$

By Theorem 6.3.5, we have a chain homotopy equivalence of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

$$\Omega^*_{(2)}(T_1, E_1^\rho) \simeq C^*_{(2)}(T_1, \rho). \tag{4.2.31}$$

Applying Corollary 4.1.38, along with the Equalities 4.2.30, 4.2.28 and 4.2.29, $\Omega^*_{(2)}(T_1, E_1^\rho)$ is a Fredholm complex and we have for each $0 \le p \le m$

$$b^p_{(2)}(\Omega^*_{(2)}(T_1, E_1^\rho), \Gamma) = \sum_{j=1}^{k} b^p_{(2)}(C^*_{(2)}(T_1^j, \rho_0^j), \Gamma_0^j), \tag{4.2.32}$$

$$\alpha_p(\Omega^*_{(2)}(T_1, E_1^\rho), \Gamma) \ge k \cdot \min_{j=1,\dots,k} \alpha_p(C^*_{(2)}(T_1^j, \rho_0^j), \Gamma_0^j). \tag{4.2.33}$$

For the remainder of the proof, the isomorphism $\Gamma_0^j \cong \mathbb{Z}^{d-1}$ will be of central importance. Firstly, [52, Theorem 7.7] implies that

$$b^p_{(2)}(C^*_{(2)}(T_1^j, \rho_0^j), \Gamma_0^j) = \dim_{\mathbb{C}}(V) \cdot b^p_{(2)}(C_{(2)}(T_1^j, \mathbb{1}), \Gamma_0^j), \tag{4.2.34}$$

where $\mathbb{1} : \Gamma_0^j \to \mathbb{C}^\times$ denotes the trivial representation. From the multiplicativity of ordinary L^2-Betti numbers under coverings, see [54, Example 1.37], it follows that any G-CW complex X, whose quotient space X/G admits non-trivial self-coverings, satisfies $b^p_{(2)}(C_{(2)}(X, \mathbb{1}), G) = 0$ for all $p \ge 0$. Since the quotient space $T_1^j/\Gamma_0^j \cong [1,2] \times (S^1)^{m-1}$ clearly admits non-trivial self-coverings, we obtain that $b^p_{(2)}(\Omega^*_{(2)}(T_1, E_1^\rho), \Gamma) = 0$ as well, so the complex $\Omega^*_{(2)}(T_1, E_1^\rho)$ has trivial cohomology.
Secondly, observe that the boundary operators of $C^*_{(2)}(T_1^j, \rho_0^j)$ are matrices over $\mathbb{C}[\Gamma_0] \cong \mathbb{C}[\mathbb{Z}^d]$ (acting by right-multiplication). It is shown in [53, Theorem 1.2], that any such matrix has positive Novikov-Shubin invariant. Therefore

$$\alpha_p(C^*_{(2)}(T_1^j, \rho_0^j), \Gamma_0^j) > 0 \tag{4.2.35}$$

for each $1 \le j \le k$ and each $0 \le p \le m$, finishing the proof. $\square$

In view of Remark 4.2.9, note that in the course of the previous proof, we have in fact also shown the following:

Corollary 4.2.13. *For each $R > 0$, the pair $(\partial M_R, \rho)$ is L^2-acyclic and of determinant class.*

Proposition 4.2.14. *There exists constants $\epsilon, \alpha > 0$, such that for all $R \ge 1$ and all $0 \le p \le m$ the following hold true*

1. *For all $R \ge 1$ we have*

$$F(\Delta_p[E_R^\rho], \lambda) \le F(\Delta_p[E_1^\rho], \lambda). \tag{4.2.36}$$

2. *For all $\lambda < \epsilon^{-1}$, we have*

$$F_p(D_p[E^\rho], \lambda) < \epsilon \cdot \lambda^\alpha,$$
$$F_p(D_p[E^\rho_R], \lambda) < \epsilon \cdot \lambda^\alpha.$$

In particular, both $D_p[E^\rho]$ and $D_p[E^\rho_R]$ are Fredholm at p and have vanishing p-th cohomology.

Proof. 1. There is a flat bundle isometry $F_R : E^\rho_R \downarrow T_R \to E^\rho_1 \downarrow T_1$, as defined in Lemma 2.3.7. Consequently, by Proposition 2.2.1, we have for any $x \in T_R$ and any $\lambda \geq 0$, that

$$\operatorname{tr}(\chi_{[0,\lambda^2]}(\Delta_p[E_0])(F_R(x), F_R(x))) = \operatorname{tr}(\chi_{[0,\lambda^2]}(\Delta_p[E_R])(x, x)). \tag{4.2.37}$$

We may choose fundamental domains $\mathcal{D}_R \subseteq T_R$ and $\mathcal{D}_1 \subseteq T_1$ for the respective Γ-actions, satisfying $F_R(\mathcal{D}_R) \subseteq \mathcal{D}_1$. Therefore, we have

$$\begin{aligned} F(\Delta_p[E_R], \lambda) &= \operatorname{tr}_\Gamma(\chi_{[0,\lambda^2]}(\Delta_p[E_R])) = \int_{\mathcal{D}_R} \operatorname{tr}(\chi_{[0,\lambda^2]}(\Delta_p[E_R])(x, x))dx \\ &= \int_{\mathcal{F}_R(\mathcal{D}_R)} \operatorname{tr}(\chi_{[0,\lambda^2]}(\Delta_p[E_1])(F_R(x), F_R(x)))dx \\ &\leq \int_{\mathcal{D}_1} \operatorname{tr}(\chi_{[0,\lambda^2]}(\Delta_p[E_1])(F_R(x), F_R(x)))dx = F(\Delta_p[E_1], \lambda). \end{aligned}$$

2. Define $\beta \coloneqq \min\{\alpha(\Delta_p[X]) : 0 \leq p \leq m, X \in \{E^\rho, E^\rho_1\}\}$. Propositions 4.2.11 and 4.2.12 imply that we have $\beta > 0$, as well as both $\Delta_p[E^\rho] = \Delta^\perp_p[E^\rho]$ and $\Delta_p[E^\rho_1] = \Delta^\perp_p[E^\rho_1]$. For E^ρ, we can apply Propositions 4.2.10 and 4.2.6 to find a constant $c \geq 1$, such that for all $0 \leq p \leq m$, and all $\lambda < c^{-1}$ we have

$$F_p(D_p[E^\rho], \lambda) \leq F_p(L_p[E^\rho], c\lambda) \leq F(\Delta_p[E^\rho], \sqrt{c\lambda}) < c^\alpha \lambda^\alpha.$$

For E^ρ_R, we can use the same argument, along with assertion (2), similarly yielding

$$\begin{aligned} &F_p(D_p[E^\rho_R], \lambda) \leq F_p(L_p[E^\rho_R], c\lambda) \leq F_p(\Delta_p[E^\rho_R], \sqrt{c\lambda}) \\ &\leq F_p(\Delta_p[E^\rho_1], \sqrt{c\lambda}) \leq c^{\alpha/2}\lambda^{\alpha/2}. \end{aligned}$$

Setting $\epsilon \coloneqq \max\{c^{\alpha/2}, c\}$ and $\beta = \alpha/2 > 0$, we obtain the result.

□

For $R > 1$, denote by

$$i_{M_R} : M_R \to \mathbb{H}^n, \tag{4.2.38}$$
$$i_{C_R} : C_R \to \mathbb{H}^n, \tag{4.2.39}$$
$$i_{(T_R,-)} : T_R \to M_{R+1}, \tag{4.2.40}$$
$$i_{(T_R,+)} : T_R \to C_R. \tag{4.2.41}$$

the respective smooth inclusion maps. Each of these induces a Γ-invariant map between the corresponding twisted de Rham complexes, bounded with norm 1. Moreover, the following important result holds true. The proof for the scalar-values case, presented in the reference, carries over to our situation of bundle-valued forms without further modification:

Lemma 4.2.15. [56, Lemma 5.14] *For any $R > 1$ and any $0 \leq p \leq m$, the sequence of morphisms of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes*

$$0 \to D_p[E^\rho] \xrightarrow{j_R} D_p[E^\rho_{R-}] \oplus D_p[E^\rho_{(R-1)+}] \xrightarrow{q_R} D_p[E^\rho_{R-1}] \to 0 \tag{4.2.42}$$

is exact. Here, for smooth forms, we have

$$j_R\omega := i^*_{M_R}\omega \oplus i^*_{C_{R-1}}\omega, \tag{4.2.43}$$

$$q_R(\omega_1 \oplus \omega_2) := i^*_{T_{(R-1,-)}}\omega_1 - i^*_{T_{(R-1,+)}}\omega_2. \tag{4.2.44}$$

Lemma 4.2.16. *There exists a constant $C > 0$, such that for all $R > 1$ and all $0 \leq p \leq m$, we have*

$$F_p(D_p[E^\rho_{R-}], \lambda) \leq F_p(D_p[E^\rho], C \cdot \lambda^{1/4}) + F_p(D_p[E^\rho_{R-1}], C \cdot \lambda^{1/2}) \quad \textit{for } 0 \leq \lambda \leq C^{-1}. \tag{4.2.45}$$

Proof. Consider the exact sequence

$$0 \to D_p[E^\rho] \xrightarrow{j_R} D_p[E^\rho_{R-}] \oplus D_p[E^\rho_{(R-1)+}] \xrightarrow{q_R} D_p[E^\rho_{R-1}] \to 0 \tag{4.2.46}$$

from Lemma 4.2.15. Because of Proposition 4.2.14, the outer two complexes are Fredholm at p and, moreover, have vanishing p-cohomology. In particular, we can apply Proposition 4.1.35 to obtain that

$$F_p(D_p[E^\rho_{R-}], \lambda) \leq F_p(D_p[E^\rho], c_1(R) \cdot \lambda) + F_p(D_p[E^\rho_{R-1}], c_1(R) \cdot \lambda), \tag{4.2.47}$$

for all $\lambda < c_2(R)$, where $c_1(R)$ and $c_2(R)$ are constants given by rational expressions of the norms of q_R, j_R, the differential on $D_p[E^\rho_{R-}] \oplus D_p[E^\rho_{(R-1)+}]$, and their respective inverses. Using the flat bundle isometries given in Lemma 2.3.7, one can now proceed analogously as in [55, Lemma 6.6] to show that these norms are bounded from above by universal constants independent of $R > 1$, thus proving the lemma. □

Proposition 4.2.17. *There exists constants $C, \beta > 0$, such that for all $0 \leq p \leq m$, the following hold:*

1. *The Hilbert $\mathcal{N}(\Gamma)$-cochain complex $\Omega^*_{(2)}(M_R, E^\rho_{R-})$ has vanishing p-th homology. Equivalently, we have*

$$\Delta_p[E^\rho_{R-}] = \Delta^\perp_p[E^\rho_{R-}]. \tag{4.2.48}$$

2. *For all $R > 1$, we have a uniform bound on the spectral density functions as follows*

$$F_p(\Delta_p[E_{R-}], \lambda) \leq C\lambda^\beta \quad \textit{for } 0 \leq \lambda \leq C^{-1}. \tag{4.2.49}$$

Proof. First, observe that the p-th cohomology of $\Omega^*(M_R, E^\rho_{R-})$ is isomorphic to the p-th cohomology of $L_p[E^\rho_{R-}]$. To show that the latter is trivial, we only need to show that

$$F_p(L_p[E^\rho_{R-}], 0) = 0. \tag{4.2.50}$$

Using Lemma 4.2.16, together with both Proposition 4.2.10 and Proposition 4.2.6, we obtain constants $\alpha, c, C_1, C_2 > 0$ independent of $R > 1$, such that

$$\begin{aligned}
&F_p(D_p[E^\rho_{R-}], \lambda) \leq F_p(D_p[E^\rho], c\lambda^{1/4}) + F_p(D_p[E^\rho_{R-1}], c\lambda^{1/2}) \\
&\leq F_p(L_p[E^\rho], cC_1\sqrt{2}\lambda^{1/4}) + F_p(L_p[E^\rho_{R-1}], cC_1\sqrt{2}\lambda^{1/2}) \\
&\leq F_p(\Delta^\perp_p[E^\rho], \sqrt{cC_1\sqrt{2}}\lambda^{1/8}) + F_p(\Delta^\perp_p[E^\rho_{R-1}], \sqrt{cC_1\sqrt{2}}\lambda^{1/4}). \\
&\leq 2(cC_1\sqrt{2})^{\alpha/2}\lambda^{\alpha/8}
\end{aligned}$$

for $\lambda \leq \min\{1, c^{-1}, C_1^{-1}, C_2, \frac{1}{\sqrt{2}C_2}\}$ and all $0 \leq p \leq m$. In particular, since $F(L_p[E^\rho_{R-}], 0) \leq F_p(D_p[E^\rho_{R-}], 0)$ by Proposition 4.2.8, Equation 4.2.50 immediately follows from the above computation.
Applying now again Proposition 4.2.6 and Proposition 4.2.10 together with the above inequality, we compute

$$
\begin{aligned}
&F_p(\Delta_p[E^\rho_{R-}], \lambda) = F_p(\Delta_p^\perp[E^\rho_{R-}], \lambda) \leq F_p(D_p[E^\rho_{R-}], C_1\lambda^2) + F_{p-1}(D_{p-1}[E^\rho_{R-}], C_1\lambda^2) \\
&\leq 4(cC_1^{3/2}\sqrt{2})^{\alpha/2}\lambda^{\alpha/4}
\end{aligned}
$$

for $\lambda \leq \min\{1, c^{-1}, C_1^{-1}, C_2, \frac{1}{\sqrt{2}C_2}, C_1^{-1}C_2, \frac{1}{C_1 C_2\sqrt{2}}\} \coloneqq C'$.
Setting $C \coloneqq \max\{(C')^{-1}, 4(cC_1^{3/2}\sqrt{2})^{\alpha/2}\}$ and $\beta \coloneqq \alpha/4$, we obtain the desired result. □

Neatly summarized, we get the following result:

Corollary 4.2.18. *For each $R > 0$, the pair $(M_R, \rho|_{M_R})$ is L^2-acyclic and of determinant class. In fact, there exists a uniform constant $\beta > 0$* ***independent of*** *R, so that for each $0 \leq k \leq n$, one has $\alpha_k^{An}(M_R, \rho) \geq \beta$.*

4.2.4 Proof of large-time convergence

For the next result, choose a nested sequence

$$\cdots \subset \mathcal{F}_{R-1} \subset \mathcal{F}_R \subset F_{R+1} \subset \ldots \tag{4.2.51}$$

inside $\mathbb{H}^n$, where $\mathcal{F}_R$ is a compact fundamental domain for the Γ-action on M_R and $\mathcal{F} \coloneqq \bigcup_{R>1} \mathcal{F}_R$ is a finite-volume fundamental domain for the Γ-action on $\mathbb{H}^n$. Thus, we have in particular

$$\operatorname{Vol}(\mathcal{F}_R) < \operatorname{Vol}(\mathcal{F}) < \infty, \tag{4.2.52}$$

$$\lim_{R\to\infty} \operatorname{Vol}(\mathcal{F} \setminus \mathcal{F}_R) = 0. \tag{4.2.53}$$

Proposition 4.2.19. *For all $t \geq 1$, we have*

$$\lim_{R\to\infty} \operatorname{tr}_\Gamma(e^{-t\Delta_p^\perp[E^\rho_{R-}]}) = \operatorname{tr}_\Gamma(e^{-t\Delta_p^\perp[E^\rho]}). \tag{4.2.54}$$

Proof. Propositions 4.2.11 and 4.2.17 tell us that both $\Delta_p^\perp[E^\rho_{R-}] = \Delta_p[E^\rho_{R-}]$ and $\Delta_p[E^\rho]^\perp = \Delta_p[E^\rho]$. Moreover, since $R \geq 2$, we may apply Theorem 3.5.6 to find appropriate constants $c, C_1, C_2 > 0$ independent of R and t, so that

$$|\operatorname{tr}(e^{-t\Delta_p[E^\rho_{R-}]})(x,x) - \operatorname{tr}(e^{-t\Delta_p[E^\rho]})(x,x)| \leq C_1 e^{-\frac{R^2}{C_2 t}}$$

for all $x \in \mathcal{F}_{R/2}$. Together with 4.2.52 and 4.2.53, this implies that

$$
\begin{aligned}
&|\operatorname{tr}_\Gamma(e^{-t\Delta_p[E^\rho_{R-}]}) - \operatorname{tr}_\Gamma(e^{-t\Delta_p[E^\rho]})| = |\int_{\mathcal{F}_R} \operatorname{tr}(e^{-t\Delta_p[E^\rho_{R-}]})(x,x)dx - \int_{\mathcal{F}} \operatorname{tr}(e^{-t\Delta_p[E^\rho]})(x,x)| \\
&\leq \int_{\mathcal{F}_{R/2}} |\operatorname{tr}(e^{-t\Delta_p[E^\rho_{R-}]})(x,x) - \operatorname{tr}(e^{-t\Delta_p[E^\rho]})(x,x)|dx + \int_{\mathcal{F}_R - \mathcal{F}_{R/2}} |\operatorname{tr}(e^{-t\Delta_p[E^\rho_{R-}]})(x,x)|dx \\
&+ \int_{\mathcal{F}-\mathcal{F}_{R/2}} |\operatorname{tr}(e^{-t\Delta_p[E^\rho]})(x,x)|dx \leq \operatorname{Vol}(\mathcal{F}_{R/2})C_1 e^{-\frac{R^2}{C_2 t}} + \operatorname{Vol}(\mathcal{F} \setminus \mathcal{F}_{R/2})c. \xrightarrow{R\to\infty} 0.
\end{aligned}
$$

□

Lemma 4.2.20. *There exists a positive function $G \in C^0(1,\infty) \cap L^1(1,\infty)$, such that for all $t \geq 1$ and all sufficiently large $R >> 0$, we have*

$$t^{-1}\,\mathrm{tr}_\Gamma(e^{-t\Delta_p^\perp[E_{R^-}]}) \leq G(t). \tag{4.2.55}$$

Proof. Throughout the proof, we will abbreviate

$$F_R^p(\lambda) := F_p(\Delta_p[E_{R^-}^\rho], \lambda). \tag{4.2.56}$$

Since $\Delta_p^\perp = \Delta_p$ is Fredholm by Proposition 4.2.2, we can apply Lemma 4.1.17 to conclude that one has for any $R > 0$, any $t \geq 1$ and arbitrary $\epsilon > 0$

$$\begin{aligned}
\mathrm{tr}_\Gamma(e^{-t\Delta_p^\perp}[E^{R^-}]) &= \int_0^\infty e^{-t\lambda} dF_R^p(\lambda) = \int_0^\epsilon e^{-t\lambda} dF_R^p(\lambda) + \int_\epsilon^\infty e^{-t\lambda} dF_R(\lambda) \\
&\overset{F_R^p(0)=0}{=} t\int_0^\epsilon e^{-t\lambda} F_R^p(\lambda) d\lambda + e^{-t\epsilon} F_R^p(\epsilon) + \int_\epsilon^\infty e^{-t\lambda} dF_R(\lambda) \\
&\overset{t>1}{\leq} t\int_0^\epsilon e^{-t\lambda} F_R^p(\lambda) d\lambda + e^{-t\epsilon} F_R^p(\epsilon) + e^{-t\epsilon}\int_\epsilon^\infty e^{-\lambda+\epsilon} dF_R(\lambda) \\
&= t\int_0^\epsilon e^{-t\lambda} F_R^p(\lambda) d\lambda + e^{-t\epsilon} F_R^p(\epsilon) + e^{-t\epsilon} e^\epsilon\,\mathrm{tr}_\Gamma(e^{-\Delta_p^\perp[E^{R^-}]}),
\end{aligned}$$

and therefore in particular

$$t^{-1}\,\mathrm{tr}_\Gamma(e^{-t\Delta_p^\perp[E_{R^-}^\rho]}) \leq \int_0^\epsilon e^{-tx} F_R^p(\lambda) d\lambda + \frac{e^{-t\epsilon}}{t}(F_R^p(\epsilon) + e^\epsilon\,\mathrm{tr}_\Gamma(e^{-\Delta_p^\perp[E_{R^-}^\rho]})). \tag{4.2.57}$$

By Proposition 4.2.19 we find some $\delta > 0$, such that for all $R >> 0$, we have $\mathrm{tr}_\Gamma(e^{-\Delta_p^\perp[E_{R^-}^\rho]}) \leq \mathrm{tr}_\Gamma(e^{-\Delta_p^\perp[E^\rho]}) + \delta$. Similarly, by Proposition 4.2.17, we can choose $\epsilon, \beta > 0$ independently of R and small enough, such that $F_R^p(\lambda) \leq \epsilon^{-1}\lambda^\beta$ for all $\lambda \leq \epsilon$. From this, it becomes obvious that the function

$$G(t) := \epsilon^{-1}\int_0^\epsilon e^{-t\lambda}\lambda^\beta d\lambda + \frac{e^{-t\epsilon}}{t}(\epsilon^{-1+\beta} + e^\epsilon(\mathrm{tr}_\Gamma(e^{-\Delta_p^\perp[E^\rho]}) + \delta)) \tag{4.2.58}$$

satisfies the assertions of our lemma. ∎

Using Proposition 4.2.19, Lemma 4.2.20 and Lebesgue's theorem of dominated convergence, we finally obtain the main result of this section:

Theorem 4.2.21 (Large-time convergence)**.** *We have*

$$\lim_{R\to\infty}\int_1^\infty t^{-1}\,\mathrm{tr}_\Gamma(e^{-t\Delta_p[E_{R^-}^\rho]})dt = \int_1^\infty t^{-1}\,\mathrm{tr}_\Gamma(e^{-t\Delta_p[E^\rho]})dt. \tag{4.2.59}$$

In particular, we obtain

$$\lim_{R\to\infty}\sum_{p=0}^n\int_1^\infty t^{-1}\,\mathrm{tr}_\Gamma(e^{-t\Delta_p[E_{R^-}^\rho]})dt = \sum_{p=0}^n\int_1^\infty t^{-1}\,\mathrm{tr}_\Gamma(e^{-t\Delta_p[E^\rho]})dt. \tag{4.2.60}$$

4.3 Small-time convergence

4.3.1 Small-time asymptotics on bundles of bounded geometry

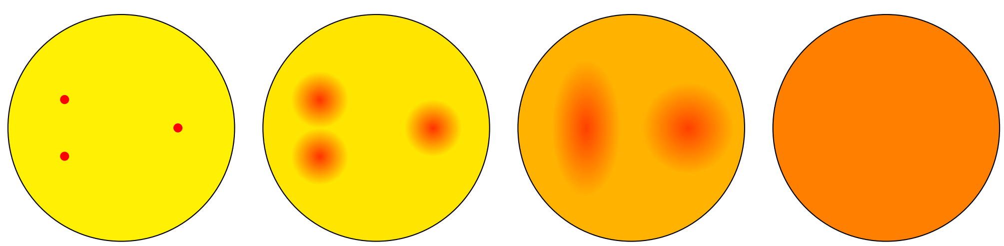
In this section, we will deal with the small time asymptotics of the heat equation on a vector bundle over a manifold. As a start, we will provide the physical context, out of which these investigations originally arose:

If we interpret a Riemannian manifold (M, g) as a closed system, we can further interpret a smooth function $f : M \to \mathbb{R}$ as a momentary assignment of *heat* to each point of $x \in M$. The natural physical question of how the initial heat f distributes over time inside the closed system M translates in mathematical terms to finding a solution $F : M \times [0, \infty) \to \mathbb{R}$ to the partial differential equation

$$\frac{\partial}{\partial t} F(x,t)|_{t=t_0} = -\Delta F(x, t_0), \tag{4.3.1}$$

$$F(x, 0) = f(x), \tag{4.3.2}$$

the so-called *heat equation*. Here, Δ is the Hodge-Laplacian on functions $C^\infty(M, \mathbb{R})$ constructed from the Riemannian metric g. Roughly speaking, this equation sheds light on the fact that the rate of change over time of the temperature at any given point $x \in M$ is proportional to the difference of the average temperature in a neighborhood of $x \in M$ to the value of the temperature at x. The latter difference is encoded in the Laplacian, which in turn depends on the Riemannian metric g. Below we have sketched the heat evolution on a closed system, modeled on a *euclidean disc* with absolute boundary conditions (in physical terms, with vanishing outward heat flow at the boundary), after applying heat to three different areas. As t approaches ∞, the heat will evenly distribute among the manifold, i.e. converge to a *constant* function.



Spectral theory then tells us that, provided that Δ is essentially self-adjoint, the heat equation is uniquely solved by $F(\ .\ , t) = e^{-t\Delta} f$. Moreover, the theory of elliptic operators tells us that $e^{-t\Delta}$ has an *integral kernel* $e^{-t\Delta}(x, y)$, the so-called *heat kernel*, which means that the solution $F(x,t)$ can be written in more concrete terms as the integral expression $F(x,t) = (e^{-t\Delta} f)(x) = \int_M e^{-t\Delta}(y, x) f(y) dy$. Classically, the heat kernel is constructed from a parametrix of the elliptic differential operator $\Delta + \frac{\partial}{\partial t}$ on $M \times \mathbb{R}$, cf. [26].

It should be noted that finding an explicit expression for the heat kernel $e^{-t\Delta}(x, y)$, (i.e. for a parametrix for $\Delta + \frac{\partial}{\partial t}$), and thus an explicit formula for the solution F, is a fruitless endeavor on a general Riemannian manifold M. In this section, we are only interested in rough asymptotics of the heat distribution for small time $t > 0$. With that in mind, the heat equation tells us that the rate of change over time depends on the previous values of F and on the local geometry. In fact, one can derive that for small time $t > 0$, the solution $F(x, t)$ is *essentially* only influenced by f and local geometric data on $\mathrm{supp}(f)$. Namely, although the heat equation does *not* have the finite propagation property of the wave equation 3.5.19, it holds that for small time $t > 0$, the contribution to the heat flow at a given $x \in M$ coming from the temperature at another $y \in M$ is negligible if the Riemannian distance $d_g(x, y)$ is not small. At least theoretically, one should therefore be able to approximate the heat operator $e^{-t\Delta}$ for small time $t > 0$ sufficiently well by *local quantities*. This is adequately reflected in the small-time asymptotics of the heat kernel near the diagonal, stated in the forthcoming result.

Replacing f by a general differential form with values in a flat, Hermitian vector bundle $E \downarrow M$ and the Laplacian of functions by the corresponding Hodge-Laplacian yields the generalized heat equation that we have implicitly already studied to a great extend. Provided that M is compact, the locality of the resulting heat operator $e^{-t\Delta_p[E]}$ for small time $t > 0$ is made precise by the following result, originally due to Greiner, see [41, Theorem 2.6.1]. In the form stated below, the proof can be found in [40, Theorem 1.11.4].

Theorem 4.3.1 (Asymptotic expansion: Compact case)**.** *Let (M, g) be an n-dimensional compact, oriented Riemannian manifold and let $E \downarrow M$ be a flat Hermitian vector bundle over M.*
Then, for each $0 \leq p \leq n$ and each $i \in \mathbb{N}$, there exist forms $\alpha_i[E] \in \Omega^n(M)$ and $\beta_i[E|_{\partial M}] \in \Omega^{n-1}(\partial M)$, such that the following holds
For each $t > 0$ and each $f \in C^\infty(M, \mathbb{R})$, the (modified) heat operator $f \cdot e^{-t\Delta_p[E]}$ is of trace class. Additionally, for small time $t \to 0$, it has the following asymptotic expansion.

$$\operatorname{tr}(f \cdot e^{-t\Delta_p[E]}) = \sum_{i=0}^{n} t^{-(n-i)/2} \left(\int_M f \cdot \alpha_i[E] + \int_{\partial M} f \cdot \beta_i[E_{\partial M}] \right) + \mathcal{O}(t^{1/2}). \tag{4.3.3}$$

Furthermore, both $\alpha_i[E]$ and $\beta_i[E]$ are ***invariant under local bundle isometries*** *in the way described as follows: Suppose that (M', g') is another n-dimensional compact, oriented Riemannian manifold and $E' \downarrow M'$ a flat Hermitian bundle over M'. Then, if $U \subseteq M$ and $V \subseteq M$ are open subsets, such that there exists a flat bundle isometry $E_U \downarrow U \xrightarrow{\mathrm{F}} E'_V \downarrow V$, we have for all $i \in \mathbb{N}$*

$$\begin{aligned} \alpha_i[E] &\equiv \alpha_i[E'] \circ F && \text{on } U, \\ \beta_i[E_{\partial M}] &\equiv \beta_i[E'_{\partial M'}] \circ F && \text{on } U \cap \partial M. \end{aligned}$$

We wish to extend the result onto the non-compact setting.

Theorem 4.3.2 (Asymptotic expansion: Cocompact case)**.** *Let (M, g) be an n-dimensional oriented Riemannian manifold, let $(E, h) \downarrow M$ be a flat trivial Hermitian bundle over M. Suppose that $(E, h) \downarrow M$ is H-equivariant for some subgroup $H \subseteq \operatorname{Isom}^+(M, g)$ that contains a uniform lattice.*
Then, for each $0 \leq p \leq n$ and each $i \in \mathbb{N}$, there exists forms $\alpha_i[M] \in \Omega^n(M)$ and $\beta_i[E_{\partial M}] \in \Omega^{n-1}(\partial M)$, such that the following holds:
For each $t > 0$, each uniform lattice $\Gamma \subseteq H$ and each Γ-invariant function $f \in C^\infty(M, \mathbb{R})$, the operator $f \cdot e^{-t\Delta_p[E]}$ is of Γ-trace class. Additionally, for $t \to 0$, we have the following asymptotic expansion:

$$\operatorname{tr}_\Gamma(f \cdot e^{-t\Delta_p[E_M]}) = \sum_{i=0}^{n} t^{-(n-i)/2} \left(\int_{\mathcal{F}} f \cdot \alpha_i[E] + \int_{\partial \mathcal{F}} f \cdot \beta_i[E_{\partial M}] \right) + \mathcal{O}(t^{1/2}), \tag{4.3.4}$$

where

- *$\mathcal{F}$ is a fundamental domain for the Γ-action on M, and*

- *$\partial\mathcal{F}$ is a fundamental domain for the induced Γ-action on ∂M.*

Furthermore, both $\alpha_i[E]$ and $\beta_i[E]$ are invariant under local bundle isometries in the same way as described in Theorem 4.3.1.

Proof. Let $(\hat{M} := M/\Gamma, \hat{g})$ be the compact quotient Riemannian manifold, with $\hat{g}$ the quotient metric induced by g (which is well-defined, since the action of Γ leaves g invariant). Similarly, let $\hat{E} \downarrow \hat{M}$ be the flat complex vector bundle with total space $\hat{E} := E/\Gamma$, obvious projection map and the induced (flat) connection. Denote by $\pi : E \downarrow M \to \hat{E} \downarrow \hat{M}$ the corresponding quotient map. Since Γ was assumed to be compatible with $E \downarrow M$, the Hermitian form h descends to a Hermitian form $\hat{h}$ on $\hat{E} \downarrow \hat{M}$.
The metric structure that we endowed $\hat{E} \downarrow \hat{M}$ with now ensures that there exists a constant $K > 0$ such that, for any $x \in M$, the restriction
$\pi : E|_{B_{2K}(x)} \downarrow B_{2K}(x) \to \hat{E}|_{B_{2K}(\pi(x))} \downarrow B_{2K}(\pi(x))$ is a flat bundle isometry. Here, $B_{2K}(x)$ denotes the metric ball around x of Radius $2K$, similarly $B_{2K}(\pi(x))$. For fixed $x \in M$ and $\hat{x} := \pi(x)$, let $N \subseteq M$ be a closed connected Riemannian submanifold, satisfying $B_K(x) \subseteq N \subseteq B_{2K}(x)$ and let $\hat{N} := \pi(N) \subseteq \hat{M}$ be its diffeomorphic image in $\hat{M}$. Then $\pi : E|_N \downarrow N \to \hat{E}|_{\hat{N}} \downarrow \hat{N}$ is a flat bundle isometry. In particular, we obtain

$$||\operatorname{tr}(e^{-t\Delta[E|_N]}(x,x)) - \operatorname{tr}(e^{-t\Delta[\hat{E}|_{\hat{N}}]}(\hat{x},\hat{x}))|| = 0. \tag{4.3.5}$$

Moreover, we have $d_N(x) = d_{\hat{N}}(\hat{x}) \leq 2K$. Therefore, by Theorem 3.5.6, we obtain constants $C_1, C_2 > 0$ independent of x and N, such that following equations hold.

$$||\operatorname{tr}(e^{-t\Delta[E]} - e^{-t\Delta[E|_N]})(x,x)|| < C_1/2e^{-\frac{1}{C_2 t}}, \tag{4.3.6}$$

$$||\operatorname{tr}(e^{-t\Delta[\hat{E}]} - e^{-t\Delta[\hat{E}|_{\hat{N}}]})(\hat{x},\hat{x})|| < C_1/2e^{-\frac{1}{C_2 t}}. \tag{4.3.7}$$

Therefore, by the triangle inequality, we obtain

$$||\operatorname{tr}(e^{-t\Delta[E]})(x,x) - \operatorname{tr}(e^{-t\Delta[\hat{E}]})(\hat{x},\hat{x})|| < C_1 e^{-\frac{1}{C_2 t}}. \tag{4.3.8}$$

Applying Theorem 4.3.1 to $\hat{E} \downarrow \hat{M}$ yields density functions $\alpha_i[\hat{E}]$ and $\beta_i[\hat{E}]$ for each $i \in \mathbb{N}$, such that for $t \to 0$, we have the expansion

$$\operatorname{tr}(e^{-t\Delta_p[\hat{E}]}) = \sum_{i=0}^{k} t^{-(n-i)/2} \left(\int_{\hat{M}} \alpha_i[\hat{E}](x)\, dx + \int_{\partial\hat{M}} \beta_i[\hat{E}_{\partial M}](x)\, dx \right) + \mathcal{O}(t^{(k-n+1)/2}). \tag{4.3.9}$$

Defining $\alpha_i[E] := \alpha_i[\hat{E}] \circ \pi$, $\beta_i[E] := \beta_i[\hat{E}] \circ \pi$ and using the fact that $e^{-\frac{1}{C_2 t}} \in \mathcal{O}(t^{(k-n+1)/2})$ for any $k \in \mathbb{N}$ yields the asymptotic expansion for $\operatorname{tr}_\Gamma(e^{-t\Delta_p[\hat{E}]})$.
We have to show that the densities defined this way do not depend on the particular choice of compatible uniform lattice in H. For that purpose, let $\Lambda \subset H$ be another such lattice, let $E \downarrow M$ be the Hermitian quotient bundle over $\overline{M} := M/\Lambda$ constructed as above, with densities $\alpha_i[E]$ and $\beta_i[E]$ corresponding to the p-th Laplacian $\Delta_p[E]$, provided by Theorem 4.3.1. For $\overline{\pi} : M \to \overline{M}$ the quotient map, $\overline{\alpha}_i[E] := \alpha_i[E] \circ \overline{\pi}$ and $\overline{\beta}_i[E] := \beta_i[E] \circ \overline{\pi}$, our goal is to show that

$$\alpha_i[E] = \overline{\alpha}_i[E], \tag{4.3.10}$$

$$\beta_i[E] = \overline{\beta}_i[E]. \tag{4.3.11}$$

For fixed $x \in M$, we can choose a neighborhood $U \ni x$ such that both $\pi : E|_U \downarrow U \to \hat{E}_{\pi(U)} \downarrow \pi(U)$ and $\overline{\pi} : E|_U \downarrow U \to \overline{E}_{\overline{\pi}(U)} \downarrow \overline{\pi}(U)$ are flat bundle isometries. Therefore, the composition $\overline{\pi} \circ \pi^{-1} : \hat{E}|_{\pi(U)} \downarrow$

$\pi(U) \to \overline{E}|_{\overline{\pi}(U)} \downarrow \overline{\pi}(U)$ is also a well-defined flat bundle isometry. Applying the local bundle isometric invariance properties to $\alpha_i[\hat{E}]$ and $\alpha_i[\overline{E}]$, as stated in Theorem 4.3.1, Equations 4.3.10 and 4.3.11 now directly follow. The local bundle isometric invariance of $\alpha_i[E]$ and $\beta_i[E]$ is shown similarly, finishing the proof.

This proofs the result in case that $f \equiv 1$. The case for general Γ-invariant $f \in C(M, \mathbb{R})$ follows from observing that the integral kernel of $f \cdot e^{-t\Delta_p[E]}$ is given by $f(x) \cdot e^{-t\Delta_p[E]}(y, x)$ and repeating the argument from above. ☐

4.3.2 The asymptotic expansion of the hyperbolic heat kernel

Recall the following notions from Section 2.3: Let $\mathbb{H}^n$ be hyperbolic n-space, $G := \mathrm{Isom}^+(\mathbb{H}^n)$ the group of orientation-preserving isometries on $\mathbb{H}^n$ and let

$$\Gamma \subset G \tag{4.3.12}$$

a torsion-free, non-uniform lattice. Let $\rho : G \to GL(V)$ be a complex, finite-dimensional irreducible representation of G and let $E^\rho \downarrow \mathbb{H}^n$ be the associated flat bundle over $\mathbb{H}^n$, equipped with the canoncial G-equivariant metric h^ρ. For each $R > 0$, recall the associated, Γ-invariant bundles

$$\begin{aligned} &E^\rho_{R-} \downarrow M_R, \\ &E^\rho_{R+} \downarrow C_R, \\ &E^\rho_{R} \downarrow T_R, \end{aligned}$$

obtained by restriction of E^ρ to the respective base space, and the Γ-regularized traces

$$\mathrm{tr}_\Gamma(e^{-t\Delta_p[E^\rho]}) = \int_{\mathcal{F}} \mathrm{tr}(e^{-t\Delta_p[E^\rho]}(x, x)) d\mu_g(x), \tag{4.3.13}$$

$$\mathrm{tr}_\Gamma(e^{-t\Delta_p[E^\rho_{R-}]}) = \int_{\mathcal{F}_R} \mathrm{tr}(e^{-t\Delta_p[E^\rho_{R-}]}(x, x)) d\mu_g(x), \tag{4.3.14}$$

where as before, $d\mu_g(x)$ denotes the volume form on $\mathbb{H}^n$ induced by the hyperbolic metric g. Here, we can and have chosen the Γ-fundamental domains $\mathcal{F}$, respectively $\mathcal{F}_R$ for the Γ-action on $\mathbb{H}^n$, respectively M_R, so that for each $R > 0$

1. $M_R \cap \mathcal{F} = \mathcal{F}_R$, so that $\mathcal{F} = \bigcup_{R \geq 0} \mathcal{F}_R$.
2. $\partial M_R \cap \mathcal{F} = \partial \mathcal{F}_R$ is a fundamental domain for the Γ-action on ∂M_R.
3. There exists a finite family $(\mathcal{G}_j)_{j \in J}$ with each $\mathcal{G}_j \subseteq \mathbb{R}^{n-1}$ a compact euclidean submanifold, such that for $0 \leq R < S < \infty$, we have
    (a) $\mathcal{F}_S \setminus \mathcal{F}_R = [R, S] \times \coprod_{j \in J} \mathcal{G}_j$,
    (b) $\partial \mathcal{F}_R = \{R\} \times \coprod_{j \in J} \mathcal{G}_j$.

Corollary 4.3.3. *For fixed $p \in \mathbb{N}$ and each $i = 0, \ldots, n$, there exist **constants** $a_i, b_i \in \mathbb{C}$, such that for $t \to 0$, we have*

$$\mathrm{tr}_\Gamma(e^{-t\Delta_p[E^\rho]}) = \mathrm{Vol}(\mathcal{F}) \sum_{i=0}^{n} t^{-(n-i)/2} a_i + \mathcal{O}(t^{1/2}), \tag{4.3.15}$$

and, for each $R > 0$, we have for $t \to 0$

$$\operatorname{tr}_\Gamma(e^{-t\Delta_p[E^\rho_{R^-}]}) = \sum_{i=0}^{n} t^{-(n-i)/2}(\operatorname{Vol}(\mathcal{F}_R)a_i + \operatorname{Vol}(\partial\mathcal{F}_R)b_i) + \mathcal{O}(t^{1/2}). \tag{4.3.16}$$

Proof. Since Γ acts cocompactly on each M_R, the previous theorem provides us with asymptotic expansions of $\operatorname{tr}_\Gamma(e^{-t\Delta_p[E^\rho_{R^-}]})$ for $t \to 0$, with respective density functions $\alpha_i[E^\rho_{R^-}]$ and $\beta_i[E^\rho_{R^-}]$ which *a priori* are non-constant and depend on R. However, $\mathbb{H}^n$ is a homogeneous space. In particular, any two points $x, y \in \mathbb{H}^n$ have arbitrarily small isometric neighborhoods. Hence, by Corollary 2.3.5, there exists arbitrarily small neighborhoods $U \ni x$ and $V \ni y$ and a bundle isometry $F : E^\rho_U \downarrow U \to E^\rho_V \downarrow V$ such that $y = F(x)$. Since $\alpha_i[E^\rho_{R^-}]$ and $\beta_i[E^\rho_{R^-}]$ are invariant under local bundle isometries, we obtain both $\alpha_i[E^\rho_{R^-}] \equiv a_i$ and $\beta_i[E^\rho_{R^-}] \equiv b_i$, with constants $a_i, b_i \in \mathbb{C}$ independent of R. Equation 4.3.16 now follows. To obtain Equation 4.3.15, we apply the previous theorem to $E^\rho \downarrow \mathbb{H}^n$ and *some* uniform lattice $\Lambda \subset \operatorname{Isom}(\mathbb{H}^n)$ (note that $\Gamma \subset \operatorname{Isom}(\mathbb{H}^n)$ is *not* uniform, by assumption). The same argument as before then yields

$$\operatorname{tr}_\Lambda(e^{-t\Delta_p[E^\rho]}) = \operatorname{Vol}(\hat{\mathcal{F}}) \sum_{i=0}^{n} t^{-(n-i)/2}a_i + \mathcal{O}(t^{1/2}), \tag{4.3.17}$$

where $\hat{F}$ is some fundamental domain for the Λ-action on $\mathbb{H}^n$. From Corollary 2.3.9, we obtain

$$\operatorname{tr}_\Gamma(e^{-t\Delta_p[E^\rho]}) = \operatorname{Vol}(\mathcal{F})/\operatorname{Vol}(\hat{\mathcal{F}})\operatorname{tr}_\Lambda(e^{-t\Delta_p[E^\rho]}). \tag{4.3.18}$$

Equation 4.3.15 now immediately follows from the previous two Equations. ☐

In order to streamline the notation, we will set

$$\alpha_i := \operatorname{Vol}(\mathcal{F})a_i \ \ , \ \ \alpha_i^R := \operatorname{Vol}(\mathcal{F}_R)a_i \ \ , \ \ \beta_i^R := \operatorname{Vol}(\partial\mathcal{F}_R)b_i. \tag{4.3.19}$$

whenever $p = 0, \dots, n$ is clear from the context.

Corollary 4.3.4. *The Γ-equivariant Fredholm bundles $E^\rho \downarrow \mathbb{H}^n$ and $E^\rho_{R^-} \downarrow M_R$ for each $R > 0$ are ζ-regular. Namely, for any fixed $0 \le p \le n$ the integral expressions*

$$\zeta_p^R(s) := \Gamma(s)^{-1} \int_0^1 t^{s-1} \operatorname{tr}_\Gamma(e^{-t\Delta_p[E^\rho_{R^-}]^\perp})dt, \tag{4.3.20}$$

$$\zeta_p(s) := \Gamma(s)^{-1} \int_0^1 t^{s-1} \operatorname{tr}_\Gamma(e^{-t\Delta_p[E^\rho]^\perp})dt, \tag{4.3.21}$$

determine holomorphic functions for sufficiently large $s >> 0$, each admitting a meromorphic extension onto all of $\mathbb{C}$ that is regular at 0. In fact, it holds that

$$\begin{aligned}\frac{d}{ds}\zeta_p^R(s)|_{s=0} &= \int_0^1 \left(\operatorname{tr}_\Gamma(e^{-t\Delta_p[E^\rho_{R^-}]}) - \sum_{i=0}^{n} t^{-(n-i)/2}(\alpha_i^R + \beta_i^R)\right) \frac{dt}{t} \\ &+ \sum_{i=0}^{n} c(i,n)(\alpha_i^R + \beta_i^R),\end{aligned} \tag{4.3.22}$$

$$\frac{d}{ds}\zeta_p(s)|_{s=0} := \int_0^1 \left(\operatorname{tr}_\Gamma(e^{-t\Delta_p[E^\rho]}) - \sum_{i=0}^{n} t^{-(n-i)/2}\alpha_i\right) \frac{dt}{t} + \sum_{i=0}^{n} c(i,n)\alpha_i, \tag{4.3.23}$$

where

$$c(i,n) := \begin{cases} -\frac{n-i}{2} & i \neq n, \\ \frac{d\Gamma}{ds}|_{s=1} & i = n. \end{cases} \tag{4.3.24}$$

Proof. First, Propositions 4.2.11 and 4.2.17 imply that we have $\Delta_p[E^\rho] = \Delta_p[E^\rho]^\perp$, as well as $\Delta_p[E^\rho_{R-}] = \Delta_p[E^\rho_{R-}]^\perp$ for each $R > 0$. Then the result follows from the asymptotic expansions as outlined in Corollary 4.3.3, along with elementary complex analysis and well-known properties of the inverse Gamma function, see [55, Lemma 2.36] for additional details. ☐

Since $\operatorname{tr}(e^{-t\Delta_p}(x,x))$ is constant on $\mathbb{H}^n$, we obtain from Equation 4.3.15 the following

Corollary 4.3.5. *There exists a constant $C > 0$, such that for all $x \in \mathbb{H}^n$, we have for $t \to 0$*

$$|\operatorname{tr}(e^{-t\Delta_p[E^\rho]}(x,x)) - \sum_{i=0}^{n} t^{-(n-i)/2} a_i| \leq C t^{1/2}. \tag{4.3.25}$$

Corollary 4.3.6. *For $R \geq 1$ and $t \to 0$, we have*

$$\begin{aligned}&\int_{\mathcal{F}_R \setminus \mathcal{F}_{R-1}} \operatorname{tr}(e^{-t\Delta_p[E^\rho_{R-}]}(x,x))\, d\mu_g(x) - \sum_{i=0}^{n} t^{-(n-i)/2}(\operatorname{Vol}(\mathcal{F}_R \setminus \mathcal{F}_{R-1}) a_i + \operatorname{Vol}(\partial\mathcal{F}_R) b_i)\\ &\in \mathcal{O}(t^{1/2}).\end{aligned} \tag{4.3.26}$$

Proof. We can write

$$\int_{\mathcal{F}_R \setminus \mathcal{F}_{R-1}} \operatorname{tr}(e^{-t\Delta_p[E^\rho_{R-}]}(x,x))\, d\mu_g(x) - \sum_{i=0}^{n} t^{-(n-i)/2}(\operatorname{Vol}(\mathcal{F}_R \setminus \mathcal{F}_{R-1}) a_i + \operatorname{Vol}(\partial\mathcal{F}_R) b_i)$$

$$= \operatorname{tr}_\Gamma(e^{-t\Delta_p[E^\rho_{R-}]}) - \sum_{i=0}^{n} t^{-(n-i)/2}(\operatorname{Vol}(\mathcal{F}_R) a_i + \operatorname{Vol}(\partial F_R) b_i) \tag{4.3.27}$$

$$- \int_{\mathcal{F}_{R-1}} \operatorname{tr}(e^{-t\Delta_p[E^\rho_{R-}]}(x,x)) - \sum_{i=0}^{n} t^{-(n-i)/2} a_i\, d\mu_g(x). \tag{4.3.28}$$

4.3.27 is in $\mathcal{O}(t^{1/2})$ by Corollary 4.3.3. Next, observe that $d_{M_R}(x) \geq 1$ for any $x \in \mathcal{F}_{R-1}$. Therefore, applying Theorem 3.5.6 to $M = \mathbb{H}^n$, $N = M_R$ and $D = 1$, we find constants $C, \kappa > 0$, such that for all $x \in \mathcal{F}_{R-1}$, we have

$$|\operatorname{tr}(e^{-t\Delta_p[E^\rho_{R-}]} - e^{-t\Delta_p[E^\rho]})(x,x)| < C e^{-2/(\kappa t)} \tag{4.3.29}$$

Since $e^{-2/(\kappa t)} \in \mathcal{O}(t^{1/2})$, we can apply Corollary 4.3.5 and obtain that 4.3.28 is also in $\mathcal{O}(t^{1/2})$. The result follows. ☐

Proof of small-time convergence

Theorem 4.3.7 (Small-time convergence)**.** *For each $0 \leq p \leq n$, we have*

$$\lim_{R\to\infty} \frac{d}{ds}\zeta_p^R(s)|_{s=0} = \frac{d}{ds}\zeta_p(s)|_{s=0}. \tag{4.3.30}$$

Proof. First, observe that $\lim_{R\to\infty}\operatorname{Vol}(\mathcal{F}_R) = \operatorname{Vol}(\mathcal{F})$ and, since $\partial\mathcal{F}_R$ is a flat subspace of $\partial M_R \cong \mathbb{R}^{n-1}$ which is equipped with the flat metric $e^{-2R}dx$, we also have $\lim_{R\to\infty}\operatorname{Vol}(\partial\mathcal{F}_R) = 0$. We deduce that $\lim_{R\to\infty}(\alpha_i - \alpha_i^R) = \lim_{R\to\infty}\beta_i^R = 0$ for each $i = 0, \ldots, n$. Therefore, the statement of the theorem will follow once we show that for

$$\Sigma[R] \coloneqq \int_0^1 \operatorname{tr}_\Gamma(e^{-t\Delta_p[E^\rho]}) - \operatorname{tr}_\Gamma(e^{-t\Delta_p[E^\rho_{R-}]}) - \sum_{i=0}^{n} t^{-(n-i)/2}(\alpha_i - \alpha_i^R - \beta_i^R)\frac{dt}{t}, \tag{4.3.31}$$

we have

$$\lim_{R\to\infty} \Sigma[R] = 0. \tag{4.3.32}$$

Recall that $\operatorname{tr}_\Gamma(e^{-t\Delta_p[E^\rho]}) = \operatorname{Vol}(\mathcal{F})\operatorname{tr}(e^{-t\Delta_p[E^\rho]}(x,x))$ for any $x \in \mathcal{F}$. For fixed $R > 2$, we can therefore decompose $\Sigma[R] = \Sigma_1[R] + \Sigma_2[R] + \Sigma_3[R] - \Sigma_4[R]$, with

$$\Sigma_1[R] \coloneqq \operatorname{Vol}(\mathcal{F} \setminus \mathcal{F}_{R-1}) \int_0^1 \operatorname{tr}(e^{-t\Delta_p[E^\rho]}(x,x)) - \sum_{i=0}^{n} t^{-(n-i)/2}\alpha_i \frac{d}{dt}, \tag{4.3.33}$$

$$\Sigma_2[R] \coloneqq \int_0^1 \int_{\mathcal{F}_{R/2}} \operatorname{tr}((e^{-t\Delta_p[E^\rho]} - e^{-t\Delta_p[E^\rho_{R-}]})(x,x))d\mu_g(x)\frac{d}{dt}, \tag{4.3.34}$$

$$\Sigma_3[R] \coloneqq \int_0^1 \int_{\mathcal{F}_{R-1}\setminus\mathcal{F}_{R/2}} \operatorname{tr}((e^{-t\Delta_p[E^\rho]} - e^{-t\Delta_p[E^\rho_{R-}]})(x,x))d\mu_g(x)\frac{d}{dt}, \tag{4.3.35}$$

$$\Sigma_4[R] \coloneqq \int_0^1 \int_{\mathcal{F}_R\setminus\mathcal{F}_{R-1}} \operatorname{tr}((e^{-\Delta_p[E^\rho_{R-}]})(x,x))d\mu_g(x) \tag{4.3.36}$$

$$-\sum_{i=0}^{n} t^{-(n-i)/2}(\operatorname{Vol}(\mathcal{F}_R \setminus \mathcal{F}_{R-1})a_i + \operatorname{Vol}(\partial\mathcal{F}_R)b_i)\frac{dt}{t}. \tag{4.3.37}$$

This splitting is allowed, since each one of these integrals converges. Convergence of $\Sigma_1[R]$ follows from Corollary 4.3.5, convergence of $\Sigma_2[R]$ and $\Sigma_3[R]$ each follows from Theorem 3.5.6, and finally, convergence of $\Sigma_4[R]$ follows from Corollary 4.3.6. Our strategy now is to show that $\lim_{R\to\infty} \Sigma_i[R]$ for each $i = 1, \ldots, 4$. For this purpose, observe first that we may apply Corollary 4.3.5 to obtain a constant $C > 0$ independent of R, such that

$$|\Sigma_1[R]| \leq C \operatorname{Vol}(\mathcal{F} \setminus \mathcal{F}_{R-1}) \int_0^1 t^{-1/2}dt = 2C \operatorname{Vol}(\mathcal{F} \setminus \mathcal{F}_{R-1}) \xrightarrow{R\to\infty} 0. \tag{4.3.38}$$

Secondly, observe that for any $R > 0$, the bundle map

$$\operatorname{tr} : \pi_1^*(E^*) \otimes \pi_2^*(E) \downarrow M_R \times M_R \to \mathbb{C} \times M_R \times M_R \downarrow M_R \times M_R \tag{4.3.39}$$

is uniformly bounded (with respect to the canonical constant Hermitian metric on $\mathbb{C} \times M_R \times M_R \downarrow M_R \times M_R$) by a constant independent of R. Now $d_{\mathcal{F}_R}(x) \geq 1$ for any $x \in \mathcal{F}_{R-1}$ and $d_{\mathcal{F}_R}(x) \geq R/2 > 1$ for any $x \in \mathcal{F}_{R/2}$. Thus, we can now apply Theorem 3.5.6.(2) with $D = 1$ and obtain constants C_1, C_2, independent of R, such that

$$|\operatorname{tr}((e^{-t\Delta_p[E^\rho]} - e^{-t\Delta_p[E^\rho_{R-}]})(x,x))| < C_1 e^{-R^2/(2tC_2)} \quad \forall x \in \mathcal{F}_{R/2}, \tag{4.3.40}$$

$$|\operatorname{tr}((e^{-t\Delta_p[E^\rho]} - e^{-t\Delta_p[E^\rho_{R-}]})(x,x))| < C_1 e^{-2/(tC_2)} \quad \forall x \in \mathcal{F}_{R-1}. \tag{4.3.41}$$

Heuristically, these estimates imply that the integrand in $\Sigma_2[R]$ decays exponentially fast in R, while the volume of the domain of integration $\mathcal{F}_{R/2}$ is, of course, uniformly bounded by the volume of $\mathcal{F}$. On the other hand, the integrand in $\Sigma_3[R]$ is uniformly bounded for all R, while the volume of the domain of integration, $\mathcal{F}_{R-1} \setminus \mathcal{F}_{R/2}$ decays exponentially fast. Put into action, we obtain

$$|\Sigma_2[R]| \leq C_1 \operatorname{Vol}(\mathcal{F}_{R/2}) \int_0^1 e^{-R^2/(2tC_2)} \frac{dt}{t} \leq C_1 \operatorname{Vol}(\mathcal{F}_{R/2}) \int_0^1 e^{-R^2/(2tC_2)} \frac{dt}{t^2} \tag{4.3.42}$$
$$\leq \operatorname{Vol}(\mathcal{F}) \frac{2C_1C_2}{R^2} e^{-R^2/(2C_2)} \xrightarrow{R\to\infty} 0,$$
$$|\Sigma_3[R]| \leq C_1 \operatorname{Vol}(\mathcal{F}_{R-1} \setminus \mathcal{F}_{R/2}) \int_0^1 e^{-2/(tC_2)} \frac{dt}{t} \leq \frac{C_1C_2}{2} \operatorname{Vol}(\mathcal{F}_{R-1} \setminus \mathcal{F}_{R/2}) e^{-2/C_2} \tag{4.3.43}$$
$$\leq \frac{C_1C_2}{2} e^{-(2+R)/C_2} (R/2-1) \xrightarrow{R\to\infty} 0.$$

The proof of $\lim_{R\to\infty} |\Sigma_4[R]| = 0$ requires a little more work. For that purpose, first define the horoball $\mathbb{H}^n_R \coloneqq (-\infty, R] \times \mathbb{R}^{n-1} \subset \mathbb{H}^n$, so that the restriction of the hyperbolic metric on $\mathbb{H}^n_R$ is of the warped product form $dr^2 + e^{-2r}dx^2$. Denote by $\Delta_p[\mathbb{H}^n_R]$ the Bochner-Laplace operator corresponding to the flat restriction bundle $E^\rho|_{\mathbb{H}^n_R}$ over $\mathbb{H}^n_R$. Observe now that $E^\rho|_{\mathbb{H}^n_R} \downarrow \mathbb{H}^n_R$ still satisfies the assumptions of Corollary 3.3.15. We may therefore apply Proposition 3.4.2 and obtain for each $t > 0$ a heat kernel $e^{-t\Delta_p[\mathbb{H}^n_R]}(x,y)$. This allows us to decompose $\Sigma_4[R]$ as the sum of the following convergent integrals

$$\Sigma_4[R] = \overbrace{\int_0^1 \int_{\mathcal{F}_R\setminus\mathcal{F}_{R-1}} \operatorname{tr}((e^{-t\Delta_p[E^\rho_{R^-}]} - e^{-t\Delta_p[M_R\setminus M_{R-2}]})(x,x)) d\mu_g(x) \frac{dt}{t}}^{\Sigma_{4,1}[R]}, \tag{4.3.44}$$
$$+ \overbrace{\int_0^1 \int_{\mathcal{F}_R\setminus\mathcal{F}_{R-1}} \operatorname{tr}((e^{-t\Delta_p[M_R\setminus M_{R-2}]} - e^{-t\Delta_p[\mathbb{H}^n_R]})(x,x)) d\mu_g(x) \frac{dt}{t}}^{\Sigma_{4,2}[R]}, \tag{4.3.45}$$
$$+ \overbrace{\int_0^1 \int_{\mathcal{F}_R\setminus\mathcal{F}_{R-1}} \operatorname{tr}(e^{-t\Delta_p[\mathbb{H}^n_R]}(x,x)) d\mu_g(x) - \sum_{i=0}^n t^{-(n-i)/2} (\operatorname{Vol}(\mathcal{F}_R \setminus \mathcal{F}_{R-1}) a_i + \operatorname{Vol}(\partial\mathcal{F}_R) b_i) \frac{dt}{t}}^{\Sigma_{4,3}[R]}. \tag{4.3.46}$$

The first two summands converge, as is shown in the course of the next paragraph, and therefore also the third. To deal with the first two summands, we can apply Theorem 3.5.6 in the following fashion: For $\Sigma_{4,1}[R]$, we put M_R in the role of the ambient Manifold and regard $M_R \setminus M_{R-2}$ as a submanifold. In this setting, we have $d_{M_R\setminus M_{R-2}}(x) \geq 1$ for any $x \in \mathcal{F}_R \setminus \mathcal{F}_{R-1}$. Therefore, Theorem 3.5.6 provides us with constants $D_1, D_2 > 0$, such that

$$|\operatorname{tr}((e^{-t\Delta_p[E^\rho_{R^-}]} - e^{-t\Delta_p[M_R\setminus M_{R-2}]})(x,x))| < D_1 e^{\frac{2}{D_2 t}}. \tag{4.3.47}$$

Moreover, as M_R and M_0 have isometric neighborhoods (in $\mathbb{H}^n$), both D_1 and D_2 can be chosen independently of R. This implies that

$$|\Sigma_{4,1}[R]| \leq \operatorname{Vol}(\mathcal{F}_R \setminus \mathcal{F}_{R-1}) \frac{D_1D_2}{2} e^{\frac{2}{D_2}} \xrightarrow{R\to\infty} 0. \tag{4.3.48}$$

Replacing M_R by $\mathbb{H}^n_R$ in the role of the ambient manifold, an analogous argument yields

$$\lim_{R\to\infty} |\Sigma_{4,2}[R]| = 0. \tag{4.3.49}$$

Next, observe that by our choice of $\mathcal{F}_R$, we have

$$\operatorname{Vol}(\mathcal{F}_R \setminus \mathcal{F}_{R-1}) = e^{-2R+2} \operatorname{Vol}(\mathcal{F}_1 \setminus \mathcal{F}_0), \tag{4.3.50}$$
$$\operatorname{Vol}(\partial\mathcal{F}_R) = e^{-2R+2} \operatorname{Vol}(\partial\mathcal{F}_1). \tag{4.3.51}$$

Secondly, recall that $\mathbb{R}^{n-1}$ has transitive isometry group, and observe that any isometry $i : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ extends to an isometry $i \times \mathbb{1} : \mathbb{H}^n_R \to \mathbb{H}^n_R$. Moreover, we have an isometry

$$\begin{aligned} &I_R : \mathbb{H}^n_R \to \mathbb{H}^n_1, \\ &I_R(u, y) \coloneqq (u + 1 - R, e^{R-1} y). \end{aligned} \tag{4.3.52}$$

Using Corollary 2.3.6 and Proposition 2.2.1, we therefore obtain a smooth map $h : (0, \infty) \times (-\infty, 1] \to \mathbb{C}$, such that for $x = (u, v) \in (-\infty, \infty) \times \mathbb{R}^{n-1}$, we have

$$\operatorname{tr}(e^{-t\Delta[\mathbb{H}^n_1]}(x, x)) = h(t, u), \qquad \text{if } x \in \mathbb{H}^n_1, \tag{4.3.53}$$

$$\operatorname{tr}(e^{-t\Delta[\mathbb{H}^n_R]}(x, x)) = h(t, u + 1 - R), \qquad \text{if } x \in \mathbb{H}^n_R. \tag{4.3.54}$$

This implies that

$$\begin{aligned} &\int_{\mathcal{F}_R \setminus \mathcal{F}_{R-1}} \operatorname{tr}(e^{-t\Delta[\mathbb{H}^n_R]}(x, x)) dx = \int_{\mathcal{G}} \int_{R-1}^{R} h(t, u - R + 1) e^{-2u} du dy \\ &= \int_{\mathcal{G}} \int_0^1 h(t, u) e^{-2(u+R-1)} du dy = e^{-2R+2} \int_{\mathcal{G}} \int_0^1 h(t, u) e^{-2u} du dy \\ &= e^{-2R+2} \int_{\mathcal{F}_1 \setminus \mathcal{F}_0} \operatorname{tr}(e^{-t\Delta[\mathbb{H}^n_1]})(x, x) dx. \end{aligned} \tag{4.3.55}$$

Equations 4.3.50, 4.3.51 and 4.3.55 now yield the equality

$$\begin{aligned} \Sigma_{4,3}[R] = e^{-2R+2} \Bigg(&\int_0^1 \int_{\mathcal{F}_1 \setminus \mathcal{F}_0} \operatorname{tr}(e^{-t\Delta_p[M_1]}(x, x))) dx \\ &- \sum_{i=0}^{n} t^{-(n-i)/2} (\operatorname{Vol}(\mathcal{F}_1 \setminus \mathcal{F}_0) a_i + \operatorname{Vol}(\partial \mathcal{F}_1) b_i) \frac{dt}{t} \Bigg) \xrightarrow{R \to \infty} 0. \end{aligned} \tag{4.3.56}$$

Therefore, $\lim_{R\to\infty} \Sigma_4[R] = 0$ follows from Equations 4.3.48, 4.3.49 and 4.3.56 and Corollary 4.3.6, finally finishing the proof of the theorem.

$\square$

Chapter 5

Combinatorial torsion

In this chapter, we will shed light on the *combinatorial* L^2-Torsion invariants that one can assign to a reasonable space and a representation of its fundamental group: Given a finite connected CW-complex K with $\Gamma := \pi_1(K)$ its fundamental group, the associated cellular cochain chain complex $C^*(\widetilde{K})$ of the universal cover $\widetilde{K}$ (cf. Section 5.1) can be seen as the algebraic foundation for all (known) combinatorial L^2-invariants that can reasonably be assigned to K. Crucially, the action of Γ on $\widetilde{K}$ by deck transformations endows $C^*(\widetilde{K})$ with the structure of a free, finitely-generated $\mathbb{Z}[\Gamma]$-module cochain complex. Therefore, given any finite-dimensional, complex representation $\rho : \Gamma \to \mathrm{GL}(V)$, the *twisted* complex $C^*(\widetilde{K}, \rho) := C^*(\widetilde{K}) \otimes_{\mathbb{C}} V$ can be regarded as a free, finitely-generated $\mathbb{C}[\Gamma]$-module cochain complex. Moreover, there is a designated class of Γ-invariant complex basis for $C^*(\widetilde{K}, \rho)$, so called *admissible pairs* (see Definition 5.2.1), each giving rise to a unique inner product structure. Taking the corresponding L^2-completion $C^*_{(2)}(\widetilde{K}, \rho)$ and employing the theory of group von Neumann algebras, one is now able to define for each $p \in \mathbb{N}$ the p-th twisted L^2-Betti number $b^{Top}_{(2),p}(\widetilde{K}, \rho) \in \mathbb{R}_{\geq 0}$. It is invariantly defined, in the sense that it does not depend on the specific choice of inner product structure that was made in the process. Furthermore, provided that (M, ρ) is *det-L^2-acyclic*, meaning that all L^2-Betti numbers vanish and the boundary operators fulfill the technical *determinant class* condition, one can define an L^2-torsion element $T_{(2)}(C^*(\widetilde{K}, \rho)) \in \mathbb{R}_{>0}$ of the associated complex. Although det-L^2-acyclicity is again an invariant property, we show that $T_{(2)}(C^*(\widetilde{K}, \rho))$ itself does, in general, depend on the choice of inner product. However, provided that ρ is *unimodular*, we prove that the *cellular L^2-torsion* $T^{CW}_{(2)}(K, \rho) := T_{(2)}(C^*(\widetilde{K}, \rho))$ is invariantly defined, see Corollary 5.2.10. In this manner, it was also considered in earlier publications, such as in a joint work by Carey, Braverman and Farber [16]. However, unlike in previous publications, we establish that this twisted L^2-torsion is a homeomorphism invariant (Corollary 5.3.12) and, under the assumption that Γ satisfies the *Farrell-Jones conjecture*, even a homotopy invariant (Definition 5.3.14). As such, we obtain a *topological L^2-torsion* $T^{Top}_{(2)}(K, \rho)$. It can be seen as a strict generalization of ordinary topological L^2-torsion $T^{Top}_{(2)}(K, \mathbf{1}_{\mathbb{C}})$, defined with respect to the *trivial representation* $\mathbf{1}_{\mathbb{C}} : \Gamma \to \mathbb{C}$, and first introduced by Mathai in [63]. As a prime application, the latter statement allows one to naturally define a twisted topological L^2-torsion for locally symmetric spaces and unimodular representations of their fundamental group, as will be done in Section 5.5.

5.1 Preliminaries

Throughout this chapter, we will study different quantities attached to a finite, connected CW-complex X and a complex, finite-dimensional representation $\rho : \pi_1(X) \to \mathrm{GL}(V)$. Many of these quantities will prove to be quite rigid under certain transformations of X. As a key intermediate step in all of these instances, the fundamental group $\pi_1(X)$ will always be identified with the deck group $\mathrm{deck}(p_X) =: \Gamma_X$ of a universal covering map $p_X : \widetilde{X} \to X$. Although there is no *canonical* universal covering space, and thus also no canonical representative of $\pi_1(X)$ as the deck group of such covering, the quantities yet to be defined will turn out to be unaffected by different choices of universal cover.
As a first general step, it is essential that we analyze the following situation (all spaces are assumed to be path-connected, locally path-connected and semi-locally simply connected). Given a map $f : X \to Y$ between spaces, elementary covering theory tells us that, picking some $x \in X$ and setting $y := f(x)$, there is a surjection from the set of pairs $\{(\widetilde{x}, \widetilde{y}) : \widetilde{x} \in p_X^{-1}(x) \text{ and } \widetilde{y} \in p_Y^{-1}(y)\}$ to the set of maps $\{\widetilde{f} \colon \widetilde{X} \to \widetilde{Y} : \widetilde{f} \text{ is a lift of } f\}$. Namely, the surjection assigns to each pair $(\widetilde{x}, \widetilde{y})$ the *unique* lift $\widetilde{f} \colon \widetilde{X} \to \widetilde{Y}$ of f with the property that $\widetilde{f}(\widetilde{x}) = \widetilde{y}$. Since the deck groups are in one-to-one correspondence *to*, and act transitively *on* each fiber, *any* choice of lift $\widetilde{f} \colon \widetilde{X} \to \widetilde{Y}$ of f therefore gives rise to a group homomorphism

$$\widetilde{f}_* \colon \Gamma_X \to \Gamma_Y, \tag{5.1.1}$$

whose image is uniquely determined by the identity

$$\widetilde{f}_*(\gamma) \circ \widetilde{f} = \widetilde{f} \circ \gamma \qquad \gamma \in \Gamma_X. \tag{5.1.2}$$

For a different lift $f' : \widetilde{X} \to \widetilde{Y}$, it also follows from the above correspondence that we find an element $\alpha \in \Gamma_X$ such that $\widetilde{f} = f' \circ \alpha$. This is clearly equivalent to $f' = \widetilde{f} \circ \alpha^{-1} = \widetilde{f}_*(\alpha)^{-1} \circ \widetilde{f}$, therefore also equivalent to $\widetilde{f} = \widetilde{f}_*(\alpha) \circ f'$. Consequently, we obtain that $f'_* = \widetilde{f}_*(\alpha)^{-1} \cdot \widetilde{f}_* \cdot \widetilde{f}_*(\alpha) : \Gamma_X \to \Gamma_Y$, i.e. *the homomorphisms f'_* and $\widetilde{f}_*$ are the same up to conjugacy.* Less trivial to see, but still true, is the following statement:

Lemma 5.1.1. *Let $f_t : X \to Y$ be a (not necessarily based) homotopy between path-connected, locally path-connected and semi-locally simply connected spaces. Let $\widetilde{f}_t : \widetilde{X} \to \widetilde{Y}$ be a lift of f_t onto the universal covering space. Then, for all $t \in [0, 1]$, we have*

$$(\widetilde{f}_t)_* = (\widetilde{f}_0)_* \colon \Gamma_X \to \Gamma_Y \tag{5.1.3}$$

Proof. It suffices to show that the assignment $t \mapsto (\widetilde{f}_t)_*$ is locally constant. Assuming the contrary, we find $s \in [0, 1]$ and a sequence $(s_n)_{n\in\mathbb{N}} \subseteq [0, 1]$ with $\lim_{n\to\infty} s_n = s$ and $(\widetilde{f}_{s_n})_* \neq (\widetilde{f}_s)_*$ for all $n \in \mathbb{N}$. This means that for each $n \in \mathbb{N}$, there exists some $\gamma_n \in \mathrm{deck}(p_X)$ with $\delta_n := (\widetilde{f}_s)_*(\gamma_n) \neq (\widetilde{f}_{s_n})_*(\gamma_n) =: \delta'_n$.
By the defining Property 5.1.2 of the deck group elements δ_n and δ'_n, we have $\delta'_n \circ \widetilde{f}_s \neq \delta_n \circ \widetilde{f}_s = \widetilde{f}_s \circ \gamma_n$. Because all of the above maps are lifts of the same map $f_s : X \to Y$ and each such lift is uniquely determined by the image of one point, it follows from the above that for any arbitrary fixed $x \in X$, one has $(\delta'_n \circ \widetilde{f}_s \circ \gamma_n^{-1})(x) \neq \widetilde{f}_s(x)$ for all $n \in \mathbb{N}$.
Since the subset $\bigcup_{n\in\mathbb{N}}(\delta'_n \circ \widetilde{f}_s \circ \gamma_n^{-1})(x) \subseteq Y$ is discrete and does not contain $\widetilde{f}_s(x)$, and since the assignment $t \mapsto \widetilde{f}_t$ is continuous, one finds an $\epsilon > 0$ and open subsets $U \supseteq \bigcup_{n\in\mathbb{N}}(\delta'_n \circ \widetilde{f}_{(s-\epsilon,s+\epsilon)} \circ \gamma_n^{-1})(x)$ and $V \ni \widetilde{f}_s(x)$ with $U \cap V = \emptyset$.
Note that there exists some $N \in \mathbb{N}$, such that $s_n \in (s-\epsilon, s+\epsilon)$ for all $n \geq N$, hence also $(\delta'_n \circ \widetilde{f}_{s_n} \circ \gamma_n^{-1})(x) \in$

U for all $n \geq N$. However, since $\delta'_n = (\widetilde{f}_{s_n})_*(\gamma_n)$, we must have $(\delta'_n \circ \widetilde{f}_{s_n} \circ \gamma_n^{-1})(x) = \widetilde{f}_{s_n}(x) \xrightarrow{n\to\infty} \widetilde{f}_s(x)$, implying that $(\delta'_n \circ \widetilde{f}_{s_n} \circ \gamma_n^{-1})(x) \in V$ for sufficiently large n, a contradiction. $\square$

These results allow us for extract from any given map $f : X \to Y$ a group homomorphsim $f_* := \widetilde{f}_* \colon \Gamma_X \to \Gamma_Y$, where $\widetilde{f} \colon \widetilde{X} \to \widetilde{Y}$ is any lift of f. Note that f_* is only well-defined up to conjugacy, which is why for any representation $\rho : \Gamma_Y \to \mathrm{GL}(V)$, the pull-back $\rho \circ f_*$ is also only well-defined up to conjugacy. However, this will not affect any of the results yet to be stated, as we will show that the invariants yet to be defined are impervious to conjugacy of the underlying representation. From Lemma 5.1.1, we now also deduce:

Corollary 5.1.2. *Let $f, g : X \to Y$ be two homotopic maps. Then, up to conjugacy, we have $f_* = g_* \colon \Gamma_X \to \Gamma_Y$.*

5.2 Cellular torsion of a pair

Let (X, Y) be a CW-pair. Throughout, we assume X to be finite of dimension n and connected. Let $p : \widetilde{X} \to X$ be a universal covering map (fixed throughout, unless stated otherwise) and set $\widetilde{Y} := p^{-1}(Y)$. Then $(\widetilde{X}, \widetilde{Y})$ is also a CW-pair (with the obvious lifted CW-structure). Let

$$\Gamma := \mathrm{deck}(p)$$

be the deck group of the covering projection. Then Γ acts freely and cellularly on the pair $(\widetilde{X}, \widetilde{Y})$. Consider the relative cellular chain complexes $C_*(\widetilde{X}, \widetilde{Y})$ and $C_*(X, Y)$, whose underlying modules can be identified with the free abelian groups generated by all cells of $\widetilde{X}$, respectively X, not entirely contained in $\widetilde{Y}$, respectively Y. Observe that the cellular Γ-action on $(\widetilde{X}, \widetilde{Y})$ endows $C_*(\widetilde{X}, \widetilde{Y})$ with a canonical $\mathbb{Z}[\Gamma]$-module structure, under which the differential $\partial : C_*(\widetilde{X}, \widetilde{Y}) \to C_{*-1}(\widetilde{X}, \widetilde{Y})$ becomes $\mathbb{Z}[\Gamma]$-linear. This allows us to define the *equivariant cellular cochain complex*

$$C^*(\widetilde{X}, \widetilde{Y}) = \hom_{\mathbb{Z}[\Gamma]}(C_*(\widetilde{X}, \widetilde{Y}), \mathbb{Z}[\Gamma]) \tag{5.2.1}$$

with differential $\delta \colon C^*(\widetilde{X}, \widetilde{Y}) \to C^{*+1}(\widetilde{X}, \widetilde{Y})$ the map dual to ∂.

For each $0 \leq k \leq n$, we choose a subset $E_k \subseteq C_k(\widetilde{X}, \widetilde{Y})$ of oriented representatives of k-cells not entirely contained in $\widetilde{Y}$, one for each Γ-orbit. It is easy to see that this forms a $\mathbb{Z}[\Gamma]$-basis for the module $C_k(\widetilde{X}, \widetilde{Y})$. We let $E^k \subseteq C^k(\widetilde{X}, \widetilde{Y})$ be the *dual basis* to E_k, and define

$$\begin{aligned} E &:= \bigcup_{k=0}^{n} E^k \subseteq C^*(\widetilde{X}, \widetilde{Y}) \ , \ j_k := |E^k| = \#\ k\text{-cells in } X \text{ not entirely contained in } Y \ , \\ j &:= |E| = \sum_{k=0}^{n} j_k. \end{aligned} \tag{5.2.2}$$

We call such graded $\mathbb{Z}[\Gamma]$-basis E constructed this way an *admissible basis* for the $\mathbb{Z}[\Gamma]$-cochain complex $C^*(\widetilde{X}, \widetilde{Y})$. In particular, we see that *each $\mathbb{Z}[\Gamma]$-module $C^k(\widetilde{X}, \widetilde{Y})$ is free of rank j_k.*

Now consider additionally a representation $\rho : \Gamma \to \mathrm{GL}(V)$ of Γ over some finite-dimensional complex vector space V. A choice of basis $B \subseteq V$ (which we assume to always be *ordered*, from now on) yields an isomorphism

$$\phi_B : V \to \mathbb{C}^m, \tag{5.2.3}$$

where $m = \dim_{\mathbb{C}}(V)$, identifying B with the (ordered) standard basis of $\mathbb{C}^m$, as well as a unique inner product $\langle\ ,\ \rangle_B$ on V, with respect to which B is an orthonormal basis and so that $\psi_B, (V, \langle\ ,\ \rangle_B) \to (\mathbb{C}^m, \langle\ ,\ \rangle_m)$ is an isometry of Hilbert spaces. Here, as everywhere else, $\langle\ ,\ \rangle_m$ denotes the canonical inner product on $\mathbb{C}^m$, which we will always assume $\mathbb{C}^m$ to be equipped with, unless specifically stated otherwise.

With this in mind, consider the twisted cochain complex

$$C^*(\widetilde{X}, \widetilde{Y}, \rho) \coloneqq C^*(\widetilde{X}, \widetilde{Y}) \otimes_{\mathbb{Z}} V \tag{5.2.4}$$

with twisted differentials

$$\delta \otimes_{\mathbb{Z}} \mathbb{1}_V : C^*(\widetilde{X}, \widetilde{Y}, \rho) \to C^{*+1}(\widetilde{X}, \widetilde{Y}, \rho). \tag{5.2.5}$$

Note that the $\mathbb{C}$-multiplication of the left factor turns each $C^k(\widetilde{X}, \widetilde{Y}, \rho)$ into a complex vector space. This and the representation ρ allow us to define a left-$\mathbb{C}[\Gamma]$-module structure on $C^*(\widetilde{X}, \widetilde{Y}, \rho)$ via the natural $\mathbb{C}[\Gamma]$-extension of the Γ-action, as defined below on elementary tensors

$$g.(\sigma \otimes v) \coloneqq (g.\sigma) \otimes \rho(g)v. \tag{5.2.6}$$

We wish to define an inner product on $C^*(\widetilde{X}, \widetilde{Y}, \rho)$ compatible with the previously defined Γ-action. For this, picking an *admissible* basis $[E, B]$ will be essential:

Definition 5.2.1. Let (X, Y) be a CW-pair with X finite and connected, and let $\rho : \Gamma \to V$ be a finite-dimensional complex representation. A tuple $[E, B]$, where

- $B \subset V$ is a $\mathbb{C}$-basis for V, and
- $E = \bigcup_{k=0}^n E^k$ is an admissible basis for $C^*(\widetilde{X}, \widetilde{Y})$

is called an *admissible pair for* $C^*(\widetilde{X}, \widetilde{Y}, \rho)$.

Let $[E, B]$ be an admissible pair for $C^*(\widetilde{X}, \widetilde{Y}, \rho)$ and consider the unique inner product $\langle\ ,\ \rangle_B$ on V, with respect to which B is an orthonormal basis. Together with E, we define $\langle\ .\ \rangle_{[E,B]}$ to be the unique inner product on the complex vector space $C^*(\widetilde{X}, \widetilde{Y}, \rho)$, with respect to which the set

$$\{g.(e \otimes b) : g \in G, e \in E, b \in B\} \tag{5.2.7}$$

is an orthonormal basis. On the resulting inner product space, denoted by

$$C^*(\widetilde{X}, \widetilde{Y}, \rho, [E, B]), \tag{5.2.8}$$

it is clear that the Γ-action defined in 5.2.6 is by isometries.

We wish to compare the norms coming from two different choices of admissible pairs $[E, B]$ and $[E', B']$. For this, we first observe that

$$||g.e \otimes \rho(g) \cdot v||_{[E,B]} = ||v||_B \tag{5.2.9}$$

for any $v \in V$, any $e \in E$ and any $g \in \Gamma$. Without loss of generality, we may assume that $E^k = \{e_1, \ldots, e_{j_k}\}$ and $(E^k)' = \{e'_1, \ldots, e'_{j_k}\}$ are ordered in such a way that for each $1 \leq i \leq j_k$, $e_i \in E_k$ and $e'_i \in (E_k)'$ lie in the same Γ-orbit. In other words, for each $1 \leq i \leq j_k$, there exists a unique $h_i \in \Gamma$ satisfying

$$e_i = \pm h_i.e'_i. \tag{5.2.10}$$

Consequently, we get for any $e_i \in E^k$ and any $v \in V$, that

$$\begin{aligned}||e_i \otimes v||_{[E',B']} &= ||h_i.e_i' \otimes v||_{[E',B']} = ||\rho(h_i^{-1})v||_{B'} \\ &\leq c \cdot |\rho(h_i^{-1})| \cdot ||v||_B = c \cdot |\rho(h_i^{-1})| \cdot ||e_i \otimes v||_{[E,B]},\end{aligned}$$

where $c > 0$ is an appropriate constant depending only on the equivalent norms $||\ ,\ ||_B$ and $||\ ,\ ||_{B'}$ on the finite-dimensional vector space V and $|\ ,\ |$ is an appropriate norm on $\mathrm{GL}(V)$. Analogously, we get for any $e_i' \in (E^k)'$, that

$$||e_i' \otimes v||_{[E,B]} \leq c \cdot |\rho(h_i)| \cdot ||e_i' \otimes v||_{[E',B']}. \tag{5.2.11}$$

As X is compact, we have $|E| = |E'| < \infty$, and so

$$\sup\{|\rho(h_i)|, |\rho(h_i^{-1})| : 1 \leq i \leq j_k\ ,\ 0 \leq k \leq n\} < \infty, \tag{5.2.12}$$

from which we finally deduce the following important property:

Lemma 5.2.2. *For any two admissible pairs $[E, B]$ and $[E', B']$, the identity map*

$$\mathbb{1} : C^*(\widetilde{X}, \widetilde{Y}, \rho, [E, B]) \to C^*(\widetilde{X}, \widetilde{Y}, \rho, [E', B']) \tag{5.2.13}$$

is a bounded morphism of inner product spaces.

A fixed admissible pair $[E, B]$ also gives rise to isometric isomorphism of inner product spaces and left-$\mathbb{C}[\Gamma]$-modules

$$\Psi_{E,B} : \bigoplus_{k=0}^{n} C^k(\widetilde{X}, \widetilde{Y}, \rho, [E, B]) \to \bigoplus_{k=0}^{n} \bigoplus_{e \in E^k} \mathbb{C}[\Gamma]_e \otimes_{\mathbb{C}} \mathbb{C}^m \cong (\mathbb{C}[\Gamma])^{j \cdot m}, \tag{5.2.14}$$

as the unique $\mathbb{C}[\Gamma]$-linear extension of the assignment

$$\Psi_{E,B}(e \otimes v) := 1_e \otimes \phi_B(v) \in \mathbb{C}[\Gamma]_e \otimes_{\mathbb{C}} \mathbb{C}^m, \tag{5.2.15}$$

with $e \in E$ and $v \in V$, where now

- $\mathbb{C}[\Gamma]_e := \mathbb{C}[\Gamma]$ is a copy of $\mathbb{C}[\Gamma]$ with $1_e = 1$ its unit element, and
- the $\mathbb{C}[\Gamma]$-action on $\bigoplus_{k=0}^{n} \bigoplus_{e \in E^k} \mathbb{C}[\Gamma]_e \otimes_{\mathbb{C}} \mathbb{C}^m \cong (\mathbb{C}[\Gamma])^{j \cdot m}$ is the direct sum of the left-factor actions given by $g.(h \otimes v) := (gh) \otimes v$ on elementary tensors. Moreover,
- the inner product structure on $\bigoplus_{k=0}^{n} \bigoplus_{e \in E^k} \mathbb{C}[\Gamma]_e \otimes_{\mathbb{C}} \mathbb{C}^m$ is the direct sum of the canonical inner products on each factor.

We wish to apply to $C^*(\widetilde{X}, \widetilde{Y}, \rho, [E, B])$ the theory of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes of finite type that we have developed in Section 4.1. However, although $C^*(\widetilde{X}, \widetilde{Y}, \rho, [E, B])$ is already a complex inner product space, equipped with an isometric Γ-action, it is only complete as a metric space if Γ is a finite group. To remedy this, we construct the following twisted cochain complex:

$$C^*_{(2)}(\widetilde{X}, \widetilde{Y}, \rho, [E, B]) := l^2(\Gamma) \otimes_{\mathbb{C}[\Gamma]} C^*(\widetilde{X}, \widetilde{Y}, \rho, [E, B]), \tag{5.2.16}$$

$$\delta^*_\rho := \mathbb{1}_{l^2(\Gamma)} \otimes_{\mathbb{C}[\Gamma]} (\delta \otimes_{\mathbb{Z}} \mathbb{1}_{\mathbb{C}^m}) : C^*_{(2)}(\widetilde{X}, \widetilde{Y}, \rho, [E, B]) \to C^{*+1}_{(2)}(\widetilde{X}, \widetilde{Y}, \rho, [E, B]). \tag{5.2.17}$$

With the choice of $[E, B]$, as before, and equipped with the left $\mathcal{N}(\Gamma)$-module structure inherited from $l^2(\Gamma)$, each $C^k_{(2)}(\widetilde{X}, \widetilde{Y}, \rho, [E, B])$ becomes a finitely generated free Hilbert $\mathcal{N}(\Gamma)$-module, isometrically isomorphic to $l^2(\Gamma)^{j_k} \otimes_{\mathbb{C}} \mathbb{C}^m \cong l^2(\Gamma)^{j_k \cdot m}$ via

$$\Psi^{(2)}_{E,B} \coloneqq \mathbb{1}_{l^2(\Gamma)} \otimes_{\mathbb{C}[\Gamma]} \Psi_{E,B} : C^*_{(2)}(\widetilde{X}, \widetilde{Y}, \rho) \to l^2(\Gamma) \otimes_{\mathbb{C}[\Gamma]} \bigoplus_{i=1}^{j_k} \mathbb{C}[\Gamma]_i \otimes_{\mathbb{C}} \mathbb{C}^m \cong l^2(\Gamma)^{j_k} \otimes_{\mathbb{C}} \mathbb{C}^m . \quad (5.2.18)$$

Furthermore, it is apparent that under the identification $\Psi_{E,B}$, the twisted differential δ^*_ρ becomes an element of $\mathrm{Mat}(l \cdot m, \mathbb{C}[\Gamma])$ with $l \coloneqq \sum_{k=0}^n l_k$. In particular, *$\delta^*_\rho$ is a bounded morphism of Hilbert $\mathcal{N}(\Gamma)$-modules and $C^*_{(2)}(\widetilde{X}, \widetilde{Y}, \rho, [E, B])$ is a Hilbert $\mathcal{N}(\Gamma)$-cochain complex of finite type.* Observe that, although the specific Hilbert space structure on $C^*(\widetilde{X}, \widetilde{Y}, \rho, [E, B])$ depends on the particular choice of $[E, B]$, *the underlying Hilbertian $\mathcal{N}(\Gamma)$-module structure does not.* Namely, it follows directly from Lemma 5.2.2 that the identity map is an isomorphism of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

$$\mathbb{1} : C^*_{(2)}(\widetilde{X}, \widetilde{Y}, \rho, [E, B]) \to C^*_{(2)}(\widetilde{X}, \widetilde{Y}, \rho, [E', B']), \quad (5.2.19)$$

which is of course generally not an isometry. Together with Corollary 4.1.38, Corollary 4.1.32 and Proposition 4.1.40, we conclude that all of the following objects and features are well-defined invariants of the triple (X, Y, ρ).

Definition 5.2.3. Let (X, Y) be a CW-pair and let $\rho : \Gamma \to V$ be a complex, finite-dimensional representation of $\Gamma \coloneqq \pi_1(X)$. Further, let $0 \leq k \leq n$.

- The number $b^{Top}_{(2),k}(X, Y, \rho) \coloneqq b^{(2)}_k(C^*_{(2)}(\widetilde{X}, \widetilde{Y}, \rho, [E, B])) \in \mathbb{R}_{\geq 0}$ is called the *k-th topological L^2-Betti number of* (X, Y, ρ). We set $b^{Top}_{(2),k}(X, \rho) \coloneqq b^{Top}_{(2),k}(X, \emptyset, \rho)$.
- The element $\alpha^{Top}_k(X, Y, \rho) \coloneqq \alpha_k(C^*_{(2)}(\widetilde{X}, \widetilde{Y}, \rho, [E, B])) \in \mathbb{R}_{\geq 0} \cup \{\infty^+\}$ is called the *k-th topological Novikov-Shubin invariant of* (X, Y, ρ). We set $\alpha^{Top}_k(X, \rho) \coloneqq \alpha^{Top}_k(X, \emptyset, \rho)$.
- (X, Y, ρ) has the *determinant class property* if the cochain complex $C^k_{(2)}(\widetilde{X}, \widetilde{Y}, \rho; [B, E])$ is of determinant class, i.e. if for any $0 \leq k \leq n$ and all boundary morphisms $\delta^k_\rho : C^k_{(2)}(\widetilde{X}, \widetilde{Y}, \rho, [E, B]) \to C^{k+1}_{(2)}(\widetilde{X}, \widetilde{Y}, [E, B])$, we have
$$\det\nolimits_\Gamma(\delta^k_\rho) \neq 0. \quad (5.2.20)$$
- (X, Y, ρ) is *det-L^2-acyclic* if it has the determinant class property and $b^{Top}_{(2),k}(X, Y, \rho) = 0$ for all $0 \leq k \leq n$. We say that the tuple (X, ρ) is det-L^2-acyclic if the triple $(X, \emptyset, \rho)$ is det-L^2-acyclic.

Here, $[E, B]$ is an arbitrary choice of admissible pair for (X, Y, ρ).

The index *Top* for $b^{Top}_{(2),*}$ and a^{Top} instead of *CW* is used in order to underline the fact that these quantities are in fact *homotopy invariants* of spaces admitting a finite CW-structure, as shown in Corollary 5.3.7. We remark that, although the det-L^2-acyclicity property is in general hard to verify, it is the immediate consequence of stronger properties that are somewhat easier to verify, and will be satisfied in many of our applications. The whole situation is summarized in the next lemma:

Lemma 5.2.4. *Let (X, Y) be a CW-pair and let $\rho : \Gamma \to V$ be a finite-dimensional, complex representation. Consider the following properties*

1. *The cellular cochain complex* $C^*(\widetilde{X},\widetilde{Y})$ *of* $\mathbb{Z}[\Gamma]$*-modules is* ***algebraically acyclic****, that is*

$$\ker(\delta^*) = \operatorname{im}(\delta^*). \tag{5.2.21}$$

2. *For each* $0 \leq k \leq n$*, we have both* $b^{Top}_{(2),k}(X,Y,\rho) = 0$ *and* $\alpha^{Top}_k(X,Y,\rho) > 0$*.*

3. (X,Y,ρ) *is det-*L^2*-acyclic.*

Then

$$(1) \Rightarrow (2) \Rightarrow (3). \tag{5.2.22}$$

Proof. $(1) \Rightarrow (2)$: Because $C^*(\widetilde{X},\widetilde{Y})$ is algebraically acyclic and it is a *free* $\mathbb{Z}[\Gamma]$-module-cochain complex, it is also contractible. This means that there exists a *chain contraction*

$$c^* : C^*(\widetilde{X},\widetilde{Y}) \to C^{*-1}(\widetilde{X},\widetilde{Y}), \tag{5.2.23}$$

a $\mathbb{Z}[\Gamma]$-linear map satisfying

$$c^{n+1} \circ \delta^n + \delta^{n-1} \circ c^n = \mathbb{1}_{C^n(\widetilde{X},\widetilde{Y})} \tag{5.2.24}$$

for every $n \in \mathbb{N}$. For any admissible pair $[E,B]$ of (X,Y,ρ), the map

$$c^*_\rho \coloneqq \mathbb{1}_{l^2(\Gamma)} \otimes_{\mathbb{C}[\Gamma]} (c \otimes_{\mathbb{C}} \mathbb{1}_V) : C^*_{(2)}(\widetilde{X},\widetilde{Y},\rho,[E,B]) \to C^{*-1}_{(2)}(\widetilde{X},\widetilde{Y},\rho,[E,B]) \tag{5.2.25}$$

is bounded and $\mathbb{C}[\Gamma]$-linear. Moreover, as the two functors involved in transforming $C^*(\widetilde{X},\widetilde{Y})$ into $C^*_{(2)}(\widetilde{X},\widetilde{Y},\rho,[E,B])$ are both additive, we must have

$$c^{n+1}_\rho \circ \delta^n_\rho + \delta^{n-1}_\rho \circ c^n_\rho = \mathbb{1}_{C^n_{(2)}(\widetilde{X},\widetilde{Y},\rho,[E,B])} \tag{5.2.26}$$

for every $n \in \mathbb{N}$. In other words, $C^*_{(2)}(\widetilde{X},\widetilde{Y},\rho,[E,B])$ is contractible as a cochain complex of Hilbert $\mathcal{N}(\Gamma)$-modules. The conclusion $(1) \Rightarrow (2)$ then follows from [54, Lemma 2.18].
$(2) \Rightarrow (3)$ follows from Proposition 4.1.24. ☐

Just like the Betti-numbers and Novikov-Shubin invariants, the next number will involve a choice of $[E,B]$ and, unlike the previous quantities, will in general depend on the choice of $[E,B]$.

Definition 5.2.5. Let (X,Y) be a CW-pair, let $\rho : \Gamma \to V$ be a representation and let $[E,B]$ be an admissible pair for (X,Y,ρ). Suppose that $(X,Y,\rho,[E,B])$ is det-L^2-acyclic. Then, we define the *cellular* L^2*-torsion* $T^{CW}_{(2)}(X,Y,\rho)[E,B]$ *the quintuple* $(X,Y,\rho,[E,B])$ as

$$\log(T^{CW}_{(2)}(X,Y,\rho))[E,B] \coloneqq \log\Big(T(C^*_{(2)}(\widetilde{X},\widetilde{Y},\rho,[E,B]))\Big) = \sum_{k=0}^{n}(-1)^{k+1}\log(\det\nolimits_\Gamma(\delta^k_\rho)) \in \mathbb{R}\,. \tag{5.2.27}$$

As before, we set $T^{CW}_{(2)}(X,\rho)[E,B] \coloneqq T^{CW}_{(2)}(X,\emptyset,\rho)[E,B]$.

As hinted towards in the introduction, these numerical invariants are unaffected when replacing a representation ρ by some conjugate.

Lemma 5.2.6 (Invariance under conjugacy)**.** *Let* (X,Y) *be a* CW*-pair, and let* $\rho,\rho' : \Gamma \to V$ *be two conjugate representations, that is, there exists some* $\alpha \in \mathrm{GL}(V)$*, such that* $\rho' = \alpha^{-1}\cdot\rho\cdot\alpha$*. Then, for any* $0 \leq k \leq n$*, we have*

1. $b^{Top}_{(2),k}(X,Y,\rho) = b^{Top}_{(2),k}(X,Y,\rho')$,

2. $\alpha^{Top}_k(X,Y,\rho) = \alpha^{Top}_k(X,Y,\rho')$,

3. (X,Y,ρ) *is det-*L^2*-acyclic if and only if* (X,Y,ρ') *is det-*L^2*-acyclic,*

4. *In case that* (X,Y,ρ) *is det-*L^2*-acyclic, we have for any choice of admissible pair* $[E,B]$*, that* $T^{CW}_{(2)}(X,Y,\rho)[E,B] = T^{CW}_{(2)}(X,Y,\rho')[E,B]$.

Proof. The isomorphism $F^* : C^*(\widetilde{X},\widetilde{Y},\rho) \to C^*(\widetilde{X},\widetilde{Y},\rho')$, defined on elementary tensors $e\otimes v$ via $F(e\otimes v) \coloneqq e\otimes(\alpha\cdot v\cdot\alpha^{-1})$ is obviously Γ-equivariant, and extends in the obvious fashion to an isomorphism of Hilbert $\mathcal{N}(\Gamma)$-modules $F^* : C^*_{(2)}(\widetilde{X},\widetilde{Y},\rho,[E,B]) \to C^*_{(2)}(\widetilde{X},\widetilde{Y},\rho',[E,B])$. To verify $1-3$, we now simply apply Proposition 4.1.32 and Corollary 4.1.38. To verify 4, we use Proposition 4.1.40 together with the fact that $\det_\Gamma(F^k) = 1$ for all $0 \le k \le n$. □

Our intermediate goal is to provide a large class of examples (X,Y,ρ), for which such an L^2-torsion can be defined also independently of a choice $[E,B]$. To do so, we first need to quantify the quotient $\log\left(\frac{T^{CW}_{(2)}(X,Y,\rho)[E,B]}{T^{CW}_{(2)}(X,Y,\rho)[E',B']}\right)$ for two distinct admissible pairs $[E,B],[E',B']$ of (X,Y,ρ). The following auxiliary lemma is essential:

Lemma 5.2.7. *Let* $[E,B]$ *and* $[E',B']$ *be two admissible pairs for* (X,Y,ρ) *and let* $M^{B'}_B \coloneqq \psi_{B'}\circ\psi_B^{-1} \in \mathrm{GL}_m(\mathbb{C})$ *be the basechange isomorphism from* B *to* B'*. For fixed* $0\le k\le n$*, denote by*

$$\mathbb{1} : C^k_{(2)}(\widetilde{X},\widetilde{Y},\rho,[E,B]) \to C^k_{(2)}(\widetilde{X},\widetilde{Y},\rho,[E',B'])$$

the identity map to the underlying Hilbertian $\mathcal{N}(\Gamma)$*-module. Then, for each* $1\le i\le j_k$*, there exists group elements* $g_i\in\Gamma$*, such that*

$$\det_\Gamma(\mathbb{1}) = \prod_{i=1}^{j_k} |\det(M^{B'}_B)|\cdot|\det(\rho(g_i))| = |\det(M^{B'}_B)|^{j_k}\cdot\prod_{i=1}^{j_k}|\det(\rho(g_i))|,$$

where det *denotes the usual determinant of endomorphisms over finite-dimensional complex vector spaces.*

Proof. Consider the following diagram

$$\bigoplus_{i=1}^{j_k} l^2(\Gamma)_i\hat{\otimes}\,\mathbb{C}^m \xrightarrow{(\phi^{(2)}_{[E,B]})^{-1}} C^k_{(2)}(\widetilde{X},\widetilde{Y},\rho,[E,B]) \xrightarrow{\mathbb{1}} C^k_{(2)}(\widetilde{X},\widetilde{Y},\rho,[E',B']) \xrightarrow{\phi^{(2)}_{[E',B']}} \bigoplus_{i=1}^{j_k} l^2(\Gamma)_i\hat{\otimes}\,\mathbb{C}^m\,. \tag{5.2.28}$$

Here, both the left-hand and the right-hand arrow are isometries of Hilbert $\mathcal{N}(\Gamma)$-modules. Setting $F^{[E',B']}_{[E,B]} \coloneqq \phi^{(2)}_{[E',B']}\circ\mathbb{1}\circ(\phi^{(2)}_{[E,B]})^{-1}$, we therefore obtain by Proposition 4.1.14 (1) and (2), that

$$\det_\Gamma(\mathbb{1}) = \det_\Gamma(F^{[E',B']}_{[E,B]}). \tag{5.2.29}$$

For $1\le i\le j_k$, $v\in\mathbb{C}^m$ arbitrary and $e\in\Gamma$ the unit, consider the tensor $e\otimes v\in l^2(\Gamma)_i\otimes\mathbb{C}^m$. Then, one verifies by direct computation that there exists some element $h_i\in\Gamma$ and some $1\le l_i\le j_k$, such that

$$F^{[E',B']}_{[E,B]}(e_i\otimes v) = h_i\otimes(\psi_{B'}\circ\rho(h_i^{-1})\circ\psi_B^{-1}(v)) \in l^2(\Gamma)_{l_i}\otimes\mathbb{C}^m, \tag{5.2.30}$$

and $l_i \ne l_{i'}$ for $i\ne i'$ (compare with the arguments preceding Equation 5.2.10). After reordering summands (which is done via some matrix of determinant 1), we may assume without loss of generality that $l_i = i$ for all $1 \leq i \leq j_k$. Consequently, with respect to the direct sum decomposition described above, we get that

$$F_{[E,B]}^{[E',B']} = \begin{pmatrix} i^*\big(r_{h_1} \otimes \psi_{B'} \circ \rho(h_1^{-1}) \circ \psi_B^{-1}\big) & 0 & 0 & \dots \\ 0 & i^*\big(r_{h_2} \otimes \psi_{B'} \circ \rho(h_2^{-1}) \circ \psi_B^{-1}\big) & 0 & \dots \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & \dots & i^*\big(r_{h_{j_k}} \otimes \psi_{B'} \circ \rho(h_{j_k}^{-1}) \circ \psi_B^{-1}\big). \end{pmatrix}$$

Here, $r_{h_i} \in \mathfrak{B}_\Gamma(l^2(\Gamma)_i)$ denotes right-multiplication by the group element h_i, while $i^*(r_{h_i} \otimes \psi_{B'} \circ \rho(h_1^{-1}) \circ \psi_B^{-1})$ denotes the extension onto $\mathfrak{B}_\Gamma(l^2(\Gamma)_i \hat{\otimes} \mathbb{C}^m)$ of the tensor product of morphisms inside the parentheses. Applying Proposition 4.1.14 (1) and (3), Lemma 4.1.15 (with $\Gamma \coloneqq \Gamma$ and $\Lambda \coloneqq \{0\} \subseteq \Gamma$), along with Lemma 4.1.12, and setting $g_i \coloneqq h_i^{-1}$, we then compute

$$\begin{aligned} \det{}_\Gamma(F_{[E,B]}^{[E',B']}) &= \prod_{i=1}^{j_k} \det{}_\Gamma\left(i^*(r_{g_i^{-1}} \otimes \psi_{B'} \circ \rho(g_i) \circ \psi_B^{-1})\right) = \prod_{i=1}^{j_k} \det{}_\Gamma(r_{g_i^{-1}}) \cdot \det{}_{\{0\}}(\psi_{B'} \circ \rho(g_i) \circ \psi_B^{-1}) \\ &= \prod_{i=1}^{j_k} \det{}_{\{0\}}(\psi_{B'} \circ \rho(g_i) \circ \psi_B^{-1}) = \prod_{i=1}^{j_k} |\det(\psi_{B'} \circ \rho(g_i) \circ \psi_B^{-1})| = \prod_{i=1}^{j_k} |\det(M_B^{B'})| \cdot |\det(\rho(g_i))|. \end{aligned}$$

□

Together with Proposition 4.1.40, we arrive at the central comparison result of this section.

Proposition 5.2.8. *Let $[E,B]$ and $[E',B']$ be two choices of admissible pairs on $C^*_{(2)}(\widetilde{X}, \widetilde{Y}, \rho)$. Assume that (X,Y,ρ) is det-L^2-acyclic. Then, for each $0 \leq k \leq n$ and each $0 \leq i \leq j_k$, there exist elements $g_{ik} \in \Gamma$, such that*

$$\log\left(\frac{T_{(2)}^{CW}(X,Y,\rho)[E,B]}{T_{(2)}^{CW}(X,Y,\rho)[E',B']}\right) = \sum_{k=0}^{n} \sum_{i=1}^{j_k} (-1)^k \log(|\det(\rho(g_{ik}))|). \tag{5.2.31}$$

Proof. Using Lemma 5.2.7, together with Proposition 4.1.40 and the fact that ρ is L^2-acyclic, we obtain

$$\log\left(\frac{T_{(2)}^{CW}(X,Y,\rho)[E,B]}{T_{(2)}^{CW}(X,Y,\rho)[E',B']}\right) = \sum_{k=0}^{n} (-1)^k j_k \log(|\det(M_B^{B'})|) + \sum_{k=0}^{n} \sum_{i=1}^{j_k} (-1)^k \log(|\det(\rho(g_{ik}))|) \tag{5.2.32}$$

for appropriate $g_{ik} \in \Gamma$. Again, because ρ is L^2-acyclic, we have $0 = \sum_{k=0}^{n} (-1)^k b_{(2),k}^{Top}(X,Y,\rho) = \dim(\rho) \cdot \chi(X,Y)$. Therefore

$$0 = \chi(X,Y) = \sum_{k=0}^{n} (-1)^k j_k. \tag{5.2.33}$$

Combined with the previous equation, the result now follows. □

The result highlights the fact that $T_{(2)}^{CW}(X,Y,\rho)[E,B]$ does not depend on the choice of basis B. However, with no conditions on the representation ρ itself, choosing different lifts of cells E and E' might produce a non-trivial anomaly of torsion elements, which is due to the the possibly non-trivial sum $\sum_{k=0}^{n} \sum_{i=1}^{j_k} (-1)^k \log(|\det(\rho(g_{ik}))|)$. In order to achieve vanishing of this sum, we must also impose additional conditions on ρ, motivating the next definition.

Definition 5.2.9. A finite-dimensional, complex representation $\rho : \Gamma \to V$ is called *unimodular* if

$$|\det(\rho(g))| = 1 \; \forall g \in \Gamma. \tag{5.2.34}$$

Summarizing this investigation now readily yields:

Corollary 5.2.10. *Let (X,Y) be a CW-pair and let $\rho : \Gamma \to V$ be a unimodular representation, so that (X,Y,ρ) is det-L^2-acyclic. Then, for any two bases $B, B' \subseteq V$ and any two admissible bases E, E' of $C^*(\widetilde{X},\widetilde{Y})$, we get $T^{CW}_{(2)}(X,Y,\rho)[E,B] = T^{CW}_{(2)}(X,Y,\rho)[E',B']$. We call*

$$T^{CW}_{(2)}(X,Y,\rho) \coloneqq T^{CW}_{(2)}(X,Y,\rho)[E,B] \tag{5.2.35}$$

the ***cellular L^2-torsion*** *of the triple (X,Y,ρ).*

Remark 5.2.11. Notice that the existence of a det-L^2-acyclic representation $\rho : \Gamma \to \mathrm{GL}(V)$ on (X,Y) requires the vanishing of the relative Euler Characteristic $\chi(X,Y)$. That is why we will assume $\chi(X,Y) = 0$ from now on, until the end of the chapter and unless explicitly stated otherwise.

Remark 5.2.12. It should be said that cellular L^2-torsion, at least the way it is defined above, is studied almost exclusively for *unitary* representations $\rho : \Gamma \to O(V)$, cf. [23], [22], [21], [55], [54] or [46]. In fact, all L^2-invariants of pairs $(X,\rho) = (X,\emptyset,\rho)$ with ρ an arbitrary finite-dimensional, unitary representation can directly be derived from the *basic* pair $(X,\mathbb{1}_{\mathbb{C}})$, where $\mathbb{1}_{\mathbb{C}} : \Gamma \to \mathbb{C}^\times$ is the trivial representation. Namely, in this instance, it is well-known (see e.g [52, Theorem 4.1]) that

1. one has $b^{Top}_{(2),k}(X,\rho) = \dim(\rho) \cdot b^{Top}_{(2),k}(X,\mathbb{1}_{\mathbb{C}})$, and
2. (X,ρ) is det-L^2-acyclic if and only if $(X,\mathbb{1})$ is det-L^2-acyclic. In this case, one obtains that

$$T^{CW}_{(2)}(X,\rho) = \dim(\rho) \cdot T^{CW}_{(2)}(X,\mathbb{1}_{\mathbb{C}}). \tag{5.2.36}$$

Remark 5.2.13. For any fixed $t \in \mathbb{R}^+$ and any group homomorphism $\phi : \Gamma \to (\mathbb{R},+)$, we obtain a 1-dimensional representation $\rho_\phi[t] : \Gamma \to \mathbb{C}^\times$ over $\mathbb{C}$ via $\gamma \mapsto t^{\phi(\gamma)}$, which is of course in general not unimodular. However, under the assumption that $\chi(X,Y) = 0$ and that the triple $(X,Y,\rho_\phi[t])$ is det-L^2-acyclic, Proposition 5.2.8 tells us that for any two admissible basis pairs $[E,B]$, $[E',B']$ the corresponding L^2-torsion elements satisfy $T^{CW}_{(2)}(X,Y,\rho_\phi[t])[E,B] = t^r \cdot T^{CW}_{(2)}(X,Y,\rho_\phi[t])[E',B']$ for some $r \in \mathbb{R}$. Thus, if one declares two functions in t to be equivalent if they satisfy a relation as above, one obtains a well-defined equivalence class $T^{CW}_{(2)}(X,Y,\rho_\phi[t])$ of cellular torsion functions in t, independent of the choice of admissible basis. If $(X,Y) = (X,\emptyset)$ is a CW-structure on a prime 3-manifold with empty or toroidal boundary, the corresponding torsion function is called the *L^2-Alexander torsion function.* Originally introduced by Dubois, Friedl and Lück, the study of its properties as well as relations to other 3-manifold invariants has attracted some attention in recent years, see for example [31] and [32].

5.3 Topological torsion

So far, we have been able to construct an invariant $T^{CW}_{(2)}(X,Y,\rho)$, called the *cellular L^2-torsion* of a finite CW-pair (X,Y) with $\chi(X,Y) = 0$ and a finite-dimensional complex representation $\rho : \pi_1(X) \coloneqq \Gamma \to \mathrm{GL}(V)$. This should raise the question of whether in certain instances, the dependency of a specific CW-structure can be dropped entirely. For example, one could ask whether for a given compact topological space M admitting two CW-structures X and Y, one always has $T^{CW}_{(2)}(X,\rho) = T^{CW}_{(2)}(Y,\rho)$. This would allows us to define a *topological L^2-torsion* $T^{Top}_{(2)}(M,\rho)$ for any compact topological space

M admitting *some* CW-structure and a finite-dimensional complex, det-L^2-acyclic unimodular representation $\rho\colon \pi_1(X) \to \mathrm{GL}(V)$. Indeed, we will show the more general statement that for any two finite CW-complexes X, Y with $\chi(X) = 0$ and a (not necessarily cellular) homeomorphism $f : X \to Y$, one has $T^{CW}_{(2)}(X, \rho \circ f_*) = T^{CW}_{(2)}(Y, \rho)$, where $\rho : \pi_1(Y) \to \mathrm{GL}(V)$ is a representation as before and $f_* : \pi_1(X) \to \pi_1(Y)$ is the induced map on fundamental groups.
A fundamental tool for dealing with this and related kinds of problems comes in form of a classic invariant, the so-called *Whitehead Torsion*, which will be the subject of the next section, first defined for cochain complexes, then for cellular homotopy equivalences.

5.3.1 Whitehead torsion of a cochain complex

Let R be a ring with unit 1_R. Throughout, we also assume that R has the *IBN* (invariant basis number) property, that is, we have $R^m \cong R^n$ if and only if $n = m$. Let $n \in \mathbb{N}$ and let $\mathrm{GL}(n, R)$ be the group of invertible $n \times n$-matrices over R. We obtain an induced diagram of embeddings

$$R^\times = \mathrm{GL}(1, R) \to \mathrm{GL}(2, R) \to \mathrm{GL}(3, R) \to \dots, \tag{5.3.1}$$

where each arrow sends a matrix $A \in \mathrm{GL}(n, R)$ to the matrix

$$\begin{pmatrix} A & 0 \\ 0 & 1_R \end{pmatrix} \in \mathrm{GL}(n+1, R).$$

We set $\mathrm{GL}(R) \coloneqq \varinjlim \mathrm{GL}(n, R)$ to be the colimit group of this sequence. Let $E(R)$ be the subgroup generated by all *elementary matrices* of $\mathrm{GL}(R)$, that is, all matrices of the form $\mathbb{1} + r \cdot E_{ij}$ for $i \neq j$, with $r \in R$. Here, E_{ij} is the matrix with every entry zero except for the (i, j)'th entry which equals 1_R. Observe that every upper, respectively lower triangular matrix with $1_R's$ on the diagonal lies in $E(R)$. More generally, it is well-known, see for example [66, Lemma 1.1], that

$$E(R) = [\mathrm{GL}(R) : \mathrm{GL}(R)]. \tag{5.3.2}$$

Define the *first K-group of R* by

$$K_1(R) \coloneqq \mathrm{GL}(R)/E(R), \tag{5.3.3}$$

as well as the *reduced first K-group of R*

$$\widetilde{K}_1(R) \coloneqq K_1(R)/ < -1_R > . \tag{5.3.4}$$

where $< -1_R >$ is the subgroup generated by the image of $-1_R \in R^\times \cong \mathrm{GL}(1, R)$ in $K_1(R)$.
Consider the special case $R = \mathbb{Z}[\Gamma]$ for some countable group Γ. Observe that there is a natural homomorphism $\iota : \Gamma \to \widetilde{K}_1(\mathbb{Z}[\Gamma])$ with image

$$\iota(\Gamma) \trianglelefteq \widetilde{K_1}(\mathbb{Z}[\Gamma]), \tag{5.3.5}$$

which arises by post-composing the projection $\mathrm{GL}(\mathbb{Z}[\Gamma]) \twoheadrightarrow \widetilde{K}_1(\mathbb{Z}[\Gamma])$ to the canonical map $\widetilde{\iota} : \Gamma \to \mathrm{GL}(\mathbb{Z}[\Gamma])$, identifying each element $\gamma \in \Gamma$ with the invertible 1×1 matrix (γ).
A cochain complex $\dots \xrightarrow{c^{-1}} C^{-1} \xrightarrow{c^0} C^0 \to C^1 \xrightarrow{c^1} \dots$, where each C^n is a finitely generated free R-module and each map c^n is R-linear is called an *free R-module cochain complex.* Any such cochain complex gives to a unique differential graded R-module (C^*, c^*), where $C^* \coloneqq \bigoplus_{k \in \mathbb{Z}} C^k$ with degree-1

differential $c^* := \bigoplus_{k\in\mathbb{Z}} c^k : C^* \to C^{*+1}$. From now on, we will identify a free R-module cochain complex with its corresponding differential graded R-module (C^*, c^*). We call (C^*, c^*) *finite* if there exists some $K \in \mathbb{N}$ such that $C^k = 0$ whenever $|k| \geq K$.

We call (C^*, c^*) *based* if it is equipped with an (ordered) *graded* R-basis $E \subseteq C^*$, such that $E^k := E \cap C^k$ is an ordered R-basis for the free submodule C^k. Via the fixed choice of E, we find for each $k \in \mathbb{N}$ some $j_k \in \mathbb{N}_0$, so that we can identify C^k with R^{j_k} and consequently $c^k : R^{j_k} \to R^{j_{k+1}}$ with an appropriate matrix over R (acting by right multiplication). We will write $C^*[E]$ for the corresponding based cochain complex Let $(C^*, c^*) = (C^*[E], c^*)$ now be such a based cochain complex (with fixed basis E throughout) that is additionally *acyclic*, i.e. satisfies $\ker(c^*) = \operatorname{im}(c^*)$. Then, one can define a torsion element $\tau(C^*) \in \widetilde{K}_1(R)$, which can be constructed as follows: Since (C^*, c^*) is acyclic, of finite length and the underlying R-modules are free, there exists an R-linear chain contraction $\gamma^* : C_* \to C_{*-1}$, i.e. a left-multiplication matrix satisfying $\gamma^{*+1}c^* + c^{*-1}\gamma^* = \mathbb{1}_{C_*}$. Set $C^{odd} := \bigoplus_{k\in\mathbb{N}} R^{j_{2k+1}}$ and $C^{even} := \bigoplus_{k\in\mathbb{N}} R^{j_{2k}}$ and observe that we have matrices

$$(\gamma^{2*+1} + c^{2*+1}) : C^{odd} \to C^{even},$$
$$(\gamma^{2*} + c^{2*}) : C^{even} \to C^{odd}.$$

Then $(\gamma^{2*+1} + c^{2*+1})(\gamma^{2*} + c^{2*}) - \mathbb{1}_{C^{even}} = \gamma^{2*+1}\gamma^{2*}$ and $(\gamma^{2*} + c^{2*})(\gamma^{2*+1} + c^{2*+1}) - \mathbb{1}_{C^{odd}} = \gamma^{2*}\gamma^{2*+1}$. It follows that the endomorphisms $(\gamma^{2*+1} + c^{2*+1})(\gamma^{2*} + c^{2*})$ and $(\gamma^{2*} + c^{2*})(\gamma^{2*+1} + c^{2*+1})$ are upper triangular matrices with 1's on their diagonal, and therefore determine elements in $E(R)$. Moreover, the matrix $(\gamma^{2*+1} + c^{2*+1})$ must be invertible as well, and therefore determines an element in $\mathrm{GL}(R)$.

Definition 5.3.1. The **Whitehead torsion** of C^* is then defined to be the class

$$\tau(C^*) := [\gamma^{2*+1} + c^{2*+1}] = -[\gamma^{2*} + c^{2*}] \in \widetilde{K}_1(R). \tag{5.3.6}$$

Since there is no *canonical* choice of chain contraction, we must show that $[\gamma^{2*+1} + c^{2*+1}] = [\delta^{2*+1} + c^{2*+1}]$ for any other choice of chain contraction $\delta^* : C^* \to C^{*-1}$. For this purpose, define

$$\mu^* := (\gamma^{*-1} - \delta^{*-1})\gamma^* : C^* \to C^{*-2}.$$

Analogously as before, one computes that both $\mathbb{1}_{C^{odd}} + \mu^{2*+1}$ and $(\gamma^{2*+1} + c^{2+1})(\mathbb{1}_{C^{odd}} + \mu^{2*+1})(\delta^{2*} + c^{2*})$ are triangular square matrices with 1's on the diagonal, and therefore elements of $E(R)$, from which follows that

$$[\gamma^{2*+1} + c^{2+1}] = -[\delta^{2*} + c^{2*}] = [\delta^{2*+1} + c^{2*+1}].$$

5.3.2 Whitehead torsion of a homotopy equivalence

Let X and Y be two connected CW-complexes, let $f : X \to Y$ be a cellular map between them, and let $f_* : C_*(X) \to C_*(Y)$ be the map of chain complexes induced by f. The **mapping cylinder** $\mathrm{Cyl}_*(f)$ is defined to be the $\mathbb{Z}$-module chain complex with k-th differential

$$C_k(Y) \oplus C_k(X) \oplus C_{k-1}(X) \xrightarrow{\begin{pmatrix} \delta_k^Y & 0 & f_{k-1} \\ 0 & \delta_k^X & \mathbb{1} \\ 0 & 0 & -\delta_{k-1}^X \end{pmatrix}} C_{k-1}(Y) \oplus C_{k-1}(X) \oplus C_{k-2}(X). \tag{5.3.7}$$

Further, the **mapping cone** $\mathrm{Cone}_*(f)$ (relative to X) is defined to be the quotient of $\mathrm{Cyl}_*(f)$ by the copy of $C_*(X)$. Therefore, its k-th differential is of the form

$$C_k(Y) \oplus C_{k-1}(X) \xrightarrow{\begin{pmatrix} \delta_k^Y & f_{k-1} \\ 0 & -\delta_{k-1}^X \end{pmatrix}} C_{k-1}(Y) \oplus C_{k-2}(X). \tag{5.3.8}$$

There are obvious split exact sequences of $\mathbb{Z}$-chain complexes

$$0 \to C_*(X) \xrightarrow{i_*^X} \mathrm{Cyl}_*(f) \xrightarrow{p_*^X} \mathrm{Cone}_*(f) \to 0, \tag{5.3.9}$$

$$0 \to C_*(Y) \xrightarrow{i_*^Y} \mathrm{Cyl}_*(f) \xrightarrow{p_*^Y} \mathrm{Cone}_*(\mathbb{1}_X) \to 0. \tag{5.3.10}$$

Observe that the map

$$\gamma_* \coloneqq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : \mathrm{Cone}_*(\mathbb{1}_X) \to \mathrm{Cone}_{*+1}(\mathbb{1}_X)$$

defines a chain contraction on $\mathrm{Cone}_*(\mathbb{1}_X)$. Thus, $\mathrm{Cone}_*(\mathbb{1}_X)$ is always algebraically acyclic. More generally, the following correspondence is well-known:

Proposition 5.3.2. [66, Section 7] *Let $f : X \to Y$ be a cellular map between CW-complexes. Then, if f is a homotopy equivalence,* $\mathrm{Cone}_*(f)$ *is algebraically acyclic.*

Now assume that X and Y are finite, connected CW-complexes and that $f : X \to Y$ is a cellular homotopy equivalence. Let $p_X : \widetilde{X} \to Y$ and $p_Y : \widetilde{Y} \to Y$ be the corresponding universal covering maps and set

$$\Gamma \coloneqq \mathrm{deck}(p_X)\ ,\ \Gamma_Y \coloneqq \mathrm{deck}(p_Y). \tag{5.3.11}$$

Then f lifts (non-uniquely) to a cellular homotopy equivalence $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$, further inducing a group isomorphism $\widetilde{f}_* : \Gamma \to \Gamma_Y$ as defined in the beginning of this chapter. In particular, Γ acts both cellularly on $\widetilde{X}$ and (by push forward through f_*) on $\widetilde{Y}$ with finitely many orbits. Permutation of cells then gives rise to both on $C_*(\widetilde{X})$ *and* on $C_*(\widetilde{Y})$ a structure of free and finite $\mathbb{Z}[\Gamma]$-module chain complexes, so that the map $\widetilde{f}_* : C_*(\widetilde{X}) \to C_*(\widetilde{Y})$ becomes $\mathbb{Z}[\Gamma]$-linear. By the definitions laid out in 5.3.7 and 5.3.8, each of the three chain complexes $\mathrm{Cyl}_*(\widetilde{X})$, $\mathrm{Cone}_*(\widetilde{f})$ and $\mathrm{Cone}_*(\mathbb{1}_{\widetilde{X}})$ inherits the structure of a free and finite $\mathbb{Z}[\Gamma]$-module chain complex, so that, with respect to those structures, the maps from 5.3.9 and 5.3.10 are $\mathbb{Z}[\Gamma]$-linear.
We denote by $C^*(\widetilde{X})$, $C^*(\widetilde{Y})$, $\mathrm{Cyl}^*(\widetilde{f})$, $\mathrm{Cone}^*(\widetilde{f})$ and $\mathrm{Cone}^*(\mathbb{1}_{\widetilde{X}})$ the free and finite $\mathbb{Z}[\Gamma]$-*co*chain complexes that are obtained by applying the contravariant *dualization* functor $\hom_{\mathbb{Z}[\Gamma]}(\ .\ ,\mathbb{Z}[\Gamma])$ to the corresponding chain complexes. Observe that, since $\hom_{\mathbb{Z}[\Gamma]}(\ .\ ,\mathbb{Z}[\Gamma])$ is an *additive* functor, the split exact sequences 5.3.9 and 5.3.10 dualize to split exact sequences of $\mathbb{Z}[\Gamma]$-cochain complexes

$$0 \to \mathrm{Cone}^*(\widetilde{f}) \xrightarrow{p^*_{\widetilde{X}}} \mathrm{Cyl}^*(\widetilde{f}) \xrightarrow{i^*_{\widetilde{X}}} C^*(\widetilde{X}) \to 0, \tag{5.3.12}$$

$$0 \to \mathrm{Cone}^*(\mathbb{1}_{\widetilde{X}}) \xrightarrow{p^*_{\widetilde{Y}}} \mathrm{Cyl}^*(\widetilde{f}) \xrightarrow{i^*_{\widetilde{Y}}} C^*(\widetilde{Y}) \to 0. \tag{5.3.13}$$

Let $E_X = \bigcup_{k=0}^{n_X} E_X^k$, respectively $E_Y = \bigcup_{k=0}^{n_Y} E_Y^k$ be a fixed admissible $\mathbb{Z}[\Gamma]$-basis for the cellular complex $C^*(\widetilde{X})$, respectively a fixed admissible $\mathbb{Z}[\Gamma_Y]$ for the cellular complex $C^*(\widetilde{Y})$, constructed as in 5.2.2 . Since $\widetilde{f}_* : \Gamma \to \Gamma_Y$ is an isomorphism, E_Y will there automatically also be a $\mathbb{Z}[\Gamma]$-basis of $C^*(\widetilde{Y})$.

Formulas 5.3.7 – 5.3.8, along with the canonical isomorphism $\hom_{\mathbb{Z}[\Gamma]}(A \oplus B, \mathbb{Z}[\Gamma]) \cong \hom_{\mathbb{Z}[\Gamma]}(A, \mathbb{Z}[\Gamma]) \oplus \hom_{\mathbb{Z}[\Gamma]}(B, \mathbb{Z}[\Gamma])$ for each pair of $\mathbb{Z}[\Gamma]$-modules A and B, allow us construct out of E_X and E_Y $\mathbb{Z}[\Gamma]$-bases

$$E^f_{\text{cyl}} = \bigcup_{k=0}^{n_X+n_Y} E^k_X \dot\cup E^k_Y \dot\cup E^{k-1}_X \subseteq \operatorname{Cyl}^*(\widetilde{f})\ ,\ E^f_{\text{cone}} = \bigcup_{k=0}^{n_X+n_Y} E^k_Y \dot\cup E^{k-1}_X \subseteq \operatorname{Cone}^*(\widetilde{f}), \tag{5.3.14}$$

$$E^{\mathbb{1}}_{\text{cone}} = \bigcup_{k=0}^{n_X+1} E^k_X \dot\cup E^{k-1}_X \subseteq \operatorname{Cone}^*(\mathbb{1}_{\widetilde{X}}) \tag{5.3.15}$$

on $\operatorname{Cyl}^*(\widetilde{f})$, $\operatorname{Cone}^*(\widetilde{f})$ and $\operatorname{Cone}^*(\mathbb{1}_{\widetilde{X}})$, called *the bases compatible with* $[E_X, E_Y]$. We denote by $\operatorname{Cyl}^*(\widetilde{f})[E^f_{\text{cyl}}]$, $\operatorname{Cone}^*(\widetilde{f})[E^f_{\text{cone}}]$ and $\operatorname{Cone}^*(\mathbb{1}_{\widetilde{X}})[E^{\mathbb{1}}_{\text{cone}}]$ the corresponding *based* complexes.
By Proposition 5.3.2, the relative *chain* complex $\operatorname{Cone}_*(\widetilde{f})$ is algebraically acyclic. Since it is a free and finite $\mathbb{Z}[\Gamma]$-chain complex, we can find a $\mathbb{Z}[\Gamma]$-linear chain contraction $\gamma_* : \operatorname{Cone}_*(\widetilde{f}) \to \operatorname{Cone}_{*-1}(\widetilde{f})$. Using again the additivity of the functor $\hom_{\mathbb{Z}[\Gamma]}(\ .\ , \mathbb{Z}[\Gamma])$ the dual

$$\gamma^* : \operatorname{Cone}^*(\widetilde{f}) \to \operatorname{Cone}^{*+1}(\widetilde{f}) \tag{5.3.16}$$

determines a $\mathbb{Z}[\Gamma]$-linear chain contraction on $\operatorname{Cone}^*(\widetilde{f})$, proving that also $\operatorname{Cone}^*(\widetilde{f})$ is also algebraically acyclic. With all this in mind, we can define the based Whitehead torsion $\tau(f)[E_X, E_Y] \in \widetilde{K}_1(\mathbb{Z}[\Gamma])$.

Definition 5.3.3. Let $f : X \to Y$ be a cellular homotopy equivalence between two finite, connected CW-complexes. Further, let E_X and E_Y be two admissible pairs for $C^*(\widetilde{X})$, respectively $C^*(\widetilde{Y})$. The (based) **Whitehead torsion** $\tau(f)$ of f is defined as the Whitehead-torsion

$$\tau(f)[E_X, E_Y] \coloneqq \tau(\operatorname{Cone}^*(\widetilde{f})[E^f_{\text{cone}}]) \in \widetilde{K}_1(\mathbb{Z}[\Gamma]), \tag{5.3.17}$$

where E^f_{cone} is the basis on $\operatorname{Cone}^*(\widetilde{f})$ compatible with $[E_X, E_Y]$.

For completion, we remark that there also is a (probably more commonly used) *unbased* version of Whitehead torsion of a cellular homotopy equivalence $f : X \to Y$, living in the *Whitehead group*

$$\operatorname{Wh}(\Gamma) \coloneqq \widetilde{K}_1(\mathbb{Z}[\Gamma])/\iota(\Gamma) \tag{5.3.18}$$

of Γ. Here, $\iota(\Gamma) \trianglelefteq \widetilde{K}_1(\mathbb{Z}[\Gamma])$ is the subgroup described in the paragraph containing Equation 5.3.5. This torsion is well-defined, since any admissible pair $[\bar{E}_X, \bar{E}_Y]$ can be, up to sign, translated via appropriate elements of Γ to any other admissible pair $[\bar{E}'_X, \bar{E}'_Y]$, implying that the corresponding based Whitehead torsion elements of f differ by an element in $\iota(\Gamma)$ (See [66, Section 7] for more details). Together with the fact that *homotopic* cellular homotopy equivalences $f, g : X \to Y$ satisfy $\tau(f) - \tau(g) \in \iota(\Gamma)$, we can now make the following:

Definition 5.3.4. A (not necessarily cellular) homotopy equivalence $f : X \to Y$ between finite CW-complexes is **simple** if $\tau(f_{cell}) \in \iota(\Gamma)$ (i.e. if the unbased $\tau(f_{cell}) = 0 \in \operatorname{Wh}(\Gamma)$), where $\Gamma = \pi_1(X)$ and f_{cell} is some cellular approximation of f. Two finite CW-complexes X and Y are *simple homotopy equivalent* if one can finite a simple homotopy equivalence between them.

Within our current context of comparing homotopy equivalent CW-complexes via appropriate maps, let us also shortly talk about the topological importance of simple homotopy equivalences. Namely, it is well-established (see [28]) that two finite CW-complexes X and Y are simple homotopy equivalent if and only if that Y can be constructed out of X by a finite composition of very explicit cellular maps, so-called *elementary expansions* and *elementary collapses* of cells. To provide examples of homotopy equivalent, yet not simple homotopy equivalent spaces, we construct for a pair (p,q) of coprime integers the 3-dimensional *lens space* $L(p,q)$, which arises as a quotient of the 3-sphere $S^3 \subset \mathbb{C}^2$ by the free $\mathbb{Z}_p$-action generated by the map

$$(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2).$$

Notice that $\pi_1(L(p,q)) \cong \mathbb{Z}_p$.
Then, it is known that

- $L(p,q)$ and $L(p,q')$ are homotopy equivalent iff $qq' \equiv \pm n^2 \mod p$ for some $n \in \mathbb{N}$, and
- $L(p,q)$ and $L(p,q')$ are simple homotopy equivalent, in fact homeomorphic, iff $q \equiv \pm (q')^{\pm 1} \mod p$,

see [80] and [17]. Implicitly in this classification also lies a criterion for the non-vanishing of $\mathrm{Wh}(\Gamma)$ if Γ is finite-cyclic of a certain type. Indeed, we see that for any integer p, for which there exist two integers q and q' both coprime to p, satisfying $qq' \cong \pm n^2 \mod p$ for some n, but are not related by $q \equiv \pm(q')^{\pm 1}$ mod p, we must have $\mathrm{Wh}(\mathbb{Z}_p) \neq 0$. For example, we see by the above that the spaces $L(7,3)$ and $L(7,1)$ are homotopy equivalent (since $3 \cdot 1 = 3 \equiv -2^2 \mod 7$) but not simple homotopy equivalent, which implies that $\mathrm{Wh}(\mathbb{Z}_7) \neq 0$.
Let us also remark that all known examples of homotopy equivalent CW-complexes X and Y that are not simple homotopy equivalent contain torsion in their fundamental group. Indeed, it is a well-known conjecture, confirmed for a large class of groups and a consequence of the stronger *Farrell-Jones Conjecture*, that $\mathrm{Wh}(\Gamma) = 0$ for any torsion-free group Γ, which would imply that any two homotopy equivalent finite CW-complexes with torsion-free fundamental group are in fact simple homotopy equivalent. We refer to [6] for a comprehensive summary of the current status of this and closely related problems.
Perhaps not surprisingly, though highly non-trivial to prove is the following celebrated theorem by Chapman, which will be one of the main two ingredients in establishing the topological invariance of L^2-Torsion:

Theorem 5.3.5. [24] *A homeomorphism $f: X \to Y$ between finite CW-complexes is a simple homotopy equivalence.*

5.3.3 L^2-Whitehead torsion of a homotopy equivalence

Now assume that we are given a finite-dimensional, complex representation $\rho : \Gamma_Y \to V$. Given some choice of admissible pairs E_X on $\widetilde{X}$ and E_Y on $\widetilde{Y}$, as well as basis $B \subseteq V$, our goal now is to obtain an explicit formula for the anomaly of the twisted L^2-torsion $T^{CW}_{(2)}(X, \rho \circ f_*)[E_X, B] - T^{CW}_{(2)}(Y, \rho)[E_Y, B]$ in terms of a twisted L^2-torsion defined over $\mathrm{Cone}^*(\widetilde{f})$.
For this, first consider the twisted cochain complexes $C^*(\widetilde{X}, \rho \circ f_*)$ and $C^*(\widetilde{Y}, \rho)$, constructed as in 5.2.4-5.2.5. Recall that they are both cochain complexes of free, finitely-generated $\mathbb{C}[\Gamma]$-modules (the diagonal action of Γ on $C^*(\widetilde{Y}, \rho)$ is given by $g.(e \otimes v) := f_*(g).e \otimes \rho(f_*(g)) \cdot v$ for $g \in \Gamma$, $e \in C^*(\widetilde{Y})$ and $v \in V$). Further, define the twisted cochain complexes of $\mathbb{C}[\Gamma]$-modules

$$\mathrm{Cyl}^*(\widetilde{f}, \rho) := \mathrm{Cyl}^*(\widetilde{f}) \otimes_{\mathbb{Z}} V, \tag{5.3.19}$$

$$\mathrm{Cone}^*(\widetilde{f}, \rho) := \mathrm{Cone}^*(\widetilde{f}) \otimes_{\mathbb{Z}} V, \tag{5.3.20}$$

$$\mathrm{Cone}^*(\mathbb{1}_{\widetilde{X}}, \rho) := \mathrm{Cone}^*(\mathbb{1}_{\widetilde{X}}) \otimes_{\mathbb{Z}} V, \tag{5.3.21}$$

with differentials twisted by $\otimes_{\mathbb{Z}} \mathbb{1}_V$ and *diagonal* Γ-action defined on elementary tensors by

$$\gamma.(e \otimes v) := \gamma.e \otimes \rho(f_*(\gamma)) \cdot v. \tag{5.3.22}$$

Since $\otimes_{\mathbb{Z}} \mathbb{1}_V$ is an additive functor and since both $C^*(\widetilde{X}, \rho \circ f_*)$ and $C^*(\widetilde{Y}, \rho)$ are free and finite $\mathbb{C}[\Gamma]$-module cochain complexes, the same properties hold true for the complexes 5.3.19–5.3.21. Moreover, the split exact sequences 5.3.12 and 5.3.13 tensor up to split exact sequences of $\mathbb{C}[\Gamma]$-cochain complexes

$$0 \to \mathrm{Cone}(\widetilde{f}, \rho) \xrightarrow{p_X^*} \mathrm{Cyl}(\widetilde{f}, \rho) \xrightarrow{i_X^*} C^*(\widetilde{X}, \rho \circ f_*) \to 0, \tag{5.3.23}$$

$$0 \to \mathrm{Cone}(\mathbb{1}_{\widetilde{X}}, \rho) \xrightarrow{p_Y^*} \mathrm{Cyl}(\widetilde{f}, \rho) \xrightarrow{i_Y^*} C^*(\widetilde{Y}, \rho) \to 0. \tag{5.3.24}$$

Given admissible pairs E_X on $C^*(\widetilde{X})$, respectively E_Y on $C^*(\widetilde{Y})$, consider the induced compatible bases $E^f_{\mathrm{cyl}} \subseteq \mathrm{Cyl}(\widetilde{f})$, $E^{\mathbb{1}}_{\mathrm{cone}} \subseteq \mathrm{Cone}(\mathbb{1}_{\widetilde{X}})$ and $E^f_{\mathrm{cone}} \subseteq \mathrm{Cone}(\widetilde{f})$. Together with a fixed $\mathbb{C}$-basis $B \subseteq V$, these determine $\mathbb{C}[\Gamma]$-bases

$$\begin{aligned}
&[E^f_{\mathrm{cyl}}, B] := \{e \otimes b : e \in E^f_{\mathrm{cyl}}, b \in B\} \subseteq \mathrm{Cyl}(\widetilde{f}, \rho),\\
&[E^{\mathbb{1}}_{\mathrm{cone}}, B] := \{e \otimes b : e \in E^{\mathbb{1}}_{\mathrm{cone}}, b \in B\} \subseteq \mathrm{Cone}^*(\mathbb{1}_{\widetilde{X}}, \rho),\\
&[E^f_{\mathrm{cone}}, B] := \{e \otimes b : e \in E^f_{\mathrm{cone}}, b \in B\} \subseteq \mathrm{Cone}^*(\widetilde{f}, \rho)
\end{aligned}$$

for the respective $\mathbb{C}[\Gamma]$-cochain complexes. Each of these bases determines a unique Γ-invariant complex inner product on the underlying complex vector spaces, with respect to which the corresponding Γ-invariant subsets $\Gamma.[E^f_{\mathrm{cyl}}, B]$, $\Gamma.[E^{\mathbb{1}}_{\mathrm{cone}}, B]$ and $\Gamma.[E^f_{\mathrm{cone}}, B]$ form an orthonormal basis. We denote the resulting inner product spaces by $\mathrm{Cyl}^*(\widetilde{f}, \rho)[E^f_{\mathrm{cyl}}, B]$, $\mathrm{Cone}^*(\widetilde{f}, \rho)[E^f_{\mathrm{cone}}, B]$ and $\mathrm{Cone}^*(\mathbb{1}_{\widetilde{X}}, \rho)[E^{\mathbb{1}}_{\mathrm{cone}}, B]$. Note that, after endowing each cochain complex with the previously described inner products, the chain maps from 5.3.23–5.3.24 are all partial isometries.
Finally, we tensor up to our inner product spaces to obtain the three the Hilbert $\mathcal{N}(\Gamma)$-cochain complexes of finite type

$$\mathrm{Cyl}^*_{(2)}(\widetilde{f}, \rho)[E^f_{\mathrm{cyl}}, B] := l^2(\Gamma) \otimes_{\mathbb{C}[\Gamma]} \mathrm{Cyl}^*(\widetilde{f}, \rho)[E^f_{\mathrm{cyl}}, B], \tag{5.3.25}$$

$$\mathrm{Cone}^*_{(2)}(\widetilde{f}, \rho)[E^f_{\mathrm{cone}}, B] := l^2(\Gamma) \otimes_{\mathbb{C}[\Gamma]} \mathrm{Cone}^*(\widetilde{f}, \rho)[E^f_{\mathrm{cone}}, B], \tag{5.3.26}$$

$$\mathrm{Cone}^*_{(2)}(\mathbb{1}_{\widetilde{X}}, \rho)[E^{\mathbb{1}}_{\mathrm{cone}}, B] := l^2(\Gamma) \otimes_{\mathbb{C}[\Gamma]} \mathrm{Cone}^*(\mathbb{1}_{\widetilde{X}}, \rho)[E^{\mathbb{1}}_{\mathrm{cone}}, B], \tag{5.3.27}$$

with differentials twisted by $\mathbb{1}_{l^2(\Gamma)\otimes_{\mathbb{C}[\Gamma]}}$. Because $l^2(\Gamma)\otimes_{\mathbb{C}[\Gamma]}$ is also an additive functor, the next proposition is an immediate consequence of the previous discussion:

Proposition 5.3.6. *In the notation established above, the following results hold true:*

1. *The Hilbert $\mathcal{N}(\Gamma)$-module cochain complexes $\mathrm{Cone}^*_{(2)}(\widetilde{f}, \rho)[E^f_{cone}, B]$ and $\mathrm{Cone}^*_{(2)}(\mathbb{1}_{\widetilde{X}}, \rho)[E^{\mathbb{1}}_{cone}, B]$ are contractible. Moreover,*

2. *there are split exact sequences of Hilbert $\mathcal{N}(\Gamma)$-module cochain complexes*

$$0 \to \mathrm{Cone}^*_{(2)}(\widetilde{f}, \rho)[E^f_{cone}, B] \xrightarrow{\iota_1} \mathrm{Cyl}^*_{(2)}(f, \rho)[E^f_{cyl}, B] \xrightarrow{\pi_1} C^*_{(2)}(\widetilde{X}, \rho \circ f_*)[E_X, B] \to 0, \tag{5.3.28}$$

$$0 \to \mathrm{Cone}^*_{(2)}(\mathbb{1}_{\widetilde{X}}, \rho)[E^{\mathbb{1}}_{cone}, B] \xrightarrow{\iota_1} \mathrm{Cyl}^*_{(2)}(f, \rho)[E^f_{cyl}, B] \xrightarrow{\pi_1} C^*_{(2)}(\widetilde{Y}, \rho)[E_Y, B] \to 0, \tag{5.3.29}$$

where all maps involved are partial isometries.

Corollary 5.3.7. *Let $f : X \to Y$ be a cellular homotopy equivalence between finite, connected CW-complexes and let $\rho : \Gamma_Y \to V$ be a finite-dimensional, complex representation. Then, for all $k \in \mathbb{N}_0$, it holds that*

$$b^{Top}_{(2),k}(X, \rho \circ f_*) = b^{Top}_{(2),k}(Y, \rho), \tag{5.3.30}$$

$$\alpha^{Top}_k(X, \rho \circ f_*) = \alpha^{Top}_k(Y, \rho). \tag{5.3.31}$$

Proof. In the situation of the previous proposition, we use the simplified notation

$$A^* := \mathrm{Cone}^*_{(2)}(\widetilde{f}, \rho)[E^f_{\mathrm{cone}}, B]\ ,\ B^* := \mathrm{Cyl}^*_{(2)}(f, \rho)[E^f_{\mathrm{cyl}}, B]\ ,\ C^* := C^*_{(2)}(\widetilde{X}, \rho \circ f_*, [E_X, B]),$$
$$D^* := \mathrm{Cone}^*_{(2)}(\mathbb{1}_{\widetilde{X}}, \rho)[E^{\mathbb{1}}_{\mathrm{cone}}, B]\ ,\ E^* := C^*_{(2)}(\widetilde{Y}, \rho, [E_Y, B]).$$

Recall that we have short exact sequences of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

$$0 \to A^* \xrightarrow{\iota_1} B^* \xrightarrow{\pi_1} C^* \to 0 \tag{5.3.32}$$

$$0 \to D^* \xrightarrow{\iota_2} B^* \xrightarrow{\pi_2} E^* \to 0, \tag{5.3.33}$$

which by [54, Theorem 1.21] each induce long, weakly exact sequences in L^2-cohomology

$$\ldots \xrightarrow{\partial_1^{k-1}} H^k(A^*) \xrightarrow{H^k(\iota_1)} H^k(B^*) \xrightarrow{H^k(\pi_1)} H^k(C^*) \xrightarrow{\partial_1^k} H^{k+1}(A^*) \ldots, \tag{5.3.34}$$

$$\ldots \xrightarrow{\partial_2^{k-1}} H^k(D^*) \xrightarrow{H^k(\iota_2)} H^k(B^*) \xrightarrow{H^k(\pi_2)} H^k(E^*) \xrightarrow{\partial_2^k} H^{k+1}(D^*) \ldots. \tag{5.3.35}$$

Since both A^* and D^* are contractible, we have $H^k(A^*) = H^k(D^*) = 0$ for each k. In particular, both $H^k(\pi_1)$ and $H^k(\pi_2)$ are weak isomorphisms, implying that

$$b_k^{(2)}(C^*) = b_k^{(2)}(B^*) = b_k^{(2)}(E^*) \tag{5.3.36}$$

for each $k \in \mathbb{N}$.
Secondly, observe that since $H^k(A^*) = H^{k+1}(A^*) = 0$ for each $k \in \mathbb{N}$, both maps $H^k(\iota_1)$, ∂_1^k are trivial, so that $\alpha(H^k(\iota_1)) = \alpha(\partial_1^k) = \infty^+$ for each $k \in \mathbb{N}_0$. Moreover, since A^* is contractible, we also have $\alpha_k(A^*) = \infty^+$ by [54, Lemma 2.18]. Applying [54, Theorem 2.20] to 5.3.32, we obtain that

$$\alpha_k(B^*) = \alpha_k(C^*)$$

for each $k \in \mathbb{N}_0$. Analogously, one shows that $\alpha_k(B^*) = \alpha_k(E^*)$ for each $k \in \mathbb{N}_0$, finishing the proof. □

Most importantly, Proposition 5.3.6 also allows us to make the next definition:

Definition 5.3.8. Let $f : X \to Y$ be a cellular homotopy equivalence between finite, connected CW-complexes and let $\rho : \Gamma_Y \to V$ be a finite-dimensional, complex representation. For an admissible pair E_X and E_Y on $\widetilde{X}$, respectively $\widetilde{Y}$, as well as $\mathbb{C}$-basis $B \subseteq V$, we define the (based) **twisted L^2-Whitehead Torsion of** f as

$$\tau_{(2)}(f, \rho, [E_X, E_Y, B]) := T\left(\mathrm{Cone}^*_{(2)}(\widetilde{f}, \rho)[E^f_{\mathrm{cone}}, B]\right), \tag{5.3.37}$$

where E^f_{cone} is the basis on $\mathrm{Cone}^*_{(2)}(\widetilde{f}, \rho)$ compatible with $[E_X, E_Y]$.

Lemma 5.3.9. *In the above situation, one has*

$$T\left(\mathrm{Cone}^*_{(2)}(\mathbb{1}_{\widetilde{X}}, \rho)[E^{\mathbb{1}}_{cone}, B]\right) = 1. \tag{5.3.38}$$

Proof. One easily verifies that, under the identification $\mathrm{Cone}^k_{(2)}(\mathbb{1}_{\widetilde{X}}, \rho \circ f_*)[E^{\mathbb{1}}_{\mathrm{cone}}, B] \cong C^k(\widetilde{X}, \rho \circ f_*) \oplus C^{k-1}(\widetilde{X}, \rho \circ f_*)$, the k-th differential $\delta^k : \mathrm{Cone}^k_{(2)}(\mathbb{1}_{\widetilde{X}}, \rho \circ f_*)[E^{\mathbb{1}}_{\mathrm{cone}}, B] \to \mathrm{Cone}^{k+1}_{(2)}(\mathbb{1}_{\widetilde{X}}, \rho \circ f_*)[E^{\mathbb{1}}_{\mathrm{cone}}, B]$ takes the form

$$\delta^k = \begin{pmatrix} (-1)^{k+1}\delta^k_X & 0 \\ \mathbb{1}_{C^k(\widetilde{X}, \rho \circ f_*)} & \delta^{k-1}_X \end{pmatrix}, \tag{5.3.39}$$

where $\delta^k_X := \delta^k_{\rho \circ f_*} : C^k(\widetilde{X}, \rho \circ f_*) \to C^{k+1}(\widetilde{X}, \rho \circ f_*)$ is the corresponding twisted differential.
Consequently, one sees that the map $\gamma^* : \mathrm{Cone}^*_{(2)}(\mathbb{1}_{\widetilde{X}}, \rho \circ f_*)[E^{\mathbb{1}}_{\mathrm{cone}}, B] \to \mathrm{Cone}^{*-1}_{(2)}(\mathbb{1}_{\widetilde{X}}, \rho \circ f_*)[E^{\mathbb{1}}_{\mathrm{cone}}, B]$ defined by

$$\gamma^k := \begin{pmatrix} 0 & \mathbb{1}_{C^{k-1}(\widetilde{X}, \rho \circ f_*)} \\ 0 & 0 \end{pmatrix} \tag{5.3.40}$$

is an explicit chain contraction for $\mathrm{Cone}^*_{(2)}(\mathbb{1}_{\widetilde{X}}, \rho \circ f_*)[E^{\mathbb{1}}_{\mathrm{cone}}, B]$.
To streamline the notation, we will abbreviate by

$$C^* := C^*(\widetilde{X}, \rho \circ f_*)\ ,\ D^* := \mathrm{Cone}^*_{(2)}(\mathbb{1}_{\widetilde{X}}, \rho \circ f_*)[E^{\mathbb{1}}_{\mathrm{cone}}, B].$$

Define

$$D_{odd} := \bigoplus_{k \in 2\mathbb{N}-1} D^k\ ,\ D_{ev} := \bigoplus_{k \in 2\mathbb{N}_0} D^k,$$
$$(\gamma^* + \delta^*)_{odd} := (\gamma^* + \delta^*)|_{D_{odd}} : D_{odd} \to D_{ev}.$$

From the isometric isomorphism $D^k \cong C^k \oplus C^{k-1}$, one obtains isometric isomorphisms

$$D_{odd} \cong D_{ev} \cong \bigoplus_{k=0}^{n} C^k. \tag{5.3.41}$$

Under this identification, one computes that

$$(\gamma^* + \delta^*)_{odd} = \begin{pmatrix} \mathbb{1}_{C^n} & \delta^{n-1}_X & 0 & 0 & \dots \\ 0 & \mathbb{1}_{C^{n-1}} & \delta^{n-2}_X & 0 & \dots \\ 0 & 0 & \mathbb{1}_{C^{n-2}} & \delta^{n-3}_X & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \dots & \dots & 0 & \mathbb{1}_{C^0} \end{pmatrix}. \tag{5.3.42}$$

The result now follows from Proposition 4.1.14 and [54, Lemma 3.41], since

$$T(D^*) = \det_\Gamma((\gamma^* + \delta^*)_{odd}) = 1. \tag{5.3.43}$$

□

The relevance of the L^2-Whitehead torsion lies in the fact that it precisely describes the anomaly of L^2-torsion between homotopy equivalent complexes, as highlighted in the next central result:

Corollary 5.3.10. *Let $f : X \to Y$ be a cellular homotopy equivalence between finite, connected CW-complexes and let $\rho : \Gamma_Y \to V$ be a finite-dimensional, complex representation. Then, the tuple $(X, \rho \circ f_*)$ is det-L^2-acyclic if and only if (Y, ρ) is det-L^2-acylic. In this case, one gets for any pair of admissible pair E_X and E_Y on $\widetilde{X}$, respectively $\widetilde{Y}$, that*

$$\frac{T^{CW}_{(2)}(Y, \rho, [E_Y, B])}{T^{CW}_{(2)}(X, \rho \circ f_*, [E_X, B])} = \tau_{(2)}(f, \rho, [E_X, E_Y, B]). \tag{5.3.44}$$

Proof. Recall from Proposition 5.3.6 that there are split exact sequences of Hilbert $\mathcal{N}(\Gamma)$-module cochain complexes

$$0 \to \mathrm{Cone}^*_{(2)}(\widetilde{f},\rho)[E^f_{\mathrm{cone}},B] \xrightarrow{\iota_1} \mathrm{Cyl}^*_{(2)}(f,\rho)[E^f_{\mathrm{cyl}},B] \xrightarrow{\pi_1} C^*_{(2)}(\widetilde{X},\rho\circ f_*,[E_X,B]) \to 0, \tag{5.3.45}$$
$$0 \to \mathrm{Cone}^*_{(2)}(\mathbb{1}_{\widetilde{X}},\rho)[E^{\mathbb{1}}_{\mathrm{cone}},B]) \xrightarrow{\iota_2} \mathrm{Cyl}^*_{(2)}(f,\rho)[E^f_{\mathrm{cyl}},B] \xrightarrow{\pi_2} C^*_{(2)}(\widetilde{Y},\rho,[E_Y,B]) \to 0, \tag{5.3.46}$$

where all maps involved are partial isometries. The same proposition also yields that both $C^*_{(2)}(M_{\widetilde{f}},X,\rho,[E^f_{\mathrm{cone}},B])$ and $\mathrm{Cone}^*_{(2)}(\mathbb{1}_{\widetilde{X}},\rho)[E^{\mathbb{1}}_{\mathrm{cone}},B]$ are always det-L^2-acyclic. From [54, Theorem 3.35], we then conclude that

$$\begin{aligned}
&(X,f_*\circ\rho) \text{ is det-}L^2\text{-acyclic } \Leftrightarrow C^*_{(2)}(\widetilde{X},\rho\circ f_*,[E_X,B]) \text{ is det-}L^2\text{-acyclic}\\
&\Leftrightarrow \mathrm{Cyl}^*_{(2)}(f,\rho)[E^f_{\mathrm{cyl}},B] \text{ is det-}L^2\text{-acyclic} \Leftrightarrow C^*_{(2)}(\widetilde{Y},\rho,[E_Y,B]) \text{ is det-}L^2\text{-acyclic}\\
&\Leftrightarrow (Y,\rho) \text{ is det-}L^2\text{-acyclic.}
\end{aligned}$$

Moreover, along with Proposition 5.3.9, we also get in the det-L^2-acyclic case, that

$$\begin{aligned}
T\left(\mathrm{Cyl}^*_{(2)}(f,\rho)[E^f_{\mathrm{cyl}},B]\right) &= T^{CW}_{(2)}(Y,\rho,[E_Y,B])\cdot T\left(\mathrm{Cone}^*_{(2)}(\mathbb{1}_{\widetilde{X}},\rho)[E^{\mathbb{1}}_{\mathrm{cone}},B])\right) = T^{CW}_{(2)}(Y,\rho,[E_Y,B]),\\
T\left(\mathrm{Cyl}^*_{(2)}(f,\rho)[E^f_{\mathrm{cyl}},B]\right) &= T^{CW}_{(2)}(X,\rho\circ f_*,[E_X,B])\cdot \tau_{(2)}(f,\rho,[E_X,E_Y,B]).
\end{aligned}$$

The result now follows. ☐

With regards to topological invariance, the last component that we need is the concrete connection between the L^2-Whitehead torsion of a homotopy equivalence $f\colon X\to Y$ complexes X and Y and its ordinary Whitehead torsion as described in the previous section.
For that purpose, let $\rho:\Gamma\to\mathrm{GL}(V)$ be a finite-dimensional, complex representation and let $B\subseteq V$ be an ordered basis of V, which gives rise to the isomorphism $\phi_B\colon V\to\mathbb{C}^m$ and the *based* representation $\rho_B\colon\Gamma\to\mathrm{GL}(m,\mathbb{C})$. Out of ρ_B, we construct a the group homomorphsim

$$\Lambda^{\rho_B}\colon\ \mathrm{GL}(\mathbb{Z}[\Gamma])\to\mathrm{GL}(\mathbb{C}[\Gamma]), \tag{5.3.47}$$
$$\Big(\sum_{g\in\Gamma}\lambda^g_{ij}\cdot g\Big)^{l=1\dots s}_{i=1\dots r}\mapsto\Big(\sum_{g\in\Gamma}\lambda^g_{ij}\cdot\rho_B(g)\cdot g\Big)^{j=1\dots s}_{i=1\dots r}. \tag{5.3.48}$$

In view of the canonical group embedding $\mathrm{GL}(\mathbb{C}[\Gamma])\hookrightarrow\mathrm{GL}(\mathcal{N}(\Gamma))$, let $\det_\Gamma\colon\mathrm{GL}(\mathbb{C}[\Gamma])\to\mathbb{R}_{>0}$ be the group homomorphism, obtained by restricting the Fuglede-Kadison determinant onto elements of $\mathrm{GL}(\mathbb{C}[\Gamma])$. From the definition of Λ^{ρ_B}, it is evident that $\Lambda^{\rho_B}(E(\mathbb{Z}[\Gamma]))\subseteq E(\mathbb{C}[\Gamma])$. Since also $\det_\Gamma(A)=\det_\Gamma(-A)$ for any $A\in\mathrm{GL}(\mathbb{C}[\Gamma])$, we obtain a homomorphism $\overline{\Lambda}^{\rho_B}\colon\widetilde{K}_1(\mathbb{Z}[\Gamma])\to\mathbb{R}_{>0}$ that fits into the commutative diagram below

$$\begin{array}{ccc}
\mathrm{GL}(\mathbb{Z}[\Gamma]) & \xrightarrow{\ \pi\ } & \widetilde{K}_1(\mathbb{Z}[\Gamma])\\
\big\downarrow{\scriptstyle\Lambda^{\rho_B}} & & \big\downarrow{\scriptstyle\overline{\Lambda^{\rho_B}}}\\
\mathrm{GL}(\mathbb{C}[\Gamma]) & \xrightarrow{\ \det_\Gamma\ } & \mathbb{R}_{>0}
\end{array}\ . \tag{5.3.49}$$

Let $\mathrm{Cone}^*(\widetilde{f})[E^{\mathbb{1}}_{\mathrm{cone}}]$ be the finite, based, acyclic $\mathbb{Z}[\Gamma]$-module cochain complex that we have defined previously. Further, let $\delta^*\colon\mathrm{Cone}^*(\widetilde{f})[E^{\mathbb{1}}_{\mathrm{cone}}]\to\mathrm{Cone}^{*+1}(\widetilde{f})[E^{\mathbb{1}}_{\mathrm{cone}}]$ be the boundary operator and let $\gamma^*\colon\mathrm{Cone}^*(\widetilde{f})[E^{\mathbb{1}}_{\mathrm{cone}}]\to\mathrm{Cone}^{*-1}(\widetilde{f})[E^{\mathbb{1}}_{\mathrm{cone}}]$ be a choice of chain contraction. With aid of the basis $E^{\mathbb{1}}_{\mathrm{cone}}$, both morphisms can be identified with appropriate square matrices in $\mathrm{Mat}(\mathbb{Z}[\Gamma])$. Now let $\mathrm{Cone}^*_{(2)}(\widetilde{f},\rho)[E^{\mathbb{1}}_{\mathrm{cone}},B]$ be the other finite, acyclic Hilbert $\mathcal{N}(\Gamma)$-module cochain complex that was previously defined. Using the admissible basis $[E^Y,B]$, it is clear that the matrix $\Lambda^{\rho_B}(\delta^*)$ is the boundary operator on $\mathrm{Cone}^*_{(2)}(\widetilde{f},\rho)[E^{\mathbb{1}}_{\mathrm{cone}},B]$ and that $\Lambda^{\rho_B}(\gamma^*)\in\mathrm{Mat}(\mathbb{C}[\Gamma])$ determines a chain contraction on $\mathrm{Cone}^*_{(2)}(\widetilde{f},\rho)[E^{\mathbb{1}}_{\mathrm{cone}},B]$. By [54, Lemma 3.41], we therefore get

$$\tau_{(2)}(f,\rho,E_X,E_Y.B)=\det\nolimits_\Gamma\left(\Lambda^{\rho_B}(\gamma^*+\delta^*)|_{odd}\right). \tag{5.3.50}$$

Combining 5.3.49 and 5.3.50 with Definitions 5.3.1 and 5.3.4, we arrive at the following relation between the ordinary based and the L^2-Whitehead Torsion.

Lemma 5.3.11. *In the notation as above, the equality*

$$\tau_{(2)}(f,\rho,E_X,E_Y,B)=\overline{\Lambda^{\rho_B}}(\tau(f,E_X,E_Y)) \tag{5.3.51}$$

holds. In particular, if f is a simple homotopy equivalence, there exists some $g\in\Gamma$, such that

$$\tau_{(2)}(f,\rho,E_X,E_Y,B)=|\det(\rho_B(g))|. \tag{5.3.52}$$

Theorem 5.3.12. *Let M be a connected topological space admitting a finite CW-structure and let $\rho:\Gamma\coloneqq\pi_1(M)\to V$ be a unimodular representation of Γ. Then,*

1. *for any two CW-structures X,Y on M, the tuple (X,ρ) is det-L^2-acyclic if and only if (Y,ρ) is det-L^2-acyclic. In this case, one obtains that*

$$T^{CW}_{(2)}(X,\rho)=T^{CW}_{(2)}(Y,\rho), \tag{5.3.53}$$

allowing us to define

$$T^{Top}_{(2)}(M,\rho)\coloneqq T^{CW}_{(2)}(X,\rho) \tag{5.3.54}$$

to be the ***topological L^2-torsion*** *of the tuple (M,ρ).*

2. *If N is another space admitting a finite CW-structure and $f:N\to M$ either a homeomorphism or a general simple homotopy equivalence, then (M,ρ) is det-L^2-acyclic if and only if $(N,\rho\circ f_*)$ det-L^2-acylic. In this case, we obtain*

$$T^{Top}(N,\rho)=T^{Top}(N,\rho\circ f_*).$$

Proof. 1 : First, note that since X and Y are cell-structures of the same ambient space M, lifting to cell-structures $\widetilde{X}$ and $\widetilde{Y}$ on $\widetilde{M}$, we have a canonical identification of deck groups $\Gamma=\Gamma_Y=\Gamma_X$. Now choose for the identity homeomorphism $\mathrm{id}_M\colon X\to Y$ some cellular approximation $f\simeq\mathrm{id}_M:X\to Y$, of which we further pick a lift $\widetilde{f}\colon\widetilde{X}\to\widetilde{Y}$. Let $\widetilde{f}_*\colon\Gamma\to\Gamma$ be the induced group automorphism, as defined in Equation 5.1.1. Since $f\simeq\mathbb{1}_M$, we must also have (up to conjugacy) $\widetilde{f}_*=(\mathbb{1}_{\widetilde{M}})_*=\mathbb{1}_\Gamma$ by Corollary 5.1.2. Corollary 5.3.10 therefore implies that (X,ρ) is det-L^2-acyclic if and only if (Y,ρ) is det-L^2-acyclic. In this case, since ρ is unimodular, we can further apply Corollary 5.2.10 to obtain for any choice E_X and E_Y of admissible pairs on $C^*(\widetilde{X})$, respectively $C^*(\widetilde{Y})$, that

$$\frac{T^{CW}_{(2)}(X,\rho)}{T^{CW}_{(2)}(Y,\rho)}=\frac{T^{CW}_{(2)}(X,\rho,[E_X,B])}{T^{CW}_{(2)}(Y,\rho,[E_Y,B])}\overset{5.3.37}{=}\tau_{(2)}(f,\rho,E_X,E_Y,B)\overset{5.3.51}{=}\overline{\Lambda^{\rho_B}}(\tau(f,E_X,E_Y)). \tag{5.3.55}$$

By definition of $\overline{\Lambda^{\rho_B}}$, we have $\overline{\Lambda^{\rho_B}}(\tau(f,E_X,E_Y))=\det_\Gamma(\Lambda^{\rho_B}A)$ for any representative $A\in\mathrm{GL}(\mathbb{Z}[\Gamma])$ of $\tau(f,E_X,E_Y)\in\widetilde{K}_1(\mathbb{Z}[\Gamma])$. Since f is the cellular approximation of a homeomorphim, it follows from

Theorem 5.3.5 that we may choose A as a 1×1 matrix of the form (g) for some $g \in \Gamma$. Applying Lemma 5.3.11, we therefore obtain

$$\overline{\Lambda^{\rho_B}}(\tau(f, E_X, E_Y)) = |\det(\rho_B(g))| = 1, \tag{5.3.56}$$

since ρ_B is unimodular. The result readily follows.
2 : Choosing CW-structures X and Y on M and N respectively, the argument is completely analogous as the one presented in 1. □

Remark 5.3.13. Observe that assertion 2 of the above theorem shows in particular that $T^{Top}_{(2)}(M, \rho)$ does **not** depend on the particular choice of universal cover $p : \widetilde{M} \to M$, i.e. on the particular choice of representative of the fundamental group $\pi_1(M)$ as the deck group of such covering map. That is why henceforth, we will simply talk about a representation $\rho : \pi_1(M) \to \mathrm{GL}(V)$ of the fundamental group without specifying a representative deck group for $\pi_1(M)$.

As it will become relevant later on, we now also introduce a topological L^2-torsion for certain *non-compact* spaces. Explicitly:

Definition 5.3.14. Suppose that M is a connected topological space, such that

1. $\mathrm{Wh}(\pi_1(M)) = 0$, and
2. M has the homotopy type of a finite CW-complex.

Given any unimodular representation $\rho : \pi_1(M) \to \mathrm{GL}(V)$, let K be a finite CW-complex and $i : K \to M$ a homotopy equivalence. Assuming that the pair $(X, \rho \circ i_*)$ is det-L^2-acyclic, we define the topological L^2-torsion of the Tuple (M, ρ) as

$$T^{Top}_{(2)}(M, \rho) := T^{Top}_{(2)}(X, \rho \circ i_*). \tag{5.3.57}$$

Remark 5.3.15. The large pool of spaces satisfying these assumptions contains in particular all non-positively curved locally symmetric spaces of finite volume, cf. [36, Theorem 0.10] and [4, Theorem 13.1].

Of course, we have to show that these definitions do not depend on the choice of $i : X \to M$. Therefore, assume that Y is another space admitting a finite CW-structure and homotopy equivalent to M via a map $j : Y \to M$. In choosing a homotopy inverse $k : M \to Y$ of j, we obtain a homotopy equivalence $f := k \circ i : \; : X \to Y$, whose induced map on fundamental groups fits into a diagram that is commutative up to conjugation

$$\begin{tikzcd}
 & \mathrm{GL}(V) & \\
 & \pi_1(M) \arrow[u, "\rho"] & \\
\pi_1(X) \arrow[ur, "i_*"] \arrow[rr, "f_*"] & & \pi_1(Y). \arrow[ul, "j_*"']
\end{tikzcd}$$

Observe that the condition $\mathrm{Wh}(\pi_1(M)) = 0$ ensures that f is a simple homotopy equivalence. We may therefore apply Theorem 5.3.12 and Lemma 5.2.6 to obtain that

$$T^{Top}_{(2)}(X, \rho \circ i_*) = T^{Top}_{(2)}(X, \rho \circ j_* \circ f_*) = T^{Top}_{(2)}(Y, \rho \circ j_*).$$

If M is compact, both notions of topological torsions $T^{Top}_{(2)}(M, \rho)$ introduced this section coincide.

5.4 The Morse-Smale torsion

We now return to the realm of smooth, compact manifolds. On such a manifold M, given a det-L^2-acyclic representation $\rho : \pi_1(M) \to V$, a choice of bundle metric h on the flat bundle $E_\rho \downarrow M$ associated to ρ, as well as a smooth Morse function $f : M \to \mathbb{R}$ satisfying the *Morse-Smale transversality conditions*, we will define the so-called L^2*-Morse-Smale torsion* $T^{MS}_{(2)}(M, \rho, h, f) \in \mathbb{R}_{>0}$. Although its definition is more involved than the topological torsion, it has the advantage of being more easily compared to the L^2-analytic torsion $T^{An}_{(2)}(M, \rho, g, h)$ that we studied in previous chapters. This comparison will be the subject of the next chapter. Moreover, in all our relevant instances, namely when ρ is unimodular and h is a *unimodular metric*, we show that it coincides with the corresponding topological torsion $T^{Top}_{(2)}(M, \rho)$, defined in the previous section.
For a given smooth function $f : M \to \mathbb{R}$, we introduce the following objects:

1. We set $Cr(f) = \{p \in M : Df_p = 0\} \subseteq M$ to be the subset of all critical points of f, and
2. for each $p \in Cr(f)$, the *index* $\mathrm{ind}(p) \in \{0, \dots, m\}$ is defined as the number of negative eigenvalues of its Hessian Hf_p (independent of the choice of coordinates at p). Lastly,
3. for each $0 \le k \le m$, we set $Cr_k(f) = \{p \in Cr(f) : \mathrm{ind}(p) = k\} \subseteq Cr(f)$ to be the subset of critical points of index k.

f is *non-degenerate* at a point $p \in M$ if the corresponding Hessian Hf_p has (in an arbitrary choice of coordinates) only non-zero eigenvalues.

Definition 5.4.1 (Morse function)**.** Let M be a smooth manifold, $a, b \in \mathbb{R}$ with $a \le b$. A smooth function $f : M \to [a, b]$ is called a **Morse-function** if f is non-degenerate at each of its critical points. In case that $\partial M \neq \emptyset$, we demand that additionally one of the following two (mutually exclusive) conditions hold:

1. f is **of type I**, that is
   (a) the restriction $f|_{\partial M}$ of f to ∂M is non-degenerate at each of its critical points.
   (b) There exists a collar neighborhood $U \supset \partial M$ of ∂M, a parametrization $G : \partial M \times [0, \epsilon) \to U$ of ∂M satisfying $G(x, 0) = x$, such that $(f \circ G)(x, t) = f|_{\partial M}(x) + t^2$.
2. f is **of type II**, that is,
   (a) one has $\partial M \subseteq f^{-1}(\{a, b\})$ with $\partial_- M \coloneqq f^{-1}(a) \cap \partial M$ and $\partial_+ M \coloneqq f^{-1}(b) \cap \partial M$, and
   (b) There exists a collar neighborhood $U \supset \partial M$ of ∂M, a parametrization $G : \partial M \times [0, \epsilon) \to U$ of ∂M satisfying $G(x, 0) = x$, such that
$$(f \circ G)(x, t) = \begin{cases} b - t & x \in \partial_+ M, \\ a + t & x \in \partial_- M. \end{cases}$$

Observe that for a type I Morse-function f on M, the restriction $f|_{\partial M}$ is a Morse-function on the (closed) boundary, such that $Cr(f|_{\partial M}) = Cr(f) \cap \partial M$. Conversely, for a type II Morse-function, the restriction $f|_{\partial M}$ is locally-constant and $Cr(f)$ is a subset of the interior of M. If f is a Morse-function, it follows easily that $Cr(f)$ is a discrete subset of M. In particular, if M is compact, this means that $Cr(f)$ must be finite.

Now assume additionally that M comes equipped with some Riemannian metric g. Recall that the *gradient* $\nabla_g f \in \Gamma(M, TM)$ of f with respect to g is the vector field uniquely determined by

$$\langle \nabla_g f, X \rangle_g = Df(X)$$

for any other vector field $X \in \Gamma(M, TM)$. For an appropriate neighborhood $U \subseteq M \times \mathbb{R}$ of $M \times \{0\}$, the gradient then generates the *negative gradient flow* $\phi : U \to M$ of f. It is the unique 1-parameter family of diffeomorphisms of M whose infinitesimal generator is the negative gradient $-\nabla_g f$. This means that it is the unique solution on U to the differential equation

$$\begin{aligned}&\frac{\partial}{\partial t}\phi(t,x)|_{t=t_0} = -\nabla_g f(\phi(t_0,x)),\\&\phi(0,x) = x\end{aligned}$$

for each $x \in M$. By construction, one can show that the set $Cr(f)$ of critical points is precisely the set of stationary points for the gradient flow ϕ.

Assuming that ϕ can be extended *globally*, so that $U = M \times \mathbb{R}$ (automatically true if M is compact), the requirement that f has compact range guarantees that for any $y \in M$, the limits $\lim_{t\to\pm\infty}\phi(y,t)$ exist, are distinct, and lie $Cr(f)$ if f is of type I and in $Cr(f) \cup \partial M$ if of f is of type II. Visually speaking, this means that $Cr(f)$ (respectively $Cr(f) \cup \partial M$, if f is of type II) is precisely the set of sources and sinks for the gradient flow ϕ, that the source of each non-constant flow line is always different from its sink and that any point in M emanates from a source and eventually flows into a sink. For fixed $y \in M$, the *flow line* at y is the map $\gamma_y := \phi(\ .\ , y) : \mathbb{R} \to M$. Observe that for $p \in Cr(f)$, γ_p is the constant map. This allows us to further define define for each $p \in Cr(f)$ the *stable manifold* $W^+(p) \subseteq M$ and the *unstable manifold* $W^-(p) \subset M$ via

$$W^+(p) := \{x \in M : \lim_{t\to\infty}\gamma_x(t) = p\}, \tag{5.4.1}$$

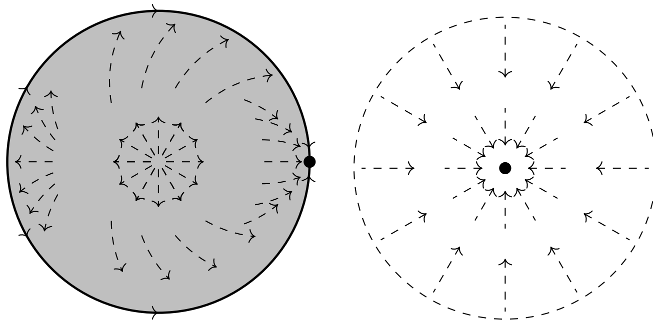
$$W^-(p) := \{x \in M : \lim_{t\to-\infty}\gamma_x(t) = p\}. \tag{5.4.2}$$

If f is of type I, it follows that

$$M = \dot{\bigcup_{p\in Cr(f)}} W^+(p) = \dot{\bigcup_{p\in Cr(f)}} W^-(p), \tag{5.4.3}$$

i.e. the stable, respectively unstable, manifolds form a partition of M.

Example 5.4.2. Below, we have sketched the gradient vector fields coming from a type I Morse function (left), as well as a type II Morse function (right) on the closed unit disc $D^2 = \{|x| \leq 1\} \subset \mathbb{C}$. In the first case, there are precisely three critical points, 0, 1 and -1. The corresponding unstable manifolds are the interior of D^2, $\{1\}$, and $\partial D^2 \setminus \{1\}$ (as such, they form a partition of D^2). In the second case, 0 is the only critical point, with corresponding unstable manifold being only $\{0\}$ itself and $\partial D^2 = \partial_+ D^2$.



Intuitively, $W^+(p)$, respectively $W^-(p)$, may be regarded as the set of points that, under the negative gradient flow ϕ, flow *into*, respectively *away from*, p. As their name suggests, both $W^+(p)$ and $W^-(p)$ are indeed submanifolds of M. Specifically, it is well-known, see for example [91], that

1. in case that f is of type I and $p \notin \partial M$, or if f is of type II and $\partial_+ M = \emptyset$, $W^+(p)$ is diffeomorphic to $\mathbb{R}^{n-\mathrm{ind}(p)}$ and disjoint from ∂M, while
2. $W^-(p)$ is always diffeomorphic to $\mathbb{R}^{\mathrm{ind}(p)}$ if either f is of type I (in this case, $W^-(p) \subseteq \partial M$ if $p \in \partial M$) or f is of type II and $\partial_- M = \emptyset$ (in this case, one always has $W^-(p) \cap \partial M = \emptyset$).

Since $\lim_{t\to-\infty} \gamma_y(t) \neq \lim_{t\to\infty} \gamma_y(t)$ for every $y \in M \setminus Cr(f)$, we get that $W^-(p) \cap W^+(p) = \{p\}$, which can be phrased as follows: *The stable and unstable manifold of the same critical point p intersect precisely at the* 0*-dimensional submanifold* $\{p\}$. More generally, we would like for any pair $p, q \in Cr(f)$ a non-empty intersection $W^-(p) \cap W^+(q)$ (i.e. the set of all flow lines between a fixed source and sink) always to be a submanifold. While this is false in the very general setting, it will always be fulfilled if ϕ satisfies the so-called *Smale-transversality condition*, which is part of the next definition

Definition 5.4.3. Let M be a smooth manifold. A pair (f, g), where $f : M \to \mathbb{R}$ is a Morse function and g is a Riemannian metric on M is called a *Morse-Smale pair* if

1. the gradient flow ϕ of $-\nabla_g f$ is globally defined over $M \times \mathbb{R}$,
2. for each $p \in Cr_k(f)$, there exist only finitely many $y \in Cr_{k+1}(f)$ such that $W^-(y) \cap W^+(p) \neq \emptyset$,
3. (local triviality) Each $p \in Cr_k(f) \cap \mathrm{int}(M)$ admits a coordinate chart $\phi_p : U_p \to \mathbb{R}^n$, such that
   (a) The pull-back $\phi_p^*(g_{eucl})$ of the euclidean metric on $\mathbb{R}^n$ equals g,
   (b) f has *normal form on* U_p, that is $f \circ \phi_p^{-1}(p_1, \ldots, p_n) = f(p) - p_1^2 - \cdots - p_{\mathrm{ind}(p)}^2 + p_{\mathrm{ind}(p)+1}^2 + \cdots + p_n^2$.
4. ϕ satisfies the *Smale-transversality condition*, that is, for any two $p, q \in Cr(f)$ and any $x \in W^-(q) \cap W^+(p)$, one has
$$T_x W^+(p) + T_x W^-(q) = T_x M. \tag{T}$$
5. If f is of type I, the restriction $(f|_{\partial M}, g_{\partial M})$ also satisfies assertions $(1) - (4)$,
6. The parametrization $G : \partial M \times [0, \epsilon) \to U$ of the collar neighborhood U of ∂M from Definition 5.4.1 is the Riemannian exponential boundary map induced by g, so that the pull-back metric $G^*(g)$ is of product form $g|_{\partial M} \oplus dt^2$.

Finally, a Morse-Smale pair (f, g) is *of type I/II* if f is of type I/II.

It is a classic result that any compact manifold admits a Morse-Smale pair (f, g), both of type I and type II, see for example [92], [75] or [3]. By considering appropriate lifts, the same is therefore true for any manifold M that admits a compact manifold quotient.
The essential property of a Morse-Smale pair (f, g) is that for any pair $p, q \in Cr(f)$, the intersection

$$\mathcal{N}(p, q) \coloneqq W^-(p) \cap W^+(q) \tag{5.4.4}$$

is a (possibly empty) submanifold of of dimension $\operatorname{ind}(p) - \operatorname{ind}(q)$. In the case $\operatorname{ind}(p) = \operatorname{ind}(q) + 1$, this means that the union of all flow lines emanating from p and flowing into q form the 1-dimensional submanifold $\mathcal{N}(p, q)$.
We now describe the construction of the (ordinary) *Morse-Smale* complex on a compact manifold M, provided that we have fixed a Morse-Smale pair (f, g) on M . Namely, choosing orientations O_p on $W(p)$ for each $p \in Cr(f)$, we explain now how such a choice gives rise to numbers

$$n(p, q) \in \mathbb{Z} \tag{5.4.5}$$

for each pair $p, q \in Cr(f)$, such that the following is satisfied:

$$n(p, q) = 0 \text{ if } \mathcal{N}(p, q) = \emptyset \text{ or } \operatorname{ind}(p) \neq \operatorname{ind}(q) + 1. \tag{MS1}$$

$$\forall\, 0 \leq k \leq m,\ \forall\, p \in Cr_k(f),\ \forall\, q \in Cr_{k+2}(f) : \sum_{z \in Cr_{k+1}(f)} n(q, z) \cdot n(z, p) = 0. \tag{MS2}$$

If either $\mathcal{N}(p, q) = \emptyset$ or $\operatorname{ind}(p) \neq \operatorname{ind}(q) + 1$, we set $n(p, q) \coloneqq 0$, so that MS1 is automatically satisfied. In the remaining case, it follows by compactness of M that $\mathcal{N}(p, q)$ is the non-empty disjoint union of finitely many flow lines of ϕ, the gradient flow determined by the pair (f, g). Let $\Gamma(p, q)$ to be the set of those flow lines and fix one such $\gamma \in \Gamma(p, q)$. Recall that it is an embedding $\gamma : \mathbb{R} \to M$ with $\lim_{t \to \infty} \gamma(t) = q$ and $\lim_{t \to -\infty} \gamma(t) = p$. Observe that for each $t \in \mathbb{R}$, the negative gradient $-\nabla_g f_{\gamma(t)}$ is non-vanishing, and therefore a basis for the 1-dimensional vector space $T_{\gamma(t)}\gamma \subseteq T_{\gamma(t)}M$. In particular, it determines a natural orientation $[-\nabla_g f_{\gamma(t)}]$ on $T_{\gamma(t)}\gamma$.
To proceed, we need the following auxiliary lemma:

Lemma 5.4.4. *Let*

$$0 \to V_1 \hookrightarrow V_2 \xrightarrow{\pi} V_3 \to 0 \tag{5.4.6}$$

be a short exact sequence of finite-dimensional $\mathbb{R}$*-vector spaces. Then, two choices of orientations on two of the vector spaces canonically determine an orientation on the third. Namely, given two orientations* $[B_i]$ *of* V_i *and* $[B_j] \in V_j$ *for* $i, j \in \{1, 2, 3\}$*, there exists a unique orientation* $[B_k] \in V_k$ *with* $k \in \{1, 2, 3\} \setminus \{i, j\}$*, such that, for any choice of split* $\iota : V_3 \to V_2$*, we have* $[B_1 \cup \iota(B_3)] = [B_2]$*.*

Proof. This is an immediate consequence from the observation that any two splits $\iota, \iota' : V_3 \to V_2$ of Γ differ by a translation along the subspace V_1. In particular, for any two bases $B_1 \subseteq V_1$, $B_3 \subseteq V_3$, the sets $B_1 \cup \iota(B_3)$ and $B_1 \cup \iota'(B_3)$ both are bases of V_2, such that the corresponding base change matrix $M^{B_1 \cup \iota'(B_3)}_{B_1 \cup \iota(B_3)}$ is a transvection. We conclude that $[B_1 \cup \iota(B_3)] = [B_1 \cup \iota'(B_3)]$, from which the result readily follows. $\square$

Observe that we have a for each $t \in \mathbb{R}$ a short exact sequence

$$0 \to T_{\gamma(t)}\gamma \hookrightarrow T_{\gamma(t)}W^-(p) \to T_{\gamma(t)}W^-(p)/T_{\gamma(t)}\gamma \to 0.$$

By the previous lemma, the orientations $[-\nabla_g f(\gamma(t))]$ on $T_{\gamma(t)}\gamma$ and $O_p\gamma(t) \subseteq T_{\gamma(t)}W^-(p)$, the restriction of our initial global choice O_p on $W^-(p)$, canonically determine a compatible orientation $O_1\gamma(t)$ on $T_{\gamma(t)}W^-(p)/T_{\gamma(t)}\gamma$.
Due to the transversality condition T, the inclusions $T_{\gamma(t)}W^-(p) \hookrightarrow T_{\gamma(t)}M$ and $T_qW^-(q) \hookrightarrow T_qM$ induce canonical isomorphisms

$$T_{\gamma(t)}W^-(p)/T_{\gamma(t)}\gamma \cong T_{\gamma(t)}M/T_{\gamma(t)}W^+(q), \tag{5.4.7}$$

$$T_qW^-(q) \cong T_qM/T_qW^+(q). \tag{5.4.8}$$

Moreover, since $W^+(q)$ is contractible, the bundle $TM/TW^+(q) \downarrow W^+(q)$ trivial. Using this, along with the canonical isomorphisms 5.4.7 and 5.4.8, we see that the orientation $O_q(q)$ of $T_qW^-(q)$ canonically determines an orientation $O_2\gamma(t)$ on $T_{\gamma(t)}M/T_{\gamma(t)}$. We define

$$n_\gamma(p,q) := \begin{cases} +1 & \text{if } O_1\gamma(t) = O_2\gamma(t), \\ -1 & \text{if } O_1\gamma(t) \neq O_2\gamma(t), \end{cases} \tag{5.4.9}$$

$$n(p,q) = \sum_{\gamma \in \Gamma(p,q)} n_\gamma(p,q). \tag{5.4.10}$$

The number $n_\gamma(p,q)$, and therefore also $n(p,q)$, does not depend on the choice of $t \in \mathbb{R}$. It is a classic result, shown for example in [87, Chapter 4], that the numbers $n(p,q)$ satisfy MS2. Also, it is clear from the construction that choosing the opposite orientation $-O_p$ has the effect of changing the sign of $n(p,q)$. For $p \in Cr(f)$, we denote by $[O_p]$ the free abelian group generated by the orientation O_p on $W^-(p)$.

Definition 5.4.5. The **Morse-Smale complex** $C^*(M, \nabla_g f)$ associated to a Morse-Smale pair (f,g) is defined as the following cochain complex of free abelian groups

$$C^*(M,\nabla_g f) := \bigoplus_{p \in Cr(f)} [O_p]\ ,\ C^k(M,\nabla_g f) := \bigoplus_{p \in Cr_k(f)} [O_p] \tag{5.4.11}$$

with boundary map

$$\partial : C^*(M,\nabla_g f) \to C^{*+1}(M,\nabla_g f)$$

being the unique $\mathbb{Z}$-linear extension of the assignment

$$\partial[O_p] := \sum_{q \in Cr(f)} n(q,p) \cdot [O_q]. \tag{5.4.12}$$

Due to MS1-MS2, it is evident that ∂ is well-defined and satisfies $\partial^2 = 0$. Note that $C^*(M, \nabla_g f)$ is independent of the explicit choice of O_p.
Let $\widetilde{M}$ be the universal cover of a compact manifold M with covering map $\pi : \widetilde{M} \to M$ and let $\Gamma :=$ deck$(p) \cong \pi_1(M)$. For a choice of Morse-Smale pair (f,g) on M, let $\widetilde{f} := f \circ \pi$, let $\widetilde{g}$ be the pull back of g, and define for each pair $p, q \in Cr(\widetilde{f})$ the integer

$$n(p,q) := n(\pi(p), \pi(q)), \tag{5.4.13}$$

where $n(\pi(p), \pi(q))$ is defined with respect to (f, g) as in 5.4.10. Clearly, the integers $n(p, q)$ then also satisfy properties $(MS1)$ and $(MS2)$, along with the additional invariance property

$$n(p, q) = n(\gamma.p, \gamma.p) \quad \forall \gamma \in \Gamma. \tag{MS3}$$

Lastly, for each $p \in Cr(\widetilde{f})$, let O_p be the orientation on $W^-(p)$ that is the pullback of the orientation $O_{\pi(p)}$ on $W^-(\pi(p))$ via π.

Definition 5.4.6. The **lifted Morse-Smale complex** $C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f})$ is the cochain complex of free abelian groups

$$C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}) := \bigoplus_{p \in Cr(\widetilde{f})} [O_p] \ , \ C^k(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}) := \bigoplus_{p \in Cr_k(\widetilde{f})} [O_p] \tag{5.4.14}$$

with boundary map

$$\partial : C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}) \to C^{*+1}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f})$$

being the unique $\mathbb{Z}$-linear extension of the assignment

$$\partial[O_p] := \sum_{q \in Cr(\widetilde{f})} n(q, p) \cdot [O_q]. \tag{5.4.15}$$

It follows from MS1 and MS2 that the corresponding differential δ is still well-defined and satisfies $\delta^2 = 0$. Also, observe that the assignment $O_p \mapsto O_{\pi(p)}$ for each $p \in Cr(f)$ gives rise to a $\mathbb{Z}$-linear map of cochain complexes

$$C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}) \to C^*(M, \nabla_g f). \tag{5.4.16}$$

Observe that whenever Γ is infinite, the modules $C^k(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f})$ are not finitely generated over $\mathbb{Z}$, in contrast to the modules $C^k(M, \nabla_g f)$. That is why the ordinary Morse-Smale complex $C^*(M, \nabla_g f)$ might seem better suited for direct computations. However, the essential feature that sets $C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f})$ apart from $C^*(M, \nabla_g f)$ is the intrinsic Γ-action on it, given by permutation of the fibers.

Definition 5.4.7. Given a finite-dimensional, complex representation $\rho : \Gamma \to \mathrm{GL}(V)$, the *twisted Morse-Smale complex* is the following cochain complex of complex vector spaces:

$$C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho) := C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}) \otimes_{\mathbb{Z}} V \tag{5.4.17}$$

$$\partial^\rho := \partial \otimes \mathbb{1}_V : C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho) \to C^{*+1}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho). \tag{5.4.18}$$

This complex comes equipped with a natural Γ-action, given by

$$\gamma.([O_p] \otimes v) := [O_{\gamma.p}] \otimes \rho(\gamma)v \tag{5.4.19}$$

on elementary tensors.

With respect to this action, it is due to property MS3 that the differential ∂^ρ is Γ-equivariant, which allows us to regard $C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho)$ as a $\mathbb{C}[\Gamma]$-module cochain complex. Moreover, it is easily verified that a choice of basis $B \subseteq V$, together with a choice of representatives P, one for each Γ-orbit of each $p \in Cr(f)$ yields a $\mathbb{C}[\Gamma]$-basis $\{q \otimes b : q \in P, b \in B\}$ of $C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho)$. We have thus shown that $C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho)$ **is a free and finite $\mathbb{C}[\Gamma]$-module cochain complex**.

We wish to define an inner product on $C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho)$ that is compatible with the Γ-action. Although there are several ways to proceed, we will derive an inner product on $C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho)$ from a choice of Hermitian form h on the flat quotient bundle $M \times V/\Gamma \downarrow M/\Gamma$. This way, one is able to efficiently compare the L^2-Morse-Smale torsion yet to be defined with the corresponding L^2-analytic torsion, as will be done in the next chapter.
Recall that the metric h lifts to a unique Γ-*equivariant metric* $\widetilde{h}$ on the bundle $M \times V$. This means that we can identify $\widetilde{h}$ with a smooth map $\widetilde{h} : M \to \mathrm{GL}(V, \overline{V^*})$ such that

1. for each $p \in M$, $\widetilde{h}(p) := \widetilde{h}_p$ is a *Riesz*-Isomorphism. Equivalently, for each $p \in M$, the map

$$\langle v, w\rangle_{\widetilde{h}_p} := \widetilde{h}_p(w)(v) \tag{5.4.20}$$

   is a complex inner product on V.

2. For each $\gamma \in \Gamma$ and each pair $v, w \in V$, one has

$$\langle v, w\rangle_{\widetilde{h}_p} = \langle \rho(\gamma)v, \rho(\gamma)w\rangle_{\widetilde{h}_{\gamma.p}}. \tag{5.4.21}$$

For each $p \in M$, we choose an orthonormal basis

$$B_p[h] := \{b_1^{(p)}, \dots, b_m^{(p)}\} \subset V \tag{5.4.22}$$

of the inner product space $(V, \widetilde{h}_p)$.
Since $\widetilde{h}$ is Γ-equivariant, we may assume without loss of generality that we have chosen the bases $B_p[h]$ to satisfy a compatibility of the form

$$B_{\gamma.p}[h] = \rho(\gamma) \cdot B_p[h]. \tag{5.4.23}$$

This allows us to define an inner product $\langle\ ,\ \rangle_h$ on $C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho)$, which is uniquely determined by specifying its orthonormal basis as the set $\{[O_p] \otimes b_i^{(p)} : p \in Cr(f), i = 1, \dots, n\}$. We declare $C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h})$ to be the inner product space $(C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho), \langle\ ,\ \rangle_h)$. Again because of Γ-equivariance of $\widetilde{h}$, is evident that the Γ-action on $C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h})$ it inherits from $C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho)$ is by isometries. Set

$$j := \#Cr(f)\ ,\ j_k := \#Cr_k(f) \tag{5.4.24}$$

and fix a choice of representatives

$$P \subseteq Cr(f), \tag{5.4.25}$$

$$P_k := P \cap Cr_k(f), \tag{5.4.26}$$

one for each Γ-orbit for each $p \in Cr(\widetilde{f})$. For each $p \in Cr(\widetilde{f})$, we let

$$\psi_p[h] \colon V \to \mathbb{C}^m \tag{5.4.27}$$

be the isometric isomorphism of inner product spaces that sends the (ordered) basis $B_p[h]$ to the (ordered) standard basis of $\mathbb{C}^m$. Because of 5.4.23, it is clear that

$$\psi_p[h] = \psi_{\gamma.p}[h] \circ \rho(\gamma) \tag{5.4.28}$$

for each $\gamma \in \Gamma$. This further allows us to construct a $\mathbb{C}[\Gamma]$-isomorphism and an isometry of inner product spaces

$$\Psi_P[h] : \bigoplus_{k=0}^{n} C^k(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h}) \to \bigoplus_{k=0}^{n} \bigoplus_{p \in P_k} \mathbb{C}[\Gamma]_p \otimes \mathbb{C}^m \cong \mathbb{C}[\Gamma]^{jm}, \tag{5.4.29}$$

determined by the assignment

$$[O_{\gamma.p}] \otimes v \mapsto \gamma \cdot 1_p \otimes \psi_{\gamma.p}[h](v) \in \mathbb{C}[\Gamma]_p \otimes \mathbb{C}^n, \tag{5.4.30}$$

for each $p \in P$, each $\gamma \in \Gamma$ and each $v \in V$. Evidently, this assignment satisfies the identity $\gamma.\Psi_P[h]([O_p]\otimes v) = \Psi_P[h]([O_{\gamma.p}] \otimes \rho(\gamma) \cdot v)$ for each $p \in Cr(\widetilde{f})$ and each $v \in V$ because of 5.4.28, and is therefore extendable to a unique $\mathbb{C}[\Gamma]$-linear bijection. Here,

1. for each $p \in P$, $\mathbb{C}[\Gamma]_p$ is a copy of $\mathbb{C}[\Gamma]$, with $1_p = 1$ its unit element,
2. the inner product on $\bigoplus_{k=0}^{n} \bigoplus_{p \in P_k} \mathbb{C}[\Gamma]_p \otimes \mathbb{C}^m$ is the canonical one, i.e. the direct sum of the tensor products of the respective canonical inner products on each factor, and
3. the Γ-action on $\bigoplus_{k=0}^{n} \bigoplus_{p \in P_k} \mathbb{C}[\Gamma]_p \otimes \mathbb{C}^m$ is the direct sum of the left-factor actions, given on elementary tensors by $\gamma.(g \otimes v) \coloneqq (\gamma g) \otimes v$.

It is easy to see that under the identification $\Psi_P[h]$, the differential ∂^ρ becomes a square matrix over $\mathbb{C}[\Gamma]$ of size $j \cdot m$, that is

$$\Psi_P[h] \circ \partial^\rho \circ \Psi_P[h]^{-1} \in \mathrm{Mat}(jm, \mathbb{C}[\Gamma]). \tag{5.4.31}$$

Note that the inner product structure on $C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h})$ only depends on h, and not the choice of P and basis $B_p[h] \subseteq V$.
Just as for the cellular complex, we wish to apply to $C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h})$ the theory of Hilbert $\mathcal{N}(\Gamma)$-modules of finite type. However, just as for the cellular complex, $C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h})$ is complete as an inner product space if and only if Γ is a finite group. To remedy this, we proceed as in the case of the cellular complex:

Definition 5.4.8. The *L^2-Morse-Smale complex* $C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h})$ is defined as

$$C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h}) \coloneqq l^2(\Gamma) \otimes_{\mathbb{C}[\Gamma]} C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h}), \tag{5.4.32}$$

$$\partial^\rho_{(2)} \coloneqq \mathbb{1}_{l^2(\Gamma)} \otimes \partial^\rho : C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h}) \to C^{*+1}_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h}). \tag{5.4.33}$$

Using 5.4.31 and the isometry

$$\Psi^{(2)}_P[h] \coloneqq \mathbb{1}_{l^2(\Gamma)} \otimes \Psi_P[h] : C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h}) \to l^2(\Gamma) \otimes_{\mathbb{C}[\Gamma]} \mathbb{C}[\Gamma]^{lm} \cong l^2(\Gamma)^{lm}, \tag{5.4.34}$$

it becomes apparent that $C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h})$ is a *Hilbert $\mathcal{N}(\Gamma)$-cochain complex of finite type.*

Definition 5.4.9. Let M be a compact manifold, (f, g) a Morse-Smale pair on M, $\rho : \Gamma \coloneqq \pi_1(M) \to \mathrm{GL}(V)$ a finite-dimensional, complex representation and h a Γ-equivariant Hermitian form on $\widetilde{M} \times V \downarrow \widetilde{M}$. Further, suppose that the Hilbert $\mathcal{N}(\Gamma)$-cochain complex $C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h})$ is det-L^2-acyclic. Then, the **L^2-Morse-Smale torsion** associated with the quadruple $(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h})$ is defined as

$$\log\left(T^{MS}_{(2)}(M, \nabla_g f, \rho, h)\right) \coloneqq \log\left(T\left(C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h})\right)\right) = \sum_{k=0}^{n} (-1)^{k+1} \log(\det{}_\Gamma(\partial^\rho_{(2)})). \tag{5.4.35}$$

This L^2-cochain complex, along with its induced torsion element, has made numerous appearances throughout the literature, see for example [22],[21], where only unitary representations ρ were considered, or [102] for a detailed study of $C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho)$ for arbitrary representations ρ.
However, what lacks in the existing literature is a direct comparison of the twisted L^2-Morse-Smale torsion $T^{MS}_{(2)}(M, \nabla_g f, \rho, h)$ with the based/unbased topological L^2-torsion of (M, ρ) that we have previously defined. The relations between the non-L^2-versions of the respective torsion elements are well-known, see for example [12, Chapter 1] or [66, Theorem 9.3]. Our comparison result will be based on the fact that a Morse-Smale pair (f, g) naturally determines a CW-structure on the compact quotient M, whose open cells are in $1:1$-correspondence to $Cr(f)$ and such that its cell-attaching maps are completely determined by the integers $n(p, q)$ defined in 5.4.5.

Theorem 5.4.10. [77, Theorems 3.8, 3.9] *Let M be a compact manifold and (f, g) a Morse-Smale pair on M. Then,*

1. *if (f, g) is of type I, the unstable manifolds $\{W^-(p) : p \in Cr(f)\}$ are the open cells of a CW-structure X_f on M, so that $\partial X_f := X_f \cap \partial M$ is a CW-structure on ∂M. Moreover, the $\mathbb{Z}$-linear extension of the assignment $W^-(p) \mapsto [O_p]$ induces isomorphisms of (relative) $\mathbb{Z}$-cochain complexes*

$$F_{f,g} : C^*(X_f) \to C^*(M, \nabla_g f), \tag{5.4.36}$$

$$F^{\partial M}_{f,g} : C^*(X_f, \partial X_f) \to \bigoplus_{p \in Cr(f)\, p \notin \partial M} [O_p]. \tag{5.4.37}$$

2. *If (f, g) is of type II and Y is any CW-structure on $\partial_- M$, the open cells of Y, together with the unstable manifolds $\{W^-(p) : p \in Cr(f)\}$ form a CW-complex $X_f := Y \sqcup \bigcup_{p \in Cr(f)} W^-(p)$ with Y a subcomplex, so that the inclusion of pairs $(X_f, Y) \hookrightarrow (M, \partial_- M)$ is a homotopy equivalence. Moreover, the $\mathbb{Z}$-linear extension of the assignment $W^-(p) \mapsto [O_p]$ induces isomorphisms of $\mathbb{Z}$-cochain complexes*

$$F_{f,g} : C^*(X_f, Y) \to C^*(M, \nabla_g f). \tag{5.4.38}$$

Remark 5.4.11. For the comparsion results of the next chapter, we will exclusively focus on type II Morse-Smale pairs. This is because the techniques that we will employ require that the critical points of the relevant Morse function f all lie in the interior of M.

Throughout the rest of this section, we assume that f is either of type I or of type II with $\partial_- M = \emptyset$. Let $\widetilde{X}_f \subseteq \widetilde{M}$ be the lift of the CW-complex X_f on $\widetilde{M}$, so that the open cells of $\widetilde{X}_f$ are precisely the unstable manifolds $\{W^-(p) : p \in Cr(\widetilde{f})\}$. Because of the previous Theorem, 5.4.13 and the subsequent Definition 5.4.15 of the boundary operator $\partial : C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}) \to C^{*+1}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f})$, it is clear that the assignment $[O_p] \mapsto W^-(p)$ extends uniquely to an isomorphism of $\mathbb{Z}[\Gamma]$-cochain complexes $\widetilde{F}_{f,g} : C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}) \to C^*(\widetilde{X}_f)$. Note that $\widetilde{F}_{f,g}$ also fits into the commutative diagram of $\mathbb{Z}$-cochain complexes

$$\begin{tikzcd} C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}) \arrow[r, "\widetilde{F}_{f,g}"] \arrow[d, "{[O_p] \mapsto [O_{\pi(p)}]}"] & C^*(\widetilde{X}_f) \arrow[d, "W^-(p) \mapsto W^-(\pi(p))"] \\ C^*(M, \nabla_g f) \arrow[r, "F_{f,g}"] & C^*(X_f) \end{tikzcd}.$$

Choosing an admissible basis pair $[E, B]$ for the twisted cellular complex $C^*(\widetilde{X}_f, \rho)$ (cf. Definition 5.2.1), the map

$$F^{\rho}_{f,g} \coloneqq \widetilde{F}_{f,g} \otimes \mathbb{1}_V : C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h}) \to C^*(\widetilde{X}_f, \rho)[E, B], \tag{5.4.39}$$

is therefore an isomorphism of $\mathbb{C}[\Gamma]$-cochain complexes. We claim that it is also bounded, when regarded as a linear map between the underlying inner product spaces. Consequently, we would obtain an isomorphism of Hilbert $\mathcal{N}(\Gamma)$ cochain-complexes

$$F^{\rho,(2)}_{f,g} : \mathbb{1}_{l^2(\Gamma)} \otimes F^{\rho}_{f,g} : C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h}) \to C^*_{(2)}(\widetilde{X}_f, \rho)[E, B], \tag{5.4.40}$$

$$F^{\rho,(2)}_{f,g}[k] \coloneqq F^{\rho,(2)}_{f,g}|_{C^k_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h})}. \tag{5.4.41}$$

To prove the claim, first observe that by Lemma 5.2.2, we are free to choose whatever set E of cell-orbit representatives we like. Specifically, we choose a fundamental domain $\mathcal{F} \subseteq \widetilde{M}$ for the Γ-action on $\widetilde{M}$, representatives $P \coloneqq Cr(\widetilde{f}) \cap \mathcal{F}$ and cell representatives $E_f \coloneqq \{W^-(p) : p \in P\}$ that induce by 5.2.14 and 5.4.29 isometries of inner product spaces

$$\Psi_{E_f,B} : C^*(\widetilde{X}_f, \rho)[E_f, B] \to \bigoplus_{k=0}^{n} \bigoplus_{p \in p_k} \mathbb{C}[\Gamma] \otimes \mathbb{C}^m,$$

$$\Psi_P[h] : C^*(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h}) \to \bigoplus_{k=0}^{n} \bigoplus_{p \in p_k} \mathbb{C}[\Gamma] \otimes \mathbb{C}^m\,.$$

From the explicit formulas of $\Psi_P[h]$ and $\Psi_{E,B}$, one easily sees that the map $\Psi_{E_f,B} \circ F^{\rho}_{f,g} \circ (\Psi_P[h])^{-1}$ is a diagonal matrix of the form

$$\Psi_{E_f,B} \circ F^{\rho}_{f,g} \circ\ (\Psi_P[h])^{-1} = \Big(\mathbb{1}_{\mathbb{C}[\Gamma]} \otimes M^B_{B_p[h]}\Big)_{p \in P}, \tag{5.4.42}$$

which proves our claim. Together with Proposition 4.1.14 and Lemma 4.1.15, we conclude that

$$\log\Big(\det\nolimits_\Gamma(F^{\rho,(2)}_{f,g})\Big) = \sum_{p \in P} \log|\det(M^B_{B_p[h]})|, \tag{5.4.43}$$

$$\log\Big(\det\nolimits_\Gamma(F^{\rho,(2)}_{f,g}[k])\Big) = \sum_{p \in P_k} \log|\det(M^B_{B_p[h]})|. \tag{5.4.44}$$

Due to Proposition 4.1.40, we can now summarize our investigation as follows

Corollary 5.4.12. *Let M be a compact manifold, (f, g) a Morse-Smale pair on M, $\rho : \Gamma \coloneqq \pi_1(M) \to \mathrm{GL}(V)$ a finite-dimensional, complex representation and h a Γ-equivariant Hermitian form on $\widetilde{M} \times V \downarrow \widetilde{M}$. Further, let $B \subseteq V$ be a fixed basis for V. Then, there exists a Γ-CW-complex $\widetilde{X}_f \subseteq \widetilde{M}$, along with a choice of representatives E_f of cells for $\widetilde{X}_f$, one for each Γ-orbit, such that the following results hold:*

1. *If f is of type I with, one has $\widetilde{X}_f = \widetilde{M}$.*

2. *If f is of type II with $\partial_- M = \emptyset$, the inclusion $\widetilde{X}_f \hookrightarrow \widetilde{M}$ is a Γ-homotopy equivalence.*

3. *The Hilbert $\mathcal{N}(\Gamma)$-cochain complexes $C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h})$ and $C^*_{(2)}(\widetilde{X}_f, \rho)[E_f, B]$ are isomorphic.*

4. *The complex $C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \rho, \widetilde{h})$ is det-L^2-acyclic if and only if $C^*_{(2)}(\widetilde{X}_f, \rho)[E_f, B]$ is det-L^2-acyclic. In this case, one has*

$$\log\left(\frac{T^{MS}_{(2)}(M, \nabla_g f, \rho, h)}{T^{CW}_{(2)}(X_f, \rho)[E_f, B]}\right) = \sum_{k=0}^{n} (-1)^k \sum_{p \in P_k} \log|\det(M^B_{B_p[h]})|. \tag{5.4.45}$$

This result emphasizes the fact that the anomaly of (based) cellular torsion and the Morse-Smale torsion is trivial if one can find an Hermitian form h and a fixed basis $B \subseteq V$ that are appropriately *compatible* with each other:

Lemma 5.4.13. *Let V be a finite-dimesional real vector space of dimension m and let $h \in \mathrm{GL}(V, V^*)$. For a fixed (ordered) basis $B \subseteq V$ and its dual basis $B^* \subseteq \overline{V^*}$, let $\phi_B : \mathbb{R}^m \to V$ and $\phi_{B^*} : \mathbb{R}^m \to V^*$ be the isomorphisms identifying the ordered standard basis with the ordered basis B, respectively B^*. Set*

$$h^B := \phi_{B^*}^{-1} \circ h \circ \phi_B \in \mathrm{GL}(m, \mathbb{R}). \tag{5.4.46}$$

Then, for any other basis $C \subseteq V$, it holds that

$$\det(h^C) = \det(h^B) \cdot \det(M_C^B)^2, \tag{5.4.47}$$

Here, as everywhere else, $M_C^B := \phi_B^{-1} \circ \phi_C \in \mathrm{GL}(m, \mathbb{R})$ denotes the base change matrix.

Proof. Immediately follows from the identities $(M_{B^*}^{C^*})^t = (M_C^B)$ and $h^C = M_{B^*}^{C^*} \circ h^B \circ M_C^B$. □

Definition 5.4.14 (Unimodular metric)**.** Let M be a compact Riemannian manifold and let $\rho : \Gamma := \pi_1(M) \to V$ be a representation on M. Further, let f be a Morse function on M and let $\widetilde{M}$ be the universal cover of M. A metric h on the associated flat bundle $E_\rho \downarrow M$ is called *unimodular*, if for any basis $B \subseteq V$ one has

$$\det(\widetilde{h}^B) \equiv c, \tag{5.4.48}$$

for some constant $c > 0$. Here, $\widetilde{h}$ is the lift of h on the trivial bundle $\widetilde{M} \times V \downarrow \widetilde{M}$.

Observe that by the previous lemma, unimodularity of a form h needs only be checked for *some* arbitrary basis $B \subseteq V$. In the next subsection, we will show that *any unimodular bundle admits a unimodular metric*. For now, let us formulate the most important consequence that can be derived from the existence of a unimodular metric.

Theorem 5.4.15. *Let M be a compact Riemannian manifold, and let $\rho : \Gamma = \pi_1(M) \to V$ be a det-L^2-acyclic unimodular representation on M. Further, let (f, g) be a Morse-Smale pair on M and let h be a unimodular metric on the associated flat bundle $E_\rho \downarrow M$. Lastly, assume either that*

(a) *f is of type I, or*

(b) $\mathrm{Wh}(\Gamma) = 0$ *and f is of type II with $\partial_- M = \emptyset$.*

Then, we get an equality of torsion elements

$$T_{(2)}^{MS}(M, \nabla_g f, \rho, h) = T_{(2)}^{Top}(M, \rho). \tag{5.4.49}$$

Proof. (1): Choose as B an orthonormal basis on the inner product space $(V, \widetilde{h}_{p_0})$, where $p_0 \in Cr(\widetilde{f})$ is arbitrarily picked. Since h is unimodular, it follows that

$$\det(\widetilde{h}_p^B) = \det(\widetilde{h}_{p_0}^B) = 1$$

for any other $p \in Cr(\widetilde{f})$. Lemma 5.4.13 then further implies that

$$\det(M^B_{B_p[h]})^2 = \det(\widetilde{h}^B_p)^{-1} \det(\widetilde{h}^{B_p[h]}_p) = 1.$$

Choose as in Theroem 5.4.10 and Corollary 5.4.12 the CW-complex $X_f \subseteq M$, its Γ-equivariant lift $\widetilde{X}_f \subseteq \widetilde{M}$ and the $\mathbb{Z}[\Gamma]$-basis of cells E_f on $C^*(\widetilde{X}_f)$. It then follows that

$$\log\left(\frac{T^{MS}_{(2)}(M, \nabla_g f, \rho, h)}{T^{CW}_{(2)}(X,\rho)[E_f, B]}\right) = \sum_{k=0}^{n}(-1)^k \sum_{p\in P_k} \log|\det(M^B_{B_p[h]})| = 0.$$

Assuming either one of the conditions (a) or (b), it further follows that the Γ-homotopy equivalence $\widetilde{X}_f \hookrightarrow \widetilde{M}$ is simple. Now since $\rho : \Gamma \to \mathrm{GL}(V)$ is unimodular, we may apply Theorem 5.2.10, followed by Theorem 5.3.12 to conclude that

$$T^{CW}_{(2)}(X,\rho)[E_f, B] = T^{Top}_{(2)}(M,\rho).$$

□

5.4.1 Unimodular metrics

The goal of this subsection is to show that any flat bundle $E_\rho \downarrow M$ over a compact manifold M, associated to a *unimodular* representation $\rho : \pi_1(M) \to \mathrm{GL}(V)$, admits a unimodular metric, and to give equivalent characterizations for such metrics in terms of differential forms. The latter part will prove to be very valuable when comparing the L^2-analytic with the L^2-topological torsion of the pair (M,ρ), as will be done in the last chapter.

Unless specifically stated otherwise, all appearing complex vector bundles/vector spaces are considered as real bundles/vector spaces with regards to their natural underlying ***real*** *scalar multiplication:* Let M be a compact manifold, $\rho : \Gamma := \pi_1(M) \to \mathrm{GL}(V)$ a finite-dimensional, complex representation of dimension m and $E_\rho \downarrow M$ the associated flat bundle over M associated to ρ (of real dimension $2m$). Let $E^*_\rho \downarrow M$ be the dual bundle and let $\Lambda^{2m}E^*_\rho \downarrow M$ the real $2m$-th exterior power of E^*_ρ. Observe that since E_ρ is the underlying real bundle of a complex bundle, it is orientable. Equivalently $\Lambda^{2m}E^*_\rho \downarrow M$ is the trivial $\mathbb{R}$-bundle over M.

Given a Riemannian metric $h \in \mathrm{GL}(E_\rho, E^*_\rho)$ and a fixed orientation, we obtain an oriented atlas of E_ρ (i.e. a cover of M by local trivializations of E_ρ so that the transitions functions have positive determinant), further allowing us to construct the **volume form**

$$\sigma_h \in \Gamma(M, \Lambda^{2m}E^*_\rho) \tag{5.4.50}$$

induced by h. It is the section uniquely determined by the equality

$$(\Phi_U^{-1} \circ \sigma_h)(x) = \det\left(h_x^{\phi_U^x(B)}\right) b^1 \wedge \cdots \wedge b^{2m} \tag{5.4.51}$$

for each $x \in M$. Here, $B = \{b_1, \ldots, b_{2m}\} \subseteq V$ is a basis of V with $B^* = \{b^1, \ldots, b^{2m}\} \subseteq \overline{V^*}$ its dual basis,

$$\begin{tikzcd} U \times V \arrow[rr, "\phi_U"] \arrow[dr, "pr_1"'] & & (E_\rho)|_U \arrow[dl] \\ & U & \end{tikzcd} \qquad \phi_U^y := \phi_U(y, \,.\,) : V \to (E_\rho)_y \tag{5.4.52}$$

is a flat trivialization of E_ρ over U, which naturally induces a flat trivialization on $\Lambda^{2m}E_\rho^*$ over U

$$\begin{tikzcd} U \times \Lambda^{2m}\overline{V^*} \arrow[rr, "\Phi_U"] \arrow[dr, "pr_1"'] & & (\Lambda^{2m}E_\rho^*)|_U \arrow[dl] \\ & U & \end{tikzcd}$$

To see that σ_h is well-defined, one has to show that a different choice of basis $C \subseteq V$, as well as a different choice of flat trivialization over a point x yields the same element $\sigma_h(x)$. This follows from Lemma 5.4.13, together with the fact that, due to orientability of E_ρ, the determinant of a (locally constant) transition function of two flat trivializations, which have been chosen to lie in the same orientation class, is positive. Let ∇ be the flat canonical connection on E_ρ. Observe that the ∇ canonically induces a flat *dual* connection ∇^* on the dual bundle $E_\rho^* \downarrow M$, which in turn induces a flat connection $\Lambda^{2m}\nabla^*$ on the real line bundle $\Lambda^{2m}E_\rho^*$, the $2m$-th exterior power of E_ρ^*.
The connection ∇, along with its dual connection ∇^*, together also induce a canonical connection ∇^{Hom} on the homomorphism bundle $\mathrm{Hom}(E_\rho, E_\rho^*) \downarrow M$. For a Hermitian form h on $E_\rho \downarrow M$, this allows us to define a 1-Form $\omega(\rho, h)$, taking values in the endomorphism bundle $\mathrm{End}(E_\rho)$, as well as a $\mathbb{C}$-valued 1-form $\theta(h) = \theta(\rho, h)$ via

$$\omega(\rho, h) := h^{-1}\nabla^{\mathrm{Hom}}h \in \Omega^1(M, \mathrm{End}(E_\rho)), \tag{5.4.53}$$

$$\theta(\rho, h) := \mathrm{tr}\,(\omega(\rho, h)) \in \Omega^1(M). \tag{5.4.54}$$

As highlighted in the next lemma, $\theta(h)$ measures precisely the flatness of the volume form σ_h.

Lemma 5.4.16. *Let $E_\rho \downarrow M$ be the flat bundle over a compact manifold associated to a finite-dimensional, complex representation $\rho : \Gamma := \pi_1(M) \to \mathrm{GL}(V)$. Let h be a Hermitian form on $E_\rho \downarrow M$ and let $\widetilde{h} : C^\infty(\widetilde{M}, \mathrm{GL}(V, \overline{V^*}))$ be its lift to $\widetilde{M}$. Then, the following assertions are equivalent*

1. *The volume form σ_h is flat, i.e. one has $\Lambda^m\nabla^*\sigma_h = 0$*
2. *$\theta(\rho, h) = 0$,*
3. *h is unimodular.*

Proof. (1) $\Leftrightarrow$ (3) : Observe that the condition $\Lambda^m\nabla^*\sigma_h = 0$ can be checked locally. Since the flat connection ∇ on E_ρ pulls back to the trivial connection on $\widetilde{M} \times V \downarrow \widetilde{M}$, the condition $\Lambda^m\nabla^*\sigma_h = 0$ is equivalent to the equality $D\sigma_{\widetilde{h}} \equiv 0$, where $\sigma_{\widetilde{h}} \in C^\infty(\widetilde{M}, \Lambda^{2m}V^*)$, denotes the volume form of the lifted metric $\widetilde{h}$. Fixing a real basis $B = \{b_1, \ldots, b_{2m}\}$ with $B^* = \{b^1, \ldots, b^{2m}\}$ the dual basis, one has by definition

$$\sigma_{\widetilde{h}}(x) = \det(\widetilde{h}^B(x)) \cdot b^1 \wedge \cdots \wedge b^{2m}$$

for each $x \in \widetilde{M}$, from which the equivalence $D\sigma_{\widetilde{h}} \equiv 0 \Leftrightarrow \det(\widetilde{h}^B(x)) \equiv c$ readily follows.
(1) $\Leftrightarrow$ (2) : The vanishing of the two forms is a local condition. Therefore, similar as in the previous paragraph, we will fix a flat (local) trivialization $\phi_U : U \times V \to E^\rho|_U$ and identify σ_h, respectively $\theta(h, \rho)$ with their corresponding pullbacks under ϕ_U, i.e. elements in $C^\infty(U, \Lambda^{2m}V^*)$, respectively $\Omega^1(U)$. This way, if $U \cong \mathbb{R}^n$ is additionally a coordinate neighborhood of M (so that $\Omega^1(U)$ can further be identified with $C^\infty(U, \mathbb{R}^n)$), one verifies via direct computation that for $x = (x_1, \ldots, x_n) \in U$ and $B \subseteq V$ a fixed

basis with $B^* = \{b^1, \ldots, b^{2m}\}$ its dual basis, one has

$$\sigma_h(x) = \det(h^B(x)) \cdot b^1 \wedge \cdots \wedge b^{2m} \in C^\infty(U, \Lambda^{2m}V^*), \tag{5.4.55}$$

$$\theta(h,\rho)(x) = \sum_{i=1}^{n} \operatorname{tr}(h^B(x)^{-1}\frac{\partial}{\partial x_i}h^B(x)) \cdot dx_i \in \Omega^1(U) \cong C^\infty(U, \mathbb{R}^n). \tag{5.4.56}$$

The result, i.e. the equivalence $D\sigma_h \equiv 0 \Leftrightarrow \theta(h,\rho) \equiv 0$ now follows from *Jacobi's formula*

$$\frac{\partial}{\partial x_i}\det(A(x)) = \det(A(x)) \cdot \operatorname{tr}(A(x)^{-1}\frac{\partial}{\partial x_i}A(x)) \tag{5.4.57}$$

[61, pp. 149-150], which holds true for any smooth function $A \in C^\infty(\mathbb{R}^n, \mathrm{GL}(m,\mathbb{R}))$. □

Lemma 5.4.17. *Any flat bundle $E_\rho \downarrow M$ over a connected manifold M induced by a unimodular representation $\rho : \Gamma \to \mathrm{GL}(V)$ admits a unimodular metric h. In fact, for any basepoint $x_0 \in M$ and* ***any*** *choice of Hermitian metric $h \in \mathrm{GL}(E_\rho, \overline{E_\rho^*})$, there exists a smooth function $f_{x_0} \in C^\infty(M, \mathbb{R}_{>0})$ with the property that*

(a) $f_{x_0}(x_0) = 1$,

(b) *for $y \in M$ with h unimodular in a connected neighborhood $U \ni y$ of y, one has $f|_U \equiv f_{x_0}(y)$,*

(c) *$f \cdot h$ is unimodular.*

Proof. Consider the volume form $\sigma_h \in \Gamma(\Lambda^{2m}E_\rho^*)$ induced by h. For any point $y \in M$, choose a smooth curve $\gamma_y : [0,1] \to M$ with $\gamma_y(0) = x_0$ and $\gamma_y(1) = 1$. Let $P_{\gamma_y} : \Lambda^{2m}E_\rho^*(x_0) \to \Lambda^{2m}E_\rho^*(y)$ be the parallel transport along γ_y with respect to the flat connection $\Lambda^{2m}\nabla^*$. Since σ_h is nowhere-vanishing and P_{γ_y} preserves orientations, it follows that there exists some constant $c(y) > 0$, such that

$$P_{\gamma_y}(\sigma_h(x_0)) = c(y) \cdot \sigma_h(y). \tag{5.4.58}$$

We claim that the choice of $c(y)$ does only depend on y, and not on the explicit path γ_y. Assuming the claim, multiplying the function $f_{x_0} : M \to \mathbb{R}_{>0}$ with $f_{x_0}(y) \coloneqq c(y)$ to h, the induced volume form $\sigma_{f \cdot h} = f \cdot \sigma_h$ then clearly is flat, which is why $f \cdot h$ is unimodular by the previous lemma, so that $(a)-(c)$ are satisfied.

To prove the claim, assume that $\gamma_y' : [0,1] \to M$ is another path with initial point x_0 and end point y. Let $(\gamma_y')^{-1} \cdot \gamma_y$ be the concatenation of the two paths. It is a loop based at x_0, and therefore determines an element $\beta \in \pi_1(M, x_0)$. The corresponding parallel transports $P_{\gamma_y'} : \Lambda^{2m}E_\rho^*(x_0) \to \Lambda^{2m}E_\rho^*(y)$ and $P_{(\gamma_y')^{-1}\cdot\gamma_y} : \Lambda^{2m}E_\rho^*(x_0) \to \Lambda^{2m}E_\rho^*(x_0)$ then fit into a diagram of isomorphisms

$$\begin{tikzcd} & \Lambda^{2m}E_\rho^*(y) & \\ \Lambda^{2m}E_\rho^*(x_0) \arrow[ur, "P_{\gamma_y}"] \arrow[rr, "P_{(\gamma_y')^{-1}\cdot\gamma_y}"] & & \Lambda^{2m}E_\rho^*(x_0) \arrow[ul, "P_{\gamma_y'}"'] \end{tikzcd}$$

Moreover,

$$P_{(\gamma_y')^{-1}\cdot\gamma_y}(\omega_h(x_0)) = |\det(\rho(\beta))|^2 \cdot \omega_h(x_0) \overset{\rho \text{ unimodular}}{=} \omega_h(x_0).$$

It follows that we have an equality of parallel transports $P_{\gamma_y'} = P_{\gamma_y}$, readily implying the claim. □

For the last chapter, we will also need the existence of the following, more refined version of constructing a unimodular metric out of a given, partially defined metric.

Corollary 5.4.18. *Let $E \downarrow M$ be a flat, unimodular bundle over a connected manifold M and $U = \bigsqcup_{i\in I} U_i \subseteq M$ a subset with each U_i open and connected. Let $x_0 \in \mathrm{Int}(M \setminus U)$ and $x_i \in U_i$ for each $i \in I$ be chosen basepoints with curves $c_i \subseteq M$ connecting x_0 to x_i. Further, let $\widetilde{h}_0$ be a Hermitian metric on E_{x_0} and $\widetilde{h}_i$ a Hermitian metric on E_{x_i} satisfying*

$$\det(\widetilde{h}_i \cdot P^*_{c_i}(\widetilde{h_0})^{-1}) = 1, \tag{5.4.59}$$

where $P_{c_i} : \mathrm{GL}(E_{x_0}, \overline{E^*_{x_0}}) \to \mathrm{GL}(E_{x_i}, \overline{E^*_{x_i}})$ denotes the parallel transport along the curve c_i. Then, for any unimodular metric $\bigsqcup h_i$ on $E|_U$ extending $\bigsqcup \widetilde{h}_i$, there exists a global unimodular metric h on E further extending $\bigsqcup h_i \sqcup \widetilde{h}_0$.

Proof. Choose some extension h' of $\bigsqcup h_i \sqcup \widetilde{h}_0$ on M. With $f_{x_0} : M \to \mathbb{R}_{>0}$ the smooth function defined as in the previous lemma, the form $h \coloneqq f_{x_0} \cdot h'$ is the desired unimodular extension of $\bigsqcup h_i \sqcup \widetilde{h}_0$. □

5.5 Applications to the representation bundle $E^\rho \downarrow \mathbb{H}^n$

Let $\mathbb{H}^n$ be hyperbolic n-space for n odd, let $G \coloneqq \mathrm{Isom}^+(\mathbb{H}^n)$ and let $E^\rho \downarrow \mathbb{H}^n$ be the flat, canonical, Hermitian bundle associated to an irreducible, complex, m-dimensional representation $\rho : G_\mathbb{C} \to \mathrm{GL}(V)$. For a fixed non-uniform lattice $\Gamma \subset G$, $R \geq 0$ and $w > 0$, let M_R, C_R and T_R be the complete, Γ-invariant submanifolds associated to it, as constructed in Section 2.3. In the same section, we have constructed a G-equivariant metric h_ρ on $E^\rho \downarrow \mathbb{H}^n$.

Lemma 5.5.1. *The metric h_ρ is unimodular.*

Proof. For a point $x \in \mathbb{H}^n$ and a fixed ordered basis $B \subseteq V$, the matrix $h^B_\rho(x) \in \mathrm{GL}_m(\mathbb{C}) \cong \mathrm{GL}_{2m}(\mathbb{R})$ is defined as

$$h^B_\rho(x) = \phi^{-1}_{B^*} \circ h_\rho(x) \circ \phi_B \in \mathrm{GL}_m(\mathbb{C}) \cong \mathrm{GL}_{2m}(\mathbb{R}), \tag{5.5.1}$$

where $\phi_B : \mathbb{C}^n \to V$ is the isomorphism identifying the ordered standard basis of $\mathbb{C}^m$ with the ordered basis B. In order to show that h_ρ is unimodular, it suffices to show by definition that for any two points $x, y \in \mathbb{H}^n$, one has

$$\det(h^B_\rho(x)) = \det(h^B_\rho(y)). \tag{5.5.2}$$

To this effect, since G acts transitively on $\mathbb{H}^n$, there exists $\gamma \in G$ with $y = \gamma.x$. Since h_ρ is G-equivariant, it follows that

$$h_\rho(x) = \dot{\rho}(\gamma) \circ h_\rho(y) \circ \rho(\gamma), \tag{5.5.3}$$

where $\dot{\rho} : G_\mathbb{C} \to \mathrm{GL}(\overline{V^*})$ is the representation contragredient to ρ. As ρ is unimodular, one has

$$|\det(\rho(\gamma))| = |\det(\dot{\rho}(\gamma))| = 1. \tag{5.5.4}$$

The desired result 5.5.2 now can readily be derived from 5.5.1, 5.5.3 and 5.5.4. □

Theorem 5.5.2. *The unbased topological L^2-torsion of the pair $(\Gamma\backslash\mathbb{H}^n, \rho)$ can be defined. The same is true for $(\Gamma\backslash M_R, \rho)$, for any fixed (arbitrary) $R > 0$.*
Further, let (f, g) be a Morse-Smale pair on the compact quotient $\Gamma\backslash M_R$ which is either of type I or of type II with $\partial_- M_R = \emptyset$. Let $(\widetilde{f}, \widetilde{g})$ be the lift of (f, g) to M_R. Then, the associated L^2-Morse-Smale cochain complex $C^*_{(2)}(M_R, \nabla_{\widetilde{g}}\widetilde{f}, \rho, h_\rho)$ is det-L^2-acyclic. Finally, we have an equality of L^2-torsions

$$T^{MS}_{(2)}(\Gamma\backslash M_R, \nabla_g f, \rho, h_\rho) = T^{Top}_{(2)}(\Gamma\backslash M_R, \rho) = T^{Top}_{(2)}(\Gamma\backslash\mathbb{H}^n, \rho). \tag{5.5.5}$$

Proof. Let X be some CW-structure on $\Gamma\backslash M_R$ and $\widetilde{X}$ its lift onto M_R. Corollary 4.2.18 and Theorem 6.3.5 together show that the the associated cellular L^2-cochain complex $C^*_{(2)}(\widetilde{X}, \rho)$ is det-L^2-acyclic. Moreover, since Γ is the fundamental group of the complete, non-positively curved, locally symmetric space $\Gamma\backslash\mathbb{H}^n$, one has $\mathrm{Wh}(\Gamma) = 0$ by [36, Proposition 0.10]. Now observe that the inclusion $\Gamma\backslash M_R \hookrightarrow \Gamma\backslash\mathbb{H}^n$ is a homotopy equivalence, which is why $\chi(\Gamma\backslash\mathbb{H}^n) = \chi(\Gamma\backslash M_R) = 0$, since $\Gamma\backslash M_R$ is odd-dimensional with toroidal boundary. Therefore, the pair $(\Gamma\backslash\mathbb{H}^n, \rho)$ satisfies all assumptions of Definition 5.3.14, which is why the unbased topological L^2-torsion $T^{Top}_{(2)}(\Gamma\backslash\mathbb{H}^n, \rho)$ can be defined, so that

$$T^{Top}_{(2)}(\Gamma\backslash\mathbb{H}^n, \rho) = T^{Top}_{(2)}(\Gamma\backslash M_R, \rho). \tag{5.5.6}$$

Finally, recall that have shown in the previous lemma that h_ρ is unimodular. Together with $\mathrm{Wh}(\Gamma) = 0$ and $\chi(\Gamma\backslash M_R) = 0$, it follows by Theorem 5.4.15 that for a Morse-Smale pair (f, g) on $\Gamma\backslash M_R$, such that f either of type I or of type II with $\partial_- M = \emptyset$, one has

$$T^{MS}_{(2)}(\Gamma\backslash M_R, \nabla_g f, \rho, h_\rho) = T^{Top}_{(2)}(\Gamma\backslash M_R, \rho), \tag{5.5.7}$$

as desired. □

Chapter 6

The L^2-Cheeger-Müller theorem on manifolds with boundary

The goal of this chapter is to prove that for a compact manifold-with-boundary M and a unimodular representation $\rho : \pi_1(M) \to \mathrm{GL}(V)$, the combinatorial L^2-invariants that we have defined in the previous chapter *essentially* agree with their analytic counterparts, as introduced in Chapter 4. For the associated Betti-Numbers and Novikov-Shubin invariants, the relevant result is Theorem 6.3.5, which states an honest equality of the combinatorial and analytic versions without any further conditions, holding even if ρ is not unimodular. For the associated L^2-torsions, however, unimodularity of the representation ρ becomes essential. Only then, we can find a *unimodular* metric h on the associated flat bundle $E_\rho \downarrow M$ with associated flat volume form σ_h. Given some Riemannian metric g on M, we will show that flatness of this form implies that the anomaly $\log(T^{An}_{(2)}(M, \rho, g, h)) - \log(T^{Top}_{(2)}(M, \rho))$ depends only on the restriction of g near ∂M and the dimension of the representation ρ. This will also provide us with the final ingredient in the proof of Corollary C and Theorem E, which are carried out in the last section of this chapter.

6.1 Preliminaries

By a **system** $\mathcal{D} = (E \downarrow M, g, h, X)$, we will always mean a set of data consisting of a flat, complex vector bundle $E \downarrow M$ over a smooth manifold M, along with a Riemannian metric g on M, a Hermitian form h on E and X either a vector field or a complex-valued function over M.

Given a uniform lattice $\Gamma < \mathrm{Isom}(M, g)$, such a system $\mathcal{D}$ is called **Γ-invariant** if in addition, the isometric action of Γ on (M, g) leaves X invariant and extends to an action of bundle isometries on the metric bundle $(E, h) \downarrow (M, g)$. Observe that Γ-invariant systems on M are precisely the lifts of systems defined over the compact quotient M/Γ.

Throughout this chapter, we will frequently form *products* of systems: Given for $i = 1, 2$ two systems $(E_i \downarrow M_i, g_i, h_i, X_i)$ with X_i either both vector fields or functions, one obtains a new system $(E_1 \hat{\otimes} E_2 \downarrow M_1 \times M_2, g_1 \oplus g_2, h_1 \hat{\otimes} h_2, X_1 + X_2)$, where $M_1 \times M_2$ is the product manifold equipped with the (direct) sum metric $g_1 \oplus g_2$, $X_1 + X_2$ is the sum of the two vector fields or functions, and

- $E_1 \hat{\otimes} E_2 \downarrow M_1 \times M_2$ is defined to be the flat tensor product bundle $\pi_1^* E_1 \otimes \pi_2^* E_2 \downarrow M_1 \times M_2$, where $\pi_i : M_1 \times M_2 \to M_i$ denotes the projection onto the i-th factor. Here, the flat structure we choose is the canonical one induced by its flat factors $\pi_i^* E_i$. Moreover,
- $h_1 \hat{\otimes} h_2 \coloneqq \pi_1^* h_1 \otimes \pi_2^* h_2$ is the tensor product of the respective pullback Hermitian forms.

The main focus of our attention will be **Morse-Smale systems**, which are by definition systems $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ with (f, g') a Morse-Smale pair.

Definition 6.1.1. A Morse-Smale system of the form $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ will be called a **type II Morse-Smale system** if (f, g') is a type II Morse-Smale pair with absolute boundary conditions. Recall from Definitions 5.4.1 and 5.4.3 that this means that the following conditions are satisfied:

(II_1) For any $0 \le k \le n$ and any $p \in Cr_k(f)$, there exists (pairwise disjoint) coordinate neighborhoods $\phi_p : U_p \to \mathbb{R}^n$ of p disjoint from ∂M, with $\phi_p(p) = 0$ and such that we have $(f \circ \phi_p^{-1})(x_1, \ldots, x_n) = f(p) - \frac{1}{2}(x_1^2 + \ldots x_k^2) + \frac{1}{2}(x_{k+1}^2 + \cdots + x_n^2)$.

(II_2) The pullback metric $\phi_p^*(g_{\mathbb{R}^n})$ of the standard Euclidean metric on $\mathbb{R}^n$ equals $g'|_{U_p}$.

(II_3) There exists a collar neighborhood U of ∂M, disjoint from $\bigcup_{p \in Cr(f)} U_p$, along with a diffeomorphism $\psi_{g'} : \partial M \times [0, \epsilon) \to U$, coming from the normal exponential map induced by g', so that $(f \circ \phi)(p, t) = b - t$ with $b = \max(f) \in \mathbb{Z}$ (In particular $\partial M \subseteq f^{-1}(b)$ and $Cr(f) \cap \partial M = \emptyset$).

A type II Morse-Smale system is **of product form**, if

(P_1) *g is a product near ∂M*: There exists a collar neighborhood V of ∂M that is the diffeomorphic image of the normal exponential map $\psi_g : \partial M \times [0, \epsilon) \to V$ induced by g, such that $\psi_g^*(g|_V) = g|_{\partial M} \oplus dt^2$, where dt^2 denotes the standard Euclidean metric on $\mathbb{R}$.

(P_2) The isometry ψ_g further extends to a flat bundle isometry

$$\Psi : (E|_{\partial M} \hat{\otimes} E_{\mathbb{C}} \downarrow \partial M \times [0, 1), h|_{\partial M} \hat{\otimes} 1_{\mathbb{C}}) \to (E|_V \downarrow V, h_V).$$

Here, $E_{\mathbb{C}} \downarrow [0, 1)$ is the trivial 1-dimensional vector bundle over $[0, 1)$ (with trivial connection), $E|_{\partial M} \hat{\otimes} E_{\mathbb{C}} \downarrow \partial M \times [0, 1)$ denotes the flat, complex product bundle as introduced in the previous paragraph and $h_{\mathbb{C}}$ denotes the canonical constant Hermitian form on $E_{\mathbb{C}}$.

A type II Morse-Smale system of product form is called *weakly admissible*, if

(A_1) M is compact.

(A_2) One has $g \equiv g'$ near $Cr(f)$ and outside from a neighborhood of ∂M.

(A_3) For each $p \in Cr(f)$, the isometric embedding $\phi_p : (U_p, g'|_{U_p}) \to (\mathbb{R}^n, g'_{\mathbb{R}^n})$ extends to flat bundle isometry $\Phi_p : (E|_{U_p}, h|_{U_p}) \to (\mathbb{C}^m \times \mathbb{R}^n, h_{\mathbb{C}^m})$. Here, as everywhere else, $h_{\mathbb{C}^m}$ denotes the ordinary (constant) inner product on $\mathbb{C}^m$.

Finally, a weakly admissible system $\mathcal{D}$ is called **admissible** if the following extra compatibility condition is satisfied:

(A_4) $h|_{\partial M}$ is unimodular.

A Γ-invariant system $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ that is the lift of an admissible, respectively weakly admissible system on the compact quotient M/Γ is called **Γ-admissible**, respectively **weakly Γ-admissible**.

Observe that a weakly admissible system is a Morse-Smale system on a compact manifold M with special *local* conditions on the Riemannian metric g and Hermitian form h near ∂M and the critical points of f, while for an admissible system, we additionally demand a *global* condition on $h|_{\partial M}$. In particular, it follows from the discussion laid out the previous chapter that any flat bundle $E \downarrow M$ over a compact manifold fits into some weakly admissible system $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$, which can be chosen admissible if and only if the restriction bundle $E|_{\partial M} \downarrow \partial M$ is unimodular.
We now describe for a general Morse-Smale system $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ with M compact the construction of the **relative L^2-torsion** $\mathcal{R}(\mathcal{D}) \in \mathbb{R}$, provided that $E \downarrow M$ is determinant class. Let

$$C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}'} \widetilde{f}, \widetilde{E}, \widetilde{h}) \coloneqq C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}'} \widetilde{f}, \rho, \widetilde{h}) \tag{6.1.1}$$

be the L^2-Morse-Smale complex defined as in Lemma 5.4.6 (because the comparsion with the analytic torsion is key for this chapter, we suppress the representation ρ from the notation and replace it by the flat bundle $\widetilde{E}$ associated to ρ). Assuming that f has range $[a, b]$, we define

$$\partial_- M \colon = \begin{cases} f^{-1}(a) \cap \partial M & f \text{ is of type II}, \\ \emptyset & \text{else}. \end{cases} \tag{6.1.2}$$

Note that $\partial_- M = \emptyset$ whenever $\mathcal{D}$ is admissible. We now let $W^*_{l-*}(\widetilde{M}, \widetilde{\partial_- M}, \widetilde{E}, \widetilde{g}, \widetilde{h})$ be the (relative) Sobolev cochain complex defined in as Proposition 4.2.2 for $l > 3n/2 + 1$ and set

$$\mathrm{Int}^* : \mathcal{W}^*_{l-*}(\widetilde{M}, \widetilde{\partial_- M}, \widetilde{g}, \widetilde{E}, \widetilde{h}) \to C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}'} \widetilde{f}, \widetilde{E}, \widetilde{h}), \tag{6.1.3}$$

$$\mathrm{Int}^k(\sigma) \coloneqq \sum_{p \in Cr_k(\widetilde{f})} \left(\int_{W^-(p)} \sigma \right) \otimes [p] \qquad \sigma \in \mathcal{W}^k_{l-k} \tag{6.1.4}$$

to be the $\mathbb{C}[\Gamma]$-equivariant map given by integration of Sobolev forms over unstable manifolds. Here, the integral $\int_{W^-(p)} \sigma \in E_p$ makes sense, since

- $\omega \in C^1 \cap L^2$ by the Sobolev inequality, which is why the left-hand side is finite. Moreover,
- we can, and do, identify $\sigma|_{W^-(p)}$ with an E_p-valued C^1-form over $W^-(p)$ under a flat, *canonical* bundle trivialization $E|_{W^-(p)} \cong E_p \times W^-(p)$. This trivialization is induced by parallel transports along curves starting at p and entirely contained within $W^-(p)$ (since $W^-(p)$ is contractible, the result does not depend on the explicit choice of curves).

By a result of Laudenbach [12, Appendix, Proposition 6], Int^* is a cochain map. Let $\pi^* : \ker(\delta^*_{MS}) \to H^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}'} \widetilde{f}, \widetilde{E}, \widetilde{h})$ be the projection of the kernel of the L^2-Morse-Smale boundary operator onto the corresponding L^2-Morse-Smale homology. By a theorem of Dodziuk [29], extended by Schick [85] to manifolds with boundary and by Shubin [90] to non-unitary bundles, the map

$$\Theta^* : \mathcal{H}^*_{(2)}(\widetilde{M}, \widetilde{\partial_- M}, \widetilde{g}, \widetilde{E}, \widetilde{h}) \to H^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}'} \widetilde{f}, \widetilde{E}, \widetilde{h}), \tag{6.1.5}$$

defined as the restriction of $\pi^* \circ \mathrm{Int}^*$ onto the closed subspace of $\mathcal{W}^*_{l-*}(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h})$ of (relative) L^2-harmonic forms is an isomorphism of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules. Define the **metric L^2-torsion** $T^{Met}_{(2)}(E \downarrow M, g, h, \nabla_{g'} f) \in \mathbb{R}_{\geq 0}$ of the system $(E \downarrow M, g, h, \nabla_{g'} f)$ as

$$\log T^{Met}_{(2)}(E \downarrow M, g, h, \nabla_{g'} f) \coloneqq \sum_{k=0}^{\infty} (-1)^k \log \det{}_\Gamma(\Theta^k) = \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k \log \det{}_\Gamma((\Theta^k)^* \Theta^k). \tag{6.1.6}$$

Note that since Θ^k is an isomorphism, $\log \det_\Gamma(\Theta^k)$ is always well-defined. Assuming that $E \downarrow M$ is of analytic determinant class, we define the **Ray-Singer L^2 Torsion** $T^{RS}_{(2)}(E \downarrow M, g, h, \nabla_{g'} f) \in \mathbb{R}_{\geq 0}$ as

$$\log T^{RS}_{(2)}(E \downarrow M, g, h, \nabla_{g'} f) \coloneqq \log \left(\frac{T^{An}_{(2)}(E \downarrow M, \partial_- M, g, h)}{T^{Met}_{(2)}(E \downarrow M, g, h, \nabla_{g'} f)} \right), \tag{6.1.7}$$

where $T^{An}_{(2)}(E \downarrow M, \partial_- M, g, h) \in \mathbb{R}_{>0}$ is the analytic torsion that was introduced in 4.2.3. Of course, if $\mathcal{H}^*(\widetilde{M}, \widetilde{\partial_- M}, \widetilde{g}, \widetilde{E}, \widetilde{h}) = 0$, i.e. if $E \downarrow M$ is L^2-acyclic, then $T^{Met}_{(2)}(E \downarrow M, g, h, \nabla_{g'} f) = 1$, so that $T^{RS}_{(2)}(E \downarrow M, g, h, \nabla_{g'} f) = T^{An}_{(2)}(E \downarrow M, \partial_- M, g, h)$. Assuming that $E \downarrow M$ is also of combinatorial determinant class, the L^2-Morse-Smale Torsion $T^{MS}_{(2)}(E \downarrow M, h, \nabla_{g'}, f)$ (see 5.4.9) is well-defined. This allows us to define the **relative L^2-torsion** $\mathcal{R}(\mathcal{D}) \in \mathbb{R}$ of the corresponding Morse-Smale system $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ as

$$\mathcal{R}(\mathcal{D}) \colon = \log \left(\frac{T^{RS}_{(2)}(E \downarrow M, g, h, \nabla_{g'} f)}{T^{MS}_{(2)}(M, \nabla_{g'} f, E, h)} \right). \tag{6.1.8}$$

We will show in Theorem 6.3.5 that the condition $E \downarrow M$ being of analytic determinant class is equivalent to $E \downarrow M$ being of combinatorial determinant class. Therefore, we are justified to say that $E \downarrow M$ is *of determinant class* whenever either determinant class condition (and therefore both) is satisfied.

Remark 6.1.2. It should be mentioned that the relative torsion $\mathcal{R}(\mathcal{D}) \in \mathbb{R}$ can be defined even if the corresponding bundle $E \downarrow M$ is not of determinant class. In that case, the individual terms $T^{RS}_{(2)}(E \downarrow M, g, h, \nabla_{g'} f)$ and $T^{MS}_{(2)}(E \downarrow M, g, h, \nabla_{g'} f)$ are not real numbers, but non-vanishing vectors in the same orientation class of a particular 1-dimensional real vector space. Therefore, their quotient yields a *positive* real number, which is why $\mathcal{R}(\mathcal{D})$, the logarithm of the quotient as above, is still well-defined. It can be shown that the main Theorem 6.1.5 still holds in this case. We refer to [20], [102] and [16] for a detailed study of L^2-torsion without the determinant class conditions.

Our goal is to derive an explicit formula the relative torsion $\mathcal{R}(\mathcal{D})$ of a given Morse-Smale System $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ for which h is a unimodular metric. We will do so by first finding a formula in case that h is not necessarily unimodular, but $\mathcal{D}$ is admissible.

In order to formulate these results, we need to introduce the notion of a *local quantity*: Let M be a smooth manifold of dimension n. The *orientation bundle* $\mathcal{O}_M \downarrow M$ is the real line bundle, whose fiber $\mathcal{O}_{M x}$ at a given $x \in M$ is the real vector space generated by the set of two orientations on the tangent space $T_x M$, subject to the (sole) relation $[-B] + [B] = 0 \in \mathcal{O}_x M$ for any basis $B \subseteq T_x M$. It has the natural structure of a flat vector bundle over M and is isomorphic to the trivial line bundle if and only if M is orientable. In particular, we obtain a twisted de Rham complex $\Omega^*(M, \mathcal{O}_M)$. The top-dimensional forms in $\Omega^n(M,\mathcal{O}_M)$ are called **densities** over M. Perhaps **the** essential feature of the orientation bundle is that densities can be integrated over M. Namely, there exists a well-defined $\mathbb{R}$-linear integration map

$$\int_M : \Omega^n(M,\mathcal{O}_M) \to \mathbb{R} \tag{6.1.9}$$

that coincides with the usual integration whenever M is orientable (and thus $\mathcal{O}_M \cong \mathbb{R} \times M$), cf. [15, pp. 85-88]. Moreover, given any smooth embedding $f : M \to N$ between two manifolds, the pullback bundle $f^*\mathcal{O}_N$ can canonically be identified with $\mathcal{O}_M$. Therefore, any such map induces a pullback $f^* : \Omega^*(N,\mathcal{O}_N) \to \Omega^*(M,\mathcal{O}_M)$ that is in fact even a chain map.
Given two systems $\mathcal{D}_i = (E_i \downarrow M_i, g_i, h_i, X_i)$, an isometry $\phi : (M_1, g_1) \to (M_2, g_2)$ between the underlying Riemannian manifolds that satisfies $\phi^* X_2 = X_1$ and extends to a flat bundle isometry $\Phi : (E_1, h_1) \to (E_2, h_2)$ is called an *isomorphism* between the systems.

Definition 6.1.3 (Local Quantity)**.** An assignment of a form $\alpha = \alpha(\mathcal{D}) \in Y$, where either $Y = \Omega^n(M,\mathcal{O}_M)$, or $Y = \Omega^{n-1}(\partial M, \mathcal{O}_{\partial M})$ for any system $\mathcal{D} = (E \downarrow M, g, h, X)$ is called a **local quantity** of $\mathcal{D}$ if it satisfies the following compatibility conditions:

1. For any open subset $U \subseteq M$, it holds that $\alpha(\mathcal{D}|_U) = \alpha(\mathcal{D})|_U$.
2. If $\phi : M_1 \to M_2$ is an isomorphism between two systems $\mathcal{D}_i = (E_i \downarrow M_i, g_i, h_i, X_i)$ (for $i = 1, 2$), then $\phi^*\alpha(\mathcal{D}_2) = \alpha(\mathcal{D}_1)$.

For any system $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ with (f, g') a Morse-Smale pair, we will now construct a local quantity of the derived system $\mathcal{D} = (E|_{M\setminus Cr(f)} \downarrow M \setminus Cr(f), g, h, \nabla_{g'} f)$ that constitutes an integral part in the analysis of the anomaly between L^2-Ray Singer and Morse-Smale torsion.
First off, as carefully explained and constructed by Bismut and Zhang in [12, Section 3], the Levi-Civita connection of the Riemannian metric g gives rise to the **Mathai-Quillen Current**

$$\Psi(M, g) \in \Omega^{n-1}(TM \setminus M, \mathcal{O}_{TM}). \tag{6.1.10}$$

Here, we have identified $M \subseteq TM$ with its zero section inside TM. For the Morse-Smale pair (f, g'), the corresponding gradient $\nabla_{g'} f$ thus determines a smooth *embedding* $\nabla_{g'} f : M \setminus Cr(f) \to TM \setminus M$. As explained in the previous paragraph, it follows that the pullback $\nabla_{g'} f^* \Psi(M, g)$ yields an element of $\Omega^{n-1}(M \setminus Cr(f), \mathcal{O}_M)$. Wedging with the 1-form $\theta(h) \in \Omega^1(M)$ as defined in 5.4.53, we obtain a density over $M \setminus Cr(f)$

$$\theta(h) \wedge \nabla_{g'} f^* \Psi(M, g) \in \Omega^n(M \setminus Cr(f), \mathcal{O}_M). \tag{6.1.11}$$

This allows us to, at least formally, define the integral

$$\int_M \theta(h) \wedge \nabla_{g'} f^* \Psi(M, g) := \int_{M\setminus Cr(f)} \theta(h) \wedge \nabla_{g'} f^* \Psi(M, g). \tag{6.1.12}$$

Note that since $M\setminus Cr(f)$ is not compact (unless $Cr(f) = \emptyset$), the integral need *a priori* not converge. That this indeed always case has been shown in [12], as an immediate consequence of their main result. Moreover, one can verify either from its explicit construction as done in [12] or immediately from [20, Section 4], that $\theta(h) \wedge \nabla_{g'} f^* \Psi(M, g)$ is a local quantity of the system $\mathcal{D} = (E|_{M\setminus Cr(f)} \downarrow M \setminus Cr(f), g, h, \nabla_{g'} f)$, as claimed. The theorem that we wish to generalize is the following result by Zhang:

Theorem 6.1.4. [102, Zhang, '04] *Let $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ be a system with (f, g') a Morse-Smale pair and M **closed**. Then*

$$\mathcal{R}(\mathcal{D}) = -\frac{1}{2} \int_M \theta(h) \wedge \nabla_{g'} f^* \Psi(M, g). \tag{6.1.13}$$

With aid of the above theorem, we will derive a similar result in case that M is odd-dimensional with non-empty boundary:

Theorem 6.1.5. *Let $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ be a type II Morse-Smale system of product form, where M is an odd-dimensional manifold and $h|_{\partial M}$ is unimodular. Further, assume that both $E \downarrow M$ and $E|_{\partial M} \downarrow \partial M$ are of determinant class. Then*

$$\mathcal{R}(\mathcal{D}) = -\frac{\log 2}{4} \chi(\partial M) \dim(E) - \frac{1}{2} \int_M \theta(h) \wedge \nabla_{g'} f^* \Psi(TM, g). \tag{6.1.14}$$

Remark 6.1.6. Similarly as in the unitary case (cf. [21, Theorem 4.1]), there is also a version of Theorem 6.1.5 for relative/mixed, instead of absolute boundary conditions as we assume here throughout. The proof presented here carries over to this case with only minor modifications. Although not relevant for this thesis, this generalization will provide to be useful when one wants to extend the glueing formula [21, Theorem 4.3] to non-unitary bundles, which could in turn be used for future computational purposes.

Example 6.1.7. Set $I = [a, b]$, and let $E_{\mathbb{C}} \coloneqq \mathbb{C} \times I$ be the trivial 1-dimensional complex vector bundle over I. As metrics, we choose g_0 to be the standard Euclidean metric and h_0 the canonical constant Hermitian form, i.e $\langle z, z' \rangle_{h_0(x)} \coloneqq z\overline{z'}$ for any $x \in I$ and any pair $z, z' \in \mathbb{C}$. Further, we choose as Morse-function a smooth map $f_0 : [a, b] \to \mathbb{R}$ satisfying

- $f_0(x) \coloneqq \frac{1}{2}(x - (b + a)/2)^2$ away from a neighborhood of $\{a, b\}$,
- $f_0(a + t\epsilon) = f_0(b - t\epsilon) = b - t\epsilon$ for all $t \in [0, 1]$ and some small $\epsilon > 0$, and so that
- $(b + a)/2$ is the only critical point of f_0.

One now easily verifies that $\mathcal{D}_I \coloneqq (E_{\mathbb{C}} \downarrow I, g_0, h_0, \nabla_{g_0'} f_0)$ is an admissible system and that $E_{\mathbb{C}} \downarrow I$ is of determinant class. In fact, we can directly compute the corresponding analytic and combinatorial torsion elements. This computation will also be essential for the proof of Theorem 6.1.5. Firstly, since f_0 has by construction only one critical point, the corresponding Morse-Smale complex has only one non-trivial chain module, immediately implying that

$$\log T_{(2)}^{MS}(I, g_0, h_0, \mathcal{F}, f_0) = 0. \tag{6.1.15}$$

Similarly, it follows that the de Rham integration map

$$\mathrm{Int}^* : \Omega^*(I, \mathcal{F}) \to C_{MS}^*(I, g_0, h_0, \mathcal{F}, f_0) = \mathbb{C} \otimes \left[\frac{b + a}{2} \right]$$

is only non-trivial on $\Omega^0(I, \mathcal{F}) \cong C^\infty(I, \mathbb{C})$, on which it is defined by

$$\mathrm{Int}^0(f) = f\left(\frac{b + a}{2} \right) \otimes \left[\frac{b + a}{2} \right].$$

Therefore, the isomorphism

$$\Theta^0 : \mathcal{H}^0(I, \mathcal{F}) \to \mathbb{C} \otimes \left[\frac{b + a}{2} \right],$$

obtained by simply restricting Int^0 to the space of harmonic, i.e. *constant*, functions, maps the function $f \equiv c$ to $c \otimes [\frac{b+a}{2}]$. Since the inner product on $\mathbb{C} \otimes [\frac{b+a}{2}]$ in the canonical one determined by h_0 and the inner product $\mathcal{H}^0(I, \mathcal{F})$ is induced by integration over the interval $I = [a, b]$, it follows that the adjoint

$$(\Theta^0)^* : \mathbb{C} \otimes \left[\frac{b+a}{2}\right] \to \mathcal{H}^0(I, \mathcal{F})$$

sends $c \cdot [\frac{b+a}{2}]$ to the constant function $f \equiv c(b-a)^{-1}$. Therefore, the composition $(\Theta^0)^*\Theta^0$ is simply scalar multiplication by $(b-a)^{-1}$, from which we deduce that

$$\log T^{Met}_{(2)}(\mathcal{D}_I) = \frac{1}{2} \log \left(\det((\Theta^0)^* \Theta^0) \right) = -\frac{1}{2} \log(b-a). \tag{6.1.16}$$

In order to compute the analytic torsion, observe first that, under the isometric identification $\Omega^1(I, \mathcal{F}) \cong C^\infty(I, \mathbb{C})$ with $f(x)dx \mapsto f(x)$, the Laplacian Δ_1 defined over $\Omega^1(I, \mathcal{F})$ corresponds to the closure of the elliptic operator $-\frac{\partial^2}{\partial x^2}$ with initial domain $\{g \in C^\infty : g' \equiv 0 \text{ on } \{a, b\}\}$. It is well-known, see for example [94, Section 4.2] for each $n \in \mathbb{N}_0$ that

$$\mathrm{spec}(\Delta_1) = \mathrm{spec}(-\frac{\partial^2}{\partial x^2}) = \{\frac{n^2\pi^2}{l^2} : n \in \mathbb{N}_0\},$$

with $l \coloneqq b - a$ (and eigenspace of $n^2\pi^2/l^2$ the $\mathbb{C}$-span of $\cos(n\pi/l(x-a))$. Therefore, the Zeta function $\zeta_{\Delta_1}(s)$ of Δ_1 satisfies

$$\zeta_{\Delta_1}(s) = \sum_{n=1}^{\infty} \left(\frac{l}{n\pi}\right)^{2s} = \left(\frac{l}{\pi}\right)^{2s} \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{2s} = \left(\frac{l}{\pi}\right)^{2s} \cdot \zeta(2s),$$

where ζ denotes the ordinary Riemann Zeta-function. Applying the well-known equalities $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, we can thus compute

$$\log T^{An}_{(2)}(\mathcal{D}_I) = \frac{1}{2} \zeta'_{\Delta_1}(0) = -\frac{1}{2} \left(\log(2) + \log(b-a) \right). \tag{6.1.17}$$

From 6.1.15–6.1.17, we get

$$\mathcal{R}(\mathcal{D}_I) = -\frac{\log 2}{2} = -\frac{\log 2}{4} \chi(\{a, b\}) - \frac{1}{2} \int_a^b \overbrace{\theta(h_0)}^{=0} \wedge (\nabla_{g_0'} f_0)^* \Psi(TI, g_0). \tag{6.1.18}$$

The main part of this chapter is devoted to the proof of 6.1.5. We will adapt the techniques and strategy developed by Burghelea, Friedlander and Kappeler in [21] to our situation of non-unitary bundles, together with employing several known anomaly results that have been shown since. We remark that Theorem 6.1.5 has also recently been verfied in an (as of now) unpublished paper by Guangxiang Su, employing techniques and methods different from the ones that we are using. Theorem 6.1.5, together with the main results established by Brüning and Ma in [18], Zhang and Ma in [60], and Zhang in [102], are then used to prove the next key result of this thesis:

Theorem 6.1.8. *Let (M, g) be a compact, connected, odd-dimensional Riemannian manifold with $Wh(\pi_1(M)) = 0$. Then there exists a density $B(g) \in \Omega^{n-1}(\partial M, \mathcal{O}_{\partial M})$ with $B(g) \equiv 0$ when g is product-like near ∂M, such that the following holds:*
Let $\rho : \pi_1(M) \to \mathrm{GL}(V)$ be a complex, finite-dimensional representation, such that

(a) *ρ is unimodular and det-L^2-acyclic,*

(b) *the restriction $E_\rho|_{\partial M} \downarrow \partial M$ of the flat bundle $E_\rho \downarrow M$ associated to ρ is of determinant class.*

Then, for any choice of unimodular metric h on E_ρ, one has

$$\log\left(\frac{T^{An}_{(2)}(E_\rho \downarrow M, g, h)}{T^{Top}_{(2)}(M,\rho)}\right) = \frac{1}{2}\dim(\rho)\int_{\partial M} B(g). \tag{6.1.19}$$

In particular, for $i = 1, 2$ and any two representations $\rho_i : \pi_1(M) \to \mathrm{GL}(V_i)$ satisfying the above assertions, it follows that

$$\dim(\rho_2)\log\left(\frac{T^{An}_{(2)}(E_{\rho_1} \downarrow M, g, h_1)}{T^{Top}_{(2)}(M,\rho_1)}\right) = \dim(\rho_1)\log\left(\frac{T^{An}_{(2)}(E_{\rho_2} \downarrow M, g, h_2)}{T^{Top}_{(2)}(M,\rho_2)}\right), \tag{6.1.20}$$

for any choice of unimodular metric h_i on $E_{\rho_i} \downarrow M$.

Remark 6.1.9. Observe that the statement is vacuous in the case that M possesses no such representations. In particular, this is true whenever $\chi(M) \neq 0$, since then, no representation can be L^2-acyclic (cf. [54, Theorem 1.35]).

Proof. Let ρ be a representation satisfying the assumptions from the theorem. By the previous remark, we must have

$$0 = \chi(M) = \frac{1}{2}\chi(\partial M), \tag{6.1.21}$$

where the last equality follows since M is odd-dimensional and compact.
Choose a Morse function f on M of type II, along a Riemannian metric g' on M that is a product near ∂M and so that (f, g') is a Morse-Smale pair. By Lemma 5.4.18, we may also choose a unimodular metric h' with $h'|_{\partial M} \equiv h|_{\partial M}$ and so that $\mathcal{D} = (E_\rho \downarrow M, g', h', f)$ becomes an admissible system (in particular, h' is of product form near ∂M). First, since h' is unimodular, $E_\rho \downarrow M$ is det-L^2-acyclic and $\mathrm{Wh}(\pi_1(M)) = 0$, we obtain from Theorem 5.4.15 that

$$T^{MS}_{(2)}(E_\rho \downarrow M, h', \nabla_{g'} f) = T^{Top}_{(2)}(M,\rho). \tag{6.1.22}$$

Furthermore, we can apply 6.1.21 and Theorem 6.1.5 to this situation and obtain

$$\log\left(\frac{T^{An}_{(2)}(E_\rho \downarrow M, g', h', f)}{T^{MS}_{(2)}(E_\rho \downarrow M, h', \nabla_{g'} f)}\right) = \mathcal{R}(\mathcal{D}) = 0. \tag{6.1.23}$$

.

Next, choose a type I Morse function $f' : M \to \mathbb{R}$ on M. As $E_\rho \downarrow M$ is by assumption L^2-acyclic, we have $T^{An}_{(2)}(E_\rho \downarrow M, g, h) = T^{RS}_{(2)}(E_\rho \downarrow M, g, h, f')$ and analogously $T^{An}_{(2)}(E_\rho \downarrow M, g', h') = T^{RS}_{(2)}(E_\rho \downarrow M, g', h', f')$. Moreover, by the main result of [60], we have the equality of Ray-Singer anomalies

$$\log\left(\frac{T^{An}_{(2)}(E_\rho \downarrow M, g, h)}{T^{An}_{(2)}(E_\rho \downarrow M, g', h')}\right) = \log\left(\frac{T^{RS}_{(2)}(E_\rho \downarrow M, g, h, f')}{T^{RS}_{(2)}(E_\rho \downarrow M, g', h', f')}\right) = \log\left(\frac{T^{RS}(E_\rho \downarrow M, g, h, f')}{T^{RS}(E_\rho \downarrow M, g', h', f')}\right). \tag{6.1.24}$$

Here, $T^{RS}(E_\rho \downarrow M, g', h')$ is the (ordinary) Ray-Singer-metric as originally introduced in [12, Definition 2.2] and first extended to manifolds with boundary in [19]. Further, it is shown in [18, Theorem 3.4] that there exists a density $B(g) \in \Omega^{n-1}(\partial M, \mathcal{O}_{\partial M})$ with $B(g) \equiv 0$ whenever g is also product-like near ∂M, so that

$$\log\left(\frac{T^{RS}(E_\rho \downarrow M, g, h, f')}{T^{RS}(E_\rho \downarrow M, g', h', f')}\right) = \frac{1}{2}\dim(\rho)\int_{\partial M} B(g). \tag{6.1.25}$$

The density $B(g)$ is constructed as in [18, Page 1103]. It depends only on the local geometry of $(\partial M, g|_{\partial M})$ inside (M, g).
Using 6.1.22 – 6.1.25 , we finally obtain

$$\log\left(\frac{T^{An}_{(2)}(E_\rho \downarrow M, g, h)}{T^{Top}_{(2)}(M,\rho)}\right) = \log\left(\frac{T^{An}_{(2)}(E_\rho \downarrow M, g, h)}{T^{An}_{(2)}(E_\rho \downarrow M, g', h')}\right) + \log\left(\frac{T^{An}_{(2)}(E_\rho \downarrow M, g', h')}{T^{MS}_{(2)}(E_\rho \downarrow M, h', \nabla_{g'} f)}\right)$$
$$= \log\left(\frac{T^{RS}(E_\rho \downarrow M, g, h, f')}{T^{RS}(E_\rho \downarrow M, g', h', f')}\right) = \frac{1}{2}\dim(\rho)\int_{\partial M} B(g), \tag{6.1.26}$$

as desired. □

6.2 Product formulas, determinant class and subdivisions

As hinted towards in the introduction, given two Morse-Smale systems $\mathcal{D}_i = (E_i \downarrow M_i, g_i, h_i, \nabla_{g_i'} f_i)$ for $i = 1, 2$, an integral part of our methods will involve considering the product system $\mathcal{D}_1 \times \mathcal{D}_2 = (E_1 \hat{\otimes} E_2 \downarrow M_1 \times M_2, g_1 \times g_2, h_1 \hat{\otimes} h_2, \nabla_{g_1' \times g_2'}(f_1 + f_2))$ and derive meaningful information of $\mathcal{D}_1 \times \mathcal{D}_2$ in terms of $\mathcal{D}_1$ and $\mathcal{D}_2$, and vice versa. Throughout, we assume exclusively that M_1 has non-empty boundary and M_2 has empty boundary. In this case, a problem that we have to address is that *a product of two type II Morse-Smale systems need not be a type II Morse-Smale system anymore.*
The problem is due to the fact that the Morse function $f_1 + f_2$ is not necessarily of shape (II_3) as in Definition 6.1.1 anymore (in particular, it is not necessarily constant on the boundary $\partial(M_1 \times M_2) = \partial M_1 \times M_2$). This can be remedied by deforming $f_1 + f_2$ in a sufficiently small neighborhood of $\partial M_1 \times M_2$ to be of the type II shape as described in Definition 6.1.1, which can be arranged in such a way that the resulting Morse function, denoted henceforth by $\underline{f_1 + f_2}$, equals $f_1 + f_2$ outside of a small neighborhood of $\partial M_1 \times M_2$, has the same critical points as $f_1 + f_2$, the same gradient trajectories with respect to $\nabla_{g_1' + g_2'}$ and the same *unstable cells*. We denote the resulting *modified product system* by

$$\underline{\mathcal{D}_1 \times \mathcal{D}_2} := (E_1 \hat{\otimes} E_2 \downarrow M_1 \times M_2, g_1 \times g_2, h_1 \hat{\otimes} h_2, \nabla_{g_1' \times g_2'}(\underline{f_1 + f_2})), \tag{6.2.1}$$

and observe that $\underline{\mathcal{D}_1 \times \mathcal{D}_2}$ is of product form, respectively weakly admissible whenever both $\mathcal{D}_1$ and $\mathcal{D}_2$ are of product form, respectively weakly admissible. Moreover, under the assumption that both M_1 and M_2 are compact, it follows immediately from the construction of $\underline{f_1 + f_2}$ that the Morse-Smale cochain complexes corresponding to $\underline{\mathcal{D}_1 \times \mathcal{D}_2}$ and $\mathcal{D}_1 \times \mathcal{D}_2$ are the same (as Hilbert $\mathcal{N}(\Gamma)$-cochain complexes). This immediately implies that

$$\log T^{Met}_{(2)}(\mathcal{D}_1 \times \mathcal{D}_2) = \log T^{Met}_{(2)}(\underline{\mathcal{D}_1 \times \mathcal{D}_2}). \tag{6.2.2}$$

In case that $E \downarrow M$ is of determinant class, we also get

$$\log T^{MS}_{(2)}(\mathcal{D}_1 \times \mathcal{D}_2) = \log T^{MS}_{(2)}(\underline{\mathcal{D}_1 \times \mathcal{D}_2}), \tag{6.2.3}$$
$$\log T^{An}_{(2)}(\mathcal{D}_1 \times \mathcal{D}_2) = \log T^{An}_{(2)}(\underline{\mathcal{D}_1 \times \mathcal{D}_2}). \tag{6.2.4}$$

Still, to obtain an admissible system from two admissible systems $\mathcal{D}_1$ and $\mathcal{D}_2$, we need to ensure that $h_1 \hat{\otimes} h_2$ is unimodular near $\partial M_1 \times M_2$, which can only be guaranteed if we assume additionally that h_2 is (globally) unimodular. For our purposes, this will provide no restriction at all, since we will always form products, where $E_2 \downarrow M_2$ is in fact a unitary bundle and h_2 is an associated unitary (and flat) metric. Summarizing, we have the following:

Lemma 6.2.1. *For, $i = 1, 2$, let $\mathcal{D}_i = (E_i \downarrow M_i, g_i, h_i, \nabla_{g_i'} f_i)$ be two type II Morse-Smale systems with $\partial M_1 \neq \emptyset$ and $\partial M_2 = \emptyset$. Then the modified product system system $\underline{\mathcal{D}_1 \times \mathcal{D}_2}$ as in 6.2.1 is also a type II Morse-Smale system. Moreover, if both $\mathcal{D}_1$ and $\mathcal{D}_2$ are additionally of product form/weakly admissible, then also $\underline{\mathcal{D}_1 \times \mathcal{D}_2}$ is of product form/weakly admissible. Lastly, if both $\mathcal{D}_1$ and $\mathcal{D}_2$ are admissible, so that h_2 is globally unimodular, then $\underline{\mathcal{D}_1 \times \mathcal{D}_2}$ is also admissible.*

The first product formula that we state is as follows:

Proposition 6.2.2 (Product Formula 1)**.** *For $i = 1, 2$, let $\mathcal{D}_i = (E_i \downarrow M_i, g_i, h_i, \nabla_{g_i'} f_i)$ be two type II Morse-Smale systems with M_1 compact, $\partial M_1 \neq \emptyset$ and with M_2 closed. Then the type II Morse-Smale system $\underline{\mathcal{D}_1 \times \mathcal{D}_2}$ is also of determinant class and we get*

1. $\log T^{An}_{(2)}(\underline{\mathcal{D}_1 \times \mathcal{D}_2}) = \chi(M_1, E_1) \log T^{An}_{(2)}(\mathcal{D}_2) + \log T^{An}_{(2)}(\mathcal{D}_1)\chi(M_2, E_2)$,
2. $\log T^{Met}_{(2)}(\underline{\mathcal{D}_1 \times \mathcal{D}_2}) = \chi(M_1, E_1) \log T^{Met}_{(2)}(\mathcal{D}_2) + \log T^{Met}_{(2)}(\mathcal{D}_1)\chi(M_2, E_2)$,
3. $\log T^{MS}_{(2)}(\underline{\mathcal{D}_1 \times \mathcal{D}_2}) = \chi(M_1, E_1) \log T^{MS}_{(2)}(\mathcal{D}_2) + \log T^{MS}_{(2)}(\mathcal{D}_1)\chi(M_2, E_2)$,
4. $\mathcal{R}(\underline{\mathcal{D}_1 \times \mathcal{D}_2}) = \chi(M_1, E_1)\mathcal{R}(\mathcal{D}_2) + \mathcal{R}(\mathcal{D}_1)\chi(M_2, E_2)$.

Proof. $(1)-(3)$: If we replace $\underline{\mathcal{D}_1 \times \mathcal{D}_2}$ by the genuine product system $\mathcal{D}_1 \times \mathcal{D}_2$, the equalities are well-known. Namely, the proofs presented in [21, Proposition 1.21, Proposition 4.2] can be copied line by line, after changing the definition of $\Lambda^{-,q}(M, E)$ to be the C^∞-closure of $d_q^*\left(\Omega^{q+1}(M, \partial M, E)\right)$. Now apply 6.2.2-6.2.4. (4) is an immediate consequence of $(1)-(3)$. $\square$

In addition, we will need to analyze the behavior under taking products of the local quantities introduced in the previous section. Here, the assumption that the Hermitian forms are unimodular at the boundary becomes essential.
For this, note first that we have a natural embedding $\Omega^*(M_1) \otimes \Omega^*(M_2) \hookrightarrow \Omega^*(M_1 \times M_2)$ (which is dense under the natural C^∞-topology). By passing to local trivializations over coordinate charts, one easily sees that the 1-form $\theta(h_1 \hat{\otimes} h_2)$ lies in $\Omega^*(M_1) \otimes \Omega^*(M_2)$ and is of the form

$$\theta(h_1 \hat{\otimes} h_2) = \theta(h_1) \otimes \dim(E_2) + \dim(E_1) \otimes \theta(h_2). \tag{6.2.5}$$

Furthermore, it has been shown in [20, pages 63-64] (see also [12, Chapter 4] or [13, Theorem 2.7] for additional details) that

$$\begin{aligned} &\nabla_{g_1' \times g_2'}(f_1 + f_2)^* \Psi(T(M_1 \times M_2), g_1 \times g_2)) = (\nabla_{g_1'} f_1)^* \Psi(TM_1, g_1) \otimes e(TM_2, g_2) \\ &+ e(TM_1, g_1) \otimes (\nabla_{g_2'} f_2)^* \Psi(TM_2, g_2) \end{aligned} \tag{6.2.6}$$

on $M_1 \times M_2 \setminus Cr(f_1 + f_2) = M_1 \times M_2 \setminus Cr(f_1) \times C(f_2)$. Here, for a Riemannian manifold (M, g), the **Euler form** $e(M, g) \in \Omega^{\dim(M)}(M, \mathcal{O}_M)$ is a density defined using Chern-Weil theory. It has the property that $e(M, g) \equiv 0$ whenever M is odd-dimensional. Moreover, if M is closed, it is a representative of the Euler class of the tangent bundle $TM \downarrow M$. By the Gauss-Chern-Bonnett theorem, it then follows that

$$\int_M e(M, g) = \chi(M), \tag{6.2.7}$$

if M is closed. We refer [18, Page 1103] for an explicit formula for $e(M,g)$.
Combining 6.2.5 with 6.2.6, we get

$$\theta(h_1\hat{\otimes}h_2)\wedge\nabla_{g_1'\times g_2'}(f_1+f_2)^*\Psi(T(M_1\times M_2),g_1\times g_2)=\theta(h_1)\wedge(\nabla_{g_1'}f_1)^*\Psi(TM_1,g_1)\otimes\dim(E_2)e(TM_2,g_2)$$
$$+\dim(E_1)e(TM_1,g_1)\otimes\theta(h_2)\wedge(\nabla_{g_2'}f_2)^*\Psi(TM_2,g_2) \tag{6.2.8}$$

on $M_1\times M_2\setminus Cr(f_1\times f_2)$. Here, we have used that $\theta(h_i)\wedge e(TM_i,g_i)\in\Omega^{\dim(M_i)+1}(M_i,\mathcal{O}_{M_i})=\{0\}$ for both $i=1,2$.

Lemma 6.2.3 (Product Formula 2)**.** *For $i=1,2$, let $\mathcal{D}_i:=(E_i\downarrow M_i,g_i,h_i,\nabla_{g_i'}f_i)$ be two type II Morse-Smale systems of product form, so that both $h_1|_{\partial M}$ and h_2 are unimodular. Then it holds that*

$$\begin{aligned}&\theta(h_1\hat{\otimes}h_2)\wedge\nabla_{g_1'\times g_2'}(\underline{f_1+f_2})^*\Phi(T(M_1\times M_2),g_1\times g_2),\\&=\theta(h_1)\wedge(\nabla_{g_1'}f_1)^*\Psi(TM_1,g_1)\otimes\dim(E_2)\cdot e(TM_2,g_2)\end{aligned} \tag{6.2.9}$$

on all of $M\setminus Cr(\underline{f_1+f_2})$. In particular, if either M_2 is odd-dimensional or h_1 is also unimodular, then

$$\theta(h_1\hat{\otimes}h_2)\wedge\nabla_{g_1'\times g_2'}(\underline{f_1+f_2})^*\Phi(T(M_1\times M_2),g_1\times g_2)=0. \tag{6.2.10}$$

Proof. Due to the assumption that $h_1|_{\partial M_1}$ and h_2 both are unimodular, it follows from 6.2.5 that $h_1|_{\partial M_1}\hat{\otimes}h_2$ determines a unimodular metric on the restriction bundle $E|_{\partial(M_1\times M_2)}=E|_{\partial M_1\times M_2}$. Since the system $\mathcal{D}_1$ is of product form, this allows us to choose a small neighborhood U of ∂M_1, so that $\theta(h_1)\equiv 0$ on U. Together with Equation 6.2.5 and $\theta(h_2)\equiv 0$ everywhere on M_2, we deduce that

$$\theta(h_1\hat{\otimes}h_2)\equiv 0 \qquad \text{on } U\times M_2. \tag{6.2.11}$$

By choosing U smaller, if necessary, we also have by construction $\underline{f_1+f_2}=f_1+f_2$ on $(M_1\setminus U)\times M_2$, and therefore the equality of gradients

$$\nabla_{g_1'\times g_2'}(\underline{f_1+f_2})=\nabla_{g_1'\times g_2'}(f_1+f_2) \qquad \text{on } (M_1\setminus U)\times M_2. \tag{6.2.12}$$

The result now follows from 6.2.11, 6.2.12 and the product formula 6.2.8. □

Apart from considering products of systems, we will also have to investigate in the anomaly of the relative torsion that arises when changing the metrics of a given system. In fact, we will only look at anomalies under the assumption that the metrics are left unchanged in a neighborhood of ∂M. The proposition below covers this situation, generalizing [20, Proposition 5.1,5.2] onto odd-dimensional Manifolds with boundary with product metrics near ∂M.

Proposition 6.2.4 (Metric anomaly with boundary conditions)**.** *Let $\mathcal{D}_i=(E\downarrow M,g_i,h_i,\nabla_g f)$ for $i=1,2$ be two Morse-Smale Systems with M odd-dimensional, such that either*

1. *near ∂M, $g_1\equiv g_2$ are of product form and $h_1|_{\partial M}\equiv h_2|_{\partial M}$, or*
2. *near ∂M, g_1 and g_2 are of product form (although possibly distinct) and $h_1|_{\partial M}\equiv h_2|_{\partial M}$ is unimodular.*

Then

$$\mathcal{R}(\mathcal{D}_1)-\mathcal{R}(\mathcal{D}_2)=\sum_{p\in Cr(f)}(-1)^{\mathrm{ind}(p)}\log\left(\det(h_1(p)^{-1}\circ h_2(p))\right). \tag{6.2.13}$$

Proof. First, observe that

$$\mathcal{R}(\mathcal{D}_1) - \mathcal{R}(\mathcal{D}_2) = \log\left(\frac{T^{An}_{(2)}(\mathcal{D}_1)}{T^{An}_{(2)}(\mathcal{D}_2)}\right) + \log\left(\frac{T^{Met}_{(2)}(\mathcal{D}_2)}{T^{Met}_{(2)}(\mathcal{D}_1)}\right) + \log\left(\frac{T^{MS}_{(2)}(\mathcal{D}_2)}{T^{MS}_{(2)}(\mathcal{D}_1)}\right). \tag{6.2.14}$$

Furthermore, we have

$$\frac{T^{Met}_{(2)}(\mathcal{D}_2)}{T^{Met}_{(2)}(\mathcal{D}_1)} = \sum_{k=0}^{n}(-1)^k \log\left(\frac{\det_\Gamma(\Theta_2^k)}{\det_\Gamma(\Theta_1^k)}\right), \tag{6.2.15}$$

where $\Theta_i^* : \mathcal{H}^*(\widetilde{M}, \widetilde{g_i}, \widetilde{E}, \widetilde{h_i}) \to H^*_{(2)}(\widetilde{M}, E, h_i, \nabla_g f)$ are the isomorphisms of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules as defined in 6.1.5. We let

$$\mathbb{1}^*_{[h_1,h_2]} : H^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \widetilde{E}, \widetilde{h_1}) \to H^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \widetilde{E}, \widetilde{h_2}) \tag{6.2.16}$$

be the isomorphism of Hilbert $\mathcal{N}(\Gamma)$-modules induced by the (not necessarily unitary) identity map $\mathbb{1}^*_{[h_2,h_1]} : C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \widetilde{E}, \widetilde{h_2}) \to C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \widetilde{E}, \widetilde{h_1})$. Also, we let

$$\tau^* : \mathcal{H}^*(\widetilde{M}, \widetilde{g_2}, \widetilde{E}, \widetilde{h_2}) \to \mathcal{H}^*(\widetilde{M}, \widetilde{g_1}, \widetilde{E}, \widetilde{h_1}) \tag{6.2.17}$$

be the isomorphism of Hilbert $\mathcal{N}(\Gamma)$-modules making the diagram below commute.

$$\begin{array}{ccc} \mathcal{H}^*(\widetilde{M}, \widetilde{g_1}, \widetilde{E}, \widetilde{h_1}) & \xrightarrow{\Theta_1^*} & H^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \widetilde{E}, \widetilde{h_1}) \\ {\scriptstyle \tau^*}\big\uparrow & & \big\downarrow{\scriptstyle \mathbb{1}^*_{[h_1,h_2]}} \\ \mathcal{H}^*(\widetilde{M}, \widetilde{g_1}, \widetilde{E}, \widetilde{h_2}) & \xrightarrow{\Theta_2^*} & H^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \widetilde{E}, \widetilde{h_2}) \end{array} \tag{6.2.18}$$

By Proposition 4.1.14, it follows that

$$\det_\Gamma(\tau^*)\det_\Gamma(\mathbb{1}^*_{[h_1,h_2]}) = \det_\Gamma(\Theta_1^*)^{-1}\det_\Gamma(\Theta_2^*). \tag{6.2.19}$$

Therefore, Equation 6.2.15 decomposes into

$$\log\left(\frac{T^{Met}_{(2)}(\mathcal{D}_2)}{T^{Met}_{(2)}(\mathcal{D}_1)}\right) = \sum_{k=0}^{n}(-1)^k \log\det(\tau^k) + \sum_{k=0}^{n}(-1)^k \log\det(\mathbb{1}^k_{[h_1,h_2]}). \tag{6.2.20}$$

By Proposition 4.1.40, we have

$$\sum_{k=0}^{n}(-1)^k \log\det(\mathbb{1}^k_{[h_1,h_2]}) + \log\left(\frac{T^{MS}_{(2)}(\mathcal{D}_2)}{T^{MS}_{(2)}(\mathcal{D}_1)}\right) = \sum_{p\in Cr(f)}(-1)^{\mathrm{ind}(p)}\log\left(\det(h_1(p)^{-1}\circ h_2(p))\right). \tag{6.2.21}$$

For the remaining term, it is due to the main Theorem of [60] that we have an equality

$$\log\left(\frac{T^{An}_{(2)}(\mathcal{D}_1)}{T^{An}_{(2)}(\mathcal{D}_2)}\right) + \sum_{k=0}^{n}(-1)^k \log\det(\tau^k) = \log\left(\frac{T^{RS}(\mathcal{D}_1)}{T^{RS}(\mathcal{D}_2)}\right). \tag{6.2.22}$$

Here, $T^{RS}(\mathcal{D}_i)$ denotes the *Ray-Singer Torsion element* as originally defined in [12, Definition 2.2]. It is shown in [18, Theorem 3.4] that, under the conditions that M is odd-dimensional and either one of the two assertions mentioned in the statement of the proposition is satisfied, one has

$$\log\left(\frac{T^{RS}(\mathcal{D}_1)}{T^{RS}(\mathcal{D}_2)}\right) = 0. \tag{6.2.23}$$

The result direct follows from 6.2.14 and 6.2.20-6.2.23. □

Definition 6.2.5 (Subdivision)**.** Let M be a compact manifold and for $i = 0, 1$, let (f_i, g_i) be a Morse-Smale pair. Then (f_1, g_1) is called a **subdivision** of (f_0, g_0) if all of the following conditions are satisfied

1. $Cr_p(f_0) \subseteq Cr_p(f_1) \subseteq \bigcup_{x \in Cr(f_0)} W_x^-(f_0)$ for each $0 \leq p \leq n$,
2. $W_x^-(f_1) \subseteq W_x^-(f_0)$ for each $x \in Cr(f_0)$,
3. $W_x^-(f_0) = \bigcup_{y \in Cr(f_1) \cap W_x^-(f_0)} W_y^-(f_1)$, and
4. $g_0 \equiv g_1$ near $Cr(f_0) \cup \partial M$ and and $f_0 \equiv f_1$ near ∂M.

We now describe the effect on the relative torsion under taking taking subdivisions. For that, let M be a compact manifold, let (f_i, g_i) be a Morse-Smale pair on M for $i = 0, 1$, so that (f_1, g_1) is a subdivision of (f_0, g_0). Let h be Hermitian form on a flat bundle $E \downarrow M$. By definition, there exists for each $y \in Cr(f_1)$ a unique $x \in Cr(f_0)$ satisfying $y \in W_x^-(f_0)$. Let $\widetilde{h}(y) \in \mathrm{GL}(E_y, \overline{E_y^*})$ be the Hermitian metric on E_y obtained by parallel transport of the metric $h(x) \in \mathrm{GL}(E_x, \overline{E_x^*})$ along a curve connecting x and y that is entirely contained within $W_x^-(f_0)$. Note that since $W_x^-(f)$ is simply-connected, the resulting metric doesn't depend on the particular choice of curve. Note also that $\widetilde{h}(y) = h(y)$ whenever h is a unitary metric.
For each $y \in Cr(f_1)$, define

$$\omega(y) := \log \det(\widetilde{h}(y)^{-1} \circ h(y)) \in \mathbb{R}_{\geq 0}\,. \tag{6.2.24}$$

Observe that $\omega \equiv 0$ whenever h is a unimodular metric. The proof of the following statement for closed manifolds is laid out in [20, Proposition 5.3] and carries over to general compact manifolds without further modification:

Proposition 6.2.6. *In the above situation, we have*

$$\mathcal{R}(E \downarrow M, g, h, \nabla_{g_0} f_0) - \mathcal{R}(E \downarrow M, g, h, \nabla_{g_1} f_1) = \sum_{y \in Cr(f_1)} (-1)^{\mathrm{ind}(y)} \omega(y). \tag{6.2.25}$$

Corollary 6.2.7 (Relative Torsion under subdivision)**.** *Let $\mathcal{D}_0 = (E \downarrow M, g_0, h_0, \nabla_{g_0'} f_0)$ be a weakly admissible system with M odd-dimensional and let (f_1, g_1') be a subdivision of (f_0, g_0'). Then one finds a Riemannian metric g_1 on M and an Hermitian form h_1 with $g_1 \equiv g_0$ and $h_1 \equiv h_0$ near ∂M on E, so that $\mathcal{D}_1 = (E \downarrow M, g_1, h_1, \nabla_{g_1'} f_1)$ is a weakly admissible system, satisfying*

$$\mathcal{R}(\mathcal{D}_0) = \mathcal{R}(\mathcal{D}_1). \tag{6.2.26}$$

Proof. For each $y \in Cr(f_1)$, there exists by the definition of a subdivision a unique $x \in Cr(f_0)$, such that $y \in W_x^-(f_0)$. As above, we let $\widetilde{h_1}(y) \in \mathrm{GL}(E_y, \overline{E_y}^*)$ be the Hermitian metric on the fiber E_y obtained by parallel transport of the Hermitian metric $h_0(x) \in \mathrm{GL}(E_x, \overline{E_x^*})$ along a curve between x and y contained entirely within $W_x^-(f)$. With

$$\omega(y) := \log \det(\widetilde{h}_1(y)^{-1} \circ h_0(y)),$$

we obtain from Proposition 6.2.6

$$\mathcal{R}(E \downarrow M, g_0, h_0, \nabla_{g_0'} f_0) = \mathcal{R}(E \downarrow M, g_0, h_0, \nabla_{g_1'} f_1) + \sum_{y \in Cr(f_1)} (-1)^{\mathrm{ind}(y)} \omega(y). \tag{6.2.27}$$

In order to construct the metric h_1, choose small disjoint open coordinate neighborhoods $U_y \supset V_y \ni y$ around each $y \in Cr(f_1)$, each also disjoint from a neighborhood of the boundary, such that $\overline{V_y} \subset U_y$. Define the Hermitian form $h_1 \in \mathrm{GL}(E, \overline{E^*})$ to be an extension of the metrics $\bigcup_{y \in Cr(f_1)} \widetilde{h}_1(y)$ that is parallel on $\bigcup_{y \in Cr(f_1)} V_y$ and equal to h_0 on $M \setminus \bigcup_{y \in Cr(f_1) \setminus Cr(f_0)} U_y$. Lastly, choose a Riemannian metric satisfying $g_1 \equiv g_1'$ near $Cr(f_1)$ and $g_1 \equiv g_0$ near ∂M (in particular, g_1 is also of product form near ∂M). By construction of the metrics h_1 and g_1, the system $\mathcal{D}_1 = (E \downarrow M, g_1, h_1, \nabla_{g_1'} f_1)$ is weakly admissible. Moreover, an application of Proposition 6.2.4 gives

$$\mathcal{R}(E \downarrow M, g_0, h_0, \nabla_{g_1'} f_1) = \mathcal{R}(E \downarrow M, g_1, h_1 . \nabla_{g_1'} f_1) + \sum_{y \in Cr(f_1)} (-1)^{\mathrm{ind}(y)+1} \omega(y). \tag{6.2.28}$$

The result now follows from 6.2.27 and 6.2.28. □

The proof of the last result of this section can be found in [21, Proposition 3.7]

Proposition 6.2.8 (Determinant Class under Glueing). *For $i = 1, 2$, let $(E_i \downarrow M_i)$ be two flat, complex bundles over a compact manifold, satisfying $E_1|_{\partial M_1} \downarrow \partial M_1 = E_2|_{\partial M_2} \downarrow \partial M_2$. Assume that both $E_i \downarrow M_i$ and $E_i|_{\partial M_i} \downarrow \partial M_i$ are of determinant class. Then the flat bundle $E \downarrow M$ with $E \coloneqq E_1 \cup_{\partial E_1} E_2$ and $M \coloneqq M_1 \cup_{\partial M_1} M_2$ is of determinant class as well.*

6.3 Witten-deformation of the De Rham complex

Throughout this section, we fix a Morse-Smale system $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$. For any parameter $t \in \mathbb{R}_{\geq 0}$, the *Witten-deformation* d_t of the exterior derivative d on $\Omega^*(M, E)$ is defined as

$$d_t \coloneqq e^{-tf} d e^{tf} = d + t df \wedge : \Omega^\bullet(M, E) \to \Omega^{\bullet+1}(M, E). \tag{6.3.1}$$

Observe that $d_t^2 = 0$ for any $t \in \mathbb{R}_{\geq 0}$, which is why we can regard the pair $\Omega_t^\bullet(M, E) \coloneqq (\Omega^\bullet(M, E), d_t)$ as a cochain complex. In this context, it is evident that the multiplication map $\omega \mapsto e^{tf} \cdot \omega$ becomes an isomorphism of cochain complexes

$$e^{tf} : \Omega_t^\bullet(M, E) \to \Omega_0^\bullet(M, E) = \Omega^\bullet(M, E). \tag{6.3.2}$$

Fixing a Riemannian metric g on M and an Hermitian form h on $E \downarrow M$ allows us to further construct the formal adjoint

$$\delta_t = -\#^{-1} \circ d_t^* \circ \# : \Omega^*(M, E, g, h) \to \Omega^{\bullet-1}(M, E, g, h), \tag{6.3.3}$$

of d_t, where

$$d_t^* : \Omega^\bullet(M, \overline{E^*}) \to \Omega^{\bullet-1}(M, \overline{E^*}) \tag{6.3.4}$$

is the differential on $\Omega^*(M, \overline{E^*})$ *dual* to d_t and

$$\# : \Omega^\bullet(M, E, g, h) \to \Omega^{n-\bullet}(M, E^*, g, h) \tag{6.3.5}$$

is the Hodge-#-operator constructed from the metrics g and h. For the remainder of this subsection, the Riemannian metric g and Hermitian form h are fixed, which is why we will simply denote by $\Omega^\bullet(M, E)$

the inner product space $\Omega^\bullet(M,E,g,h)$. With $i:\partial M\to M$ the inclusion map, we further set

$$\Delta_{*,t} \coloneqq \delta_t^{*+1}d_t^* + d_t^{*-1}\delta_t^* : \Omega^*(M,E)\to\Omega^*(M,E), \tag{6.3.6}$$

$$\Omega^*(M,\partial M,E) \coloneqq \{\omega\in\Omega_c^*(M,E) : i^*\#\omega = 0\}, \tag{6.3.7}$$

$$\Omega_{2,t}^*(M,\partial M,E) : \{\omega\in\Omega_c^*(M,E) : i^*\#\omega = i^*\#d_t\omega = 0\}, \tag{6.3.8}$$

$$\mathcal{H}_t^*(M,\partial M,E) \coloneqq \{\omega\in\Omega^*(M,E)\cap\Omega_{(2)}^*(M,E) : i^*\#\omega = d_t\omega = \delta_t\omega = 0\}, \tag{6.3.9}$$

$$\Delta_{*,t}[E] \coloneqq \Delta_t|_{\Omega_{2,t}^*(M,\partial M,E)}. \tag{6.3.10}$$

As shown in [12, Proposition 5.5], one has

$$\Delta_{k,t} = \Delta_{k,0} + t^2|\nabla_g f| + tL^k, \tag{6.3.11}$$

where L^k is a degree-0 differential operator (i.e. a bundle endomorphism) on $\Omega^k(M,E)$. Observe that for any $t\geq 0$, d_t and δ_t are all elliptic differential operators of order 1, while $\Delta_{*,t}[E]$ is an elliptic differential operator of order 2 that is symmetric with respect to the inner product on $\Omega_t^*(M,E)$ induced by g and h. Moreover, just as in the case $t=0$, one verifies that all three operators are closeable when regarded as unbounded operators on the L^2-completion $\Omega_{(2)}^\bullet(M,E)$. We define the *$L^2$-Witten-de Rham complex* of the system $\mathcal{D}$ as

$$\Omega_{(2),t}^\bullet(M,E) \coloneqq (\Omega_{(2)}^\bullet(M,E), d_t), \tag{6.3.12}$$

where we identify d_t with its minimal L^2-closure. Similarly, for fixed $l\geq n/2+1$, we define the *Witten-de Rham-Sobolev complex* of the system $\mathcal{D}$ as

$$\mathcal{W}_{l-\bullet,t}^\bullet(M,E) \coloneqq (\mathcal{W}_{l-\bullet}^\bullet(M,E), d_t), \tag{6.3.13}$$

where now, we identify d_t with its *bounded* extension $d_t : W_{l-\bullet}^\bullet(M,E)\to W_{l-\bullet-1}^{\bullet+1}(M,E)$. The closed, symmetric operator $\Delta_t[E] : \Omega_{(2),t}^*(M,E)\to\Omega_{(2),t}^*(M,E)$ is called the *Witten-Laplacian* associated to the system $\mathcal{D} = (E\downarrow M, g, h, \nabla'_{g'}f)$.

In the situations relevant for our results, namely if $\mathcal{D}$ is a Γ-invariant Morse-Smale system, $\Delta_t[E]$ will also always be a self-adjoint operator, which is established as follows: For fixed $m\in\mathbb{N}$, we define the a bounded boundary differential operator

$$\iota_t\colon \Omega_t^\bullet(M,E)\to\Omega^\bullet(\partial M, E|_{\partial M})$$
$$\iota_t(\omega) \coloneqq \hat{\#}^{-1}i^*\#\omega + \hat{\#}^{-1}i^*\#d_t\omega.$$

One now verifies that these differential operators all fit together to define an *elliptic*, formally self-adjoint boundary value problem of order $2m$ (compare with [21, Page 50])

$$\mathcal{B}_{m,t} \coloneqq (\Delta_t^m, \iota_t, \ldots, \iota_t\Delta_t^{m-1}). \tag{6.3.14}$$

Since all operators involved are Γ-equivariant, $\mathcal{B}_{m,t}$ is the lift of a unique elliptic boundary value problem over the compact quotient M/Γ. Thus, $\mathcal{B}_{m,t}$ is therefore *uniformly elliptic*, cf. Corollary 3.3.15. Using the very same methods and arguments as in Section 3.4, we can conclude as follows:

Proposition 6.3.1. *Let $\mathcal{D} = (E\downarrow M, g, h, \nabla_{g'}f)$ be a Γ-invariant Morse-Smale system. Then, for any $t\geq 0$, the following holds:*

1. *The cochain complex $\Omega_{(2),t}^\bullet(M,E)$ is a Hilbert $\mathcal{N}(\Gamma)$-cochain complex. Moreover, the multiplication map from 6.3.2 extends to an isomorphism of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes*

$$e^{tf} : \Omega_{(2),t}^\bullet(M,E)\to\Omega_{(2)}^\bullet(M,E). \tag{6.3.15}$$

2. *For $l \geq n/2+1$, the cochain complex $\mathcal{W}^{\bullet}_{l-\bullet,t}(M,E)$ is a Hilbert $\mathcal{N}(\Gamma)$-module cochain complex with bounded differentials. Moreover, the multiplication map from 6.3.2 extends to an isomorphism of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes*

$$e^{tf} : W^{\bullet}_{l-\bullet,t}(M,E) \to W^{\bullet}_{l-\bullet}(M,E). \tag{6.3.16}$$

3. *For any $m \in \mathbb{N}_0$ and each $0 \leq k \leq n$, the (minimal) closure of the unbounded operator $\Delta^m_{k,t}[E] : \Omega^k_{(2)}(M,E) \to \Omega^k_{(2)}(M,E)$, which we also denote by $\Delta^m_{k,t}[E]$, is a positive, self-adjoint morphism of Hilbert $\mathcal{N}(\Gamma)$-modules, so that*

$$\begin{aligned}&\mathrm{dom}(\Delta^m_{k,t}[E]) \coloneqq \{\omega \in W^k_{2m,t}(M,E) \colon \iota_t\omega = \ldots \iota_t\Delta^{m-1}_{k,t}\omega = 0\},\\ &\mathcal{H}^k_t(M,\partial M,E) = \ker(\Delta_{k,t}[E]).\end{aligned}$$

4. *For any $m \in \mathbb{N}_0$ and each $0 \leq k \leq n$, we have the orthogonal Hodge-decomposition of Hilbert $\mathcal{N}(\Gamma)$-subcomplexes*

$$\mathcal{W}^k_{2m,t}(E) = \ker(\Delta_{k,t}[E]) \oplus \overline{d_t\Omega^{k-1}_c(M,E)}^{2m} \oplus \overline{\delta_t\Omega^{k+1}(M,\partial M,E)}^{2m}. \tag{6.3.17}$$

Here, $\overline{A}^m$ denotes the $\mathcal{W}_{2m}$-closure of a given subspace $A \subseteq W^*_{2m}(E)$.

By Statement 3 of the previous theorem, the operator $\Delta_{*,t}[E]$ admits a spectral projection for each interval in $\mathbb{R}_{\geq 0}$. This allows us for $\lambda \geq 0$ to define the partial isometry

$$P^*_{[0,\lambda)}(t) : \Omega^{\bullet}_{(2)}(M,E) \to \Omega^*_{(2)}(M,E)$$

as the spectral projection of $\Delta_{*,t}[E]$ associated to the half-open interval $[0,\lambda)$. We define the submodules

$$\Omega^{\bullet}_{Sm,t}(M,E) \coloneqq \bigoplus_{k=0}^{n} \mathrm{im}\left(P^*_{[0,1)}(t)\right), \tag{6.3.18}$$

$$\Omega^{\bullet}_{La,t}(M,E) \coloneqq \left(\Omega^{\bullet}_{t,Sm}(M,E)\right)^{\perp} = \bigoplus_{k=0}^{n} \mathrm{im}\left(\mathbb{1} - P^*_{[0,1)}(t)\right). \tag{6.3.19}$$

Then

Proposition 6.3.2. (*a*) *We have an orthogonal decomposition of Hilbert $\mathcal{N}(\Gamma)$-module* ***subcomplexes***

$$(\Omega^{\bullet}_{(2)}(M,E), d_t) = (\Omega^{\bullet}_{Sm,t}(M,E), d_t) \oplus (\Omega^{\bullet}_{La,t}(M,E), d_t). \tag{6.3.20}$$

(*b*) *The Hilbert $\mathcal{N}(\Gamma)$-cochain complex $(\Omega^{\bullet}_{Sm,t}(M,E), d_t)$ is of finite type.*

(*c*) *One has $\Omega^*_{Sm,t}(M,E) \subseteq \bigcap_{l=0}^{\infty} \mathcal{W}^*_l(M,E)$. In particular, $\Omega^*_{Sm,t}(M,E)$ consists of smooth forms.*

(*d*) *The inclusion $i_t : \Omega^{\bullet}_{Sm,t}(M,E) \to \Omega^{\bullet}_{(2)}(M,E)$ is a chain homotopy equivalence of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes with chain homotopy inverse $P^{\bullet}_t(1) : \Omega^{\bullet}_{(2)}(M,E) \to \Omega^{\bullet}_{Sm,t}(M,E)$.*

Proof. (*a*) For a subset $I \subseteq \mathbb{R}^+$, we denote by $\Delta_t[E] = \bigoplus_{k=0}^n \Delta_{k,t}[E]$ the *total* Laplacian, by $P_I(\Delta_t[E])$ the spectral projection of $\Delta_t[E]$ corresponding to (the indicator function of) I and set $\Omega_I(E) \coloneqq \mathrm{im}(P_I(\Delta_t[E])$. We claim that $(\Omega_I(E), d_t)$ is always a subcomplex of Hilbert $\mathcal{N}(\Gamma)$-modules. Applied to $I = [0,1)$ and $I = [1,\infty)$, (*a*) then clearly follows.

First, $\Omega_I(E)$ is clearly a closed, Γ-invariant subspace, since it is the image of a Γ-equivariant projection.

To see that $\Omega_I(E)$ is a subcomplex, it suffices to show that

$$d_t P_I(\Delta_t[E]) = P_I(\Delta_t[E]) d_t. \tag{6.3.21}$$

To prove this, consider first the left-handed polar decomposition $d_t = u\,|d_t| : \ker(d_t)^\perp \to \overline{\mathrm{im}(d_t)}$ of d_t.

With this restriction of domain and range (implicit throughout) the operator is injective with dense image, therefore u is an honest unitary, and $|d_t| = \sqrt{\delta_t d_t}$.

Note furthermore that under the Hodge decomposition

$$\Omega^\bullet_{(2)}(M,E)M) = \ker(d_t)^\perp \oplus \ker(\Delta_t[E]) \oplus \overline{\mathrm{im}(d_t)},$$

the Laplacian $\Delta_t[E]$ decomposes as direct sum of self-adjoint operators $\delta_t d_t \oplus 0 \oplus d_t \delta_t$, implying that

$$P_I(\Delta_t[E]) = P_I(\delta_t d_t) \oplus P_I(d_t \delta_t). \tag{6.3.22}$$

Therefore, to prove Equation 6.3.21, it suffices to show that $d_t \chi_I(\delta_t d_t) = P_I(d_t\delta_t) d_t$. Now, using the above polar decomposition for d_t, we get $\delta_t = |d_t|\, u^*$, so

$$P_I(d_t \delta_t) = P_I(u\,|d_t|^2\, u^*) = u \chi_I(|d_t|^2) u^*.$$

Consequently, using the fact that the spectral projections for any positive operator f commute with $\sqrt{f}$, we compute

$$\begin{aligned} &d_t \chi_I(\delta_t d_t) = u\,|d_t|\, P_I(|d_t|^2) = u P_I(|d_t|^2)\,|d_t| \\ &= u P_I(|d_t|^2) u^* u\,|d_t| = P_I(d_t\delta_t) d_t. \end{aligned}$$

(b) By the spectral theorem, we have for each $0 \le k \le n$ that

$$\begin{aligned} &\Omega^k_{Sm,t}(M,E) = \mathrm{im}(P_{[0,1)}(\Delta_{k,t}[E])) \subseteq \mathrm{dom}(\Delta_{k,t}[E]) \subseteq \{\omega \in \mathcal{W}^k_2(M,E) : i^*\#\omega = i^*\#_t d\omega = 0\} \\ &\subseteq \mathcal{W}^k_2(M,E). \end{aligned}$$

Moreover, since the spectral projections of $\Delta_{k,t}[E]$ commute with $\Delta_{k,t}[E]$, it also follows that

$$\Delta_{k,t}[E]\Omega^k_{Sm,t}(M,E) \subseteq \Omega^k_{Sm,t}(M,E).$$

Inductively, one concludes that $\Omega^k_{Sm,t}(M,E) \subseteq \mathrm{dom}(\Delta^m_{k,t}[E]) \subseteq W^k_{2m}(E)$ for each $m \in \mathbb{N}$, from which (c) follows.

(c) The proof in case that $\partial M = \emptyset$, which can be found in [90], can be adapted word-by-word when $\partial M \neq \emptyset$.

□

Now assume additionally that the Γ-invariant system $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ is also weakly Γ-admissible. Recall from the axioms laid out in Definition 6.1.1 that Γ-admissibility implies that we can choose for each $p \in Cr(f)$ radii $r_p > 0$, coordinate charts $\phi_p : B_{r_p}(0) \xrightarrow{\cong} U_p \subseteq \mathbb{R}^n$ disjoint from ∂M with

$B_{r_p}(0) \coloneqq \{x \in \mathbb{R}^n : ||x|| < r_p\}$ and $\phi_p(0) = p$, along with a flat bundle isomorphism $\Phi_p : B_{r_p}(0) \times \mathbb{C}^m \xrightarrow{\cong} E|_{U_p}$ that fit into the commutative diagram

$$
\begin{array}{ccc}
B_{r_p}(0) \times \mathbb{C}^m & \xrightarrow[\cong]{\Phi_p} & E|_{U_p} \\
\downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \pi_E} \\
B_{r_p}(0) & \xrightarrow[\cong]{\phi_p} & U_p,
\end{array}
\tag{6.3.23}
$$

and such that all of the following conditions hold:

(H_1) The pullback metric $\phi_p^*(g|_{U_p})$ equals the Euclidean metric on $\mathbb{R}^n$.

(H_2) The pullback Hermitian form $\Phi_p^*(h|_{U_p})$ equals the standard inner product on $\mathbb{C}^m$.

(H_3) One has

$$(f \circ \phi_p)(x_1, \ldots, x_n) = f(p) - \frac{1}{2}\sum_{i=1}^{\mathrm{ind}(p)} x_i^2 + \frac{1}{2}\sum_{i=\mathrm{ind}(p)+1}^{n} x_i^2.$$

(H_4) The above choices are Γ-invariant, i.e. $\gamma.U_p = U_{\gamma.p}$, $r_p = r_{\gamma.p}$, $\gamma \circ \phi_p = \phi_{\gamma.p}$ and $\gamma \circ \Phi_p = \Phi_{\gamma.p}$ for each $p \in Cr(f)$ and each $\gamma \in \Gamma$.

Together with the global form 6.3.11 of the Witten-Laplacian $\Delta_{*,t}$, it is *precisely* due to this Γ-invariant shape of f and metric bundle $(E, h) \downarrow (M, g)$ near $Cr(f)$ that Burghelea et al. were able to prove the next theorem. With the aid of properties $(H_1) - (H_4)$, their proof from [21, Section 3.3] can be adapted, word by word, to our situation of non-unitary bundles without any further modification.

Theorem 6.3.3. *Let $(E \downarrow M, g, h, \nabla_{g'} f)$ be a weakly Γ-admissible system. Then, for each $t \geq 0$, there exists an isometric embedding of Hilbert $\mathcal{N}(\Gamma)$-modules*

$$J^\bullet(t) \coloneqq \bigoplus_{k=0}^{n} J^k(t) : C^\bullet_{(2)}(M, \nabla_{g'} f, E, h) \to \Omega^\bullet_{(2)}(M, E),$$

Moreover, for large $t >> 0$, the composition

$$Q(t) \coloneqq P^\bullet_{[0,1)}(t) \circ J^\bullet(t) : C^\bullet_{(2)}(M, \nabla_{g'} f, E, h) \to \Omega^\bullet_{Sm,t}(M, E)$$

is an isomorphism of Hilbert $\mathcal{N}(\Gamma)$-modules.

We stress the fact that the map of Hilbert $\mathcal{N}(\Gamma)$-modules $J^*(t)$ from the previous theorem (and therefore also the isomorphism $Q^*(t)$) is in general **not** a map of **cochain complexes**. This is why the maps $Q^*(t)$ alone cannot be used to reach our desired conclusion, namely that the complexes $C^\bullet_{(2)}(M, \nabla_{g'} f, E, h)$ and $\Omega^\bullet_{Sm,t}(M, E, g, h)$ are chain homotopy equivalent. In spite of this, it still follows that for sufficiently large $t >> 0$, the isomorphism $Q^\bullet(t)$ can be used to define the isometry

$$I^\bullet(t) \coloneqq Q^\bullet(t)\left(Q^\bullet(t)^* Q^\bullet(t)\right)^{-1/2} : C^\bullet_{(2)}(M, \nabla_{g'} f, E, h) \to \Omega^\bullet_{Sm,t}(M, E, g, h). \tag{6.3.24}$$

Moreover, since $(E \downarrow M, g, h, \nabla_{g'} f)$ is the lift of an admissible system with deck group Γ, there are also isomorphisms of Hilbert $\mathcal{N}(\Gamma)$-modules for $t > 0$:

$$S^k(t) : C^k_{(2)}(M, \nabla_{g'} f, E, h) \to C^k_{(2)}(M, \nabla_{g'} f, E, h), \tag{6.3.25}$$

$$\lambda_p \otimes [p] \mapsto (\pi/t)^{\frac{n-2k}{4}} e^{-tf(p)} \cdot \lambda_p \otimes [p] \qquad p \in Cr_k(f), \tag{6.3.26}$$

$$S^\bullet(t) \coloneqq \bigoplus_{k=0}^{n} S^k(t) : C^\bullet_{(2)}(M, \nabla_{g'} f, E, h) \to C^\bullet_{(2)}(M, \nabla_{g'} f, E, h). \tag{6.3.27}$$

Here, we have used the fact that f is Γ-equivariant, hence in particular satisfies $f(\gamma.x) = f(x)$ for any $x \in M$.
Observe that because of Proposition 6.3.2,(b), we may regard $\Omega^\bullet_{Sm,t}(M,E)$ as a subcomplex of the Sobolev complex $\mathcal{W}^\bullet_{l-\bullet,t}(M,E)$ where $l > n/2+1$. This allows us to define the morphism of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

$$F^\bullet(t) := \mathrm{Int}^\bullet \circ e^{tf} : \Omega^\bullet_{Sm,t}(M,E,g,h) \to C^\bullet_{(2)}(M, \nabla_{g'} f, E, h), \tag{6.3.28}$$

as restricting to the subcomplex $\Omega^\bullet_{Sm,t}(M,E)$ the composition of the isomorphism

$$e^{tf} : \mathcal{W}^\bullet_{l-\bullet,t}(M,E) \to \mathcal{W}^\bullet_{l-\bullet}(M,E)$$

from Proposition 6.3.1,(a) with the integration map

$$\mathrm{Int}^\bullet : \mathcal{W}^\bullet_{l-\bullet}(M,E) \to C^\bullet_{(2)}(M, \nabla_{g'} f, E, h),$$

defined as in 6.1.3. Just as before, the proof of the next theorem, laid out for unitary bundles in [21, Section 3.3], can be adapted to our setting without any modifications:

Theorem 6.3.4. *Under the previous assumptions, we obtain for large $t >> 0$, that*

$$S^\bullet(t) \circ F^\bullet(t) \circ I^\bullet(t) = \mathbb{1} + \mathcal{O}(t^{-1}). \tag{6.3.29}$$

Consequently, for large $t >> 0$, the map $F^\bullet(t)$ is an isomorphism of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes.

Finally, we arrive at the following very important intermediate result

Theorem 6.3.5. *Let M be a compact manifold, let $\rho : \pi_1(M) \to \mathrm{GL}(V)$ be some finite-dimensional, complex representation and let $E := E_\rho \downarrow M$ be the associated flat complex vector bundle over M. Choose some CW structure X on M with $\widetilde{X}$ the lifted structure on $\widetilde{M}$ and let $C^*_{(2)}(\widetilde{X}, \rho)$ be the associated cellular L^2-cochain complex. Further, let $\widetilde{E} \downarrow \widetilde{M}$ be the lifted bundle on the universal cover, and let $\Omega^*_{(2)}(\widetilde{M}, \widetilde{E})$ be the L^2-cochain complex with absolute boundary conditions (with inner product constructed with respect to some choice of lifted metrics). Then there is a L^2-chain homotopy equivalence of Hilbert-$\mathcal{N}(\Gamma)$ cochain complexes*

$$\Omega^*_{(2)}(\widetilde{M}, \widetilde{E}) \simeq C^*_{(2)}(\widetilde{X}, \rho). \tag{6.3.30}$$

In particular, we obtain:

1. *For each $0 \le k \le n$, it holds that $b^{An}_{(2),k}(M,\rho) = b^{Top}_{(2),k}(M,\rho)$.*
2. *For each $0 \le k \le n$, it holds that $\alpha^{An}_k(M,\rho) = \alpha^{Top}_k(M,\rho)$.*
3. *(M,ρ) is of analytic determinant class if and only if it is of combinatorial determinant class.*

Proof. Choose a type II Morse-Smale function $f : M \to \mathbb{R}$, along with a Riemannian metric g on M and an Hermitian form h on E_ρ, so that the system $(E_\rho \downarrow M, g, h, \nabla_{g'} f)$ is weakly admissible. Consequently, the *lifted* system $(\widetilde{E}_\rho \downarrow \widetilde{M}, \widetilde{g}, \widetilde{h}, \nabla_{\widetilde{g}'} \widetilde{f})$ is weakly Γ-admissible.
Recall from Proposition 6.3.1 that for any $t \ge 0$, there is an isomorphism of Hilbert $\mathcal{N}(\Gamma)$-complexes

$$e^{tf} : \Omega^\bullet_{(2)}(\widetilde{M}, \widetilde{E}_\rho, \widetilde{g}, \widetilde{h}) \xrightarrow{\cong} \Omega^\bullet_{(2),t}(\widetilde{M}, \widetilde{E}_\rho, \widetilde{g}, \widetilde{h}). \tag{6.3.31}$$

Moreover, due to Proposition 6.3.2,3, the spectral projection of $\Delta_t[\widetilde{E}_\rho]$ associated to $[0,1)$ determines a chain homotopy equivalence of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

$$P(t) : \Omega^\bullet_{(2),t}(\widetilde{M}, \widetilde{E}_\rho, \widetilde{g}, \widetilde{h}) \xrightarrow{\simeq} \Omega^\bullet_{Sm,t}(\widetilde{M}, \widetilde{E}_\rho, \widetilde{g}, \widetilde{h}). \tag{6.3.32}$$

Next, for sufficiently large $t >> 0$, Theorem 6.3.4 implies that there is an isomorphism of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

$$F(t) : \Omega^\bullet_{Sm,t}(\widetilde{M}, \widetilde{E}_\rho, \widetilde{g}, \widetilde{h}) \xrightarrow{\cong} C^\bullet_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}'}\widetilde{f}, \widetilde{E}_\rho, \widetilde{h}). \tag{6.3.33}$$

Let X be any CW-structure on M. Then Corollary 5.4.12 implies that there is a chain homotopy equivalence of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

$$G : C^\bullet_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}'}\widetilde{f}, \widetilde{E}_\rho, \widetilde{h}) \xrightarrow{\simeq} C^\bullet_{(2)}(\widetilde{X}, \rho). \tag{6.3.34}$$

6.3.31-6.3.34 imply that the two Hilbert $\mathcal{N}(\Gamma)$-cochain complexes $\Omega^\bullet_{(2)}(\widetilde{M}, \widetilde{E}_\rho, \widetilde{g}, \widetilde{h})$ and $C^\bullet_{(2)}(\widetilde{X}, \rho)$ are chain homotopy equivalent. $\square$

6.4 Asymptotic expansions

Let $\mathcal{D} = (E \downarrow M, g, h, \nabla_g f)$ be a weakly admissible system with $\Gamma \coloneqq \pi_1(M)$ and let $\widetilde{\mathcal{D}} \coloneqq (\widetilde{E} \downarrow \widetilde{M}, \widetilde{g}, \widetilde{h}, \nabla_{\widetilde{g}}\widetilde{f})$ be the Γ-invariant lift of $\mathcal{D}$ (throughout this subsection, we assume that $g = g'$). We set $b \coloneqq f^{-1}(\partial M)$. For $t \geq 0$, let $\Omega^\bullet_{(2),t}(\widetilde{M}, \widetilde{E})$ be the Witten-deformed complex defined in the previous section (with metric induced by $\widetilde{g}$ and $\widetilde{h}$ implicit, in order to simplify notation) with Witten-deformed Laplacian $\Delta_{*,t}[\widetilde{E}] : \Omega^*_{(2),t}(\widetilde{M}, \widetilde{E}) \to \Omega^*_{(2),t}(\widetilde{M}, \widetilde{E})$. the orthogonal decomposition into the small and large subcomplex. Further, we define $\Theta^*(t) : \ker(\Delta_{*,t}[E]) \to H^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \widetilde{E}, \widetilde{h})$ to be the isomorphim of finitely-generated Hilbert $\mathcal{N}(\Gamma)$-modules that is the composition $\Theta^* \cdot e^{t\widetilde{f}}$, where $\Theta^* : \ker(\Delta_{*,0}[E]) \to H^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \widetilde{E}, \widetilde{h})$ is the isomorphism from 6.1.5. Introduce

$$\mathrm{Vol}(\mathcal{D})(t) \coloneqq \prod_{k=0}^{n} \det{}_\Gamma(\Theta^k(t))^{(-1)^k}. \tag{6.4.1}$$

Observe that

$$\mathrm{Vol}(\mathcal{D})(0) = T^{Met}_{(2)}(E \downarrow M, g, h, \nabla_g f). \tag{6.4.2}$$

Moreover, recall the orthogonal decomposition of subcomplexes

$$\Omega^\bullet_{(2),t}(\widetilde{M}, \widetilde{E}) = \Omega^\bullet_{Sm,t}(\widetilde{M}, \widetilde{E}) \oplus \Omega^\bullet_{La,t}(\widetilde{M}, \widetilde{E}),$$

which implies the following: Provided that $E \downarrow M$ is of determinant class, the torsion elements $T^{An}_{(2)}(\mathcal{D})(t), T^{Sm}_{(2)}(\mathcal{D})(t)$ and $T^{La}_{(2)}(\mathcal{D})(t)$ of the complexes $\Omega^\bullet_{(2),t}(\widetilde{M}, \widetilde{E})$, $\Omega^\bullet_{Sm,t}(\widetilde{M}, \widetilde{E})$, respectively $\Omega^\bullet_{La,t}(\widetilde{M}, \widetilde{E})$ are all well-defined positive real numbers, so that

$$T^{An}_{(2)}(\mathcal{D})(0) = T^{An}_{(2)}(M, E, g, h), \tag{6.4.3}$$

$$T^{An}_{(2)}(\mathcal{D})(t) = T^{Sm}_{(2)}(\mathcal{D})(t) \cdot T^{La}_{(2)}(\mathcal{D})(t). \tag{6.4.4}$$

A function $F : \mathbb{R} \to \mathbb{R}$ is said to admit an *asymptotic expansion*, if there exists an integer $N \in \mathbb{N}$ and constants $(a_j)_{j=0}^N, (b_j)_{j=0}^N$ such that for $t \to +\infty$

$$F(t) = \sum_{j=0}^{N} (a_j + b_j \log(t))\, t^j + o(1). \tag{6.4.5}$$

The coefficient a_0 in the expansion is called the *free term* of F and is denoted by $\mathrm{FT}(F)$.

Proposition 6.4.1 (Asymptotic expansion for the analytic torsion). *There exists a constant $C \in \mathbb{R}$, such that the following holds: For any weakly admissible system $\mathcal{D} = (E \downarrow M, g, h, \nabla_g f)$ of determinant class with M odd-dimensional, the function $\log T^{An}_{(2)}(\mathcal{D})(t) - \log Vol(\mathcal{D})(t)$ admits the asymptotic expansion.*

$$\log T^{An}_{(2)}(E \downarrow M, g, h) - \log Vol(\mathcal{D})(0) + Ct \dim(E)\chi(\partial M). \tag{6.4.6}$$

Proof. For $0 \leq p \leq n$, recall that the Witten-Laplacian $\Delta_{p,t}[\widetilde{E}]$ of the lifted system $\widetilde{\mathcal{D}} = (\widetilde{E} \downarrow \widetilde{M}, \widetilde{g}, \widetilde{h}, \nabla_{\widetilde{g}} \widetilde{f})$ is the lift $\widetilde{\Delta_{p,t}[E]}$ of the Witten-Laplacian corresponding to $\mathcal{D}$. For $t, \lambda \geq 0$, introduce the function

$$\Theta_{\mathcal{D}}(\lambda, t) = \sum_{p=0}^{n} (-1)^p \operatorname{tr}_\Gamma(\widetilde{f} \cdot e^{-\lambda \Delta_{p,t}[\widetilde{E}]}). \tag{6.4.7}$$

It follows from Theorem 4.3.2 that for fixed $t \geq 0$, $\Theta_{\mathcal{D}}(\lambda, t)$ is smooth in λ, and that for each $0 \leq k \leq n$, and each $t \in \mathbb{R}_{\geq 0}$, there exists Γ-equivariant differential forms

$$\alpha^t_k(\mathcal{D}) \in \Omega^n(\widetilde{M}), \tag{6.4.8}$$

$$\beta^t_k(\mathcal{D}) \in \Omega^{n-1}(\widetilde{\partial M}), \tag{6.4.9}$$

that are local quantities of the lifted system $\widetilde{\mathcal{D}}$, depend smoothly on t, satisfy $\alpha^t_k(\mathcal{D}) \equiv 0$ whenever k is odd, and such that for $\lambda \to 0$, we have

$$\Theta_{\mathcal{D}}(t, \lambda) = \sum_{k=0}^{n} \lambda^{\frac{-n+k}{2}} (a^t_k(\mathcal{D}) + b^t_k(\mathcal{D})) + \mathcal{O}(\lambda^{1/2}), \tag{6.4.10}$$

$$a^t_k(\mathcal{D}) := \int_{\mathcal{F}} f \cdot \alpha^t_k(\mathcal{D}) \ \ , \ \ b^t_k(\mathcal{D}) := \int_{\partial \mathcal{F}} b \cdot \beta^t_k(\mathcal{D}), \tag{6.4.11}$$

where $\mathcal{F}$ is a fundamental domain for the Γ-action on $\widetilde{M}$, such that $\partial\mathcal{F} := \mathcal{F} \cap \widetilde{\partial M}$ is a fundamental domain for the Γ-action on $\widetilde{\partial M}$. Moreover, the very same arguments employed in [22, pp. 821-824] can now be applied to show that

$$\frac{d}{dt}\left(\log T^{An}_{(2)}(\mathcal{D})(t) - \log Vol(\mathcal{D})(t)\right)\Big|_{t=t_0} = a^{t_0}_n(\mathcal{D}) + b^{t_0}_n(\mathcal{D}) \overset{n \text{ odd}}{=} b^{t_0}_n(\mathcal{D}). \tag{6.4.12}$$

Therefore, in order to show the proposition, it remains to find a constant $C > 0$ independent of $\mathcal{D}$ or t satisfying

$$b^t_n(\mathcal{D}) = C \dim(E)\chi(\partial M) \qquad \forall t \in \mathbb{R}_{\geq 0}\,. \tag{6.4.13}$$

For this, the assumption that the metric bundle $(E, h) \downarrow (M, g)$ is of product structure near ∂M is essential.

First, we will show that for a simple system $\mathcal{D}_0$ we now construct, $b^t_n(\mathcal{D}_0)$ is completely independent of t: Let $E_{\mathbb{C}} \downarrow [a, b]$ be the trivial complex line bundle over $[a, b]$. We equip $[a, b]$ with the standard Euclidean metric $1_{\mathbb{R}}$ and the bundle $E_{\mathbb{C}}$ with the constant (parallel) Hermitian form $1_{\mathbb{C}}$. Further we choose a Morse function $f_0(x) : [a, b] \to \mathbb{R}$ on $[a, b]$ in such a way that $f_0(a + s\epsilon) = f_0(b - s\epsilon) = b - s\epsilon$ for all $s \in [0, 1]$ and some small $\epsilon > 0$. These choices form the admissible system $\mathcal{D}_0 = (E_{\mathbb{C}} \downarrow [a, b], 1_{\mathbb{R}}, 1_{\mathbb{C}}, \frac{d}{dx} f_0)$.

We wish to compute the density $\beta^t_1(\mathcal{D}_0)$ that appears in the asymptotic expansion of the theta function

$$\Theta_{\mathcal{D}_0}(\lambda, t) = \operatorname{tr}(f_0 \cdot e^{-\lambda \Delta_{0,t}[E_{\mathbb{C}}]}) - \operatorname{tr}(f_0 \cdot e^{-\lambda \Delta_{1,t}[E_{\mathbb{C}}]}). \tag{6.4.14}$$

For this, observe first that the function $G(x) := b + a - x$ gives rise to a bundle isometry of $E_{\mathbb{C}} \downarrow [a, b]$ to itself that also satisfies $f_0(x) = f_0(b + a - x)$ for all x near a. Therefore, G is an isomorphism between the restricted systems $\mathcal{D}_0|_{[a,a+\epsilon)}$ and $\mathcal{D}_0|_{(b-\epsilon,b]}$. Consequently, since $\beta_1^t(\mathcal{D}_0)$ is a local quantity, it follows that $\beta_1^t(\mathcal{D}_0)(a) = \beta_1^t(\mathcal{D}_0)(b)$ for all $t \in \mathbb{R}$. Next, observe that the two Witten-Laplacians $\Delta_{1,t}[E_\mathbb{C}]$ and $\Delta_{0,t}[E_\mathbb{C}]$ are elliptic operators of order 2 over a 1-dimensional manifold, which is why $\beta_1^t(\mathcal{D}_0)$ depends only on the principal symbol of the $\Delta_{i,t}[E_\mathbb{C}]$ with $i = 0, 1$. Lastly, under the natural isometry

$$\begin{aligned}&\Omega^1([a,b],\mathbb{C}) \cong \Omega^0([a,b],\mathbb{C}),\\ &f(x)dx \mapsto f(x),\end{aligned}$$

the Witten-Laplacian $\Delta_{1,t}[E_\mathbb{C}]$ of 1-forms can be identified with the Witten-Laplacian $\Delta_{0,t}[E_\mathbb{C}]$ of 0-forms, which takes the form

$$\Delta_{0,t}[E_\mathbb{C}] = -\frac{d^2}{dx^2} + t^2$$

near b. Since the principal symbol of $\Delta_{0,t}[E_\mathbb{C}]$ is independent of t, we finally obtain

$$\beta_1^t(\mathcal{D}_0) \equiv C', \tag{6.4.15}$$

$$b_1^t(\mathcal{D}_0) = \int_{\{a,b\}} f_0 \cdot \beta_1^t(\mathcal{D}_0) = b\left(\beta_1^t(\mathcal{D}_0)(a) + \beta_1^t(\mathcal{D}_0)(b)\right) = 2bC'. \tag{6.4.16}$$

for some constant $C' \in \mathbb{R}$.

Consider the weakly admissible system $\mathcal{D}_1 = (E|_{\partial M} \otimes E_\mathbb{C} \downarrow \partial M \times [a,b], g|_{\partial M} \otimes 1_\mathbb{R}, h|_{\partial M}\hat{\otimes} 1_\mathbb{C}, F)$, where $F : \partial M \times [a,b] \to \mathbb{R}$ is the Morse function defined by $F(p,x) := f_0(x)$ and we let $\widetilde{\mathcal{D}_1}$ be its lift to $\widetilde{\partial M} \times [a,b]$. We use the abbreviations

$$\Delta_{*,t} := \Delta_{*,t}[\widetilde{E|_{\partial M}}\hat{\otimes} E_\mathbb{C}]\ ,\ \Delta_{*,t}[0] := \Delta_{*,t}[E_\mathbb{C}]\ ,\ \Delta_* := \Delta_*[\widetilde{E|_{\partial M}}]$$

and observe that we have a orthogonal sum decomposition of the Laplacian

$$\Delta_{p,t} = \bigoplus_{q=0}^{p} (\Delta_q \otimes \mathbb{1}_{E_\mathbb{C}}) + \left(\mathbb{1}_{\widetilde{E_{\partial M}}} \otimes \Delta_{p-q,t}[0]\right).$$

From this, we deduce the decomposition of heat operators

$$e^{-\lambda \Delta_{p,t}} = \bigoplus_{q=0}^{p} e^{-\lambda\Delta_q} \otimes e^{-\lambda\Delta_{p-q,t}[0]}. \tag{6.4.17}$$

Using the Hodge-decomposition, one verifies as in [22, Proposition 1.21, Proposition 4.2] that

$$\sum_{q=0}^{n} (-1)^q \operatorname{tr}_\Gamma(e^{-\lambda\Delta_q}) = \chi(\partial M, E) = \dim(E)\chi(\partial M), \tag{6.4.18}$$

Applying Equations 6.4.14 and 6.4.17–6.4.18, we compute

$$\begin{aligned}\Theta_{\mathcal{D}_1}(\lambda,t) &= \sum_{p=0}^{n} (-1)^p \operatorname{tr}_\Gamma(\widetilde{F} \cdot e^{-\lambda(\Delta_{p,t})})\\ &= \sum_{q,r} (-1)^{q+r} \operatorname{tr}_\Gamma\left(e^{-\lambda(\Delta_q)} \otimes f_0 \cdot e^{-\lambda\Delta_{r,t}[0]}\right)\\ &= \sum_q (-1)^q \operatorname{tr}_\Gamma(e^{-\lambda\Delta_q}) \cdot \sum_r (-1)^r \operatorname{tr}(f_0 \cdot e^{-\lambda\Delta_{r,t}[0]})\\ &= \dim(E)\chi(\partial M) \cdot \Theta_{\mathcal{D}_0}(\lambda,t).\end{aligned} \tag{6.4.19}$$

In particular, for any $t \geq 0$, the functions $\Theta_{\mathcal{D}_1}(\lambda, t)$ and $\dim(E)\chi(\partial M)\Theta_{\mathcal{D}_0}(\lambda, t)$ have the same asymptotic expansions for $\lambda \to 0$, therefore also the same constant terms

$$b_n^t(\mathcal{D}_1) = \dim(E)\chi(\partial M)b_1^t(\mathcal{D}_0) \overset{6.4.16}{=} \dim(E)\chi(\partial M)2bC'. \tag{6.4.20}$$

Furthermore, using the locality of densities, one deduces that

$$\beta_n^t(\mathcal{D})(x) = \beta_n^t(\mathcal{D}_1)(x, b) = \beta_n^t(\mathcal{D}_1)(x, a)$$

for all $x \in \partial M \cong \partial M \times \{b\} \cong \partial M \times \{a\}$, which implies that

$$b_n^t(\mathcal{D}) = \int_{\partial M} b\beta_n^t(\mathcal{D}) = \int_{\partial M \times \{b\}} b\beta_n^t(\mathcal{D}_1) = \frac{1}{2}\int_{\partial M \times \{a,b\}} b\beta_n^t(\mathcal{D}_1) = \frac{1}{2}b_n^t(\mathcal{D}_1) = bC' \dim(E)\chi(\partial M).$$

With $C \coloneqq bC'$, Equation 6.4.13 is finally proven. □

Proposition 6.4.2 (Asymptotic expansion for the small torsion)**.** *For any weakly admissible system $\mathcal{D} = (E \downarrow M, g, h, \nabla_g f)$ of determinant class with $n \coloneqq \dim(M)$ and $m_k \coloneqq \#Cr_k(f)$, the function $\log T_{(2)}^{Sm}(\mathcal{D})(t) - \log Vol(\mathcal{D})(t)$ admits the asymptotic expansion*

$$\log T_{MS}^{(2)}(M, E, h, \nabla_g f) + \dim(E)\left(\sum_{k=0}^{n}(-1)^k m_k \frac{n-2k}{4}\log(\pi/t) + t(-1)^{k+1}\sum_{x \in Cr_k(f)} f(x)\right) + o(1). \tag{6.4.21}$$

Proof. For large $t >> 0$, there exists by Theorem 6.3.4 an isomorphism of finitely generated Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

$$F^\bullet(t) : \Omega_{Sm,t}^\bullet(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h}) \to C_{(2)}^\bullet(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \widetilde{E}, \widetilde{h}). \tag{6.4.22}$$

From Proposition 4.1.40, it then follows that

$$\log T_{(2)}^{Sm}(M, E, g, h, f)(t) - \log \mathrm{Vol}(t) = \log T_{MS}^{(2)}(M, E, h, \nabla_g f) - \sum_{k=0}^{n}(-1)^k \log \det\nolimits_\Gamma F^k(t). \tag{6.4.23}$$

Recall also from Theorem 6.3.4 the formula $S^k(t) \circ F^k(t) \circ I^k(t) = \mathbb{1}_{C_{(2)}^k} + O(t^{-1})$, where $I^k(t)$ is the isometry from 6.3.24 and $S^k(t)$ is the scaling isomorphism from 6.3.25. Consequently, by the multiplicativity of the Fuglede-Kadison determinant in this setting, it holds that

$$\log \det\nolimits_\Gamma F^k(t) = -\log \det\nolimits_\Gamma S^k(t) + o(1). \tag{6.4.24}$$

From the explicit formula of $S^k(t)$, along with Proposition 4.1.40, we obtain

$$\det\nolimits_\Gamma S^k(t) = \left(\prod_{x \in Cr_k(f)} (\pi/t)^{\frac{n-2k}{4}} e^{-tf(p)}\right)^{\dim(E)}. \tag{6.4.25}$$

The result now is an immediate consequence of 6.4.23 – 6.4.25. □

Corollary 6.4.3 (Asymptotic expansion for the large torsion)**.** *Let $\mathcal{D} = (E \downarrow M, g, h, \nabla_g f)$ be a weakly admissible system of determinant class with M odd-dimensional. The, the following assertions hold*

1. *The function $\log T^{La}_{(2)}(\mathcal{D})(t)$ admits an asymptotic expansion. More precisely, there exists a polynomial $\Phi(\mathcal{D})(t) : \mathbb{R} \to \mathbb{R}$ in t and $\log(t)$, such that for $t \to \infty$*

$$\log T^{La}_{(2)}(\mathcal{D})(t) = R(\mathcal{D}) + \Phi(\mathcal{D})(t) + o(1). \tag{6.4.26}$$

Finally, for any arbitrary small neighborhood U of $Cr(f) \cup \partial M$, the polynomial $\Phi(\mathcal{D})$ depends only on the isomorphism class of the system $\mathcal{D}^f|_U := (E|_U \downarrow U, g|_U, h|_U, f|_U)$.

2. *Suppose that $\mathcal{D}_1 = (E_1 \downarrow M_1, g_1, h_1, \nabla_{g_1} f_1)$ is another weakly admissible system, such that there exists neighborhoods $U \subseteq M$ of $Cr(f) \cup \partial M$ and $U_1 \subset M_1$ of $Cr(f_1) \cup \partial M_1$ with the property that the derived systems $\mathcal{D}^f|_U := (E|_U \downarrow U, g|_U, h|_U, f|_U)$ and $\mathcal{D}_1^{f_1}|_{U_1} := (E_1|_{U_1} \downarrow U_1, g|_{U_1}, h|_{U_1}, f_1|_{U_1})$ are isomorphic (in particular $\#Cr_k(f) = \#Cr_k(f_1)$ for each $0 \leq k \leq n$). Then*

$$R(\mathcal{D}) - R(\mathcal{D}_1) = FT\left(\log T^{La}_{(2)}(\mathcal{D})\right) - FT\left(\log T^{La}_{(2)}(\mathcal{D}_1)\right). \tag{6.4.27}$$

3. *Under the assumptions of (2), there exists local quantities $\alpha(\mathcal{D}) \in \Omega^n(M \setminus Cr(f), \mathcal{O}_M)$ and $\alpha(\mathcal{D}_1) \in \Omega^n(M_1 \setminus Cr(f_1), \mathcal{O}_{M_1})$ of the derived systems systems $\mathcal{D}|_{M\setminus Cr(f)}$ and $\mathcal{D}_1|_{M_1\setminus Cr(f_1)}$, such that one has*

$$FT\left(\log T^{La}_{(2)}(\mathcal{D})\right) - FT\left(\log T^{La}_{(2)}(\mathcal{D}_1)\right) = \int_{M\setminus Cr(f)} \alpha(\mathcal{D}) - \int_{M\setminus Cr(f_1)} \alpha(\mathcal{D}_1). \tag{6.4.28}$$

Proof.

1. We have $\log T^{An}_{(2)}(\mathcal{D})(t) = \log T^{Sm}_{(2)}(\mathcal{D})(t) + \log T^{La}_{(2)}(\mathcal{D})(t)$, hence also in particular

$$\left(\log T^{An}_{(2)}(\mathcal{D})(t) - \log \mathrm{Vol}(\mathcal{D})(t)\right) - \left(\log T^{Sm}_{(2)}(\mathcal{D})(t) - \log \mathrm{Vol}(\mathcal{D})(t)\right) = \log T^{La}_{(2)}(\mathcal{D})(t).$$

Since the left-hand side of the equation admits an asymptotic expansion, given by the sum of the explicit formulas 6.4.6 and 6.4.21, the result follows.

2. Observe that $\mathrm{FT}\left(\log T^{La}_{(2)}(\mathcal{D})\right) = R(\mathcal{D}) + \mathrm{FT}(\Phi(\mathcal{D}))$ and analogously $\mathrm{FT}\left(\log T^{La}_{(2)}(\mathcal{D}_1)\right) = R(\mathcal{D}_1) + \mathrm{FT}(\Phi(\mathcal{D}_1))$. Since the systems $\mathcal{D}^f|_U$ and $\mathcal{D}^{f_1}|_{U_1}$ are isomorphic by assumption, assertion (1) implies that $\Phi(\mathcal{D}) \equiv \Phi(\mathcal{D}_1)$ and the result follows.

3. In case that $\partial M = \emptyset$, this is proven in [22, Theorem B, Section 6.2] for unitary bundles (whose proof is also referred to in [20, Proposition 4.2] for arbitrary flat bundles). The same proof works without any modifications in the case that $\partial M \neq \emptyset$.

□

6.5 Proof of Theorem 6.1.5

Proposition 6.5.1. *For $i = 1, 2$, let $\mathcal{D}_i = (M_i, E_i, g_i, h_i, \nabla_{g_i} f_i)$ be two weakly admissible systems satisfying the assumptions of Corollary 6.4.3.2. Moreover, assume that there exists a flat bundle $E_3 \downarrow M_3$ with M_3 compact, satisfying*

1. *$(E_3|_{\partial M_3}) \downarrow \partial M_3 = E_i|_{\partial M_i} \downarrow \partial M_i$, and*

2. *the bundle* $\overline{E_i} \downarrow N_i$ *is of determinant class, where* $N_i := M_3 \cup_{\partial M_3} M_i$ *and* $\overline{E_i} := E_3 \cup_{E_3|_{\partial M_3}} E_i$.

Then

$$\mathcal{R}(\mathcal{D}_1) + \frac{1}{2}\int_{M_1} \theta(h_1) \wedge (\nabla_{g_1} f_1)^* \Psi(TM_1, g_1) = \mathcal{R}(\mathcal{D}_2) + \frac{1}{2}\int_{M_2} \theta(h_2) \wedge (\nabla_{g_2} f_2)^* \Psi(TM_2, g_2). \quad (6.5.1)$$

Proof. Choose a smooth function $f_3 : M_3 \to \mathbb{R}$ on M_3 with $f_3|_{\partial M_3} = f_i|_{\partial M_i}$ for $i = 1, 2$ and such that the function $\overline{f}_i := f_3 \cup_{\partial M_3} f_i : N_i \to \mathbb{R}$ is a Morse function. Furthermore, choose a Riemannian metric g_3 on M_3 with $g_3|_{\partial M_3} = g_i|_{\partial M_i}$ for $i = 1, 2$, such that for the metric $\overline{g}_i = g_3 \cup_{\partial M_3} g_i$ on N_i, the pair $(\overline{f}_i, \overline{g}_i)$ is a Morse-Smale pair (since N_i is closed, there is no distinction between type I and type II). Lastly, choose a Hermitian form h_3 on the flat bundle $E_3 \downarrow M_3$ with $h_3|_{\partial M_3} = h_i|_{\partial M_i}$ for $i = 1, 2$ with $\overline{h}_i := h_3 \cup_{\partial M_3} h_i$, such that the system

$$\overline{\mathcal{D}_i} := (\overline{E_i} \downarrow N_i, \overline{g}_i, \overline{h}_i, \nabla_{\overline{g}_i} \overline{f}_i) \quad (6.5.2)$$

is weakly admissible. By construction, the pair $\overline{\mathcal{D}_i}$ also satisfies the assumptions of Corollary 6.4.3.2. Applying Corollary 6.4.3.3, we can find densities α_i on $M_i \setminus Cr(f_i)$ and $\overline{\alpha_i}$ on $N_i \setminus Cr(\overline{f_i})$, so that

$$R(\mathcal{D}_1) - R(\mathcal{D}_2) = \int_{M_1 \setminus Cr(f_1)} \alpha_1 - \int_{M_2 \setminus Cr(f_2)} \alpha_2, \quad (6.5.3)$$

$$R(\overline{\mathcal{D}_1}) - R(\overline{\mathcal{D}_2}) = \int_{N_1 \setminus Cr(\overline{f_1})} \overline{\alpha_1} - \int_{N_2 \setminus Cr(\overline{f_2})} \overline{\alpha_2}. \quad (6.5.4)$$

Since the densities are local quantities, it follows from the chosen metrics on the respective bundles that $\alpha_i = \overline{\alpha_i}|_{M_i}$ and $\overline{\alpha_1}|_{M_3} = \overline{\alpha_2}|_{M_3}$. Moreover, since $Cr(\overline{f_i}) \cap M_i = Cr(f_i)$ by construction, we get from 6.5.3 and 6.5.4

$$R(\mathcal{D}_1) - R(\mathcal{D}_2) = R(\overline{\mathcal{D}}_1) - R(\overline{\mathcal{D}}_2). \quad (6.5.5)$$

As N_i is closed, we can apply [102, Theorem 4.2] and obtain

$$R(\overline{\mathcal{D}_i}) = \frac{1}{2}\int_{N_i} \theta(\overline{E_i}, \overline{h_i}) \wedge (\nabla_{\overline{g_i}} \overline{f}_i)^* \Psi(TN_i, \overline{g_i}), \quad (6.5.6)$$

As mentioned in the introduction, the n-form $\theta(\overline{E_i}, \overline{h_i}) \wedge (\nabla_{\overline{g_i}} \overline{f}_i)^* \Psi(TN_i, \overline{g_i})$ is a local quantity. In particular, it follows both that $\theta(\overline{E_i}, \overline{h_i}) \wedge (\nabla_{\overline{g_i}} \overline{f_i})^* \Psi(TN_i, \overline{g_i})|_{M_i} = \theta(E_i, h_i) \wedge (\nabla_{g_i} f_i)^* \Psi(TM_i, g_i)$ and that $\theta(\overline{E_1}, \overline{h_1}) \wedge (\nabla_{\overline{g_1}} \overline{f}_1)^* \Psi(TN_1, \overline{g_1})|_{M_3} = \theta(\overline{E_2}, \overline{h_2}) \wedge (\nabla_{\overline{g_2}} \overline{f}_2)^* \Psi(TN_2, \overline{g_2})|_{M_3}$. Therefore

$$\begin{aligned} &\int_{N_1} \theta(\overline{E_1}, \overline{h_1}) \wedge (\nabla_{\overline{g_1}} \overline{f}_1)^* \Psi(TN_1, \overline{g_1}) - \int_{N_2} \theta(\overline{E_2}, \overline{h_2}) \wedge (\nabla_{\overline{g_2}} \overline{f}_2)^* \Psi(TN_2, \overline{g_2}) \\ &= \int_{M_1} \theta(E_1, h_1) \wedge (\nabla_{g_1} f_1)^* \Psi(TM_1, g_1) - \int_{M_2} \theta(E_2, h_2) \wedge (\nabla_{g_2} f_2)^* \Psi(TM_2, g_2). \quad (6.5.7) \end{aligned}$$

Equation 6.5.1 now is an immediate consequence of 6.5.5 – 6.5.7. ☐

Theorem 6.5.2. *Assume that* $\mathcal{D}_i = (E_i \downarrow M_i, g_i, h_i, \nabla_{g_i'} f_i)$ *are two admissible systems with* M_i *odd-dimensional,* $(\partial M_1, g_1|_{\partial M_1}) = (\partial M_2, g_2|_{\partial M_2})$ *and* $(E_1|_{\partial M_1}, h_1|_{\partial M_1}) = (E_2|_{\partial M_2}, h_2|_{\partial M_2})$*. Then, if both* $E_i \downarrow M_i$ *and* $E_i|_{\partial M_i} \downarrow M_i$ *are of determinant class, we get*

$$\mathcal{R}(\mathcal{D}_1) + \frac{1}{2}\int_{M_1} \theta(E_1, h_1) \wedge (\nabla_{g_1'} f_1)^* \Psi(TM_1, g_1) = \mathcal{R}(\mathcal{D}_2) + \frac{1}{2}\int_{M_2} \theta(E_2, h_2) \wedge (\nabla_{g_2'} f_2)^* \Psi(TM_2, g_2).$$

Proof. We consider different cases:

Case 1: The systems $\mathcal{D}_i$ satisfy the hypotheses of Corollary 6.4.3.2:

Consider the admissible system $\mathcal{D}_{S^2} : (E_{\mathbb{C}}^{S^2} \downarrow S^2, g, h, \nabla_g f)$ with $E_{\mathbb{C}}^{S^2} \downarrow S^2$ the the trivial complex line bundle over S^2, (f, g) some Morse-Smale pair on S^2 and h a parallel metric on $E_{\mathbb{C}}^{S^2}$. Since S^2 is simply-connected, the system $\mathcal{D}_{S^2}$ is of determinant class. It follows from Proposition 6.2.2 that that also the modified product systems $\underline{\mathcal{D}_i \times \mathcal{D}_{S^2}}$ are of determinant class, so that

$$\mathcal{R}(\underline{\mathcal{D}_i \times \mathcal{D}_{S^2}}) = 2\mathcal{R}(\mathcal{D}_i), \tag{6.5.8}$$

where we have used that $\chi(S^2) = 2$, as well as the well-known fact that $\mathcal{R}(\mathcal{D}_{S^2}) = 0$, which follows for example also from [102, Theorem 4.2].

Next, consider the trivial complex line bundle $E_{\mathbb{C}}^{D^3} \downarrow D^3$. Since D^3 is simply-connected, it is of determinant class. Moreover, since $E|_{\partial M_1} \downarrow \partial M_1$ is of determinant class by assumption and ∂M_1 is closed, it follows again from Proposition 6.2.2 that the product bundle $E|_{\partial M_1} \hat{\otimes} E_{\mathbb{C}}^{D^3} \downarrow \partial M_1 \times D^3$, as well as its restriction to $\partial(\partial M_1 \times D^3) = \partial M_1 \times \partial D^3$, is of determinant class. Now observe that by construction, the identification $\partial D^3 \cong S^2$ induces an isomorphism of flat bundles $E_1 \hat{\otimes} E_{\mathbb{C}}^{D^3}|_{\partial M_1 \times \partial D^3} \downarrow \partial M_1 \times \partial D^3 \cong E_i \hat{\otimes} E_{\mathbb{C}}^{S^2}|_{\partial M_i \times S^2} \downarrow \partial M_i \times S^2$ for $i = 1, 2$. Just as in Proposition 6.5.1, we can therefore define for $i = 1, 2$

$$\begin{aligned} N_i &:= M_i \times S_2 \cup_{\partial M_1 \times S^2} \partial M_1 \times D^3, \\ \overline{E_i} &:= E_i \hat{\otimes} E_{\mathbb{C}}^{S^2} \cup_{E_i|_{\partial M_i} \hat{\otimes} E_{\mathbb{C}}^{S^2}} E_1 \hat{\otimes} E_{\mathbb{C}}^{D^3}. \end{aligned}$$

By Proposition 6.2.8, it follows that $\overline{E_i} \downarrow N_i$ is of determinant class. Hence, the modified product systems $\underline{\mathcal{D}_i \times \mathcal{D}_{S^2}}$ satisfy also the assumptions of Proposition 6.5.1, from which we get

$$\begin{aligned} &\mathcal{R}(\underline{\mathcal{D}_1 \times \mathcal{D}_{S^2}}) + \frac{1}{2} \int_{M_1 \times S^2} \theta(h_1 \hat{\otimes} h) \wedge \nabla_{g_1 \times g}(\underline{f_1 + f})^* \Psi \left(T(M_1 \times S^2), g_1 \times g \right) \\ &= \mathcal{R}(\underline{\mathcal{D}_2 \times \mathcal{D}_{S^2}}) + \frac{1}{2} \int_{M_2 \times S^2} \theta(h_2 \hat{\otimes} h) \wedge (\nabla_{g_2 \times g}(\underline{f_2 + f}))^* \Psi \left(T(M_2 \times S^2), g_2 \times g \right). \end{aligned} \tag{6.5.9}$$

Applying the product formula 6.2.3, we obtain for $i = 1, 2$

$$\theta(h_i \hat{\otimes} h) \wedge \nabla_{g_i \times g}(\underline{f_i + f})^* \Psi \left(T(M_i \times S^2), g_i \times g \right) = (\theta(h_i) \wedge (\nabla_{g_i} f_i)^* \Psi(TM_i, g_i)) \otimes e(TS^2, g),$$

Since $e(TS^2, g)$ is a representative of the rational Euler class of TS^2, we obtain that $\int_{S^2} e(TS^2, g) = \chi(S^2) = 2$. Together with the previous equation, this implies for $i = 1, 2$, that

$$\int_{M_i \times S^2} \theta(h_i \hat{\otimes} h) \wedge \nabla_{g_i \times g}(\underline{f_i + f})^* \Psi \left(T(M_i \times S^2), g_i \times g \right) = 2 \int_{M_i} \theta(h_i) \wedge (\nabla_{g_i} f_i)^* \Psi(TM_i, g_i). \tag{6.5.10}$$

The result now follows from 6.5.8 – 6.5.10.

Case 2: The systems $\mathcal{D}_i$ don't satisfy the hypotheses of Corollary 6.4.3.2:

Since the $\mathcal{D}_i$ are by assumption admissible, we find a neighborhood U of ∂M, such $\theta(h_i) \equiv 0$ on U and $g_i \equiv g_i'$ on $M \setminus U$, which is why $\theta(h_i) \wedge (\nabla_{g_i'} f_i)^* \Psi(TM_i, g_i) = \theta(h_i) \wedge (\nabla_{g_i} f_i)^* \Psi(TM_i, g_i)$ on *all of* M. Moreover, since both g_i' and g_i are of product form near ∂M_i and $h_i|_{\partial M_i}$ is unimodular, it follows from Proposition 6.2.4 that $\mathcal{R}(\mathcal{D}_i) = \mathcal{R}(E_i \downarrow M_i, g_i', h_i, \nabla_{g_i'} f_i)$. Therefore, we may assume without loss of generality that $g_i \equiv g_i'$ on all of M.

Now since the M_i are odd-dimensional with $\partial M_1 = \partial M_2$, we have $\chi(M_1) = \chi(M_2)$. Using this, one proceeds as in [22, Section 6] to show that there exist subdivisions $(\overline{f_i}, \overline{g_i})$ of (f_i, g_i) (with $\overline{g_i} = g_i$ near ∂M_i), neighborhoods U_i of $Cr(\overline{f_i}) \cup \partial M_i$ and an isometry $\theta : (U_1, \overline{g_1}) \to (U_2, \overline{g_2})$ satisfying $\theta(Cr(\overline{f_1})) =$

$Cr(\overline{f_2})$, $\theta(M_1) = M_2$ and $\overline{f_2} \circ \theta = \overline{f_1}$. By Lemma 6.2.7, one additionally finds a Hermitian form $\overline{h_i}$ on the bundle $E_i \downarrow M_i$ (with $h_i = \overline{h_i}$ near ∂M_i) so that $\overline{\mathcal{D}_i} \coloneqq (E_i \downarrow M_i, \overline{g_i}, \overline{h_i}, \nabla_{\overline{g_i}} \overline{f_i})$ is an admissible system, satisfying

$$\mathcal{R}(\mathcal{D}_i) = \mathcal{R}(\overline{\mathcal{D}_i}). \tag{6.5.11}$$

Moreover, since the new systems $\overline{\mathcal{D}_i}$ now also satisfy the assertions of Corollary 6.4.3.2, we can apply Case 1 to them and are finished. □

Proof of Theorem 6.1.5: Let $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ be an Morse-Smale system of product form, M odd-dimensional, so that $E|_{\partial M} \downarrow \partial M$ is also of determinant class. After pertubing the metric g outside from a neighborhood of ∂M, it is because of Proposition 6.2.15 that we may assume without loss of generality that $g \equiv g'$ outside from a neighborhood of ∂M, i.e. that $\mathcal{D}$ is admissible.
Choose a Morse-Smale pair $(\hat{f}, \hat{g})$ on ∂M. Then

$$\mathcal{D}' \coloneqq (E|_{\partial M} \downarrow \partial M, g|_{\partial M}, h|_{\partial M}, \nabla_{\hat{g}} \hat{f})$$

is a Morse-Smale system of determinant class. Since ∂M is closed, we have by [102, Theorem 4.2]

$$\mathcal{R}(\mathcal{D}') = -\frac{1}{2} \int_{\partial M} \theta(h|_{\partial M}) \wedge (\nabla_{\hat{g}} \hat{f})^* \Psi(T\partial M, g|_{\partial M}) = 0, \tag{6.5.12}$$

where the last equality follows from the assumption that $h_{\partial M}$ is unimodular, i.e. $\theta(h|_{\partial M}) \equiv 0$.
Now recall the trivial system $\mathcal{D}_0 = (E_{\mathbb{C}} \downarrow I, g_0, h_0, \nabla_{g_0} f_0)$ over the interval $I = [a, b]$ that we have defined in 6.1.7 and its relative torsion

$$\mathcal{R}(\mathcal{D}_0) = -\frac{\log 2}{2}. \tag{6.5.13}$$

Since ∂M is closed and $\partial I = \{a, b\}$, we can form the modified product system

$$\underline{\mathcal{D}' \times \mathcal{D}_0} = (E_I \downarrow \partial M \times I, g_I, h_I, \nabla_{\hat{g}_I} \hat{f}_I), \tag{6.5.14}$$

with $E_I \coloneqq E_{\partial M} \hat{\otimes} E_{\mathbb{C}}$, $g_I \coloneqq g_{\partial M} \times g_0$, $\hat{g}_I \coloneqq \hat{g} \times g_0$, $h_I \coloneqq h|_{\partial M} \hat{\otimes} h_0$ and $\hat{f}_I$ the sum of the Morse functions $\hat{f} + f_0$ that is appropriately modified near the boundary $\partial M \times \{a, b\}$, so that $\underline{\mathcal{D}' \times \mathcal{D}_0}$ is a type II Morse-Smale system. By Proposition 6.2.2, this system is of determinant class as well and satisfies

$$\mathcal{R}(\underline{\mathcal{D}' \times \mathcal{D}_0}) = \mathcal{R}(\mathcal{D}') - \frac{\log 2}{2} \chi(\partial M, E) \overset{6.5.12}{=} -\frac{\log 2}{2} \chi(\partial M) \dim(E). \tag{6.5.15}$$

Moreover, as $\theta(h_0) \equiv 0$ and $\theta(h|_{\partial M}) = 0$ by assumption, we retrieve from the product formula 6.2.5 the equality

$$\theta(h_I) = \theta(h|_{\partial M} \hat{\otimes} h_0) = 0. \tag{6.5.16}$$

Notice that $\underline{\mathcal{D}' \times \mathcal{D}_0}$ is not necessarily an admissible system. This is due to the fact that neither is g_I trivial nor h_I parallel near $Cr(\hat{f}_I)$. However, since $Cr(\hat{f}_I)$ is disjoint from $\partial M \times \{a, b\}$, we can pertube the metrics outside of a small neighborhood of ∂M to produce metrics $\widetilde{g_I}$ and $\widetilde{h_I}$, so that $\widetilde{h_I}$ is parallel near $Cr(f_I)$, and that we have $\widetilde{g_I} \equiv \hat{g}_I$ outside of a neighborhood of ∂M and near $Cr(\hat{f}_I)$. By Lemma 5.4.18, the pertubation of the Hermitian form h_I can be performed in such way that still, we have

$$\theta(\widetilde{h_I}) \equiv 0, \tag{6.5.17}$$

$$\widetilde{h_I}(p) = h_I(p), \qquad p \in Cr(\hat{f}_I). \tag{6.5.18}$$

For the resulting admissible system $\mathcal{D}_I := (E_I \downarrow \partial M \times I, \widetilde{g_I}, \widetilde{h_I}, \nabla_{\hat{g}_I} \hat{f}_I)$, we obtain from Proposition 6.2.15 that

$$\mathcal{R}(\mathcal{D}_I) = \mathcal{R}(\underline{\mathcal{D}' \times \mathcal{D}_0}) = -\frac{\log 2}{2} \chi(\partial M) \dim(E). \tag{6.5.19}$$

Observe now that by construction, $\mathcal{D}_I$ and the disjoint union $\mathcal{D} \sqcup \mathcal{D} := (E \downarrow M \sqcup E \downarrow M, g \sqcup g, h \sqcup h, \nabla_{g'} f \sqcup \nabla_{g'} f)$ of $\mathcal{D}$ with itself are two admissible systems satisfying the hypotheses of Theorem 6.5.2. This allows us to finally conclude as follows:

$$\begin{aligned}
&2\mathcal{R}(\mathcal{D}) = \mathcal{R}(\mathcal{D} \sqcup \mathcal{D}) \\
&\overset{6.5.8}{=} \mathcal{R}(\mathcal{D}_I) - \int_M \theta(h) \wedge (\nabla_g f)^* \Psi(TM, g) + \frac{1}{2} \int_{\partial M \times I} \theta(\widetilde{h_I}) \wedge (\nabla_{\hat{g}_I} \hat{f}_I)^* \Psi(T(\partial M \times I), \widetilde{g_I}) \\
&\overset{6.5.17}{=} \mathcal{R}(\mathcal{D}_I) - \int_M \theta(h) \wedge (\nabla_g f)^* \Psi(TM, g) \\
&\overset{6.5.19}{=} -\frac{\log 2}{2} \chi(\partial M) \dim(E) - \int_M \theta(h) \wedge (\nabla_g f)^* \Psi(TM, g).
\end{aligned} \tag{6.5.20}$$

This finishes the proof of Theorem 6.1.5. $\square$

6.6 Proof of Corollary C and Theorem E

Together with the work done in the previous chapters, we are finally ready to prove Corollary C and Theorem E. To refresh our memory, we start by re-introducing the broader mathematical realm, from which these two results emerged (cf. Section 1.3):

A **linear algebraic group G** over $\mathbb{Q}$ is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ for some $n \in \mathbb{N}$ that is the zero locus of a set of polynomials in the n^2 variables with coefficients in $\mathbb{Q}$. We set G to be the identity component of $\mathbf{G}(\mathbb{R}) = \mathbf{G} \cap \mathrm{GL}(n, \mathbb{R})$. Then G is a real Lie group, which we assume to be semi-simple without compact factors. For $K \subseteq G$ a maximal compact subgroup, the quotient space $X := G/K$ then has the natural structure of a non-positively curved globally symmetric space: As a smooth manifold, X is diffeomorphic to $\mathbb{R}^d$ for appropriate $d \in \mathbb{N}$ and there exists a canonical Riemannian metric g on X of non-positive sectional curvature, unique up to a positive scalar, turning the transitive action of G on X into an action by *isometries*. The **fundamental rank** $\delta(G)$ of G is the non-negative integer $\delta(G) := \mathrm{rank}_{\mathbb{C}}(G) - \mathrm{rank}_{\mathbb{C}}(K) \in \mathbb{N}_0$.

Let $\rho \colon G \to \mathrm{GL}(V)$ be a complex, finite-dimensional irreducible representation. Such ρ gives rise to the G-equivariant bundle $E^\rho := X \times V \downarrow X$, on the total space of which G acts diagonally via $\gamma.(x, v) := (\gamma.x, \rho(\gamma)v)$. Due to [64, Lemma 3.1], E^ρ can be equipped with a canonical G-equivariant Hermitian metric h_ρ, unique up to a positive scalar. For each degree $0 \leq p \leq d$, the pair of metrics (g, h_ρ) induce on the associated E_ρ-valued de Rham complex $\Omega^*(X, E_\rho)$ Hodge-Laplacians $\Delta_p : \Omega^p(X, E_\rho) \to \Omega^p(X, E_\rho)$. For $t > 0$, let $e^{-t\Delta_p}(x, y) : X \times X \to \mathrm{End}(V)$ be the associated heat kernel. Due to G-equivariance of the pair (g, h_ρ), one has $e^{-t\Delta_p}(x, y) = e^{-t\Delta_p}(\gamma.x, \gamma.y)$ for each $\gamma \in G$. It follows that there exists a smooth, non-negative, monotonically decreasing function $H^p(\rho, t)$ in $t > 0$ which satisfies $\mathrm{tr}(e^{-t\Delta_p}(x, x)) \equiv H^p(\rho, t)$. Therefore, we can define for each $0 \leq p \leq d$ the non-negative real number

$$b^p(\rho) := \lim_{t \to \infty} H^p(\rho, t). \tag{6.6.1}$$

It vanishes precisely when there are no harmonic, L^2-integrable p-forms in $\Omega^p(X, E_\rho)$. We say that ρ is L^2*-acyclic* if and only if $b^p(\rho) = 0$ for each $0 \leq p \leq d$. From the collection of functions $H^p(\rho, t)$ a number

$\tau_{(2)}(\rho) \in \mathbb{R}_{>0}$ is constructed as follows: By [9, Lemma 3.8, Proposition 5.2],

1. we have $H^p(\rho, t) \in \mathcal{O}(t^{-\dim(X)/2})$ for $t \to 0$. It follows that for $s \in \mathbb{C}$ with $\Re(s) >> 0$, there exists $\epsilon > 0$ such the expression $\zeta^p_\rho(s) := \Gamma(s)^{-1} \int_0^\epsilon t^{s-1}(H^p(\rho, t) - b^p(\rho))dt$ determines a holomorphic function, that extends to a meromorphic function on all of $\mathbb{C}$ which is regular at 0.

2. We have $\int_\epsilon^\infty t^{-1}(H^p(\rho, t) - b^p(\rho))dt < \infty$ for each $0 \le p \le d$.

3. The real number

$$\tau_{(2)}(\rho) := \frac{1}{2}\sum_{p=0}^{d}(-1)^{p+1}p\left(\frac{d}{ds}\zeta_\rho(s)|_{s=0} + \int_\epsilon^\infty t^{-1}(H^p(\rho, t) - b^p(\rho))dt\right)$$

   is well-defined by assertions 1 and 2. Moreover, there exists a *positive* number $c(\rho) > 0$, satisfying

$$\tau_{(2)}(\rho) = \begin{cases} 0 & \text{if } \delta(G) \neq 1, \\ (-1)^{\frac{d-1}{2}} c(\rho) & \text{if } \delta(G) = 1. \end{cases} \tag{6.6.2}$$

The number $\tau_{(2)}(\rho)$ does not depend on the normalization constant of the Hermitian form h_ρ, and changes by the factor C^{-d} when scaling the Riemannian metric g by the factor $C > 0$. In particular, given a lattice $\Gamma < G$, it follows that the positive number

$$T^{An}_{(2)}(\Gamma, \rho) := \exp(\mathrm{Vol}(\Gamma) \cdot \tau_{(2)}(\rho)) \in \mathbb{R}_{>0}\,.$$

does not depend on the normalization constants of g and h_ρ. Here, $\mathrm{Vol}(\Gamma)$ denotes the Riemannian volume of a fundamental domain for the Γ-action on X.

In the case that Γ is torsion-free, the bundle $E^\rho \downarrow X$ descends to a flat bundle $\Gamma\backslash E^\rho \downarrow \Gamma\backslash X$ over the locally symmetric quotient space $\Gamma\backslash X$, which is precisely the flat bundle associated to the restricted representation $\rho|_\Gamma$. To avoid cumbersome notation, we denote the respective quotient metrics on $\Gamma\backslash E^\rho \downarrow \Gamma\backslash X$ also by g and h_ρ on $\Gamma\backslash E^\rho \downarrow \Gamma\backslash X$. It is now easily verified from the definitions that the flat bundle $(\Gamma\backslash E^\rho \downarrow \Gamma\backslash X)$ is analytically L^2-acyclic if and only if $b^p(\rho) = 0$ for each $0 \le p \le d$, and that we have

$$T^{An}_{(2)}(\Gamma\backslash X, \rho|_\Gamma, g, h_\rho) = T^{An}_{(2)}(\Gamma, \rho). \tag{6.6.3}$$

Since by assumption, G is a connected semi-simple Lie group with finite center and no compact factors and $\Gamma < G$ is a torsion-free lattice, the quotient manifold $\Gamma\backslash X$, although not necessarily compact, is always a CW-model for the classifying space $B\Gamma$. That is because $X \cong \mathbb{R}^d$ is contractible. Notably, however, it is not finite CW-model whenever Γ is not uniform. Regardless, it is known, cf. [4, Theorem 13.1], that $\Gamma\backslash X$ is always the interior of a compact manifold with boundary, which we denote by $\overline{\Gamma\backslash X}$. As such, a given CW-structure on $\overline{\Gamma\backslash X}$ always serves as a *finite* CW-model for $B\Gamma$.

Identifying Γ with the fundamental group of $\overline{\Gamma\backslash X}$ under the homotopy equivalent inclusion $\Gamma\backslash X \hookrightarrow \overline{\Gamma\backslash X}$, choosing a finite CW-structure on $\overline{\Gamma\backslash X}$ and some basis on the representation space V, we can form the L^2-cochain complex $C^*_{(2)}(Y, \rho)$, defined over a preferred universal cover Y of $\overline{\Gamma\backslash X}$ (equipped with the induced Γ-CW structure). It is a finite cochain complex of Hilbert $\mathcal{N}(\Gamma)$-modules. The pair (Γ, ρ) is said to be *det-L^2-acyclic* if the combinatorial complex $C^*_{(2)}(Y, \rho)$ is det-L^2-acyclic as in Definition 5.2.3. Since G is semi-simple, ρ is unimodular (see for example [73, Lemma 4.3]). Thus, if (Γ, ρ) is det-L^2-acyclic, we can define by Theorem 5.3.12 the topological L^2-torsion

$$T^{Top}_{(2)}(\Gamma, \rho) = T^{Top}_{(2)}(\overline{\Gamma\backslash X}, \rho) \in \mathbb{R}_{>0}\,. \tag{6.6.4}$$

Since $\mathrm{Wh}(\Gamma) = 0$ (see [36, Proposition 0.10]), $T^{Top}_{(2)}(\Gamma, \rho)$ is in fact a *homotopy invariant* of the space $\Gamma\backslash X$: We may have chosen *any* finite CW-model of $B\Gamma$ to define the same number $T^{Top}_{(2)}(\Gamma, \rho)$ in the above fashion. This follows from the arguments succeeding Definition 5.3.14.

Corollary C. *In the above situation, suppose that ρ is L^2-acyclic and that $\Gamma < G$ is a* ***uniform*** *lattice. Then the pair (Γ, ρ) is det-L^2-acyclic. Moreover, we have an equality of L^2-torsion elements*

$$T^{Top}_{(2)}(\Gamma, \rho) = T^{An}_{(2)}(\Gamma, \rho). \tag{6.6.5}$$

Proof. Since $\Gamma\backslash X$ is compact, we have $\overline{\Gamma\backslash X} = \Gamma\backslash X$. Therefore, we may choose $Y := X$ as the universal cover of $\overline{\Gamma\backslash X}$ and obtain by Theorem 6.3.5 a chain homotopy equivalence between the combinatorial L^2-cochain complex $C^*_{(2)}(X, \rho)$ and the twisted L^2-de Rham complex $\Omega^*_{(2)}(X, E^\rho)$ (as complexes of Hilbert $\mathcal{N}(\Gamma)$-modules). Since det-L^2-acyclicity is a chain-homotopy invariant of Hilbert cochain complexes by Corollaries 4.1.32 and 4.1.38, and det-L^2-acyclicity of Hilbert $\mathcal{N}(\Gamma)$-cochain complex $\Omega^*_{(2)}(X, E^\rho, h_\rho)$ is satisfied if and only if ρ is L^2-acyclic, the first assertion follows. For the second assertion, first note that h_ρ is unimodular. This can be deduced from the fact that h_ρ is G-equivariant and the underlying representation ρ is unimodular. The proof is completely analogous to the one carried out in Section 5.5 for the special case $G = SO_0(d, 1)$, and will be therefore be omitted. Finally, taking Equation 6.6.3 into account, we may apply Theorem 6.1.8 to obtain the equality $T^{Top}_{(2)}(\Gamma, \rho) = T^{An}_{(2)}(\Gamma, \rho)$. □

Theorem E. *Let $G := SO_0(n, 1)$ with n odd, $\rho : G \to \mathrm{GL}(V)$ be an irreducible, finite-dimensional, complex representation and $\Gamma < G$ a torsion-free lattice. Then the following holds:*

1. *The pair (Γ, ρ) is det-L^2-acyclic.*
2. *One has*

$$T^{An}_{(2)}(\Gamma, \rho) = T^{Top}_{(2)}(\Gamma, \rho). \tag{6.6.6}$$

Proof. Let $M_R \subseteq \mathbb{H}^n$ $(R \in \mathbb{R}_{\geq 0})$ be the the exhaustion of complete, Γ-invariant hyperbolic submanifolds constructed as in Section 2.3. Firstly, Equation 6.6.3 and Theorem 2.3.13 imply that

$$T^{An}_{(2)}(\Gamma, \rho) = \lim_{R\to\infty} T^{An}_{(2)}(\Gamma\backslash M_R, \rho, g, h_\rho). \tag{6.6.7}$$

Secondly, we can apply Corollary 4.2.18 in order to see that the L^2-de Rham complex $\Omega^*_{(2)}(M_R, E^\rho, g, h_\rho)$ (with absolute boundary conditions) is det-L^2-acyclic. Due to Theorem 6.3.5, the same must therefore be true for the cellular cochain complex associated to a Γ-CW structure of M_R. Since $\Gamma\backslash M_R$ is a finite CW-model for $\Gamma\backslash \mathbb{H}^n$, we now obtain by definition that also (Γ, ρ) is det-L^2-acyclic. Moreover, this allows us to apply Theorem 5.5.2 in order to obtain an equality of topological L^2-torsions

$$T^{Top}_{(2)}(\Gamma, \rho) = T^{Top}_{(2)}(\Gamma\backslash M_R, \rho) \tag{6.6.8}$$

for *all* $R > 0$. Thirdly, we have both that $\mathrm{Wh}(\Gamma) = 0$ by [36, Proposition 0.10], and that the pair $(\partial M_R, \rho)$ is of determinant class by Corollary 4.2.13. Therefore, if we denote by $\mathbb{1} : G_{\mathbb{C}} \to \mathbb{C}$ the trivial representation, we may apply Theorem 6.1.8 to obtain that

$$\log\left(\frac{T^{An}_{(2)}(\Gamma\backslash M_R, \rho, g, h_\rho)}{T^{Top}_{(2)}(\Gamma\backslash M_R, \rho)}\right) = \dim(\rho)\cdot\log\left(\frac{T^{An}_{(2)}(\Gamma\backslash M_R, \mathbb{1}, g, h_{\mathbb{1}})}{T^{Top}_{(2)}(\Gamma\backslash M_R, \mathbb{1})}\right). \tag{6.6.9}$$

Finally, the main result of [55] implies that

$$\lim_{R\to\infty} \log\left(\frac{T^{An}_{(2)}(\Gamma\backslash M_R, \mathbb{1}, g, h_{\mathbb{1}})}{T^{Top}_{(2)}(\Gamma\backslash M_R, \mathbb{1})}\right) = 0. \tag{6.6.10}$$

The equality $T^{An}_{(2)}(\Gamma, \rho) = T^{Top}_{(2)}(\Gamma, \rho)$ is now a direct consequence of Equations 6.6.7–6.6.10. $\square$

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