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Breather solutions of the cubic Klein–Gordon equation

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Abstract

We obtain real-valued, time-periodic and radially symmetric solutions of the cubic Klein–Gordon equation $\partial_t^2 U - \Delta U + m^2 U = \Gamma(x) U^3$ on $\mathbb{R} \times \mathbb{R}^3$, which are weakly localized in space. Various families of such 'breather' solutions are shown to bifurcate from any given nontrivial stationary solution. The construction of weakly localized breathers in three space dimensions is, to the author's knowledge, a new concept and based on the reformulation of the cubic Klein–Gordon equation as a system of coupled nonlinear Helmholtz equations involving suitable conditions on the far field behavior.

Keywords: Klein–Gordon equation, breather, bifurcation, nonlinear Helmholtz system

Mathematics Subject Classification numbers: primary 35L71, 35B32, secondary 35B10, 35J05.

1. Introduction and main results

We construct real-valued solutions $U(t, x)$ of the cubic Klein–Gordon equation

$$\partial_t^2 U - \Delta U + m^2 U = \Gamma(x)\, U^3 \quad \text{on } \mathbb{R} \times \mathbb{R}^3 \tag{1}$$

where[1] $\Gamma \in L^\infty_{\mathrm{rad}}(\mathbb{R}^3) \cap C^1_{\mathrm{loc}}(\mathbb{R}^3)$ and $m > 0$ is a (mass) parameter. Here we restrict ourselves to the case of three space dimensions which is the most relevant one for applications in physics and which allows to use the tools established in [23]. Throughout, the notations $\partial_{1,2,3}, \nabla, \Delta$

[1] We use $f \in C^j_{\mathrm{loc}}(\mathbb{R}^3)$, $j \in \mathbb{N}_0$, as the set of all functions $f : \mathbb{R}^3 \to \mathbb{R}$ which are j times continuously differentiable (and hence in particular f and its derivatives are uniformly bounded on every compact subset of $\mathbb{R}^3$). If uniform bounds on all of $\mathbb{R}^3$ are assumed, we denote $C^j(\mathbb{R}^3)$, which is then a Banach space with the canonical norm $\|f\|_{C^j} := \sum_{l=0}^{j} \|f^{(l)}\|_\infty$

refer to differential operators acting on the space variables. The solutions we aim to construct are polychromatic, that is, they take the form

$$U(t,x) = u_0(x) + \sum_{k=1}^{\infty} 2\,\cos(\omega kt) u_k(x) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\omega kt} u_k(x) \tag{2}$$

$$\text{where}\quad u_k \in X_1 = \left\{ u \in C_{\mathrm{rad}}(\mathbb{R}^3,\mathbb{R}) \,\middle|\, \left\| (1+|\cdot|^2)^{\frac{1}{2}} u \right\|_\infty < \infty \right\} \subseteq L^4_{\mathrm{rad}}(\mathbb{R}^3), \quad u_{-k} = u_k$$

and (for simplicity) $\omega > m$.

Such solutions are periodic in time and localized as well as radially symmetric in space. They are sometimes referred to as breather solutions, cf the 'Sine-Gordon breather' in [1], equation (28). The construction of breather solutions is of particular interest since, as indicated in a study [8] on perturbations of the Sine-Gordon breather, Birnir, McKean and Weinstein conjecture that 'for the general nonlinear wave equation [author's note: in 1 + 1 dimensions], breathing [...] takes place only for isolated nonlinearities', see [8, p 1044]. This conjecture is supported by recent existence results for breathers for the 1 + 1 dimensional wave equation with specific, carefully designed potentials which we comment on below. Our results, however, indicate that the situation might be entirely different for weakly localized breathers for the Klein–Gordon equation in 1 + 3 dimensions, in the sense that such breather solutions are abundant even in 'simple' settings.

In physical literature, especially in quantum field theory, the study of the cubic Klein–Gordon equation is usually referred to as scalar ϕ^4 theory, which describes scalar particles of mass m with a pure self-interaction quantified by (usually constant positive) $\Gamma \equiv \Gamma_0$; for a number of more specific applications see [10, p 2]. From a more theoretical viewpoint, the special importance of ϕ^4 theory is that it serves as an exemplary case for studying phenomena in interacting quantum field theory, which is often only achieved via perturbation expansions; non-perturbative real-valued time-periodic and localized ('breather' or 'kink') solutions being also of interest but usually hard to find. For the $1+1$-dimensional ϕ^4 model, the occurrence of such solutions has been investigated [25]. Our result provides weakly localized non-perturbative solutions in (clearly physically relevant) 3 space dimensions, which to the author's best knowledge is a new contribution. Since our focus is more on the mathematical aspects of the exposition, we allow for spatially non-constant self-coupling, the physical relevance of which is not immediately clear (a suitable generalization being rather a nonlocal structure with a convolution kernel different from a Dirac delta); on the other hand, we wish to point out that it is precisely the physically relevant case of constant coupling where we can assure explicitly that all assumptions of our main existence result are satisfied. A quantum theoretical interpretation should, however, address the weak spatial decay rates of our breather solutions (which do not belong to L^2), maybe as partially bound/free states.

The result we present is part of the author's dissertation thesis [24]. We will find nonstationary breather solutions of (1) by rewriting it into an infinite system of (stationary) equations for the functions u_k. Indeed, inserting (2), a short and formal calculation leads to

$$-\Delta u_0 + m^2\, u_0 = \Gamma(x)\,(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_0, \tag{3a}$$

$$-\Delta u_k - (\omega^2 k^2 - m^2) u_k = \Gamma(x)\,(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_k \quad \text{for}\quad k \in \mathbb{Z}\backslash\{0\}. \tag{3b}$$

In fact, (3b) includes (3a), but we intend to separate the 'Schrödinger' equation characterized by $0 \notin \sigma(-\Delta + m^2)$ from the infinite number of 'Helmholtz' equations characterized by $0 \in \sigma(-\Delta - (\omega^2 k^2 - m^2))$, $k \neq 0$. (Please recall that $\omega > m$.) Our construction of breathers for (1) relies on new methods for such Helmholtz equations introduced in [23]. These are based on solution formulae obtained from classical limiting absorption principles, which give access to so-called far field properties; exploiting this, we are lead to a rich bifurcation structure. These methods will be sketched only briefly in the main body of this paper; more details will be given in section 4 at the end (which can be read independently).

The solutions we obtain bifurcate from any given stationary (radial) solution $w_0 \in X_1$, $w_0 \not\equiv 0$ of the Klein–Gordon equation (1). That is, w_0 solves the stationary nonlinear Schrödinger equation

$$-\Delta w_0 + m^2 w_0 = \Gamma(x)\, w_0^3 \quad \text{on } \mathbb{R}^3; \tag{4}$$

regarding existence of such w_0, cf remark 1(b). Let us remark briefly that all (distributional) solutions of (4) in $X_1 \subseteq L^4_{\text{rad}}(\mathbb{R}^3)$ are twice differentiable by elliptic regularity. In order to make bifurcation theory work, we impose the following nondegeneracy assumption:

$$q_0 \in X_1, \;\; -\Delta q_0 + m^2 q_0 = 3\Gamma(x)\, w_0^2\, q_0 \quad \text{on } \mathbb{R}^3 \text{ implies } q_0 \equiv 0. \tag{5}$$

We comment on this assumption in remark 1(c) below. In particular, (5) and our main result presented next hold if Γ is constant and w_0 is a (positive) ground state of (4). We now present our main result.

Theorem 1. *Let $\Gamma \in L^\infty_{rad}(\mathbb{R}^3) \cap C^1_{loc}(\mathbb{R}^3)$, $\omega > m > 0$ and assume there is some stationary solution $U^0(t,x) = w_0(x)$, $w_0 \not\equiv 0$ of the cubic Klein–Gordon equation (1), i.e. $w_0 \in X_1$ solving (4). Assume further that w_0 is nondegenerate in the sense of (5). Then for every $s \in \mathbb{N}$ there exist an open interval $J_s \subseteq \mathbb{R}$ with $0 \in J_s$ and a family $(U^\alpha)_{\alpha \in J_s} \subseteq C^2(\mathbb{R}, X_1)$ with the following properties:*

(*a*) *All U^α are time-periodic, twice continuously differentiable classical solutions of (1) of the polychromatic form (2),*

$$U^\alpha(t,x) = u_0^\alpha(x) + \sum_{k=1}^{\infty} 2\, \cos(\omega k t) u_k^\alpha(x).$$

(*b*) *The map $\alpha \mapsto (u_k^\alpha)_{k \in \mathbb{N}_0}$ is smooth in the topology of $\ell^1(\mathbb{N}_0, X_1)$ with*

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0} u_k^\alpha \not\equiv 0 \quad \textit{if and only if } k = s$$

('excitation of the sth mode'). In particular, for sufficiently small $\alpha \neq 0$, these solutions are non-stationary. Moreover, for different values of s, the families of solutions mutually differ close to U^0.

(*c*) *If we assume additionally $\Gamma(x) \neq 0$ for almost all $x \in \mathbb{R}^3$, then every nonstationary polychromatic solution U^α possesses infinitely many nonvanishing modes u_k^α.*

Remark 1.

(a) We require continuity of Γ since we use the functional analytic framework of [23]. The existence and continuity of $\nabla\Gamma$ will be exploited in proving that U^α is twice differentiable. This assumption as well as $\Gamma \neq 0$ almost everywhere in (c) might be relaxed; however, this study does not aim at the most general setting for the coefficients but rather focuses on the introduction of the setup for the existence result.

(b) The existence of stationary solutions of the Klein–Gordon equation (1) respective of solutions to (4) can be guaranteed under additional assumptions on Γ. We refer to [21], theorem I.2 and remarks I.5 and I.6 by Lions for positive (ground state) solutions and to theorem 2.1 of [4, 5] by Bartsch and Willem for bound states.

(c) We comment on the nondegeneracy property (5). In brief, the application of bifurcation from simple eigenvalues will require that the linearization of the infinite system (3a) and (3b) has a one-dimensional kernel and hence that all linearized equations only have the trivial solution—but for the sth component, giving the direction of bifurcation. As we shall see, this can be ensured for the 'Helmholtz' equation (3b) by choosing suitable resolvent-type maps to work with; for the 'Schrödinger' equation (3a), which has a classical resolvent, there is no such freedom of choice, which is why we introduce nondeceneracy as an assumption.

In some special cases, nondegeneracy properties like (5) have been verified, e.g. by Bates and Shi [6] in theorem 5.4 (6), or by Wei [29] in lemma 4.1, both assuming that w_0 is a ground state solution of (4) in the autonomous case with constant positive Γ. The existence of ground states w_0 has been shown, under these assumptions, in the famous paper by Berestycki and Lions [7, theorem 1 and example 1].

It should be pointed out that, although the quoted results discuss nondegeneracy in a setting on the Hilbert space $H^1(\mathbb{R}^3)$, the statements can be adapted to the topology of X_1, as we demonstrate in lemma 1. Generalizations of this might provide an interesting project in its own; we add some thoughts following lemma 1 concerning the difficulties in adapting the constant-case methods to non-constant Γ.

(d) The assumption $\omega > m$ on the frequency ensures that the stationary system (3) contains only one equation of Schrödinger type. This avoids further nondegeneracy assumptions on higher modes, which would not be covered by the previously mentioned results in the literature.

(e) The above result provides, locally, a multitude of families of breathers bifurcating from every given stationary solution characterized by different values of s, ω and possibly certain asymptotic parameters, see remark 2 below.

It would be natural, further, to ask for the global bifurcation picture given some trivial family $\mathcal{T} = \{(w_0, \lambda) \mid \lambda \in \mathbb{R}\}$. (Here $\lambda \in \mathbb{R}$ denotes a bifurcation parameter which in our case is not visible in the differential equation and thus will be properly introduced later.) Typically, global bifurcation theorems state that a maximal bifurcating continuum of solutions (U, λ) emanating from $\mathcal{T}$ at (w_0, λ_0) is unbounded unless it returns to $\mathcal{T}$ at some point (w_0, λ_0'), $\lambda_0' \neq \lambda_0$. In the former (desirable) case, however, a satisfactory characterization of global bifurcation structures should provide a criterion whether or not unboundedness results from another stationary solution $w_1 \neq w_0$ with $\{(w_1, \lambda) \mid \lambda \in \mathbb{R}\}$ belonging to the maximal continuum. Since it is not obvious at all whether and how such a criterion might be derived within our framework, we focus on the local result, which already adds new aspects to the state of knowledge about the existence of breather solutions summarized next.

1.1. An overview of literature

Polychromatic solutions. The results in theorem 1 can and should be compared with recent findings on breather (that is to say, time-periodic and spatially localized) solutions of the wave equation with periodic potentials $V(x), q(x) = c \cdot V(x) \geqslant 0$,

$$V(x)\partial_t^2 U - \partial_x^2 U + q(x)U = \Gamma(x)U^3 \quad \text{on } \mathbb{R} \times \mathbb{R}. \tag{6}$$

Such breather solutions have been constructed by Schneider *et al*, see theorem 1.1 in [9], and Hirsch and Reichel, see theorem 1.3 in [16], respectively. In brief, the main differences to the results in this article are that the authors of [9, 16] consider a setting in one space dimension and obtain strongly spatially localized solutions, which requires a comparably huge technical effort. We give some details: Both existence results are established using a polychromatic ansatz, which reduces the time-dependent equation to an infinite set of stationary problems with periodic coefficients, see [9], p 823, respectively [16], equation (1.2). The authors of [9] apply spatial dynamics and center manifold reduction; their ansatz is based on a very explicit choice of the coefficients q, V, Γ. The approach in [16] incorporates more general potentials and nonlinearities and is based on variational techniques. It provides ground state solutions, which are possibly 'large'—in contrast to our local bifurcation methods, which only yield solutions close to a given stationary one as described in theorem 1, i.e. with a typically 'small' time-dependent contribution.

Periodicity of the potentials in (6) is explicitly required since it leads to the occurrence of spectral gaps when analyzing the associated differential operators of the stationary equations. In contrast to the Helmholtz methods introduced here, the authors both of [9] and of [16] strive to construct the potentials in such way that 0 lies in the aforementioned spectral gaps, and moreover that the distance between 0 and the spectra has a positive lower bound. This is realized by assuming a certain 'roughness' of the potentials, referring to the step potential defined in theorem 1.1 of [9] and to the assumptions (P1)–(P3) in [16] which allow potentials with periodic spikes modeled by Dirac delta distributions, periodic step potentials or some specific, non-explicit potentials in $H^r_{\mathrm{rad}}(\mathbb{R})$ with $1 \leqslant r < \frac{3}{2}$ (see [16], lemma 2.8).

Let us summarize that the methods for constructing breather solutions of (6) outlined above can handle periodic potentials but require irregularity, are very restrictive concerning the form of the potentials and involve a huge technical effort in analyzing spectral properties based on Floquet–Bloch theory. The Helmholtz ansatz presented in this article provides a technically elegant and short approach suitable for constant potentials; in the context of breather solutions, it is new in the sense that it provides breathers with slow decay, it provides breathers on the full space $\mathbb{R}^3$, and it provides breathers for simple (constant) potentials.

The Klein–Gordon equation as a Cauchy problem. Possibly due to its relevance in physics, there is a number of classical results in the literature concerning the nonlinear Klein–Gordon equation. The fundamental difference to the results in this article is that the vast majority of these concerns the Cauchy problem of the Klein–Gordon equation, i.e.

$$\begin{aligned} &\partial_t^2 U - \Delta U + m^2\, U = \pm U^3 \quad \text{on } [0,\infty) \times \mathbb{R}^3 \\ &U(0,x) = f(x),\ \partial_t U(0,x) = g(x) \quad \text{on } \mathbb{R}^3 \end{aligned} \tag{7}$$

for suitable initial data $f, g : \mathbb{R}^3 \to \mathbb{R}$. Usually, the dependence of the nonlinearity on U is much more general (allowing also derivatives of U) and the space dimension is not restricted to $N = 3$. On the other hand, most results in the literature only concern the autonomous case, which is why we set in this discussion $\Gamma \equiv \pm 1$.

An overview of the state of knowledge towards the end of the 1970s can be found e.g. in [28] by Strauss, who discusses among other topics global existence for $\Gamma \equiv -1$ (theorem 1.1), regularity and uniqueness (theorem 1.2), blow-up (theorem 1.4) and scattering (theorem 4.1). In the first-mentioned result, which is originally due to Jörgens, global existence of distributional solutions with locally as well as globally finite energy is proved provided $\Gamma \equiv -1$. Following a classical strategy for evolution problems, local existence is shown by means of a fixed point iteration, and global existence can be obtained by an iteration argument based on energy conservation. For $\Gamma \equiv +1$, theorem 1.4 due to Keller and Levine demonstrates the existence of blow-up solutions. During the following decade, Klainerman [18, 19] and Shatah [26, 27] independently developed new techniques leading to significant improvements in the study of uniqueness questions and of the asymptotic behavior of solutions as $t \to \infty$. These results work in settings with high regularity and admit more general nonlinearities with growth assumptions for small arguments, which includes the cubic one as a special case.

The relation to our results is not straightforward since the bifurcation methods automatically provide solutions U^α which exist globally in time irrespective of the sign (or even of a possible x-dependence) of Γ and which do not decay as $t \to \infty$, and there is no special emphasis on the role of the initial values $U^\alpha(0, x), \nabla U^\alpha(0, x)$ along the bifurcating branches. Our methods instead focus on several global properties of the solutions $U^\alpha(t, x)$ such as periodicity in time and localization as well as decay rates in space, i.e. the defining properties of breathers.

1.2. Discussion and research perspectives

In this article, we prove an existence result for 'breather' solutions of the cubic Klein–Gordon equation

$$\partial_t^2 U - \Delta U + m^2 U = \Gamma(x) U^3 \quad \text{on } \mathbb{R} \times \mathbb{R}^3$$

for any mass $m > 0$ of the form $U(t, x) = \sum_{k \in \mathbb{Z}} 2\cos(\omega k t)\, u_k(x)$ with any $\omega > m$ and coefficients $u_k \in X_1$. Our existence result comes with a sufficient condition, which we explicitly prove to be valid for constant Γ. As it is the intention of this article to keep the amount of technical work low and to present the main ideas—a new way to find breathers using ideas from the field of Helmholtz equations—in more clarity, we decided not to consider generalizations to other space dimensions $N \geqslant 2$, which we believe to be straightforward by choosing the proper fundamental solutions of the stationary Helmholtz equation, imposing proper decay rates X_q and considering suitable powers in the nonlinearity. Not surprisingly, the case $N = 1$ is an exception since then fundamental solutions of the Helmholtz equation are periodic and in particular not localized.

Another natural generalization is the zero-mass case, resulting in the classical wave equation

$$\partial_t^2 U - \Delta U = \Gamma(x) U^3 \quad \text{on } \mathbb{R} \times \mathbb{R}^3.$$

In the framework we use, this causes problems concerning the lack of a resolvent respectively of a limiting absorption principle in X_1 applicable to the component u_0 solving (3a). A different setup for the 0th component, however, will (in the best case) make the mixed convolution terms harder to handle and increase the amount of technical details, which is why we did not follow this line of thought. Another idea used in [9, 16] to avoid 0th components is to study breathers which are not even but odd in time, e.g. $U(t, x) = \sum_{k \in \mathbb{Z}\setminus\{0\}} 2\ \sin(\omega k t)\, u_k(x)$. Unfortunately, the idea of bifurcation from a nontrivial constant-in-time solution (which is at the heart of this article) then has to be abandoned and another 'trivial' solution would have to be identified.

A second look into [9, 16] suggests further that it might be interesting to consider a periodic potential term (added to the mass term), thus establishing better comparability with the existence results in $1 + 1$ dimensions and addressing possible applications in the field of periodic structures such as photonic crystals. A direct application of the ideas in this article in a non-constant periodic setting would require, firstly, suitable limiting absorption principles for

Helmholtz equations with periodic potentials; secondly, a thorough study of the associated fundamental solutions and their asymptotic properties, which we heavily rely on in creating a framework with simple eigenvalues; and, thirdly, compactness properties as required by most bifurcation techniques, which are essentially ensured in the constant-coefficient case by imposing radial symmetry. Concerning the former, suitable limiting absorption techniques for periodic potentials have recently been developed [22], essentially using the Floquet–Bloch transform in place of the Fourier transform. Both latter issues might be challenges worth an independent project.

However, we strongly believe that there is another, more promising, way towards weakly localized breathers both for the nonlinear wave equation and for periodic potentials:

Apart from bifurcation methods, nonlinear Helmholtz equations and systems can also be discussed in a 'dual' variational framework as introduced by Evéquoz and Weth [13]. This might offer another way to analyze the system (3) leading to 'large' breathers in the sense that they are not close to a given stationary solution as the ones constructed in theorem 1. Furthermore, such an ansatz would also allow to investigate the wave equation, e.g. in the odd-in-time framework just mentioned, since it does not require to have a zero mode present and thus allows to circumvent in an elegant way the lack of a limiting absorption principle. Another strong point of the variational method is certainly that it is not bound to the assumption of radial symmetry. This makes an extension to periodic potentials (added to the mass term) more realistic using the limiting absorption techniques for periodic potentials in [22]. In addition, in a variational discussion there is some hope that the inevitable loss of compactness might be compensated by some variant of the well-established concentration compactness technique, see e.g. the nonvanishing theorem in [13].

Another extension might strive to include nonlocal problems of the form

$$\partial_t^2 U - \Delta U + m^2 U = U \cdot \left(G * U^2\right)$$

with a suitable kernel G modelling, in the above-mentioned ϕ^4 theory, the self-interaction of particles beyond point interactions, which is usually the first step to take when introducing higher-order terms in particle physics. Point interactions correspond to $G(x) = \Gamma(x)\delta(x)$ with a Dirac delta, thus producing equation (1).

Finally, generalizations of lemma 1 (which we essentially quote from the literature) concerning the nondegeneracy condition in the case of constant Γ might be of interest. Since such results also lead to uniqueness statements for ground state solutions of nonlinear equations, cf [11], we believe this to be relevant beyond the investigation of the existence of breather solutions where it appears only as a technical tool. For some more detailed thoughts in that direction, please go to the comments following lemma 1.

1.3. Conventions, strategy, and organization of the article

Throughout, we denote the convolution in $\mathbb{R}^3$ by the symbol * and use $\star$ in the convolution algebra ℓ^1. Extending the notation defined in (2), for $q \geqslant 0$, we let

$$X_q := \left\{ u \in C_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{R}) \mid \|u\|_{X_q} < \infty \right\} \qquad \text{with } \|u\|_{X_q} := \sup_{x \in \mathbb{R}^3} (1 + |x|^2)^{q/2} |u(x)|,$$

$$\mathcal{X}_q := \ell^1_{\mathrm{sym}}(\mathbb{Z}, X_q) \qquad \text{with } \|\mathbf{u}\|_{\mathcal{X}_q} := \|(u_k)_k\|_{\mathcal{X}_q} := \sum_{k \in \mathbb{Z}} \|u_k\|_{X_q}.$$

We look for polychromatic solutions as in (2) with coefficients $\mathbf{u} = (u_k)_{k\in\mathbb{Z}} \in \mathcal{X}_1$ which bifurcate from some (given) stationary solution henceforth denoted by

$$\mathbf{w} = (\delta_{k,0} w_0)_{k\in\mathbb{Z}} = (\ldots, 0, w_0, 0, \ldots)$$

with $w_0 \in X_1 \cap C^2_{\text{loc}}(\mathbb{R}^3)$ fixed according to equation (4). The components u_k solve the countably infinite Schrödinger–Helmholtz system (3a) and (3b). This is what motivates the use of the Banach space X_1, which prescribes the natural decay rate for solutions of Helmholtz equations as in (3b). The (linear) theory of these equations is, for the reader's convenience, summarized in the final section 4. In general, such equations can be solved using suitable limiting absorption principles. This will be motivated briefly in section 4 and the resulting solution formulae will be provided; however, given our regularity assumptions, these formulae are pointwise well-defined convolutions with explicit kernels, and the differentiability and decay properties stated in section 4 can be verified, alternatively, by direct computation.

Let us now give some details on the strategy we follow in order to prove the main result.

- Very first, we verify in proposition 1 that the convolution terms in the infinite-dimensional system (3a) and (3b) are well-defined in the Banach spaces we have just introduced.
- Intending to apply bifurcation techniques, we have to analyze the linearized version of (3a) and (3b). A collection of results concerning the linearized setting is summarized in proposition 2, referring to the more detailed discussion in the final section and/or in [23], where a two-component system of such type is discussed (which, however, cannot be traced back to polychromatic solutions of a time-dependent problem as in this article).
- We then present a suitable setup for bifurcation theory; in particular, we introduce a bifurcation parameter which is not visible in the differential equation but appears in the so-called far field of the functions u_k, i.e. in the leading-order contribution as $|x| \to \infty$. This is made more explicit in remark 2(a).
- In proposition 3 we discuss in what way solutions of the infinite stationary system (3a) and (3b) provide polychromatic, classical solutions of the Klein–Gordon equation (1). Indeed, regarding differentiability, we will see that the choice of suitable asymptotic conditions will ensure uniform convergence and hence smoothness properties of the infinite sums defining the polychromatic states.
- Finally, in proposition 4, we essentially verify the assumptions of the Crandall–Rabinowitz bifurcation theorem.

After that, closing section 2, we are able to give a very short proof of theorem 1. The auxiliary results will be proved in section 3. As announced, the final section 4 provides some more details on the theory of linear Helmholtz equations in X_1.

2. The proof of theorem 1

2.1. The functional-analytic setting

We start with an auxiliary result concerning the convolution terms on the right-hand side of (3a) and (3b) in the spaces $\mathcal{X}_1$ respectively $\mathcal{X}_3$ defined in the previous section.

Proposition 1. *The convolution of sequences $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)} \in \mathcal{X}_1$ is well-defined in a pointwise sense and satisfies $\mathbf{u}^{(1)} \star \mathbf{u}^{(2)} \star \mathbf{u}^{(3)} \in \mathcal{X}_3$. Moreover, we have the estimate*

$$\left\|\mathbf{u}^{(1)} \star \mathbf{u}^{(2)} \star \mathbf{u}^{(3)}\right\|_{\mathcal{X}_3} \leqslant \left\|\mathbf{u}^{(1)}\right\|_{\mathcal{X}_1} \left\|\mathbf{u}^{(2)}\right\|_{\mathcal{X}_1} \left\|\mathbf{u}^{(3)}\right\|_{\mathcal{X}_1}.$$

Having convinced ourselves that the system (3a) and (3b) is well-defined, we now rewrite it using $\mathbf{u} = \mathbf{w} + \mathbf{v}$ with $\mathbf{w} = (\ldots, 0, w_0, 0, \ldots)$; then,

$$\begin{aligned} &- \Delta v_k - (\omega^2 k^2 - m^2)\, v_k \\ &\quad = \Gamma(x) \cdot \left[((\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}))_k - \delta_{k,0} w_0^3 \right] \quad \text{on } \mathbb{R}^3 \end{aligned}. \tag{8}$$

We will find solutions of this system of differential equations by solving instead a system of coupled convolution equations which, for $k \notin \{0, \pm s\}$, have the form $v_k = \mathcal{R}_{\mu_k}^{\tau_k}[f_k]$. Here f_k represents the right-hand side of (8), $\mu_k := \omega^2 k^2 - m^2$, and the coefficients $\tau_k \in (0, \pi)$ will have to be chosen properly according to a nondegeneracy condition. The convolution operators

$$\mathcal{R}_\mu^\tau = \frac{\sin(|\cdot|\sqrt{\mu} + \tau)}{4\pi\, \sin(\tau) |\cdot|} * : X_3 \to X_1 \quad (\mu > 0,\ \ 0 < \tau < \pi)$$

can be viewed as resolvent-type operators for the Helmholtz equation $(-\Delta - \mu)v = f$ on $\mathbb{R}^3$ involving an asymptotic condition on the far field of the solution v, namely

$$|x|\, v(x) \sim \sin(|x|\sqrt{\mu} + \tau) + O\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty.$$

Such conditions are required since the homogeneous Helmholtz equation $(-\Delta - \mu)v = 0$ has smooth nontrivial solutions in X_1 (known as Herglotz waves), which are all multiples of

$$\tilde{\Psi}_\mu(x) := \frac{\sin(|x|\sqrt{\mu})}{4\pi|x|} \quad (x \neq 0).$$

We refer to section 4, more precisely lemma 5, for details; the case $\tau = 0$ requires a larger technical effort and is presented in lemma 6. This involves linear functionals $\alpha^{(\mu)}, \beta^{(\mu)} \in X_1'$ which, essentially, yield the coefficients of the sine respectively cosine terms in the asymptotic expansion above. Relying on these tools and notations, we summarize the relevant facts on the linearized versions of the Helmholtz equation (3b) in the following proposition.

Proposition 2. *Let $w_0 \in X_1$ be a solution of equation (4) with $\Gamma \in L^\infty_{rad}(\mathbb{R}^3) \cap C_{loc}(\mathbb{R}^3)$ and $\omega > m > 0$; define $\mu_k := \omega^2 k^2 - m^2$. For every $k \in \mathbb{Z}\backslash\{0\}$, there exists (up to a multiplicative constant) a unique nontrivial and radially symmetric solution $q_k \in X_1$ of*

$$-\Delta q_k - \mu_k\, q_k = 3\, \Gamma(x) w_0^2(x)\, q_k \quad \text{on } \mathbb{R}^3. \tag{9a}$$

It is twice continuously differentiable and satisfies, for some $c_k \neq 0$ and $\sigma_k \in [0, \pi)$,

$$q_k(x) = c_k \cdot \frac{\sin(|x|\,\sqrt{\mu_k} + \sigma_k)}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty. \tag{9b}$$

The equations (9a) and (9b) are equivalent to the convolution identities

$$\begin{cases} q_k = 3\, \mathcal{R}_{\mu_k}^{\sigma_k}[\Gamma w_0^2\, q_k] = 3\, \left(\mathcal{R}_{\mu_k}^{\pi/2}[\Gamma w_0^2\, q_k] + \cot(\sigma_k) \tilde{\Psi}_{\mu_k} * [\Gamma w_0^2\, q_k] \right) & \text{if } \sigma_k \in (0, \pi), \\ q_k = 3\, \mathcal{R}_{\mu_k}^{\pi/2}[\Gamma w_0^2\, q_k] + \left(\alpha^{(\mu_k)}(q_k) + \beta^{(\mu_k)}(q_k) \right) \cdot \tilde{\Psi}_{\mu_k} & \text{if } \sigma_k = 0. \end{cases}$$

For all $k \in \mathbb{Z}$, $\cos(\sigma_k)\, \beta^{(\mu_k)}(q_k) = \sin(\sigma_k)\, \alpha^{(\mu_k)}(q_k)$.

The existence statement and the asymptotic properties in (9) can be proved using the Prüfer transformation, see [23], proposition 6; the statements in the second part are consequences of lemmas 5 and 6 in the final section 4. For these results to apply we have assumed initially that Γ is continuous and bounded, whence $3\,\Gamma w_0^2 \in X_2$.

We now present the general assumptions valid throughout the following construction and the proof of theorem 1. Before focusing on the technical details, let us shortly explain how the introduction of the bifurcation parameter we are about to give in equations (12a) and (12b) below relates to the Helmholtz resolvents above. (For a more general statement concerning those, please also have a look at lemmas 4 and 5 in the appendix A.) Essentially, we will exploit the fact that a differential equation $-\Delta q - \mu q = f$ with $\mu > 0, f \in X_3$ (e.g. (9a)) has a multitude of (radial) solutions given by

$$q = \mathcal{R}_{\mu}^{\pi/2}[f] + c\,\tilde{\Psi}_{\mu}, \quad c \in \mathbb{R}$$

where the second term can be written as a convolution $\tilde{\Psi}_{\mu} * [f]$ unless that is identically zero. The bifurcation parameter λ we introduce below will only affect the constant c above and therefore changes in the bifurcation parameter (which are hard to trace explicitly) do not affect the differential equation solved. Moreover, when including asymptotic conditions such as (9b), the choice of the bifurcation parameter determines the number of nontrivial solutions of the associated equation (9a). This fact will be made use of when verifying the assumptions of the Crandall–Rabinowitz theorem, and the careful choice of asymptotic parameters is precisely what we prepare next.

We let σ_k for $k \in \mathbb{Z}\backslash\{0\}$ as in proposition 2 above and fix $s \in \mathbb{N}$, recalling that we aim to 'excite the sth mode' in the sense of theorem 1(b). With this, let us introduce

$$\tau_{\pm s} := \sigma_{\pm s}, \quad \tau_k := \begin{cases} \dfrac{\pi}{4} & \text{if } \sigma_k \neq \dfrac{\pi}{4}, \\ \dfrac{3\pi}{4} & \text{if } \sigma_k = \dfrac{\pi}{4} \end{cases} \quad \text{for } k \in \mathbb{Z}\backslash\{0, \pm s\}, \tag{10}$$

see also remark 2(b). Thus in particular $\tau_k \neq \sigma_k$ for $k \in \mathbb{Z}\backslash\{0, \pm s\}$, and we conclude from the uniqueness statement in proposition 2 the nondegeneracy property

$$k \in \mathbb{Z}\backslash\{0, \pm s\}, \quad q \in X_1, \quad q = 3\,\mathcal{R}_{\mu_k}^{\tau_k}[\Gamma w_0^2\, q] \;\Rightarrow\; q \equiv 0; \tag{11a}$$

for the 0th mode, using the resolvent $\mathcal{P}_{\mu_0} = (-\Delta + \mu_0)^{-1} : X_3 \to X_1$ (see lemma 7), the corresponding property is assumed in (5):

$$q \in X_1, \quad q = 3\,\mathcal{P}_{\mu_0}[\Gamma w_0^2\, q] \;\Rightarrow\; q \equiv 0. \tag{11b}$$

We now introduce a map the zeros of which provide solutions of the system (8). Throughout, we use the shorthand notation

$$\mathbf{u} = \mathbf{v} + \mathbf{w} \quad \text{for } \mathbf{v} \in \mathcal{X}_1 \text{ and the stationary solution } \mathbf{w} = (\ldots, 0, w_0, 0, \ldots).$$

As above, we have to distinguish the cases $\tau_s \in (0, \pi)$ and $\tau_s = 0$. (In the following, please recall that we consider some fixed $s \neq 0$.) For $0 < \tau_{\pm s} < \pi$, we introduce $F:\ \mathcal{X}_1 \times \mathbb{R} \to \mathcal{X}_1$ via

$$F(\mathbf{v},\lambda)_k := v_k - \begin{cases} \mathcal{P}_{\mu_0}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_0 - \Gamma\,w_0^3\right] & k=0,\\ \mathcal{R}_{\mu_s}^{\pi/2}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & \\ \quad +(\cot(\tau_{\pm s})-\lambda)\tilde{\Psi}_{\mu_s}*\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & k=\pm s,\\ \mathcal{R}_{\mu_k}^{\tau_k}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\right] & \text{else.}\end{cases} \tag{12a}$$

Similarly, if $\sigma_s = 0$, we define $G:\ \mathcal{X}_1\times\mathbb{R}\to\mathcal{X}_1$ by

$$G(\mathbf{v},\lambda)_k := v_k - \begin{cases} \mathcal{P}_{\mu_0}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_0 - \Gamma\,w_0^3\right] & k=0,\\ \mathcal{R}_{\mu_s}^{\pi/2}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & \\ \quad +(1-\lambda)\left(\alpha^{(\mu_s)}(v_{\pm s})+\beta^{(\mu_s)}(v_{\pm s})\right)\tilde{\Psi}_{\mu_s} & k=\pm s,\\ \mathcal{R}_{\mu_k}^{\tau_k}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\right] & \text{else.}\end{cases} \tag{12b}$$

The following result collects some basic properties of the maps F and G and the polychromatic states related to their zeros.

Proposition 3. *Let $s\in\mathbb{N}$ and $(\tau_k)_{k\in\mathbb{Z}}$ be chosen as in (10). The maps $F, G:\mathcal{X}_1\times\mathbb{R}\to\mathcal{X}_1$ are well-defined and smooth with $F(\mathbf{0},\lambda)=G(\mathbf{0},\lambda)=\mathbf{0}$ for all $\lambda\in\mathbb{R}$. Further, if $F(\mathbf{v},\lambda)=\mathbf{0}$ respectively $G(\mathbf{v},\lambda)=\mathbf{0}$ for some $\mathbf{v}\in\mathcal{X}_1$, $\lambda\in\mathbb{R}$, then $\mathbf{v}$ solves the stationary system (8) and*

$$U(t,x) := w_0(x) + v_0(x) + \sum_{k=1}^{\infty} 2\,\cos(\omega k t) v_k(x) \quad (t\in\mathbb{R}, x\in\mathbb{R}^3)$$

defines a twice continuously differentiable, classical solution $U\in C^2(\mathbb{R},X_1)$ of the Klein–Gordon equation (1).

Again, the proof can be found in section 3. We will even show that $U\in C^\infty(\mathbb{R},X_1)$. For the derivatives of F respectively G with respect to the Banach space component $\mathbf{v}\in\mathcal{X}_1$, we will verify the following explicit formulas: Letting $\mathbf{q}\in\mathcal{X}_1$ and abbreviating $\mathbf{u}:=\mathbf{v}+\mathbf{w}$,

$$(DF(\mathbf{v},\lambda)[\mathbf{q}])_k = q_k - \begin{cases} 3\,\mathcal{P}_{\mu_0}[\Gamma(\mathbf{q}\star\mathbf{u}\star\mathbf{u})_0] & k=0,\\ 3\,\mathcal{R}_{\mu_s}^{\pi/2}\left[\Gamma(\mathbf{q}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & \\ \quad +3\,(\cot(\tau_s)-\lambda)\tilde{\Psi}_{\mu_s}*\left[\Gamma(\mathbf{q}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & k=\pm s,\\ 3\,\mathcal{R}_{\mu_k}^{\tau_k}\left[\Gamma(\mathbf{q}\star\mathbf{u}\star\mathbf{u})_k\right] & \text{else;}\end{cases} \tag{13a}$$

$$(DG(\mathbf{v},\lambda)[\mathbf{q}])_k = q_k - \begin{cases} 3\,\mathcal{P}_{\mu_0}[\Gamma(\mathbf{q}\star\mathbf{u}\star\mathbf{u})_0] & k=0,\\ 3\,\mathcal{R}_{\mu_s}^{\pi/2}\left[\Gamma(\mathbf{q}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & \\ \quad +(1-\lambda)\left(\alpha^{(\mu_s)}(q_{\pm s})+\beta^{(\mu_s)}(q_{\pm s})\right)\tilde{\Psi}_{\mu_s} & k=\pm s,\\ 3\,\mathcal{R}_{\mu_k}^{\tau_k}\left[\Gamma(\mathbf{q}\star\mathbf{u}\star\mathbf{u})_k\right] & \text{else.}\end{cases} \tag{13b}$$

Remark 2.

(a) As earlier announced, we now see that the bifurcation parameter λ appears only in the asymptotic expansions of the sth components $v_{\pm s}$ of the solutions and not in the differential equation (1). More precisely, abbreviating $\mathbf{f_u} := \Gamma\ (\mathbf{u} \star \mathbf{u} \star \mathbf{u})$, the identities $F(\mathbf{v}, \lambda) = \mathbf{0}$ respectively $G(\mathbf{v}, \lambda) = \mathbf{0}$ imply

$$v_s = \mathcal{R}^{\pi/2}_{\mu_s}[(\mathbf{f_u})_s] + (\cot(\tau_{\pm s}) - \lambda)\tilde{\Psi}_{\mu_s} * (\mathbf{f_u})_s,$$
$$v_s = \mathcal{R}^{\pi/2}_{\mu_s}[(\mathbf{f_u})_s] + (1 - \lambda)\left(\alpha^{(\mu_s)}(v_{\pm s}) + \beta^{(\mu_s)}(v_{\pm s})\right)\tilde{\Psi}_{\mu_s},$$

respectively. In either case, this yields $-\Delta v_s - \mu_s v_s = (\mathbf{f_u})_s$, and after some calculations using lemmas 4 and 6 one finds for all suitable λ, respectively, as $|x| \to \infty$

$$v_s(x) = \sqrt{\frac{\pi}{2}}\,\widehat{(\mathbf{f_u})_s}(\sqrt{\mu_s})\left[\frac{\cos(|x|\sqrt{\mu_s})}{|x|} + (\cot(\tau_{\pm s}) - \lambda)\frac{\sin(|x|\sqrt{\mu_s})}{|x|}\right] + O\left(\frac{1}{|x|^2}\right),$$
$$v_s(x) = \sqrt{\frac{\pi}{2}}\,\widehat{(\mathbf{f_u})_s}(\sqrt{\mu_s})\left[\frac{\cos(|x|\sqrt{\mu_s})}{|x|} + \frac{1-\lambda}{\lambda}\frac{\sin(|x|\sqrt{\mu_s})}{|x|}\right] + O\left(\frac{1}{|x|^2}\right).$$

In either case, writing $\cot(\tau(\lambda)) := \cot(\tau_{\pm s}) - \lambda$ respectively $\cot(\tau(\lambda)) := \frac{1-\lambda}{\lambda}$, this yields via the angle sum identities

$$v_s(x) = \sqrt{\frac{\pi}{2}}\,\widehat{(\mathbf{f_u})_s}(\sqrt{\mu_s})\frac{\sin(|x|\sqrt{\mu_s} + \tau(\lambda))}{\sin(\tau(\lambda))|x|} + O\left(\frac{1}{|x|^2}\right),$$

and it is in this sense that we claim the bifurcation parameter to be a phase parameter in the far field. This being only a rough sketch, let us mention rather briefly that there is no singularity in the latter term as $\lambda \to 0$, since then the evaluation of the Fourier transform in the prefactor vanishes, too, and thus compensates.

(b) The choice of the parameters τ_k in equation (10) is far from unique. Indeed, one could instead consider any configuration satisfying

$$\tau_k = \tau_{-k} \neq \sigma_k \text{ for all } k \in \mathbb{Z}\backslash\{\pm s\}, \quad \{\tau_k \mid k \in \mathbb{Z}\backslash\{\pm s\}\} \subseteq (\delta, \pi - \delta)$$

for some $\delta \in \left(0, \frac{\pi}{2}\right)$. The former condition is required for the nondegeneracy statement (11a), and the latter will be used to obtain uniform decay estimates in the proof of proposition 3, see lemma 3.

However, as in [23], the question whether another choice of τ_k leads to different bifurcating families is still open. Hence we discuss only the explicit choice in (10).

In the so-established framework, we intend to apply the Crandall–Rabinowitz bifurcation theorem. The next result shows that its assumptions are satisfied.

Proposition 4 (Simplicity and transversality). *Let $s \in \mathbb{N}$ and $(\tau_k)_{k\in\mathbb{Z}}$ be chosen as in* (10). *The linear operator $DF(\mathbf{0}, 0) : \mathcal{X}_1 \to \mathcal{X}_1$ is 1-1-Fredholm with a kernel of the form*

$$\ker DF(\mathbf{0}, 0) = \operatorname{span}\{\mathbf{q}\} \quad \textit{where } q_k \neq 0 \textit{ if and only if } k = \pm s.$$

Moreover, the transversality condition is satisfied, that is,

$$\partial_\lambda DF(\mathbf{0},0)[\mathbf{q}] \notin \operatorname{ran} DF(\mathbf{0},0).$$

A corresponding statement holds true for $DG(\mathbf{0},0) : \mathcal{X}_1 \to \mathcal{X}_1$.

2.2. The proof of theorem 1

Let us fix some $s \in \mathbb{N}$, and choose $(\tau_k)_{k\in\mathbb{Z}}$ as in (10). We introduce the trivial family $\mathcal{T} := \{(\mathbf{0},\lambda) \in \mathcal{X}_1 \times \mathbb{R} \mid \lambda \in \mathbb{R}\}$.

Step 1. ***Proof of*** **(a).**

By proposition 3, the maps F respectively G are smooth and vanish on the trivial family $\mathcal{T}$. In view of proposition 4, the Crandall–Rabinowitz theorem (cf [12, theorem 1.7]) shows that $(\mathbf{0},0) \in \mathcal{T}$ is a bifurcation point for $F(\mathbf{v},\lambda) = 0$ respectively $G(\mathbf{v},\lambda) = 0$ and provides an open interval $J_s \subseteq \mathbb{R}$ containing 0 and a smooth curve

$$J_s \to \mathcal{X}_1 \times \mathbb{R}, \quad \alpha \mapsto (\mathbf{v}^\alpha, \lambda^\alpha) = \big((v_k^\alpha)_{k\in\mathbb{Z}}, \lambda^\alpha\big)$$

of zeros of F respectively G (we do not denote its dependence on s) with $\mathbf{v}^0 = \mathbf{0}, \lambda^0 = 0$ as well as $\frac{\mathrm{d}}{\mathrm{d}\alpha}\big|_{\alpha=0}\mathbf{v}^\alpha = \mathbf{q}$ where $\mathbf{q}$ is a nontrivial element of the kernel of $DF(\mathbf{0},0)$ respectively $DG(\mathbf{0},0)$. We let $\mathbf{u}^\alpha := \mathbf{v}^\alpha + \mathbf{w}$ and define polychromatic states U^α as in (a). Then U^α is a classical solution of the cubic Klein–Gordon equation (1) due to proposition 3 since $F(\mathbf{v}^\alpha,\lambda^\alpha) = 0$ respectively $G(\mathbf{v}^\alpha,\lambda^\alpha) = 0$. By their very definition, the solutions U^α are time-periodic with period $2\pi/\omega$ (maybe less). This proves (a).

Step 2. ***Proof of*** **(b).**

Since F respectively G are smooth, so is the map $J_s \to \mathcal{X}_1 \times \mathbb{R}$, $\alpha \mapsto (\mathbf{v}^\alpha, \lambda^\alpha)$. By proposition 4, $q_k \neq 0$ if and only if $k = \pm s$, which implies that only the $\pm s$th components of

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\bigg|_{\alpha=0} \mathbf{u}^\alpha = \frac{\mathrm{d}}{\mathrm{d}\alpha}\bigg|_{\alpha=0} \mathbf{v}^\alpha = \mathbf{q}$$

do not vanish. For sufficiently small nonzero values of α, the solutions U^α are thus nonstationary. In particular, the direction of bifurcation changes when changing the value of s, and the associated bifurcating curves are, at least locally, mutually different.

Step 3. ***Proof of*** **(c).**

We show finally that, under the additional assumption that $\Gamma(x) \neq 0$ for almost all $x \in \mathbb{R}^3$, every non-stationary solution

$$U^\alpha(t,x) = w_0(x) + v_0^\alpha(x) + \sum_{k=1}^{\infty} 2\ \cos(\omega k t)\, v_k^\alpha(x)$$

in fact possesses infinitely many nontrivial coefficients v_k^α. Indeed, assuming the contrary, we can choose a maximal $r > 0$ (since U^α is non-stationary) with $v_r^\alpha \not\equiv 0$ or equivalently $u_r^\alpha = v_r^\alpha + w_r = v_r^\alpha \not\equiv 0$. But then,

$$v_{3r}^\alpha = \sum_{l+m+n=3r} \mathcal{R}_{\mu_{3r}}^{\tau_{3r}}[\Gamma\, u_l^\alpha\, u_m^\alpha\, u_n^\alpha] = \mathcal{R}_{\mu_{3r}}^{\tau_{3r}}[\Gamma\,(v_r^\alpha)^3] \not\equiv 0$$

since the convolution identity implies $-\Delta v_{3r}^\alpha - \mu_{3r} v_{3r}^\alpha = \Gamma\,(v_r^\alpha)^3$, and $\Gamma\,(v_r^\alpha)^3 \not\equiv 0$ since $\Gamma(x) \neq 0$ almost everywhere by assumption. This contradicts the maximality of r. □

2.3. The proof of remark 1(c)

Finally, as announced in remark 1(c), we verify the nondegeneracy assumption (11b) respectively (5) for constant positive Γ.

Lemma 1 (Nondegeneracy, à la Bates and Shi [6]). *Let $\Gamma \equiv \Gamma_0$ for some $\Gamma_0 > 0$, and assume that $w_0 \in C^2_{rad}(\mathbb{R}^3)$ is a radially symmetric solution of (4) the profile of which satisfies $w_0(r) > 0$, $w_0'(r) < 0$ for all $r > 0$, and both $w_0(r)$ and $w_0'(r)$ decay exponentially as $r \to \infty$. Then the nondegeneracy property (5) holds, i.e. for any radial, twice differentiable $q_0 \in X_1$*

$$-\Delta q_0 + m^2 q_0 = 3\Gamma_0\, w_0^2\, q_0 \quad \text{on } \mathbb{R}^3 \text{ implies } \; q_0 \equiv 0.$$

As hinted at in remark 1(c), the existence of a suitable positive and radially exponentially decaying ground state solution $w_0 \in C^2_{\mathrm{rad}}(\mathbb{R}^3)$ of the autonomous nonlinear equation

$$-\Delta w_0 + m^2 w_0 = \Gamma_0 w_0^3$$

has been proved in a classical paper by Berestycki and Lions [7, theorem 1 and example 1]. For such w_0, at least, lemma 1 guarantees nondegeneracy.

Lemma 1 can be proved closely following the line of argumentation by Bates and Shi [6], theorem 5.4 (6). The main difference is that they state the nondegeneracy result as a spectral property of the operator $-\Delta + m^2 + 3\Gamma_0 w_0^2 : H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ whereas we cannot use the Hilbert space setting but discuss solutions in X_1. However, the technique of Bates and Shi (and also of Wei's proof in [29]) is based on an expansion at a fixed radius $r > 0$ in terms of the eigenfunctions of the Laplace–Beltrami operator on $L^2(\mathbb{S}^2)$. This provides coefficients depending on r, and the conclusions are obtained from the analysis of these profiles on an ODE level using results due to Kwong and Zhang [20] built in turn on work by Coffman [11]. These ideas apply in the topology of X_1 in the very same way since they entirely focus on the profiles and on the question whether these are localized or not; for details, cf [24], (proof of) lemma 4.11.

Coffman's result [11] for constant positive Γ is part of a uniqueness result for nonlinear ground states. The technique relies on a thorough analysis of the ODE initial value problems satisfied by the profiles (here for $\Gamma \equiv 1$ and $m = 1$),

$$w'' + \frac{2}{r}w' - w + w^3 = 0, \quad w(0) = \alpha, \quad w'(0) = 0 \quad \text{resp.}$$

$$q'' + \frac{2}{r}q' - q + 3w^2\, q = 0, \quad q(0) = 1, \quad q'(0) = 0$$

with $q = \frac{\mathrm{d}}{\mathrm{d}\alpha}w$. Here α is a parameter which makes it possible to establish a shooting-type argumentation, and it is shown that whenever α is such that w is a ground state of the nonlinear equation (i.e. in particular positive and localized), this produces an exponentially unbounded solution q of the linearized problem and hence guarantees nondegeneracy in X_1 (and in H^2, respectively). For details, we refer essentially to lemma 4.2(ii) in [11], the proof of which relies (among other aspects) on a series of identities (4.16)–(4.20) obtained by further differentiation of combinations of the differential equations for w, q in such way that the resulting right-hand sides have a sign.

Striving to include non-constant (but still radially symmetric) Γ, differentiation of Γ leads to additional terms, and the required positivity respectively negativity properties then lead to conditions intertwining Γ and the ground state solution $w = w_0$, which in turn depends on Γ occurring in the cubic equation (see [11], proof of lemma 4.4). This is why we strongly feel that there is no straightforward generalization of this technique to non-constant Γ, which is supported by the fact that the (few) non-autonomous generalizations mentioned in [20] do not cover this case. In view of the length and complexity of the proof even for the autonomous problem, it might certainly be a challenging project of its own with applications beyond the study of breather solutions, e.g. concerning the uniqueness of ground states as in [11].

Anyway, a perturbative strategy might provide a family of close-to-constant $\Gamma(x)$ satisfying the nondegeneracy condition. This could be based on the viewpoint taken in the related statements in [6, 29] proving (in a non-radial H^2 setting with constant Γ) that $\eta = 3$ is an isolated eigenvalue of the (weighted) eigenvalue problem $-\Delta q + q = \eta w^2 q$ with three linearly independent, non-radial eigenfunctions; this should persist for 'almost constant' $\Gamma(x)$ on the right-hand side.

3. Proofs of the auxiliary results

Proof of proposition 1. Let $\mathbf{u}^{(j)} = (u_k^{(j)})_{k\in\mathbb{Z}} \in \mathcal{X}_1$ for $j = 1, 2, 3$. We find the following chain of inequalities

$$\begin{aligned}
\left\|\mathbf{u}^{(1)} \star \mathbf{u}^{(2)} \star \mathbf{u}^{(3)}\right\|_{\mathcal{X}_3} &= \sum_{k\in\mathbb{Z}} \left\|(\mathbf{u}^{(1)} \star \mathbf{u}^{(2)} \star \mathbf{u}^{(3)})_k\right\|_{X_3} \\
&\leqslant \sum_{k\in\mathbb{Z}} \sum_{\substack{l,m,n\in\mathbb{Z}\\ l+m+n=k}} \left\|u_l^{(1)}\, u_m^{(2)}\, u_n^{(3)}\right\|_{X_3} \\
&\leqslant \sum_{k\in\mathbb{Z}} \sum_{\substack{l,m,n\in\mathbb{Z}\\ l+m+n=k}} \left\|u_l^{(1)}\right\|_{X_1} \left\|u_m^{(2)}\right\|_{X_1} \left\|u_n^{(3)}\right\|_{X_1} \\
&= \left\|\left(\left\|u_l^{(1)}\right\|_{X_1}\right)_{l\in\mathbb{Z}} \star \left(\left\|u_m^{(2)}\right\|_{X_1}\right)_{m\in\mathbb{Z}} \star \left(\left\|u_n^{(3)}\right\|_{X_1}\right)_{n\in\mathbb{Z}}\right\|_{\ell^1(\mathbb{Z})} \\
&\leqslant \left\|\left(\left\|u_l^{(1)}\right\|_{X_1}\right)_{l\in\mathbb{Z}}\right\|_{\ell^1(\mathbb{Z})} \left\|\left(\left\|u_m^{(2)}\right\|_{X_1}\right)_{m\in\mathbb{Z}}\right\|_{\ell^1(\mathbb{Z})} \left\|\left(\left\|u_n^{(3)}\right\|_{X_1}\right)_{n\in\mathbb{Z}}\right\|_{\ell^1(\mathbb{Z})} \\
&= \left\|\mathbf{u}^{(1)}\right\|_{\mathcal{X}_1} \left\|\mathbf{u}^{(2)}\right\|_{\mathcal{X}_1} \left\|\mathbf{u}^{(3)}\right\|_{\mathcal{X}_1},
\end{aligned}$$

where finally Young's inequality for convolutions in $\ell^1(\mathbb{Z})$ has been applied. Since the latter term is finite, we infer $\mathbf{u}^{(1)} \star \mathbf{u}^{(2)} \star \mathbf{u}^{(3)} \in \mathcal{X}_3$. □

Proof of proposition 3. **Step 1.** ***Decay estimates.***

The proof of proposition 3 requires convergence properties in order to handle the infinite series in the definition of $U(t, x)$, which we first provide in the following two lemmas.

Lemma 2. *The convolution operators* $\mathcal{R}_\mu^\tau : X_3 \to X_1$ *satisfy for* $\tau \in (0, \pi)$ *and* $\mu > 0$

$$\forall\, f \in X_3 \qquad \begin{aligned}
\left\|\mathcal{R}_\mu^\tau[f]\right\|_{X_1} &\leqslant \frac{C}{\sin(\tau)} \left(1 + \frac{1}{\sqrt{\mu}}\right) \cdot \|f\|_{X_3}, \\
\left\|\mathcal{R}_\mu^\tau[f]\right\|_{L^4(\mathbb{R}^3)} &\leqslant \frac{C}{\sqrt[4]{\mu}\,\sin(\tau)} \cdot \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}.
\end{aligned}$$

The fact that a power of μ appears in the denominator is crucial since it will finally provide the convergence and regularity of the polychromatic sums where $\mu = \mu_k = \omega^2 k^2 - m^2$ for $k \in \mathbb{Z}$.

The proof of lemma 2 relies, via rescaling, on the respective estimates for $\mu = 1$. These can be found in [23], pp 1038–1039 for the X_3–X_1 estimate and in [13], theorem 2.1 for the $L^{4/3}$–L^4 estimate.

Lemma 3. *Let $\Gamma \in L^\infty_{rad}(\mathbb{R}^3) \cap C^1_{loc}(\mathbb{R}^3)$ and assume $\mathbf{u} = (u_k)_{k\in\mathbb{Z}} \in \mathcal{X}_1$ is a sequence of C^2_{loc} functions which satisfy the following system of convolution equations:*

$$u_k = \mathcal{R}^{\tau_k}_{\mu_k}[\Gamma\,(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_k] \quad \textit{for all } k \in \mathbb{Z} \textit{ with } |k| > s$$

where $\mu_k = \omega^2 k^2 - m^2$ and $\tau_k \in (\delta, \pi - \delta)$ for some $\omega > m, \delta \in \left(0, \frac{\pi}{2}\right)$. Then there holds:

(*a*) *For every $\alpha \geqslant 0$, there exists a constant $C_\alpha \geqslant 0$ with*

$$\|u_k\|_{L^4(\mathbb{R}^3)} + \|\Gamma\,(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_k\|_{L^4(\mathbb{R}^3)} \leqslant C_\alpha \cdot (k^2+1)^{-\frac{\alpha}{2}} \quad (k \in \mathbb{Z}).$$

(*b*) *For every ball $B = B_R(0) \subseteq \mathbb{R}^3$ and $\alpha \geqslant 0$ there exists a constant $D_\alpha(B) \geqslant 0$ with*

$$|u_k(x)| + |\nabla u_k(x)| + |D^2 u_k(x)| \leqslant D_\alpha(B) \cdot (k^2+1)^{-\frac{\alpha}{2}} \quad (k \in \mathbb{Z}, x \in B).$$

(*c*) *For every $\alpha \geqslant 0$, there exists a constant $E_\alpha \geqslant 0$ with*

$$\|u_k\|_{X_1} \leqslant E_\alpha \cdot (k^2+1)^{-\frac{\alpha}{2}} \quad (k \in \mathbb{Z}).$$

The proof of lemma 3 can be found in detail also in the author's PhD thesis [24], lemma 4.13. Though, aiming for a self-contained exposition, we present it in full detail. Part (a) is proved by iteratively using the scaling properties in lemma 2; the remainder parts are then derived via local elliptic regularity estimates and suitable integral representations. In case the reader is not interested in these rather technical estimates, it is possible to immediately continue with step 2 instead.

As $u_k \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$ for all $k \in \mathbb{Z}$ by assumption, it is straightforward to find constants as in the lemma for a finite number of elements $u_{-s}, \ldots, u_s$. Hence it is sufficient to study those $k \in \mathbb{Z}$ with $|k| > s$; for these, we have $\mu_k = k^2\omega^2 - m^2 \geqslant c_s(k^2+1)$ for some positive $c_s > 0$ depending on the parameters ω and m.

Proof of (a).

The decay estimates of arbitrary order in k we aim to prove essentially go back to the $L^{4/3}$–L^4 scaling property stated in lemma 2 above. Indeed, due to $\delta < \tau_k < \pi - \delta$, it provides $C_1 = C_1(\|\Gamma\|_\infty, \delta, \omega, m, s) \geqslant 0$ with

$$\|u_k\|_{L^4(\mathbb{R}^3)} \leqslant \frac{C_1}{(k^2+1)^{\frac{1}{4}}} \|(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_k\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \quad \text{for all } k \in \mathbb{Z}. \tag{14a}$$

With that, assuming $\sum_{k\in\mathbb{Z}} (k^2+1)^{\frac{\alpha}{2}} \|u_k\|_{L^4(\mathbb{R}^3)} < \infty$ for some $\alpha \geqslant 0$ (which is trivially satisfied for $\alpha = 0$ since $\mathbf{u} \in \mathcal{X}_1$), one can iterate as follows

$$\sum_{k\in\mathbb{Z}}(k^2+1)^{\frac{\alpha+1/2}{2}}\ \|u_k\|_{L^4(\mathbb{R}^3)} \overset{(14a)}{\leqslant} C_1 \sum_{k\in\mathbb{Z}}(k^2+1)^{\frac{\alpha}{2}}\ \|(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}$$
$$\leqslant C_1 \sum_{k\in\mathbb{Z}}\sum_{l+m+n=k}((l+m+n)^2+1)^{\frac{\alpha}{2}}\|u_l\|_{L^4(\mathbb{R}^3)}\,\|u_m\|_{L^4(\mathbb{R}^3)}\,\|u_n\|_{L^4(\mathbb{R}^3)}$$
$$\leqslant 2^{\alpha}\,C_1 \sum_{k\in\mathbb{Z}}\sum_{l+m+n=k}\Big[(l^2+1)^{\frac{\alpha}{2}}\|u_l\|_{L^4(\mathbb{R}^3)}\,(m^2+1)^{\frac{\alpha}{2}}$$
$$\times\ \|u_m\|_{L^4(\mathbb{R}^3)}\,(n^2+1)^{\frac{\alpha}{2}}\|u_n\|_{L^4(\mathbb{R}^3)}\Big]$$
$$= 2^{\alpha}\,C_1\left(\sum_{k\in\mathbb{Z}}(k^2+1)^{\frac{\alpha}{2}}\|u_k\|_{L^4(\mathbb{R}^3)}\right)^3$$
$$< \infty.$$

This shows the first part of the estimate in (a), and we may set for $\beta \geqslant 0$

$$C(\beta) := \sup_{j\in\mathbb{Z}}\|u_j\|_{L^4(\mathbb{R}^3)}(1+j^2)^{\frac{\beta}{2}} < \infty. \tag{14b}$$

The second part follows by combining the former with the interpolation estimate

$$\|u_l u_m u_n\|_{L^4(\mathbb{R}^3)} \leqslant \|u_l\|_{L^{12}(\mathbb{R}^3)}\|u_m\|_{L^{12}(\mathbb{R}^3)}\|u_n\|_{L^{12}(\mathbb{R}^3)}$$
$$\leqslant \left[\|u_l\|_{L^4(\mathbb{R}^3)}\|u_m\|_{L^4(\mathbb{R}^3)}\|u_n\|_{L^4(\mathbb{R}^3)}\right]^{\frac{1}{3}}\left[\|u_l\|_{\infty}\|u_m\|_{\infty}\|u_n\|_{\infty}\right]^{\frac{2}{3}}$$
$$\leqslant \left[\|u_l\|_{L^4(\mathbb{R}^3)}\|u_m\|_{L^4(\mathbb{R}^3)}\|u_n\|_{L^4(\mathbb{R}^3)}\right]^{\frac{1}{3}}\ \|\mathbf{u}\|^2_{\mathcal{X}_1},$$

which finally yields with (14b), with the inequality $1+(l+m+n)^2 \leqslant 2(1+l^2)(1+m^2)(1+n^2)$ valid for all $l, m, n \in \mathbb{Z}$, and with $C_2 := \sum_{j\in\mathbb{Z}}\frac{1}{1+j^2}$ the second part of the asserted estimate

$$\|\Gamma(\mathbf{u}*\mathbf{u}*\mathbf{u})_k\|_{L^4(\mathbb{R}^3)} \leqslant \|\Gamma\|_{\infty}\sum_{l+m+n=k}\|u_l u_m u_n\|_{L^4(\mathbb{R}^3)}$$
$$\leqslant \|\Gamma\|_{\infty}\left[\sum_{l+m+n=k}\left[\|u_l\|_{L^4(\mathbb{R}^3)}\|u_m\|_{L^4(\mathbb{R}^3)}\|u_n\|_{L^4(\mathbb{R}^3)}\right]^{\frac{1}{3}}\right]\ \|\mathbf{u}\|^2_{\mathcal{X}_1}$$
$$\leqslant \|\Gamma\|_{\infty}C(3\alpha+6)\left[\sum_{l+m+n=k}\left((l^2+1)(m^2+1)(n^2+1)\right)^{-\frac{\alpha}{2}-1}\right]\ \|\mathbf{u}\|^2_{\mathcal{X}_1}$$
$$\leqslant \|\Gamma\|_{\infty}C(3\alpha+6)\left[\sum_{l+m+n=k}\frac{2^{\frac{\alpha}{2}}(k^2+1)^{-\frac{\alpha}{2}}}{(l^2+1)(m^2+1)(n^2+1)}\right]\ \|\mathbf{u}\|^2_{\mathcal{X}_1}$$
$$\leqslant \|\Gamma\|_{\infty}C(3\alpha+6)\,2^{\frac{\alpha}{2}}\ C_2^3\ \|\mathbf{u}\|^2_{\mathcal{X}_1}\ \cdot(k^2+1)^{-\frac{\alpha}{2}}.$$

Proof of (b).

The local estimate in (b) can be derived from the global L^4 bounds in (a) using elliptic regularity, which first provides estimates in $W^{2,4}_{\mathrm{loc}}(\mathbb{R}^3)$ and then in suitable Hölder spaces.

We fix some ball $\tilde{B} = B_{\tilde{R}}(0) \supseteq B$. For $|k| > s$, we set $g_k := \mu_k u_k + \Gamma(\mathbf{u}^*\mathbf{u}^*\mathbf{u})_k$ where again $\mu_k = k^2\omega^2 - m^2 \geqslant c_s(k^2+1)$, and lemma 4 guarantees that $u_k = \mathcal{R}_{\mu_k}^{\tau_k}[\Gamma(\mathbf{u}*\mathbf{u}*\mathbf{u})_k]$ is the (unique strong resp. classical) solution of the boundary value problem

$$\begin{cases} -\Delta\varphi = g_k & \text{on } \tilde{B}, \\ \varphi = u_k(\tilde{R}) & \text{on } \partial\tilde{B}. \end{cases} \tag{14c}$$

This rearrangement ensures that the constants arising in elliptic regularity estimates, which we take from the book by Gilbarg and Trudinger [14], do not depend on k resp. μ_k (but possibly on the balls $B, \tilde{B}$ and on $\alpha \geqslant 0$), which we shall indicate in this part writing $\lesssim$. First, we apply L^4 estimates, cf [14, theorem 9.11] and obtain a local $W^{2,4}$ bound

$$\|u_k\|_{W^{2,4}(\tilde{B})} \lesssim \|g_k\|_{L^4(\mathbb{R}^3)} + \|u_k\|_{L^4(\mathbb{R}^3)}.$$

The Sobolev embedding $W^{2,4}(\tilde{B}) \hookrightarrow C^1(\overline{\tilde{B}})$ as well as part (a) then yield

$$\|u_k\|_{C^1(\overline{\tilde{B}})} \lesssim \|g_k\|_{L^4(\mathbb{R}^3)} + \|u_k\|_{L^4(\mathbb{R}^3)} \lesssim (k^2+1)^{-\frac{\alpha}{2}}.$$

Assuming differentiability of Γ, we infer $\|g_k\|_{C^{0,\gamma}(\overline{\tilde{B}})} \lesssim (k^2+1)^{-\frac{\alpha}{2}}$ for some fixed $\gamma \in (0,1)$, and thus the Schauder interior estimates [14, corollary 6.9] imply

$$\|u_k\|_{C^{2,\gamma}(\overline{B})} \lesssim \|g_k\|_{C^{0,\gamma}(\overline{\tilde{B}})} + \|u_k\|_{C^0(\overline{\tilde{B}})} \lesssim (k^2+1)^{-\frac{\alpha}{2}},$$

which provides in particular the asserted local C^2 bound.

Proof of (c).

The estimate (c) in the X_1 norm essentially uses the explicit representations (given $f \in X_3$)

$$\begin{aligned} \mathcal{R}_\mu^\tau[f](x) &= \int_{\mathbb{R}^3} \frac{\sin(|x-y|\sqrt{\mu_k}+\tau_k)}{4\pi|x-y|\sin(\tau_k)} \cdot f(y)\,\mathrm{d}y \\ &= \frac{\sin(|x|\sqrt{\mu_k}+\tau_k)}{4\pi|x|\sin(\tau_k)} \int_{B_{|x|}(0)} \frac{\sin(|y|\sqrt{\mu_k})}{|y|\sqrt{\mu_k}} f(y)\,\mathrm{d}y \\ &\quad + \frac{\sin(|x|\sqrt{\mu_k})}{4\pi|x|\sin(\tau_k)} \int_{\mathbb{R}^3\setminus B_{|x|}(0)} \frac{\sin(|y|\sqrt{\mu_k}+\tau_k)}{|y|\sqrt{\mu_k}} f(y)\,\mathrm{d}y. \end{aligned} \tag{14d}$$

Starting here with $f = \Gamma(\mathbf{u}^*\mathbf{u}^*\mathbf{u})_k$ and $\mathcal{R}_\mu^\tau[f] = u_k$ for $|k| > s$, it is essentially Hölder's inequality and the estimates in (a) which yield (c). We also exploit again that the choice of asymptotic conditions guarantees a positive lower bound $|4\pi\ \sin(\tau_k)| \geqslant \tilde{\delta}$. For instance, the first line in (14d) yields

$$
\begin{aligned}
|u_k(x)| &\leqslant \frac{\|\Gamma\|_\infty}{\tilde{\delta}} \sum_{l+m+n=k} \int_{\mathbb{R}^3} \frac{|u_l(y)u_m(y)u_n(y)|}{|x-y|}\,\mathrm{d}y \\
&\leqslant \frac{\|\Gamma\|_\infty}{\tilde{\delta}} \sum_{l+m+n=k} \left[\int_{\mathbb{R}^3\setminus B_1(x)} \frac{|u_l(y)u_m(y)u_n(y)|}{|x-y|}\,\mathrm{d}y + \int_{B_1(x)} \frac{|u_l(y)u_m(y)u_n(y)|}{|x-y|}\,\mathrm{d}y\right] \\
&\leqslant \frac{\|\Gamma\|_\infty}{\tilde{\delta}} \sum_{l+m+n=k} \left[\|u_l\|_{L^4(\mathbb{R}^3)}\|u_m\|_{L^4(\mathbb{R}^3)}\|u_n\|_{L^4(\mathbb{R}^3)} \cdot \left(\int_{\mathbb{R}^3\setminus B_1(0)} \frac{\mathrm{d}y}{|y|^4}\right)^{\frac{1}{4}}\right. \\
&\quad \left. + \|u_l\|_{L^{12}(\mathbb{R}^3)}\|u_m\|_{L^{12}(\mathbb{R}^3)}\|u_n\|_{L^{12}(\mathbb{R}^3)} \cdot \left(\int_{B_1(0)} \frac{\mathrm{d}y}{|y|^{\frac{4}{3}}}\right)^{\frac{3}{4}}\right],
\end{aligned}
$$

and with the very same ideas as in the final step of part (a), i.e., applying (14b), this yields uniform upper bounds

$$
|u_k(x)| \leqslant E'_\alpha \cdot (k^2+1)^{-\frac{\alpha}{2}}.
$$

It remains to estimate $|x\, u_k(x)|$ for $|x| \geqslant 1$, which is achieved in a similar way but using the final identity in (14d):

$$
\begin{aligned}
|x\, u_k(x)| &\leqslant \frac{\|\Gamma\|_\infty}{\tilde{\delta}} \sum_{l+m+n=k} \left[\int_{B_{|x|}(0)} \frac{|\sin(|y|\sqrt{\mu_k})|}{|y|\sqrt{\mu_k}} |u_l(y)u_m(y)u_n(y)|\,\mathrm{d}y \right. \\
&\quad \left. + \int_{\mathbb{R}^3\setminus B_{|x|}(0)} \frac{|u_l(y)u_m(y)u_n(y)|}{|y|\sqrt{\mu_k}}\,\mathrm{d}y\right] \\
&\leqslant \frac{\|\Gamma\|_\infty}{\tilde{\delta}} \mu_k^{-\frac{3}{8}} \cdot \left[\left(\int_{B_{|x|\sqrt{\mu_k}}(0)} \frac{\sin^4(|y|)}{|y|^4}\mathrm{d}y\right)^{\frac{1}{4}} + \left(\int_{\mathbb{R}^3\setminus B_{|x|\sqrt{\mu_k}}(0)} \frac{\mathrm{d}y}{|y|^4}\right)^{\frac{1}{4}}\right] \\
&\quad \cdot \left[\sum_{l+m+n=k} \|u_l\|_{L^4(\mathbb{R}^3)}\|u_m\|_{L^4(\mathbb{R}^3)}\|u_n\|_{L^4(\mathbb{R}^3)}\right],
\end{aligned}
$$

and using the result and the ideas in (a) as above, this yields a uniform bound

$$
|x\, u_k(x)| \leqslant E''_\alpha \cdot (k^2+1)^{-\frac{\alpha}{2}}.
$$

Both results together provide the desired bound in X_1 asserted in the lemma. □

Step 2. ***Mapping properties of F respectively G*****.**

For $\lambda \in \mathbb{R}$ and $\mathbf{v} \in \mathcal{X}_1$, we set $\mathbf{u} := \mathbf{w} + \mathbf{v}$ and recall the defining equations (12a) and (12b):

$$
F(\mathbf{v},\lambda)_k := v_k - \begin{cases}
\mathcal{P}_{\mu_0}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_0 - \Gamma\, w_0^3\right] & k=0, \\
\mathcal{R}_{\mu_s}^{\pi/2}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & \\
\quad +(\cot(\tau_{\pm s})-\lambda)\tilde{\Psi}_{\mu_s} * \left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & k=\pm s, \\
\mathcal{R}_{\mu_k}^{\tau_k}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\right] & \text{else;}
\end{cases}
$$

$$G(\mathbf{v},\lambda)_k := v_k - \begin{cases} \mathcal{P}_{\mu_0}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_0 - \Gamma\, w_0^3\right] & k=0,\\ \mathcal{R}_{\mu_s}^{\pi/2}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & \\ \quad +(1-\lambda)\left(\alpha^{(\mu_s)}(v_{\pm s}) + \beta^{(\mu_s)}(v_{\pm s})\right)\tilde{\Psi}_{\mu_s} & k=\pm s,\\ \mathcal{R}_{\mu_k}^{\tau_k}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\right] & \text{else.} \end{cases}$$

Our main concern will be convergence of the infinite sums related to the space $\mathcal{X}_1 = \ell^1_{\mathrm{sym}}(\mathbb{Z}, X_1)$. Noticing that F and G only differ in the $\pm s$th component, and that the scalar parameter λ only appears as a multiplicative factor, we solely discuss smoothness of the map $F(\,\cdot\,,\lambda) : \mathcal{X}_1 \to \mathcal{X}_1$ with $\lambda \in \mathbb{R}$ fixed.

The main tool is the following uniform norm estimate for the operators appearing in the components of F. Recalling that $\tau_k \in \{\frac{\pi}{4}, \frac{3\pi}{4}\}$ for $k \neq 0, \pm s$ by (10), lemma 2 above (for $k \neq 0, \pm s$) as well as the continuity properties stated in lemmas 4 and 7 (for $k = \pm s$ and $k = 0$, respectively) provide a constant $C_0 = C_0(\lambda, \tau_s, \omega, m) > 0$ with

$$\begin{aligned} &\left\|\mathcal{R}_{\mu_k}^{\tau_k}\right\|_{\mathcal{L}(X_3,X_1)} \leqslant C_0 \quad (k \in \mathbb{Z}\backslash\{\pm s\}),\\ &\left\|\mathcal{R}_{\mu_s}^{\pi/2}\right\|_{\mathcal{L}(X_3,X_1)} \leqslant \frac{C_0}{2}, \quad \left\|(\cot(\tau_{\pm s}) - \lambda)\,\tilde{\Psi}_{\mu_s} *\right\|_{\mathcal{L}(X_3,X_1)} \leqslant \frac{C_0}{2},\\ &\left\|\mathcal{P}_{\mu_0}\right\|_{\mathcal{L}(X_3,X_1)} \leqslant C_0. \end{aligned} \tag{15}$$

Since Γ is assumed to be continuous and bounded and $\mathbf{u} = \mathbf{v} + \mathbf{w} \in \mathcal{X}_1$, proposition 1 implies that $\Gamma\;(\mathbf{u}\star\mathbf{u}\star\mathbf{u}) \in \mathcal{X}_3$. Thus every component $F(\mathbf{v},\lambda)_k$ is a well-defined element of X_1, and we estimate

$$\begin{aligned} \|F(\mathbf{v},\lambda)\|_{\mathcal{X}_1} &= \sum_{k\in\mathbb{Z}} \|F(\mathbf{v},\lambda)_k\|_{X_1}\\ &\overset{(15)}{\leqslant} \|\mathbf{v}\|_{\mathcal{X}_1} + C_0\left\|\Gamma w_0^3\right\|_{X_3} + C_0 \sum_{k\in\mathbb{Z}} \|\Gamma(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\|_{X_3}\\ &\overset{\text{Prop. 1}}{\leqslant} \|\mathbf{v}\|_{\mathcal{X}_1} + C_0\|\Gamma\|_\infty \|w_0\|_{X_1}^3 + C_0\|\Gamma\|_\infty\|\mathbf{u}\|_{\mathcal{X}_1}^3. \end{aligned}$$

This is finite, hence $F(\mathbf{v},\lambda) \in \mathcal{X}_1$ as asserted. Since $F(\cdot,\lambda)$ is a combination of continuous linear operators and polynomials in the convolution algebra, essentially the same estimates can be used to show differentiability (to arbitrary order); one thus obtains in particular (13a).

Step 3. ***Solution properties of $u_k(x)$.***

First of all, recalling that $\mathbf{w} = (\ldots, 0, w_0, 0, \ldots)$ and hence $(\mathbf{w}\star\mathbf{w}\star\mathbf{w})_k = \delta_{k,0}\, w_0^3$ for $k \in \mathbb{Z}$, one can immediately see that $F(\mathbf{0},\lambda) = G(\mathbf{0},\lambda) = \mathbf{0}$ for all $\lambda \in \mathbb{R}$. Let us now assume that $F(\mathbf{v},\lambda) = 0$ respectively $G(\mathbf{v},\lambda) = 0$ for some $\mathbf{v} \in \mathcal{X}_1$ and $\lambda \in \mathbb{R}$. Again, we define $\mathbf{u} := \mathbf{v} + \mathbf{w}$, and summarize

$$\begin{aligned} u_0 - w_0 = v_0 &= \mathcal{P}_{\mu_0}\left[\Gamma(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_0 - \Gamma\, w_0^3\right],\\ u_{\pm s} = v_{\pm s} &= \mathcal{R}_{\mu_s}^{\pi/2}\left[\Gamma(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right]\\ &\quad + \begin{cases} (\cot(\tau_s) - \lambda)\tilde{\Psi}_{\mu_s} * \left[\Gamma(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & \text{if } \sigma_s \neq 0\,(F \text{ case}),\\ (1-\lambda)\left(\alpha^{(\mu_s)}(v_{\pm s}) + \beta^{(\mu_s)}(v_{\pm s})\right)\tilde{\Psi}_{\mu_s} & \text{if } \sigma_s = 0\,(G \text{ case}), \end{cases}\\ u_k = v_k &= \mathcal{R}_{\mu_k}^{\tau_k}\left[\Gamma(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\right] \quad (k \in \mathbb{Z}\backslash\{0, \pm s\}). \end{aligned}$$

By choice of τ_k in equation (10), we observe in particular that the requirements of lemma 3 are satisfied with any $\delta < \frac{\pi}{4}$, which we will rely on throughout the subsequent steps. But first, according to lemmas 4 and 7, $v_k, u_k \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$ satisfy the differential equations

$$-\Delta v_k - \mu_k v_k = \Gamma(x)\ \left[(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_k - \delta_{k,0} w_0^3\right] \quad \text{on}\, \mathbb{R}^3$$

or equivalently, in view of $\mathbf{w} = (\dots, 0, w_0, 0, \dots)$, of (4) and of $\mu_k = \omega^2 k^2 - m^2$,

$$-\Delta u_k - (\omega^2 k^2 - m^2) u_k = \Gamma(x)\,(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_k \quad \text{on}\, \mathbb{R}^3. \tag{16}$$

We now define formally for $t \in \mathbb{R}, x \in \mathbb{R}^3$

$$U(t,x) := w_0(x) + v_0(x) + \sum_{k=1}^{\infty} 2\ \cos(\omega k t)\, v_k(x) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\omega k t}\, u_k(x). \tag{17}$$

Since by assumption $\mathbf{u} = \mathbf{v} + \mathbf{w} \in \ell^1(\mathbb{Z}, X_1)$, the Weierstrass M-test asserts that the sum in (17) converges in X_1 uniformly with respect to $t \in \mathbb{R}$, and hence the map $t \mapsto U(t,\cdot)$ is continuous as a map from $\mathbb{R}$ to X_1. We next show stronger regularity properties of $U(t,x)$.

Step 4. *Differentiability of U(t, x)*.

We prove that the map $t \mapsto U(t,\cdot)$, when interpreted as a map from $\mathbb{R}$ to X_1, possesses two continuous time derivatives given by

$$\partial_t U(t,\ \cdot) = \sum_{k\in\mathbb{Z}} \mathrm{i}\omega k\, \mathrm{e}^{\mathrm{i}\omega k t}\, u_k, \quad \partial_t^2 U(t,\ \cdot) = \sum_{k\in\mathbb{Z}} -\omega^2 k^2\, \mathrm{e}^{\mathrm{i}\omega k t}\, u_k.$$

Indeed, term-by-term differentiation is justified since the sums above as well as in (17) converge in X_1 uniformly with respect to time. This is a consequence of the Weierstraß M-test and the decay estimate in lemma 3(c). Hence, as asserted, the map $t \mapsto U(t,\cdot)$ is twice continuously differentiable as a map from $\mathbb{R}$ to X_1—the same strategy yields in fact C^∞ regularity in time.

Similarly, the local regularity estimate in lemma 3(b) implies $U \in C^2(\mathbb{R} \times B)$ for every given ball $B = B_R(0) \subseteq \mathbb{R}^3$ again via term-by-term differentiation. Since the radius of the ball B is arbitrary, we conclude for $t \in \mathbb{R}$ and all $x \in \mathbb{R}^3$

$$\begin{aligned}
\left[\partial_t^2 - \Delta + m^2\right] U(t,x) &= \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\omega k t}\ \left[-\omega^2 k^2 - \Delta + m^2\right] u_k(x) \\
&\overset{(16)}{=} \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\omega k t}\, \Gamma(x) \sum_{l+m+n=k} u_l(x)\, u_m(x)\, u_n(x) \\
&= \Gamma(x) \left(\sum_{l\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\omega l t}\, u_l(x)\right) \left(\sum_{m\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\omega m t}\, u_m(x)\right) \\
&\quad \times \left(\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\omega n t}\, u_n(x)\right) \\
&= \Gamma(x)\, U(t,x)^3
\end{aligned}$$

where the re-ordering of the summation is justified by absolute convergence of the sums. Thus U is shown to be a classical solution of the Klein–Gordon equation (1). □

Proof of proposition 4. We prove the statement for the map F and then comment on the aspects that differ in case of G. Using formula (13a), we find for $k \in \mathbb{Z}$ and $\mathbf{q} \in \mathcal{X}_1$, recalling that $w_k = 0$ for $k \in \mathbb{Z}\backslash\{0\}$ and that $\mathcal{R}_{\mu_s}^{\tau_s} = \mathcal{R}_{\mu_s}^{\pi/2} + \cot(\tau_s)\,\tilde{\Psi}_{\mu_s}*$,

$$DF(\mathbf{0},0)[\mathbf{q}]_k = q_k - 3\,\mathcal{R}_{\mu_k}^{\tau_k}[\Gamma\,(\mathbf{q}\star\mathbf{w}\star\mathbf{w})_k] = q_k - 3\,\mathcal{R}_{\mu_k}^{\tau_k}\left[\Gamma\,w_0^2\cdot q_k\right],$$

$$DF(\mathbf{0},0)[\mathbf{q}]_0 = q_0 - 3\,\mathcal{P}_{\mu_0}[\Gamma\,(\mathbf{q}\star\mathbf{w}\star\mathbf{w})_0] = q_0 - 3\,\mathcal{P}_{\mu_0}\left[\Gamma\,w_0^2\cdot q_0\right].$$

For $\mathbf{q} \in \ker DF(\mathbf{0},0)$, and in view of the choice of τ_k in (10), the nondegeneracy properties (11) imply $q_k \equiv 0$ for $k \in \mathbb{Z}, k \neq \pm s$. Since $\tau_{\pm s} = \sigma_s$ in (10), proposition 2 guarantees the existence of a nontrivial solution $q_s \in X_1$ of

$$q_s = 3\,\mathcal{R}_{\mu_s}^{\tau_s}\left[\Gamma\,w_0^2\cdot q_s\right] \tag{18}$$

which is unique up to a multiplicative factor. Hence $\ker DF(\mathbf{0},0)$ has the asserted form. (We recall here that we consider the subspace of symmetric sequences, whence $q_{-s} = q_s$.) Further, by lemmas 4 and 7 in the final section 4, the operators

$$X_1 \to X_1, \quad \begin{cases} q_k \mapsto q_k - 3\,\mathcal{R}_{\mu_k}^{\tau_k}\left[\Gamma\,w_0^2\cdot q_k\right] & (k\neq 0) \\ q_0 \mapsto q_0 - 3\,\mathcal{P}_{\mu_0}\left[\Gamma\,w_0^2\cdot q_0\right] \end{cases}$$

are linear compact perturbations of the identity and so $DF(\mathbf{0},0)$ is 1-1-Fredholm. In order to verify transversality, we compute for $k \in \mathbb{Z}$ and $\mathbf{q} \in \ker DF(\mathbf{0},0)\backslash\{\mathbf{0}\}$

$$\partial_\lambda DF(\mathbf{0},0)[\mathbf{q}]_k = \begin{cases} 3\,\tilde{\Psi}_{\mu_s} * [\Gamma\,w_0^2\,q_s], & k = \pm s, \\ 0, & \text{else}. \end{cases}$$

Assuming for contradiction that $\partial_\lambda DF(\mathbf{0},0)[\mathbf{q}] = DF(\mathbf{0},0)[\mathbf{p}]$ for some $\mathbf{p} \in \mathcal{X}_1$, we infer in particular that the component p_s satisfies the convolution identity

$$p_s - 3\,\mathcal{R}_{\mu_s}^{\tau_s}\left[\Gamma\,w_0^2\cdot p_s\right] = 3\,\tilde{\Psi}_{\mu_s} * [\Gamma\,w_0^2\cdot q_s] \tag{19}$$

and hence, following lemmas 4 and 5

$$-\Delta p_s - \mu_s p_s = 3\,\Gamma(x)\,w_0^2(x)\,p_s \quad \text{on } \mathbb{R}^3,$$

which is also nontrivially solved by q_s as a consequence of (18). Due to the uniqueness statement in proposition 2, this implies that $p_s = c\cdot q_s$ for some $c \in \mathbb{R}$. But then, applying (18) to (19), we obtain $\tilde{\Psi}_{\mu_s} * [\Gamma\,w_0^2\cdot q_s] = 0$. Hence by the asymptotic expansion in lemma 4

$$\widehat{\Gamma w_0^2 q_s}(\sqrt{\mu_s}) = 0$$

and therefore, due to $q_s = 3\,\mathcal{R}_{\mu_s}^{\tau_s}[\Gamma\,w_0^2\,q_s]$ and lemma 5,

$$q_s(x) = O\left(\frac{1}{|x|^2}\right) \text{ as } |x| \to \infty.$$

This contradicts proposition 2 stating that the leading-order term as $|x| \to \infty$ of a nontrivial solution q_s of $-\Delta q_s - \mu_s q_s = 3\,\Gamma(x)\,w_0^2(x)\,q_s$ cannot vanish.

In the case $\tau_s = 0$, we see as above that $\mathbf{q} \in \ker DG(\mathbf{0}, 0)$ if and only if $q_k = 0$ for $k \neq \pm s$, and that $q_s = q_{-s}$ can be chosen to be the (nontrivial) solution of

$$q_s = 3\,\mathcal{R}_{\mu_s}^{\pi/2}\left[\Gamma\, w_0^2 \cdot q_s\right] + \alpha^{(\mu_s)}(q_s)\,\tilde{\Psi}_{\mu_s} \quad \text{with} \;\; \beta^{(\mu_s)}(q_s) = 0. \tag{20}$$

Similarly, $DG(\mathbf{0}, 0)$ is 1-1-Fredholm. We again assume for contradiction that there is $\mathbf{p} \in \mathcal{X}_1$ with $\partial_\lambda DG(\mathbf{0}, 0)[\mathbf{q}] = DG(\mathbf{0}, 0)[\mathbf{p}]$, which implies in particular

$$p_s - 3\,\mathcal{R}_{\mu_s}^{\pi/2}\left[\Gamma\, w_0^2 \cdot p_s\right] - \left(\alpha^{(\mu_s)}(p_s) + \beta^{(\mu_s)}(p_s)\right)\tilde{\Psi}_{\mu_s} = \alpha^{(\mu_s)}(q_s)\tilde{\Psi}_{\mu_s} \tag{21}$$

with $\beta^{(\mu_s)}(q_s) = 0$. Thus, according to lemma 4, p_s solves the differential equation

$$-\Delta p_s - \mu_s p_s = 3\,\Gamma(x)\, w_0^2(x)\, p_s \quad \text{on}\, \mathbb{R}^3,$$

which is also solved by q_s, see equation (20). As before, the uniqueness property in proposition 2 implies $p_s = c \cdot q_s$ for some $c \in \mathbb{R}$, and inserting this into the identity (21), comparison with (20) yields $\alpha^{(\mu_s)}(q_s) = 0$. Since also $\beta^{(\mu_s)}(q_s) = 0$, we infer from the definition of the functionals $\alpha^{(\mu_s)}$, $\beta^{(\mu_s)}$ preceding lemma 6 that, again, $q_s(x) = O(1/|x|^2)$, contradicting proposition 2. □

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Appendix. Stationary linear Helmholtz and Schrödinger equations

Given $\mu > 0$, we study aspects of the solution theory of the linear equations

$$-\Delta u \pm \mu u = f \quad \text{on}\, \mathbb{R}^3. \tag{22}$$

In the case of a '+', equation (22) is said to be a Schrödinger equation. Given any right-hand side $f \in L^2(\mathbb{R}^3)$, a unique solution $u \in H^2(\mathbb{R}^3)$ can be obtained by applying the resolvent $(-\Delta + \mu)^{-1}$, which can be calculated explicitly by applying the Fourier transform

$$u = (-\Delta + \mu)^{-1} f = \int_{\mathbb{R}^3} \frac{\hat{f}(\xi)}{|\xi|^2 + \mu}\, \mathrm{e}^{\mathrm{i}\langle\,\cdot\,,\xi\rangle}\, \frac{\mathrm{d}\xi}{(2\pi)^{3/2}}.$$

In the case of a Helmholtz equation, i.e. of a '$-$' sign in (22), this is not possible since $\mu > 0$ belongs to the essential spectrum of $-\Delta$ on $\mathbb{R}^3$. A well-established strategy to find solutions in spaces other than $L^2(\mathbb{R}^3)$ is known as limiting absorption principle(s). The idea is to replace

μ by $\mu + \mathrm{i}\varepsilon$, apply an L^2-resolvent, and pass to the limit $\varepsilon \to 0$ in a suitable topology, i.e. formally

$$u = \lim_{\varepsilon \searrow 0} (-\Delta - (\mu + \mathrm{i}\varepsilon))^{-1} f = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^3} \frac{\hat{f}(\xi)}{|\xi|^2 - (\mu + \mathrm{i}\varepsilon)} \, \mathrm{e}^{\mathrm{i}\langle \,\cdot\, , \xi \rangle} \, \frac{\mathrm{d}\xi}{(2\pi)^{3/2}}.$$

Using tools from harmonic analysis, such a construction of solutions of linear inhomogeneous Helmholtz equations has been successfully done by Agmon [2] in weighted L^2 spaces, and by Kenig *et al* [17] as well as Gutiérrez [15] in certain pairs of L^p spaces. The resolvent-type operator is, then, for sufficiently nice f, given by a convolution

$$u = \frac{\mathrm{e}^{\mathrm{i}|\,\cdot\,|\sqrt{\mu}}}{4\pi|\,\cdot\,|} * f.$$

Such studies are completed by characterizations of the so-called Herglotz waves, i.e. the solutions of the homogeneous equation $-\Delta u - \mu u = 0$ on the respective spaces, see e.g. [3].

We study the case of (real-valued, radial) functions $f \in X_3, u \in X_1$ with the Banach spaces

$$X_q := \left\{ v \in C_{\mathrm{rad}}(\mathbb{R}^3) \big| \ \|v\|_{X_q} := \left\| (1 + |\cdot|^2)^{\frac{q}{2}} v \right\|_\infty < \infty \right\}, \quad q \in \{1, 3\}.$$

These have been successfully applied in solving systems of cubic Helmholtz equations in [23]. Let us again point out that the decay rate prescribed in X_1 is the natural one for solutions of Helmholtz equations on the full space $\mathbb{R}^3$. Such solutions of the Helmholtz equation

$$-\Delta u - \mu u = f \quad \text{on } \mathbb{R}^3 \tag{23}$$

can be obtained using convolution operators with kernels $\Psi_\mu, \tilde{\Psi}_\mu$ given by

$$\Psi_\mu(x) = \frac{\cos(|x|\sqrt{\mu})}{4\pi|x|}, \quad \tilde{\Psi}_\mu(x) = \frac{\sin(|x|\sqrt{\mu})}{4\pi|x|} \quad (x \in \mathbb{R}^3 \backslash \{0\}).$$

Here $\Psi_\mu, \tilde{\Psi}_\mu$ are radial solutions of the homogeneous Helmholtz equation on $\mathbb{R}^3 \backslash \{0\}$. We notice that $\tilde{\Psi}_\mu$ extends to a smooth solution of $-\Delta u - \mu u = 0$ in X_1 and it is, up to constant multiples, the only one. Moreover, the following holds:

Lemma 4 ([23], proposition 4). *The convolution operators $f \mapsto \Psi_\mu * f$, $f \mapsto \tilde{\Psi}_\mu * f$ are well-defined, linear and compact as operators from X_3 to X_1. Moreover, given $f \in X_3$, the functions $w := \Psi_\mu * f$ and $\tilde{w} := \tilde{\Psi}_\mu * f$ belong to $X_1 \cap C^2_{loc}(\mathbb{R}^3)$ and satisfy*

$$-\Delta w - \mu w = f \quad on\, \mathbb{R}^3, \quad w(x) = 4\pi \sqrt{\frac{\pi}{2}} \, \hat{f}(\sqrt{\mu}) \ \Psi_\mu(x) + O\left(\frac{1}{|x|^2}\right);$$

$$-\Delta \tilde{w} - \mu \tilde{w} = 0 \quad on\, \mathbb{R}^3, \quad \tilde{w}(x) = 4\pi \sqrt{\frac{\pi}{2}} \, \hat{f}(\sqrt{\mu}) \ \tilde{\Psi}_\mu(x).$$

Here $\hat{f}(\sqrt{\mu})$ refers to the profile of the Fourier transform on $\mathbb{R}^3$. Working in a radial setting with strongly decaying inhomogeneities $f \in X_3$, the properties in the previous lemma (and in the following ones) can be verified immediately by explicit calculations and need not be derived from suitable limiting absorption principles; for details, we refer to the earlier article [23].

The study of conditions guaranteeing uniqueness of solutions of (23) in X_1 involves the characterization of Herglotz waves in X_1, which are all multiples of $\tilde{\Psi}_\mu$. As in [23], inspired by the analysis of the so-called far field of solutions of Helmholtz equations in scattering theory, we impose asymptotic conditions governing the leading-order contribution of $u(x)$ as $|x| \to \infty$. For $\tau \in (0, \pi)$, we introduce

$$\mathcal{R}_\mu^\tau[f] = \Psi_\mu * f + \cot(\tau)\,\tilde{\Psi}_\mu * f = \frac{\sin(|\cdot|\sqrt{\mu} + \tau)}{4\pi\,\sin(\tau)\,|\cdot|} * f.$$

Then, using the above lemma 4, one obtains:

Lemma 5 ([23], corollary 5). *Let $\tau \in (0, \pi)$ and $\mu > 0$. Then the operator $\mathcal{R}_\mu^\tau : X_3 \to X_1$ is well-defined, linear and compact. Moreover, given $f \in X_3$, we have $u = \mathcal{R}_\mu^\tau[f]$ if and only if $u \in C^2_{loc}$ with*

$$-\Delta u - \mu u = f \quad \text{on } \mathbb{R}^3, \quad u(x) = c \cdot \frac{\sin(|x|\sqrt{\mu} + \tau)}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty$$

for some $c \in \mathbb{R}$, and in this case $c = \frac{1}{\sin(\tau)}\sqrt{\frac{\pi}{2}}\,\hat{f}(\sqrt{\mu})$.

Handling the case of far field conditions with $\tau = 0$ is somewhat more delicate since the existence of the solution $\tilde{\Psi}_\mu$ (which satisfies exactly this condition) excludes an analogous uniqueness statement. For proving theorem 1, the following setting is suitable. First, by the Hahn–Banach theorem, we define continuous linear functionals $\alpha^{(\mu)}, \beta^{(\mu)} \in X_1'$ with the property that, for $u \in X_1$ with

$$u(x) = \alpha_u \cdot \tilde{\Psi}_\mu(x) + \beta_u \cdot \Psi_\mu(x) + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty,$$

we have $\alpha^{(\mu)}(u) = \alpha_u$ and $\beta^{(\mu)}(u) = \beta_u$, cf [23], equation (13) and the following explanations. Then, the following analogue of lemma 5 holds.

Lemma 6. *Given $f \in X_3$, we have $u = \mathcal{R}_\mu^{\pi/2}[f] + (\alpha^{(\mu)}(u) + \beta^{(\mu)}(u)) \cdot \tilde{\Psi}_\mu$ if and only if $u \in C^2_{loc}$ with*

$$-\Delta u - \mu u = f \quad \text{on } \mathbb{R}^3, \quad u(x) = c \cdot \frac{\sin(|x|\sqrt{\mu})}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty$$

for some $c \in \mathbb{R}$. In this case, $\beta^{(\mu)}(u) = 0$.

These results will allow to handle the nonlinear Helmholtz equations in (3b); for the proofs, we refer to the corresponding parts of [23]. Nonlinear Schrödinger equations such as

$$-\Delta u + \mu u = f \quad \text{on } \mathbb{R}^3 \tag{24}$$

for some $\mu > 0$ can also be discussed in a similar setting, which is certainly neither optimal nor most elegant but perfectly suitable for our purpose as another analogue of lemma 4.

Lemma 7. *Let $\mu > 0$. Then the operator*

$$\mathcal{P}_\mu : X_3 \to X_1, \quad f \mapsto \frac{e^{-|\cdot|\sqrt{\mu}}}{4\pi|\cdot|} * f$$

is well-defined, linear and compact. Moreover, given $f \in X_3$*, we have* $u := \mathcal{P}_\mu[f] \in X_3 \cap C^2_{loc}(\mathbb{R}^3)$*, and u is a solution in* X_1 *of*

$$-\Delta u + \mu u = f \quad on\ \mathbb{R}^3.$$

For details on the proof, which is similar to that of lemma 4 but with less difficulties due to the strongly localized kernel, cf [24], lemma 4.10.

Let us remark that, in the Schrödinger case, we do not obtain a family of possible 'resolvent-type' operators as $\mathcal{R}_1^\tau = \mathcal{R}_1 + \cot(\tau)\widetilde{\mathcal{R}}_1$, $0 < \tau < \pi$, in the Helmholtz case. This is due to the fact that the homogeneous Schrödinger equation $-\Delta u + \mu u = 0$ has no smooth and localized nontrivial solution in X_1. In particular, a major consequence in our study of Klein–Gordon breathers is that we have to impose nondegeneracy of w_0 as an assumption rather than, as in the Helmholtz case, generate it by choosing an appropriate resolvent $\mathcal{R}_1^\tau$, see equations (10) and (11a) in the proof of theorem 1.

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