Karlsruhe Institute of Technology

Wave phenomena
analysis and numerics

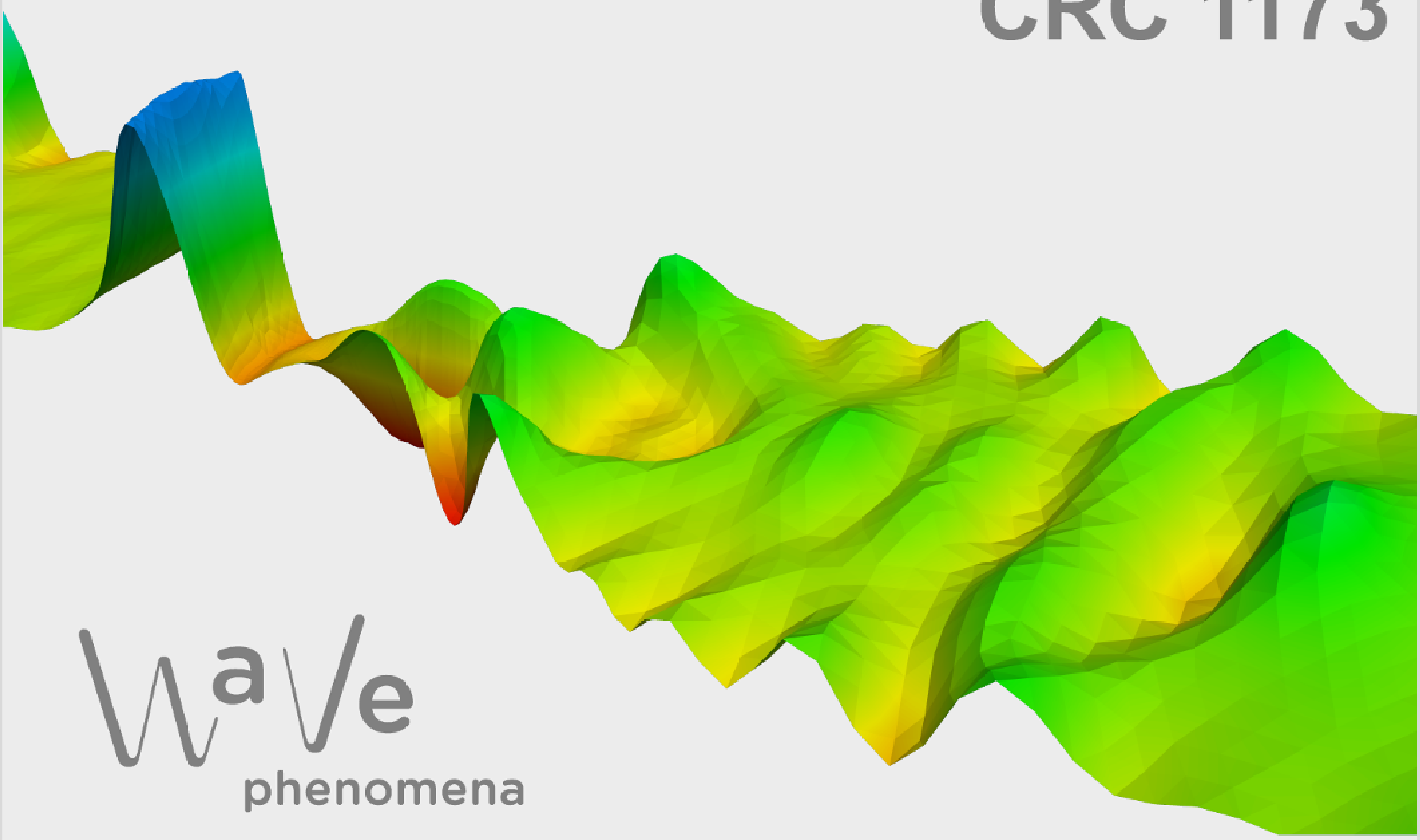
# Some non-homogeneous Gagliardo–Nirenberg inequalities and application to a biharmonic non-linear Schrödinger equation

Antonio J. Fernández, Louis Jeanjean,
Rainer Mandel, Mihai Mariş

CRC Preprint 2020/27, October 2020

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173



Wave phenomena

**Participating universities**

Universität Stuttgart

EBERHARD KARLS UNIVERSITÄT TÜBINGEN

TU WIEN

**Funded by**

DFG

ISSN 2365-662X

# Some non-homogeneous Gagliardo-Nirenberg inequalities and application to a biharmonic non-linear Schrödinger equation

Antonio J. Fernández, Louis Jeanjean,
Rainer Mandel and Mihai Mariş

**Abstract**

We study the standing waves for a fourth-order Schrödinger equation with mixed dispersion that minimize the associated energy when the $L^2-$norm (the *mass*) is kept fixed. We need some non-homogeneous Gagliardo-Nirenberg-type inequalities and we develop a method to prove such estimates that should be useful elsewhere. We prove optimal results on the existence of minimizers in the *mass-subcritical* and *mass-critical* cases. In the *mass supercritical* case we show that global minimizers do not exist, and we investigate the existence of local minimizers. If the mass does not exceed some threshold $\mu_0 \in (0, +\infty)$, our results on "best" local minimizers are also optimal.

## 1 Introduction

We consider the biharmonic non-linear Schrödinger equation with mixed dispersion

$$\text{(BNLS)} \qquad i\partial_t \psi + \alpha \Delta^2 \psi + \beta \Delta \psi + \gamma |\psi|^{2\sigma} \psi = 0 \qquad \text{in } \mathbf{R} \times \mathbf{R}^N,$$

where $\alpha, \sigma > 0$ and $\beta, \gamma \in \mathbf{R}$, $\gamma \neq 0$. This equation has been introduced by Karpman and Shagalov in [12] and [13] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr non-linearity; see also [10]. It has also been used to describe the motion of a vortex filament in an incompressible fluid ([11]). The equation received considerable attention since then.

By simple scaling it is possible to get rid of the parameters $\alpha, \beta, \gamma$. Indeed, if $\beta \neq 0$, taking $\psi(t, x) = a\tilde{\psi}\left(\frac{t}{c}, \frac{x}{b}\right)$ where $a = \alpha^{-\frac{1}{2\sigma}} |\beta/2|^{\frac{1}{\sigma}} |\gamma|^{-\frac{1}{2\sigma}}$, $b = \alpha^{\frac{1}{2}} |\beta/2|^{-\frac{1}{2}}$, and $c = 4\alpha|\beta|^{-2}$, we see that $\psi$ solves the above equation if and only if $\tilde{\psi}$ solves (after dropping " ˜ ")

$$(1.1) \qquad i\psi_t + \Delta^2 \psi + 2\epsilon \Delta \psi + \vartheta |\psi|^{2\sigma} \psi = 0 \qquad \text{in } \mathbf{R} \times \mathbf{R}^N,$$

where $\epsilon = \text{sgn}(\beta) \in \{-1, 0, 1\}$ and $\vartheta = \text{sgn}(\gamma) \in \{-1, 1\}$. By analogy to the usual non-linear Schrödinger equation, the case $\gamma > 0$ (or $\vartheta = 1$) is called *defocusing,* and the case $\gamma < 0$ (or $\vartheta = -1$) is called *focusing.*

Equations (BNLS) and (1.1) are Hamiltonian. Two important quantities are conserved by the flow associated to (1.1): the "mass" $\|\psi(t, \cdot)\|_{L^2}^2$, and the "energy"

$$E(\psi) = \int_{\mathbf{R}^N} |\Delta \psi|^2 \, dx - 2\epsilon \int_{\mathbf{R}^N} |\nabla \psi|^2 \, dx + \frac{\vartheta}{\sigma + 1} \int_{\mathbf{R}^N} |\psi|^{2\sigma+2} \, dx.$$

The natural "energy space" associated to (1.1) is $H^2(\mathbf{R}^N)$. Equation (1.1) is *mass-critical* for $\sigma = \frac{4}{N}$, and *energy-critical* when $N \geqslant 5$ and $\sigma = \frac{4}{N-4}$ (this corresponds to $2\sigma + 2 = 2^{**}$, where $2^{**} = \frac{2N}{N-4}$ is the Sobolev exponent satisfying $\|u\|_{L^{2^{**}}} \leqslant C\|\Delta u\|_{L^2}$ for any $u \in H^2(\mathbf{R}^N)$).

The Cauchy problem for (1.1) has been considered in several articles; see [22] and references therein. In the energy-subcritical case (that is, $N \leqslant 4$ and $\sigma \in (0, \infty)$, or $N \geqslant 5$ and $0 < \sigma < \frac{4}{N-4}$), B. Pausader proved local existence in $H^2(\mathbf{R}^N)$ as well as the conservation of mass and energy in all cases (see Proposition 4.1 p. 204 in [22]).

In the defocusing case ($\vartheta = 1$) B. Pausader also proved global existence for any $\epsilon \in \{-1, 0, 1\}$ and all initial data (Corollary 4.1 (a) p. 205 in [22]), and scattering provided that $\epsilon \leqslant 0$, $N \geqslant 5$ and $\frac{4}{N} < \sigma < \frac{4}{N-4}$. In low dimensions $1 \leqslant N \leqslant 4$, scattering has been proved in [23] (see Theorem 1.1 p. 2177 in [23]) provided that $\epsilon \in \{-1, 0\}$ and $\sigma > \frac{4}{N}$. The latter condition can be weakened to $\sigma > \frac{2}{N}$ if $\epsilon = -1$; this is due to the fact that Strichartz estimates are better for $\epsilon = -1$.

In the focusing case ($\vartheta = -1$), global existence holds provided that $\sigma$ is energy-subcritical and the initial data is sufficiently small in $H^2(\mathbf{R}^N)$, or $\sigma < \frac{4}{N}$ and the initial data is arbitrary, or $\sigma = \frac{4}{N}$ and the initial data is sufficiently small in $L^2(\mathbf{R}^N)$ (Corollary 4.1 (b)-(d) p. 205 in [22]). Global existence in the critical case in also shown for radial initial data (Theorem 1.1 p. 198 in [22]) and for arbitrary small data, as well as scattering for radial data if $\epsilon \in \{-1, 0\}$.

Equation (1.1) admits an important class of special solutions, the *standing waves*. These are solutions of the form $\psi(t, x) = e^{-i\omega t}u(x)$, where $\omega \in \mathbf{R}$ and $u$ is a complex-valued function. They appear as a balance between non-linearity and dispersion and are supposed to play an important role in the dynamics. The standing wave profile $u$ satisfies the equation

$$\Delta^2 u + 2\epsilon\Delta u + \omega u + \vartheta|u|^{2\sigma}u = 0 \qquad \text{in } \mathbf{R}^N. \tag{1.2}$$

Solutions of (1.2) are critical points of the *action*

$$\mathscr{S}_\omega(u) := \int_{\mathbf{R}^N} |\Delta u|^2 - 2\epsilon|\nabla u|^2 + \omega|u|^2 + \frac{\vartheta}{\sigma + 1}|\psi|^{2\sigma+2}\,dx = E(u) + \omega\|u\|^2_{L^2}. \tag{1.3}$$

Taking into account the Hamiltonian structure of (1.1), it is natural to search for standing waves as minimizers (or local minimizers) of the energy when the $L^2-$norm is kept fixed. By a standard application of the approach laid down by T. Cazenave and P.-L. Lions [9], the set of solutions obtained in this way is orbitally stable. Studying the behaviour of the energy with respect to the mass and to the scaling gives an insight into possible blow-up scenarios. We refer to [6] and [3] for blow-up results.

In the case $\epsilon = -1$, the existence of standing waves has been investigated in several papers (see [2], [3], [4], [5]) by using various methods, including minimisation of the energy at fixed mass (see Theorem 1.1 p. 3050 in [4]). Some qualitative properties of these solutions as well as the orbital stability of the set of minimizers have also been established.

In the case $\epsilon = -1$, it has been observed in [5], Theorem 1.1 and in [4], Theorem 1.2 that it is possible to minimize $\int_{\mathbf{R}^N} |\Delta u|^2 - 2|\nabla u|^2 + \omega|u|^2\,dx$ under the constraint $\int_{\mathbf{R}^N} |u|^{2\sigma+2}\,dx = 1$ provided that $\omega > 1$. Although this approach gives the existence of standing waves, it is not completely satisfactory because the considered quantities are not conserved by the flow of (1.1), and consequently it does not give much information about the dynamics of (1.1).

The case $\epsilon = 1$ (corresponding to $\beta > 0$ in (BNLS)) is more difficult and, as far as we know, there are no satisfactory results in the literature concerning the minimisation of the energy at fixed $L^2-$norm. Our aim is to clarify this situation in the focusing case ($\vartheta = -1$ in (1.1) or $\gamma < 0$ in (BNLS)). In the sequel we will always assume that $\epsilon = 1$, although most of our results are still valid if $\epsilon = 0$ or if $\epsilon = -1$. Rewriting our proofs with $\epsilon = -1$ would give alternate proofs of some results in [2], [3], [4], [5].

To be more precise, we focus our attention on the minimisation problem

$$(\mathcal{P}_m) \qquad \begin{gathered} \text{minimize } E(u) := \int_{\mathbf{R}^N} |\Delta u|^2\,dx - 2\int_{\mathbf{R}^N} |\nabla u|^2\,dx - \frac{1}{\sigma+1}\int_{\mathbf{R}^N} |u|^{2\sigma+2}\,dx \\ \text{in the set } S(m) := \left\{ u \in H^2(\mathbf{R}^N) \;\middle|\; \int_{\mathbf{R}^N} |u|^2\,dx = m \right\}. \end{gathered}$$

We denote

$$E_{min}(m) := \inf\{E(u) \mid u \in S(m)\}. \tag{1.4}$$

The basic properties of the function $E_{min}$ are given in Proposition 3.1. In particular, we show that $E_{min}(m)$ is finite for any $m > 0$ if $N\sigma < 4$, and $E_{min}(m) = -\infty$ for any $m > 0$ if

$N\sigma > 4$ (of course, this is related to the fact that (1.1) is mass-critical for $\sigma = \frac{4}{N}$). If $N\sigma = 4$, there exists some $k_* > 0$ such that $E_{min}(m)$ is finite for $m \in (0, k_*)$ and $E_{min}(m) = -\infty$ if $k \geqslant k_*$. A simple scaling argument shows that we have always $E_{min}(m) \leqslant -m$. If $E_{min}(m) = -m$, the minimisation problem $(\mathcal{P}_m)$ does not have solutions, and all minimizing sequences converge weakly to zero. If $E_{min}(m) < -m$, it is shown in Theorem 3.4 that there exist minimizers for $(\mathcal{P}_m)$ and that all minimizing sequences are pre-compact (after translation), which gives the orbital stability of the set of minimizers by the flow associated to (1.1). If $0 < \sigma \leqslant \frac{4}{N}$, there exists $m_0 \geqslant 0$ such that $E_{min}(m) = -m$ for $m \in (0, m_0]$ and $E_{min}(m) < -m$ for $m > m_0$. It is an important question whether $m_0 = 0$ or $m_0 > 0$. Notice that the presence of standing waves prevents scattering for (1.1). Therefore, if $m_0 = 0$ we cannot expect a scattering theory for solutions of (1.1) having small $L^2-$norm.

It is easily seen that for any $u \in H^2(\mathbf{R}^N)$ with $\|u\|_{L^2}^2 = m$ we have

$$\begin{aligned} E(u) + \|u\|_{L^2}^2 &= \|\Delta u + u\|_{L^2}^2 - \tfrac{1}{\sigma+1}\|u\|_{L^{2\sigma+2}}^{2\sigma+2} = \|\Delta u + u\|_{L^2}^2 \left(1 - \tfrac{1}{\sigma+1} \tfrac{\|u\|_{L^{2\sigma+2}}^{2\sigma+2}}{\|\Delta u + u\|_{L^2}^2}\right) \\ &= \|\Delta u + u\|_{L^2}^2 \left(1 - \tfrac{m^\sigma}{\sigma+1} \tfrac{\|u\|_{L^{2\sigma+2}}^{2\sigma+2}}{\|u\|_{L^2}^{2\sigma}\|\Delta u + u\|_{L^2}^2}\right) = \|\Delta u + u\|_{L^2}^2 \left(1 - \tfrac{m^\sigma}{\sigma+1}\mathcal{Q}(u)^{2\sigma+2}\right), \end{aligned} \tag{1.5}$$

where

$$\mathcal{Q}(u) = \frac{\|u\|_{L^{2\sigma+2}}}{\|u\|_{L^2}^{\frac{\sigma}{\sigma+1}} \|\Delta u + u\|_{L^2}^{\frac{1}{\sigma+1}}}.$$

Let $M = \sup\{\mathcal{Q}(u) \mid H^2(\mathbf{R}^N), u \neq 0\}$. If $M$ is finite, it follows from (1.5) that $E(u) + \|u\|_{L^2}^2 \geqslant 0$ for any $u$ satisfying $\|u\|_{L^2}^2 = m$ provided that $m$ is small enough, so that $1 - \frac{m^\sigma}{\sigma+1}M^{2\sigma+2} \geqslant 0$. This shows that $E_{min}(m) \geqslant -m$ for sufficiently small $m$, and consequently $(\mathcal{P}_m)$ does not admit minimizers for small $m$. If $M = \infty$, then for any $m > 0$ we may find $u \in H^2(\mathbf{R}^N)$, $u \neq 0$ such that $1 - \frac{m^\sigma}{\sigma+1}\mathcal{Q}(u)^{2\sigma+2} < 0$. Then taking $v = \frac{\sqrt{m}}{\|u\|_{L^2}}u$ we see that $\|v\|_{L^2} = m$, $Q(v) = Q(u)$, and (1.5) gives $E(v) + \|v\|_{L^2}^2 < 0$, which implies $E_{min}(m) < -m$. Notice that $M$ is finite if and only if the Gagliardo-Nirenberg-type inequality

$$\|u\|_{L^{2\sigma+2}} \leqslant C\|u\|_{L^2}^{\frac{\sigma}{\sigma+1}} \|\Delta u + u\|_{L^2}^{\frac{1}{\sigma+1}}$$

holds true for all $u \in H^2(\mathbf{R}^N)$. Obviously, $M$ is the best possible constant in this inequality.

We will study slightly more general inequalities, namely we will investigate whether there exists $C > 0$ such that

$$\|u\|_{L^p} \leqslant C\|u\|_{L^2}^\kappa \| \, |D|^s u - u\|_{L^2}^{1-\kappa} \qquad \text{for all } u \in H^s(\mathbf{R}^N),$$

where $p \in (2, \infty)$, $\kappa \in (0,1)$ and $|D|^s$ is the Fourier integral operator given by $|D|^s u = \mathcal{F}^{-1}(|\cdot|^s \mathcal{F}(u))$. This leads us to study the boundedness on $H^s(\mathbf{R}^N) \setminus \{0\}$ of the quotient

$$Q_\kappa(u) = \frac{\|u\|_{L^p}}{\|u\|_{L^2}^\kappa \| \left(|D|^s - 1\right) u\|_{L^2}^{1-\kappa}}.$$

We obtain the following result.

**Theorem 1.1** *Let $N \in \mathbf{N}^*$, $p \in (2, \infty)$, $\kappa \in (0,1)$, and $s > 0$. Then $Q_\kappa$ is bounded on $H^s(\mathbf{R}^N) \setminus \{0\}$ if and only if*

$$\kappa \geqslant \frac{1}{2} \quad \text{and} \quad \frac{N}{s}\left(\frac{1}{2} - \frac{1}{p}\right) \leqslant 1 - \kappa \leqslant \frac{N+1}{2}\left(\frac{1}{2} - \frac{1}{p}\right). \tag{1.6}$$

To prove Theorem 1.1 we are led to develop an original approach, based on the Hausdorff-Young inequality in space dimension $N = 1$, and on the Tomas-Stein inequality in higher dimensions. This method is not limited to the study of $Q_\kappa$ here above. It is much more general and can be used to prove non-homogeneous Gagliardo-Nirenberg inequalities of the form

$$\|u\|_{L^p} \leqslant C\|P_1(D)u\|_{L^2}^\kappa \|P_2(D)u\|_{L^2}^{1-\kappa},$$

where $P_1(D)$ and $P_2(D)$ are Fourier integral operators defined by $P_i(D)(u) = \mathcal{F}^{-1}(P_i(\cdot)\mathcal{F}(u))$. See Remark 2.8. Some quantitative variants are also available: see Remark 2.9.

Using Theorem 1.1 with $s = 2$, $p = 2\sigma + 2$, and $\kappa = \frac{\sigma}{\sigma+1}$ we infer that the quotient $\mathcal{Q}$ in (1.5) is bounded on $H^2(\mathbf{R}^N) \setminus \{0\}$ if and only if $\max\left(1, \frac{4}{N+1}\right) \leqslant \sigma \leqslant \frac{4}{N}$ (see Proposition 3.3).

As a matter of fact, our method works in the simpler case when $\epsilon = -1$ in (1.2). Proceeding as in (1.5) we write $E(u) = \left(\|\Delta u\|_{L^2}^2 + 2\|\nabla u\|_{L^2}^2\right)\left(1 - \frac{\|u\|_{L^2}^{2\sigma}}{\sigma+1}\mathcal{Q}(u)^{2\sigma+2}\right)$, where $\mathcal{Q}(u) = \frac{\|u\|_{L^{2\sigma+2}}}{\|u\|_{L^2}^{\frac{\sigma}{\sigma+1}}\left(\|\Delta u\|_{L^2}^2 + 2\|\nabla u\|_{L^2}^2\right)^{\frac{1}{\sigma+1}}}$, and we need to study the boundedness of the quotient $\mathcal{Q}$ to decide whether $E_{min}(m) < 0$ for all $m > 0$ or $E_{min}(m) = 0$ for small $m$. In this way it is possible to give an alternate (and shorter) proof of Theorem 1.1 p. 5030 in [4].

Having at hand Theorem 1.1, we establish the existence of solutions to the problem $(\mathcal{P}_m)$ under optimal assumptions. We use some ideas in [18] and [19], but all our proofs are self-contained and elementary. The next Theorem summarizes our main results on the existence of minimizers for $(\mathcal{P}_m)$.

**Theorem 1.2** *Let $N \in \mathbf{N}^*$. Let $E_{min}$ be as in (1.4). The following assertions hold true.*

*(i) If $0 < \sigma < \max\left(1, \frac{4}{N+1}\right)$ and $\sigma < \frac{4}{N}$ we have $-\infty < E_{min}(m) < -m$ for all $m > 0$.*

*(ii) If $\max\left(1, \frac{4}{N+1}\right) \leqslant \sigma < \frac{4}{N}$, there exists $m_0 > 0$ (given by (3.16)) such that $E_{min}(m) = -m$ for all $m \in (0, m_0]$ and $-\infty < E_{min}(m) < -m$ for any $m > m_0$.*

*(iii) If $\sigma = \frac{4}{N}$, let $m_0 = 0$ if $\sigma < 1$ and let $m_0$ be as in (3.16) if $\sigma \geqslant 1$. Let $k_*$ be as in Proposition 3.1 (vi). Then we have $m_0 < k_*$ and $E_{min}(m) = -m$ for all $m \in (0, m_0]$, $-\infty < E_{min}(m) < -m$ for $m \in (m_0, k_*)$ and $E_{min}(m) = -\infty$ for $m \geqslant k_*$.*

*(iv) If $\sigma > \frac{4}{N}$ we have $E_{min}(m) = -\infty$ for all $m > 0$.*

*Problem $(\mathcal{P}_m)$ admits solutions whenever $-\infty < E_{min}(m) < -m$; moreover, any minimizing sequence for $(\mathcal{P}_m)$ has a subsequence that converges strongly in $H^2(\mathbf{R}^N)$ modulo translations. Minimizers of $(\mathcal{P}_m)$ solve (1.2) for some $\omega > 1$.*

*Problem $(\mathcal{P}_m)$ does not admit minimizers if $m_0 > 0$ and $m \in (0, m_0)$.*

If $\epsilon = 1$ and $\vartheta = -1$, as we assume throughout this paper, writing $\omega = 1 + c$ equation (1.2) becomes

$$\Delta^2 u + 2\Delta u + (1+c)u - |u|^{2\sigma}u = 0 \qquad \text{in } \mathbf{R}^N. \tag{1.7}$$

As already mentioned, solutions of (1.7) are critical points of the *action* functional

$$\begin{aligned} S_c(u) &:= \int_{\mathbf{R}^N} |\Delta u|^2 - 2|\nabla u|^2 + (1+c)|u|^2 - \frac{1}{\sigma+1}|\psi|^{2\sigma+2}\,dx \\ &= E(u) + (1+c)\|u\|_{L^2}^2 = T_c(u) - \frac{1}{\sigma+1}\int_{\mathbf{R}^N}|\psi|^{2\sigma+2}\,dx, \end{aligned} \tag{1.8}$$

where $T_c(u) := \int_{\mathbf{R}^N} |\Delta u|^2 - 2|\nabla u|^2 + (1+c)|u|^2\,dx$. A classical approach to find solutions for (1.8) is to show that $t(c) := \inf\{T_c(u) \mid u \in H^2(\mathbf{R}^N),\ \int_{\mathbf{R}^N}|u|^{2\sigma+2}\,dx = 1\}$ is achieved. In Theorem 3.7 we prove that minimizers for $t(c)$ exist for any $c > 0$ and any $\sigma \in (0, \infty)$ if $N \leqslant 4$, respectively any $\sigma \in (0, \frac{4}{N-4})$ if $N \geqslant 5$. Moreover, if $u$ is a minimizer for $t(c)$ then $v := t(c)^{\frac{1}{2\sigma}}u$ solves (1.7) and for any other solution $w \in H^2(\mathbf{R}^N)$ of (1.7) we have $S_c(v) \leqslant S_c(w)$ (see Proposition 3.9); we say that $v$ is a *minimum action solution* of (1.7). Therefore equation (1.7) admits minimum action solutions for any energy-subcritical $\sigma$ and for any $c > 0$. The next result shows that minimizers given by Theorem 1.2 are minimum action solutions for (1.7):

**Theorem 1.3** *Assume that $0 < \sigma \leqslant \frac{4}{N}$. Let $u$ be a minimizer for problem $(\mathcal{P}_m)$, as given by Theorem 1.2. The following properties hold true:*

*(i) There exists some $c = c(u) > 0$ such that $u$ is a minimum action solution for (1.7). Furthermore, any minimum action solution of (1.7) with $c = c(u)$ is also a minimizer for $(\mathcal{P}_m)$.*

*(ii) If $m_1 < m_2$, the function $u_1$ solves $(\mathcal{P}_{m_1})$ and $u_2$ solves $(\mathcal{P}_{m_2})$, then $c(u_1) < c(u_2)$.*

*(iii) If $0 < \sigma < \frac{4}{N}$, we have $c(u) \longrightarrow \infty$ as $m \longrightarrow \infty$.*

*(iv) If $0 < \sigma < \max\left(1, \frac{4}{N+1}\right)$ and $\sigma \leqslant \frac{4}{N}$ we have $c(u) \longrightarrow 0$ as $m \longrightarrow 0$. If $u_m$ is any solution of the minimisation problem $(\mathcal{P}_m)$, denote $v_m = \frac{v_m}{\sqrt{m}} = \frac{v_m}{\|u_m\|_{L^2}}$, so that $\|v_m\|_{L^2} = 1$. Then we have*

$$\|\Delta v_m\|_{L^2} \longrightarrow 1, \quad \|\nabla v_m\|_{L^2} \longrightarrow 1, \quad \|(\Delta + 1)v_m\|_{L^2} \longrightarrow 0 \qquad \textit{as } m \longrightarrow 0,$$

*and $\|v_m\|_{L^p} \longrightarrow 0$ for any $p \in (2, \infty)$ if $N \geqslant 4$, respectively for any $p \in (2, 2^{**})$ if $N \geq 5$.*

It is proven in Proposition 3.8 that $t(c) \leqslant C\sqrt{c}$ as $c \longrightarrow 0$ and Corollary 3.13 below shows that for any energy-subcritical $\sigma > 0$, we have $t(c) \sim c^{1-\frac{N\sigma}{4(\sigma+1)}}$ as $c \longrightarrow \infty$. The behaviour of minimum energy solutions of (1.7) (and, in particular, the behaviour of minimizers for the problem $(\mathcal{P}_m)$ as $m \longrightarrow \infty$ in the case $0 < \sigma < \frac{4}{N}$) is described in Proposition 3.12 and Corollary 3.13: after rescaling and translation, they converge to minimizers of the functional $K(u) := \int_{\mathbf{R}^N} |\Delta u|^2 + |u|^2 \, dx$ under the constraint $\int_{\mathbf{R}^N} |u|^{2\sigma+2} \, dx = 1$.

In the case $\sigma > \frac{N}{4}$ we have $E_{min}(m) = -\infty$ for any $m > 0$ and the minimization problem $(\mathcal{P}_m)$ does not make sense. In this case we investigate the existence of *local* minimizers of $E$ when the $L^2-$norm is kept fixed. By *local minimizer* we mean a function $u \in H^2(\mathbf{R}^N)$ such that there exists an open set $\mathcal{U} \subset H^2(\mathbf{R}^N)$ having the property that $u \in \mathcal{U}$ and $E(u) = \inf\{E(v) \mid v \in \mathcal{U} \text{ and } \|v\|_{L^2} = \|u\|_{L^2}\}$.

We find an open set $\mathcal{O} \subset H^2(\mathbf{R}^N)$ (described in (4.4)) such that any possible local minimizer of $E$ at fixed $L^2-$norm must belong to $\mathcal{O}$. The set $\mathcal{O} \cup \{0\}$ is star-shaped and is an open neighbourhood of the origin in $H^2(\mathbf{R}^N)$, and $\mathcal{O}$ is unbounded in $H^2(\mathbf{R}^N)$. We denote

$$\tilde{E}_{min}(m) = \inf\{E(u) \mid u \in \mathcal{O} \text{ and } \|u\|_{L^2}^2 = m\}.$$

The problem of finding minimizers for $\tilde{E}_{min}(m)$ (which are "best possible" local minimizers of $E$ when the $L^2-$norm is fixed) in the mass-supercritical case $\sigma > \frac{4}{N}$ is much harder than finding global minimizers for $E_{min}$ in the subcritical and critical cases. To the best of our knowledge this problem has not been addressed in the literature. Understanding the behaviour of $E$ with respect to the $L^2-$norm is also an important step in understanding the dynamics associated to (1.1). Our main results in the case $\sigma > \frac{4}{N}$ are given below.

**Theorem 1.4** *Suppose that $\sigma > \frac{4}{N}$ and $\sigma < \infty$ if $N \leqslant 4$, respectively $\sigma < \frac{4}{N-4}$ if $N \geqslant 5$. The following assertions are true.*

*(i) Assume that $N \geqslant 5$ and $\frac{4}{N} < \sigma < 1$. Then we have $\tilde{E}_{min}(m) < -m$ for any $m > 0$.*

*(ii) If $\frac{4}{N} < \sigma$ and $\sigma \geqslant 1$, there exists $m_0 > 0$ such that $\tilde{E}_{min}(m) = -m$ for any $m \in (0, m_0]$ and the infimum $\tilde{E}_{min}(m)$ is not achieved for any $m \in (0, m_0)$.*

*(iii) Assume that $0 < m < \mu_0$, where $\mu_0$ is given by (4.6), and $\tilde{E}_{min}(m) < -m$. Then $\tilde{E}_{min}(m)$ is achieved and any minimizing sequence for $\tilde{E}_{min}(m)$ has a subsequence that converges strongly in $H^2(\mathbf{R}^N)$ modulo translations.*

*(iv) Any minimizer for $\tilde{E}_{min}(m)$ solves (1.8) for some $c = c(u)$ satisfying*

$$0 < c < -1 + \frac{(N\sigma - 2)^2}{N\sigma(N\sigma - 4)} + \frac{8(N\sigma - 2)}{N(N\sigma - 4)^2}.$$

*Moreover, if $N \geqslant 5$ and $\frac{4}{N} < \sigma < 1$ we have $c(u) \longrightarrow 0$ as $m \longrightarrow 0$.*

Any solution $u$ of (1.7) provided by Theorem 1.4 must satisfy (4.9), and consequently there is some explicit constant $C > 0$ such that $\|u\|_{H^2} \leqslant C\|u\|_{L^2}$. Therefore if $N \geqslant 5$ and $\frac{4}{N} < \sigma < 1$ equation (1.1) admits standing waves with small $H^2-$norm and this rules out a scattering theory for small solutions of (1.1). It is an open question whether small solutions of (1.1) scatter or not in the remaining cases.

In the case $\sigma > \frac{4}{N}$, the least energy solutions of (1.7) given by Proposition 3.9 have small $L^2-$norm as $c \longrightarrow \infty$, but they have large $H^2-$norm and do not belong to the set $\mathcal{O}$ (see Remark 4.11 and Corollary 3.13). Thus we have two types of interesting standing waves with small $L^2-$norm: the minimum action solutions for $c \longrightarrow \infty$, and the local minimizers provided by Theorem 1.4.

Let us compare our results to similar results in the cases $\epsilon = -1$ and $\epsilon = 0$. Let us consider the problem $(\mathcal{P}_m)$ with $E$ replaced by

$$E(u) = \int_{\mathbf{R}^N} |\Delta u|^2\, dx - 2\epsilon \int_{\mathbf{R}^N} |\nabla u|^2\, dx - \frac{1}{\sigma+1} \int_{\mathbf{R}^N} |u|^{2\sigma+2}\, dx. \tag{1.9}$$

We define $E_{min}$ as in (1.4). Then Theorem 1.1 p. 5030 in [4] and Theorem 1.2 p. 2170 in [2] give the following result:

**Theorem.** ([4, 2]) *Assume that $\epsilon = -1$. Then:*

*(i) If $0 < \sigma < \frac{2}{N}$, we have $-\infty < E_{min}(m) < 0$ for any $m > 0$.*

*(ii) If $\frac{2}{N} \leqslant \sigma < \frac{4}{N}$, there exists $m_{cr} > 0$, depending on $\sigma$ and $N$, such that $E_{min}(m) = 0$ for any $m \leqslant m_{cr}$ and $-\infty < E_{min}(m) < 0$ for any $m > m_{cr}$.*

*(iii) If $\sigma = \frac{4}{N}$, there exists $m_{cr} > 0$ such that $E_{min}(m) = 0$ for any $m \leqslant m_{cr}$ and $E_{min}(m) = -\infty$ for $m > m_{cr}$, and $E_{min}(m)$ is never achieved.*

*(iv) If $\sigma > \frac{4}{N}$ we have $E_{min}(m) = -\infty$ for all $m > 0$.*

*The problem $(\mathcal{P}_m)$ admits solutions whenever $-\infty < E_{min}(m) < 0$. Moreover, all minimizing sequences have subsequences that converge strongly in $H^2(\mathbf{R}^N)$ (modulo translations).*

Quite remarkably, Proposition 2.8 (ii) p. 5038 in [4] shows that for $\sigma \in \left(\frac{2}{N}, \frac{4}{N}\right)$ and for $m = m_{cr}$, problem $(\mathcal{P}_{m_{cr}})$ admits solutions despite the fact that there exist minimizing sequences that do not have any convergent subsequence (modulo translations). In fact, proceeding as in (1.5) it is easily seen that in this case one has

$$\frac{\sigma+1}{m_{cr}^{\sigma}} = \sup \left\{ \frac{\|u\|_{L^{2\sigma+2}}^{2\sigma+2}}{\|u\|_{L^2}^{2\sigma} \left(\|\Delta u\|_{L^2}^2 + 2\|\nabla u\|_{L^2}^2\right)} \;\Big|\; u \in H^2(\mathbf{R}^N) \setminus \{0\} \right\}$$

and that $u$ is a minimizer for $(\mathcal{P}_{m_{cr}})$ if and only if $\|u\|_{L^2}^2 = m_{cr}$ and $u$ is an optimal function for the non-homogeneous Gagliardo-Nirenberg inequality

$$\|v\|_{L^{2\sigma+2}} \leqslant C\|v\|_{L^2}^{\frac{\sigma}{\sigma+1}} \left(\|\Delta v\|_{L^2}^2 + 2\|\nabla v\|_{L^2}^2\right)^{\frac{1}{2\sigma+2}}.$$

We stress that local minimizers of the energy at fixed mass are specific to the case $\epsilon = 1$. Such solutions do *not* exist if $\epsilon \leqslant 0$, see Remark 4.2. If $\epsilon = -1$ it is shown in [2], Theorem 1.3 p. 2171 that one can minimize $E$ in the set $\{u \in H^2(\mathbf{R}^N) \mid \|u\|_{L^2}^2 = m \text{ and } P_1(u) = 0\}$ for some values of $m > 0$, where $P_1$ is a Pohozaev-type functional given in (3.39). The minimizers found in [2] are minimum action solutions of (1.2), but are not minimizers of $E$ at fixed $L^2-$norm. They correspond to solutions given by Theorem 3.7 and Proposition 3.9 below. The instability by blow-up of such minimum action solutions has been proven in [2], Theorem 1.1 provided that they are radial and $\frac{4}{N} \leqslant \sigma \leqslant 4$ (and $\sigma < \frac{4}{N-4}$ if $n \geqslant 5$), and that instability result is an indication that those solutions cannot be local minimizers of the energy at fixed $L^2-$norm.

The case $\epsilon = 0$ is much simpler. Proceeding as in (1.5) we find

$$E(u) = \|\Delta u\|_{L^2}^2 \left(1 - \frac{\|u\|_{L^2}^{2\sigma}}{\sigma+1} \frac{\|u\|_{L^{2\sigma+2}}^{2\sigma+2}}{\|u\|_{L^2}^{2\sigma}\|\Delta u\|_{L^2}^2}\right) = \|\Delta u\|_{L^2}^2 \left(1 - \frac{m^\sigma}{\sigma+1} \mathcal{Q}_0(u)^{2\sigma+2}\right) \tag{1.10}$$

for any $u \in H^2(\mathbf{R}^N)$ such that $\|u\|_{L^2}^2 = m$, where $\mathcal{Q}_0(u) = \frac{\|u\|_{L^{2\sigma+2}}}{\|u\|_{L^2}^{\frac{\sigma}{\sigma+1}}\|\Delta u\|_{L^2}^{\frac{1}{\sigma+1}}}$. A simple scaling argument shows that the quotient $\mathcal{Q}_0$ is unbounded on $H^2(\mathbf{R}^N)\setminus\{0\}$ if $\sigma \neq \frac{4}{N}$. With the above notation we get:

(i) $-\infty < E_{min}(m) < 0$ for any $m > 0$ if $0 < \sigma < \frac{4}{N}$.

(ii) If $\sigma = \frac{4}{N}$, there exists $m_{cr} > 0$ such that $E_{min}(m) = 0$ for any $m \leqslant m_{cr}$ and $E_{min}(m) = -\infty$ for $m > m_{cr}$. $E_{min}(m_{cr})$ is achieved by some optimal function for the Sobolev-Gagliardo-Nirenberg inequality (3.3), and $E_{min}(m)$ is never achieved if $m \neq m_{cr}$.

(iii) $E_{min}(m) = -\infty$ if $\sigma > \frac{4}{N}$.

Assertions (i) and (iii) are proven exactly as statements (v) and (i), respectively, in Proposition 3.1 below, and (ii) follows from (1.10). The existence of minimizers for any $m > 0$ in case (i) is standard (one may use a simplified version of the proof of Theorem 3.4).

If $\epsilon \leqslant 0$, equation (1.2) has infinitely many solutions that can be obtained by using topological methods (see, e.g., Theorem 1.4 p. 2172 in [2]). This is presumably true for $\epsilon = 1$, too. In this paper we focus on standing waves that minimize the energy at fixed mass, which are the most important for the dynamical study of (1.1).

This paper is organized as follows. In the next section we develop a method to deal with non-homogeneous Gagliardo-Nirenberg inequalities and we prove Theorem 1.1. We separate the cases $N = 1$ (when a simple argument based on the Hausdorff-Young inequality is sufficient, see Theorem 2.3) and $N \geqslant 2$ (when a more involved argument relying on the Tomas-Stein inequality is needed, see Theorem 2.6). Examples 2.4 and 2.7 show that the results we obtain are optimal for the operator $|D|^s - 1$. Extensions to more general operators are indicated in Remarks 2.8 and 2.9.

In Section 3 we consider the problem $(\mathcal{P}_m)$ and we prove Theorems 1.2 and 1.3. Statements (i)-(iv) in Theorem 1.2 follow from Propositions 3.1 and 3.3. The existence of minimizers and the pre-compactness of minimizing sequences are given by Theorem 3.4. Theorem 1.3 follows from Propositions 3.5 and 3.10 (see also Remark 3.2 and Proposition 3.3). Some asymptotic properties of minimum action solutions as $c \longrightarrow \infty$ are given in Proposition 3.12 and Corollary 3.13.

In Section 4 we consider the more delicate problem of minimizing the energy at fixed $L^2-$norm in the set $\mathcal{O}$ when $\sigma > \frac{4}{N}$ and we prove Theorem 1.4. Statement (i) and the first part of (ii) in Theorem 1.4 follow from Lemma 4.4, and the second assertion in (ii) follows from Remark 4.9. Part (iii) is Theorem 4.8. For (iv), see Remark 4.10.

# 2 A class of non-homogeneous Gagliardo-Nirenberg inequalities

Let $(X, \mathcal{A}, \mu)$ be an arbitrary measure space and let $f$ be a complex-valued measurable function on $X$. We say that $f \not\equiv 0$ if the set $\{x \in X \mid f(x) \neq 0\}$ has positive measure. Obviously, if $f \not\equiv 0$ then $\int_X |f|^\alpha \, d\mu > 0$ for any $\alpha > 0$. If $f_1, f_2$ are measurable and $f_1 f_2 \not\equiv 0$, then $f_1 \not\equiv 0$ and $f_2 \not\equiv 0$. We use the convention $\frac{f_1(x)}{f_2(x)} = 0$ if $f_1(x) = f_2(x) = 0$.

The following elementary lemma will be very useful in the sequel.

**Lemma 2.1** *Let $(X, \mathcal{A}, \mu)$ be an arbitrary measure space, let $q \in [1, 2)$ and $\kappa \in (0, 1)$. Consider three measurable functions $w, w_1, w_2 : X \longrightarrow \mathbf{C}$ such that $ww_1 \not\equiv 0$, $ww_2 \not\equiv 0$ and $w = 0$ on the set $\{x \in X \mid w_1(x) = 0$ and $w_2(x) = 0\}$. Let*

$$M_1 = \sup \left\{ \frac{\|\varphi w\|_{L^q}}{\|\varphi w_1\|_{L^2}^{\kappa} \cdot \|\varphi w_2\|_{L^2}^{1-\kappa}} \;\middle|\; \begin{array}{l} \varphi \text{ is measurable, } \varphi w_i \in L^2(X), \\ \varphi w_i \not\equiv 0 \text{ for } i = 1, 2 \end{array} \right\}, \text{ and}$$

$$M_2 = \sup_{t>0} \left( t^{\frac{1-\kappa}{2}} \left\| \frac{w}{(w_1^2 + t|w_2|^2)^{\frac{1}{2}}} \right\|_{L^{\frac{2q}{2-q}}} \right).$$

*Then $M_1 \leqslant (1-\kappa)^{\frac{\kappa-1}{2}} \kappa^{-\frac{\kappa}{2}} M_2$. Moreover, if $(X, \mathcal{A}, \mu)$ is $\sigma-$finite or if there exists $t_* > 0$ such that $w\left(w_1^2 + t_*|w_2|^2\right)^{-\frac{1}{2}} \in L^{\frac{2q}{2-q}}(X)$, then $M_1 = (1-\kappa)^{\frac{\kappa-1}{2}} \kappa^{-\frac{\kappa}{2}} M_2$.*

**Remark 2.2**

(i) Let $A_i = \{x \in X \mid w_i(x) = 0\}$ for $i = 1, 2$. We have $\left\| \frac{w}{\left(w_1^2 + t|w_2|^2\right)^{\frac{1}{2}}} \right\|_{L^{\frac{2q}{2-q}}(X)} \geqslant \left\| \frac{w}{|w_1|} \right\|_{L^{\frac{2q}{2-q}}(A_2)}$, and $\left\| \frac{w}{\left(w_1^2 + t|w_2|^2\right)^{\frac{1}{2}}} \right\|_{L^{\frac{2q}{2-q}}(X)} \geqslant t^{-\frac{1}{2}} \left\| \frac{w}{|w_2|} \right\|_{L^{\frac{2q}{2-q}}(A_1)}$. We infer that if $M_2 < \infty$, then necessarily $w = 0$ a.e. on $A_1 \cup A_2$ (here we use the assumption that $w = 0$ on $A_1 \cap A_2$ and the convention $\frac{0}{0} = 0$).

(ii) For any fixed $a, b > 0$, the function $g(t) = \frac{t^{1-\kappa}}{a+tb} = \frac{1}{at^{\kappa-1} + bt^{\kappa}}$ achieves its maximum on $(0, \infty)$ at $t_{max} = \frac{(1-\kappa)a}{\kappa b}$ and $g(t_{max}) = \kappa^{\kappa}(1-\kappa)^{1-\kappa} a^{-\kappa} b^{\kappa-1}$. Hence for any $x \in X$ such that $w_1(x) w_2(x) \neq 0$ we have $\max\limits_{t>0} \left[ t^{\frac{1-\kappa}{2}} \left( |w_1(x)|^2 + t|w_2(x)|^2 \right)^{-\frac{1}{2}} \right] = \kappa^{\frac{\kappa}{2}} (1-\kappa)^{\frac{1-\kappa}{2}} |w_1(x)|^{-\kappa} |w_2(x)|^{\kappa-1}$. We have thus a sufficient condition for the finiteness of $M_2$, namely $M_2 < \infty$ if $w = 0$ a.e. on the set $A_1 \cup A_2$ and $w|w_1|^{-\kappa}|w_2|^{\kappa-1} \in L^{\frac{2q}{2-q}}(X)$.

(iii) Assume that there is $t_* > 0$ such that $w\left(|w_1|^2 + t_*|w_2|^2\right)^{-\frac{1}{2}} \in L^{\frac{2q}{2-q}}(X)$. We have

$$\min\left(1, \frac{t}{t_*}\right) \leqslant \frac{|w_1|^2 + t|w_2|^2}{|w_1|^2 + t_*|w_2|^2} \leqslant \max\left(1, \frac{t}{t_*}\right) \qquad \text{whenever } (w_1, w_2) \neq (0,0). \tag{2.1}$$

Since $w = 0$ if $(w_1, w_2) = (0, 0)$, we infer that $w\left(|w_1|^2 + t|w_2|^2\right)^{-\frac{1}{2}} \in L^{\frac{2q}{2-q}}(X)$ for any $t > 0$. Now let

$$F(t) = \left\| \frac{t^{\frac{1-\kappa}{2}} w}{(w_1^2 + t|w_2|^2)^{\frac{1}{2}}} \right\|_{L^{\frac{2q}{2-q}}}^{\frac{2q}{2-q}} = \int_X \frac{|w(x)|^{\frac{2q}{2-q}}}{(t^{\kappa-1}|w_1(x)|^2 + t^{\kappa}|w_2(x)|^2)^{\frac{q}{2-q}}}\, d\mu.$$

Clearly, $M_2$ is finite if and only if $F$ is bounded from above. For any $x \in X$, the mapping

$$t \longmapsto \frac{|w(x)|^{\frac{2q}{2-q}}}{(t^{\kappa-1}|w_1(x)|^2 + t^{\kappa}|w_2(x)|^2)^{\frac{q}{2-q}}}$$

is continuous on $(0, \infty)$. Then the estimates (2.1) and the dominated convergence theorem imply that $F$ is continuous on $(a, b)$ for any $0 < a < t_* < b$, hence $F$ is continuous on $(0, \infty)$. In order to show that $M_2$ is finite, we only have to prove that $F$ is bounded in a neighbourhood of zero and in a neighbourhood of infinity.

(iv) If $(X, \mathcal{A}, \mu)$ is $\sigma-$finite and $M_1 < \infty$, the second part of Lemma 2.1 implies that we must have $w(|w_1|^2 + t|w_2|^2)^{-\frac{1}{2}} \in L^{\frac{2q}{2-q}}(X)$ for all $t > 0$.

*Proof of Lemma 2.1.* First notice that for any given $A, B > 0$, the function $f(t) = At^{\kappa-1} + Bt^{\kappa}$ achieves its minimum on $(0, \infty)$ at $t_{min} = \frac{(1-\kappa)A}{\kappa B}$ and $f(t_{min}) = (1-\kappa)^{\kappa-1}\kappa^{-\kappa} A^{\kappa} B^{1-\kappa}$.

Let $\varphi$ be a measurable function such that $\varphi w_i \in L^2(X)$ and $\varphi w_i \not\equiv 0$ for $i = 1, 2$. Using the previous observation with $A = \|\varphi w_1\|_{L^2}^2$, $B = \|\varphi w_2\|_{L^2}^2$ and $t_{min} = \frac{(1-\kappa)\|\varphi w_1\|_{L^2}^2}{\kappa \|\varphi w_2\|_{L^2}^2}$ we get

$$\begin{aligned}
&\|\varphi w_1\|_{L^2}^{2\kappa} \cdot \|\varphi w_2\|_{L^2}^{2(1-\kappa)} = (1-\kappa)^{1-\kappa}\kappa^{\kappa} \left( t_{min}^{\kappa-1} \|\varphi w_1\|_{L^2}^2 + t_{min}^{\kappa} \|\varphi w_2\|_{L^2}^2 \right) \\
&= (1-\kappa)^{1-\kappa}\kappa^{\kappa} t_{min}^{\kappa-1} \int_X |\varphi|^2 \left( |w_1|^2 + t_{min}|w_2|^2 \right) d\mu \\
&= (1-\kappa)^{1-\kappa}\kappa^{\kappa} t_{min}^{\kappa-1} \left\| \varphi \left( |w_1|^2 + t_{min}|w_2|^2 \right)^{\frac{1}{2}} \right\|_{L^2}^2.
\end{aligned} \tag{2.2}$$

Hölder's inequality implies that for any two measurable functions $f, g$ defined on $X$ there holds

$$\|fg\|_{L^q} \leqslant \|f\|_{L^2} \|g\|_{L^{\frac{2q}{2-q}}}. \tag{2.3}$$

Using (2.3) with $f = \varphi\left(|w_1|^2 + t_{min}|w_2|^2\right)^{\frac{1}{2}}$ and $g = w\left(|w_1|^2 + t_{min}|w_2|^2\right)^{-\frac{1}{2}}$ we obtain

$$\|\varphi w\|_{L^q} \leqslant \left\|\varphi \left(|w_1|^2 + t_{min}|w_2|^2\right)^{\frac{1}{2}}\right\|_{L^2} \cdot \left\| w\left(|w_1|^2 + t_{min}|w_2|^2\right)^{-\frac{1}{2}} \right\|_{L^{\frac{2q}{2-q}}}. \tag{2.4}$$

From (2.2) and (2.4) we get

$$\frac{\|\varphi w\|_{L^q}}{\|\varphi w_1\|_{L^2}^{\kappa}\cdot\|\varphi w_2\|_{L^2}^{1-\kappa}} \leqslant (1-\kappa)^{\frac{\kappa-1}{2}}\kappa^{-\frac{\kappa}{2}} t_{min}^{\frac{1-\kappa}{2}} \left\| w\left(|w_1|^2+t_{min}|w_2|^2\right)^{-\frac{1}{2}}\right\|_{L^{\frac{2q}{2-q}}}$$
$$\leqslant (1-\kappa)^{\frac{\kappa-1}{2}}\kappa^{-\frac{\kappa}{2}} M_2.$$

Since the above chain of inequalities holds for any measurable function $\varphi$ such that $\varphi w_i \in L^2(X)$ and $\varphi w_i \not\equiv 0$ for $i=1,\,2$, taking the supremum we get $M_1 \leqslant (1-\kappa)^{\frac{\kappa-1}{2}}\kappa^{-\frac{\kappa}{2}} M_2$.

Next, assume that there is $t_*>0$ such that $w\left(|w_1|^2+t_*|w_2|^2\right)^{-\frac{1}{2}} \in L^{\frac{2q}{2-q}}(X)$. By Remark 2.2 (iii) we have $w\left(|w_1|^2+t|w_2|^2\right)^{-\frac{1}{2}} \in L^{\frac{2q}{2-q}}(X)$ for any $t>0$.

If a measurable function $\varphi$ satisfies $\varphi w_i \in L^2(X)$ and $\varphi w_i \not\equiv 0$ for $i=1,\,2$, it is obvious that (2.2) holds if we replace $t_{min}$ by any $t>0$ and the first "=" by "$\leqslant$." In other words, we have

$$\|\varphi w_1\|_{L^2}^{\kappa}\cdot\|\varphi w_2\|_{L^2}^{1-\kappa} \leqslant (1-\kappa)^{\frac{1-\kappa}{2}}\kappa^{\frac{\kappa}{2}} t^{\frac{\kappa-1}{2}} \left\|\varphi\left(|w_1|^2+t|w_2|^2\right)^{\frac{1}{2}}\right\|_{L^2} \qquad \text{for any } t>0$$

and consequently

$$\frac{\|\varphi w\|_{L^q}}{\|\varphi w_1\|_{L^2}^{\kappa}\cdot\|\varphi w_2\|_{L^2}^{1-\kappa}} \geqslant (1-\kappa)^{\frac{\kappa-1}{2}}\kappa^{-\frac{\kappa}{2}} t^{\frac{1-\kappa}{2}} \frac{\|\varphi w\|_{L^q}}{\left\|\varphi\left(|w_1|^2+t|w_2|^2\right)^{\frac{1}{2}}\right\|_{L^2}} \qquad \text{for all } t>0. \tag{2.5}$$

For any $z\in\mathbf{C}$ we denote $\mathrm{sgn}(z)=0$ if $z=0$ and $\mathrm{sgn}(z)=\frac{z}{|z|}$ if $z\neq 0$. Let $\psi = \overline{\mathrm{sgn}(w)}|w|^{\frac{q}{2-q}}\left(|w_1|^2+t|w_2|^2\right)^{-\frac{1}{2-q}}$. Then $\psi w_i \not\equiv 0$ because $ww_i\not\equiv 0$, and

$$|\psi w_1|^2 \leqslant |w|^{\frac{2q}{2-q}}\left(|w_1|^2+t|w_2|^2\right)^{-\frac{q}{2-q}}, \qquad |\psi w_2|^2 \leqslant \frac{1}{t}|w|^{\frac{2q}{2-q}}\left(|w_1|^2+t|w_2|^2\right)^{-\frac{q}{2-q}}$$

hence $\psi w_i\in L^2(X)$ for $i=1,2$. It is easily seen that

$$\|\psi w\|_{L^q} = \left\|w\left(|w_1|^2+t|w_2|^2\right)^{-\frac{1}{2}}\right\|_{L^{\frac{2q}{2-q}}}^{\frac{2}{2-q}},$$

and

$$\left\|\psi(|w_1|^2+t|w_2|^2)^{\frac{1}{2}}\right\|_{L^2} = \left\|w\left(|w_1|^2+t|w_2|^2\right)^{-\frac{1}{2}}\right\|_{L^{\frac{2q}{2-q}}}^{\frac{q}{2-q}}.$$

Using (2.5) with $\varphi=\psi$, we discover

$$M_1 \geqslant \frac{\|\psi w\|_{L^q}}{\|\psi w_1\|_{L^2}^{\kappa}\cdot\|\psi w_2\|_{L^2}^{1-\kappa}} \geqslant (1-\kappa)^{\frac{\kappa-1}{2}}\kappa^{-\frac{\kappa}{2}} t^{\frac{1-\kappa}{2}} \left\|w\left(|w_1|^2+t|w_2|^2\right)^{-\frac{1}{2}}\right\|_{L^{\frac{2q}{2-q}}}. \tag{2.6}$$

Since (2.6) holds for any $t>0$ we infer that $M_1 \geqslant (1-\kappa)^{\frac{\kappa-1}{2}}\kappa^{-\frac{\kappa}{2}} M_2$.

Finally assume that $(X,\mathcal{A},\mu)$ is $\sigma-$finite. Consider a collection of sets $(X_n)_{n\geqslant 1}\subset\mathcal{A}$ such that $\mu(X_n)<\infty$, $X_n\subset X_{n+1}$ for all $n$ and $\cup_{n\geqslant 1}X_n = X$. Fix $t>0$ and denote $A_n = \left\{x\in X \mid \left|w(x)(|w_1(x)|^2+t|w_2(x)|^2)^{-\frac{1}{2}}\right|\leqslant n\right\}\cap X_n$. Then $A_n\subset A_{n+1}$, $\mu(A_n)\leqslant\mu(X_n)<\infty$ for any $n$ and $\cup_{n\geqslant 1}A_n = X$. Let $\psi_n = \overline{\mathrm{sgn}(w)}|w|^{\frac{q}{2-q}}\left(|w_1|^2+t|w_2|^2\right)^{-\frac{1}{2-q}}\mathbb{1}_{A_n}$. For all $n$ sufficiently large we have $ww_i\mathbb{1}_{A_n}\not\equiv 0$, and consequently $\psi_n w_i\not\equiv 0$.

As above we see that

$$|\psi_n w_1|^2 \leqslant |w|^{\frac{2q}{2-q}}\left(|w_1|^2+t|w_2|^2\right)^{-\frac{q}{2-q}}\mathbb{1}_{A_n} \leqslant n^{\frac{2q}{2-q}}\mathbb{1}_{A_n} \quad \text{and}$$

$$|\psi_n w_2|^2 \leqslant \frac{1}{t}|w|^{\frac{2q}{2-q}}\left(|w_1|^2+t|w_2|^2\right)^{-\frac{q}{2-q}}\mathbb{1}_{A_n} \leqslant \frac{1}{t}n^{\frac{2q}{2-q}}\mathbb{1}_{A_n},$$

hence $\psi_n w_i\in L^2(X)$. Proceeding as in the proof of (2.6) with $\psi_n$ instead of $\psi$ we get

$$M_1 \geqslant (1-\kappa)^{\frac{\kappa-1}{2}}\kappa^{-\frac{\kappa}{2}} t^{\frac{1-\kappa}{2}} \left\|w\left(|w_1|^2+t|w_2|^2\right)^{-\frac{1}{2}}\right\|_{L^{\frac{2q}{2-q}}(A_n)} \qquad \text{for all } n \text{ sufficiently large.}$$

Then letting $n \longrightarrow \infty$ and using the monotone convergence theorem we find

$$M_1 \geqslant (1-\kappa)^{\frac{\kappa-1}{2}} \kappa^{-\frac{\kappa}{2}} t^{\frac{1-\kappa}{2}} \left\| w_1 \left(1 + t|w_2|^2\right)^{-\frac{1}{2}} \right\|_{L^{\frac{2q}{2-q}}}.$$

Since this is true for any $t > 0$, we have $M_1 \geqslant (1-\kappa)^{\frac{\kappa-1}{2}} \kappa^{-\frac{\kappa}{2}} M_2$ and Lemma 2.1 is proven. □

We consider the Fourier transform defined by $\mathcal{F}(u)(\xi) = \widehat{u}(\xi) = \int_{\mathbf{R}^N} e^{-ix.\xi} u(x)\, dx$ if $u \in L^1(\mathbf{R}^N)$, and extended as usually to tempered distributions. We consider the Fourier integral operator $|D|^s - 1$ defined by

$$(|D|^s - 1)u = \mathcal{F}^{-1}\left((|\cdot|^s - 1)\widehat{u}\right).$$

The space $H^s(\mathbf{R}^N)$ is defined by $H^s(\mathbf{R}^N) = \{u \in \mathcal{S}'(\mathbf{R}^N) \mid (1 + |\cdot|^2)^{\frac{s}{2}} \widehat{u} \in L^2(\mathbf{R}^N)\}$. Given a tempered distribution $u$, we have $u \in H^s(\mathbf{R}^N)$ if and only if $\widehat{u} \in L^2(\mathbf{R}^N)$ and $(|\cdot|^s - 1)\widehat{u} \in L^2(\mathbf{R}^N)$. Moreover, by Plancherel's identity we have

$$\|u\|_{L^2(\mathbf{R}^N)} = \frac{1}{(2\pi)^{\frac{N}{2}}} \|\widehat{u}\|_{L^2(\mathbf{R}^N)} \quad \text{and} \quad \|(|D|^s - 1)u\|_{L^2(\mathbf{R}^N)} = \frac{1}{(2\pi)^{\frac{N}{2}}} \|(|\cdot|^s - 1)\widehat{u}\|_{L^2(\mathbf{R}^N)}. \tag{2.7}$$

Let $p \in (2, \infty)$ and let $\kappa \in (0,1)$. Define

$$Q_\kappa(u) = \frac{\|u\|_{L^p}}{\|u\|_{L^2}^{\kappa} \|(|D|^s - 1)u\|_{L^2}^{1-\kappa}} \qquad \text{for all } u \in H^s(\mathbf{R}^N) \setminus \{0\}, \tag{2.8}$$

and

$$M := \sup_{u \in H^s(\mathbf{R}^N) \setminus \{0\}} Q_\kappa(u). \tag{2.9}$$

We will investigate whether $M$ is finite. In the one-dimensional case we have the following:

**Theorem 2.3** *Assume that $N = 1$, $s \in (0, \infty)$, $p \in (2, \infty)$, and $\kappa \in (0,1)$. The supremum $M$ in (2.9) is finite if and only if $\frac{1}{2s} \leqslant \frac{(1-\kappa)p}{p-2} \leqslant \frac{1}{2}$.*

If $N = 1$, the condition in Theorem 2.3 is equivalent to condition (1.6) in Theorem 1.1.

*Proof.* Let $N \in \mathbf{N}^*$. Since $p > 2$, by the Hausdorff-Young Theorem (see, e.g., Therem 1.2.1 p. 6 in [1]) we have

$$\|u\|_{L^p(\mathbf{R}^N)} \leqslant (2\pi)^{\frac{N}{p} - N} \|\widehat{u}\|_{L^{p'}(\mathbf{R}^N)}, \qquad \text{where } p' = \frac{p}{p-1}. \tag{2.10}$$

Taking into account Plancherel's identity and the fact that $u \in H^s(\mathbf{R}^N)$ if and only if $\widehat{u} \in L^2(\mathbf{R}^N)$ and $(|\cdot|^s - 1)\widehat{u} \in L^2(\mathbf{R}^N)$, we infer that $M \leqslant (2\pi)^{\frac{N}{p} - \frac{N}{2}} M_3$ , where

$$M_3 := \sup \left\{ \frac{\|\varphi\|_{L^{p'}}}{\|\varphi\|_{L^2}^{\kappa} \left\| (|\cdot|^s - 1)\, \varphi \right\|_{L^2}^{1-\kappa}} \;\middle|\; \varphi,\ (|\cdot|^s - 1)\, \varphi \in L^2(\mathbf{R}^N) \setminus \{0\} \right\}. \tag{2.11}$$

Hence $M$ is finite if $M_3$ is finite. To prove the finiteness of $M_3$ we may use Lemma 2.1 in $\mathbf{R}^N$ endowed with the Lebesgue measure, with $w = w_1 = 1$ and $w_2(\xi) = |\xi|^s - 1$. By Lemma 2.1, it suffices to show that $M_4 := \sup\limits_{t>0} \left( t^{\frac{1-\kappa}{2}} \left\| \frac{1}{\left(1 + t(|\cdot|^s - 1)^2\right)^{\frac{1}{2}}} \right\|_{L^{\frac{2p'}{2-p'}}} \right)$ is finite. Notice that $\frac{2p'}{2-p'} = \frac{2p}{p-2}$. Given any $p > 2$ and any $t > 0$, the function $\xi \longmapsto \frac{1}{\left(1 + t(|\cdot|^s - 1)^2\right)^{\frac{1}{2}}}$ belongs to $L^{\frac{2p}{p-2}}(\mathbf{R}^N)$ if and only if $\frac{2sp}{p-2} > N$ (which is equivalent to $p(N - 2s) < 2N$). Let

$$F(t) = \left\| \frac{t^{\frac{1-\kappa}{2}}}{\left(1 + t\left(|\cdot|^s - 1\right)^2\right)^{\frac{1}{2}}} \right\|_{L^{\frac{2p}{p-2}}}^{\frac{2p}{p-2}} = \int_{\mathbf{R}^N} \frac{1}{\left(t^{\kappa-1} + t^{\kappa}\left(|\xi|^s - 1\right)^2\right)^{\frac{p}{p-2}}}\, d\xi. \tag{2.12}$$

In view of Remark 2.2 (iii), $F$ is continuous on $(0,\infty)$ provided that $\frac{2sp}{p-2} > N$.

Assume now that $N = 1$ and $\frac{1}{2s} \leqslant \frac{(1-\kappa)p}{p-2} \leqslant \frac{1}{2}$. Then $\frac{2sp}{p-2} \geqslant \frac{1}{1-\kappa} > 1$, hence $F$ is continuous on $(0,\infty)$ and we need only to check that $F$ is bounded in a neighbourhood of zero and of infinity. We have

$$F(t) = 2\int_0^\infty f(r,t)\,dr, \qquad \text{where} \qquad f(r,t) = \frac{1}{\left(t^{\kappa-1} + t^\kappa \left(r^s - 1\right)^2\right)^{\frac{p}{p-2}}}.$$

For any fixed $A > 0$ we have

$$\int_0^A |f(r,t)|\,dr \leqslant \int_0^A t^{\frac{(1-\kappa)p}{p-2}}\,dr = At^{\frac{(1-\kappa)p}{p-2}} \longrightarrow 0 \quad \text{as } t \longrightarrow 0.$$

Choose $A \geqslant 2^{\frac{1}{s}}$, so that $r^s > r^s - 1 > \frac{1}{2}r^s$ for $r > A$. We have then

$$\frac{1}{\left(t^{\kappa-1} + t^\kappa r^{2s}\right)^{\frac{p}{p-2}}} < f(r,t) < \frac{1}{\left(t^{\kappa-1} + \frac{1}{4}t^\kappa r^{2s}\right)^{\frac{p}{p-2}}} \qquad \text{for any } r > A.$$

Using the change of variable $r = t^{-\frac{1}{2s}}y$ we get

$$0 < \int_A^\infty f(r,t)\,dr < \int_A^\infty \frac{1}{\left(t^{\kappa-1} + \frac{1}{4}t^\kappa r^{2s}\right)^{\frac{p}{p-2}}}\,dr = t^{\frac{(1-\kappa)p}{p-2} - \frac{1}{2s}} \int_{t^{\frac{1}{2s}}A}^\infty \frac{1}{\left(1 + \frac{1}{4}y^{2s}\right)^{\frac{p}{p-2}}}\,dy.$$

We conclude that $F$ is bounded in a neighbourhood of zero if $\frac{(1-\kappa)p}{p-2} \geqslant \frac{1}{2s}$.

Let us study the behaviour of $F$ as $t \longrightarrow \infty$. We have

$$0 < \int_2^\infty f(r,t)\,dr < \int_2^\infty \frac{1}{\left(t^\kappa \left(r^s - 1\right)^2\right)^{\frac{p}{p-2}}}\,dr = t^{-\frac{\kappa p}{p-2}} \int_2^\infty \frac{1}{(r^s - 1)^{\frac{2p}{p-2}}}\,dr \longrightarrow 0 \text{ as } t \longrightarrow \infty.$$

There exist two positive constants $c_2$, $c_2$ such that $c_1|y| \leqslant \big||1+y|^s - 1\big| \leqslant c_2|y|$ for all $y \in [-1,1]$. Using the change of variable $r = 1 + y$, the above estimate and then the change of variable $y = t^{-\frac{1}{2}}z$ we get

$$\int_0^2 f(r,t)\,dr = \int_{-1}^1 \frac{1}{\left(t^{\kappa-1} + t^\kappa \big||1+y|^s - 1\big|^2\right)^{\frac{p}{p-2}}}\,dy \leqslant \int_{-1}^1 \frac{1}{\left(t^{\kappa-1} + t^\kappa c_1^2 y^2\right)^{\frac{p}{p-2}}}\,dy$$

$$= t^{-\frac{1}{2} + \frac{(1-\kappa)p}{p-2}} \int_{-t^{\frac{1}{2}}}^{t^{\frac{1}{2}}} \frac{1}{\left(1 + c_1^2 z^2\right)^{\frac{p}{p-2}}}\,dz$$

and we infer that $F$ is bounded in a neighbourhood of infinity if $\frac{(1-\kappa)p}{p-2} \leqslant \frac{1}{2}$.

So far we have proved that $M_3$ is finite (and consequently $M$ is finite) if $\frac{1}{2s} \leqslant \frac{(1-\kappa)p}{p-2} \leqslant \frac{1}{2}$. The fact that $M = \infty$ if one of these two inequalities is not satisfied follows from the next example. $\square$

**Example 2.4** Given $\tau > 0$, we define $u_\tau(x) := e^{ix - \frac{x^2}{2\tau^2}}$. It is clear that $u \in \mathcal{S}(\mathbf{R})$ and direct computations give

$$\widehat{u}_\tau(\xi) = \sqrt{2\pi}\tau e^{-\frac{\tau^2(\xi-1)^2}{2}}, \quad \|u_\tau\|_{L^2}^2 = \sqrt{\pi}\tau \quad \text{and} \quad \|u_\tau\|_{L^p}^p = \sqrt{\frac{2\pi}{p}}\tau. \tag{2.13}$$

Using Plancherel's formula we get

$$\begin{aligned} \|(|D|^s - 1)u_\tau\|_{L^2}^2 &= \frac{1}{2\pi}\|(|\cdot|^s - 1)\widehat{u}_\tau\|_{L^2}^2 \\ &= \tau^2 \int_{\mathbf{R}} (|\xi|^s - 1)^2 e^{-\tau^2(\xi-1)^2}\,d\xi \\ &= \tau^{1-2s} \int_{\mathbf{R}} (|\tau + x|^s - \tau^s)^2 e^{-x^2}\,dx \sim \begin{cases} \tau^{-1} & \text{as } \tau \to \infty, \\ \tau^{1-2s} & \text{as } \tau \to 0. \end{cases} \end{aligned} \tag{2.14}$$

Thus we obtain

$$Q_\kappa(u_\tau) \sim \frac{\tau^{\frac{1}{p}}}{\tau^{\frac{\kappa}{2}}\tau^{\frac{\kappa-1}{2}}} = \tau^{\frac{1}{p}+\frac{1}{2}-\kappa} \longrightarrow \infty \quad \text{as } \tau \to \infty \quad \text{if } \frac{1}{p}+\frac{1}{2}-\kappa > 0, \tag{2.15}$$

while in the limit $\tau \to 0$ we have

$$Q_\kappa(u_\tau) \sim \frac{\tau^{\frac{1}{p}}}{\tau^{\frac{\kappa}{2}}\tau^{\frac{(1-2s)(1-\kappa)}{2}}} = \tau^{\frac{1}{p}-\frac{\kappa}{2}-\frac{(1-2s)(1-\kappa)}{2}} \longrightarrow \infty \quad \text{if } \frac{1}{p}-\frac{\kappa}{2}-\frac{(1-2s)(1-\kappa)}{2} < 0. \tag{2.16}$$

Notice that for $p > 2$ and $\kappa \in (0,1)$, the inequality $\frac{1}{p}+\frac{1}{2}-\kappa > 0$ is equivalent to $\frac{(1-\kappa)p}{p-2} > \frac{1}{2}$ and the inequality $\frac{1}{p}-\frac{\kappa}{2}-\frac{(1-2s)(1-\kappa)}{2} < 0$ is equivalent to $\frac{(1-\kappa)p}{p-2} < \frac{1}{2s}$.

**Remark 2.5** Let $F$ be as in (2.12). Using polar coordinates in $\mathbf{R}^N$ and proceeding as in the proof of Theorem 2.3, one can prove that $F$ is bounded near zero if and only if $\frac{(1-\kappa)p}{p-2} \geqslant \frac{N}{2s}$ and $F$ is bounded near infinity if and only if $\frac{(1-\kappa)p}{p-2} \leqslant \frac{1}{2}$. By Lemma 2.1, $M_3$ is finite if and only if $F$ is bounded, and a necessary condition for the boundedness of $F$ would be $\frac{N}{2s} \leqslant \frac{1}{2}$, or equivalently $N \leqslant s$. Thus any attempt to show that $M < \infty$ by proving that $M_3 < \infty$ will fall short from providing the optimal range of parameters $(\kappa, s)$ for which the supremum in (2.9) is finite, given by Theorem 2.6. The Hausdorff-Young inequality (that we have used successfully in dimension one) is not sufficiently accurate in higher dimensions and a more subtle argument is needed. See also the second part of Remark 2.8.

The next theorem gives optimal conditions for the finiteness of $M$ in any space dimension $N \geqslant 2$. Its proof is based on the Tomas-Stein Theorem, which asserts that for $p \geqslant \frac{2N+2}{N-1}$, there exists a positive constant $C_{TS}$ depending only on $p$ and on $N$ such that for any $\varphi \in L^2(\mathbb{S}^{N-1}, d\sigma)$ there holds

$$\|\widehat{f\,d\sigma}\|_{L^p(\mathbf{R}^N)} \leqslant C_{TS}\|f\|_{L^2(\mathbb{S}^{N-1})} \tag{2.17}$$

(see, e.g., Theorem 7.1 p. 45 in [24]). Here $\mathbb{S}^{N-1} = \{\omega \in \mathbf{R}^N \mid |\omega| = 1\}$ is the unit sphere in $\mathbf{R}^N$, $\sigma$ is the usual surface measure on the unit sphere, $L^2(\mathbb{S}^{N-1}, d\sigma)$ is the space of measurable functions defined on the unit sphere which are square integrable with respect to the surface measure, and $\widehat{f\,d\sigma}$ is the Fourier transform of the measure $f\,d\sigma$, given by

$$\widehat{f\,d\sigma}(\xi) = \int_{\mathbb{S}^{N-1}} f(\omega)e^{-i\omega.\xi}\,d\sigma(\omega) \qquad \text{for any } \xi \in \mathbf{R}^N \text{ and any } f \in L^1(\mathbb{S}^{N-1}, d\sigma).$$

**Theorem 2.6** *Let $N \in \mathbf{N}$, $N \geqslant 2$, $p \in (2,\infty)$, $\kappa \in (0,1)$, and $s > 0$. Then $Q_\kappa$ is bounded on $H^s(\mathbf{R}^N) \setminus \{0\}$ (that is, $M$ in (2.9) is finite) if and only if*

$$\kappa \geqslant \frac{1}{2} \quad \textit{and} \quad \frac{N}{s}\left(\frac{1}{2}-\frac{1}{p}\right) \leqslant 1-\kappa \leqslant \frac{N+1}{2}\left(\frac{1}{2}-\frac{1}{p}\right). \tag{2.18}$$

Observe that the last condition in (2.18) implies that $s \geqslant \frac{2N}{N+1}$ and it is equivalent to $\frac{2(N+1)}{N+4\kappa-3} \leqslant p \leqslant \frac{2N}{N-2(1-\kappa)s}$ if $2(1-\kappa)s < N$, respectively to $\frac{2(N+1)}{N+4\kappa-3} \leqslant p$ if $2(1-\kappa)s \geqslant N$.

*Proof.* Assume that (2.18) hold and assume also in a first stage that $p \geqslant \frac{2(N+1)}{N-1}$, so that we may apply the Tomas-Stein Theorem. Let $u \in \mathcal{S}(\mathbf{R}^N)$. Using the Fourier inversion formula and passing to polar coordinates in $\mathbf{R}^N$ we get

$$u(x) = \frac{1}{(2\pi)^N}\int_0^\infty r^{N-1}\int_{\mathbb{S}^{N-1}} \widehat{u}(r\omega)e^{i(rx).\omega}d\sigma(\omega)\,dr = \frac{1}{(2\pi)^N}\int_0^\infty r^{N-1}\widehat{\widehat{u}(r\cdot)\,d\sigma}(-rx)\,dr.$$

Using Minkowski's inequality in integral form (see, e.g., Theorem 2.4 p. 47 in [16]), then the

Tomas-Stein inequality we get

$$\begin{aligned}\|u\|_{L^p(\mathbf{R}^N)} &\leqslant \frac{1}{(2\pi)^N}\int_0^\infty r^{N-1}\big\|\widehat{\widehat{u}(r\cdot)\,d\sigma}(-r\cdot)\big\|_{L^p(\mathbf{R}^N)}\,dr\\ &= \frac{1}{(2\pi)^N}\int_0^\infty r^{N-1-\frac{N}{p}}\big\|\widehat{\widehat{u}(r\cdot)\,d\sigma}\big\|_{L^p(\mathbf{R}^N)}\,dr\\ &\leqslant \frac{C_{TS}}{(2\pi)^N}\int_0^\infty r^{N-1-\frac{N}{p}}\big\|\widehat{u}(r\cdot)\big\|_{L^2(\mathbb{S}^{N-1})}\,dr.\end{aligned}$$

Denoting $z_u(r) = r^{\frac{N-1}{2}}\left(\int_{\mathbb{S}^{N-1}}|\widehat{u}(r\omega)|^2\,d\sigma(\omega)\right)^{\frac12}$, we have proved that there exists $C>0$ depending only on $p$ and on $N$ such that

$$\|u\|_{L^p(\mathbf{R}^N)} \leqslant C\int_0^\infty r^{\frac{N-1}{2}-\frac{N}{p}}z_u(r)\,dr \qquad \text{for all } u\in\mathcal{S}(\mathbf{R}^N) \text{ and for all } p\geqslant \frac{2N+2}{N-1}. \tag{2.19}$$

On the other hand, using Fourier's inversion formula and polar coordinates in $\mathbf{R}^N$ we have

$$\|u\|_{L^2}^2 = \frac{1}{(2\pi)^N}\int_0^\infty\int_{\mathbb{S}^{N-1}}|\widehat{u}(r\omega)|^2\,d\sigma(\omega)\,dr = \frac{1}{(2\pi)^N}\int_0^\infty z_u^2(r)\,dr \tag{2.20}$$

and

$$\begin{aligned}\|(|D|^s-1)u\|_{L^2}^2 &= \frac{1}{(2\pi)^N}\big\|(|\cdot|^s-1)\widehat{u}\big\|_{L^2}^2\\ &= \frac{1}{(2\pi)^N}\int_0^\infty r^{N-1}(r^s-1)^2\int_{\mathbb{S}^{N-1}}|\widehat{u}(r\omega)|^2\,d\sigma(\omega)\,dr = \frac{1}{(2\pi)^N}\int_0^\infty (r^s-1)^2z_u^2(r)\,dr.\end{aligned} \tag{2.21}$$

From (2.19) - (2.21) it follows that there is $C>0$ such that for all $u\in\mathcal{S}(\mathbf{R}^N)$ we have

$$Q_\kappa(u)\leqslant C\frac{\int_0^\infty r^{\frac{N-1}{2}-\frac{N}{p}}z_u(r)\,dr}{\|z_u\|_{L^2(0,\infty)}^\kappa\left(\int_0^\infty (r^s-1)^2z_u^2(r)\,dr\right)^{\frac{1-\kappa}{2}}}. \tag{2.22}$$

Notice that $z_u\in L^2(0,\infty)$ and $(|\cdot|^s-1)\,z_u\in L^2(0,\infty)$ by (2.20) and (2.21). We use Lemma 2.1 in $(0,\infty)$ endowed with the usual Lebesgue measure and we take $w(r)=r^{\frac{N-1}{2}-\frac{N}{p}}$, $w_1(r)=1$ and $w_2(r)=r^s-1$. We get

$$\sup_{\varphi\in L^2(0,\infty)\setminus\{0\}}\frac{\|w\varphi\|_{L^1(0,\infty)}}{\|\varphi\|_{L^2(0,\infty)}^\kappa\|w_2\varphi\|_{L^2(0,\infty)}^{1-\kappa}}\leqslant C\sup_{t>0}\left(t^{\frac{1-\kappa}{2}}\left\|\frac{w}{(1+t|w_2|^2)^{\frac12}}\right\|_{L^2(0,\infty)}\right). \tag{2.23}$$

Let

$$G(t):=\left\|\frac{t^{\frac{1-\kappa}{2}}w}{(1+t|w_2|^2)^{\frac12}}\right\|_{L^2(0,\infty)}^2 = \int_0^\infty\frac{r^{N-1-\frac{2N}{p}}}{t^{\kappa-1}+t^\kappa(r^s-1)^2}\,dr =: \int_0^\infty g(r,t)\,dr. \tag{2.24}$$

Since $N-2s-\frac{2N}{p}<0$ (because $\frac{N}{s}\left(\frac12-\frac1p\right)\leqslant 1-\kappa<1$), we have $g(\cdot,t)\in L^1(0,\infty)$ and then Remark 2.2 (iii) implies that $G$ is continuous on $(0,\infty)$.

For any fixed $A>0$ we have

$$0<\int_0^A g(r,t)\,dr < t^{1-\kappa}\int_0^A r^{N-1-\frac{2N}{p}}\,dr = t^{1-\kappa}\frac{A^{N-\frac{2N}{p}}}{N-\frac{2N}{p}}\longrightarrow 0 \qquad \text{as } t\longrightarrow 0.$$

We have $r^s-1>\frac12 r^s$ if $r>2^{\frac1s}$. Taking $A\geqslant 2^{\frac1s}$ and using the change of variable $r=t^{-\frac{1}{2s}}y$ we find

$$0<\int_A^\infty g(r,t)\,dr<\int_A^\infty\frac{r^{N-1-\frac{2N}{p}}}{t^{\kappa-1}+\frac14 t^\kappa r^{2s}}\,dr = t^{1-\kappa-\frac{1}{2s}(N-\frac{2N}{p})}\int_{t^{\frac{1}{2s}}A}^\infty\frac{y^{N-1-\frac{2N}{p}}}{1+\frac14 y^{2s}}\,dy.$$

We infer that $G$ is bounded as $t \longrightarrow 0$ if $1-\kappa \geqslant \frac{N}{s}\left(\frac{1}{2}-\frac{1}{p}\right)$.

It is clear that

$$0 < \int_2^\infty g(r,t)\,dr < t^{-\kappa}\int_2^\infty \frac{r^{N-1-\frac{2N}{p}}}{(r^s-1)^2}\,dr \longrightarrow 0 \qquad \text{as } t \longrightarrow \infty.$$

There is $C>0$ such that $r^{N-1-\frac{2N}{p}} \leqslant C$ for $r\in[0,2]$ because $N-1-\frac{2N}{p}>0$ (recall that $p \geqslant \frac{2N+2}{N-1}$). There is $c_1>0$ such that $(|1+y|^s-1)^2 \geqslant c_1 y^2$ for all $y\in[-1,1]$. Using the change of variable $r=1+y$, the previous observations, then the change of variable $z=t^{\frac{1}{2}}y$ we get

$$0 < \int_0^2 g(r,t)\,dr \leqslant \int_{-1}^1 \frac{C}{t^{\kappa-1}+t^\kappa(|1+y|^s-1)^2}\,dy \leqslant \int_{-1}^1 \frac{C}{t^{\kappa-1}+t^\kappa c_1^2 y^2}\,dy = \int_{-\sqrt{t}}^{\sqrt{t}} \frac{Ct^{\frac{1}{2}-\kappa}}{1+c_1^2 z^2}\,dz.$$

We conclude that $G$ is bounded as $t\longrightarrow\infty$ if $\kappa \geqslant \frac{1}{2}$.

We have thus proved that if (2.18) and the additional assumption $p \geqslant \frac{2(N+1)}{N-1}$ hold, the function $G$ is bounded on $(0,\infty)$ and therefore the supremum on the right hand side of (2.23) is finite. Then (2.22) and (2.23) imply that there exists $C(\kappa)>0$ such that $Q_\kappa(u)\leqslant C(\kappa)$ for any $u\in\mathcal{S}(\mathbf{R}^N)\setminus\{0\}$. Since $u\longmapsto Q_\kappa(u)$ is continuous on $H^s(\mathbf{R}^N)\setminus\{0\}$ and $\mathcal{S}(\mathbf{R}^N)\setminus\{0\}$ is dense in $H^s(\mathbf{R}^N)$, we infer that $Q_\kappa(u)\leqslant C(\kappa)$ for any $u\in H^s(\mathbf{R}^N)\setminus\{0\}$.

It remains to consider the case $\frac{2(N+1)}{N+4\kappa-3} \leqslant p < \frac{2(N+1)}{N-1}$. We proceed by interpolation. Denote $q := \frac{2(N+1)}{N-1}$. We see that $\frac{N+1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)=\frac{1}{2}$. Since $2<p<q$, there is some $\theta\in(0,1)$ such that $\frac{1}{p}=\frac{\theta}{2}+\frac{1-\theta}{q}$. We have

$$\theta = \left(\frac{1}{p}-\frac{1}{q}\right)\left(\frac{1}{2}-\frac{1}{q}\right)^{-1} = (N+1)\left(\frac{1}{p}-\frac{1}{q}\right) \leqslant (N+1)\left(\frac{N+4\kappa-3}{2(N+1)}-\frac{N-1}{2(N+1)}\right) = 2\kappa-1.$$

Since $\kappa<1$, the above inequality gives $\theta<\kappa$. By Hölder's inequality we have

$$\|u\|_{L^p} \leqslant \|u\|_{L^2}^\theta \cdot \|u\|_{L^q}^{1-\theta}$$

and then for any $u\in H^s(\mathbf{R}^N)\setminus\{0\}$ we find

$$Q_\kappa(u) \leqslant \frac{\|u\|_{L^2}^\theta\cdot\|u\|_{L^q}^{1-\theta}}{\|u\|_{L^2}^\kappa \|(|D|^s-1)u\|_{L^2}^{1-\kappa}} = \left(\tilde{Q}_{\tilde{\kappa}}(u)\right)^{1-\theta}, \tag{2.25}$$

where $\tilde{\kappa}=\frac{\kappa-\theta}{1-\theta}$, so that $1-\tilde{\kappa}=\frac{1-\kappa}{1-\theta}$, and $\tilde{Q}_{\tilde{\kappa}}(u)=\frac{\|u\|_{L^q}}{\|u\|_{L^2}^{\tilde{\kappa}}\|(|D|^s-1)u\|_{L^2}^{1-\tilde{\kappa}}}$. Notice that the inequality $\theta<2\kappa-1$ implies that $\tilde{\kappa}\geqslant\frac{1}{2}$ and then we get $1-\tilde{\kappa}\leqslant\frac{1}{2}=\frac{N+1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)$. Using (2.18) and the fact that $\frac{1}{2}-\frac{1}{p}=(1-\theta)\left(\frac{1}{2}-\frac{1}{q}\right)$, we get

$$1-\tilde{\kappa}=\frac{1-\kappa}{1-\theta} \geqslant (1-\theta)^{-1}\frac{N}{s}\left(\frac{1}{2}-\frac{1}{p}\right)=\frac{N}{s}\left(\frac{1}{2}-\frac{1}{q}\right).$$

Thus we see that (2.18) is satisfied with $q$ and $\tilde{\kappa}$ instead of $p$ and $\kappa$, respectively. From the first part of the proof we infer that $\tilde{Q}_{\tilde{\kappa}}$ is bounded from above on $H^s(\mathbf{R}^N)\setminus\{0\}$, and then (2.25) implies that $Q_\kappa$ is also bounded.

So far we have proved that $Q_\kappa$ is bounded on $H^s(\mathbf{R}^N)\setminus\{0\}$ if (2.18) hold. Now let us show that (2.18) is necessary for the boundedness of $Q_\kappa$. Let $u\in\mathcal{S}(\mathbf{R}^N)$, $u\neq0$. For $\tau>0$ let $u_\tau(x)=u\left(\frac{x}{\tau}\right)$. A simple computation gives $\|u_\tau\|_{L^q}=\tau^{\frac{N}{q}}\|u\|_{L^q}$ for any $q\in[1,\infty)$ and

$$\|(|D|^s-1)u\|_{L^2}^2=\frac{\tau^{N-2s}}{(2\pi)^N}\int_{\mathbf{R}^N}\left(|\xi|^{2s}-2\tau^s|\xi|^s+\tau^{2s}\right)|\widehat{u}(\xi)|^2\,d\xi.$$

Thus we find

$$Q_\kappa(u_\tau) = (2\pi)^{\frac{(1-\kappa)N}{2}} \tau^{\frac{N}{p} - \frac{\kappa N}{2} - (1-\kappa)(\frac{N}{2}-s)} \frac{\|u\|_{L^p}}{\|u\|_{L^2}^\kappa \left(\int_{\mathbf{R}^N} (|\xi|^{2s} - 2\tau^s|\xi|^s + \tau^{2s}) |\widehat{u}(\xi)|^2 \, d\xi\right)^{\frac{1-\kappa}{2}}}.$$

If $Q_\kappa(u_\tau)$ remains bounded as $\tau \longrightarrow 0$ we must have $\frac{N}{p} - \frac{\kappa N}{2} - (1-\kappa)(\frac{N}{2} - s) \geqslant 0$ and this is equivalent to $1 - \kappa \geqslant \frac{N}{s}\left(\frac{1}{2} - \frac{2}{p}\right)$.

The next example shows that $Q_\kappa$ is not bounded if $\kappa < \frac{1}{2}$ or if $\frac{N+1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) < 1 - \kappa$. $\square$

**Example 2.7** We consider a variant of Knapp's example related to the Tomas-Stein inequality (see, e.g., [24] p. 46). For small $\delta > 0$, let $S_\delta = \{\omega = (\omega_1, \dots, \omega_N) \in \mathbb{S}^{N-1} \mid \omega_N > 1 - \delta^2\}$. It is easily seen that there exist positive constants $C_1$, $C_2$ such that for all $\delta \in (0, \frac{1}{10})$, say, we have

$$C_1 \delta^{N-1} \leqslant \sigma(S_\delta) \leqslant C_2 \delta^{N-1},$$

where $\sigma$ is the surface measure on $\mathbb{S}^{N-1}$. For small $\varepsilon > 0$, $\delta > 0$ we define $v_{\varepsilon,\delta} : \mathbf{R}^N \longrightarrow \mathbf{R}$ by

$$v_{\varepsilon,\delta}(\xi) = \begin{cases} 1 & \text{if } 1 - \varepsilon < |\xi| < 1 + \varepsilon \text{ and } \frac{\xi}{|\xi|} \in S_\delta, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$u_{\varepsilon,\delta} = \mathcal{F}^{-1}(v_{\varepsilon,\delta}), \qquad \text{that is } u_{\varepsilon,\delta}(x) = \frac{1}{(2\pi)^N} \int_{1-\varepsilon}^{1+\varepsilon} r^{N-1} \int_{S_\delta} e^{ix.(r\omega)} \, d\sigma(\omega) \, dr. \tag{2.26}$$

Since $\widehat{u_{\varepsilon,\delta}} = v_{\varepsilon,\delta}$ is bounded and compactly supported, we have $u_{\varepsilon,\delta} \in H^s(\mathbf{R}^N)$ for all $s$. By Plancherel's identity we get

$$\|u_{\varepsilon,\delta}\|_{L^2}^2 = \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} |v_{\varepsilon,\delta}(\xi)|^2 \, d\xi = \frac{1}{(2\pi)^N} \int_{1-\varepsilon}^{1+\varepsilon} r^{N-1} \sigma(S_\delta) \, dr \sim \varepsilon \delta^{N-1} \tag{2.27}$$

and

$$\begin{aligned} \| (|D|^s - 1) \, u_{\varepsilon,\delta}\|_{L^2}^2 &= \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} (|\xi|^s - 1)^2 |v_{\varepsilon,\delta}(\xi)|^2 \, d\xi \\ &= \frac{1}{(2\pi)^N} \int_{1-\varepsilon}^{1+\varepsilon} r^{N-1} (r^s - 1)^2 \sigma(S_\delta) \, dr \sim \varepsilon^3 \delta^{N-1}. \end{aligned} \tag{2.28}$$

Let $e_N = (0, \dots, 0, 1) \in \mathbf{R}^N$. It is clear that

$$|u_{\varepsilon,\delta}(x)| = |e^{-ix.e_N} u_{\varepsilon,\delta}(x)| = \frac{1}{(2\pi)^N} \Big| \int_{1-\varepsilon}^{1+\varepsilon} r^{N-1} \int_{S_\delta} e^{ix.(r\omega - e_N)} \, d\sigma(\omega) \, dr \Big|.$$

Let $A_{\varepsilon,\delta} = \left\{x \in \mathbf{R}^N \;\middle|\; |x.(r\omega - e_N)| \leqslant \frac{\pi}{3} \text{ for all } r \in (1-\varepsilon, 1+\varepsilon) \text{ and all } \omega \in S_\delta\right\}$. For any $x \in A_{\varepsilon,\delta}$, $r \in (1-\varepsilon, 1+\varepsilon)$, and $\omega \in S_\delta$ we have $\mathrm{Re}(e^{ix.(r\omega - e_N)}) \geqslant \frac{1}{2}$, hence

$$\begin{aligned} |u_{\varepsilon,\delta}(x)| &\geqslant \frac{1}{(2\pi)^N} \int_{1-\varepsilon}^{1+\varepsilon} r^{N-1} \int_{S_\delta} \mathrm{Re}\left(e^{ix.(r\omega - e_N)}\right) d\sigma(\omega) \, dr \\ &\geqslant C \int_{1-\varepsilon}^{1+\varepsilon} r^{N-1} \sigma(S_\delta) \, dr \geqslant C \varepsilon \delta^{N-1} \end{aligned}$$

for some $C > 0$ independent of $\varepsilon$ and $\delta$. We infer that

$$\|u_{\varepsilon,\delta}\|_{L^p(\mathbf{R}^N)} \geqslant \|u_{\varepsilon,\delta}\|_{L^p(A_{\varepsilon,\delta})} \geqslant C \varepsilon \delta^{N-1} |A_{\varepsilon,\delta}|^{\frac{1}{p}}. \tag{2.29}$$

We will find a lower bound for $|A_{\varepsilon,\delta}|$. Denote $x = (x', x_N)$, $\omega = (\omega', \omega_N)$ where $x', \omega' \in \mathbf{R}^{N-1}$, and assume that $\varepsilon \leqslant 1$. We have

$$|x.(r\omega - e_N)| = |x.(r\omega', r\omega_N - 1)| \leqslant r|x'.\omega'| + |x_N(r\omega_N - 1)|.$$

For $\omega = (\omega', \omega_N) \in S_\delta$ and $r \in [1-\varepsilon, 1+\varepsilon]$ we have $|\omega'| < \sqrt{2}\delta$ and $|r\omega_N - 1| \leqslant 2\delta^2 + \varepsilon$, hence $r|x'.\omega'| \leqslant \frac{\pi}{6}$ if $|x'| \leqslant \frac{\sqrt{2}\pi}{24}\delta^{-1}$ and $|x_N(r\omega_N - 1)| \leqslant \frac{\pi}{6}$ if $|x_N| \leqslant \frac{\pi}{6}(2\delta^2+\varepsilon)^{-1}$. We conclude that

$$\left\{ (x', x_N) \in \mathbf{R}^{N-1} \times \mathbf{R} \;\Big|\; |x'| \leqslant \frac{\sqrt{2}\pi}{24}\delta^{-1},\ |x_N| \leqslant \frac{\pi}{6}(2\delta^2+\varepsilon)^{-1} \right\} \subset A_{\varepsilon,\delta}.$$

Hence there exists $C > 0$ independent of $\varepsilon$ and $\delta$ such that $|A_{\varepsilon,\delta}| \geqslant \frac{C}{\delta^{N-1}(\delta^2+\varepsilon)}$ and (2.29) gives

$$\|u_{\varepsilon,\delta}\|_{L^p(\mathbf{R}^N)} \geqslant C\varepsilon\delta^{(N-1)(1-\frac{1}{p})}(\delta^2+\varepsilon)^{-\frac{1}{p}}. \tag{2.30}$$

From (2.27), (2.28), and (2.30) we obtain

$$Q_\kappa(u_{\varepsilon,\delta}) \geqslant C\delta^{(N-1)(\frac{1}{2}-\frac{1}{p})}\varepsilon^{\kappa-\frac{1}{2}}(\delta^2+\varepsilon)^{-\frac{1}{p}}. \tag{2.31}$$

Fix $\delta_0 \in (0, \frac{1}{10})$ and let $\varepsilon \longrightarrow 0$. If $Q_\kappa(u_{\varepsilon,\delta_0})$ remains bounded, (2.31) implies that $\kappa \geqslant \frac{1}{2}$.

Putting $\varepsilon = \delta^2$ in (2.31) we get $Q_\kappa(u_{\delta^2,\delta}) \geqslant C\delta^{(N-1)(\frac{1}{2}-\frac{1}{p})+2\kappa-1-\frac{2}{p}}$. If $Q_\kappa(u_{\delta^2,\delta})$ remains bounded as $\delta \longrightarrow 0$ we must have $(N-1)(\frac{1}{2}-\frac{1}{p}) + 2\kappa - 1 - \frac{2}{p} \geqslant 0$, and this is equivalent to $\frac{N+1}{2}\left(\frac{1}{2}-\frac{1}{p}\right) \geqslant 1-\kappa$.

**Remark 2.8** The method used in the proof of Theorem 2.6 is very flexible and can be used to prove non-homogeneous Gagliardo-Nirenberg inequalities of the form

$$\|u\|_{L^p} \leqslant C\|P_1(D)u\|_{L^2}^{\kappa}\|P_2(D)u\|_{L^2}^{1-\kappa}, \tag{2.32}$$

where $N \geqslant 2$, $p \geqslant \frac{2N+2}{N-1}$ and $P_1(D)$, $P_2(D)$ are Fourier integral operators defined by

$$P_i(D)(u) = \mathcal{F}^{-1}\left(P_i(\cdot)\widehat{u}\right), \quad i = 1, 2.$$

Assuming that there exist non-negative functions $p_1, p_2 : [0,\infty) \longrightarrow \mathbf{R}_+$ such that $|P_i(\xi)| \geqslant p_i(|\xi|)$ for all $\xi \in \mathbf{R}^N$, $i = 1, 2$ and proceeding as in (2.20) and (2.21) we get for all $u \in \mathcal{S}(\mathbf{R}^N)$

$$\|P_i(D)u\|_{L^2}^2 \geqslant \frac{1}{(2\pi)^N}\int_0^\infty p_i^2(r)z_u^2(r)\,dr, \qquad \text{where } z_u(r) = r^{\frac{N-1}{2}}\|\widehat{u}(r\cdot)\|_{L^2(\mathbb{S}^{N-1})}. \tag{2.33}$$

In order to prove the inequality (2.32) in some function space $\mathscr{X}$ (typically $\mathscr{X} = H^s(\mathbf{R}^N)$, but other spaces might be considered), one needs to show the continuity of the $L^p-$norm and of the operators $P_1(D)$ and $P_2(D)$ on $\mathscr{X}$, as well as the density of $\mathcal{S}(\mathbf{R}^N)$ in $\mathscr{X}$. Then, taking into account (2.19) and (2.33) and denoting $w(r) = r^{\frac{N-1}{2}-\frac{N}{p}}$, it suffices to show that

$$\sup\left\{ \frac{\|w\varphi\|_{L^1(0,\infty)}}{\|p_1\varphi\|_{L^2(0,\infty)}^{\kappa}\|p_2\varphi\|_{L^2(0,\infty)}^{1-\kappa}} \;\Big|\; p_i\varphi \in L^2(0,\infty)\setminus\{0\},\, i = 1, 2 \right\}$$

is finite. To do this, by Lemma 2.1 and Remark 2.2 (iii) it suffices to prove that the function

$$H(t) := \left\| \frac{t^{\frac{1-\kappa}{2}}w}{(p_1^2 + t|p_2|^2)^{\frac{1}{2}}} \right\|_{L^2(0,\infty)}^2 = \int_0^\infty \frac{r^{N-1-\frac{2N}{p}}}{t^{\kappa-1}p_1^2(r) + t^\kappa p_2^2(r)}\,dr \tag{2.34}$$

is bounded on $(0,\infty)$.

If $N = 1$ or if the Tomas-Stein inequality is not available (for instance, if $2 < p < \frac{2N+2}{N-1}$), one may try to use the Hausdorff-Young Theorem to prove the inequality (2.32), as in the proof of Theorem 2.3. Indeed, to establish (2.32) it suffices to show that the supremum

$$\sup\left\{ \frac{\|\varphi\|_{L^{p'}}}{\|P_1\varphi\|_{L^2}^{\kappa}\|P_2\varphi\|_{L^2}} \;\Big|\; P_i\varphi \in L^2(\mathbf{R}^N)\setminus\{0\},\ i = 1,\, 2 \right\}$$

is finite, and by Lemma 2.1 this amounts to proving that the function

$$K(t) := \left\| \frac{t^{\frac{1-\kappa}{2}}}{(P_1^2 + tP_2^2)^{\frac{1}{2}}} \right\|_{L^{\frac{2p}{p-2}}}^{\frac{2p}{p-2}} = \int_{\mathbf{R}^N} \frac{1}{(t^{\kappa-1}|P_1(\xi)|^2 + t^{\kappa}|P_2(\xi)|^2)^{\frac{p}{p-2}}} \, d\xi \tag{2.35}$$

is bounded on $(0, \infty)$. However, we expect the approach based on the Hausdorff-Young inequality to give weaker results than the approach based on the Tomas-Stein inequality. For instance, if $P_1$ and $P_2$ are radial and non-negative (that is, if $P_i(\xi) = p_i(|\xi|) \geqslant 0$), it is easily seen that the boundedness of the function $K$ implies the boundedness of the function $H$, but the converse might not be true. See Remark 2.5.

As a matter of fact, the Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|_{L^p} \leqslant C\|u\|_{L^2}^{1-\frac{N(p-2)}{4sp}} \|(-\Delta)^s u\|_{L^2}^{\frac{N(p-2)}{4sp}}$$

(with $2 < p < \frac{2N}{N-4s}$ if $s < \frac{N}{4}$) can be proven by using our method and the Hausdorff-Young inequality; in this case $P_1 = 1$ and $P_2(\xi) = |\xi|^{2s}$, and the integral in $K(t)$ is easily evaluated using polar coordinates and the change of variables $r = t^{-\frac{1}{4s}} y$. For $s = 1$ and $p = 2\sigma + 2$, this gives (3.3).

**Remark 2.9** More quantitative variants of (2.32) can be proved, too. For instance, in some applications it is useful to dispose of inequalities of the form (2.32) under the additional constraint $\|P_2(D)u\|_{L^2} \leqslant R\|P_1(D)u\|_{L^2}$, where $R > 0$ is given. To obtain such inequalities we may use a slight modification of Lemma 2.1.

With the notation and the assumptions in Lemma 2.1, let

$$M_1^R = \sup \left\{ \frac{\|\varphi w\|_{L^q}}{\|\varphi w_1\|_{L^2}^{\kappa} \cdot \|\varphi w_2\|_{L^2}^{1-\kappa}} \;\middle|\; \begin{array}{l} \varphi \text{ is measurable, } \varphi w_i \in L^2(X) \text{ for } i = 1, 2, \\ \text{and } 0 < \|\varphi w_2\|_{L^2} \leqslant R\|\varphi w_1\|_{L^2} \end{array} \right\},$$

$$M_2^a = \sup_{t > a} \left( t^{\frac{1-\kappa}{2}} \left\| \frac{w}{(w_1^2 + t|w_2|^2)^{\frac{1}{2}}} \right\|_{L^{\frac{2q}{2-q}}} \right).$$

Then we have $M_1^R \leqslant (1-\kappa)^{\frac{\kappa-1}{2}} \kappa^{-\frac{\kappa}{2}} M_2^{\frac{(1-\kappa)}{\kappa R^2}}$.

To prove the above statement we use again the observation that for any $A, B > 0$, the function $f(t) = At^{\kappa-1} + Bt^{\kappa}$ achieves its minimum on $(0, \infty)$ at $t_{min} = \frac{(1-\kappa)A}{\kappa B}$ and $f(t_{min}) = (1-\kappa)^{\kappa-1}\kappa^{-\kappa}A^{\kappa}B^{1-\kappa}$. If $\varphi$ is a measurable function satisfying $\varphi w_i \in L^2(X)$ for $i = 1, 2$ and $0 < \|\varphi w_2\|_{L^2} \leqslant R\|\varphi w_1\|_{L^2}$, taking $A = \|\varphi w_1\|_{L^2}^2$, $B = \|\varphi w_2\|_{L^2}^2$ and $t_{min} = \frac{(1-\kappa)\|\varphi w_1\|_{L^2}^2}{\kappa\|\varphi w_2\|_{L^2}^2}$, we see that (2.2) holds and, moreover, $t_{min} \geqslant \frac{(1-\kappa)}{\kappa R^2}$. Then we use (2.2) and (2.4) and we proceed exactly as in the proof of Lemma 2.1.

Assume that $P_1$ and $P_2$ are radial, that is $P_i(\xi) = p_i(|\xi|)$ for $i = 1, 2$. Then we have equality in (2.33) and the condition $\|P_2(D)u\|_{L^2} \leqslant R\|P_1(D)u\|_{L^2}$ is equivalent to $\|p_2(|\cdot|)\widehat{u}\|_{L^2(\mathbf{R}^N)} \leqslant R\|p_1(|\cdot|)\widehat{u}\|_{L^2(\mathbf{R}^N)}$ and to $\|p_2 z_u\|_{L^2(0,\infty)} \leqslant R\|p_1 z_u\|_{L^2(0,\infty)}$. We infer that

$$\sup \left\{ \frac{\|u\|_{L^p}}{\|P_1(D)u\|_{L^2}^{\kappa}\|P_2(D)u\|_{L^2}^{1-\kappa}} \;\middle|\; u \in \mathcal{S}(\mathbf{R}^N), 0 < \|P_2(D)u\|_{L^2} \leqslant R\|P_2(D)u\|_{L^2} \right\} \tag{2.36}$$

is finite if one of the functions $H$ or $K$ defined in (2.34) and in (2.35) is bounded on $[\frac{(1-\kappa)}{\kappa R^2}, \infty)$. If $H(t)$ (respectively $K(t)$) is finite for some $t > 0$, it suffices to verify the boundedness of $H$ (respectively of $K$) in a neighbourhood of infinity. Of course, having explicit bounds on $H$ or on $K$ on the interval $[\frac{(1-\kappa)}{\kappa R^2}, \infty)$ would provide explicit bounds on the supremum in (2.36).

Remark 2.9 enables us to state the following quantitative variant of Theorems 2.3 and 2.6.

**Corollary 2.10** *Let $Q_\kappa$ be as in (2.8). The supremum*

$$\sup \left\{ Q_\kappa(u) \;\middle|\; u \in H^s(\mathbf{R}^N) \setminus \{0\} \text{ and } \|(|D|^s - 1)u\|_{L^2} \leqslant R\|u\|_{L^2} \right\} \tag{2.37}$$

*is finite for any fixed $R > 0$ if*

$$\frac{1}{p} > \frac{1}{2} - \frac{s}{N}, \qquad \kappa \geqslant \frac{1}{2} \qquad \textit{and} \qquad 1-\kappa \leqslant \frac{N+1}{2}\left(\frac{1}{2} - \frac{1}{p}\right). \tag{2.38}$$

*Proof.* We have already seen in the proofs of Theorems 2.3 and 2.6 that the functions $F$ given by (2.12) and $G$ given by (2.24) are well-defined and continuous on $(0, \infty)$ if $\frac{1}{p} > \frac{1}{2} - \frac{s}{N}$.

In the proof of Theorem 2.3 it is shown that $F$ is bounded in a neighbourhood of infinity if $1 - \kappa \leqslant \frac{1}{2} - \frac{1}{p}$, and Remark 2.9 above implies Corollary 2.10 in dimension $N = 1$.

Assume that $N \geqslant 2$. In the case $p \geqslant \frac{2(N+1)}{N-1}$, it is shown in the proof of Theorem 2.6 that the function $G$ is bounded near infinity if $\kappa \geqslant \frac{1}{2}$ and this proves Corollary 2.10. In the case $\frac{2(N+1)}{N+4\kappa-3} \leqslant p < \frac{2(N+1)}{N-1}$, the conclusion follows from the case $p = \frac{2(N+1)}{N-1}$ by interpolation, using (2.25). □

# 3 Global minimisation of the energy at fixed $L^2-$norm

In this section we study the minimisation problem $(\mathcal{P}_m)$. Recall that $E_{min}$ has been introduced in (1.4). Scaling properties of various terms appearing in $E$ will be important. It is easily seen that for any function $u \in H^2(\mathbf{R}^N)$ and for any $a, b > 0$, letting $u_{a,b}(x) = au\left(\frac{x}{b}\right)$ we have

$$\begin{aligned}
&\int_{\mathbf{R}^N} |\Delta u_{a,b}|^2\,dx = a^2 b^{N-4}\int_{\mathbf{R}^N} |\Delta u|^2\,dx, \quad \int_{\mathbf{R}^N} |\nabla u_{a,b}|^2\,dx = a^2 b^{N-2}\int_{\mathbf{R}^N} |\nabla u|^2\,dx,\\
&\int_{\mathbf{R}^N} |u_{a,b}|^{2\sigma+2}\,dx = a^{2\sigma+2} b^{N}\int_{\mathbf{R}^N} |u|^2\,dx, \qquad \int_{\mathbf{R}^N} |u_{a,b}|^2\,dx = a^2 b^N \int_{\mathbf{R}^N} |u|^2\,dx.
\end{aligned} \tag{3.1}$$

Using the Plancherel Theorem we have for all $u \in H^2(\mathbf{R}^N)$

$$\|\Delta u\|_{L^2}^2 = \frac{1}{(2\pi)^N}\||\xi|^2\widehat{u}\|_{L^2}^2 = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} |\xi|^4 |\widehat{u}(\xi)|^2\,d\xi \quad \text{and} \quad \left\|\frac{\partial^2 u}{\partial x_j \partial x_k}\right\|_{L^2}^2 = \frac{1}{(2\pi)^N}\left\|\xi_j\xi_k\widehat{u}\right\|_{L^2}^2.$$

It is then obvious that $\left\|\frac{\partial^2 u}{\partial x_j \partial x_k}\right\|_{L^2} \leqslant \|\Delta u\|_{L^2}$. We have also the interpolation inequality

$$\|\nabla u\|_{L^2}^2 = \frac{1}{(2\pi)^N}\||\cdot|\widehat{u}\|_{L^2}^2 \leqslant \frac{1}{(2\pi)^N}\||\cdot|^2\widehat{u}\|_{L^2}\|\widehat{u}\|_{L^2} = \|\Delta u\|_{L^2}\|u\|_{L^2}. \tag{3.2}$$

Notice that we have *strict* inequality in (3.2), except for $u = 0$.

We denote $2^{**} = \infty$ if $N \leqslant 4$ and $2^{**} = \frac{2N}{N-4}$ if $N \geqslant 5$. It is well-known (see, e.g., [8] section 9.3) that $H^2(\mathbf{R}^N) \subset L^\infty(\mathbf{R}^N)$ if $N \leqslant 3$, $H^2(\mathbf{R}^N) \subset L^p(\mathbf{R}^N)$ for any $p \in [2, \infty)$ if $N = 4$ and $H^2(\mathbf{R}^N) \subset L^{2^{**}}(\mathbf{R}^N)$ if $N \geqslant 5$. Moreover, in the latter case we have the Sobolev inequality $\|u\|_{L^{2^{**}}} \leqslant C_S\|\Delta u\|_{L^2}$ for any $u \in H^2(\mathbf{R}^N)$.

For any $\sigma \in [0, \frac{2^{**}}{2} - 1)$ we have the Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant B\|\Delta u\|_{L^2}^{\frac{\sigma N}{2}}\|u\|_{L^2}^{2+2\sigma-\frac{\sigma N}{2}} \qquad \text{for all } u \in H^2(\mathbf{R}^N), \tag{3.3}$$

where $B$ is independent of $u$ (see e.g. [21] or the end of Remark 2.8). We denote by $B(N, \sigma)$ the best possible value of the constant $B$ in (3.3), namely

$$B(N,\sigma) = \sup_{u \in H^2(\mathbf{R}^N),\ u \neq 0} \frac{\|u\|_{L^{2\sigma+2}}^{2\sigma+2}}{\|\Delta u\|_{L^2}^{\frac{\sigma N}{2}}\|u\|_{L^2}^{2+2\sigma-\frac{\sigma N}{2}}}. \tag{3.4}$$

It is also well known that there exist optimal functions for (3.4); that is, the supremum in (3.4) is, in fact, a maximum (see, e.g., Example 3.10 in [19]).

The properties of the function $E_{min}$ will be crucial in the sequel. They are summarized in the next Proposition and in the remark following it.

**Proposition 3.1** *The function $m \longmapsto E_{min}(m)$ has the following properties:*

*(i) If $\sigma N > 4$ we have $E_{min}(m) = -\infty$ for all $m > 0$.*

*For the following statements we assume that $0 < \sigma N \leqslant 4$. We have:*

*(ii) The function $E_{min}$ is concave on $(0, \infty)$.*

*(iii) For any $m > 0$ there holds $E_{min}(m) \leqslant -m$.*

*(iv) $\lim_{m \downarrow 0} E_{min}(m) = 0$ and $\lim_{m \downarrow 0} \dfrac{E_{min}(m)}{m} = -1$.*

*(v) If $0 < \sigma N < 4$ we have $E_{min}(m) > -\infty$ for all $m > 0$ and there exist $A \in \mathbf{R}$, $B > 0$ such that $E_{min}(m) < Am - Bm^{\sigma+1}$ (thus, in particular, $\dfrac{E_{min}(m)}{m} \longrightarrow -\infty$ as $m \longrightarrow \infty$). Moreover, for any $k_1,\ k_2 > 0$ the set $\{u \in H^2(\mathbf{R}^N) \mid \|u\|_{L^2} \leqslant k_1 \text{ and } E(u) \leqslant k_2\}$ is bounded in $H^2(\mathbf{R}^N)$.*

*(vi) Assume that $\sigma N = 4$. Let $B(N, \sigma)$ be as in (3.4) and let $k_* = (\sigma+1)^{\frac{1}{\sigma}} B(N,\sigma)^{-\frac{1}{\sigma}}$.*
*Then $E_{min}(m)$ is finite for any $m \in (0, k_*)$ and $E_{min}(m) = -\infty$ if $m \geqslant k_*$.*
*In addition, for any $k_1 < k_*$ and any $k_2 > 0$, the set $\big\{u \in H^2(\mathbf{R}^N) \mid \|u\|_{L^2}^2 \leqslant k_1 \text{ and } E(u) \leqslant k_2\big\}$ is bounded in $H^2(\mathbf{R}^N)$.*

**Remark 3.2** The function $E_{min}$ is finite and concave on $(0, \infty)$ if $\sigma N < 4$, respectively on $(0, k_*)$ if $\sigma N = 4$, hence it is continuous and admits left and right derivatives at any point of these intervals. We denote by $E'_{min,\ell}(m)$ and $E'_{min,r}(m)$, respectively, the left and right derivatives of $E_{min}$ at $m$. The functions $E'_{min,\ell}$ and $E'_{min,r}$ are nonincreasing, we have $E'_{min,\ell}(m) \geqslant E'_{min,r}(m)$ for all $m$ and equality must occur at all but countably many $m$'s. Proposition 3.1 (iv) implies that

$$\lim_{m \downarrow 0} E'_{min,\ell}(m) = \sup_{m>0} E'_{min,\ell}(m) = \lim_{m \downarrow 0} E'_{min,r}(m) = \sup_{m>0} E'_{min,r}(m) = -1.$$

Let

$$m_0 := \sup\{m > 0 \mid E_{min}(m) = -m\}. \tag{3.5}$$

It is clear that $E_{min}(m) = -m$ on $(0, m_0)$ and $E_{min}(m) < -m$ on $(m_0, \infty)$. If $m > m_0$ and $E_{min}(m) > -\infty$ we must have $E'_{min,\ell}(m) < -1$. If $\sigma N < 4$ we have $m_0 < \infty$ and $\lim_{m \to \infty} E'_{min,\ell}(m) = \lim_{m \to \infty} E'_{min,r}(m) = -\infty$ because $\lim_{m \to \infty} \frac{E_{min}(m)}{m} = -\infty$ by Proposition 3.1 (v).

*Proof of Proposition 3.1.* (i) Let $m > 0$. Choose $u \in H^2(\mathbf{R}^N)$ satisfying $\|u\|_{L^2}^2 = m$. We use (3.1) with $u_{a,b} = au\left(\frac{\cdot}{b}\right)$ and $a = t^{\frac{N}{4}}$, $b = t^{-\frac{1}{2}}$. It is obvious that $\|u_{t^{N/4},\, t^{-1/2}}\|_{L^2}^2 = \|u\|_{L^2}^2 = m$ for any $t > 0$, and consequently $E_{min}(m) \leqslant E(u_{t^{N/4},\, t^{-1/2}})$ for all $t$. From (3.1) we have

$$E(u_{t^{N/4},\, t^{-1/2}}) = t^2 \int_{\mathbf{R}^N} |\Delta u|^2 \, dx - 2t \int_{\mathbf{R}^N} |\nabla u|^2 \, dx - t^{\frac{N\sigma}{2}} \int_{\mathbf{R}^N} |u|^{2\sigma+2} \, dx. \tag{3.6}$$

If $N\sigma > 4$, letting $t \longrightarrow \infty$ we discover $E_{min}(m) \leqslant \lim_{t \to \infty} E(u_{t^{N/4},\, t^{-1/2}}) = -\infty$.

(ii) It is obvious that $u \in S(m)$ if and only if there exists $v \in S(1)$ such that $u = \sqrt{m} v$. Hence for any $m > 0$ we have

$$\begin{aligned} E_{min}(m) &= \inf\{E(\sqrt{m} v) \mid v \in S(1)\} \\ &= \inf\left\{ m \left( \int_{\mathbf{R}^N} |\Delta v|^2 \, dx - 2 \int_{\mathbf{R}^N} |\nabla v|^2 \, dx \right) - \frac{m^{\sigma+1}}{\sigma+1} \int_{\mathbf{R}^N} |v|^{2\sigma+2} \, dx \;\Big|\; v \in S(1) \right\}. \end{aligned}$$

For any $A \in \mathbf{R}$ and any $B \geqslant 0$ the function $m \longmapsto Am - Bm^{\sigma+1}$ is concave on $(0, \infty)$. The infimum of a family of concave functions is also a concave function and statement (ii) follows.

(iii) Let $m > 0$ and $\varepsilon > 0$. Choose a function $\eta \in C_c^\infty(\mathbf{R}^N)$ such that $\|\eta\|_{L^2}^2 = (2\pi)^N m$ and the support of $\eta$ is contained in the annulus $B(0,1) \setminus B(0, 1-\varepsilon)$. Let $u = \mathcal{F}^{-1}(\eta)$. Then $u \in \mathcal{S}(\mathbf{R}^N)$ and $\|u\|_{L^2}^2 = \frac{1}{(2\pi)^N}\|\eta\|_{L^2}^2 = m$. Using the basic properties of the Fourier transform, Plancherel's formula and the fact that $0 \leqslant \left(|\xi|^2 - 1\right)^2 \leqslant 4\varepsilon^2$ on the support of $\eta$ we get

$$\int_{\mathbf{R}^N} |\Delta u|^2\, dx - 2\int_{\mathbf{R}^N} |\nabla u|^2\, dx + \int_{\mathbf{R}^N} |u|^2\, dx = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} \left(|\xi|^4 - 2|\xi|^2 + 1\right)|\widehat{u}(\xi)|^2\, d\xi$$

$$= \frac{1}{(2\pi)^N}\int_{B(0,1)\setminus B(0,1-\varepsilon)} \left(1 - |\xi|^2\right)^2 |\eta(\xi)|^2\, d\xi \leqslant \frac{4\varepsilon^2}{(2\pi)^N}\int_{\mathbf{R}^N} |\eta(\xi)|^2\, d\xi = 4\varepsilon^2 m.$$

We infer that

$$E_{min}(m) + m \leqslant E(u) + \|u\|_{L^2}^2 \leqslant \int_{\mathbf{R}^N} |\Delta u|^2\, dx - 2\int_{\mathbf{R}^N} |\nabla u|^2\, dx + \int_{\mathbf{R}^N} |u|^2\, dx \leqslant 4\varepsilon^2 m,$$

that is $E_{min}(m) \leqslant -m + 4\varepsilon^2 m$. Since $\varepsilon > 0$ is arbitrary, (iii) follows.

(iv) Consider first the case $0 < \sigma N < 4$. Let $0 < \varepsilon < 1$. Using the Gagliardo-Nirenberg-Sobolev inequality (3.3) and Young's inequality ($|ab| \leqslant \frac{|a|^p}{p} + \frac{|b|^q}{q}$ if $\frac{1}{p} + \frac{1}{q} = 1$) with exponents $p = \frac{4}{\sigma N}$ and $q = \frac{4}{4 - \sigma N}$, we get for any $v \in H^2(\mathbf{R}^N)$

$$\frac{1}{\sigma+1}\|v\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant B\frac{1}{\sigma+1}\|\Delta v\|_{L^2}^{\frac{\sigma N}{2}}\|v\|_{L^2}^{2+2\sigma-\frac{\sigma N}{2}} \leqslant \varepsilon\|\Delta v\|_{L^2}^2 + C_1(\varepsilon)\|v\|_{L^2}^{\frac{8(\sigma+1)-2\sigma N}{4-\sigma N}},$$

where $C_1(\varepsilon)$ is independent of $v$. It follows that

$$E(v) \geqslant (1-\varepsilon)\|\Delta v\|_{L^2}^2 - 2\|\nabla v\|_{L^2}^2 - C_1(\varepsilon)\|v\|_{L^2}^{\frac{8(\sigma+1)-2\sigma N}{4-\sigma N}}. \tag{3.7}$$

Using Plancherel's formula we get

$$(1-\varepsilon)\|\Delta v\|_{L^2}^2 - 2\|\nabla v\|_{L^2}^2 + \frac{1}{1-\varepsilon}\|v\|_{L^2}^2 = \frac{1-\varepsilon}{(2\pi)^N}\int_{\mathbf{R}^N}\left(|\xi|^2 - \frac{1}{1-\varepsilon}\right)^2 |\widehat{v}(\xi)|^2\, d\xi \geqslant 0. \tag{3.8}$$

Notice that the inequality in (3.8) is strict if $u \neq 0$. From (3.7) and (3.8) we get

$$E(v) \geqslant -\frac{1}{1-\varepsilon}\|v\|_{L^2}^2 - C_1(\varepsilon)\|v\|_{L^2}^{\frac{8(\sigma+1)-2\sigma N}{4-\sigma N}} \qquad \text{for all } v \in H^2(\mathbf{R}^N). \tag{3.9}$$

Taking the infimum in (3.9) over all $v \in H^2(\mathbf{R}^N)$ satisfying $\|v\|_{L^2}^2 = m$ we discover

$$E_{min}(m) \geqslant -\frac{m}{1-\varepsilon} - C_1(\varepsilon)m^{\frac{4(\sigma+1)-\sigma N}{4-\sigma N}} \qquad \text{for any } m > 0. \tag{3.10}$$

From (iii) and (3.10) it follows that $E_{min}(m) \longrightarrow 0$ as $m \longrightarrow 0$ and $-\dfrac{1}{1-\varepsilon} \leqslant \liminf\limits_{m\downarrow 0}\dfrac{E_{min}(m)}{m} \leqslant \limsup\limits_{m\downarrow 0}\dfrac{E_{min}(m)}{m} \leqslant -1$. Since $\varepsilon$ is arbitrary, (iv) is proven.

Next consider the case $\sigma N = 4$. The Gagliardo-Nirenberg-Sobolev inequality (3.3) becomes

$$\|v\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant B\|\Delta v\|_{L^2}^2\|v\|_{L^2}^{2\sigma}. \tag{3.11}$$

Let $0 < \varepsilon < 1$. Using (3.11), for any $v \in H^2(\mathbf{R}^N)$ satisfying $\frac{B}{\sigma+1}\|v\|_{L^2}^{2\sigma} \leqslant \varepsilon$ we get

$$E(v) \geqslant (1-\varepsilon)\|\Delta v\|_{L^2}^2 - 2\|\nabla v\|_{L^2}^2$$

and then using (3.8) we obtain $E(v) \geqslant -\frac{1}{1-\varepsilon}\|v\|_{L^2}^2$. This gives $E_{min}(m) \geqslant -\frac{m}{1-\varepsilon}$ for all $m > 0$ satisfying $\frac{B}{\sigma+1}m^{2\sigma} \leqslant \varepsilon$. Taking into account (iii), statement (iv) is now obvious.

(v) Using (3.10) with $\varepsilon = \frac{1}{2}$, say, it is clear that $E_{min}(m) > -\infty$ for any $m > 0$.

Fix $u \in H^2(\mathbf{R}^N)$ such that $\|u\|_{L^2} = 1$. Let $A = \int_{\mathbf{R}^N} |\Delta u|^2\,dx - 2\int_{\mathbf{R}^N} |\Delta u|^2\,dx$ and $B = \frac{1}{\sigma+1}\int_{\mathbf{R}^N} |u|^{2\sigma+2}\,dx$. For all $m > 0$ we have $\|m^{\frac{1}{2}}u\|_{L^2}^2 = m$, hence $E_{min}(m) \leqslant E(u) = Am - Bm^{\sigma+1}$.

If $\|u\|_{L^2} \leqslant k_1$ and $E(u) \leqslant k_2$, using (3.2) and (3.3) we get

$$k_1 \geqslant \|\Delta u\|_{L^2}^2 - 2k_1\|\Delta u\|_{L^2} - \frac{B}{\sigma+1}k_1^{2+2\sigma-\frac{\sigma N}{2}}\|\Delta u\|_{L^2}^{\frac{\sigma N}{2}}.$$

Since $\frac{\sigma N}{2} < 2$, the above inequality implies that $\|\Delta u\|_{L^2}$ is bounded. Then (3.2) and the inequality $\left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|_{L^2} \leqslant \|\Delta u\|_{L^2}$ imply that $\|u\|_{H^2(\mathbf{R}^N)}$ is bounded.

(vi) Assume $\sigma N = 4$. By (3.2) and (3.3) we have for all $u \in H^2(\mathbf{R}^N)$

$$(3.12)\qquad \begin{aligned} E(u) &\geqslant \|\Delta u\|_{L^2}^2 - 2\|u\|_{L^2}\|\Delta u\|_{L^2} - \tfrac{B(N,\sigma)}{\sigma+1}\|\Delta u\|_{L^2}^2\|u\|_{L^2}^{2\sigma} \\ &= \left(1 - \tfrac{B(N,\sigma)}{\sigma+1}\|u\|_{L^2}^{2\sigma}\right)\|\Delta u\|_{L^2}^2 - 2\|u\|_{L^2}\|\Delta u\|_{L^2}. \end{aligned}$$

Let $k_1 < k_* = (\sigma+1)^{\frac{1}{\sigma}}B(N,\sigma)^{-\frac{1}{\sigma}}$. Let $\tau(k_1) = 1 - \frac{B(N,\sigma)}{\sigma+1}k_1^{\sigma} > 0$. For any $u \in H^2(\mathbf{R}^N)$ such that $\|u\|_{L^2}^2 \leqslant k_1$, by (3.12) we get

$$(3.13)\qquad E(u) \geqslant \tau(k_1)\|\Delta u\|_{L^2}^2 - 2k_1\|\Delta u\|_{L^2} \geqslant \min_{s\in\mathbf{R}}(\tau(k_1)s^2 - 2k_1 s) = -\frac{k_1^2}{\tau(k_1)}.$$

We infer that $E_{min}(m) \geqslant -\frac{k_1^2}{\tau(k_1)} > -\infty$ for all $m \in (0, k_1]$. Since $k_1 < k_*$ was arbitrary, we see that $E_{min}$ is finite on $(0, k_*)$. Moreover, if $\|u\|_{L^2}^2 \leqslant k_1 < k_*$ and $E(u) \leqslant k_2$ then (3.13) implies that $\|\Delta u\|_{L^2}$ is bounded and arguing as in part (v) we see that $\|u\|_{H^2(\mathbf{R}^N)}$ is bounded.

Let $Q$ be an optimal function for the Gagliardo-Nirenberg-Sobolev inequality (3.3) with $\sigma = \frac{4}{N}$ such that $\|Q\|_{L^2} = k_*^{\frac{1}{2}} = (\sigma+1)^{\frac{1}{2\sigma}}B(N,\sigma)^{-\frac{1}{2\sigma}}$. Such a function $Q$ exists because whenever $u$ is an optimal function for (3.3), the rescaled functions $u_{a,b}(\cdot) = au\left(\frac{\cdot}{b}\right)$are optimal functions, too. We have

$$\frac{1}{\sigma+1}\|Q\|_{L^{2\sigma+2}}^{2\sigma+2} = \frac{1}{\sigma+1}B(N,\sigma)\|\Delta Q\|_{L^2}^2\|Q\|_{L^2}^{2\sigma} = \|\Delta Q\|_{L^2}^2.$$

For $t > 0$ let $u_t(x) = t^{\frac{N}{4}}Q(t^{\frac{1}{2}}x)$. From (3.1) and (3.6) it follows that $\|u_t\|_{L^2}^2 = \|Q\|_{L^2}^2 = k_*$ and $E(u_t) = -2t\|\nabla Q\|_{L^2}^2$. Letting $t \longrightarrow \infty$ we discover $E_{min}(k_*) = -\infty$. If $m > k_*$, using the test function $m^{\frac{1}{2}}k_*^{-\frac{1}{2}}u_t$ and letting $t \longrightarrow \infty$ we find $E_{min}(m) = -\infty$. □

As we will see later, problem $(\mathcal{P}_m)$ admits solutions if and only if $-\infty < E_{min}(m) < -m$. We have already seen that $E_{min}(m) = -\infty$ for all $m$ if $\sigma > \frac{4}{N}$. Proposition 3.3 gives necessary and sufficient conditions to have $E_{min}(m) < -m$ whenever $\sigma \in (0, \frac{4}{N}]$. Its proof relies on the functional inequalities proved in Section 2 and on the test functions constructed there.

**Proposition 3.3** *Let $N \in \mathbf{N}^*$. We have:*

*(i) If $0 < \sigma < \max\left(1, \frac{4}{N+1}\right)$, then $E_{min}(m) < -m$ for all $m > 0$.*

*(ii) If $\max\left(1, \frac{4}{N+1}\right) \leqslant \sigma \leqslant \frac{4}{N}$, there exists $m_0 > 0$ such that $E_{min}(m) = -m$ for any $m \in (0, m_0]$ and $E_{min}(m) < -m$ for any $m > m_0$. Moreover, $m_0$ is given by (3.16) below.*

*(iii) Assume that $\sigma = \frac{4}{N}$ and let $k_*$ be as in Proposition 3.1 (vi). Let $m_0$ be as in (3.16). Then $m_0 < k_*$ and we have $E_{min}(m) = -m$ if $m \in (0, m_0]$, respectively $-\infty < E_{min}(m) < -m$ if $m \in (m_0, k_*)$.*

*Proof.* (i) Assume that $N = 1$ and $0 < \sigma < 2$. Fix $m > 0$ and let $u_\tau$ be as in Example 2.4. Denote $v_\tau = \pi^{-\frac{1}{4}}\tau^{-\frac{1}{2}}m^{\frac{1}{2}}u_\tau$. By (2.13) we have $\|v_\tau\|_{L^2}^2 = m$, $\|v_\tau\|_{L^{2\sigma+2}}^{2\sigma+2} = (\sigma+1)^{-\frac{1}{2}}\pi^{-\frac{\sigma}{2}}m^{\sigma+1}\tau^{-\sigma}$ and (2.14) gives $\|(\Delta+1)v_\tau\|_{L^2}^2 \sim Cm\tau^{-2}$ as $\tau \longrightarrow \infty$. For $\tau$ sufficiently large we have $E(v_\tau) + \|v_\tau\|_{L^2}^2 = \|(\Delta+1)v_\tau\|_{L^2}^2 - \frac{1}{\sigma+1}\|v_\tau\|_{L^{2\sigma+2}}^{2\sigma+2} < 0$, and this implies $E_{min}(m) < -m$.

If $N \geqslant 2$, for small $\varepsilon, \delta > 0$ let $u_{\varepsilon,\delta}$ be as in (2.26). Denote $w_{\varepsilon,\delta} = \frac{\sqrt{m}}{\|u_{\varepsilon,\delta}\|_{L^2}}u_{\varepsilon,\delta}$, so that $\|w_{\varepsilon,\delta}\|_{L^2}^2 = m$. By (2.27) and (2.28) we have $\|(\Delta+1)w_{\varepsilon,\delta}\|_{L^2}^2 \sim m\varepsilon^2$, while (2.27) and (2.30) give $\|w_{\varepsilon,\delta}\|_{L^{2\sigma+2}}^{2\sigma+2} \geqslant Cm^{\sigma+1}\varepsilon^{\sigma+1}\delta^{\sigma(N-1)}(\delta^2+\varepsilon)^{-1}$ for some $C > 0$.

If $\sigma \in (0,1)$, fix a small $\delta_0 > 0$ and observe that

$$E(w_{\varepsilon,\delta_0}) + \|w_{\varepsilon,\delta_0}\|_{L^2}^2 = \|(\Delta+1)w_{\varepsilon,\delta_0}\|_{L^2}^2 - \frac{1}{\sigma+1}\|w_{\varepsilon,\delta_0}\|_{L^{2\sigma+2}}^{2\sigma+2} < 0 \tag{3.14}$$

if $\varepsilon$ is sufficiently small, hence $E_{min}(m) + m < 0$.

If $\sigma \in \left(0, \frac{4}{N+1}\right)$, taking $\varepsilon = \delta^2$ it follows from the above estimates that $\|(\Delta+1)w_{\delta^2,\delta}\|_{L^2}^2 \sim m\delta^4$ and $\|w_{\delta^2,\delta}\|_{L^{2\sigma+2}}^{2\sigma+2} \geqslant Cm^{\sigma+1}\delta^{\sigma(N+1)}$ for some $C > 0$ and any small $\delta > 0$. As in (3.14), this implies $E(w_{\delta^2,\delta}) + \|w_{\delta^2,\delta}\|_{L^2}^2 < 0$ for sufficiently small $\delta$, and consequently $E_{min}(m) + m < 0$.

(ii) It is easy to see that for any $u \in H^2(\mathbf{R}^N)$ we have

$$E(u)+\|u\|_{L^2}^2 = \|(\Delta+1)u\|_{L^2}^2 - \frac{1}{\sigma+1}\|u\|_{L^{2\sigma+2}}^{2\sigma+2} = \|(\Delta+1)u\|_{L^2}^2\left(1 - \frac{\|u\|_{L^2}^{2\sigma}}{\sigma+1}\frac{\|u\|_{L^{2\sigma+2}}^{2\sigma+2}}{\|u\|_{L^2}^{2\sigma}\|(\Delta+1)u\|_{L^2}^2}\right).$$

Let $\kappa = \frac{\sigma}{\sigma+1}$ and $Q_\kappa(u) = \dfrac{\|u\|_{L^{2\sigma+2}}}{\|u\|_{L^2}^{\kappa}\|(\Delta+1)u\|_{L^2}^{1-\kappa}}$ (see (2.8)). The above equality can be written as

$$E(u) + \|u\|_{L^2}^2 = \|(\Delta+1)u\|_{L^2}^2\left(1 - \frac{\|u\|_{L^2}^{2\sigma}}{\sigma+1}Q_\kappa(u)^{2\sigma+2}\right) \qquad \text{for all } u \in H^2(\mathbf{R}^N) \setminus \{0\}. \tag{3.15}$$

We use the results in Section 2 with $s = 2$, $p = 2\sigma+2$ and $\kappa = \frac{\sigma}{\sigma+1}$.

If $N = 1$, condition $\frac{1}{2s} \leqslant \frac{(1-\kappa)p}{p-2} \leqslant \frac{1}{2}$ in Theorem 2.3 is equivalent to $4 \geqslant \sigma \geqslant 2$. Hence $Q_\kappa$ is bounded from above if $\sigma \in [2,4]$.

If $N \geqslant 2$, the condition $\kappa \geqslant \frac{1}{2}$ in (2.18) is equivalent to $\sigma \geqslant 1$, $\frac{N}{s}\left(\frac{1}{2} - \frac{1}{p}\right) \leqslant 1-\kappa$ is equivalent to $\sigma \leqslant \frac{4}{N}$, and $1-\kappa \leqslant \frac{N+1}{2}\left(\frac{1}{2} - \frac{1}{p}\right)$ is equivalent to $\sigma \geqslant \frac{4}{N+1}$. By Theorem 2.6, $Q_\kappa$ is bounded from above if and only if $\max\left(1, \frac{4}{N+1}\right) \leqslant \sigma \leqslant \frac{4}{N}$.

Whenever $Q_\kappa$ is bounded from above, let $M = \sup\limits_{u \in H^s(\mathbf{R}^N)\setminus\{0\}} Q_\kappa(u)$, as in (2.9), and let

$$m_0 = (\sigma+1)^{\frac{1}{\sigma}}M^{-\frac{2\sigma+2}{\sigma}}. \tag{3.16}$$

If $m \in (0, m_0]$, using (3.15) we infer that $E(u) + \|u\|_{L^2}^2 \geqslant 0$ for any $u \in H^2(\mathbf{R}^N)$ satisfying $\|u\|_{L^2}^2 = m$, hence $E_{min}(m) + m \geqslant 0$. Then Proposition 3.1 (iii) implies $E_{min}(m) = -m$.

If $m > m_0$, we have $(\sigma+1)^{\frac{1}{2\sigma+2}}m^{-\frac{\sigma}{2\sigma+2}} < M$. Choose $u \in H^2(\mathbf{R}^N)$, $u \neq 0$, such that $Q_\kappa(u) > (\sigma+1)^{\frac{1}{2\sigma+2}}m^{-\frac{\sigma}{2\sigma+2}}$. Let $v = \frac{\sqrt{m}}{\|u\|_{L^2}}u$, so that $\|v\|_{L^2}^2 = m$ and $Q_\kappa(v) = Q_\kappa(u)$. From (3.15) we get $E(v) + \|v\|_{L^2}^2 < 0$, hence $E_{min}(m) < -m$.

(iii) Taking into account (3.16) and the expression of $k_*$ in Proposition 3.1 (vi), the inequality $m_0 < k_*$ is equivalent to $B(N,\sigma) < M^{2\sigma+2}$, where $B(N,\sigma)$ is given by (3.4). Denote by $\mathcal{Q}(u)$ the quotient appearing in (3.4). Let $u_*$ be an optimal function for (3.4). Then $u_\tau = u_*\left(\frac{\cdot}{\sqrt{\tau}}\right)$ is also an optimal function for (3.4), that is $\mathcal{Q}(u_\tau) = B(N,\sigma)$ for all $\tau > 0$. The conclusion follows if we find $\tau > 0$ such that $\mathcal{Q}(u_\tau) < Q_\kappa(u_\tau)^{2\sigma+2}$, and this is equivalent to $\|(\Delta+1)u_\tau\|_{L^2}^2 < \|\Delta u_\tau\|_{L^2}^2$, or using Plancherel's theorem, $\int_{\mathbf{R}^N}\left(|\xi|^2 - \tau\right)^2|\widehat{u}_*|^2(\xi)\,d\xi < \int_{\mathbf{R}^N}|\xi|^4|\widehat{u}_*|^2(\xi)\,d\xi$. The last inequality can be written as

$$-2\tau\int_{\mathbf{R}^N}|\xi|^2|\widehat{u}_*|^2(\xi)\,d\xi + \tau^2\int_{\mathbf{R}^N}|\widehat{u}_*|^2(\xi)\,d\xi < 0$$

and holds true if $0 < \tau < \frac{2\| |\cdot|\widehat{u}_*\|^2_{L^2}}{\|\widehat{u}_*\|^2_{L^2}}$. We have thus shown that $m_0 < k_*$. The rest follows from part (ii) and Proposition 3.1 (vi). $\Box$

The next Theorem establishes the existence of minimizers for the problem $(\mathcal{P}_m)$ as well as the pre-compactness modulo translations of all minimizing sequences.

**Theorem 3.4** *Assume that $N\sigma < 4$ and $m > 0$ is such that $E_{min}(m) < -m$.*

*Then for any sequence $(u_n)_{n\geqslant 1} \subset H^2(\mathbf{R}^N)$ satisfying $M(u_n) \longrightarrow m$ and $E(u_n) \longrightarrow E_{min}(m)$ there exist a subsequence, still denoted $(u_n)_{n\geqslant 1}$, a sequence of points $(x_n)_{n\geqslant 1} \subset \mathbf{R}^N$ and a function $u \in H^2(\mathbf{R}^N)$ such that $u_n(\cdot + x_n) \longrightarrow u$ strongly in $H^2(\mathbf{R}^N)$.*

*In particular, there exists a solution $u \in H^2(\mathbf{R}^N)$ to the minimization problem $(\mathcal{P}_m)$.*

*The same conclusion holds if $N\sigma = 4$, $0 < m < k_*$ (where $k_*$ is as in Proposition 3.1 (vi)), and $E_{min}(m) < -m$.*

*Proof.* Let $(u_n)_{n\geqslant 1}$ be a minimizing sequence. It follows from Proposition 3.1 (v) or (vi) that $(u_n)_{n\geqslant 1}$ is bounded in $H^2(\mathbf{R}^N)$.

Using (3.8) with $\varepsilon = 0$ we infer that for any $u \in H^2(\mathbf{R}^N)$ there holds

$$\begin{aligned} \int_{\mathbf{R}^N} |u|^{2\sigma+2}\,dx &= (\sigma+1)\int_{\mathbf{R}^N} |\Delta u|^2 - 2|\nabla u|^2 + |u|^2\,dx - (\sigma+1)(E(u) + \|u\|^2_{L^2}) \\ &\geqslant -(\sigma+1)(E(u) + \|u\|^2_{L^2}). \end{aligned} \tag{3.17}$$

Choose $\sigma' > \sigma$ such that $2\sigma' + 2 < 2^{**}$. We denote by $\mathcal{L}^N$ the Lebesgue measure in $\mathbf{R}^N$. Using Hölder's inequality and the Sobolev embedding we get for any $u \in H^2(\mathbf{R}^N)$ and for any $t > 0$,

$$\begin{aligned} &\textstyle\int_{\mathbf{R}^N} |u|^{2\sigma+2}\,dx = \int_{\{|u|<t\}} |u|^{2\sigma+2}\,dx + \int_{\{|u|\geqslant t\}} |u|^{2\sigma+2}\,dx \\ &\leqslant t^{2\sigma}\textstyle\int_{\{|u|<t\}} |u|^2\,dx + \left(\int_{\{|u|\geqslant t\}} |u|^{2\sigma'+2}\,dx\right)^{\frac{\sigma+1}{\sigma'+1}} \mathcal{L}^N\left(\{|u| \geqslant t\}\right)^{1-\frac{\sigma+1}{\sigma'+1}} \\ &\leqslant t^{2\sigma}\|u\|^2_{L^2} + (C_S\|u\|_{H^2})^{2\sigma+2}\,\mathcal{L}^N\left(\{|u| \geqslant t\}\right)^{1-\frac{\sigma+1}{\sigma'+1}}. \end{aligned} \tag{3.18}$$

Choose $\delta > 0$ such that $2\delta < -(\sigma+1)(E_{min}(m)+m)$ (this is possible because $E_{min}(m) < -m$). Since $\|u_n\|^2_{L^2} \longrightarrow m$ and $E(u_n) \longrightarrow E_{min}(m)$, (3.17) implies that $\displaystyle\int_{\mathbf{R}^N} |u_n|^{2\sigma+2}\,dx > 2\delta$ for all $n$ sufficiently large. Choose $t_0 > 0$ such that $t_0^{2\sigma}(m+1) = \delta$. Using (3.18) for $u_n$ and the boundedness of $(u_n)_{n\geqslant 1}$ in $H^2(\mathbf{R}^N)$, we infer that there exists a constant $a > 0$, independent of $n$, such that $\mathcal{L}^N\left(\{|u_n| \geqslant t_0\}\right) \geqslant a$ for all sufficiently large $n$. Using Lieb's Lemma (see Lemma 6 p. 447 in [15] or Appendix 2 in [19]) we infer that there exists a constant $b > 0$, independent of $n$, and for each $n$ large there exists $x_n \in \mathbf{R}^N$ such that

$$\mathcal{L}^N\left(\left\{x \in B(x_n,1) \;\middle|\; |u_n| \geqslant \frac{t_0}{2}\right\}\right) \geqslant b.$$

From now on we replace $u_n$ by $u_n(\cdot + x_n)$, which is still a minimizing sequence and satisfies $\mathcal{L}^N\left(\left\{x \in B(0,1) \;\middle|\; |u_n| \geqslant \frac{t_0}{2}\right\}\right) \geqslant b$. Since $(u_n)_{n\geqslant 1}$ is bounded in $H^2(\mathbf{R}^N)$ there exists a subsequence, still denoted $(u_n)_{n\geqslant 1}$, and there is $u \in H^2(\mathbf{R}^N)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \qquad \text{weakly in } H^2(\mathbf{R}^N), \\ &u_n \longrightarrow u \qquad \text{in } L^p_{loc}(\mathbf{R}^N) \text{ for } 1 \leqslant p < 2^{**} \text{ and a.e.} \end{aligned} \tag{3.19}$$

It is clear that $\int_{B(0,1)} |u_n|^p\,dx \geqslant b\left(\frac{t_0}{2}\right)^p$ for all $n$ sufficiently large. Take any $p \in [1, 2^{**})$ and pass to the limit to get $\int_{B(0,1)} |u|^p\,dx \geqslant b\left(\frac{t_0}{2}\right)^p$. In particular, we infer that $u \neq 0$.

Let $m_1 = \|u\|^2_{L^2}$. It is clear that $0 < m_1 \leqslant \liminf_{n\to\infty} \|u_n\|^2_{L^2} = m$. We will show that $m_1 = m$. We argue by contradiction and we assume that $m_1 < m$. The weak convergence in a Hilbert space gives as $n \longrightarrow \infty$

$$\begin{aligned} \|u_n\|_{L^2}^2 &= \|u\|_{L^2}^2 + \|u_n - u\|_{L^2}^2 + o(1), \\ \|\nabla u_n\|_{L^2}^2 &= \|\nabla u\|_{L^2}^2 + \|\nabla(u_n - u)\|_{L^2}^2 + o(1), \\ \|\Delta u_n\|_{L^2}^2 &= \|\Delta u\|_{L^2}^2 + \|\Delta(u_n - u)\|_{L^2}^2 + o(1). \end{aligned} \tag{3.20}$$

Using the Brezis-Lieb Lemma (see e.g. Lemma 4.6 p. 10 in [14]) we get

$$\int_{\mathbf{R}^N} |u_n|^{2\sigma+2}\,dx = \int_{\mathbf{R}^N} |u|^{2\sigma+2}\,dx + \int_{\mathbf{R}^N} |u_n - u|^{2\sigma+2}\,dx + o(1). \tag{3.21}$$

From (3.20) and (3.21) we get

$$E(u_n) = E(u) + E(u_n - u) + o(1) \qquad \text{as } n \longrightarrow \infty. \tag{3.22}$$

It is obvious that $E(u) \geqslant E_{min}(m_1)$ and $E(u_n - u) \geqslant E_{min}(\|u_n - u\|_{L^2}^2)$. By (3.20) we have $\|u_n - u\|_{L^2}^2 \longrightarrow m - m_1$. The function $E_{min}$ is continuous on $(0, \infty)$ if $0 < \sigma N < 4$, respectively on $(0, k_*)$ if $\sigma N = 4$, and passing to the limit in (3.22) we get

$$E_{min}(m) \geqslant E_{min}(m_1) + E_{min}(m - m_1). \tag{3.23}$$

Since $E_{min}$ is concave and $E_{min}(\eta) \longrightarrow 0$ as $\eta \longrightarrow 0$ we have $E_{min}(m_1) \geqslant \frac{m_1}{m} E_{min}(m)$ and $E_{min}(m - m_1) \geqslant \frac{m - m_1}{m} E_{min}(m)$. Moreover, equality may occur in one of these inequalities if and only if $E_{min}$ is linear on $(0, m]$. Summing up and comparing to (3.23) we infer that necessarily we have $E_{min}(m_1) = \frac{m_1}{m} E_{min}(m)$ and $E_{min}$ must be linear on $(0, m]$. Then Proposition 3.1 (iv) implies that $E_{min}(\eta) = -\eta$ for any $\eta \in (0, m]$, and in particular $E_{min}(m) = -m$, contradicting the fact that $E_{min}(m) < -m$. This contradiction shows that necessarily $m_1 = m$.

Since $u_n \rightharpoonup u$ weakly in $L^2(\mathbf{R}^N)$ and $\|u_n\|_{L^2}^2 \longrightarrow m = \|u\|_{L^2}^2$ we infer that $u_n \longrightarrow u$ strongly in $L^2(\mathbf{R}^N)$. Using (3.2) and (3.3) for $u_n - u$ we infer that $\nabla u_n \longrightarrow \nabla u$ strongly in $L^2(\mathbf{R}^N)$ and $u_n \longrightarrow u$ strongly in $L^{2\sigma+2}(\mathbf{R}^N)$. The weak convergence $u_n \rightharpoonup u$ in $H^2(\mathbf{R}^N)$ gives $\|\Delta u\|_{L^2}^2 \leqslant \liminf_{n\to\infty} \|\Delta u_n\|_{L^2}^2$, and consequently we get $E(u) \leqslant \liminf_{n\to\infty} E(u_n) = E_{min}(m)$. On the other hand we have $E(u) \geqslant E_{min}(m)$ because $\|u\|_{L^2}^2 = m$. Therefore $E(u) = E_{min}(m)$ and $u$ solves the problem $(\mathcal{P}_m)$. Moreover, we have $\|\Delta u_n\|_{L^2}^2 \longrightarrow \|\Delta u\|_{L^2}^2$. Since $\Delta u_n \rightharpoonup \Delta u$ weakly in $L^2(\mathbf{R}^N)$, we infer that $\Delta u_n \longrightarrow \Delta u$ strongly in $L^2(\mathbf{R}^N)$. The inequality $\left\| \frac{\partial^2 v}{\partial x_i \partial x_j} \right\|_{L^2} \leqslant \|\Delta v\|_{L^2}$ for any $v \in H^2(\mathbf{R}^N)$ implies that $u_n \longrightarrow u$ strongly in $H^2(\mathbf{R}^N)$ and Theorem 3.4 is proven. □

**Proposition 3.5** *Assume that $\sigma N \leqslant 4$, $m > 0$ and $u \in H^2(\mathbf{R}^N)$ is a solution of the minimization problem $(\mathcal{P}_m)$. Then there exists $c = c(u) > 0$ such that $u$ satisfies the equation*

$$\Delta^2 u + 2\Delta u + (1 + c)u - |u|^{2\sigma} u = 0 \qquad \text{in } H^{-2}(\mathbf{R}^N). \tag{3.24}$$

*Moreover, we have:*

*(i) $1 + c \in [-E'_{min,\ell}(m), -E'_{min,r}(m)]$.*

*(ii) If $m_0 = 0$ (where $m_0$ is given by (4.6)), then $c(u) \longrightarrow 0$ as $m \longrightarrow 0$.*

*(iii) If $\sigma N < 4$ we have $c(u) \longrightarrow \infty$ as $m \longrightarrow \infty$.*

*(iv) If $m > m_0$ and $E'_{min,\ell}(m) > E'_{min,r}(m)$, there exist at least two solutions $u_1$ and $u_2$ for the problem $(\mathcal{P}_m)$ such that $1 + c(u_1) = -E'_{min,\ell}(m)$ and $1 + c(u_2) = -E'_{min,r}(m)$.*

*(v) If $m_1 < m_2$, the function $u_1$ solves $(\mathcal{P}_{m_1})$ and $u_2$ solves $(\mathcal{P}_{m_2})$, then $c(u_1) < c(u_2)$.*

*(vi) If $m_0 > 0$, problem $(\mathcal{P}_m)$ does not admit solutions for any $m \in (0, m_0)$.*

*Proof.* Since $E$ and $M(u) := \|u\|_{L^2}^2$ are $C^1$ functionals on $H^2(\mathbf{R}^N)$, the existence of a Lagrange multiplier $\lambda_u \in \mathbf{R}$ such that $E'(u) = \lambda_u M'(u)$ in $H^{-2}(\mathbf{R}^N)$ is standard. Then we have

$$\Delta^2 u + 2\Delta u - \lambda_u u - |u|^{2\sigma} u = 0 \qquad \text{in } H^{-2}(\mathbf{R}^N). \tag{3.25}$$

We claim that $\lambda \in [E'_{min,r}(m), E'_{min,\ell}(m)]$. We have $\|(1\pm t)u\|_{L^2}^2 = (1\pm t)^2 m$, hence $E(u \pm tu) \geqslant E_{min}\left((1\pm t)^2 m\right)$ and therefore

$$2\lambda_u m = 2\lambda_u \|u\|_{L^2}^2 = \lambda_u M'(u).u = E'(u).u = \lim_{t\downarrow 0} \frac{E(u+tu) - E(u)}{t}$$

$$\geqslant \lim_{t\downarrow 0} \frac{E_{min}\left((1+t)^2 m\right) - E_{min}(m)}{t} = 2mE'_{min,r}(m).$$

We conclude that $\lambda_u \geqslant E'_{min,r}(m)$. Proceeding similarly with $1-t$ instead of $1+t$ we get $-\lambda_u \geqslant -E'_{min,\ell}(m)$ and the claim is proven. Denoting $c(u) = -\lambda_u - 1$, statement (i) follows.

Taking the $H^{-2} - H^2$ duality product of (3.25) and of $u$ we get

$$\int_{\mathbf{R}^N} |\Delta u|^2\,dx - 2\int_{\mathbf{R}^N} |\nabla u|^2\,dx - \lambda_u \int_{\mathbf{R}^N} |u|^2\,dx - \int_{\mathbf{R}^N} |u|^{2\sigma+2}\,dx = 0. \tag{3.26}$$

Using (3.26) and the identities $\|u\|_{L^2}^2 = m$, $E(u) = E_{min}(m)$ we get

(3.27)

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} = \frac{\sigma+1}{\sigma}\left(E_{min}(m) - \lambda_u m\right) \quad \text{and} \quad \int_{\mathbf{R}^N} |\Delta u|^2 - 2|\nabla u|^2 dx = \frac{\sigma+1}{\sigma} E_{min}(m) - \frac{\lambda_u}{\sigma} m.$$

Since $E_{min}(m) \leqslant -m$ for all $m > 0$, the first part in (3.27) implies that for any $m > 0$ and for any solution $u$ of $(\mathcal{P}_m)$ we must have $\lambda_u < -1$, that is $c(u) > 0$. If $m_0 > 0$, $m \in (0, m_0)$ and $u$ is a solution to the problem $(\mathcal{P}_m)$ by (i) we should have $\lambda_u = -1$, a contradiction. Thus (vi) is proven.

Consider $m > m_0$ such that $E_{min}(m)$ is finite. Take an increasing sequence $(m_n)_{n\geqslant 2}$ in $(m_0, m)$ such that $m_n \longrightarrow m$. For each $n$, let $u_n$ be a solution of the minimization problem $(\mathcal{P}_{m_n})$ (the existence of $u_n$ is guaranteed by Theorem 3.4). Then $(u_n)_{n\geqslant 2}$ is a minimizing sequence for $(\mathcal{P}_m)$. Using Theorem 3.4 again we see that there exists a subsequence, still denoted $(u_n)_{n\geqslant 2}$, and there exists a solution $u_1$ of the problem $(\mathcal{P}_m)$ such that $u_n \longrightarrow u_1$ strongly in $H^2(\mathbf{R}^N)$. Identity (3.26) and the strong convergence in $H^2(\mathbf{R}^N)$ imply that $\lambda_{u_1} = \lim_{n\to\infty} \lambda_{u_n}$. On the other hand, (i) and the basic properties of concave functions imply that $\lambda_{u_n} \longrightarrow E'_{min,\ell}(m)$. Thus we have $\lambda_{u_1} = E'_{min,\ell}(m)$. Taking a decreasing sequence $m_n \longrightarrow m$ and proceeding similarly we see that there exists a solution $u_2$ of $(\mathcal{P}_m)$ such that $\lambda_{u_2} = E'_{min,r}(m)$. This proves (iv).

To prove (v) we argue by contradiction and we assume that there are $m_1 < m_2$ and there are solutions $u_1$ and $u_2$ of $(\mathcal{P}_{m_1})$ and of $(\mathcal{P}_{m_2})$, respectively, such that $c(u_1) = c(u_2)$. Since $-1 - c(u_1) \geqslant E'_{min,r}(m_1) \geqslant E'_{min,\ell}(m_2) \geqslant -1 - c(u_2)$, we see that $E'_{min,r}(m_1) = E'_{min,\ell}(m_2)$, and this implies that $E_{min}$ is affine on $[m_1, m_2]$. Hence there exist $\lambda < -1$ and $B \in \mathbf{R}$ such that $E_{min}(m) = \lambda m + B$ for any $m \in [m_1, m_2]$. For any $m \in (m_1, m_2)$, Theorem 3.4 gives the existence of a solution $u$ to the problem $(\mathcal{P}_m)$ and statement (i) above implies that $\lambda_u = \lambda$, or equivalently $c(u) = c(u_1) = c(u_2)$. Fix $m_3 \in (m_1, m_2)$ and let $u$ be a minimizer for $(\mathcal{P}_{m_3})$. The first part of (3.27) gives $\|u\|_{L^{2\sigma+2}}^{2\sigma+2} = \frac{\sigma+1}{\sigma} B$, hence $B > 0$. Using $\sqrt{t}u$ as test function and taking (3.27) into account we get for $t$ sufficiently close to 1,

$$\begin{aligned} \lambda t m_3 + B = E_{min}(tm_3) \leqslant E(\sqrt{t}u) &= t\textstyle\int_{\mathbf{R}^N} |\Delta u|^2 - 2|\nabla u|^2 dx - \frac{t^{\sigma+1}}{\sigma+1}\int_{\mathbf{R}^N} |u|^{2\sigma+2}\,dx \\ &= \lambda t m_3 + B + \frac{\sigma+1}{\sigma} B\left(t - \frac{t^{\sigma+1}}{\sigma+1} - \frac{\sigma}{\sigma+1}\right). \end{aligned} \tag{3.28}$$

Since $B > 0$ and $t - \frac{t^{\sigma+1}}{\sigma+1} - \frac{\sigma}{\sigma+1} < 0$ for $t \neq 1$, (3.28) gives a contradiction. This proves (v).

All other statements in Proposition 3.5 are obvious. □

**Proposition 3.6** *Assume that* $0 < \sigma < \max(1, \frac{4}{N+1})$ *and* $\sigma \leqslant \frac{4}{N}$. *For any* $m > 0$ *let* $u_m$ *be any solution of the minimisation problem* $(\mathcal{P}_m)$, *as given by Theorem 3.4 and Proposition 3.3, and let* $c_m = c(u_m)$ *be the Lagrange multiplier given by Proposition 3.5, so that* $(u_m, c_m)$ *solve (3.24). Denote* $v_m = \frac{v_m}{\sqrt{m}} = \frac{v_m}{\|u_m\|_{L^2}}$, *so that* $\|v_m\|_{L^2} = 1$. *Then we have*

$$\|\Delta v_m\|_{L^2} \longrightarrow 1, \quad \|\nabla v_m\|_{L^2} \longrightarrow 1, \quad \|(\Delta+1)v_m\|_{L^2} \longrightarrow 0 \qquad \text{as } m \longrightarrow 0, \tag{3.29}$$

*and* $\|v_m\|_{L^p} \longrightarrow 0$ *for any* $p \in (2,\infty)$ *if* $N \geqslant 4$, *respectively for any* $p \in (2, 2^{**})$ *if* $N \geq 5$.

*Proof.* The $H^{-2} - H^2$ duality product of (3.24) with $u_m$ gives $T_{c_m}(u_m) = \int_{\mathbf{R}^N} |u_m|^{2\sigma+2}\,dx$. Using (3.2) and (3.3) we get

$$\|\Delta u_m\|_{L^2}^2 - 2\|\Delta u_m\|_{L^2}\|u_m\|_{L^2} + (1+c_m)\|u_m\|_{L^2} - B(N,\sigma)\|\Delta u_m\|_{L^2}^{\frac{\sigma N}{2}}\|u_m\|_{L^2}^{2\sigma+2-\frac{\sigma N}{2}} \leqslant 0.$$

Dividing the above inequality by $\|u_m\|_{L^2}^2$ we see that $\|\Delta v_m\|_{L^2}^2 = \frac{\|\Delta u_m\|_{L^2}^2}{\|u_m\|_{L^2}^2}$ remains bounded if $m$ is sufficiently small. Using (3.3) again we find $\|u_m\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant C\|u_m\|_{L^2}^{2\sigma+2}$, and then dividing the equality $T_{c_m}(u_m) = \int_{\mathbf{R}^N} |u_m|^{2\sigma+2}\,dx$ by $\|u_m\|_{L^2}^2$ and letting $m \longrightarrow 0$ we infer that $\|(\Delta+1)v_m\|_{L^2} \longrightarrow 0$ as $m \longrightarrow 0$.

It is proven in Remark 3.11 below that $u_m$ satisfies the identity $P_{c_m}(u_m) = 0$, where $P_c$ is given in (3.38). Dividing this identity by $\|u_m\|_{L^2}^2$ and letting $m \longrightarrow 0$ we obtain

$$\frac{N-4}{N}\|\Delta v_m\|_{L^2}^2 - 2\frac{N-2}{N}\|\nabla v_m\|_{L^2}^2 + \|v_m\|_{L^2}^2 \longrightarrow 0 \quad \text{as } m \longrightarrow 0.$$

The above convergence and the fact that $\|(\Delta+1)v_m\|_{L^2} \longrightarrow 0$ and $\|v_m\|_{L^2} = 1$ give (3.29).

We claim that for any sequence $m_n \longrightarrow 0$ and for any sequence $(x_n)_{n\geqslant 1} \subset \mathbf{R}^N$, the only possible weak limit in $H^2(\mathbf{R}^N)$ of $v_{m_n}(\cdot + x_n)$ is zero. Indeed, assume that in $H^2(\mathbf{R}^N) \rightharpoonup w \neq 0$ weakly in $H^2(\mathbf{R}^N)$. Then by weak convergence we have

$$\|(\Delta+1)w\|_{L^2} \leqslant \liminf_{n\to\infty} \|(\Delta+1)v_{m_n}(\cdot + x_n)\|_{L^2} = \liminf_{n\to\infty} \|(\Delta+1)v_{m_n}\|_{L^2} = 0,$$

and then Plancherel's identity implies $(|\cdot|^2 - 1)\widehat{w} = 0$ a.e. in $\mathbf{R}^N$, thus $w = 0$, a contradiction. The claim is thus proven. It is then standard to show that $v_m \longrightarrow 0$ strongly in $L^p(\mathbf{R}^N)$ for all $p \in (2,\infty)$ if $N \geqslant 4$, respectively for all $p \in (2, 2^{**})$ if $N \geq 5$ (see, e.g., Lemma 6.1 in [20]). □

So far we have solved the global minimization problem $(\mathcal{P}_m)$ in $H^2(\mathbf{R}^N)$ in the case $\sigma N \leqslant 4$ and we have shown that any solution satisfies (3.24) for some $c > 0$. Obviously, $u \in H^2(\mathbf{R}^N)$ solves (3.24) if and only if $u$ is a critical point of the following functional, called *action*:

$$S_c(u) = \int_{\mathbf{R}^N} |\Delta u|^2\,dx - 2\int_{\mathbf{R}^N} |\nabla u|^2\,dx + (1+c)\int_{\mathbf{R}^N} |u|^2\,dx - \frac{1}{\sigma+1}\int_{\mathbf{R}^N} |u|^{2\sigma+2}\,dx.$$

At this stage it is not clear that given any $c > 0$, there exists $m > 0$ and a solution $u$ of $(\mathcal{P}_m)$ such that $c(u) = c$. We will show that for any $c > 0$ and for any $\sigma > 0$ satisfying $2\sigma + 2 < 2^{**}$, equation (3.24) has solutions and, moreover, it has solutions minimizing the action $S_c$ among all solutions (these are called *minimum action solutions* or *ground states*). Moreover, we will show that all minimizers of a problem $(\mathcal{P}_m)$ are ground states. To this end we introduce another family of minimization problems.

Let $c \geqslant 0$. We consider the minimization problem

$$(\mathcal{T}_c) \qquad \begin{aligned} &\text{minimize } T_c(u) := \int_{\mathbf{R}^N} |\Delta u|^2\,dx - 2\int_{\mathbf{R}^N} |\nabla u|^2\,dx + (1+c)\int_{\mathbf{R}^N} |u|^2\,dx \\ &\text{in the set } U := \left\{ u \in H^2(\mathbf{R}^N) \;\middle|\; \int_{\mathbf{R}^N} |u|^{2\sigma+2}\,dx = 1 \right\}. \end{aligned}$$

We denote $t(c) = \inf\{T_c(u) \mid u \in U\}$. It is clear that

$$S_c(u) = T_c(u) - \frac{1}{\sigma+1}\int_{\mathbf{R}^N} |u|^{2\sigma+2}\,dx = E(u) + (1+c)\int_{\mathbf{R}^N} |u|^2\,dx. \tag{3.30}$$

**Theorem 3.7** *Assume that* $0 < \sigma < \infty$ *if* $N \geqslant 4$ *and* $0 < \sigma < \frac{4}{N-4}$ *if* $N > 4$. *Then for any* $c > 0$ *we have* $t(c) > 0$ *and the minimization problem* $(\mathcal{T}_c)$ *admits solutions. Moreover, for any sequence* $(u_n)_{n\geqslant 1} \subset H^2(\mathbf{R}^N)$ *satisfying* $\int_{\mathbf{R}^N} |u_n|^{2\sigma+2}\,dx \longrightarrow 1$ *and* $T_c(u_n) \longrightarrow t(c)$ *there exist a subsequence* $(u_{n_k})_{k\geqslant 1}$, *a sequence* $(x_k)_{k\geqslant 1} \subset \mathbf{R}^N$ *and a minimizer* $u$ *for* $(\mathcal{T}_c)$ *such that* $u_{n_k}(\cdot + x_k) \longrightarrow u$ *strongly in* $H^2(\mathbf{R}^N)$.

*Proof.* The proof is standard, so we only sketch it. Fix $c > 0$. Then fix $\varepsilon > 0$ such that $\frac{1}{1-\varepsilon} + \varepsilon < 1 + c$. Using (3.8) we get $T_c(v) \geqslant \varepsilon \|\Delta v\|_{L^2}^2 + \varepsilon \|v\|_{L^2}^2$ for any $v \in H^2(\mathbf{R}^N)$, and then it is clear that $T_c^{\frac{1}{2}}$ is a norm on $H^2(\mathbf{R}^N)$ and that it is equivalent to the usual norm. By the Sobolev embedding there exists $K_c > 0$ such that $\|v\|_{L^{2\sigma+2}} \leqslant K_c T_c^{\frac{1}{2}}(v)$, thus $t(c) \geqslant K_c^{-2} > 0$. For any $v \in H^2(\mathbf{R}^N)$, $v \neq 0$ we have $\frac{v}{\|v\|_{L^{2\sigma+2}}} \in U$, hence $T_c\left(\frac{v}{\|v\|_{L^{2\sigma+2}}}\right) \geqslant t(c)$ and this gives

$$T_c(v) \geqslant t(c)\|v\|_{L^{2\sigma+2}}^2 \qquad \text{for any } v \in H^2(\mathbf{R}^N). \tag{3.31}$$

Let $(u_n)_{n\geqslant 1}$ be a sequence as in Theorem 3.7. It is obvious that $(u_n)_{n\geqslant 1}$ is bounded in $H^2(\mathbf{R}^N)$. We choose $\sigma' > \sigma$ such that $2\sigma' + 2 < 2^{**}$ and we use (3.18) for $u_n$ to infer that there exists constants $t_0$, $a > 0$, independent of $n$, such that $\mathcal{L}^N(\{|u_n| \geqslant t_0\}) \geqslant a$ for all sufficiently large $n$. Then Lieb's Lemma implies that there exists a constant $b > 0$, independent of $n$, and for each $n$ large there exists $x_n \in \mathbf{R}^N$ such that $\mathcal{L}^N\left(\left\{x \in B(x_n, 1) \;\middle|\; |u_n| \geqslant \frac{t_0}{2}\right\}\right) \geqslant b$. We replace $u_n$ by $u_n(\cdot + x_n)$, which is still a minimizing sequence and satisfies $\int_{B(0,1)} |u_n|^p\,dx \geqslant b\left(\frac{t_0}{2}\right)^p$ for all $n$. Since $(u_n)_{n\geqslant 1}$ is bounded in $H^2(\mathbf{R}^N)$ there is a subsequence, still denoted $(u_n)_{n\geqslant 1}$, and there is $u \in H^2(\mathbf{R}^N)$ such that (3.19) holds. The convergence $u_n \longrightarrow u$ in $L^p_{loc}$ for $1 \leqslant p < 2^{**}$ gives $\int_{B(0,1)} |u|^p\,dx \geqslant b\left(\frac{t_0}{2}\right)^p$, and therefore $u \neq 0$. Denote $\eta = \int_{\mathbf{R}^N} |u|^{2\sigma+2}\,dx > 0$. By Fatou's Lemma we get $\eta \leqslant 1$. It is obvious that (3.20) and (3.21) hold. By (3.21) we have $\int_{\mathbf{R}^N} |u_n - u|^{2\sigma+2}\,dx \longrightarrow 1 - \eta$ and then using (3.20) and (3.31) we find

$$T_c(u_n) = T_c(u_n - u) + T_c(u) + o(1) \geqslant t(c)\|u_n - u\|_{L^{2\sigma+2}}^2 + t(c)\|u\|_{L^{2\sigma+2}}^2 + o(1).$$

Letting $n \longrightarrow \infty$ in the above inequality we obtain $1 \geqslant (1-\eta)^{\frac{1}{\sigma+1}} + \eta^{\frac{1}{\sigma+1}}$ and this implies that $\eta = 1$, that is $u \in U$. Then we must have $T_c(u) \geqslant t(c)$. On the other hand, $T_c(u) \leqslant \liminf_{n\to\infty} T_c(u_n) = t(c)$ by weak convergence, and therefore $T_c(u) = t(c) = \lim_{n\to\infty} T_c(u_n)$. Since $T_c$ is a norm on $H^2(\mathbf{R}^N)$ we infer that $u_n \longrightarrow u$ strongly in $H^2(\mathbf{R}^N)$, as desired. ☐

**Proposition 3.8** *The mapping* $c \longmapsto t(c)$ *is strictly increasing on* $(0, \infty)$ *and there is* $C > 0$ *such that* $t(c) \leqslant C\sqrt{c}$ *for all sufficiently small* $c$. *In particular we have* $t(c) \longrightarrow 0$ *as* $c \longrightarrow 0$.

*Proof.* Let $0 < c_1 < c_2$. Let $u$ be a minimizer for the problem $(\mathcal{T}_{c_2})$. We have $u \in U$ and $t(c_2) = T_{c_2}(u) > T_{c_1}(u) \geqslant t(c_1)$. Hence the mapping $c \longmapsto t(c)$ is strictly increasing.

Let $u_{\varepsilon,\delta}$ be as in (2.26). Fix $\delta_0 = \frac{1}{20}$ and let $v_c = u_{\sqrt{c},\delta_0}$. By (2.27), (2.28) and (2.30) we have

$$\|v_c\|_{L^2}^2 \leqslant C_1 c^{\frac{1}{2}}, \qquad \|\Delta v_c\|_{L^2}^2 - 2\|\nabla v_c\|_{L^2}^2 + \|v_c\|_{L^2}^2 \leqslant C_2 c^{\frac{3}{2}} \quad \text{and} \quad \|v_c\|_{L^{2\sigma+2}} \geqslant C_3 c^{\frac{1}{2}}$$

for some $C_1, C_2, C_3 > 0$, so that $T_c(v_c) \leqslant C_4 c^{\frac{3}{2}}$. Using (3.31) we see that $t(c) \leqslant \frac{T_c(v_c)}{\|v_c\|_{L^{2\sigma+2}}^2} \leqslant C\sqrt{c}$. ☐

**Proposition 3.9** *Let* $u$ *be any minimizer for the problem* $(\mathcal{T}_c)$. *Then* $v := t(c)^{\frac{1}{2\sigma}} u$ *is a solution of (3.24). Moreover, for any solution* $w \in H^2(\mathbf{R}^N)$ *of (3.24) we have* $S_c(w) \geqslant S_c(v) = \frac{\sigma}{\sigma+1} t(c)^{\frac{\sigma+1}{\sigma}}$. *In other words,* $v$ *is a least action solution of (3.24).*
*Conversely, if* $\tilde{v}$ *is any least action solution of (3.24) then* $\tilde{u} := t(c)^{-\frac{1}{2\sigma}} \tilde{v}$ *is a solution of* $(\mathcal{T}_c)$.

*Proof.* Assume that $u$ solves $(\mathcal{T}_c)$. The functionals $T_c$ and $u \longmapsto \|u\|_{L^{2\sigma+2}}^{2\sigma+2}$ are $C^1$ on $H^2(\mathbf{R}^N)$, and consequently there exists a Lagrange multiplier $\kappa \in \mathbf{R}$ such that

$$\Delta^2 u + 2\Delta u + (1+c)u = \kappa |u|^{2\sigma} u \qquad \text{in } H^{-2}(\mathbf{R}^N). \tag{3.32}$$

Taking the $H^{-2}-H^2$ duality product of (3.32) with $u$ we get $T_c(u)=\kappa\int_{\mathbf{R}^N}|u|^{2\sigma+2}\,dx$, which implies that $t(c)=\kappa$. Denoting $v=t(c)^{\frac{1}{2\sigma}}u=\kappa^{\frac{1}{2\sigma}}u$, it is clear that $v$ solves (3.24). We have

$$S_c(v)=t(c)^{\frac{1}{\sigma}}T_c(u)-\frac{1}{\sigma+1}t(c)^{\frac{\sigma+1}{\sigma}}\int_{\mathbf{R}^N}|u|^{2\sigma+2}\,dx=\frac{\sigma}{\sigma+1}t(c)^{\frac{\sigma+1}{\sigma}}. \tag{3.33}$$

Let $w\neq 0$ be an arbitrary solution of (3.24). The duality product of (3.24) with $w$ gives

$$T_c(w)=\int_{\mathbf{R}^N}|w|^{2\sigma+2}\,dx. \tag{3.34}$$

From (3.34) and (3.31) we obtain $T_c(w)\leqslant t(c)^{-\sigma-1}T_c(w)^{\sigma+1}$ and this implies $T_c(w)\geqslant t(c)^{\frac{\sigma+1}{\sigma}}$. (By (3.34) we have also the lower bound $\int_{\mathbf{R}^N}|w|^{2\sigma+2}\,dx=T_c(w)\geqslant t(c)^{\frac{\sigma+1}{\sigma}}$.) Using again (3.34) we find

$$S_c(w)=T_c(w)-\frac{1}{\sigma+1}\int_{\mathbf{R}^N}|w|^{2\sigma+2}\,dx=\frac{\sigma}{\sigma+1}T_c(w)\geqslant\frac{\sigma}{\sigma+1}t(c)^{\frac{\sigma+1}{\sigma}}=S_c(v).$$

Conversely, let $\tilde{v}$ be a least action solution of (3.24). Then we have $S_c(\tilde{v})=S_c(v)=\frac{\sigma}{\sigma+1}t(c)^{\frac{\sigma+1}{\sigma}}$. On the other hand, by (3.34) we get $S_c(\tilde{v})=\frac{\sigma}{\sigma+1}T_c(\tilde{v})=\frac{\sigma}{\sigma+1}\|\tilde{v}\|_{L^{2\sigma+2}}^{2\sigma+2}$. We conclude that $T_c(\tilde{v})=\|\tilde{v}\|_{L^{2\sigma+2}}^{2\sigma+2}=t(c)^{\frac{\sigma+1}{\sigma}}$ and then one immediately checks that $\tilde{u}=t(c)^{-\frac{1}{2\sigma}}\tilde{v}$ is a minimizer for the problem $(\mathcal{T}_c)$. □

**Proposition 3.10** *Assume that $u\in H^2(\mathbf{R}^N)$ is a solution of the minimization problem $(\mathcal{P}_m)$ for some $m>0$ and that $u$ solves (3.24). Then:*

*(i) $u$ is a minimum action solution of (3.24).*

*(ii) If $v$ is any minimum action solution of (3.24) we have $\|v\|_{L^2}^2=m$ and $v$ is a minimizer for $(\mathcal{P}_m)$.*

*Proof.* Since $u$ solves $(\mathcal{P}_m)$ and (3.24), using Proposition 3.9 we get

$$E_{min}(m)+(1+c)m=E(u)+(1+c)\|u\|_{L^2}^2=S_c(u)\geqslant\frac{\sigma}{\sigma+1}t(c)^{\frac{\sigma+1}{\sigma}}. \tag{3.35}$$

Let $v$ be an arbitrary minimum action solution for (3.24) (the existence of such solutions follows from Theorem 3.7 and Proposition 3.9). From the proof of Proposition 3.9 we know that $T_c(v)=\int_{\mathbf{R}^N}|v|^{2\sigma+2}\,dx=t(c)^{\frac{\sigma+1}{\sigma}}$. Denote $m'=\|v\|_{L^2}^2$. For any $a>0$ we have $\|a^{\frac12}v\|_{L^2}^2=am'$ and taking $a^{\frac12}v$ as test function we discover

$$\begin{aligned}&E_{min}(am')+(1+c)am'\leqslant E(a^{\frac12}v)+(1+c)\|a^{\frac12}v\|_{L^2}^2\\ &=T_c(a^{\frac12}v)-\frac{1}{\sigma+1}\int_{\mathbf{R}^N}|a^{\frac12}v|^{2\sigma+2}\,dx=\left(a-\frac{a^{\sigma+1}}{\sigma+1}\right)t(c)^{\frac{\sigma+1}{\sigma}}.\end{aligned} \tag{3.36}$$

The mapping $a\longmapsto\varphi(a):=a-\frac{a^{\sigma+1}}{\sigma+1}$ reaches its maximum value on $(0,\infty)$ only at $a=1$ and the maximum is $\varphi(1)=\frac{\sigma}{\sigma+1}$. Comparing (3.35) and (3.36) we get

$$\frac{\sigma}{\sigma+1}t(c)^{\frac{\sigma+1}{\sigma}}\leqslant E_{min}(m)+(1+c)m\leqslant\varphi\left(\frac{m}{m'}\right)t(c)^{\frac{\sigma+1}{\sigma}}\leqslant\frac{\sigma}{\sigma+1}t(c)^{\frac{\sigma+1}{\sigma}}.$$

We infer that we must have equality throughout in the above sequence of inequalities. Therefore $S_c(u)=\frac{\sigma}{\sigma+1}t(c)^{\frac{\sigma+1}{\sigma}}$ and $u$ is a minimum action solution for (3.24). Moreover, we must have $m=m'$, that is any minimum action solution $v$ of (3.24) satisfies $\|v\|_{L^2}^2=m$. Then we find $E(v)=S_c(v)-(1+c)\|v\|_{L^2}^2=S_c(u)-(1+c)\|u\|_{L^2}^2=E(u)=E_{min}(m)$, and consequently $v$ solves $(\mathcal{P}_m)$. □

**Remark 3.11** (Some integral identities) Taking the $H^{-2}-H^2$ duality product of (3.24) with $u$ we see that any solution $u \in H^2(R^N)$ of (3.24) satisfies the identity $N_c(u) = 0$, where

$$N_c(u) = \int_{\mathbf{R}^N} |\Delta u|^2\, dx - 2\int_{\mathbf{R}^N} |\nabla u|^2\, dx + (1+c)\int_{\mathbf{R}^N} |u|^2\, dx - \int_{\mathbf{R}^N} |u|^{2\sigma+2}\, dx. \tag{3.37}$$

Any solution $u \in H^2(R^N)$ of (3.24) satisfies the identity $P_c(u) = 0$, where

$$\begin{aligned} P_c(u) = \frac{N-4}{N}\int_{\mathbf{R}^N} |\Delta u|^2\, dx - 2\frac{N-2}{N}\int_{\mathbf{R}^N} |\nabla u|^2\, dx + (1+c)\int_{\mathbf{R}^N} |u|^2\, dx \\ -\frac{1}{\sigma+1}\int_{\mathbf{R}^N} |u|^{2\sigma+2}\, dx. \end{aligned} \tag{3.38}$$

The functionals $N_c$ and $P_c$ are the Nehari and Pohozaev functionals, respectively, and $N_c(u) = 0$ and $P_c(u) = 0$ are the Nehari and Pohozaev (or Derrick-Pohozaev) identities. The Pohozaev identity expresses the behaviour of the action functional $S_c$ with respect to dilations: for any $u \in H^2(\mathbf{R}^N)$ we have $P_c(u) = \frac{d}{dt}_{|t=1} S_c\left(u\left(\frac{\cdot}{t}\right)\right)$, and consequently one expects $P_c(u) = 0$ for any critical point of $S_c$. To give a formal proof of this fact, one first uses a bootstrap argument to prove some regularity of solutions of (3.24) ($u \in H^3(\mathbf{R}^N)$ is enough). Then consider a cut-off function $\chi \in C_c^\infty(\mathbf{R}^N)$ such that $\chi = 1$ on $B(0,1)$ and $\text{supp}(\chi) \subset B(0,2)$, take the $H^{-2} - H^2$ duality product of (3.24) with $\chi\left(\frac{\cdot}{n}\right)\sum_{j=1}^N x_j \frac{\partial u}{\partial x_j}$ and integrate by parts, then let $n \longrightarrow \infty$. See Lemma 2.1 in [3] for details.

Two other functionals are of interest:

$$P_1(u) = \frac{N}{4}\left(N_c(u) - P_c(u)\right) = \int_{\mathbf{R}^N} |\Delta u|^2\, dx - \int_{\mathbf{R}^N} |\nabla u|^2\, dx - \frac{N\sigma}{4(\sigma+1)}\int_{\mathbf{R}^N} |u|^{2\sigma+2}\, dx \tag{3.39}$$

and

$$\begin{aligned} P_2(u) &= \frac{1}{\sigma}N_c(u) - \frac{\sigma+1}{\sigma}P_c(u) \\ &= \left(\frac{4(\sigma+1)}{N\sigma} - 1\right)\int_{\mathbf{R}^N} |\Delta u|^2\, dx - 2\left(\frac{2(\sigma+1)}{N\sigma} - 1\right)\int_{\mathbf{R}^N} |\nabla u|^2\, dx - (1+c)\int_{\mathbf{R}^N} |u|^2\, dx. \end{aligned}$$

Obviously, any solution $u \in H^2(\mathbf{R}^N)$ of (3.24) satisfies $P_1(u) = P_2(u) = 0$. If $u \in H^2(\mathbf{R}^N)$ satisfies $N_c(u) = 0$ and $P_i(u) = 0$ for some $i \in \{1, 2\}$, then $P_c(u) = P_1(u) = P_2(u) = 0$.

Given $u \in H^2(\mathbf{R}^N)$ and $t > 0$, we denote

$$u_t(x) = t^{\frac{N}{4}} u(t^{\frac{1}{2}} x) \qquad \text{and} \qquad u^t(x) = t^{\frac{N}{2\sigma+2}} u(tx). \tag{3.40}$$

By (3.1) we have $\|u_t\|_{L^2} = \|u\|_{L^2}$ and $\|u^t\|_{L^{2\sigma+2}} = \|u\|_{L^{2\sigma+2}}$ for all $t > 0$.

One has $\frac{d}{dt}\left(S_c\left(u_t\right)\right) = \frac{d}{dt}\left(E\left(u_t\right)\right) = \frac{2}{t}P_1(u_t)$. If the mapping $t \longmapsto E(u_t)$ (or, equivalently, $t \longmapsto S_c(u_t)$) achieves a local minimum or a local maximum at $t = 1$ we must have $\frac{d}{dt}_{|t=1} E\left(u_t\right) = 0$ and this gives $P_1(u) = 0$.

If $t \longmapsto T_c(u^t)$ (or, equivalently, $t \longmapsto S_c(u^t)$) achieves a local minimum at $t = 1$ we must have $\frac{d}{dt}_{|t=1} T_c\left(u^t\right) = 0$ and this gives $P_2(u) = 0$.

We will study the behaviour of minimum action solutions of (3.24) as $c \longrightarrow \infty$. To do this we use once again the scaling properties of functionals. Given $c > 0$ and $v \in H^2(\mathbf{R}^N)$, we denote

$$K_c(v) = \int_{\mathbf{R}^N} |\Delta v|^2\, dx - \frac{2}{\sqrt{1+c}}\int_{\mathbf{R}^N} |\nabla v|^2\, dx + \int_{\mathbf{R}^N} |v|^2\, dx \tag{3.41}$$

and $K(v) = \int_{\mathbf{R}^N} |\Delta v|^2\, dx + \int_{\mathbf{R}^N} |v|^2\, dx$. We consider the minimisation problems

$(\mathcal{A}_c)$ minimize $K_c(v)$ in $H^2(\mathbf{R}^N)$ under the constraint $\int_{\mathbf{R}^N} |v|^{2\sigma+2}\, dx = 1$,

$$(\mathcal{A}) \qquad \text{minimize } K(v) \text{ in } H^2(\mathbf{R}^N) \text{ under the constraint } \int_{\mathbf{R}^N} |v|^{2\sigma+2}\,dx = 1.$$

Let $c > 0$. Take $b = (1+c)^{-\frac{1}{4}}$. Using (3.1) we see that for any $u \in H^2(\mathbf{R}^N)$ we have

$$T_c(u_{a,b}) = a^2(1+c)^{1-\frac{N}{4}} K_c(u) \qquad \text{and} \qquad \|u_{a,b}\|_{L^{2\sigma+2}}^{2\sigma+2} = a^{2\sigma+2}(1+c)^{-\frac{N}{4}} \|u\|_{L^{2\sigma+2}}^{2\sigma+2}.$$

Now choose $a$ such that $a^{2\sigma+2}(1+c)^{-\frac{N}{4}} = t(c)^{\frac{\sigma+1}{\sigma}}$, that is $a = (1+c)^{\frac{N}{8(\sigma+1)}} t(c)^{\frac{1}{2\sigma}}$. With choice of $a$ and $b$ and using Proposition 3.9 we see that $u_{a,b}$ is a minimum action solution for (3.24) if and only if $u$ is a minimizer for $(\mathcal{A}_c)$. Theorem 3.7 and Proposition 3.9 give the existence of minimizers for $(\mathcal{A}_c)$ for any $c > 0$. The existence of minimizers for $(\mathcal{A})$ is standard (see, e.g., Example 3.10 in [19]), as well as the pre-compactness of any minimizing sequence modulo translations. We have the following:

**Proposition 3.12** *Let $(c_n)_{n\geqslant 1}$ be any sequence of positive numbers such that $c_n \longrightarrow \infty$. Assume that for each $n$, $v_n$ is a minimizer for the problem $(\mathcal{A}_{c_n})$. There exists a subsequence $(c_{n_k})_{k\geqslant 1}$, a sequence of points $(x_k)_{k\geqslant 1} \subset \mathbf{R}^N$ and a minimizer $v$ for the problem $(\mathcal{A})$ such that $v_{n_k} \longrightarrow v$ strongly in $H^2(\mathbf{R}^N)$.*

*Proof.* It suffices to show that $(v_n)_{n\geqslant 1}$ is a minimizing sequence for the problem $(\mathcal{A})$. Then the conclusion of Proposition 3.12 is a consequence of the pre-compactness modulo translations of minimizing sequences for problem $(\mathcal{A})$. Let

$$I = \inf\{K(u) \mid u \in H^2(\mathbf{R}^N) \text{ and } \|u\|_{L^{2\sigma+2}} = 1\}. \tag{3.42}$$

From (3.3) we have $I > 0$. [1]

Since $\|v_n\|_{L^{2\sigma+2}} = 1$ for any $n$, all we have to do is to show that $K(v_n) \longrightarrow I$ as $n \longrightarrow \infty$. From (3.8) we have

$$K_c(u) < \left(1 - \frac{1}{\sqrt{1+c}}\right) K(u) \qquad \text{for any } u \in H^2(\mathbf{R}^N),\ u \neq 0.$$

Let $Q$ be a minimizer for the problem $(\mathcal{A})$ and let $v_c$ be a minimizer for $(\mathcal{A}_c)$. Taking $Q$ as test function in $(\mathcal{A}_c)$ we get $K_c(v_c) \leqslant K_c(Q)$ and taking $v_c$ as test function in $(\mathcal{A})$ we obtain $K(Q) \leqslant K(v_c)$, hence

$$\left(1 - \frac{1}{\sqrt{1+c}}\right) K(Q) \leqslant \left(1 - \frac{1}{\sqrt{1+c}}\right) K(v_c) < K_c(v_c) \leqslant K_c(Q) < K(Q). \tag{3.43}$$

Using (3.43) we infer that $K(v_n)$ is bounded, thus $(v_n)_{n\geqslant 1}$ is bounded in $H^2(\mathbf{R}^N)$. Moreover, the above inequality implies that $\lim_{n\to\infty} K(v_n) = K(Q) = I$ and the conclusion of Proposition 3.12 follows. ☐

**Corollary 3.13** *Let $I$ be as in (3.42). For any $c > 0$ we have*

$$\left(1 - \frac{1}{\sqrt{1+c}}\right)(1+c)^{1-\frac{N\sigma}{4(\sigma+1)}} I < t(c) < (1+c)^{1-\frac{N\sigma}{4(\sigma+1)}} I. \tag{3.44}$$

*Moreover, if $u_c$ is any minimum action solution of (3.24) we have*

$$(1+c)^{\frac{N}{4}-\frac{1}{\sigma}} \int_{\mathbf{R}^N} |u_c|^2\,dx \longrightarrow \frac{4(\sigma+1) - N\sigma}{4(\sigma+1)} I^{\frac{\sigma+1}{\sigma}}, \tag{3.45}$$

$$(1+c)^{\frac{N}{4}-\frac{1}{\sigma}-1} \int_{\mathbf{R}^N} |\Delta u_c|^2\,dx \longrightarrow \frac{N\sigma}{4(\sigma+1)} I^{\frac{\sigma+1}{\sigma}} \qquad \textit{and} \tag{3.46}$$

$$(1+c)^{\frac{N}{4}-\frac{1}{\sigma}-1} \int_{\mathbf{R}^N} |u_c|^{2\sigma+2}\,dx \longrightarrow I^{\frac{\sigma+1}{\sigma}} \qquad \textit{as } c \longrightarrow \infty. \tag{3.47}$$

[1] It can be proved that $I = \dfrac{4(\sigma+1)}{N\sigma}\left(\dfrac{4(\sigma+1)}{N\sigma} - 1\right)^{\frac{N\sigma}{4(\sigma+1)}-1} B(N,\sigma)^{-\frac{1}{\sigma+1}}$, where $B(N,\sigma)$ is as in (3.4), and that minimizers for $(\mathcal{A})$ are optimal functions for (3.3), but we will not make use of this fact. See [19] for details.

*Proof.* If $Q$ is a minimizer for $(\mathcal{A})$ and $Q^t(x) = t^{\frac{N}{2\sigma+2}}Q(tx)$ is as in (3.40), the mapping $t \longmapsto K(Q^t)$ achieves its minimum on $(0,\infty)$ at $t=1$, hence $\frac{t}{dt}_{|t=1}K(Q^t) = 0$ and this gives

$$\left(4 - \frac{N\sigma}{\sigma+1}\right)\int_{\mathbf{R}^N} |\Delta Q|^2\,dx - \frac{N\sigma}{\sigma+1}\int_{\mathbf{R}^N} |Q|^2\,dx = 0.$$

From this identity and the fact that $K(Q) = I$ we get

$$\int_{\mathbf{R}^N} |\Delta Q|^2\,dx = \frac{N\sigma}{4(\sigma+1)}I \qquad \text{and} \qquad \int_{\mathbf{R}^N} |Q|^2\,dx = \frac{4(\sigma+1)-N\sigma}{4(\sigma+1)}I.$$

Notice that the above identities hold for *any* minimizer of $(\mathcal{A})$. For $c > 0$, let $v_c$ be any minimizer for the problem $(\mathcal{A}_c)$. Then Proposition 3.12 and the previous identities imply that

$$(3.48) \quad \int_{\mathbf{R}^N} |\Delta v_c|^2\,dx \longrightarrow \frac{N\sigma}{4(\sigma+1)}I \quad \text{and} \quad \int_{\mathbf{R}^N} |v_c|^2\,dx = \frac{4(\sigma+1)-N\sigma}{4(\sigma+1)}I \quad \text{as } c \longrightarrow \infty.$$

Given $c > 0$, let $u_c$ be a minimum action solution of (3.24). Let $a = (1+c)^{\frac{N}{8(\sigma+1)}}t(c)^{\frac{1}{2\sigma}}$, $b = (1+c)^{-\frac{1}{4}}$, and let $v_c = (u_c)_{a^{-1},b^{-1}} = \frac{1}{a}u_c(\cdot)$. We have already seen that $v_c$ is a minimizer for problem $(\mathcal{A}_c)$. We have $u_c = (v_c)_{a,b}$ and

$$t(c)^{\frac{\sigma+1}{\sigma}} = T_c(u_c) = a^2(1+c)^{1-\frac{N}{4}}K_c(v_c) = (1+c)^{1-\frac{N\sigma}{4(\sigma+1)}}t(c)^{\frac{1}{\sigma}}K_c(v_c).$$

From the above equality and (3.43) we get (3.44). We have also

$$\int_{\mathbf{R}^N} |u_c|^2\,dx = a^2b^N\int_{\mathbf{R}^N} |v_c|^2\,dx = (1+c)^{-\frac{N\sigma}{4(\sigma+1)}}t(c)^{\frac{1}{\sigma}}\int_{\mathbf{R}^N} |v_c|^2\,dx \qquad \text{and}$$

$$\int_{\mathbf{R}^N} |\Delta u_c|^2\,dx = a^2b^{N-4}\int_{\mathbf{R}^N} |\Delta v_c|^2\,dx = (1+c)^{1-\frac{N\sigma}{4(\sigma+1)}}t(c)^{\frac{1}{\sigma}}\int_{\mathbf{R}^N} |\Delta v_c|^2\,dx.$$

Then taking into account (3.48) we obtain (3.45) and (3.46). Recall that $\int_{\mathbf{R}^N} |u_c|^{2\sigma+2}\,dx = t(c)^{\frac{\sigma+1}{\sigma}}$, and consequently (3.47) follows from (3.44). ☐

**Remark 3.14** We have $1 + \frac{1}{\sigma} - \frac{N}{4} > 0$ because $2+2\sigma < 2^{**}$, and (3.46) implies that we have always $\|\Delta u_c\|_{L^2} \longrightarrow \infty$ as $c \longrightarrow \infty$. On the contrary, from (3.45) we see that $\|u_c\|_{L^2} \longrightarrow \infty$ if $N\sigma < 4$ and $\|u_c\|_{L^2} \longrightarrow 0$ if $N\sigma > 4$. In the case $N\sigma = 4$, (3.45) implies that $\|u_c\|_{L^2} \longrightarrow (\sigma+1)^{\frac{1}{\sigma}}B(N,\sigma)^{-\frac{1}{\sigma}} = k_*$, where $k_*$ is as in Proposition 3.1 (vi).

**Remark 3.15** For any $\sigma > 0$ such that $2\sigma+2 < 2^{**}$, the functional $S_c$ has a mountain-pass geometry. Indeed, we have

$$S_c(u) = T_c(u) - \frac{1}{\sigma+1}\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \geqslant T_c(u) - \left(\frac{T_c(u)}{t(c)}\right)^{\sigma+1}.$$

The mapping $\varphi(t) := t - \frac{1}{\sigma+1}\left(\frac{t}{t(c)}\right)^{\sigma+1}$ is increasing on $[0, t(c)^{\frac{\sigma+1}{\sigma}}]$, decreasing on $[t(c)^{\frac{\sigma+1}{\sigma}}, \infty)$, and $\varphi\left(t(c)^{\frac{\sigma+1}{\sigma}}\right) = \frac{\sigma}{\sigma+1}t(c)^{\frac{\sigma+1}{\sigma}} > 0$. Denoting $B_c := \{u \in H^2(\mathbf{R}^N) \mid T_c(u) < t(c)^{\frac{\sigma+1}{\sigma}}\}$, we have:

- $S_c(u) \geqslant \varphi(T_c(u)) \geqslant 0$ for any $u \in B_c$ and $\inf\limits_{u\in B_c} S_c(u) = S_c(0) = 0$.
- $\inf\{S_c(u) \mid u \in H^2(\mathbf{R}^N) \text{ and } T_c(u) = t(c)^{\frac{\sigma+1}{\sigma}}\} = \frac{\sigma}{\sigma+1}t(c)^{\frac{\sigma+1}{\sigma}} > 0$.
- $\lim\limits_{t\to\infty} S_c(tu) = -\infty$ for any $u \neq 0$.

Let $\Gamma := \{\gamma : [0,1] \longrightarrow H^2(\mathbf{R}^N) \mid \gamma \text{ is continuous}, \gamma(0) = 0 \text{ and } S_c(\gamma(1)) < 0\}$. It is obvious that for any $\gamma \in \Gamma$ there exists $s \in (0,1)$ such that $T_c(\gamma(s)) = t(c)^{\frac{\sigma+1}{\sigma}}$ and therefore

$$i_c := \inf_{\gamma\in\Gamma}\left(\sup_{s\in[0,1]} S_c(\gamma(s))\right) \geqslant \varphi\left(t(c)^{\frac{\sigma+1}{\sigma}}\right) = \frac{\sigma}{\sigma+1}t(c)^{\frac{\sigma+1}{\sigma}} > 0.$$

On the other hand, let $u_c$ be a minimum action solution of (3.24), as given by Theorem 3.7 and Proposition 3.9. We have

$$S_c(\tau^{\frac{1}{2}} u_c) = \tau t(c)^{\frac{\sigma+1}{\sigma}} - \frac{\tau^{\sigma+1}}{\sigma+1} t(c)^{\frac{\sigma+1}{\sigma}}$$

and we see that $\max_{\tau>0} S_c(\tau^{\frac{1}{2}} u_c) = S_c(u_c) = \dfrac{\sigma}{\sigma+1} t(c)^{\frac{\sigma+1}{\sigma}}$. We conclude that necessarily $i_c = \frac{\sigma}{\sigma+1} t(c)^{\frac{\sigma+1}{\sigma}}$, that for $a > 0$ sufficiently large the mapping $\tau \longmapsto \tau^{\frac{1}{2}} a u_c$ is an optimal path in $\Gamma$, and that $u_c$ is a "mountain-pass solution" of (3.24).

Conversely, if $u$ is any critical point of $S_c$ at the mountain-pass level $i_c$ (that is, $S_c(u) = i_c$), by Proposition 3.9 we know that $u$ is a minimum action solution of (3.24).

# 4 Local minimization in the case $N\sigma > 4$

Throughout this section we assume that $\sigma > \frac{4}{N}$ and $2\sigma + 2 < 2^{**}$, that is $\sigma < \infty$ if $N \leqslant 4$ and $\sigma < \frac{4}{N-4}$ if $N \geqslant 5$. By Proposition 3.1 (i) we have $E_{min}(m) = -\infty$ for any $m > 0$. We will investigate the existence of *local* minimizers of $E$ when the $L^2-$norm is kept fixed. By *local minimizer* we mean a function $u \in H^2(\mathbf{R}^N)$ such that there exists an open set $\mathcal{U} \subset H^2(\mathbf{R}^N)$ such that $u \in \mathcal{U}$ and $E(u) = \inf\{E(v) \mid v \in \mathcal{U} \text{ and } \|v\|_{L^2} = \|u\|_{L^2}\}$.

For any $u \in H^2(\mathbf{R}^N)$ let $u_t(x) = t^{\frac{N}{4}} u(t^{\frac{1}{2}} x)$ be as in (3.40). We denote

$$\varphi_u(t) = E(u_t) = t^2 \int_{\mathbf{R}^N} |\Delta u|^2 \, dx - 2t \int_{\mathbf{R}^N} |\nabla u|^2 \, dx - \frac{t^{\frac{N\sigma}{2}}}{\sigma+1} \int_{\mathbf{R}^N} |u|^{2\sigma+2} \, dx,$$

and

$$D(u) = \int_{\mathbf{R}^N} |\Delta u|^2 \, dx - \frac{N\sigma}{4(\sigma+1)} \left(\frac{N\sigma}{2} - 1\right) \int_{\mathbf{R}^N} |u|^{2\sigma+2} \, dx. \tag{4.1}$$

The behaviour of the function $\varphi_u$ inspired the local minimization approach developed below. For later use we state here the following elementary lemma.

**Lemma 4.1** *Let $a$, $b$, $c > 0$ and define $f : [0, \infty) \longrightarrow \mathbf{R}$ by $f(t) = at^2 - 2bt - ct^{\frac{N\sigma}{2}}$. We have:*

*(i) The second derivative $f''$ is decreasing. There exists a unique $t_{infl} > 0$ such that $f''(t_{infl}) = 0$, and it is given by $t_{infl} = \left(\frac{8a}{N\sigma(N\sigma-2)c}\right)^{\frac{2}{N\sigma-4}}$.*

*(ii) The derivative $f'$ is increasing on $[0, t_{infl}]$ and decreasing on $[t_{infl}, \infty)$, and we have $f'(t_{infl}) > 0$ if and only if $a^{1-\frac{N\sigma}{2}} b^{\frac{N\sigma}{2}-2} c < \frac{8}{N\sigma(N\sigma-2)} \left(\frac{N\sigma-4}{N\sigma-2}\right)^{\frac{N\sigma}{2}-2}$.*

*For the next statements we assume that $f'(t_{infl}) > 0$.*

*(iii) There exist a unique $t_1 \in (0, t_{infl})$ and a unique $t_2 \in (t_{infl}, \infty)$ such that $f'(t_1) = 0$ and $f'(t_2) = 0$. The map $f$ is decreasing on $[0, t_1]$, increasing on $[t_1, t_2]$, decreasing on $[t_2, \infty)$ and reaches its minimum value on $[0, t_2]$ at $t_1$.*

*(iv) For $t_2 \leqslant t' < t''$ we have $f(t'') - f(t') \leqslant \frac{1}{2}(t'' - t')^2 f''(t_2)$.*

*(v) We have $f(t_{infl}) - f(t_1) = h\left(\frac{t_{infl}}{t_1}\right) t_1^{\frac{N\sigma}{2}} c$, where*

$$h(s) = \tfrac{1}{2}\left(\tfrac{N\sigma}{2} + 1\right)\left(\tfrac{N\sigma}{2} - 2\right) s^{\frac{N\sigma}{2}} - \tfrac{N\sigma}{2}\left(\tfrac{N\sigma}{2} - 1\right) s^{\frac{N\sigma}{2}-1} + \tfrac{N\sigma}{4}\left(\tfrac{N\sigma}{2} - 1\right) s^{\frac{N\sigma}{2}-2} + \tfrac{N\sigma}{2}(s-1) + 1.$$

*The function $h$ satisfies $h(1) = h'(1) = h''(1) = 0$ and*

$$h''(s) = \tfrac{N\sigma}{4}\left(\tfrac{N\sigma}{2} - 1\right)\left(\tfrac{N\sigma}{2} - 2\right) s^{\frac{N\sigma}{2}-4}(s-1)\left[\left(\tfrac{N\sigma}{2} + 1\right)s - \left(\tfrac{N\sigma}{2} - 3\right)\right],$$

*thus $h$ is positive, increasing and convex on $(1, \infty)$.*

*Proof.* This is simple Calculus. We have

$$f'(t) = 2at - 2b - \frac{N\sigma}{2} c t^{\frac{N\sigma}{2}-1} \qquad \text{and} \qquad f''(t) = 2a - \frac{N\sigma}{2}\left(\frac{N\sigma}{2} - 1\right) c t^{\frac{N\sigma}{2}-2}.$$

Statements (i), (ii), (iii) are obvious. For (iv) we use the fact that $f''$ is decreasing on $[0,\infty)$ and $f' < 0$ on $(t_2, \infty)$. We have:

$$f(t'') - f(t') = \int_{t'}^{t''} \left( f'(t') + \int_{t'}^{s} f''(\tau)\, d\tau \right) ds \leqslant \int_{t'}^{t''} \int_{t'}^{s} f''(t_2)\, d\tau\, ds = \frac{1}{2}(t'' - t')^2 f''(t_2).$$

(vi) Let $s = \frac{t_{infl}}{t_1}$. Recall that $s > 1$ because $t_1 < t_{infl}$. From the identity $f''(t_{infl}) = f''(t_1 s) = 0$ we get $a = \frac{N\sigma}{4}\left(\frac{N\sigma}{2} - 1\right) c t_1^{\frac{N\sigma}{2}-2} s^{\frac{N\sigma}{2}-2}$. Replacing this into the identity $f'(t_1) = 0$ we obtain $b = \frac{N\sigma}{4} c t_1^{\frac{N\sigma}{2}-1}\left[\left(\frac{N\sigma}{2} - 1\right) s^{\frac{N\sigma}{2}-2} - 1\right]$. Replacing these values of $a$ and $b$ into $f(t_1 s) - f(t_1)$ we get the announced identity. The properties of the function $h$ are obtained by direct computation. □

Recall that the functional $P_1$ has been introduced in (3.39). We have

(4.2)

$$\varphi_u'(t) = 2t \int_{\mathbf{R}^N} |\Delta u|^2\, dx - 2\int_{\mathbf{R}^N} |\nabla u|^2\, dx - \frac{N\sigma t^{\frac{N\sigma}{2}-1}}{2(\sigma+1)} \int_{\mathbf{R}^N} |u|^{2\sigma+2}\, dx = \frac{2P_1(u_t)}{t} \quad \text{and}$$

$$\varphi_u''(t) = 2\int_{\mathbf{R}^N} |\Delta u|^2\, dx - \frac{N\sigma t^{\frac{N\sigma}{2}-2}}{2(\sigma+1)}\left(\frac{N\sigma}{2} - 1\right) \int_{\mathbf{R}^N} |u|^{2\sigma+2}\, dx = \frac{2}{t^2} D(u_t).$$

For any $u \neq 0$ there exists a unique $t_{u,infl} > 0$ such that $\varphi_u''(t_{u,infl}) = 0$. It is given by

$$t_{u,infl} = \left( \frac{8(\sigma+1)}{N\sigma(N\sigma - 2)} \int_{\mathbf{R}^N} |\Delta u|^2\, dx \right)^{\frac{2}{N\sigma-4}} \left( \int_{\mathbf{R}^N} |u|^{2\sigma+2}\, dx \right)^{-\frac{2}{N\sigma-4}}. \tag{4.3}$$

We have $\varphi_u'' > 0$ on $(0, t_{u,infl})$ and $\varphi_u'' < 0$ on $(t_{u,infl}, \infty)$, hence $\varphi_u'$ is increasing on $(0, t_{u,infl}]$ and decreasing on $[t_{u,infl}, \infty)$, therefore reaches its maximum value at $t_{u,infl}$. If $\varphi_u'(t_{u,infl}) \leqslant 0$, the mapping $\varphi_u$ is (strictly) decreasing on $(0,\infty)$ and consequently none of the functions $(u_t)_{t>0}$ can be a local minimizer of $E$ when the $L^2-$norm is kept fixed. If $\varphi_u'(t_{u,infl}) > 0$, it is easily seen that there exist a unique $t_{u,1} \in (0, t_{u,infl})$ and a unique $t_{u,2} \in (t_{u,infl}, \infty)$ such that $\varphi_u'(t_{u,1}) = \varphi_u'(t_{u,2}) = 0$. We have $\varphi_u' < 0$ on $(0, t_{u,1}) \cup (t_{u,2}, \infty)$ and $\varphi_u' > 0$ on $(t_{u,1}, t_{u,2})$, therefore $\varphi_u$ is decreasing on $(0, t_{u,1}]$, increasing on $[t_{u,1}, t_{u,2}]$ and decreasing on $[t_{u,2}, \infty)$. It is now clear that among the functions $(u_t)_{t>0}$, the only one that could eventually be a local minimizer of $E$ when the $L^2-$norm is fixed is $u_{t_{u,1}}$. If $u$ is a local minimizer of $E$ at constant $L^2-$norm, we must have $t_{u,1} = 1$ and $1 < t_{u,infl} < t_{u,2}$, thus necessarily $D(u) > 0$. The above discussion indicates that it is natural to look for local minimizers of $E$ at fixed $L^2-$norm in the set

$$\begin{aligned} \mathcal{O} &= \{u \in H^2(\mathbf{R}^N) \mid u \neq 0,\ t_{u,infl} > 1 \text{ and } \varphi_u'(t_{u,infl}) > 0\} \\ &= \{u \in H^2(\mathbf{R}^N) \mid u \neq 0,\ D(u) > 0 \text{ and } \varphi_u'(t_{u,infl}) > 0\}. \end{aligned} \tag{4.4}$$

It is clear that $u \longmapsto t_{u,infl}$ and $u \longmapsto P_1(u_{t_{u,infl}})$ are continuous on $H^2(\mathbf{R}^N) \setminus \{0\}$ (see (4.3)), hence $\mathcal{O}$ is open. Given any $u \in H^2(\mathbf{R}^N) \setminus \{0\}$, using Lemma 4.1 (ii) we see that $\varphi_u'(t_{u,infl}) > 0$ if and only if

$$H(u) := \frac{\left(\int_{\mathbf{R}^N} |\Delta u|^2\, dx\right)^{\frac{N\sigma}{2}-1}}{\int_{\mathbf{R}^N} |u|^{2\sigma+2}\, dx \cdot \left(\int_{\mathbf{R}^N} |\nabla u|^2\, dx\right)^{\frac{N\sigma}{2}-2}} > \frac{N\sigma}{4(\sigma+1)}\left(\frac{N\sigma}{2} - 1\right)\left(\frac{N\sigma - 2}{N\sigma - 4}\right)^{\frac{N\sigma}{2}-2}. \tag{4.5}$$

Using (3.2) (with strict inequality because $u \neq 0$) and (3.3) we have

$$\int_{\mathbf{R}^N} |u|^{2\sigma+2}\, dx \cdot \left( \int_{\mathbf{R}^N} |\nabla u|^2\, dx \right)^{\frac{N\sigma}{2}-2} < B(N,\sigma) \|\Delta u\|_{L^2}^{N\sigma-2} \|u\|_{L^2}^{2\sigma}.$$

Therefore $H(u) > \frac{1}{B(N,\sigma)\|u\|_{L^2}^{2\sigma}}$. Denote

$$\mu_0 = B(N,\sigma)^{-\frac{1}{\sigma}} \left[ \frac{N\sigma}{4(\sigma+1)} \left( \frac{N\sigma}{2} - 1 \right) \right]^{-\frac{1}{\sigma}} \left( \frac{N\sigma - 2}{N\sigma - 4} \right)^{\frac{2}{\sigma} - \frac{N}{2}}. \tag{4.6}$$

We infer that (4.5) holds for any $u \in H^2(\mathbf{R}^N) \setminus \{0\}$ satisfying $\|u\|_{L^2}^2 \leqslant \mu_0$.

Using (3.3) we see that $D(u) \geqslant \|\Delta u\|_{L^2}^2 - \frac{N\sigma}{4(\sigma+1)} \left( \frac{N\sigma}{2} - 1 \right) B(N,\sigma) \|\Delta u\|_{L^2}^{\frac{N\sigma}{2}} \|u\|_{L^2}^{2\sigma+2-\frac{N\sigma}{2}}$ for any $u$, hence $D(u) > 0$ if $u \neq 0$ and $1 > \frac{N\sigma}{4(\sigma+1)} \left( \frac{N\sigma}{2} - 1 \right) B(N,\sigma) \|\Delta u\|_{L^2}^{\frac{N\sigma}{2}-2} \|u\|_{L^2}^{2\sigma+2-\frac{N\sigma}{2}}$. Let

$$\mathcal{O}_1 = \left\{ u \in H^2(\mathbf{R}^N) \;\middle|\; u \neq 0, \; \|\Delta u\|_{L^2}^{\frac{N\sigma}{2}-2} \|u\|_{L^2}^{2\sigma+2-\frac{N\sigma}{2}} < \frac{8(\sigma+1)}{N\sigma(N\sigma-2)B(N,\sigma)} \text{ and } \|u\|_{L^2}^2 < \mu_0 \right\}.$$

Obviously, $\mathcal{O}_1 \cup \{0\}$ is an open neighbourhood of 0 in $H^2(\mathbf{R}^N)$ and $\mathcal{O}_1 \subset \mathcal{O}$. It follows immediately from the definition of $D$ and from (4.5) that $\mathcal{O} \cup \{0\}$ is "star-shaped": for all $u \in \mathcal{O}$ and for all $a \in (0,1)$ we have $au \in \mathcal{O}$.

For any $m > 0$ we denote

$$\tilde{E}_{min}(m) = \inf\{E(u) \mid u \in \mathcal{O} \text{ and } \|u\|_{L^2}^2 = m\}. \tag{4.7}$$

It is obvious that $(u_t)_s = u_{ts}$. If $u \in \mathcal{O}$ and $t > 0$ we have $u_t \in \mathcal{O}$ if and only if $t < t_{u,infl}$, and $t_{u_t,infl} = \frac{t_{u,infl}}{t}$, $t_{u_t,i} = \frac{t_{u,i}}{t}$ for $i = 1, 2$. If $u \in \mathcal{O}$ satisfies $\|u\|_{L^2}^2 = m$, the previous discussion shows that $\min\{E(u_t) \mid 0 < t < t_{u,infl}\} = E(u_{t_{u,1}})$ and $u_{t_{u,1}}$ is the only function among $(u_t)_{0<t<t_{u,infl}}$ where $P_1$ vanishes. We have thus proved that

$$\tilde{E}_{min}(m) = \inf\{E(u) \mid u \in \mathcal{O} \text{ and } \|u\|_{L^2}^2 = m \text{ and } P_1(u) = 0\}. \tag{4.8}$$

**Remark 4.2** If $\sigma > \frac{4}{N}$ and $E$ is as in (1.9) with $\epsilon \leqslant 0$, there do not exist non-trivial minimizers of $E$ at fixed $L^2-$norm. Indeed, let $u \in H^2(\mathbf{R}^N) \setminus \{0\}$, let $u_t = t^{\frac{N}{4}} u(t^{\frac{N}{2}} \cdot)$, as in (3.40), and let $\varphi_u(t) = E(u_t)$ as above. There exists a unique $t_{u,infl} > 0$ such that $\varphi_u''(t_{u,infl}) = 0$ and it is given by (4.3). We have $\varphi_u'' > 0$ on $(0, t_{u,infl})$ and $\varphi_u'' < 0$ on $(t_{u,infl}, \infty)$. There exists a unique $t_u > 0$ such that $\varphi_u'(t_u) = 0$ and we have $t_u > t_{u,infl}$, $\varphi_u' > 0$ on $(0, t_u)$ and $\varphi_u' < 0$ on $(t_u, \infty)$. Therefore $\varphi_u$ is increasing on $(0, t_u)$, decreasing on $(t_u, \infty)$, it achieves its global maximum at $t = t_u$ and it has no local minimum on $(0, \infty)$. The previous discussion shows that no function $u \in H^2(\mathbf{R}^N) \setminus \{0\}$ can be a local minimizer of the energy at fixed mass.

**Lemma 4.3** *The following assertions hold true:*

*(i) For any $m > 0$, the set $\{u \in \mathcal{O} \mid \|u\|_{L^2}^2 = m$ and $P_1(u) = 0\}$ is not empty (thus $\tilde{E}_{min}(m) < \infty$), and $\tilde{E}_{min}(m) \geqslant -\frac{(N\sigma-2)^2}{N\sigma(N\sigma-4)} m$.*

*(ii) For all $m > 0$ and all $d, e \in \mathbf{R}$, the set*

$$\left\{ u \in H^2(\mathbf{R}^N) \mid D(u) \geqslant d, \; \|u\|_{L^2}^2 \leqslant m \text{ and } E(u) \leqslant e \right\}$$

*is bounded in $H^2(\mathbf{R}^N)$.*

*(iii) $\tilde{E}_{min}$ is sub-additive: $\tilde{E}_{min}(m_1 + m_2) \leqslant \tilde{E}_{min}(m_1) + \tilde{E}_{min}(m_2)$ for any $m_1, m_2 > 0$.*

*(iv) $\tilde{E}_{min}(m) \leqslant -m$ for any $m > 0$.*

*(v) $\tilde{E}_{min}$ is decreasing and continuous on $(0, \infty)$ and $\tilde{E}_{min}(m) \longrightarrow 0$ as $m \longrightarrow 0$.*

*(vi) Let $m > 0$. Assume that $(u_n)_{n \geqslant 1}$ is a bounded sequence in $H^2(\mathbf{R}^N)$ such that $\|u_n\|_{L^2}^2 \longrightarrow m$ and $E(u_n) \longrightarrow e$ as $n \longrightarrow \infty$, where $e \leqslant -m$. Then we have $\liminf_{n\to\infty} \|\Delta u_n\|_{L^2}^2 > 0$. In addition, if $e < -m$ then we have $\liminf_{n\to\infty} \|u_n\|_{L^{2\sigma+2}}^{2\sigma+2} > 0$.*

*(vii) If $u \in H^2(\mathbf{R}^N)$ satisfies $D(u) > 0$ and $P_1(u) = 0$, we have*

$$\frac{N\sigma - 2}{N\sigma - 4} \|u\|_{L^2}^2 > \|\nabla u\|_{L^2}^2 > \frac{N\sigma - 4}{N\sigma - 2} \|\Delta u\|_{L^2}^2 > \frac{N\sigma}{4(\sigma+1)} \left( \frac{N\sigma}{2} - 2 \right) \int_{\mathbf{R}^N} |u|^{2\sigma+2} \, dx. \tag{4.9}$$

*Proof.* (i) If $m \leqslant \mu_0$ (where $\mu_0$ is as in (4.6)), we have seen that any $u \in H^2(\mathbf{R}^N)$ with $\|u\|_{L^2}^2 = m$ satisfies (4.5), and then $u_{t_{u,1}} \in \mathcal{O}$, $\|u_{t_{u,1}}\|_{L^2}^2 = m$ and $P_1(u_{t_{u,1}}) = 0$. If $m > \mu_0$, choose an integer $n$ such that $\frac{m}{n} < \mu_0$, and take $v \in C_c^\infty(\mathbf{R}^N)$ such that $\|v\|_{L^2}^2 = \frac{m}{n}$. Let $w = v_{t_{v,1}}$, so that $w \in C_c^\infty(\mathbf{R}^N)$, $\|w\|_{L^2}^2 = \frac{m}{n}$, $P_1(w) = 0$ and $D(w) > 0$. Choose $R > 0$ such that $\mathrm{supp}(w) \subset B(0,R)$, then choose $x_0 \in \mathbf{R}^N$ such that $|x_0| > 2R$. Let $u = w + w(\cdot + x_0) + w(\cdot + 2x_0) + \cdots + w(\cdot + (n-1)x_0)$. Then we have $\|u\|_{L^2}^2 = n\|w\|_{L^2}^2 = m$, $P_1(u) = nP_1(w) = 0$ and $D(u) = nD(w) > 0$. From (4.2) we see that $\varphi_u'(t) > 0$ if $t > 1$ and $t$ is close to 1, hence $\varphi_u'(t_{u,infl}) > 0$ and therefore $u \in \mathcal{O}$.

We use (4.8) to obtain a lower bound for $\tilde{E}_{min}(m)$. Let $u \in H^2(\mathbf{R}^N)$ such that $\|u\|_{L^2}^2 = m$ and $P_1(u) = 0$. From the identity $P_1(u) = 0$ we obtain

$$\frac{1}{\sigma+1}\int_{\mathbf{R}^N} |u|^{2\sigma+2}\,dx = \frac{4}{N\sigma}\int_{\mathbf{R}^N} |\Delta u|^2\,dx - \frac{4}{N\sigma}\int_{\mathbf{R}^N} |\nabla u|^2\,dx.$$

Replacing this into $E(u)$ and using (3.2) we get

$$E(u) = \left(1 - \tfrac{4}{N\sigma}\right)\int_{\mathbf{R}^N} |\Delta u|^2\,dx - \left(2 - \tfrac{4}{N\sigma}\right)\int_{\mathbf{R}^N} |\nabla u|^2\,dx$$

$$\geqslant \left(1 - \tfrac{4}{N\sigma}\right)\|\Delta u\|_{L^2}^2 - \left(2 - \tfrac{4}{N\sigma}\right)\|\Delta u\|_{L^2}\|u\|_{L^2} \geqslant \inf_{s>0}\left\{\left(1 - \tfrac{4}{N\sigma}\right)s^2 - \left(2 - \tfrac{4}{N\sigma}\right)m^{\frac{1}{2}}s\right\}$$

$$= -\tfrac{(N\sigma-2)^2}{N\sigma(N\sigma-4)}m.$$

The above estimate is true for any $u$ satisfying $\|u\|_{L^2}^2 = m$ and $P_1(u) = 0$, and (i) follows from (4.8).

(ii) From $D(u) \geqslant d$ we get $\frac{1}{\sigma+1}\int_{\mathbf{R}^N} |u|^{2\sigma+2}\,dx \leqslant \frac{8}{N\sigma(N\sigma-2)}\int_{\mathbf{R}^N} |\Delta u|^2\,dx - \frac{8d}{N\sigma(N\sigma-2)}$. Using this inequality, the bound $E(u) \leqslant e$, then (3.2) and the fact that $\|u\|_{L^2}^2 \leqslant m$ we find

$$e \geqslant E(u) = \left(1 - \tfrac{8}{N\sigma(N\sigma-2)}\right)\int_{\mathbf{R}^N} |\Delta u|^2\,dx - 2\int_{\mathbf{R}^N} |\nabla u|^2\,dx + \tfrac{8d}{N\sigma(N\sigma-2)}$$

$$\geqslant \left(1 - \tfrac{8}{N\sigma(N\sigma-2)}\right)\|\Delta u\|_{L^2}^2 - 2m^{\frac{1}{2}}\|\Delta u\|_{L^2} + \tfrac{8d}{N\sigma(N\sigma-2)}.$$

Notice that $1 - \frac{8}{N\sigma(N\sigma-2)} > 0$ because $N\sigma > 4$, and the above inequality implies that $\|\Delta u\|_{L^2}$ is bounded. Since $\|u\|_{L^2}^2 \leqslant m$, we infer that $\|u\|_{H^2}$ is bounded.

(iii) Fix $m_1, m_2 > 0$ and $\varepsilon > 0$. Using the density of $C_c^\infty(\mathbf{R}^N)$ in $H^2(\mathbf{R}^N)$, it is easily seen that for $i \in \{1,2\}$ there exist $u_i \in C_c^\infty(\mathbf{R}^N) \cap \mathcal{O}$ such that $\|u_i\|_{L^2}^2 = m_i$ and $E(u_i) < \tilde{E}_{min}(m_i) + \frac{\varepsilon}{2}$. We may assume that $P_1(u_i) = 0$ for $i = 1,2$ (otherwise we replace $u_i$ by $(u_i)_{t_{u_i,1}}$). Choose $R > 0$ so large that $\mathrm{supp}(u_i) \subset B(0,R)$ for $i = 1,2$. Choose $x_0 \in \mathbf{R}^N$ such that $|x_0| > 2R$ and define $u = u_1 + u_2(\cdot + x_0)$. It is obvious that $\|u\|_{L^2}^2 = \|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2 = m_1 + m_2$, $D(u) = D(u_1) + D(u_2) > 0$ and $P_1(u) = P_1(u_1) + P_1(u_2) = 0$. This implies that $P_1(u_t) > 0$ for $t > 1$ and $t$ close to 1, and we infer that $\varphi_u'(t_{u,infl}) > 0$ and consequently $u \in \mathcal{O}$. Then we have

$$\tilde{E}_{min}(m_1 + m_2) \leqslant E(u) = E(u_1) + E(u_2) \leqslant \tilde{E}_{min}(m_1) + \tilde{E}_{min}(m_2) + \varepsilon.$$

Since $\varepsilon$ is arbitrary, the conclusion follows.

(iv) Let $m > 0$ and $\varepsilon > 0$. Let $u$ be the function constructed in the proof of Proposition 3.1 (iii). Since $\mathrm{supp}(\widehat{u}) \subset B(0,1) \setminus B(0,1-\varepsilon)$, we have $\|\Delta u\|_{L^2}^2 \leqslant \|u\|_{L^2}^2$. Then using (3.3) and the fact that $N\sigma > 4$ we get $\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant B(N,\sigma)\|\Delta u\|_{L^2}^{\frac{N\sigma}{2}}\|u\|_{L^2}^{2+2\sigma-\frac{N\sigma}{2}} \leqslant B(N,\sigma)\|\Delta u\|_{L^2}^2\|u\|_{L^2}^{2\sigma}$, and consequently

$$D(u) \geqslant \|\Delta u\|_{L^2}^2\left(1 - \frac{N\sigma(N\sigma-2)B(N,\sigma)}{8(\sigma+1)}\|u\|_{L^2}^{2\sigma}\right).$$

Denote $m_1 = \min\left(\mu_0, \left(\frac{8(\sigma+1)}{N\sigma(N\sigma-2)B(N,\sigma)}\right)^{\frac{1}{2\sigma}}\right)$. If $m < m_1$ we have $D(u) > 0$ by the above inequality. It is obvious that $u$ satisfies (4.5) because $\|u\|_{L^2}^2 < \mu_0$, hence $u \in \mathcal{O}$. In the proof of Proposition 3.1 (iii) we have shown that $E(u) \leqslant -\|u\|_{L^2}^2 + 4\varepsilon^2 m$, thus $\tilde{E}_{min}(m) \leqslant E(u) \leqslant -m + 4\varepsilon^2 m$. Since $\varepsilon > 0$ is arbitrary, assertion (iv) is proven in the case $m < m_1$.

If $m \geqslant m_1$, choose $n \in \mathbf{N}^*$ such that $\frac{m}{n} < m_1$. Using the sub-additivity of $\tilde{E}_{min}$ we get

$$\tilde{E}_{min}(m) \leqslant n\tilde{E}_{min}\left(\frac{m}{n}\right) \leqslant -m.$$

(v) Form (i) and (iv) we get $\tilde{E}_{min}(m) \longrightarrow 0$ as $m \longrightarrow 0$. If $0 < m_1 < m_2$, by (iii) and (iv) we have $\tilde{E}_{min}(m_2) \leqslant \tilde{E}_{min}(m_1) + \tilde{E}_{min}(m_2 - m_1) \leqslant \tilde{E}_{min}(m_1) - (m_2 - m_1)$, thus $\tilde{E}_{min}$ is decreasing.

Fix $M > 0$. By (ii), the set $\{u \in \mathcal{O} \mid \|u\|_{L^2}^2 \leqslant M \text{ and } E(u) \leqslant 0\}$ is bounded in $H^2(\mathbf{R}^N)$. Using the Sobolev embedding we see that there exists $K = K(M, N, \sigma) > 0$ such that for any $u$ in the above set we have $\frac{1}{\sigma+1}\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant K$. It is easily seen that for any $u \in \mathcal{O}$ and any $a \in (0,1)$ we have $au \in \mathcal{O}$. Let $0 < m_1 < m_2 \leqslant M$ and denote $a = \left(\frac{m_1}{m_2}\right)^{-\frac{1}{2}}$. Let $u \in \mathcal{O}$ such that $\|u\|_{L^2}^2 = m_2$ and $E(u) < 0$. We have $au \in \mathcal{O}$, $\|au\|_{L^2}^2 = m_1$ and consequently

$$\tilde{E}_{min}(m_1) \leqslant E(au) = a^2 E(u) + \frac{a^2 - a^{2\sigma+2}}{\sigma+1}\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant a^2 E(u) + (a^2 - a^{2\sigma+2})K.$$

Taking the infimum in the above inequality we find

$$\tilde{E}_{min}(m_1) \leqslant \frac{m_1}{m_2}\tilde{E}_{min}(m_2) + \left(\frac{m_1}{m_2} - \frac{m_1^{\sigma+1}}{m_2^{\sigma+1}}\right) K.$$

Thus $0 < \tilde{E}_{min}(m_1) - \tilde{E}_{min}(m_2) \leqslant \left(\frac{m_1}{m_2} - 1\right)\tilde{E}_{min}(m_2) + \left(\frac{m_1}{m_2} - \frac{m_1^{\sigma+1}}{m_2^{\sigma+1}}\right) K$. Using (i) we infer that $\tilde{E}_{min}$ is continuous on $(0, M)$. Since $M$ is arbitrary, (v) is proven.

(vi) Let $\ell = \liminf_{n\to\infty}\|u_n\|_{L^{2\sigma+2}}^{2\sigma+2}$. If $\ell = 0$, there is a subsequence $(u_{n_k})_{k\geqslant 1}$ such that $\|u_n\|_{L^{2\sigma+2}}^{2\sigma+2} \longrightarrow 0$ and using (3.8) with $\varepsilon = 0$ we get $\limsup_{k\to\infty} E(u_{n_k}) \geqslant -m$. Since $E(u_{n_k}) \longrightarrow e \leqslant m$ we infer that necessarily $e = -m$. Moreover, using again (3.8) we have

$$\int_{\mathbf{R}^N} \left(|\xi|^2 - 1\right)^2 |\widehat{u}_{n_k}(\xi)|^2 \, d\xi = (2\pi)^N \left(E(u_{n_k}) + \|u_{n_k}\|_{L^2}^2 + \frac{1}{\sigma+1}\|u_{n_k}\|_{L^{2\sigma+2}}^{2\sigma+2}\right) \longrightarrow 0$$

as $k \longrightarrow \infty$. Using Plancherel's formula, the Cauchy-Schwarz inequality, the above convergence and the boundedness of $(u_n)_{n\geqslant 1}$ in $H^2(\mathbf{R}^N)$ we get

$$\begin{aligned} \left| \|\Delta u_{n_k}\|_{L^2}^2 - \|u_{n_k}\|_{L^2}^2 \right| &\leqslant \tfrac{1}{(2\pi)^N} \int_{\mathbf{R}^N} \left| |\xi|^4 - 1 \right| |\widehat{u}_{n_k}(\xi)|^2 \, d\xi \\ &\leqslant \tfrac{1}{(2\pi)^N} \left( \int_{\mathbf{R}^N} \left(|\xi|^2 - 1\right)^2 |\widehat{u}_{n_k}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \left( \int_{\mathbf{R}^N} \left(|\xi|^2 + 1\right)^2 |\widehat{u}_{n_k}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \longrightarrow 0 \end{aligned}$$

as $k \longrightarrow \infty$ and we conclude that $\lim_{k\to\infty} \|\Delta u_{n_k}\|_{L^2}^2 = \lim_{k\to\infty} \|u_{n_k}\|_{L^2}^2 = m$.

If $\ell > 0$, from (3.3) and the fact that $\|u_n\|_{L^2}$ is bounded it follows that there exist $\eta > 0$ and $n_0 \in \mathbf{N}$ such that $\|\Delta u_n\|_{L^2} \geqslant \eta$ for all $n \geqslant n_0$, thus $\liminf_{n\to\infty} \|\Delta u_n\|_{L^2}^2 \geqslant \eta^2$.

Obviously, our arguments hold for any subsequence of $(u_n)_{n\geqslant 1}$. We infer that there cannot be a subsequence $(u_{n_j})_{j\geqslant 1}$ satisfying $\|\Delta u_{n_j}\|_{L^2} \longrightarrow 0$ as $j \longrightarrow \infty$, and this implies that $\liminf_{n\to\infty} \|\Delta u_n\|_{L^2}^2 > 0$.

It follows from the above arguments that in the case $e < -m$ we must have $\ell > 0$ and the second assertion in (vi) is now clear.

(vii) Assume that $u \in H^2(\mathbf{R}^N)$ satisfies $D(u) > 0$ and $P_1(u) = 0$. From $D(u) > 0$ we get $\int_{\mathbf{R}^N} |\Delta u|^2 \, dx > \frac{N\sigma}{4(\sigma+1)}\left(\frac{N\sigma}{2} - 1\right)\int_{\mathbf{R}^N} |u|^{2\sigma+2} \, dx$, which is the last inequality in (4.9). Replacing this into $P_1(u) = 0$ we obtain the second inequality in (4.9). Then the second inequality in (4.9) and (3.2) give $\|u\|_{L^2} > \frac{N\sigma-4}{N\sigma-2}\|\Delta u\|_{L^2}$. Combining this with (3.2) we get the first inequality in (4.9). □

**Lemma 4.4** *(i) If $N \geqslant 5$ and $\frac{4}{N} < \sigma < 1$, we have $\tilde{E}_{min}(m) < -m$ for any $m > 0$.*

*(ii) If $\frac{4}{N} < \sigma$ and $\sigma \geqslant 1$, there exists $m_0 > 0$ such that $\tilde{E}_{min}(m) = -m$ for any $m \in (0, m_0]$.*

*Proof.* (i) Let $m > 0$. We use the same test functions as in the proof of Proposition 3.3, constructed in Example 2.7. For small $\varepsilon, \delta > 0$ let $u_{\varepsilon,\delta}$ be as in (2.26) and let $w_{\varepsilon,\delta} = \frac{\sqrt{m}}{\|u_{\varepsilon,\delta}\|_{L^2}} u_{\varepsilon,\delta}$, so that $\|w_{\varepsilon,\delta}\|_{L^2}^2 = m$. Fix $\delta_0 \in (0, \frac{1}{10})$. We have already seen in the proof of Proposition 3.3 that $E(w_{\varepsilon,\delta_0}) + \|w_{\varepsilon,\delta_0}\|_{L^2}^2 < 0$ for all sufficiently small $\varepsilon$ (cf. (3.14)).

The conclusion of Lemma 4.4 (i) follows if we show that $w_{\varepsilon,\delta_0} \in \mathcal{O}$ for all sufficiently small $\varepsilon$. Since $\text{supp}(\widehat{w}_{\varepsilon,\delta_0}) = \text{supp}(\widehat{u}_{\varepsilon,\delta_0}) \subset \overline{B}(0, 1+\varepsilon) \setminus B(0, 1-\varepsilon)$, we have $(1-\varepsilon)^2 \|w_{\varepsilon,\delta_0}\|_{L^2} \leqslant \|\Delta w_{\varepsilon,\delta_0}\|_{L^2} \leqslant (1+\varepsilon)^2 \|w_{\varepsilon,\delta_0}\|_{L^2}$ and $(1-\varepsilon)\|w_{\varepsilon,\delta_0}\|_{L^2} \leqslant \|\nabla w_{\varepsilon,\delta_0}\|_{L^2} \leqslant (1+\varepsilon)\|w_{\varepsilon,\delta_0}\|_{L^2}$.

By the Hausdorff-Young inequality (2.10) and Hölder's inequality we have

$$\|w_{\varepsilon,\delta_0}\|_{L^{2\sigma+2}} \leqslant C\|\widehat{w}_{\varepsilon,\delta_0}\|_{L^{\frac{2\sigma+2}{2\sigma+1}}} \leqslant C\|\widehat{w}_{\varepsilon,\delta_0}\|_{L^2} \cdot |\text{supp}(\widehat{w}_{\varepsilon,\delta_0})|^{\frac{2\sigma+1}{2\sigma+2} - \frac{1}{2}} \leqslant C\sqrt{m}\left(\varepsilon \delta_0^{N-1}\right)^{\frac{\sigma}{2\sigma+2}}.$$

Since $\delta_0$ is fixed, we have $\|w_{\varepsilon,\delta}\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant C m^{\sigma+1} \varepsilon^{\sigma}$ and therefore $D(w_{\varepsilon,\delta_0}) \geqslant (1-\varepsilon)^2 m - C m^{\sigma+1}\varepsilon^{\sigma} > 0$ if $\varepsilon$ is small enough. Moreover, if $H$ is given by (4.5) we have $H(w_{\varepsilon,\delta_0}) \geqslant C m^{-\sigma} \varepsilon^{-\sigma} \longrightarrow \infty$ as $\varepsilon \longrightarrow 0$ and we conclude that $w_{\varepsilon,\delta_0} \in \mathcal{O}$ for all sufficiently small $\varepsilon$.

(ii) Recall that by (3.15) we have

$$E(u) + \|u\|_{L^2}^2 = \|(\Delta+1)u\|_{L^2}^2 \left(1 - \frac{\|u\|_{L^2}^{2\sigma}}{\sigma+1} Q_\kappa(u)^{2\sigma+2}\right) \qquad \text{for any } u \in H^2(\mathbf{R}^N) \setminus \{0\}, \tag{4.10}$$

where $\kappa = \frac{\sigma}{\sigma+1}$ and $Q_\kappa$ is given in (2.8).

Lemma 4.3 (vii) implies that there exists $R_0 > 0$ such that $\|(\Delta+1)u\|_{L^2} \leqslant R_0 \|u\|_{L^2}$ for any $u \in H^2(\mathbf{R}^N)$ satisfying $D(u) > 0$ and $P_1(u) = 0$. Since $\frac{4}{N} < \sigma$ and $\sigma \geqslant 1$, condition (2.38) is satisfied with $s = 2$, $p = 2\sigma + 2$ and $\kappa = \frac{\sigma}{\sigma+1}$. Then Corollary 2.10 implies that there exists $M > 0$ such that $Q_\kappa(u) \leqslant M$ for any $u$ as above. Using (4.8) and (4.10) we infer that $\tilde{E}_{min}(m) + m \geqslant 0$ if $0 < m \leqslant (\sigma+1)^{\frac{1}{\sigma}} M^{-\frac{2\sigma+2}{\sigma}}$. The conclusion follows from this inequality and Lemma 4.3 (iv). $\square$

**Lemma 4.5** *Let $(u_n)_{n\geqslant 1} \subset \mathcal{O}$ be a sequence satisfying*

*(a) $P_1(u_n) \longrightarrow 0$,*

*(b) $\|u_n\|_{L^2}^2 \longrightarrow m$ as $n \longrightarrow \infty$ and $m < \mu_0$, where $\mu_0$ is given by (4.6), and*

*(c) there exists $k > 0$ such that $\|\Delta u_n\|_{L^2} \geqslant k$ for all $n$.*

*Then $\liminf_{n\to\infty} D(u_n) > 0$. Moreover, if $\|\Delta u_n\|_{L^2}$ is bounded then we have $\liminf_{n\to\infty} t_{u_n,infl} > 1$.*

*Proof.* We have $D(u_n) > 0$ for all $n$ because $u_n \in \mathcal{O}$. We argue by contradiction and we assume that there is a subsequence, still denoted $(u_n)_{n\geqslant 1}$, such that $D(u_n) \longrightarrow 0$. We have

$$\frac{N\sigma}{4(\sigma+1)} \int_{\mathbf{R}^N} |u_n|^{2\sigma+2}\,dx = \frac{2}{N\sigma-2}\left(\int_{\mathbf{R}^N} |\Delta u_n|^2\,dx - D(u_n)\right). \tag{4.11}$$

Using this identity in the expression of $P_1(u_n)$ we get

$$P_1(u_n) = \frac{N\sigma-4}{N\sigma-2}\int_{\mathbf{R}^N} |\Delta u_n|^2\,dx - \int_{\mathbf{R}^N} |\nabla u_n|^2\,dx + \frac{2}{N\sigma-2} D(u_n). \tag{4.12}$$

From the equality above and (3.2) we obtain

$$\|\Delta u_n\|_{L^2}\|u_n\|_{L^2} > \|\nabla u_n\|_{L^2}^2 = \frac{N\sigma-4}{N\sigma-2}\|\Delta u_n\|_{L^2}^2 + \frac{2}{N\sigma-2} D(u_n) - P_1(u_n).$$

The last inequality, assumptions (a) and (b) and the fact that $D(u_n) \longrightarrow 0$ imply that $\|\Delta u_n\|_{L^2}$ is bounded. We rewrite the last inequality in the form

$$\|\Delta u_n\|_{L^2} < \frac{N\sigma-2}{N\sigma-4}\left(\|u_n\|_{L^2} - \frac{2}{N\sigma-2}\frac{D(u_n)}{\|\Delta u_n\|_{L^2}} + \frac{P_1(u_n)}{\|\Delta u_n\|_{L^2}}\right). \tag{4.13}$$

Using the definition of $D$ (see (4.1)) and (3.3) we get

$$\|\Delta u_n\|_{L^2}^2 - D(u_n) = \frac{N\sigma(N\sigma-2)}{8(\sigma+1)}\|u_n\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant \frac{N\sigma(N\sigma-2)B(N,\sigma)}{8(\sigma+1)}\|\Delta u_n\|_{L^2}^{\frac{N\sigma}{2}}\|u_n\|_{L^2}^{2\sigma+2-\frac{N\sigma}{2}}.$$

Dividing by $\|\Delta u_n\|_{L^2}^2$ and using (4.13) we discover

$$1 - \frac{D(u_n)}{\|\Delta u_n\|_{L^2}^2} \leqslant \frac{N\sigma(N\sigma-2)B(N,\sigma)}{8(\sigma+1)}\|\Delta u_n\|_{L^2}^{\frac{N\sigma}{2}-2}\|u_n\|_{L^2}^{2\sigma+2-\frac{N\sigma}{2}}$$

$$< \frac{N\sigma(N\sigma-2)B(N,\sigma)}{8(\sigma+1)}\left[\frac{N\sigma-2}{N\sigma-4}\left(\|u_n\|_{L^2} - \frac{2}{N\sigma-2}\frac{D(u_n)}{\|\Delta u_n\|_{L^2}} + \frac{P_1(u_n)}{\|\Delta u_n\|_{L^2}}\right)\right]^{\frac{N\sigma}{2}-2}\|u_n\|_{L^2}^{2\sigma+2-\frac{N\sigma}{2}}.$$

Letting $n \longrightarrow \infty$ in the above inequality and using assumptions (b), (c) and (4.6) we obtain $1 \leqslant \frac{m^\sigma}{\mu_0^\sigma}$, contradicting the fact that $m < \mu_0$. We have thus proved that $\liminf_{n\to\infty} D(u_n) > 0$.

Since $u_n \in \mathcal{O}$ we have $t_{u_n,infl} > 1$ for each $n$. We argue again by contradiction for the second part of Lemma 4.5 and we assume that there is a subsequence, still denoted $(u_n)_{n\geqslant 1}$, such that $t_{u_n,infl} \longrightarrow 1$ as $n \longrightarrow \infty$. By (4.3) we have $D(u_n) = \left(1 - t_{u_n,infl}^{-\frac{N\sigma}{2}+2}\right)\|\Delta u_n\|_{L^2}^2$ and the boundedness of $\|\Delta u_n\|_{L^2}$ implies that $D(u_n) \longrightarrow 0$, contradicting assumption (c). □

**Lemma 4.6** *Assume that $m < \mu_0$, where $\mu_0$ is given by (4.6). Suppose that the sequence $(u_n)_{n\geqslant 1} \subset H^2(\mathbf{R}^N)$ satisfies $\|u_n\|_{L^2}^2 \longrightarrow m$ and $D(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.*

*Then we have $\liminf_{n\to\infty} E(u_n) \geqslant \tilde{E}_{min}(m)$.*

*Moreover, if $\tilde{E}_{min}(m) < -m$ we have $\liminf_{n\to\infty} E(u_n) > \tilde{E}_{min}(m)$.*

Notice that in Lemma 4.6 we do not assume that $(u_n)_{n\geqslant 1} \subset \mathcal{O}$.

*Proof.* The sequence $(u_n)_{n\geqslant 1}$ is bounded in $H^2(\mathbf{R}^N)$ by Lemma 4.3 (ii). By Lemma 4.3 (iv) we have $\tilde{E}_{min}(m) \leqslant -m$, so the conclusion of Lemma 4.6 is obvious if $\liminf_{n\to\infty} E(u_n) \geqslant -m$. Form now on we only consider the case when $\liminf_{n\to\infty} E(u_n) < -m$. Passing to a subsequence we may assume that $E(u_n) \longrightarrow e < -m$ as $n \longrightarrow \infty$ and that $\|u_n\|_{L^2}^2 < \mu_0$ for all $n \geqslant 1$, so that $\varphi'_{u_n}(t_{u_n,infl}) > 0$ and $t_{u_n,1}$, $t_{u_n,2}$ do exist.

It follows from Lemma 4.3 (vi) that there exist $\eta > 0$ and $n_0 \in \mathbf{N}$ such that $\|\Delta u_n\|_{L^2} \geqslant \eta$ for all $n \geqslant n_0$. Using (4.11) and (3.3) we get for $n \geqslant n_0$,

$$\frac{N\sigma}{4(\sigma+1)}B(N,\sigma)\|\Delta u_n\|_{L^2}^{\frac{N\sigma}{2}-2}\|u_n\|_{L^2}^{2\sigma+2-\frac{N\sigma}{2}} \geqslant \frac{2}{N\sigma-2}\left(1 - \frac{1}{\eta^2}D(u_n)\right)$$

Then using (4.12), (3.2) and the above inequality we obtain for $n \geqslant n_0$

$$P_1(u_n) \geqslant \|\Delta u_n\|_{L^2}\left(\frac{N\sigma-4}{N\sigma-2}\|\Delta u_n\|_{L^2} - \|u_n\|_{L^2}\right) + \frac{2}{N\sigma-2}D(u_n)$$

$$\geqslant \|\Delta u_n\|_{L^2}\left(\frac{N\sigma-4}{N\sigma-2}\left(\frac{8(\sigma+1)}{N\sigma(N\sigma-2)B(N,\sigma)}\left(1 - \frac{1}{\eta^2}D(u_n)\right)\right)^{\frac{2}{N\sigma-4}}\|u_n\|_{L^2}^{-\frac{4\sigma+4-N\sigma}{N\sigma-4}} - \|u_n\|_{L^2}\right) + \frac{2D(u_n)}{N\sigma-2}.$$

Letting $n \longrightarrow \infty$ and using the fact that $D(u_n) \longrightarrow 0$ we discover

$$(4.14) \qquad \liminf_{n\to\infty} P_1(u_n) \geqslant \eta\left(\frac{N\sigma-4}{N\sigma-2}\left(\frac{8(\sigma+1)}{N\sigma(N\sigma-2)B(N,\sigma)}\right)^{\frac{2}{N\sigma-4}} m^{-\frac{2\sigma+2-\frac{N\sigma}{2}}{N\sigma-4}} - m^{\frac{1}{2}}\right).$$

Since $0 < m < \mu_0$, where $\mu_0$ is given by (4.6), the right-hand side of (4.14) is equal to $\eta m^{\frac{1}{2}}\left(\left(\frac{\mu_0}{m}\right)^{\frac{2\sigma}{N\sigma-4}} - 1\right)$ and this quantity is positive. We conclude that there exists $n_1 \in \mathbf{N}$ such that $P_1(u_n) > 0$ for all $n \geqslant n_1$. This means that $t_{u_n,1} < 1 < t_{u_n,2}$ for all $n \geqslant n_1$.

We denote $v_n = (u_n)_{t_{u_n,1}}$, so that $v_n \in \mathcal{O}$, $\|v_n\|_{L^2} = \|u_n\|_{L^2}$, $P_1(v_n) = 0$ and $E(v_n) \leqslant E(u_n)$ for each $n \geqslant n_1$ (recall that $t \longmapsto E(u_t)$ is increasing on $[t_{u,1}, t_{u,2}]$). By Lemma 4.3

(ii), the sequence $(v_n)_{n\geqslant 1}$ is bounded in $H^2(\mathbf{R}^N)$. Since $\tilde{E}_{min}(\|v_n\|_{L^2}^2) \leqslant E(v_n) \leqslant E(u_n)$, passing to the limit and using the continuity of $\tilde{E}_{min}$ (see Lemma 4.3 (v)) we get

$$\tilde{E}_{min}(m) \leqslant \liminf_{n\to\infty} E(v_n) \leqslant \limsup_{n\to\infty} E(v_n) \leqslant \lim_{n\to\infty} E(u_n) = e. \tag{4.15}$$

We show that if $\tilde{E}_{min}(m) < -m$, then at least one inequality in (4.15) is strict. This clearly implies the conclusion of Lemma 4.6. We assume that equality occurs in the first two inequalities in (4.15), which means precisely that $E(v_n) \longrightarrow \tilde{E}_{min}(m) < -m$. We show that in this case the last inequality in (4.15) must be strict. Denote $\ell := \liminf_{n\to\infty} \|v_n\|_{L^{2\sigma+2}}^{2\sigma+2}$. Using Lemma 4.3 (vi) we see that $\ell > 0$ and there exist $\eta_1 > 0$ such that $\|\Delta v_n\|_{L^2} \geqslant \eta_1$ for all sufficiently large $n$. Now we may apply Lemma 4.5 to $(v_n)_{n\geqslant 1}$ and we infer that $\liminf_{n\to\infty} D(v_n) > 0$ and $\liminf_{n\to\infty} t_{v_n,infl} > 1$.

Denote $s_n = (t_{u_n,1})^{-1}$, so that $u_n = (v_n)_{s_n}$. Recall that $t_{u_n,1} < 1$, hence $s_n > 1$ for all $n \geqslant n_1$. We have

$$D(u_n) = D((v_n)_{s_n}) = s_n^2 \left( \int_{\mathbf{R}^N} |\Delta v_n|^2 \, dx - s_n^{\frac{N\sigma}{2}-2} \frac{N\sigma(N\sigma-2)}{8(\sigma+1)} \int_{\mathbf{R}^N} |v_n|^{2\sigma+2} \, dx \right).$$

Since $D(u_n) \longrightarrow 0$, the second factor in the expression of $D((v_n)_{s_n})$ here above must tend to 0, and from (4.3) and the fact that $\ell > 0$ we infer that $r_n := \frac{s_n}{t_{v_n,infl}} \longrightarrow 1$ as $k \longrightarrow \infty$. Using the boundedness of $(u_n)_{n\geqslant 1}$ in $H^2(\mathbf{R}^N)$ we obtain then

$$E(u_n) - E((v_n)_{t_{v_n,infl}}) = E(u_n) - E((u_n)_{r_n^{-1}}) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{4.16}$$

From Lemma 4.1 (v) we have

$$E((v_n)_{t_{v_n,infl}}) - E(v_n) = \frac{h(t_{v_n,infl})}{\sigma+1} \int_{\mathbf{R}^N} |v_n|^{2\sigma+2} \, dx. \tag{4.17}$$

Fix $t_*$ such that $1 < t_* < \liminf_{n\to\infty} t_{v_n,infl}$. From (4.16) and (4.17) we see that $E(u_n) - E(v_n) > \frac{\ell}{2\sigma+2} h(t_*)$ for all sufficiently large $n$. Therefore the last inequality in (4.15) is strict and the conclusion of Lemma 4.6 follows. ☐

**Lemma 4.7** *Let $\mu_0$ be as in (4.6). Denote $m_0 = \inf\{m \in (0, \mu_0] \mid \tilde{E}_{min}(m) < -m\}$. Then:*

*(i) The mapping $m \longmapsto \frac{\tilde{E}_{min}(m)}{m}$ is non-increasing on $(0, \mu_0]$, and it is decreasing on $(m_0, \mu_0]$.*

*(ii) If $m \in (0, \mu_0]$ satisfies $\tilde{E}_{min}(m) < -m$, then for any $m' \in (0, m)$ we have*

$$\tilde{E}_{min}(m) < \tilde{E}_{min}(m') + \tilde{E}_{min}(m - m').$$

*Proof.* It is easy to see that for any $u \in \mathcal{O}$ and any $a \in (0, 1)$ we have $au \in \mathcal{O}$.

Assume that $u \in H^2(\mathbf{R}^N)$ satisfies $D(u) > 0$, $P_1(u) = 0$ $\|u\|_{L^2}^2 < \mu_0$. We show that for any $a \in \left[1, \frac{\mu_0}{\|u\|_{L^2}^2}\right]$ we have $a^{\frac{1}{2}} u \in \mathcal{O}$. Since $\|a^{\frac{1}{2}} u\|_{L^2}^2 = a\|u\|_{L^2}^2 \leqslant \mu_0$, the function $a^{\frac{1}{2}} u$ automatically satisfies (4.5) and we only have to prove that $D(a^{\frac{1}{2}} u) > 0$. Using (3.3) and the fact that $\|u\|_{L^2} > \frac{N\sigma-4}{N\sigma-2} \|\Delta u\|_{L^2}$ (see (4.9)) we have

$$\begin{aligned}
D(a^{\frac{1}{2}} u) &= a\|\Delta u\|_{L^2}^2 - \tfrac{N\sigma}{4(\sigma+1)} \left(\tfrac{N\sigma}{2} - 1\right) a^{\sigma+1} \|u\|_{L^{2\sigma+2}}^{2\sigma+2} \\
&\geqslant a\|\Delta u\|_{L^2}^2 \left(1 - \tfrac{N\sigma}{4(\sigma+1)} \left(\tfrac{N\sigma}{2} - 1\right) a^\sigma B(N,\sigma) \|\Delta u\|_{L^2}^{\frac{N\sigma}{2}-2} \|u\|_{L^2}^{2\sigma+2-\frac{N\sigma}{2}}\right) \\
&> a\|\Delta u\|_{L^2}^2 \left(1 - \tfrac{N\sigma}{4(\sigma+1)} \left(\tfrac{N\sigma}{2} - 1\right) B(N,\sigma) \left(\tfrac{N\sigma-2}{N\sigma-4}\right)^{\frac{N\sigma}{2}-2} a^\sigma \|u\|_{L^2}^{2\sigma}\right).
\end{aligned}$$

The last expression is non-negative if $a\|u\|_{L^2}^2 \leqslant \mu_0$ by (4.6), hence $a^{\frac{1}{2}} u \in \mathcal{O}$. Thus we have

$$\tilde{E}_{min}(a\|u\|_{L^2}^2) \leqslant E(a^{\frac{1}{2}} u) = aE(u) + \frac{a - a^{\sigma+1}}{\sigma+1} \int_{\mathbf{R}^N} |u|^{2\sigma+2} \, dx \text{ for any } a \in \left(0, \frac{\mu_0}{\|u\|_{L^2}^2}\right]. \tag{4.18}$$

Let $m \in (0, \mu_0)$. Take a minimising sequence $(u_n)_{n\geqslant 1} \subset \mathcal{O}$ such that $\|u_n\|_{L^2} = m$, $P_1(u_n) = 0$ and $E(u_n) \longrightarrow \tilde{E}_{min}(m)$. By (3.8) we have $\frac{1}{\sigma+1}\|u_n\|_{L^{2\sigma+2}}^{2\sigma+2} \geqslant -(E(u_n) + \|u_n\|_{L^2}^2)$ for each $n$. Using this in (4.18) and letting $n \longrightarrow \infty$ we obtain

$$\tilde{E}_{min}(am) \leqslant a\tilde{E}_{min}(m) + (a^{\sigma+1} - a)(\tilde{E}_{min}(m) + m) \text{ for any } a \in \left[1, \frac{\mu_0}{m}\right]$$

or equivalently

$$\frac{\tilde{E}_{min}(am)}{am} \leqslant \frac{\tilde{E}_{min}(m)}{m} + (a^{\sigma} - 1)\left(\frac{\tilde{E}_{min}(m)}{m} + 1\right) \text{ for any } a \in \left[1, \frac{\mu_0}{m}\right]. \tag{4.19}$$

Since $\tilde{E}_{min}(m) \leqslant -m$ (see Lemma 4.3 (iv)), conclusion (i) of Lemma 4.7 follows easily from (4.19).

(ii) It follows from the continuity of $\tilde{E}_{min}$ that $\tilde{E}_{min}(m') < -m'$ for $m'$ in a neighbourhood of $m$, and using part (i) we infer that $\frac{\tilde{E}_{min}(m)}{m} < \frac{\tilde{E}_{min}(m')}{m'}$ for all $m' \in (0, m)$. In particular, for $m' \in (0, m)$ we have $\frac{\tilde{E}_{min}(m)}{m} < \frac{\tilde{E}_{min}(m')}{m'}$ and $\frac{\tilde{E}_{min}(m)}{m} < \frac{\tilde{E}_{min}(m-m')}{m-m'}$. Combining the last two inequalities we get (ii). ☐

**Theorem 4.8** *Assume that $0 < m < \mu_0$ and $\tilde{E}_{min}(m) < -m$. Then $\tilde{E}_{min}(m)$ is achieved.*

*Moreover, for any sequence $(u_n)_{n\geqslant 1} \subset \mathcal{O}$ satisfying $\|u_n\|_{L^2}^2 \longrightarrow m$ and $E(u_n) \longrightarrow \tilde{E}_{min}(m)$ there exist a subsequence $(u_{n_k})_{k\geqslant 1}$, a sequence of points $(x_k) \subset \mathbf{R}^N$ and $u \in \mathcal{O}$ such that $u_{n_k}(\cdot + x_k) \longrightarrow u$ strongly in $H^2(\mathbf{R}^N)$. (Then, obviously, $\|u\|_{L^2}^2 = m$ and $E(u) = \tilde{E}_{min}(m)$.)*

*Proof.* Let $(u_n)_{n\geqslant 1}$ be a sequence as in Theorem 4.8. It follows from Lemma 4.3 (ii) that $(u_n)_{n\geqslant 1}$ is bounded in $H^2(\mathbf{R}^N)$. Lemma 4.3 (vi) implies that there exist $\delta > 0$ and $\ell > 0$ such that $\|\Delta u_n\|_{L^2} \geqslant \delta$ and $\|u_n\|_{L^{2\sigma+2}}^{2\sigma+2} \geqslant \ell$ for all sufficiently large $n$.

Proceeding exactly as in the proof of Theorem 3.4 we see that there exists a subsequence, still denoted $(u_n)_{n\geqslant 1}$, there exist points $x_n \in \mathbf{R}^N$ and there is $u \in H^2(\mathbf{R}^N)$, $u \neq 0$ such that after replacing $u_n$ by $u_n(\cdot + x_n)$, (3.19) holds. Then the weak convergence $u_n \rightharpoonup u$ gives (3.20), while Brezis-Lieb Lemma and the fact that $u_n \longrightarrow u$ a.e. give (3.21).

We denote $v_n = (u_n)_{t_{u_n,1}}$. Then we have $v_n \in \mathcal{O}$ for all $n$, $\|v_n\|_{L^2} = \|u_n\|_{L^2}$, $P_1(v_n) = 0$, and $\tilde{E}_{min}(\|v_n\|_{L^2}^2) \leqslant E(v_n) \leqslant E(u_n)$ for all $n$, thus

$$E(v_n) \longrightarrow \tilde{E}_{min}(m) < -m \qquad \text{as } n \longrightarrow \infty. \tag{4.20}$$

Lemma 4.3 (ii) implies that $(v_n)_{n\geqslant 1}$ is bounded in $H^2(\mathbf{R}^N)$, and by Lemma 4.3 (vi) there exist $\tilde{\delta} > 0$ and $\tilde{\ell} > 0$ such that $\|\Delta v_n\|_{L^2} \geqslant \tilde{\delta}$ and $\|u_n\|_{L^{2\sigma+2}}^{2\sigma+2} \geqslant \tilde{\ell}$ for all sufficiently large $n$. We have $\|\Delta v_n\|_{L^2} = t_{u_n,1}\|\Delta u_n\|_{L^2}$, and $\|\Delta v_n\|_{L^2}$ as well as $\|\Delta u_n\|_{L^2}$ are bounded and stay away from zero, thus the sequence $(t_{u_n,1})_{n\geqslant 1}$ is bounded and stays away from zero. We infer that there exist $t_1 \in (0, \infty)$ such that after passing to a subsequence of $(u_n)_{n\geqslant 1}$, still denoted the same, we have $t_{u_n,1} \longrightarrow t_1$ as $n \longrightarrow \infty$. It is easy to see that $(u_n)_{t_{u_n,1}} \rightharpoonup u_{t_1} \neq 0$ as $n \longrightarrow \infty$. Let $v = u_{t_1}$. Then $v \neq 0$ and $v_n \rightharpoonup v$ weakly in $H^2(\mathbf{R}^N)$. Passing eventually to further subsequences of $(u_n)_{n\geqslant 1}$ and of $(v_n)_{n\geqslant 1}$, still denoted the same, we may assume in addition that $v_n \longrightarrow v$ in $L^p_{loc}(\mathbf{R}^N)$ for any $1 \leqslant p < 2^{**}$, and almost everywhere. It is then clear that (3.20) and (3.21) hold with $v_n$ and $v$ instead of $u_n$ and $u$, respectively.

Our strategy is as follows. We will show firstly that $\|v\|_{L^2}^2 = m$. Then we prove that $v_n \longrightarrow v$ strongly in $H^2(\mathbf{R}^N)$ and that $v \in \mathcal{O}$. Finally we show that necessarily $t_1 = 1$ (thus $u = v$) and that $u_n \longrightarrow u$ strongly in $H^2(\mathbf{R}^N)$.

To carry out the first step of our plan we argue by contradiction and we assume that $\|v\|_{L^2}^2 < m$. Then (3.20) implies that $\|v_n - v\|_{L^2}^2 \longrightarrow m - \|v\|_{L^2}^2 \in (0, m)$ as $n \longrightarrow \infty$. By (3.20) and (3.21) we have

$$E(v_n) = E(v) + E(v_n - v) + o(1), \tag{4.21}$$

$$D(v_n) = D(v) + D(v_n - v) + o(1), \quad \text{and} \tag{4.22}$$

$$(4.23) \qquad 0 = P_1(v_n) = P_1(v) + P_1(v_n - v) + o(1) \qquad \text{as } n \longrightarrow \infty.$$

Passing to a further subsequence if necessary, we may assume that

$$\int_{\mathbf{R}^N} |\Delta v_n - \Delta v|^2 \, dx \longrightarrow a, \quad \|\nabla v_n - \nabla v\|_{L^2}^2 \longrightarrow b \quad \text{and} \quad \frac{1}{\sigma+1} \int_{\mathbf{R}^N} |v_n - v|^{2\sigma+2} \, dx \longrightarrow c$$

as $n \longrightarrow \infty$, where $a$, $b$, $c \geqslant 0$. Notice that $\liminf_{n\to\infty} D(v_n) > 0$ by Lemma 4.5, and using (4.22) and passing to the limit we infer that $D(v) + a - \frac{N\sigma}{4}\left(\frac{N\sigma}{2} - 1\right) c > 0$. Thus at least one of the quantities $D(v)$ and $a - \frac{N\sigma}{4}\left(\frac{N\sigma}{2} - 1\right) c$ must be positive. There are several possibilities and we analyse all of them, showing that in each case we get a contradiction.

*Case 1.* $D(v) > 0$ and $a - \frac{N\sigma}{4}\left(\frac{N\sigma}{2} - 1\right) c > 0$. In this case we have $v \in \mathcal{O}$ and $D(v_n - v) > 0$ (thus $v_n - v \in \mathcal{O}$) for all sufficiently large $n$, hence $E(v) \geqslant \tilde{E}_{min}(\|v\|_{L^2}^2)$ and $E(v - v_n) \geqslant \tilde{E}_{min}(\|v - v_n\|_{L^2}^2)$. Using (4.21), (4.20) and the continuity of $\tilde{E}_{min}$ we get

$$(4.24) \qquad \tilde{E}_{min}(m) \geqslant \tilde{E}_{min}(\|v\|_{L^2}^2) + \tilde{E}_{min}(m - \|v\|_{L^2}^2)$$

and this contradicts Lemma 4.7 (ii).

*Case 2.* $D(v) > 0$ and $a - \frac{N\sigma}{4}\left(\frac{N\sigma}{2} - 1\right) c = 0$. We have $v \in \mathcal{O}$ and $D(v_n - v) \longrightarrow 0$, and Lemma 4.6 implies $\liminf_{n\to\infty} E(v_n - v) \geqslant \tilde{E}_{min}(m - \|v\|_{L^2}^2)$. Proceeding as in the first case we get (4.24), and this is in contradiction with Lemma 4.7 (ii).

*Case 3.* $D(v) = 0$ and $a - \frac{N\sigma}{4}\left(\frac{N\sigma}{2} - 1\right) c > 0$. As in Case 1, for all sufficiently large $n$ we have $D(v_n - v) > 0$, hence $v_n - v \in \mathcal{O}$ and we find $\liminf_{n\to\infty} E(v_n - v) \geqslant \tilde{E}_{min}(m - \|v\|_{L^2}^2)$. We have $t_{v,infl} = 1$, hence $E(v) > E(v_{t_{v,1}}) \geqslant \tilde{E}_{min}(\|v\|_{L^2}^2)$ and using (4.21) we get (4.24) (with strict inequality), contradicting again Lemma 4.7 (ii).

*Case 4.* $a - \frac{N\sigma}{4}\left(\frac{N\sigma}{2} - 1\right) c < 0$. In this case we have necessarily $D(v) > 0$, hence $v \in \mathcal{O}$ and $E(v) \geqslant \tilde{E}_{min}(\|v\|_{L^2}^2)$. We distinguish two subcases:

*Subcase A.* There is a subsequence $(v_{n_k})_{k\geqslant 1}$ such that $P_1(v_{n_k} - v) \geqslant 0$, that is $t_{(v_{n_k}-v),1} \leqslant 1 \leqslant t_{(v_{n_k}-v),2}$. For all $k$ sufficiently large we have $\|v_{n_k} - v\|_{L^2}^2 < \mu_0$ and then $(v_{n_k} - v)_{t_{(v_{n_k}-v),1}} \in \mathcal{O}$, hence

$$E(v_{n_k} - v) \geqslant E\left((v_{n_k} - v)_{t_{(v_{n_k}-v),1}}\right) \geqslant \tilde{E}_{min}\left(\|v_{n_k} - v\|_{L^2}^2\right).$$

Letting $k \longrightarrow \infty$ we discover $\liminf_{k\to\infty} E(v_{n_k} - v) \geqslant \tilde{E}_{min}\left(m - \|v\|_{L^2}^2\right)$. Then using (4.20) and (4.21) for the subsequence $(v_{n_k})_{k\geqslant 1}$ we infer that (4.24) holds and we reach a contradiction as in the previous cases.

*Subcase B.* $P_1(v_n - v) < 0$ for all sufficiently large $n$. To simplify notation, let $w_n = v_n - v$. Since $v_n$ satisfies (3.19), we have $w_n \rightharpoonup 0$ weakly in $H^2(\mathbf{R}^N)$ and $w_n \longrightarrow 0$ strongly in $L^p(\mathbf{R}^N)$ for all $p \in [1, 2^{**})$ and almost everywhere, and it is clear that for any fixed $t > 0$, the sequence $((w_n)_t)_{n\geqslant 1}$ has the same properties. We fix $t > 1$ (and $t$ sufficiently close to 1) such that

$$D(v) + at^2 - \frac{N\sigma}{4}\left(\frac{N\sigma}{2} - 1\right) ct^{\frac{N\sigma}{2}} > 0.$$

Such $t$ exists because $D(v) + a - \frac{N\sigma}{4}\left(\frac{N\sigma}{2} - 1\right) c > 0$. Let $\tilde{v}_n = v + (w_n)_t$. The weak convergence $(w_n)_t \rightharpoonup 0$ weakly in $H^2(\mathbf{R}^N)$ gives

$$\|\tilde{v}_n\|_{L^2}^2 = \|v\|_{L^2}^2 + \|(w_n)_t\|_{L^2}^2 + o(1) = \|v\|_{L^2}^2 + \|w_n\|_{L^2}^2 + o(1) = \|v_n\|_{L^2}^2 + o(1) = m + o(1),$$

$$\|\nabla \tilde{v}_n\|_{L^2}^2 = \|\nabla v\|_{L^2}^2 + \|\nabla (w_n)_t\|_{L^2}^2 + o(1),$$

$$\|\Delta \tilde{v}_n\|_{L^2}^2 = \|\Delta v\|_{L^2}^2 + \|\Delta (w_n)_t\|_{L^2}^2 + o(1).$$

Since $(w_n)_t \longrightarrow 0$ a.e. and $\|(w_n)_t\|_{L^{2\sigma+2}}^{2\sigma+2}$ is bounded, using Brezis-Lieb Lemma we have

$$\|\tilde{v}_n\|_{L^{2\sigma+2}}^{2\sigma+2} = \|v\|_{L^{2\sigma+2}}^{2\sigma+2} + \|(w_n)_t\|_{L^{2\sigma+2}}^{2\sigma+2} + o(1) = \|v\|_{L^{2\sigma+2}}^{2\sigma+2} + t^{\frac{N\sigma}{2}} \|w_n\|_{L^{2\sigma+2}}^{2\sigma+2} + o(1).$$

In particular, we infer that

$$E(\tilde{v}_n) = E(v) + E((w_n)_t) + o(1) \qquad \text{as } n \longrightarrow \infty. \tag{4.25}$$

It follows from the above that $D(\tilde{v}_n) \longrightarrow D(v) + at^2 - \frac{N\sigma}{4}\left(\frac{N\sigma}{2} - 1\right) ct^{\frac{N\sigma}{2}} > 0$ as $n \longrightarrow \infty$, hence $D(\tilde{v}_n) > 0$ and consequently $\tilde{v}_n \in \mathcal{O}$ for all sufficiently large $n$. This implies $E(\tilde{v}_n) \geqslant \tilde{E}_{min}(\|\tilde{v}\|^2_{L^2})$, and letting $n \longrightarrow \infty$ and using the continuity of $\tilde{E}_{min}$ we get

$$\liminf_{n\to\infty} E(\tilde{v}_n) \geqslant \tilde{E}_{min}(m). \tag{4.26}$$

On the other hand, from (4.21) and (4.25) we get

$$E(\tilde{v}_n) - E(v_n) = E((w_n)_t) - E(w_n) + o(1).$$

For all sufficiently large $n$ (so that $P_1(w_n) \leqslant 0$), using (4.2) and Lemma 4.1 (iv) we obtain

$$E((w_n)_t) - E(w_n) \leqslant 2(t-1)P_1(w_n) - (t-1)^2 D(w_n) \leqslant (t-1)^2 D(w_n).$$

We infer that

$$\begin{aligned} \limsup_{n\to\infty} E(\tilde{v}_n) &\leqslant \lim_{n\to\infty} E(v_n) + (t-1)^2 \lim_{n\to\infty} D(w_n) \\ &= \tilde{E}_{min}(m) + (t-1)^2 \left(a - \tfrac{N\sigma}{4}\left(\tfrac{N\sigma}{2} - 1\right) c\right) < \tilde{E}_{min}(m) \end{aligned} \tag{4.27}$$

and this is in contradiction with (4.26).

*Case 5.* $D(v) < 0$. This case is very similar to Case 4, and a bit simpler. We have necessarily $a - \frac{N\sigma}{4}\left(\frac{N\sigma}{2} - 1\right) c > 0$, thus $D(v_n - v) > 0$ and consequently $v_n - v \in \mathcal{O}$ for all sufficiently large $n$, and we find $E(v_n - v) \geqslant \tilde{E}_{min}(\|v_n - v\|^2_{L^2})$, which implies $\liminf_{n\to\infty} E(v_n - v) \geqslant \tilde{E}_{min}(m - \|v\|^2_{L^2})$.

If $P_1(v) \geqslant 0$ we have $t_{1,v} \leqslant 1 \leqslant t_{2,v}$, hence $E(v) \geqslant E(v_{t_{v,1}}) \geqslant \tilde{E}_{min}(\|v\|^2_{L^2})$ and we find that (4.24) holds.

If $P_1(v) < 0$ we may choose $t > 1$ such that $D(v_t) + a - \frac{N\sigma}{4}\left(\frac{N\sigma}{2} - 1\right) c > 0$. Denoting $v_n^\sharp = v_t + v_n - v$ we see that $\|v_n^\sharp\|^2_{L^2} \longrightarrow m$ and $D(v_n^\sharp) > 0$ if $n$ is sufficiently large. Thus $v_n^\sharp \in \mathcal{O}$ for all large $n$ and then it is easy to see that $\liminf_{n\to\infty} E(v_n^\sharp) \geqslant \tilde{E}_{min}(m)$. On the other hand,

$$\lim_{n\to\infty} E(v_n^\sharp) = E(v_t) + \lim_{n\to\infty} E(v_n - v) < E(v) + \lim_{n\to\infty} E(v_n - v) = \lim_{n\to\infty} E(v_n) = \tilde{E}_{min}(m),$$

which is a contradiction.

Cases 1-5 here above cover all possible situations and all of them lead to a contradiction. We have thus proved that $\|v\|^2_{L^2} = m$. Now let us prove that $v_n \longrightarrow v$ in $H^2(\mathbf{R}^N)$ and $v \in \mathcal{O}$. The weak convergence $v_n \rightharpoonup v$ in $L^2(\mathbf{R}^N)$ and the convergence of norms $\|v_n\|^2_{L^2} \longrightarrow m = \|v\|^2_{L^2}$ imply that $v_n \longrightarrow v$ strongly in $L^2(\mathbf{R}^N)$. Then (3.2), (3.3) and the boundedness of $v_n$ in $H^2(\mathbf{R}^N)$ imply that $v_n \longrightarrow v$ in $L^{2\sigma+2}(\mathbf{R}^N)$ and $\nabla v_n \longrightarrow \nabla v$ in $L^2(\mathbf{R}^N)$. Since $\Delta v_n \rightharpoonup \Delta v$ in $L^2(\mathbf{R}^N)$ we have $\|\Delta v_n\|^2_{L^2} = \|\Delta v\|^2_{L^2} + \|\Delta v_n - \Delta v\|^2_{L^2} + o(1)$, therefore $E(v_n) = E(v) + \|\Delta v_n - \Delta v\|^2_{L^2} + o(1)$. We have $E(v_n) \longrightarrow \tilde{E}_{min}(m)$ and we infer that $\|\Delta v_n - \Delta v\|^2_{L^2}$ converges in $\mathbf{R}$. It is clear that $D(v_n) = D(v) + \|\Delta v_n - \Delta v\|^2_{L^2} + o(1)$. Recall that by Lemma 4.5 we have $\liminf_{n\to\infty} D(v_n) > 0$, hence $D(v) + \lim_{n\to\infty} \|\Delta(v_n - v)\|^2_{L^2} > 0$. We may thus choose $t \in (0,1)$ such that $D(v) + t^2\|\Delta(v_n - v)\|^2_{L^2} > 0$ for all $n$ sufficiently large and we denote $\tilde{v}_n = v + t(v_n - v)$. Since $v_n \longrightarrow v$ in $L^2 \cap L^{2\sigma+2}(\mathbf{R}^N)$ we have $\tilde{v}_n \longrightarrow v$ in $L^2 \cap L^{2\sigma+2}(\mathbf{R}^N)$. Similarly we get $\nabla \tilde{v}_n \longrightarrow \nabla v$ in $L^2(\mathbf{R}^N)$. Since $\Delta v_n - \Delta v \rightharpoonup 0$ we get $\|\Delta \tilde{v}_n\|^2_{L^2} = \|\Delta v\|^2_{L^2} + t^2\|\Delta(v_n - v)\|^2_{L^2} + o(1)$ and then $D(\tilde{v}_n) = D(v) + t^2\|\Delta(v_n - v)\|^2_{L^2} + o(1)$. Therefore $\|\tilde{v}_n\|^2_{L^2} < \mu_0$ and $D(\tilde{v}_n) > 0$ for all sufficiently large $n$, which implies that $\tilde{v}_n \in \mathcal{O}$ and consequently $E(\tilde{v}_n) \geqslant \tilde{E}_{min}(\|\tilde{v}_n\|^2_{L^2})$ for all large $n$. Letting $n \longrightarrow \infty$ we get

$$\liminf_{n\to\infty} E(\tilde{v}_n) \geqslant \tilde{E}_{min}(m).$$

On the other hand we have

$$E(\tilde{v}_n) = E(v) + t^2\|\Delta(v_n - v)\|_{L^2}^2 + o(1) = E(v_n) + (t^2-1)\|\Delta(v_n - v)\|_{L^2}^2 + o(1)$$

and letting $n \longrightarrow \infty$ we find

$$\tilde{E}_{min}(m) \leqslant \liminf_{n\to\infty} E(\tilde{v}_n) = \tilde{E}_{min}(m) + (t^2-1)\lim_{n\to\infty}\|\Delta(v_n - v)\|_{L^2}^2.$$

We conclude that necessarily $\|\Delta v_n - \Delta v\|_{L^2} \longrightarrow 0$. Since $\|v_n - v\|_{L^2} \longrightarrow 0$, this implies that $v_n \longrightarrow v$ in $H^2(\mathbf{R}^N)$, as desired. Then $D(v) = \lim_{n\to\infty} D(v_n)$ and Lemma 4.5 implies that $D(v) > 0$, hence $v \in \mathcal{O}$. Moreover, we have $E(v) = \lim_{n\to\infty} E(v_n) = \tilde{E}_{min}(m)$, hence $v$ minimizes $E$ in the set $\{w \in \mathcal{O} \mid \|w\|_{L^2}^2 = m\}$, and $P_1(v) = \lim_{n\to\infty} P_1(v_n) = 0$.

Recall that $v_n = (u_n)_{t_{u_n,1}}$ and $t_{u_n,1} \longrightarrow t_1 \in (0,\infty)$ as $n \longrightarrow \infty$. Then we have $u_n = (v_n)_{t_{u_n,1}^{-1}}$. Since $v_n \longrightarrow v$ in $H^2(\mathbf{R}^N)$, it is easy to show that $u_n \longrightarrow v_{t_1}^{-1}$ in $H^2(\mathbf{R}^N)$. This implies that $E(u_n) \longrightarrow E(v_{t_1}^{-1})$, that is $E(v_{t_1}^{-1}) = \tilde{E}_{min}(m)$. We have $D(u_n) > 0$ for all $n$ and we infer that $D(v_{t_1}^{-1}) \geqslant 0$; in other words, ${t_1}^{-1} \leqslant t_{v,infl}$. Therefore $0 < {t_1}^{-1} \leqslant t_{v,infl}$ and $E(v_{t_1}^{-1}) = E(v) = \tilde{E}_{min}(m)$. Since $t \longmapsto E(v_t)$ reaches its minimum on $[0, t_{v,infl}]$ only at $t = 1$, we infer that necessarily $t_1 = 1$, thus $u = v$ and $u_n \longrightarrow u$ strongly in $H^2(\mathbf{R}^N)$. This completes the proof of Theorem 4.8. □

**Remark 4.9** If there exists $m_0 > 0$ such that $\tilde{E}_{min}(m) = -m$ on $(0, m_0]$, it is easily seen that $\tilde{E}_{min}(m)$ is not achieved for $m \in (0, m_0)$. Indeed, if $u \in \mathcal{O}$ is a minimizer for $\tilde{E}_{min}(m)$ then $\sqrt{a}u \in \mathcal{O}$ for $a > 1$ and $a$ close to 1 and we get $\tilde{E}_{min}(am) \leqslant E(\sqrt{a}u) < aE(u) = -am$, contradicting the fact that $\tilde{E}_{min}(am) = -am$.

**Remark 4.10** Let $u$ be a minimizer for $\tilde{E}_{min}(m)$, as given by Theorem 4.8. It is obvious that $P_1(u) = 0$ and $u$ satisfies (4.9). In particular, we have $\|u\|_{H^2}^2 \leqslant Cm = C\|u\|_{L^2}^2$, where $C$ depends only on $N$ and on $\sigma$.

Since $u$ minimizes $E$ at constant $L^2-$norm in the open set $\mathcal{O} \subset H^2(\mathbf{R}^N)$, it is standard to see that there exists a Lagrange multiplier $\lambda_u$ such that (3.25) holds. Taking the $H^{-2} - H^2$ duality product of (3.25) with $u$ we see that $u$ satisfies (3.26) and this integral identity can be written as

$$E(u) - \frac{\sigma}{\sigma+1}\|u\|_{L^{2\sigma+2}}^{2\sigma+2} = \lambda_u\|u\|_{L^2}^2.$$

Since $0 < \frac{\sigma}{\sigma+1}\|u\|_{L^{2\sigma+2}}^{2\sigma+2} < \frac{8(N\sigma-2)}{N(N\sigma-4)^2}\|u\|_{L^2}^2$ (see (4.9), we infer that

$$-1 \geqslant \frac{\tilde{E}_{min}(m)}{m} > \lambda_u > \frac{\tilde{E}_{min}(m)}{m} - \frac{8(N\sigma-2)}{N(N\sigma-4)^2}.$$

Denoting $\lambda_u = -1 - c(u)$ and using the above estimate and Lemma 4.3 (i) we see that $u$ satisfies (3.24) with $0 < c(u) < -1 + \frac{(N\sigma-2)^2}{N\sigma(N\sigma-4)} + \frac{8(N\sigma-2)}{N(N\sigma-4)^2}$. Thus we have an explicit bound on Lagrange multipliers associated to local minimizers provided by Theorem 4.8.

Using (3.8) with $\varepsilon = 0$ and (3.26) we get $c(u)\|u\|_{L^2}^2 < \|u\|_{L^{2\sigma+2}}^{2\sigma+2}$. Then using (3.3) and (4.9) we see that there is $C > 0$, depending only on $N$ and $\sigma$, such that $\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant C\|u\|_{L^2}^{2\sigma+2}$. These estimates give $c(u) \leqslant C\|u\|_{L^2}^{2\sigma} = Cm^\sigma$ and we conclude that necessarily $c(u) \longrightarrow 0$ as $m \longrightarrow 0$.

**Remark 4.11** Let $u_c$ be a minimum action solution of (3.24) as provided by Theorem 3.7 and Proposition 3.9. We have already seen (cf. (3.45) and Remark 3.14) that in the case $N\sigma > 4$ we have $\|u_c\|_{L^2}^2 \longrightarrow 0$ as $c \longrightarrow \infty$. Using (3.46) and (3.47) we see that as $c \longrightarrow \infty$,

$$(1+c)^{\frac{N}{4}-\frac{1}{\sigma}-1}D(u_c) \longrightarrow \frac{N\sigma}{4(\sigma+1)}\left(2 - \frac{N\sigma}{2}\right)I^{\frac{\sigma+1}{\sigma}} < 0.$$

Therefore for sufficiently large $c$ we have $\|u_c\|_{L^2}^2 < \mu_0$ and $D(u_c) < 0$, hence $u_c \notin \mathcal{O}$. We conclude that if $c$ is large enough, $u_c$ cannot be a local minimizer of $E$ when the $L^2-$norm is kept fixed. (See Remark 3.15 for another interesting variational characterization of $u_c$.)

We have thus proved that there are at least two types of small $L^2-$ norm solutions for equation (3.24):

(a) the minimizers in $\mathcal{O}$ of the energy $E$ at fixed $L^2-$ norm. They exist for any $L^2-$norm smaller than $\mu_0$. Their Lagrange multipliers are bounded, and their $H^2-$norm is controlled by their $L^2-$norm because they satisfy (4.9).

(b) the minimum action solutions $u_c$ for large values of the Lagrange multiplier $c$. These solutions have large $H^2-$norm: although $\|u_c\|_{L^2} \longrightarrow 0$, we have $\|\Delta u_c\|_{L^2} \longrightarrow \infty$ (see Remark 3.14). We were no able to show that the set $\{\|u_c\|^2_{L^2} \mid c > 0\}$ contains an interval of the form $(0, a)$ with $a > 0$.

**Remark 4.12** Some related results have been obtained in [17]. The authors have worked in the space of radial functions $H^2_{rad} = \{u \in H^2(\mathbf{R}^N) \mid u \text{ is radially symmetric}\}$ and for $m$ sufficiently small they proved the existence of two solutions of (3.24) with $L^2-$norm equal to $m$. The first one is a local minimizer, and the second one is a mountain-pass type solution. The associated Lagrange multipliers are not explicit (they are part of the problem).

It is an open question whether the minimizers provided by Theorems 3.4, 3.7 and 4.8 are or not radially symmetric (some partial results if $\sigma$ is an integer can be found in [7]). We could have worked in $H^2_{rad}$, too. All our arguments are valid when working in this space, and most proofs become much simpler. In this way we get the analogues of Theorems 3.4, 3.7 and 4.8 in $H^2_{rad}$, which give the existence of radial solutions to (3.24). We do not know whether the energies of solutions in $H^2_{rad}$ are higher or not than the energies of the corresponding solutions in $H^2(\mathbf{R}^N)$. Our main motivation is to understand the existence and the properties of standing waves to a fourth-order non-linear Schrödinger equation. Since $E$ and the $L^2-$norm are conserved quantities by that equation, the set of travelling waves that we obtain is orbitally stable. When working in $H^2_{rad}$ one can get stability only with respect to radial perturbations.

# Acknowledgments

R.M. acknowledges funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Project-ID 258734477 - SFB 1173.

# References

[1] J. Bergh, J. Löfström, *Interpolation spaces. An introduction,* Grundlehren der mathematischen Wissenschaften 223, Springer-Verlag, 1976.

[2] D. Bonheure, J.-B. Casteras, T. Gou, and L. Jeanjean, *Normalized solutions to the mixed dispersion nonlinear Schrödinger equation in the mass critical and supercritical regime,* Transactions of the AMS, Vol. 372, No. 3, pp. 2167-2212.

[3] D. Bonheure, J.-B. Casteras, T. Gou, and L. Jeanjean, *Strong instability of ground states to a fourth order Schrödinger equation,* Int. Math. Res. Notices 17, 2019, pp. 5299-5315.

[4] D. Bonheure, J.-B. Casteras, E. Moreira dos Santos, R. Nascimento, *Orbitally stable standing waves of a mixed dispersion nonlinear Schrödinger equation* SIAM J. Math. Anal., Vol. 50, No. 5, 2018, pp. 5027-5071.

[5] D. Bonheure and R. Nascimento, *Waveguide solutions for a nonlinear Schrödinger equation with mixed dispersion*, in Contributions to nonlinear elliptic equations and systems, Progr. Nonlinear Differential Equations Appl. Series, Vol. 86, pp. 31-53, Birkhäuser/Springer, Cham, 2015.

[6] T. Boulenger and E. Lenzmann, *Blowup for Biharmonic NLS,* Annales scientifiques de l'ENS, série 4, vol. 50, no. 3 (2017), pp. 503-544.

[7] L. Bugiera, E. Lenzmann, A. Schikorra, J. Sok, *On symmetry of traveling solitary waves for dispersion generalized NLS,* Nonlinearity 33 (2020), pp. 2797-2819.

[8] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations,* Springer, Universitext, 2011.

[9] T. Cazenave, P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations,* Commun. Math. Phys. 85, (4) (1982), pp. 549-561.

[10] G. Fibich, B. Ilan, and G. Papanicolaou, *Self-focusing with fourth-order dispersion,* SIAM J. Appl. Math. 62, No. 4, 2002, pp. 1437 - 1462.

[11] Y. Fukumoto and H. K. Mofatt, 2000 *Motion and expansion of a viscous vortex ring: I. A higher-order asymptotic formula for the velocity,* J. Fluid. Mech. 417, 2000, pp. 1 - 45

[12] V. I. Karpman, *Stabilization of soliton instabilities by higher-order dispersion: Fourth-order nonlinear Schrödinger-type equations*, Phys. Rev. E 53, No. 2, pp. R1336–R1339, 1996, American Physical Society, doi 10.1103/PhysRevE.53.R1336.

[13] V. I. Karpman, A. G. Shagalov, *Stability of solitons described by nonlinear Schrödinger-type equations with higher-order dispersion*, Physica D. Nonlinear Phenomena 144, 2000, No. 1-2, pp. 194 - 210, doi 10.1016/S0167-2789(00)00078-6.

[14] O. Kavian, *Introduction à la théorie des points critiques et applications aux problèmes elliptiques,* Springer-Verlag, Paris, 1993.

[15] E. H. Lieb, *On the lowest eigenvalue of the Laplacian for the intersection of two domains,* Invent. Math. 74 (1983), pp. 441-448.

[16] E. H. Lieb, M. Loss, *Analysis. Second edition,* Graduate Studies in Mathematics 14, AMS, 2001.

[17] X. Luo, T. Yang, *Normalised solutions for a fourth-order Schrödinger equation with positive second-order dispersion coefficient,* preprint, arXiv: 1908.03079.

[18] M. Mariş, *On some minimisation problems in* $\mathbf{R}^N$*,* in New Trends in Differential Equations, Control Theory and Optimization, Ed. V. Barbu, C. Lefter and I. Vrabie, World Scientific (2016), pp. 216-231.

[19] M. Mariş, *On some minimisation problems in* $\mathbf{R}^N$*: Pohozaev constraints, Lagrange multipliers and concentration-compactness revisited,* preprint.

[20] M. Mariş, *Profile decomposition for sequences of Borel measures,* preprint.

[21] L. Nirenberg, *On elliptic partial differential equations,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (3), 13 (1959), pp. 115-162.

[22] B. Pausader, *Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case,* Dynamics of PDE, Vol. 4, No.3 , pp. 197-225, 2007.

[23] B. Pausader and S. Xia, *Scattering theory for the fourth-order Schrödinger equation in low dimensions,* Nonlinearity 26 (2013), pp. 2175-2191.

[24] T. H. Wolff, *Lectures on Harmonic Analysis,* University Lecture Series Vol. 29, AMS, 2003.

Antonio J. Fernández

Department of Mathematical Sciences, University of Bath,
Bath BA2 7AY, United Kingdom
ajf77@bath.ac.uk

Louis Jeanjean

Laboratoire de Mathématiques (UMR 6623), Université Bourgogne Franche-Comté,
16 Route de Gray, 25030 Besançon Cedex, France
louis.jeanjean@univ-fcomte.fr

Rainer Mandel

Institute for Analysis, Karlsruhe Institute of Technology,
Englerstraße 2 D-76131 Karlsruhe, Germany
rainer.mandel@kit.edu

Mihai Mariş

Institut de Mathématiques de Toulouse (UMR 5219), Université de Toulouse , UPS &
Institut Universitaire de France
118, Route de Narbonne, 31062 Toulouse, France
mihai.maris@math.univ-toulouse.fr