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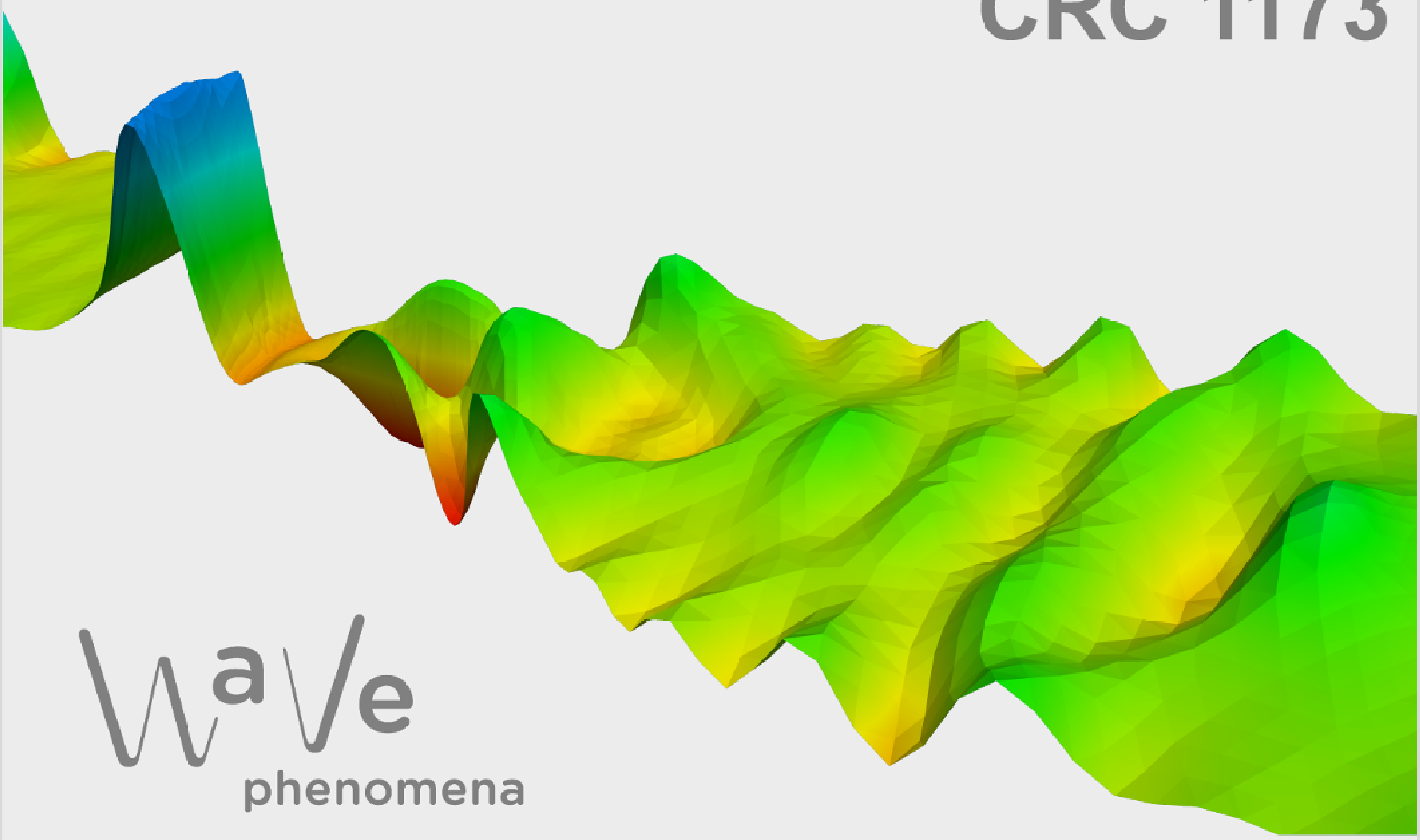

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LEAST ENERGY SOLUTIONS TO A COOPERATIVE SYSTEM OF SCHRÖDINGER EQUATIONS WITH PRESCRIBED L^2-BOUNDS: AT LEAST L^2-CRITICAL GROWTH

JAROSŁAW MEDERSKI AND JACOPO SCHINO

Abstract. We look for least energy solutions to the cooperative systems of coupled Schrödinger equations

$$\begin{cases} -\Delta u_i + \lambda_i u_i = \partial_i G(u) \quad \text{in } \mathbb{R}^N,\ N \geq 3, \\ u_i \in H^1(\mathbb{R}^N), & i \in \{1,\dots,K\} \\ \int_{\mathbb{R}^N} |u_i|^2\, dx \leq \rho_i^2 \end{cases}$$

with $G \geq 0$, where $\rho_i > 0$ is prescribed and $(\lambda_i, u_i) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ is to be determined, $i \in \{1,\dots,K\}$. Our approach is based on the minimization of the energy functional over a linear combination of the Nehari and Pohožaev constraints intersected with the product of the closed balls in $L^2(\mathbb{R}^N)$ of radii ρ_i, which allows to provide general growth assumptions on G and to know in advance the sign of the corresponding Lagrange multipliers. We assume that G has at least L^2-critical growth at 0 and Sobolev subcritical growth at infinity. The more assumptions we make on G, N, and K, the more can be said about the minimizers of the energy functional. In particular, if $K = 2$, $N \in \{3,4\}$, and G satisfies further assumptions, then $u = (u_1, u_2)$ is normalized, i.e., $\int_{\mathbb{R}^N} |u_i|^2\, dx = \rho_i^2$ for $i \in \{1,2\}$.

Introduction

We consider the following system of autonomous nonlinear Schrödinger equations of gradient type

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \partial_1 G(u) \\ \cdots \\ -\Delta u_K + \lambda_K u_K = \partial_K G(u) \end{cases} \text{in } \mathbb{R}^N \tag{1.1}$$

with $u = (u_1,\dots,u_K)\colon \mathbb{R}^N \to \mathbb{R}^K$, which arises in different areas of mathematical physics. In particular, the system (1.1) describes the propagation of solitons, which are special nontrivial solitary wave solutions $\Phi_j(x,t) = u_j(x)e^{-\mathrm{i}\lambda_j t}$ to a system of time-dependent Schrödinger equations of the form

$$\mathrm{i}\frac{\partial \Phi_j}{\partial t} - \Delta \Phi_j = g_j(\Phi) \quad \text{for } j = 1,\dots,K, \tag{1.2}$$

where, for instance, g_j are responsible for the nonlinear polarization in a photonic crystal [2,33] and λ_j are the external electric potentials.

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Another field of application is condensed matter physics, where (1.1) comes from the system of coupled Gross-Pitaevski equations (1.2) with nonlinearities of the form

$$g_j(\Phi) = \left(\sum_{k=1}^{K} \beta_{j,k}|\Phi_k|^2\right)\Phi_j \quad \text{for } j = 1, \dots, K.$$

The following L^2-bounds for Φ will be studied:

$$\int_{\mathbb{R}^N} |\Phi_j(t,x)|^2 \, dx = \rho_j^2 \quad \text{and} \quad \int_{\mathbb{R}^N} |\Phi_j(t,x)|^2 \, dx \le \rho_j^2.$$

Problems with prescribed masses ρ_j^2 (the former constraint) appear in nonlinear optics, where the mass represents the power supply, and in the theory of Bose-Einstein condensates, where it represents the total number of atoms (see [1,16,18,26,29,31,37]). Prescribing the masses make sense also because they are conserved quantities in the corresponding evolution equation (1.2) together with the energy (see the functional J below), cf. [12,13]. As for the latter constraint, we propose it as a model for some experimental situations, when the power supply provided can oscillate without exceeding a given value.

Recall that a general class of autonomous systems of Schrödinger equations was studied by Brezis and Lieb in [11] and using a constrained minimization method they showed the existence of a *least energy solution*, i.e., a nontrivial solution with the minimal energy. Their method using rescaling arguments does not apply with the L^2-bounds.

Our aim is to provide a general class of nonlinearities and to find solutions to the nonlinear Schrödinger problems

$$\begin{cases} -\Delta u_i + \lambda_i u_i = \partial_i G(u) & \text{in } \mathbb{R}^N, \ N \ge 3, \\ u_i \in H^1(\mathbb{R}^N), \\ \int_{\mathbb{R}^N} |u_i|^2 \, dx \le \rho_i^2 \end{cases} \qquad \text{for every } i \in \{1, \dots, K\} \tag{1.3}$$

and

$$\begin{cases} -\Delta u_i + \lambda_i u_i = \partial_i G(u) & \text{in } \mathbb{R}^N, \ N \ge 3, \\ u_i \in H^1(\mathbb{R}^N), \\ \int_{\mathbb{R}^N} |u_i|^2 \, dx = \rho_i^2 \end{cases} \qquad \text{for every } i \in \{1, \dots, K\}, \tag{1.4}$$

where $\rho = (\rho_1, \dots, \rho_K) \in (0,\infty)^K$ is prescribed and $(\lambda, u) \in \mathbb{R}^K \times H^1(\mathbb{R}^N)^K$ is the unknown.

Let us introduce the sets

$$\mathcal{D} := \left\{ u \in H^1(\mathbb{R}^N)^K \ : \ \int_{\mathbb{R}^N} |u_i|^2 \, dx \le \rho_i^2 \text{ for every } i \in \{1, \dots, K\} \right\},$$
$$\mathcal{S} := \left\{ u \in H^1(\mathbb{R}^N)^K \ : \ \int_{\mathbb{R}^N} |u_i|^2 \, dx = \rho_i^2 \text{ for every } i \in \{1, \dots, K\} \right\}$$

and note that $\mathcal{S} \subset \partial\mathcal{D}$.

We shall provide suitable assumptions under which the solutions to (1.3) (resp. (1.4)) are critical points of the energy functional $J\colon H^1(\mathbb{R}^N)^K \to \mathbb{R}$ defined as

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} G(u) \, dx$$

restricted to the constraint $\mathcal{D}$ (resp. $\mathcal{S}$) with Lagrange multipliers $\lambda_i \in \mathbb{R}$, i.e., they are critical points of

$$H^1(\mathbb{R}^N)^K \ni u \mapsto J(u) + \frac{1}{2}\sum_{i=1}^K \lambda_i \int_{\mathbb{R}^N} |u_i|^2\,dx \in \mathbb{R}$$

for some $\lambda = (\lambda_1, \dots, \lambda_K) \in \mathbb{R}^K$. Let us recall that, under mild assumptions on G, see [11, Theorem 2.3], every critical point of the functional above belongs to $W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)^K$ for all $q < \infty$ and satisfies the Pohožaev [9, 21, 30, 32]

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx = 2^* \int_{\mathbb{R}^N} G(u) - \frac{1}{2}\sum_{i=1}^K \lambda_i |u_i|^2\,dx$$

and Nehari

$$J'(u)(u) + \sum_{i=1}^K \lambda_i \int_{\mathbb{R}^N} |u_i|^2\,dx = 0$$

identities. By a linear combination of the two equalities above it is easily checked that every solution satisfies

$$M(u) := \int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{N}{2}\int_{\mathbb{R}^N} H(u)\,dx = 0,$$

where $H(u) := \langle g(u), u\rangle - 2G(u)$ ($\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^K$) and $g := \nabla G$, see e.g. [21]. Hence we introduce the constraint

$$\mathcal{M} := \left\{u \in H^1(\mathbb{R}^N)^K \setminus \{0\} : M(u) = 0\right\},$$

which contains all the nontrivial solutions to (1.3) or (1.4) and does not depend on λ. Observe that every nontrivial solution to (1.3) belongs to $\mathcal{M} \cap \mathcal{D}$ and every (nontrivial) solution to (1.4) belongs to $\mathcal{M} \cap \mathcal{S} \subset \mathcal{M} \cap \mathcal{D}$. By a *ground state solution* to (1.3) we mean a nontrivial solution which minimizes J among all the nontrivial solutions. In particular, if (λ, u) solves (1.3) and $J(u) = \inf_{\mathcal{M}\cap\mathcal{D}} J$, then (λ, u) is a ground state solution (cf. Theorems 1.1 and 1.2). By a *ground state solution* to (1.4) we mean that (λ, u) solves (1.4) and $J(u) = \inf_{\mathcal{M}\cap\mathcal{D}} J$ (cf. Theorems 1.2 and 1.3). Note that this is more than just requiring $J(u) = \inf_{\mathcal{M}\cap\mathcal{S}} J$, which, on the other hand, appears as a more "natural" requirement.

Working with the set $\mathcal{D}$ instead of the set $\mathcal{S}$ for a system of Schrödinger equations seems to be new and has, among others, a specific advantage related to the sign of the Lagrange multipliers λ_i. We begin by showing why this issue is important. First of all, from a physical point of view there are situations, e.g. concerning the eigenvalues of equations describing the behaviour of ideal gases, where the chemical potentials λ_i have to be positive, see e.g. [26,31]. In addition, from a mathematical point of view the (strict) positivity of such Lagrange multipliers often plays an important role in the strong convergence of minimizing sequences in $L^2(\mathbb{R}^N)$, see e.g. [5, Lemma 3.9]; finally, the nonnegativity is used in some of the proofs below, e.g. the one of Lemma 2.9 (a). The aforementioned advantage is as follows: in [14] Clarke proved that, in a minimization problem, Lagrange multipliers related to a constraint given by inequalities have a sign, i.e., $\lambda_i \geq 0$; therefore it is enough to rule out the case $\lambda_i = 0$ in order to prove that $\lambda_i > 0$ for every $i \in \{1, \dots, K\}$; note that ruling out the case $\lambda_i = 0$ is simpler than ruling out the case $\lambda_i \leq 0$, cf. the proof of Theorem 1.2 (b). The nonnegativity/positivity of the Lagrange multipliers of (1.4) has often been obtained by means of involved tools (or at the very minimum in a not-so-straightforward way), such as stronger variants of Palais-Smale sequences in the spirit of [21] as in [5, Lemma 3.6, proof of Theorem 1.1] or preliminary properties of the ground state energy map $\rho \mapsto \inf_{\mathcal{M}\cap\mathcal{S}} J$ as in [23, Lemma 2.1, proof of Lemma 4.5]. Our argument, based on [14], is simple, does not seem to be exploited in the theory of normalized solutions, and is demonstrated in Proposition A.1 in an abstract way for future applications, e.g. for different operators in the normalized solutions setting like the fractional Laplacian [24, 28].

Recall that, when $K = 1$ and

$$G(u) = \frac{1}{p}|u|^p, \quad 2 < p < 2^*,\ p \neq 2_N := 2 + \frac{4}{N} \tag{1.5}$$

(1.4) is equivalent to the corresponding problem with fixed $\lambda > 0$ (and without the L^2-bound) via a scaling-type argument. This approach fails in the case of nonhomogeneous nonlinearities or when $K \geq 2$. In the L^2-subcritical case, i.e., when $G(u) \sim |u|^p$ with $2 < p < 2_N$, one can obtain the existence of a global minimizer by minimizing directly on $\mathcal{S}$, cf. [27, 35]. In the L^2-critical ($p = 2_N$) and the L^2-supercritical and Sobolev-subcritical ($2_N < p < 2^* := \frac{2N}{N-2}$) cases this method does not work; in particular, if $p > 2_N$ in (1.5), then $\inf_{\mathcal{S}} J = -\infty$. The purpose of this work is to find general growth conditions on G in the spirit of Berestycki, Lions [9] and Brezis, Lieb [11], and to provide a direct approach to obtain ground state solutions to (1.3), (1.4), and similar elliptic problems.

The problem (1.4) for one equation was studied by Jeanjean [21] and by Bartsch and Soave [6, 7] with a general nonlinear term satisfying the following condition of Ambrosetti-Rabinowitz type: there exist $\frac{4}{N} < a \leq b < 2^* - 2$ such that

$$0 < aG(u) \leq H(u) \leq bG(u) \text{ for } u \in \mathbb{R} \setminus \{0\}. \tag{1.6}$$

In [21] the author used a mountain pass argument, while in [6, 7] a mini-max approach in $\mathcal{M}$ based on the σ-homotopy stable family of compact subsets of $\mathcal{M}$ and the Ghoussoub minimax principle [19] were adopted. The same topological principle has been recently applied to the system (1.4) with particular power-like nonlinearities, e.g. in [4–7], and by Jeanjean and Lu [22] for $K = 1$ and a general nonlinearity without (1.6), but with L^2-supercritical growth.

We stress that the lack of compactness of the embedding $H^1_{\text{rad}}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$ causes troubles in the analysis of L^2-supercritical problems and makes the argument quite involved, see e.g. [6, 7, 21]. A possible strategy to recover the compactness of Palais-Smale sequences, at least when $K = 1$, is to show that the ground state energy map is nonincreasing with respect to $\rho > 0$ and decreasing in a subinterval of $(0, \infty)$, see e.g. [8, 22].

In our approach we do not work in H^1_{rad}, with Palais-Smale sequences, or with (1.6), nor the monotonicity of the ground state energy map is required, so that we avoid the mini-max approach in $\mathcal{M}$ involving a technical topological argument based on [19], which has been recently intensively exploited by many authors e.g. in [4–7, 22–24, 28, 34].

In particular, we work with a weaker version of (1.6), see the condition (A5) below and we admit L^2-critical growth at 0. We make use of a minimizing sequence of $J|_{\mathcal{M}\cap\mathcal{D}}$ and we are able to consider a wide class of nonlinearities G. In the first part of this work we adapt the techniques of [10] to the system (1.3), which ensure that the minimum of J on $\mathcal{M} \cap \mathcal{D}$ is attained. If G is even, we exploit the Schwartz rearrangement $u^* := (u_1^*, \ldots, u_K^*)$ of $(|u_1|, \ldots, |u_K|)$ because, if $u \in \mathcal{M} \cap \mathcal{D}$, then u^* can be projected onto the same set without increasing the energy. Next, we point out that dealing with systems (1.3) and (1.4) one has to involve more tools in order to find a ground state $u \in \mathcal{M} \cap \partial\mathcal{D}$ and some additional restrictions imposed on G, N, or K will be required. In particular, if we want to ensure that the Lagrange multipliers are positive and $u \in \mathcal{S}$, we use the elliptic regularity results contained in [9, 11], the Liouville type result [20], and Proposition A.1. Finally, a multi-dimensional version of the strict monotonicity of the ground state energy map is simply obtained in Proposition 2.13 as a consequence of our approach.

For $2 < p \le 2^*$, let $C_{N,p} > 0$ be the optimal constant in the *Gagliardo-Nirenberg inequality*

$$|u|_p \le C_{N,p} |\nabla u|_2^{\delta_p} |u|_2^{1-\delta_p} \quad \text{for } u \in H^1(\mathbb{R}^N), \tag{1.7}$$

where $\delta_p = N\left(\frac{1}{2} - \frac{1}{p}\right)$ and $\delta_p p > 2$ (resp. $\delta_p p = 2$, $\delta_p p < 2$) if and only if $p > 2_N$ (resp. $p = 2_N$, $p < 2_N$). Here and in what follows we denote by $|u|_k$ the L^k-norm of u, $1 \le k \le \infty$.

We set $h := \nabla H$ and consider the following assumptions:

(A0) g and h are continuous and there exists $\tilde{c} > 0$ such that $|h(u)| \le \tilde{c}(|u| + |u|^{2^*-1})$.

(A1) $\eta := \limsup_{u \to 0} \dfrac{G(u)}{|u|^{2_N}} < \infty$.

(A2) $\lim_{|u| \to \infty} \dfrac{G(u)}{|u|^{2_N}} = \infty$.

(A3) $\lim_{|u| \to \infty} \dfrac{G(u)}{|u|^{2^*}} = 0$.

(A4) $2_N H(u) \le \langle h(u), u \rangle$.

(A5) $\dfrac{4}{N} G \le H \le (2^* - 2) G$.

(A6) There exists $\zeta \in \mathbb{R}^N$ such that $H(\zeta) > 0$.

Note that (A5) implies $G, H \ge 0$ and that, if (A2) and (A5) hold, then so does (A6). Note also that J and M are of class $\mathcal{C}^1$ if (A0) is satisfied. For every $u \in H^1(\mathbb{R}^N)^K$ such that $\int_{\mathbb{R}^N} H(u)\,dx > 0$ we define

$$R := R_u := \sqrt{\frac{N \int_{\mathbb{R}^N} H(u)\,dx}{2 \int_{\mathbb{R}^N} |\nabla u|^2\,dx}} > 0$$

and note that $u(R\cdot) \in \mathcal{M}$.

Assuming also (A6), similarly to [9, page 325] for every $r > 0$ we can construct $w \in H_0^1(B_r)^K \cap L^\infty(B_r)^K$, where B_r stands for the ball of radius r, such that $\int_{\mathbb{R}^N} H(w)\,dx > 0$, therefore $\mathcal{M} \ne \emptyset$. Moreover $\mathcal{M}$ is a $\mathcal{C}^1$-manifold, since $M'(u) \ne 0$ for $u \in \mathcal{M}$, cf. [32]. As a matter of fact, if $M'(u) = 0$, then u solves $-\Delta u = \frac{N}{4} h(u)$ and satisfies the Pohožaev identity $\int_{\mathbb{R}^N} |\nabla u|^2\,dx = 2^* \frac{N}{4} \int_{\mathbb{R}^N} H(u)\,dx$. If $M(u) = 0$, then we infer $u = 0$.

We introduce the following relation:

> Let $f_1, f_2 \colon \mathbb{R}^K \to \mathbb{R}$. Then $f_1 \preceq f_2$ if and only if $f_1 \le f_2$ and for every $\varepsilon > 0$ there exists $u \in \mathbb{R}^K$, $|u| < \varepsilon$, such that $f_1(u) < f_2(u)$,

and for better outcomes we need a stronger variant of (A4), denoted (A4,$\preceq$), where the inequality $\le$ is replaced with $\preceq$.

From now on we assume the following condition

$$2^* C_{N,2_N}^{2_N} \eta \left(\sum_{i=1}^K \rho_i^2 \right)^{2/N} < 1, \tag{1.8}$$

and the first main result concerning (1.3) reads as follows.

Theorem 1.1. *Suppose (A0)–(A5) and* (1.8) *hold.*
(a) There exists $u \in \mathcal{M} \cap \mathcal{D}$ *such that* $J(u) = \inf_{\mathcal{M}\cap\mathcal{D}} J$. *Moreover* u *is a* K*-tuple of radial, nonnegative, and radially nonincreasing functions provided that* G *is of the form*

$$G(u) = \sum_{i=1}^{K} G_i(u_i) + \sum_{j=1}^{L} \beta_j \prod_{i=1}^{K} |u_i|^{r_{i,j}}, \tag{1.9}$$

where $L \geq 1$, $G_i \colon \mathbb{R} \to [0,\infty)$ *is even,* $r_{i,j} > 1$ *or* $r_{i,j} = 0$, $\beta_j \geq 0$, $2_N \leq \sum_{i=1}^{K} r_{i,j} < 2^*$, *and for every* j *there exists* $i_1 \neq i_2$ *such that* $r_{i_1,j} > 1$ *and* $r_{i_2,j} > 1$.
(b) If, moreover, (A4,$\preceq$*) holds, then* u *is of class* $\mathcal{C}^2$ *and there exists* $\lambda = (\lambda_1, \dots . \lambda_K) \in [0,\infty)^K$ *such that* (λ, u) *is a ground state solution to* (1.3).

Notice that (A1) allows G to have L^2-critical growth $G(u) \sim |u|^{2_N}$ at 0, but (A2) excludes the same behaviour at infinity. Moreover, (A3) rules out a Sobolev-critical growth $G(u) \sim |u|^{2^*}$ at infinity, but at 0 such a behaviour is possible. Finally, the *pure* L^2*-critical* case for $|u|$ small is ruled out by (A4,$\preceq$), i.e., $G(u)$ cannot be of the form (1.9) with $G_i(u) = \alpha_i |u|^{2_N}$, $\alpha_i \geq 0$, and $\sum_{i=1}^{K} r_{i,j} = 2_N$ for every j.

Here and later on, when we say G is of the form (1.9), we also mean the additional conditions on G_i, β_j, and $r_{i,j}$ listed in Theorem 1.1 (a). Observe that G of the form (1.9) satisfies (A4) if and only if G_i satisfies the scalar variant of (A4) for all $i \in \{1,\dots,K\}$. If, in addition, G_i satisfies (A4,$\preceq$) for some i, then G satisfies (A4,$\preceq$) as well.

More can be said if $N \in \{3,4\}$.

Theorem 1.2. *Assume that (A0)–(A3), (A4,*$\preceq$*), (A5), and* (1.8) *are satisfied,* G *is of the form* (1.9), *and* $N \in \{3,4\}$. *Then there exist* $u \in \mathcal{M} \cap \partial\mathcal{D}$ *of class* $\mathcal{C}^2$ *and* $\lambda = (\lambda_1,\dots,\lambda_K) \in [0,\infty)^K$ *such that* (u,λ) *is a ground state solution to* (1.3). *In addition, each* u_i *is radial, nonnegative, and radially nonincreasing. Moreover for every* $i \in \{1,\dots,K\}$ *either* $u_i = 0$ *or* $\int_{\mathbb{R}^N} |u_i|^2\,dx = \rho_i^2$ *and, if* $u_i \neq 0$, *then* $\lambda_i > 0$ *and* $u_i > 0$. *In particular, if* $u \in \mathcal{S}$, *then* $\lambda \in (0,\infty)^K$ *and* (λ, u) *is a ground state solution to* (1.4).

Note that the obtained ground state solution u belongs to $\partial\mathcal{D}$, i.e., at least one of the L^2-bounds must be the equality $\int_{\mathbb{R}^N} |u_i|^2\,dx = \rho_i^2$. In particular, ground states solutions can be semitrivial.

If $K = 2$, $L = 1$, and the coefficient of the coupling term is large, then we find ground state solutions to (1.4).

Theorem 1.3. *Assume that (A0)–(A3), (A4,*$\preceq$*), (A5), and* (1.8) *are satisfied,* $N \in \{3,4\}$, $K = 2$, *and* $L = 1$. *If* G *is of the form* (1.9), *each* G_i *is nondecreasing, and* $r_{1,1} + r_{2,1} > 2_N$, *then for every sufficiently large* $\beta_1 > 0$ *there exists a ground state solution* $(u,\lambda) \in \mathcal{S} \times (0,\infty)^2$ *to* (1.4). *Moreover, each component of* u *is positive, radial, radially nonincreasing and of class* $\mathcal{C}^2$.

Observe that, if in Theorem 1.3 $G_i(t) = \mu_i |t|^{p_i}/p_i$ for some $\mu_i > 0$ and $p_i \in (2_N, 2^*)$, $i \in \{1,2\}$, then clearly $\eta = 0$ in (1.8) and this result was very recently obtained by Li and Zou in [23, Theorem 1.3], again, unlike this paper, by means of the involved topological argument due to Ghoussoub [19], cf. [4–7, 22, 24, 28, 34]. If $\eta > 0$, the result seems to be new and we obtain a ground state solution to (1.4) for sufficiently small $|\rho|$, see (1.8). Furthermore, to our knowledge, this is the first result about normalized solutions to a system of Schrödinger equations where the nonlinearity is rather general, in particular not (entirely) of power-type. As for possible examples of scalar functions G_1, G_2 we refer to (E1)–(E4) in [10]; in particular, we can deal with

$$G_i(u) = \frac{\mu_i}{p_i}|u_i|^{p_i} + \frac{\nu_i}{2_N}|u_i|^{2_N}, \mu_i, \nu_i > 0, i \in \{1,2\}, \text{ where } \eta = \frac{\max\{\nu_1,\nu_2\}}{2_N} > 0.$$

2. The proof

Lemma 2.1. *Let $f_1, f_2 \in \mathcal{C}(\mathbb{R}^K)$ and assume there exists $C > 0$ such that $|f_1(u)| + |f_2(u)| \le C(|u|^2 + |u|^{2^*})$ for every $u \in \mathbb{R}^K$. Then $f_1 \preceq f_2$ if and only if $f_1 \le f_2$ and*

$$\int_{\mathbb{R}^N} f_1(u) - f_2(u)\,dx < 0$$

for every $u \in H^1(\mathbb{R}^N)^K \setminus \{0\}$.

Proof. We argue similarly as in the case $K = 1$ provided in [10, Lemma 2.1]. □

We will always assume that (A0) holds. Recall that (A6) holds if both (A2) and (A5) do. Lemmas 2.2–2.5 and 2.7 are variants of the results contained in [10,22] with some improvements and adapted to the system of equations.

Lemma 2.2. *If (A1), (A3), (A5), (A6), and (1.8) hold, then* $\inf\{|\nabla u|_2^2 : u \in \mathcal{M} \cap \mathcal{D}\} > 0$.

Proof. Recall that, if $p \in [2, 2^*]$, then

$$\big||u|\big|_p = |u|_p \text{ and } \big|\nabla|u|\big|_2 \le |\nabla u|_2 \text{ for every } u \in H^1(\mathbb{R}^N)^K.$$

For every $\varepsilon > 0$ there exists c_ε such that for every $u \in \mathcal{M} \cap \mathcal{D}$

$$\begin{aligned}|\nabla u|_2^2 &= \frac{N}{2}\int_{\mathbb{R}^N} H(u)\,dx \le 2^*\big(c_\varepsilon|u|_{2^*}^{2^*} + (\varepsilon+\eta)|u|_{2_N}^{2_N}\big) = 2^*\big(c_\varepsilon\big||u|\big|_{2^*}^{2^*} + (\varepsilon+\eta)\big||u|\big|_{2_N}^{2_N}\big)\\ &\le 2^*\left(c_\varepsilon C_{N,2^*}^{2^*}\big|\nabla|u|\big|_2^{2^*} + (\varepsilon+\eta)C_{N,2_N}^{2_N}\left(\sum_{i=1}^K \rho_i^2\right)^{2/N}\big|\nabla|u|\big|_2^2\right)\\ &\le 2^*\left(c_\varepsilon C_{N,2^*}^{2^*}|\nabla u|_2^{2^*} + (\varepsilon+\eta)C_{N,2_N}^{2_N}\left(\sum_{i=1}^K \rho_i^2\right)^{2/N}|\nabla u|_2^2\right)\end{aligned}$$

i.e.,

$$0 \le 2^* c_\varepsilon C_{N,2^*}^{2^*}|\nabla u|_2^{2^*} + \left(2^*(\varepsilon+\eta)C_{N,2_N}^{2_N}\left(\sum_{i=1}^K \rho_i^2\right)^{2/N} - 1\right)|\nabla u|_2^2$$

Taking ε sufficiently small so that

$$2^*(\varepsilon+\eta)C_{N,2_N}^{2_N}\left(\sum_{i=1}^K \rho_i^2\right)^{2/N} < 1$$

we conclude. □

For $u \in H^1(\mathbb{R}^N)^K \setminus \{0\}$ and $s > 0$ define $s \star u(x) := s^{N/2}u(sx)$ and $\varphi(s) := J(s \star u)$.

Lemma 2.3. *Assume that (A1)–(A5) hold and let $u \in H^1(\mathbb{R}^N)^K \setminus \{0\}$ such that*

$$\eta < \frac{|\nabla u|_2^2}{2|u|_{2_N}^{2_N}}. \tag{2.1}$$

Then there exist $a = a(u) > 0$ and $b = b(u) \geq a$ such that each $s \in [a, b]$ is a global maximizer for φ and φ is increasing on $(0, a)$ and decreasing on (b, ∞). Moreover $s \star u \in \mathcal{M}$ if and only if $s \in [a, b]$, $M(s \star u) > 0$ if and only if $s \in (0, a)$ and $M(s \star u) < 0$ is and only if $s > b$. If (A4,$\preceq$) holds, then $a = b$.

Note that (1.8) implies (2.1) provided that $u \in \mathcal{D}$.

Proof. Notice that from (A1)

$$\varphi(s) = \int_{\mathbb{R}^N} \frac{s^2}{2}|\nabla u|^2 - \frac{G(s^{N/2}u)}{s^N}\,dx \to 0$$

as $s \to 0^+$ and from (A2) $\lim_{s\to\infty} \varphi(s) = -\infty$. From (A1) and (A3) for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$G(u) \leq (\varepsilon + \eta)|u|^{2_N} + c_\varepsilon |u|^{2^*},$$

therefore,

$$\varphi(s) \geq s^2\Big(\int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 - (\eta+\varepsilon)|u|^{2_N}\,dx\Big) - c_\varepsilon s^{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,dx > 0$$

for sufficiently small ε and s. It follows that there exists an interval $[a, b] \subset (0, \infty)$ such that $\varphi|_{[a,b]} = \max \varphi$. Moreover

$$\varphi'(s) = s\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{N}{2}\frac{H(s^{N/2}u)}{s^{N+2}}\,dx$$

and the function

$$s \in (0, \infty) \mapsto \int_{\mathbb{R}^N} \frac{H(s^{N/2}u)}{s^{N+2}}\,dx$$

is nondecreasing (resp. increasing) due to (A4) (resp. (A4,$\preceq$) and Lemma 2.1) and tends to ∞ as $s \to \infty$ due to (A2) and (A5). There follows that $\varphi'(s) > 0$ if $s \in (0, a)$ and $\varphi'(s) < 0$ if $s > b$ and that $a = b$ if (A4,$\preceq$) holds. Finally, observe that

$$s\varphi'(s) = \int_{\mathbb{R}^N} s^2|\nabla u|^2 - \frac{N}{2}\frac{H(s^{N/2}u)}{s^N}\,dx = M(s \star u).$$

□

Lemma 2.4. *If (A1)–(A5) and (1.8) are verified, then J is coercive on $\mathcal{M} \cap \mathcal{D}$.*

Proof. First of all note that, if $u \in \mathcal{M}$, then due to (A5)

$$J(u) = J(u) - \frac{1}{2}M(u) = \int_{\mathbb{R}^N} \frac{N}{4}H(u) - G(u)\,dx \geq 0$$

and so, a fortiori, J is nonegative on $\mathcal{M} \cap \mathcal{D}$. Let $(u^{(n)}) \subset \mathcal{M} \cap \mathcal{D}$ such that $\|u^{(n)}\| \to \infty$, i.e., $\lim_n |\nabla u^{(n)}|_2 = \infty$, and define

$$s_n := |\nabla u^{(n)}|_2^{-1} > 0 \quad \text{and} \quad w^{(n)} := s_n \star u^{(n)}.$$

Note that $s_n \to 0$, $|w_i^{(n)}|_2 = |u_i^{(n)}|_2 \le \rho_i$ for $i \in \{1, \dots, K\}$, and $|\nabla w^{(n)}|_2^2 = 1$, in particular $(w^{(n)})$ is bounded in $H^1(\mathbb{R}^N)^K$. Suppose by contradiction that

$$\limsup_n \max_{y \in \mathbb{R}^N} \int_{B(y,1)} |w^{(n)}|^2 \, dx > 0.$$

Then there exist $(y^{(n)}) \subset \mathbb{R}^N$ and $w \in H^1(\mathbb{R}^N)^K$ such that, up to a subsequence, $w^{(n)}(\cdot + y^{(n)}) \rightharpoonup w \neq 0$ in $H^1(\mathbb{R}^N)^K$ and $w^{(n)}(\cdot + y^{(n)}) \to w$ a.e. in $\mathbb{R}^N$. Thus, owing to (A2),

$$\begin{aligned}
0 \le \frac{J(u^{(n)})}{|\nabla u^{(n)}|_2^2} &\le \frac{1}{2} - \int_{\mathbb{R}^N} \frac{G(u^{(n)})}{|\nabla u^{(n)}|_2^2} \, dx = \frac{1}{2} - s_n^{N+2} \int_{\mathbb{R}^N} G\big(u^{(n)}(s_n x)\big) \, dx \\
&= \frac{1}{2} - s_n^{N+2} \int_{\mathbb{R}^N} G(s_n^{-N/2} w^{(n)}) = \frac{1}{2} - \int_{\mathbb{R}^N} \frac{G(s_n^{-N/2} w^{(n)})}{|s_n^{-N/2} w^{(n)}|^{2_N}} |w^{(n)}|^{2_N} \, dx \\
&= \frac{1}{2} - \int_{\mathbb{R}^N} \frac{G\big(s_n^{-N/2} w^{(n)}(x + y^{(n)})\big)}{|s_n^{-N/2} w^{(n)}(x + y^{(n)})|^{2_N}} |w^{(n)}(x + y^{(n)})|^{2_N} \, dx \to -\infty.
\end{aligned}$$

It follows that

$$\lim_n \max_{y \in \mathbb{R}^N} \int_{B(y,1)} |w^{(n)}|^2 \, dx = 0$$

and so, from Lions' Lemma [27], $w^{(n)} \to 0$ in $L^{2_N}(\mathbb{R}^N)^K$. Since

$$s_n^{-1} \star w^{(n)} = u^{(n)} \in \mathcal{M},$$

Lemma 2.3 yields

$$J(u^{(n)}) = J(s_n^{-1} \star w^{(n)}) \ge J(s \star w^{(n)}) = \frac{s^2}{2} - s^N \int_{\mathbb{R}^N} G\big(s^{N/2} w^{(n)}(s\cdot)\big) \, dx$$

for every $s > 0$. Taking into account that

$$\lim_n \int_{\mathbb{R}^N} G\big(s^{N/2} w^{(n)}(s\cdot)\big) \, dx = 0,$$

we have that $\liminf_n J(u^{(n)}) \ge s^2/2$ for every $s > 0$, i.e., $\lim_n J(u^{(n)}) = \infty$. □

Lemma 2.5. *If (A1)–(A5) and (1.8) are verified, then* $c := \inf_{\mathcal{M} \cap \mathcal{D}} J > 0$.

Proof. We prove that there exists $\alpha > 0$ such that

$$|\nabla u|_2 \le \alpha \Rightarrow J(u) \ge \frac{|\nabla u|_2^2}{2N}. \tag{2.2}$$

From (1.7) and (1.8), for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^N} G(u) \, dx &\le c_\varepsilon C_{N,2^*}^{2^*} |\nabla u|_2^{2^*} + (\varepsilon + \eta) C_{N,2_N}^{2_N} \left(\sum_{i=1}^K \rho_i^2 \right)^{2/N} |\nabla u|_2^2 \\
&\le \left(c_\varepsilon C_{N,2^*}^{2^*} |\nabla u|_2^{2^*-2} + \varepsilon C_{N,2_N}^{2_N} \left(\sum_{i=1}^K \rho_i^2 \right)^{2/N} + \frac{1}{2} - \frac{1}{N} \right) |\nabla u|_2^2.
\end{aligned}$$

Choosing

$$\varepsilon = \frac{1}{4NC_{N,2_N}^{2_N}\left(\sum_{i=1}^K \rho_i^2\right)^{2/N}} \quad \text{and} \quad \alpha = \frac{1}{(4Nc_\varepsilon C_{N,2^*}^{2^*})^{\frac{1}{2^*-2}}}$$

we obtain, provided $|\nabla u|_2 \le \alpha$,

$$\int_{\mathbb{R}^N} G(u)\,dx \le \left(\frac{1}{2} - \frac{1}{2N}\right)|\nabla u|_2^2$$

and so $J(u) \ge \frac{|\nabla u|_2^2}{2N}$. Now take $u \in \mathcal{M} \cap \mathcal{D}$ and $\alpha > 0$ such that (2.2) holds and define

$$s := \frac{\alpha}{|\nabla u|_2} \quad \text{and} \quad w := s \star u.$$

Clearly $|w_i|_2 = |u_i|_2 \le \rho_i$ for $i \in \{1, \dots, K\}$ and $|\nabla w|_2 = \alpha$, whence in view of Lemma 2.3

$$J(u) \ge J(w) \ge \frac{|\nabla w|_2^2}{2N} = \frac{\alpha^2}{2N} > 0. \qquad \square$$

From now on, $c > 0$ will stand for the infimum of J over $\mathcal{M} \cap \mathcal{D}$. In view of Lemma 2.4, any minimizing sequence $(u^{(n)}) \subset \mathcal{M} \cap \mathcal{D}$ such that $J(u^{(n)}) \to c > 0$ is bounded. By the standard concentration-compactness argument [27], $u^{(n)} \rightharpoonup \tilde{u}$ for some $\tilde{u} \ne 0$ up to a subsequence and up to translations. It is not clear, however, if $J(\tilde{u}) = c$ or $\tilde{u} \in \mathcal{M} \cap \mathcal{D}$. Note that we can find $R > 0$ such that $\tilde{u}(R\cdot) \in \mathcal{M}$ and in order to ensure that $J(\tilde{u}) = c$ and $\tilde{u} \in \mathcal{D}$ we need to know that $R \ge 1$. The latter crucial condition requires the profile decomposition analysis of $(u^{(n)})$ provided by the following lemma.

Lemma 2.6. *Let $(u^{(n)}) \subset H^1(\mathbb{R}^N)^K$ be bounded. Then there exist sequences $(\tilde{u}^{(i)})_{i=0}^\infty \subset H^1(\mathbb{R}^N)^K$ and $(y^{(i,n)})_{i=0}^\infty \subset \mathbb{R}^N$ such that $y^{(0,n)} = 0$, $\lim_n |y^{(i,n)} - y^{(j,n)}| = 0$ if $i \ne j$, and for every $i \ge 0$ and every $F\colon \mathbb{R}^N \to \mathbb{R}$ of class $\mathcal{C}^1$ such that*

$$\lim_{u\to 0} \frac{F(u)}{|u|^2} = \lim_{|u|\to\infty} \frac{F(u)}{|u|^{2^*}} = 0$$

there holds

$$u^{(n)}(\cdot + y^{(i,n)}) \rightharpoonup \tilde{u}^{(i)} \text{ as } n \to \infty \tag{2.3}$$

$$\lim_n \int_{\mathbb{R}^N} |\nabla u^{(n)}|^2\,dx = \sum_{j=0}^i \int_{\mathbb{R}^N} |\nabla \tilde{u}^{(j)}|^2\,dx + \lim_n \int_{\mathbb{R}^N} |\nabla v^{(i,n)}|^2\,dx \tag{2.4}$$

$$\limsup_n \int_{\mathbb{R}^N} F(u^{(n)})\,dx = \sum_{i=0}^\infty \int_{\mathbb{R}^N} F(\tilde{u}^{(i)})\,dx, \tag{2.5}$$

where $v^{(i,n)}(x) := u^{(n)}(x) - \sum_{j=0}^i \tilde{u}^{(j)}(x - y^{(j,n)})$.

Proof. We argue similarly as in the case $K = 1$ provided in [30, Theorem 1.4]. $\square$

Lemma 2.7. *If (A1)–(A5) and (1.8) hold, then c is attained.*

Proof. Let $(u^{(n)}) \subset \mathcal{M} \cap \mathcal{D}$ such that $\lim_n J(u^{(n)}) = c$. Then $(u^{(n)})$ is bounded due to Lemma 2.4 and, in view of Lemma 2.6, we find $(\tilde{u}^{(i)})_{i=0}^\infty \subset H^1(\mathbb{R}^N)^K$ and $(y_n^{(i,n)})_{i=0}^\infty \subset \mathbb{R}^N$ such that (2.3)–(2.5) hold. Let

$$I := \{i \geq 0 : \tilde{u}^{(i)} \neq 0\}$$

and suppose by contradiction that $I = \emptyset$. Then, since $u^{(n)} \in \mathcal{M} \cap \mathcal{D}$, there holds

$$\lim_n \int_{\mathbb{R}^N} |\nabla u^{(n)}| \, dx = \lim_n \frac{N}{2} \int_{\mathbb{R}^N} H(u^{(n)}) \, dx = 0$$

owing to (2.5), which contradicts Lemma 2.2. Now we prove that

$$\frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}^{(i)}) \, dx \geq \int_{\mathbb{R}^N} |\nabla \tilde{u}^{(i)}|^2 \, dx \tag{2.6}$$

for some $i \in I$. Assume by contradiction that

$$\frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}^{(i)}) \, dx < \int_{\mathbb{R}^N} |\nabla \tilde{u}^{(i)}|^2 \, dx$$

for every $i \in I$. Then from (2.4) and (2.5) we have

$$\begin{aligned} \limsup_n \frac{N}{2} \int_{\mathbb{R}^N} H(u^{(n)}) \, dx &= \limsup_n \int_{\mathbb{R}^N} |\nabla u^{(n)}|^2 \geq \sum_{i=0}^\infty \int_{\mathbb{R}^N} |\nabla \tilde{u}^{(i)}|^2 \\ &> \sum_{i=0}^\infty \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}^{(i)}) \, dx = \limsup_n \frac{N}{2} \int_{\mathbb{R}^N} H(u^{(n)}) \, dx, \end{aligned}$$

a contradiction. Let $\tilde{u} = \tilde{u}^{(i)}$ satisfy (2.6) for some $i \in I$. Then there exists $R > 0$ such that $\tilde{u}(R\cdot) \in \mathcal{M}$ and again from (2.6) we indeed know that $R \geq 1$, whence $\tilde{u}(R\cdot) \in \mathcal{D}$. Hence Fatou's Lemma yields

$$\begin{aligned} c &\leq J\big(\tilde{u}(R\cdot)\big) = J\big(\tilde{u}(R\cdot)\big) - \frac{1}{2} M\big(\tilde{u}(R\cdot)\big) \, dx = \frac{1}{R^N} \int_{\mathbb{R}^N} \frac{N}{4} H(\tilde{u}) - G(\tilde{u}) \, dx \\ &\leq \liminf_n \int_{\mathbb{R}^N} \frac{N}{4} H(u^{(n)}) - G(u^{(n)}) \, dx = \liminf_n J(u^{(n)}) - \frac{1}{2} M(u^{(n)}) = \liminf_n J(u^{(n)}) = c, \end{aligned}$$

i.e., $R = 1$ and $J(\tilde{u}) = c$. □

For $f \colon \mathbb{R}^N \to \mathbb{R}$ measurable we denote by f^* the Schwartz rearrangement of $|f|$. Likewise, if $A \subset \mathbb{R}^N$ is measurable, we denote by A^* the Schwartz rearrangement of A [9, 25].

Lemma 2.8. *Assume that (A1)–(A5) and (1.8) are verified and G is of the form (1.9). Then c is attained by a K-tuple of radial, nonnegative and radially nonincreasing functions.*

Proof. Let $\tilde{u} \in \mathcal{M} \cap \mathcal{D}$ such that $J(\tilde{u}) = c$ be given by Lemma 2.7. For every $j = 1, \dots, K$ let u_j be the Schwartz rearrangement of $|\tilde{u}_j|$ and denote $u := (u_1, \dots, u_K)$. Let $a = a(u)$ be determined by Lemma 2.3. In view of the properties of the Schwartz rearrangement [9, 25], we obtain

$$M(1 \star u) = M(u) \leq M(\tilde{u}) = 0,$$

therefore in view of Lemma 2.3 we have that $a \leq 1$ and, consequently, $M(a \star \tilde{u}) \geq 0$. Let

$$d := \frac{N}{2} \max_{j=1,\dots,L} \Big(\sum_{i=1}^K r_{i,j} - 2 \Big) \geq 2.$$

Then

$$\begin{aligned} c &\le J(a\star u) = J(a\star u) - \frac{1}{d}M(a\star u)\\ &= \int_{\mathbb{R}^N}\sum_{i=1}^K a^2\Big(\frac12-\frac1d\Big)|\nabla u_i|^2 + \frac{1}{a^N}\Big(\frac{N}{2d}H_i(a^{N/2}u_i) - G_i(a^{N/2}u_i)\Big)\,dx\\ &\quad - \frac{1}{a^N}\sum_{j=1}^L \beta_j\Big(1-\frac{N}{2d}\Big(\sum_{i=1}^K r_{i,j}-2\Big)\Big)\prod_{i=1}^K |a^{N/2}u_i|^{r_{i,j}}\\ &\le \int_{\mathbb{R}^N}\sum_{i=1}^K a^2\Big(\frac12-\frac1d\Big)|\nabla \tilde u_i|^2 + \frac{1}{a^N}\Big(\frac{N}{2d}H_i(a^{N/2}|\tilde u_i|) - G_i(a^{N/2}|\tilde u_i|)\Big)\,dx\\ &\quad - \frac{1}{a^N}\sum_{j=1}^L \beta_j\Big(1-\frac{N}{2d}\Big(\sum_{i=1}^K r_{i,j}-2\Big)\Big)\prod_{i=1}^K |a^{N/2}\tilde u_i|^{r_{i,j}}\\ &= J(a\star\tilde u) - \frac1d M(a\star\tilde u) \le J(a\star\tilde u)\le J(\tilde u) = c, \end{aligned}$$

i.e., $J(a\star u) = c$. □

Lemma 2.9. *(a) Assume that (A1)–(A3), (A4,$\preceq$), (A5), and (1.8) hold and let $u\in\mathcal{M}\cap\mathcal{D}$ such that $J(u)=c$ and u_i is radial, nonnegative, and radially nonincreasing for every $i\in\{1,\dots,K\}$. Then u is of class $\mathcal{C}^2$.*
(b) If, in addition, $N\in\{3,4\}$ and G is of the form (1.9), then $u\in\partial\mathcal{D}$. Moreover, for every $i\in\{1,\dots,K\}$ either $u_i=0$ or $|u_i|_2=\rho_i$.

Proof. *(a)* In Proposition A.1 we set $f=J$, $\phi_i(v)=|v_i|_2^2-\rho_i^2$, $1\le i\le m=K$, $\psi_1(v)=M(v)$, $n=1$, $v\in\mathcal{H}=H^1(\mathbb{R}^N)^K$. Then there exist $(\lambda_1,\dots,\lambda_K)\in[0,\infty)^K$ and $\sigma\in\mathbb{R}$ such that

$$-(1-2\sigma)\Delta u_i + \lambda_i u_i = \partial_i G(u) - \sigma\frac{N}{2}\partial_i H(u) \tag{2.7}$$

for every $i\in\{1,\dots,K\}$ and u satisfies the Nehari identity

$$(1-2\sigma)\int_{\mathbb{R}^N}|\nabla u|^2\,dx + \sum_{i=1}^K\int_{\mathbb{R}^N}\lambda_i|u_i|^2\,dx + \int_{\mathbb{R}^N}\sigma\frac{N}{2}\langle h(u),u\rangle - \langle g(u),u\rangle\,dx = 0. \tag{2.8}$$

If $\sigma=\frac12$, then (A4,$\preceq$), (A5), and (2.8) yield

$$\begin{aligned} 0 &\ge \int_{\mathbb{R}^N}\frac{N}{4}\langle h(u),u\rangle - \langle g(u),u\rangle\,dx = \int_{\mathbb{R}^N}\frac{N}{4}\langle h(u),u\rangle - H(u) - 2G(u)\,dx\\ &> \int_{\mathbb{R}^N}\frac{N}{2}H(u) - 2G(u)\,dx \ge 0, \end{aligned}$$

a contradiction. Hence $\sigma\ne\frac12$ and u satisfies also the Pohožaev identity

$$(1-2\sigma)\int_{\mathbb{R}^N}|\nabla u|^2\,dx + \frac{2^*}{2}\sum_{i=1}^K\int_{\mathbb{R}^N}\lambda_i|u_i|^2\,dx + 2^*\int_{\mathbb{R}^N}\sigma\frac{N}{2}H(u) - G(u)\,dx = 0. \tag{2.9}$$

Combining (2.8) and (2.9) we obtain

$$(1-2\sigma)\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\frac{N}{2}\int_{\mathbb{R}^N}\sigma N\Big(\frac{1}{2}\langle h(u),u\rangle-H(u)\Big)-H(u)\,dx=0$$

and, using the fact that $u\in\mathcal{M}$,

$$(1-2\sigma)\int_{\mathbb{R}^N}H(u)\,dx+\int_{\mathbb{R}^N}\sigma N\Big(\frac{1}{2}\langle h(u),u\rangle-H(u)\Big)-H(u)\,dx=0,$$

that is

$$\sigma\int_{\mathbb{R}^N}\langle h(u),u\rangle-2_N H(u)\,dx=0,$$

which together with (A4,$\preceq$) yields $\sigma=0$. In view of [11, Theorem 2.3], $u\in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)^K$ for all $q<\infty$, hence $u\in\mathcal{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)^K$ for all $\alpha<1$. Then, arguing as in the proof of [9, Lemma 1], we have that u is of class $\mathcal{C}^2$.

(b) Suppose by contradiction that $\lambda_1=\dots=\lambda_K=0$, which is the case when $|u_i|<\rho_i$ for every i. From (2.8) and (2.9) (with $\sigma=0$) there follows

$$\int_{\mathbb{R}^N}\langle g(u),u\rangle-2^*G(u)\,dx=0. \tag{2.10}$$

In view of (A5)

$$2^*G\big(u(x)\big)=\langle g\big(u(x)\big),u(x)\rangle \tag{2.11}$$

for every $x\in\mathbb{R}^N$. Since G_i satisfies (A5), we get $2^*G_i(u_i(x))\ge g_i(u_i(x))u_i(x)$ for all $i\in\{1,\dots,K\}$ and note that

$$2^*\sum_{j=1}^L\beta_j\prod_{i=1}^K|u_i(x)|^{r_{i,j}}\ge\sum_{j=1}^L\beta_j\sum_{k=1}^K r_{k,j}\prod_{i=1}^K|u_i(x)|^{r_{i,j}},$$

hence, from (2.11), the equalities above are actually equalities. On the other hand, for every $j\in\{1,\dots,L\}$, $\sum_{i=1}^K r_{i,j}<2^*$, which yields $\beta_j=0$ or $\prod_{i=1}^K|u_i(x)|^{r_{i,j}}=0$ for every $x\in\mathbb{R}^N$, thus

$$2^*G_i(u_i(x))=g_i(u_i(x))u_i(x)$$

for every $i\in\{1,\dots,K\}$ and every $x\in\mathbb{R}^N$.

Now fix $i\in\{1,\dots,K\}$ such that $u_i\neq0$. Since $u_i\in H^1(\mathbb{R}^N)$, there exists an open interval $I\subset\mathbb{R}$ such that $0\in\overline{I}$ and $2^*G_i(s)=g_i(s)s$ for $s\in\overline{I}$. Then $G(s)=G(1)|s|^{2^*}$ for $s\in\overline{I}$ and u_i solves $-\Delta u_i=(2^*G(1))|u_i|^{2^*-2}u_i$. Hence u_i is an Aubin-Talenti instanton, up to scaling and translations, which is not L^2-integrable because $N\in\{3,4\}$, see [3,36].

Suppose that there exists $\nu\in\{1,\dots,K-1\}$ such that, up to changing the order, $|u_i|_2<\rho_i$ for every $i\in\{1,\dots,\nu\}$ and $|u_i|_2=\rho_i$ for every $i\in\{\nu+1,\dots,K\}$. From Proposition A.1 there exist $0=\lambda_1=\dots=\lambda_\nu\le\lambda_{\nu+1},\dots,\lambda_K$ and $\sigma\in\mathbb{R}$ such that

$$\begin{cases}-(1-2\sigma)\Delta u_i=\partial_iG(u)-\sigma\frac{N}{2}\partial_iH(u) & \text{for every } i\in\{1,\dots,\nu\}\\ -(1-2\sigma)\Delta u_i+\lambda_iu_i=\partial_iG(u)-\sigma\frac{N}{2}\partial_iH(u) & \text{for every } i\in\{\nu+1,\dots,K\}\end{cases} \tag{2.12}$$

and as before we obtain $\sigma=0$ and u is of class $\mathcal{C}^2$. Since G_i satisfies the scalar variant of (A5), $(0,\infty)\ni s\mapsto G_i(s)/s^{2_N}\in\mathbb{R}$ is nondecreasing, hence G_i is nondecreasing as well for all i. Then, the first ν equations in (2.12) with $\sigma=0$ yield $-\Delta u_i\ge0$ for $i\in\{1,\dots,\nu\}$.

Since $u \in L^{\frac{N}{N-2}}(\mathbb{R}^N)^K$ as $N \in \{3,4\}$, [20, Lemma A.2] implies $u_i = 0$ for every $i \in \{1,\dots,\nu\}$. Notice that we have proved that $\lambda_i = 0$ implies that $u_i = 0$. □

Remark 2.10. *We point out that in addition to assumptions of Lemma 2.9, i.e., (A1)–(A3), (A4,$\preceq$), (A5), and (1.8) hold, $u \in \mathcal{M} \cap \mathcal{D}$, and $J(u) = c$, we can show that $u \in \partial\mathcal{D}$ for any dimension $N \geq 3$ provided that $H \preceq (2^* - 2)G$ holds. Indeed, observe that (2.10) contradicts $H \preceq (2^* - 2)G$ and Lemma 2.1.*

Proof of Theorem 1.1. Statement (a) follows from Lemmas 2.7 and 2.8. Now we prove statement (b). From Lemma 2.9 (a), u is of class $\mathcal{C}^2$, while from Proposition A.1 there exist $(\lambda_1,\dots,\lambda_K) \in [0,\infty)^K$ and $\sigma \in \mathbb{R}$ such that (2.7) holds and $\sigma = 0$ as in the proof of Lemma 2.9 (a). □

Proof of Theorem 1.2. It follows from Lemma 2.9 (b), Theorem 1.1 (b), and the maximum principle [17, Lemma IX.V.1] (the implication $u_i \neq 0 \Rightarrow \lambda_i > 0$ is proved as in the proof of Lemma 2.9 (b)). □

Lemma 2.11. *Suppose that $K = 2$, $L = 1$ and the assumptions in Lemma 2.9 (b) hold. If $r_{1,1} + r_{2,1} > 2_N$ and β_1 is sufficiently large, then $u \in \mathcal{S}$.*

Proof. Since $L = 1$, we denote β_1, $r_{1,1}$, $r_{2,1}$ by β, r_1, r_2 respectively. Suppose by contradiction that $u_1 = 0$ or $u_2 = 0$, say $u_1 = 0$, which implies that $|u_2|_2 = \rho_2$. We want to find a suitable $w \in \mathcal{S}$ such that

$$J(a \star w) < c = J(0,u_2), \tag{2.13}$$

where $a = a(w)$ is defined in Lemma 2.3 (note that $a(w) = b(w)$ because (A4,$\preceq$) holds), which is impossible. First we show that c does not depend on β. Consider the functional

$$J_*\colon v \in H^1(\mathbb{R}^N) \mapsto \int_{\mathbb{R}^N} \frac{1}{2}|\nabla v|^2 - G_2(v)\,dx \in \mathbb{R}$$

and the sets

$$\mathcal{D}_* := \left\{ v \in H^1(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} |v|^2\,dx \leq \rho_2^2 \right\},$$
$$\mathcal{M}_* := \left\{ v \in H^1(\mathbb{R}^N) \setminus \{0\} \ : \ \int_{\mathbb{R}^N} |v|^2\,dx = \frac{N}{2}\int_{\mathbb{R}^N} H_2(v) \right\}.$$

Observe that $J(0,v) = J_*(v)$ for $v \in H^1(\mathbb{R}^N)$. Moreover $(0,v) \in \mathcal{D}$ if and only if $v \in \mathcal{D}_*$, and $(0,v) \in \mathcal{M}$ if and only if $v \in \mathcal{M}_*$. In particular,

$$c = J(0,u_2) = J_*(u_2) \geq \inf_{\mathcal{M}_* \cap \mathcal{D}_*} J_* = \inf\{J(0,v) : (0,v) \in \mathcal{M} \cap \mathcal{D}\} \geq c,$$

i.e., $c = \inf_{\mathcal{M}_* \cap \mathcal{D}_*} J_*$, and the claim follows because J_*, $\mathcal{D}_*$, and $\mathcal{M}_*$ do not depend on β.

In view of Theorem 1.2 for $K = 1$ (in this case $\beta_j = 0$ for all j), there exists $\bar{v} \in \mathcal{M}_* \cap \mathcal{S}_*$ such that

$$J_*(\bar{v}) = \inf_{\mathcal{M}_* \cap \mathcal{D}_*} J_* = c = \inf_{\mathcal{M}_* \cap \partial\mathcal{D}_*} J_*.$$

Note that $\bar{v}$ does not depend on β. Define $w = (w_1, w_2) := \left(\frac{\rho_1}{\rho_2}\bar{v}, \bar{v}\right)$. From Lemma 2.3, $a = a_\beta$ is implicitly defined by

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla w|^2\, dx = \frac{N}{2}\int_{\mathbb{R}^N} &\frac{G_1'(a_\beta^{N/2}w_1)a_\beta^{N/2}w_1 - 2G_1(a_\beta^{N/2}w_1)}{a_\beta^{N+2}} + \frac{G_2'(a_\beta^{N/2}w_2)a_\beta^{N/2}w_2 - 2G_2(a_\beta^{N/2}w_2)}{a_\beta^{N+2}}\\ &+ \beta(r_1+r_2-2)a_\beta^{N(r_1+r_2-2)/2-2}w_1^{r_1}w_2^{r_2}\, dx\\ \geq \beta(r_1+r_2-2)&a_\beta^{N(r_1+r_2-2)/2-2}\frac{N}{2}\int_{\mathbb{R}^N} w_1^{r_1}w_2^{r_2}\, dx,\end{aligned}$$

hence there exist $C>0$ not depending on β such that

$$0 < \beta a_\beta^{N(r_1+r_2-2)/2-2} \leq C, \tag{2.14}$$

whence

$$\lim_{\beta\to\infty} a_\beta = 0. \tag{2.15}$$

Since $a_\beta \star w \in \mathcal{M}$, we have from (A5)

$$\begin{aligned}J(a_\beta\star w) &= \int_{\mathbb{R}^N} \frac{N}{4}H(a_\beta\star w) - G(a_\beta\star w)\, dx \leq \frac{2}{N-2}\int_{\mathbb{R}^N} G(a_\beta\star w)\, dx\\ &= \frac{2}{N-2}\int_{\mathbb{R}^N}\frac{G_1(a_\beta^{N/2}w_1)+G_2(a_\beta^{N/2}w_2)}{a_\beta^N}\, dx + \frac{2\beta a_\beta^{N(r_1+r_2-2)/2}}{N-2}\int_{\mathbb{R}^N} w_1^{r_1}w_2^{r_2}\, dx,\end{aligned}$$

therefore (2.13) holds true for sufficiently large β owing to (A1), (2.14), and (2.15). ☐

Remark 2.12. *The proof of Lemma 2.11 shows that, under the assumptions of Theorem 1.3, every ground state solution (u,λ) to (1.3) is such that $u \in \mathcal{S}$, hence a ground state solution to (1.4).*

Proof of Theorem 1.3. It follows from Lemma 2.11 and Theorem 1.2. ☐

Now we investigate the behaviour of the ground state energy with respect to ρ. For $\rho = (\rho_1,\dots,\rho_K) \in (0,\infty)^K$ we denote

$$\begin{aligned}\mathcal{D}(\rho) &:= \left\{u \in H^1(\mathbb{R}^N)^K : \int_{\mathbb{R}^N} |u_i|^2\, dx \leq \rho_i^2 \text{ for every } i\in\{1,\dots,K\}\right\}\\ \mathcal{S}(\rho) &:= \left\{u \in H^1(\mathbb{R}^N)^K : \int_{\mathbb{R}^N} |u_i|^2\, dx = \rho_i^2 \text{ for every } i\in\{1,\dots,K\}\right\}\\ c(\rho) &:= \inf\{J(u) : u\in\mathcal{M}\cap\mathcal{D}(\rho)\}.\end{aligned}$$

Proposition 2.13. *Assume that (A0)–(A5) and (1.8) are satisfied. Then c is continuous and*

$$\lim_{\rho\to 0^+} c(\rho) = \infty,$$

where $\rho\to 0^+$ means $\rho_i \to 0^+$ for every $i\in\{1,\dots,K\}$.
If, moreover, every ground state solution to (1.3) belongs to $\mathcal{S}(\rho)$ (e.g. if the assumptions of Theorem 1.3 are satisfied), then c is decreasing in the following sense: if $\rho,\rho' \in (0,\infty)^K$ are such that $\rho_i \geq \rho_i'$ for every $i\in\{1,\dots,K\}$ and $\rho_j > \rho_j'$ for some $j\in\{1,\dots,K\}$, then $c(\rho) < c(\rho')$.

Proof. Fix $\rho \in (0,\infty)^K$ and let $\rho^{(n)} \to \rho$. For every n let $u^{(n)} \in \mathcal{M} \cap \mathcal{D}(\rho^{(n)}) \subset \mathcal{M} \cap \mathcal{D}(2\rho)$ such that $J(u^{(n)}) = c(\rho^{(n)}) \le c(\rho/2)$. In view of 2.4, $(u^{(n)})$ is bounded and so, arguing as in Lemma 2.7, there exists $u \in \mathcal{D}(\rho) \setminus \{0\}$ such that, up to subsequences and translations, $u^{(n)} \rightharpoonup u$ in $H^1(\mathbb{R}^N)^K$, $u^{(n)} \to u$ a.e. in $\mathbb{R}^N$, and $R \ge 1$, where $R = R_u > 0$ is such that $u(R\cdot) \in \mathcal{M}$. Fatou's Lemma and (A4) yield

$$\begin{aligned} c(\rho) \le J\big(u(R\cdot)\big) &= \frac{1}{R^N}\int_{\mathbb{R}^N} \frac{N}{4}H(u) - G(u)\,dx \le \int_{\mathbb{R}^N} \frac{N}{4}H(u) - G(u)\,dx \\ &\le \liminf_n \int_{\mathbb{R}^N} \frac{N}{4}H(u^{(n)}) - G(u^{(n)})\,dx = \liminf_n J(u^{(n)}) = \liminf_n c(\rho^{(n)}). \end{aligned}$$

Now let $w \in \mathcal{M} \cap \mathcal{D}(\rho)$ such that $J(w) = c(\rho)$. Denote $w_i^{(n)} := \rho_i^{(n)} w_i/\rho_i$ and consider $w^{(n)} = (w_1^{(n)}, \dots, w_K^{(n)}) \in \mathcal{D}(\rho^{(n)})$. Due to Lemma 2.3, for every n there exists $s_n > 0$ such that $s_n \star w^{(n)} \in \mathcal{M}$. Note that

$$\frac{N}{2}\int_{\mathbb{R}^N} \frac{H\big(s_n^{N/2}(\rho_1^{(n)} w_1/\rho_1, \dots, \rho_K^{(n)} w_K/\rho_K)\big)}{s_n^{N+2}}\,dx = \int_{\mathbb{R}^N} |\nabla w^{(n)}|^2\,dx \to \int_{\mathbb{R}^N} |\nabla w|^2\,dx. \tag{2.16}$$

If $\limsup_n s_n = \infty$, then from (A2) and (A5) the right hand side of (2.16) tends to ∞ up to a subsequence, which is a contradiction. If $\liminf_n s_n = 0$, then from (A1), (A3), (A5) and (1.8) and arguing as in Lemma 2.2 we obtain that the limit superior of the right hand side of (2.16) is less than $|\nabla w|_2^2$, which is again a contradiction. There follows that, up to a subsequence, $s_n \to s$ for some $s > 0$ and $s \star w \in \mathcal{M}$. In view of Lemma 2.3,

$$\limsup_n c(\rho^{(n)}) \le \lim_n J(s_n \star w_n) = J(s \star w) = J(w) = c(\rho)$$

and the continuity of c is proved.

Let $\rho^{(n)} \to 0^+$ and $u^{(n)} \in \mathcal{M} \cap \mathcal{D}(\rho^{(n)})$ such that $J(u^{(n)}) = c(\rho^{(n)})$. Denote $s_n := |\nabla u^{(n)}|_2^{-1}$ and $w^{(n)} := s_n \star u^{(n)}$ and note that $s_n^{-1} \star w^{(n)} = u^{(n)} \in \mathcal{M}$, $|\nabla w^{(n)}|_2 = 1$ and

$$|w^{(n)}|_2^2 = |u^{(n)}|_2^2 = \sum_{i=1}^K \big(\rho_i^{(n)}\big)^2 \to 0$$

as $n \to \infty$. In particular $\big(w^{(n)}\big)$ is bounded in $L^{2^*}(\mathbb{R}^N)^K$ and so

$$|w^{(n)}|_{2_N} \le |w^{(n)}|_2^{\frac{2}{N+2}} |w^{(n)}|_{2^*}^{\frac{N}{N+2}} \to 0$$

as $n \to \infty$. Then, in view of (A1) and (A3), for every $s > 0$

$$\lim_n \int_{\mathbb{R}^N} \frac{G(s^{N/2} w^{(n)})}{s^{-N}}\,dx = 0$$

and, consequently,

$$J(u^{(n)}) = J(s_n^{-1} \star w^{(n)}) \ge J(s \star w^{(n)}) = \frac{s^2}{2} - \int_{\mathbb{R}^N} \frac{G(s^{N/2} w^{(n)})}{s^{-N}}\,dx = \frac{s^2}{2} + o(1),$$

whence $\lim_n J(u^{(n)}) = \infty$.

Now assume that every ground state solution to (1.3) belongs to $\mathcal{S}(\rho)$ and let ρ, ρ' as in the statement. Let $u \in \mathcal{M} \cap \mathcal{S}(\rho)$ and $u' \in \mathcal{M} \cap \mathcal{S}(\rho') \subset \mathcal{M} \cap \mathcal{D}(\rho) \setminus \mathcal{S}(\rho)$ such that $J(u) = c(\rho)$ and $J(u') = c(\rho')$. Clearly $c(\rho) \le c(\rho')$. If $c(\rho) = c(\rho')$, then $c(\rho) = J(u')$, with $u' \in \mathcal{M} \cap \mathcal{D}(\rho) \setminus \mathcal{S}(\rho)$, which is a contradiction. □

Appendix A. Sign of Lagrange multipliers

The following result concerns the sign of a Lagrange multiplier when the corresponding constraint is given by an inequality and the critical point of the restricted functional is a minimizer. The result is related with Clarke's [14, Theorem 1], however it is not clear whether we can apply it directly in our situation.

Proposition A.1. *Let $\mathcal{H}$ be a real Hilbert space and $f, \phi_i, \psi_j \in \mathcal{C}^1(\mathcal{H})$, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$. Suppose that for every*

$$x \in \bigcap_{i=1}^{m} \phi_i^{-1}(0) \cap \bigcap_{j=1}^{n} \psi_j^{-1}(0)$$

the differential

$$\big(\phi_i'(x), \psi_j'(x)\big)_{1\leq i\leq m, 1\leq j\leq n} \colon \mathcal{H} \to \mathbb{R}^{m+n}$$

is surjective. If $\bar{x} \in \mathcal{H}$ minimizes f over

$$\{x \in \mathcal{H} : \phi_i(x) \leq 0 \text{ for every } i = 1, \ldots, m \text{ and } \psi_j(x) = 0 \text{ for every } j = 1, \ldots, n\},$$

then there exist $(\lambda_i)_{i=1}^m \in [0, \infty)^m$ and $(\sigma_j)_{j=1}^n \in \mathbb{R}^n$ such that

$$f'(\bar{x}) + \sum_{i=1}^{m} \lambda_i \phi_i'(\bar{x}) + \sum_{j=1}^{n} \sigma_j \psi_j'(\bar{x}) = 0.$$

Proof. Fix $\varepsilon > 0$ and define the functional $F \colon \mathcal{H} \to [0, \infty)$ as

$$F(x) := \max_{1\leq i\leq m, 1\leq j\leq n} \{f(x) - f(\bar{x}) + \varepsilon, \phi_i(x), |\psi_j(x)|\}.$$

and observe that F is locally Lipschitz and bounded from below by 0. Since $F(\bar{x}) = \varepsilon$, in view of the Ekeland variational principle [15, Theorem 1.1] there exists $z = z_\varepsilon \in \mathcal{H}$ such that

$$\begin{aligned} \|\bar{x} - z\| &\leq \sqrt{\varepsilon}, \\ F(x) + \sqrt{\varepsilon}\,\|x - z\| &\geq F(z) \quad \forall x \in \mathcal{H}. \end{aligned}$$

From [14, Propositions 6, 8] there follows that $0 \in \partial F(z) + \sqrt{\varepsilon}\,\partial\|\cdot - z\|(z)$, where ∂ stands for the generalized gradient [14, Definition 1]. Hence, there exists $\xi = \xi_\varepsilon \in \partial F(z)$ such that $-\xi \in \sqrt{\varepsilon}\,\partial\|\cdot - z\|(z)$. In view of [14, Propositions 1, 9], $\|\xi\| \leq \sqrt{\varepsilon}$ and ξ lies in the convex hull of $f(z) - f(\bar{x}) + \varepsilon$, $\phi_i(z)$, and $|\psi_j(z)|$, i.e., there exists $\tau, \lambda_1, \ldots, \lambda_m, \hat{\sigma}_1, \ldots, \hat{\sigma}_n \geq 0$ depending on ε, such that $\tau + \lambda_1 + \cdots + \lambda_m + \hat{\sigma}_1 + \cdots + \hat{\sigma}_n = 1$,

$$\xi \in \left(\tau f'(z) + \sum_{i=1}^{m} \lambda_i \phi_i'(z) + \sum_{j=1}^{n} \hat{\sigma}_j \partial|\psi_j|(z)\right),$$

and $\lambda_i = 0$ (resp. $\hat{\sigma}_j = 0$) if $\phi_i(z) \leq 0$ (resp. $\psi_j(z) = 0$).

For every $j \in \{1, \ldots, n\}$ such that $\psi_j(z) \neq 0$ we have

$$\partial|\psi_j|(z) = \{\operatorname{sign}\big(\psi_j(z)\big)\psi_j'(z)\}.$$

If $j \in \{1, \ldots, n\}$ is as before, we define $\sigma_j := \operatorname{sign}\big(\psi_j(z)\big)\hat{\sigma}_j$, otherwise we define $\sigma_j := 0$. In particular, we have

$$\sum_{j=1}^{n} \hat{\sigma}_j \partial|\psi_j|(z) = \left\{\sum_{j=1}^{n} \sigma_j \psi_j'(z)\right\}.$$

Summing up, we obtain the following: for every $\varepsilon > 0$ there exist $\tau \ge 0$, $(\lambda_i)_{i=1}^m \in [0,\infty)^m$, $(\sigma_j)_{j=1}^n \in \mathbb{R}^n$ and $z \in B(\bar{x}, \sqrt{\varepsilon})$ such that

$$\xi := \tau f'(z) + \textstyle\sum_{i=1}^m \lambda_i \phi_i'(z) + \sum_{j=1}^n \sigma_j \psi_j'(z) \in B(0, \sqrt{\varepsilon}),$$
$$\tau + \textstyle\sum_{i=1}^m \lambda_i + \sum_{j=1}^n |\sigma_j| = 1.$$

Letting $\varepsilon \to 0^+$ we get

$$\tau f'(\bar{x}) + \sum_{i=1}^m \lambda_i \phi_i'(\bar{x}) + \sum_{j=1}^n \sigma_j \psi_j'(\bar{x}) = 0 \tag{A.1}$$

for some $\tau \ge 0$, $(\lambda_i)_{i=1}^m \in [0,\infty)^m$, $(\sigma_j)_{j=1}^n \in \mathbb{R}^n$ such that

$$\tau + \sum_{i=1}^m \lambda_i + \sum_{j=1}^n |\sigma_j| = 1.$$

Suppose by contradiction that $\tau = 0$, whence

$$\sum_{i=1}^m \lambda_i \phi_i'(\bar{x}) + \sum_{j=1}^n \sigma_j \psi_j'(\bar{x}) = 0. \tag{A.2}$$

If $\phi_i(\bar{x}) < 0$ for some $i \in \{1,\dots,m\}$, then of course $\lambda_i = 0$, hence, up to considering a (possibly empty) subset of $\{1,\dots,m\}$ in (A.2), we can assume that $\phi_1(\bar{x}) = \dots = \phi_{m_0}(\bar{x}) = 0$ and $\lambda_{m_0+1} = \ldots = \lambda_m = 0$ for some $0 \le m_0 \le m$, where $m_0 = 0$ denotes that $\lambda_i = 0$ for all $i \in \{1,\dots,m\}$, whereas $m_0 = m$ denotes $\phi_1(\bar{x}) = \dots = \phi_m(\bar{x}) = 0$. Then the differential

$$\big(\phi_1'(\bar{x}), \dots, \phi_{m_0}'(\bar{x}), \psi_1'(\bar{x}), \dots, \psi_n'(\bar{x})\big) \colon \mathcal{H} \to \mathbb{R}^{m_0+n}$$

is surjective and so, for every $i \in \{1,\dots,m_0\}$ (resp. $j \in \{1,\dots,n\}$), we can choose $y \in \mathcal{H}$ such that $\phi_i'(\bar{x})(y) \ne 0$, $\phi_k'(\bar{x})(y) = 0$ for every $k \in \{1,\dots,m_0\} \setminus \{i\}$ and $\psi_j'(\bar{x})(y) = 0$ for every $j \in \{1,\dots,n\}$ (resp. $\psi_j'(\bar{x})(y) \ne 0$, $\psi_k'(\bar{x})(y) = 0$ for every $k \in \{1,\dots,n\} \setminus \{j\}$ and $\phi_i'(\bar{x})(y) = 0$ for every $i \in \{1,\dots,m_0\}$). This and (A.2) imply $\lambda_i = 0$ for every $i \in \{1,\dots,m_0\}$ and $\sigma_j = 0$ for every $j \in \{1,\dots,n\}$, a contradiction. We can thus divide both sides of (A.1) by τ and, up to relabelling λ_i and σ_j ($i \in \{1,\dots,m_0\}$, $j \in \{1,\dots,n\}$), conclude the proof. □

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