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Exact results for Z_m^{OS} and Z_2^{OS} with two mass scales and up to three loops

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ABSTRACT: We consider the on-shell mass and wave function renormalization constants Z_m^{OS} and Z_2^{OS} up to three-loop order allowing for a second non-zero quark mass. We obtain analytic results in terms of harmonic polylogarithms and iterated integrals with the additional letters $\sqrt{1-\tau^2}$ and $\sqrt{1-\tau^2}/\tau$ which extends the findings from ref. [1] where only numerical expressions are presented. Furthermore, we provide terms of order $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon)$ at two- and three-loop order which are crucial ingredients for a future four-loop calculation. Compact results for the expansions around the zero-mass, equal-mass and large-mass cases allow for a fast high-precision numerical evaluation.

KEYWORDS: NLO Computations

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1 Introduction and notation

Once quantum corrections to quantities, which involve heavy quarks, are computed to higher orders in perturbation theory the renormalization of the mass and wave function has to be performed. The corresponding renormalization constants are usually denoted by Z_m^{OS} and Z_2^{OS} , respectively. They are defined through

$$\begin{aligned} m^0 &= Z_m^{\text{OS}} m^{\text{OS}}, \\ \psi^0 &= \sqrt{Z_2^{\text{OS}}} \psi^{\text{OS}}, \end{aligned} \tag{1.1}$$

where m^0 and ψ^0 stand for the bare quark mass and wave function. The superscript “OS” refers to the on-shell scheme, which for QCD corrections is used synonymous to the pole scheme.

Within QCD, analytic results for both renormalization constants are available up to three loops [2–9]. At four-loop order [10–13] semi-analytic methods were used. Starting from two loops there are contributions with closed quark loops, which can either be massless, have the mass of the external quark (m_1), or have a different mass (m_2). Sample Feynman diagrams of this type can be found in figure 1. The case $m_i \neq 0$ ($i = 1, 2$) and $m_1 \neq m_2$ was considered in refs. [3, 4] and [1] at two- and three-loop orders (see also refs. [14, 15]). In this work we re-consider these contributions to Z_m^{OS} and Z_2^{OS} up to three-loop order and provide analytic results including $\mathcal{O}(\epsilon)$ terms. In ref. [1] only expansions for $m_2/m_1 \rightarrow 0$ and numerical results have been provided up to the constant term in ϵ .

In the Standard Model the largest quark mass ratio comes from charm and bottom quarks. In fact, with increasing experimental precision in B physics observables it becomes more and more important to include charm mass effects even at higher orders in perturbation theory. A prominent example is the semileptonic decay $b \rightarrow c\ell\nu$ which is currently known to $\mathcal{O}(\alpha_s^2)$ [16–18]. The extension to $\mathcal{O}(\alpha_s^3)$, necessary to determine $|V_{cb}|$ with a relative uncertainty of about 1%, requires the two-mass renormalization constants considered in this paper. They must be known analytically in all three limits $m_2/m_1 \rightarrow 0$, $m_2/m_1 \rightarrow 1$ and $m_2/m_1 \rightarrow \infty$. Indeed, to renormalize the mass and wave function of the

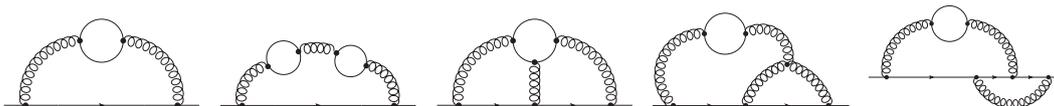


Figure 1. Sample Feynman diagrams contributing to Z_m^{OS} and Z_2^{OS} . Straight and curly lines represent quarks and gluons, respectively. The fermions in the closed loop may have a different mass from the external fermion.

bottom field, one has $m_1 = m_b$ and $m_2 = m_c$. To include the effects of the lighter charm quark, one considers Z_2^{OS} and Z_m^{OS} in the regime $m_2/m_1 = m_c/m_b \lesssim 1$. Vice versa, for the renormalization of the charm field, one has to set $m_1 = m_c$ and $m_2 = m_b$. Therefore effects of the heavier bottom quark are covered by the opposite limit $m_2/m_1 = m_b/m_c \gtrsim 1$.

In addition, the future analysis of the data from the MUonE experiment [19, 20] will likely require the knowledge of the dominant α_{em}^3 corrections to muon-electron scattering [21–23]. As it often happens for QED processes, it is not possible to set the electron mass $m_e = 0$ in the evaluation of virtual corrections, since charged leptons are experimentally distinguishable from their collinear photon radiation (contrary to what happens in QCD to quarks in jets). Therefore in QED, keeping finite fermion masses is often a necessity in order to regularize collinear singularities. Even in the case when finite electron mass effects are restored via *massification* [24] of virtual amplitudes computed for $m_e = 0$, the procedure employs the lepton’s wave-function renormalization constant with finite m_e effects to the relevant order in α_{em} . The results presented in this paper thus apply also to such processes, with the proper translation of QCD color factors to QED.

A further, at first sight not obvious, application of the three-loop two-mass result for Z_m^{OS} is the renormalon analysis of the relation between the top quark pole and $\overline{\text{MS}}$ mass [25, 26] with the aim to determine the ultimate uncertainty of the top-quark pole mass. Note that the typical loop momentum at order α_s^{n+1} scales as $m_t e^{-n}$ and thus light quark mass effects are important at higher loop orders although at first sight m_b/m_t and m_c/m_t seem to be negligibly small.

It is convenient to introduce the variable

$$x = \frac{m_2}{m_1}. \tag{1.2}$$

We furthermore adopt the notation from [1] and write ($i = m, 2$)

$$\begin{aligned} Z_i^{\text{OS}} = & 1 + \frac{\alpha_s(\mu)}{\pi} \left(\frac{e^{\gamma_E}}{4\pi}\right)^{-\epsilon} \delta Z_i^{(1)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left(\frac{e^{\gamma_E}}{4\pi}\right)^{-2\epsilon} \delta Z_i^{(2)} \\ & + \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \left(\frac{e^{\gamma_E}}{4\pi}\right)^{-3\epsilon} \delta Z_i^{(3)} + \mathcal{O}(\alpha_s^4), \end{aligned} \tag{1.3}$$

where α_s denotes the $\overline{\text{MS}}$ renormalized strong coupling constant defined in n_f -flavour QCD, and μ is the renormalization constant; γ_E is the Euler-Mascheroni constant. We decompose

Quark type	Mass	Colour Label
external	$m_1 > 0$	n_h
internal, massive	$m_2 > 0$	n_m
internal, massless	0	n_l

Table 1. Conventions adopted in this work for closed quark loops.

the coefficients $\delta Z_i^{(k)}$ according to the different colour factors and obtain

$$\begin{aligned}
 \delta Z_i^{(1)} &= C_F Z_i^F \\
 \delta Z_i^{(2)} &= C_F^2 Z_i^{FF} + C_F C_A Z_i^{FA} + C_F T_F n_l Z_i^{FL} + C_F T_F n_h Z_i^{FH} + C_F T_F n_m Z_i^{FM}(x) \\
 \delta Z_i^{(3)} &= C_F^3 Z_i^{FFF} + C_F^2 C_A Z_i^{FFA} + C_F C_A^2 Z_i^{FAA} + C_F T_F n_l (C_F Z_i^{FFL} + C_A Z_i^{FAL} \\
 &\quad + T_F n_l Z_i^{FLL} + T_F n_h Z_i^{FHL} + T_F n_m Z_i^{FML}(x)) \\
 &\quad + C_F T_F n_h (C_F Z_i^{FFH} + C_A Z_i^{FAH} + T_F n_h Z_i^{FHH} + T_F n_m Z_i^{FMH}(x)) \\
 &\quad + C_F T_F n_m (C_F Z_i^{FFM}(x) + C_A Z_i^{FAM}(x) + T_F n_m Z_i^{FMM}(x)) , \tag{1.4}
 \end{aligned}$$

with the $SU(N_c)$ colour factors $C_F = (N_c^2 - 1)/(2N_c)$, $C_A = N_c$ and $T_F = 1/2$. We have introduced the quantities n_l , n_h and n_m to label closed quark loops with mass zero, m_1 and m_2 , respectively, see also table 1. We have $n_f = n_l + n_m + n_h = n_l + 1 + 1$ active quark flavours. Note that only the terms proportional to n_m and n_m^2 have a non-trivial dependence on x . This is the main subject of the present paper.

For the quark mass renormalization constant we also introduce the ratio

$$z_m = \frac{Z_m^{\text{OS}}}{Z_m^{\overline{\text{MS}}}} \tag{1.5}$$

which is finite since both Z_m^{OS} and the $\overline{\text{MS}}$ renormalization constant $Z_m^{\overline{\text{MS}}}$ only contain ultra-violet poles, which cancel in the ratio. Note that Z_2^{OS} contains both ultra-violet and infra-red poles. z_m has an analogous perturbative expansion as Z_m^{OS} and Z_2^{OS} in eq. (1.3).

In ref. [1] the n_m -dependent terms of Z_m^{OS} and Z_2^{OS} were computed up to three loops. At two-loop order analytic results were obtained. However, at three-loop order, for the complicated master integrals only an expansion for $x \rightarrow 0$ could be obtained. For larger values of x a numerical evaluation was necessary. For most practical purposes this is sufficient. However, in some cases analytic expressions or expansions are useful. In this work we extend the result of [1] in the following aspects:

- We extend the ϵ expansion by one order both at two and three loops, which is necessary input for a future four-loop calculation of the n_m terms of the on-shell renormalization constants.
- We provide analytic results in terms of iterated integrals with the letters $\tau, 1 - \tau, 1 + \tau, \sqrt{1 - \tau^2}, \sqrt{1 - \tau^2}/\tau$. They are present both in the $\alpha_s^3 \epsilon^0$ and $\alpha_s^3 \epsilon^1$ terms. Note that in [1] for some of the colour factors even for the $1/\epsilon$ poles only numerical results were available.

- We provide 26 terms in an analytic expansions both for $x \rightarrow 0$, $x \rightarrow 1$ and $x \rightarrow \infty$ (i.e. up to order x^{25} , $(1-x)^{25}$ and $1/x^{25}$). In ref. [1], for the three-loop term only the expansion for $x \rightarrow 0$ up to x^8 was considered.

In the next section we briefly describe the approach which we use to obtain the analytic results and the expansions in the various limits. In section 3 we discuss our results for the renormalization constants and give our conclusions. In the appendix we provide details to the supplementary material attached to this paper.

2 Technicalities

We base our three-loop calculation on intermediate expressions obtained in ref. [1]. In particular, we use the results where Z_m^{OS} and Z_2^{OS} are expressed in terms of the 28 master integrals shown in figure 2. They are obtained from three integral families introduced in refs. [1, 27]. We have implemented the integral families in LiteRed [28, 29] and redone the reduction to the 28 master integrals shown in figure 2. After using the additional relation [27]

$$M_{17} = -\frac{1}{x^2}M_{15} - \frac{2d-5}{2x^2}M_{14} - \frac{2-d}{4x^2}M_4, \quad (2.1)$$

one ends up with 27 master integrals, which have to be computed.

We utilize LiteRed [28, 29] to derive a closed system of differential equations. Out of the 27 master integrals 23 can be solved in terms of harmonic polylogarithms (HPLs). Their analytic results are given in [27]. There it was also noted that four master integrals (M_{20} , M_{21} , M_{22} , M_{23}) cannot be expressed in terms of HPLs at higher orders in $\epsilon = (4-d)/2$. For these integrals expansions around $x=0$ were obtained and numerical values for larger values of x were calculated. In this work we obtain analytic results for the missing master integrals and, furthermore, extend all master integrals by one order in ϵ such that we obtain the renormalization constants to $\mathcal{O}(\epsilon^2)$ at two and to $\mathcal{O}(\epsilon)$ at three loops, respectively. This will be a crucial input for a future four-loop calculation of the renormalization constants.

We solve the coupled system of differential equations using the algorithmic approach presented in ref. [30]. For the convenience of the reader we outline in the following the main steps of this approach. The differential equation can be written in the form

$$\frac{d\vec{M}(x, \epsilon)}{dx} = \mathcal{A}(x, \epsilon) \cdot \vec{M}(x, \epsilon), \quad (2.2)$$

where $\vec{M}(x, \epsilon)$ is the vector of our 27 master integrals. It can be chosen such that the matrix \mathcal{A} is in upper-block diagonal form, i.e. the diagonal elements are square matrices with possible non-vanishing entries to the left.

The square matrices on the diagonal represent coupled sets of master integrals, which only depend on themselves and integrals from lower sectors. One can therefore solve the system successively starting from simpler systems and insert the solutions as inhomogeneities into the more involved ones. In total we find one 3×3 and seven 2×2 systems. The x dependence of seven of the remaining ten master integrals factorizes.

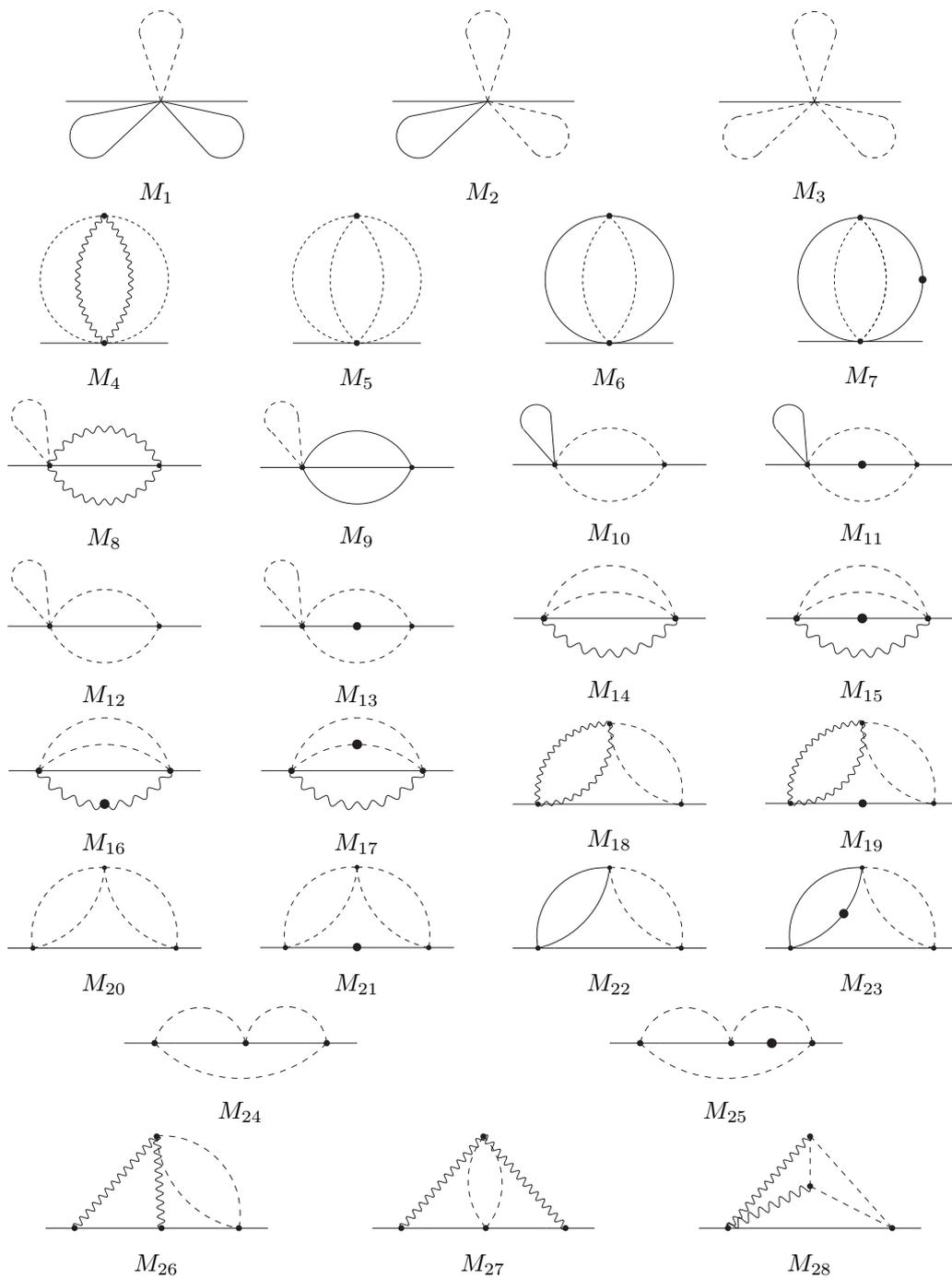


Figure 2. Three-loop master integrals after the LiteRed reduction. Solid and dashed lines represent propagators with mass m_1 and m_2 . Wavy lines stand for massless particles. Note that there is a relation between M_4 , M_{14} , M_{15} and M_{17} (see text).

We decouple the systems of differential equations with the package `OreSys` [31], which is based on `Sigma` [32, 33], to obtain a single differential equation of higher order for one of the master integrals in the system. Furthermore, `OreSys` provides rules to construct the other master integrals from the solution of the differential equation. The higher order differential equations are then expanded in ϵ and iteratively solved order by order. To solve the differential equations we make use of the solver implemented in `HarmonicSums` [34–46], which is particularly well suited to find solutions in terms of iterated integrals. In a first step we consider the homogeneous part of the differential equation and try to write it in factorized form. If it fully factorizes into first order factors the solution in terms of iterated integrals can be obtained in a straightforward way. If second order factors remain Kovacic’s algorithm [47] is used to find all solutions of the differential equation, which can be written in terms of iterated integrals. In our case the homogenous solutions in terms of iterated integrals exist and thus also the particular solutions can be expressed in terms of iterated integrals. The construction and simplification of the homogeneous and particular solution is automated in `HarmonicSums`. To fully solve the differential equations we still need to fix the integration constants multiplying the homogeneous solutions. Boundary values for all integrals at $x = 0$ and $x = 1$ can be extracted from the on-shell integrals given in ref. [48]. To fix all integration constants we need both limits since for some master integrals the homogeneous solutions vanish at $x = 0$ or $x = 1$.

Four master integrals cannot be expressed in terms of usual HPLs. We want to illustrate this for the system of differential equations of the integrals M_{22} and M_{23} . After decoupling the homogenous differential equations for the master integral M_{22} we obtain

$$M_{22}'' = \frac{1 - 4x^2}{x(1 - x^2)} M_{22}' + \frac{4}{1 - x^2} M_{22}, \tag{2.3}$$

where $d = 4$ has been used in the coefficients since the ϵ -dependent terms enter the inhomogeneous part. Equation (2.3) has the two solutions

$$\begin{aligned} M_{22}^{(1)} &= x^2(4 - x^2), \\ M_{22}^{(2)} &= (2 - 3x^2 + x^4)\sqrt{1 - x^2} + x^2(4 - x^2) \left[1 + I \left(\left\{ \frac{\sqrt{1 - \tau^2}}{\tau} \right\}, x \right) \right], \end{aligned} \tag{2.4}$$

where I denotes a generalized iterated integral over the specified integration kernels indicated in the curly brackets with the help of the variable τ , i.e.

$$I \left(\left\{ g(\tau), \vec{h}(\tau) \right\}, x \right) = \int_0^x dt g(t) I \left(\left\{ \vec{h}(\tau) \right\}, t \right). \tag{2.5}$$

Note that a regularization is needed for letters which lead to divergent expressions for $t \rightarrow 0$. This is in complete analogy to HPLs [49]. Equation (2.4) illustrates that one has to introduce the new letter $\sqrt{1 - \tau^2}/\tau$ in order to solve the differential equation. Analogously the system of master integrals M_{20} and M_{21} introduces the letter $\sqrt{1 - \tau^2}$. After fixing the boundary conditions it turns out that for all master integrals the generalized letters are

only needed from $\mathcal{O}(\epsilon)$ onwards. Note that the $\mathcal{O}(\epsilon)$ terms enter the finite contribution of Z_m^{OS} and Z_2^{OS} .

Since the additional letters only introduce one square root it is possible to rationalize the letters with a suitable variable transformation. One possibility is the so-called trigonometric substitution

$$x = \frac{2y}{1+y^2}, \tag{2.6}$$

which introduces the letters

$$\frac{1}{1+\tau^2}, \quad \frac{\tau}{1+\tau^2}. \tag{2.7}$$

Iterated integrals over these kinds of letters have been studied in ref. [37]; the corresponding iterated integrals are called cyclotomic HPLs.

Alternatively one can factor the polynomial over the complex numbers and introduce Goncharov polylogarithms [50] with letters taken from the set of the 4th root of unity. For example one has

$$\begin{aligned} I\left(\left\{\sqrt{1-\tau^2}\right\}, x\right) &= \frac{y(1-y^2)}{(1+y^2)^2} + I\left(\left\{\frac{1}{1+\tau^2}\right\}, y\right) \\ &= \frac{y(1-y^2)}{(1+y^2)^2} + \frac{i}{2} [G(i, y) - G(-i, y)], \end{aligned} \tag{2.8}$$

but also

$$H_1(x) = 2H_1(y) + 2I\left(\left\{\frac{\tau}{1+\tau^2}\right\}, y\right), \tag{2.9}$$

which shows that the variable transformation in eq. (2.6) converts HPLs with argument x into cyclotomic HPLs with argument y . Note, however, that the transformation in eq. (2.6) significantly increases the complexity of the rational functions in the differential equations. Thus, we have chosen to solve them in the variable x . However, eq. (2.6) is needed to fix the boundary conditions at $x = 1$, since this requires the evaluation of the iterated integrals at this point. The corresponding results up to weight 5 are conveniently obtained by transforming the iterated integrals to cyclotomic HPLs for which the values at $x = 1$ are known up to weight 6 [37]. This leads to relations like

$$I\left(\left\{\frac{1}{\tau}, \sqrt{1-\tau^2}, \frac{1}{1-\tau}\right\}, 1\right) = \frac{5}{4} + C\left(\frac{1}{2} + l_2\right) - \frac{3l_2}{2} + \frac{l_2^2\pi}{4} - \frac{\pi^2}{8} - \frac{\pi^3}{96} + \frac{1}{2}cs_1, \tag{2.10}$$

with $l_2 = \log(2)$, Catalan's constant

$$C = \sum_{i=1}^{\infty} \frac{(-1)^i}{(2i+1)^2} \approx 0.915966, \tag{2.11}$$

and a further cyclotomic constant

$$cs_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sum_{k=1}^n \frac{(-1)^k}{1+2k} \approx 0.330798. \tag{2.12}$$

Note that for the evaluation of individual functions at argument $x = 1$ several cyclotomic constants appear. However, in our final result, all but C cancel. Analytic results for all master integrals are provided in the supplementary material attached to this paper.

Up to $\mathcal{O}(\alpha_s^2)$ all renormalization constants can be expressed in terms of HPLs to all orders in ϵ . This changes at $\mathcal{O}(\alpha_s^3)$. Here only the pole terms can be expressed in terms of HPLs. Note that even the ϵ^{-1} terms of Z_2^{OS} has not been known analytically before. At $\mathcal{O}(\alpha_s^3\epsilon^0)$ HPLs up to weight 5 and 14 iterated integrals with the two additional letters introduced above up to weight 4 contribute. For the next term in the ϵ expansion HPLs up to weight 6 and 100 additional iterated integrals up to weight 5 are needed. Note also that starting from $\mathcal{O}(\alpha_s^3\epsilon^0)$ the prefactors of iterated itegrals, HPLs and transcendental numbers become analytics functions in x rather than rational functions, since also here $\sqrt{1-x^2}$ appears.

For fast and precise numerical evaluations we provide expansions around $x = 0$, $x = 1$ and $x \rightarrow \infty$. The expansions around $x = 0$ can be obtained from

$$\frac{d}{dx} I(\{w_1(\tau), \dots, w_n(\tau)\}, x) = w_1(x) I(\{w_2(\tau), \dots, w_n(\tau)\}, x) . \quad (2.13)$$

For the expansion around $x = 1$ we first map the argument of the iterated integrals to $1-x$. This can be achieved iteratively with the formula

$$I(\{w_1(\tau), \dots, w_n(\tau)\}, x) = I(\{w_1(\tau), \dots, w_n(\tau)\}, 1) - \int_0^{1-x} dt w_1(1-t) I(\{w_2(\tau), \dots, w_n(\tau)\}, 1-t) , \quad (2.14)$$

which can be easily proven from the integral representation. In our case this step does not introduce new letters, but introduces the iterated integrals at argument $x = 1$. The same constants were already needed to fix boundary conditions for the differential equations. Afterwards we can expand easily around $1-x$.

The expansion for $x \rightarrow \infty$ is more involved since the letters involving square roots develop a branch cut for $x > 1$. Thus, in a first step we have to construct the analytic continuation for the iterated integrals, i.e., the relations for the corresponding functions with argument $x < 1$. We use differential equations to do this. Let us for illustration consider an iterated integral of weight one. Then we have

$$\frac{d}{dx} I\left(\left\{\frac{\sqrt{1-\tau^2}}{\tau}\right\}, x\right) = \frac{\sqrt{1-x^2}}{x} . \quad (2.15)$$

Now we change the variable to $z = 1/x$ and find

$$\frac{d}{dz} f(z) = -i \frac{\sqrt{1-z^2}}{z^2} , \quad (2.16)$$

where $f(z)$ is the analytic continuation of the iterated integral in eq. (2.15). We assume $0 < z < 1$ in accordance with $x > 1$. Note that in our case the change of variables again does not introduce new letters. The differential equation can be easily solved by integrating

the right hand side over z and fixing the integration constant for $x = z = 1$. This again only requires the knowledge of the iterated integrals at argument $x = 1$. For our example, we obtain

$$I\left(\left\{\frac{\sqrt{1-\tau^2}}{\tau}\right\}, x\right) = l_2 - 1 + i\left[\frac{(1-z^2)^{3/2}}{z} + 2I\left(\left\{\sqrt{1-\tau^2}\right\}, z\right) - \frac{\pi}{2}\right]. \quad (2.17)$$

For higher weights one can proceed iteratively, since the derivative of an iterated integral of weight w with respect to its argument only depends on iterated integrals of weight $w - 1$.

Note that the analytic continuation of the individual iterated integrals introduces imaginary parts (cf. eq. (2.17)). However, after inserting the analytic continuations for all iterated integrals into the expressions for Z_m^{OS} and Z_2^{OS} all imaginary parts cancel analytically and the expansion around $1/x = z = 0$ can be obtained in a straightforward way.

3 Results and conclusions

In this section we briefly discuss our results for Z_m^{OS} , z_m and Z_2^{OS} . After inserting the exact master integrals into the corresponding amplitudes we renormalize the quark masses m_1 and m_2 in the on-shell scheme, the strong coupling constant in the $\overline{\text{MS}}$ scheme and expand in ϵ such that we obtain results up to ϵ^2 at two-loop and ϵ^1 at three-loop order. Whereas the two-loop results are still quite compact (see, e.g., eqs. (15) and (28) of ref. [1]), the three-loop expressions are too big to be printed. Instead we provide the analytic expressions in the supplementary material attached to this paper. We also provide transformation rules which map the iterated integrals introduced in the previous section to Goncharov polylogarithms which can be evaluated numerically with the help of `GiNaC` [51]. Note that our final three-loop result contains iterated integrals up to weight five and six in the ϵ^0 and ϵ^1 term.

More compact expressions are obtained after expanding for $x \rightarrow 0$, $x \rightarrow 1$ or $x \rightarrow \infty$. For illustration we show for the n_m dependent terms of z_m , which we define via

$$z_m^M = z_m - z_m(n_m = 0), \quad (3.1)$$

the first three expansion terms at two and three loops. To keep the expressions compact we specify the colour factors to QCD, i.e. $C_A = 3$, $C_F = 4/3$ and $T_F = 1/2$. Furthermore we set $\mu = m_1$, $n_h = 1$ and restrict ourselves to the ϵ^0 term. For $x \rightarrow 0$ we obtain

$$\begin{aligned} z_m^M = & \left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{71}{144} + \frac{\pi^2}{18} - \frac{\pi^2 x}{6} + x^2 \right] \\ & + \left(\frac{\alpha_s}{\pi}\right)^3 \left[\frac{40715}{3888} + \frac{941\pi^2}{648} - \frac{61\pi^4}{1944} + \frac{695}{216}\zeta_3 - \frac{8a_4}{27} + \frac{11\pi^2 l_2}{81} - \frac{2}{81}\pi^2 l_2^2 \right. \\ & - \frac{l_2^4}{81} + n_l \left(-\frac{2353}{11664} - \frac{13\pi^2}{162} - \frac{7}{27}\zeta_3 - \frac{2x^2}{9} + x \left(\frac{7\pi^2}{27} - \frac{2}{9}\pi^2 l_2 - \frac{1}{9}\pi^2 l_x \right) \right) \\ & + x \left(-\frac{29513\pi^2}{2430} - \frac{13\pi^3}{162} + \frac{1199\pi^2 l_2}{81} + \frac{31\pi^2 l_x}{18} \right) + x^2 \left(\frac{62}{9} + \frac{13\pi^2}{12} \right. \\ & \left. + \left(4 + \frac{3\pi^2}{2} - \frac{\pi^4}{12} \right) l_x + \frac{11}{2}\zeta_3 - \frac{3}{4}\pi^2 \zeta_3 - \frac{5}{2}\zeta_5 \right) \Big] + \mathcal{O}(\epsilon, x^3, \alpha_s^4), \quad (3.2) \end{aligned}$$

with $l_x = \log(x)$, $a_4 = \text{Li}_4(1/2)$ and ζ_n is Riemann's zeta function. For $x \rightarrow 1$ we have

$$\begin{aligned}
 z_m^M = & \left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{143}{144} - \frac{\pi^2}{9} + \left(-\frac{4}{3} + \frac{2\pi^2}{9}\right)y + \left(1 - \frac{\pi^2}{12}\right)y^2 \right] \\
 & + \left(\frac{\alpha_s}{\pi}\right)^3 \left[\frac{74141}{3888} - \frac{67127\pi^2}{9720} - \frac{41\pi^4}{972} - \frac{619}{216}\zeta_3 + \frac{1}{4}\pi^2\zeta_3 - \frac{5}{4}\zeta_5 - \frac{8a_4}{27} \right. \\
 & + \frac{640\pi^2 l_2}{81} + \frac{1}{81}\pi^2 l_2^2 - \frac{l_2^4}{81} + n_l \left(-\frac{5917}{11664} + \frac{13\pi^2}{324} + \frac{2}{27}\zeta_3 + \left(\frac{20}{27} - \frac{10\pi^2}{81}\right)y \right. \\
 & \left. \left. + y^2 \left(-\frac{4}{9} - \frac{\pi^2}{36} - \frac{7}{12}\zeta_3 + \frac{\pi^2 l_2}{6} \right) \right) + y \left(-\frac{5473}{243} + \frac{49738\pi^2}{3645} + \frac{979\pi^4}{19440} + \frac{839}{162}\zeta_3 \right. \right. \\
 & \left. \left. - \frac{1}{2}\pi^2\zeta_3 + \frac{5}{2}\zeta_5 + \frac{140a_4}{27} - \frac{1274}{81}\pi^2 l_2 - \frac{35}{162}\pi^2 l_2^2 + \frac{35l_2^4}{162} \right) \right. \\
 & \left. + y^2 \left(\frac{2665}{162} - \frac{85549\pi^2}{9720} - \frac{979\pi^4}{12960} + \frac{473}{216}\zeta_3 + \frac{1}{4}\pi^2\zeta_3 - \frac{5}{4}\zeta_5 - \frac{70a_4}{9} + \frac{107\pi^2 l_2}{9} \right. \right. \\
 & \left. \left. + \frac{35}{108}\pi^2 l_2^2 - \frac{35l_2^4}{108} \right) \right] + \mathcal{O}(\epsilon, y^3, \alpha_s^4), \tag{3.3}
 \end{aligned}$$

with $y = 1 - x$ and for $x \rightarrow \infty$ we have

$$\begin{aligned}
 z_m^M = & \left(\frac{\alpha_s}{\pi}\right)^2 \left[-\frac{89}{432} + \frac{13}{18}l_z - \frac{1}{3}l_z^2 + \frac{1}{225}z^2(19 - 20l_z) \right] \\
 & + \left(\frac{\alpha_s}{\pi}\right)^3 \left[-\frac{119}{1296} - \frac{103\pi^4}{6480} + \frac{157}{288}\zeta_3 + \frac{4a_4}{9} - \frac{1}{54}\pi^2 l_2^2 + \frac{l_2^4}{54} \right. \\
 & + \left(\frac{5755}{648} + \frac{2\pi^2}{9} + \frac{14}{9}\zeta_3 \right) l_z + \frac{2}{27}\pi^2 l_2 l_z - \frac{287l_z^2}{108} - l_z^3 + z^2 \left(-\frac{3207593}{17496000} \right. \\
 & \left. + \frac{667\pi^2}{77760} - \frac{29}{648}\zeta_3 + \frac{887117l_z}{1166400} - \frac{5809l_z^2}{6480} \right) + n_l \left(-\frac{1327}{11664} + \frac{2\zeta_3}{27} \right. \\
 & \left. \left. - \left(\frac{1}{12} + \frac{\pi^2}{27} \right) l_z + \frac{2l_z^3}{27} + z^2 \left(\frac{529}{10125} - \frac{2\pi^2}{405} - \frac{2l_z}{75} + \frac{4l_z^2}{135} \right) \right) \right] + \mathcal{O}(\epsilon, z^3, \alpha_s^4), \tag{3.4}
 \end{aligned}$$

with $z = 1/x$.

Expansion terms up to order x^{25} , $(1-x)^{25}$ and $(1/x)^{25}$, also to higher order in ϵ , can be found in the supplementary material attached to this paper. It is interesting to note that in the $(1-x)$ and $(1/x)$ expansion only the usual transcendental numbers as ζ_n , $\log(2)$ and $\text{Li}_n(1/2)$ appear. On the other hand for $x \rightarrow 0$ we observe in the $\mathcal{O}(\alpha_s^3\epsilon)$ term Catalan's constant, see eq. (2.11). Note that the expansions of the individual iterated integrals show a more complicated structure.

Depending on the application it is advantageous to transform either m_1 or m_2 or both to the $\overline{\text{MS}}$ scheme. For this purpose it is convenient to introduce the variables

$$x = \frac{m_2^{\text{OS}}}{m_1^{\text{OS}}}, \quad x_f(\mu_f) = \frac{\overline{m}_2(\mu_f)}{m_1^{\text{OS}}}, \quad x_q(\mu) = \frac{m_2^{\text{OS}}}{\overline{m}_1(\mu)}, \quad x_{fq}(\mu_f, \mu) = \frac{\overline{m}_2(\mu_f)}{\overline{m}_1(\mu)}, \tag{3.5}$$

where μ_f is the renormalization scale of the quark mass m_2 and μ is the common renormalization scale of m_1 and α_s . For z_m explicit transformation rules to the various schemes can

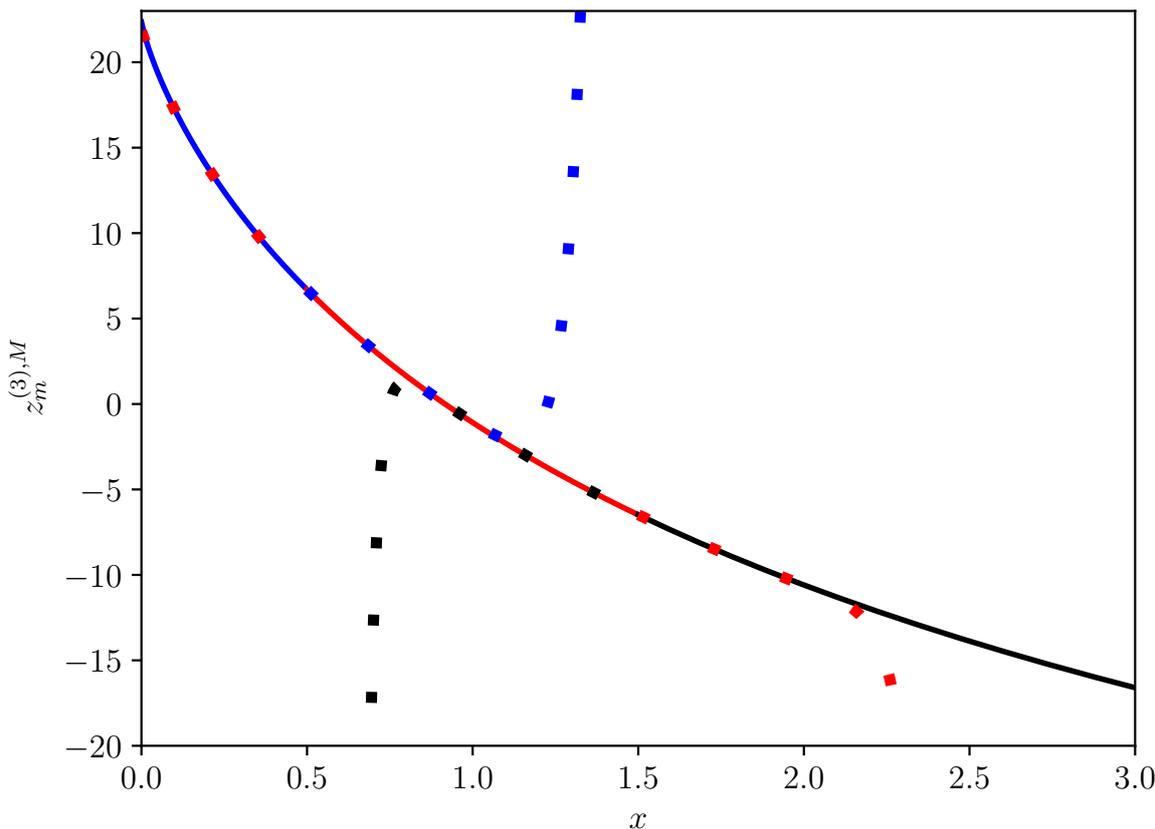


Figure 3. $z_m^{(3)}$ as a function of x . The expansions for $x \rightarrow 0$, $x \rightarrow 1$ and $x \rightarrow \infty$ are shown as solid lines in the regions where the respective expansion is used for the numerical evaluation of z_m in the supplementary material. Outside this region dotted lines are used.

be found in ref. [1]. In the supplementary material attached to this paper we provide for z_m and Z_2^{OS} different variants for the expansions in the three limits $x \rightarrow 0$, $x \rightarrow 1$ and $x \rightarrow \infty$.

We update the `Mathematica` routines provided in ref. [1] for the numerical evaluation of z_m and Z_2^{OS} . In the supplementary material attached to this paper one finds the functions `zmmum[x,m1,mu1,mu2[,scheme]]` and `Z2OSnum[x,m1,mu1,mu2[,scheme]]` (see appendix) which implement the expansion for $x \rightarrow 0$, $x \rightarrow 1$ and $x \rightarrow \infty$. We switch between the first two expansions at $x = 1/2$ and between the latter two at $x = 3/2$. The justification for this choice is illustrated in figure 3, where we show the expansions for $z_m^{(3),M}$ for $\epsilon = 0$, $n_l = 3$, $n_h = n_m = 1$ and $\mu = m_1$. In the regions where the expansions converge ($x < 1/2$, $1/2 < x < 3/2$, $3/2 < x$ for $x \rightarrow 0$, $x \rightarrow 1$ and $x \rightarrow \infty$, respectively) we plot solid lines and outside these region we switch to dotted lines. One observes that both around $x = 1/2$ and $x = 3/2$ there is a large overlap among at least two expansion, which justifies that we use the expansion results to construct the functions `zmmum[x,m1,mu1,mu2[,scheme]]` and `Z2OSnum[x,m1,mu1,mu2[,scheme]]`. Let us also mention that we observe an agreement with the exact result to at least 8 digits over the whole range in x . Similar results are obtained for the $\mathcal{O}(\epsilon)$ term of z_m and for Z_2^{OS} .

To summarize, we have obtained analytic results of all 27 master integrals which are needed to obtain the three-loop contributions for the on-shell renormalization constants Z_m^{OS} and Z_2^{OS} with dependence on two different quark masses, m_1 and m_2 . Our final result includes terms of $\mathcal{O}(\epsilon)$, which are relevant for a future four-loop calculation. Furthermore, we have obtained 26 expansion terms for three cases $m_1 \gg m_2$, $m_1 \approx m_2$ and $m_1 \ll m_2$.

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A README for supplementary material

Together with this paper we provide the following files which contain analytic expressions for the various quantities in `Mathematica` format:

- `master.m` contains the 27 master integrals $M_1, \dots, M_{16}, M_{18}, \dots, M_{28}$ as `resM1, \dots, resM28`. Note that M_{17} is obtained from eq. (2.1).
- `zmZmZ2.m` contains exact results for the expressions `zmos`, `ZmOS` and `Z2OS` in the on-shell scheme. Here the variable `x` corresponds to `xOSOS`.
- `expansions/` is a directory which contains the expansions for z_m and Z_2^{OS} in the three limits $x \rightarrow 0$, $x \rightarrow 1$ and $x \rightarrow \infty$ for various combinations of on-shell and $\overline{\text{MS}}$ masses for m_1 and m_2 .
- `zmZ2_eval.m` provides the functions `zmnum[x,m1,mu1,mu2[,scheme]]` and `Z2OSnum[x,m1,mu1,mu2[,scheme]]` which can be used for the numerical evaluation of z_m and Z_2^{OS} . In the case of `zmnum` the option `scheme` may take the values "OSOS", "MSOS", "OSMS" and "MSMS", where the first (last) two letters refer to the scheme of m_2 (m_1). In the case of `Z2OSnum` the values "OSOS" and "MSOS" are allowed. Depending on the value of x and the specified scheme the corresponding results from `expansions/` are loaded.
- `toGINAC.m` provides rules which maps the iterated integrals `GL[{...}, x]` and HPLs `H[... , x]` to Goncharov polylogarithms which allows for a numerical evaluation with `GiNaC` [51].

For the meaning of the symbols we refer to table 2. The exact expressions in `zmZmZ2.m` contain in addition the iterated integrals `GL[{...}, x] = I({...}, x)` (cf. eq. (2.5)) and the HPLs (`H[... , x]`), both in the notation of `HarmonicSums` [34–46].

api	lmm1	cf	ca	tr	nl	nm	nh
$\alpha_s(\mu)/\pi$	$\log(\mu^2/(m_1^{\text{OS}})^2)$	C_F	C_A	T_F	n_l	n_m	n_h
x	xOSOS	xMSOS	xOSMS	xMSMS	mu1	mu2	
x	x	$x_f(\mu_f)$	$x_q(\mu_f)$	$x_{fq}(\mu_f, \mu)$	μ	μ_f	
m1OS	m1MS	m2OS	m2MS				
m_1^{OS}	$\overline{m}_1(\mu)$	m_2^{OS}	$\overline{m}_2(\mu_f)$				

Table 2. Meaning of the symbols used in the `Mathematica` expressions.

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