Explicit Port-Hamiltonian Formulation of Bond Graphs with Dependent Storages

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Abstract: Explicit port-Hamiltonian systems (PHSs) are the starting point for many powerful controller and observer design methods. It is well-known that explicit PHSs can be formulated on the basis of bond graphs. Indeed, the port-Hamiltonian formulation of bond graphs *without* dependent storages has been well investigated. However, little effort has been made towards bond graphs *with* dependent storages. This is a problem as dependent storages frequently occur in models from many engineering fields. In this paper, we address the explicit port-Hamiltonian formulation of bond graphs with dependent storages. Our idea is to express the port-Hamiltonian dynamics and output as functions of only the system inputs and *independent* storages. The main result is a rigorous and constructive method to formulate bond graphs containing dependent storages as explicit PHSs. An acadamic example illustrates and verifies our method.

Keywords: Port-Hamiltonian systems; bond graphs; dependent storages; state-space models; model generation

1. INTRODUCTION

The theory of port-Hamiltonian systems (PHSs) provides powerful controller and observer design methods for nonlinear systems. Most of these methods base on a system model in form of an *explicit* input-state-output PHS [1] , see e.g., Ortega et al. (2008); Dörfler et al. (2009); Venkatraman and van der Schaft (2010); Vincent et al. (2016); van der Schaft (2016). Bond graphs are a reasonable starting point for the structured and automated derivation of such a model (Duindam et al., 2009; Donaire and Junco, 2009; Lopes, 2016; Pfeifer et al., 2019). Recently, it has been shown that a bond graph with *independent sources* and *independent storages* can always be formulated as an explicit PHS (Pfeifer et al., 2019). The independence of sources is necessary for the existence of such a PHS; however, the independence of storages is part of a sufficient condition. It would be desirable to relax this sufficient condition to the case of *dependent* storages.

Bond graphs with dependent storages can be found in many engineering fields as multibody systems (Breedveld, 2018), mechatronic systems (Karnopp et al., 2012, p. 190), and electric circuits (Najnudel et al., 2018) to name a few. There exist various strategies to resolve the dependence of storages by modifying the bond graph (Karnopp et al., 2012, p. 609). These strategies base on the addition or elimination of certain bond graph elements. However, such measures alter the physics of the model which is why they are often undesirable. In the literature, there has been made little effort to the port-Hamiltonian formulation of bond graphs with dependent storages. Golo et al. (2003) have shown that a bond graph (that possibly contains dependent storages) can be formulated as an *implicit* PHS. However, the transfer from an implicit to an explicit formulation is not trivial. In particular, the existence of an explicit formulation of an implicit PHS is not guaranteed (Pfeifer et al., 2019). Donaire and Junco (2009) addressed the derivation of an *explicit* PHS from a bond graph possibly containing dependent storages. The work of Donaire and Junco (2009) is inspiring but has several shortcomings: (i) no output equation for the PHS is derived; (ii) the existence of some inverse matrices is not discussed (e.g., Equation (14) in Donaire and Junco (2009)); (iii) the approach is restricted to non-feedthrough systems. Apart from bond graphs, Najnudel et al. (2018) addressed the derivation of PHSs from electric circuits containing dependent storages in terms of two capacitors in parallel or two inductances in series. Of course, these two scenarios are special cases with respect to all possible structures of dependent storages that can occur in a system.

As can be seen from above, the literature lacks a rigorous method for the systematic derivation of explicit PHSs from bond graphs with dependent storages. In this paper, we bridge this gap by a rigorous and constructive explicit

[1] In this paper, we focus on continuous-time, real-valued, finite-dimensional PHS.

port-Hamiltonian formulation for such systems. The remainder of this paper is organized as follows: The problem under consideration is defined in Section 2. In Section 3, we present, discuss, and prove the main result of this paper. The result from Section 3 is illustrated through an academic example in Section 4. Section 5 provides the conclusion of this paper.

Notation: Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a square matrix. We write $\boldsymbol{A} \succ 0$ and $\boldsymbol{A} \succeq 0$ if $\boldsymbol{A}$ is positive definite or positive semi-definite, respectively. Spec($\boldsymbol{A}$) is the spectrum of $\boldsymbol{A}$, i.e., the set of its eigenvalues. Identity matrices are denoted as $\boldsymbol{I}$; zero matrices are written as $\boldsymbol{0}$. Throughout this paper, we omit the time-dependence "(t)" of variables and vectors in the notation.

2. PROBLEM DESCRIPTION

In this paper, we consider bond graphs in the generalized bond graph framework (Duindam et al., 2009, p. 24) with the following types of elements: storages (C), modulated resistors (R), sources of flow (Sf), sources of effort (Se), 0-junctions (0), 1-junctions (1), modulated transformers (TF) and modulated gyrators (GY). Concerning the storages, we differ between *independent* storages ($\mathrm{C_i}$) and *dependent* storages ($\mathrm{C_d}$).

Definition 1. An *independent storage* is a C-type element which can be described by an energy variable independent of the energy variables of the other storage elements.

Definition 2. A *dependent storage* is a C-type element whose energy variable is dependent on the energy variable of an independent storage (Borutzky, 2010, p. 108).

Suppose a bond graph which consists of n elements that are connected by m bonds. Each bond $j \in \{1,\ldots,m\}$ carries a flow $\boldsymbol{f}_j \in \mathbb{R}$ and an effort $\boldsymbol{e}_j \in \mathbb{R}$. We assume the following properties to ensure that the bond graph is non-degenerate: (i) the bond graph is weakly connected; (ii) each element of type C, R, Sf, and Se is connected by exactly one bond to an element of type 0, 1, TF, or GY; (iii) all sources are independent, i.e., the junction structure does *not* imply a dependency between flows of Sf and/or efforts of Se elements. Moreover, we assume the bond graph to have the following property: (iv) modulation of elements of type R, TF, and GY can be expressed only in dependence on states of C-type elements and constant parameters. Bond graphs that do not have this property lead in general to mathematical models in form of differential-algebraic equations (DAEs) (Borutzky, 2010, p. 159). Thus, bond graphs without property (iv) in general cannot be formulated in a purely explicit representation. Next, we define an important part of the bond graph, viz. the junction structure.

Definition 3. The *junction structure* of a bond graph is the sub-graph which contains all elements of type 0, 1, TF, and GY as well as the bond connecting these elements.

Consider a bond graph that has the properties (i) to (iv) from above. Define the set $\mathbb{E} := \{\mathrm{C_i}, \mathrm{C_d}, \mathrm{R}, \mathrm{Sf}, \mathrm{Se}\}$. For each $\alpha \in \mathbb{E}$, the number of elements of type α in the bond graph is given by n_α. Moreover, define $n_\mathrm{E} := \sum_{\alpha\in\mathbb{E}} n_\alpha$. For each $\alpha \in \mathbb{E}$ we collect the flows of all bonds that are connected to an element of type α in the vector $\boldsymbol{f}_\alpha \in \mathbb{R}^{n_\alpha}$. Correspondingly, for each $\alpha \in \mathbb{E}$ the vector $\boldsymbol{e}_\alpha \in \mathbb{R}^{n_\alpha}$ collects the efforts of all bonds that are connected to an element of type α. Recently, it was shown that the junction structure of such a bond graph can be described by a Dirac structure in input-output form (Pfeifer et al., 2019):

$$\mathcal{D} = \{ (\begin{pmatrix} \boldsymbol{f}_\mathrm{C} \\ \boldsymbol{f}_\mathrm{R} \\ \boldsymbol{f}_\mathrm{Sf} \\ \boldsymbol{f}_\mathrm{Se} \end{pmatrix}, \begin{pmatrix} \boldsymbol{e}_\mathrm{C} \\ \boldsymbol{e}_\mathrm{R} \\ \boldsymbol{e}_\mathrm{Sf} \\ \boldsymbol{e}_\mathrm{Se} \end{pmatrix}) \in \mathbb{R}^{n_\mathrm{E}} \times \mathbb{R}^{n_\mathrm{E}} \mid \begin{pmatrix} \boldsymbol{y}_\mathrm{C_i} \\ \boldsymbol{y}_\mathrm{C_d} \\ \boldsymbol{y}_\mathrm{R} \\ \boldsymbol{y}_\mathrm{P} \end{pmatrix} = \underbrace{\begin{pmatrix} \boldsymbol{Z}_\mathrm{C_iC_i}(\boldsymbol{x}) & -\boldsymbol{Z}_\mathrm{C_iC_d}(\boldsymbol{x}) & -\boldsymbol{Z}_\mathrm{C_iR}(\boldsymbol{x}) & -\boldsymbol{Z}_\mathrm{C_iP}(\boldsymbol{x}) \\ \boldsymbol{Z}^\top_\mathrm{C_iC_d}(\boldsymbol{x}) & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{Z}_\mathrm{C_dP}(\boldsymbol{x}) \\ \boldsymbol{Z}^\top_\mathrm{C_iR}(\boldsymbol{x}) & \boldsymbol{0} & \boldsymbol{Z}_\mathrm{RR}(\boldsymbol{x}) & -\boldsymbol{Z}_\mathrm{RP}(\boldsymbol{x}) \\ \boldsymbol{Z}^\top_\mathrm{C_iP}(\boldsymbol{x}) & \boldsymbol{Z}^\top_\mathrm{C_dP}(\boldsymbol{x}) & \boldsymbol{Z}^\top_\mathrm{RP}(\boldsymbol{x}) & \boldsymbol{Z}_\mathrm{PP}(\boldsymbol{x}) \end{pmatrix}}_{\boldsymbol{Z}(\boldsymbol{x})} \begin{pmatrix} \boldsymbol{u}_\mathrm{C_i} \\ \boldsymbol{u}_\mathrm{C_d} \\ \boldsymbol{u}_\mathrm{R} \\ \boldsymbol{u}_\mathrm{P} \end{pmatrix} \} \tag{1}$$

where $\boldsymbol{Z}(\boldsymbol{x}) = -\boldsymbol{Z}^\top(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathcal{X}$ and

$$\boldsymbol{u}_\mathrm{C_i} = \boldsymbol{e}_\mathrm{C_i}, \boldsymbol{u}_\mathrm{C_d} = -\boldsymbol{f}_\mathrm{C_d}, \boldsymbol{u}_\mathrm{R} = \begin{pmatrix} \boldsymbol{e}_\mathrm{R,1} \\ -\boldsymbol{f}_\mathrm{R,2} \end{pmatrix}, \boldsymbol{u}_\mathrm{P} = \begin{pmatrix} \boldsymbol{f}_\mathrm{Sf} \\ \boldsymbol{e}_\mathrm{Se} \end{pmatrix},$$
$$\boldsymbol{y}_\mathrm{C_i} = -\boldsymbol{f}_\mathrm{C_i}, \boldsymbol{y}_\mathrm{C_d} = \boldsymbol{e}_\mathrm{C_d}, \boldsymbol{y}_\mathrm{R} = \begin{pmatrix} -\boldsymbol{f}_\mathrm{R,1} \\ \boldsymbol{e}_\mathrm{R,2} \end{pmatrix}, \boldsymbol{y}_\mathrm{P} = \begin{pmatrix} \boldsymbol{e}_\mathrm{Sf} \\ \boldsymbol{f}_\mathrm{Se} \end{pmatrix}. \tag{2}$$

In (2), $\left(\boldsymbol{f}^\top_\mathrm{R,1}\ \boldsymbol{f}^\top_\mathrm{R,2}\right)^\top$ is a splitting of $\boldsymbol{f}_\mathrm{R}$ and $\left(\boldsymbol{e}^\top_\mathrm{R,1}\ \boldsymbol{e}^\top_\mathrm{R,2}\right)^\top$ is a corresponding splitting of $\boldsymbol{e}_\mathrm{R}$.

Remark 4. In (1), the zero blocks in the matrix $\boldsymbol{Z}(\boldsymbol{x})$ are due to the fact that energy variables of *dependent* storages are (by definition) functions only of energy variables of *independent storages* and inputs from sources (Wellstead, 1979, p. 226).

Remark 5. The minus signs in (2) stem from the standard bond orientation rules (Borutzky, 2010, p. 59) in which bonds are incoming to storages and resistors and outgoing from sources of flow and effort.

Remark 6. The Dirac structure in (1) can be obtained by following causal paths in the *causalized* bond graph (Donaire and Junco, 2009). Alternatively, Pfeifer et al. (2019) provide a fully automatable method for the determination of (1) from given bond graphs.

Before we state the problem studied in this paper, let us specify the constitutive relations of the C-type and R-type elements. For the C-type elements, we consider constitutive relations of the form (Borutzky, 2010, p. 357):

$$\begin{pmatrix} \boldsymbol{f}_\mathrm{C_i} \\ \boldsymbol{f}_\mathrm{C_d} \end{pmatrix} \overset{(2)}{=} \begin{pmatrix} -\boldsymbol{y}_\mathrm{C_i} \\ -\boldsymbol{u}_\mathrm{C_d} \end{pmatrix} = \begin{pmatrix} \dot{\boldsymbol{x}}_\mathrm{i} \\ \dot{\boldsymbol{x}}_\mathrm{d} \end{pmatrix}, \tag{3a}$$

$$\begin{pmatrix} \boldsymbol{e}_\mathrm{C_i} \\ \boldsymbol{e}_\mathrm{C_d} \end{pmatrix} \overset{(2)}{=} \begin{pmatrix} \boldsymbol{u}_\mathrm{C_i} \\ \boldsymbol{y}_\mathrm{C_d} \end{pmatrix} = \begin{pmatrix} \frac{\partial V_\mathrm{i}}{\partial \boldsymbol{x}_\mathrm{i}}(\boldsymbol{x}_\mathrm{i}) \\ \frac{\partial V_\mathrm{d}}{\partial \boldsymbol{x}_\mathrm{d}}(\boldsymbol{x}_\mathrm{d}) \end{pmatrix}. \tag{3b}$$

In (3), $\boldsymbol{x}_\mathrm{i} \in \mathbb{R}^{n_\mathrm{C_i}}$, $\boldsymbol{x}_\mathrm{d} \in \mathbb{R}^{n_\mathrm{C_d}}$ are the *energy states* and $V_\mathrm{i}\colon \mathbb{R}^{n_\mathrm{C_i}} \to \mathbb{R}_{\geq 0}$, $V_\mathrm{d}\colon \mathbb{R}^{n_\mathrm{C_d}} \to \mathbb{R}_{\geq 0}$ are (nonlinear) *storage functions* of the independent and dependent storage elements, respectively. The overall energy in the system is given by the composite storage function $V(\boldsymbol{x}_\mathrm{i}, \boldsymbol{x}_\mathrm{d}) = V_\mathrm{i}(\boldsymbol{x}_\mathrm{i}) + V_\mathrm{d}(\boldsymbol{x}_\mathrm{d})$.

For the R-type elements, we assume constitutive relations of the form (Duindam et al., 2009, p. 64):

$$\boldsymbol{f}_\mathrm{R} = \boldsymbol{D}(\boldsymbol{x})\,\boldsymbol{e}_\mathrm{R}. \tag{4}$$

The matrix $\boldsymbol{D}(\boldsymbol{x})$ is a positive definite diagonal matrix which collects the (possibly state-dependent) dissipation terms of the individual R-type elements. Note that we can write (4) equivalently in input-output form[2]:

$$\boldsymbol{u}_{\mathrm{R}} = -\tilde{\boldsymbol{R}}(\boldsymbol{x})\,\boldsymbol{y}_{\mathrm{R}}, \tag{5}$$

where $\tilde{\boldsymbol{R}}(\boldsymbol{x})$ is again diagonal and positive definite and $\boldsymbol{u}_{\mathrm{R}}$, $\boldsymbol{y}_{\mathrm{R}}$ as in (2).

Problem 7. The problem considered in this paper is to formulate a bond graph described by (1), (3), and (4) as explicit input-state-output PHS of the following form (Duindam et al., 2009, p. 71):

$$\dot{\boldsymbol{x}} = [\boldsymbol{J}(\boldsymbol{x}) - \boldsymbol{R}(\boldsymbol{x})]\,\tfrac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x}) + [\boldsymbol{G}(\boldsymbol{x}) - \boldsymbol{P}(\boldsymbol{x})]\,\boldsymbol{u}, \tag{6a}$$

$$\boldsymbol{y} = [\boldsymbol{G}(\boldsymbol{x}) + \boldsymbol{P}(\boldsymbol{x})]^{\top}\,\tfrac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x}) + [\boldsymbol{M}(\boldsymbol{x}) + \boldsymbol{S}(\boldsymbol{x})]\,\boldsymbol{u}. \tag{6b}$$

In (6), $\boldsymbol{x} \in \mathcal{X}$, $\boldsymbol{u} \in \mathbb{R}^{p}$, and $\boldsymbol{y} \in \mathbb{R}^{p}$ are the *state vector*, the *input vector*, and the *output vector*, respectively. The state-space $\mathcal{X}$ is a real vector space of dimension q. The Hamiltonian is a non-negative function $H\colon \mathcal{X} \to \mathbb{R}_{\geq 0}$. The matrices $\boldsymbol{J}(\boldsymbol{x})$, $\boldsymbol{R}(\boldsymbol{x}) \in \mathbb{R}^{q\times q}$, $\boldsymbol{G}(\boldsymbol{x})$, $\boldsymbol{P}(\boldsymbol{x}) \in \mathbb{R}^{q\times p}$, $\boldsymbol{M}(\boldsymbol{x})$, $\boldsymbol{S}(\boldsymbol{x}) \in \mathbb{R}^{p\times p}$ have to satisfy $\boldsymbol{J}(\boldsymbol{x}) = -\boldsymbol{J}^{\top}(\boldsymbol{x})$, $\boldsymbol{M}(\boldsymbol{x}) = -\boldsymbol{M}^{\top}(\boldsymbol{x})$, and

$$\boldsymbol{Q}_s(\boldsymbol{x}) := \begin{pmatrix} \boldsymbol{R}(\boldsymbol{x}) & \boldsymbol{P}(\boldsymbol{x}) \\ \boldsymbol{P}^{\top}(\boldsymbol{x}) & \boldsymbol{S}(\boldsymbol{x}) \end{pmatrix} = \boldsymbol{Q}_s^{\top}(\boldsymbol{x}) \succeq 0,\ \forall \boldsymbol{x} \in \mathcal{X}. \tag{7}$$

3. MAIN RESULT

In Section 3.1 and 3.2, we present and discuss the main result of this paper, respectively. Section 3.3 then provides the proof of this main result.

3.1 Presentation of the Main Result

We first make two assumptions and then state the main result in Theorem 10.

Assumption 8. In (1), we have $\boldsymbol{Z}_{\mathrm{C_dP}}(\boldsymbol{x}) = \boldsymbol{0}$ for all $\boldsymbol{x} \in \mathcal{X}$ and $\boldsymbol{Z}_{\mathrm{C_iC_d}}(\boldsymbol{x}) = \boldsymbol{Z}_{\mathrm{C_iC_d}} = \text{const}$.

Assumption 9. The storage functions in (3) are of the form $V_{\mathrm{i}}(\boldsymbol{x}_{\mathrm{i}}) = \frac{1}{2}\boldsymbol{x}_{\mathrm{i}}^{\top}\boldsymbol{D}_{\mathrm{i}}\boldsymbol{x}_{\mathrm{i}}$ and $V_{\mathrm{d}}(\boldsymbol{x}_{\mathrm{d}}) = \frac{1}{2}\boldsymbol{x}_{\mathrm{d}}^{\top}\boldsymbol{D}_{\mathrm{d}}\boldsymbol{x}_{\mathrm{d}}$ where $\boldsymbol{D}_{\mathrm{i}} = \boldsymbol{D}_{\mathrm{i}}^{\top} \succ 0$ and $\boldsymbol{D}_{\mathrm{d}} = \boldsymbol{D}_{\mathrm{d}}^{\top} \succ 0$.

Assumptions 8 and 9 will be discussed in Section 3.2. Theorem 10 now presents the main result of this paper.

Theorem 10. Given a bond graph described by (1), (3), and (4). Let Assumptions 8 and 9 hold. Then, the bond graph can be formulated as an explicit PHS of the form (6). The input, state, and output of the PHS are given as $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{P}}$, $\boldsymbol{x} = \boldsymbol{x}_{\mathrm{i}}$, and $\boldsymbol{y} = \boldsymbol{y}_{\mathrm{P}}$, respectively, with the Hamiltonian being $H(\boldsymbol{x}) = V_{\mathrm{i}}(\boldsymbol{x}_{\mathrm{i}})$. The matrices of the PHS are calculated as given in (8) on the next page with

$$\tilde{\boldsymbol{K}}(\boldsymbol{x}) = \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}(\boldsymbol{x})\,\boldsymbol{Z}_{\mathrm{RR}}(\boldsymbol{x})\right)^{-1}, \tag{9a}$$

$$\tilde{\boldsymbol{L}} = \left(\boldsymbol{I} + \boldsymbol{Z}_{\mathrm{C_iC_d}}\boldsymbol{D}_{\mathrm{d}}^{-1}\boldsymbol{Z}_{\mathrm{C_iC_d}}^{\top}\boldsymbol{D}_{\mathrm{i}}\right)^{-1}. \tag{9b}$$

3.2 Discussion of the Main Result

Theorem 10 states that under Assumptions 8 and 9 an explicit PHS can be obtained from a bond graph that may or may not contain dependent storages. This is remarkable as (in bond graph theory) dependent storages are known to lead to mathematical models in the form of DAEs (Borutzky, 2010, pp. 142-143). Moreover, Theorem 10 is independent of the specific form of the skew-symmetric matrix $\boldsymbol{Z}_{\mathrm{RR}}(\boldsymbol{x})$ in (1). This is also noteworthy as the case $\boldsymbol{Z}_{\mathrm{RR}}(\boldsymbol{x}) \neq \boldsymbol{0}$ (i.e., the case of *dependent resistors*) is also known to lead to DAE models (Borutzky, 2010, p. 134). In the port-Hamiltonian formulation of Theorem 10, the state vector consists of the states of the *independent* energy storages. This is in line with the literature, where the order of an ODE description of a bond graph is known to be equal to the number of independent storages (Borutzky, 2010, p. 119). The *dependent* storages do not contribute to the system state as they are no "free" storages. Later, we will derive an explicit expression which relates the states of the dependent storages with the states of the independent storages, viz. equation (14). It is important to note that our approach retains the dependent storages in the bond graph. This is in contrast to some approaches from the literature in which the dependent storages are dissolved into the independent storages, e.g., Najnudel et al. (2018). The choice of inputs and outputs in Theorem 10 is plausible (Pfeifer et al., 2019, Property 2.5): the inputs of the PHS are the flows of Sf-type elements and the efforts of Se-type elements; the outputs are the respective conjugated variables.

Besides its theoretical contribution, the result of Theorem 10 is also of practical interest. The calculation rules for the matrices of the PHS in (8a)-(8f) can be easily implemented in a computer algebra system. By this, the result of this paper can be directly integrated into tools aiming at an automated generation of port-Hamiltonian models from bond graphs, e.g., Falaize and Hélie (2019); Pfeifer et al. (2019). Equations (8a)-(8f) require the existence of the matrices $\tilde{\boldsymbol{K}}(\boldsymbol{x})$ and $\tilde{\boldsymbol{L}}$ in (9a) and (9b), respectively. In Section 3.3, we will show that these matrices indeed always exist.

Next, we argue that Assumptions 8 and 9 are not very restrictive. Assumption 8 requires that the dependent storages are static functions of only the independent storages. Indeed, this is the most common kind of dependent storages (Borutzky, 2010, pp. 142ff.). Assumption 9 demands the storage functions of the C-type elements to be positive definite and quadratic. Positive definiteness implies non-negativity which is a natural property of an energy storage function. Moreover, quadratic functions are the most common form of storage functions as they imply a linear relation between the states and the efforts of a storage element (Borutzky, 2010, p. 45). Nonlinear relations of states and efforts can be found for example in relativistic mechanics.

In conclusion, Theorem 10 provides a constructive and rigorous method to formulate a wide class of bond graphs with or without dependent storages as explicit PHSs.

3.3 Proof of the Main Result

In this section, we give the proof of Theorem 10. In the first part of the proof, we show that the matrices $\tilde{\boldsymbol{K}}(\boldsymbol{x})$ in (9a) and $\tilde{\boldsymbol{L}}$ in (9b) always exist. In the second part, we

[2] The negative sign in (5) stems from the opposite signs of the flows in the vectors $(\boldsymbol{f}_{\mathrm{R}}, \boldsymbol{e}_{\mathrm{R}})$ and $(\boldsymbol{u}_{\mathrm{R}}, \boldsymbol{y}_{\mathrm{R}})$ (see (2)).

$$\boldsymbol{J}(\boldsymbol{x}) = \frac{1}{2}\left(\boldsymbol{Z}_{\mathrm{C_iR}}(\boldsymbol{x})\,\tilde{\boldsymbol{R}}(\boldsymbol{x})\,\tilde{\boldsymbol{K}}^\top(\boldsymbol{x})\,\boldsymbol{Z}_{\mathrm{C_iR}}^\top(\boldsymbol{x})\,\tilde{\boldsymbol{L}}^\top - \tilde{\boldsymbol{L}}\boldsymbol{Z}_{\mathrm{C_iR}}(\boldsymbol{x})\,\tilde{\boldsymbol{K}}(\boldsymbol{x})\,\tilde{\boldsymbol{R}}(\boldsymbol{x})\,\boldsymbol{Z}_{\mathrm{C_iR}}^\top(\boldsymbol{x}) - \boldsymbol{Z}_{\mathrm{C_iC_i}}(\boldsymbol{x})\,\tilde{\boldsymbol{L}}^\top - \tilde{\boldsymbol{L}}\boldsymbol{Z}_{\mathrm{C_iC_i}}(\boldsymbol{x})\right) \tag{8a}$$

$$\boldsymbol{R}(\boldsymbol{x}) = \frac{1}{2}\left(\boldsymbol{Z}_{\mathrm{C_iR}}(\boldsymbol{x})\,\tilde{\boldsymbol{R}}(\boldsymbol{x})\,\tilde{\boldsymbol{K}}^\top(\boldsymbol{x})\,\boldsymbol{Z}_{\mathrm{C_iR}}^\top(\boldsymbol{x})\,\tilde{\boldsymbol{L}}^\top + \tilde{\boldsymbol{L}}\boldsymbol{Z}_{\mathrm{C_iR}}(\boldsymbol{x})\,\tilde{\boldsymbol{K}}(\boldsymbol{x})\,\tilde{\boldsymbol{R}}(\boldsymbol{x})\,\boldsymbol{Z}_{\mathrm{C_iR}}^\top(\boldsymbol{x}) - \boldsymbol{Z}_{\mathrm{C_iC_i}}(\boldsymbol{x})\,\tilde{\boldsymbol{L}}^\top + \tilde{\boldsymbol{L}}\boldsymbol{Z}_{\mathrm{C_iC_i}}(\boldsymbol{x})\right) \tag{8b}$$

$$\boldsymbol{G}(\boldsymbol{x}) = \frac{1}{2}\left(\boldsymbol{I} + \tilde{\boldsymbol{L}}\right)\boldsymbol{Z}_{\mathrm{C_iP}}(\boldsymbol{x}) - \frac{1}{2}\left(\boldsymbol{Z}_{\mathrm{C_iR}}(\boldsymbol{x})\,\tilde{\boldsymbol{R}}(\boldsymbol{x})\,\tilde{\boldsymbol{K}}^\top(\boldsymbol{x}) - \tilde{\boldsymbol{L}}\boldsymbol{Z}_{\mathrm{C_iR}}(\boldsymbol{x})\,\tilde{\boldsymbol{K}}(\boldsymbol{x})\,\tilde{\boldsymbol{R}}(\boldsymbol{x})\right)\boldsymbol{Z}_{\mathrm{RP}}(\boldsymbol{x}) \tag{8c}$$

$$\boldsymbol{P}(\boldsymbol{x}) = \frac{1}{2}\left(\boldsymbol{I} - \tilde{\boldsymbol{L}}\right)\boldsymbol{Z}_{\mathrm{C_iP}}(\boldsymbol{x}) - \frac{1}{2}\left(\boldsymbol{Z}_{\mathrm{C_iR}}(\boldsymbol{x})\,\tilde{\boldsymbol{R}}(\boldsymbol{x})\,\tilde{\boldsymbol{K}}^\top(\boldsymbol{x}) + \tilde{\boldsymbol{L}}\boldsymbol{Z}_{\mathrm{C_iR}}(\boldsymbol{x})\,\tilde{\boldsymbol{K}}(\boldsymbol{x})\,\tilde{\boldsymbol{R}}(\boldsymbol{x})\right)\boldsymbol{Z}_{\mathrm{RP}}(\boldsymbol{x}) \tag{8d}$$

$$\boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{Z}_{\mathrm{PP}}(\boldsymbol{x}) + \frac{1}{2}\boldsymbol{Z}_{\mathrm{RP}}^\top(\boldsymbol{x})\left(\tilde{\boldsymbol{K}}(\boldsymbol{x})\,\tilde{\boldsymbol{R}}(\boldsymbol{x}) - \tilde{\boldsymbol{R}}(\boldsymbol{x})\,\tilde{\boldsymbol{K}}^\top(\boldsymbol{x})\right)\boldsymbol{Z}_{\mathrm{RP}}(\boldsymbol{x}) \tag{8e}$$

$$\boldsymbol{S}(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{Z}_{\mathrm{RP}}^\top(\boldsymbol{x})\left(\tilde{\boldsymbol{K}}(\boldsymbol{x})\,\tilde{\boldsymbol{R}}(\boldsymbol{x}) + \tilde{\boldsymbol{R}}(\boldsymbol{x})\,\tilde{\boldsymbol{K}}^\top(\boldsymbol{x})\right)\boldsymbol{Z}_{\mathrm{RP}}(\boldsymbol{x}) \tag{8f}$$

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prove the main statement of Theorem 10. For the proofs of the existence of the matrices $\tilde{\boldsymbol{K}}(\boldsymbol{x})$ and $\tilde{\boldsymbol{L}}$ we use the *Rayleigh quotient*.

Definition 11. Given a quadratic matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and a non-zero vector $\boldsymbol{x} \in \mathbb{R}^n$. The *Rayleigh quotient* $\rho(\boldsymbol{A}, \boldsymbol{x})$ is defined as $\frac{\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}}$ (Deuflhard and Hohmann, 1995, p. 273). The set of all Rayleigh quotients over non-zero vectors

$$W(\boldsymbol{A}) \coloneqq \{\rho(\boldsymbol{A}, \boldsymbol{x}) \,|\, \boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}\} \tag{10}$$

is called *numerical range* of $\boldsymbol{A}$.

We have $\mathrm{Spec}\,(\boldsymbol{A}) \subseteq W(\boldsymbol{A})$ and for a symmetric matrix $\boldsymbol{A}$ we have $W(\boldsymbol{A}) = [\lambda_{\min}, \lambda_{\max}]$ by the min-max Theorem (also known as Courant-Fischer Theorem) (Deuflhard and Hohmann, 1995, Lemma 8.29, p. 273), where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and the largest eigenvalues of $\boldsymbol{A}$.

Now, we have all prerequisites to show that $\tilde{\boldsymbol{K}}(\boldsymbol{x})$ in (9a) always exists.

Lemma 12. Let $\boldsymbol{X} \in \mathbb{R}^{p\times p}$ be a symmetric positive definite matrix and $\boldsymbol{Y} \in \mathbb{R}^{p\times p}$ be any matrix whose numerical range $W(\boldsymbol{Y})$ is contained in $[0, \infty)$, i.e., $\boldsymbol{Y}$ has only non-negative eigenvalues. Then the matrices $\boldsymbol{I} + \boldsymbol{Y}\boldsymbol{X}$ and $\boldsymbol{I} + \boldsymbol{X}\boldsymbol{Y}$ are regular.

Proof. We only give the proof for $\boldsymbol{I} + \boldsymbol{X}\boldsymbol{Y}$ as the proof for $\boldsymbol{I}+\boldsymbol{Y}\boldsymbol{X}$ is the same except for $\boldsymbol{X}$ and $\boldsymbol{Y}$ interchanged. Without loss of generality we may assume $\boldsymbol{X}$ to be diagonal. Indeed, since $\boldsymbol{X}$ is a symmetric and real matrix, there exists (by the Spectral Theorem) an orthogonal matrix $\boldsymbol{T}$ such that $\boldsymbol{T}\boldsymbol{X}\boldsymbol{T}^\top$ is diagonal. Moreover, $\boldsymbol{I} + \boldsymbol{X}\boldsymbol{Y}$ is invertible if and only if $\boldsymbol{T}(\boldsymbol{I} + \boldsymbol{X}\boldsymbol{Y})\boldsymbol{T}^\top = \boldsymbol{I} + (\boldsymbol{T}\boldsymbol{X}\boldsymbol{T}^\top)(\boldsymbol{T}\boldsymbol{Y}\boldsymbol{T}^\top) = \boldsymbol{I} + \tilde{\boldsymbol{X}}\tilde{\boldsymbol{Y}}$ is invertible, where $\tilde{\boldsymbol{X}} = \boldsymbol{T}\boldsymbol{X}\boldsymbol{T}^\top$ is diagonal and positive definite and the numerical range of $\tilde{\boldsymbol{Y}} = \boldsymbol{T}\boldsymbol{Y}\boldsymbol{T}^\top$ is contained in $[0, \infty)$. Thus, we can assume $\boldsymbol{X}$ to be diagonal in the remainder of the proof.

The matrix $\boldsymbol{I} + \boldsymbol{X}\boldsymbol{Y}$ is regular if and only if 0 is not an eigenvalue, that is if -1 is not an eigenvalue of $\boldsymbol{X}\boldsymbol{Y}$. We will show that $\boldsymbol{X}\boldsymbol{Y}$ has only eigenvalues in $[0, \infty)$. Let $\sqrt{\boldsymbol{X}}$ be the diagonal matrix which is a square root of $\boldsymbol{X}$, i.e. $\sqrt{\boldsymbol{X}}\sqrt{\boldsymbol{X}} = \sqrt{\boldsymbol{X}}\sqrt{\boldsymbol{X}}^\top = \boldsymbol{X}$. Such a matrix exists and is invertible since $\boldsymbol{X}$ is diagonal and positive definite. Because the spectrum of a matrix is invariant under conjugation, we have

$$\begin{aligned}\mathrm{Spec}\,(\boldsymbol{X}\boldsymbol{Y}) &= \mathrm{Spec}\left(\sqrt{\boldsymbol{X}}^{-1}\boldsymbol{X}\boldsymbol{Y}\sqrt{\boldsymbol{X}}\right)\\ &= \mathrm{Spec}\left(\sqrt{\boldsymbol{X}}\boldsymbol{Y}\sqrt{\boldsymbol{X}}\right) = \mathrm{Spec}\left(\sqrt{\boldsymbol{X}}\boldsymbol{Y}\sqrt{\boldsymbol{X}}^\top\right)\\ &\subseteq W\left(\sqrt{\boldsymbol{X}}\boldsymbol{Y}\sqrt{\boldsymbol{X}}^\top\right) \subseteq W(\boldsymbol{Y}) \cdot (0, \infty) \subseteq [0, \infty).\end{aligned} \tag{11}$$

In (11), the second to last inclusion holds since

$$\frac{\boldsymbol{x}^\top\sqrt{\boldsymbol{X}}\boldsymbol{Y}\sqrt{\boldsymbol{X}}^\top\boldsymbol{x}}{\boldsymbol{x}^\top\boldsymbol{x}} = \underbrace{\frac{\boldsymbol{x}^\top\sqrt{\boldsymbol{X}}\boldsymbol{Y}\sqrt{\boldsymbol{X}}^\top\boldsymbol{x}}{\boldsymbol{x}^\top\sqrt{\boldsymbol{X}}\sqrt{\boldsymbol{X}}^\top\boldsymbol{x}}}_{\in W(\boldsymbol{Y})} \cdot \underbrace{\frac{\boldsymbol{x}^\top\sqrt{\boldsymbol{X}}\sqrt{\boldsymbol{X}}^\top\boldsymbol{x}}{\boldsymbol{x}^\top\boldsymbol{x}}}_{\in W(\boldsymbol{X})} \tag{12}$$

and $W(\boldsymbol{X}) \subseteq (0, \infty)$ by the min-max Theorem. Thus, -1 is not an eigenvalue of $\boldsymbol{X}\boldsymbol{Y}$ and $\boldsymbol{I} + \boldsymbol{X}\boldsymbol{Y}$ is invertible.

Corollary 13. The matrix $\tilde{\boldsymbol{K}}(\boldsymbol{x})$ in (9a) always exists for all $\boldsymbol{x} \in \mathcal{X}$.

Proof. Recall that in (9a) we have $\tilde{\boldsymbol{R}}(\boldsymbol{x}) = \tilde{\boldsymbol{R}}^\top(\boldsymbol{x}) \succ 0$ and $\boldsymbol{Z}_{\mathrm{RR}}(\boldsymbol{x}) = -\boldsymbol{Z}_{\mathrm{RR}}^\top(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathcal{X}$. The Rayleight quotient of a skew-symmetric matrix is always 0. Then, the statement from Corollary 13 follows directly from Lemma 12.

Corollary 14. The matrix $\tilde{\boldsymbol{L}}(\boldsymbol{x})$ in (9b) always exists.

Proof. Recall that $\boldsymbol{D}_{\mathrm{d}} \succ 0$ and thus $\boldsymbol{D}_{\mathrm{d}}^{-1} \succ 0$ which also implies $\boldsymbol{D}_{\mathrm{d}}^{-1} \succeq 0$. Hence, $\boldsymbol{Z}_{\mathrm{C_iC_d}}\boldsymbol{D}_{\mathrm{d}}^{-1}\boldsymbol{Z}_{\mathrm{C_iC_d}}^\top$ is positive semi-definite. The Rayleight quotient of a positive semi-definite matrix is always ≥ 0, i.e., $W(\boldsymbol{Z}_{\mathrm{C_iC_d}}\boldsymbol{D}_{\mathrm{d}}^{-1}\boldsymbol{Z}_{\mathrm{C_iC_d}}^\top) \subseteq [0, \infty)$. As $\boldsymbol{D}_{\mathrm{i}}$ is symmetric and positive definite, the claim follows from Lemma 12.

Corollaries 13 and 14 show that the matrices in (9a) and (9b) always exist, respectively. Next, we give the proof of the main part of Theorem 10.

Proof. The proof follows three steps: (i) In (1), we eliminate variables that belong to resistive elements and dependent storages; (ii) we decompose the equation system obtained from (i) into a symmetric and a skew-symmetric part and insert the constitutive relations of the independent storages from (3); (iii) we prove (7). For the sake of releasing notational burden, we will suppress the argument $\boldsymbol{x}$ to the matrices during the proof.

(i) Let assumption 8 hold. Substituting the third line from the equation system of (1) into (5) gives

$$\begin{aligned}\boldsymbol{u}_{\mathrm{R}} &= -\tilde{\boldsymbol{R}}\left(\boldsymbol{Z}_{\mathrm{C_iR}}^\top\boldsymbol{u}_{\mathrm{C_i}} + \boldsymbol{Z}_{\mathrm{RR}}\,\boldsymbol{u}_{\mathrm{R}} - \boldsymbol{Z}_{\mathrm{RP}}\,\boldsymbol{u}_{\mathrm{P}}\right)\\ \Leftrightarrow \boldsymbol{u}_{\mathrm{R}} &= -\tilde{\boldsymbol{K}}\,\tilde{\boldsymbol{R}}\,\boldsymbol{Z}_{\mathrm{C_iR}}^\top\boldsymbol{u}_{\mathrm{C_i}} + \tilde{\boldsymbol{K}}\,\tilde{\boldsymbol{R}}\,\boldsymbol{Z}_{\mathrm{RP}}\,\boldsymbol{u}_{\mathrm{P}}\end{aligned} \tag{13}$$

with $\tilde{\boldsymbol{K}}$ as in (9a). According to Corollary 13, $\tilde{\boldsymbol{K}}$ always exists. Now, we use an idea of Wellstead (1979) (pp. 226-227) to eliminate the variables that belong to dependent storages. In addition to assumption 8, let assumption 9 hold. With (3b), the second line of the equation system of (1) reads:

$$\begin{aligned} \boldsymbol{y}_{\mathrm{C_d}} &= \boldsymbol{Z}_{\mathrm{C_iC_d}}^{\top} \boldsymbol{u}_{\mathrm{C_i}} \\ \Leftrightarrow \frac{\partial V_{\mathrm{d}}}{\partial \boldsymbol{x}_{\mathrm{d}}}(\boldsymbol{x}_{\mathrm{d}}) &= \boldsymbol{Z}_{\mathrm{C_iC_d}}^{\top} \left(\frac{\partial V_{\mathrm{i}}}{\partial \boldsymbol{x}_i}(\boldsymbol{x}_i) \right) \\ \Leftrightarrow \quad \boldsymbol{D}_{\mathrm{d}}\, \boldsymbol{x}_{\mathrm{d}} &= \boldsymbol{Z}_{\mathrm{C_iC_d}}^{\top} \boldsymbol{D}_{\mathrm{i}}\, \boldsymbol{x}_{\mathrm{i}} \\ \Leftrightarrow \quad \boldsymbol{x}_{\mathrm{d}} &= \boldsymbol{D}_{\mathrm{d}}^{-1} \boldsymbol{Z}_{\mathrm{C_iC_d}}^{\top} \boldsymbol{D}_{\mathrm{i}}\, \boldsymbol{x}_{\mathrm{i}}. \end{aligned} \tag{14}$$

By differentiating (14) with respect to time and using (3a), we obtain

$$\begin{aligned} &\Leftrightarrow \dot{\boldsymbol{x}}_{\mathrm{d}} = \boldsymbol{D}_{\mathrm{d}}^{-1} \boldsymbol{Z}_{\mathrm{C_iC_d}}^{\top} \boldsymbol{D}_{\mathrm{i}}\, \dot{\boldsymbol{x}}_{\mathrm{i}} \\ &\Leftrightarrow \; \boldsymbol{0} = -\boldsymbol{u}_{\mathrm{C_d}} + \boldsymbol{D}_{\mathrm{d}}^{-1} \boldsymbol{Z}_{\mathrm{C_iC_d}}^{\top} \boldsymbol{D}_{\mathrm{i}}\, \boldsymbol{y}_{\mathrm{C_i}}\,. \end{aligned} \tag{15}$$

Insertion of (13) into the first line of the equation system in (1) gives

$$\begin{aligned} \boldsymbol{y}_{\mathrm{C_i}} = &\left(\boldsymbol{Z}_{\mathrm{C_iC_i}} + \boldsymbol{Z}_{\mathrm{C_iR}} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{C_iR}}^{\top} \right) \boldsymbol{u}_{\mathrm{C_i}} - \boldsymbol{Z}_{\mathrm{C_iC_d}} \boldsymbol{u}_{\mathrm{C_d}} \\ &+ \left(-\boldsymbol{Z}_{\mathrm{C_iP}} - \boldsymbol{Z}_{\mathrm{C_iR}} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{RP}} \right) \boldsymbol{u}_{\mathrm{P}}. \end{aligned} \tag{16}$$

Equations (15) and (16) can be written in matrix-vector form:

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{Z}_{\mathrm{C_iC_d}} \\ -\boldsymbol{D}_{\mathrm{d}}^{-1} \boldsymbol{Z}_{\mathrm{C_iC_d}}^{\top} \boldsymbol{D}_{\mathrm{i}} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{y}_{\mathrm{C_i}} \\ \boldsymbol{u}_{\mathrm{C_d}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Z}_{\mathrm{C_iC_i}} + \boldsymbol{Z}_{\mathrm{C_iR}} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{C_iR}}^{\top} & -\boldsymbol{Z}_{\mathrm{C_iP}} - \boldsymbol{Z}_{\mathrm{C_iR}} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{RP}} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{\mathrm{C_i}} \\ \boldsymbol{u}_{\mathrm{P}} \end{pmatrix}. \tag{17}$$

Next, we use Corollary 19 from Appendix A to invert the 2×2 block matrix on the left-hand side of equation (17). The Schur complement of this matrix is given by (Lu and Shiou, 2002):

$$\boldsymbol{L} \coloneqq \boldsymbol{I} + \boldsymbol{Z}_{\mathrm{C_iC_d}} \boldsymbol{D}_{\mathrm{d}}^{-1} \boldsymbol{Z}_{\mathrm{C_iC_d}}^{\top} \boldsymbol{D}_{\mathrm{i}}. \tag{18}$$

From Corollary 14 it follows that $\boldsymbol{L}$ is always regular. Thus, the inverse matrix of $\boldsymbol{L}$ exists and is given by $\tilde{\boldsymbol{L}} \coloneqq \boldsymbol{L}^{-1}$ as in (9b). By applying Corollary 19, we can then write (17) equivalently as

$$\begin{pmatrix} \boldsymbol{y}_{\mathrm{C_i}} \\ \boldsymbol{u}_{\mathrm{C_d}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Z}_1 & \boldsymbol{Z}_2 \\ \boldsymbol{Z}_3 & \boldsymbol{Z}_4 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{\mathrm{C_i}} \\ \boldsymbol{u}_{\mathrm{P}} \end{pmatrix} \tag{19}$$

with

$$\begin{aligned} \boldsymbol{Z}_1 &= \tilde{\boldsymbol{L}} \left(\boldsymbol{Z}_{\mathrm{C_iC_i}} + \boldsymbol{Z}_{\mathrm{C_iR}} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{C_iR}}^{\top} \right), \\ \boldsymbol{Z}_2 &= \tilde{\boldsymbol{L}} \left(-\boldsymbol{Z}_{\mathrm{C_iP}} - \boldsymbol{Z}_{\mathrm{C_iR}} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{RP}} \right), \\ \boldsymbol{Z}_3 &= \boldsymbol{D}_{\mathrm{d}}^{-1} \boldsymbol{Z}_{\mathrm{C_iC_d}}^{\top} \boldsymbol{D}_{\mathrm{i}} \tilde{\boldsymbol{L}} \left(\boldsymbol{Z}_{\mathrm{C_iC_i}} - \boldsymbol{Z}_{\mathrm{C_iR}} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{C_iR}}^{\top} \right), \\ \boldsymbol{Z}_4 &= \boldsymbol{D}_{\mathrm{d}}^{-1} \boldsymbol{Z}_{\mathrm{C_iC_d}}^{\top} \boldsymbol{D}_{\mathrm{i}} \tilde{\boldsymbol{L}} \left(-\boldsymbol{Z}_{\mathrm{C_iP}} - \boldsymbol{Z}_{\mathrm{C_iR}} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{RP}} \right). \end{aligned}$$

Equation (19) will pave the way to the state differential equation (6a) of the PHS. However, we also require an expression for the output equation (6b). To this end, we insert (13) into the fourth line of the equation system of (1) and obtain

$$\begin{aligned} \boldsymbol{y}_{\mathrm{P}} = &\left(\boldsymbol{Z}_{\mathrm{C_iP}}^{\top} - \boldsymbol{Z}_{\mathrm{RP}}^{\top} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{C_iR}}^{\top} \right) \boldsymbol{u}_{\mathrm{C_i}} \\ &+ \left(\boldsymbol{Z}_{\mathrm{PP}} + \boldsymbol{Z}_{\mathrm{RP}}^{\top} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{RP}} \right) \boldsymbol{u}_{\mathrm{P}}. \end{aligned} \tag{20}$$

The first line of the equation system in (19) and equation (20) can be written together in matrix-vector form as

$$\begin{pmatrix} \boldsymbol{y}_{\mathrm{C_i}} \\ \boldsymbol{y}_{\mathrm{P}} \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{L}} \left(\boldsymbol{Z}_{\mathrm{C_iC_i}} + \boldsymbol{Z}_{\mathrm{C_iR}} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{C_iR}}^{\top} \right) & -\tilde{\boldsymbol{L}} \left(\boldsymbol{Z}_{\mathrm{C_iP}} + \boldsymbol{Z}_{\mathrm{C_iR}} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{RP}} \right) \\ \boldsymbol{Z}_{\mathrm{C_iP}}^{\top} - \boldsymbol{Z}_{\mathrm{RP}}^{\top} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{C_iR}}^{\top} & \boldsymbol{Z}_{\mathrm{PP}} + \boldsymbol{Z}_{\mathrm{RP}}^{\top} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{RP}} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{\mathrm{C_i}} \\ \boldsymbol{u}_{\mathrm{P}} \end{pmatrix}. \tag{21}$$

Note that (21) is independent of variables that belong to resistive elements and dependent storages.

(ii) The matrix in (21) can be decomposed into a skew-symmetric and a symmetric part. Using this decomposition, (21) can be equivalently written as

$$\begin{pmatrix} \boldsymbol{y}_{\mathrm{C_i}} \\ \boldsymbol{y}_{\mathrm{P}} \end{pmatrix} = \Bigg[\underbrace{\begin{pmatrix} -\boldsymbol{J} & -\boldsymbol{G} \\ \boldsymbol{G}^{\top} & \boldsymbol{M} \end{pmatrix}}_{=:\boldsymbol{Q}_{ss}} + \underbrace{\begin{pmatrix} \boldsymbol{R} & \boldsymbol{P} \\ \boldsymbol{P}^{\top} & \boldsymbol{S} \end{pmatrix}}_{=:\boldsymbol{Q}_{s}} \Bigg] \begin{pmatrix} \boldsymbol{u}_{\mathrm{C_i}} \\ \boldsymbol{u}_{\mathrm{P}} \end{pmatrix} \tag{22}$$

with $\boldsymbol{J}$, $\boldsymbol{R}$, $\boldsymbol{G}$, $\mathbf{P}$, $\boldsymbol{M}$, $\boldsymbol{S}$ as in (8) and $\boldsymbol{Q}_{ss} = -\boldsymbol{Q}_{ss}^{\top}$, $\boldsymbol{Q}_{s} = \boldsymbol{Q}_{s}^{\top}$. By inserting the identities of the *independent variables* from (3) into (22), we finally obtain the explicit PHS (6) with $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{P}}$, $\boldsymbol{x} = \boldsymbol{x}_{\mathrm{i}}$, $\boldsymbol{y} = \boldsymbol{y}_{\mathrm{P}}$ and Hamiltonian $H(\boldsymbol{x}) = V_{\mathrm{i}}(\boldsymbol{x}_{\mathrm{i}})$.

(iii) In the last step, we show that (7) holds. Let us merge (22) with the second line of the equation system in (1):

$$\begin{pmatrix} \boldsymbol{y}_{\mathrm{C_i}} \\ \boldsymbol{y}_{\mathrm{P}} \\ \boldsymbol{y}_{\mathrm{C_d}} \end{pmatrix} = \Bigg[\underbrace{\begin{pmatrix} -\boldsymbol{J} & -\boldsymbol{G} & \boldsymbol{0} \\ \boldsymbol{G}^{\top} & \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}}_{=:\tilde{\boldsymbol{Q}}_{ss}} + \underbrace{\begin{pmatrix} \boldsymbol{R} & \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{P}^{\top} & \boldsymbol{S} & \boldsymbol{0} \\ \boldsymbol{Z}_{\mathrm{C_iC_d}}^{\top} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}}_{=:\tilde{\boldsymbol{Q}}_{s}} \Bigg] \begin{pmatrix} \boldsymbol{u}_{\mathrm{C_i}} \\ \boldsymbol{u}_{\mathrm{P}} \\ \boldsymbol{u}_{\mathrm{C_d}} \end{pmatrix}. \tag{23}$$

Note that $\tilde{\boldsymbol{Q}}_{ss}$ is skew-symmetric but $\tilde{\boldsymbol{Q}}_{s}$ is *not* symmetric. Nevertheless, $\tilde{\boldsymbol{Q}}_{s} \succeq 0$ implies $\boldsymbol{Q}_{s} \succeq 0$. In the following, we show that $\tilde{\boldsymbol{Q}}_{s} \succeq 0$:

$$\begin{aligned} &\left(\boldsymbol{u}_{\mathrm{C_i}}^{\top} \; \boldsymbol{u}_{\mathrm{P}}^{\top} \; \boldsymbol{u}_{\mathrm{C_d}}^{\top} \right) \tilde{\boldsymbol{Q}}_{s} \begin{pmatrix} \boldsymbol{u}_{\mathrm{C_i}} \\ \boldsymbol{u}_{\mathrm{P}} \\ \boldsymbol{u}_{\mathrm{C_d}} \end{pmatrix} \\ &= \left(\boldsymbol{u}_{\mathrm{C_i}}^{\top} \; \boldsymbol{u}_{\mathrm{P}}^{\top} \; \boldsymbol{u}_{\mathrm{C_d}}^{\top} \right) \left(\tilde{\boldsymbol{Q}}_{ss} + \tilde{\boldsymbol{Q}}_{s} \right) \begin{pmatrix} \boldsymbol{u}_{\mathrm{C_i}} \\ \boldsymbol{u}_{\mathrm{P}} \\ \boldsymbol{u}_{\mathrm{C_d}} \end{pmatrix} \\ &\overset{(23)}{=} \left(\boldsymbol{u}_{\mathrm{C_i}}^{\top} \; \boldsymbol{u}_{\mathrm{P}}^{\top} \; \boldsymbol{u}_{\mathrm{C_d}}^{\top} \right) \begin{pmatrix} \boldsymbol{y}_{\mathrm{C_i}} \\ \boldsymbol{y}_{\mathrm{P}} \\ \boldsymbol{y}_{\mathrm{C_d}} \end{pmatrix} \\ &\overset{(1)}{=} -\boldsymbol{y}_{\mathrm{R}}^{\top} \boldsymbol{u}_{\mathrm{R}} \overset{(5)}{=} \boldsymbol{y}_{\mathrm{R}}^{\top} \tilde{\boldsymbol{R}} \boldsymbol{y}_{\mathrm{R}} \geq 0. \end{aligned} \tag{24}$$

Hence, we have $\tilde{\boldsymbol{Q}}_{s} \succeq 0$ which implies $\boldsymbol{Q}_{s} \succeq 0$. This concludes the proof of Theorem 10.

Remark 15. By inserting (3) into the second line of (19), we yield an explicit expression for the dynamics of the dependent states:

$$\begin{aligned} \dot{\boldsymbol{x}}_{\mathrm{d}} = &\boldsymbol{D}_{\mathrm{d}}^{-1} \boldsymbol{Z}_{\mathrm{C_iC_d}}^{\top} \boldsymbol{D}_{\mathrm{i}} \tilde{\boldsymbol{L}} \left(\boldsymbol{Z}_{\mathrm{C_iC_i}} - \boldsymbol{Z}_{\mathrm{C_iR}} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{C_iR}}^{\top} \right) \frac{\partial V_{\mathrm{i}}}{\partial \boldsymbol{x}_{\mathrm{i}}}(\boldsymbol{x}_{\mathrm{i}}) \\ &- \boldsymbol{D}_{\mathrm{d}}^{-1} \boldsymbol{Z}_{\mathrm{C_iC_d}}^{\top} \boldsymbol{D}_{\mathrm{i}} \tilde{\boldsymbol{L}} \left(\boldsymbol{Z}_{\mathrm{C_iP}} + \boldsymbol{Z}_{\mathrm{C_iR}} \tilde{\boldsymbol{K}} \tilde{\boldsymbol{R}} \boldsymbol{Z}_{\mathrm{RP}} \right) \boldsymbol{u}_{\mathrm{P}}. \end{aligned} \tag{25}$$

Remark 16. With (14), the total energy in the system can be given as a function only of the *independent* states $\boldsymbol{x}_{\mathrm{i}}$:

$$
\begin{aligned}
V(\boldsymbol{x}_\mathrm{i},\boldsymbol{x}_\mathrm{d}) &= V_\mathrm{i}(\boldsymbol{x}_\mathrm{i}) + V_\mathrm{d}(\boldsymbol{x}_\mathrm{d})\\
&= \tfrac{1}{2}\boldsymbol{x}_\mathrm{i}^\top \boldsymbol{D}_\mathrm{i}\boldsymbol{x}_\mathrm{i} + \tfrac{1}{2}\boldsymbol{x}_\mathrm{d}^\top \boldsymbol{D}_\mathrm{d}\boldsymbol{x}_\mathrm{d}\\
&= \tfrac{1}{2}\boldsymbol{x}_\mathrm{i}^\top \boldsymbol{D}_\mathrm{i}\left(\boldsymbol{I} + \boldsymbol{Z}_{\mathrm{C_iC_d}}\boldsymbol{D}_\mathrm{d}^{-1}\boldsymbol{Z}_{\mathrm{C_iC_d}}^\top \boldsymbol{D}_\mathrm{i}\right)\boldsymbol{x}_\mathrm{i}\\
&\overset{(18)}{=} \tfrac{1}{2}\boldsymbol{x}_\mathrm{i}^\top \boldsymbol{D}_\mathrm{i}\boldsymbol{L}\boldsymbol{x}_\mathrm{i} = V(\boldsymbol{x}_\mathrm{i}).
\end{aligned} \tag{26}
$$

4. EXAMPLE

In this section, the result from Section 3 is illustrated through an academic example. Consider the bond graph in Figure 1.

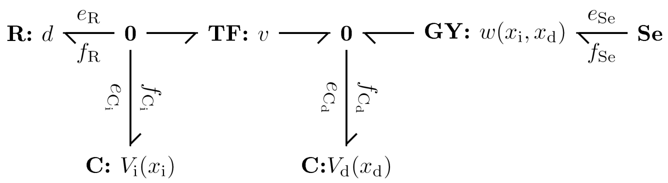

Fig. 1. Exemplary bond graph with dependent storage

The C-type element with storage function $V_\mathrm{i}(\boldsymbol{x}_\mathrm{i})$ is considered as *independent* storage element; the C-type element with storage function $V_\mathrm{d}(\boldsymbol{x}_\mathrm{d})$ is a *dependent* storage element. The respective storage functions are given by $V_\mathrm{i}(x_\mathrm{i}) = x_\mathrm{i}^2/(2c_\mathrm{i})$ and $V_\mathrm{d}(x_\mathrm{d}) = x_\mathrm{d}^2/(2c_\mathrm{d})$ where $c_\mathrm{i} > 0$, $c_\mathrm{d} > 0$. The constitutive relation of the R-type element is specified by $f_\mathrm{R} = d\, e_\mathrm{R}$ where $d > 0$. The transformer TF has a constant transformation ratio $v > 0$ and the gyrator GY is state-modulated with an arbitrary gyration ratio $w(x_\mathrm{i}, x_\mathrm{d}) > 0$ for all $x_\mathrm{i}, x_\mathrm{d} \in \mathbb{R}$.

The junction structure of the bond graph can be described by a Dirac structure of the form (1):

$$
\mathcal{D} = \{(\begin{pmatrix} f_{\mathrm{C_i}}\\ f_{\mathrm{C_d}}\\ f_\mathrm{R}\\ f_\mathrm{Se}\end{pmatrix}, \begin{pmatrix} e_{\mathrm{C_i}}\\ e_{\mathrm{C_d}}\\ e_\mathrm{R}\\ e_\mathrm{Se}\end{pmatrix}) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid
$$

$$
\begin{pmatrix} -f_{\mathrm{C_i}}\\ e_{\mathrm{C_d}}\\ e_\mathrm{R}\\ f_\mathrm{Se}\end{pmatrix} = \begin{pmatrix} 0 & -v & -1 & -\frac{v}{w(x_\mathrm{i},x_\mathrm{d})}\\ v & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ \frac{v}{w(x_\mathrm{i},x_\mathrm{d})} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_{\mathrm{C_i}}\\ -f_{\mathrm{C_d}}\\ -f_\mathrm{R}\\ e_\mathrm{Se}\end{pmatrix} \}. \tag{27}
$$

Obviously, Assumptions 8 and 9 are satisfied. Hence, from Theorem 10 we known that the bond graph from Figure 1 permits an explicit port-Hamiltonian formulation of the form (6) with $u = e_\mathrm{Se}$, $x = x_\mathrm{i}$, $y = f_\mathrm{Se}$, and $H(x) = V_\mathrm{i}(x_\mathrm{i})$. By using the calculation rules (8), we obtain the following explicit PHS:

$$
\dot{x} = -\underbrace{\left(\tfrac{d\, c_\mathrm{i}}{c_\mathrm{i} + v^2\, c_\mathrm{d}}\right)}_{=\boldsymbol{R}(x)=\boldsymbol{R}} \frac{\partial H}{\partial x}(x) + \underbrace{\left(\tfrac{v\, c_\mathrm{i}}{w(x)(c_\mathrm{i} + v^2\, c_\mathrm{d})}\right)}_{=\boldsymbol{G}(x)-\boldsymbol{P}(x)} u, \tag{28a}
$$

$$
y = \underbrace{\left(\tfrac{v}{w(x)}\right)}_{=(\boldsymbol{G}(x)+\boldsymbol{P}(x))^\top} \frac{\partial H}{\partial x}(x), \tag{28b}
$$

with $\boldsymbol{J}(\boldsymbol{x})$, $\boldsymbol{M}(\boldsymbol{x})$, and $\boldsymbol{S}(\boldsymbol{x})$ being zero. Note that (28) indeed fulfills (7).

Remark 17. The function $w(x) = w(x_\mathrm{i})$ in (28) can be obtained from the gyration ratio $w(x_\mathrm{i}, x_\mathrm{d})$ by substituting x_d with (14).

5. CONCLUSION

In this paper, we presented a method which transfers a large class of bond graphs with dependent storages into explicit PHSs. The idea is to express the port-Hamiltonian dynamics and output as functions of only system inputs and *independent* storages. The method gives calculation rules for the matrices of the explicit PHS and can thus be easily implemented in a computer algebra system. The existence of all involved matrices is rigorously proven. Future work will focus on the extension of our method to non-quadratic storage functions, i.e., a relaxation of Assumption 9.

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Appendix A. INVERSES OF 2 × 2 BLOCK MATRICES

Lemma 18. (Lu and Shiou (2002)). Consider a non-singular 2×2 square block matrix

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix}, \tag{A.1}$$

where $\boldsymbol{A}$ and $\boldsymbol{D}$ are square matrices. Assume $\boldsymbol{D}$ is non-singular. Then the matrix (A.1) is invertible if and only if the Schur complement $\boldsymbol{L} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}$ of $\boldsymbol{D}$ is invertible, and the inverse is given by

$$\begin{pmatrix} \boldsymbol{L}^{-1} & -\boldsymbol{L}^{-1}\boldsymbol{B}\boldsymbol{D}^{-1} \\ -\boldsymbol{D}^{-1}\boldsymbol{C}\boldsymbol{L}^{-1} & \boldsymbol{D}^{-1} + \boldsymbol{D}^{-1}\boldsymbol{C}\boldsymbol{L}^{-1}\boldsymbol{B}\boldsymbol{D}^{-1} \end{pmatrix}. \tag{A.2}$$

Corollary 19. Consider the situation of Lemma 18 with $\boldsymbol{A}$ and $\boldsymbol{D}$ being identity matrices. Then, the matrix (A.1) is invertible if and only if $\boldsymbol{L} := \boldsymbol{I} - \boldsymbol{B}\boldsymbol{C}$ is invertible, and the inverse is given by

$$\begin{pmatrix} \boldsymbol{L}^{-1} & -\boldsymbol{L}^{-1}\boldsymbol{B} \\ -\boldsymbol{C}\boldsymbol{L}^{-1} & \boldsymbol{I} + \boldsymbol{C}\boldsymbol{L}^{-1}\boldsymbol{B} \end{pmatrix}. \tag{A.3}$$

Proof. Follows directly from Lemma 18 under $\boldsymbol{A} = \boldsymbol{I}$ and $\boldsymbol{D} = \boldsymbol{I}$.