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SHORT COMMUNICATION

Implications of the principle of effective stress

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Abstract

While Terzaghi justified his principle of effective stress for water-saturated soil empirically, it can be derived by means of the neutrality of the mineral with respect to changes of the pore water pressure p_w. This principle works also with dilating shear bands arising beyond critical points of saturated grain fabrics, and with patterns of shear bands as relics of critical phenomena. The shear strength of over-consolidated clay is explained without effective cohesion, which results also from swelling up to decay, while rapid shearing of water-saturated clay can lead to a cavitation of pore water. The p_w-neutrality is also confirmed by triaxial tests with sandstone samples, while Biot's relation with a reduction factor for p_w is contestable. An effective stress tensor is heuristically legitimate also for soil and rock with relics of critical phenomena, particularly for critical points with a Mohr–Coulomb condition. Therein, the p_w-neutrality of the solid mineral determines the interaction of solid fabric and pore water, but numerical models are questionable due to fractal features.

Keywords Effective stress · Interaction of solid fabric and pore water · Pore pressure neutrality of mineral · Shear bands and cracks

1 Introduction

Terzaghi [24] proposed the principle of effective stress in a paper for the first international conference on soil mechanics and foundation engineering. This short report, which constitutes Terzaghi's most important contribution to soil mechanics, is inspiring although some of his arguments are contestable. He states that the principal stress components of a 'mass of earth' are sums of 'solid phase' components and the 'neutral stress' of the pore water for full saturation, writing $n'_I = n_I - n_w, n'_{II} = n_{II} - n_w$ and $n'_{III} = n_{III} - n_w$, so that shear stresses are transmitted only by the solid. Referring to experiments with water-saturated sand, clay and concrete, he writes that the stress of the solid is effective for 'compression, distortion and failure' of such 'porous materials' independently of the 'neutral stress' n_w, and that the solid is not compressed by n_w. Writing the principle in modern terms

$$\sigma_{ij} = \sigma'_{ij} + p_w \delta_{ij} \tag{1}$$

with tensors σ_{ij} and σ'_{ij} of total and solid fabric or effective stress, respectively, pore water pressure p_w and unit tensor δ_{ij}, I derive it in Sect. 2 from the neutrality of the mineral with respect to changes of p_w.

This argument is at variance with the theory of mixtures with partial pressures and (1) cannot be justified by means of minute contact flats in a grain fabric. I show also in Sect. 2 that the argument of Shao et al. [21] for (1) is tautological, while the thermodynamic derivation by Jiang et al. [15] is legitimate for an elastic range if the mineral density is p_w-independent. Jiang et al. [15] use the elastic strain as state variable of the solid fabric, which is conjugated with an elastic stress via an elastic energy as potential. Jiang and Liu [13] propose that outside the elastic range, the solid fabric stress is smaller than the elastic stress by a factor which grows with the kinetic energy of jiggling grains. In a recent paper [8], I propose instead a state variable for the intensity of spatial fluctuations already at equilibrium. Thus, the observed response of water-saturated sand with many reversals is captured for the stable range up to its verge; therein, (1) captures the interaction of solid and water as the p_w-neutrality of the mineral is independent of spatial fluctuations.

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Employing notions of materials science, Terzaghi [24] defines failure as shear rupture in the sense of Mohr and derives directions of shear planes from tangents of effective stress circles. He shows that over-consolidated clay has an effective cohesion c' depending on the actual void ratio e, and an effective friction angle ϕ' depending on σ', while the shearing resistance without preloading is proportional to σ'. He does not call states of stationary shearing critical as proposed by Casagrande [4] for sand, and later by Roscoe [19] also for clay. Jiang and Liu [13] propose an elastic energy with critical or saddle points which are equivalent to a Mohr–Coulomb condition with $c' = 0$. I modified it so that a grain fabric can also be critical for an upper bound of e; with water saturation and (1), sand then collapses into a mush [9].

Roscoe said to me 1969 'uniform critical states are an Eldorado, a golden country where you never get' and that therefore the book of Schofield and Wroth [20] appeared too early. I explain in Sect. 3 that evolutions of shear band patterns are critical phenomena in the sense of Sornette [22], namely successions of critical points. With them, soil is no more a material with local action, nor a continuum with Cauchy stress and deformations via differentiable displacements [8]. A dead end can be circumvented by spatially averaged stresses, deformations and pore pressures, and with them, (1) is still legitimate as the p_w-neutrality of the mineral is not impaired by spatial fluctuations. Such an approach works apparently well with so slow changes of e that excess pore pressures are avoided, but a closer inspection shows that remoulded clay after consolidation has no effective cohesion. I show also that the rupture of rapidly sheared saturated clay can be explained with a cavitation of pore water, which can occur also with a capillary entry by shrinkage. Details of such critical phenomena, which do not impair the p_w-neutrality of minerals, elude as yet mathematical treatment.

Terzaghi [24] states that his principle is valid likewise for concrete and marble. However, rock is no more a material with shear bands and cracks as relics of critical phenomena because these can diverge in a fractal way (Sect. 4). Differently from sand and clay, sedimentary rock has cohesion due to condensate bridges, and therefore a wider elastic range. Biot [3] derived a variant of (1) with a reduction factor for p_w by means of the elastic energy of solid and pore water as potential. I show in Sect. 4 that this argument is contestable and not supported by triaxial 'jacketed' and 'unjacketed' tests. Multi-stage triaxial tests with water-saturated sandstone [16] prove the validity of (1) with overall stresses, which can again be attributed to the p_w-neutrality of minerals. This is a perspective for critical phenomena of geomatter with pore water from geotechnical operations to bigger parts of the lithosphere (Sect. 5).

2 Water-saturated soil in the stable range

A one-dimensional precursor of (1), which can be written

$$\sigma = \sigma' + p_w, \tag{2}$$

was proposed by Terzaghi [23] for the consolidation of saturated clay. This is a stabilization so that limit states are not attained. Assuming the solid fabric of mineral particles as linearly elastic, and combining its compression with Darcy's law, the conservation of pore water mass leads to a diffusion equation for p_w. This first mathematical model of an interaction of a porous solid with pore water was attacked by Fillunger [6], but apart from a factor 2 and the misnomer 'dissipation of pore pressure' instead of 'diffusion of pore water', it is consistent for a linear range [7].

The total pressure of a mixture of fluids or gases is the sum of its partial pressures, which are proportional to their mass fractions. Such an approach is at variance with (2) as this equation does not describe a mixture, but a porous solid interacting with its pore water. Therefore, attempts with the theory of mixtures [5] are questionable as they imply reduction factors for solid and fluid pressures depending on their mass fractions. A reduction factor for p_w just below 1 for contact flats between solid particles in a fabric under stress is likewise misleading: Terzaghi [24] refers to experiments with pressures up to some hundred atmospheres so that the area fraction of contact flats in a wavy plane through the solid fabric is not always minute. His principle must have another physical reason.

Shao et al. [21] present an overview of publications concerning (1). They take over the notion 'skeleton' from Terzaghi [23], which he later replaced by 'porous solid' [24]. The second notion is more adequate as a fabric of solid particles has no hinges like a skeleton. Shao et al. [21] derive (1) from differential equations of static equilibrium (including gravity) with total stress σ_{ij}, and for the 'skeleton' with stress σ'_{ij} and gradient of pore water pressure $\nabla_i p_w$. The difference of both equations leads to (1), so this equation is equivalent to taking $\nabla_i p_w$ as the specific seepage force transferred to the 'skeleton' by the pore water. This argument is tautological, and it does not justify Terzaghi's principle as no properties of solid and water are taken into account.

On the other hand, the *p_w-neutrality of the mineral* is sufficient for $\sigma'_{ij} = \sigma_{ij} - p_w\delta_{ij}$ by (1) to be the stress tensor of the solid fabric. In an experiment with a volume element of water-saturated soil, a quasi-static change of $\frac{1}{3}\sigma_{ii}$ and of p_w by the same amount would thus not change the state of the solid fabric, in particular its stress σ'_{ij} and void ratio e. In an equivalent experiment, σ'_{ij} and e of a water-saturated sample fixed in a device are not changed by changing p_w. This is a summary of Terzaghi's [24] experiments with a precise substitute of his 'neutral stress' p_w and his 'incompressibility of porous materials'. It implies that elastic compliances and activation energies (for mutual dislocations) of mineral crystallites do not depend on the pressure p_w of surrounding water in the pore system. The latter is interconnected ($\pi o \rho o \varsigma$ means passage), so we exclude micro-cavities between crystallites which would reduce activation energies and make the mineral compressible. Extremely high pressures p_w could compress crystallites and increase their activation energies, but minerals are apparently p_w-neutral for p_w from ca -20 MPa (cavitation) to almost 100 MPa, beyond the compression of crystallites could matter.

Jiang et al. [15] propose a *thermodynamic* argument for more or less saturated soils, considering particularly the effective stress. I focus here on the case of full saturation, i.e. $S_r = 1$. The authors consider a solid fabric at equilibrium as a 'frozen' mixture without structure for which a mean-field approach suffices. Changes of equilibrium states are considered in a stable elastic range. (1) is obtained as limiting case of $\sigma'_{ij} = \sigma_{ij} - p_T \delta_{ij}$ with a 'thermodynamic' pressure p_T which equals p_w for $S_r = 1$. Therein, the potential energy $w = w^e + w^w$ of solid and water per unit of fabric volume is a function of the gross mass densities $\bar{\rho}_s \equiv (1-n)\rho_s$ and $\bar{\rho}_w \equiv n\rho_w$ with the pore fraction $n = 1/(1+e)$, and of the elastic strain ϵ^e_{ij} of the solid fabric. So its differential $dw = \mu_s d\bar{\rho}_s + \mu_w d\bar{\rho}_w - \sigma^e_{ij} d\epsilon^e_{ij}$ is total with the chemical potentials $\mu_s \equiv \partial w/\partial \rho_s$ and $\mu_w \equiv \partial w/\partial \rho_w$ and the effective stress $\sigma'_{ij} \equiv \sigma^e_{ij} = -\partial w^e/\partial \epsilon^e_{ij}$ (negative as pressure and extension are positive).

Jiang et al. [15] state that only thus the conservation laws and the second law of thermodynamics can be satisfied, but call their derivation incomplete as (i) convective nonlinearity and dissipative terms are omitted, (ii) the uniqueness of separation is not proven, and (iii) the solid mineral density ρ_s is assumed to be independent of elastic strain ϵ^e_{ij}. (iv) Does not matter as the additivity of masses and energies implies the one of partial pressures as long as diffusive transfers between solid and water are negligible. (5) Belongs to the p_w-neutrality of the mineral as this implies a constant ρ_s so that it is not changed by elastic deformations.

requires a more elaborate consideration. While 'convective non-linearity' may be left aside for small deformations, dissipative terms arise beyond the elastic range. For this case, Jiang and Liu [13] propose a reduction of the effective stress as against the elastic one by

$$\sigma'_{ij} = (1-\alpha)\sigma^e_{ij} = (1-\alpha)\partial w^e/\partial \epsilon^e_{ij} \tag{3}$$

with an $\alpha \leq$ ca 0.8 increasing with a granular temperature T_g which represents the kinetic intensity of jiggling grains (pressure and contraction positive). Equations for the evolution of the elastic strain ϵ^e_{ij} with the strain rate $\dot{\epsilon}_{ij}$, including a relaxation by T_g, constitute a realistic constitutive model from elastic to hypoplastic behaviour. Therein, the specific elastic energy w^e depends on ϵ^e_{ij} and the void ratio e so that a wide range of differential stiffness is captured, and Onsager's symmetry relation for the entropy production is extended for a non-thermal entropy related to T_g.

This theory does not work for the vicinity of equilibria with $T_g \rightarrow 0$; therefore, hysteresis and ratcheting are not properly captured. As visible with an assembly of photoelastic discs, e.g. in Fig. 1 by Behringer et al. [2], intergranular forces are transmitted at equilibrium via chains with spatial fluctuations. Therefore, a fraction of the elastic energy is entropy-like and does not take part as potential for the solid stress σ'_{ij}. This reduction can again be captured with (3), but now α is proportional to the intensity χ of the *force-roughness* [9] which is visible, e.g. in Fig. 1. Taking over otherwise relations from Jiang and Liu [13], except w^e for high e (Sect. 3) and proposing a rate-independent evolution of χ, I capture thus experimental findings of Wichtmann [26] with water-saturated sand in triaxial tests without and with reversals for the stable range. This works because driven attractors of grain fabrics, i.e. asymptotic responses to strain paths without and with reversals (shakedown and ratcheting, [7]), are captured.

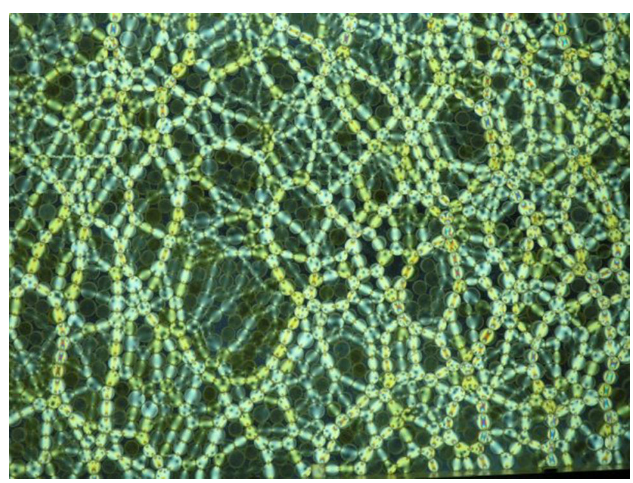

Fig. 1 Force chains in a fabric of photoelastic discs [2]

Therein, the p_w-neutrality of the grain mineral, which justifies (1), is valid independently of the force roughness.

Turning again to clay, we consider *shrinkage* with full saturation in the stable range. Leaving aside gravity, a spherical lump of initially soft clay exposed to the air shrinks as the negative p_w (suction) equals the isotropic fabric pressure, i.e. $p' = -p_w$ with $p = 0$ by (1) (relative to the atmospheric pressure). This kind of consolidation [23] comes to an end when the negative p_w reaches the Kelvin suction determined by the vapour pressure of the surrounding air, or when $-p_w$ reaches the capillary entry pressure of the solid fabric [7]. After a capillary entry, the clay lump is no more fully saturated, and shrinkage cracks can arise at critical points (Sect. 3).

Swelling occurs after placing a consolidated lump of clay in water. This reverse consolidation [23] starts at the free surface where the suction first disappears and leads to a dissolution of the solid fabric if the osmotic repulsion of the clay particles equals the attraction, which is achieved with remoulded clay and the same pH $\approx$ 7 of free and pore water without salt. Thus, the expansion of the solid fabric is elastic up to its decay, like with sand but with bigger changes of e. This means that water-saturated remoulded clay after consolidation has no effective cohesion in a physical sense, i.e. its isotropic tensile strength is solely due to the pore water (Sect. 3). This statement is at variance with Terzaghi's [24] view which prevails until present.

3 Water-saturated soil with critical phenomena

The specific elastic energy w^e of a grain fabric, as proposed by Jiang and Liu [13] and modified in my recent paper [9], has *critical points* which are equivalent to critical states with a Mohr–Coulomb condition via (3). One of them is visible in Fig. 2 as a saddle point with two

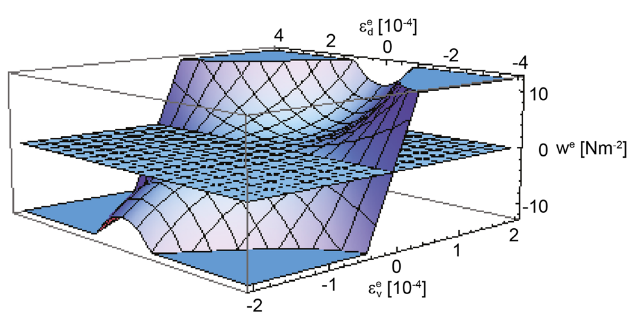

Fig. 2 Specific elastic energy versus volumetric and deviatoric invariants of elastic strain of a grain fabric near a critical point with a tangential plane

invariants of elastic strain ϵ_{ij}^e and a certain void ratio e. (With a third invariant, the hexagonal Mohr–Coulomb cone is better approximated, as proposed by Jiang and Liu [14], but then critical points are no more visible.) A shear band arises in a grain fabric at such a critical point, but remains a single one at best in a thin layer sheared between two rough plates.

X-ray photographs exhibit an evolving pattern of shear bands in a biaxial setup with dry sand before one of them dominates, and this is also obtained by numerical simulations with polar quantities [7]. Such successions of critical points are *critical phenomena* which produce rather fractal patterns [22]. The evolution of shear bands is driven by boundary conditions, and it leaves back relics which can be erased by cyclic shearing with constant pressure. Thus, samples in biaxial or triaxial setups can lose their initial uniformity of elastic strain ϵ_{ij}^e, stress by (3) and void ratio e, but can regain it by cyclic shearing with small amplitude and constant pressure p'. In other words, a strange attractor with fractal features can be followed by a cyclic driven attractor [7].

Shear band patterns can also arise in *water-saturated sand bodies*, but the interaction of grain fabric and pore water enables a greater variety of critical phenomena. The p_w-neutrality of the mineral is valid as it does not depend on spatial fluctuations of elastic strain and void ratio, but these notions of continuum models get questionable as spatial and temporal distributions are no more differentiable and as the principle of local action gets lost with diverging shear bands [8]. All the more so as loose water-saturated sand can collapse into a mush. (Liquefaction is a misnomer as sand mush does not flow like a liquid.) The onset of this critical phenomenon is captured by an additional critical point of w^e with respect to the void ratio e at its upper bound [9]. On the other hand, Casagrande's critical states are rather chaotic successions of granular critical phenomena, which confirms Roscoe's statement that uniform critical states cannot be attained (Sect. 1).

Due to the p_w-neutrality of the mineral, the principle of effective stress is still legitimate with spatial fluctuations from critical phenomena, but in a heuristic version of (1), viz.

$$\hat{\sigma}_{ij} = \hat{\sigma}'_{ij} + \hat{p}_w \delta_{ij} \tag{4}$$

with *overall values* of stress tensors and pore water pressure. Therein, $\hat{\sigma}'_{ij}$ is related to an overall elastic strain $\hat{\epsilon}_{ij}^e$ by (3) with an α which increases with an overall force roughness $\hat{\chi}$. $\hat{\sigma}_{ij}$ can be observed at the surface of a sample, so it is a spatial average with dilated shear bands. $\hat{p}_w$ equals p_w at an endplate for states of rest, but not during the evolution of shear bands as p_w drops therein due to dilation. The sample is no more a volume element with uniform stress and elastic strain in Cauchy's sense, and the relation (3) of effective stress and elastic strain is contestable as gradients are no more strictly given for lack of differentiability [8]. Fluctuating distributions cannot be averaged out like with materials for which shear bands have to be confined: shear bands in sand bodies can diverge together with the force roughness so that there are no representative volume elements. Therefore, sand with shear bands, which are ubiquitous in the ground, is no more a material which can fail to meet technical requirements. Nevertheless, a Mohr–Coulomb limit condition with $\hat{\sigma}'_{ij}$ is empirically legitimate, also with localized dilation, but numerical simulations with it are inevitably imprecise.

Shear bands arise also in *water-saturated clay* if their elastic energy attains a critical point. Terzaghi [24] refers to drained experiments with differently consolidated clay, which were later published by Hvorslev [12], and derives an increasing effective cohesion c' for an increasing over-consolidation, while the direction of shear bands is nearly given by the tangent to the actual effective stress circle. This c', determined with such tangents by extrapolation to $\sigma' = 0$, is at variance with the lack of mutual attraction of clay particles in water concluded from swelling (Sect. 2). Hvorslev [12] observed single shear bands in ring shear tests with thin clay layers, and patterns of shear bands with samples in a triaxial device. A critical point of the fabric of clay particles can at best be concluded from a single shear band if p_w is hydrostatic, but uniform critical states of clay are illusory as Roscoe said (Sect. 1), the more so as p_w is not hydrostatic except permanent states of rest because of the low permeability. The principle of effective stress (4) with overall values is a cruder approximation than with sand as non-uniformities of stress, void ratio and pore pressure are more marked.

Shear tests with *thin layers* of remoulded and consolidated saturated clay by Balthasar et al. [1] exhibit features beyond those outlined by Terzaghi [24], as shown in Fig. 3. Ca 3 mm thin plates of clay were produced by consolidation in an oedometer with pressures p_c from 2 to 14 MPa and placed between filter plates in a shear apparatus (a). They were coherent by suction $-p_w \approx p_c$ without capillary entry. This suggests $c' \propto p_c$ as proposed by Terzaghi and Hvorslev, but the tensile strength is solely in the pore water. Pulling or bending such a plate can lead to a *cavitation* of the pore water at a critical point in the sense of Griffith [10], i.e. a saddle point of the sum of elastic energy of the solid fabric and surface energy of pore water for a growing crack. This loss of stability can no more be captured with specific energies as in the mean-field theory of Jiang and Liu [13].

A single shear band arises and is partly exposed in a sheared thin layer, and therein, clay particles are aligned, and their fabric is dilated (Fig. 3b). The overall shearing resistance $\hat{\tau}$ is not only higher for higher p_c, but rises and drops with changes of the shear rate by orders of magnitude (c). The initial jumps of $\hat{\tau}$ can be attributed to temporal jumps of p_w in the sample as it cannot dilate or contract immediately despite minute drainage lengths, which indicates jumps of the effective pressure $\hat{\sigma}'$ for a constant total pressure due to a *nonlinear viscosity* of the clay fabric. The asymptotic shear stress $\hat{\tau}$ after the equalization of $\hat{p}_w$ is proportional to $\hat{\sigma}'$ (d), i.e. the frictional resistance $\hat{\tau}_c = \hat{\sigma}' \tan \phi'$ is rate-independent. Tests with a constant $\hat{\tau} > \hat{\tau}_c$ and over-consolidation lead to a sudden rupture after creep, which can be attributed to a cavitation after localized shearing with dilation which is delayed as the access of pore water.

This evaluation works with the principle of effective stress in the version (4) with overall quantities. Two-dimensional finite element simulations with a visco-hypoplastic constitutive relation and (1) enable a more detailed insight [1], but therein, the thickness of shear bands is given by the element size and not by the size of clay particles, and the sudden cavitation cannot be captured. Three-dimensional numerical simulations of triaxial tests with many shear bands and cracks are still out of reach. Shear bands and cracks as relics of critical phenomena, typical of stiff fissured clay, can at best be captured by means of 'undisturbed' samples, let alone enclosed gas and sandwich formations of clay and fine sand. The p_w-neutrality of the mineral is valid despite this complexity, but calculations with overall quantities are inevitably imprecise.

4 Water-saturated porous rock

Terzaghi [24] points to experiments with concrete and marble which confirm his principle, although without showing how a sufficiently uniform pore water pressure p_w was achieved and registered with a very low permeability. Certainly, the samples were rather uniform, so the effective stress by (1) is determined by boundary values of samples and legitimate together with the p_w-neutrality of the mineral. Different to remoulded and consolidated water-saturated clay, concrete and sandstone have an effective cohesion c', defined as isotropic tensile strength, due to mineral condensate bridges of the grains. A Mohr–Coulomb condition with effective stress can capture limit states as critical states with ϕ' and c' for shear ruptures, while cracking can be captured by the concept of Griffith [10] and his followers. The objective of materials science is to confine shear bands and cracks, but as both are not likewise

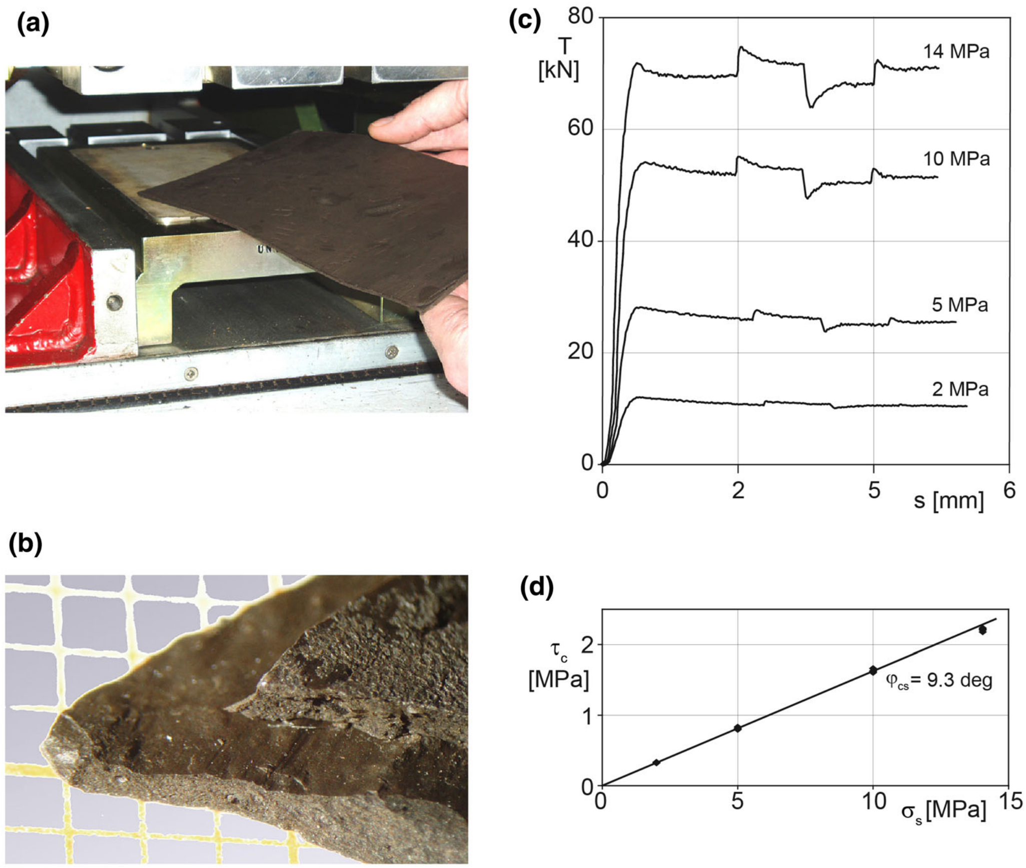

Fig. 3 Thin layer shear tests with a plastic clay [1]: **a** placing a precompressed clay disc with suction upon a filter plate (movable loading plate lifted), **b** shear band after an experiment (square width 1 mm), **c** resistance to shearing with velocities suddenly increased by factor 10 and reduced by factor 100 under different pressures, **d** shearing resistance versus pressure (except just after a sudden change of velocity)

confined in rock, this is not a material in general. So how far is an overall effective stress by (4) legitimate?

Biot [3] proposes energy-based relations for fluid-saturated porous rock. I confine in the sequel to an isotropic solid with pore water and use notations as elsewhere in this paper. Biot's elastic energy w^e (per unit volume of the porous fabric) depends on the elastic strain ϵ^e_{ij} of the fabric and on the volume change $\zeta \equiv n\nabla_i(u_{wi} - u_{si})$ of the pore water due to its relative displacement $u_{wi} - u_{si}$ and volume fraction n. Taking the differential dw^e as total, Biot's $w^e(\epsilon^e_{ij}, \zeta)$ is the potential of elastic stress by $\sigma^e_{ij} = \partial w^e/\partial\epsilon^e_{ij}$ and of pore water pressure by $p_w = \partial w^e/\partial\zeta$. Confining to a linear range and therefore representing w^e only with second-order terms in ϵ^e_{ij} and ζ, w^e has quadratic terms in ϵ^e_{ij} and ζ and a mixed term $\alpha\epsilon^e_v\zeta$ with a *coupling factor* $0 \le \alpha < 1$, wherein $\epsilon^e_v \equiv \frac{1}{3}\epsilon^e_{ii}$ denotes the elastic volume change of the solid fabric. This leads to $p_w = K_w(\zeta - \alpha\epsilon^e_v)$ with the net volume change $\zeta - \alpha\epsilon^e_v$ of pore water and its compression modulus K_w. Taking over Terzaghi's relation (1) with an effective stress σ'_{ij} 'for slip and failure' of the solid, Biot derives

$$\sigma_{ij} = \sigma^e_{ij} + \alpha p_w \delta_{ij} \tag{5}$$

with an elastic stress σ^e_{ij} determined by a linear isotropic relation with the net elastic strain $\epsilon^e_{ij} + (\zeta - \alpha\epsilon^e_v)\delta_{ij}$ of the fabric and his coupling factor α.

Biot's argument is contestable as his specific volume change ζ of pore water is not an objective state variable like its density ρ_w. Moreover, the mixed term $\alpha\epsilon^e\zeta$ in his elastic energy $w^e(\epsilon^e_{ij}, \zeta)$ cannot occur with the correct decomposition $w^e = w^e_s(\epsilon^e_{ij}) + w^e_w(n\rho_w)$, as employed, for example, by Jiang et al. [15]. Applying (5) to an isotropic compression without drainage leads to

$$\alpha = 1 - K'/K_s \tag{6}$$

with the compression moduli of the porous fabric K' and of the mineral K_s. This relation is apparently reasonable, but misleading as are Biot's variable ζ and his mixed term $\alpha\epsilon^e_v\zeta$. Without the latter Biot's distinction of Terzaghi's σ'_{ij} and his stress, σ^e_{ij} related to a 'net elastic strain' disappears.

Nur and Byerlee [17] take over the elastic response of solid fabric and pore water from Biot [3] and study further arguments for (5). They point out that a solid with a single cavity would be compressible, and thereafter, they consider connected pores ($\pi o \rho o \varsigma$=passage). Their derivation of (5) by means of the internal surface of the pore system is not tractable. Their experiments with a sandstone, dry in a metal sleeve or water-saturated without confining membrane, seemingly speak for (5), while experiments with a granite indicate a strongly nonlinear compressive response with sleeve and a stiffer linear one without sleeve. Nur and Byerlee [17] conclude that Biot's (5) with (6) is questionable except with $\alpha = 1$ for an incompressible mineral. The latter is equivalent to the p_w-neutrality of the mineral for justifying (1), which is practically equivalent to (5) and (6) with $K_s \gg K'$ due to the pore system except extremely low pore fractions. Attempts to calibrate K' and K_s with 'jacketed' (dry) and 'unjacketed' (saturated without mould) compression tests are contestable: dry rock samples have a higher internal surface energy than wet ones, and wet rock samples without a mould can attain substantial p_w-gradients.

Biot's theory was the first attempt to capture the interaction of rock with pore water with an energy-based approach, and his famous two pressure waves can in fact arise as the compressibility of solid mineral and pore water is not generally negligible and as the one of a porous fabric can be far bigger. However, two P-waves can be achieved without the contestable quantities ζ and α. As outlined in Sect. 2, Jiang et al. [15] obtain Terzaghi's relation (1) for full saturation and constant densities ρ_s and ρ_w of solid mineral and water. For these authors, the elastic stress σ^e_{ij} agrees with the effective one σ'_{ij}. Another difference of Biot's σ^e_{ij} by (5) and σ'_{ij} arises by (3) (wherein α is not the same as in (5)!) with constant ρ_s and ρ_w outside the elastic range due to fluctuations, and then, the elastic energy w^e of the solid engulfs critical points.

All these mean-field approaches assume an *amorphous fabric* so that cracks and shear bands are excluded, while Lempp et al. [16] focus on structural changes of the solid fabric in multi-stage triaxial tests with water-saturated sandstone. Initially, intact samples were repeatedly brought to limit states by increasing the overall amount of stress deviator $|\hat{\sigma}_1 - \hat{\sigma}_3|$ and/or pore water pressure $\hat{p}_w$, avoiding disintegration by means of a servo-control. Each limit state was approached quasi-statically with an overall dilation so that $\hat{p}_w$ came close to p_w at the endplates. The subsequent confined collapse due to a drop of $|\hat{\sigma}_1 - \hat{\sigma}_3|$ occurred with an audible noise and a contraction of the pore system so that p_w at the endplates rose suddenly, but less than $\hat{p}_w$ as the spatial distribution of p_w was no more uniform. After several such stages, the rubber mould was removed and the cylindrical surface was photographed, e.g. Fig. 4. One can see shear bands in a rather fractal pattern, and cracks aligned with them, as *relics of critical phenomena* which

Fig. 4 Unrolled combined photographs of a sandstone sample after a multi-stage triaxial test [16]

impair the initial uniformity. In contrast with these findings, one-stage tests with $\hat{\sigma}'_3 \to 0$ or $\hat{\sigma}'_1 \to 0$ led to axial splitting or horizontal discing, respectively, whereas with a

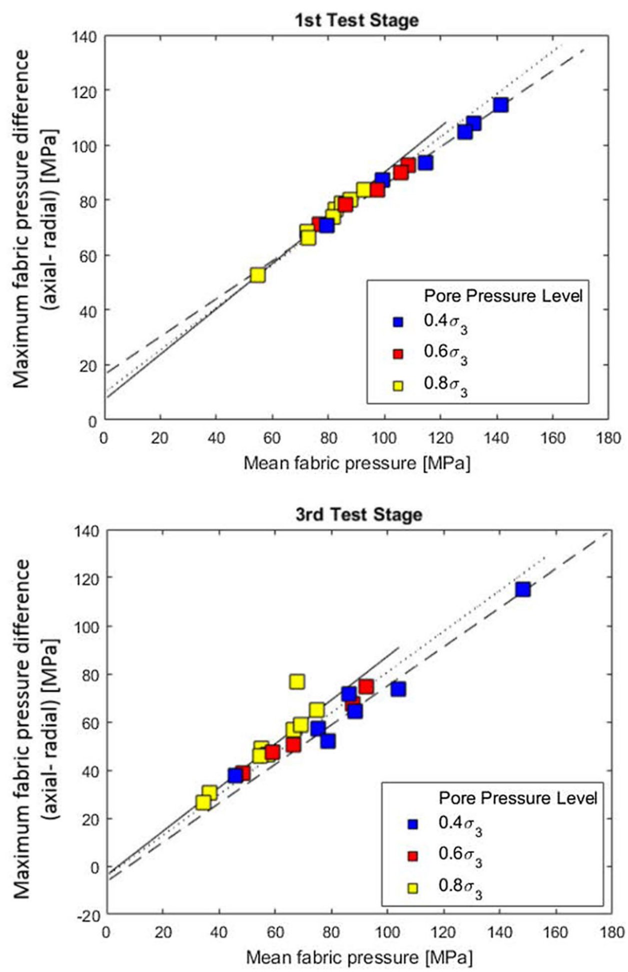

Fig. 5 Limit stress states of water-saturated sandstone samples, attained in multi-stage triaxial tests (modified from Lempp et al. [16]). Deviators of overall stress versus spatial mean overall pressure minus pore pressure at one endplate

sufficient $\hat{p}' \equiv \frac{1}{3}(\hat{\sigma}'_1 + 2\hat{\sigma}'_3)$, shear band patterns arose without cracks.

Taking 22 intact samples with the same orientation from a homogeneous block, limit stress states were attained and plotted as $|\hat{\sigma}_1 - \hat{\sigma}_3|$ versus $\hat{p}'$ for successive test stages, e.g. Fig. 5 with axial shortening. The first limit states are captured with a straight line which corresponds to a Mohr–Coulomb condition with $\phi' \approx 50^0$ by tangents of stress circles and $c' \approx 15$ MPa by extrapolation to $p' = 0$ (upper half of Fig. 5). As this finding is obtained with different ratios p_w/σ_3, it confirms Terzaghi's principle for the shear rupture of a material. Limit states after two previous ones (lower half) can be captured with a Mohr–Coulomb condition with almost the same ϕ', but with $c' \approx 0$ and a bigger scattering. This finding can be attributed to shear bands and cracks from previous test stages which damage condensation bridges, while the p_w-neutrality of the mineral is not impaired by more marked spatial fluctuations with relics of critical phenomena.

The findings of Lempp et al. [16] deepen the understanding of *tectonic critical phenomena* beyond Sornette 22, but rise several questions. A Mohr–Coulomb condition with overall stresses helps explain tectonically active parts of the lithosphere with the World Stress Map, but therein, stresses depend on the mesh size [11], which speaks for fractal spatial distributions so that the size of volume elements with the same centre influences overall stresses. The limit of $|\hat{\sigma}_1 - \hat{\sigma}_3|/\hat{p}'$ (or an invariant substitute) for $c' \rightarrow 0$ suits to the p_w-independence of the mineral and the independence of solid friction forces on the spatial distribution of solid bridges [18]. On the other hand, the isotropic tensile strength of the solid fabric dwindles with a growing size of a volume element and is spatially and temporally variable. The succession of driven dilation and spontaneous contraction in multi-stage triaxial tests is the clue to seismogenic chain reactions in the lithosphere, but we are far from a consistent mechanical model for them.

The issue gets more complex for rock with very low permeability and for compounds of it with more permeable formations. We can leave aside limestone or granite without cracks which are materials with a so low porosity that one can hardly speak of a pore system. Layers of mudstone and clay smears from them in faults can work as fluid seals as long as they are not interrupted by tectonic faulting [25]. The permeability of sedimentary rock is orders of magnitude bigger with cracks and shear bands, or joints and faults in the large, but then pore fluids are no more captured by Darcy's law and the continuity equation for lack of differentiability [8]. The interaction of solid fabric and pore water is still determined by the p_w-neutrality of minerals, but constitutive relations and balance equations with overall quantities are more questionable with relics of critical phenomena which cannot be swept out like with soils.

5 Conclusions

Terzaghi's principle of effective stress for water-saturated soil can be derived with the neutrality of the mineral with respect to changes of the pore water pressure p_w. It is thermodynamically correct for the stable elastic range of grain fabrics if the mineral density is p_w-independent, wherein the effective stress tensor is related to the elastic strain tensor via a specific elastic energy. A reduction of the effective stress tensor in this relation was proposed for jiggling grains, but a recently proposed reduction for spatially fluctuating force chains can better capture shakedown and ratcheting of water-saturated sand. The principle of effective stress was first proposed by Terzaghi for the consolidation of water-saturated clay, which occurs also for shrinkage prior to capillary entry and cracking. Different to Terzaghi, I conclude from swelling under water after consolidation that reconstituted fabrics of clay particles have no effective cohesion.

Shear bands arise in sand when the elastic energy of the grain fabric attains a critical point with respect to its elastic strain. Rather fractal patterns of shear bands arise with dilation as critical phenomena, with them spatial fluctuations diverge so that Cauchy stress fields and gradients are contestable. The principle of effective stress is still legitimate for saturated sand with shear bands by using spatial averages of stress, pore pressure and void fraction as the p_w-neutrality of the mineral is not impaired by spatial fluctuations. This works also for saturated clay if the viscosity of the solid fabric is taken into account. Different to Terzaghi's and Hvorslev's standpoint, the resistance to drained shearing after over-consolidation can be explained without an effective cohesion, while the total cohesion without drainage can be attributed to suction which can lead to a cavitation of pore water.

Biot's effective stress with a coupling factor for reducing p_w, derived with a specific energy of porous rock and its pore water, is contestable and misleading. His relation tends to the one of Terzaghi as the elastic compression modulus of the mineral exceeds by far the one of the fabrics except very small pore fractions. Triaxial tests with saturated sandstone confirm Terzaghi's relation with effective friction angle and cohesion in a Mohr–Coulomb condition for intact samples. Including cracks and shear bands as relics of previous critical phenomena and employing overall quantities, the same condition works with nearly the same friction angle, but with dwindling cohesion and bigger scattering. Successions of driven localized dilation and spontaneous rapid contraction of the pore system resemble slow and fast tectonic evolutions in the lithosphere, such critical phenomena elude as yet numerical simulations because of their fractality.

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