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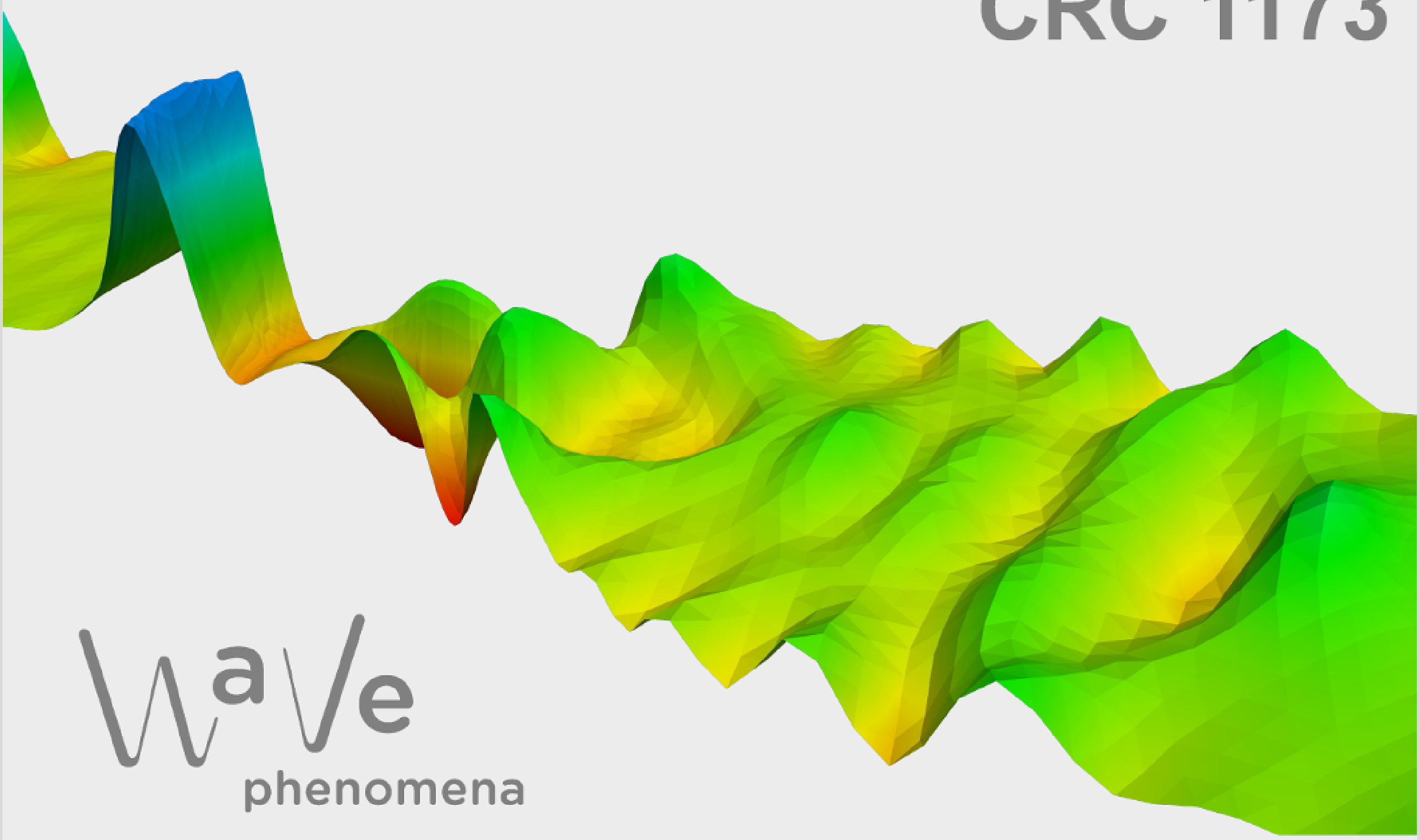

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A LIMITING ABSORPTION PRINCIPLE FOR HELMHOLTZ SYSTEMS AND TIME-HARMONIC ISOTROPIC MAXWELL'S EQUATIONS

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ABSTRACT. In this work we investigate the $L^p - L^q$-mapping properties of the resolvent associated with the time-harmonic isotropic Maxwell operator. As spectral parameters close to the spectrum are also covered by our analysis, we obtain an $L^p - L^q$-type Limiting Absorption Principle for this operator. Our analysis relies on new results for Helmholtz systems with zero order non-Hermitian perturbations. Moreover, we provide an improved version of the Limiting Absorption Principle for Hermitian (self-adjoint) Helmholtz systems.

1. Introduction

The propagation of electromagnetic waves in continuous three-dimensional media is governed by the *Maxwell's equations*. In absence of free charges their macroscopic formulation reads as follows

$$\partial_t \mathcal{D} - \nabla \times \mathcal{H} = -\mathcal{J}, \qquad \partial_t \mathcal{B} + \nabla \times \mathcal{E} = 0, \qquad \nabla \cdot \mathcal{D} = \nabla \cdot \mathcal{B} = 0, \tag{1}$$

with $\mathcal{D}, \mathcal{H}, \mathcal{B}, \mathcal{E}, \mathcal{J} \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^3$. Constitutive relations that specify the connections between the electric displacement $\mathcal{D}$ and the electric field $\mathcal{E}$ and between the magnetic flux density $\mathcal{B}$ and the magnetic field $\mathcal{H}$ are necessary for meaningful applications of this model. In general, these relations need not be simple, but in the physically realistic scenario where ferro-electric and ferro-magnetic materials are discarded and where the fields are weak enough, the material laws may be assumed to obey the following *linear* relations:

$$\mathcal{D} = \varepsilon(x)\mathcal{E}, \qquad \mathcal{B} = \mu(x)\mathcal{H}. \tag{2}$$

Here ε and μ embody the *permittivity* respectively the *permeability* of the medium. In general anisotropic materials, where the interaction of fields and matter not only depends on the position in the material but also on the direction of the fields, these quantities are mathematically represented as tensors. In this paper we will be exclusively concerned with the case of *isotropic* (*i.e.* direction-independent) media where ε and μ are *scalar*-valued functions on $\mathbb{R}^3$. For a more detailed description of Maxwell's equations we refer the reader to [17, 23].

We will focus on *monochromatic* waves only, *i.e.*, electromagnetic fields $\mathcal{D}$ and $\mathcal{B}$ that are periodic functions of time with the same frequency $\omega \in \mathbb{R} \setminus \{0\}$, more specifically $\mathcal{E}(x,t) := e^{i\omega t}E(x)$, $\mathcal{D}(x,t) := e^{i\omega t}D(x)$, $\mathcal{B}(x,t) := e^{i\omega t}B(x)$, $\mathcal{H}(x,t) := e^{i\omega t}H(x)$, $\mathcal{J}(x,t) := e^{i\omega t}J(x)$ for vector fields $E, D, B, H, J \colon \mathbb{R}^3 \to \mathbb{C}^3$. This gives rise to the following time-harmonic analogue of Maxwell's equations (1) once the linear constitutive relations from (2) are imposed:

$$i\omega\varepsilon E - \nabla \times H = -J, \qquad i\omega\mu H + \nabla \times E = 0. \tag{3}$$

In this paper we are interested in the following more general model

$$i\zeta\varepsilon E - \nabla \times H = -J_{\mathrm{e}}, \qquad i\zeta\mu H + \nabla \times E = J_{\mathrm{m}}, \tag{4}$$

where $\zeta \in \mathbb{C}$ and where both electric and magnetic current densities J_{e} and J_{m} are included. Allowing for spectral parameters $\zeta \in \mathbb{C} \setminus \mathbb{R}$ reflects the so-called Ohm's law for conducting media, which asserts that the current J induced by the electric field E can be described (in linear approximation) by $J = \sigma E + J_{\mathrm{e}}$, where

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$\sigma\colon \mathbb{R}^3 \to \mathbb{R}$ represents the *conductivity* and J_{e} is the external current density. Thus, plugging in Ohm's law into (3) one gets that the first equation in (3) can be rewritten as

$$i(\omega\varepsilon - i\sigma)E - \nabla \times H = -J_{\mathrm{e}},$$

which motivates the interest in the model (4).

The main purpose of this paper is to prove an L^p-type Limiting Absorption Principle for the time-harmonic Maxwell's equations (4). Roughly speaking, proving a Limiting Absorption Principle means proving existence and continuity of the resolvent operator up to the essential spectrum. In the context of the Maxwell system (4) this translates into studying the boundedness of solutions (E_ζ, H_ζ) of (4) with $\mathrm{Im}(\zeta) \neq 0$ and characterizing their limits as $\mathrm{Im}(\zeta) \to 0^\pm$. In this paper we shall prove the following result.

Theorem 1. *Let $\omega \in \mathbb{R} \setminus \{0\}$ and assume that $1 \leq p, \tilde{p}, q \leq \infty$ satisfy*

$$\frac{2}{3} < \frac{1}{p} \leq 1, \qquad \frac{1}{6} < \frac{1}{q} < \frac{1}{3}, \qquad \frac{1}{2} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{3}, \qquad 0 \leq \frac{1}{\tilde{p}} - \frac{1}{q} \leq \frac{1}{3}. \tag{5}$$

Moreover assume that there are $\varepsilon_\infty, \mu_\infty > 0$ such that

(A1) $\varepsilon, \mu \in W^{1,\infty}(\mathbb{R}^3)$ are uniformly positive,

(A2) $|\nabla(\varepsilon\mu)| + |\varepsilon\mu - \varepsilon_\infty\mu_\infty| + |\nabla\varepsilon|^2 + |\nabla\mu|^2 + |D^2\varepsilon| + |D^2\mu| \in L^{\frac{3}{2}}(\mathbb{R}^3) + L^2(\mathbb{R}^3)$,

(A3) $|\nabla(\varepsilon\mu)| + |\varepsilon\mu - \varepsilon_\infty\mu_\infty| \in L^{\frac{q_1}{q_1-2}}(\mathbb{R}^3) + L^{\frac{q_2}{q_2-2}}(\mathbb{R}^3)$ where $q \in [q_1, q_2]$ and $(p, \tilde{p}, q_1), (p, \tilde{p}, q_2)$ satisfy (5).

Then for all divergence-free vector fields $J_{\mathrm{e}}, J_{\mathrm{m}} \in L^p(\mathbb{R}^3; \mathbb{C}^3) \cap L^{\tilde{p}}(\mathbb{R}^3; \mathbb{C}^3)$, there are weak solutions $(E^\pm_\omega, H^\pm_\omega) \in L^q(\mathbb{R}^3; \mathbb{C}^6) \cap H^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{C}^6)$ of the time-harmonic Maxwell system (4) with $\zeta = \omega$ satisfying

$$\|(E^\pm_\omega, H^\pm_\omega)\|_q \leq C(\omega)\big(\|(J_{\mathrm{e}}, J_{\mathrm{m}})\|_p + \|(J_{\mathrm{e}}, J_{\mathrm{m}})\|_{\tilde{p}}\big)$$

where $\omega \mapsto C(\omega)$ is continuous on $\mathbb{R} \setminus \{0\}$. Moreover the following holds:

(i) We have $(E_\zeta, H_\zeta) \to (E^\pm_\omega, H^\pm_\omega)$ in $L^q(\mathbb{R}^3; \mathbb{C}^6) \cap H^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{C}^6)$ as $\zeta \to \omega \pm i0$ where $(E_\zeta, H_\zeta) \in H^1(\mathbb{R}^3; \mathbb{C}^6)$ is the unique weak solution solution of (4) with divergence-free vector fields $J^\zeta_{\mathrm{e}}, J^\zeta_{\mathrm{m}} \in L^p(\mathbb{R}^3; \mathbb{C}^3) \cap L^{\tilde{p}}(\mathbb{R}^3; \mathbb{C}^3) \cap L^2(\mathbb{R}^3; \mathbb{C}^6)$ converging to $J_{\mathrm{e}}, J_{\mathrm{m}}$ in $L^p(\mathbb{R}^3; \mathbb{C}^3) \cap L^{\tilde{p}}(\mathbb{R}^3; \mathbb{C}^3)$, respectively.

(ii) The function $u^\pm_\omega := (\varepsilon^{\frac{1}{2}} E^\pm_\omega, \mu^{\frac{1}{2}} H^\pm_\omega) \in L^q(\mathbb{R}^3; \mathbb{C}^6)$ solves the Helmholtz system

$$(\Delta + \omega^2 \varepsilon_\infty\mu_\infty)u^\pm_\omega + \mathcal{V}(\omega)u^\pm_\omega = \mathcal{L}_1(\omega)\tilde{J} + \mathcal{L}_2\tilde{J}$$

where $\mathcal{V}(\omega), \mathcal{L}_1(\omega), \mathcal{L}_2, \tilde{J}$ are defined at the beginning of Section 3. More precisely, $u^\pm_\omega$ satisfies the integral equation (45).

(iii) If additionally $J_{\mathrm{e}}, J_{\mathrm{m}} \in L^q(\mathbb{R}^3; \mathbb{C}^3)$, then $(E^\pm_\omega, H^\pm_\omega) \in W^{1,q}(\mathbb{R}^3; \mathbb{C}^6)$.

Remark 2.

(a) We shall not provide explicit values for the constants $C(\omega)$, but content ourselves with proving estimates that are uniform on compact subsets of $\mathbb{R} \setminus \{0\}$. This is indeed sufficient for the existence of a map $\omega \mapsto C(\omega)$ that is continuous on $\mathbb{R} \setminus \{0\}$.

(b) The convergence in $H^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{C}^6)$ is stated for simplicity. By standard elliptic regularity theory, convergence holds in $W^{2,r}_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{C}^6)$ where $r \geq 1$ depends on the local regularity of $\varepsilon, \mu, J_{\mathrm{e}}, J_{\mathrm{m}}$.

(c) If the currents $J^\zeta_{\mathrm{e}}, J^\zeta_{\mathrm{m}}$ also converge in $L^q(\mathbb{R}^3; \mathbb{C}^3)$ then one finds $(E_\zeta, H_\zeta) \to (E^\pm_\omega, H^\pm_\omega)$ in $W^{1,q}(\mathbb{R}^3; \mathbb{C}^6)$.

In the context of time-harmonic Maxwell's equations the Limiting Absorption Principle only few results are available. Picard, Weck and Witsch proved a Limiting Absorption Principle in weighted L^2-spaces (similar to [1]) for time-harmonic Maxwell's equations in an exterior domain with boundary conditions $\nu \wedge E = 0$, see [32, Theorem 2.10]. Since this result is based on Fredholm's Alternative, the frequencies $\omega \in \mathbb{R} \setminus \{0\}$ are assumed not to belong to a discrete (possibly empty) set of eigenvalues. As in Agmon's fundamental paper [1] about the perturbed Helmholtz equation, the permittivity ε and permeability μ are assumed to be isotropic and to decay to some positive constants at infinity faster than $|x|^{-1}$. Despite some quantitative differences, this is similar to our assumptions (A1),(A2),(A3). A similar result in the anisotropic case was obtained by Pauly [31, Theorem 3.5]. We note that these results also apply to discontinuous ε, μ, which indicates that

(A1) may be relaxed. In a recent work [28, Theorem 2.1] by Nguyen and Sil, still in a weighted L^2-framework, the Limiting Absorption Principle is studied in the case of anisotropic sign-changing coefficients that are used to describe metamaterials. As far as we can see, our contribution is the first dealing with L^p-estimates for time-harmonic Maxwell's equations. All the aforementioned results relate to the three-dimensional Maxwell operator. A Limiting Absorption Principle in periodic 1D waveguides can be found in the recent preprint by De Nittis, Moscolari, Richard and Tiedra de Aldecoa [29]. Further relevant tools for Limiting Absorption Principles such as Carleman inequalities or Unique Continuation results can be found in [8, 30].

In view of the Limiting Absorption Principle from Theorem 1, it is expected that embedded eigenvalues of the Maxwell operator do not exist under the assumptions of Theorem 1. In the Fredholm theoretical approaches from [1, 32] this is even a necessary condition for the Limiting Absorption Principle to hold. Using Carleman estimates by Koch and Tataru [22], we obtain the following result.

Theorem 3. *Assume (A1),(A2) for some $\varepsilon_\infty, \mu_\infty > 0$, $\zeta \in \mathbb{C}$ and let $(E,H) \in H^1_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{C}^6)$ be a weak solution of the homogeneous ($J_{\mathrm{e}} = J_{\mathrm{m}} = 0$) time-harmonic Maxwell system* (4) *that satisfies* $(1+|x|)^{\tau_1-\frac12}(|E|+|H|) \in L^2(\mathbb{R}^3)$ *for some $\tau_1 > 0$. Then $E \equiv H \equiv 0$.*

In the proofs of Theorem 1 and 3 we will use the nontrivial fact that each solution (E,H) of the Maxwell system (4) gives rise to a solution $(\tilde{E},\tilde{H}) := (\varepsilon^{\frac12}E, \mu^{\frac12}H)$ of a linear Helmholtz system with complex-valued zeroth order perturbations that are non-Hermitian in general. The tools that we will need in the analysis of this particular system are inspired from the theory for Helmholtz systems with Hermitian perturbations that we will develop first. For the sake of simplicity we restrict our attention to the case $n \geq 3$.

Theorem 4. *Let $n,m \in \mathbb{N}, n \geq 3$, $\zeta \in \mathbb{C}\setminus\mathbb{R}$. Assume $V = \overline{V}^T \in L^{\frac{n}{2}}(\mathbb{R}^n;\mathbb{C}^{m\times m}) + L^{\frac{n+1}{2}}(\mathbb{R}^n;\mathbb{C}^{m\times m})$ and that $1 \leq p,q \leq \infty$ satisfy*

$$\frac{n+1}{2n} < \frac{1}{p} \leq 1, \qquad \frac{(n-1)^2}{2n(n+1)} < \frac{1}{q} < \frac{n-1}{2n}, \qquad \frac{2}{n+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}.$$

Then $R(\zeta) := (\Delta I_m + V(x) + \zeta I_m)^{-1} \colon L^p(\mathbb{R}^n;\mathbb{C}^m) \to L^q(\mathbb{R}^n;\mathbb{C}^m)$ exists as a bounded linear operator and extends by pointwise convergence to the positive half-axis via

$$R(\lambda \pm i0)f := \lim_{\zeta\to\lambda\pm i0} R(\zeta)f, \qquad \text{in } L^q(\mathbb{R}^n;\mathbb{C}^m), \tag{6}$$

for any $f \in L^p(\mathbb{R}^n;\mathbb{C}^m)$ and $\lambda > 0$.

Remark 5.

(a) We will actually prove a slightly stronger result than Theorem 4. We will show that all conclusions mentioned in this theorem are true assuming $V = \overline{V}^T \in L^{\frac{n}{2}}(\mathbb{R}^n;\mathbb{C}^{m\times m}) + L^{\tilde{\kappa}}(\mathbb{R}^n;\mathbb{C}^{m\times m})$ with $\frac{n}{2} \leq \tilde{\kappa} \leq \frac{n+1}{2}$ where the condition $\frac{(n-1)^2}{2n(n+1)} < \frac1q < \frac{n-1}{2n}$ is replaced by the weaker one $\frac{n+1}{2n} - \frac{1}{\tilde{\kappa}} < \frac1q < \frac{n-1}{2n}$. In other words, Theorem 4 corresponds to the special case $\tilde{\kappa} = \frac{n+1}{2}$ of this stronger result.

(b) We stress that we do not require any sign / smallness condition on V. This demands the assumption $\zeta \in \mathbb{C}\setminus\mathbb{R}$ which ensures the existence of $R(\zeta)$.

(c) The limit in (6) is a pointwise limit and it is natural to ask whether this convergence also holds in the uniform operator topology. Ideas related to this question can be found in [14, p.46].

(d) The two-dimensional case $n = 2$ can in principle be discussed using the same techniques. We expect that the same statements hold for $V = \overline{V}^T \in L^{\kappa}(\mathbb{R}^2;\mathbb{C}^{m\times m}) + L^{\frac32}(\mathbb{R}^2;\mathbb{C}^{m\times m})$ for some $\kappa > 1$.

We stress that, even in the scalar case $m = 1$, our Theorem 4 improves earlier results in this direction. Goldberg and Schlag [10] were the first to go beyond the Hilbert space framework in which, since Agmon's work [1], the Limiting Absorption Principles for self-adjoint Schrödinger operators were studied. They proved an L^p-type Limiting Absorption Principle that inspired our Theorem 4. For $n = 3$ they showed

$$\sup_{0<\delta<1,\,\lambda\geq\lambda_0} \|R(\lambda + i\delta)\|_{\frac43\to 4} \leq C(\lambda_0, V)\lambda^{-\frac14}, \qquad \lambda_0 > 0, \tag{7}$$

provided that $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, $p > \frac{3}{2}$. In [10, Proposition 1.3] it is even stated that the assumption on V can be considerably weakened to $V \in L^{\frac{3}{2}+\varepsilon}(\mathbb{R}^3) + L^{2-\varepsilon}(\mathbb{R}^3), \varepsilon > 0$, as soon as embedded eigenvalues for Schrödinger operators with such pertubations do not exist. The latter was meanwhile proved by Koch and Tataru [22, Theorem 3]. Huang, Yao and Zheng [14] generalized the result from [10] to the higher-dimensional case. Indeed, they proved that the estimate

$$\sup_{0<\delta<1,\lambda\geq\lambda_0} \|R(\lambda+i\delta)\|_{\frac{2(n+1)}{n+3}\to\frac{2(n+1)}{n-1}} \leq C(\lambda_0,V)\lambda^{-\frac{1}{n+1}}, \qquad \lambda_0 > 0,$$

holds for all potentials $V \in L^{\frac{n}{2}}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $p > \frac{n}{2}$ and all $n \in \mathbb{N}, n \geq 3$. The most recent result in this direction is due to Ionescu and Schlag [16] where a new Limiting Absorption Principle was proved for a much larger class of potentials than the ones covered by the aforementioned results. As our Theorem 4, this result covers all $V \in L^{\frac{n}{2}}(\mathbb{R}^n) + L^{\frac{n+1}{2}}(\mathbb{R}^n)$ as one can check from (1.19) in [16]. In their Theorem 1.3 (d), the resolvent estimate

$$\sup_{\lambda\in I,\,0<\delta\leq 1} \|R(\lambda\pm i\delta)\|_{X\to X^*} \leq C(I,V)$$

is proved where $I \subset \mathbb{R}\setminus\{0\}$ is a compact set that does not intersect the set of nonzero eigenvalues. For the precise definition of the Banach space X we refer to [16, p.400]. We emphasize that all the above estimates are self-dual in the sense that they bound the operator norms of the resolvents acting between some Banach space and the corresponding dual space. In this respect, our result from Theorem 4 is more general than [16]. Concerning Limiting Absorption Principles for Helmholtz equations ($m = 1$) in other settings and under different assumptions we would like to mention the papers [6, 7] (Morrey-Campanato spaces) and [34] for dissipative Helmholtz operators, [27] (sign-changing coefficients), [4, 25, 33] (periodic potentials) and [5, 26] (critical potentials).

Let us now briefly comment on the proof of Theorem 4. As anticipated (see again Proposition 1.3 in [10]), in the proof of a Limiting Absorption Principle, excluding embedded eigenvalues usually represents the discriminating step where the hypotheses on the potential come into play. In this regards, Ionescu-Jerison in [15, Theorem 2.5] showed that for all $\varepsilon > 0$ there are $V \in L^{\frac{n+1}{2}+\varepsilon}(\mathbb{R}^n)$ such that the scalar Schrödinger operator $\Delta + V$ has embedded eigenvalues with rapidly decaying eigenfunctions. Thus, the exponent $\frac{n+1}{2}$ in our assumption is optimal. (We stress that as far as asymptotic decay conditions are investigated, a higher exponent in the Lebesgue space allows to cover a wider class of perturbations.) On the other hand, the optimality of the exponent $\frac{n}{2}$ is not entirely clear, even though it is known that standard properties of Schrödinger operators like semi-boundedness need not hold for potentials with lower integrability. In [19] [21, Theorem 1.a)] it is shown that 0 can be an embedded eigenvalue when potentials in the class $L^\kappa_{\mathrm{loc}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $\kappa < \frac{n}{2}$ are considered, see also [18, Remark 6.5]. Up to the authors' knowledge, a counterexample for non-zero eigenvalues is not known.

As customary, a basic tool for ruling out embedded eigenvalues is a suitable Carleman estimate. In our case, due to the weak and almost optimal conditions $V = \overline{V}^T \in L^{\frac{n}{2}}(\mathbb{R}^n;\mathbb{C}^{m\times m}) + L^{\frac{n+1}{2}}(\mathbb{R}^n;\mathbb{C}^{m\times m})$, we need to use the fine Carleman estimate for scalar Schrödinger operators provided by Koch and Tataru in [22] (see also (23) below), which allows to cover this wide class of potentials. We stress that the possibility to use a scalar Carleman estimate in our vector-valued setting only works because the chosen weight in the Carleman bound in [22, Proposition 4] *does not* depend on the solution itself. Indeed, this fact ultimately permits to sum up the estimates obtained for the components and to get anestimate for the full vector field. Analogue results for Helmholtz systems with first order perturbations cannot be obtained in this way since the weights in the corresponding Carleman estimates from [22] (see Theorem 8 and Theorem 11) depend on the solution itself. Hence, it is not guaranteed that one Carleman weight works for all components, which is why systems with first order perturbations appear to be more difficult.

The rest of the paper is organized as follows: Section 2 is devoted to the proof of the Limiting Absorption Principle for Helmholtz systems with Hermitian coefficients stated in Theorem 4. The aforementioned relation between Maxwell's equations (4) and Helmholtz systems will be discussed in Section 3. Here we also provide the proof of Theorem 3 about the absence of eigenvalues for the Maxwell system (4). Finally, in Section 4, we prove the most involved result of the paper, namely the Limiting Absorption Principle for Maxwell's equations (4) from Theorem 1.

We conclude this introduction with the main notations used in this paper.

Notations.
* For $Z \in \{\mathbb{R}, \mathbb{C}, \mathbb{R}^m, \mathbb{C}^m, \mathbb{R}^{m\times m}, \mathbb{C}^{m\times m}\}$ we shall shortly write $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R}^n;Z)}$.
* $\mathcal{B}(X,Y)$ denotes the Banach spaces of bounded linear operators between Banach spaces X, Y equipped with the standard operator norm.
* We write $V \in L^{[p_1,p_2]}(\mathbb{R}^n; Z) := L^{p_1}(\mathbb{R}^n; Z) + L^{p_2}(\mathbb{R}^n; Z)$ if V can be decomposed as $V = V_1 + V_2$, with $V_1 \in L^{p_1}(\mathbb{R}^n; Z)$ and $V_2 \in L^{p_2}(\mathbb{R}^n; Z)$.
* χ_B represents the indicator of a measurable subset $B \subset \mathbb{R}^n$.
* $\zeta \to \lambda \pm i0$ means $\zeta \to \lambda$ with $\pm\,\mathrm{Im}(\zeta) \searrow 0$.
* I_m denotes the identity matrix in $\mathbb{R}^{m\times m}$, $m \in \mathbb{N}$.
* The notation I is used for the identity operator in some function space
* We use the notation $\lesssim$ where we want to indicate that we have an inequality $\leq$ up to a constant factor which does not depend on the relevant parameters.
* $H^1(\mathrm{curl}; \mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3; \mathbb{C}^3) \colon\ \mathrm{curl}\, u \in L^2(\mathbb{R}^3; \mathbb{C}^3)\}$.
* We adopt the following definition for the Fourier transform

$$\widehat{f}(\xi) := \mathcal{F}(f)(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\,\mathrm{d}x.$$

2. The LAP for Helmholtz systems – Proof of Theorem 4

This section is concerned with the proof of Theorem 4 that relies on a well-known perturbative argument based on Fredholm operator theory. This strategy has its origin in the pioneering work by Agmon [1] where it was used to establish a Limiting Absorption Principle for Schrödinger operators acting between weighted L^2-spaces. Since then, this technique has permeated many works in the subject. We summarize Agmon's approach as follows. Consider a reference operator H_0 and let H be a suitable perturbation of H_0, let $\zeta \in \mathbb{C}$. The first step is to prove the existence of a right inverse $R_0(\zeta)$ for the operator $H_0 + \zeta$ satisfying an estimate of the form

$$\|R_0(\zeta) f\|_{X_1} \leq C(\zeta)\|f\|_{X_2}, \tag{8}$$

where X_1, X_2 are Banach spaces. In Agmon's paper, for spectral parameters $\zeta := \lambda > 0$ and the Laplacian $H_0 = \Delta$ such right inverses are constructed via the classical Limiting Absorption Principle for Helmholtz equations, namely by investigating the mapping properties of the resolvents $R_0(\zeta)$ as $\zeta \to \lambda \pm i0, \lambda > 0$, see Theorem 4.1 [1]. This is a nontrivial task given that every such λ belongs to the essential spectrum of the (negative) Laplacian and therefore no such limits can exist when $X_1 = X_2 = L^2(\mathbb{R}^n)$. In [1, 2] this was circumvented by introducing suitable and, as a matter of fact, optimal weighted L^2-spaces such that the operators $R_0(\zeta)$ converge in $\mathcal{L}(X_2, X_1)$ as $\zeta \to \lambda \pm i0$ (with different limits). In order to extend the estimate (8) to the perturbed operator H one assumes that $V := H - H_0$ is a relatively compact perturbation of H_0, meaning that the linear operator $K(\zeta) := -R_0(\zeta)V$ is compact on X_1. In view of the formula

$$H + \zeta = (H_0 + \zeta)(I - K(\zeta))$$

a right inverse $R(\zeta)$ for the operator $H + \zeta$ is given by

$$R(\zeta) := (I - K(\zeta))^{-1} R_0(\zeta)$$

as soon as $I - K(\zeta) \colon X_1 \to X_1$ is bijective. By Fredholm theory, it suffices to verify injectivity, which is the most delicate part of the argument. Once this is achieved, one obtains the desired estimate

$$\|R(\zeta) f\|_{X_1} \leq C(\zeta)\|(I - K(\zeta))^{-1}\|_{X_1 \to X_1}\|f\|_{X_2}. \tag{9}$$

We stress that a good control of the right hand side with respect to ζ will be of central interest in the following.

In our context the reference operator H_0 and its perturbation H are the free and the perturbed matrix-valued Schrödinger operators, namely

$$H_0 := \Delta, \qquad H := \Delta + V(x),$$

where $V = \overline{V}^T \in L^{[\frac{n}{2},\frac{n+1}{2}]}(\mathbb{R}^n;\mathbb{C}^{m\times m})$ and $m \in \mathbb{N}$. Here, the Laplacian Δ acts as a diagonal operator on each of the m components and $\zeta \in \mathbb{C}\setminus\mathbb{R}_{\geq 0}$ or $\zeta = \lambda \pm i0, \lambda > 0$ as we explain below. According to the general strategy described above, to get an analogue of estimate (9) under our assumptions, we need to accomplish the following three steps:

Step 1: Provide $L^p - L^q$ estimates for $R_0(\zeta)$.
Step 2: Show that the linear operator $K(\zeta) = -R_0(\zeta)V\colon L^q(\mathbb{R}^n;\mathbb{C}^m) \to L^q(\mathbb{R}^n;\mathbb{C}^m)$ is compact.
Step 3: Prove the injectivity of the Fredholm operator $I - K(\zeta)\colon L^q(\mathbb{R}^n;\mathbb{C}^m) \to L^q(\mathbb{R}^n;\mathbb{C}^m)$.

We will see that Step 1 is essentially available in the literature. Only minor modifications will be needed to pass from the scalar to the vector-valued framework. To accomplish Step 2, which is rather standard, we will use the local compactness of Sobolev embeddings. So the main difficulty is to achieve Step 3. It will be accomplished with the aid of Carleman estimates by Koch and Tataru [22] and by exploiting the fact that V is Hermitian.
Our results from Theorem 4 even provide the uniform bounds in $\mathbb{C}\setminus\mathbb{R}_{\geq 0}$

$$C(\zeta) := \|R_0(\zeta)\|_{p\to q} \lesssim |\zeta|^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q}-\frac{2}{n})} \qquad \text{and} \qquad \|(I-K(\zeta))^{-1}\|_{q\to q} \lesssim 1$$

as well as continuity properties of $\zeta \mapsto K(\zeta)$ and $\zeta \mapsto (I-K(\zeta))^{-1}$ needed for the proof of (6). The following subsections are devoted to the proof of the aforementioned facts.

2.1. **$L^p - L^q$ estimates for $R_0(\zeta)$.** In the *scalar* case, optimal $L^p - L^q$ resolvent estimates for $n \geq 3$ are originally due to Kenig, Ruiz and Sogge [20, Theorem 2.3] in the selfdual case $q = p'$ and to Gutiérrez in [13, Theorem 6] in the general case. For the precise asymptotics with respect to ζ, which results from rescaling, we refer to [24, p.1419].

Theorem 6 (Kenig-Ruiz-Sogge, Gutiérrez)**.** *Let $m = 1, n \in \mathbb{N}, n \geq 3$ and assume $\zeta \in \mathbb{C}\setminus\mathbb{R}_{\geq 0}$. Then, for $1 \leq p, q \leq \infty$ such that*

$$\frac{n+1}{2n} < \frac{1}{p} \leq 1, \qquad 0 \leq \frac{1}{q} < \frac{n-1}{2n}, \qquad \frac{2}{n+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}, \tag{10}$$

$R_0(\zeta)$ is a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n;\mathbb{C})$ satisfying

$$\|R_0(\zeta)f\|_q \lesssim |\zeta|^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q}-\frac{2}{n})}\|f\|_p. \tag{11}$$

Moreover, there are bounded linear operators $R_0(\lambda\pm i0) : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n;\mathbb{C})$ such that $R_0(\zeta)f \to R_0(\lambda\pm i0)f$ as $\zeta \to \lambda \pm i0$ for all $f \in L^p(\mathbb{R}^n)$ and

$$\|R_0(\lambda \pm i0)f\|_q \lesssim \lambda^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q}-\frac{2}{n})}\|f\|_p \qquad (\lambda > 0). \tag{12}$$

Proof. The estimate (11) is available in the literature mentioned above. The existence of a bounded linear operator $R_0(\lambda \pm i0)$ with $R_0(\zeta)f \rightharpoonup R_0(\lambda \pm i0)f$ as $\zeta \to \lambda \pm i0$ follows from the uniform boundedness of the functions $R_0(\zeta)f$ in $L^q(\mathbb{R}^n)$ for ζ near λ (see (11)) and the continuity of Cauchy type integrals as in [1, Theorem 4.1]. We indicate how to prove that this convergence in fact holds in the strong sense. By density of test functions and (11) it suffices to prove $R_0(\zeta)f - R_0(\tilde{\zeta})f \to 0$ for test functions $f \in C_c^\infty(\mathbb{R}^n)$ as $\zeta, \tilde{\zeta} \to \zeta_0 \in \mathbb{C}\setminus\{0\}, \operatorname{Im}(\zeta)\operatorname{Im}(\tilde{\zeta}) > 0$. For simplicity we only consider the case $\operatorname{Im}(\zeta), \operatorname{Im}(\tilde{\zeta}) > 0$ Here we can use $R_0(\zeta)f - R_0(\tilde{\zeta})f = (G_\zeta - G_{\tilde{\zeta}}) * f$ where, according to [14, p.46], we have for $\zeta = \mu^2 \neq 0, \operatorname{Re}(\mu), \operatorname{Im}(\mu) > 0$ and $\tilde{\zeta} = \tilde{\mu}^2$ sufficiently close to ζ with $\operatorname{Re}(\tilde{\mu}), \operatorname{Im}(\tilde{\mu}) > 0$,

$$|G_\zeta(z) - G_{\tilde{\zeta}}(z)| \lesssim \begin{cases} |\mu - \tilde{\mu}||z|^{3-n} & ,\text{if } |z| \leq |\mu|^{-1} \\ |\mu - \tilde{\mu}||\mu|^{\frac{n-3}{2}}|z|^{\frac{3-n}{2}} & ,\text{if } |\mu|^{-1} \leq |z| \leq |\mu - \tilde{\mu}|^{-1} \\ |\mu|^{\frac{n-3}{2}}|z|^{\frac{1-n}{2}} & ,\text{if } |z| \geq |\mu - \tilde{\mu}|^{-1}. \end{cases}$$

So Young's convolution inequality implies in view of $q > \frac{2n}{n-1}$

$$\begin{aligned}\|R_0(\zeta)f - R_0(\tilde{\zeta})f\|_q &\lesssim |\mu-\tilde{\mu}|\, \||z|^{3-n}\chi_{|z|\leq|\mu|^{-1}}\|_1\|f\|_q \\ &+ |\mu-\tilde{\mu}|\, \||z|^{\frac{3-n}{2}}\chi_{|\mu|^{-1}\leq|z|\leq|\mu-\tilde{\mu}|^{-1}}\|_q\|f\|_1 \\ &+ \||z|^{\frac{1-n}{2}}\chi_{|z|\geq|\mu-\tilde{\mu}|^{-1}}\|_q\|f\|_1 \\ &\lesssim |\mu-\tilde{\mu}| + |\mu-\tilde{\mu}|\cdot|\mu-\tilde{\mu}|^{\frac{n-3}{2}-\frac{n}{q}} + |\mu-\tilde{\mu}|^{\frac{n-1}{2}-\frac{n}{q}} \\ &= |\mu-\tilde{\mu}| + |\mu-\tilde{\mu}|^{\frac{n-1}{2}-\frac{n}{q}}.\end{aligned}$$

Hence, $(R_0(\zeta)f)$ is a Cauchy sequence in L^q and thus converges. Since the limit must coincide with the weak limit, we get the conclusion. □

The conditions (10) on (p,q) are optimal for the uniform estimates (12), *cf.* [24, p.1419]. For any fixed $\zeta \in \mathbb{C}\setminus\mathbb{R}_{\geq 0}$, however, the estimate (11) actually holds for a larger range of exponents, which is due to the improved properties of the Fourier symbol $1/(|\xi|^2-\zeta)$ and related Bessel potential estimates. We refer to [24] for more details about sharp $L^p - L^q$ resolvent estimates of the form (11).
Theorem 6 extends in an obvious way to the system case that we shall need in the following.

Corollary 7 (Step 1). *Let $m,n\in\mathbb{N}, n\geq 3$ and assume $\zeta\in\mathbb{C}\setminus\mathbb{R}_{\geq 0}$. Then, for $1\leq p,q\leq\infty$ as in (10), $R_0(\zeta)$ is a bounded linear operator from $L^p(\mathbb{R}^n;\mathbb{C}^m)$ to $L^q(\mathbb{R}^n;\mathbb{C}^m)$ satisfying*

$$\|R_0(\zeta)f\|_q \lesssim |\zeta|^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q}-\frac{2}{n})}\|f\|_p.$$

Moreover, there are bounded linear operators $R_0(\lambda\pm i0): L^p(\mathbb{R}^n;\mathbb{C}^m)\to L^q(\mathbb{R}^n;\mathbb{C}^m)$ such that $R_0(\zeta)f\to R_0(\lambda\pm i0)f$ as $\zeta\to\lambda\pm i0$ for all $f\in L^p(\mathbb{R}^n;\mathbb{C}^m)$. Furthermore, (12) holds.

2.2. **Compactness of $K(\zeta)$.** We first proceed in greater generality by proving the boundedness and compactness of $K(\zeta)$ as an operator from $L^{q_1}(\mathbb{R}^n;\mathbb{C}^m)$ to $L^{q_2}(\mathbb{R}^n;\mathbb{C}^m)$ for suitable q_1,q_2, as we will use this more general result later. The proof of Step 2 then follows from the particular choice $q_1=q_2=q$, see Corollary 9 below. In order to simplify the notation in the proofs, we will write $L^s := L^s(\mathbb{R}^n;\mathbb{C}^m)$, $L^s_{\mathrm{loc}} := L^s_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{C}^m)$, etc.

Proposition 8. *Let $n,m\in\mathbb{N}, n\geq 3$ and suppose that $V\in L^{[\kappa,\tilde{\kappa}]}(\mathbb{R}^n;\mathbb{C}^{m\times m})$ where $1\leq\kappa\leq\tilde{\kappa}<\infty$. Then, for $\zeta\in\mathbb{C}\setminus\mathbb{R}_{\geq 0}$ or $\zeta=\lambda\pm i0, \lambda>0$, the operator $K(\zeta)=-R_0(\zeta)V\colon L^{q_1}(\mathbb{R}^n;\mathbb{C}^m)\to L^{q_2}(\mathbb{R}^n;\mathbb{C}^m)$ is compact provided that the following conditions hold for $q_1,q_2\in[1,\infty]$:*

$$\frac{n+1}{2n}-\frac{1}{\tilde{\kappa}}<\frac{1}{q_1}\leq 1-\frac{1}{\kappa},\qquad 0\leq\frac{1}{q_2}<\frac{n-1}{2n},\qquad \frac{2}{n+1}-\frac{1}{\tilde{\kappa}}\leq\frac{1}{q_1}-\frac{1}{q_2}\leq\frac{2}{n}-\frac{1}{\kappa}. \tag{13}$$

Moreover,

$$\|K(\zeta)\|_{q_1\to q_2} \lesssim \inf_{V=V_1+V_2}\left[|\zeta|^{\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2}+\frac{1}{\kappa}-\frac{2}{n})}\|V_1\|_\kappa + |\zeta|^{\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2}+\frac{1}{\tilde{\kappa}}-\frac{2}{n})}\|V_2\|_{\tilde{\kappa}}\right]. \tag{14}$$

Furthermore, $u_j\rightharpoonup u, \zeta_j\to\zeta$ implies $K(\zeta_j)u_j\to K(\zeta)u$. In particular, the operators $K(\zeta)$ depend continuously on $\zeta\in\mathbb{C}\setminus\mathbb{R}_{\geq 0}$ in the uniform operator topology and we have $K(\zeta)\to K(\lambda\pm i0)$ as $\zeta\to\lambda\pm i0$.

Proof. We begin with proving *boundedness* of $K(\zeta)\colon L^{q_1}\to L^{q_2}$. We use $V=V_1+V_2$, where $V_1\in L^\kappa(\mathbb{R}^n;\mathbb{C}^{m\times m}), V_2\in L^{\tilde{\kappa}}(\mathbb{R}^n;\mathbb{C}^{m\times m})$. This implies $K(\zeta)=-R_0(\zeta)V_1-R_0(\zeta)V_2$ so that Corollary 7 gives

$$\|K(\zeta)f\|_{q_2}\leq\|R_0(\zeta)\|_{p\to q_2}\|V_1f\|_p+\|R_0(\zeta)\|_{\tilde{p}\to q_2}\|V_2f\|_{\tilde{p}}$$

whenever the tuples (p,q_2) and $(\widetilde{p},q_2)$ satisfy the conditions in (10). In view of (13) and $\kappa\leq\tilde{\kappa}$ these conditions are satisfied if we choose $p,\tilde{p}$ according to $\frac{1}{p}=\frac{1}{\kappa}+\frac{1}{q_1}$, $\frac{1}{\tilde{p}}=\frac{1}{\tilde{\kappa}}+\frac{1}{q_1}$. So Hölder's inequality and Corollary 7 give

$$\|K(\zeta)f\|_{q_2}\leq\left(\|R_0(\zeta)\|_{p\to q_2}\|V_1\|_\kappa+\|R_0(\zeta)\|_{\tilde{p}\to q_2}\|V_2\|_{\tilde{\kappa}}\right)\|f\|_{q_1}$$

$$\begin{aligned}&\lesssim \left(|\zeta|^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q_2}-\frac{2}{n})}\|V_1\|_\kappa + |\zeta|^{\frac{n}{2}(\frac{1}{\tilde{p}}-\frac{1}{q_2}-\frac{2}{n})}\|V_2\|_{\tilde\kappa}\right)\|f\|_{q_1}\\ &= \left(|\zeta|^{\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2}+\frac{1}{\kappa}-\frac{2}{n})}\|V_1\|_\kappa + |\zeta|^{\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2}+\frac{1}{\tilde\kappa}-\frac{2}{n})}\|V_2\|_{\tilde\kappa}\right)\|f\|_{q_1},\end{aligned}$$

which proves the claimed boundedness as well as (14).

Next we show that $u_j \rightharpoonup u, \zeta_j \to \zeta$ implies $K(\zeta_j)u_j \to K(\zeta)u$. Here, $\zeta \in \mathbb{C}\setminus\mathbb{R}_{\geq 0}$ or $\zeta = \lambda \pm i0, \lambda > 0$. Notice that this fact and Corollary 7 imply the compactness of $K(\zeta)$ (choose $\zeta_j := \zeta$) as well as the existence of a continuous extension of $\zeta \mapsto K(\zeta)$ in $\mathcal{B}(L^{q_1}; L^{q_2})$ to the closed upper resp. lower complex half-planes. We argue similarly as in [10, Lemma 3.1]. As a starting point we reduce our analysis to the case of bounded and compactly supported V. The potentials $V_l := V\chi_{\{|V|\leq l,\,|x|\leq l\}}$ are bounded, compactly supported and satisfy $\|V_l - V\|_{[\kappa,\tilde\kappa]} \to 0$ as $l \to \infty$ because of $1 \leq \kappa, \tilde\kappa < \infty$. So the corresponding operators $K_l(\zeta_j) := -R_0(\zeta_j)V_l$ satisfy

$$\sup_{j\in\mathbb{N}}\|K_l(\zeta_j) - K(\zeta_j)\|_{\mathcal{B}(L^{q_1};L^{q_2})} = \sup_{j\in\mathbb{N}}\|R_0(\zeta_j)[V - V_l]\|_{\mathcal{B}(L^{q_1};L^{q_2})} \to 0 \qquad \text{as } l \to \infty$$

because of (14). Hence, it is sufficient to prove the claim for bounded V with compact support.

We first prove $K(\zeta_j)u_j \to K(\zeta)u$ in $L^{q_2}_{\mathrm{loc}}$. To this end, it suffices to prove the uniform boundedness of the operators $K(\zeta_j)\colon L^{q_1} \to W^{2,q_1}(B;\mathbb{C}^m)$ with respect to j for any given bounded ball $B \subset \mathbb{R}^n$. Indeed, the embedding $W^{2,q_1}(B;\mathbb{C}^m) \hookrightarrow L^{q_2}(B;\mathbb{C}^m)$ is compact due to $\frac{1}{q_1} - \frac{1}{q_2} \leq \frac{2}{n} - \frac{1}{\kappa} < \frac{2}{n}$ and the Rellich-Kondrachov Theorem, which implies $K(\zeta_j)u_j \to v$ in $L^{q_2}(B;\mathbb{C}^m)$ for some v. This and $u_j \rightharpoonup u$ implies $v = K(\zeta)u$ because of

$$\int_B v\phi\,\mathrm{d}x = \lim_{j\to\infty}\int_B K(\zeta_j)u_j\phi\,\mathrm{d}x = \lim_{j\to\infty}\int_B u_jK(\zeta_j)\phi\,\mathrm{d}x = \int_B uK(\zeta)\phi\,\mathrm{d}x = \int_B K(\zeta)u\phi\,\mathrm{d}x$$

for all $\phi \in C^\infty(B;\mathbb{C}^m)$. Here we used $K(\zeta_j)\phi \to K(\zeta)\phi$ in $L^{q_1'}(B;\mathbb{C}^m)$, which in turn follows as in the proof of Corollary 7. So we conclude $K(\zeta_j)u_j \to K(\zeta)u$ in $L^{q_2}_{\mathrm{loc}}$ once we have proved the uniform boundedness of $K(\zeta_j)\colon L^{q_1} \to W^{2,q_1}(B;\mathbb{C}^m)$.

To prove this let $f \in L^{q_1}$ be arbitrary. Then $w_j := K(\zeta_j)f$ satisfies the elliptic system

$$\Delta w_j = (\Delta + \zeta_j)w_j - \zeta_j w_j = -Vf - \zeta_j w_j \quad \text{in } 2B.$$

From elliptic interior regularity estimates and the mapping properties of $K(\zeta_j)$ stated in Corollary 7 and Hölder's inequality we obtain

$$\|w_j\|_{W^{2,q_1}(B;\mathbb{C}^m)} \lesssim \|-Vf + \zeta_j w_j\|_{L^{q_1}(2B;\mathbb{C}^m)} \leq \left(\|V\|_\infty\|f\|_{q_1} + |\zeta_j||2B|^{\frac{1}{q_1}-\frac{1}{q_2}}\|w_j\|_{q_2}\right) \lesssim \|f\|_{q_1},$$

which is what we had to prove. (Notice that this estimate is uniform with respect to j whereas uniformity with respect to $|B|$ is not needed.)

To conclude it is sufficient to show $\sup_j \|\chi_{\mathbb{R}^n\setminus B}K(\zeta_j)\|_{q_1\to q_2} \to 0$ as $B \nearrow \mathbb{R}^n$. To this end we use

$$K(\zeta_j)f(x) = -\int_{\mathbb{R}^n} G_{\zeta_j}(x-y)V(y)f(y)\,\mathrm{d}y,$$

where $G_{\zeta_j}(z)$ is the integral kernel of the resolvent operator $R_0(\zeta_j)$, which is explicitly given in terms of Bessel functions. We use the bound $\sup_j |G_{\zeta_j}(z)| \lesssim |z|^{\frac{1-n}{2}}$ for $|z| \geq 1$, see (2.21),(2.25) in [20]. Recalling that V is assumed to be bounded and compactly supported we infer for $M := \operatorname{supp} V$

$$|K(\zeta_j)f(x)| \lesssim \int_M |x-y|^{\frac{1-n}{2}}|V(y)||f(y)|\,\mathrm{d}y \lesssim |x|^{\frac{1-n}{2}}\|V\|_{q_1'}\|f\|_{q_1} \quad \text{if } \operatorname{dist}(x,M) \geq 1.$$

This yields for large enough balls B

$$\|\chi_{\mathbb{R}^n\setminus B}K(\zeta_j)f\|_{q_2} \lesssim \|V\|_{q_1'}\|f\|_{q_1}\Big(\int_{\mathbb{R}^n\setminus B}|x|^{\frac{q_2(1-n)}{2}}\,\mathrm{d}x\Big)^{\frac{1}{q_2}}$$

and the conclusion follows due to $q_2 > \frac{2n}{n-1}$. □

The second step now results from considering the special case $q_1 = q_2 = q$ in Proposition 8.

Corollary 9 (Step 2)**.** *Let $n, m \in \mathbb{N}$, $n \geq 3$ and assume that $V \in L^{[\kappa,\tilde{\kappa}]}(\mathbb{R}^n; \mathbb{C}^{m\times m})$ where $\frac{n}{2} \leq \kappa \leq \tilde{\kappa} \leq \frac{n+1}{2}$. Then, for $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ or $\zeta = \lambda \pm i0, \lambda > 0$, the operator $K(\zeta) = -R_0(\zeta)V : L^q(\mathbb{R}^n; \mathbb{C}^m) \to L^q(\mathbb{R}^n; \mathbb{C}^m)$ is compact provided that*

$$\frac{n+1}{2n} - \frac{1}{\tilde{\kappa}} < \frac{1}{q} < \frac{n-1}{2n}. \tag{15}$$

Furthermore, $u_j \rightharpoonup u, \zeta_j \to \zeta$ implies $K(\zeta_j)u_j \to K(\zeta)u$. In particular, the operators $K(\zeta)$ depend continuously on $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ in the uniform operator topology and we have $K(\zeta) \to K(\lambda \pm i0)$ as $\zeta \to \lambda \pm i0$.

2.3. **Injectivity of $I - K(\zeta)$.** We now prove the injectivity of the Fredholm operator $I - K(\zeta) : L^q(\mathbb{R}^n; \mathbb{C}^m) \to L^q(\mathbb{R}^n; \mathbb{C}^m)$ for q as in (15). So we have to show that

$$u - K(\zeta)u = 0, \qquad u \in L^q(\mathbb{R}^n; \mathbb{C}^m) \tag{16}$$

implies $u = 0$. As a starting point, using a bootstrapping procedure, we show that solutions of (16) display both more local integrability and better decay at infinity.

Proposition 10. *Let $n, m \in \mathbb{N}$, $n \geq 3$, q as in (15) and assume $V \in L^{[\kappa,\tilde{\kappa}]}(\mathbb{R}^n; \mathbb{C}^{m\times m})$ where $\frac{n}{2} \leq \kappa \leq \tilde{\kappa} \leq \frac{n+1}{2}$. Then any solution $u \in L^q(\mathbb{R}^n; \mathbb{C}^m)$ of (16) belongs to*

$$L^r(\mathbb{R}^n; \mathbb{C}^m) \cap H^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{C}^m) \text{ for all } r \in \Big(\frac{2n}{n-1}, \frac{2n}{n-3}\Big) \quad \text{when } \kappa = \frac{n}{2},$$
$$L^r(\mathbb{R}^n; \mathbb{C}^m) \cap H^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{C}^m) \text{ for all } r \in \Big(\frac{2n}{n-1}, \infty\Big] \quad \text{when } \kappa > \frac{n}{2}.$$

Moreover, for any given such r, q we have $\|u\|_r \lesssim \|u\|_q$.

Proof. We write again $L^s := L^s(\mathbb{R}^n; \mathbb{C}^m)$. As a starting point we show that any given solution $u \in L^q$ of (16) belongs to L^r for all $r \in (\frac{2n}{n-1}, q)$, in other words u displays a *better decay at infinity.* We give a proof of this fact distinguishing between $\tilde{\kappa} < \frac{n+1}{2}$ and the limiting case $\tilde{\kappa} = \frac{n+1}{2}$. Let us first consider $\tilde{\kappa} < \frac{n+1}{2}$. Define $\frac{1}{q_0} < \frac{1}{q_1} < \cdots < \frac{1}{q_j} < \frac{1}{q_{j+1}} < \cdots < \frac{n-1}{2n}$ by

$$\frac{1}{q_0} := \frac{1}{q}, \qquad \frac{1}{q_{j+1}} := \min\left\{\frac{1}{2}\Big(\frac{1}{q_j} + \frac{n-1}{2n}\Big), \frac{1}{q_j} - \frac{2}{n+1} + \frac{1}{\tilde{\kappa}}\right\} \qquad (j \in \mathbb{N}_0). \tag{17}$$

Since we are assuming $\tilde{\kappa} < \frac{n+1}{2}$, at each iteration we indeed get a smaller Lebesgue exponent, namely $\frac{n-1}{2n} > \frac{1}{q_{j+1}} > \frac{1}{q_j}$. We claim that the tuple (q_j, q_{j+1}) satisfies the conditions (13) in Proposition 8, thus $K(\zeta)$ maps L^{q_j} to $L^{q_{j+1}}$. Indeed, the chain of inequalities

$$\frac{n+1}{2n} - \frac{1}{\tilde{\kappa}} \overset{(15)}{<} \frac{1}{q} = \frac{1}{q_0} \leq \frac{1}{q_j} < \frac{n-1}{2n} \leq 1 - \frac{2}{n} \leq 1 - \frac{1}{\kappa}$$

implies that the first two inequalities in (13) hold. The second two inequalities in (13) result from $0 \leq \frac{1}{q_j} < \frac{1}{q_{j+1}} \leq \frac{1}{2}\big(\frac{1}{q_j} + \frac{n-1}{2n}\big) < \frac{n-1}{2n}$, while the third condition in (13) holds due to $\frac{1}{q_j} < \frac{1}{q_{j+1}} \leq \frac{1}{q_j} - \frac{2}{n+1} + \frac{1}{\tilde{\kappa}}$ and $0 \leq \frac{2}{n} - \frac{1}{\kappa}$. So Proposition 8 may be applied iteratively to the equation (16) and we obtain $u \in L^{q_j}$ for all $j \in \mathbb{N}_0$. From $\frac{1}{q_j} \nearrow \frac{n-1}{2n}$ we infer $u \in L^r$ for all $\frac{2n}{n-1} < r < q$ by interpolation as well as $\|u\|_r \lesssim \|u\|_q$.

Now we consider the limiting case $\tilde{\kappa} = \frac{n+1}{2}$. Observe that, according to the previous definition (17) of q_{j+1}, in this limiting situation there would be no decay gain at each iteration because of $q_{j+1} = q_j$. We can circumvent this by choosing a decomposition $V = V_1 + V_2$, where $V_1 \in L^{\frac{n}{2}}(\mathbb{R}^n; \mathbb{C}^{m\times m})$ and $V_2 \in L^{\frac{n+1}{2}}(\mathbb{R}^n; \mathbb{C}^{m\times m})$ has a small $L^{\frac{n+1}{2}}$−norm. Indeed, if $V = \tilde{V}_1 + \tilde{V}_2$ with $\tilde{V}_1 \in L^{\kappa}(\mathbb{R}^n; \mathbb{C}^{m\times m})$ and $\tilde{V}_2 \in L^{\frac{n+1}{2}}(\mathbb{R}^n; \mathbb{C}^{m\times m})$, then choose

$$V_1 := \tilde{V}_1 \chi_{|\tilde{V}_1| > \varepsilon} + \tilde{V}_2 \chi_{\varepsilon < |\tilde{V}_2| < \varepsilon^{-1}}, \qquad V_2 := \tilde{V}_1 \chi_{|\tilde{V}_1| \leq \varepsilon} + \tilde{V}_2 \chi_{|\tilde{V}_2| \leq \varepsilon} + \tilde{V}_2 \chi_{|\tilde{V}_2| \geq \varepsilon^{-1}}$$

for sufficiently small $\varepsilon > 0$. We use this observation in order to justify a similar iteration as above, this time for exponents $\frac{1}{q_0} < \frac{1}{q_1} < \cdots < \frac{1}{q_j} < \frac{1}{q_{j+1}} < \cdots < \frac{n-1}{2n}$ given by

$$\frac{1}{q_0} := \frac{1}{q}, \qquad \frac{1}{q_{j+1}} := \min\left\{\frac{1}{2}\Big(\frac{1}{q_j} + \frac{n-1}{2n}\Big), \frac{1}{q_j} - \frac{2}{n+1} + \frac{2}{n}\right\} \qquad (j \in \mathbb{N}_0).$$

We have to show that $u = K(\zeta)u, u \in L^{q_j}$ implies $u \in L^{q_{j+1}}$. Having done this, we conclude from $\frac{1}{q_j} \nearrow \frac{n-1}{2n}$ and interpolation that $u \in L^r$ for all $\frac{2n}{n-1} < r < q$ as well as $\|u\|_r \lesssim \|u\|_q$, which then finishes the proof of our first claim.

So assume $u \in L^{q_j}$. A suitable choice for $\varepsilon > 0$ above, which depends on j, leads to $V = V_1 + V_2$, $V_1 \in L^{\frac{n}{2}}(\mathbb{R}^n; \mathbb{C}^{m\times m})$ with

$$C_j \|V_2\|_{\frac{n+1}{2}} < \frac{1}{2} \tag{18}$$

where C_j denotes the operator norm of $R_0(\zeta)\colon L^{s_j} \to L^{q_{j+1}}$ where s_j is defined via $\frac{1}{s_j} - \frac{1}{q_{j+1}} = \frac{2}{n+1}$. Observe that this operator norm is finite due to Corollary 7 because of $\frac{n+1}{2n} < \frac{1}{s_j} \leq 1$, which in turn is a consequence of

$$1 - \frac{2}{n+1} \geq \frac{n-1}{2n} > \frac{1}{q_{j+1}} > \frac{1}{q_j} > \ldots > \frac{1}{q} \overset{(15)}{>} \frac{n+1}{2n} - \frac{1}{\tilde{\kappa}} = \frac{n+1}{2n} - \frac{2}{n+1} \qquad (j \in \mathbb{N})$$

We introduce the auxiliary operator $T := I + R_0(\zeta)V_2 : L^{q_{j+1}} \to L^{q_{j+1}}$. Using (18) we find that T is bounded and invertible due to Corollary 15. Indeed,

$$\|R_0(\zeta)V_2 v\|_{q_{j+1}} \leq C_j \|V_2 v\|_{s_j} \leq C_j \|V_2\|_{\frac{n+1}{2}} \|v\|_{q_{j+1}} \leq \frac{1}{2}\|v\|_{q_{j+1}}.$$

So T has a bounded inverse $T^{-1} : L^{q_{j+1}} \to L^{q_{j+1}}$. Since u satisfies (16), we have

$$\int_{\mathbb{R}^n} u\phi \,\mathrm{d}x = \int_{\mathbb{R}^n} K(\zeta)u\phi \,\mathrm{d}x = -\int_{\mathbb{R}^n} R_0(\zeta)V_1 u\phi \,\mathrm{d}x - \int_{\mathbb{R}^n} R_0(\zeta)V_2 u\phi \,\mathrm{d}x$$

for any given $\phi \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^m)$. Since each of these integrals is finite, we find

$$\int_{\mathbb{R}^n} Tu\phi \,\mathrm{d}x = -\int_{\mathbb{R}^n} R_0(\zeta)V_1 u\phi \,\mathrm{d}x \tag{19}$$

and thus

$$\Big|\int_{\mathbb{R}^n} Tu\phi \,\mathrm{d}x\Big| \leq \Big|\int_{\mathbb{R}^n} R_0(\zeta)V_1 u\phi \,\mathrm{d}x\Big| \leq \|R_0(\zeta)V_1 u\|_{q_{j+1}} \|\phi\|_{q'_{j+1}} \lesssim \|u\|_{q_j} \|\phi\|_{q'_{j+1}} \tag{20}$$

by Proposition 8. So we may invoke the dual characterization of the Lebesgue norms $\|\cdot\|_{q_{j+1}}$ by taking the supremum over all $\phi \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^m)$, $\|\phi\|_{q'_{j+1}} = 1$ and using the density of the test functions in $L^{q'_{j+1}}$, to get $\|Tu\|_{q_{j+1}} \leq \|u\|_{q_j}$. Then the boundedness of T^{-1} gives $\|u\|_{q_{j+1}} \lesssim \|u\|_{q_j}$, which is what we had to prove.

Next we prove *higher integrability* of u. In the case $\kappa > \frac{n}{2}$ classical Moser iteration implies $u \in L^\infty$ and the claim follows by interpolation. So it remains to prove $u \in L^r$ for $r \in (q, \frac{2n}{n-3})$ in the limiting case $\kappa = \frac{n}{2}$. Again we use an iteration scheme and define $\frac{1}{q_0} > \frac{1}{q_1} > \cdots > \frac{1}{q_j} > \frac{1}{q_{j+1}} > \cdots > \frac{n-3}{2n}$ by

$$\frac{1}{q_0} := \frac{1}{q}, \qquad \frac{1}{q_{j+1}} := \min\left\{\frac{1}{2}\Big(\frac{1}{q_j} + \frac{n-3}{2n}\Big), \frac{1}{q_j} - \frac{2}{n} + \frac{2}{n+1}\right\} \qquad (j \in \mathbb{N}_0).$$

As above, it suffices to show that $u = K(\zeta)u, u \in L^{q_j}$ implies $u \in L^{q_{j+1}}$. Similar as above we find $V = V_1 + V_2$ with $V_1 \in L^{\frac{n+1}{2}}(\mathbb{R}^n; \mathbb{C}^{m\times m})$ and $C_j \|V_2\|_{\frac{n}{2}} < \frac{1}{2}$, where now C_j denotes the operator norm of $R_0(\zeta)\colon L^{s_j}(\mathbb{R}^n) \to L^{q_{j+1}}(\mathbb{R}^n)$ with $\frac{1}{s_j} - \frac{1}{q_{j+1}} = \frac{2}{n}$. Observe that Corollary 7 ensures the finiteness of this norm because of $\frac{n+1}{2n} < \frac{1}{s_j} \leq 1$, which is a consequence of $\frac{n-1}{2n} > \frac{1}{q} > \frac{1}{q_{j+1}} > \frac{n-3}{2n}$. (This is the reason for our assumption $r < \frac{2n}{n-3}$.) For instance, choose $V_2 := V\chi_{|V|>R_j}$ for $R_j > 0$ large enough and $V_1 := V - V_2$. As before we find that $T := I + R_0(\zeta)V_2 : L^{q_{j+1}} \to L^{q_{j+1}}$ is bounded, invertible and that (19), (20) hold. So we conclude

$Tu \in L^{q_{j+1}}$ and hence $u \in L^{q_{j+1}}$ for all $j \in \mathbb{N}_0$, whence $u \in L^r$ for all $r \in (q, \frac{2n}{n-3})$ and $\|u\|_r \lesssim \|u\|_q$ by interpolation.

Finally, we discuss the H^1_{loc}-regularity. In the case $\kappa = \frac{n}{2}$ solutions u of (16) satisfy $\Delta u = -(\zeta + V)u$ in $\mathbb{R}^n$ in the distributional sense. Since $\zeta + V \in L^{\frac{n}{2}}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{C}^{m\times m})$ and $u \in L^r_{\mathrm{loc}}$ for all $r < \frac{2n}{n-3}$, we conclude $\Delta u \in L^s_{\mathrm{loc}}$ for all $s \in [1, \frac{2n}{n+1})$. So Calderón-Zygmund estimates yield $u \in W^{2,s}_{\mathrm{loc}}$ for those exponents and thus $u \in H^1_{\mathrm{loc}}$ by Sobolev's Embedding Theorem, which finishes the proof for $\kappa = \frac{n}{2}$. In the case $\kappa > \frac{n}{2}$ one obtains similarly $u \in W^{2,\kappa}_{\mathrm{loc}} \subset H^1_{\mathrm{loc}}$ and the proof is finished. $\square$

With these improved integrability and regularity properties of solutions at hand, we may turn towards the injectivity of $I - K(\zeta)$. As in the paper by Goldberg and Schlag [10] we will discuss separately the case $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and the more involved situation $\zeta = \lambda \pm i0, \lambda > 0$. Observe that if u is a solution to (16), then it solves the corresponding eigenvalue equation

$$(\Delta + \zeta)u + Vu = 0, \qquad u \in L^q(\mathbb{R}^n; \mathbb{C}^m). \tag{21}$$

To prove injectivity of $I - K(\zeta)$ one is thus lead to prove the absence of embedded eigenvalues for the Schrödinger operator $\Delta + V$. However, for $\zeta = \lambda > 0$, the Helmholtz equation (21) may possess nontrivial solutions: consider for instance $V = 0$ and ordinary Herglotz waves u with pointwise decay rate $|u(x)| \lesssim (1 + |x|)^{(1-n)/2}$. As a consequence, some extra decay condition at infinity coming from the integral representation of u from (16) has to be used in order to deduce $u \equiv 0$. In [10] the authors managed to prove that in the case $n = 3, m = 1$ solutions $u \in L^4(\mathbb{R}^3; \mathbb{C})$ of (16) even satisfy $(1 + |\cdot|)^{\tau_1 - \frac{1}{2}} u \in L^2(\mathbb{R}^3; \mathbb{C})$ for some $\tau_1 > 0$ so that a fundamental result by Ionescu-Jerison [15] on the absence of embedded eigenvalues allows to conclude $u \equiv 0$. The ideas presented in [10] are not limited to $n = 3$, but carry over to general dimensions $n \geq 2$ in a straightforward manner. In order to avoid redundancy we only state the (scalar) results that generalize Lemma 2.3 and Proposition 2.4 from this paper.

Proposition 11 (Goldberg-Schlag). *Let $n \in \mathbb{N}, n \geq 2$ and assume $f \in L^p(\mathbb{R}^n)$ where $1 \leq p \leq \frac{2(n+1)}{n+3}$. Then we have for $|t| < \frac{1}{2}$ and $\gamma := \frac{1}{2}\min\{1, \frac{n+1}{p} - \frac{n+3}{2}\}$ the estimate*

$$\|\hat{f}((1+t)\cdot)\|_{L^2(S_\lambda)} \lesssim |t|^\gamma \|f\|_p$$

provided that $\hat{f}$ vanishes identically on the unit sphere in $\mathbb{R}^n$.

Proposition 12 (Goldberg-Schlag). *Let $n \in \mathbb{N}, n \geq 2$ and $f \in L^p(\mathbb{R}^n)$ for $\max\{1, \frac{2n}{n+4}\} \leq p \leq \frac{2(n+1)}{n+3}$, $(n,p) \neq (4,1)$. Then we have for all $\tau_1 < \frac{1}{2}\min\{1, \frac{n+1}{p} - \frac{n+3}{2}\}$ the estimate*

$$\|(1 + |\cdot|)^{\tau_1 - \frac{1}{2}} R_0(\lambda \pm i0) f\|_2 \lesssim \|f\|_p$$

provided that $\hat{f}$ vanishes identically on the unit sphere in $\mathbb{R}^n$.

Clearly, Proposition 12 generalizes to systems simply by considering each component separately. This is how we will deduce $(1+|\cdot|)^{\tau_1 - \frac{1}{2}} u \in L^2(\mathbb{R}^n; \mathbb{C}^m)$ for some $\tau_1 > 0$, which is crucial for the absence of embedded eigenvalues that we will prove in Theorem 14. Here we use the symbol τ_1 in order to keep with the notation introduced in [22, Theorem 3]. A simplified version of their result reads as follows.

Theorem 13 (Koch-Tataru). *Let $n \in \mathbb{N}, n \geq 3$ suppose that $V \in L^{[\frac{n}{2}, \frac{n+1}{2}]}(\mathbb{R}^n), \lambda > 0$. Let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfy $(\Delta + \lambda)u + Vu = 0$ in $\mathbb{R}^n$ and $|x|^{\tau_1 - \frac{1}{2}} u \in L^2(\mathbb{R}^n)$ for some $\tau_1 > 0$. Then $u \equiv 0$.*

This is indeed a special case of [22, Theorem 3] because potentials $V \in L^{[\frac{n}{2}, \frac{n+1}{2}]}(\mathbb{R}^n)$ satisfy assumption A2 from [22]. This is due to the embeddings

$$\begin{aligned} &L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^n) + L^{\frac{2n}{n+2}}(\mathbb{R}^n) \hookrightarrow W^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}(\mathbb{R}^n), \\ &L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n) \cap L^{\frac{2n}{n-2}}(\mathbb{R}^n) \hookleftarrow W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}(\mathbb{R}^n). \end{aligned} \tag{22}$$

Closely following the strategy outlined in [22, p.424] we generalize this result to systems of the form (21). The underlying idea of proving the absence of embedded eigenvalues, already on a scalar level, is to use suitable L^p-Carleman estimates in order to prove that the corresponding eigenfunction decays exponentially at infinity (Step 3.1). Using this information one then shows that it is actually compactly supported (Step 3.2). A standard unique continuation argument finally allows to conclude that the eigenfunction is necessarily trivial (Step 3.3). This strategy is carried out in the proof of the following result which represents the analogue for systems of Theorem 13 above.

Theorem 14. *Let $n,m\in\mathbb{N}, n\geq 3$, assume $V\in L^{[\frac{n}{2},\frac{n+1}{2}]}(\mathbb{R}^n;\mathbb{C}^{m\times m})$ and let $u\in H^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{C}^m)$ be a solution of (21) for $\zeta\in\mathbb{R}_{>0}$ satisfying $|x|^{\tau_1-\frac12}u\in L^2(\mathbb{R}^n;\mathbb{C}^m)$. Then $u\equiv 0$.*

Proof. The key point is the L^p-Carleman estimate proved in [22]. More precisely, introducing the Carleman weight $h_\epsilon(t)$ such that $h_\epsilon'(t)=\tau_1+(\tau e^{\frac{t}{2}}-\tau_1)\frac{\tau^2}{\tau^2+\epsilon e^t}$, $\epsilon>0$, Koch and Tataru proved in [22, Proposition 4] the estimate

$$\|e^{h_\epsilon(\ln|x|)}v\|_{\frac{2n}{n-2}}+\|e^{h_\epsilon(\ln|x|)}v\|_{\frac{2(n+1)}{n-1}}\lesssim\inf_{(\Delta+\zeta)v=f+g}\|e^{h_\epsilon(\ln|x|)}f\|_{\frac{2n}{n+2}}+\|e^{h_\epsilon(\ln|x|)}g\|_{\frac{2(n+1)}{n+3}} \tag{23}$$

for all v supported in $\mathbb{R}^n\setminus B_1$ such that $|x|^{\tau_1-\frac12}v\in L^2(\mathbb{R}^n)$. Notice that (23) is uniform with respect to $0<\epsilon\leq\epsilon_0$ and $\tau\geq\tau_0>0$ for some $\varepsilon_0,\tau_0>0$. (In [22] the estimate (23) is proved even for more general classes of Helmholtz-type operators also allowing for long-range perturbations. Moreover, the corresponding estimate (7) in that paper is formulated with even stronger norms on the left and weaker norms on the right, as one can check using the embeddings (22).)

Step 3.1: Exponential decay. As anticipated we first show that u decays at infinity faster than $e^{-\tau|x|^{1/2}}$ in some integrated sense. In order to apply (23) for this purpose, we need to localize the support of u to a spatial region far from the origin. So we pick a non-negative bump function $\phi\in C^\infty(\mathbb{R}^n)$ such that $\phi(x)=1$ for $|x|>2R$ and $\phi(x)=0$ for $|x|\leq R$ for a sufficiently large R to be chosen later and define $v:=\phi u$. Then v solves

$$(\Delta+\zeta)v=[(\Delta\phi)u+2\nabla\phi\cdot\nabla u+V_1v]+V_2v.$$

Now we apply the scalar Carleman estimate (23) to each component of the system. Summing up the resulting estimates one gets

$$\begin{aligned}
&\|e^{h_\epsilon(\ln|x|)}v\|_{\frac{2n}{n-2}}+\|e^{h_\epsilon(\ln|x|)}v\|_{\frac{2(n+1)}{n-1}}\\
&\lesssim\|e^{h_\epsilon(\ln|x|)}\left((\Delta\phi)u+2\nabla\phi\cdot\nabla u+V_1v\right)\|_{\frac{2n}{n+2}}+\|e^{h_\epsilon(\ln|x|)}V_2v\|_{\frac{2(n+1)}{n+3}}\\
&\lesssim_R\|e^{h_\epsilon(\ln|x|)}\left(|u|+|\nabla u|\right)\chi_{B_{2R}\setminus B_R}\|_{\frac{2n}{n+2}}\\
&+\|e^{h_\epsilon(\ln|x|)}V_1v\|_{\frac{2n}{n+2}}+\|e^{h_\epsilon(\ln|x|)}V_2v\|_{\frac{2(n+1)}{n+3}}\\
&\lesssim\|e^{h_\epsilon(\ln|x|)}\left(|u|+|\nabla u|\right)\chi_{B_{2R}\setminus B_R}\|_{\frac{2n}{n+2}}\\
&+\|V_1\chi_{\mathbb{R}^n\setminus B_R}\|_{\frac{n}{2}}\|e^{h_\epsilon(\ln|x|)}v\|_{\frac{2n}{n-2}}+\|V_2\chi_{\mathbb{R}^n\setminus B_R}\|_{\frac{n+1}{2}}\|e^{h_\epsilon(\ln|x|)}v\|_{\frac{2(n+1)}{n-1}}.
\end{aligned}$$

Here we used that ϕ is supported in $\mathbb{R}^n\setminus B_R$ and $|\nabla\phi|,\Delta\phi$ are supported in $\overline{B_{2R}}\setminus B_R$. For R sufficiently large we can absorb the potential-dependent terms on the right-hand side and get

$$\|e^{h_\epsilon(\ln|x|)}v\|_{\frac{2n}{n-2}}+\|e^{h_\epsilon(\ln|x|)}v\|_{\frac{2(n+1)}{n-1}}\lesssim_R\|e^{h_\epsilon(\ln|x|)}(|u|+|\nabla u|)\chi_{B_{2R}\setminus B_R}\|_{\frac{2n}{n+2}}.$$

Since $h_\epsilon(\ln|x|)\nearrow\tau|x|^{1/2}$ as $\epsilon\to0^+$, the Monotone Convergence Theorem implies

$$\|e^{\tau|x|^{1/2}}v\|_{\frac{2n}{n-2}}+\|e^{\tau|x|^{1/2}}v\|_{\frac{2(n+1)}{n-1}}\lesssim_R\|e^{\tau|x|^{1/2}}(|u|+|\nabla u|)\chi_{B_{2R}\setminus B_R}\|_{\frac{2n}{n+2}}. \tag{24}$$

Since the right side is finite by Proposition 10 (notice that the presence of the exponential factor is irrelevant as we are localized in a bounded region), v and hence u have exponential decay at infinity.

Step 3.2: Compact support. From (24) we even infer that u is supported in $\overline{B_{2R}}$. Indeed, $|x|^{1/2} \leq \sqrt{2R}$ on $B_{2R} \setminus B_R$ implies

$$e^{-\tau\sqrt{2R}}\Big(\|e^{\tau|x|^{1/2}}v\|_{\frac{2n}{n-2}} + \|e^{\tau|x|^{1/2}}v\|_{\frac{2(n+1)}{n-1}}\Big) \lesssim_R \|(|u|+|\nabla u|)\chi_{B_{2R}\setminus B_R}\|_{\frac{2n}{n+2}}.$$

Letting τ go to infinity shows that v and hence u vanishes identically outside $\overline{B_{2R}}$.

Step 3.3: Triviality. In virtue of the conclusions provided by Step 3.2, proving the triviality of u reduces to proving the weak unique continuation property for the differential inequality $|\Delta u| \leq |V||u|$. At this stage we only need that our potential $V \in L^{[\frac{n}{2},\frac{n+1}{2}]}(\mathbb{R}^n;\mathbb{C}^{m\times m})$ belongs to $L^{\frac{n}{2}}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{C}^{m\times m})$, as only local properties of the solution u are investigated. So [18, Theorem 6.3] applies and we obtain $u \equiv 0$. □

Now we are in the position to prove the injectivity of $I - K(\zeta)$.

Corollary 15 (Step 3)**.** *Let $n, m \in \mathbb{N}$, $n \geq 3$ and assume $V = \overline{V}^T \in L^{[\kappa,\tilde\kappa]}(\mathbb{R}^n;\mathbb{C}^{m\times m})$ where $\frac{n}{2} \leq \kappa \leq \tilde\kappa \leq \frac{n+1}{2}$. Then, for $\zeta \in \mathbb{C}\setminus\mathbb{R}_{\geq 0}$ or $\zeta = \lambda \pm i0, \lambda > 0$, the operator $I - K(\zeta)\colon L^q(\mathbb{R}^n;\mathbb{C}^m) \to L^q(\mathbb{R}^n;\mathbb{C}^m)$ is bijective and satisfies*

$$\|(I-K(\zeta))^{-1}\|_{q\to q} \leq C < \infty \qquad \text{for all } \zeta \in \mathbb{C}\setminus\mathbb{R}_{\geq 0}$$

provided that q satisfies (15)*. Moreover, $\zeta \mapsto (I-K(\zeta))^{-1}$ is continuous on $\mathbb{C}\setminus\mathbb{R}_{\geq 0}$ and we have $(I - K(\zeta))^{-1} \to (I - K(\lambda \pm i0))^{-1}$ in the uniform operator topology as $\zeta \to \lambda, \mathrm{Im}(\zeta) \to 0^{\pm}$.*

Proof. Once again we write $L^s := L^s(\mathbb{R}^n;\mathbb{C}^m)$. We only consider the case $\zeta = \lambda + i0$, $\lambda > 0$ since the remaining case $\zeta \in \mathbb{C}\setminus\mathbb{R}_{\geq 0}$ is straightforward and can be handled as in equation (3.6) in [10]. We first prove that the operator is injective, so our aim is to show that any given solution u of (16) must be trivial. Proposition 10 implies $u \in L^r$ whenever $\frac{n-3}{2n} < \frac{1}{r} < \frac{n-1}{2n}$. So $V \in L^{[\kappa,\tilde\kappa]}(\mathbb{R}^n;\mathbb{C}^{m\times m}) \subset L^{[\frac{n}{2},\frac{n+1}{2}]}(\mathbb{R}^n;\mathbb{C}^{m\times m})$ implies $Vu \in L^{[p_1,p_2]}(\mathbb{R}^n;\mathbb{C}^m)$ for some p_1, p_2 satisfying

$$\frac{2}{n} + \frac{n-3}{2n} < \frac{1}{p_1} < \frac{2}{n} + \frac{n-1}{2n}, \qquad \frac{2}{n+1} + \frac{n-3}{2n} < \frac{1}{p_2} < \frac{2}{n+1} + \frac{n-1}{2n}.$$

This implies $Vu \in L^p$ where $\frac{2}{n} + \frac{n-3}{2n} < \frac{1}{p} < \frac{2}{n+1} + \frac{n-1}{2n}$. In particular, we deduce from $u \in L^r$ for $\frac{n-3}{2n} < \frac{1}{r} < \frac{n-1}{2n}$ the statement

$$u \in L^{p'},\ g := Vu \in L^p \qquad \text{whenever} \quad \frac{n+1}{2n} < \frac{1}{p} < \frac{n^2+4n-1}{2n(n+1)}.$$

So a density argument and $V = \overline{V}^T$ imply

$$0 = \mathrm{Im}\,(\langle u, Vu\rangle) \overset{(16)}{=} \mathrm{Im}\,(\langle K(\lambda+i0)u, g\rangle) = -\,\mathrm{Im}\,(\langle R_0(\lambda+i0)g, g\rangle) = -c\sqrt{\lambda}\int_{S^{n-1}} |\widehat{g}(\sqrt{\lambda}\omega)|^2 \mathrm{d}\sigma(\omega) \qquad (25)$$

for some positive number $c > 0$, *cf.* (3.7) in [10]. This implies that $\hat g$ vanishes identically on the sphere of radius $\sqrt{\lambda}$ so that Proposition 12 (choose $\frac{2n(n+1)}{n^2+4n-1} < p < \frac{2(n+1)}{n+3}$) implies

$$(1+|\cdot|)^{\tau_1-\frac12}u = (1+|\cdot|)^{\tau_1-\frac12}K(\lambda+i0)u = -(1+|\cdot|)^{\tau_1-\frac12}R_0(\lambda+i0)g \in L^2$$

provided that $0 < \tau_1 < \frac12\min\{1, \frac{n+1}{p} - \frac{n+3}{2}\}$. In view of $u \in H^1_{\mathrm{loc}}$, which is a consequence of Proposition 10, we see that the hypotheses of Theorem 14 are satisfied and we conclude $u \equiv 0$, which proves the injectivity of $I - K(\zeta)$ and hence its invertibility.

To prove the continuity of $\zeta \mapsto (I-K(\zeta))^{-1}$ from $\mathbb{C}\setminus\mathbb{R}_{\geq 0}$ to the space $\mathcal{B}(L^q;L^q)$ we use the identity

$$(I-K(\zeta_1))^{-1} - (I-K(\zeta_2))^{-1} = (I-K(\zeta_1))^{-1}(K(\zeta_1)-K(\zeta_2))(I-K(\zeta_2))^{-1} \qquad (26)$$

for $\zeta_1, \zeta_2 \in \mathbb{C}\setminus\mathbb{R}_{\geq 0}$. Hence, for all ζ_1, ζ_2 belonging to $A\setminus\mathbb{R}$ and compact sets $A \subset \mathbb{C}\setminus\mathbb{R}_{\geq 0}$ we have

$$\|(I-K(\zeta_1))^{-1} - (I-K(\zeta_2))^{-1}\|_{q\to q} \lesssim \|K(\zeta_1)-K(\zeta_2)\|_{q\to q}.$$

Here we used that the operator norms of $(I-K(\zeta_1))^{-1}, (I-K(\zeta_2))^{-1}$ are uniformly bounded on A. Indeed, assume for contradiction $\|(I-K(\zeta))^{-1}\|_{\mathcal{B}(L^q,L^q)} \to \infty$ as $\zeta \to \lambda \pm i0, \lambda > 0$. Then there exist functions

$u_\zeta \in L^q$ with $\|u_\zeta\|_q = 1$ such that $(I - K(\zeta))u_\zeta \to 0$ in L^q. Taking $u \in L^q$ with $u_\zeta \rightharpoonup u$ for some subsequence we infer $(I - K(\lambda \pm i0))u = 0$. On the other hand, Corollary 9 implies that $(K(\zeta)u_\zeta)$ converges in L^q, so $u_\zeta \to u$ in L^q. In particular $\|u\|_q = 1$ and thus $u \neq 0$, which contradicts the injectivity of $I - K(\lambda \pm i0)$. So the assumption was false, which proves that the resolvents $(I - K(\zeta_1))^{-1}, (I - K(\zeta_2))^{-1}$ are uniformly bounded. Hence, the continuity of K and (26) imply the continuity of $\zeta \mapsto (I - K(\zeta))^{-1}$. In the same way, the existence of a continuous extension of $\zeta \mapsto K(\zeta)$ to the positive half-axis in the operator norm topology provided by Corollary 9 implies the existence of a continuous extension of $\zeta \mapsto (I - K(\zeta))^{-1}$ in the operator norm topology. This and Theorem 6 finally imply that $\zeta \mapsto R(\zeta) = (I - K(\zeta))^{-1}R_0(\zeta)$ is pointwise convergent as $\zeta \to \lambda \pm i0$, $\lambda > 0$ as claimed in Theorem 4.

Finally, using a decomposition of V as in Proposition 10 with small $L^{\frac{n}{2}}$-part, the bound (14) from Proposition 8 implies

$$\|K(\zeta)\|_{q\to q} \leq \varepsilon + C_\varepsilon |\zeta|^{-\frac{1}{n+1}}$$

for all $\varepsilon > 0$, $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and some $C_\varepsilon > 0$. In particular, the norms of these operators tend to zero as $|\zeta| \to \infty$. Thus, one can choose sufficiently large R and sufficiently small ε such that $\|K(\zeta)\|_{q\to q} < \frac{1}{2}$ provided that $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and $|\zeta| \geq R$. So the Neumann series expansion for $|\zeta| \geq R$ and the uniform boundedness for $|\zeta| < R$ show that $\|(I - K(\zeta))^{-1}\| \leq C < \infty$ for all $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and the claim is proved. □

Remark 16. For a better understanding of the forthcoming sections, we stress here that the hypothesis of V being Hermitian is crucial in the proof of the injectivity, thus invertibility, of $I - K(\zeta)$. Indeed, the vanishing of $\hat{g}$ on the sphere of radius $\sqrt{\lambda}$, which is a fundamental step in the proof, strongly relies on the identity $\mathrm{Im}(\langle u, Vu \rangle) = 0$, see (25).

3. Absence of eigenvalues for Maxwell's equations – Proof of Theorem 3

In this section we prove Theorem 3. To this end we rewrite the time-harmonic Maxwell system (4) as a linear Helmholtz system without first order terms so that the result essentially follows from Theorem 14. To write down this Helmholtz system we introduce some notation. For any given ε, μ as in (A1),(A2) and $\zeta \in \mathbb{C}$ we define the 6×6 complex-valued block matrices/operators

$$\mathcal{V}(\zeta) := \begin{pmatrix} V_1(\zeta) & -i\zeta v \times \\ i\zeta v \times & V_2(\zeta) \end{pmatrix}, \quad \mathcal{L}_1(\zeta) := \begin{pmatrix} i\zeta(\varepsilon\mu)^{\frac{1}{2}} I_3 & -\frac{1}{2}\nabla(\log\varepsilon)\times \\ -\frac{1}{2}\nabla(\log\mu)\times & -i\zeta(\varepsilon\mu)^{\frac{1}{2}} I_3 \end{pmatrix}, \quad \mathcal{L}_2 := \begin{pmatrix} 0 & -\nabla\times \\ -\nabla\times & 0 \end{pmatrix}. \tag{27}$$

Here, $v := 2\nabla((\varepsilon\mu)^{\frac{1}{2}}) : \mathbb{R}^3 \to \mathbb{R}^3$ and $V_1(\zeta), V_2(\zeta) : \mathbb{R}^3 \to \mathbb{C}^{3\times 3}$ are given by

$$\begin{aligned} V_1(\zeta) &:= -\varepsilon^{-\frac{1}{2}}\Delta(\varepsilon^{\frac{1}{2}})I_3 + \nabla\nabla^T(\log\varepsilon) - \zeta^2(\varepsilon_\infty\mu_\infty - \varepsilon\mu)I_3, \\ V_2(\zeta) &:= -\mu^{-\frac{1}{2}}\Delta(\mu^{\frac{1}{2}})I_3 + \nabla\nabla^T(\log\mu) - \zeta^2(\varepsilon_\infty\mu_\infty - \varepsilon\mu)I_3, \end{aligned} \tag{28}$$

In the following Lemma we show that solutions of the time-harmonic Maxwell system give rise to solutions of a 6×6 Helmholtz system with complex-valued coefficients given by (27),(28).

Lemma 17. *Assume (A1), $\zeta \in \mathbb{C}$ and let $J_e, J_m \in L^2_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{C}^3)$ be divergence-free. Then every weak solution $(E, H) \in H^1_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{C}^6)$ of the time-harmonic Maxwell system* (4) *satisfies*

$$(\Delta + \zeta^2\varepsilon_\infty\mu_\infty)\begin{pmatrix} \tilde{E} \\ \tilde{H} \end{pmatrix} + \mathcal{V}(\zeta)\begin{pmatrix} \tilde{E} \\ \tilde{H} \end{pmatrix} = \mathcal{L}_1(\zeta)\begin{pmatrix} \tilde{J}_e \\ \tilde{J}_m \end{pmatrix} + \mathcal{L}_2\begin{pmatrix} \tilde{J}_e \\ \tilde{J}_m \end{pmatrix} \quad \text{in } \mathbb{R}^3 \tag{29}$$

in the weak sense where $(\tilde{E}, \tilde{H}) := (\varepsilon^{\frac{1}{2}}E, \mu^{\frac{1}{2}}H)$ and $(\tilde{J}_e, \tilde{J}_m) := (\mu^{\frac{1}{2}}J_e, \varepsilon^{\frac{1}{2}}J_m)$.

Proof. For notational simplicity we verify (29) in the pointwise sense assuming that classical derivatives exist. This carries over to weak solutions by moving first order derivatives to the test functions. So let (E, H) denote a weak solution of (4) as assumed. Then $(\tilde{E}, \tilde{H}) = (\varepsilon^{-\frac{1}{2}}D, \mu^{-\frac{1}{2}}B)$ where D, B are divergence-free vector fields. The latter follows from the fact that J_e, J_m are divergence-free. So we have $\nabla \times \nabla \times D =$

$\nabla(\nabla\cdot D)-\Delta D=-\Delta D$ and obtain

$$\begin{aligned}\Delta\tilde E&=\Delta(\varepsilon^{-\frac12})D+2[\nabla(\varepsilon^{-\frac12})\cdot\nabla]D+\varepsilon^{-\frac12}\Delta D\\&=\left(\frac34\varepsilon^{-2}|\nabla\varepsilon|^2-\frac12\varepsilon^{-1}\Delta\varepsilon\right)\tilde E-\varepsilon^{-\frac32}[\nabla\varepsilon\cdot\nabla]D-\varepsilon^{-\frac12}\nabla\times\nabla\times D.\end{aligned}\tag{30}$$

The second order term can be simplified with the aid of (4). The vector calculus identities

$$\begin{aligned}\nabla\times(\psi A)&=\nabla\psi\times A+\psi(\nabla\times A),\\\nabla\times(A\times C)&=A(\nabla\cdot C)-C(\nabla\cdot A)+(C\cdot\nabla)A-(A\cdot\nabla)C\end{aligned}$$

for scalar fields ψ and vector fields A,C lead to

$$\begin{aligned}\nabla\times\nabla\times D&=\nabla\times\nabla\times(\varepsilon E)\\&=\nabla\times(\nabla\varepsilon\times E+\varepsilon(\nabla\times E))\\&=\nabla\times(\nabla\varepsilon\times E+\varepsilon\,[-i\zeta\mu H+J_m])\\&=\nabla\times(\nabla\varepsilon\times E)-i\zeta\nabla(\varepsilon\mu)\times H-i\zeta\varepsilon\mu(\nabla\times H)+\nabla\times(\varepsilon J_m)\\&=\nabla\varepsilon(\nabla\cdot E)-(\Delta\varepsilon)E+(E\cdot\nabla)\nabla\varepsilon-(\nabla\varepsilon\cdot\nabla)E\\&\quad-i\zeta\nabla(\varepsilon\mu)\times H-i\zeta\varepsilon\mu(i\zeta\varepsilon E+J_e)+\nabla\times(\varepsilon^{\frac12}\tilde J_m).\end{aligned}$$

To simplify these terms we use that $D=\varepsilon E$ is divergence-free and thus $\nabla\cdot E=-\varepsilon^{-1}\nabla\varepsilon\cdot E$. Moreover,

$$(\nabla\varepsilon\cdot\nabla)E=(\nabla\varepsilon\cdot\nabla)(\varepsilon^{-1}D)=-\varepsilon^{-2}|\nabla\varepsilon|^2D+\varepsilon^{-1}(\nabla\varepsilon\cdot\nabla)D.$$

This implies

$$\begin{aligned}-\varepsilon^{-\frac12}\nabla\times\nabla\times D&=-\varepsilon^{-\frac12}\cdot\left[-\varepsilon^{-1}\nabla\varepsilon(\nabla\varepsilon\cdot E)-(\Delta\varepsilon)E+(E\cdot\nabla)\nabla\varepsilon+\varepsilon^{-2}|\nabla\varepsilon|^2D-\varepsilon^{-1}(\nabla\varepsilon\cdot\nabla)D\right]\\&\quad+i\zeta(\varepsilon\mu)^{-\frac12}\nabla(\varepsilon\mu)\times\tilde H-\zeta^2\varepsilon\mu\tilde E+i\zeta(\varepsilon\mu)^{\frac12}\tilde J_e-\varepsilon^{-\frac12}\nabla\times(\varepsilon^{\frac12}\tilde J_m)\\&=\varepsilon^{-2}\nabla\varepsilon(\nabla\varepsilon)^T\tilde E+\varepsilon^{-1}(\Delta\varepsilon)\tilde E-\varepsilon^{-1}(\tilde E\cdot\nabla)\nabla\varepsilon-\varepsilon^{-2}|\nabla\varepsilon|^2\tilde E+\varepsilon^{-\frac32}(\nabla\varepsilon\cdot\nabla)D\\&\quad+i\zeta(\varepsilon\mu)^{-\frac12}\nabla(\varepsilon\mu)\times\tilde H-\zeta^2\varepsilon\mu\tilde E+i\zeta(\varepsilon\mu)^{\frac12}\tilde J_e-\nabla\times\tilde J_m-\frac12\varepsilon^{-1}\nabla\varepsilon\times\tilde J_m.\end{aligned}$$

Combining this formula with (30) we find that the first order terms (involving D) cancel and

$$\begin{aligned}\Delta\tilde E&=\left(\frac34\varepsilon^{-2}|\nabla\varepsilon|^2-\frac12\varepsilon^{-1}\Delta\varepsilon\right)\tilde E+\varepsilon^{-2}\nabla\varepsilon(\nabla\varepsilon)^T\tilde E+\varepsilon^{-1}\Delta\varepsilon\tilde E-\varepsilon^{-1}(\tilde E\cdot\nabla)\nabla\varepsilon-\varepsilon^{-2}|\nabla\varepsilon|^2\tilde E\\&\quad+i\zeta(\varepsilon\mu)^{-\frac12}\nabla(\varepsilon\mu)\times\tilde H-\zeta^2\varepsilon\mu\tilde E+i\zeta(\varepsilon\mu)^{\frac12}\tilde J_e-\nabla\times\tilde J_m-\frac12\varepsilon^{-1}\nabla\varepsilon\times\tilde J_m\\&=\left(-\frac14\varepsilon^{-2}|\nabla\varepsilon|^2+\frac12\varepsilon^{-1}\Delta\varepsilon+\varepsilon^{-2}\nabla\varepsilon(\nabla\varepsilon)^T-\zeta^2\varepsilon\mu-\varepsilon^{-1}\nabla\nabla^T\varepsilon\right)\tilde E\\&\quad+i\zeta(\varepsilon\mu)^{-\frac12}\nabla(\varepsilon\mu)\times\tilde H+i\zeta(\varepsilon\mu)^{\frac12}\tilde J_e-\nabla\times\tilde J_m-\frac12\varepsilon^{-1}\nabla\varepsilon\times\tilde J_m\\&=\left[\varepsilon^{-\frac12}\Delta(\varepsilon^{\frac12})-\nabla\nabla^T(\log\varepsilon)+\zeta^2(\varepsilon_\infty\mu_\infty-\varepsilon\mu)\right]\tilde E-\zeta^2\varepsilon_\infty\mu_\infty\tilde E\\&\quad+2i\zeta\nabla((\varepsilon\mu)^{\frac12})\times\tilde H+i\zeta(\varepsilon\mu)^{\frac12}\tilde J_e-\nabla\times\tilde J_m-\frac12\nabla(\log\varepsilon)\times\tilde J_m.\end{aligned}$$

This corresponds to the first line in (29). To derive the second line, one proceeds in an analogous manner and subsequently derives the formulas

$$\begin{aligned}\Delta\tilde H&=\left(\frac34\mu^{-2}|\nabla\mu|^2-\frac12\mu^{-1}\Delta\mu\right)\tilde H-\mu^{-\frac32}[\nabla\mu\cdot\nabla]B-\mu^{-\frac12}\nabla\times\nabla\times B,\\\nabla\times\nabla\times B&=\nabla\mu(\nabla\cdot H)-(\Delta\mu)H+(H\cdot\nabla)\nabla\mu-(\nabla\mu\cdot\nabla)H\\&\quad+i\zeta\nabla(\varepsilon\mu)\times E+i\zeta\varepsilon\mu(-i\zeta\mu H+J_m)+\nabla\times(\mu^{\frac12}\tilde J_e),\end{aligned}$$

$$
\begin{aligned}
-\mu^{-\frac{1}{2}}\nabla\times\nabla\times B &= \mu^{-2}\nabla\mu(\nabla\mu)^T\tilde{H}+\mu^{-1}(\Delta\mu)\tilde{H}-\mu^{-1}(\tilde{H}\cdot\nabla)\nabla\mu-\mu^{-2}|\nabla\mu|^2\tilde{H}+\mu^{-\frac{3}{2}}(\nabla\mu\cdot\nabla)B \\
&\quad - i\zeta(\varepsilon\mu)^{-\frac{1}{2}}\nabla(\varepsilon\mu)\times\tilde{E}-\zeta^2\varepsilon\mu\tilde{H}-i\zeta(\varepsilon\mu)^{\frac{1}{2}}\tilde{J}_m-\nabla\times\tilde{J}_e-\frac{1}{2}\mu^{-1}\nabla\mu\times\tilde{J}_e, \\
\Delta\tilde{H} &= \Big[\mu^{-\frac{1}{2}}\Delta(\mu^{\frac{1}{2}})-\nabla\nabla^T(\log\mu)+\zeta^2(\varepsilon_\infty\mu_\infty-\varepsilon\mu)\Big]\tilde{H}-\zeta^2\varepsilon_\infty\mu_\infty\tilde{H} \\
&\quad - 2i\zeta\nabla((\varepsilon\mu)^{\frac{1}{2}})\times\tilde{E}-i\zeta(\varepsilon\mu)^{\frac{1}{2}}\tilde{J}_m-\nabla\times\tilde{J}_e-\frac{1}{2}\nabla(\log\mu)\times\tilde{J}_e.
\end{aligned}
$$

□

Remark 18.

(a) Lemma 17 allows to deduce further properties of solutions of time-harmonic Maxwell's equations from the corresponding theory for elliptic PDEs. For instance, one may deduce local regularity properties as we will do in Proposition 20. We refer to [3, Section 3] for other approaches to regularity results for time-harmonic Maxwell's equations. Further features such as Harnack inequalities or maximum principles can be proved as well. Our assumptions on the data ε,μ may however be far from optimal.

(b) It would be interesting to find a counterpart of Lemma 17 for anisotropic material laws where ε,μ are matrix-valued.

Proof of Theorem 3: Let $(E,H)\in H^1_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{C}^6)$ be a weak solution of the homogeneous ($J_{\mathrm{e}}=J_{\mathrm{m}}=0$) time-harmonic Maxwell system (4) for $\zeta\in\mathbb{C}$ and assume $(1+|x|)^{\tau_1-\frac{1}{2}}(|E|+|H|)\in L^2(\mathbb{R}^3)$ for some $\tau_1>0$. From (4) we infer $(1+|x|)^{\tau_1-\frac{1}{2}}(|\nabla\times E|+|\nabla\times H|)\in L^2(\mathbb{R}^3)$ as well as

$$\int_{\mathbb{R}^3}\frac{1}{\mu}(\nabla\times E)\cdot(\nabla\times\phi)\,\mathrm{d}x=\zeta^2\int_{\mathbb{R}^3}\varepsilon E\phi\qquad\text{for all }\phi\in C_c^\infty(\mathbb{R}^3;\mathbb{C}^3).$$

By density of test functions and $E\in H^1_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{C}^3)$ we obtain ($\phi=\chi\bar{E}$)

$$\int_{\mathbb{R}^3}\frac{1}{\mu}|\nabla\times E|^2\chi\,\mathrm{d}x+\int_{\mathbb{R}^3}\frac{1}{\mu}(\nabla\times E)\cdot(\nabla\chi\times\bar{E})\,\mathrm{d}x=\zeta^2\int_{\mathbb{R}^3}\varepsilon|E|^2\chi\,\mathrm{d}x\qquad\text{for all }\chi\in C_c^\infty(\mathbb{R}^3).\tag{31}$$

We choose $\chi=\chi^*(\cdot/R)$ where $\chi^*\in C_c^\infty(\mathbb{R}^3)$ is a real-valued radially nonincreasing nonnegative function that is identically one near the origin so that $\chi^*(\cdot/R)\nearrow 1$ as $R\to\infty$. Using $|\nabla\chi^*(z)|\lesssim(1+|z|)^{-1}$ we get for $R\geq 1$

$$
\begin{aligned}
\left|\int_{\mathbb{R}^3}\frac{1}{\mu}(\nabla\times E)\cdot(\nabla\chi\times\bar{E})\,\mathrm{d}x\right| &\lesssim R^{-1}\int_{\mathbb{R}^3}|\nabla\times E||\nabla\chi^*(x/R)||E|\,dx \\
&\lesssim R^{-1}\int_{\mathbb{R}^3}|\nabla\times E||E|(1+|x|/R)^{-1}\,dx \\
&= \int_{\mathbb{R}^3}|\nabla\times E||E|\cdot(R+|x|)^{-1}\,dx \\
&\lesssim R^{-2\tau_1}\int_{\mathbb{R}^3}|\nabla\times E||E|(1+|x|)^{2\tau_1-1}\,dx \\
&\lesssim R^{-2\tau_1}\qquad(R\to\infty).
\end{aligned}
$$

In other words, the second integral in (31) vanishes as $R\to\infty$.

From this we conclude as follows. In the case $\mathrm{Im}(\zeta^2)\neq 0$ we take the imaginary part of (31) and get from the Monotone Convergence Theorem $\int_{\mathbb{R}^3}\varepsilon|E|^2=0$, hence $E=0$. In the case $\mathrm{Re}(\zeta^2)\leq 0$ we take the real part of (31) and obtain $\int_{\mathbb{R}^3}\frac{1}{\mu}|\nabla\times E|^2-\mathrm{Re}(\zeta^2)\varepsilon|E|^2=0$. Again, $E=0$. So we have $E=0$ in both cases, which then implies $H=0$ because of (4). This proves the absence of eigenvalues for all $\zeta\in\mathbb{C}\setminus\mathbb{R}\cup\{0\}$.

We now prove the claim for $\zeta\in\mathbb{R}\setminus\{0\}$. We deduce from Lemma 17 that $(\tilde{E},\tilde{H})\in H^1_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{C}^6)$ is a weak solution of the Helmholtz system (29) for $\tilde{J}_e=\tilde{J}_m=0$. After decomposing $\tilde{E},\tilde{H}$ and the coefficient matrix $\mathcal{V}(\zeta)$ into real and imaginary part, we find that $u:=(\mathrm{Re}(\tilde{E}),\mathrm{Re}(\tilde{H}),\mathrm{Im}(\tilde{E}),\mathrm{Im}(\tilde{H}))$ is a weak solution in $H^1_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^{12})$ of a real (12×12)-Helmholtz system of the form $(\Delta+\lambda)u + Vu = 0$ in $\mathbb{R}^3$ where $\lambda = \zeta^2\varepsilon_\infty\mu_\infty$ and $V \in L^{[\frac{3}{2},2]}(\mathbb{R}^3;\mathbb{R}^{12\times 12})$. The latter is a consequence of (A2). Since our assumptions imply $(1+|x|)^{\tau_1-\frac{1}{2}}u \in L^2(\mathbb{R}^3;\mathbb{R}^{12})$ and $\lambda = \zeta^2\varepsilon_\infty\mu_\infty > 0$, Theorem 14 yields $u\equiv 0$ and thus $E\equiv H\equiv 0$. This finishes the proof. □

Remark 19. In the proof of Theorem 3 we used two different approaches to treat the cases $\zeta\in\mathbb{C}\setminus\mathbb{R}\cup\{0\}$ and $\zeta\in\mathbb{R}\setminus\{0\}$. As a matter of fact, we cannot deduce the absence of eigenvalues $\zeta\in\mathbb{C}\setminus\mathbb{R}$ by a reduction to the Helmholtz-type system (29) as in the case $\zeta\in\mathbb{R}\setminus\{0\}$. Indeed, as soon as we allow $\mathrm{Im}(\lambda)\neq 0$ (recall that $\lambda=\zeta^2\varepsilon_\infty\mu_\infty$), the function $u := (\mathrm{Re}(\tilde{E}),\mathrm{Re}(\tilde{H}),\mathrm{Im}(\tilde{E}),\mathrm{Im}(\tilde{H}))$ satisfies $(\Delta+\mathrm{Re}\,\lambda)u + Wu + Vu = 0$, where W is a constant-valued 12×12-matrix given by

$$W = \begin{pmatrix} 0 & \mathrm{Im}(\lambda)I_6 \\ -\mathrm{Im}(\lambda)I_6 & 0 \end{pmatrix}.$$

The lack of decay of W rules out the possibility of applying Theorem 14 and as a consequence one could not conclude $u\equiv 0$. So the difficulty of treating complex-valued potentials cannot be resolved just by taking the real and imaginary parts of the equation. On the contrary, as we will see in the next section, the treatment of complex-valued potentials requires a more accurate analysis of the problem.

4. The LAP for Maxwell's equations – Proof of Theorem 1

This section is devoted to the proof of the Limiting Absorption Principle for the time-harmonic Maxwell system (4) stated in Theorem 1. We shall first give an overview of the main steps of the proof, the rigorous details are provided afterwards. We start by considering $(E_\zeta,H_\zeta)\in H^1(\mathbb{R}^3;\mathbb{C}^6)$, the uniquely determined solutions of the *approximating* time-harmonic Maxwell system

$$i\zeta\varepsilon E_\zeta - \nabla\times H_\zeta = -J_{\mathrm{e}}^\zeta, \qquad i\zeta\mu H_\zeta + \nabla\times E_\zeta = J_{\mathrm{m}}^\zeta, \tag{32}$$

where $\zeta\in\mathbb{C}\setminus\mathbb{R}$ and $J_{\mathrm{e}}^\zeta, J_{\mathrm{m}}^\zeta \in L^p(\mathbb{R}^3;\mathbb{C}^3)\cap L^{\tilde{p}}(\mathbb{R}^3;\mathbb{C}^3)\cap L^2(\mathbb{R}^3;\mathbb{C}^3)$ are divergence-free currents that approximate the given divergence-free currents $J_{\mathrm{e}},J_{\mathrm{m}}\in L^p(\mathbb{R}^3;\mathbb{C}^3)\cap L^{\tilde{p}}(\mathbb{R}^3;\mathbb{C}^3)$ as $\zeta\to\omega\in\mathbb{R}_{>0}$. The necessity of introducing the approximating problem (32) with square integrable currents $(J_{\mathrm{e}}^\zeta, J_{\mathrm{m}}^\zeta)$ comes from the fact that, up to our knowledge, the existence and uniqueness of solutions to the time-harmonic Maxwell system (4) for general divergence-free currents $(J_{\mathrm{e}},J_{\mathrm{m}})\in L^p(\mathbb{R}^3;\mathbb{C}^6)\cap L^{\tilde{p}}(\mathbb{R}^3;\mathbb{C}^6)$ are not known.

In order to prove the Limiting Absorption Principle, the main task is to prove the convergence of the sequence (E_ζ,H_ζ) in $L^q(\mathbb{R}^3;\mathbb{C}^6)$ as $\zeta\to\omega\pm i0$. Notice that Sobolev's Embedding Theorem implies $(E_\zeta,H_\zeta)\in H^1(\mathbb{R}^3;\mathbb{C}^6)\subset L^q(\mathbb{R}^3;\mathbb{C}^6)$ due to $3<q<6$. From Lemma 17 one infers that the functions

$$u_\zeta := (\tilde{E}_\zeta,\tilde{H}_\zeta) := (\varepsilon^{\frac{1}{2}}E_\zeta, \mu^{\frac{1}{2}}H_\zeta) \tag{33}$$

are solutions of the Helmholtz system

$$(\Delta+\zeta^2\varepsilon_\infty\mu_\infty)u_\zeta + \mathcal{V}(\zeta)u_\zeta = \mathcal{L}_1(\zeta)\tilde{J}^\zeta + \mathcal{L}_2\tilde{J}^\zeta, \tag{34}$$

where $\mathcal{V}(\zeta),\mathcal{L}_1(\zeta)$ and $\mathcal{L}_2$ are defined in (27) and $\tilde{J}^\zeta = (\tilde{J}_{\mathrm{e}}^\zeta,\tilde{J}_{\mathrm{m}}^\zeta) := (\mu^{\frac{1}{2}}J_{\mathrm{e}}^\zeta, \varepsilon^{\frac{1}{2}}J_{\mathrm{m}}^\zeta)$. Due to the explicit relation between the spectral parameters in the Maxwell and Helmholtz systems, the limiting case $\zeta=\omega\pm i0, \omega\in\mathbb{R}\setminus\{0\}$ in the Maxwell system corresponds to the limiting case in the Helmholtz system $\zeta=\lambda\pm\mathrm{sign}(\omega)i0$, $\lambda=\omega^2\varepsilon_\infty\mu_\infty>0$. The boundedness assumption on ε,μ from (A1) implies that, as soon as we are able to provide a uniform bound of $\|u_\zeta\|_q$ as $|\mathrm{Im}(\zeta)|\to 0$, a corresponding bound also holds true for $\|(E_\zeta,H_\zeta)\|_q$.

In order to prove such bounds for $\|u_\zeta\|_q$ we need to investigate the vectorial Helmholtz type operators

$$\Delta+\zeta^2\varepsilon_\infty\mu_\infty I_m + \mathcal{V}(\zeta),$$

that we may rewrite as $(\Delta+\zeta^2\varepsilon_\infty\mu_\infty)(I-\mathcal{K}(\zeta))$ where

$$\begin{aligned} \mathcal{K}(\zeta) &:= -R_0\big(\zeta^2\varepsilon_\infty\mu_\infty\big)\mathcal{V}(\zeta) \qquad && (\zeta\in\mathbb{C}\setminus\mathbb{R}) \\ \mathcal{K}(\omega\pm i0) &:= -R_0\big((\omega\pm i0)^2\varepsilon_\infty\mu_\infty\big)\mathcal{V}(\omega) \qquad && (\omega\in\mathbb{R}\setminus\{0\}) \end{aligned} \tag{35}$$

and $(\omega \pm i0)^2\varepsilon_\infty\mu_\infty := \omega^2\varepsilon_\infty\mu_\infty \pm \operatorname{sign}(\omega)i0$. Since the potential $\mathcal{V}(\zeta)$ is in general not Hermitian (see (27)), not even for $\zeta = \omega \pm i0, \omega \in \mathbb{R}\setminus\{0\}$, we cannot verify the sufficient condition $\operatorname{Im}(\langle u, \mathcal{V}(\zeta)u\rangle) = 0$ for the bijectivity of $I - \mathcal{K}(\zeta)$, see the proof of Corollary 15 and Remark 16. Nevertheless, in Proposition 22 we will quantify the potential lack of injectivity of the operator $I - \mathcal{K}(\zeta)$ through the following injectivity estimates

$$\|u\|_{q_1} + \|u\|_{q_2} \lesssim \|(I - \mathcal{K}(\zeta))u\|_{q_1} + \|(I - \mathcal{K}(\zeta))u\|_{q_2} + |\operatorname{Im}(\langle u, \mathcal{V}(\zeta)u\rangle)|^{1/2} \tag{36}$$

for all $u = (u_e, u_m) \in L^{q_1}(\mathbb{R}^3;\mathbb{C}^6) \cap L^{q_2}(\mathbb{R}^3;\mathbb{C}^6)$ and $|\operatorname{Re}(\zeta)| \geq \delta > 0$. Here,

$$\operatorname{Im}(\langle u, \mathcal{V}(\zeta)u\rangle) = \operatorname{Im}(\zeta^2)\int_{\mathbb{R}^3}(\varepsilon\mu - \varepsilon_\infty\mu_\infty)|u|^2\,\mathrm{d}x - 2\operatorname{Re}(\zeta)\int_{\mathbb{R}^3} v\cdot\operatorname{Re}(u_m \times \overline{u}_e)\,\mathrm{d}x. \tag{37}$$

We remark that it is for proving (36) that the additional integrability assumptions from (A3) enter the proof of Theorem 1.

The estimate (36) is valid for any $u = (u_e, u_m) \in L^q(\mathbb{R}^3;\mathbb{C}^6)$, no matter whether u is a solution of the Helmholtz system (34) or not. On the other hand, if we consider the family u_ζ of solutions to (34) defined in (33), the following representation formula in terms of the resolvent R_0 of the Laplacian is available

$$(I - \mathcal{K}(\zeta))u_\zeta = R_0\big(\zeta^2\varepsilon_\infty\mu_\infty\big)\big[\mathcal{L}_1(\zeta)\tilde{J}^\zeta + \mathcal{L}_2\tilde{J}^\zeta\big]. \tag{38}$$

Thus the injectivity estimate (36),(37) yields

$$\begin{aligned}\|u_\zeta\|_{q_1} + \|u_\zeta\|_{q_2} &\lesssim \|R_0\big(\zeta^2\varepsilon_\infty\mu_\infty\big)\big[\mathcal{L}_1(\zeta)\tilde{J}^\zeta + \mathcal{L}_2\tilde{J}^\zeta\big]\|_{q_1} + \|R_0\big(\zeta^2\varepsilon_\infty\mu_\infty\big)\big[\mathcal{L}_1(\zeta)\tilde{J}^\zeta + \mathcal{L}_2\tilde{J}^\zeta\big]\|_{q_2} \\ &\quad + \left|\operatorname{Im}(\zeta^2)\int_{\mathbb{R}^3}(\varepsilon\mu - \varepsilon_\infty\mu_\infty)|u_\zeta|^2 - 2\operatorname{Re}(\zeta)\int_{\mathbb{R}^3} v\cdot\operatorname{Re}(u_m^\zeta \times \overline{u}_e^\zeta)\right|^{1/2}.\end{aligned} \tag{39}$$

As soon as (39) is proved, the final step will be to bound the right-hand side of (39) in terms of $\|J\|_p + \|J\|_{\tilde{p}}$ and other terms depending on $\|u_\zeta\|_{q_1} + \|u_\zeta\|_{q_2}$ that become negligible as $\zeta \to \omega \pm i0$. Estimating the first two terms only requires minor modifications of Corollary 7 and it is explicitly performed in Proposition 28. On the other hand, in order to estimate the third term, the structure of Maxwell's equations (32) comes into play. Indeed, in order to get the claimed bound, we shall not only make use of the explicit expression of u_ζ in terms of (E_ζ, H_ζ) from (33), but also that (E_ζ, H_ζ) solves Maxwell's equations (32). Combining the previous steps one gets the following uniform estimate

$$\|E_\zeta\|_q + \|H_\zeta\|_q \lesssim \|J\|_p + \|J\|_{\tilde{p}} + o(1) \qquad \text{as } \zeta \to \omega \pm i0. \tag{40}$$

From (40), in a rather standard way (see Subsection 4.3), one obtains the Limiting Absorption Principle contained in Theorem 1.

In the following $p, \tilde{p}, q, q_1, q_2$ are chosen as in Theorem 1. We will use the notation from (27) and (28). In the proofs we will write $L^s := L^s(\mathbb{R}^3;\mathbb{C}^6)$ and similarly for $W^{1,s}, L^s_{\mathrm{loc}}$ etc.

4.1. **Injectivity estimates.** First we recall from Proposition 10 (with $\kappa = \frac{n}{2} = \frac{3}{2}, \tilde{\kappa} = \frac{n+1}{2} = 2$) the regularity and integrability properties of $L^q(\mathbb{R}^3;\mathbb{C}^6)$-solutions to $(I - \mathcal{K}(\zeta))u = 0$. Using the definitions (27),(35) and assumption (A2) we get the following result.

Proposition 20. *Assume (A2) and let q satisfy $3 < q < 6$. Moreover assume $(I - \mathcal{K}(\zeta))u = 0$ for some $u \in L^q(\mathbb{R}^3;\mathbb{C}^6)$ where $\zeta \in \mathbb{C}\setminus\mathbb{R}$ or $\zeta = \omega \pm i0$, $\omega \in \mathbb{R}\setminus\{0\}$. Then any solution $u \in L^q(\mathbb{R}^3;\mathbb{C}^6)$ of $(I - \mathcal{K}(\zeta))u = 0$ belongs to $L^r(\mathbb{R}^3;\mathbb{C}^6) \cap H^1_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{C}^6)$ for all $r \in (3,\infty)$. Moreover, for any given such r, q we have $\|u\|_r \lesssim \|u\|_q$.*

As in the Helmholtz case these integrability properties are actually better for $\zeta \in \mathbb{C}\setminus\mathbb{R}$ where we even have $u \in H^1(\mathbb{R}^3;\mathbb{C}^6)$.
Next we present the crucial scalar condition ensuring the injectivity of $I - \mathcal{K}(\zeta) = I + R_0(\zeta^2\varepsilon_\infty\mu_\infty)[\mathcal{V}(\zeta)\cdot]$. It comes as no surprise that the condition to guarantee injectivity only involves $\operatorname{Im}(\langle u, \mathcal{V}(\zeta)u\rangle)$.

Proposition 21. *Let (A2) hold and assume $3 < q < 6$. Moreover assume $(I - \mathcal{K}(\zeta))u = 0$ for some $u = (u_e, u_m) \in L^q(\mathbb{R}^3; \mathbb{C}^6)$ where $\zeta \in \mathbb{C}$ with $\operatorname{Im}(\zeta^2) \neq 0$ or $\zeta = \omega \pm i0$, $\omega \in \mathbb{R} \setminus \{0\}$. Then*

$$\operatorname{Im}(\zeta^2) \int_{\mathbb{R}^3} (\varepsilon\mu - \varepsilon_\infty \mu_\infty)|u|^2 \,\mathrm{d}x - 2 \operatorname{Re}(\zeta) \int_{\mathbb{R}^3} v \cdot \operatorname{Re}(u_m \times \overline{u}_e) \,\mathrm{d}x = 0 \qquad \Longleftrightarrow \qquad u = 0. \tag{41}$$

Proof. We only need to prove the implication "$\Longrightarrow$", so we assume that the integral is zero. Notice that the latter is well-defined due to the integrability properties of u obtained in the previous proposition. From the definition of $\mathcal{V}(\zeta)$ given in (27) and (28) one has

$$\begin{aligned}
\operatorname{Im}\left(\int_{\mathbb{R}^3} \overline{u} \cdot \mathcal{V}(\zeta) u \,\mathrm{d}x\right) &= \operatorname{Im}\left(\int_{\mathbb{R}^3} \overline{u}_e V_1(\zeta) u_e + \overline{u}_m V_2(\zeta) u_m + i\zeta\big(\overline{u}_e \cdot (v \times u_m) - \overline{u}_m \cdot (v \times u_e)\big) \,\mathrm{d}x\right) \\
&= -\operatorname{Im}(\zeta^2) \int_{\mathbb{R}^3} (\varepsilon_\infty \mu_\infty - \varepsilon\mu)|u|^2 \,\mathrm{d}x - \operatorname{Im}\left(i\zeta \int_{\mathbb{R}^3} v \cdot (u_m \times \overline{u}_e) - v \cdot (u_e \times \overline{u}_m) \,\mathrm{d}x\right) \\
&= \operatorname{Im}(\zeta^2) \int_{\mathbb{R}^3} (\varepsilon\mu - \varepsilon_\infty \mu_\infty)|u|^2 \,\mathrm{d}x - \operatorname{Im}\left(i\zeta \int_{\mathbb{R}^3} v \cdot 2 \operatorname{Re}(u_m \times \overline{u}_e) \,\mathrm{d}x\right) \\
&= \operatorname{Im}(\zeta^2) \int_{\mathbb{R}^3} (\varepsilon\mu - \varepsilon_\infty \mu_\infty)|u|^2 \,\mathrm{d}x - 2 \operatorname{Re}(\zeta) \int_{\mathbb{R}^3} v \cdot \operatorname{Re}(u_m \times \overline{u}_e) \,\mathrm{d}x \\
&= 0.
\end{aligned}$$

Hence we get

$$\begin{aligned}
0 &= \operatorname{Im}\left(\int_{\mathbb{R}^3} \overline{u} \cdot \mathcal{V}(\zeta) u \,\mathrm{d}x\right) \\
&= \operatorname{Im}\left(\int_{\mathbb{R}^3} \overline{\mathcal{K}(\zeta) u} \cdot \mathcal{V}(\zeta) u \,\mathrm{d}x\right) \\
&\overset{(35)}{=} -\operatorname{Im}\left(\int_{\mathbb{R}^3} \overline{R_0(\zeta^2 \varepsilon_\infty \mu_\infty)[\mathcal{V}(\zeta) u]} \cdot \mathcal{V}(\zeta) u \,\mathrm{d}x\right).
\end{aligned}$$

For $\operatorname{Im}(\zeta^2) \neq 0$ we deduce as in equation (3.6) of [10] that $\mathcal{V}(\zeta) u \equiv 0$ and hence $u \equiv \mathcal{K}(\zeta) u \equiv 0$ by (35). In the case $\zeta = \omega \pm i0, \omega \in \mathbb{R} \setminus \{0\}$ we get as in the proof of Corollary 15 for $\lambda = \omega^2 \varepsilon_\infty \mu_\infty > 0$

$$\begin{aligned}
0 &= \operatorname{Im}\left(\int_{\mathbb{R}^3} \overline{\mathcal{K}(\omega \pm i0) u} \cdot \mathcal{V}(\omega) u\right) \\
&= -\operatorname{Im}\left(\int_{\mathbb{R}^3} \overline{R_0(\lambda \pm \operatorname{sign}(\omega) i0)[\mathcal{V}(\omega) u]} \cdot \mathcal{V}(\omega) u\right) \\
&= \mp \operatorname{sign}(\omega) c \int_{S_\lambda} |\widehat{\mathcal{V}(\omega) u}|^2 \, d\sigma_\lambda
\end{aligned}$$

for some $c \neq 0$. Hence, $\widehat{\mathcal{V}(\omega) u} = 0$ on S_λ in the L^2-trace sense. Moreover, $\mathcal{V}(\omega) u \in L^p$ for some (sufficiently large) $p \in [1, \frac{4}{3})$ by assumption (A2) and Proposition 20. So Proposition 12 implies for some $\tau_1 > 0$

$$(1 + |\cdot|)^{\tau_1 - \frac{1}{2}} u = (1 + |\cdot|)^{\tau_1 - \frac{1}{2}} \mathcal{K}(\omega \pm i0) u = -(1 + |\cdot|)^{\tau_1 - \frac{1}{2}} R_0(\lambda \pm \operatorname{sign}(\omega) i0)[\mathcal{V}(\omega) u] \in L^2.$$

Given that u solves the homogeneous Helmholtz system (21) we deduce from Theorem 14 $u = 0$. $\square$

From this fact we deduce our injectivity estimates. We take the condition $\operatorname{Im}(\zeta^2) \neq 0$ or $\zeta = \omega \pm i0, \omega \in \mathbb{R} \setminus \{0\}$ from the previous proposition into account by restricting our attention to spectral parameters ζ with nontrivial real parts. Recall from Theorem 6 and Proposition 8 that this implies continuity properties of $\zeta \mapsto \mathcal{K}(\zeta)$.

Proposition 22. *Assume (A2),(A3). Then, for any given compact subset $K \subset \{\zeta \in \mathbb{C} : \operatorname{Re}(\zeta) \neq 0\}$ we have for all $\zeta \in K \setminus \mathbb{R}$ and all $u = (u_e, u_m) \in L^{q_1}(\mathbb{R}^3; \mathbb{C}^6) \cap L^{q_2}(\mathbb{R}^3; \mathbb{C}^6)$*

$$\|u\|_{q_1} + \|u\|_{q_2} \lesssim \|(I - \mathcal{K}(\zeta)) u\|_{q_1} + \|(I - \mathcal{K}(\zeta)) u\|_{q_2}$$

$$+\left|\mathrm{Im}(\zeta^2)\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)|u|^2\,\mathrm{d}x-2\,\mathrm{Re}(\zeta)\int_{\mathbb{R}^3}v\cdot\mathrm{Re}(u_m\times\overline{u}_e)\,\mathrm{d}x\right|^{1/2}.$$

Proof. We argue by contradiction and assume that there are sequences $(\zeta^j)\subset K$ and $(u^j)\subset L^{q_1}\cap L^{q_2}$ such that $\|u^j\|_{q_1}+\|u^j\|_{q_2}=1$ and

$$\begin{aligned}&\|(I-\mathcal{K}(\zeta^j))u^j\|_{q_1}+\|(I-\mathcal{K}(\zeta^j))u^j\|_{q_2}\to 0,\\ &\left|\mathrm{Im}((\zeta^j)^2)\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)|u^j|^2\,\mathrm{d}x-2\,\mathrm{Re}(\zeta^j)\int_{\mathbb{R}^3}v\cdot\mathrm{Re}(u_m^j\times\overline{u}_e^j)\,\mathrm{d}x\right|^{1/2}\to 0.\end{aligned}\tag{42}$$

We choose subsequences such that $\zeta^j\to\zeta^*\in K$ and $u^j\rightharpoonup u^*$ in L^{q_1} and L^{q_2}. In the case $\zeta^*\in\mathbb{R}\setminus\{0\},\mathrm{Im}(\zeta^j)\to 0^+$ we will write $\mathcal{K}(\zeta^*)$ instead of $\mathcal{K}(\zeta^*+i0)$ for notational simplicity. Clearly similar arguments apply in the case $\mathrm{Im}(\zeta^j)\to 0^-$. The second part of Corollary 9 and (35) imply $\mathcal{K}(\zeta^j)u^j\to\mathcal{K}(\zeta^*)u^*$ so that $\|(I-\mathcal{K}(\zeta^j))u^j\|_{q_1}+\|(I-\mathcal{K}(\zeta^j))u^j\|_{q_2}\to 0$ gives $(I-\mathcal{K}(\zeta^*))u^*=0$ and thus $u^j\to u^*$ in L^q_1 and in L^{q_2}. From the second part of (42) we want to deduce that the injectivity condition (41) holds for (u^*,ζ^*). To verify this we write $|\varepsilon_\infty\mu_\infty-\varepsilon\mu|+|v|=m_1+m_2$ where $m_1\in L^{\frac{q_1}{q_1-2}}(\mathbb{R}^3),m_2\in L^{\frac{q_2}{q_2-2}}(\mathbb{R}^3)$. This is possible due to assumption (A3). Then

$$\begin{aligned}&\left|\mathrm{Im}((\zeta^j)^2)\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)|u^j|^2\,\mathrm{d}x-\mathrm{Im}((\zeta^*)^2)\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)|u^*|^2\,\mathrm{d}x\right|\\ &\le|(\zeta^j)^2-(\zeta^*)^2|\int_{\mathbb{R}^3}(m_1+m_2)|u^j|^2\,\mathrm{d}x+|\zeta^*|^2\int_{\mathbb{R}^3}(m_1+m_2)||u^j|^2-|u^*|^2|\,\mathrm{d}x\\ &\lesssim|(\zeta^j)^2-(\zeta^*)^2|(\|m_1\|_{\frac{q_1}{q_1-2}}\|u^j\|_{q_1}^2+\|m_2\|_{\frac{q_2}{q-2}}\|u^j\|_{q_2}^2)\\ &+\|m_1\|_{\frac{q_1}{q_1-2}}\|u^j-u\|_{q_1}\||u^j|+|u|\|_{q_1}+\|m_2\|_{\frac{q_2}{q_2-2}}\|u^j-u\|_{q_2}\||u^j|+|u|\|_{q_2}\\ &\lesssim(\|m_1\|_{\frac{q_1}{q_1-2}}+\|m_2\|_{\frac{q_2}{q_2-2}})(|(\zeta^j)^2-(\zeta^*)^2|+\|u_j-u\|_{q_1}+\|u_j-u\|_{q_2})\\ &=o(1)\qquad(j\to\infty).\end{aligned}$$

Analogous computations yield

$$\mathrm{Re}(\zeta^j)\int_{\mathbb{R}^3}v\cdot\mathrm{Re}(u_m^j\times\overline{u}_e^j)\,\mathrm{d}x\to\mathrm{Re}(\zeta^*)\int_{\mathbb{R}^3}v\cdot\mathrm{Re}(u_m^*\times\overline{u}_e^*)\,\mathrm{d}x\qquad(j\to\infty).$$

So (41) holds and Proposition 21 implies $u^*=0$. This however contradicts $u^j\to u^*=0$ and $\|u^j\|_{q_1}+\|u^j\|_{q_2}=1$. So the assumption was false, which proves the claim. ☐

4.2. **Bounds for E_ζ,H_ζ.** Proposition 22 makes it possible to bound the L^q-norm of solutions $u_\zeta:=(u_e^\zeta,u_m^\zeta):=(\tilde{E}_\zeta,\tilde{H}_\zeta)$ of the Helmholtz system (29) with $\zeta\in\mathbb{C}\setminus\mathbb{R}$ in terms of J as soon as we find suitable bounds for

$$\mathrm{Im}(\zeta^2)\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)|u_\zeta|^2\,\mathrm{d}x-2\,\mathrm{Re}(\zeta)\int_{\mathbb{R}^3}v\cdot\mathrm{Re}(u_m^\zeta\times\overline{u}_e^\zeta)\,\mathrm{d}x$$

Those are provided in the next proposition.

Proposition 23. *Let the assumptions (A1),(A2),(A3) hold. Then, for any given $\zeta\in\mathbb{C}\setminus\mathbb{R}$ the solutions $u_\zeta:=(u_e^\zeta,u_m^\zeta):=(\tilde{E}_\zeta,\tilde{H}_\zeta):=(\varepsilon^{\frac12}E_\zeta,\mu^{\frac12}H_\zeta)\in L^q(\mathbb{R}^3;\mathbb{C}^6)$ of the Helmholtz system* (29) *satisfy*

$$\begin{aligned}\int_{\mathbb{R}^3}v\cdot\mathrm{Re}(u_m^\zeta\times\overline{u_e^\zeta})\,\mathrm{d}x&=\mathrm{Im}(\zeta)\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)|u_\zeta|^2\,\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)\,\mathrm{Re}\left(\mu^{-\frac12}\overline{J_{\mathrm{m}}^\zeta}\cdot u_m^\zeta-\varepsilon^{-\frac12}J_{\mathrm{e}}^\zeta\cdot\overline{u_e^\zeta}\,\mathrm{d}x\right).\end{aligned}$$

In particular, for $\zeta \in K \setminus \mathbb{R}$ and any compact set $K \subset \mathbb{C}$,

$$\begin{aligned}&\left|\mathrm{Im}(\zeta^2)\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)|u_\zeta|^2\,\mathrm{d}x - 2\,\mathrm{Re}(\zeta)\int_{\mathbb{R}^3} v\cdot \mathrm{Re}(u_m^\zeta\times\overline{u_e^\zeta})\,\mathrm{d}x\right| \\ &\lesssim |\,\mathrm{Im}(\zeta)|(\|u_\zeta\|_{q_1}+\|u_\zeta\|_{q_2})^2 + (\|J_{\mathrm{e}}^\zeta\|_p+\|J_{\mathrm{m}}^\zeta\|_p)(\|u_\zeta\|_{q_1}+\|u_\zeta\|_{q_2}).\end{aligned} \tag{43}$$

Proof. We recall from (28) the identity

$$v = 2\nabla((\varepsilon\mu)^{1/2}) = (\varepsilon\mu)^{-\frac12}\nabla(\varepsilon\mu) = (\varepsilon\mu)^{-\frac12}\nabla(\varepsilon\mu-\varepsilon_\infty\mu_\infty).$$

Then integration by parts gives

$$\begin{aligned}&\int_{\mathbb{R}^3} v\cdot\mathrm{Re}(u_m^\zeta\times\overline{u_e^\zeta})\,\mathrm{d}x\\
&=\int_{\mathbb{R}^3}\nabla(\varepsilon\mu-\varepsilon_\infty\mu_\infty)\cdot\mathrm{Re}\left(H_\zeta\times\overline{E_\zeta}\,\mathrm{d}x\right)\\
&=-\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)\,\left[\nabla\cdot\mathrm{Re}\left(H_\zeta\times\overline{E_\zeta}\right)\right]\,\mathrm{d}x\\
&=-\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)\,\mathrm{Re}\left((\nabla\times H_\zeta)\cdot\overline{E_\zeta}-(\overline{\nabla\times E_\zeta})\cdot H_\zeta\right)\,\mathrm{d}x\\
&=-\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)\,\mathrm{Re}\left((i\zeta\varepsilon E_\zeta+J_{\mathrm{e}}^\zeta)\cdot\overline{E}_\zeta-\overline{(-i\zeta\mu H_\zeta+J_{\mathrm{m}}^\zeta)}\cdot H_\zeta\right)\,\mathrm{d}x\\
&=-\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)\,\mathrm{Re}\left(i\zeta\varepsilon|E_\zeta|^2+J_{\mathrm{e}}^\zeta\cdot\overline{E_\zeta}-i\overline{\zeta}\mu|H_\zeta|^2-\overline{J_{\mathrm{m}}^\zeta}\cdot H_\zeta\right)\,\mathrm{d}x\\
&=\mathrm{Im}(\zeta)\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)\,\left(\varepsilon|E_\zeta|^2+\mu|H_\zeta|^2\right)\,\mathrm{d}x+\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)\,\mathrm{Re}\left(\overline{J_{\mathrm{m}}^\zeta}\cdot H_\zeta-J_{\mathrm{e}}^\zeta\cdot\overline{E_\zeta}\,\mathrm{d}x\right)\\
&=\mathrm{Im}(\zeta)\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)|u_\zeta|^2\,\mathrm{d}x+\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)\,\mathrm{Re}\left(\mu^{-\frac12}\overline{J_{\mathrm{m}}^\zeta}\cdot u_m^\zeta-\varepsilon^{-\frac12}J_{\mathrm{e}}^\zeta\cdot\overline{u_e^\zeta}\right)\,\mathrm{d}x.\end{aligned}$$

So Hölder's inequality and (A3) imply with m_1, m_2 as in the proof of Proposition 22

$$\begin{aligned}\left|2\,\mathrm{Re}(\zeta)\int_{\mathbb{R}^3} v\cdot\mathrm{Re}(u_m^\zeta\times\overline{u_e^\zeta})\,\mathrm{d}x\right| &\lesssim |\,\mathrm{Im}(\zeta)|(\|m_1\|_{\frac{q_1}{q_1-2}}\|u_\zeta\|_{q_1}^2+\|m_2\|_{\frac{q_2}{q_2-2}}\|u_\zeta\|_{q_2}^2)\\ &\quad+\int_{\mathbb{R}^3}(m_1+m_2)(|J_{\mathrm{m}}^\zeta||u_m^\zeta|+|J_{\mathrm{e}}^\zeta||u_e^\zeta|)\,\mathrm{d}x\end{aligned}$$

We have $m_1 \in L^{\frac{q_1}{q_1-2}}(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ and $m_2 \in L^{\frac{q_2}{q_2-2}}(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ by (A1), $J_{\mathrm{m}}^\zeta, J_{\mathrm{e}}^\zeta \in L^p\cap L^{\tilde p}$ and $u_m^\zeta, u_e^\zeta \in L^{q_1}\cap L^{q_2}$. Our assumptions on $p,\tilde p, q_1, q_2$ from the theorem imply

$$\begin{aligned}\frac{1}{\infty}+\frac{1}{\tilde p}+\frac{1}{q_1}\le 1\le\frac{q_1-2}{q_1}+\frac1p+\frac{1}{q_1},\\ \frac{1}{\infty}+\frac{1}{\tilde p}+\frac{1}{q_2}\le 1\le\frac{q_2-2}{q_2}+\frac1p+\frac{1}{q_2}.\end{aligned}$$

So we can find suitable exponents for Hölder's inequality and thus obtain

$$\begin{aligned}\left|2\,\mathrm{Re}(\zeta)\int_{\mathbb{R}^3} v\cdot\mathrm{Re}(u_m^\zeta\times\overline{u_e^\zeta})\,\mathrm{d}x\right| &\lesssim |\,\mathrm{Im}(\zeta)|(\|u_\zeta\|_{q_1}+\|u_\zeta\|_{q_2})^2\\ &\quad+(\|J_{\mathrm{e}}^\zeta\|_p+\|J_{\mathrm{m}}^\zeta\|_p+\|J_{\mathrm{e}}^\zeta\|_{\tilde p}+\|J_{\mathrm{m}}^\zeta\|_{\tilde p})(\|u_\zeta\|_{q_1}+\|u_\zeta\|_{q_2}).\end{aligned}$$

Moreover, $\mathrm{Im}(\zeta^2) = 2\,\mathrm{Re}(\zeta)\,\mathrm{Im}(\zeta)$ gives

$$\left|\mathrm{Im}(\zeta^2)\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)|u_\zeta|^2\,\mathrm{d}x\right| \lesssim |\,\mathrm{Im}(\zeta)|(\|u_\zeta\|_{q_1}+\|u_\zeta\|_{q_2})^2$$

so that (43) is proved. $\square$

Combining this fact and the injectivity estimates from Proposition 22 we obtain uniform bounds for the solutions (E_ζ, H_ζ) provided that $|\operatorname{Im}(\zeta)|$ is sufficiently small.

Corollary 24. *Let the assumptions (A1),(A2),(A3) hold and let $K \subset \mathbb{C}$ be compact. Then, for $|\operatorname{Im}(\zeta)|$ sufficiently small, any solution (E_ζ, H_ζ) of the time-harmonic Maxwell system* (4) *for $\zeta \in K \setminus \mathbb{R}$ satisfies*

$$\begin{aligned}\|E_\zeta\|_q + \|H_\zeta\|_q &\lesssim \|R_0(\zeta^2\varepsilon_\infty\mu_\infty)[\mathcal{L}_1(\zeta)\tilde{J}^\zeta + \mathcal{L}_2\tilde{J}^\zeta]\|_{q_1} \\ &\quad + \|R_0(\zeta^2\varepsilon_\infty\mu_\infty)[\mathcal{L}_1(\zeta)\tilde{J}^\zeta + \mathcal{L}_2\tilde{J}^\zeta]\|_{q_2} + \|J^\zeta\|_p + \|J^\zeta\|_{\tilde{p}}.\end{aligned} \tag{44}$$

Proof. We define $u_\zeta := (\tilde{E}_\zeta, \tilde{H}_\zeta) := (\varepsilon^{\frac12}E_\zeta, \mu^{\frac12}H_\zeta)$. By Lemma 17 these functions solve the Helmholtz system (29) and hence satisfy the representation formula (38). So Proposition 22 and Proposition 23 give

$$\begin{aligned}\|u_\zeta\|_{q_1} + \|u_\zeta\|_{q_2} &\lesssim \|(I-\mathcal{K}(\zeta))u_\zeta\|_{q_1} + \|(I-\mathcal{K}(\zeta))u_\zeta\|_{q_2} \\ &\quad + \left|\operatorname{Im}(\zeta^2)\int_{\mathbb{R}^3}(\varepsilon\mu-\varepsilon_\infty\mu_\infty)|u_\zeta|^2\,\mathrm{d}x - 2\operatorname{Re}(\zeta)\int_{\mathbb{R}^3} v\cdot\operatorname{Re}(u_m^\zeta\times\overline{u}_e^\zeta)\,\mathrm{d}x\right|^{1/2} \\ &\lesssim \|R_0(\zeta^2\varepsilon_\infty\mu_\infty)\big[\mathcal{L}_1(\zeta)\tilde{J}^\zeta + \mathcal{L}_2\tilde{J}^\zeta\big]\|_{q_1} + \|R_0(\zeta^2\varepsilon_\infty\mu_\infty)\big[\mathcal{L}_1(\zeta)\tilde{J}^\zeta + \mathcal{L}_2\tilde{J}^\zeta\big]\|_{q_2} \\ &\quad + \sqrt{|\operatorname{Im}(\zeta)|}(\|u_\zeta\|_{q_1} + \|u_\zeta\|_{q_2}) + (\|J_{\mathrm{e}}^\zeta\|_p + \|J_{\mathrm{m}}^\zeta\|_p)^{\frac12}(\|u_\zeta\|_{q_1} + \|u_\zeta\|_{q_2})^{\frac12}.\end{aligned}$$

This and $\|u_\zeta\|_q \lesssim \|u_\zeta\|_{q_1} + \|u_\zeta\|_{q_2}$ yields the corresponding bound for u_ζ provided that $|\operatorname{Im}(\zeta)|$ is sufficiently small. Assumption (A1) implies $\|E_\zeta\|_r + \|H_\zeta\|_r \lesssim \|u_\zeta\|_r$ for $r \in \{q_1, q_2\}$ and (44) follows. □

4.3. **Proof of the Limiting Absorption Principle.** We first prove the existence of the functions (E_ζ, H_ζ) the bounds for which we provided above. We recall that it is defined as the unique solution in $H^1(\mathbb{R}^3;\mathbb{C}^6) \subset L^q(\mathbb{R}^3;\mathbb{C}^6)$ of the time-harmonic Maxwell system (4) with divergence-free currents $J_{\mathrm{e}}^\zeta, J_{\mathrm{m}}^\zeta$ lying in $L^p(\mathbb{R}^3;\mathbb{C}^3) \cap L^{\tilde{p}}(\mathbb{R}^3;\mathbb{C}^3) \cap L^2(\mathbb{R}^3;\mathbb{C}^3)$ that converge to $J_{\mathrm{e}}, J_{\mathrm{m}}$, respectively. (The reason for considering $J_{\mathrm{e}}^\zeta, J_{\mathrm{m}}^\zeta$ instead of $J_{\mathrm{e}}, J_{\mathrm{m}}$ is because the existence of $L^q(\mathbb{R}^3;\mathbb{C}^6)$-solutions (E_ζ, H_ζ) for the currents $J_{\mathrm{e}}, J_{\mathrm{m}} \in L^p(\mathbb{R}^3;\mathbb{C}^3) \cap L^{\tilde{p}}(\mathbb{R}^3;\mathbb{C}^3)$ is not clear.) In the next proposition we first show that divergence-free currents $J_{\mathrm{e}}, J_{\mathrm{m}} \in L^p(\mathbb{R}^3;\mathbb{C}^3) \cap L^{\tilde{p}}(\mathbb{R}^3;\mathbb{C}^3)$ can be approximated by a sequence of divergence-free currents $J_{\mathrm{e}}^\zeta, J_{\mathrm{m}}^\zeta \in L^p(\mathbb{R}^3;\mathbb{C}^3) \cap L^{\tilde{p}}(\mathbb{R}^3;\mathbb{C}^3) \cap L^2(\mathbb{R}^3;\mathbb{C}^3)$.

Proposition 25. *Let $p, \tilde{p} \in (1,\infty)$ and assume $J_{\mathrm{e}}, J_{\mathrm{m}} \in L^p(\mathbb{R}^3;\mathbb{C}^3) \cap L^{\tilde{p}}(\mathbb{R}^3;\mathbb{C}^3)$ to be divergence-free. Then there are divergence-free vector fields $J_{\mathrm{e}}^\zeta, J_{\mathrm{m}}^\zeta \in L^p(\mathbb{R}^3;\mathbb{C}^3) \cap L^{\tilde{p}}(\mathbb{R}^3;\mathbb{C}^3) \cap L^2(\mathbb{R}^3;\mathbb{C}^3)$ satisfying $(J_{\mathrm{e}}^\zeta, J_{\mathrm{m}}^\zeta) \to (J_{\mathrm{e}}, J_{\mathrm{m}})$ in $L^p(\mathbb{R}^3;\mathbb{C}^3) \cap L^{\tilde{p}}(\mathbb{R}^3;\mathbb{C}^3)$ as $\zeta \to \omega \in \mathbb{R}\setminus\{0\}$.*

Proof. The map $\Pi\colon f \mapsto f - \mathcal{F}^{-1}(|\xi|^{-2}\xi\xi^T\hat{f})$ is a bounded linear operator from $L^p \cap L^{\tilde{p}}$ to the divergence-free vector fields in $L^p \cap L^{\tilde{p}}$. This is a consequence of the $L^p \cap L^{\tilde{p}}$-boundedness of Riesz transforms. So for any given $f \in L^p \cap L^{\tilde{p}}$ we can choose $(f_n) \subset \mathcal{S}$ such that f_n converges to f in $L^p \cap L^{\tilde{p}}$. The sequence (Πf_n) then has the desired properties. □

Next we show that for $J_{\mathrm{e}}^\zeta, J_{\mathrm{m}}^\zeta$ as in Proposition 25 there are uniquely determined solutions (E_ζ, H_ζ) in $H^1(\mathbb{R}^3;\mathbb{C}^6)$. In the proof we will need the following result for $r = 2$.

Proposition 26. *Assume (A1) and $\zeta \in \mathbb{C}, r \in (1,\infty)$. Then every solution of* (4) *satisfies*

$$\|\nabla E\|_r + \|\nabla H\|_r \lesssim (1+|\zeta|)(\|E\|_r + \|H\|_r) + \|J_{\mathrm{e}}\|_r + \|J_{\mathrm{m}}\|_r$$

Proof. Since $D = \varepsilon E$ and $B = \mu H$ are divergence-free, we have

$$\nabla\cdot E = \varepsilon^{-1}\nabla\varepsilon\cdot E, \qquad \nabla\cdot H = \mu^{-1}\nabla\mu\cdot H.$$

This and [35, Theorem 1.1] imply

$$\begin{aligned}\|\nabla E\|_r + \|\nabla H\|_r &\lesssim \|\nabla\times E\|_r + \|\nabla\cdot E\|_r + \|\nabla\times H\|_r + \|\nabla\cdot H\|_r \\ &\lesssim \|-i\zeta\mu H + J_{\mathrm{m}}\|_r + \|\varepsilon^{-1}\nabla\varepsilon\cdot E\|_r + \|i\zeta\varepsilon E + J_{\mathrm{e}}\|_r + \|\mu^{-1}\nabla\mu\cdot H\|_r \\ &\lesssim (1+|\zeta|)(\|E\|_r + \|H\|_r) + \|J_{\mathrm{e}}\|_r + \|J_{\mathrm{m}}\|_r.\end{aligned}$$

□

Proposition 27. *Assume (A1). Then, for* $\zeta \in \mathbb{C}$ *with* $\mathrm{Re}(\zeta), \mathrm{Im}(\zeta) \neq 0$*, there is a unique solution* $(E_\zeta, H_\zeta) \in H^1(\mathbb{R}^3; \mathbb{C}^6)$ *of* (4) *for the divergence-free currents given by* $J_\mathrm{e}^\zeta, J_\mathrm{m}^\zeta \in L^2(\mathbb{R}^3; \mathbb{C}^3)$*.*

Proof. The existence and uniqueness of such a solution $(E_\zeta, H_\zeta) \in H(\mathbb{R}^3; \mathrm{curl}) \times H(\mathbb{R}^3; \mathrm{curl})$ can be proved as in [9, Section 7.4]. From $\varepsilon, \mu \in W^{1,\infty}(\mathbb{R}^3)$ by (A1) and Proposition 26 for $r = 2$ we obtain $(E_\zeta, H_\zeta) \in H^1(\mathbb{R}^3; \mathbb{C}^6)$. □

The preceding propositions ensure that the sequences of solutions (E_ζ, H_ζ) we were speaking of really exist in the space $H^1(\mathbb{R}^3; \mathbb{C}^6)$ and in particular in $L^q(\mathbb{R}^3; \mathbb{C}^6)$ for all $q \in (3, 6)$ by Sobolev's Embedding Theorem. In Corollary 24 we showed that (E_ζ, H_ζ) remain bounded once we have bounds for suitable Lebesgue-norms of $\tilde{J}^\zeta$ and $R_0(\zeta^2 \varepsilon_\infty \mu_\infty)\big[\mathcal{L}_1(\zeta)\tilde{J}^\zeta + \mathcal{L}_2 \tilde{J}^\zeta\big]$ which are independent of $\mathrm{Im}(\zeta)$. As mentioned earlier, this can be achieved rather easily with the aid of Theorem 6 and a suitable modification of it when first order derivates are involved, see Theorem 31 in the Appendix.

Proposition 28. *Assume (A1) and let* $K \subset \{\zeta \in \mathbb{C} : \mathrm{Re}(\zeta) \neq 0\}$ *be compact. Then, for* $\zeta \in K \setminus \mathbb{R}$ *and* $\tilde{J}^\zeta := (\mu^{\frac{1}{2}} J_\mathrm{e}^\zeta, \varepsilon^{\frac{1}{2}} J_\mathrm{m}^\zeta)$ *as above, we have*

$$\|R_0(\zeta^2 \varepsilon_\infty \mu_\infty)\big[\mathcal{L}_1(\zeta)\tilde{J} + \mathcal{L}_2 \tilde{J}^\zeta\big]\|_{q_1} + \|R_0(\zeta^2 \varepsilon_\infty \mu_\infty)\big[\mathcal{L}_1(\zeta)\tilde{J} + \mathcal{L}_2 \tilde{J}^\zeta\big]\|_{q_2} \lesssim \|J^\zeta\|_p + \|J^\zeta\|_{\tilde{p}}.$$

Proof. To bound the term involving $\mathcal{L}_1$ we use Theorem 6. Since $\|\mathcal{L}_1(\zeta)\|_\infty \lesssim 1 + |\zeta| \lesssim 1$ by the definition of $\mathcal{L}_1$ from (28) and assumption (A1) we get

$$\|R_0(\zeta^2 \varepsilon_\infty \mu_\infty)\big[\mathcal{L}_1(\zeta)\tilde{J}^\zeta\big]\|_{q_1} \lesssim \|\mathcal{L}_1(\zeta)\tilde{J}^\zeta\|_p \lesssim \|\tilde{J}^\zeta\|_p \lesssim \|J^\zeta\|_p.$$

The estimate for the term involving $\mathcal{L}_2$ corresponds to the special case $n = 3, m = 6$ in Theorem 31:

$$\|R_0(\zeta^2 \varepsilon_\infty \mu_\infty)\big[\mathcal{L}_2 \tilde{J}^\zeta\big]\|_{q_1} \lesssim \|\tilde{J}^\zeta\|_p + \|\tilde{J}^\zeta\|_{\tilde{p}} \lesssim \|J^\zeta\|_p + \|J^\zeta\|_{\tilde{p}}.$$

Since the same holds for q_1 replaced by q_2, this proves the claim. □

Now we are in the position to prove the Limiting Absorption Principle for time-harmonic Maxwell's equations (4).

Proof of Theorem 1: In order to prove Theorem 1, it suffices to combine the auxiliary results that we established above. So assume (A1),(A2),(A3) and let p, q and $J_\mathrm{e}, J_\mathrm{m} \in L^p \cap L^{\tilde{p}}$ be given as in the theorem. We prove the existence of the solutions $(E_\omega^\pm, H_\omega^\pm)$ with the desired properties by proving the convergence of the solutions (E_ζ, H_ζ) as outlined in part (i) of the theorem. To reduce the notation we only consider the limit $\zeta \to \omega + i0$.

Proof of (i): For $\zeta \in \mathbb{C} \setminus \mathbb{R}$ with $\mathrm{Im}(\zeta) > 0, \mathrm{Re}(\zeta) \neq 0$ let $J_\mathrm{e}^\zeta, J_\mathrm{m}^\zeta \in L^p \cap L^{\tilde{p}} \cap L^2$ be the divergence-free approximating sequence whose existence is ensured by Proposition 25. Let then (E_ζ, H_ζ) denote the unique H^1-solutions of the corresponding inhomogeneous time-harmonic Maxwell system (4) from Proposition 27. Corollary 24 yields for small $|\mathrm{Im}(\zeta)|$

$$\|E_\zeta\|_q + \|H_\zeta\|_q \lesssim \|R_0(\zeta^2 \varepsilon_\infty \mu_\infty)\big[\mathcal{L}_1(\zeta)\tilde{J}^\zeta + \mathcal{L}_2 \tilde{J}^\zeta\big]\|_{q_1} + \|R_0(\zeta^2 \varepsilon_\infty \mu_\infty)\big[\mathcal{L}_1(\zeta)\tilde{J}^\zeta + \mathcal{L}_2 \tilde{J}^\zeta\big]\|_{q_2} + \|J^\zeta\|_p + \|J^\zeta\|_{\tilde{p}}$$

where $\tilde{J}^\zeta := (\mu^{\frac{1}{2}} J_\mathrm{e}^\zeta, \varepsilon^{\frac{1}{2}} J_\mathrm{m}^\zeta)$. So Proposition 28 implies

$$\|E_\zeta\|_q + \|H_\zeta\|_q \lesssim \|J^\zeta\|_p + \|J^\zeta\|_{\tilde{p}} = \|J\|_p + \|J\|_{\tilde{p}} + o(1) \quad \text{as } \zeta \to \omega + i0,$$

which proves that the sequence of approximate solutions (E_ζ, H_ζ) is bounded in L^q. So a subsequence of (u_ζ) defined via $u_\zeta := (\tilde{E}_\zeta, \tilde{H}_\zeta) := (\varepsilon^{\frac{1}{2}} E_\zeta, \mu^{\frac{1}{2}} H_\zeta)$ converges weakly to some $u_\omega^+ =: (\tilde{E}_\omega^+, \tilde{H}_\omega^+)$ in L^q. Defining $(E_\omega^+, H_\omega^+) := (\varepsilon^{-\frac{1}{2}} \tilde{E}_\omega^+, \mu^{-\frac{1}{2}} \tilde{H}_\omega^+)$ we thus obtain a weak solution of the time-harmonic Maxwell system (4) (for $\zeta = \omega$) that satisfies

$$\|E_\omega^+\|_q + \|H_\omega^+\|_q \lesssim \|\tilde{E}_\omega^+\|_q + \|\tilde{H}_\omega^+\|_q \lesssim \|J\|_p + \|J\|_{\tilde{p}}.$$

In the first estimate assumption (A1) is used. This proves the existence of the solution (E_ω^+, H_ω^+) along with the corresponding norm estimate. To conclude the proof of (i) we need to show that for any given approximations $J_\mathrm{e}^\zeta, J_\mathrm{m}^\zeta$ as above the full sequence (u_ζ) converges to u_ω^+.

So let $(\zeta_j),(\tilde{\zeta}_j)$ sequences converging to $\omega+i0$ and let $J^{1,\zeta_j},J^{2,\tilde{\zeta}_j}$ be divergence-free currents converging to J. Let $u_\omega^1,u_\omega^2\in L^q$ denote the corresponding weak limits, i.e., $u_{\zeta_j}\rightharpoonup u_\omega^1, u_{\tilde{\zeta}_j}\rightharpoonup u_\omega^2$. We need to show $u_\omega^1=u_\omega^2$. From (38) we infer $(I-\mathcal{K}(\zeta_j))u_{\zeta_j}=f^1_{\zeta_j}$ and $(I-\mathcal{K}(\tilde{\zeta}_j))u_{\tilde{\zeta}_j}=f^2_{\tilde{\zeta}_j}$ where

$$\begin{aligned} f^1_\zeta &:= R_0\big(\zeta^2\varepsilon_\infty\mu_\infty\big)\big[\mathcal{L}_1(\zeta)\tilde{J}^{1,\zeta}+\mathcal{L}_2\tilde{J}^{1,\zeta}\big],\\ f^2_\zeta &:= R_0\big(\zeta^2\varepsilon_\infty\mu_\infty\big)\big[\mathcal{L}_1(\zeta)\tilde{J}^{2,\zeta}+\mathcal{L}_2\tilde{J}^{2,\zeta}\big],\\ f^+_\omega &:= R_0\big((\omega+i0)^2\varepsilon_\infty\mu_\infty\big)\big[\mathcal{L}_1(\omega)\tilde{J}+\mathcal{L}_2\tilde{J}\big]. \end{aligned}$$

Using first Proposition 28 and $\tilde{J}^{1,\zeta_j}\to\tilde{J}$, then the uniform boundedness of the resolvents $R_0(\zeta_j)$ and $\mathcal{L}_1(\zeta_j)\to\mathcal{L}_1(\omega)$ in L^∞, and finally the pointwise convergence $R_0(\zeta_j^2\varepsilon_\infty\mu_\infty)\to R_0((\omega+i0)^2\varepsilon_\infty\mu_\infty)$ we infer

$$\begin{aligned} f^1_{\zeta_j} &= R_0\big(\zeta_j^2\varepsilon_\infty\mu_\infty\big)\big[\mathcal{L}_1(\zeta_j)\tilde{J}^{1,\zeta_j}+\mathcal{L}_2\tilde{J}^{1,\zeta_j}\big]\\ &= R_0\big(\zeta_j^2\varepsilon_\infty\mu_\infty\big)\big[\mathcal{L}_1(\zeta_j)\tilde{J}+\mathcal{L}_2\tilde{J}\big]+o(1)\\ &= R_0\big(\zeta_j^2\varepsilon_\infty\mu_\infty\big)\big[\mathcal{L}_1(\omega)\tilde{J}+\mathcal{L}_2\tilde{J}\big]+o(1)\\ &= R_0\big((\omega+i0)^2\varepsilon_\infty\mu_\infty\big)\big[\mathcal{L}_1(\omega)\tilde{J}+\mathcal{L}_2\tilde{J}\big]+o(1)\\ &= f^+_\omega+o(1). \end{aligned}$$

Since the same argument applies to $f^2_{\tilde{\zeta}_j}$, we get $f^1_{\zeta_j}-f^2_{\tilde{\zeta}_j}\to f^+_\omega-f^+_\omega=0$. Using the continuity and compactness properties of $\mathcal{K}$ from Corollary 9 we infer that $w:=u^1_\omega-u^2_\omega$ satisfies

$$\begin{aligned} w &= u_{\zeta_j}-u_{\tilde{\zeta}_j}+o_w(1)\\ &= \mathcal{K}(\zeta_j)u_{\zeta_j}-\mathcal{K}(\zeta_j)u_{\tilde{\zeta}_j}+f^1_{\zeta_j}-f^2_{\tilde{\zeta}_j}+o_w(1)\\ &= \mathcal{K}(\omega+i0)u^1_\omega-\mathcal{K}(\omega+i0)u^2_\omega+o_w(1)\\ &= \mathcal{K}(\omega+i0)w+o_w(1). \end{aligned}$$

Here, $o_w(1)$ stands for a null sequence in the weak topology in L^q. The function $(\tilde{w}_{\mathrm{e}},\tilde{w}_{\mathrm{m}})$ given by $(\varepsilon^{1/2}\tilde{w}_{\mathrm{e}},\mu^{1/2}\tilde{w}_{\mathrm{m}}):=w$ is a weak solution of the homogeneous time-harmonic Maxwell system (4) for $\zeta=\omega$ and $\tilde{J}=0$. Repeating the computations in Proposition 23 in the limiting case $\mathrm{Im}(\zeta)=0$ one finds

$$2\omega\int_{\mathbb{R}^3} v\cdot\mathrm{Re}(w_m\times\overline{w_e})\,\mathrm{d}x=0.$$

So Proposition 21 gives $w=0$. This proves that all possible weak limits coincide. Hence, the standard subsequence-of-subsequence argument ensures that all approximating sequences weakly converge to the same limit as $\zeta\to\omega+i0$.

To finish the proof of (i) it remains to show that this convergence also holds in the strong sense and hence, by elliptic regularity theory, in $H^1_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{C}^6)$. To this end we recall from above $(I-\mathcal{K}(\zeta_j))u_{\zeta_j}=f_{\zeta_j}$. From the definition of $\mathcal{K}$ and the second part of Proposition 8 we get $\mathcal{K}(\zeta_j)u_{\zeta_j}\to\mathcal{K}(\omega+i0)u^+_\omega$ in L^q. Moreover, we showed above $f_{\zeta_j}\to f^+_\omega$ in L^q as $j\to\infty$. Hence, we conclude that (u_{ζ_j}) converges in L^q. Since the limit necessarily coincides with the weak limit, we finally obtain $u_{\zeta_j}\to u^+_\omega$ as $\zeta_j\to\omega+i0$. This proves (i) as well as

$$(I-\mathcal{K}(\omega\pm i0))u^\pm_\omega=R_0\big((\omega\pm i0)^2\varepsilon_\infty\mu_\infty\big)\big[\mathcal{L}_1(\omega)\tilde{J}+\mathcal{L}_2\tilde{J}\big]. \tag{45}$$

In particular, u^+_ω solves the Helmholtz system mentioned in the theorem.

We finally prove part (iii) of the theorem, so we assume that the divergence-free currents $(J_{\mathrm{e}},J_{\mathrm{m}})$ lie in the smaller space $L^p\cap L^q\subset L^p\cap L^{\tilde{p}}$ (because $p<\tilde{p}<q$). From above we get a solution $(E,H)\in L^q$ of (4) which, according to Proposition 26, satisfies

$$\|\nabla E\|_q+\|\nabla H\|_q\lesssim(1+|\zeta|)(\|E\|_q+\|H\|_q)+\|J_{\mathrm{e}}\|_q+\|J_{\mathrm{m}}\|_q\lesssim\|J\|_p+\|J\|_q.$$

Hence we conclude $E, H \in W^{1,q}(\mathbb{R}^3;\mathbb{C}^3)$ as claimed. $\square$

5. Appendix

This appendix is devoted to the proof of Theorem 31 (see below) that we needed in the proof of the Limiting Absorption Principle for Maxwell's equations. In order to do that we need the following classical result on Fourier multipliers.

Theorem 29 (Mikhlin-Hörmander)**.** *Let $n \in \mathbb{N}, 1 < r < \infty$. For $k := \lfloor \frac{n}{2} \rfloor + 1$ assume that $m \in C^k(\mathbb{R}^n)$ satisfies $|\partial_\alpha m(\xi)| \le A|\xi|^{-|\alpha|}$ for all multi-indices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$. Then*

$$\|\mathcal{F}^{-1}(m\mathcal{F}f)\|_r \le C_{n,r}A\|f\|_r$$

Proof. The result is a particular case of Theorem 6.2.7 in [11]. $\square$

We also need some boundedness properties of Bessel potentials.

Theorem 30. *Assume $n \in \mathbb{N}, n \ge 2$. Let $\mathcal{J}_1$ be the Bessel potential of order 1 defined as*

$$\mathcal{J}_1 f := \mathcal{F}^{-1}\left(\frac{1}{\sqrt{1+|\xi|^2}}\hat{f}\right), \qquad \textit{alternatively,} \qquad \mathcal{J}_1 f := G * f := \mathcal{F}^{-1}\left(\frac{1}{\sqrt{1+|\xi|^2}}\right) * f,$$

Assume $1 \le p \le q \le \infty$ be such that

$$0 \le \frac{1}{p} - \frac{1}{q} \le \frac{1}{n}, \qquad \left(\frac{1}{p}, \frac{1}{q}\right) \notin \left\{\left(1, 1-\frac{1}{n}\right), \left(\frac{1}{n}, 0\right)\right\}.$$

Then $\mathcal{J}_1$ is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

Proof. From [12, Corollary 1.2.6 (a),(b)] and interpolation we have

$$\mathcal{J}_1 \colon L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n), \qquad \text{if } 1 < p \le q < \infty, \quad 0 \le \frac{1}{p} - \frac{1}{q} \le \frac{1}{n}.$$

Thus, it remains to study the case $p = 1$ and the case $q = \infty$. Let us start with the case $p = 1$. Again from [12, Corollary 1.2.6 (b)] we have that $\mathcal{J}_1 \colon L^1(\mathbb{R}^n) \to L^{q,\infty}(\mathbb{R}^n)$, when $p = 1$ and for $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$. Moreover, from [12, Corollary 1.2.6 (a)], we know that $\mathcal{J}_1 \colon L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$. Thus, Marcinkiewicz's interpolation theorem gives that $\mathcal{J}_1 \colon L^1(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, with $1 - \frac{1}{q} < \frac{1}{n}$. We continue with the case $q = \infty$. We know from the proof of [12, Corollary 1.2.6 (b)] that the kernel G satisfies $|G(x)| \lesssim |x|^{1-n}$ if $|x| \le 2$ and $|G(x)| \lesssim e^{-\frac{|x|}{2}}$ if $|x| \ge 2$. Using Young's convolution inequality we thus get

$$\|\mathcal{J}_1 f\|_\infty = \|G * f\|_\infty \le \|G\|_{p'}\|f\|_p, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

The proof is concluded once one observes that $\|G\|_{p'} < \infty$ for $0 \le \frac{1}{p} < \frac{1}{n}$. $\square$

With these results at hands, we are in the position to prove the following estimates that complement Theorem 6.

Theorem 31. *Let $m, n \in \mathbb{N}, n \ge 3$ and assume $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\ge 0}$. Then, for $1 \le p, \tilde{p}, q \le \infty$ such that*

$$\begin{aligned} &1 \ge \frac{1}{p} > \frac{n+1}{2n}, \qquad 0 \le \frac{1}{q} < \frac{n-1}{2n}, \qquad \frac{2}{n+1} \le \frac{1}{p} - \frac{1}{q} \le \frac{2}{n}, \\ &0 \le \frac{1}{\tilde{p}} - \frac{1}{q} \le \frac{1}{n}, \qquad \left(\frac{1}{\tilde{p}}, \frac{1}{q}\right) \notin \left\{\left(1, 1-\frac{1}{n}\right), \left(\frac{1}{n}, 0\right)\right\}, \end{aligned} \tag{46}$$

$R_0(\zeta)\partial_j f$ is a bounded linear operator from $L^p(\mathbb{R}^n;\mathbb{C}^m)$ to $L^q(\mathbb{R}^n;\mathbb{C}^m)$ satisfying

$$\|R_0(\zeta)\partial_j f\|_q \lesssim |\zeta|^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q}-\frac{1}{n})}\|f\|_p + |\zeta|^{\frac{n}{2}(\frac{1}{\tilde{p}}-\frac{1}{q}-\frac{1}{n})}\|f\|_{\tilde{p}}, \qquad j = 1, 2, \ldots, n. \tag{47}$$

Moreover, there are bounded linear operators $R_0(\lambda\pm i0)\partial_j : L^p(\mathbb{R}^n;\mathbb{C}^m)\to L^q(\mathbb{R}^n;\mathbb{C}^m)$ such that $R_0(\zeta)\partial_j f\to R_0(\lambda\pm i0)\partial_j f$, $j=1,2,\dots,n$ as $\zeta\to\lambda\pm i0, \lambda\in\mathbb{R}_{>0}$ for all $f\in L^p(\mathbb{R}^n;\mathbb{C}^m)$ and

$$\|R_0(\lambda\pm i0)\partial_j f\|_q \lesssim |\lambda|^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q}-\frac{1}{n})}\|f\|_p + |\lambda|^{\frac{n}{2}(\frac{1}{\tilde{p}}-\frac{1}{q}-\frac{1}{n})}\|f\|_{\tilde{p}} \qquad (\lambda>0).$$

Proof. We first isolate the singularity (in Fourier space) of the Fourier multiplier $\frac{1}{|\xi|^2-\zeta}$. In order to do that we introduce the cut-off function $\chi\in C_0^\infty(\mathbb{R}^n)$ with $\chi(\xi)=1$ whenever $|\xi|\le 2$ and we define $\chi_\zeta(\xi):=\chi(|\zeta|^{-\frac{1}{2}}\xi)$. We then write

$$R_0(\zeta)(\partial_j f) = \mathcal{F}^{-1}\left(\frac{\chi_\zeta(\xi)(i\xi_j\hat{f}(\xi))}{|\xi|^2-\zeta}\right) + \mathcal{F}^{-1}\left(\frac{(1-\chi_\zeta(\xi))(i\xi_j\hat{f}(\xi))}{|\xi|^2-\zeta}\right) \qquad j=1,2,\dots,n. \tag{48}$$

Observe that χ_ζ is nontrivial in a neighborhood of the sphere of radius ζ, on the contrary $1-\chi_\zeta$ vanishes in the same neighborhood. In other words, the singularity of the multiplier affects only the first term of the right-hand side of (48). The latter can be estimated with the aid of Theorem 6. More specifically, one has

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\left(\frac{\chi_\zeta(\xi)(i\xi_j\hat{f}(\xi))}{|\xi|^2-\zeta}\right)\right\|_q &= |\zeta|^{-\frac{1}{2}-\frac{n}{2q}}\left\|\mathcal{F}^{-1}\left(\frac{\chi(\xi)\xi_j\mathcal{F}[f(|\zeta|^{-1/2}\cdot)](\xi)}{|\xi|^2-\frac{\zeta}{|\zeta|}}\right)\right\|_q \\
&\lesssim |\zeta|^{-\frac{1}{2}-\frac{n}{2q}}\left\|\mathcal{F}^{-1}\left(\chi(\xi)\xi_j\mathcal{F}[f(|\zeta|^{-1/2}\cdot)](\xi)\right)\right\|_p \\
&\lesssim |\zeta|^{-\frac{1}{2}-\frac{n}{2q}}\left\|\mathcal{F}^{-1}\left(\chi(\xi)\xi_j\right)*f(|\zeta|^{-1/2}\cdot)\right\|_p \\
&\lesssim |\zeta|^{-\frac{1}{2}-\frac{n}{2q}}\|f(|\zeta|^{-1/2}\cdot)\|_p \\
&\lesssim |\zeta|^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q}-\frac{1}{n})}\|f\|_p.
\end{aligned}$$

Here we used Young's convolution inequality and that $\mathcal{F}^{-1}(\chi(\xi)\xi_j)$ is integrable (being a Schwartz function).

We now turn to the estimate of the second term in the sum in (48). We shall use that

$$m(\xi) := \frac{(1-\chi(\xi))\xi_j\sqrt{1+|\xi|^2}}{|\xi|^2-\zeta/|\zeta|}, \qquad j=1,2,\dots,n,$$

is a $L^r(\mathbb{R}^n)-L^r(\mathbb{R}^n)$ multiplier for $1<r<\infty$, due to the simplified version of the Mikhlin-Hörmander Theorem stated in Theorem 29. Recall that χ satisfies $1-\chi\equiv 0$ on a neighborhood of the unit sphere. Notice that, by (46) we have $0<\frac{1}{q}<1$ or $0<\frac{1}{\tilde{p}}<1$. In the case $0<\frac{1}{q}<1$ we use the above observation for $r=q$ and Theorem 30 implies

$$\begin{aligned}
\mathcal{F}^{-1}\left(\frac{(1-\chi_\zeta(\xi))(i\xi_j\hat{f}(\xi))}{|\xi|^2-\zeta}\right) &= |\zeta|^{-\frac{1}{2}-\frac{n}{2q}}\left\|\mathcal{F}^{-1}\left(\frac{(1-\chi(\xi))\xi_j\mathcal{F}[f(|\zeta|^{-1/2}\cdot)](\xi)}{|\xi|^2-1}\right)\right\|_q \\
&= |\zeta|^{-\frac{1}{2}-\frac{n}{2q}}\left\|\mathcal{F}^{-1}\left(m(\xi)\mathcal{F}\left(\mathcal{F}^{-1}\left(\frac{1}{\sqrt{|\xi|^2+1}}\mathcal{F}[f(|\zeta|^{-1/2}\cdot)](\xi)\right)\right)\right)\right\|_q \\
&= |\zeta|^{-\frac{1}{2}-\frac{n}{2q}}\left\|\mathcal{F}^{-1}\left(m(\xi)\mathcal{F}\left(\mathcal{J}_1[f(|\zeta|^{-1/2}\cdot)]\right)\right)\right\|_q \\
&\lesssim |\zeta|^{-\frac{1}{2}-\frac{n}{2q}}\left\|\mathcal{J}_1[f(|\zeta|^{-1/2}\cdot)]\right\|_q \\
&\lesssim |\zeta|^{-\frac{1}{2}-\frac{n}{2q}}\|f(|\zeta|^{-1/2}\cdot)\|_{\tilde{p}} \\
&\le |\zeta|^{\frac{n}{2}(\frac{1}{\tilde{p}}-\frac{1}{q}-\frac{1}{n})}\|f\|_{\tilde{p}}.
\end{aligned}$$

In the complementary case $0<\frac{1}{\tilde{p}}<1$, we use the Mikhlin-Hörmander Theorem 29 for $r=\tilde{p}$ and proceed similarly. Plugging the two previous bounds in (48) gives (47). The final part of Theorem 31 can be proved as in Theorem 6. □

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