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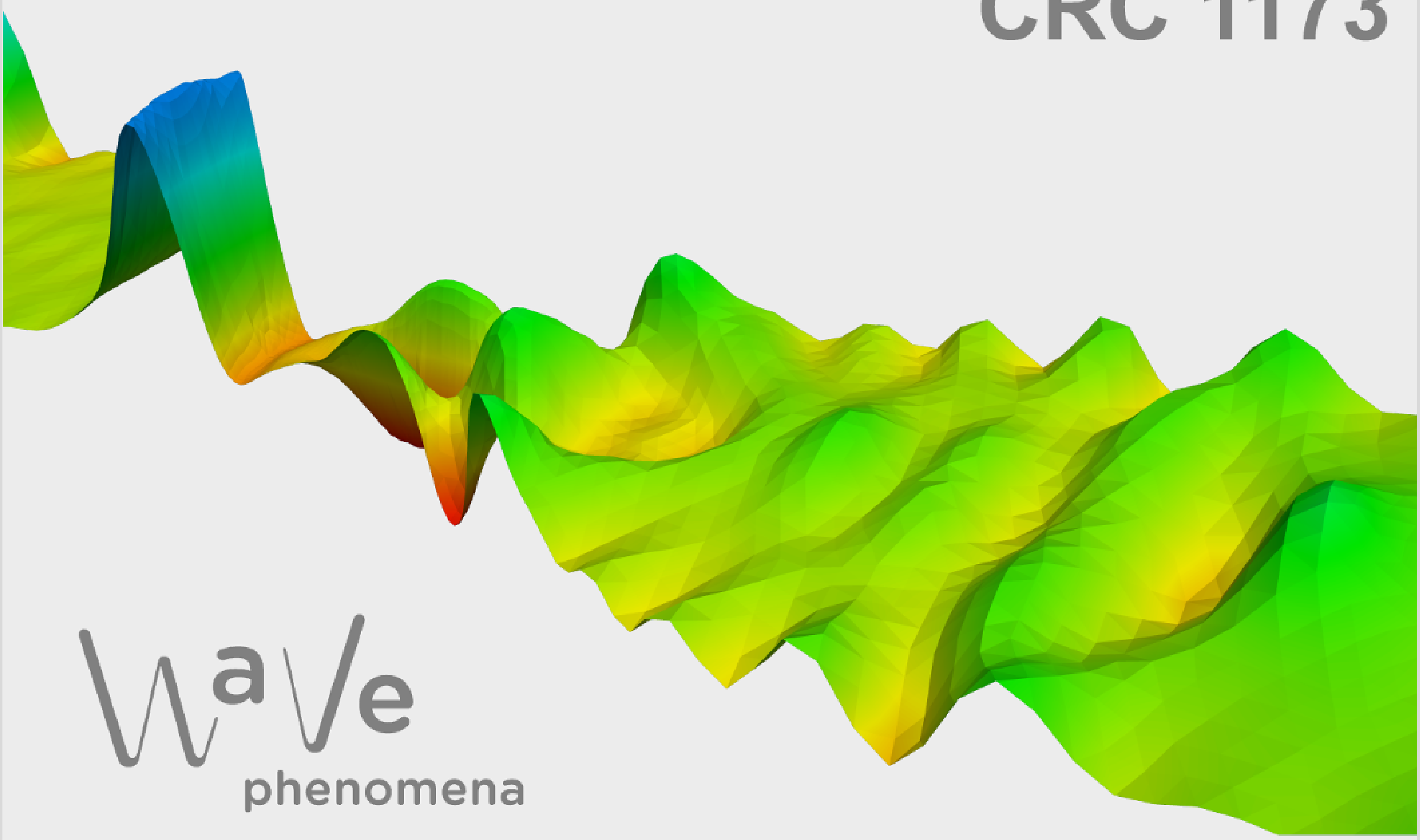

Wave phenomena

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ON THE ASYMPTOTIC DYNAMICS OF 2-D MAGNETIC QUANTUM SYSTEMS

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Abstract. In this work we provide results on the long time localisation in space (dynamical localisation) of certain two-dimensional magnetic quantum systems. The underlying Hamiltonian may have the form $H = H_0 + W$, where H_0 is rotationally symmetric, has dense point spectrum, and W is a perturbation that breaks the rotational symmetry. In the latter case, we also give estimates for the growth of the angular momentum operator in time.

Contents

1. Introduction

Consider a charged quantum particle subject to a time independent electro-magnetic field profile. The system may be described through a self-adjoint operator H with domain $\mathcal{D}(H)$ in a Hilbert space $\mathcal{H}$. If we assume that the system is initially in a state $\varphi \equiv \varphi(0) \in \mathcal{D}(H)$ then, according to the Schrödinger equation, the state of the system at time t, $\varphi(t)$, is given by $e^{-iHt/\hbar}\varphi$ (here $\hbar > 0$ is Plank's constant devided by 2π). A fundamental question is whether the system remains localized for long times and, if not, what the speed of the wavepackage spreading is in terms of the electromagnetic field profile.

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This phenomena can be investigated, for instance, by looking at the long-time behavior of the expected radius of the state

$$\langle\varphi(t), |\mathbf{x}|\, \varphi(t)\rangle\,, \quad \text{for} \quad t \gg 1\,. \tag{1}$$

This, in turn, can sometimes be estimated if one has information on the spectral quality of the underlying Hamiltonian (see, e.g., [8, 22] and [15]). Let us assume that the initial state belongs to a finite energy region $I \subset \mathbb{R}$ (with $|I| < \infty$) of $\mathcal{H}$, i.e., $\varphi = E_I(H)\varphi$, with $E_I(H)$ being the spectral projection of H on I. Then, one can easily check that if the spectrum of the Hamiltonian is a discrete set, then the system remains localized in the sense that

$$\sup_{t\in\mathbb{R}}\langle\varphi(t), |\mathbf{x}|\, \varphi(t)\rangle \leqslant C\, \|\varphi\|^2\,,$$

for some constant $C > 0$. Moreover, in dimension one, it is known that if the spectrum is absolutely continuous the wavefunction spreading is ballistic in time average. More precisely, there is a constant $c > 0$ such that

$$\frac{1}{T}\int_0^T \langle\varphi(t), |\mathbf{x}|\, \varphi(t)\rangle \geqslant cT\,, \quad T > 1\,.$$

(See [15] for this and more general results of this type). However, if the spectrum is dense point or singular continuous there is very little one can say a priori (see [9]). Indeed, for H having point spectrum, it is only known in general that the system exhibits a sub-ballistic dynamical behaviour [21], i.e.,

$$\lim_{t\to\infty}\langle\varphi(t), |\mathbf{x}|\, \varphi(t)\rangle/t \to 0\,.$$

Moreover, there are examples of Hamiltonians with pure point spectrum where the spreading rate is arbitrarily close to ballistic [9], i.e., for any $\varepsilon > 0$

$$\limsup_{t\to\infty}\langle\varphi(t), |\mathbf{x}|\, \varphi(t)\rangle/t^{1-\varepsilon} \to \infty\,,$$

for a large class of initial data φ.

In the work at hand, we shed some more light on this problem for the cases, (a) when $H = H_0$ is the two-dimensional magnetic Schrödinger operator with a radially symmetric magnetic field B and has dense point spectrum and (b) when $H = H_0 + W$, with H_0 as before and W being an electric perturbation, smooth in the angular variable, that decays at infinity. Our conditions include the cases when

$$\int_0^r B(s) s ds = \lambda r^\sigma\,, \quad \lambda > 0, \sigma \geqslant 1\,.$$

Using the same arguments as in [17] one can easily show that for $\sigma \in (1, 2)$ the spectrum of H_0 is dense pure point. Moreover, if $\sigma = 1$ there is a *mobility edge* at energy λ^2, i.e., the spectrum is dense pure point on $[0, \lambda^2)$ and purely absolutely continuous on (λ^2, ∞). It follows directly from Theorem 1.5 below that when $\sigma \in (1, 2)$ the dynamics generated by H_0 is localized in time, provided the initial data is sufficiently smooth. Moreover, we show an anlogous result for the case $\sigma = 1$, whenever $\varphi = E_{[0,\lambda^2)}(H)\varphi$ (see Section 5). Similar results have been obtained for Dirac operators in [5].

The problem for $\sigma \in (1,2)$ becomes much more delicate if we turn on the electric perturbation W. In this case we do not even know the quality of the spectrum. Indeed, through the perturbation, certain spectral subspaces may cease to be pure point and continuous spectrum (presumably singular) may appear (see e.g., [10] and [8]). In this case, we provide estimates on the wave package spreading in terms of the decay rate of W. In particular, we show that if W decays exponentially fast, then the expected radius of the system grows at most logarithmically fast in time. Moreover, if $W = \mathcal{O}(1/|\mathbf{x}|^p)$ for some $p > 2\sigma$, then $\langle \varphi(t), |\mathbf{x}|\, \varphi(t)\rangle$ grows at most as t^θ with $\theta = (p-\sigma)^{-1} < 1$ (see Theorem 1.8, below).

In order to prove that, we show on the one hand, that one can control the growth of the radius (1) in terms of the expected, time-dependent, angular momentum. (This is the actual content of Theorem 1.5.) On the other hand, in Theorem 1.7, we provide estimates on the growth in time of the angular momentum operator in terms of the decay rate of W. As an essential tool, we use certain novel tunnelling estimates (see Theorem 1.3) which are in turn derived from fairly general exponential decay estimates for the spectral projections $E_I(H)$ given in Theorm 3.1.

The above discussion roughly summarizes our main results. We emphasize that we are not aware of other localisation bounds of this type in such situations (perturbed dense point spectrum) for deterministic systems. Notice, however, that the subject is frequently addressed in the realm of random Schrödinger operators. Although in these cases the randomness of the potential plays a fundamental role in the proofs of localisation (see, e.g., [13, 2]).

This paper is organized as follows: In the rest of this section we describe precisely the model and state most of our main results. We show theorems 1.5 and 1.7 in Section 2. In Section 3 we state and prove the exponential decay estimates for the spectral projections Theorem 3.1. Finally, in Section 4, we apply the latter theorem to the model at hand and show the tunnelling estimates stated in Theorem 1.3.

1.1. **The model and main results.** Let us introduce the Hamiltonian H_0 of a quantum particle moving in $\mathbb{R}^2$ that is interacting with a magnetic field $\mathbf{B}$ pointing perpendicularly to the plane. We denote by $\mathbf{A} = (A_1, A_2) : \mathbb{R}^2 \to \mathbb{R}^2$ a magnetic vector potential associated to the magnetic field through $\mathbf{B} = (\partial_1 A_2 - \partial_2 A_1)\hat{x}_3$. Throughout this work we use units such that $\hbar = 2m = 1$, where m is the mass of the particle. For $\mathbf{A} \in L^2_{\rm loc}(\mathbb{R}^2, \mathbb{R}^2)$ we define the sesquilinear form

$$q_0(\varphi,\psi) = \int_{\mathbb{R}^2} \overline{(-i\nabla - \mathbf{A}(\mathbf{x}))\varphi(\mathbf{x})}(-i\nabla - \mathbf{A}(\mathbf{x}))\psi(\mathbf{x})\mathrm{d}\mathbf{x}\,, \qquad \varphi,\psi \in \mathcal{D}(q_0)\,, \tag{2}$$

with domain

$$\mathcal{D}(q_0) = \left\{\varphi \in L^2(\mathbb{R}^2) \,|\, (-i\partial_j - A_j)\varphi \in L^2(\mathbb{R}^2)\,, j \in \{1,2\}\right\}\,. \tag{3}$$

It is well known [20] that $C_0^\infty(\mathbb{R}^2)$ is a form core for q_0. We denote by H_0 the self-adjoint operator corresponding to q_0 and by $\mathcal{D}(H_0) \subset \mathcal{D}(q_0)$ its domain.

We are interested in the particular case in which H_0 describes the dynamics of a particle in a rotationally symmetric magnetic field $\mathbf{B}(\mathbf{x}) = B(|\mathbf{x}|)\hat{x}_3$. We choose the Poincaré gauge where (here $r := |\mathbf{x}|$, as usual)

$$\mathbf{A}(\mathbf{x}) = A(r)\hat{\theta} = \frac{\Phi(r)}{r^2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad \text{with} \quad \Phi(r) = A(r)r = \int_0^r B(s)s\,\mathrm{d}s\,. \tag{4}$$

We will show that this choice of vector potential is locally square integrable, whenever the magnetic field is, see Lemma A.1 in the appendix. Notice that $\Phi(r)$ is, up to factor of 2π, the magnetic flux through a disc of radius $r > 0$ centered at the origin.

One can show, see the discussion in Appendix A, that the quadratic form q_0 corresponding to H_0 is given by

$$q_0(\varphi,\psi) = \big\langle \partial_r\varphi, \partial_r\psi\big\rangle + \big\langle \frac{1}{r}(\Phi - L)\varphi, \frac{1}{r}(\Phi - L)\psi\big\rangle \tag{5}$$

for all $\varphi,\psi \in \mathcal{D}(q_0)$ since the magnetic flux Φ is radial. Here $\partial_r = \frac{x}{|x|}\cdot\nabla$ is the radial derivative and $L = -i(x_1\partial_2 - x_2\partial_1)$ is the generator of rotations in $\mathbb{R}^2$.

Identifying the underlying Hilbert space $L^2(\mathbb{R}^2)$ with $\mathcal{H} := L^2(\mathbb{R}^+\times\mathbb{S}^1, r\mathrm{d}r\,\mathrm{d}\theta)$ through the transformation

$$\mathcal{U} : L^2(\mathbb{R}^2) \to \mathcal{H}\,, \tag{6}$$

with $\psi \mapsto \mathcal{U}\psi = \tilde{\psi}$, where $\tilde{\psi}(r,\theta) = \psi(r\cos\theta, r\sin\theta)$, we define the self-adjoint angular momentum operator $J = \mathcal{U}L\mathcal{U}^{-1}$. It is easy to see that

$$J\widetilde{\varphi} := -i\frac{\partial}{\partial\theta}\widetilde{\varphi} \tag{7}$$

when $\widetilde{\varphi} = \mathcal{U}\varphi$. In this coordinates we have, for any $\varphi,\psi \in \mathcal{D}(q_0) = \mathcal{Q}(H_0)$,

$$q_0(\varphi,\psi) = \big\langle \partial_r\tilde{\varphi}, \partial_r\tilde{\psi}\big\rangle_{\mathcal{H}} + \big\langle \tfrac{1}{r}(\Phi - J)\tilde{\varphi}, \tfrac{1}{r}(\Phi - J)\tilde{\psi}\big\rangle_{\mathcal{H}}\,, \tag{8}$$

where $\tilde{\varphi} = \mathcal{U}\varphi$ and $\tilde{\psi} = \mathcal{U}\psi$ with the unitary $\mathcal{U} : L^2(\mathbb{R}^2) \to \mathcal{H}$ given in (6).

Notice that the spectrum of J coincides with the set of integers $\mathbb{Z}$. We define P_j to be the eigenprojection onto the subspace of $\mathcal{H}$ with fixed angular momentum $j \in \mathbb{Z}$. It is well known that the family $(P_j)_{j\in\mathbb{Z}}$ gives a complete decomposition of the Hilbert space $\mathcal{H}$ into subspaces which diagonalize q_0: Expanding in Fourier series we have $\mathcal{U}\varphi(r,\theta) = (2\pi)^{-1/2}\sum_{j\in\mathbb{Z}}\varphi_j(r)e^{ij\theta}$ and similarly for ψ and using (8) gives

$$\begin{aligned} q_0(\varphi,\psi) &= \sum_{j\in\mathbb{Z}}\Big(\big\langle \partial_r\varphi_j, \partial_r\psi_j\big\rangle_{L^2(\mathbb{R}_+,rdr)} + \big\langle \tfrac{1}{r}(\Phi - j)\varphi_j, \tfrac{1}{r}(\Phi - j)\psi_j\big\rangle_{L^2(\mathbb{R}_+,rdr)}\Big) \\ &= \sum_{j\in\mathbb{Z}}\Big(\big\langle \partial_r\varphi_j, \partial_r\psi_j\big\rangle_{L^2(\mathbb{R}_+,rdr)} + \big\langle \varphi_j, V_j\psi_j\big\rangle_{L^2(\mathbb{R}_+,rdr)}\Big) \\ &=: \big\langle \partial_r\varphi, \partial_r\psi\big\rangle + \big\langle \varphi, V\psi\big\rangle \end{aligned} \tag{9}$$

where $V_j(r) := \frac{1}{r^2}(\Phi(r) - j)^2$ is the *effective potential.* We will frequently identify φ_j with $P_j\mathcal{U}\varphi$ and similarly for ψ_j.

We consider electric perturbations of H_0 through a potential W which is not necessarily rotationally symmetric but satisfies the following smoothness condition in the angular variable.

Condition 1. *There are constants $a > 0$, $0 < \zeta \leqslant 1$, and a function $v \in L^2(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ such that for all $j \in \mathbb{Z}$ and almost every $r > 0$*

$$\left|\widehat{W}(r,j)\right| \leqslant b(r) e^{-a|j|^\zeta}\,, \tag{10}$$

where $b(|\mathbf{x}|) = v(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^2$ and, for $j \in \mathbb{Z}$,

$$\widehat{W}(r,j) \coloneqq \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} W(r,\theta) e^{-ij\theta} \mathrm{d}\theta\,, \quad \text{for a.e.} \quad r > 0\,.$$

is the Fourier transform of the potential W in the angular variable.

Remarks 1.1. (i) The above condition is an analyticity condition in terms of the Gevrey scale in the angular variable θ. In particular, if $\zeta = 1$, the above condition is precisely the analyticity W in θ, for almost every $r > 0$, which is also clear from the familiar Paley–Wiener theorem. Such an analyticity condition is not unusual, see [11] and [19].

(ii) In view of the diamagnetic inequality, see Theorem 2.4 and Theorem 2.5 from [3], we see that W is infinitesimally H_0-bounded in the operator and form sense. In particular, we have that for any $\varepsilon > 0$ there exist constants $C(\varepsilon) > 0$ such that

$$\|W\varphi\|^2 \leqslant \varepsilon \|H_0\varphi\|^2 + C(\varepsilon)\|\varphi\|^2\,, \quad \varphi \in \mathcal{D}(H_0) \tag{11}$$

and

$$\left|\int_{\mathbb{R}^2} W(\mathbf{x}) |\varphi(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\right| \leqslant \varepsilon\, q_0[\varphi] + C(\varepsilon)\|\varphi\|^2\,, \quad \varphi \in \mathcal{D}(q_0)\,. \tag{12}$$

(iii) Notice that the diamagnetic inequality (form bounded with respect to the nonmagnetic kinetic energy implies the same for the magnetic kinetic energy) in this gauge, is an easy consequence of our analysis of the magnetic quadratic form, see Appendix A: Lemma A.5 and Remark A.6.

In the work at hand we study the dynamics of a quantum particle governed by the Hamiltonian

$$H\varphi \coloneqq H_0\varphi + W\varphi\,, \qquad \varphi \in \mathcal{D}(H) = \mathcal{D}(H_0)\,. \tag{13}$$

In view of Remark 1.1.ii and the Kato-Rellich theorem the operator H is bounded from below and self-adjoint.

For φ, ψ in the form domain of W, $\mathcal{Q}(W)$, one checks that as quadratic forms

$$\langle \varphi, W\psi \rangle = \sum_{j,k\in\mathbb{Z}} \left\langle \varphi_j, \widehat{W}(\cdot, j-k)\psi_k \right\rangle_{L^2(\mathbb{R}_+, rdr)} \tag{14}$$

and thus the quadratic form of the magnetic Schrödinger operator $H = H_0 + W$ is given by

$$\begin{aligned} q(\varphi,\psi) &= \langle \varphi, H\psi\rangle \\ &= \sum_{j\in\mathbb{Z}} \Big(\langle \partial_r\varphi_j, \partial_r\psi_j\rangle_{L^2(\mathbb{R}_+,rdr)} + \langle \varphi_j, V_j\psi_j\rangle_{L^2(\mathbb{R}_+,rdr)}\Big) \\ &\quad + \sum_{j,k\in\mathbb{Z}} \langle \varphi_j, \widehat{W}(\cdot, j-k)\psi_k\rangle_{L^2(\mathbb{R}_+,rdr)} \\ &= \langle \partial_r\varphi, \partial_r\psi\rangle + \langle \varphi, V\psi\rangle + \langle \varphi, W\psi\rangle\,. \end{aligned} \tag{15}$$

see also the discussion in Appendix A for more details.

In order to state our results in a concise way we will work separately with the following two conditions on the magnetic flux Φ given in (4).

Condition 2. *Let $\Phi \in L^2_{\mathrm{loc}}(\mathbb{R}^+, dr/r)$ such that there are constants $\lambda_+ < \infty$ and $\sigma_+ > 1$ such that*

$$|\Phi(r)| \leqslant \lambda_+(1 + r^{\sigma_+})\,, \tag{16}$$

for all $r > 0$.

Condition 3. *Let $\Phi \in L^2_{\mathrm{loc}}(\mathbb{R}^+, dr/r)$ be such that there are constants $r_0 > 1$, $\lambda_- > 0$ and $\sigma_- > 1$ such that*

$$|\Phi(r)| \geqslant \lambda_-\, r^{\sigma_-} \quad r \geqslant r_0\,. \tag{17}$$

Remark 1.2. It is interesting to consider the model case where the magnetic field is bounded and asymptotically decays to zero as

$$B(r) \sim r^{-\alpha}, \qquad \text{for } r \to \infty\,, \tag{18}$$

for some $\alpha < 1$. Here conditions 2 and 3 are fulfilled with $\sigma_+ = \sigma_- = 2 - \alpha$.

Our results concern the time evolution of a state φ with energy on a bounded interval $I \subset \mathbb{R}$. In order to state them we introduce some notation that will be used throughout this work. Let $e_0 := \inf \mathrm{spec}(H)$, be the infimum of the spectrum of H and let $E_0 \in (e_0, \infty)$ be a fixed constant. We set

$$I := [e_0, E_0]$$

and denote by $E_I(H)$ the spectral projection of H onto the interval I. Let $U(t) := e^{-itH}$ be the time evolution operator associated to H. For any initial state $\varphi \in L^2(\mathbb{R}^2)$ we denote by $\varphi(t) := U(t)\varphi$ the state of the system at time t.

We are now ready to state our main results.

Theorem 1.3. *Assume that Condition 1 is satisfied.*

i) (*Interior tunnelling estimates*). *Under Condition 2 there exist constants $c_+ \in (0,1]$ and $\delta_+ > 0$ such that*

$$\sum_{j\in\mathbb{Z}} e^{\delta_+|j|^\zeta} \|\mathbb{1}_{[0,c_+|j|^{\zeta/\sigma_+}]}(|\mathbf{x}|)P_jE_I(H)\|^2 < \infty\,. \tag{19}$$

ii) (Exterior tunnelling estimates). Under Condition 3 there exist constants $c_- \geqslant r_0$ and $\delta_- > 0$ such that

$$\sum_{j\in\mathbb{Z}} \|\mathbb{1}_{[c_-|j|^{\zeta/\sigma_-},\infty)}(|\mathbf{x}|)\, e^{\delta_-|\mathbf{x}|^{\zeta\sigma_-}}\, P_j E_I(H)\|^2 < \infty\,. \tag{20}$$

Here $\mathbb{1}_A$ stands for the indicator function of the set A and σ_- and σ_+ are the parameters given in condition 2 and 3.

Remarks 1.4. (i) The interior and exterior tunnelling bounds above show *strong decay* of the spectral projection $E_I(H)$ for finite energy intervals I into the classically forbidden region. They are derived from the exponential decay of the energy projections described in Section 3. Remarkably, these bounds are valid in a regime, where the unperturbed operator H_0 has dense point spectrum.

(ii) The bound (19) also immediately implies that there exists a positive constant C such that

$$\left\|\mathbb{1}_{[0,c_+|j|^{\zeta/\sigma_+}]}(|\mathbf{x}|)\, P_j E_I(H)\right\| \leqslant C\, e^{-\delta_+|j|^{\zeta}}\,, \qquad j\in\mathbb{Z}\,. \tag{21}$$

(iii) Consider the example of Remark 1.2. Theorem 1.3 indicates that a wave-function with energies in the interval I and angular momentum $j\in\mathbb{Z}$, $P_jE_I(H)\varphi$, is essentially localized in the annulus between $c_+|j|^{1/(2-\alpha)}$ and $c_-|j|^{1/(2-\alpha)}$. This is in fact the scale where the classically allowed region (see (35) below) for $P_jE_I(H)\varphi$ is located. For further details see Section 4, where the proof of Theorem 1.3 is given.

Our next theorem states that the expectation of $|\mathbf{x}|$ in time, is dominated by the expectation of the angular momentum operator in time to certain power. This power depends on the behavior of the magnetic flux far from the origin, see Condition 3.

Theorem 1.5 ($J(t)$ controls $x(t)$)**.** *Let H be the Hamiltonian defined in (13) and assume that conditions 1 and 3 are satisfied. Then, for any $\nu > 0$, there exists a constant $C > 0$ such that for all initial states $\varphi \in E_I(H)L^2(\mathbb{R}^2)$*

$$\big\langle\varphi(t), |\mathbf{x}|^{\nu}\varphi(t)\big\rangle \leqslant C\left(\|\varphi\|^2 + \big\langle\varphi(t), |J|^{\zeta\nu/\sigma_-}\varphi(t)\big\rangle\right)\,, \quad \text{for} \quad t\in\mathbb{R}\,. \tag{22}$$

Remarks 1.6. (i) Assume further that W is rotationally symmetric. Then, the time evolution $U(t)$ commutes with $|J|$ (i.e., angular momentum is conserved), hence Theorem 1.5 implies dynamical localization for any $\varphi\in\mathcal{D}(|J|^{\zeta\nu/(2\sigma_-)})$, i.e.,

$$\sup_{t\geqslant 0}\big\langle\varphi(t), |\mathbf{x}|^{\nu}\varphi(t)\big\rangle < \infty\,.$$

(ii) The main point of Theorem 1.5 is the dynamical estimate when $E_0 \geqslant 0$. If $E_0 < 0$ then one can easily prove dynamical localization since the spectrum of H below zero is discrete.

In order to formulate the next theorem we define the symmetric and non-symmetric parts of the potential W by writting

$$W = W_{\rm s} + W_{\rm ns}\,,$$

where $W_{\rm s}$ is the radially symmetric part of W given, for almost all $r > 0$, by

$$W_{\rm s}(r) := \frac{1}{2\pi} \int_0^{2\pi} W(r, \theta) \mathrm{d}\theta \,.$$

If we assume some further decay in space for $W_{\rm ns}$, we obtain bounds for the expectation of the angular momentum in time. We state our results for two different classes of decay of $W_{\rm ns}$.

Theorem 1.7 (Bounds on $J(t)$)**.** *Assume that conditions 1 and 2 are satisfied.*

(*i*) *Suppose that for $p > \sigma_+/\zeta$*

$$W_{\rm ns}(\mathbf{x}) = \mathcal{O}\left(\frac{1}{|\mathbf{x}|^p}\right), \qquad |\mathbf{x}| \to \infty \,. \tag{23}$$

Then, for any $0 < \beta < (\zeta p - \sigma_+)/\sigma_+$ there exists $C > 0$ such that for all initial states $\varphi \in E_I(H)L^2(\mathbb{R}^2)$ we have

$$\| |J|^{\beta/2} \varphi(t) \|^2 \leqslant \| |J|^{\beta/2} \varphi \|^2 + C\, t^{\gamma\beta} \, \|\varphi\|^2 \,, \qquad t > 1 \,, \tag{24}$$

where $\gamma = \frac{\sigma_+}{\zeta p - \sigma_+}$.

(*ii*) *Suppose that there exists $\mu > 0$ and $s > 0$ such that*

$$W_{\rm ns}(\mathbf{x}) = \mathcal{O}\big(\exp(-\mu |\mathbf{x}|^s)\big), \qquad |\mathbf{x}| \to \infty \,. \tag{25}$$

Then, for any $\beta > 0$, there exists $C > 0$ such that for all initial states $\varphi \in E_I(H)L^2(\mathbb{R}^2)$ we have

$$\| |J|^{\beta/2} \varphi(t) \|^2 \leqslant \| |J|^{\beta/2} \varphi \|^2 + C\left(\ln(t)^{\theta\beta} + 1\right) \|\varphi\|^2 \,, \qquad t > 1 \,, \tag{26}$$

where $\theta = 1/\min\{\zeta, \zeta s/\sigma_+\}$.

The proofs of Theorem 1.5 and Theorem 1.7 are given in Section 2. Let us emphasize that while Theorem 1.5 uses Condition 3 through the exterior tunnelling estimate (20), Theorem 1.7 requires Condition 2 in order to apply (19). The next result is a direct combination of Theorem 1.5 and Theorem 1.7.

Theorem 1.8 (Bounds on $x(t)$)**.** *Assume that conditions 1 and 2, and 3 are satisfied with $0 < \zeta \leqslant 1$, $1 < \sigma_- \leqslant \sigma_+$. Then*

(*i*) *Assume that $W_{\rm ns}(\mathbf{x}) = \mathcal{O}(\frac{1}{|\mathbf{x}|^p})$, as $|\mathbf{x}| \to \infty$, for some $p > \sigma_+/\zeta$. Then, for any $0 < \nu < \frac{\sigma_-}{\zeta}(\frac{\zeta p - \sigma_+}{\sigma_+})$, there exists $C > 0$ such that, for all $\varphi \in E_I(H)L^2(\mathbb{R}^2)$, we have*

$$\big\langle \varphi(t), |\mathbf{x}|^\nu \varphi(t) \big\rangle \leqslant C\left(\big\langle \varphi, |J|^{\zeta\nu/\sigma_-} \varphi \big\rangle + t^{\varepsilon_p} \|\varphi\|^2 \right) \,, \qquad t > 1 \,. \tag{27}$$

where $\varepsilon_p = \frac{\zeta}{\sigma_-}(\frac{\sigma_+}{\zeta p - \sigma_+})\nu < 1$.

(*ii*) *Suppose that there exists $\mu > 0$ and $s > 0$ such that $W_{\rm ns}(\mathbf{x}) = \mathcal{O}(\exp(-\mu|\mathbf{x}|^s)$, as $|\mathbf{x}| \to \infty$. Then, for any $\nu > 0$, there exists $C > 0$ such that, for all $\varphi \in E_I(H)L^2(\mathbb{R}^2)$, we have*

$$\big\langle \varphi(t), |\mathbf{x}|^\beta \varphi(t) \big\rangle \leqslant C\left(\big\langle \varphi, |J|^{\zeta\nu/\sigma_-} \varphi \big\rangle + \ln(t)^{\theta_s} \|\varphi\|^2 \right) \,, \qquad t > 1 \,, \tag{28}$$

where $\theta_s = \frac{1}{\sigma_- \min\{1, s/\sigma_+\}}\, \nu$.

2. From tunnelling estimates to dynamical bounds

We start with the proof of Theorem 1.7 and consider Theorem 1.5 at the end of this section. The proof of Theorem 1.7 is based upon certain dynamical bounds, that use Heisenberg's equation for $J(t)$, combined with the tunnelling estimates given in Remark 1.4.ii. Before proceeding with the proof of Theorem 1.7 we establish the following.

Lemma 2.1. *Assume that conditions 1 and 2 are satisfied. Then, there is a constant* $C_{\mathrm{ns}} \in (0,\infty)$ *such that*

$$\|E_I(H)W_{\mathrm{ns}}\| \leqslant C_{\mathrm{ns}}\,.$$

Proof. In view of Equation (10), for $j=0$, and Remark 1.1.ii we see that the inequality (11) holds for W_{ns} as well as for W_{s}. This implies that $\mathcal{D}(W_{\mathrm{ns}}) \supseteq \mathcal{D}(H_0) = \mathcal{D}(H)$. By the Closed Graph Theorem, we have that $W_{\mathrm{ns}}(H+\lambda)^{-1}$ is a bounded operator, for some $\lambda > -e_0$. We conclude by observing that

$$\|W_{\mathrm{ns}}E_I(H)\| \leqslant \left\|W_{\mathrm{ns}}(H+\lambda)^{-1}\right\| \, \|(H+\lambda)E_I(H)\| < \infty\,.$$

$\square$

Proof of Theorem 1.7. For any $\varphi \in E_I(H)L^2(\mathbb{R}^2)$ and $M>0$ we have

$$\left\||J|^{\beta/2}\varphi(t)\right\|^2 \leqslant M^\beta \|\varphi\|^2 + \sum\nolimits_{|j|>M} |j|^\beta \|P_j\varphi(t)\|^2\,. \tag{29}$$

Using Heisenberg's evolution equation we get

$$\|P_j\varphi(t)\|^2 = \|P_j\varphi\|^2 + i\int_0^t \langle \varphi(s), [W,P_j]\,\varphi(s)\rangle\,\mathrm{d}s\,.$$

Notice that $[W,P_j]=[W_{\mathrm{ns}},P_j]$. The above equation combined with (29) yields

$$\begin{aligned}\left\||J|^{\beta/2}\varphi(t)\right\|^2 &\leqslant \left\||J|^{\beta/2}\varphi\right\|^2 + M^\beta\|\varphi\|^2 + \sum_{|j|>M}|j|^\beta \int_0^t |\langle\varphi(s),[W_{\mathrm{ns}},P_j]\,\varphi(s)\rangle|\,\mathrm{d}s \\ &\leqslant \left\||J|^{\beta/2}\varphi\right\|^2 + M^\beta\|\varphi\|^2 + 2\sum_{|j|>M}|j|^\beta \int_0^t |\langle\varphi(s),W_{\mathrm{ns}}P_j\varphi(s)\rangle|\,\mathrm{d}s \\ &\leqslant \left\||J|^{\beta/2}\varphi\right\|^2 + M^\beta\|\varphi\|^2 + 2t\sum_{|j|>M}|j|^\beta \|E_I(H)W_{\mathrm{ns}}P_jE_I(H)\|\,\|\varphi\|^2\,.\end{aligned} \tag{30}$$

Using Lemma 2.1 we can estimate the norm appearing in the last sum as

$$\begin{aligned}&\|E_I(H)W_{\mathrm{ns}}P_jE_I(H)\| \\ &\leqslant \left\|E_I(H)W_{\mathrm{ns}}\mathbb{1}_{(0,c_+|j|^{\zeta/\sigma_+})}(|\mathbf{x}|)P_jE_I(H)\right\| + \left\|E_I(H)W_{\mathrm{ns}}\mathbb{1}_{(c_+|j|^{\zeta/\sigma_+},\infty)}(|\mathbf{x}|)P_jE_I(H)\right\| \\ &\leqslant \|E_I(H)W_{\mathrm{ns}}\|\,\left\|\mathbb{1}_{(0,c_+|j|^{\zeta/\sigma_+})}(|\mathbf{x}|)P_jE_I(H)\right\| + \left\|W_{\mathrm{ns}}\mathbb{1}_{(c_+|j|^{\zeta/\sigma_+},\infty)}(|\mathbf{x}|)\right\| \\ &\leqslant C\,\|E_I(H)W_{\mathrm{ns}}\|\,e^{-\delta|j|^\zeta} + \left\|W_{\mathrm{ns}}\mathbb{1}_{(c_+|j|^{\zeta/\sigma_+},\infty)}(|\mathbf{x}|)\right\|\,,\end{aligned} \tag{31}$$

where in the last step we used Remark 1.4.ii.

We now study separately the two cases depending on the decay rate of the potential. *Case* (i): Assume that $W_{\rm ns}$ decays as in (23). Then for all sufficiently large $M>0$

$$\left\| W_{\rm ns} \mathbb{1}_{(c_+|j|^{\zeta/\sigma_+},\infty)}(|\mathbf{x}|)\right\| \lesssim |j|^{-\zeta p/\sigma_+}\,, \quad |j|>M\,. \tag{32}$$

Combining this bound with (31), using that $e^{-\delta|j|^\zeta} \leqslant |j|^{-\zeta p/\sigma_+}$ for large $|j|$ this implies

$$\sum_{|j|>M} |j|^\beta \,\|E_I(H)W_{\rm ns}P_jE_I(H)\| \lesssim \sum_{|j|>M} |j|^{\beta-\zeta p/\sigma_+} \lesssim M^{\beta+1-\zeta p/\sigma_+}\,. \tag{33}$$

Since $\beta-\zeta p/\sigma_+<-1$. This together with (30) implies that there exists a constant $C>0$ such that

$$\left\||J|^{\beta/2}\varphi(t)\right\|^2 \leqslant \left\||J|^{\beta/2}\varphi\right\|^2 + M^\beta\,\|\varphi\|^2 + CtM^{\beta+1-\zeta p/\sigma_+}\,\|\varphi\|^2\,.$$

The theorem holds provided we pick $M(t)=t^{\frac{\sigma_+}{\zeta p-\sigma_+}}$.
Case (ii): We now turn to the case when $W_{\rm ns}$ decays as in (25). Then analogously as above we conclude for all $|j|$ large enough

$$\|E_I(H)W_{\rm ns}P_jE_I(H)\| \lesssim \exp(-\delta|j|^\zeta)+\exp(-\mu(c_+)^s|j|^{s\zeta/\sigma_+}) \lesssim \exp(-\eta|j|^\kappa)\,,$$

for $\kappa:=\min\{\zeta, s\zeta/\sigma_+\}$ and $\eta=\min\{\delta,\mu(c_+)^s\}>0$. Hence

$$\sum_{|j|>M} |j|^\beta\,\|E_I(H)W_{\rm ns}P_jE_I(H)\| \lesssim \exp(-\tfrac{\eta}{2}M^\kappa)\,. \tag{34}$$

for all large enough $M>0$. This together with (30) yields

$$\left\||J|^{\beta/2}\varphi(t)\right\|^2 \leqslant \left\||J|^{\beta/2}\varphi\right\|^2 + M^\beta\,\|\varphi\|^2 + 2Ct\exp(-\tfrac{\eta}{2}M^\kappa)\,\|\varphi\|^2\,.$$

for some constant $C>0$. Finally, the claim follows by picking $M(t)=(2\ln(t)/\eta)^{1/\kappa}$. □

Remark 2.2. The proof of Theorem 1.5 is a rigorous implementation of the intuition that the component of the wave–function with angular momentum j, $P_j\varphi$, moves under the influence of an effective potential V_j whose classical region $\{x\in\mathbb{R}^2 : V_j(x)\leqslant E\}$ is concentrated inside an annular region of inner radius $\sim |j|^{1/\sigma_+}$ and outer radius $\sim |j|^{1/\sigma_-}$ (see Theorem 1.3 and Remark 1.4.iii).

Proof of Theorem 1.5: Let $t\geqslant 0$ and consider the following splitting

$$\left\||\mathbf{x}|^{\nu/2}\varphi(t)\right\|^2 = \sum_{j\in\mathbb{Z}}\left\|\mathbb{1}_{[0,c_-|j|^{\zeta/\sigma_-}]}\,|\mathbf{x}|^{\nu/2}P_jE_I(H)\varphi(t)\right\|^2 + \sum_{j\in\mathbb{Z}}\left\|\mathbb{1}_{[c_-|j|^{\zeta/\sigma_-},\infty)}\,|\mathbf{x}|^{\nu/2}P_jE_I(H)\varphi(t)\right\|^2$$

where c_- is the constant given by Theorem 1.3. For notational simplicity, we will drop the argument in the indicator function and simply write $\mathbb{1}_A\equiv\mathbb{1}_A(|\mathbf{x}|)$ in the following. The first sum may be estimated in terms of the expectation values of the angular momentum as

$$\sum_{j\in\mathbb{Z}}\left\|\mathbb{1}_{[0,c_-|j|^{\zeta/\sigma_-}]}\,|\mathbf{x}|^{\nu/2}P_jE_I(H)\varphi(t)\right\|^2 \leqslant c_-^\nu\,\left\||J|^{\nu\zeta/(2\sigma_-)}\varphi(t)\right\|^2\,.$$

As for the second sum, we may use Equation (20) from Theorem 1.3 to obtain

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \left\| \mathbb{1}_{[c_-|j|^{\zeta/\sigma_-},\infty)} \, |\mathbf{x}|^{\nu/2} P_j E_I(H)\, \varphi(t) \right\|^2 \\ \leqslant \left\| |\mathbf{x}|^{\nu/2} e^{-\delta|\mathbf{x}|^{\zeta\sigma_-}} \right\|^2 \sum_{j\in\mathbb{Z}} \| \mathbb{1}_{[c_-|j|^{\zeta/\sigma_-},\infty)} \, e^{\delta|\mathbf{x}|^{\zeta\sigma_-}} \, P_j E_I(H) \|^2 \, \|\varphi\|^2 \,.\end{aligned}$$

This completes the proof of the theorem. □

3. Exponential decay of the energy projections

An essential tool in our approach are certain exponential decay estimates for the spectral projections $E_I(H)$ in the variables j and r. They enable us to control the tunnelling effect away from the classically allowed region. Note that since the perturbation W is not rotationally symmetric, H and J cannot be simultaneously diagonalized. This combined with the fact that the unperturbed operator has dense point spectrum makes such decay bounds trickier to deal with.

It turns out that it suffices to work with the classical region associated to the unperturbed magnetic operator H_0, which is well defined for fixed angular momentum j. The analysis in this section is given for general magnetic fields which are rotationally symmetric such that the magnetic vector potential vector potential in the Poincaré gauge is locally square integrable.

For a given energy $E > 0$ and fixed $j \in \mathbb{Z}$ we define the *classically allowed region* for angular momentum $j \in \mathbb{Z}$ as the set

$$\mathcal{C}_j(E) := \{ r \in \mathbb{R}^+ \, : \, V_j(r) \leqslant E \} \,. \tag{35}$$

where $V = (V_j)_{j\in\mathbb{Z}}$ is the effective potential. Moreover, let $\chi_j(E) : \mathbb{R}^+ \to [0,1]$, with $\chi_j(E) = 1$ on $\mathcal{C}_j(E)$ and $\chi_j(E) = 0$ otherwise, be the indicator function on $\mathcal{C}_j(E)$. We also set $\chi_j^{\perp}(E) := 1 - \chi_j(E)$.

Recall the constants $a > 0$ and $\zeta \in (0,1]$ are defined through Condition 1 and write $\xi(a,\zeta) = \sum_{m\in\mathbb{Z}} e^{-\frac{a}{2}|m|^\zeta}$. Since v is a radial potential, Lemma A.5 in Appendix A shows that for any $a > 0$ and $0 < \zeta \leqslant 1$ there exist a c_0, such that

$$\big\langle \partial_r \varphi, \partial_r \varphi \big\rangle - \xi(a,\zeta) \big\langle \varphi, v\varphi \big\rangle \geqslant -c_0 \, \|\varphi\|^2 \,, \tag{36}$$

for all φ in the quadratic form domain of the magnetic Schrödinger operator.

We denote by $PC^1_{\mathrm{bd}}(\mathbb{R}^+,\mathbb{R})$ the set of *bounded* functions $f : \mathbb{R}^+ \to \mathbb{R}$ that are continuous and piecewise continuously differentiable. We say that a sequence of non-negative functions $F = (F_j)_{j\in\mathbb{Z}}$ is in $PC^1_{\mathrm{bd}}(\mathbb{R}^+,\mathbb{R})$ whenever $F_j \in PC^1_{\mathrm{bd}}(\mathbb{R}^+,\mathbb{R})$ and

$$\|F\|_\infty := \sup_{j\in\mathbb{Z}} \sup_{r>0} |F_j(r)| < \infty \,. \tag{37}$$

For such sequences we write $e^{\pm F} = \sum_{j\in\mathbb{Z}} e^{\pm F_j} P_j$, which are bounded operators satisfying $\|e^{\pm F}\| \leqslant e^{\|F\|_\infty}$. Moreover, Lemma B.1 shows that if also F' is bounded, i.e.,

$$\|F'\|_\infty := \sup_{j\in\mathbb{Z}} \sup_{r>0} |F_j'(r)| < \infty \tag{38}$$

then $e^{\pm F}\varphi$ is in $\mathcal{D}(q_0) = \mathcal{D}(q)$, the domain of the magnetic Schrödinger, whenever $\varphi \in \mathcal{D}(q_0)$. We also set $\chi(\tilde{E}) := \sum_{j\in\mathbb{Z}} \chi_j(\tilde{E}) P_j$.

Theorem 3.1 (Exponential decay of energy projections)**.** *Let H be the perturbed magnetic Schrödinger operator defined by the quadratic form (15). Given $\delta_0 > 0$ and $E_0 \geqslant 0$ put*

$$\tilde{E} := E_0 + c_0 + \delta_0\,. \tag{39}$$

with c_0 from (36) above. Then, for any sequence of weight functions $F = (F_j)_{j\in\mathbb{Z}}$ in $PC^1_{bd}(\mathbb{R}^+,\mathbb{R})$ that satisfy

$$(F')^2 \leqslant V - \widetilde{E}\chi^\perp(\widetilde{E})\,, \tag{40}$$

$$\|e^F\chi(\widetilde{E})\|_\infty < \infty\,, \tag{41}$$

$$\sup_{r>0} |F_j(r) - F_k(r)| \leqslant \tfrac{a}{2}\,|j-k|^\zeta\,, \text{ for all } j,k \in \mathbb{Z}\,, \tag{42}$$

there exists $C = C(\delta_0) > 0$ such that

$$\left\|e^F\, E_I(H)\right\| \leqslant C\|e^F\chi(\widetilde{E})\|_\infty\,. \tag{43}$$

If $W = 0$ the bound (43) holds without the requirement of (42).

Moreover, the bound (43) extends to all $F = (F_j)_{j\in\mathbb{Z}}$ in $PC^1(\mathbb{R}^+,\mathbb{R})$ satisfying conditions (40), (41), and (42) without requiring that F is bounded.

Remarks 3.2. (i) Note that the right hand side of (43) stays finite as long as e^F is bounded on $\operatorname{supp}(\chi(\widetilde{E})) = \cup_{j\in\mathbb{Z}} \operatorname{supp}(\chi_j(\widetilde{E}))$. Thus, we may approximate unbounded weight functions F, which may grow in both variables $r > 0$ and $j \in \mathbb{Z}$, by bounded ones and deduce from (43) that $\left\|e^F\, E_I(H)\right\|$ is *finite* as long as F is bounded on the support of $\chi(\widetilde{E})$. In particular, we will choose a family (F_j) such that it provides exponential decay estimates of $E_I(H)$ away from the classical regions $\mathcal{C}_j(\widetilde{E})$. The difference $\widetilde{E} - E_0 > 0$, is a price we pay for defining the classical region with respect to the operator H_0 instead of H.

(ii) In the case when $W = 0$ we may choose $c_0 = 0$ and we have almost optimality in the energy $\widetilde{E} = E_0 + \delta_0$.

(iii) Our proof of Theorem 3.1 borrows ideas from [12] which are, in turn, inspired by the proof of exponential decay in QED systems given in [4] and the beautiful approach to exponential bounds for eigenfunctions in [1].

As a first step we define a more convenient (smooth) version of the spectral projection E_I, denoted by $g_\triangle(H)$, which is given in formula (47) below. The constants $\widetilde{E}, E_0, \delta_0$ and c_0 are as in Theorem 3.1.

Set $\triangle := [e_0 - \delta_0/2, E_0 + \delta_0/2]$, where $e_0 = \inf \sigma(H)$. Consider $g_\triangle \in C_0^\infty(\mathbb{R}, [0,1])$ such that $\operatorname{supp} g_\triangle \subseteq \triangle$ and $g_\triangle|_I = 1$. Since $E_I(H) = g_\triangle(H)E_I(H)$ it is enough to prove the bound (43) with $E_I(H)$ replaced by $g_\triangle(H)$.

Next, we use the almost analytic functional calculus (see [7]) to write $g_\triangle(H)$ in terms of an integral over the resolvent of H. We denote by $\tilde{g}_\triangle$ an almost analytic extension of $g_\triangle$ with the property that $\operatorname{supp}(\tilde{g}_\triangle)$ is a compact subset of $\triangle + i\mathbb{R}$ and

$$|\partial_{\bar{z}}\tilde{g}_\triangle(z)| = \mathcal{O}(|\mathrm{Im}(z)|), \quad \mathrm{Im}(z) \to 0\,. \tag{44}$$

One can give a straightforward explicit construction of $\tilde{g}_\triangle$ with the above properties, however, see [7, 12] for details. Then, we have the formula

$$g_\triangle(H) = -\frac{1}{\pi}\int (z-H)^{-1}\,\partial_{\bar{z}}\tilde{g}_\triangle(z)\,\mathrm{d}x\,\mathrm{d}y\,. \tag{45}$$

for any self-adjoint operator H. We work with the comparison operator, more precisely, with the quadratic form corresponding to

$$\widetilde{H} \coloneqq H + \widetilde{E}\chi(\widetilde{E})\,,$$

Notice that $\widetilde{H}$ is just H except that it is boosted by $\widetilde{E}$ in (a neighborhood of) the classical region, i.e., as quadratic forms

$$\langle \varphi, \widetilde{H}\varphi\rangle = \langle \partial_r\varphi, \partial_r\varphi\rangle + \langle \varphi, (V + \widetilde{E}\chi(\widetilde{E}))\varphi\rangle + \langle \varphi, W\varphi\rangle \tag{46}$$

for all $\varphi \in \mathcal{D}(q_0)$

It is easy to check, see Lemma 3.3 below, that $\widetilde{H} \geqslant E_0 + \delta_0$. Since $\operatorname{supp} g_\triangle \subset \triangle$, we have that $g_\triangle(\widetilde{H}) = 0$ and, therefore, by the resolvent identity and (45)

$$g_\triangle(H) = g_\triangle(H) - g_\triangle(\widetilde{H}) = -\frac{1}{\pi}\int (z-\widetilde{H})^{-1}(\widetilde{H}-H)(z-H)^{-1}\,\frac{\partial\tilde{g}_\triangle}{\partial\bar{z}}\,\mathrm{d}x\,\mathrm{d}y\,. \tag{47}$$

Lemma 3.3. *For the operator $\widetilde{H}$ defined above one has a quadratic forms*

$$\widetilde{H} \geqslant E_0 + \delta_0\,. \tag{48}$$

Furthermore, if the sequence of non-negative functions $F = (F_j)_{j\in\mathbb{Z}}$ in $PC^1_{bd}(\mathbb{R}^+, \mathbb{R})$ satisfies (40), (42), and F' is bounded, then

$$\sup_{z\in\operatorname{supp}(\tilde{g}_\triangle)} \|e^F(z-\widetilde{H})^{-1}e^{-F}\| \leqslant 2/\delta_0\,. \tag{49}$$

where $\delta_0 > 0$ is the parameter from Theorem 3.1.

Remark 3.4. Again, it is essential, that the right hand side of (49) does not depend on any a-priori bound on $\|F\|_\infty$.

Proof. In order to show (48) notice that from (46) we get

$$\begin{aligned}\langle \varphi, \widetilde{H}\varphi\rangle &= \langle \partial_r\varphi, \partial_r\varphi\rangle + \langle \varphi, (V + \tilde{E}\chi(\tilde{E}))\varphi\rangle + \langle \varphi, W\varphi\rangle \\ &\geqslant \langle \partial_r\varphi, \partial_r\varphi\rangle + \langle \varphi, (V + \tilde{E}\chi(\tilde{E}))\varphi\rangle - \xi(2a,\zeta)\langle \varphi, v\varphi\rangle \\ &\geqslant \langle \varphi, (V + \tilde{E}\chi(\tilde{E}) - c_0)\varphi\rangle = \langle \varphi, (V - \tilde{E}\chi^\perp(\tilde{E}) + \widetilde{E} - c_0)\varphi\rangle\end{aligned} \tag{50}$$

where we also used the bound (98) from Lemma B.3 and then (36). Since by assumption $V \geqslant \widetilde{E}\chi_j^{\perp}$ and $\widetilde{E} - c_0 = \delta_0$, the bound (48) follows.

Next we turn to the proof of (49). Since $\|F\|_\infty, \|F'\|_\infty < \infty$, we know that $e^{\pm F}$ are bounded operators $L^2(\mathbb{R}^2)$ which by Lemma B.1 also leave the form domain of H invariant.

Moreover, the operator $\tilde{H}_F := e^F \widetilde{H} e^{-F}$, or better its associated quadratic form, is well defined, see the discussion in Appendix B. It is easy to check that $\widetilde{H}_F - z$ is invertible if z is such that

$$\operatorname{Re}\langle \varphi, (\widetilde{H}_F - z)\varphi\rangle \geqslant \tfrac{\delta_0}{2}\|\varphi\|^2 , \qquad \varphi \in \mathcal{D}(\widetilde{H}_F)\,, \tag{51}$$

and that then also

$$\left\|(\widetilde{H}_F - z)^{-1}\right\| \leqslant \frac{2}{\delta_0}$$

since clearly $\operatorname{Re}\langle \varphi, (\widetilde{H}_F - z)\varphi\rangle \leqslant \left\|(\widetilde{H}_F - z)\varphi\right\| \|\varphi\|$. If (51) holds, then it is also straightforward to show

$$e^F(\widetilde{H} - z)^{-1}e^{-F} = (\widetilde{H}_F - z)^{-1}\,.$$

Thus, to show (49) it suffices to prove that, for any $z \in \operatorname{supp}\tilde{g}_\triangle$, (51) holds. For this we use the exponentially twisted version of (46) provided by (94) in Appendix B which shows that as quadratic forms

$$\begin{aligned}
\operatorname{Re}\langle\varphi, e^F H_0 e^{-F}\varphi\rangle &= \big\langle \partial_r\varphi, \partial_r\varphi\big\rangle + \big\langle \varphi, (V + \widetilde{E}\chi(\widetilde{E}) - (F')^2)\varphi\big\rangle + \operatorname{Re}\big\langle e^F\varphi, We^{-F}\varphi\big\rangle \\
&\geqslant \big\langle \partial_r\varphi, \partial_r\varphi\big\rangle + \big\langle \varphi, (V + \widetilde{E}\chi(\widetilde{E}) - (F')^2))\varphi\big\rangle - \xi(a,\zeta)\big\langle\varphi, v\varphi\big\rangle \\
&\geqslant \big\langle \varphi, (V + \widetilde{E}\chi(\widetilde{E}) - (F')^2 - c_0)\varphi\big\rangle \\
&= \big\langle \varphi, (V - \widetilde{E}\chi^{\perp}(\widetilde{E}) + \widetilde{E} - (F')^2 - c_0)\varphi\big\rangle
\end{aligned}$$

where we also used Lemma B.3 and (36). Thus, using also (40) we get

$$\begin{aligned}
\operatorname{Re}\langle\varphi, e^F H_0 e^{-F}\varphi\rangle &\geqslant \big\langle \varphi, (V - \widetilde{E}\chi^{\perp}(\widetilde{E}) + \widetilde{E} - (F')^2 - c_0)\varphi\big\rangle \\
&= \big\langle \varphi, (V - \widetilde{E}\chi^{\perp}(\widetilde{E}) - (F')^2 + E_0 + \delta_0)\varphi\big\rangle \\
&\geqslant (E_0 + \delta_0)\|\varphi\|^2
\end{aligned}$$

which implies (51) since for all $z \in \operatorname{supp}(\widetilde{g}_\Delta)$ we have $\operatorname{Re} z \leqslant E_0 + \delta_0/2$. This finishes the proof of the lemma. $\square$

Proof of Theorem 3.1. Using equation (47) we may write

$$e^F g_\triangle(H) = -\frac{\widetilde{E}}{\pi}\int e^F(z - \widetilde{H})^{-1}e^{-F}\, e^F\chi\widetilde{E}(z - H)^{-1}\,\frac{\partial\tilde{g}_\triangle}{\partial\bar{z}}\,\mathrm{d}x\,\mathrm{d}y\,, \tag{52}$$

so

$$\left\|e^F g_\triangle(H)\right\| \leqslant \frac{\widetilde{E}}{\pi}\left\|e^F\chi(\widetilde{E})\right\| \int \left\|e^F(z-\widetilde{H})^{-1}e^{-F}\right\| \left\|(z-H)^{-1}\right\| |\partial_{\bar z}\tilde g_\triangle(z)| \,\mathrm{d}x\,\mathrm{d}y\,, \tag{53}$$

$$\leqslant \left\|e^F\chi(\widetilde{E})\right\| \frac{2\widetilde{E}}{\pi\delta_0}\int \frac{|\partial_{\bar z}\tilde g_\triangle(z)|}{|y|}\,dxdy \tag{54}$$

where we also used the standard bound $\|(z-H)^{-1}\| \leqslant 1/|\mathrm{Im}(z)|$. Since for the almost analytic extension the integral above is finite, this proves (43) when F and F' are bounded.

Now assume that F is positive but unbounded and satisfies (40), (41), (42), and F' bounded. Define the family of bounded functions $F_n = (F_{j,n})_{j\in\mathbb{Z}}$ given by

$$F_{j,n} := \frac{F_j}{1+\frac{1}{n}F_j}\,, \quad n\in\mathbb{N}\,, j\in\mathbb{Z}\,. \tag{55}$$

Notice that $|F'_{j,n}| = (1+\frac{1}{n}F_j)^{-2}|F'_{j,n}|$ is bounded uniformly in $j\in\mathbb{Z}$. Clearly, for each $j\in\mathbb{Z}$, the sequence $(F_{j,n})$ is increasing in $n\in\mathbb{N}$ and converges to F_j pointwise. Moreover, one checks that for each $n\in\mathbb{N}$ we also have $|F_{j,n}-F_{k,n}| \leqslant |F_j - F_k|$, hence $(F_{j,n})_{j\in\mathbb{Z}}$ satisfies conditions (40), (41) and (42) for each $n\in\mathbb{N}$. Moreover, notice that for any $n\in\mathbb{N}$, the estimate $F_{j,n}\leqslant n$ holds uniformly in $j\in\mathbb{Z}$.

By what we just showed, this implies that the bound (43) holds when F is replaced by F_n, but since the right hand side of (43) is uniform in $n\in\mathbb{N}$, the Monotone Convergence Theorem shows that for any $\varphi\in L^2(\mathbb{R}^2)$,

$$\left\|e^F g_\triangle(H)\varphi\right\|^2 = \lim_{n\to\infty}\left\|e^{F^{(n)}}g_\triangle(H)\varphi\right\|^2 \leqslant C^2_{\delta_0}\|\varphi\|^2\,.$$

This finishes the proof. □

4. The tunnelling bounds

In this section we apply Theorem 3.1 to derive the interior and exterior tunnelling bounds from Theorem 1.3. To do so, we need to construct suitable sequences of weights $(F_j)_{j\in\mathbb{Z}}$ that satisfy the requirements of Theorem 3.1.

In order to verify (40) it is important to estimate the value of the effective potential $V_j(r) = (j-\Phi(r))^2/r^2$ in the classically forbidden regions. One can get an intuition by considering the case $\Phi(r) = r^\sigma$, $\sigma>1$. It is easy to see that the classically allowed region is either empty, or it is contained in an interval $[r_-|j|^{1/\sigma}, r_+|j|^{1/\sigma}]$, for some $r_+ > r_- > 0$. Conditions 2 and 3 allow us to obtain estimates for $V_j - E$, to the left and to the right of the classically allowed region of the corresponding effective potential, respectively. This is shown in the next lemma, for any energy $E>0$.

Lemma 4.1. *i) Under Condition 2 there exists a constant $j_0>0$ and, given $E>0$, a constant $\varepsilon_E>0$ such that, for any $j\in\mathbb{Z}$ with $|j|\geqslant j_0$ and $r\leqslant\varepsilon_E|j|^{1/\sigma_+}$*

$$V_j(r) - E \geqslant |j|^{2\frac{\sigma_+-1}{\sigma_+}}\,. \tag{56}$$

ii) Under Condition 3 there exist for any $E > 0$, a constant $\eta_E > 1$ such that, for any $j \in \mathbb{Z}$ and $r \geqslant \eta_E(1+|j|)^{1/\sigma_-}$,

$$V_j(r) - E \geqslant \lambda_-^2 r^{2(\sigma_- - 1)}\,, \tag{57}$$

where $\sigma_\mp, \lambda_\mp$ are parameters defined in Condition 2 and 3.

Proof. i): From Condition 2 one sees that for any $\varepsilon > 0$ we have for all $r \leqslant \varepsilon |j|^{1/\sigma_+}$

$$|\Phi(r)| \leqslant \lambda_+(1 + r^{\sigma_+}) \leqslant \lambda_+\big(1 + \varepsilon^{\sigma_+}|j|\big)\,. \tag{58}$$

Thus

$$\begin{aligned}\sqrt{V_j(r)} &= \frac{1}{r}\,|j - \Phi(r)| \geqslant \frac{1}{r}\,(|j| - |\Phi(r)|) \geqslant \frac{1}{r}\big(|j| - \lambda_+(1+\varepsilon^{\sigma_+}|j|)\big) \\ &= \frac{1}{r}\big(|j|(1 - \lambda_+\varepsilon^{\sigma_+}) - \lambda_+\big)\end{aligned}$$

for all $0 < r \leqslant \varepsilon|j|^{1/\sigma_+}$. Thus, if we choose ε so small that $1 - \lambda_+\varepsilon_+^\sigma \geqslant 1/2$ we get

$$\sqrt{V_j(r)} \geqslant \frac{1}{r}\big(\frac{1}{2}|j| - \lambda_+\big) \geqslant \frac{|j|}{4r} \geqslant \frac{1}{4\varepsilon}|j|^{1-\frac{1}{\sigma_+}} \tag{59}$$

whenever $0 \leqslant r \leqslant \varepsilon|j|^{1/\sigma_+}$ and $|j| \geqslant 4\lambda_+$. So by making ε so small that also $16\varepsilon^2 \leqslant 1/(E+1)$ we get, for any $|j| \geqslant 4\lambda_+ > 0$ and $r \leqslant \varepsilon|j|^{1/\sigma_+}$,

$$V_j(r) - E \geqslant |j|^{2\frac{\sigma_+ - 1}{\sigma_+}}\left(\frac{1}{16\varepsilon^2} - E\right) \geqslant |j|^{2\frac{\sigma_+ - 1}{\sigma_+}}\,. \tag{60}$$

This shows the claim.

ii): Let $\eta \geqslant \max\{r_0, (2/\lambda_-)^{1/\sigma_-}\}$ and $j \in \mathbb{Z}$. Using Condition 3, we see that for any $r \geqslant \eta(1+|j|)^{1/\sigma_-}$

$$|\Phi(r)| \geqslant \lambda_- \eta^{\sigma_-}(1+|j|) \geqslant 2|j|\,. \tag{61}$$

Therefore,

$$\sqrt{V_j(r)} = \frac{1}{r}\,|\Phi(r) - j| \geqslant \frac{1}{r}\,(|\Phi(r)| - |j|) \geqslant \frac{|\Phi(r)|}{2r} \geqslant \frac{\lambda_-}{2} r^{\sigma_- - 1}\,. \tag{62}$$

Thus, for any $r \geqslant \eta(1+|j|)^{1/\sigma_-}$,

$$V_j(r) - E \geqslant r^{2(\sigma_- - 1)}(\lambda_-^2/4 - E/\eta)\,, \tag{63}$$

so claim follows with the choice $\lambda_0^2 = \lambda_-^2/4 - E/\eta$ by choosing $\eta > 1$ sufficiently large. □

Remark 4.2. The following simple observation is useful: If for any $j \in \mathbb{Z}$, the sets M_j are subsets of $\mathbb{R}^+$, possibly empty, and $V_j - E > 0$ on M_j, then

$$\mathbb{1}_{M_j}(r)\chi_j(E) = 0\,, \quad \text{and} \tag{64}$$

$$(V_j(r) - E)\chi_j^\perp(E) \geqslant (V_j(r) - E)\mathbb{1}_{M_j}(r)\,. \tag{65}$$

Now we come to the

Proof of Theorem 1.3. In order to show (19) and (20) we construct two different sequences and verify that they satisfy the requirements of Theorem 3.1, equations (40), (41) and (42). Throughout this proof we abbreviate $\widetilde{E} \equiv E$ and $\chi_j(\widetilde{E}) \equiv \chi_j$.

Proof of (19). Let $\varepsilon \in (0, \varepsilon_E]$ be a constant to be fixed below. In the following ε_E and $j_0 > 0$ are the parameters from the first part of Lemma 4.1. We define, for any $j \in \mathbb{Z}$ with $|j| \geqslant j_0 + 1$,

$$F_j(r) = |j|^{\zeta(1-\sigma_+^{-1})}(\varepsilon|j|^{\zeta/\sigma_+} - r)_+ \,, \qquad r > 0\,, \tag{66}$$

where $x_+ = \max\{x, 0\}$, and $F_j(r) = 0$ for all $r > 0$ when $|j| \leqslant j_0$. Note that F_j is piecewise continuously differentiable and its derivative is given by

$$|F_j'(r)| = |j|^{\zeta(1-\sigma_+^{-1})} \mathbb{1}_{(0,\varepsilon|j|^{\zeta/\sigma_+})}(r)\,, \tag{67}$$

when $|j| \geqslant j_0 + 1$ and its derivative vanishes when $|j| \leqslant j_0$.

In view of (56) and (65) this choice of F_j clearly satisfies (40), since on $\mathbb{R}^+$ we have

$$(V_j - E)\chi_j^\perp \geqslant (V_j - E)\mathbb{1}_{(0,\varepsilon|j|^{\zeta/\sigma_+})} \geqslant |j|^{2\zeta(1-\sigma_+^{-1})}\mathbb{1}_{(0,\varepsilon|j|^{\zeta/\sigma_+})} \geqslant |F_j'|^2\,.$$

Moreover, using Remark 4.2, one sees $F_j\chi_j = 0$, for all $j \in \mathbb{Z}$, and hence we also have (41).

To show that F satisfies (42), it is enough to assume $|j| > |k|$, by symmetry. Also, if $|j|, |k| \leqslant j_0$, then $F_j = F_k = 0$, so (42) trivially holds in this case.

In the case $|j| \geqslant |k| \geqslant j_0 + 1$ we argue as follows: If $r \leqslant \varepsilon|k|^{\zeta/\sigma_+}$, then we have

$$\left(|j|^{\zeta(1-\sigma_+{}^{-1})} - |k|^{\zeta(1-\sigma_+{}^{-1})}\right) r \leqslant \varepsilon\left(|j|^\zeta - |k|^\zeta\right) \leqslant \varepsilon\,|j-k|^\zeta\,,$$

since $0 < \zeta \leqslant 1$ and the map $\mathbb{Z} \ni j \to |j|^\zeta$ obeys the triangle inequality – this is recalled in Lemma B.5 in the appendix. Thus

$$|F_j(r) - F_k(r)| = \varepsilon\left(|j|^\zeta - |k|^\zeta\right) - \left(|j|^{\zeta(1-\sigma_+{}^{-1})} - |k|^{\zeta(1-\sigma_+{}^{-1})}\right) r \leqslant \varepsilon|j-k|^\zeta\,.$$

When $r \in [\varepsilon|k|^{\zeta/\sigma_+}, \varepsilon|j|^{\zeta/\sigma_+}]$ then

$$\begin{aligned}|F_j(r) - F_k(r)| &= \varepsilon\,|j|^\zeta - |j|^{\zeta(1-\sigma_+{}^{-1})} r \leqslant \varepsilon|j|^\zeta - \varepsilon|j|^{\zeta(1-\sigma_+{}^{-1})}|k|^{\zeta/\sigma_+}\\ &\leqslant \varepsilon(|j|^\zeta - |k|^\zeta) \leqslant \varepsilon|j-k|^\zeta\,.\end{aligned}$$

Moreover, for $r > \varepsilon|j|^{\zeta/\sigma_+}$ both F_j and F_k vanish, thus (42) will hold for all $|j|, |k| \geqslant j_0 + 1$, provided we pick $\varepsilon < a/2$.

If $|j| \geqslant j_0 + 1$ and $|k| \leqslant j_0$, then $|j-k| \geqslant 1$, so

$$|j| \leqslant |j-k| + |k| \leqslant |j-k| + j_0 \leqslant |j-k| + j_0|j-k| = (j_0+1)|j-k|$$

Thus, in this case

$$|F_j(r) - F_k(r)| = |F_j(r)| \leqslant \varepsilon|j|^\zeta \leqslant \varepsilon(j_0+1)^\zeta|j-k|^\zeta\,.$$

Choosing $\varepsilon = a/(2(j_0+1)^\zeta) \leqslant a/2$ one sees that (40), (41) and (42) are satisfied by F.

Therefore, we may apply Theorem 3.1 to conclude

$$\sum_{j\in\mathbb{Z}} \|e^{F_j} P_j E_I(H)\|^2 < \infty.$$

Finally, notice that $F_j(r) \geqslant \frac{\varepsilon}{2}|j|^\zeta$ whenever $r \in (0, \frac{\varepsilon}{2}|j|^{\zeta/\sigma_+})$. Hence,

$$e^{F_j} \geqslant \mathbb{1}_{(0,\frac{\varepsilon}{2}|j|^{\zeta/\sigma_+})} e^{F_j} \geqslant \mathbb{1}_{(0,\frac{\varepsilon}{2}|j|^{\zeta/\sigma_+})} e^{\frac{\varepsilon}{2}|j|^\zeta},$$

for all $j \in \mathbb{Z}$ which yields (19).

Proof of (20): We consider another sequence of functions defined, for any $j \in \mathbb{Z}$, by

$$G_j(r) = c[\, r^{\zeta\sigma_-} - \eta^{\zeta\sigma_-}\, (1+|j|)^\zeta]_+\,, \quad r > 0\,, \tag{68}$$

for constants $c > 0$ and $\eta \geqslant \eta_E$ to be fixed below, and $0 < \zeta \leqslant 1$. Using Lemma 4.1.(ii) we have

$$\left|G_j'\right|^2 = (c\zeta\sigma_-)^2\ r^{2(\zeta\sigma_- -1)} \mathbb{1}_{(\eta(1+|j|)^{1/\sigma_-},\infty)} \leqslant (V_j - E)\mathbb{1}_{(\eta(1+|j|)^{1/\sigma_-},\infty)}\,, \tag{69}$$

provided $c\sigma_- \leqslant \lambda_-$. Thus, in view of Remark 4.2 this choice of G_j satisfies the requirements (40) and (41).

In a similar but easier fashion as above for F, one can check

$$|G_j - G_k| \leqslant c\eta^{\zeta\sigma_-}\, |j-k|^\zeta\,,$$

for any $j, k \in \mathbb{Z}$. Hence, for the choice $\eta = \eta_E$ and $c = \min\{\lambda_-/\sigma_-, (a/(2\eta_E))^{1/(\zeta\sigma_-)}\}$, one sees that (40)–(42) are satisfied by G_j. This implies $\sum_{j\in\mathbb{Z}} \|e^{G_j} P_j E_I(H)\|^2 < \infty$.

Finally, we note that for $r \geqslant \eta\,[2(1+|j|)]^{1/\sigma_-}$ we have $G_j(r) \geqslant \frac{c_2}{2} r^{\zeta\sigma_-}$. Therefore,

$$e^{G_j(r)} \geqslant \mathbb{1}_{(\eta\,[2(1+|j|)]^{\zeta/\sigma_-},\infty)} e^{G_j(r)} \geqslant \mathbb{1}_{(\eta\,(4|j|)^{\zeta/\sigma_-},\infty)}\, e^{\frac{c_2}{2} r^{\zeta\sigma_-}}\,.$$

This concludes the proof of (20).

□

5. A remark on a model with mobility edge

So far in the article, nothing has been said about the limiting case $\sigma_- = \sigma_+ = 1$. On this subsection, we give results on the localization of particles moving under such magnetic fields when no electric field is present. More precisely, we consider the situation in which the magnetic flux is given by

$$\Phi(r) = \lambda\, r\,, \qquad r > 0 \tag{70}$$

for some $\lambda > 0$, with the Hamiltonian H_0 being defined through (2). The spectral quality of this operator has already been determined, see [18] or [6, Theorem 6.2]. In particular, it is proven that $\sigma(H_0) = [0,\infty)$ and the spectrum is dense pure point in $[0,\lambda^2)$ and absolutely continuous in (λ^2,∞). For energies above λ^2 one may use the absolute continuity of the spectrum, similarly as it was first done in [15] for Schrödinger operators and then adapted in [16] to Dirac particles, with rotational symmetry to show ballistic dynamics. The long time dynamics for high energies is therefore understood; we now settle the question about dynamics for low, positive energies.

First, note that the rotational symmetry of H_0 makes the dynamics of J trivial. Therefore, to obtain an estimate on $|\mathbf{x}(t)|$ it suffices to adapt Theorem 1.5 to the present case. One may go through its proof and realize that the exterior tunnelling estimate (20) is all that is needed. We state both of these adapted results in the following.

Theorem 5.1. *Let H_0 be the Hamiltonian associated to the quadratic form (2) with $A \equiv \lambda > 0$, as given in (4). Let $E \in (0, \lambda^2)$ and $I = [0, E]$. Then, there exist constants $c_- \in (1, \infty)$ and $\delta > 0$ such that*

$$\sum_{j\in\mathbb{Z}} \|\mathbb{1}_{[c_-|j|,\infty)}(|\mathbf{x}|)\, e^{\delta|\mathbf{x}|}\, P_j E_I(H_0)\|^2 < \infty\,. \tag{71}$$

Consequently, for every $\nu > 0$ there exists a constant $C > 0$ such that for all $\varphi \in E_I(H_0)L^2(\mathbb{R}^2)$

$$\big\langle \varphi(t), |\mathbf{x}|^\nu \varphi(t)\big\rangle \leqslant C\left(\|\varphi\|^2 + \big\langle \varphi, |J|^\nu \varphi\big\rangle\right)\,, \quad \text{for} \quad t \geqslant 0\,, \tag{72}$$

holds.

We conclude that dynamical localization holds for energies $E \in (0, \lambda^2)$, provided the initial data is sufficiently regular in the angular variable, that is,

$$\sup_{t\geqslant 0}\big\langle \varphi(t), |\mathbf{x}|^\nu \varphi(t)\big\rangle < \infty\,.$$

for all $\varphi \in E_I(H_0)L^2(\mathbb{R}^2) \cap D(|J|^{\nu/2})$.

Proof of Theorem 6. We adapt the argument used to prove Theorem 1.3, i.e. we construct an explicit sequence of functions satisfying (40) and (41) and apply Theorem 3.1. Since we assume $W = 0$, we can omit (42) and take $\epsilon_0 = c_0 = 0$ during the proof. Let $\delta_0 > 0$ and $E \equiv \tilde{E}$. First, notice that the classically allowed regions are simplified to

$$C_j(E) = \begin{cases} [\,\frac{j}{\lambda+E^{1/2}}, \frac{j}{\lambda-E^{1/2}}\,] & j > 0, \\ \emptyset & j \leqslant 0. \end{cases} \tag{73}$$

Note that the proof breaks down for $E > \lambda^2$ since then this structure is lost; one has in turn $C_j(E) = [\frac{j}{\lambda+E^{1/2}}, \infty)$ for $j > 0$. Now, pick $\eta_1 = \eta_1(\lambda, E) > \frac{1}{\lambda^2 - E}$ big enough such that

$$\frac{2\lambda}{\eta_1} \leqslant \frac{1}{2}(\lambda^2 - E) \tag{74}$$

holds. Then, let $\delta_1 = \delta_1(\lambda, E, \eta_1) < \min\{\sqrt{\frac{\lambda^2-E}{2}}, \frac{a}{2\eta_1}\}$ and define $H_j(r) = \delta_1(r - \eta_1|j|)_+$. We estimate for $r \in (\eta_1|j|, \infty)$

$$V_j(r) - E = \lambda^2\left(1 - \frac{j}{\lambda r}\right)^2 - E \geqslant \lambda^2\left(1 - \frac{2}{\lambda\eta_1}\right) - E = (\lambda^2 - E) - \frac{2\lambda}{\eta_1} \geqslant \frac{\lambda^2 - E}{2}$$

where the last inequality follows from (74). Since $|H_j'|^2 = \delta_1^2\, \mathbb{1}_{(\eta_1|j|,\infty)} \leqslant \frac{\lambda^2-E}{2}\mathbb{1}_{(\eta_1|j|,\infty)}$ we have that (40) is satisfied. (41) is fulfilled in view of $\eta_1 > \frac{1}{\lambda^2-E}$ and (73). We finish the proof using Theorem 3.1, putting $c_- = 2\eta_1$ together with $\delta = \min\{\delta_1/2, \delta_0\}$ and arguing as in the end of the proof of Theorem 1.3. $\square$

Appendix A. The magnetic Schrödinger operator H_0

Here we want to review the form definition of the magnetic Schrödinger operator H_0. We have to be a little bit careful, since we want to be able to handle rotationally symmetric, but possibly singular, magnetic fields B. Recall that we choose the vector potential A in the Poincaré gauge given by (4).

Lemma A.1. *If the magnetic field $B : \mathbb{R}^2 \to \mathbb{R}$ is rotationally symmetric and locally square integrable, then the function*

$$\mathbb{R}^2 \ni x \mapsto \Phi(|x|)/|x| = \frac{1}{|x|}\int_0^{|x|} B(s)s ds$$

is locally square integrable. In particular, the magnetic vector potential A given by (4) is in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$.

Proof. This is s simple consequence of Jensen's inequality. Since $\frac{2}{r^2}\int_0^r s ds = 1$, Jensen's inequality shows

$$\Big(\frac{2\Phi(r)}{r^2}\Big)^2 = \Big(\frac{2}{r^2}\int_0^r B(s)s ds\Big)^2 \leqslant \frac{2}{r^2}\int_0^r B(s)^2 s ds$$

for all $r > 0$. Thus

$$\int_{|x|\leqslant R}\Big(\frac{\Phi(|x|)}{|x|}\Big)^2 dx = 2\pi\int_0^R \frac{r^2}{4}\Big(\frac{2\Phi(r)}{r^2}\Big)^2 r dr \leqslant \pi\int_0^R\int_0^r B(s)^2 s ds\, r dr \tag{75}$$

$$= \frac{\pi}{2}\int_0^R B(s)^2(R^2 - s^2)s ds < \infty \tag{76}$$

for all $R > 0$. By the definition (4) we have $|A(x)| = \frac{\Phi(|x|)}{|x|}$, so also the magnetic vector potential in the Poincaré gauge A is locally square integrable. □

Denote by $p = -i\nabla$ the usual momentum operator. We will need a representation of the magnetic Schrödinger operator $(p - A)^2$, when A is in the Poincaré gauge and the magnetic field is rotationally symmetric. This is well–known, but we want to include singular magnetic fields, so we have to be a bit careful.

Lemma A.2. *The quadratic form q_0 of the free magnetic Schrödinger operator $(p - A)^2$ is given by*

$$q_0(\varphi, \varphi) = \big\langle (p - A)\varphi, (p - A)\varphi\big\rangle = \big\langle \partial_r\varphi, \partial_r\varphi\big\rangle + \Big\langle \frac{1}{r}(\Phi - L)\varphi, \frac{1}{r}(\Phi - L)\varphi\Big\rangle \tag{77}$$

for all $\varphi \in \mathcal{D}(q_0)$. Here $L = x_1p_2 - x_2p_1$ is the generator of rotations, i.e., the angular momentum operator, in $L^2(\mathbb{R}^2)$, $r = |x|$, and the radial derivative is given by $\partial_r = \frac{x}{|x|}\cdot\nabla$. In particular, $\mathcal{D}(q_0) = \mathcal{D}(\partial_r) \cap \mathcal{D}(\frac{1}{r}(\Phi - L))$.

Before we show this, we collect one more result, which is needed

Lemma A.3. *The quadratic form corresponding to the kinetic energy p^2 in dimension $d \geqslant 2$ is given by*

$$\big\langle p\varphi, p\varphi \big\rangle = \big\langle \partial_r\varphi, \partial_r\varphi \big\rangle + \sum_{1\leqslant j<k\leqslant d} \big\langle \frac{1}{r}L_{j,k}\varphi, \frac{1}{r}L_{j,k}\varphi \big\rangle \tag{78}$$

for all $\varphi \in H^1(\mathbb{R}^d)$, the form domain of p^2, where $r = |x|$, $\partial_r = \frac{x}{|x|}\cdot\nabla$ is the radial derivative on $\mathbb{R}^d$ and $L_{j,k} = x_j p_k - x_k p_j$, $1 \leqslant j < k \leqslant d$ are the angular momentum generators.

In particular, $H^1(\mathbb{R}^d) = \mathcal{D}(\partial_r) \cap \bigcap_{1\leqslant j<k\leqslant d} \mathcal{D}(\frac{1}{r}L_{j,k})$.

Proof of Lemma A.2. We assume that $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, by density. Then since A is locally square integrable,

$$\begin{aligned}\big\langle \varphi, H_0\varphi \big\rangle &= \big\langle (p-A)\varphi, (p-A)\varphi \big\rangle \\ &= \big\langle p\varphi, p\varphi \big\rangle - 2\mathrm{Re}\big\langle A_1\varphi, p_1\varphi \big\rangle - 2\mathrm{Re}\big\langle A_2\varphi, p_2\varphi \big\rangle + \big\langle A\varphi, A\varphi \big\rangle\end{aligned}$$

Using the explicit form of vector potential in the gauge (4), one also sees

$$\big\langle A_1\varphi, p_1\varphi \big\rangle = -\big\langle x_2\frac{\Phi}{r^2}\varphi, p_1\varphi \big\rangle = -\big\langle \frac{\Phi}{r}\varphi, \frac{1}{r}x_2p_1\varphi \big\rangle$$

and similarly

$$\big\langle A_2\varphi, p_2\varphi \big\rangle = \big\langle \frac{\Phi}{r}\varphi, \frac{1}{r}x_1p_2\varphi \big\rangle$$

where all terms are well defined, since Lemma A.1 shows that $\frac{\phi(|x|)}{|x|}$ is locally square integrable over $\mathbb{R}^2$. Thus

$$\big\langle A_1\varphi, p_1\varphi \big\rangle + \big\langle A_2\varphi, p_2\varphi \big\rangle = \big\langle \frac{\Phi}{r}\varphi, \frac{1}{r}(x_1p_2 - x_2p_1)\varphi \big\rangle = \big\langle \frac{\Phi}{r}\varphi, \frac{1}{r}L\varphi \big\rangle$$

with $L = x_1p_2 - x_2p_1$. Since also

$$\big\langle A\varphi, A\varphi \big\rangle = \big\langle \frac{\Phi}{r}\varphi, \frac{\Phi}{r}\varphi \big\rangle$$

this yields

$$\begin{aligned}\big\langle (p-A)\varphi, (p-A)\varphi \big\rangle &= \big\langle p\varphi, p\varphi \big\rangle + 2\mathrm{Re}\big\langle \frac{\Phi}{r}\varphi, \frac{1}{r}L\varphi \big\rangle + \big\langle A\varphi, A\varphi \big\rangle \\ &= \big\langle p\varphi, p\varphi \big\rangle + 2\big\langle \frac{\Phi}{r}\varphi, \frac{1}{r}L\varphi \big\rangle + \big\langle \frac{\Phi}{r}\varphi, \frac{\Phi}{r}\varphi \big\rangle\end{aligned}$$

since the angular momentum commutes with rotationally symmetric functions, so $\big\langle \frac{\Phi}{r}\varphi, \frac{1}{r}L\varphi \big\rangle$ is real. Moreover, by Lemma A.3, and again using that L commutes with multiplication by rotationally symmetric functions, this gives

$$\begin{aligned}\big\langle (p-A)\varphi, (p-A)\varphi \big\rangle &= \big\langle \partial_r\varphi, \partial_r\varphi \big\rangle + \big\langle \frac{1}{r}L\varphi, \frac{1}{r}L\varphi \big\rangle + 2\big\langle \frac{\Phi}{r}\varphi, \frac{1}{r}L\varphi \big\rangle + \big\langle \frac{\Phi}{r}\varphi, \frac{\Phi}{r}\varphi \big\rangle \\ &= \big\langle \partial_r\varphi, \partial_r\varphi \big\rangle + \big\langle \frac{1}{r}(\Phi - L)\varphi, \frac{1}{r}(\Phi - L)\varphi \big\rangle\end{aligned}$$

which proves (77) when $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. Since $\mathcal{C}_0^\infty(\mathbb{R}^2)$ is dense in the domain of q_0 [20] and all terms on the right hand side of (77) are non–negative, a standard density argument shows that the domain of $\mathcal{D}(q_0)$ is equal to the intersection of $\mathcal{D}(\partial_r)$ and $\mathcal{D}(\frac{1}{r}(\Phi - L))$. This proves Lemma A.2.

□

Proof of Lemma A.3 . We can use the same density argument as above to see that it is enough to assume that $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. Then

$$\sum_{1\leqslant j<k\leqslant d} \langle L_{j,k}\varphi, L_{j,k}\varphi\rangle = -\sum_{1\leqslant j<k\leqslant d} \langle \varphi, (x_j\partial_k - x_k\partial_j)^2\varphi\rangle$$

Moreover,

$$\begin{aligned}
\sum_{1\leqslant j<k\leqslant d} (x_j\partial_k - x_k\partial_j)^2 &= \frac{1}{2}\sum_{j\neq k}(x_j\partial_k - x_k\partial_j)^2 \\
&= \frac{1}{2}\sum_{j\neq k}(x_j\partial_k x_j\partial_k - x_j\partial_k x_k\partial_j - x_k\partial_j x_j\partial_k + x_k\partial_j x_k\partial_j) \\
&= \frac{1}{2}\sum_{1\leqslant j,k\leqslant d}(x_j^2\partial_k^2 + x_k^2\partial_j^2 - \partial_k x_k x_j\partial_j - x_j\partial_j - \partial_j x_j x_k\partial_k - x_k\partial_k) \\
&\quad - \sum_j (x_j^2\partial_j^2 - \partial_j x_j^2\partial_j - x_j\partial_j) \\
&= |x|^2\Delta - (\nabla\cdot x)(x\cdot\nabla) + 2x\cdot\nabla = |x|^2\Delta - (x\cdot\nabla)^2 - (d-2)x\cdot\nabla
\end{aligned}$$

Thus

$$p^2 = -\Delta = -\frac{1}{|x|^2}(x\cdot\nabla)^2 - \frac{(d-2)}{|x|^2}x\cdot\nabla + \frac{1}{|x|^2}\sum_{1\leqslant j<k\leqslant d} L_{j,k}^2\,,$$

that is, as quadratic forms

$$\langle p\varphi, p\varphi\rangle = -\langle\varphi, \frac{1}{|x|^2}(x\cdot\nabla)^2\varphi\rangle - \langle\varphi, \frac{d-2}{|x|^2}(x\cdot\nabla)\varphi\rangle + \sum_{1\leqslant j<k\leqslant d}\langle\varphi, \frac{1}{|x|^2}L_{j,k}^2\varphi\rangle \tag{79}$$

at least when $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. For such φ we set

$$\psi(r,\omega) := \varphi(r\omega)\,,$$

when $r \geqslant 0$ and $|\omega| = 1$. That is, $\varphi(x) = \psi(|x|, x/|x|)$. Then clearly, $\partial_r\psi(r,\omega) = \omega\cdot\nabla\varphi(r\omega)$, so

$$x\cdot\varphi(x) = r\partial_r\psi(r,\omega) \tag{80}$$

with $r = |x|$ and $\omega = x/|x| \in S^{d-1}$. Then

$$-\langle\varphi, \frac{1}{|x|^2}(x\cdot\nabla)^2\varphi\rangle = -\int_{S^{d-1}} d\omega \int_0^\infty dr r^{d-1}\,\overline{\psi(r,\omega)}r^{-2}(r\partial_r)^2\psi(r,\omega) \tag{81}$$

Now for fixed ω, we have

$$\begin{aligned}
&-\int_0^\infty drr^{d-1}\,\overline{\psi(r,\omega)}r^{-2}(r\partial_r)^2\psi(r,\omega)\\
&\qquad\qquad= -\Big[r^{d-1}\,\overline{\psi(r,\omega)}\partial_r\psi(r,\omega)\Big]_{r=0}^{r=\infty} + \int_0^\infty dr\overline{(\partial_r(r^{d-2}\,\psi(r,\omega)))}(r\partial_r)\psi(r,\omega)\\
&\qquad\qquad= (d-2)\int_0^\infty drr^{d-3}\overline{\psi(r,\omega))}\,r\partial_r\psi(r,\omega) + \int_0^\infty drr^{d-1}|\partial_r\psi(r,\omega))|^2\,.
\end{aligned}$$

the first term in the second line above vanishes, since φ has compact support and $d\geqslant 2$. Thus

$$-\big\langle\varphi,\frac{1}{|x|^2}(x\cdot\nabla)^2\varphi\big\rangle = (d-2)\big\langle\varphi,\frac{1}{|x|^2}(x\cdot\nabla)\varphi\big\rangle + \big\langle\partial_r\varphi,\partial_r\varphi\big\rangle$$

with $\partial_r=\frac{x}{|x|}\cdot\nabla$. Using this in (79) shows

$$\begin{aligned}
\big\langle p\varphi,p\varphi\big\rangle &= \big\langle\partial_r\varphi,\partial_r\varphi\big\rangle + \sum_{1\leqslant j<k\leqslant d}\big\langle\varphi,\frac{1}{|x|^2}L_{j,k}^2\varphi\big\rangle\\
&= \big\langle\partial_r\varphi,\partial_r\varphi\big\rangle + \sum_{1\leqslant j<k\leqslant d}\big\langle\frac{1}{|x|}L_{j,k}\varphi,\frac{1}{|x|}L_{j,k}\varphi\big\rangle
\end{aligned}$$

since $L_{j,k}$ commutes with multiplication with radial functions. This proves (78), at least when $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^d)$. On the other hand, since all the terms on the right hand side of (78) are positive and $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense in the Sobolev space $H^1(\mathbb{R}^d)$, the form domain of p^2, it is an easy exercise to show that then $\partial_r\varphi,\frac{1}{|x|}L_{j,k}\varphi\in L^2(\mathbb{R}^d)$ for all $\varphi\in H^1(\mathbb{R}^d)$ and (78) holds for all $\varphi\in H^1(\mathbb{R}^d)$. $\square$

To work in polar coordinates, we identify the Hilbert space $L^2(\mathbb{R}^2)$ with the Hilbert space $\mathcal{H}=L^2(\mathbb{R}^+\times\mathbb{S}^1, r\mathrm{d}r\,\mathrm{d}\theta)$ (see (6) above) with the scalar product

$$\langle f,g\rangle_{\mathcal{H}}\equiv\langle f,g\rangle=\int_{\mathbb{R}^+\times[0,2\pi)}\overline{f(r,\theta)}g(r,\theta)\,r\mathrm{d}r\mathrm{d}\theta\,.$$

It is well-known and easy to see that the map $\mathcal{U}:L^2(\mathbb{R}^2)\to\mathcal{H}$, defined first for $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^2)$ by

$$\widetilde{\varphi}(r,\theta)=(\mathcal{U}\varphi)(r,\theta)\coloneqq\varphi(r\cos\theta,r\sin\theta)\,, \tag{82}$$

extend to a unitary operator $L^2(\mathbb{R}^2)\to\mathcal{H}$. One easily checks

$$-i\partial_\theta\widetilde{\varphi}(r,\theta)=((x_1p_2-x_2p_1)\varphi)(r\cos\theta,r\sin\theta)=(\mathcal{U}(L_{1,2}\varphi))(r,\theta) \tag{83}$$

so

$$J\coloneqq -i\frac{\partial}{\partial\theta}=\mathcal{U}L_{1,2}\mathcal{U}^* \tag{84}$$

is the self–adjoint angular momentum operator in the r,θ coordinates.

Since the complex exponentials $e^{ij\theta}$, $m \in \mathbb{Z}$ and $0 \leqslant \theta \leqslant 2\pi$, are an orthonormal basis for $L^2([0,2\pi])$, which we identify with $L^2(\mathbb{S}^1)$, one can expand every $\widetilde{\varphi} \in \mathcal{H}$ as

$$\widetilde{\varphi}(r,\theta) = \sum_{j\in\mathbb{Z}} \varphi_j(r) e^{ij\theta} \tag{85}$$

where $(\varphi_j)_{m\in\mathbb{Z}} \in L^2(\mathbb{R}_+, rdr)$ and $\|\widetilde{\varphi}\|_{\mathcal{H}}^2 = \sum_{j\in\mathbb{Z}} \|\varphi_j\|^2_{L^2(\mathbb{R},rdr)}$. Then

$$J\widetilde{\varphi}(r,\theta) = \sum_{j\in\mathbb{Z}} \varphi_j(r) j e^{ij\theta} \,. \tag{86}$$

Thus the family of corresponding eigen–projections $(P_j)_{j\in\mathbb{Z}}$ of the angular momentum operator J given by $(P_j\widetilde{\varphi})(r,\theta) = \varphi_j(r)e^{ij\theta}$ decomposes the underlying Hilbert space. We will often write $\varphi_j = P_j\mathcal{U}\varphi$, when $\varphi \in L^2(\mathbb{R}^2)$. In these coordinates we have

Proposition A.4. *Let φ be in the domain of the quadratic form q_0 corresponding to $(P-A)^2$, and expand $\widetilde{\varphi} = \mathcal{U}\varphi$ as in (85). Then*

$$q_0(\varphi,\varphi) = \sum_{j\in\mathbb{Z}} \Big(\big\langle \partial_r\varphi_j, \partial_r\varphi_j \big\rangle_{L^2(\mathbb{R}_+,rdr)} + \big\langle \varphi_j, \frac{1}{r^2}(\Phi(r)-j)^2\varphi_j \big\rangle_{L^2(\mathbb{R}_+,rdr)} \Big) \tag{87}$$

So the eigen–spaces corresponding to P_j are invariant subspaces for the unperturbed magnetic Schrödinger operator with a rotationally symmetric magnetic field, when the magnetic vector potential is in the Poincaré gauge (4).

Because of the above identity, it is convenient to recall the defintion of the effective potential, namely,

$$V_j(r) := \frac{1}{r^2}(\Phi(r)-j)^2 \,. \tag{88}$$

By polarization, Proposition A.4 shows that when φ, ψ are in the domain of the form q_0 corresponding to $(p-A)^2$ and $\widetilde{\varphi} = \mathcal{U}\varphi, \widetilde{\psi} = \mathcal{U}\psi$ are expanded as in (85) then

$$q_0(\varphi,\psi) = \sum_{j\in\mathbb{Z}} \Big(\big\langle \partial_r\varphi_j, \partial_r\psi_j \big\rangle_{L^2(\mathbb{R}_+,rdr)} + \big\langle \varphi_j, V_j\psi_j \big\rangle_{L^2(\mathbb{R}_+,rdr)} \Big) \tag{89}$$

We need one more result, concerning the form boundedness of potentials W satisfying Condition 1 with respect to the radial kinetic energy.

Lemma A.5. *Assume that v is a rotationally symmetric potential which is form bounded with respect to p^2, that is, for any $0 < \varepsilon$ there exists $C(\varepsilon) < \infty$ with*

$$|\langle \varphi, v\varphi\rangle| \leqslant \varepsilon \|\nabla\varphi\|^2 + C(\varepsilon)\|\varphi\|^2 \tag{90}$$

for all $\varphi \in \mathcal{D}(p)$. Then also

$$|\langle \varphi, v\varphi\rangle| \leqslant \varepsilon \|\partial_r\varphi\|^2 + C(\varepsilon)\|\varphi\|^2 \tag{91}$$

for all $\varphi \in \mathcal{D}(\partial_r)$, where $\partial_r = \frac{x}{|x|}\cdot\nabla$ is the radial derivative.

Proof. We expand $\widetilde{\varphi} = \mathcal{U}\varphi = \sum_{j\in\mathbb{Z}} \varphi_j e_j$, where φ_j are purely radial functions and e_j are the basis of complex exponentials. Then for a radial potential v we have

$$\langle \varphi, v\varphi \rangle = \sum_{j\in\mathbb{Z}} \langle P_j\varphi, v\varphi \rangle = \sum_{j\in\mathbb{Z}} \langle P_j^2\varphi, v\varphi \rangle = \sum_{j\in\mathbb{Z}} \langle P_j\varphi, vP_j\varphi \rangle = \sum_{j\in\mathbb{Z}} \langle \varphi_j, v\varphi_j \rangle_{L^2(\mathbb{R}_+, rdr)}$$

with the angular momentum projections P_j. Lifting each φ_j back to $L^2(\mathbb{R}^2)$, by considering it to be constant in the angular coordinate, i.e., identifying it as the function $\mathbb{R}^2 \ni x \mapsto \varphi_j(|x|)$, we see have by assumption (90)

$$\big|\langle \varphi, v\varphi \rangle\big| \leqslant \varepsilon \sum_{j\in\mathbb{Z}} \langle \nabla\varphi_j, \nabla\varphi_j \rangle + C(\varepsilon) \sum_{j\in\mathbb{Z}} \langle \varphi_j, \varphi_j \rangle$$

Now, since φ_j lifted back to $\mathbb{R}^2$ is radial, we have

$$\langle \nabla\varphi_j, \nabla\varphi_j \rangle = \langle \partial_r\varphi_j, \partial_r\varphi_j \rangle = \langle \partial_r P_j\varphi, \partial_r P_j\varphi \rangle = \langle P_j\partial_r\varphi, \partial_r\varphi \rangle$$

since each angular momentum projection P_j commutes with the radial part of the kinetic energy. We also have

$$\langle \varphi_j, \varphi_j \rangle = \langle P_j\varphi, P_j\varphi \rangle = \langle P_j\varphi, \varphi \rangle$$

so combining the above yields

$$\big|\langle \varphi, v\varphi \rangle\big| \leqslant \varepsilon \sum_{j\in\mathbb{Z}} \langle P_j\partial_r\varphi, \partial_r\varphi \rangle + C(\varepsilon) \sum_{j\in\mathbb{Z}} \langle P_j\varphi, \varphi \rangle = \varepsilon\langle \partial_r\varphi, \partial_r\varphi \rangle + C(\varepsilon)\langle \varphi, \varphi \rangle$$

which proves the claim. ∎

Remark A.6. The above result also shows that any radial potential v which is form bounded with respect to the nonmagnetic kinetic energy is also form bounded with respect to the magnetic kinetic energy with a rotationally symmetric magnetic field, with the same constants, since by Lemma A.5 we have

$$\big|\langle \varphi, v\varphi \rangle\big| \leqslant \varepsilon\langle \partial_r\varphi, \partial_r\varphi \rangle + C(\varepsilon)\langle \varphi, \varphi \rangle \leqslant \varepsilon(\langle \partial_r\varphi, \partial_r\varphi \rangle + \langle \varphi, V\varphi \rangle) + C(\varepsilon)\langle \varphi, \varphi \rangle \tag{92}$$

$$= \varepsilon q_0(\varphi, \varphi) + C(\varepsilon)\langle \varphi, \varphi \rangle \tag{93}$$

since the effective potential $V = (V_j)_{j\in\mathbb{Z}} \geqslant 0$.

Appendix B. The exponentially twisted magnetic quadratic form

Here we show that the twisted operators $e^F H e^{-F}$, or better their quadratic forms, are well behaved for a large class of weights F. Moreover, the bounds are *uniformly in* F for which $\|F'\|_\infty := \sup_{j\in\mathbb{Z}} \sup_{r>0} |F_j'(r)|$ is bounded.

We denote by Υ_K the class of sequences of functions $F = (F_j)_{j\in\mathbb{Z}}$ satisfying $\|F\|_\infty < \infty$ and $\|F'\|_\infty \leqslant K$.

Lemma B.1. *For any sequence* $F = (F_j)_{j\in\mathbb{Z}}$ *with* $F \in \Upsilon_K$ *we have* $e^{\pm F}\mathcal{D}(q_0) \subset \mathcal{D}(q_0)$. *Moreover, the quadratic form corresponding to* $T_F := e^F H_0 e^{-F} - H_0$, *that is,*

$$\langle \varphi, T_F\varphi \rangle := q_0(e^F\varphi, e^{-F}\varphi) - q_0(\varphi, \varphi)$$

is, uniformly in $F \in \Upsilon_K$, *infinitesimally form bounded with respect to* H_0.

Proof. We have

$$\partial_r(e^{\pm F_j}\varphi_j) = e^{\pm F_j}\left(\partial_r\varphi_j \pm F_j'\varphi_j)\right)$$

and since F_j and F_j' are bounded, this implies $|\partial_r(e^{\pm F_j}\varphi_j)| \lesssim |\partial_r\varphi_j| + \varphi_j|$. Since also $\sqrt{V_j}|e^{\pm F_j}\varphi_j| \lesssim \sqrt{V_j}|\varphi_j|$ one sees that $e^{\pm F}\varphi \in \mathcal{D}(q_0)$ as soon as $\varphi \in \mathcal{D}(q_0)$

As quadratic forms and using Proposition A.4 we have

$$\begin{aligned}\langle\varphi, T_F\varphi\rangle &= q_0(e^F\varphi, e^{-F}\varphi) - q_0(\varphi,\varphi) = \sum_{j\in\mathbb{Z}}\left(\langle\partial_r(e^{F_j}\varphi_j), \partial_r(e^{-F_j}\varphi_j\rangle - \langle\partial_r\varphi_j, \partial_r\varphi_j\rangle\right)\\ &= \sum_{j\in\mathbb{Z}}\left(\langle\partial_r\varphi_j + F_j'\varphi_j, \partial_r\varphi_j - F_j'\varphi_j\rangle - \langle\partial_r\varphi_j, \partial_r\varphi_j\rangle\right)\\ &= \sum_{j\in\mathbb{Z}}\left(\langle F_j'\varphi_j, \partial_r\varphi_j\rangle - \langle\partial_r\varphi_j, F_j'\varphi_j\rangle - \langle F_j'\varphi_j, F_j'\varphi_j\rangle\right)\end{aligned} \tag{94}$$

since e^{F_j} commutes with the effective potential V_j for all $j \in \mathbb{Z}$. Thus, for all $0 < \varepsilon \leqslant 1$,

$$\begin{aligned}|\langle\varphi, T_F\varphi\rangle| &\leqslant \sum_{j\in\mathbb{Z}}\left(2\left\|F_j'\varphi_j\right\|\left\|\partial_r\varphi_j\right\| - \langle F_j'\varphi_j, F_j'\varphi_j\rangle\right) \leqslant \sum_{j\in\mathbb{Z}}\left(\varepsilon\left\|\partial_r\varphi_j\right\|^2 + (\varepsilon^{-1}-1)\left\|F_j'\varphi_j\right\|^2\right)\\ &\leqslant \varepsilon\left\|\partial_r\varphi\right\|^2 + K(\varepsilon^{-1}-1)\left\|\varphi\right\|^2 \leqslant \varepsilon q_0(\varphi,\varphi) + K(\varepsilon^{-1}-1)\left\|\varphi\right\|^2\ ,\end{aligned}$$

which finishes the proof. ☐

Remark B.2. Using [14, Theorem VI.1.33] this implies that, uniformly in $F \in \Upsilon_K$,

$$\langle\varphi, e^F H_0 e^{-F}\varphi\rangle \coloneqq q_0(e^F\varphi, e^{-F}\varphi) = q_0(\varphi,\varphi) + \langle\varphi, T_F\varphi\rangle \tag{95}$$

yields a non–symmetric sectorial closed quadratic form on $\mathcal{D}(q_0)$.

To control a perturbation W which is not rotationally symmetric, we recall that the Fourier transformation of the angular variable is given through the unitary operator

$$\mathcal{F} : \mathcal{H} \to \bigoplus_{j\in\mathbb{Z}} L^2(\mathbb{R}^+, \mathrm{d}r)$$

acting as the closure of the map

$$\psi \longmapsto (\mathcal{F}\psi)_j \equiv \hat{\psi}_j \coloneqq \left(\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\psi(\,\cdot\,,\theta)e^{-ij\theta}d\theta\right)_{j\in\mathbb{Z}},$$

initially defined on $\mathcal{UC}_0^\infty(\mathbb{R}^2)$.

It is easy to check that, for any $j \in \mathbb{Z}$ and $\varphi \in \mathcal{UC}_0^\infty(\mathbb{R}^2)$,

$$[P_j\psi](r,\theta) = \hat{\psi}_j(r)e_j(\theta)\,, \quad r > 0, \theta \in [0, 2\pi)\,,$$

with $e_j(\theta) \coloneqq e^{ij\theta}/\sqrt{2\pi}$, since $P_j \coloneqq \mathbf{1} \otimes |e_j\rangle\langle e_j|$ on $L^2(\mathbb{R}^+) \otimes L^2(\mathbb{S}^1) \simeq \mathcal{H}$. Moreover, we have

$$\langle P_j\varphi, WP_k\psi\rangle_{\mathcal{H}} = \langle\hat{\varphi}_j, \widehat{W}(\cdot, j-k)\hat{\psi}_k\rangle_{L^2(\mathbb{R}^+)}\,. \tag{96}$$

Lemma B.3. *Let $F=(F_j)_{j\in\mathbb{Z}}$ be a sequence of bounded functions satisfying (42) for some $a>0$ and $0<\zeta\leqslant 1$. Then, for any $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^2)$,*

$$\left|\langle\varphi, e^F W e^{-F}\varphi\rangle\right| \leqslant \xi(a,\zeta)\langle\varphi, v\,\varphi\rangle\,. \tag{97}$$

Moreover, for any $\varphi\in C_0^\infty(\mathbb{R}^2)$,

$$|\langle\varphi, W\varphi\rangle| \leqslant \xi(2a,\zeta)\langle\varphi, v\,\varphi\rangle\,. \tag{98}$$

Here v is defined through Condition 1 and $\xi(a,\zeta):=\sum_{k\in\mathbb{Z}} e^{\frac{a}{2}|k|^\zeta}$.

Proof. We estimate using (10) for any $\varphi\in C_0^\infty(\mathbb{R}^2)$

$$\begin{aligned}
\left|\langle\varphi, e^F W e^{-F}\varphi\rangle\right| &\leqslant \sum_{j,k\in\mathbb{Z}} \left|\left\langle e^{F_j}P_j\varphi, We^{-F_k}P_k\varphi\right\rangle\right| \\
&= \sum_{j,k\in\mathbb{Z}} \left|\left\langle e^{F_j}\hat\varphi_j, \widehat{W}(\cdot, j-k)e^{-F_k}\hat\varphi_k\right\rangle_{L^2(\mathbb{R}^+)}\right| \\
&\leqslant \sum_{j,k\in\mathbb{Z}} e^{-a|j-k|^\zeta}\left\langle e^{F_j}\left|\hat\varphi_j\right|, b\,e^{-F_k}\left|\hat\varphi_k\right|\right\rangle_{L^2(\mathbb{R}^+)} \\
&\leqslant \sum_{j,k\in\mathbb{Z}} e^{-a|j-k|^\zeta/2}\left\langle \left|\hat\varphi_j\right|, b\,\left|\hat\varphi_k\right|\right\rangle_{L^2(\mathbb{R}^+)} \\
&\leqslant \sum_{j,k\in\mathbb{Z}} e^{-a|j-k|^\zeta/2}\left\|b^{1/2}\hat\varphi_j\right\|_{L^2(\mathbb{R}^+)}\left\|b^{1/2}\hat\varphi_k\right\|_{L^2(\mathbb{R}^+)}
\end{aligned}$$

where in the last two inequalities we use (42) and Cauchy-Schwarz inequality for the scalar product, respectively. We can estimate de last sums applying Young's inequality for convolutions to get

$$\left|\langle\varphi, e^F W e^{-F}\varphi\rangle\right| \leqslant \left(\sum_{k\in\mathbb{Z}} e^{-a|k|^\zeta/2}\right)\left(\sum_{j\in\mathbb{Z}}\left\|b^{1/2}\hat\varphi_j\right\|^2_{L^2(\mathbb{R}^+)}\right) = \xi(a,\zeta)\langle\varphi, v\,\varphi\rangle\,.$$

This proves (97). In the case, $F=0$, we clearly obtain the same estimate as above with $a/2$ replaced by a. This concludes the proof of the lemma. □

Proposition B.4. *Assume that W satisfies Condition 1 for some $a>0$, $0<\zeta\leqslant 1$, and $F=(F_j)_{j\in\mathbb{Z}}\subset PC^1(\mathbb{R}_+,\mathbb{R})$ functions satisfying (42) such that also $\|F\|_\infty, \|F'\|_\infty<\infty$. Then the twisted quadratic form*

$$q_F(\varphi,\varphi) = q(e^F\varphi, e^{-F}\varphi) = q_0(e^F\varphi, e^{-F}\varphi) + \left\langle e^F\varphi, We^{-F}\varphi\right\rangle \tag{99}$$

is a closed sectorial form on $\mathcal{D}(q_0)$. Moreover, we have

$$\operatorname{Re} q(e^F\varphi, e^{-F}\varphi) = \left\langle\partial_r\varphi, \partial_r\varphi\right\rangle + \left\langle\varphi, (V-(F')^2)\varphi\right\rangle + \operatorname{Re}\left\langle e^F\varphi, We^{-F}\varphi\right\rangle \tag{100}$$

as quadratic forms on $\mathcal{D}(q_0)$.

Proof. By Lemmas B.1 and B.3, the quadratic forms corresponding to T_F and $e^F We^{-F}$ are infinitesimally form bounded with respect to H_0. Thus we can apply [14][Theorem VI.1.33] to the form

$$q(e^F\varphi, e^{-F}\varphi) = q_0(\varphi,\varphi) + \langle \varphi, T_F\varphi\rangle + \langle e^F\varphi, We^{-F}\varphi\rangle .$$

to see that is it is closed sectorial form on $\mathcal{D}(q_0)$. The explicit form (85) follows from this since by (94) we have

$$\mathrm{Re}\langle \varphi, T_F\varphi\rangle = \sum_{j\in\mathbb{Z}} \mathrm{Re}\left(\langle F_j'\varphi_j, \partial_r\varphi_j\rangle - \langle \partial_r\varphi_j, F_j'\varphi_j\rangle - \langle F_j'\varphi_j, F_j'\varphi_j\rangle\right) \tag{101}$$

$$= -\sum_{j\in\mathbb{Z}} \langle F_j'\varphi_j, F_j'\varphi_j\rangle = -\langle F'\varphi, F'\varphi\rangle \tag{102}$$

□

One more result, which we need and recall here, is the (reverse) triangle inequality for $j \mapsto |j|^\zeta$, when $0 < \zeta \leqslant 1$.

Lemma B.5. *For all $j,k \in \mathbb{Z}$ we have $|j+k|^\zeta \leqslant |j|^\zeta + |k|^\zeta$ and, in particular, also $\big||j|^\zeta - |k|^\zeta\big| \leqslant |j+k|^\zeta$*

Proof. This is well–known, we give the easy argument for the convenience of the reader(s). If $\zeta = 1$, this is the usual triangle inequality. So let $0 < \zeta < 1$ and also $j,k \neq 0$. Then

$$|j+k|^\zeta \leqslant (|j|+|k|)^\zeta = \frac{|j|+|k|}{(|j|+|k|)^{1-\zeta}} = \frac{|j|}{(|j|+|k|)^{1-\zeta}} + \frac{|k|}{(|j|+|k|)^{1-\zeta}}$$
$$\leqslant \frac{|j|}{|j|^{1-\zeta}} + \frac{|k|}{|k|^{1-\zeta}} = |j|^\zeta + |k|^\zeta .$$

and with the usual trick, the reverse triangle inequality $\big||j|^\zeta - |k|^\zeta\big| \leqslant |j+k|^\zeta$ follows. □

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