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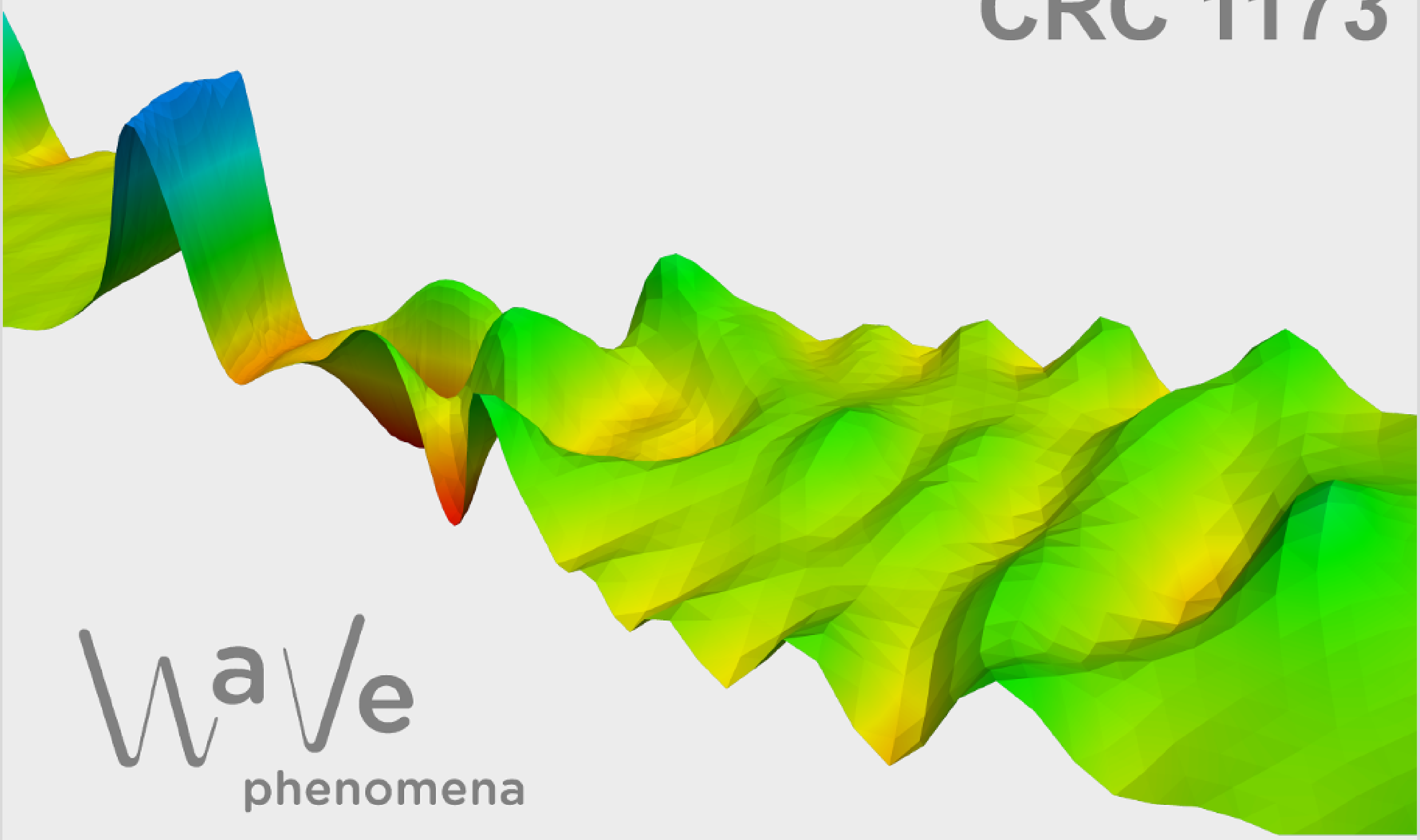

Wave phenomena

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L^p ESTIMATES FOR WAVE EQUATIONS WITH SPECIFIC $C^{0,1}$ COEFFICIENTS

DOROTHEE FREY AND PIERRE PORTAL

Abstract. Peral/Miyachi's celebrated theorem on fixed time L^p estimates with loss of derivatives for the wave equation states that the operator $(I-\Delta)^{-\frac{\alpha}{2}}\exp(i\sqrt{-\Delta})$ is bounded on $L^p(\mathbb{R}^d)$ if and only if $\alpha \geq s_p := (d-1)\left|\frac{1}{p}-\frac{1}{2}\right|$. We extend this result to operators of the form $L = -\sum_{j=1}^{d} a_j\partial_j a_j\partial_j$, for functions $x \mapsto a_i(x_i)$ that are bounded above and below, but merely Lipschitz continuous. This is below the $C^{1,1}$ regularity that is known to be necessary in general for Strichartz estimates in dimension $d \geq 2$. Our proof is based on an approach to the boundedness of Fourier integral operators recently developed by Hassell, Rozendaal, and the second author. We construct a scale of adapted Hardy spaces on which $\exp(i\sqrt{L})$ is bounded by lifting L^p functions to the tent space $T^{p,2}(\mathbb{R}^d)$, using a wave packet transform adapted to the Lipschitz metric induced by the coefficients a_j. The result then follows from Sobolev embedding properties of these spaces.

Mathematics Subject Classification (2020): Primary 42B35. Secondary 35L05, 42B30, 42B37, 35S30.

1. Introduction

In 1980, Peral [21] and Miyachi [19] proved that the operator $(I-\Delta)^{-\frac{\alpha}{2}}\exp(i\sqrt{-\Delta})$ is bounded on $L^p(\mathbb{R}^d)$ if and only if $\alpha \geq s_p := (d-1)\left|\frac{1}{p}-\frac{1}{2}\right|$. Their result was then extended to general Fourier integral operators (FIOs) in a celebrated theorem of Seeger, Sogge, and Stein [23], leading, in particular, to $L^p(\mathbb{R}^d)$ well-posedness results for wave equations with smooth variable coefficients on $\mathbb{R}^d$ or driven by the Laplace-Beltrami operator on a compact manifold. To establish well-posedness of wave equations in more complex geometric settings, many results have been obtained in the past 30 years, using extensions of Peral/Miyachi's fixed time estimates with loss of derivatives, Strichartz estimates, and/or local smoothing properties. This includes Smith's parametrix construction [25] and Tataru's Strichartz estimates [30] for wave equations on $\mathbb{R}^d$ with $C^{1,1}$ coefficients, and Müller-Seeger's extension of Peral-Miyachi's result to the sublaplacian on Heisenberg type groups [20], as well as many other important results for specific operators, such as Laplace-Beltrami operators on symmetric spaces.

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In this paper, we consider operators of the form $L = -\sum_{j=1}^{d} a_j\partial_j a_j\partial_j$, for functions $x \mapsto a_i(x_i)$ that are bounded above and below, and Lipschitz continuous. For these operators, we extend Peral/Miyachi's result by proving that $(I+L)^{-\frac{\alpha}{2}}\exp(i\sqrt{L})$ is bounded on $L^p(\mathbb{R}^d)$ for $\alpha \geq s_p := (d-1)\left|\frac{1}{p}-\frac{1}{2}\right|$. This gives, in particular, $L^p(\mathbb{R})$ well-posedness of one dimensional wave equations $\partial_t^2 u = a\frac{d}{dx}a\frac{d}{dx}u$ with Lipschitz coefficients a (a natural general result that appears to be new). Divergence form operators $\frac{d}{dx}a\frac{d}{dx}$ can also be treated by perturbation. More generally, when $s_p \leq 2$, we show well-posedness for data in $W^{s_p,p}(\mathbb{R}^d)$. See Theorem 8.6 for a precise statement. While the algebraic structure of the coefficient matrix is a serious limitation in dimension $d > 1$, the roughness of the coefficients is a satisfying and somewhat surprising feature of our result. Indeed, Strichartz estimates for wave equations are known to fail, in general, for coefficients rougher than $C^{1,1}$, see [26,27].

Our proof is based on a new approach to Seeger-Sogge-Stein's L^p boundedness theorem for FIOs, initiated by Hassell, Rozendaal, and the second author in [15], building on earlier work of Smith [24]. The approach consists in developing a scale of Hardy spaces H^p_{FIO}, that are invariant under the action of FIOs. One then shows that this scale relates to the Sobolev scale through the embedding $W^{\frac{s_p}{2},p} \subset H^p_{FIO} \subset W^{-\frac{s_p}{2},p}$, for $p \in (1,\infty)$. This is similar, in spirit, to the theory of Hardy spaces associated with operators, which has been extensively developed over the past 15 years, starting with [5,10,14] (see also the memoir [13]). In this theory, one first constructs a scale of spaces H^p_L by lifting functions from L^p to one of the tent spaces introduced by Coifman, Meyer, and Stein in [8], using the functional calculus of the operator L (rather than convolutions). One then shows that the spaces are invariant under the action of the functional calculus of L. Finally, one relates these spaces to more classical ones. For instance $H^p_\Delta(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ for all $p \in (1,\infty)$. More generally, when one considers Hodge-Dirac operators Π_B, $H^p_{\Pi_B} = L^p$ precisely for those p for which Hodge projections are L^p bounded (a result proven by McIntosh and the authors in [11]).

In the present paper, we go one step further in connecting both theories, by developing a scale of Hardy-Sobolev spaces $H^{p,s}_{FIO,a}$ on which $\exp(i\sqrt{L})$ is bounded, and proving analogues of the embedding $W^{\frac{s_p}{2},p}(\mathbb{R}^d) \subset H^{p,0}_{FIO}(\mathbb{R}^d) \subset W^{-\frac{s_p}{2},p}(\mathbb{R}^d)$ such as, for $p \in (1,2)$, $H^{p,\frac{s_p}{2}}_{FIO,a} \subset L^p$ and $(I+\sqrt{L})^{-\frac{s_p}{2}} \in B(L^p, H^{p,0}_{FIO,a})$. This gives our L^p boundedness with loss of derivatives result, and more. Indeed, one can apply the half wave group $\exp(i\sqrt{L})$ repeatedly on $H^{p,s}_{FIO,a}$, and only loose derivatives when one compares $H^{p,s}_{FIO,a}$ to classical Sobolev spaces. This allows for iterative arguments in constructing parametrices. One can also perturb the half wave group using abstract operator theory on the Banach space $H^{p,s}_{FIO,a}$.

The paper is structured as follows. In Section 3 we study the transport group generated by the commuting tuple $(a_1\partial_1, ..., a_d\partial_d) =: iD_a$. It is a representation of $\mathbb{R}^d$ on $L^2(\mathbb{R}^d)$ and a bounded group on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. The Phillips functional calculus associated with this group replaces convolutions/Fourier multipliers in the context of our Lipschitz metric. Using this calculus, we use the approach of [4] to construct an adapted scale of Hardy-Sobolev spaces in Section 4. For all integrability parameters $p \in (1,\infty)$ and regularity parameter $s \in [0,2]$, these spaces coincide with classical Sobolev spaces, thanks to the regularity properties of the heat kernel of L arising from the Lipschitz continuity of its coefficients. To go from these spaces to $H^{p,s}_{FIO,a}$, one needs to directionally refine the Littlewood-Paley decomposition, as in the proof of Seeger-Sogge-Stein's theorem. This is done in [15] using a wave packet transform defined by Fourier multipliers. In Section 5 we construct a similar wave packet transform, replacing Fourier multipliers by the Phillips calculus of the transport group. This allows us to define $H^{p,s}_{FIO,a}$ in Section 6, and to prove its embedding properties in Section 7. Finally, in Section 8, we prove that the half wave group $(\exp(it\sqrt{L}))_{t\in\mathbb{R}}$ is bounded on $H^{p,s}_{FIO,a}$ for all $1 < p < \infty$ and $s \in \mathbb{R}$. To do so, we first notice that the transport group is. We then realise that, in a given direction ω, $\exp(i\sqrt{D_a.D_a})$ is close to $\exp(-i\omega.D_a)$, when acting on an appropriate wave packet, in the sense that operators of the form $\big(\exp(i\sqrt{D_a.D_a}) - \exp(-i\omega.D_a)\big)\varphi_\omega(D_a)$ are L^p bounded.

Our approach relies heavily on algebraic properties: the wave group commutes with the wave packet localisation operators, and can be expressed in the Phillips functional calculus of a commutative group. Although our coefficients are merely Lipschitz continuous, these algebraic properties match those of the standard Euclidean wave group. In the same way as Peral-Miyachi's result for that group is a starting point for the well-posedness theory of wave equations with coefficients that are smooth enough perturbations of constant coefficients, we expect the results proven here to provide a basis for the development of a well-posedness theory of wave equations with coefficients that are smooth enough perturbations of structured Lipschitz continuous coefficients.

2. Preliminaries

We first recall (a special case of) the following Banach space valued Marcinkiewicz-Lizorkin Fourier multiplier's theorem (see [29, Theorem 4.5]).

Theorem 2.1. *(Fernandez/ Štrkalj-Weis) Let $p \in (1,\infty)$. Let $m \in C^1(\mathbb{R}^d\backslash\{0\})$ be such that, for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha|_\infty \leq 1$ there exists a constant $C = C(\alpha) > 0$ such that*

$$|\zeta^\alpha \partial_\zeta^\alpha m(\zeta)| \leq C \quad \forall \zeta \in \mathbb{R}^d \setminus \{0\}.$$

Let T_m denote the Fourier multiplier with symbol m. Then $T_m \otimes I_{L^p(\mathbb{R}^d)}$ extends to a bounded operator on $L^p(\mathbb{R}^d; L^p(\mathbb{R}^d))$.

This theorem will be combined with the following version of the Coifman-Weiss transference principle (see [17, Theorem 10.7.5]). Note that the extension of this theorem from a one parameter group to a d parameter group generated by a tuple of commuting operators is straightforward.

Theorem 2.2. *(Coifman-Weiss) Let $p \in (1,\infty)$. Let $iD_1, ..., iD_d$ generate bounded commuting groups $(\exp(itD_j))_{t\in\mathbb{R}}$ on $L^p(\mathbb{R}^d)$, and consider the d parameter group defined by*

$\exp(i\xi D) = \prod_{j=1}^{d} \exp(i\xi_j D_j)$ *for* $\xi \in \mathbb{R}^d$*. Then, for all* $\psi \in \mathcal{S}(\mathbb{R}^d)$*, we have that*

$$\| \int_{\mathbb{R}^d} \widehat{\psi}(\xi) \exp(i\xi D) f d\xi \|_{L^p(\mathbb{R}^d)} \lesssim \|T_\psi \otimes I_{L^p(\mathbb{R}^d)}\|_{B(L^p(\mathbb{R}^d;L^p(\mathbb{R}^d)))} \|f\|_{L^p(\mathbb{R}^d)} \quad \forall f \in L^p(\mathbb{R}^d).$$

To define our Hardy-Sobolev spaces, we use the tent spaces introduced by Coifman, Meyer, and Stein in [8], and used extensively in the theory of Hardy spaces associated with operators (see e.g. the memoir [13] and the references therein). These tent spaces $T^{p,2}(\mathbb{R}^d)$ are defined as follows. For $F : \mathbb{R}^d \times (0,\infty) \to \mathbb{C}$ measurable and $x \in \mathbb{R}^d$, set

$$\mathcal{A}F(x) := \left(\int_0^\infty \fint_{B(x,\sigma)} |F(y,\sigma)|^2 \, dy \frac{d\sigma}{\sigma} \right)^{1/2} \in [0,\infty].$$

Definition 2.3. *Let* $p \in [1,\infty)$*. The tent space* $T^{p,2}(\mathbb{R}^d)$ *is defined as the space of all* $F \in L^2_{\text{loc}}(\mathbb{R}^d \times (0,\infty), dx\frac{d\sigma}{\sigma})$ *such that* $\mathcal{A}F \in L^p(\mathbb{R}^d)$*, endowed with the norm*

$$\|F\|_{T^{p,2}(\mathbb{R}^d)} := \|\mathcal{A}F\|_{L^p(\mathbb{R}^d)}.$$

Recall that the tent space $T^{1,2}$ admits an atomic decomposition (see [8]) in terms of atoms A supported in sets of the form $B(c_B, r) \times [0,r]$, and satisfying

$$r^d \int_0^r \int_{\mathbb{R}^d} |A(y,\sigma)|^2 \frac{dy d\sigma}{\sigma} \leq 1.$$

Recall also that the classical Hardy space $H^1(\mathbb{R}^d)$ norm can be obtained as

$$\|f\|_{H^1(\mathbb{R}^d)} := \|(t,x) \mapsto \psi(t^2\Delta) f(x)\|_{T^{1,2}(\mathbb{R}^d)},$$

where $\psi(t^2\Delta)$ denotes the Fourier multiplier with symbol $\xi \mapsto t^2|\xi|^2 \exp(-t^2|\xi|^2)$. This is the starting point of the theory of Hardy spaces associated with operators (or equations): one replaces the Fourier multiplier by an appropriately adapted operator. To do so, one often uses the holomorphic functional calculus of a (bi)sectorial operator. The relevant theory is presented in [17]. We use it here with the following notation.

Definition 2.4. *Let* $0 < \theta < \frac{\pi}{2}$*. Define the open sector in the complex plane by*

$$S^o_{\theta+} := \{z \in \mathbb{C} \setminus \{0\} \, : \, |\arg(z)| < \theta\},$$

as well as the bisector $S^o_\theta = S^o_{\theta+} \cup S^o_{\theta-}$*, where* $S^o_{\theta-} = -S^o_{\theta+}$*. We denote by* $H(S^o_\theta)$ *the space of holomorphic functions on* S^o_θ*, and set*

$$H^\infty(S^o_\theta) := \{g \in H(S^o_\theta) \, : \, \|g\|_{L^\infty(S^o_\theta)} < \infty\},$$
$$\Psi^\beta_\alpha(S^0_\theta) := \{\psi \in H^\infty(S^o_\theta) \, : \, \exists C > 0 : |\psi(z)| \leq C|z|^\alpha (1 + |z|^{\alpha+\beta})^{-1} \, \forall z \in S^o_\theta\}$$

for every $\alpha, \beta > 0$*. We say that* $\psi \in H^\infty(S^o_\theta)$ *is non-degenerate if neither of its restrictions to* $S^o_{\theta+}$ *or* $S^o_{\theta-}$ *vanishes identically.*

For bisectorial operators D such that iD generates a bounded group on L^p, we also use the Phillips calculus defined by

$$\psi(D)f := \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\psi}(\xi)\exp(i\xi D) f d\xi,$$

for $f \in L^p$ and $\psi \in \mathcal{S}(\mathbb{R})$. See [4,18] for more information on how these two functional calculi interact in the theory of Hardy spaces associated with operators. The results in Section 4 are fundamentally inspired by these papers.

3. The transport group

For $j \in \{1, \dots, d\}$, let $a_j \in C^{0,1}(\mathbb{R})$ with $\frac{d}{dx}a_j \in L^\infty$, and assume that there exist $0 < \lambda \leq \Lambda$ such that $\lambda \leq a_j(x) \leq \Lambda$ for all $x \in \mathbb{R}$. We denote by $\widetilde{a_j} \in C^{0,1}(\mathbb{R}^d)$ the map defined by $\widetilde{a_j} : x \mapsto a_j(x_j)$. For $x \in \mathbb{R}^d$, and $j \in \{1, \dots, d\}$, the ordinary differential equation

$$\begin{cases} \dot{\chi}_j(t) = a_j(\chi_j(t)) \quad \forall t \in \mathbb{R}, \\ \chi_j(0) = x_j, \end{cases}$$

has a unique solution implicitly given by the equation:

$$t = \int_{\chi_j(0)}^{\chi_j(t)} \frac{1}{a_j(y)}\, dy \quad \forall t \in \mathbb{R}. \tag{3.1}$$

We define the corresponding flow by $\chi : (x, t_1, ..., t_d) \mapsto (\chi_1(t_1), ..., \chi_d(t_d))$, and the associated transport group by

$$[T(t_1, ..., t_d)f](x) := f(\chi(x, t_1, ..., t_d)) \quad \forall x, (t_1, ..., t_d) \in \mathbb{R}^d. \tag{3.2}$$

Theorem 3.1. *Let $p \in [1, \infty)$. $(T(t))_{t\in\mathbb{R}^d}$ is a bounded C_0-group on $L^p(\mathbb{R}^d)$, and a bounded group on $L^\infty(\mathbb{R}^d)$. It has a finite speed of propagation $\kappa > 0$ in the following sense: for all compactly supported $f \in L^p(\mathbb{R}^d)$ and all $(t_1, ..., t_d) \in \mathbb{R}^d$, we have that*

$$supp(T(t_1, ..., t_d)f) \subset \{y \in \mathbb{R}^d \ ; \ dist(y, supp(f)) \leq \kappa|(t_1, ..., t_d)|\}.$$

Moreover, for all $f \in L^p(\mathbb{R}^d)$

$$T(t_1, ..., t_d)f = \exp(\sum_{j=1}^{d} t_j \widetilde{a_j}\partial_j)f \quad \forall (t_1, ..., t_d) \in \mathbb{R}^d,$$

where $\widetilde{a_j}\partial_j$ is given with domain $W^{1,p}(\mathbb{R}^d)$.

Proof. Let $j = 1, ..., d$. The implicit equation (3.1) gives that

$$\partial_{x_j}\chi(x, t) = \frac{a_j(\chi(x, t).e_j)}{a_j(x_j)}.e_j \quad \forall x, t \in \mathbb{R}^d.$$

Therefore $x \mapsto \partial_{x_j}\chi(x, t).e_k = 0$ for $j \neq k$, and $x \mapsto \partial_{x_j}\chi(x, t).e_j$ is bounded above and below, uniformly in t, and χ is a thus a bi-Lipschitz flow. This implies that $(T(t))_{t\in\mathbb{R}}$ is a bounded group on $L^p(\mathbb{R}^d)$ for all $p \in [1, \infty]$, with finite speed of propagation. Strong continuity $\|T(t)f - f\|_p \underset{t\to 0}{\to} 0$ for $p < \infty$ follows by dominated convergence for f continuous, and then density for general f. To identify the generator, let $f \in W^{1,p}$, and note that, for all $x \in \mathbb{R}^d$,

$$\begin{aligned}\frac{\partial}{\partial_{t_j}} T(t)f(x)|_{t_j=0} &= \frac{\partial}{\partial_{t_j}} f(\chi(x,t))|_{t_j=0} = \nabla f(x) \cdot \partial_{t_j}\chi(x,t)|_{t_j=0} \\ &= a_j(x_j)\partial_{x_j} f(x).\end{aligned}$$

The result then follows from the fact that the operators $\{\tilde{a_j}\partial_j \ ; \ j = 1, ..., d\}$ commute. □

For $E, F \subset \mathbb{R}^d$ Borel sets and $\omega \in S^{d-1}$, we set $\omega.d(E,F) := \inf_{x\in E, y\in F} |\langle \omega, x-y\rangle|$.

Remark 3.2. *The specific form of the flow* $\chi : (x, t_1, ..., t_d) \mapsto (\chi_1(t_1), ..., \chi_d(t_d))$ *with* $\partial_{t_j}\chi(x,t).e_k = 0$ *for* $j \neq k$ *implies the stronger form of finite speed of propagation: There exists* $\kappa > 0$ *such that for all* $f \in L^2(\mathbb{R}^d)$*, all Borel sets* $E, F \subset \mathbb{R}^d$*, all* $\xi \in \mathbb{R}^d$ *and all* $\omega \in S^{d-1}$ *we have*

$$1_E \exp(i\xi D_a)(1_F f) = 0,$$

whenever $\kappa|\langle\omega,\xi\rangle| < \omega.d(E,F)$*. See also [18, Remark 3.6], where such a stronger statement is proven in more generality.*

We set $D_j = -i\partial_j$, $D = (D_1, \ldots, D_d)$, and denote by $iD_a = i(\tilde{a_1}D_1, ..., \tilde{a_d}D_d)$ the d-tuple of commuting unbounded operators with domain $W^{1,p}$ that generates the d-parameter C_0 group $(T(t))_{t\in\mathbb{R}^d}$ on $L^p(\mathbb{R}^d)$ for $p \in [1, \infty)$. For $p = 2$, the following lemma shows that this transport group is similar to the standard translation group.

Lemma 3.3. *There exists* $S \in B(L^2(\mathbb{R}^d))$ *such that*

$$\exp(i\xi D_a) = S^{-1}\exp(i\xi D)S \quad \forall \xi \in \mathbb{R}^d.$$

Proof. Define $b \in L^\infty(\mathbb{R}^d)$ by $b(x_1, ..., x_d) := \prod_{j=1}^d a_j(x_j)^{-1}$. Let H be the Hilbert space $L^2(\mathbb{R}^d)$ endowed with the inner product defined by

$$\langle u, v\rangle_a := \langle bu, v\rangle \quad \forall u, v \in L^2(\mathbb{R}^d),$$

and T be the identity map from $L^2(\mathbb{R}^d)$ to H. Let $j \in \{1, ..., d\}$. Note that $P_j := Te_j.D_aT^{-1}$ is self-adjoint in H, since $\partial_k\tilde{a_j} = 0$ for all $j \neq k$. Define $Q_j : u \mapsto \tilde{b_j}u$ for $b_j \in C^{1,1}(\mathbb{R})$ such that $b_j'(x) = \frac{1}{a_j(x)}$ $\forall x \in \mathbb{R}$, and $\tilde{b_j} : x \mapsto b_j(x_j)$. Then Q_j is also self-adjoint in H, and $(\exp(isQ_j))_{s\in\mathbb{R}}$ is a bounded multiplication group Moreover, since $b_j(\chi_j(t)) = b_j(\chi_j(0)) + t$ for all $t \in \mathbb{R}$ by (3.1), we have the commutation relation

$$\exp(isQ_k)\exp(itP_j) = \exp(-ist\delta_{jk})\exp(itP_j)\exp(isQ_k)$$

for all $s, t \in \mathbb{R}$. Therefore, by the Stone-von Neumann theorem, there exists a unitary map $U \in B(H, L^2(\mathbb{R}^d))$ such that, for all $j = 1, ..., d$:

$$\exp(i\xi P_j) = U^{-1}\exp(i\xi\partial_j)U \quad \forall \xi \in \mathbb{R}.$$

The result follows by taking $S = UT$. □

Remark 3.4. *Lemma 3.3 shows that the transport group* $\{\exp(i\xi D_a) \ ; \ \xi \in \mathbb{R}^d\}$ *is, algebraically, a representation of* $\mathbb{R}^d$*. This is a fundamental consequence of the specific structure of the coefficients of* D_a*. Such a representation is rough in the sense that it is generated by non-smooth differential operators. In future work, we plan to extend the methods developed in this paper in two directions: replacing* $\mathbb{R}^d$ *by other Lie groups (for* *which an appropriate Fourier multiplier theory exists), and allowing the transport group to be a sufficiently smooth perturbation of a rough representation.*

4. Hardy spaces associated with the transport group

Lemma 4.1. *There exists $C > 0$ such that, for all $\Psi \in \mathcal{S}(\mathbb{R}^d)$, all $E, F \subset \mathbb{R}^d$ Borel sets and all $\omega \in S^{d-1}$, we have that*

$$\|1_E \Psi(D_a)(1_F f)\|_2 \leq C \|1_F f\|_2 \int\limits_{\{|\xi| \geq \frac{d(E,F)}{\kappa}\} \cap \{|\langle \omega, \xi \rangle| \geq \frac{\omega . d(E,F)}{\kappa}\}} |\widehat{\Psi}(\xi)| d\xi \quad \forall f \in L^2(\mathbb{R}^d).$$

Consequently, for every $\Psi \in \mathcal{S}(\mathbb{R}^d)$ and every $M \in \mathbb{N}$, there exists $C_M > 0$ such that

$$\|1_E \Psi(\sigma D_a)(1_F f)\|_2 \leq C_M (1 + \frac{d(E,F)}{\kappa \sigma})^{-M} \|1_F f\|_2 \quad \forall f \in L^2(\mathbb{R}^d)$$

for all Borel sets $E, F \subset \mathbb{R}^d$ and all $\sigma > 0$.

Proof. Let $f \in L^2(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$. Since the group $(\exp(itD_a))_{t \in \mathbb{R}^d}$ has finite speed of propagation κ according to Theorem 3.1 and Remark 3.2, we have that

$$1_E \exp(i\xi D_a)(1_F f) = 0,$$

whenever $\kappa|\xi| < d(E,F)$ or $\kappa|\langle \omega, \xi \rangle| < \omega . d(E,F)$. Therefore, using Phillips functional calculus, we have that

$$\begin{aligned} \|1_E \Psi(D_a)(1_F f)\|_2 &\leq \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} |\widehat{\Psi}(\xi)| \|1_E \exp(i\xi D_a)(1_F f)\|_2 d\xi \\ &\leq C \|1_F f\|_2 \int\limits_{\{|\xi| \geq \frac{d(E,F)}{\kappa}\} \cap \{|\langle \omega, \xi \rangle| \geq \frac{\omega . d(E,F)}{\kappa}\}} |\widehat{\Psi}(\xi)| d\xi, \end{aligned}$$

where $C := \frac{1}{(2\pi)^d} \sup\{\|\exp(itD_a)\|_{B(L^2)} \; ; \; t \in \mathbb{R}^d\}$. The last statement then follows from a change of variables and $\Psi \in \mathcal{S}(\mathbb{R}^d)$. □

We recall the following fact, which is a corollary of the results in [6], using that the coefficients a_j are Lipschitz continuous.

Theorem 4.2. *(Auscher, McIntosh, Tchamitchian) Let $p \in (1, \infty)$. On $L^p(\mathbb{R}^d)$, the operator $-L = \sum\limits_{j=1}^{d} \widetilde{a_j} \partial_j \widetilde{a_j} \partial_j$, with domain $W^{2,p}(\mathbb{R}^d)$, generates an analytic semigroup, and has a bounded H^∞ calculus of angle 0. Moreover, $\{\exp(-tL) \; ; \; t > 0\}$ satisfies Gaussian estimates.*

Corollary 4.3. *Let $p \in (1, \infty)$, $\theta > 0$, $g \in H^\infty(S^o_{\theta+})$, and let $\Psi \in C_c^\infty(\mathbb{R}^d)$ be supported away from 0. Then there exists a constant $C > 0$ independent of g such that, for all $F \in T^{p,2}(\mathbb{R}^d)$,*

$$\|(\sigma, x) \mapsto \Psi(\sigma D_a) g(L) F(\sigma, .)(x)\|_{T^{p,2}(\mathbb{R}^d)} \leq C \|g\|_{L^\infty(S^o_{\theta+})} \|(\sigma, x) \mapsto F(\sigma, .)(x)\|_{T^{p,2}(\mathbb{R}^d)}.$$

Proof. For $M \in \mathbb{N}$, set $q_M(z) := z^M(1+z)^{-2M}$, $z \in S^o_{\theta+}$. Note that then $q_M \in \Psi^M_M(S^o_{\theta+})$. The statement for $\Psi(\sigma D_a)$ replaced by $q_M(\sqrt{\sigma}L)$ for M large enough then follows from a combination of [16, Theorem 5.2] and [16, Lemma 7.3], using Lemma 4.1 and Theorem 4.2 to check the assumptions.
On the other hand, we have by assumption $\zeta \mapsto \Psi(\zeta)q_M^{-1}(|\zeta|^2) \in \mathcal{S}(\mathbb{R}^d)$, so that an application of [16, Theorem 5.2] together with Lemma 4.1 yields the assertion. $\square$

Lemma 4.4. *Let $\alpha \in \mathbb{R}$, and non-degenerate $\Psi, \widetilde{\Psi} \in C_c^\infty(\mathbb{R}^d)$ be supported away from 0. Let $p \in [1,\infty)$. Then*

$$\|(\sigma,x) \mapsto \sigma^\alpha \Psi(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)} \sim \|(\sigma,x) \mapsto \sigma^\alpha \widetilde{\Psi}(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)},$$

for all f such that the above quantities are finite. Moreover, for $L = -\sum_{j=1}^d \tilde{a}_j\partial_j\tilde{a}_j\partial_j$, we have that

$$\|(\sigma,x) \mapsto \Psi(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)} \sim \|(\sigma,x) \mapsto \sigma^2 L\exp(-\sigma^2 L)f(x)\|_{T^{p,2}(\mathbb{R}^d)}.$$

Proof. Since

$$\|(\sigma,x) \mapsto \sigma^\alpha \Psi(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)} \sim \|(\sigma,x) \mapsto \int_0^\infty \sigma^\alpha \Psi(\sigma D_a)(\widetilde{\Psi})^2(\tau D_a)f(x)\frac{d\tau}{\tau}\|_{T^{p,2}(\mathbb{R}^d)},$$

by [16, Corollary 5.1], it suffices to show that, for all $\sigma,\tau > 0$, $(\frac{\sigma}{\tau})^\alpha\Psi(\sigma D_a)\widetilde{\Psi}(\tau D_a) = \min(\frac{\sigma}{\tau},\frac{\tau}{\sigma})^N S_{\sigma,\tau}$ for some $N > \frac{d}{2}$ and a family of operators $S_{\sigma,\tau} \in B(L^2)$ such that for every $M \in \mathbb{N}$, there exists $C_M > 0$ such that

$$\|1_E S_{\sigma,\tau}(1_F f)\|_2 \leq C_M(1+\frac{d(E,F)}{\kappa\max(\sigma,\tau)})^{-M}\|1_F f\|_2 \quad \forall f \in L^2(\mathbb{R}^d)$$

for all Borel sets $E,F \subset \mathbb{R}^d$ and all $\sigma > 0$. This follows from Lemma 4.1 using that, for all $\xi \in \mathbb{R}^d\backslash\{0\}$,

$$(\frac{\sigma}{\tau})^\alpha\Psi(\sigma\xi)\widetilde{\Psi}(\tau\xi) = (\frac{\sigma}{\tau})^{N'-\alpha}\overline{\Psi}(\sigma\xi)\underline{\widetilde{\Psi}}(\tau\xi) = (\frac{\tau}{\sigma})^{N'+\alpha}\underline{\Psi}(\sigma\xi)\overline{\widetilde{\Psi}}(\tau\xi),$$

for $\overline{\Psi} : \xi \mapsto \frac{\Psi(\xi)}{\xi^\beta}$ and $\underline{\Psi} : \xi \mapsto \xi^\beta\Psi(\xi)$ with $\beta \in \mathbb{N}^d$, $|\beta|_1 = N'$, for $N' > |\alpha|+N$. For the second statement, we first show the comparison of $\Psi(\sigma D_a)$ with $(\sigma^2 L)^M\exp(-\sigma^2 L)$ for some $M \in \mathbb{N}$, $M > \frac{d}{4}$ in the exact same way as above. For the comparison of $(\sigma^2 L)^M\exp(-\sigma^2 L)$ with $\sigma^2 L\exp(-\sigma^2 L)$, we use [11, Proposition 10.1] instead of [16, Corollary 5.1], together with the Gaussian estimates for $\exp(-tL)$ as stated in Theorem 4.2. $\square$

Theorem 4.5. *Let $s \in \mathbb{R}$, let $p \in (1,\infty)$. For all non-degenerate $\Psi \in C_c^\infty(\mathbb{R}^d)$ supported away from 0, and all $M \in \mathbb{N}$, we have that*

$$\|(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\sigma^{-s}\Psi(\sigma D_a)f(x)+1_{[1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)} \sim \|(I+\sqrt{L})^s f\|_p, \tag{4.1}$$

for all $f \in D((I+\sqrt{L})^s)$. Moreover, for $s \in [0,2]$, we have that

$$\|(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\sigma^{-s}\Psi(\sigma D_a)f(x)+1_{[1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)} \sim \|f\|_{W^{s,p}} \tag{4.2}$$

for all $f \in W^{s,p}(\mathbb{R}^d)$.

Proof. We use the Hardy space H^p_L associated with L, as defined in [9]. For all $f \in L^p \cap L^2$, we have, by Lemma 4.4,

$$\|(\sigma,x) \mapsto \Psi(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)} \sim \|f\|_{H^p_L}.$$

It is a folklore fact that $H^p_L = L^p$ for $p \in (1,\infty)$, thanks to the heat kernel bounds of $(e^{tL})_{t\geq 0}$. This result appeared in draft form in an unpublished manuscript of Auscher, Duong, McIntosh, and inspired the proofs of many similar results. For our particular L, an appropriate version of the result does not seem to have appeared in the literature. It can however be proven as follows. By [6, Theorem 4.19], the operators $tL\exp(-tL)$ have standard kernels satisfying the assumptions of [12, Theorem 4.4]. Therefore, for all $f \in L^p \cap L^2$, $f \in H^p_L$ and

$$\|f\|_{H^p_L} \lesssim \|f\|_p.$$

The reverse inequality is proven in [9, Proposition 4.2] for $p \leq 2$. Given that the above reasoning also applies to L^*, we obtain the full result by duality. Combined with Lemma 4.4, this gives the result for $s = 0$. For $s \in \mathbb{N}$, using Lemma 4.4 with an appropriate $\widetilde{\Psi} \in C^\infty_c(\mathbb{R}^d)$, we then have that

$$\begin{aligned}\|(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\sigma^{-s}\Psi(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)} &\lesssim \|(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\widetilde{\Psi}(\sigma D_a)L^{\frac{s}{2}}f(x)\|_{T^{p,2}(\mathbb{R}^d)}\\ &\lesssim \|L^{\frac{s}{2}}f\|_p \lesssim \|(I+\sqrt{L})^s f\|_p.\end{aligned}$$

We also have that

$$\|(\sigma,x) \mapsto 1_{[1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)} \lesssim \|f\|_p \lesssim \|(I+\sqrt{L})^s f\|_p.$$

For $-s \in \mathbb{N}$, we have that

$$\begin{aligned}&\|(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\sigma^{-s}\Psi(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)}\\ &\quad\lesssim \sum_{k=0}^{|s|} \|(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\sigma^{|s|}L^{\frac{k}{2}}\Psi(\sigma D_a)(I+\sqrt{L})^{-|s|}f(x)\|_{T^{p,2}(\mathbb{R}^d)}\\ &\quad\lesssim \sum_{k=0}^{|s|} \|(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\widetilde{\Psi}(\sigma D_a)(I+\sqrt{L})^{-|s|}f(x)\|_{T^{p,2}(\mathbb{R}^d)} \lesssim \|(I+\sqrt{L})^s f\|_p,\end{aligned}$$

as well as

$$\begin{aligned}&\|(\sigma,x) \mapsto 1_{[1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)}\\ &\quad\lesssim \sum_{k=0}^{|s|} \|(\sigma,x) \mapsto 1_{[1,\infty)}(\sigma)\sigma^{k}L^{\frac{k}{2}}\Psi(\sigma D_a)(I+\sqrt{L})^{-|s|}f(x)\|_{T^{p,2}(\mathbb{R}^d)}\\ &\quad\lesssim \sum_{k=0}^{|s|} \|(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\widetilde{\Psi}(\sigma D_a)(I+\sqrt{L})^{-|s|}f(x)\|_{T^{p,2}(\mathbb{R}^d)} \lesssim \|(I+\sqrt{L})^s f\|_p.\end{aligned}$$

Reverse inequalities are proven similarly, using that, for all $s \in \mathbb{R}$,

$$\|(I+\sqrt{L})^s f\|_p \sim \|(\sigma,x) \mapsto (I+\sqrt{L})^s\Psi(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)}.$$

This gives (4.1) for all $s \in \mathbb{Z}$, and the result for all $s \in \mathbb{R}$ then follows by complex interpolation of weighted tent spaces as in [1, Theorem 2.1].
To obtain (4.2) one first remarks that, for $s \in \{0,1,2\}$, the above reasoning also gives

$$\|(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\sigma^{-s}\Psi(\sigma D_a)f(x) + 1_{[1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)} \sim \sum_{m=0}^{s}\sum_{j=1}^{d} \|(\widetilde{a_j}\partial_j)^m f\|_p,$$

for all $f \in \bigcap_{m=0}^{s}\bigcap_{j=1}^{d} D\big((\widetilde{a_j}\partial_j)^m\big)$. We then notice that, for all $j = 1,...,d$, we have that $\|\partial_j f\|_p \sim \|\widetilde{a_j}\partial_j f\|_p$ for all $f \in W^{1,p}$. Moreover,

$$(\widetilde{a_j}\partial_j)^2 f = \widetilde{a_j}^2\partial_j^2 f + \widetilde{a_j}(\partial_j\widetilde{a_j})\partial_j f \quad \forall f \in W^{2,p},$$

and thus

$$\|f\|_{W^{2,p}} \sim \|f\|_p + \sum_{j=1}^{d}\|\widetilde{a_j}\partial_j f\|_p + \sum_{j=1}^{d}\|(\widetilde{a_j}\partial_j)^2 f\|_p \quad \forall f \in W^{2,p}.$$

□

Corollary 4.6. *Let $\alpha \geq 0$, $p \in (1,\infty)$, and $q \in [p,\infty)$ be such that*

$$\alpha = \frac{d}{2}(\frac{1}{p} - \frac{1}{q}).$$

Then there exists $C > 0$ such that, for all $f \in L^p(\mathbb{R}^d)$ with $L^\alpha f \in L^p(\mathbb{R}^d)$, we have that

$$\|f\|_{L^q(\mathbb{R}^d)} \leq C\|L^\alpha f\|_{L^p(\mathbb{R}^d)}.$$

Proof. For $f \in L^p(\mathbb{R}^d)$ with $L^\alpha f \in L^p(\mathbb{R}^d)$, Theorem 4.5 gives that

$$\begin{aligned}\|f\|_{L^q(\mathbb{R}^d)} &\lesssim \|(\sigma,x) \mapsto L^{-\alpha}\Psi(\sigma D_a)L^\alpha f(x)\|_{T^{q,2}(\mathbb{R}^d)}\\ &\lesssim \|(\sigma,x) \mapsto \sigma^{2\alpha}\widetilde{\Psi}(\sigma D_a)L^\alpha f(x)\|_{T^{q,2}(\mathbb{R}^d)}\end{aligned}$$

for $\widetilde{\Psi} : \xi \mapsto |\xi|^{-\alpha}\Psi(\xi)$. Using the embedding properties of weighted tent spaces proven in [1, Theorem 2.19], we have that

$$\|(\sigma,x) \mapsto \sigma^{2\alpha}\widetilde{\Psi}(\sigma D_a)L^\alpha f\|_{T^{q,2}(\mathbb{R}^d)} \lesssim \|(\sigma,x) \mapsto \widetilde{\Psi}(\sigma D_a)L^\alpha f\|_{T^{p,2}(\mathbb{R}^d)},$$

and thus

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim \|L^\alpha f\|_{L^p(\mathbb{R}^d)},$$

by Theorem 4.5.

□

5. Wave packet transform

We use a wave packet transform which is similar to the ones used in [15,22].

Let $\Psi \in C_c^\infty(\mathbb{R}^d)$ be a non-negative radial function with $\Psi(\zeta) = 0$ for $|\zeta| \notin [\frac{1}{2}, 2]$, and

$$\int_0^\infty \Psi(\sigma\zeta)^2\,\frac{d\sigma}{\sigma} = 1 \tag{5.1}$$

for $\zeta \neq 0$. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be a radial, non-negative function with $\varphi(\zeta) = 1$ for $|\zeta| \leq \frac{1}{2}$ and $\varphi(\zeta) = 0$ for $|\zeta| > 1$. These functions Ψ, φ are now fixed for the remainder of the paper.

For $\omega \in S^{d-1}$, $\sigma > 0$ and $\zeta \in \mathbb{R}^d \setminus \{0\}$, set $\varphi_{\omega,\sigma}(\zeta) := c_\sigma \varphi\left(\frac{\hat{\zeta}-\omega}{\sqrt{\sigma}}\right)$, where $c_\sigma := \left(\int_{S^{d-1}} \varphi\left(\frac{e_1 - \nu}{\sqrt{\sigma}}\right)^2 d\nu\right)^{-1/2}$. Set $\varphi_{\omega,\sigma}(0) := 0$. Set furthermore $\Psi_\sigma(\zeta) := \Psi(\sigma\zeta)$ and $\psi_{\omega,\sigma}(\zeta) := \Psi_\sigma(\zeta)\varphi_{\omega,\sigma}(\zeta)$ for $\omega \in S^{d-1}$, $\sigma > 0$ and $\zeta \in \mathbb{R}^d$. By construction, we then have

$$\int_0^\infty \int_{S^{d-1}} \psi_{\omega,\sigma}(\zeta)^2 \, d\omega \frac{d\sigma}{\sigma} = 1 \tag{5.2}$$

for all $\zeta \in \mathbb{R}^d \setminus \{0\}$, see [15, Lemma 4.1]. For $\omega \in S^{d-1}$ and $\zeta \in \mathbb{R}^d$, we moreover set

$$\varphi_\omega(\zeta) := \int_0^4 \psi_{\omega,\tau}(\zeta) \, \frac{d\tau}{\tau}.$$

For the convenience of the reader, we recall the following properties of $\psi_{\omega,\sigma}$ stated in [22, Lemma 3.2].

Lemma 5.1. *Let $\omega \in S^{d-1}$ and $\sigma \in (0,1)$. Each $\zeta \in \operatorname{supp}(\psi_{\omega,\sigma})$ satisfies*

$$\frac{1}{2\sigma} \leq |\zeta| \leq \frac{2}{\sigma}, \qquad |\hat{\zeta} - \omega| \leq 2\sqrt{\sigma}. \tag{5.3}$$

For all $\alpha \in \mathbb{N}_0^d$ and $\beta \in \mathbb{N}_0$ there exists a constant $C = C(\alpha,\beta) > 0$ such that

$$|\langle \omega, \nabla_\zeta \rangle^\beta \partial_\zeta^\alpha \psi_{\omega,\sigma}(\zeta)| \leq C\sigma^{-\frac{d-1}{4} + \frac{|\alpha|_1}{2} + \beta} \tag{5.4}$$

for all $(\zeta,\omega,\sigma) \in \mathbb{R}^d \times S^{d-1} \times (0,\infty)$. For every $N \geq 0$ there exists a constant $C_N > 0$ such that

$$|\mathcal{F}^{-1}(\psi_{\omega,\sigma})(x)| \leq C_N \sigma^{-\frac{3d+1}{4}} (1 + \sigma^{-1}|x|^2 + \sigma^{-2}\langle \omega, x\rangle^2)^{-N} \tag{5.5}$$

for all $(x,\omega,\sigma) \in \mathbb{R}^d \times S^{d-1} \times (0,\infty)$.
In particular, $\{\sigma^{\frac{d-1}{4}} \mathcal{F}^{-1}(\psi_{\omega,\sigma}) \,|\, \omega \in S^{d-1}, \, \sigma > 0\} \subseteq L^1(\mathbb{R}^d)$ is uniformly bounded.

We also recall important properties of the family $(\varphi_\omega)_{\omega \in S^{d-1}}$ from [22, Remark 3.3].

Lemma 5.2. *Let $\omega \in S^{d-1}$. By construction, $\varphi_\omega \in C^\infty(\mathbb{R}^d)$, and for $\zeta \neq 0$, $\varphi_\omega(\zeta) = 0$ for $|\zeta| < \frac{1}{8}$ or $|\hat{\zeta} - \omega| > 2|\zeta|^{-1/2}$. Moreover, for all $\alpha \in \mathbb{N}_0^d$ and $\beta \in \mathbb{N}_0$, there exists a constant $C = C(\alpha,\beta) > 0$ such that*

$$|\langle \omega, \nabla_\zeta \rangle^\beta \partial_\zeta^\alpha \varphi_\omega(\zeta)| \leq C|\zeta|^{\frac{d-1}{4} - \frac{|\alpha|_1}{2} - \beta}$$

for all $\omega \in S^{d-1}$ and $\zeta \neq 0$, and

$$|\langle \hat{\zeta}, \nabla_\zeta \rangle^\beta \partial_\zeta^\alpha \left(\int_{S^{d-1}} \varphi_\nu(\zeta)^2 \, d\nu\right)| \leq C|\zeta|^{-\frac{|\alpha|_1}{2} - \beta} \tag{5.6}$$

for all $\zeta \in \mathbb{R}^d \setminus \{0\}$.

Remark 5.3. *For $\omega = e_1$ and ζ, σ chosen as in (5.3) with $\sigma \in (0, 2^{-8})$, we have*

$$\frac{1}{4\sigma} < \zeta_1 \leq \frac{2}{\sigma}, \qquad |\zeta_j| \leq \frac{4}{\sqrt{\sigma}}, \qquad j \in \{2, \dots, d\}. \tag{5.7}$$

This follows from

$$|\hat{\zeta} - e_1|^2 = |e_1.(\hat{\zeta} - e_1)|^2 + \sum_{j=2}^{d} |e_j.(\hat{\zeta} - e_1)|^2 = |\frac{\zeta_1}{|\zeta|} - 1|^2 + \sum_{j=2}^{d} |\frac{\zeta_j}{|\zeta|}|^2,$$

thus

$$|\zeta_1 - |\zeta||^2 + \sum_{j=2}^{d} |\zeta_j|^2 \leq 4\sigma|\zeta|^2 \leq \frac{16}{\sigma},$$

which directly yields (5.7) for $j \geq 2$. The case $j = 1$ then follows from

$$\zeta_1 > |\zeta| - \frac{4}{\sqrt{\sigma}} \geq \frac{1}{2\sigma} - \frac{4}{\sqrt{\sigma}}.$$

Lemma 5.4. *For all $\sigma \in (0, 1)$, we have that*

$$\int_{S^{d-1}} \|\varphi_{\omega,\sigma}(D_a)f\|_2^2 \, d\omega \lesssim \|f\|_2^2 \quad \forall f \in L^2(\mathbb{R}^d).$$

Moreover,

$$\int_{S^{d-1}} \int_0^\infty \|\psi_{\omega,\sigma}(D_a)f\|_2^2 \frac{d\sigma}{\sigma} d\omega \lesssim \|f\|_2^2 \quad \forall f \in L^2(\mathbb{R}^d).$$

Proof. By Lemma 3.3 and Plancherel's theorem, there exists $S \in B(L^2(\mathbb{R}^d))$ such that

$$\int_{S^{d-1}} \|\varphi_{\omega,\sigma}(D_a)f\|_2^2 \, d\omega \lesssim \int_{S^{d-1}} \int_{\mathbb{R}^d} |\varphi_{\omega,\sigma}(\xi)\widehat{S(f)}(\xi)|_2^2 \, d\xi d\omega \lesssim \int_{S^{d-1}} \int_{\mathbb{R}^d} |\varphi_{\omega,\sigma}(\xi)\widehat{S(f)}(\xi)|_2^2 \, d\xi d\omega,$$

for all $f \in L^2(\mathbb{R}^d)$ and $\sigma \in (0, 1)$. Since $\int_{S^{d-1}} |\varphi_{\omega,\sigma}(\xi)|^2 d\omega = 1$ for all $\xi \neq 0$, we have that

$$\int_{S^{d-1}} \|\varphi_{\omega,\sigma}(D_a)f\|_2^2 \, d\omega \lesssim \|S(f)\|_2^2 \lesssim \|f\|_2^2.$$

The same proof, combined with (5.2), gives the second inequality. □

Definition 5.5. *We define a wave packet transform adapted to D_a, $W_a \in B(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d \times S^{d-1} \times (0, \infty); dx d\omega \frac{d\sigma}{\sigma}))$ by*

$$W_a f(\omega, \sigma, x) := 1_{(1,\infty)}(\sigma)|S^{d-1}|^{-1/2}\Psi(\sigma D_a)f(x) + 1_{[0,1]}(\sigma)\varphi_\omega(D_a)\Psi(\sigma D_a)f(x) \quad \forall f \in L^2(\mathbb{R}^d).$$

We define $\pi_a \in B(L^2(\mathbb{R}^d \times S^{d-1} \times (0, \infty); dx d\omega \frac{d\sigma}{\sigma}), L^2(\mathbb{R}^d))$ by

$$\begin{aligned} \pi_a F(x) :=& |S^{d-1}|^{-1/2} \int_{S^{d-1}} \int_1^\infty \Psi(\sigma D_a)F(\omega, \sigma, \,.\,)(x) \, \frac{d\sigma}{\sigma} d\omega \\ &+ \int_{S^{d-1}} \int_0^1 \varphi_\omega(D_a)\Psi(\sigma D_a)F(\omega, \sigma, \,.\,)(x) \, \frac{d\sigma}{\sigma} d\omega \end{aligned}$$

for all $F \in L^2(\mathbb{R}^d \times S^{d-1} \times (0,\infty); dxd\omega \frac{d\sigma}{\sigma})$.

Note that π_a is the adjoint of the operator $\bar{W}_a$, where $\bar{W}_a$ is defined as W_a with D_a replaced by D_a^*.

The following reproducing formulas follow from their analogues in [15,22] using Lemma 3.3.

Lemma 5.6. *For all* $\sigma \in (0,1)$, *and all* $f \in L^2(\mathbb{R}^d)$, *we have that*

$$\pi_a W_a f = f, \tag{5.8}$$

$$\sigma^{-\frac{d-1}{4}} \int_{S^{d-1}} \varphi_{\omega,\sigma}(D_a) f \, d\omega = C_\sigma f, \tag{5.9}$$

with constant C_σ *such that* $\sigma \mapsto C_\sigma$ *is bounded above and below.*

Proof. This follows from Lemma 3.3, and the identities (5.2) and [15, Formula (7.4)]. □

Definition 5.7. *Given* $\omega \in S^{d-1}$, *we fix vectors* $\omega_1, ..., \omega_{d-1}$ *such that* $\{\omega, \omega_1, ..., \omega_{d-1}\}$ *is an orthonormal basis of* $\mathbb{R}^d$. *We then define the parabolic (quasi) distance in the direction of* ω *by*

$$d_\omega(x,y) := \langle \omega, x-y \rangle + \sum_{j=1}^{d-1} \langle \omega_j, x-y \rangle^2 \quad \forall x,y \in \mathbb{R}^d.$$

We also define (anistropic) operators associated with this parabolic distance by

$$\Delta_{\omega^\perp} := \sum_{j=1}^{d-1} \langle \omega_j, \nabla \rangle^2, \quad L_{\omega^\perp} := -\sum_{j=1}^{d-1} \langle \omega_j, D_a \rangle^2.$$

Lemma 5.8. *(i) Let* $N \in \mathbb{N}$, $N > \frac{d+1}{2}$. *There exists* $C > 0$ *such that for all* $\sigma \in (0,1)$ *and* $\omega \in S^{d-1}$, *we have*

$$\|(1 + \sigma L_{\omega^\perp} + \sigma^2 \langle \omega, D_a \rangle^2)^{-N} f\|_{L^\infty(\mathbb{R}^d)} \leq C \sigma^{-\frac{d+1}{2}} \|f\|_{L^1(\mathbb{R}^d)}$$

for all $f \in L^1(\mathbb{R}^d)$.
(ii) For every $M \in \mathbb{N}$, *there exists* $C_M > 0$ *such that for all* $E, F \subset \mathbb{R}^d$ *Borel sets,* $\sigma \in (0,1)$ *and* $\omega \in S^{d-1}$, *we have*

$$\|1_E \psi_{\omega,\sigma}(D_a)(1_F f)\|_{L^\infty(\mathbb{R}^d)} \leq C_M \sigma^{-\frac{3d+1}{4}} (1 + \frac{d_\omega(E,F)}{\sigma})^{-M} \|1_F f\|_{L^1(\mathbb{R}^d)}$$

for all $f \in L^1(\mathbb{R}^d)$.
(iii) Let $p \in [1,\infty]$. *There exists* $C > 0$ *such that for all* $\sigma \in (0,1)$ *and* $\omega \in S^{d-1}$, *we have*

$$\|\psi_{\omega,\sigma}(D_a) f\|_{L^p(\mathbb{R}^d)} \leq C \sigma^{-\frac{d-1}{4}} \|f\|_{L^p(\mathbb{R}^d)}$$

for all $f \in L^p(\mathbb{R}^d)$.

Proof. Part (i) follows from [6, Proposition 4.3], tracking the scaling factor σ in its proof. (ii) Let $\omega \in S^{d-1}$. For given Borel sets $E, F \subseteq \mathbb{R}^d$ with $d(E,F) > 0$, let $\chi_\omega \in C^\infty(\mathbb{R}^d)$ be a function with values in $[0,1]$, $\chi_\omega(\zeta) = 0$ for $|\zeta| \leq \frac{1}{2}\kappa^{-1} d_\omega(E,F)$ and $\chi_\omega(\zeta) = 1$ for $|\zeta| \geq \kappa^{-1} d_\omega(E,F)$, and $\|\langle \omega, \nabla \rangle \chi_\omega\|_\infty + \|\Delta_{\omega^\perp} \chi_\omega\|_\infty \lesssim \frac{1}{d_\omega(E,F)}$. Lemma 4.1 implies

$$c_d 1_E \psi_{\omega,\sigma}(D_a) 1_F f = 1_E \int_{\mathbb{R}^d} \chi(\zeta) \mathcal{F}^{-1}(\psi_{\omega,\sigma})(\zeta) e^{i\zeta D_a} 1_F f \, d\zeta.$$

Now note that $(1 - \sigma\Delta_{\omega^\perp} - \sigma^2 \langle \omega, \nabla_\zeta \rangle^2) e^{i\zeta D_a} = (1 + \sigma L_{\omega^\perp} + \sigma^2 \langle \omega, D_a \rangle^2) e^{i\zeta D_a}$, thus for $N \in \mathbb{N}$,

$$e^{i\zeta D_a} = (1 + \sigma L_{\omega^\perp} + \sigma^2 \langle \omega, D_a \rangle^2)^{-N} (1 - \sigma\Delta_{\omega^\perp} - \sigma^2 \langle \omega, \nabla_\zeta \rangle^2)^N e^{i\zeta D_a}.$$

From integration by parts we then get for $j \in \{0,1\}$

$$\begin{aligned} c_d 1_E \psi_{\omega,\sigma}(D_a) 1_F f &= (1 + \sigma L_{\omega^\perp} + \sigma^2 \langle \omega, D_a \rangle^2)^{-N} \\ &\circ \int_{\mathbb{R}^d} ((1 - \sigma\Delta_{\omega^\perp} - \sigma^2 \langle \omega, \nabla_\zeta \rangle^2)^N)^* (\chi^j \cdot \mathcal{F}^{-1}(\psi_{\omega,\sigma}))(\zeta) e^{i\zeta D_a} (1_F f) \, d\zeta. \end{aligned} \tag{5.10}$$

Consider first the case $d_\omega(E,F) \leq \sigma$, for which we take $j = 0$. According to Lemma 5.1, we have $\|\mathcal{F}^{-1}(\psi_{\omega,\sigma})\|_{L^1(\mathbb{R}^d)} \lesssim \sigma^{-\frac{d-1}{4}}$. Similarly, one can check that

$$\|\zeta \mapsto (\sigma \langle \omega, \nabla_\zeta \rangle)^\beta (\sigma \Delta_{\omega^\perp})^\alpha \mathcal{F}^{-1}(\psi_{\omega,\sigma})(\zeta)\|_{L^1(\mathbb{R}^d)} \lesssim \sigma^{-\frac{d-1}{4}}$$

for all $\alpha \in \mathbb{N}_0^d$ and $\beta \in \mathbb{N}_0$. We use this estimate together with Theorem 3.1 and Part (i) to obtain for $N > \frac{d+1}{2}$

$$\|\psi_{\omega,\sigma}(D_a) f\|_{L^\infty(\mathbb{R}^d)} \lesssim \sigma^{-\frac{d-1}{4}} \|(1 + \sigma L_{\omega^\perp} + \sigma^2 \langle \omega, D_a \rangle^2)^{-N}\|_{1\to\infty} \|f\|_{L^1(\mathbb{R}^d)} \lesssim \sigma^{-\frac{3d+1}{4}} \|f\|_{L^1(\mathbb{R}^d)}.$$

In the case $d_\omega(E,F) > \sigma$, we choose $j = 1$ in (5.10). Then note that according to the choice of χ_ω, we have for $\sigma \in (0,1)$ that $\|\zeta \mapsto (\sigma \langle \omega, \nabla_\zeta \rangle)^\beta (\sigma \Delta_{\omega^\perp})^\alpha \chi(\zeta)\|_\infty \lesssim (\frac{\sigma}{d_\omega(E,F)})^{|\alpha|+\beta} \lesssim 1$, for all $\alpha \in \mathbb{N}_0^d$, $\beta \in \mathbb{N}_0$. Using the product rule, a version of (5.5) for derivatives of $\mathcal{F}^{-1}(\psi_{\omega,\sigma})$, Part (i), and an anisotropic change of variable, we obtain

$$\begin{aligned} &\|1_E \psi_{\omega,\sigma}(D_a)(1_F f)\|_\infty \\ &\lesssim \sigma^{-\frac{d+1}{2}} \|1_F f\|_1 \sup_{\substack{\alpha \in \mathbb{N}_0^d, \beta \in \mathbb{N}_0 \\ |\alpha| + 2\beta \leq N}} \int_{\{|\xi| \geq \frac{d(E,F)}{\kappa}\} \cap \{|\langle \omega, \xi \rangle| \geq \frac{\omega.d(E,F)}{\kappa}\}} |(\sigma \langle \omega, \nabla_\zeta \rangle)^\beta (\sqrt{\sigma} \partial_\zeta)^\alpha \mathcal{F}^{-1}(\psi_{\omega,\sigma})(\zeta)| \, d\zeta \\ &\lesssim \sigma^{-\frac{d+1}{2}} \sigma^{-\frac{3d+1}{4}} \|1_F f\|_1 \int_{\{|\xi| \geq \frac{d(E,F)}{\kappa}\} \cap \{|\langle \omega, \xi \rangle| \geq \frac{\omega.d(E,F)}{\kappa}\}} (1 + \sigma^{-1}|\zeta|^2 + \sigma^{-2} \langle \omega, \zeta \rangle^2)^{-\tilde{N}} \, d\zeta \\ &\lesssim \sigma^{-\frac{3d+1}{4}} (1 + \frac{d_\omega(E,F)}{\sigma})^{-(2\tilde{N}-d)} \|1_F f\|_1. \end{aligned}$$

Choosing $\tilde{N}$ large enough in (5.5) yields the result.
(iii) According to Theorem 3.1 and Lemma 5.1, we have

$$\|\psi_{\omega,\sigma}(D_a) f\|_p \lesssim \|f\|_p \int_{\mathbb{R}^d} |\mathcal{F}^{-1}(\psi_{\omega,\sigma})(\zeta)| \, d\zeta \lesssim \sigma^{-\frac{d-1}{4}} \|f\|_p.$$

□

6. The Hardy-Sobolev spaces $H^{p,s}_{FIO,a}(\mathbb{R}^d)$

In the following, we denote by $\Psi \in C_c^\infty(\mathbb{R}^d)$ the function defining the wave packet transforms from Section 5. We denote by $H^1_L(\mathbb{R}^d)$ the Hardy space associated with L as defined in [9]. Recall that for all $f \in H^1_L(\mathbb{R}^d)$, we have by Lemma 4.4,

$$\|f\|_{H^1_L(\mathbb{R}^d)} \sim \|(\sigma,x) \mapsto \Psi(\sigma D_a)f(x)\|_{T^{1,2}(\mathbb{R}^d)}.$$

Definition 6.1. *Define*

$$\mathcal{S}_1 = \{f \in H^1_L(\mathbb{R}^d) \,:\, \exists g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \;\; \exists \tau > 0 \;\; f = \Psi(\tau D_a)g\},$$

and for $p \in (1,\infty)$

$$\mathcal{S}_p = \{f \in L^p(\mathbb{R}^d) \,:\, \exists g \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \;\; \exists \tau > 0 \;\; f = \Psi(\tau D_a)g\}.$$

Lemma 6.2. *Let $p \in [1,\infty)$ and $f \in \mathcal{S}_p$. Then, for all $\omega \in S^{d-1}$, $\varphi_\omega(D_a)f \in L^p(\mathbb{R}^d)$, and, in the case $p = 1$, $\varphi_\omega(D_a)f \in H^1_L(\mathbb{R}^d)$, each with norm independent of ω.*

Proof. We have that $\varphi_\omega(D_a)f = \psi_{\omega,\tau}(D_a)g$ for some $g \in L^p(\mathbb{R}^d)$, up to a change of constants in the support conditions of $\psi_{\omega,\tau}$. By Lemma 5.8, we have $\psi_{\omega,\tau}(D_a) \in B(L^p(\mathbb{R}^d))$, and thus $\|\varphi_\omega(D_a)f\|_p \lesssim_\tau \|g\|_p$. In the case $p = 1$ we moreover have that $\psi_{\omega,\tau}(D_a)g \in R(L)$, since Ψ is supported away from 0, hence $\psi_{\omega,\tau}(D_a)g \in H^1_L(\mathbb{R}^d)$. □

Corollary 6.3. *Let $p \in [1,\infty)$, $s \in \mathbb{R}$, and $f \in \mathcal{S}_p$. Then*

$$\omega \mapsto [(\sigma,x) \mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)f(x) + 1_{[0,1]}(\sigma)\sigma^{-s}\varphi_\omega(D_a)\Psi(\sigma D_a)f(x)] \in L^p(S^{d-1}; T^{p,2}(\mathbb{R}^d)).$$

Proof. This follows from Lemma 6.2 and Theorem 4.5. □

Lemma 6.4. *Let $\widetilde{\Psi} \in C_c^\infty(\mathbb{R}^d)$ be non-degenerate and supported away from 0. Let $p \in (1,\infty)$, $s \in \mathbb{R}$, and $f \in \mathcal{S}_p$. Then, we have that*

$$\omega \mapsto [(\sigma,x) \mapsto 1_{(1,\infty)}(\sigma)\widetilde{\Psi}(\sigma D_a)f(x) + 1_{[0,1]}(\sigma)\sigma^{-s}\varphi_\omega(D_a)\widetilde{\Psi}(\sigma D_a)f(x)] \in L^p(S^{d-1}; T^{p,2}(\mathbb{R}^d)),$$

with an equivalent norm to the corresponding map in Corollary 6.3, and

$$\begin{aligned}&\|(I+\sqrt{L})^{-M}f\|_{L^p}\\ &\quad\lesssim \|\omega \mapsto [(\sigma,x) \mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)f(x) + 1_{[0,1]}(\sigma)\sigma^{-s}\varphi_\omega(D_a)\Psi(\sigma D_a)f(x)]\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))},\end{aligned}$$

for all $M \in \mathbb{N}$ such that $M \geq \frac{d-1}{4} - s$.

Proof. Let $M \in \mathbb{N}$ be such that $M \geq \frac{d-1}{4} - s$. Lemma 4.4 and Corollary 6.3 give the first part, and Corollary 4.3, Lemma 4.4 together with Theorem 4.5 give

$$\begin{aligned}\|(I+\sqrt{L})^{-M}f\|_{L^p} &\lesssim \|(\sigma,x) \mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)(I+\sqrt{L})^{-M}f(x)\|_{T^{p,2}(\mathbb{R}^d)}\\ &\quad + \|(\sigma,x) \mapsto 1_{[0,1]}(\sigma)(\sigma\sqrt{L})^M(I+\sqrt{L})^{-M}\Psi^2(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)}.\end{aligned}$$

Using Corollary 4.3 again, we then have that

$$\begin{aligned}\|(I+\sqrt{L})^{-M}f\|_{L^p} &\lesssim \|(\sigma,x) \mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)}\\ &\quad + \|(\sigma,x) \mapsto 1_{[0,1]}(\sigma)\sigma^M\Psi^2(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)}.\end{aligned}$$

We then use the reproducing formula (5.9) to obtain that

$$\begin{aligned}
&\|(I+\sqrt{L})^{-M} f\|_{L^p} \\
&\quad \lesssim \|(\sigma,x)\mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)f(x) + 1_{[0,1]}(\sigma)\int\limits_{S^{d-1}} \sigma^{M-\frac{d-1}{4}}\varphi_{\omega,\sigma}(D_a)\Psi^2(\sigma D_a)f(x)d\omega\|_{T^{p,2}(\mathbb{R}^d)} \\
&\quad \lesssim \|\omega\mapsto [(\sigma,x)\mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)f(x) + 1_{[0,1]}(\sigma)\sigma^{-s}\varphi_{\omega}(D_a)\Psi(\sigma D_a)f(x)]\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d)},
\end{aligned}$$

since $M \geq \frac{d-1}{4} - s$. □

Definition 6.5. *Let $p \in [1,\infty)$, and $s \in \mathbb{R}$. We define the space $H^{p,s}_{FIO,a}(\mathbb{R}^d)$ as the completion of $\mathcal{S}_p$ for the norm defined by*

$$\begin{aligned}
&\|f\|_{H^{p,s}_{FIO,a}(\mathbb{R}^d)} \\
&\quad := \|\omega\mapsto [(\sigma,x)\mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)f(x) + 1_{[0,1]}(\sigma)\sigma^{-s}\varphi_{\omega}(D_a)\Psi(\sigma D_a)f(x)]\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}.
\end{aligned}$$

We write $H^{p}_{FIO,a}(\mathbb{R}^d) := H^{p,0}_{FIO,a}(\mathbb{R}^d)$.

Remark 6.6. *By Lemma 6.4, we have that $H^{p}_{FIO,a}(\mathbb{R}^d)$ is a subspace of the M-th extrapolation space associated with L, and is independent of the choice of $\Psi \in C_c^\infty(\mathbb{R}^d)\backslash\{0\}$ and supported away from 0.*

Remark 6.7. *By Lemma 5.6, interpolation properties of $H^{p,s}_{FIO,a}(\mathbb{R}^d)$ follow from the interpolation properties of weighted tent spaces (see [1]) with the same proof as in [15, Proposition 6.7].*

We also have the following version of [22, Theorem 4.1].

Proposition 6.8. *Let $p \in (1,\infty)$, and $s \in \mathbb{R}$. Let $q \in C_c^\infty(\mathbb{R}^d)$ with $q(\zeta) \equiv 1$ for $|\zeta| \leq \frac{1}{8}$. Then*

$$\|f\|_{H^{p,s}_{FIO,a}(\mathbb{R}^d)} \simeq \|q(D_a)f\|_{L^p(\mathbb{R}^d)} + \left(\int_{S^{d-1}} \|\varphi_\omega(D_a)(I+\sqrt{L})^s f\|^p_{L^p(\mathbb{R}^d)}\, d\omega\right)^{1/p} \quad \forall f \in \mathcal{S}_p.$$

Proof. Let $f \in \mathcal{S}_p$. By Lemma 4.4, we can choose Ψ with an appropriate support, such that $\Psi(\sigma D_a)f = \Psi(\sigma D_a)q(D_a)f$ for all $\sigma \geq 1$, $\Psi(\sigma D_a)q(D_a) = 0$ for all $\sigma \leq \frac{1}{8}$, and $\varphi_\omega(D_a)\Psi(\sigma D_a) = 0$ for all $\sigma \geq 1$ and $\omega \in S^{d-1}$.
Then, by Theorem 4.5, we have that

$$\begin{aligned}
\|f\|_{H^{p,s}_{FIO,a}(\mathbb{R}^d)} &\lesssim \|(\sigma,x)\mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)q(D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)} \\
&\qquad + \|\omega\mapsto [(\sigma,x)\mapsto 1_{[0,1]}(\sigma)\sigma^{-s}\varphi_\omega(D_a)\Psi(\sigma D_a)f(x)]\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))} \\
&\lesssim \|q(D_a)f\|_{L^p(\mathbb{R}^d)} + \left(\int_{S^{d-1}} \|(I+\sqrt{L})^s\varphi_\omega(D_a)f\|^p_{L^p(\mathbb{R}^d)}\, d\omega\right)^{1/p}.
\end{aligned}$$

In the other direction, Theorem 4.5 and the support properties of q and Ψ give us that

$$\|q(D_a)f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{H^{p,s}_{FIO,a}(\mathbb{R}^d)} + \|(\sigma,x)\mapsto 1_{[\frac{1}{8},1]}(\sigma)\Psi(\sigma D_a)q(D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)}.$$

With the same proof as in Lemma 4.4, we then have that, for all $M \geq \frac{d-1}{4} - s$,

$$\begin{aligned}
&\|(\sigma,x) \mapsto 1_{[\frac{1}{8},1]}(\sigma)\Psi(\sigma D_a)q(D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)} \\
&\quad\lesssim \|(\sigma,x) \mapsto 1_{[\frac{1}{8},1]}(\sigma)\int_0^\infty \Psi(\sigma D_a)q(D_a)\Psi(\tau D_a)(I+\sqrt{L})^{M}(I+\sqrt{L})^{-M}f(x)\frac{d\tau}{\tau}\|_{T^{p,2}(\mathbb{R}^d)} \\
&\quad\lesssim \|(I+\sqrt{L})^{-M}f\|_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

Therefore, using Lemma 6.4, we have that $\|q(D_a)f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{H^{p,s}_{FIO,a}(\mathbb{R}^d)}$. For the second term, we use Theorem 4.5 and the support properties of Ψ again to get that

$$\begin{aligned}
&\Big(\int_{S^{d-1}} \|\varphi_\omega(D_a)(I+\sqrt{L})^s f\|^p_{L^p(\mathbb{R}^d)}\,d\omega\Big)^{1/p} \\
&\quad\lesssim \|\omega \mapsto [(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\sigma^{-s}\varphi_\omega(D_a)\Psi(\sigma D_a)f(x)]\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))} \\
&\quad\lesssim \|f\|_{H^{p,s}_{FIO,a}(\mathbb{R}^d)}.
\end{aligned}$$

□

7. Sobolev embedding properties of $H^p_{FIO,a}(\mathbb{R}^d)$

We use a variation of the arguments in [15, Section 7].
We let $m(D_a) = (I+\sqrt{L})^{-\frac{d-1}{4}}$.

Lemma 7.1. *For every $0 < \theta < \frac{\pi}{2}$ there exist $C_\theta, c_\theta > 0$ such that for all atoms $A \in T^{1,2}(\mathbb{R}^d)$, and all $s \in \mathbb{R}$*

$$\int_{S^{d-1}} \|(\sigma,x) \mapsto 1_{[0,1]}(\sigma)m(\sqrt{L})^{1+is}\psi_{\omega,\sigma}(D_a)A(\sigma,.)(x)\|_{T^{1,2}(\mathbb{R}^d)}\,d\omega \leq C_\theta e^{|s|c_\theta}. \tag{7.1}$$

Proof. Let A be a $T^{1,2}(\mathbb{R}^d)$ atom associated with a ball $B = B(c_B, r)$. Without loss of generality, we assume that $A(\sigma,.) = 0$ for all $\sigma \geq 1$.
By renormalisation, we can replace $\psi_{\omega,\sigma}(D_a)$ in (7.1) by $\Psi_\sigma(D_a)\psi_{\omega,\sigma}(D_a)$. Noting that $\|m^{is}\|_{L^\infty(S^o_\theta)} \leq ce^{|s|c_\theta}$, for $c_\theta = \frac{\theta(d-1)}{4}$, we use Corollary 4.3 to obtain for every $\omega \in S^{d-1}$ and given $\theta \in (0,\frac{\pi}{2})$

$$\begin{aligned}
&\|(\sigma,x) \mapsto 1_{[0,1]}(\sigma)m(D_a)^{1+is}\Psi_\sigma(D_a)\psi_{\omega,\sigma}(D_a)A(\sigma,.)(x)\|_{T^{1,2}(\mathbb{R}^d)} \\
&\quad= \|(\sigma,x) \mapsto 1_{[0,1]}(\sigma)L^{\frac{d-1}{8}}m(D_a)^{1+is}\Psi_\sigma(D_a)L^{-\frac{d-1}{8}}\psi_{\omega,\sigma}(D_a)A(\sigma,.)(x)\|_{T^{1,2}(\mathbb{R}^d)} \\
&\quad\leq C_\theta e^{|s|c_\theta}\|(\sigma,x) \mapsto 1_{[0,1]}(\sigma)L^{-\frac{d-1}{8}}\psi_{\omega,\sigma}(D_a)A(\sigma,.)(x)\|_{T^{1,2}(\mathbb{R}^d)},
\end{aligned}$$

with C_θ independent of $s \in \mathbb{R}$.
For $j \in \mathbb{N}^*$, and $\omega \in S^{d-1}$, define $C_{j,\omega} := \{y \in \mathbb{R}^d\ ;\ 2^{j-1}r < |\langle\omega, c_B - y\rangle| + |c_B - y|^2 \leq 2^j r\}$ and $C_{0,\omega} := \{y \in \mathbb{R}^d\ ;\ |\langle\omega, c_B - y\rangle| + |c_B - y|^2 \leq r\}$. Remark that $|C_{j,\omega}| \sim (2^j r)^{\frac{d+1}{2}}$, and that $d_\omega(C_{j,\omega}, C_{0,\omega}) > 2^{j-1}r$. Using Lemma 5.4 and Corollary 4.6 for $p = \frac{4d}{3d-1}$, we have that

$$
\begin{aligned}
&\Big(\int_{S^{d-1}} \|(\sigma,x)\mapsto 1_{C_{0,\omega}}(x)1_{[0,1]}(\sigma)L^{-\frac{d-1}{8}}\psi_{\omega,\sigma}(D_a)A(\sigma,.)(x)\|_{T^{1,2}(\mathbb{R}^d)}d\omega\Big)^2\\
&\lesssim r^{\frac{d+1}{2}}\int_{S^{d-1}}\int_0^{\min(r,1)} \|L^{-\frac{d-1}{8}}\psi_{\omega,\sigma}(D_a)A(\sigma,.)(x)\|^2_{L^2(\mathbb{R}^d)}\frac{d\sigma}{\sigma}d\omega\\
&\lesssim r^{\frac{d+1}{2}}\int_{S^{d-1}}\int_0^{\min(r,1)} \|L^{-\frac{d-1}{8}}A(\sigma,.)(x)\|^2_{L^2(\mathbb{R}^d)}\frac{d\sigma}{\sigma}d\omega\\
&\lesssim r^{\frac{d+1}{2}}\int_{S^{d-1}}\int_0^{r} \|A(\sigma,.)(x)\|^2_{L^p(\mathbb{R}^d)}\frac{d\sigma}{\sigma}d\omega\\
&\lesssim r^{\frac{d+1}{2}}r^{\frac{d-1}{2}}\int_{S^{d-1}}\int_0^{r} \|A(\sigma,.)(x)\|^2_{L^2(\mathbb{R}^d)}\frac{d\sigma}{\sigma}d\omega \lesssim r^d\|A\|^2_{T^{2,2}}\lesssim 1.
\end{aligned}
$$

Let $M > d+1$, and define $\widetilde{\Psi}:\xi\mapsto \frac{|\xi|^{-\frac{d-1}{4}}\Psi(\xi)}{(\int_0^\infty |\sigma\xi|^{-\frac{d-1}{2}}|\Psi(\sigma\xi)|^2\frac{d\sigma}{\sigma})^{\frac12}}$, and $\tilde{\psi}_{\omega,\sigma}:\xi\mapsto\varphi_{\omega,\sigma}(\xi)\widetilde{\Psi}(\sigma\xi)$.

For all $j\in\mathbb{N}^*$, we obtain from Lemma 5.8 for $\widetilde{\psi_{\omega,\sigma}}$ instead of $\psi_{\omega,\sigma}$

$$
\begin{aligned}
&\Big(\int_{S^{d-1}} \|(\sigma,x)\mapsto 1_{C_{j,\omega}}(x)1_{[0,1]}(\sigma)L^{-\frac{d-1}{8}}\psi_{\omega,\sigma}(D_a)A(\sigma,.)(x)\|_{T^{1,2}(\mathbb{R}^d)}d\omega\Big)^2\\
&\lesssim (2^jr)^{\frac{d+1}{2}}\int_{S^{d-1}}\int_0^{\min(r,1)} \sigma^{\frac{d-1}{2}}\|\widetilde{\psi_{\omega,\sigma}}(D_a)A(\sigma,.)\|^2_{L^2(C_{j,\omega})}\frac{d\sigma}{\sigma}d\omega\\
&\lesssim (2^jr)^{d+1}\int_{S^{d-1}}\int_0^{\min(r,1)} \sigma^{\frac{d-1}{2}}\|\widetilde{\psi_{\omega,\sigma}}(D_a)A(\sigma,.)\|^2_{L^\infty(C_{j,\omega})}\frac{d\sigma}{\sigma}d\omega\\
&\lesssim (2^jr)^{d+1}\int_{S^{d-1}}\int_0^{\min(r,1)} \sigma^{\frac{d-1}{2}}\sigma^{-\frac{3d+1}{2}}\left(\frac{\sigma}{2^jr}\right)^M\|A(\sigma,.)\|^2_{L^1(\mathbb{R}^d)}\frac{d\sigma}{\sigma}d\omega\\
&\lesssim r^d(2^jr)^{d+1}\int_{S^{d-1}}\int_0^{\min(r,1)} \sigma^{\frac{d-1}{2}}\sigma^{-\frac{3d+1}{2}}\left(\frac{\sigma}{2^jr}\right)^M\|A(\sigma,.)\|^2_{L^2(\mathbb{R}^d)}\frac{d\sigma}{\sigma}d\omega\\
&\lesssim 2^{-j(M-d-1)}r^d\|A\|^2_{T^{2,2}}\lesssim 2^{-j(M-d-1)}.
\end{aligned}
$$

Summing over j yields the conclusion. □

Lemma 7.2. *For all $p\in[1,2]$, and $s_p=(d-1)(\frac1p-\frac12)$, we have the continuous inclusion $H^{p,\frac{s_p}{2}}_{FIO,a}(\mathbb{R}^d)\subset H^p_L(\mathbb{R}^d)$, where $H^p_L(\mathbb{R}^d)=L^p(\mathbb{R}^d)$ for $p>1$. For $p\in(1,\infty)$, and $b:\xi\mapsto$*

$|\xi|^{\frac{d-1}{4}}m(\xi)$, we have that

$$\|(\sigma,x)\mapsto m(D_a)\Psi(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)} \lesssim \|(b(D_a)+m(D_a))f\|_{H^p_{FIO,a}(\mathbb{R}^d)} \lesssim \|f\|_{H^p_{FIO,a}(\mathbb{R}^d)},$$

for all $f \in \mathcal{S}_p$.

Proof. Let f be an H^1_L atom. We have, using the reproducing formula (5.9), that

$$\begin{aligned}\|f\|_{H^1_L} &\sim \|(\sigma,x)\mapsto \Psi(\sigma D_a)f(x)\|_{T^{1,2}(\mathbb{R}^d)}\\ &\lesssim \int_{S^{d-1}} \|(\sigma,x)\mapsto 1_{[0,1]}(\sigma)\sigma^{-\frac{d-1}{4}}\psi_{\omega,\sigma}(D_a)f(x) + 1_{[1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\|_{T^{1,2}(\mathbb{R}^d)}d\omega\\ &\lesssim \|f\|_{H^{1,\frac{d-1}{4}}_{FIO,a}(\mathbb{R}^d)},\end{aligned}$$

where the last inequality follows from the comparability of $\psi_{\omega,\sigma}$ with $\varphi_\omega\Psi_\sigma$ for $\sigma\in(0,1)$. Since $H^2_{FIO,a} = L^2$, the continuous inclusion $H^{p,\frac{sp}{2}}_{FIO,a}(\mathbb{R}^d) \subset H^p_L(\mathbb{R}^d)$ follows by interpolation. In the same way,

$$\begin{aligned}&\|(\sigma,x)\mapsto 1_{[0,1]}(\sigma)m(D_a)\Psi(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)}\\ &\quad\lesssim \int_{S^{d-1}} \|(\sigma,x)\mapsto 1_{[0,1]}(\sigma)b(D_a)\varphi_\omega(D_a)\widetilde{\Psi}(\sigma D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)}d\omega,\end{aligned}$$

for $\widetilde{\Psi}$ such that $\Psi(\xi) = |\xi|^{\frac{d-1}{4}}\widetilde{\Psi}(\xi)$ for all $\xi\in\mathbb{R}^d$. Turning to the low frequency term, we note that, for $\sigma > 1$, we have that $\Psi(\sigma\xi) = \Psi(\sigma\xi)q(\xi)$ for all $\xi\in\mathbb{R}^d$. Therefore, by Theorem 4.5 and Proposition 6.8 we have that

$$\|(\sigma,x)\mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)m(D_a)f(x)\|_{T^{p,2}(\mathbb{R}^d)} \lesssim \|m(D_a)q(D_a)f\|_{L^p(\mathbb{R}^d)} \lesssim \|m(D_a)f\|_{H^p_{FIO,a}(\mathbb{R}^d)}.$$

To conclude the proof, we use Theorem 2.1 and Theorem 2.2, along with Theorem 3.1, to show that $b(D_a)$ and $m(D_a)$ are bounded operators on $L^p(\mathbb{R}^d)$, and thus also on $H^p_{FIO,a}(\mathbb{R}^d)$, thanks to Proposition 6.8. □

Corollary 7.3. *Let $p\in(1,2]$. Then*

$$\|(I+\sqrt{L})^{-\frac{sp}{2}}f\|_{H^p_{FIO,a}(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)},$$

for all $f\in\mathcal{S}_p$.

Proof. For $z\in\mathbb{C}$ such that $Re(z)\in[0,1]$, we consider the operators defined by

$$T_z f(x,\omega,\sigma) := 1_{[0,1]}(\sigma)(I+\sqrt{L})^{-(\frac{d-1}{4})z}\psi_{\omega,\sigma}(D_a)f(x) \quad \forall f\in L^2(\mathbb{R}^d).$$

For $Re(z) = 0$, they are well defined as operators from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d\times S^{d-1}\times(0,\infty); dxd\omega\frac{d\sigma}{\sigma})$ by Lemma 5.4, with norm independent of $Im(z)$. For $Re(z) = 1$, by Lemma 7.1, T_z extends to a bounded operator from $H^1(\mathbb{R}^d)$ to $L^1(S^{d-1};T^{1,2}(\mathbb{R}^d))$ with norm bounded by $C_\theta e^{|Im(z)|c_\theta}$ for fixed $\theta>0$. Therefore, by Stein interpolation [28] with admissible growth, $T_z\in B(L^p(\mathbb{R}^d), L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))$ for $Re(z) = \frac{2}{p}-1$. To conclude the proof, we thus only have to show the low frequency estimate

$$\|(\sigma,x)\mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)(I+\sqrt{L})^{-\frac{sp}{2}}f(x)\|_{T^{p,2}(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

This follows from Theorem 4.5 and the L^p boundedness of $(I+\sqrt{L})^{-\frac{sp}{2}}$. □

8. The wave group

Theorem 8.1. *Let $p \in (1,\infty)$, and $s \in \mathbb{R}$. Then*

$$e^{it\sqrt{L}} : H^{p,s}_{FIO,a}(\mathbb{R}^d) \to H^{p,s}_{FIO,a}(\mathbb{R}^d)$$

is bounded for each $t > 0$.

For simplicity, we set $t = 1$ and $s = 0$. All the proofs extend verbatim to other values of t. The case $s \in \mathbb{R}$ is an immediate consequence of the case $s = 0$ by Proposition 6.8. For the transport group, the L^p boundedness is clear.

Lemma 8.2. *Let $p \in (1,\infty)$ and $\omega \in S^{d-1}$. Then $e^{i\omega.D_a} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ and $e^{i\omega.D_a} : H^p_{FIO,a}(\mathbb{R}^d) \to H^p_{FIO,a}(\mathbb{R}^d)$ is bounded.*

Proof. The L^p boundedness is proven in Theorem 3.1. The boundedness on $H^p_{FIO,a}(\mathbb{R}^d)$ is an immediate consequence of the L^p boundedness, by Proposition 6.8. □

For the low frequency estimate, we need the following lemma.

Lemma 8.3. *Let $p \in (1,\infty)$, let $q \in C_c^\infty(\mathbb{R}^d)$. Then $q(D_a)e^{i\sqrt{L}} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is bounded.*

Proof. Because of the compact support of q, the symbol $\zeta \mapsto q(\zeta)e^{i|\zeta|}$ clearly satisfies the Marcinkiewicz-Lizorkin multiplier condition of Theorem 2.1. The result thus follows from Theorem 2.1 and Theorem 2.2 using that D_a generates a bounded d-parameter group, as shown in Theorem 3.1. □

Proof of Theorem 8.1. For $f \in \mathcal{S}_p$, Proposition 6.8 yields

$$\|e^{i\sqrt{L}}f\|_{H^p_{FIO,a}(\mathbb{R}^d)} \lesssim \|q(D_a)e^{i\sqrt{L}}f\|_{L^p(\mathbb{R}^d)} + \left(\int_{S^{d-1}} \|\varphi_\omega(D_a)e^{i\sqrt{L}}f\|^p_{L^p(\mathbb{R}^d)}\,d\omega\right)^{1/p}.$$

For the low frequency part, recall that $q \in C_c^\infty(\mathbb{R}^d)$ with $q(\zeta) \equiv 1$ for $|\zeta| \leq \frac{1}{8}$. Choose $\tilde{q} \in C_c^\infty(\mathbb{R}^d)$ with $\tilde{q}(\zeta) \equiv 1$ on $\operatorname{supp} q$. Then $q(D_a)e^{i\sqrt{L}} = \tilde{q}(D_a)e^{i\sqrt{L}}q(D_a)$, since D_a and $\sqrt{L}$ are commuting, and $\tilde{q}(D_a)e^{i\sqrt{L}}$ is L^p bounded according to Lemma 8.3. Thus,

$$\|q(D_a)e^{i\sqrt{L}}f\|_{L^p(\mathbb{R}^d)} = \|\tilde{q}(D_a)e^{i\sqrt{L}}q(D_a)f\|_{L^p(\mathbb{R}^d)} \lesssim \|q(D_a)f\|_{L^p(\mathbb{R}^d)}.$$

Let us now consider the high frequency part. For fixed $\omega \in S^{d-1}$, we decompose

$$\varphi_\omega(D_a)e^{i\sqrt{L}} = \varphi_\omega(D_a)e^{i\omega.D_a} + \varphi_\omega(D_a)(e^{i\sqrt{L}} - e^{i\omega.D_a}).$$

The first part can be dealt with Lemma 8.2, which directly yields

$$\left(\int_{S^{d-1}} \|\varphi_\omega(D_a)e^{i\omega.D_a}f\|^p_{L^p(\mathbb{R}^d)}\,d\omega\right)^{1/p} \lesssim \|f\|_{H^p_{FIO,a}(\mathbb{R}^d)}.$$

For the second part, we use (5.8) to write

$$\varphi_\omega(D_a)(e^{i\sqrt{L}} - e^{i\omega.D_a}) = \varphi_\omega(D_a)e^{i\omega.D_a}(e^{-i\omega.D_a}e^{i\sqrt{L}} - I)\pi_a W_a.$$

Since $e^{i\omega.D_a}$ is bounded on $L^p(\mathbb{R}^d)$ by Lemma 8.2, it suffices to show that

$$\|\varphi_\omega(D_a)(e^{-i\omega.D_a}e^{i\sqrt{L}} - I)\pi_a W_a f\|_{L^p(\mathbb{R}^d)} \lesssim \|\varphi_\omega(D_a)f\|_{L^p(\mathbb{R}^d)}.$$

We can write

$$\varphi_\omega(D_a)(e^{-i\omega.D_a}e^{i\sqrt{L}} - I)\pi_a W_a = m_\omega(D_a)\varphi_\omega(D_a) + q_\omega(D_a)\varphi_\omega(D_a)$$

for the symbols

$$m_\omega(\zeta) = \tilde{\varphi}_\omega(\zeta)\tilde{m}_\omega(\zeta)\int_0^1\int_{S^{d-1}}\psi_{\nu,\sigma}(\zeta)^2\,d\nu\frac{d\sigma}{\sigma} \tag{8.1}$$

and

$$q_\omega(\zeta) = \tilde{\varphi}_\omega(\zeta)\tilde{m}_\omega(\zeta)r(\zeta)^2$$

with $\tilde{m}_\omega(\zeta) = e^{-i\omega.\zeta+i|\zeta|} - 1$, $\tilde{\varphi}_\omega \in C_c^\infty(\mathbb{R}^d)$ a function with $\tilde{\varphi}_\omega \equiv 1$ on $\operatorname{supp}\varphi_\omega$ and $\tilde{\varphi}_\omega(\zeta) = 0$ for $|\zeta| < \frac{1}{16}$ or $|\hat{\zeta} - \omega| > 4|\zeta|^{-1/2}$, and

$$r(\zeta) := \left(\int_1^\infty \Psi_\sigma(\zeta)^2\,\frac{d\sigma}{\sigma}\right)^{1/2}, \quad \zeta \neq 0,$$

and $r(0) := 1$. As noted in [15, Section 4.1], we have $r \in C_c^\infty(\mathbb{R}^d)$.
The proof will be concluded by applying Theorem 2.1, and Theorem 2.2, using Theorem 3.1. We only have to check that m_ω and q_ω satisfy the assumption of Theorem 2.1. For q_ω, this directly follows from the fact that $r \in C_c^\infty(\mathbb{R}^d)$. For m_ω, this is proven in Lemma 8.5 below. □

Remark 8.4. *Let $\omega \in S^{d-1}$. Let $\tilde{\varphi}_\omega \in C_c^\infty(\mathbb{R}^d)$ a function with $\tilde{\varphi}_\omega \equiv 1$ on $\operatorname{supp}\varphi_\omega$ and $\tilde{\varphi}_\omega(\zeta) = 0$ for $|\zeta| < \frac{1}{16}$ or $|\hat{\zeta} - \omega| > 4|\zeta|^{-1/2}$. By the choice of the cut-off function $\tilde{\varphi}_\omega$ and the support properties of φ_ω, we have the following: For all $\alpha \in \mathbb{N}_0^d$ and $\beta \in \mathbb{N}_0$, there exists a constant $C = C(\alpha,\beta) > 0$ such that*

$$|\langle\omega,\nabla_\zeta\rangle^\beta\partial_\zeta^\alpha\tilde{\varphi}_\omega(\zeta)| \le C|\zeta|^{-\frac{|\alpha|}{2}-\beta}$$

for all $\omega \in S^{d-1}$ and $\zeta \in \mathbb{R}^d \setminus \{0\}$.

Lemma 8.5. *Let $\omega \in S^{d-1}$, let m_ω be as defined in* (8.1)*. For all $\alpha \in \mathbb{N}_0^d$ with $|\alpha|_\infty \le 1$ there exists a constant $C = C(\alpha) > 0$ such that*

$$|\zeta^\alpha\partial_\zeta^\alpha m_\omega(\zeta)| \le C$$

for all $\zeta \in \mathbb{R}^d \setminus \{0\}$.

Proof. By rotational invariance it suffices to consider the case $\omega = e_1$. Let $\zeta \in \mathbb{R}^d \setminus \{0\}$. The bound $|m_{e_1}(\zeta)| \le C$ directly follows from (5.2) and the boundedness of $\tilde{m}_{e_1}$ and $\tilde{\varphi}_{e_1}$. Moreover, by the specific form of $\tilde{m}_{e_1}(\zeta) = e^{ib(\zeta)} - 1$ with $b(\zeta) = -\zeta_1 + |\zeta|$, it can easily be seen that the condition

$$|\zeta^\alpha\partial_\zeta^\alpha b(\zeta)| \le c \tag{8.2}$$

for $|\alpha|_\infty \le 1$ immediately implies $|\zeta^\alpha \partial_\zeta^\alpha \tilde{m}_{e_1}(\zeta)| \le c$ for $|\alpha|_\infty \le 1$. We check (8.2):

$$|\zeta_1 \partial_1 b(\zeta)| = |\zeta_1 \partial_1(-\zeta_1 + |\zeta|)| \le |\zeta_1||1 - \frac{\zeta_1}{|\zeta|}| = \left|\frac{\zeta_1}{|\zeta|}\right| ||\zeta| - \zeta_1|$$

$$\le ||\zeta| - \zeta_1| = |\zeta_1| \left(\sqrt{1 + \sum_{j=2}^d \frac{\zeta_j^2}{\zeta_1^2}} - 1 \right).$$

According to the support properties of $\tilde{\varphi}_{e_1}$ and $\psi_{\nu,\sigma}$, we have $|\nu - e_1| \lesssim \sqrt{\sigma}$. Thus a slight modification of (5.7) yields that there exist constants $c_1, c_2 > 0$ such that for $0 < \sigma \ll 1$, one has

$$\zeta_1 > \frac{c_1}{\sigma} \quad \text{and} \quad |\zeta_j| \le \frac{c_2}{\sqrt{\sigma}}, \quad j \in \{2, \dots, d\}, \tag{8.3}$$

on the support of m_{e_1}. Thus, for such choice of ζ,

$$|\zeta_1 \partial_1 b(\zeta)| \lesssim |\zeta_1| \left(\sqrt{1 + \frac{c}{\zeta_1}} - 1 \right).$$

This expression remains bounded for $\zeta_1 \to \infty$ or equivalently $|\zeta| \to \infty$, since replacing $h = \frac{1}{\zeta_1}$, we see that

$$\lim_{h \to 0} \frac{\sqrt{1 + ch} - 1}{h} = \frac{c}{2}.$$

Again using (8.3) and $|\zeta| \ge |\zeta_1| > \frac{c_1}{\sigma}$, we obtain for $j \in \{2, \dots, d\}$ that

$$|\zeta_j \partial_j b(\zeta)| = |\zeta_j \partial_j(-\zeta_1 + |\zeta|)| \le |\zeta_j \frac{\zeta_j}{|\zeta|}| \le c.$$

Concerning the mixed derivatives, one can inductively show that for $\alpha \in \mathbb{N}_0^d$ with $|\alpha|_\infty \le 1$ and $\alpha_1 = 0$, $|\zeta^\alpha \partial_\zeta^\alpha b(\zeta)| = |\frac{\zeta^{2\alpha}}{|\zeta|^{2|\alpha|-1}}| \le c$, for ζ as in (8.3). Finally, for $j \ne 1$,

$$|\zeta_1 \zeta_j \partial_1 \partial_j b(\zeta)| = |\zeta_1 \zeta_j \partial_1 \partial_j(-\zeta_1 + |\zeta|)| = |\zeta_1 \zeta_j|| \frac{\zeta_1 \zeta_j}{|\zeta|^3}| \le c.$$

Putting all arguments together shows (8.2). The bound $|\zeta^\alpha \partial_\zeta^\alpha \tilde{\varphi}_{e_1}(\zeta)| \le c$ follows from Remark 8.4 together with (8.3), whereas the analogous bound for the last factor in (8.1) concerning $\psi_{\nu,\sigma}$ is a consequence of (5.6) together with (8.3). ☐

Combining Corollary 7.3 with Theorem 8.1 and Theorem 4.5 then gives our main result.

Theorem 8.6. *Let $p \in (1, \infty)$ and $s_p = (d-1)|\frac{1}{p} - \frac{1}{2}|$. For each $t \in \mathbb{R}$, the operator $(I + \sqrt{L})^{-s_p} \exp(it\sqrt{L})$ is bounded on $L^p(\mathbb{R}^d)$. Moreover, if $s_p \le 2$, the operator $\exp(it\sqrt{L})$ is bounded from $W^{s_p,p}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$.*

Proof. By duality, it suffices to consider the case $p \in (1, 2)$. Let $f \in \mathcal{S}_p$. By Lemma 7.2 and Theorem 8.1, we have that

$$\| \exp(it\sqrt{L}) f \|_{L^p(\mathbb{R}^d)} \lesssim \| \exp(it\sqrt{L}) f \|_{H^{p, \frac{s_p}{2}}_{FIO,a}(\mathbb{R}^d)} \lesssim \|f\|_{H^{p, \frac{s_p}{2}}_{FIO,a}(\mathbb{R}^d)}.$$

Using Proposition 6.8, and Corollary 7.3, we then have that

$$\|\exp(it\sqrt{L})f\|_{L^p(\mathbb{R}^d)} \lesssim \|(I+\sqrt{L})^{\frac{s_p}{2}}f\|_{H^p_{FIO,a}(\mathbb{R}^d)} \lesssim \|(I+\sqrt{L})^{s_p}f\|_{L^p(\mathbb{R}^d)}.$$

For $s_p \leq 2$, Theorem 4.5 then gives $\|f\|_{W^{s_p,p}} \sim \|(I+\sqrt{L})^{s_p}f\|_{L^p(\mathbb{R}^d)}$. □

To obtain analogues of Theorem 8.1 for more general operators with Lipschitz coefficients, we plan to develop a perturbation theory in future work. Here we just give a prototype of the results that such a theory should give, in the case where $d = 1$. This case is simple because $H^p_{FIO,a} = L^p$, and Riesz transforms associated with L are L^p bounded.

Corollary 8.7. *Let $d = 1$, and $a \in C^{0,1}(\mathbb{R})$ be bounded above and below, with $\frac{d}{dx}a \in L^\infty$. Let $p \in (1,\infty)$. The operator $\tilde{L} = -\frac{d}{dx}a^2\frac{d}{dx}$ (with domain $W^{2,p}$) generates a cosine family on L^p.*

Proof. By Theorem 8.1, Lemma 7.2, and Corollary 7.3, the operator $L = \tilde{L} - (\frac{d}{dx}a)a\frac{d}{dx}$ generates a cosine family on L^p, with Kisyński space $D(\sqrt{L})$ (see [2] for the theory of cosine families). By [6, Theorem 2.36] and [3, Section 4], we have that $D(\sqrt{L}) = W^{1,p}$. Since $(\frac{d}{dx}a)a\frac{d}{dx} \in B(W^{1,p}, L^p)$, the result thus follows by [2, Corollary 3.14.13]. □

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