ADI schemes for the time integration of Maxwell equations

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No Man is an Iland, intire of it selfe;
every man is a peece of the Continent, a part of the maine;
if a Clod bee washed away by the Sea, Europe is the lesse,
as well as if a Promontorie were,
as well as if a Mannor of thy friends, or of thine owne were;
Any Man's death diminishes me, because I am involved in Mankinde;
And therefore never send to know for whom the bell tolls;
It tolls for thee.[1]

[1] From: 17th Meditation by John Donne, see page 299 in [Muel15].

Contents

Introduction

The Maxwell equations play an essential role in physics. Among other effects, they in particular describe the propagation of electromagnetic waves in media. Combining the Maxwell equations furthermore with the Lorentz force equation and Newton's second law of motion, all classical electromagnetic phenomena can be explained, see page 239 in [Jack99] for instance. Understanding and solving the Maxwell equations is also important to simulate and to design optical devices, such as wave guides or masks for photolithography, see Sections 8.2–8.4 in [Jack99], Chapter 8 in [SaTe07], and Chapter 3 in [Vett19].

As there is no general solution formula for the Maxwell equations at hand, numerical approximations to solutions play an important role. The computation of approximations is, however, in general expensive. This is due to the complicated structure of the Maxwell equations as a six-dimensional coupled system of partial differential equations on a three-dimensional domain. One possible way to compute approximations efficiently, is to split the Maxwell system into two smaller subproblems, which are easier to handle. Both subproblems are then solved separately, and the resulting solutions are afterwards combined to obtain an approximate solution of the original problem. This line of thought leads to splitting schemes, see [McQu02] (in particular Section 1 therein) for instance.

To obtain a splitting scheme, it is natural to ask, how the Maxwell system is actually split up into two parts. In this thesis, we follow the path of a class of dimension splitting schemes, which are favorable for the time integration of linear isotropic Maxwell equations on cuboidal domains. The spatial differential operator of the original problem is here split up into two new differential operators, so that only derivatives with respect to a single coordinate direction arise in each line of the new subproblems. In other words, the complicated spatial differential operator is split up according to the spatial dimensions, along which it differentiates. The two new spatial differential operators give rise to subproblems that are integrated separately. The idea of alternating direction implicit (ADI) schemes is then to apply alternatingly an explicit and an implicit time integrator to both subproblems, see [PeRa55].

An ADI method is first proposed by Peaceman and Rachford in [PeRa55] for a two-dimensional heat equation. In [Nami00, ZhCZ00], this idea is transferred to the three-dimensional linear isotropic Maxwell equations on cuboids. We call the resulting integrator Peaceman-Rachford ADI scheme. The implicit steps in the mentioned ADI schemes lead to unconditional stability. In contrast to many other implicit time integrators, these schemes furthermore have the linear complexity of explicit methods. Indeed, the arising implicit steps can be formulated in a way, in which the degrees of freedom essentially decouple. Important for us are also two ADI schemes from [ChLL10] that conserve energy under appropriate assumptions. A detailed presentation of selected results about ADI schemes for Maxwell equations is contained in Section 1.3. Note that dimension splitting schemes are also analyzed in a more general framework in [HaOs08].

Throughout this thesis, we focus on the analytical study of time-discrete approximations to linear isotropic Maxwell equations on a cuboid with perfectly conducting boundary. The approximations hereby result from different ADI splitting schemes. In particular, we do not discretize in space, so that we deal with abstract evolution equations in infinite-dimensional spaces. As our error analysis treats all arising spatial differential operators as unbounded mappings, it is however likely that the techniques and error results can be transferred to a full-discrete setting. Note also that full-discrete ADI splitting schemes are analyzed in [Nami00, ZhCZ00, ChLL10, HoKö19, Köhl18, HoKö20].

To reproduce the decay result from the first part of this thesis on a full-discrete level, we expect that special attention has to be paid to the spatial discretization method. This is for instance done in Section 4 of [ErZu09] for a one-dimensional damped wave equation, which is discretized in space by means of a mixed finite element method.

We next present the two main parts of this thesis. The first part on uniformly exponentially stable ADI schemes is a slightly extended version of the already published paper [Zeru20] by the author of this thesis. It appeared in the December 2020 issue number 492 of the Journal of Mathematical Analysis and Applications.

A uniformly exponentially stable ADI scheme

We first consider the Maxwell equations on a conductive cuboidal domain without external electric currents. Here the solutions are known to decay exponentially in time, see [Phun00, NiPi05, Elle19] and Section 1.2. The exponential decay rate is hereby uniform with respect to the initial data, and the system is said to be exponentially stable. In this respect, the physical law of the absence of magnetic monopoles plays an important role. It is part of the Maxwell system, and prescribes that the magnetic field is divergence-free. This is crucial for the decay behavior of the electromagnetic field, as the damping in the Maxwell system effects only the divergence-free parts of the magnetic field.

Our goal are time-discrete approximations to the exponentially stable Maxwell equations that preserve the uniform decay behavior. More precisely, the L^2-norm of the numerical approximations is supposed to satisfy an exponential decay estimate in time. The exponential decay rate has to be independent of the chosen starting value and the time step size. We then call the numerical scheme uniformly exponentially stable.

Note that the ADI schemes from [ZhCZ00, ChLL10] do not preserve the important divergence constraint on the magnetic field, see Section 7 in [ChLL10] and Section 6.4 in [GaZh13]. It is furthermore not clear how to uniformly control the divergence errors on large time scales. Recalling that the damping in the continuous Maxwell system only has an effect on divergence-free parts of the magnetic field, we do hence not expect that the schemes from [ZhCZ00, ChLL10] preserve the exponential decay rate uniformly with respect to the time step size.

For other wave equations, it is well known that space or time discretization can destroy exponential stability, see Section 1.2 for more details and references. In the discrete systems, the former uniform decay then typically depends on the discretization. One possible remedy is to incorporate artificial (viscous) damping into the discretization to restore the uniform exponential stability of the discrete system, see [ANVW13, ErZu09, RaTT07, TeZu03].

Compared to the setting in [ANVW13, ErZu09, RaTT07, TeZu03], the above mentioned failure of the divergence constraint causes additional difficulties. To solve them already on the continuous level, we extend the original Maxwell system by one equation, see (3.1), with the help of a mixed hyperbolic divergence cleaning technique from [DKKM02]. The rough idea is to shift spurious curl-free parts from the magnetic field to an artificial variable that receives additional damping.

We then construct the uniformly exponentially stable ADI scheme, see (3.24), as an integrator for the extended Maxwell system (3.1). To be more precise, the new spatial differential operator from the extended Maxwell system is split into six operators, to ensure that the spatial dimensions decouple in each subproblem. The desired scheme is then obtained by integrating the split system similar to [ChLL10] in time, and by incorporating additional damping from [ErZu09]. In contrast to [ErZu09], we include the artificial damping into the numerical approximation of almost every subproblem in the split system (except the part that corresponds to the damping terms in the continuous Maxwell system). Each of the artificial damping operators then involves only the current splitting operator. Altogether, we obtain a scheme that is still unconditionally stable, see Proposition 3.9. Furthermore, the implicit steps only require the solution of one-dimensional elliptic problems, see Remark 3.19.

The main result in this part is the uniform exponential stability of the new scheme, see Theorem 3.10. The proof is inspired by the proof of Theorem 1.1 in [TeZu03]. In particular, a discrete observability estimate is derived by means of a discrete multiplier technique. This technique is also applied in Section 4 of [Nica08]. It allows us to transfer arguments from the continuous setting in [NiPi05] to the time-discrete one.

In case of physically reasonable initial data, the exponentially stable ADI scheme (3.24) converges with order one to the solution of the original Maxwell system, see Theorem 6.5. We do not expect a better convergence rate, as the Maxwell system is split into six subproblems which are subsequently integrated in time. In the error analysis, we employ techniques from [EiJS19]. It is worth noting that our error statement makes only assumptions on the parameters of the continuous problem as well as on the initial data, but none on the unknown solution of the continuous problem. This is achieved by means of a rigorous regularity analysis. To the best of our knowledge, our error analysis is the first one that provides precise convergence rates for an exponentially stable scheme with artificial damping. Also the complexity of exponentially stable schemes is usually not addressed, see [ANVW13, ErZu09, RaTT07, TeZu03] for instance.

Error analysis of the Peaceman-Rachford ADI scheme for inhomogeneous Maxwell equations in heterogeneous media

In many applications, the considered medium for the Maxwell equations is not homogeneous. Instead, the material is heterogeneous. This means that it consists of several adjacent submedia with different material properties. Such a setting can be modeled by considering discontinuous material parameters in the Maxwell equations. The discontinuities are hereby located at the interfaces between different submedia. An example are Bragg-grating waveguides, see Section 1.1.

Despite the relevance of this problem, a rigorous error analysis that makes assumptions only on the model and the initial data, seems to be missing for ADI schemes in this setting. Note that [HoJS15, Eili17, EiSc18, EiSc17, EiJS19] deal with material parameters that are at least Lipschitz continuous, and that [Köhl18, HoKö20] impose a regularity assumption on the solution of the continuous Maxwell system.

It is likely that the actual convergence rate is lower in the case of heterogeneous media, compared to the setting of inhomogeneous media with regular material parameters. Indeed, a numerical example in [HoJS15] indicates that ADI schemes suffer from a loss of convergence order in the case of discontinuous material coefficients.

Here, we are concerned with linear isotropic Maxwell equations involving currents and charges on a heterogeneous cuboid. The domain consists of two homogeneous subcuboids, representing two different media. In particular, the material parameters are allowed to have a discontinuity at the interface between the two different submedia. The Maxwell system is here integrated in time by means of the Peaceman-Rachford ADI scheme from [Eili17, EiSc18, EiSc17].

In Theorem 10.7, we provide a rigorous error bound of order 3/2 for the L^2-error of the time-discrete approximations. This means a loss of order $1/2$, compared to the regular setting in [HoJS15, Eili17, EiSc17, Köhl18, HoKö20]. The crucial ingredient of our error analysis is an estimate for certain interface integrals. The inequality is derived by means of a trace method from interpolation theory. Note that the interface integrals are only present in the case of discontinuous coefficients, and that they cause the loss of convergence order. During the error analysis, we also make use of the techniques from [HoJS15, EiSc17, EiSc18].

Another important feature of our error estimate in Theorem 10.7 are the required preconditions. Similar to our findings in the first part of this thesis, we make assumptions only on the model problem and the initial data, but none on the unknown solution of the continuous Maxwell system. To achieve this goal, we do a detailed regularity analysis of the Maxwell equations in our setting. In Corollary 9.24, we establish piecewise H^2-regularity for the solutions of the considered Maxwell system (assuming appropriate initial data). Such a regularity result seems not to be proved in the literature so far. The results in [CoDN99] for instance state piecewise $H^{2-\theta}$-regularity, $\theta > 0$ arbitrary small, for the electric field, see Example (iii) after Theorem 7.1 therein. Note that this paper deals with the time-harmonic Maxwell equations without conductivity.

Our regularity analysis employs techniques from [HoJS15, EiSc17, Lemr78, Kell71]. The key issues are in particular regularity statements for elliptic transmission problems. Also interpolation theory plays an essential role in our arguments. Among other instances, it enters the description of a transmission condition for the electric field.

There is also some preliminary work on a more complicated material configuration that is not included in this thesis. In the more involved setting, each of the above subcuboids contains several other cuboids. The latter are called small subcuboids, and are not allowed to touch each other. It is furthermore assumed that the values of the material parameters do not change too much at the transition from the small subcuboids to the surrounding medium, roughly speaking. Given appropriate initial data, preliminary calculations then indicate that the L^2-error of the time-discrete approximations from the Peaceman-Rachford ADI scheme has the order $3/2 - \theta$. Here $\theta \in (0, 1/2)$ is a number that depends only on the material parameters. It increases as the jumps of the material parameters become stronger. A detailed regularity analysis is also in this setting necessary. The preliminary work suggests here piecewise $H^{2-\tilde{\theta}}$-regularity of the solutions to the Maxwell equations. The number $\tilde{\theta}$ is positive, and depends only on the material parameters. Similar to θ, the constant $\tilde{\theta}$ has to be chosen larger when the coefficients in the Maxwell equations vary stronger at the material interfaces. In case the material parameters do not change their values too strongly, this desired statement could improve the results in [CoDN99]. Note, however, that the setting in the latter publication is more general. It in particular allows complicated configurations involving arbitrary Lipschitz polyhedra. We also remark that for every positive number s, one can find a Maxwell transmission problem on a polyhedral domain, which has a solution that is not H^s-regular, see Section 7.1 in [CoDN99].

Organization

In Chapter 1, we proceed with an introduction to the Maxwell equations, the concepts of exponential stability and observability, as well as an outline of previous results on ADI schemes for Maxwell equations.

The first main part of this thesis is concerned with the analysis of a uniformly exponentially stable ADI scheme. This part is structured into the Chapters 2–6.

Chapter 2 serves as an introductory part, as it presents the considered damped Maxwell system, fixes the assumptions on the model, and recalls the basic concepts from the analysis of linear Maxwell equations.

Chapter 3 is devoted to the new exponentially stable ADI scheme. To formulate the method, the above mentioned extended Maxwell system is introduced. We then study the extended Maxwell system. In particular, we show its wellposedness, and demonstrate that the corresponding solutions also solve the original Maxwell system for appropriate initial data, roughly speaking. Afterwards, the new exponentially stable ADI scheme (3.24) is constructed. Also an energy-conserving variant is presented here, see (3.23). We close the chapter with a detailed regularity analysis of the proposed numerical methods.

In Chapter 4, we prove a uniform observability estimate for the energy conserving scheme (3.23) from Chapter 3, see Theorem 4.2. To prove the inequality, the curl-free and the divergence-free parts of the magnetic field approximations are estimated separately.

The desired uniform exponential stability of the damped scheme (3.24) is concluded in Chapter 5. One of the major tools is here the uniform observability estimate from Chapter 4.

A rigorous error result for the damped scheme (3.24) is demonstrated in Chapter 6. The main steps of the analysis are a stability estimate in an auxiliary space of H^1-regular functions, a bound on the local error, and an investigation of the error propagation.

The second main part of this thesis deals with an error analysis of the Peaceman-Rachford ADI scheme for linear Maxwell equations in heterogeneous cuboids. Chapters 7–10 deal with this topic.

The model problem as well as the main assumptions are first presented in Chapter 7. Auxiliary analytical statements are also shown here, and important function spaces are introduced.

Chapter 8 is concerned with a detailed study of selected transmission problems for the Laplacian on a cuboid. The main results yield piecewise H^2-regularity for the solutions of the transmission problems with homogeneous Dirichlet, homogeneous Neumann, or mixed boundary conditions.

In Chapter 9, our findings from Chapter 8 allow us to derive embeddings for the abstract function spaces from Chapter 7 into spaces of piecewise H^1- and H^2-regularity. We can then furthermore show that the latter spaces are state spaces for the Maxwell equations. This finally leads to the desired piecewise H^2-regularity statement for the solution of the Maxwell system in Corollary 9.24.

The Peaceman-Rachford ADI scheme is recalled in Chapter 10. The desired error estimate for the resulting time-discrete approximations is given in Theorem 10.7. In the proof, our regularity results from Chapter 9 come into play.

For the convenience of the reader, we also add an appendix at the end of this thesis. It contains useful formulas that we employ during coordinate transformations in Chapter 8. A subsequent glossary finally lists the most important objects, such as operators, functions, and spaces.

1. Background information on central topics

This chapter has an introductory character. It first explains the Maxwell equations, and describes the main assumptions in the model problems of this thesis. Furthermore, we provide results on exponential stability and observability for our Maxwell system. These statements also give a motivation for the first part of this thesis. Finally, we present an overview of selected literature on ADI splitting schemes for Maxwell equations.

1.1. Maxwell equations and model problem

Let Ω be a Lipschitz domain in $\mathbb{R}^3$. We study the Maxwell equations in their macroscopic formulation

$$\begin{aligned} \partial_t \mathbf{D}(x,t) &= \operatorname{curl} \mathbf{H}(x,t) - \mathbf{J}_\Sigma(x,t), \\ \partial_t \mathbf{B}(x,t) &= -\operatorname{curl} \mathbf{E}(x,t), \\ \operatorname{div} \mathbf{D}(x,t) &= \rho(x,t), \\ \operatorname{div} \mathbf{B}(x,t) &= 0, \end{aligned} \tag{1.1}$$

for positive times $t \geq 0$ on Ω, see equation (I.1a) in [Jack99]. This system of differential equations involves the electric displacement $\mathbf{D}$, the magnetizing field $\mathbf{H}$, the macroscopic current density $\mathbf{J}_\Sigma$, the magnetic field $\mathbf{B}$, the electric field $\mathbf{E}$, and the free charge density ρ.

The first line in (1.1) is Ampère's law, stating that time-varying electric fields and electric currents induce magnetic curls, see Section 6.1 in [Jack99]. Faraday's law is given in the second line of (1.1). It describes that time-varying magnetic fields induce electric curls, see Section 5.15 in [Jack99]. As a result, the first two lines explain the interaction of electric and magnetic fields. The divergence formula for the electric displacement field is called Coulomb's law. This equation means that the electric charge density is the source of the electric displacement. The last identity in (1.1) corresponds to the physical law of the absence of magnetic monopoles, see Section 2.1 in [KaKl19] and Section 6.11 in [Jack99].

In a given medium, the electric displacement $\mathbf{D}$ is defined in terms of the electric field $\mathbf{E}$, and the electric polarization $\mathbf{P}$ via the formula

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \tag{1.2}$$

see equation (4.21) in [Grif13]. Here, ε_0 is the constant vacuum permittivity. The magnetizing field $\mathbf{H}$ is moreover given by the identity

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}, \tag{1.3}$$

with the magnetization $\mathbf{M}$ and the vacuum permeability μ_0, see formula (6.18) in [Grif13]. Note that also μ_0 is constant. The combination of (1.2) and (1.3) is called constitutive relations. Note that (1.2) and (1.3) allow for complicated connections of $\mathbf{E}, \mathbf{H}, \mathbf{P}$, and $\mathbf{M}$. These relations are simplified in the following.

We first point out that we do not consider electro-magnetic coupling in this thesis. In other words, the electric polarization $\mathbf{P}$ only depends on the electric field $\mathbf{E}$, while the magnetization $\mathbf{M}$ is only a function of the magnetizing field $\mathbf{H}$. Furthermore, the medium is assumed to be linear, isotropic, and to respond instantaneously to the applied fields. The relation between $\mathbf{P}$ and $\mathbf{E}$, as well as $\mathbf{M}$ and $\mathbf{H}$ also have to be local in space, meaning that $\mathbf{E}(x,\cdot)$ and $\mathbf{H}(x,\cdot)$ only effect polarization and magnetization at the position x. These assumptions bring (1.2) and (1.3) into the form

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} =: \varepsilon \mathbf{E}, \qquad \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \chi_m \mathbf{H} =: \frac{1}{\mu}\mathbf{B}. \tag{1.4}$$

Hereby we use the electric and magnetic susceptibilities χ_e and χ_m, the electric permittivity ε, and the magnetic permeability μ, see Section I.4 in [Jack99], as well as Sections 4.4.1 and 6.4.1 in [Grif13]. The above assumptions imply that the latter four quantities are scalar, space dependent, and constant in time.

Note that the above presumption of linear material laws is justified for weak fields, see Sections 4.4.1 and 6.4.1 in [Grif13]. Furthermore, the proportionality between the electric polarization $\mathbf{P}$ and the electric field $\mathbf{E}$, as well as the magnetization $\mathbf{M}$ and the magnetizing field $\mathbf{H}$, is usually local in space for electromagnetic waves with frequencies in the range of visible light and beyond, see Section I.4 of [Jack99] for instance. Finally, the response of a medium to the present fields is instantaneous for a polychromatic wave with narrow frequency width, for instance. Here the permittivity and permeability are considered to be only space dependent, see Section 5.3 of [SaTe07].

It now remains to make the dependence of the electric current $\mathbf{J}_\Sigma$ on the electric field $\mathbf{E}$ more precise. The function $\mathbf{J}_\Sigma$ can be written as the sum of two currents $\mathbf{J}_c$ and $\mathbf{J}$. The first one is called conduction current, and it is given by the identity

$\mathbf{J}_c = \sigma\mathbf{E}$ with the conductivity $\sigma \geq 0$. This is a form of Ohm's law, see Section 1.1 in [BoWo09]. The latter function $\mathbf{J} = \mathbf{J}(x,t)$ is an external current, which is in our setting assumed to be known.

Denote by ν the exterior normal vector at the boundary $\partial\Omega$ of Ω. We finally equip (1.1) with perfectly conducting boundary conditions and initial conditions, see Subsection I.4.2.4.3 in [DaLi90]. Altogether, system (1.1) possesses the new representation

$$\begin{aligned}
\partial_t \mathbf{E} &= \frac{1}{\varepsilon}\operatorname{curl}\mathbf{H} - \frac{\sigma}{\varepsilon}\mathbf{E} - \frac{1}{\varepsilon}\mathbf{J} && \text{on } \Omega,\ t \geq 0,\\
\partial_t \mathbf{H} &= -\frac{1}{\mu}\operatorname{curl}\mathbf{E} && \text{on } \Omega,\ t \geq 0,\\
\operatorname{div}(\varepsilon\mathbf{E}) &= \rho && \text{on } \Omega,\ t \geq 0,\\
\operatorname{div}(\mu\mathbf{H}) &= 0 && \text{on } \Omega,\ t \geq 0,\\
\mathbf{E}\times\nu = 0, &\quad \mu\mathbf{H}\cdot\nu = 0 && \text{on } \partial\Omega,\ t \geq 0,\\
\mathbf{E}(0) = \mathbf{E}_0, &\quad \mathbf{H}(0) = \mathbf{H}_0 && \text{on } \Omega,
\end{aligned} \tag{1.5}$$

as an initial boundary value problem on Ω.

Both parts of this thesis are concerned with the analysis of time discrete approximations to (1.5). Throughout our studies, Ω is a cuboid. Due to the strong connection between the mappings $\mathbf{H}$ and $\mathbf{B}$, we call also $\mathbf{H}$ magnetic field.

The properties of the material parameters ε, μ, and σ influence the behavior of solutions of the Maxwell system (1.5) in a crucial way. In case of a positive conductivity σ and a vanishing external current $\mathbf{J}$, the solutions of (1.5) decay exponentially in time, see Section 1.2. The first part of this thesis is devoted to the reproduction of this physical phenomenon in numerical approximations, see Chapters 2–6.

Note that the material parameters ε, μ, and σ are not assumed to be continuous in (1.5). Indeed, we also deal with discontinuous coefficients. The following configuration serves as a simplified model for Bragg-grating waveguides. We here follow the presentation in Section 8.4 in [SaTe07], see in particular Figure 8.4-1 therein. The model configuration is obtained by dividing the cuboid Ω into a chain of several cuboidal layers. Each of the layers consists of a homogeneous dielectric medium, such as Silica. This means in particular that the material parameters ε, μ, and σ are constant in every layer, but change their values across an interface between two adjacent media. In the case of a Bragg-grating waveguide, one of the layers in the middle serves as a waveguide. The remaining ones consist of two alternating materials.

The rough principle of this device is as follows. An incoming wave with a certain range of angles of incidence, and a given frequency is reflected at the upper and lower boundary faces of the middle waveguide layer. The wave is reflected, as it cannot propagate into the cuboids that are adjacent to the waveguide layer. (More precisely, it has a low penetration depth into the adjacent layers.) This leads to a reflectivity of approximately unity, and a tight confinement of the wave to the waveguide. As a result, the electromagnetic wave propagates inside the waveguide into the desired direction.

The second part of this thesis analyzes the Maxwell system (1.5), as well as corresponding time discrete approximations in a similar setting. Indeed, the major difficulties in the above problem arise at the interfaces between the subcuboids. The arguments are thus of a local nature, and it suffices to focus on the case of only two adjacent cuboids. The latter configuration is studied in Chapters 7–10.

1.2. Exponential stability of a linear damped Maxwell system

We follow here in parts Section 1 in [Zeru20]. Let the external current $\mathbf{J}$ be zero in (1.5). We are interested in the long-time evolution of the energy

$$\mathscr{E}(t) = \frac{1}{2}\int_\Omega \Big((\varepsilon\mathbf{E}(x,t))\cdot\mathbf{E}(x,t) + (\mu\mathbf{H}(x,t))\cdot\mathbf{H}(x,t)\Big)\,\mathrm{d}x, \qquad t \geq 0, \tag{1.6}$$

of the Maxwell system (1.5). In case system (1.5) is considered on a C^∞-smooth domain in $\mathbb{R}^3$, ε and μ are positive numbers, and $\sigma \in L^\infty$ is uniformly positive, the energy $\mathscr{E}$ is known to decay in a uniform exponential way, see Théorème 5.1 in [Phun00]. This means that there are positive numbers C and β with

$$\mathscr{E}(t) \leq C\mathrm{e}^{-\beta t}\mathscr{E}(0), \qquad t \geq 0,$$

for appropriate initial data $(\mathbf{E}_0, \mathbf{H}_0)$. This property is called exponential stability. It is essential that C and β do not depend on the data $(\mathbf{E}_0, \mathbf{H}_0)$. The same decay statement is true on a C^2-domain with C^1-regular scalar coefficients ε, μ, and σ that are uniformly positive, see Lemma 3.1 and Theorem 4.1 in [NiPi05]. Eller considers (1.5) on a connected and bounded Lipschitz domain with uniformly positive definite L^∞-coefficients ε, μ, and σ. Eller also allows symmetric matrix-valued material parameters. In this setting, he also demonstrates the exponential stability of (1.5), see [Elle19].

The exponential stability of (1.5) is due to the conduction current $-\sigma\mathbf{E}$. Via the coupling of the differential equations in the first two lines of (1.5), this current has a damping effect on the electric and magnetic field. Note, however, that the differential equation for the time derivative of $\mathbf{H}$ only contains the curl of $\mathbf{E}$. As a result, the damping effect acts solely on the divergence-free parts of the magnetic field. The divergence constraint $\operatorname{div}(\mu\mathbf{H}) = 0$ in (1.5) is thus crucial for the exponential stability. If there were gradient parts of the magnetic field present in (1.5), these would be conserved over time. This aspect causes difficulties for the numerical approximations from ADI schemes to (1.5), as these numerical schemes are known to violate the divergence constraint on the magnetic field, see Section 7 in [ChLL10] and Section 6.4 in [GaZh13]. To deal with this issue, we introduce an extended Maxwell system in Section 3.1.

Exponential stability is often concluded by means of observability estimates, see the proof of Theorem 4.1 in [NiPi05], Theorem 2.2 in [AmTu01], and the proof of Proposition 7.4.5 in [TuWe09] for instance. We present here an observability estimate for (1.5) in the case of a C^2-domain Ω.

Fix a time $T > 0$. By Lemma 3.1 in [NiPi05], there is a positive number $C > 0$ with

$$\int_\Omega (\varepsilon |\mathbf{E}_0|^2 + \mu |\mathbf{H}_0|^2) \,\mathrm{d}x \le C \int_0^T \int_\Omega |\mathbf{E}_c|^2 \,\mathrm{d}x \,\mathrm{d}t \tag{1.7}$$

for all initial data $(\mathbf{E}_0, \mathbf{H}_0) \in H_0(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$ with

$$\mathrm{div}(\varepsilon \mathbf{E}_0) = \mathrm{div}(\mu \mathbf{H}_0) = 0 \quad \text{and} \quad \mu \mathbf{H}_0 \cdot \nu = 0 \text{ on } \partial\Omega.$$

(The space $H(\mathrm{curl}, \Omega)$ is recalled in Section 2.2.) The field $(\mathbf{E}_c, \mathbf{H}_c)$ denotes here the solution of the Maxwell system (1.5) without damping (meaning $\sigma = 0$). As above, it is important that the involved constants C, and T do not depend on the given initial data. Inequality (1.7) is called internal observability estimate for the undamped system (1.5), and the latter system is said to be exactly observable in time T, see Definition 6.1.1 in [TuWe09].

The name observability estimate reflects the following meaning of (1.7). The electric field $\mathbf{E}_c$ is assumed to be observable or measurable over the observation time $[0, T]$. Then (1.7) implies that the knowledge of $\mathbf{E}_c$ determines the solution $(\mathbf{E}_c, \mathbf{H}_c)$ of (1.5) uniquely. This is the concept of observability, see Section 1 in [Zuaz05]. To verify the latter claim, let $(\mathbf{E}_c^1, \mathbf{H}_c^1)$ and $(\mathbf{E}_c^2, \mathbf{H}_c^2)$ be two solutions of (1.5) with the same observation $\mathbf{E}_c^1 = \mathbf{E}_c^2$ on $[0, T]$. The difference $(0, \mathbf{H}_c^1 - \mathbf{H}_c^2)$ then still solves (1.5) for the initial data $(0, \mathbf{H}_c^1(0) - \mathbf{H}_c^2(0))$. Applying (1.7) to this difference, we infer that the initial data $\mathbf{H}_c^1(0)$ and $\mathbf{H}_c^2(0)$ have to coincide. As (1.5) is wellposed, see Theorem 2.2 in [NiPi05], this shows that both solutions $(\mathbf{E}_c^1, \mathbf{H}_c^1)$ and $(\mathbf{E}_c^2, \mathbf{H}_c^2)$ are equal.

There are different techniques to establish observability estimates. Among them are microlocal analysis and a multiplier method, see [BaLR92, Elle19, Phun00, Komo94, Zhan00, NiPi05] for instance.

There arise, however, difficulties when observable or exponentially stable systems are discretized in space or time. In [ZhZZ09], an observable wave equation is discretized in time with a numerically stable scheme, and the resulting time-discrete approximations do not satisfy a uniform observability estimate. Finite difference and finite element space discretizations of an observable one-dimensional wave equation are studied in [InZu99]. Also here, the discrete systems do not satisfy uniform observability estimates. The lack of uniform observability is here explained by means of non-physical high-frequency modes in the discrete systems. These numerical artifacts are not present in the original continuous system. The modes in the discrete systems are calculated in Sections 2.1 and 3.2 of [InZu99]. At high frequencies, the distance between the roots of consecutive eigenvalues of the discrete problem shrinks as the spatial grid becomes finer. This is in contrast to the continuous problem, where the gap between the roots of two consecutive eigenvalues is independent of the frequency. This phenomenon is crucial for the blow up of discrete observability estimates. In a similar way, Tébou and Zuazua explain the loss of uniform observability respectively exponential stability for finite difference space discretizations of two one-dimensional wave equations in [TeZu03]. A survey on observability of discrete approximations to wave equations, spurious oscillations, and possible cures is given in [Zuaz05].

For space semidiscretizations of linear Maxwell equations on a Yee grid from [Yee66], a similar phenomenon is known, see [Nica08]. For the continuous setting, Komornik proves a uniform observability estimate in [Komo94]. Nevertheless, Nicaise demonstrates that the observability inequality from the continous system is not uniformly valid for the spatial-discrete one.

1.3. Genesis and analysis of ADI schemes for Maxwell equations

In this thesis, we study the Maxwell system (1.5) on a cuboid, as well as time-discrete approximations from alternating direction implicit (ADI) schemes. To our knowledge, the first ADI scheme is proposed in [PeRa55] to solve the two-dimensional heat equation $\partial_t u = \partial_x^2 u + \partial_y^2 u$ on a square numerically. Peaceman and Rachford develop here the idea to approximate the solution by splitting the arising spatial differential operator according to the spatial dimensions, and to integrate both split problems separately in an alternating manner. This leads to a dimension splitting.

The resulting scheme is formulated in two steps, and both splitting operators ∂_x^2 and ∂_y^2 are discretized by means of central second order finite difference quotients. In the first half step of the scheme, the discrete counterpart of ∂_x^2 is applied to the unknown next iterate, while the discrete version of ∂_y^2 is applied to the already known current iterate. The second half step of the scheme interchanges the roles of ∂_x^2 and ∂_y^2. Altogether, the method alternates the spatial direction that is integrated implicitly in time. The above way, in which the splitting system is integrated in time, is also called Peaceman-Rachford scheme. The paper [PeRa55] also provides a stability and efficiency analysis.

To approximate the solution of linear isotropic Maxwell equations on cuboids with perfectly conducting boundary conditions, Yee presents a finite-difference time-domain (FDTD) method in [Yee66]. Being an explicit time integrator, however, it suffers from a Courant-Friedrichs-Levy (CFL) condition on the time step size, see Section 4.7 in [TaHa05]. In other words, the time step size has to be chosen sufficiently small to ensure numerical stability.

In [Nami99], Namiki combines the above ADI operator splitting idea from Peaceman and Rachford for time integration with a finite difference space discretization on the Yee cell from [Yee66]. This leads to a scheme with the name ADI-FDTD. It is used to solve the Maxwell equations for a two-dimensional transverse magnetic wave numerically. The scheme is also shown to be numerically stable, without restriction on the time step size. Afterwards, ADI splitting schemes for the linear three-dimensional Maxwell equations on cuboids are introduced in [Nami00, ZhCZ00]. The Maxwell system is here again spatially discretized by means of finite differences on a Yee grid. Both papers arrive at essentially the same operator splitting, and the split system is integrated in time in the Peaceman-Rachford way. It is furthermore shown that the schemes have linear complexity. In [LeFo03], the operator splitting from [Nami00, ZhCZ00] is furthermore integrated in time by several other higher order methods. We also note that the mentioned methods only deal with the homogeneous Maxwell equations. The efficiency of the Peaceman-Rachford ADI schemes from [Nami00, ZhCZ00] can moreover be enhanced by transforming the methods into fundamental implicit schemes, see [Tan08, Tan20]. By means of supplementary vectors, the explicit steps of the Peaceman-Rachford ADI scheme can here be simplified to involve only sums and differences of vectors. Several other implicit splitting schemes can also be implemented as fundamental implicit schemes, see [Tan08, Tan20].

If the electric current $\mathbf{J}_\Sigma$ is zero, the energy $\mathscr{E}$ in (1.6) of the Maxwell system is conserved, see Proposition 3.5 in [HoJS15]. It is then reasonable to use also an energy-conserving time-integrator for (1.5). Indeed, two energy-conserving ADI-FDTD schemes are presented in [ChLL10], employing again the operator splitting from [Nami00, ZhCZ00]. Both schemes are unconditionally stable. The L^2-convergence rate of the schemes is also analyzed, assuming that the solution of (1.5) is C^3-regular in space and time. The paper [GaLC13] further studies one of the schemes from [ChLL10], providing an error and stability analysis in discrete H^1-norms. Here, it is however required that the unknown solution of the Maxwell problem (1.5) is at least C^4-regular in space and C^3-regular in time. In the first part of this thesis, a scheme is constructed that is inspired by the first scheme of [ChLL10], see Section 3.3.

The first rigorous error analysis for an ADI scheme is provided in [HoJS15]. Hochbruck, Jahnke, and Schnaubelt study here the Maxwell system (1.5) with positive coefficients ε and μ in $W^{1,\infty} \cap W^{2,3}$. The current $\mathbf{J}_\Sigma$ is set to zero, and (1.5) is considered on the entire space $\mathbb{R}^3$, and on a cuboid with perfectly conducting boundary. Here, the Peaceman-Rachford ADI scheme from [ZhCZ00] is used to discretize in time, but no space discretization is analyzed. Based on a regularity analysis of (1.5), they derive an L^2-error estimate of order two in time. It is crucial that the error result only depends on the data but not on the unknown solution. The case of a nontrivial current $\mathbf{J}_\Sigma$ is then analyzed in [Eili17, EiSc18, EiSc17] by generalizing the scheme of [ZhCZ00] to inhomogeneous systems. In the first two publications, Eilinghoff and Schnaubelt provide a regularity analysis for (1.5), and show that the new scheme converges weakly with order two in a space that is similar to H^{-1}. We employ their scheme in the second part of this thesis. In [Eili17, EiSc17], more regular material parameters, inhomogeneities, as well as initial data are analyzed that ensure H^2-regularity of the solutions to (1.5). The new scheme from [Eili17, EiSc18] is shown to converge with order two in L^2 in the time-discrete setting. The second scheme from [ChLL10] is afterwards modified in [EiJS19] to allow also for inhomogeneities in the Maxwell system (1.5). Under appropriate regularity assumptions on the material parameters ε, μ, and σ, as well as the inhomogeneity $\mathbf{J}$, Eilinghoff, Jahnke, and Schnaubelt prove that their new scheme converges weakly with order two in a space that is similar to H^{-1}.

Recall that the above mentioned publications only analyze time-discrete problems (meaning that space is not discretized), or they discretize in space by means of finite differences. For applications with discontinuous material parameters, see Section 1.1 for instance, discrete ansatz spaces from discontinuous Galerkin (dG) space discretization schemes are however interesting. In [HoKö19, Köhl18, HoKö20], Hochbruck and Köhler investigate full discretizations of (1.5) on a cuboid (respectively more general domain in [HoKö20]), where the Maxwell equations are spatially discretized with a central-flux dG method. The spatial discrete counterpart to the Maxwell operator is again split in the ADI manner, and the time integration is performed by means of the Peaceman-Rachford method. The resulting full discrete scheme is shown to be of linear complexity, see [HoKö19, Köhl18]. This is achieved by ordering the degrees of freedom in the matrices of the implicit steps, so that only linear systems with a tridiagonal structure have to be solved implicitly. Furthermore, the bandwidth only depends on the polynomial degree of the dG ansatz space, and not on the width of the spatial mesh. In [Köhl18, HoKö20], Köhler and Hochbruck also establish rigorous error estimates for the Peaceman-Rachford ADI-dG full discretization of (1.5). These results bound the difference between the exact solution and the numerical approximation. They furthermore estimate the error between time respectively space discrete derivatives of the approximations and time respectively space derivatives of the solution. Actually, the results in [HoKö19, Köhl18, HoKö20] do not only cover full discretizations of the linear Maxwell system (1.5), but also of more general wave-type problems (with appropriate properties).

Part I.

A uniformly exponentially stable ADI scheme

2. Uniformly exponentially stable Maxwell equations and analytical preliminaries

In this chapter, we first introduce our model problem as well as the main assumptions. Afterwards, useful analytical concepts for the analysis of linear Maxwell equations are repeated. The arising spaces and notation are used throughout the thesis. Some of the constructions are further refined in Sections 7.2–7.3.

2.1. Damped Maxwell equations

In the first part of this thesis, we study the linear isotropic Maxwell system with Ohm's law

$$
\begin{aligned}
\partial_t \mathbf{E} &= \frac{1}{\varepsilon}\operatorname{curl}\mathbf{H} - \tilde{\sigma}\mathbf{E} && \text{in } Q \times [0,\infty),\\
\partial_t \mathbf{H} &= -\frac{1}{\mu}\operatorname{curl}\mathbf{E} && \text{in } Q \times [0,\infty),\\
\operatorname{div}(\mu\mathbf{H}) &= 0 && \text{in } Q \times [0,\infty),\\
\mathbf{E}\times\nu &= 0, \qquad \mu\mathbf{H}\cdot\nu = 0 && \text{on } \partial Q \times [0,\infty),\\
\mathbf{E}(0) &= \mathbf{E}_0, \qquad \mathbf{H}(0) = \mathbf{H}_0 && \text{in } Q,
\end{aligned}
\tag{2.1}
$$

on a cuboid

$$Q = (a_1^-, a_1^+) \times (a_2^-, a_2^+) \times (a_3^-, a_3^+)$$

with perfectly conducting boundary ∂Q. This system will be referred to as the *original damped Maxwell system*, as we later also introduce an extended Maxwell system, see (3.1). Here, the vector $\mathbf{E}(x,t) \in \mathbb{R}^3$ denotes the electric field, $\mathbf{H}(x,t) \in \mathbb{R}^3$ the magnetic field, $\varepsilon(x) > 0$ the electric permittivity, and $\mu(x) > 0$ the magnetic permeability. For notational convenience, we use the symbol $\tilde{\sigma}$ for the fraction σ/ε throughout this part of the thesis. The function $\sigma(x) > 0$ stands for the conductivity. Furthermore, the vector $\nu \in \mathbb{R}^3$ denotes the unit exterior normal vector at ∂Q.

Certain properties are crucial for the behavior of solutions of (2.1). We here assume

$$\begin{aligned} &\varepsilon, \tilde{\sigma} \in W^{1,\infty}(Q), \qquad && \mu \in W^{1,\infty}(Q) \cap W^{2,3}(Q), \\ &\varepsilon, \mu, \tilde{\sigma} \geq \delta > 0, && \frac{\partial \mu}{\partial \nu} = 0 \text{ on } \partial Q, \end{aligned} \tag{2.2}$$

with a positive number δ. Note that the requirements for μ are slightly stronger, due to technical reasons. More precisely, we consider an extended Maxwell system (3.1) with an additional damping term. In the latter system, we incorporate more derivatives of μ than of the other parameters, see the proof of Lemma 3.14. Let us also comment on the assumptions for $\tilde{\sigma}$. In view of the relation $\sigma = \tilde{\sigma}\varepsilon$, the assumptions in (2.2) on $\tilde{\sigma}$ are satisfied if and only if they are valid for σ. (The number δ may have to be replaced by another positive number in the version of (2.2) for σ.)

To ensure wellposedness of (2.1), requirements on the initial data $(\mathbf{E}_0, \mathbf{H}_0)$ are also essential. These are assumed to belong to the space $H_0(\text{curl}, Q) \times H(\text{curl}, Q)$ with

$$\operatorname{div}(\varepsilon \mathbf{E}_0) \in L^2(Q), \quad \operatorname{div}(\mu \mathbf{H}_0) = 0 \text{ on } Q, \quad \mu \mathbf{H}_0 \cdot \nu = 0 \text{ on } \partial Q.$$

The space $H(\text{curl}, Q)$ hereby consists of functions in $L^2(Q)^3$ whose curl exists in $L^2(Q)^3$, and $H_0(\text{curl}, Q)$ contains all functions that have additionally a vanishing tangential trace, see Section 2.2. In view of (2.1), these conditions are natural to make the posed differential equations meaningful. In this setting, the Maxwell equations (2.1) have a unique classical solution, see Proposition 2.3 in [EiSc18]. More precisely, the system is wellposed in the sense of evolution equations.

Recall also from the Introduction and Section 1.2 that (2.1) is exponentially stable. This means that the energy

$$\mathscr{E}(t) = \frac{1}{2} \int_Q \varepsilon \, |\mathbf{E}(x,t)|^2 + \mu \, |\mathbf{H}(x,t)|^2 \, \mathrm{d}x, \qquad t \geq 0,$$

satisfies the exponential decay requirement

$$\mathscr{E}(t) \leq C \mathrm{e}^{-\beta t} \mathscr{E}(0), \qquad t \geq 0, \tag{2.3}$$

see [NiPi05, Phun00, Elle19]. It is crucial that C and β are two positive numbers, independent of the initial data. Due to the positivity and boundedness assumptions in (2.2) on ε and μ, the energy $\mathscr{E}$ is equivalent to the standard L^2-norm on Q.

Our target are time discrete approximations to (2.1) that preserve the decay property (2.3) uniformly with respect to the step size.

2.2. Basic analytical framework and notation

In this section we mainly repeat analytic concepts and spaces that are important for our arguments, such as the Helmholtz decomposition and the associated domains of the curl and divergence operator.

First, we subdivide the boundary ∂Q of the open cuboid Q into the three parts

$$\Gamma_j := \{x \in \partial Q \mid x_j \in \{a_j^-, a_j^+\}\}, \qquad j \in \{1,2,3\},$$

and introduce the corresponding trace maps $\operatorname{tr}_{\Gamma_j} := \operatorname{tr}|_{\Gamma_j}$ that we need for the definition of the splitting operators.

In this respect, we also want to assign traces to functions which have weak derivatives in only one direction. Hereby we keep to Section 2 of [EiSc18]. Suppose a function $v \in L^2(Q)$ possesses a weak derivative $\partial_1 v \in L^2(Q)$. Then, v has a unique representative $\hat{v}$ in $H^1\Big((a_1^-, a_1^+), L^2((a_2^-, a_2^+) \times (a_3^-, a_3^+))\Big)$, as a consequence of the fundamental theorem for Sobolev functions. Additionally, the H^1-norm of $\hat{v}$ is bounded by a uniform constant times the L^2-norm of v and $\partial_1 v$. This leads to a well-defined continuous extension of the trace operator $\operatorname{tr}_{\Gamma_1} = \operatorname{tr}|_{\Gamma_1}$ from $H^1(Q)$ to all functions in the space $\{f \in L^2(Q) \mid \partial_1 f \in L^2(Q)\}$. We will simply write $v = 0$ on Γ_1 if $\operatorname{tr}_{\Gamma_1} v = 0$, and proceed similar for the remaining faces of Q.

We continue with the domains of the curl and divergence operators that have already been considered in Section 2.1. Define the Banach spaces

$$\begin{aligned}
H(\operatorname{curl}, Q) &:= \{\varphi \in L^2(Q)^3 \mid \operatorname{curl}\varphi \in L^2(Q)^3\}, & \|\varphi\|_{\operatorname{curl}}^2 &:= \|\varphi\|_{L^2}^2 + \|\operatorname{curl}\varphi\|_{L^2}^2\,,\\
H(\operatorname{div}, Q) &:= \{v \in L^2(Q)^3 \mid \operatorname{div} v \in L^2(Q)\}, & \|v\|_{\operatorname{div}}^2 &:= \|v\|_{L^2}^2 + \|\operatorname{div} v\|_{L^2}^2\,.
\end{aligned}$$

The subspaces $H_0(\operatorname{curl}, Q)$ and $H_0(\operatorname{div}, Q)$ are also essential, being the completion of the space of test functions $C_c^\infty(Q)^3$ with respect to the norms $\|\cdot\|_{\operatorname{curl}}$ and $\|\cdot\|_{\operatorname{div}}$, respectively. In this setting, Theorems I.2.4–I.2.6 in [GiRa86] provide the following fact. The space $C^\infty(\overline{Q})^3$ is dense in $H(\operatorname{div}, Q)$, and the normal trace operator $\gamma_n : v \mapsto v \cdot \nu|_{\partial Q}$ extends from $C^\infty(\overline{Q})^3$ in a linear and continuous way to the space $H(\operatorname{div}, Q)$, now mapping into $H^{-1/2}(\partial Q)$. In the following, we will simply write $v \cdot \nu$ instead of $\gamma_n(v)$ for $v \in H(\operatorname{div}, Q)$. As a consequence of the density and extension result, one may transfer Green's formula to $H(\operatorname{div}, Q)$, stating

$$\int_Q v \cdot \nabla\varphi \,\mathrm{d}x + \int_Q (\operatorname{div} v)\varphi \,\mathrm{d}x = \langle v \cdot \nu, \varphi\rangle_{H^{-1/2}(\partial Q)\times H^{1/2}(\partial Q)}$$

for functions $v \in H(\operatorname{div}, Q)$ and $\varphi \in H^1(Q)$. Moreover, the subspace $H_0(\operatorname{div}, Q)$ coincides with the kernel of γ_n on $H(\operatorname{div}, Q)$.

Regarding the curl operator, Theorems I.2.10–I.2.12 in [GiRa86] establish similar results. The space $C^\infty(\overline{Q})^3$ is also dense in $H(\operatorname{curl}, Q)$, and the tangential trace operator $\gamma_t : v \mapsto v \times \nu|_{\partial Q}$ has a unique linear and continuous extension $\gamma_t : H(\mathrm{curl}, Q) \to H^{-1/2}(\partial Q)$ with kernel $H_0(\mathrm{curl}, Q)$. Again, we write only $v \times \nu$ instead of $\gamma_t(v)$ for $v \in H(\mathrm{curl}, Q)$. Here, Green's formula reads

$$\int_Q (\mathrm{curl}\, v) \cdot \varphi \,\mathrm{d}x - \int_Q v \cdot \mathrm{curl}\, \varphi \,\mathrm{d}x = \langle v \times \nu, \varphi \rangle_{H^{-1/2}(\partial Q) \times H^{1/2}(\partial Q)}$$

for mappings $v \in H(\mathrm{curl}, Q)$ and $\varphi \in H^1(Q)^3$. Throughout, we will simply call the application of both Green's formulas integration by parts.

The above domains of the curl and divergence operators contain rather irregular functions, e.g., all compactly supported gradients and curl-fields in L^2, respectively.

Fortunately, one obtains subspaces of H^1 if one intersects some of the above spaces. We first define the *tangential space*

$$\begin{aligned} H_T(\mathrm{curl}, \mathrm{div}, Q) := \{\mathbf{H} \in L^2(Q)^3 \mid \ &\mathrm{curl}\, \mathbf{H} \in L^2(Q)^3,\ \mathrm{div}\, \mathbf{H} \in L^2(Q), \\ &\mathbf{H} \cdot \nu = 0 \text{ on } \partial Q\}. \end{aligned}$$

This space coincides with $H_0(\mathrm{div}, Q) \cap H(\mathrm{curl}, Q)$, and we equip it with the norm

$$\|\mathbf{H}\|_{H_T}^2 := \|\mathrm{curl}\, \mathbf{H}\|_{L^2(Q)^3}^2 + \|\mathrm{div}\, \mathbf{H}\|_{L^2(Q)}^2, \qquad \mathbf{H} \in H_T(\mathrm{curl}, \mathrm{div}, Q).$$

In fact, $H_T(\mathrm{curl}, \mathrm{div}, Q)$ is continuously embedded into $H^1(Q)^3$, which means that there is a constant $C_T > 0$ with

$$\|\mathbf{H}\|_{H^1(Q)^3}^2 \le C_T \|\mathbf{H}\|_{H_T}^2 = C_T(\|\mathrm{curl}\, \mathbf{H}\|_{L^2(Q)^3}^2 + \|\mathrm{div}\, \mathbf{H}\|_{L^2(Q)}^2) \tag{2.4}$$

for all $\mathbf{H} \in H_T(\mathrm{curl}, \mathrm{div}, Q)$, see for example Lemma I.3.6 and Theorem I.3.9 in [GiRa86]. For the analysis of the electric field $\mathbf{E}$, the *normal space*

$$\begin{aligned} H_N(\mathrm{curl}, \mathrm{div}, Q) := \{\mathbf{E} \in L^2(Q)^3 \mid \ &\mathrm{curl}\, \mathbf{E} \in L^2(Q)^3,\ \mathrm{div}\, \mathbf{E} \in L^2(Q), \\ &\mathbf{E} \times \nu = 0 \text{ on } \partial Q\} \end{aligned}$$

is useful. It is equal to the intersection $H(\mathrm{div}, Q) \cap H_0(\mathrm{curl}, Q)$. Thus, the space $H_N(\mathrm{curl}, \mathrm{div}, Q)$ is complete with respect to the natural norm

$$\|\mathbf{E}\|_{H_N}^2 := \|\mathrm{curl}\, \mathbf{E}\|_{L^2(Q)^3}^2 + \|\mathrm{div}\, \mathbf{E}\|_{L^2(Q)}^2, \qquad \mathbf{E} \in H_N(\mathrm{curl}, \mathrm{div}, Q).$$

As for the tangential space, the normal space is continuously embedded into $H^1(Q)^3$, and the analogous estimate

$$\|\mathbf{E}\|_{H^1(Q)^3}^2 \le C_N \|\mathbf{E}\|_{H_N}^2 = C_N(\|\mathrm{curl}\, \mathbf{E}\|_{L^2(Q)^3}^2 + \|\mathrm{div}\, \mathbf{E}\|_{L^2(Q)}^2) \tag{2.5}$$

is satisfied by all vectors $\mathbf{E} \in H_N(\mathrm{curl}, \mathrm{div}, Q)$ with a uniform constant $C_N > 0$, see Lemma I.3.4 and Theorem I.3.7 in [GiRa86]. In Section 9.1 we derive extended versions of the just mentioned embeddings of $H_T(\operatorname{curl}, \operatorname{div}, Q)$ and $H_N(\operatorname{curl}, \operatorname{div}, Q)$ into $H^1(Q)^3$.

The proof of an observability estimate will heavily make use of appropriate Helmholtz decompositions. One is provided by Theorem I.3.6 and Corollary I.3.4 of [GiRa86].

Theorem 2.1 (Helmholtz decomposition). *Each function $g \in L^2(Q)^3$ may be orthogonally decomposed into the sum*

$$g = \operatorname{curl} \phi + \nabla q, \tag{2.6}$$

where $q \in H^1(Q)$ is unique up to a constant, and $\phi \in H^1(Q)^3$ satisfies $\operatorname{div} \phi = 0$, $\operatorname{curl} \phi \in H_0(\operatorname{div}, Q)$ *and $\phi \times \nu = 0$ on ∂Q.*

Remark 2.2. We denote the resulting orthogonal projection onto the curl-free part of each vector according to the Helmholtz decomposition by p_∇. By p_{curl} we mean the projection onto the divergence-free part. In particular, the boundary condition

$$(p_{\mathrm{curl}}\, g) \cdot \nu = 0 \qquad \text{on } \partial Q,$$

for $g \in L^2(Q)^3$, will be used several times for integration by parts. ♢

Since operator theory accompanies us throughout our arguments, we shortly introduce the main notation as well as the concepts of extrapolation theory. Let $(E, \|\cdot\|)$ be a normed vector space. The symbol $\mathscr{B}(E)$ stands for the space of bounded linear operators on E, and the corresponding operator norm is $\|\cdot\|_{\mathscr{B}(E)}$. Let A be a linear operator on E with domain $\mathcal{D}(A)$. The graph norm of A is given by $\|x\|_{\mathcal{D}(A)}^2 := \|x\|^2 + \|Ax\|^2$ for $x \in \mathcal{D}(A)$.

Another important term is the part of an operator with respect to a linear subspace. Let $Y \subseteq E$ be a linear subspace. The part of A in Y is denoted by A_Y, has the domain

$$\mathcal{D}(A_Y) := \{y \in Y \mid y \in \mathcal{D}(A),\ Ay \in Y\},$$

and is defined via $A_Y y := Ay$ for $y \in \mathcal{D}(A_Y)$. In this context, the notation A_Y^k refers to the domain of the power of A_Y, being defined on the domain

$$\mathcal{D}(A_Y^k) := \{y \in \mathcal{D}(A_Y^{k-1}) \mid A_Y^{k-1} y \in \mathcal{D}(A_Y)\}$$

for $k \in \mathbb{N}_{\geq 2}$.

For the error analysis of the arising ADI schemes, we need classical extrapolation spaces of first order. These are introduced in the following, keeping to Section V.1.3 in [Aman95] and Section 2.10 in [TuWe09]. For this purpose, let $(E, \|\cdot\|)$ be additionally a Banach space, and let A be densely defined and closed with nonempty resolvent set $\rho(A)$. Take $\lambda \in \rho(A)$, and define the norm $\|\cdot\|_{-1,A} := \|(\lambda - A)^{-1} \cdot\|$

on E, as well as the completion E_{-1}^J of E with respect to the new norm. We call E_{-1}^A the first order extrapolation space of E with respect to A. By construction, the operator $A : (\mathcal{D}(A), \|\cdot\|) \to (E, \|\cdot\|_{-1})$ is bounded, and it may be extended in a unique continuous manner to an extrapolation operator $A_{-1} : E \to E_{-1}$. Additionally we define $E_1 := \mathcal{D}(A)$, and we equip it with the standard graph norm. Then E_1 is again a Banach space, and the part of A in E_1 is denoted by A_1. These constructions may be iterated, yielding higher order extrapolation spaces. It is essential that the definition of the above extrapolation spaces does not depend on the chosen resolvent value λ.

As a special class of extrapolation spaces, we also use fractional extrapolation spaces in some regularity proofs. They can be defined in the following way, see Section V.1.3–1.4 in [Aman95]. Assume additionally that $(E, \|\cdot\|)$ is a Hilbert space, and that A is self-adjoint and positive definite on E. Then, the fractional powers A^α are for $\alpha \in \mathbb{R}$ well-defined, and we put

$$E_\alpha^A := (\mathcal{D}(A^\alpha), \|A^\alpha \cdot\|)$$

for $\alpha > 0$, being again a Banach space. Moreover, we call the part of A in E_α^A by A_α. Now, let $\alpha \in (0, 1]$. The completion of E with respect to the norm $\|\cdot\|_{-\alpha} := \|A^{-\alpha}\cdot\|$ is denoted by $(E_{-\alpha}^A, \|\cdot\|_{-\alpha})$, and the closure of A in $E_{-\alpha}^A$ is $A_{-\alpha}$. We call $E_{-\alpha}^A$ fractional extrapolation space of E with respect to A. Then Theorem V.1.4.12 in [Aman95] states that the dual space $(E_\alpha^A)^*$ is isometrically isomorphic to $E_{-\alpha}^A$. For $\alpha \neq 0$ being an integer, the fractional extrapolation space coincides with the above classical extrapolation space.

To avoid misunderstandings about the interplay of differential operators and products, we fix the following convention. The application of a differential operator to a product of two functions without parenthesis always implies that the product rule is employed. This means for instance $\partial_x fg = \partial_x(fg) = (\partial_x f)g + (\partial_x g)f$ for functions $f, g \in H^1(\mathbb{R})$.

3. Construction of a uniformly exponentially stable ADI scheme

As announced in the introduction, we do not only add damping to the numerical time integrator to obtain uniformly exponentially stable approximations to (2.1). Instead, we also couple the Maxwell system with a damped one-dimensional differential equation by means of a new artificial variable. The new system is called extended Maxwell system. To keep consistency, we show in Section 3.1 that certain solutions of the new extended system also solve the original Maxwell equations. Afterwards, we derive the desired exponentially stable ADI scheme as a time integrator for the extended system. The actual stability result is then provided in Theorem 3.10.

As the proof of Theorem 3.10 and the error analysis for the new scheme demand for a detailed regularity analysis, we study the splitting operators in detail in Section 3.4.

3.1. An extended Maxwell system

Inspired by the mixed hyperbolic divergence cleaning technique from [DKKM02], we study the system

$$\begin{aligned}
\partial_t \mathbf{E} &= \frac{1}{\varepsilon}\operatorname{curl}\mathbf{H} - \tilde{\sigma}\mathbf{E} && \text{in } Q\times[0,\infty),\\
\partial_t \mathbf{H} &= -\frac{1}{\mu}\operatorname{curl}\mathbf{E} - \nabla\left(\frac{1}{\mu}\Phi\right) && \text{in } Q\times[0,\infty),\\
\partial_t \Phi &= -\frac{1}{\mu^2}\operatorname{div}(\mu\mathbf{H}) - \eta\Phi && \text{in } Q\times[0,\infty),\\
\mathbf{E}\times\nu &= 0, \quad \mu\mathbf{H}\cdot\nu = 0 && \text{on } \partial Q\times[0,\infty),\\
\mathbf{E}(0) &= \mathbf{E}_0, \quad \mathbf{H}(0) = \mathbf{H}_0, \quad \Phi(0) = \Phi_0 && \text{in } Q.
\end{aligned} \tag{3.1}$$

We call it *extended* or *enlarged Maxwell system*. As for the original Maxwell system (2.1), we assume that the parameters ε, μ, and $\tilde{\sigma}$ satisfy (2.2).

One of the major differences between the extended Maxwell system and the original one is the absence of the Gauss law $\operatorname{div}(\mu\mathbf{H}) = 0$. This is an important point for our analysis. On the one hand, numerical approximations for (2.1) usually do not preserve the condition $\operatorname{div}(\mu\mathbf{H}) = 0$, see [EiSc18]. On the other hand, the divergence constraint is essential for the exponential stability of (2.1). To overcome the issue of nonvanishing divergence, the new variable $\Phi = \Phi(x,t) \in \mathbb{R}$ is incorporated in (3.1). It is mainly used to couple the differential equation for the magnetic field in the second line with the damped differential equation in the third one. This leads to a damping of the divergence of the magnetic field, see [DKKM02]. We motivate this effect by means of a formal argument.

Assume that μ and η are constant, and that $(\mathbf{E}, \mathbf{H}, \Phi)$ is a solution of (3.1) which is regular enough to do the following calculations. We first take the divergence of the second line in (3.1), and differentiate once with respect to time. This leads to the identity

$$\partial_t^2 \operatorname{div}(\mu\mathbf{H}) = -\Delta\partial_t\Phi. \tag{3.2}$$

Inserting now the third line in (3.1) as an equation for $\partial_t\Phi$, we infer the formula

$$\partial_t^2 \operatorname{div}(\mu\mathbf{H}) = \tfrac{1}{\mu^2}\Delta \operatorname{div}(\mu\mathbf{H}) + \eta\Delta\Phi.$$

The last summand on the right hand side is next substituted by means of the relation $\partial_t \operatorname{div}(\mu\mathbf{H}) = -\Delta\Phi$. (The latter equation is obtained by applying the divergence operator to the second line in (3.1).) As a result, the divergence of the magnetic field satisfies the damped wave equation

$$\partial_t^2 \operatorname{div}(\mu\mathbf{H}) = \tfrac{1}{\mu^2}\Delta \operatorname{div}(\mu\mathbf{H}) - \eta\partial_t \operatorname{div}(\mu\mathbf{H}).$$

In our analysis, the vector $\nabla(\frac{1}{\mu}\Phi)$ is crucial to establish the observability of time-discrete approximations to (3.1), see the proof of Lemma 4.8 for instance.

The damping in (3.1) is caused by the terms $-\tilde{\sigma}\mathbf{E}$ and $-\eta\Phi$. To ensure that the damping is strong enough, and that the solution of (3.1) is sufficiently regular, we assume in the following that

$$\eta \in W^{1,\infty}(Q) \quad \text{with } \eta \geq \delta. \tag{3.3}$$

It is furthermore intuitive to require the initial data $(\mathbf{E}_0, \mathbf{H}_0, \Phi_0)$ to belong to the space

$$H_N(\operatorname{curl}, \operatorname{div}, Q) \times H_T(\operatorname{curl}, \operatorname{div}, Q) \times H^1(Q),$$

to make all differential expressions in (3.1) meaningful.

Although the above reasoning suggests that the solutions of (2.1) and (3.1) are entirely different, this is in general not true. Indeed, Remark 3.6 explains that the extended system reduces to the original one for physically reasonable settings.

(Otherwise, a time integrator for (3.1) would be of no use for our purposes, as we aim for approximations to (2.1).)

To derive a wellposedness and regularity statement for system (3.1), we write it as an evolution equation on appropriate state spaces. First, we study (3.1) on the space $X_{\text{ext}} = L^2(Q)^7$. This space is equipped with the weighted inner product

$$\left(\begin{pmatrix} \mathbf{E}^1 \\ \mathbf{H}^1 \\ \Phi^1 \end{pmatrix}, \begin{pmatrix} \mathbf{E}^2 \\ \mathbf{H}^2 \\ \Phi^2 \end{pmatrix} \right) := \int_Q (\varepsilon \mathbf{E}^1 \cdot \mathbf{E}^2 + \mu \mathbf{H}^1 \cdot \mathbf{H}^2 + \mu \Phi^1 \Phi^2) \, \mathrm{d}x$$

that induces the norm $\|\cdot\|$ on X_{ext}. In view of (2.2), this norm is equivalent to the standard L^2-norm on Q. In line with the reasoning in Section 2.1, the squared norm $\|\cdot\|^2$ is also called energy. To system (3.1) we associate the extended Maxwell operator

$$\begin{aligned} M_{\text{ext}} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix} &:= \begin{pmatrix} \frac{1}{\varepsilon} \operatorname{curl} \mathbf{H} - \tilde{\sigma} \mathbf{E} \\ -\frac{1}{\mu} \operatorname{curl} \mathbf{E} - \nabla(\frac{1}{\mu}\Phi) \\ -\frac{1}{\mu^2} \operatorname{div}(\mu \mathbf{H}) - \eta \Phi \end{pmatrix}, \\ \mathcal{D}(M_{\text{ext}}) &:= H_0(\operatorname{curl}, Q) \times H_T(\operatorname{curl}, \operatorname{div}, Q) \times H^1(Q). \end{aligned} \tag{3.4}$$

Observe that the definition of M_{ext} incorporates derivatives of the products of $\mu \mathbf{H}$ and $\frac{1}{\mu}\Phi$, while the domain of $\mathcal{D}(M_{\text{ext}})$ prescribes only regularity conditions on the functions $\mathbf{H}$ and Φ alone. This issue is addressed in the next remark. Note that the statement is given in Remark 3.3 of [HoJS15].

Remark 3.1. Combining assumption (2.2) with the product rule $\operatorname{div}(\mu \mathbf{H}) = (\nabla \mu) \cdot \mathbf{H} + \mu \operatorname{div} \mathbf{H}$, the function $\operatorname{div}(\mu \mathbf{H})$ is an element of $L^2(Q)$ if and only if $\operatorname{div} \mathbf{H}$ is. In a similar way, the function $\frac{1}{\mu}\Phi$ belongs to $H^1(Q)$ if and only if Φ does. Analogous statements apply to the boundary conditions of $\mathbf{H}$, as well as to the functions $\varepsilon \mathbf{E}$ and $\mathbf{E}$. These facts will be used later on, without further notice.$\Diamond$

On X_{ext}, the extended system (3.1) can be written as the Cauchy problem

$$\frac{d}{dt} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix} = M_{\text{ext}} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix}, \quad t \geq 0, \qquad \begin{pmatrix} \mathbf{E}(0) \\ \mathbf{H}(0) \\ \Phi(0) \end{pmatrix} = \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \\ \Phi_0 \end{pmatrix} \in \mathcal{D}(M_{\text{ext}}). \tag{3.5}$$

In the following, we focus on the analysis of (3.1).

To derive the wellposedness of (3.5), we will make use of the following density result. The latter is important for us because it enables us to approximate functions in $H(\operatorname{curl}, Q)$ by H^1-regular functions that vanish in normal direction on the boundary ∂Q. Although well known to experts, we give a proof for the sake of a self-contained presentation.

Lemma 3.2. *The space $H_T(\mathrm{curl}, \mathrm{div}, Q)$ is a dense subspace of $H(\mathrm{curl}, Q)$.*

Proof. By Theorem 2.10 in Section I.2 of [GiRa86], the space $C^\infty(\overline{Q})^3$ is dense in $H(\mathrm{curl}, Q)$. It consequently suffices to approximate an arbitrary element φ of $C^\infty(\overline{Q})^3$ by means of a sequence $(\varphi_n)_n$ of smooth functions with $\varphi_n \cdot \nu = 0$ on ∂Q.

For $i \in \{1, 2, 3\}$ and $n \in \mathbb{N}$, let $\chi_n^i : [a_i^-, a_i^+] \to [0, 1]$ be a smooth cut-off function with compact support in $[a_i^- + \frac{1}{2n}, a_i^+ - \frac{1}{2n}]$, and with $\chi_n^i = 1$ on $[a_i^- + \frac{1}{n}, a_i^+ - \frac{1}{n}]$. We denote the i-th component of φ by φ^i. Consider then the mapping

$$\varphi_n(x) := (\chi_n^i(x_i)\varphi^i(x))_{i=1}^3, \qquad x = (x_1, x_2, x_3) \in \overline{Q}.$$

By construction, φ_n is smooth, and satisfies the boundary condition $\varphi_n \cdot \nu = 0$ on ∂Q. By Lebesgue's dominated convergence theorem, the sequence $(\varphi_n)_n$ tends to φ and the vectors

$$\mathrm{curl}\, \varphi_n = \begin{pmatrix} \chi_n^3 \partial_2 \varphi^3 - \chi_n^2 \partial_3 \varphi^2 \\ \chi_n^1 \partial_3 \varphi^1 - \chi_n^3 \partial_1 \varphi^3 \\ \chi_n^2 \partial_1 \varphi^2 - \chi_n^1 \partial_2 \varphi^1 \end{pmatrix}$$

converge to $\mathrm{curl}\, \varphi$ in $L^2(Q)^3$ as $n \to \infty$. □

We can next deduce the wellposedness of the Cauchy problem (3.5) in X_{ext}. Note that we hereby employ arguments from the proof for Proposition 3.1 in [HoJS15]. Although the extended system is designed to incorporate strong damping on the electromagnetic field $(\mathbf{E}, \mathbf{H})$, it is also interesting to study the undamped setting $\tilde{\sigma} = \eta = 0$. Under this conditions, the energy of (2.1) is conserved, which is in line with the behavior of the original undamped Maxwell system (2.1), see Proposition 3.5 in [HoJS15].

Proposition 3.3. *Let ε, and μ satisfy (2.2). The following statements are true.*

a) In the undamped case $\tilde{\sigma} = \eta = 0$, the operator M_{ext} is skewadjoint. It consequently generates a strongly continuous group of isometries on X_{ext}.

b) Let $\tilde{\sigma}, \eta \geq 0$ be contained in $W^{1,\infty}(Q)$. Here M_{ext} is the generator of a contractive strongly continuous semigroup on X_{ext}.

In both cases, the Cauchy problem (3.5) has a unique classical solution $(\boldsymbol{E}, \boldsymbol{H}, \Phi)$ in the space $C([0, \infty), \mathcal{D}(M_{\mathrm{ext}})) \cap C^1([0, \infty), X_{\mathrm{ext}})$.

Proof. It suffices to show the skewadjointness of M_{ext} in the case $\tilde{\sigma} = \eta = 0$ to establish item a). Indeed, Stone's Theorem then provides the remainder of part a). In view of the identity

$$M_{\mathrm{ext}} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon} \mathrm{curl}\, \mathbf{H} \\ -\frac{1}{\mu} \mathrm{curl}\, \mathbf{E} - \nabla(\frac{1}{\mu}\Phi) \\ -\frac{1}{\mu^2} \mathrm{div}(\mu \mathbf{H}) \end{pmatrix} - \begin{pmatrix} \tilde{\sigma}\mathbf{E} \\ 0 \\ \eta\Phi \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix} \in \mathcal{D}(M_{\mathrm{ext}}),$$

perturbation theory for generators of semigroups, see Theorem III.2.7 in [EnNa00], implies part b). The final wellposedness statement is a consequence of standard semigroup theory. So, let $\tilde{\sigma} = \eta = 0$.

1) We first split the extended Maxwell operator into the two parts

$$M_{\text{ext}} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon} \operatorname{curl} \mathbf{H} \\ -\frac{1}{\mu} \operatorname{curl} \mathbf{E} - \frac{1}{\mu} \nabla \Phi \\ -\frac{1}{\mu} \operatorname{div} \mathbf{H} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{\mu^2} (\nabla \mu) \Phi \\ -\frac{1}{\mu^2} (\nabla \mu) \cdot \mathbf{H} \end{pmatrix} =: \tilde{M}_{\text{ext}} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix} + P \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix}$$

for

$$(\mathbf{E}, \mathbf{H}, \Phi) \in \mathcal{D}(\tilde{M}_{\text{ext}}) := \mathcal{D}(M_{\text{ext}}) = H_0(\operatorname{curl}, Q) \times H_T(\operatorname{curl}, \operatorname{div}, Q) \times H^1(Q).$$

The operator P is defined on the entire space $\mathcal{D}(P) := X_{\text{ext}}$. The regularity and positivity assumption (2.2) implies that P is bounded and skewadjoint. With perturbation theory for selfadjoint operators, we only have to show that $\tilde{M}_{\text{ext}}$ is skewadjoint, see Theorem V.4.3 in [Kato95].

2) As the space of test functions $C_c^\infty(Q)^7$ is contained in the domain of $\tilde{M}_{\text{ext}}$, the space $\mathcal{D}(\tilde{M}_{\text{ext}})$ is dense in X_{ext}. The operator $\tilde{M}_{\text{ext}}$ is also closed. To verify this claim, let $((\mathbf{E}^n, \mathbf{H}^n, \Phi^n))_n$ be a sequence in $\mathcal{D}(\tilde{M}_{\text{ext}})$ with

$$(\mathbf{E}^n, \mathbf{H}^n, \Phi^n) \to (\mathbf{E}, \mathbf{H}, \Phi) \quad \text{and} \quad \tilde{M}_{\text{ext}}(\mathbf{E}^n, \mathbf{H}^n, \Phi^n) \to (\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\Phi}), \tag{3.6}$$

as $n \to \infty$, for vectors $(\mathbf{E}, \mathbf{H}, \Phi)$ and $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\Phi})$ in X_{ext}. We now take into account that the operators div and curl are closed on their domains $H(\operatorname{div}, Q)$ and $H(\operatorname{curl}, Q)$, respectively. Recall also that the normal trace operator is bounded from $H(\operatorname{div}, Q)$ into $H^{-1/2}(\partial Q)$. As the requirement (3.6) implies the convergence of $(\operatorname{curl} \mathbf{H}^n)_n$ to $\varepsilon \tilde{\mathbf{E}}$ and of $(-\operatorname{div} \mathbf{H}^n)_n$ to $\mu \tilde{\Phi}$, we consequently infer that the functions $\operatorname{curl} \mathbf{H}$ and $\operatorname{div} \mathbf{H}$ belong to L^2, and that the relations

$$\mathbf{H} \cdot \nu = 0 \text{ on } \partial Q, \quad \tfrac{1}{\varepsilon} \operatorname{curl} \mathbf{H} = \tilde{\mathbf{E}} \quad \text{and} \quad -\tfrac{1}{\mu} \operatorname{div} \mathbf{H} = \tilde{\Phi}$$

are valid.

It remains to consider the second component of $\tilde{M}_{\text{ext}}(\mathbf{E}^n, \mathbf{H}^n, \Phi^n)$. The assumption (2.2) then implies that the sequence $(-\operatorname{curl} \mathbf{E}^n - \nabla \Phi^n)_n$ converges in $L^2(Q)$. The boundary condition $\mathbf{E} \times \nu = 0$ on ∂Q further yields that the vectors $\operatorname{curl} \mathbf{E}^n$ and $\nabla \Phi^n$ are orthogonal to each other in L^2. We hence conclude that both sequences $(\operatorname{curl} \mathbf{E}^n)_n$ and $(\nabla \Phi^n)_n$ converge in $L^2(Q)^3$. As the spaces $H_0(\operatorname{curl}, Q)$ and $H^1(Q)$ are complete, the facts $\mathbf{E} \in H_0(\operatorname{curl}, Q)$ and $\Phi \in H^1(Q)$ follow. This reasoning also yields the formula $-\frac{1}{\mu} \operatorname{curl} \mathbf{E} - \frac{1}{\mu} \nabla \Phi = \tilde{\mathbf{H}}$. As a result, $\tilde{M}_{\text{ext}}$ is closed in X_{ext}.

3) We next show that $\tilde{M}_{\mathrm{ext}}$ is skewsymmetric. To that end, let $(\mathbf{E}^1, \mathbf{H}^1, \Phi^1)$, and $(\mathbf{E}^2, \mathbf{H}^2, \Phi^2)$ in $\mathcal{D}(\tilde{M}_{\mathrm{ext}})$. Applying the boundary conditions $\mathbf{E}^1 \times \nu = \mathbf{E}^2 \times \nu = 0$ and $\mathbf{H}^1 \cdot \nu = \mathbf{H}^2 \cdot \nu = 0$ on ∂Q in an integration by parts, the identities

$$
\begin{aligned}
&\left(\tilde{M}_{\mathrm{ext}} \begin{pmatrix} \mathbf{E}^1 \\ \mathbf{H}^1 \\ \Phi^1 \end{pmatrix}, \begin{pmatrix} \mathbf{E}^2 \\ \mathbf{H}^2 \\ \Phi^2 \end{pmatrix} \right) \\
&\qquad = \int_Q \Big((\operatorname{curl} \mathbf{H}^1) \cdot \mathbf{E}^2 - (\operatorname{curl} \mathbf{E}^1) \cdot \mathbf{H}^2 - (\nabla \Phi^1) \cdot \mathbf{H}^2 - (\operatorname{div} \mathbf{H}^1) \Phi^2 \Big) \,\mathrm{d}x \\
&\qquad = \int_Q \Big(\mathbf{H}^1 \cdot \operatorname{curl} \mathbf{E}^2 - \mathbf{E}^1 \cdot \operatorname{curl} \mathbf{H}^2 + \Phi^1 \operatorname{div} \mathbf{H}^2 + \mathbf{H}^1 \cdot \nabla \Phi^2 \Big) \,\mathrm{d}x \\
&\qquad = - \left(\begin{pmatrix} \mathbf{E}^1 \\ \mathbf{H}^1 \\ \Phi^1 \end{pmatrix}, \tilde{M}_{\mathrm{ext}} \begin{pmatrix} \mathbf{E}^2 \\ \mathbf{H}^2 \\ \Phi^2 \end{pmatrix} \right)
\end{aligned}
$$

are obtained. This shows that $\tilde{M}_{\mathrm{ext}}$ is skewsymmetric.

4) It now suffices to demonstrate that $I \pm \tilde{M}_{\mathrm{ext}}$ has dense range in X_{ext} to conclude the skewadjointness of $\tilde{M}_{\mathrm{ext}}$. We therefore show that the space $C_c^\infty(Q)^7$ is contained in the range of $I \pm \tilde{M}_{\mathrm{ext}}$. Let $(\check{\mathbf{E}}, \check{\mathbf{H}}, \check{\Phi}) \in C_c^\infty(Q)^7$. The desired identity $(I \pm \tilde{M}_{\mathrm{ext}})(\mathbf{E}, \mathbf{H}, \Phi) = (\check{\mathbf{E}}, \check{\mathbf{H}}, \check{\Phi})$ is then equivalent to the system

$$
\mathbf{E} \pm \frac{1}{\varepsilon} \operatorname{curl} \mathbf{H} = \check{\mathbf{E}}, \tag{3.7}
$$

$$
\mathbf{H} \mp \frac{1}{\mu} \operatorname{curl} \mathbf{E} \mp \frac{1}{\mu} \nabla \Phi = \check{\mathbf{H}}, \tag{3.8}
$$

$$
\Phi \mp \frac{1}{\mu} \operatorname{div} \mathbf{H} = \check{\Phi}. \tag{3.9}
$$

Our target is to find a solution $(\mathbf{E}, \mathbf{H}, \Phi)$ of (3.7)–(3.9), which belongs to the domain $\mathcal{D}(\tilde{M}_{\mathrm{ext}})$. We therefore insert the first and third line into the second, and arrive at the formula

$$
\begin{aligned}
\mu \mathbf{H} + \operatorname{curl} \frac{1}{\varepsilon} \operatorname{curl} \mathbf{H} - \nabla \frac{1}{\mu} \operatorname{div} \mathbf{H} &= \mu \check{\mathbf{H}} \pm \operatorname{curl} \check{\mathbf{E}} \pm \nabla \check{\Phi} \\
&=: h \in W^{1,\infty}(Q)^3 \cap C_c(Q)^3.
\end{aligned} \tag{3.10}
$$

It is now convenient to look at the associated weak formulation

$$
\int_Q \mu \mathbf{H} \cdot v + \tfrac{1}{\varepsilon} (\operatorname{curl} \mathbf{H}) \cdot (\operatorname{curl} v) + \tfrac{1}{\mu} (\operatorname{div} \mathbf{H})(\operatorname{div} v) \,\mathrm{d}x = \int_Q h v \,\mathrm{d}x, \tag{3.11}
$$

for $v \in H_T(\operatorname{curl}, \operatorname{div}, Q) = H(\operatorname{curl}, Q) \cap H_0(\operatorname{div}, Q)$. Note now that the left hand side of (3.11) defines a bounded coercive bilinear form on the Banach space

$H_T(\mathrm{curl}, \mathrm{div}, Q)$, while the right hand side is a bounded linear form. Hence, the Lax-Milgram Lemma provides a unique function $\mathbf{H} \in H_T(\mathrm{curl}, \mathrm{div}, Q)$, satisfying (3.11) for all mappings $v \in H_T(\mathrm{curl}, \mathrm{div}, Q)$. In the next two steps, we deduce that $\mathbf{H}$ is also a strong solution of (3.10).

4.i) In a first step, we demonstrate that the function $\frac{1}{\mu}\operatorname{div}\mathbf{H}$ belongs to $H^2(Q)$. To that end, we modify the proof of Theorem 1.1 in [CoDN99]. Let ζ be a function in $H^2(Q)$ with homogeneous Neumann boundary conditions on ∂Q. We choose $v := \nabla\zeta$ in (3.11), and obtain the formula

$$\int_Q \tfrac{1}{\mu}(\operatorname{div}\mathbf{H})(\Delta\zeta)\,\mathrm{d}x = -\int_Q (\operatorname{div} h - \operatorname{div}(\mu\mathbf{H}))\zeta\,\mathrm{d}x.$$

Subtracting the term $\frac{1}{\mu}(\operatorname{div}\mathbf{H})\zeta$ on both sides, the identity

$$\int_Q \tfrac{1}{\mu}(\operatorname{div}\mathbf{H})(\Delta\zeta) - \tfrac{1}{\mu}(\operatorname{div}\mathbf{H})\zeta\,\mathrm{d}x = -\int_Q (\operatorname{div} h - \operatorname{div}(\mu\mathbf{H}) + \tfrac{1}{\mu}\operatorname{div}\mathbf{H})\zeta\,\mathrm{d}x \quad (3.12)$$

follows. To apply elliptic regularity theory, we consider the associated boundary value problem

$$\begin{aligned} -\Delta\hat{u} + \hat{u} &= \operatorname{div} h - \operatorname{div}(\mu\mathbf{H}) + \frac{1}{\mu}\operatorname{div}\mathbf{H} &&\text{in } Q,\\ \frac{\partial\hat{u}}{\partial\nu} &= 0 &&\text{on } \partial Q, \end{aligned}$$

for the Neumann Laplacian on Q. By Theorem 3.2.1.3 in [Gris85], this system has a unique solution $\hat{u} \in H^2(Q)$, satisfying the formula

$$\int_Q \hat{u}\Delta\zeta - \hat{u}\zeta\,\mathrm{d}x = -\int_Q (\operatorname{div} h - \operatorname{div}(\mu\mathbf{H}) + \tfrac{1}{\mu}\operatorname{div}\mathbf{H})\zeta\,\mathrm{d}x, \quad (3.13)$$

after integrating by parts twice. We now subtract (3.12) from (3.13) to conclude the fact

$$\langle \hat{u} - \tfrac{1}{\mu}\operatorname{div}\mathbf{H}, \Delta\zeta - \zeta\rangle_{L^2} = 0.$$

Since the operator $\Delta - I$ is invertible on $L^2(Q)$ with domain $\{\Phi \in H^2(Q) \mid \frac{\partial\Phi}{\partial\nu} = 0 \text{ on } \partial Q\}$, see Theorem 3.2.1.3 in [Gris85], the statements $\frac{1}{\mu}\operatorname{div}\mathbf{H} = \hat{u} \in H^2(Q)$ and $\frac{\partial}{\partial\nu}(\frac{1}{\mu}\operatorname{div}\mathbf{H}) = 0$ on ∂Q are valid.

4.ii) We next prove that the vector $\frac{1}{\varepsilon}\operatorname{curl}\mathbf{H}$ is an element of $H_0(\mathrm{curl}, Q)$. To do so, we subtract the first and the last summand on the left hand side of (3.11). Part 4.i) shows that $\frac{1}{\mu}\operatorname{div}\mathbf{H}$ is H^2-regular, whence an integration by parts argument leads to the relations

$$\int_Q \tfrac{1}{\varepsilon}(\operatorname{curl}\mathbf{H})\cdot(\operatorname{curl} v)\,\mathrm{d}x = \int_Q (h - \mu\mathbf{H})v - \tfrac{1}{\mu}(\operatorname{div}\mathbf{H})\operatorname{div} v\,\mathrm{d}x$$

$$= \int_Q \Big(h - \mu\mathbf{H} + \nabla(\tfrac{1}{\mu}\operatorname{div}\mathbf{H})\Big)\cdot v\,\mathrm{d}x \qquad (3.14)$$

for all vectors $v \in H_T(\mathrm{curl}, \mathrm{div}, Q)$. This implies that $\frac{1}{\varepsilon}\operatorname{curl}\mathbf{H}$ is contained in $H(\mathrm{curl}, Q)$. By Lemma 3.2, the space $H_T(\mathrm{curl}, \mathrm{div}, Q)$ is dense in $H(\mathrm{curl}, Q)$, so that the relation is even valid in $H(\mathrm{curl}, Q)$. Lemma 2.4 in Section I of [GiRa86] now yields that the vector $\frac{1}{\varepsilon}\operatorname{curl}\mathbf{H}$ belongs to $H_0(\mathrm{curl}, Q)$. Integrating now the left hand side of (3.14) by parts, the identity

$$\int_Q (\operatorname{curl}\tfrac{1}{\varepsilon}\operatorname{curl}\mathbf{H})\cdot v\,\mathrm{d}x = \int_Q (h - \mu\mathbf{H} + \nabla(\tfrac{1}{\mu}\operatorname{div}\mathbf{H}))\cdot v\,\mathrm{d}x$$

follows by density for all functions $v \in L^2(Q)^3$. Altogether, $\mathbf{H}$ solves (3.10) in strong form.

4.iii) Put $\mathbf{E} := \check{\mathbf{E}} \mp \frac{1}{\varepsilon}\operatorname{curl}\mathbf{H} \in H_0(\mathrm{curl}, Q)$ and $\Phi := \check{\Phi} \pm \frac{1}{\mu}\operatorname{div}\mathbf{H} \in H^1(Q)$. The results of part 4.ii) then imply that the vector $(\mathbf{E}, \mathbf{H}, \Phi)$ belongs to $\mathcal{D}(\tilde{M}_{\mathrm{ext}})$, and that it solves (3.7)–(3.9). □

3.2. Connection between the original and the damped Maxwell systems

Let $(\mathbf{E}, \mathbf{H}, \Phi)$ be a solution of the extended Maxwell system (3.1). As the extended system involves the new variable Φ, the solution components $(\mathbf{E}, \mathbf{H})$ do in general not solve the original system (2.1). In view of our plan to approximate the solution of (2.1) by means of time-discrete approximations to (3.1), this is not satisfactory. In this Section, we thus determine a subset of initial data $(\mathbf{E}_0, \mathbf{H}_0, \Phi_0)$, for which we can guarantee the following: If $(\mathbf{E}, \mathbf{H}, \Phi)$ solves (3.1) with initial data $(\mathbf{E}_0, \mathbf{H}_0, \Phi_0)$, then $(\mathbf{E}, \mathbf{H})$ satisfies (2.1). We reach this target by means of subspace theory for semigroups.

Define the space

$$\begin{aligned} X_{\mathrm{div}} := \{(\mathbf{E}, \mathbf{H}, \Phi) \in X_{\mathrm{ext}} \mid\ & \operatorname{div}(\mu\mathbf{H}) = 0,\ \mathbf{H}\cdot\nu = 0 \text{ on } \partial Q, \\ & \operatorname{div}\mathbf{E} \in L^2(Q),\ \Phi = 0\}. \end{aligned}$$

It is essential for the below reasoning that the definition of this space contains again the Gauss law of the absence of magnetic monopoles. A similar space is defined in equation (2.4) of [EiSc18]. To ensure that a solution $(\mathbf{E}, \mathbf{H}, \Phi)$ of the extended system (3.1) indeed satisfies this divergence constraint, we study the evolution equation (3.5) on the subspace X_{div} of X_{ext}. This leads to the definition of

$$X_{\mathrm{ext},1} := \mathcal{D}(M_{\mathrm{ext}}) \cap X_{\mathrm{div}}, \qquad (3.15)$$

with the norm

$$\left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix} \right\|_{X_{\mathrm{ext},1}}^2 := \left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix} \right\|_{\mathcal{D}(M_{\mathrm{ext}})}^2 + \|\operatorname{div}(\varepsilon \mathbf{E})\|_{L^2}^2 .$$

In view of the initial conditions for the original Maxwell system (2.1), it is physically reasonable to choose initial data for the extended Maxwell system (3.1) within the space $X_{\mathrm{ext},1}$.

Let us also mention that the space $X_{\mathrm{ext},1}$ is complete. To verify this claim, let $v_n := (\mathbf{E}^n, \mathbf{H}^n, \Phi^n)$ be a Cauchy sequence in $X_{\mathrm{ext},1}$. We recall that the extended Maxwell operator M_{ext} is closed on its domain $\mathcal{D}(M_{\mathrm{ext}})$, and that the divergence operator is closed on $H_0(\operatorname{div}, Q)$. As a result, the domain $\mathcal{D}(M_{\mathrm{ext}})$ is complete with respect to the graph norm of M_{ext}. This implies that $(v_n)_n$ converges in $\mathcal{D}(M_{\mathrm{ext}})$ to a vector $v = (\mathbf{E}, \mathbf{H}, \Phi)$. In particular, $(v_n)_n$ tends in $L^2(Q)^7$ to v. By construction of X_{div}, we also infer that $(v_n)_n$ is a Cauchy sequence in the space $V := H(\operatorname{div}, Q) \times H_0(\operatorname{div}, Q) \times \{0\}$. Since the latter space is complete, embeds into X_{ext}, and limits are unique, we infer that $(v_n)_n$ converges in V to v. This reasoning shows that v is the limit of $(v_n)_n$ in $X_{\mathrm{ext},1}$.

In the remainder of this Section, we analyze the extended Maxwell system (3.1) on the space $X_{\mathrm{ext},1}$. As a first step, the following Lemma states that functions in the space $X_{\mathrm{ext},1}$ are H^1-regular. This fact is used for the regularity analysis of the solutions of the extended Maxwell system.

Lemma 3.4. *Let ε, μ, and $\tilde{\sigma}$ satisfy (2.2). The space $X_{\mathrm{ext},1}$ embeds continuously into $H^1(Q)^7$.*

Proof. We first deal with the desired relation $X_{\mathrm{ext},1} \subseteq H^1(Q)^7$. Let $(\mathbf{E}, \mathbf{H}, \Phi) \in X_{\mathrm{ext},1}$. In view of the definition of $\mathcal{D}(M_{\mathrm{ext}})$ in (3.4), as well as the embedding of the spaces $H_N(\operatorname{curl}, \operatorname{div}, Q)$ and $H_T(\operatorname{curl}, \operatorname{div}, Q)$ into $H^1(Q)^3$, see (2.4) and (2.5), it remains to verify that $\operatorname{div} \mathbf{E}$ is an element of $L^2(Q)$. This, however, is a direct consequence of the reasoning in Remark 3.1 and the precondition $\operatorname{div}(\varepsilon \mathbf{E}) \in L^2(Q)$. Altogether, the space $X_{\mathrm{ext},1}$ is a subspace of $H^1(Q)^7$.

We now deal with the asserted embedding property. Note that (2.2) is in the following applied without further notice. The product rule for the divergence operator and the Gauss law lead to the formulas

$$\operatorname{div} \mathbf{H} = -\frac{1}{\mu}(\nabla \mu) \cdot \mathbf{H}, \qquad \operatorname{div}(\varepsilon \mathbf{E}) = \varepsilon \operatorname{div} \mathbf{E} + (\nabla \varepsilon) \cdot \mathbf{E}.$$

With Young's inequality we then conclude the relations

$$0 = \|\operatorname{div} \mathbf{H}\|_{L^2}^2 - \|\tfrac{1}{\mu}(\nabla \mu) \cdot \mathbf{H}\|_{L^2}^2 \geq \|\operatorname{div} \mathbf{H}\|_{L^2}^2 - \frac{\|\nabla \mu\|_\infty^2}{\delta^2} \|\mathbf{H}\|_{L^2}^2 , \quad (3.16)$$

$$\begin{aligned}\|\operatorname{div}(\varepsilon\mathbf{E})\|_{L^2}^2 &= \|\varepsilon\operatorname{div}\mathbf{E}\|_{L^2}^2 + 2(\varepsilon\operatorname{div}\mathbf{E}, (\nabla\varepsilon)\cdot\mathbf{E})_{L^2} + \|(\nabla\varepsilon)\cdot\mathbf{E}\|_{L^2}^2 \\ &\geq \frac{1}{2}\|\varepsilon\operatorname{div}\mathbf{E}\|_{L^2}^2 - 2\|\nabla\varepsilon\|_{L^\infty}^2\|\mathbf{E}\|_{L^2}^2 .\end{aligned} \tag{3.17}$$

Inequalities (3.16) and (3.17) imply that we can control the L^2-norm of $\operatorname{div}\mathbf{E}$ and $\operatorname{div}\mathbf{H}$ by means of the norm of $(\mathbf{E},\mathbf{H},\Phi)$ in $X_{\mathrm{ext},1}$. With the fact $\Phi = 0$, we furthermore infer the relations

$$\begin{aligned}\left\| M_{\mathrm{ext}}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\\ \Phi\end{pmatrix}\right\|^2 &\geq \|\tfrac{1}{\sqrt{\varepsilon}}\operatorname{curl}\mathbf{H} - \tilde{\sigma}\sqrt{\varepsilon}\mathbf{E}\|_{L^2}^2 + \|\tfrac{1}{\sqrt{\mu}}\operatorname{curl}\mathbf{E}\|_{L^2}^2 \\ &\geq \frac{1}{2\|\mu\|_\infty}\|\operatorname{curl}\mathbf{H}\|_{L^2}^2 - 3\|\tilde{\sigma}\|_\infty^2\|\varepsilon\|_\infty\|\mathbf{E}\|_{L^2}^2 + \frac{1}{\|\mu\|_\infty}\|\operatorname{curl}\mathbf{E}\|_{L^2}^2.\end{aligned}$$

Altogether, we can hence estimate the norms of $\mathbf{E}$ in $H_N(\operatorname{curl},\operatorname{div},Q)$ and of $\mathbf{H}$ in $H_T(\operatorname{curl},\operatorname{div},Q)$ by the norm of $(\mathbf{E},\mathbf{H},\Phi)$ in $X_{\mathrm{ext},1}$. Taking now (2.4) and (2.5) into account, we infer the desired statement. □

To employ subspace theory for semigroups on $X_{\mathrm{ext},1}$, we deal with the part of M_{ext} in $X_{\mathrm{ext},1}$, denoted by $M_{\mathrm{ext},1}$. Parts of operators on subspaces are introduced in Section 2.2. The domain $\mathcal{D}(M_{\mathrm{ext},1})$ then satisfies the equation

$$\mathcal{D}(M_{\mathrm{ext},1}) = \mathcal{D}(M_{\mathrm{ext}}^2)\cap X_{\mathrm{ext},1}.$$

Indeed, the definition of $X_{\mathrm{ext},1}$ justifies the inclusion from left to right. For the reverse inclusion, let $(\mathbf{E},\mathbf{H},\Phi)\in\mathcal{D}(M_{\mathrm{ext}}^2)\cap X_{\mathrm{ext},1}$, and set $(\check{\mathbf{E}},\check{\mathbf{H}},\check{\Phi}) := M_{\mathrm{ext}}(\mathbf{E},\mathbf{H},\Phi)$. We next combine the identity $X_{\mathrm{ext},1} = \mathcal{D}(M_{\mathrm{ext}})\cap X_{\mathrm{div}}$ with the fact that $(\check{\mathbf{E}},\check{\mathbf{H}},\check{\Phi})$ is an element of $\mathcal{D}(M_{\mathrm{ext}})$. Consequently, we only need to show that the vector $(\check{\mathbf{E}},\check{\mathbf{H}},\check{\Phi})$ is an element of X_{div}. For this purpose, we apply the product rule for the divergence operator. It provides the identity

$$\begin{aligned}\operatorname{div}\check{\mathbf{E}} &= \operatorname{div}(\tfrac{1}{\varepsilon}\operatorname{curl}\mathbf{H} - \tilde{\sigma}\mathbf{E}) \\ &= -\frac{1}{\varepsilon^2}(\nabla\varepsilon)\cdot\operatorname{curl}\mathbf{H} - (\nabla\tilde{\sigma})\cdot\mathbf{E} - \tilde{\sigma}\operatorname{div}\mathbf{E},\end{aligned}$$

so that $\operatorname{div}\check{\mathbf{E}}$ is contained in $L^2(Q)$. The function $\check{\mathbf{H}}$ further satisfies the Gauss law $\operatorname{div}(\mu\check{\mathbf{H}}) = 0$, since $\Phi = 0$. The boundary condition $\mu\check{\mathbf{H}}\cdot\nu = 0$ on ∂Q is valid, as the vector $(\check{\mathbf{E}},\check{\mathbf{H}},\check{\Phi})$ is contained in $\mathcal{D}(M_{\mathrm{ext}})$. The identity $\check{\Phi} = \frac{1}{\mu^2}\operatorname{div}(\mu\mathbf{H}) - \eta\Phi = 0$ is finally true because $(\mathbf{E},\mathbf{H},\Phi)$ is an element of $X_{\mathrm{ext},1}$. Altogether, $(\check{\mathbf{E}},\check{\mathbf{H}},\check{\Phi})$ is an element of $X_{\mathrm{ext},1}$, and $(\mathbf{E},\mathbf{H},\Phi)$ is contained in $\mathcal{D}(M_{\mathrm{ext},1})$.

Employing arguments from the proof of Proposition 2.3 in [EiSc18], we next deduce the wellposedness of (3.5) as an evolution equation on $X_{\mathrm{ext},1}$. This has two crucial consequences for our subsequent analysis, provided that the initial data for (3.5) is chosen in $X_{\mathrm{ext},1}$.

The first consequence is the H^1-regularity for the solutions to the extended Maxwell system, see Lemma 3.4. The second one is a direct relationship between the solutions of the Maxwell systems (2.1) and (3.1), see Remark 3.6.

Proposition 3.5. *Let $\varepsilon, \mu, \tilde{\sigma}$, and η satisfy (2.2) and (3.3). The part $M_{\mathrm{ext},1}$ of M_{ext} is the generator of a C_0-semigroup $(\mathrm{e}^{tM_{\mathrm{ext},1}})_{t\geq 0}$ on $X_{\mathrm{ext},1}$. This operator family is the restriction of $(\mathrm{e}^{tM_{\mathrm{ext}}})_{t\geq 0}$ to $X_{\mathrm{ext},1}$, and it obeys the bound*

$$\|\mathrm{e}^{tM_{\mathrm{ext},1}}\|_{\mathscr{B}(X_{\mathrm{ext},1})} \leq C_{\mathrm{stab},1}(1+t), \qquad t \geq 0,$$

with a positive number $C_{\mathrm{stab},1}$.

Proof. 1) We restrict the family $(\mathrm{e}^{tM_{\mathrm{ext},1}})_{t\geq 0}$ to the space $X_{\mathrm{ext},1}$, and demonstrate that it is a C_0-semigroup on $X_{\mathrm{ext},1}$. Note first that the semigroup property is immediate by construction. It hence suffices to show that $(\mathrm{e}^{tM_{\mathrm{ext},1}})_{t\geq 0}$ leaves $X_{\mathrm{ext},1}$ invariant, and that it is strongly continuous on $X_{\mathrm{ext},1}$.

Let $(\mathbf{E}_0, \mathbf{H}_0, \Phi_0) \in \mathcal{D}(M_{\mathrm{ext},1})$. We consider the extended Maxwell system (3.1), and denote the solution of the original system (2.1) for initial data $(\mathbf{E}_0, \mathbf{H}_0)$ by $(\mathbf{E}, \mathbf{H})$. The divergence constraint in (2.1) implies the fact $\operatorname{div}(\mu \mathbf{H}(t)) = 0$ for all $t \geq 0$, whence $(\mathbf{E}, \mathbf{H}, 0)$ is the unique classical solution of the extended system (3.1). Proposition 3.3 then yields the identity $(\mathbf{E}(t), \mathbf{H}(t), 0) = \mathrm{e}^{tM_{\mathrm{ext}}}(\mathbf{E}_0, \mathbf{H}_0, \Phi_0)$ for $t \geq 0$. By Proposition 2.3 of [EiSc18] and Remark 3.1, the mapping $\operatorname{div}(\mathbf{E}(t))$ is contained in $L^2(Q)$ for $t \geq 0$. As a result, $(\mathbf{E}(t), \mathbf{H}(t), 0)$ is an element of $X_{\mathrm{ext},1}$, and the family $(\mathrm{e}^{tM_{\mathrm{ext}}})_{t\geq 0}$ leaves $X_{\mathrm{ext},1}$ invariant.

To show the desired strong continuity in $X_{\mathrm{ext},1}$, we note that $(\mathbf{E}(t), \mathbf{H}(t), 0)$ tends to $(\mathbf{E}_0, \mathbf{H}_0, \Phi_0)$ as $t \to 0$ in the topology of $\mathcal{D}(M_{\mathrm{ext}})$, see Proposition 3.3. The statements of Proposition 2.3 in [EiSc18] moreover establish the convergence of $\operatorname{div}(\varepsilon \mathbf{E}(t))$ to $\operatorname{div}(\varepsilon \mathbf{E}_0)$ as $t \to 0$ in $L^2(Q)$. Altogether, the vector $(\mathbf{E}(t), \mathbf{H}(t), 0)$ tends to $(\mathbf{E}_0, \mathbf{H}_0, \Phi_0)$ in $X_{\mathrm{ext},1}$ as $t \to 0$. Combining the above results, we conclude that the family $(\mathrm{e}^{tM_{\mathrm{ext}}}|_{X_{\mathrm{ext},1}})_{t\geq 0}$ is a C_0-semigroup on $X_{\mathrm{ext},1}$, which is generated by $M_{\mathrm{ext},1}$, see Subsection II.2.3 of [EnNa00].

2) To control the norm of the operator $\mathrm{e}^{tM_{\mathrm{ext},1}}$ on $X_{\mathrm{ext},1}$, we use the statements of Proposition 2.3 in [EiSc18] together with the contractivity of $(\mathrm{e}^{tM_{\mathrm{ext}}})_{t\geq 0}$ on X_{ext}. In this way, we derive the estimates

$$\begin{aligned}
\|\operatorname{div}(\varepsilon \mathbf{E}(t))\|_{L^2} &\leq \|\operatorname{div}(\varepsilon \mathbf{E}_0)\|_{L^2} + \tilde{C} t \, \|(\mathbf{E}_0, \mathbf{H}_0, 0)\|_{L^2}, \\
\|(\mathbf{E}(t), \mathbf{H}(t), \Phi_0)\|^2_{\mathcal{D}(M_{\mathrm{ext}})} &= \|(\mathbf{E}_0, \mathbf{H}_0, \Phi_0)\|^2 + \|\mathrm{e}^{tM_{\mathrm{ext}}} M_{\mathrm{ext}}(\mathbf{E}_0, \mathbf{H}_0, \Phi_0)\|^2 \\
&\leq \|(\mathbf{E}_0, \mathbf{H}_0, \Phi_0)\|^2_{\mathcal{D}(M_{\mathrm{ext}})},
\end{aligned}$$

involving a uniform constant $\tilde{C} > 0$. This shows the desired linear growth restriction. □

The following remark is crucial for the error analysis of the desired exponentially stable ADI scheme. It shows that the extended Maxwell system (3.1) reduces to the original system (2.1) for initial data $(\mathbf{E}_0, \mathbf{H}_0, \Phi_0)$ in $X_{\mathrm{ext},1}$.

Remark 3.6. A direct consequence of Proposition 3.5 is the following wellposedness result. Let $(\mathbf{E}_0, \mathbf{H}_0, \Phi_0) \in \mathcal{D}(M_{\mathrm{ext},1})$. System (3.1) has a unique classical solution $(\mathbf{E}, \mathbf{H}, \Phi)$ that is an element of the space $C([0,\infty), \mathcal{D}(M_{\mathrm{ext},1})) \cap C^1([0,\infty), X_{\mathrm{ext},1})$. By Lemma 3.4, the vector $(\mathbf{E}(t), \mathbf{H}(t), \Phi(t))$ is then an element of $H^1(Q)^7$ for $t \geq 0$. Moreover, the reasoning in part 1) of the proof for Proposition 3.5 demonstrates that the mapping $(\mathbf{E}, \mathbf{H})$ is the classical solution of the Maxwell system (2.1). $\Diamond$

3.3. Two splitting schemes for the extended Maxwell system

We now construct two splitting schemes for the time integration of the extended Maxwell system (3.1). Hereby, we follow the procedure in Section 2.2 of [HoJS15] to deal with the common parts in (2.1) and (3.1). We first split the curl-operator into the difference

$$\operatorname{curl} = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} = \mathscr{C}_1 - \mathscr{C}_2,$$

employing the two first-order differential operators

$$\mathscr{C}_1 := \begin{pmatrix} 0 & 0 & \partial_2 \\ \partial_3 & 0 & 0 \\ 0 & \partial_1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathscr{C}_2 := \begin{pmatrix} 0 & \partial_3 & 0 \\ 0 & 0 & \partial_1 \\ \partial_2 & 0 & 0 \end{pmatrix}$$

on their maximal domains

$$\mathcal{D}(\mathscr{C}_j) := \{u \in L^2(Q)^3 \mid \mathscr{C}_j u \in L^2(Q)^3\}, \qquad j \in \{1,2\}.$$

For later symmetry considerations, the integration by parts rule

$$(\mathscr{C}_2 u, v)_{L^2} = -(u, \mathscr{C}_1 v)_{L^2} \tag{3.18}$$

is important. It is valid for functions $u = (u_i)_{i=1}^3 \in \mathcal{D}(\mathscr{C}_2)$ and $v = (v_i)_{i=1}^3 \in \mathcal{D}(\mathscr{C}_1)$ with

$$(\mathrm{tr}_{\Gamma_2} u_1)(\mathrm{tr}_{\Gamma_2} v_3) = 0 = (\mathrm{tr}_{\Gamma_3} u_2)(\mathrm{tr}_{\Gamma_3} v_1) = (\mathrm{tr}_{\Gamma_1} u_3)(\mathrm{tr}_{\Gamma_1} v_2),$$

see Section 4.3 of [HoJS15]. The arising trace condition is meaningful, since both functions u and v have the required partial regularity by definition of $\mathcal{D}(\mathscr{C}_1)$ and $\mathcal{D}(\mathscr{C}_2)$, see also the considerations in Section 2.2.

The part associated to the original Maxwell system (2.1) with $\tilde{\sigma} = 0$ is then treated in the sum

$$\begin{pmatrix} 0 & \frac{1}{\varepsilon}\operatorname{curl} & 0 \\ -\frac{1}{\mu}\operatorname{curl} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = A + B,$$

with the two splitting operators

$$A := \begin{pmatrix} 0 & \frac{1}{\varepsilon}\mathscr{C}_1 & 0 \\ \frac{1}{\mu}\mathscr{C}_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B := \begin{pmatrix} 0 & -\frac{1}{\varepsilon}\mathscr{C}_2 & 0 \\ -\frac{1}{\mu}\mathscr{C}_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.19}$$

These are defined on the domains

$$\begin{aligned} \mathcal{D}(A) := \{(\mathbf{E},\mathbf{H},\Phi) \in X_{\text{ext}} \mid (\mathscr{C}_1\mathbf{H},\mathscr{C}_2\mathbf{E},\Phi) \in X_{\text{ext}},\ &\mathbf{E}_1 = 0 \text{ on } \Gamma_2,\ \mathbf{E}_2 = 0 \text{ on } \Gamma_3, \\ &\mathbf{E}_3 = 0 \text{ on } \Gamma_1\}, \\ \mathcal{D}(B) := \{(\mathbf{E},\mathbf{H},\Phi) \in X_{\text{ext}} \mid (\mathscr{C}_2\mathbf{H},\mathscr{C}_1\mathbf{E},\Phi) \in X_{\text{ext}},\ &\mathbf{E}_1 = 0 \text{ on } \Gamma_3,\ \mathbf{E}_2 = 0 \text{ on } \Gamma_1, \\ &\mathbf{E}_3 = 0 \text{ on } \Gamma_2\}. \end{aligned}$$

By construction of the domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$, only the boundary condition for the electric field is incorporated. Moreover, this boundary condition is distributed onto both domains. The imposed partial regularity hereby ensures that all arising traces are well-defined, see Section 2.2. Note that the boundary condition for the magnetic field is treated below.

We furthermore stress that the operators A and B essentially coincide with their counterparts in [HoJS15]. (In our case, we only add zero entries in the operator matrix to account for an additional variable in the extended space $X_{\text{ext}} = L^2(Q)^7$.) Consequently Lemma 4.3 from [HoJS15] yields the following statement, which is crucial for the unconditional stability of the schemes in the first part of this thesis.

Lemma 3.7. *Let ε and μ satisfy (2.2). The operators A and B are skewadjoint on X_{ext}. This implies that the inverse $(I - \tau L)^{-1}$ is a contraction, and that the Cayley-Transform*

$$S_\tau(L) := (I + \tfrac{\tau}{2}L)(I - \tfrac{\tau}{2}L)^{-1}$$

is an isometry on X_{ext} for all $\tau > 0$ and $L \in \{A, B\}$.

Let $i \in \{1, 2, 3\}$, and let $\mathbf{e}_i \in \mathbb{R}^3$ be the i-th standard unit vector. So far, we only take the parts of the extended Maxwell system (3.1) into account that also arise in the original Maxwell system (2.1). To deal with the ingredients concerning the new artificial variable Φ, the operator

$$D_i \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix} := \begin{pmatrix} 0 \\ -\partial_i(\frac{1}{\mu}\Phi)\mathbf{e}_i \\ -\frac{1}{\mu^2}\partial_i(\mu\mathbf{H}_i) \end{pmatrix} \tag{3.20}$$

with domain

$$\begin{aligned} \mathcal{D}(D_i) := L^2(Q)^3 &\times \{\mathbf{H} \in L^2(Q)^3 \mid \partial_i\mathbf{H}_i \in L^2(Q),\ \mathbf{H}_i = 0 \text{ on } \Gamma_i\} \\ &\times \{\Phi \in L^2(Q) \mid \partial_i\Phi \in L^2(Q)\} \end{aligned}$$

is introduced. Now, the boundary condition for the magnetic field is incorporated. Similar to the definition of the domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$, this boundary condition is distributed onto the domains of the operators D_1, D_2, and D_3.

With the above splitting operators, the extended Maxwell operator M_{ext} from (3.4) is split into the sum

$$\begin{pmatrix} -\tilde{\sigma} & \frac{1}{\varepsilon}\operatorname{curl} & 0 \\ -\frac{1}{\mu}\operatorname{curl} & 0 & -\nabla(\frac{1}{\mu}\cdot) \\ 0 & -\frac{1}{\mu^2}\operatorname{div}(\mu\cdot) & -\eta \end{pmatrix} = \begin{pmatrix} -\tilde{\sigma} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\eta \end{pmatrix} + A + B + D_1 + D_2 + D_3$$

on the intersection

$$\mathcal{D}(A) \cap \mathcal{D}(B) \cap \mathcal{D}(D_1) \cap \mathcal{D}(D_2) \cap \mathcal{D}(D_3) \subseteq \mathcal{D}(M_{\text{ext}}).$$

To formulate the below time integration schemes, we demonstrate in the next result that certain resolvent operators of the splitting operators D_i are well-defined. Furthermore, we provide here the basis for our stability analysis in Section 6.1. The statement is central, and it will frequently be used in our arguments.

Lemma 3.8. *Let $i \in \{1, 2, 3\}$, and let ε and μ satisfy (2.2). The operator D_i is skewadjoint on X_{ext}. This implies that the Cayley-Transform $S_\tau(D_i) := (I + \frac{\tau}{2}D_i)(I - \frac{\tau}{2}D_i)^{-1}$ is well-defined and an isometry on X_{ext} for $\tau > 0$.*

Proof. 1) We only consider the case $i = 1$. All others are covered by similar arguments. Since the domain $\mathcal{D}(D_1)$ contains the space $C_c^\infty(Q)^7$, the operator D_1 is densely defined on X_{ext}.

The operator D_1 is furthermore closed. To verify this claim, let $((\mathbf{E}^n, \mathbf{H}^n, \Phi^n))_n$ be a sequence in $\mathcal{D}(D_1)$ with

$$(\mathbf{E}^n, \mathbf{H}^n, \Phi^n) \to (\mathbf{E}, \mathbf{H}, \Phi), \quad \text{and} \quad D_1(\mathbf{E}^n, \mathbf{H}^n, \Phi^n) \to (\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\Phi}), \quad n \to \infty,$$

for two vectors $(\mathbf{E}, \mathbf{H}, \Phi)$ and $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\Phi})$ in X_{ext}. By definition of D_1 in (3.20), the relations $\tilde{\mathbf{E}} = 0$ and $\tilde{\mathbf{H}}_2 = \tilde{\mathbf{H}}_3 = 0$ are valid. With the product rule, we furthermore infer the convergence statements

$$\mathbf{H}_1^n \to \mathbf{H}_1, \quad \text{and} \quad \frac{1}{\mu^2}\partial_1(\mu \mathbf{H}_1^n) \to -\tilde{\Phi}, \quad n \to \infty.$$

Assumption (2.2) on μ now implies that also the sequence $(\partial_1 \mathbf{H}_1^n)_n$ converges in $L^2(Q)$. As the partial derivative ∂_1 is closed in $L^2(Q)$ with its domain

$$\mathcal{D}_0(\partial_1) = \{\phi \in L^2(Q) \mid \partial_1\phi \in L^2(Q),\ \phi = 0 \text{ on } \Gamma_1\},$$

we conclude that $\mathbf{H}_1$ is an element of $\mathcal{D}_0(\partial_1)$. This reasoning also shows that $\frac{1}{\mu^2}\partial_1(\mu\mathbf{H}_1) = -\tilde{\Phi}$. Similar arguments lead to the facts $\partial_1\Phi \in L^2(Q)$ and $-\partial_1(\frac{1}{\mu}\Phi) = \tilde{\mathbf{H}}_1$. This means that D_1 is closed.

2) We next show that D_1 is skewsymmetric. Let $(\mathbf{E}^1, \mathbf{H}^1, \Phi^1)$ and $(\mathbf{E}^2, \mathbf{H}^2, \Phi^2)$ be elements of $\mathcal{D}(D_1)$. Using the zero boundary condition $\mathbf{H}_1^1 = \mathbf{H}_1^2 = 0$ on Γ_1 in an integration by parts, we arrive at the identities

$$\begin{aligned}
\left(D_1 \begin{pmatrix} \mathbf{E}^1 \\ \mathbf{H}^1 \\ \Phi^1 \end{pmatrix}, \begin{pmatrix} \mathbf{E}^2 \\ \mathbf{H}^2 \\ \Phi^2 \end{pmatrix} \right) &= -\int_Q (\mu\partial_1(\tfrac{1}{\mu}\Phi^1)\mathbf{H}_1^2 + \tfrac{1}{\mu}\partial_1(\mu\mathbf{H}_1^1)\Phi^2)\,\mathrm{d}x \\
&= \int_Q (\tfrac{1}{\mu}\Phi^1\partial_1(\mu\mathbf{H}_1^2) + \mu\mathbf{H}_1^1\partial_1(\tfrac{1}{\mu}\Phi^2))\,\mathrm{d}x \\
&= -\left(\begin{pmatrix} \mathbf{E}^1 \\ \mathbf{H}^1 \\ \Phi^1 \end{pmatrix}, D_1 \begin{pmatrix} \mathbf{E}^2 \\ \mathbf{H}^2 \\ \Phi^2 \end{pmatrix} \right).
\end{aligned}$$

As a result, D_1 is skewsymmetric, and thus also dissipative.

3) To show the skewadjointness relation $D_1^* = -D_1$, it is now sufficient to demonstrate that the adjoint operator D_1^* is extended by $-D_1$. Let $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\Phi}) \in \mathcal{D}(D_1^*)$, and abbreviate $(\hat{\mathbf{E}}, \hat{\mathbf{H}}, \hat{\Phi}) := D_1^*(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\Phi})$. Since D_1^* is the adjoint operator of D_1, the formula

$$\begin{aligned}
\left(\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix}, \begin{pmatrix} \hat{\mathbf{E}} \\ \hat{\mathbf{H}} \\ \hat{\Phi} \end{pmatrix} \right) &= \left(D_1 \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \\ \tilde{\Phi} \end{pmatrix} \right) \\
&= -\int_Q (\mu\partial_1(\tfrac{1}{\mu}\Phi)\tilde{\mathbf{H}}_1 + \tfrac{1}{\mu}\partial_1(\mu\mathbf{H}_1)\tilde{\Phi})\,\mathrm{d}x \qquad (3.21)
\end{aligned}$$

is true for every vector $(\mathbf{E}, \mathbf{H}, \Phi)$ in $\mathcal{D}(D_1)$. In the following, we want to deduce that $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\Phi})$ is also contained in the domain of D_1. Therefore, we first consider the case $\mathbf{E} = \mathbf{H} = 0$. Equation (3.21) then takes the form

$$-\int_Q \mu\partial_1(\tfrac{1}{\mu}\Phi)\tilde{\mathbf{H}}_1\,\mathrm{d}x = \int_Q \mu\Phi\hat{\Phi}\,\mathrm{d}x$$

for every function $\Phi \in L^2(Q)$ with $\partial_1 \Phi \in L^2(Q)$. (The last condition is equivalent to the property $(0,0,\Phi) \in \mathcal{D}(D_1)$.) As a result, the function $\partial_1(\mu\tilde{\mathbf{H}}_1)$ belongs to $L^2(Q)$, and the relations $\frac{1}{\mu^2}\partial_1(\mu\tilde{\mathbf{H}}_1) = \hat{\Phi}$ as well as $\mu\tilde{\mathbf{H}}_1 = 0$ on Γ_1 are valid.

So, it remains to show that $\tilde{\Phi}$ fulfills the conditions for the statement $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\Phi}) \in \mathcal{D}(D_1)$. We hence consider the vector $(\mathbf{E}, \mathbf{H}, \Phi) \in \mathcal{D}(D_1)$ with $\Phi = 0$, $\mathbf{E} = 0$, and $\mathbf{H}_2 = \mathbf{H}_3 = 0$ in (3.21). This time, the equation

$$-\int_Q \tfrac{1}{\mu}\partial_1(\mu\mathbf{H}_1)\tilde{\Phi}\,\mathrm{d}x = \int_Q \mu\mathbf{H}_1\hat{\mathbf{H}}_1\,\mathrm{d}x$$

follows for all $\mathbf{H}_1 \in L^2(Q)$ with $\partial_1\mathbf{H}_1 \in L^2(Q)$ and $\mathbf{H}_1 = 0$ on Γ_1. We thus deduce that $\partial_1(\frac{1}{\mu}\tilde{\Phi})$ is contained in $L^2(Q)$, and that $\hat{\mathbf{H}}_1 = \partial_1(\frac{1}{\mu}\tilde{\Phi})$. The regularity and positivity assumption (2.2) on μ finally yields that $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\Phi})$ is an element of $\mathcal{D}(D_1)$. Altogether, D_1 is skewadjoint. □

As a consequence of Lemmas 3.7–3.8, the right complex half plane is contained in the resolvent sets of the splitting operators A, B, D_1, D_2, and D_3. This enables us to introduce operators that cause an artificial damping effect in our time integration scheme. The damping effect is called artificial, because it is not present in the continuous problem (3.1).

We incorporate the damping in our time integration scheme by means of an operator from Section 2.3 of [ErZu09]. For $\tau > 0$ and $L \in \{A, B, D_1, D_2, D_3\}$, consider the operator $I - \frac{\tau^3}{4}L^2(I - \frac{\tau^2}{4}L^2)^{-1}$. The arising inverse operator is well-defined, since the squared operator L^2 is negative selfadjoint by Lemmas 3.7–3.8. Taking also the identity $I - \frac{\tau^3}{4}L^2(I - \frac{\tau^2}{4}L^2)^{-1} = (I - \frac{\tau^2+\tau^3}{4}L^2)(I - \frac{\tau^2}{4}L^2)^{-1}$ into account, the same results further imply that the inverse

$$V_\tau(L) := \Big(I - \frac{\tau^3}{4}L^2\Big(I - \frac{\tau^2}{4}L^2\Big)^{-1}\Big)^{-1} = \Big(I - \frac{\tau^2}{4}L^2\Big)\Big(I - \frac{\tau^2+\tau^3}{4}L^2\Big)^{-1} \tag{3.22}$$

is a bounded mapping on X_{ext}.

The notation $V_\tau(L)$ here stems from the interpretation of the mapping $\frac{\tau^2}{4}L^2(I - \frac{\tau^2}{4}L^2)^{-1}$ as a viscosity operator in Section 2.3 of [ErZu09]. (The setting in [ErZu09] is different from the current one. There, the operator L is additionally assumed to have a compact resolvent. This makes it possible to work with an orthonormal basis of eigenfunctions.) Note that the operator $V_\tau(L)$ is obtained by applying the implicit Euler scheme with step size $\tau > 0$ to the evolution equation $w' = \frac{\tau^2}{4}L^2(I - \frac{\tau^2}{4}L^2)^{-1}w$, posed on X_{ext}. We deduce in Section 5.1, see formula (5.3), that $V_\tau(L)$ is indeed a damping operator, meaning that its application reduces energy.

The above preparations allow to formulate the following two schemes for the approximation of the extended Maxwell system (3.1). Let $n \in \mathbb{N}_0$, and let $\tau > 0$

denote the fixed time step size. Focus first on the case without damping, where $\tilde{\sigma} = \eta = 0$ and the energy of system (3.1) is conserved. It is then natural to approximate the solution of (3.1) by a scheme that also conserves energy. Starting with initial data $(\mathbf{E}_c^0, \mathbf{H}_c^0, \Phi_c^0)$, the solution $(\mathbf{E}(t), \mathbf{H}(t), \Phi(t))$ of (3.1) at time $t = (n+1)\tau$ is approximated by

$$\begin{pmatrix} \mathbf{E}_c^{n+1} \\ \mathbf{H}_c^{n+1} \\ \Phi_c^{n+1} \end{pmatrix} = S_\tau(D_3)S_\tau(D_2)S_\tau(D_1)S_\tau(B)S_\tau(A) \begin{pmatrix} \mathbf{E}_c^{n} \\ \mathbf{H}_c^{n} \\ \Phi_c^{n} \end{pmatrix}. \tag{3.23}$$

Recall that $S_\tau(L)$ denotes the Cayley-Transform for $L \in \{A, B, D_1, D_2, D_3\}$. We use here the subscript c to stress that the conservative undamped problem is considered. This method is inspired by an energy conserving ADI scheme in [ChLL10]. Scheme (3.23) is in this thesis only employed to derive that the below scheme (3.24) provides uniformly exponentially stable iterates. Applying Lemmas 3.7–3.8 to each Cayley-Transform in (3.23), we immediately conclude that the scheme conserves energy. In particular, the scheme is unconditionally stable on $X_{\text{ext}} = L^2(Q)^7$.

Consider now the extended Maxwell system (3.1) with damping. In other words, $\tilde{\sigma}$ and η are assumed to satisfy the strict positivity and regularity assumptions (2.2) and (3.3). We denote the initial data by $(\mathbf{E}^0, \mathbf{H}^0, \Phi^0)$. The solution $(\mathbf{E}(t), \mathbf{H}(t), \Phi(t))$ of (3.1) at time $t = (n+1)\tau$ is approximated via the method

$$\begin{aligned} \begin{pmatrix} \mathbf{E}^{n+1} \\ \mathbf{H}^{n+1} \\ \Phi^{n+1} \end{pmatrix} &= \begin{pmatrix} \mathrm{e}^{-\tau\tilde{\sigma}} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathrm{e}^{-\tau\eta} \end{pmatrix} \prod_{i=1}^{3} \Big(S_\tau(D_i)V_\tau(D_i) \Big) \\ &\quad \cdot S_\tau(B)V_\tau(B)S_\tau(A)V_\tau(A) \begin{pmatrix} \mathbf{E}^{n} \\ \mathbf{H}^{n} \\ \Phi^{n} \end{pmatrix}. \end{aligned} \tag{3.24}$$

The product sign means here that the arising operators are concatenated such that their indices decrease from left to right. During the error analysis in Chapter 6, we assume that the initial data are chosen exactly. This means that the start values of (3.24) and (3.1) coincide. Note, however, that this restriction on the starting values is not necessary for our first main Theorem 3.10.

The spectral properties of the splitting operators imply that also the scheme (3.24) is unconditionally stable.

Proposition 3.9. *Let $\varepsilon, \mu, \tilde{\sigma}$, and η satisfy (2.2) and (3.3). Then the scheme (3.24) is unconditionally stable on $X_{\text{ext}} = L^2(Q)^7$.*

Proof. Let $L \in \{A, B, D_1, D_2, D_3\}$, $\tau > 0$, and $n \in \mathbb{N}$. By Lemmas 3.7–3.8, the operator $S_\tau(L)$ is contractive on X_{ext}. We next deal with the damping operator

$V_\tau(L)$, and use that L is skewadjoint, see Lemmas 3.7–3.8. Let $y \in X_{\mathrm{ext}}$, and set $x := V_\tau(L)y = (I - \frac{\tau^3}{4}L^2(I - \frac{\tau^2}{4}L^2)^{-1})^{-1}y$. A computation reveals the relations

$$\begin{aligned}
\|y\|^2 &= \left\| \Big(I - \frac{\tau^3}{4}L^2(I - \frac{\tau^2}{4}L^2)^{-1}\Big)x \right\|^2 \\
&= \|x\|^2 - \frac{\tau^3}{2}\Big(L^2(I - \frac{\tau^2}{4}L^2)^{-1}x, x\Big) + \frac{\tau^6}{16}\left\| L^2(I - \frac{\tau^2}{4}L^2)^{-1}x \right\|^2 \\
&= \|x\|^2 + \frac{\tau^3}{2}\Big(L(I + \frac{\tau}{2}L)^{-1}x, L(I + \frac{\tau}{2}L)^{-1}x\Big) + \frac{\tau^6}{16}\left\| L^2(I - \frac{\tau^2}{4}L^2)^{-1}x \right\|^2 \\
&\geq \|x\|^2 = \|V_\tau(L)y\|^2.
\end{aligned}$$

This means that also $V_\tau(L)$ is contractive.

As the functions $\tilde\sigma$ and η are positive, the operator matrix

$$\begin{pmatrix} \mathrm{e}^{-\tau\tilde\sigma} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathrm{e}^{-\tau\eta} \end{pmatrix}$$

has norm one. The above statements now imply that the scheme (3.24) is stable, independent of the step size $\tau > 0$. □

We next present the main result of the first part of this thesis. It states that the time-discrete approximations from (3.24) are uniformly exponentially stable. The statement uses the number

$$\begin{aligned}
\kappa_Y :=& \frac{\|\nabla\varepsilon\|_\infty + 2\|\nabla\mu\|_\infty(1 + \frac{\|\nabla\mu\|_\infty}{\delta})}{2\delta^2} \\
&+ C_S\frac{1+\delta}{\delta^3}(\|\partial^2\mu\|_{L^3} + \tfrac{2}{\delta}\|\nabla\mu\|_\infty^2)\|\mu\|_\infty^{1/2} \geq 0. \qquad (3.25)
\end{aligned}$$

Here, $C_S > 0$ is the Sobolev constant of the embedding $H^1(Q) \hookrightarrow L^6(Q)$, the symbol $\partial^2\mu$ denotes the Hessian of μ, and $\delta > 0$ is the number from (2.2) and (3.3).

Theorem 3.10. *Let $\varepsilon, \mu, \tilde\sigma$, and η satisfy (2.2) and (3.3). Let further $\zeta \in (0,1)$ be fixed, and let $(\boldsymbol{E}^n, \boldsymbol{H}^n, \Phi^n)$ be the iterates of (3.24) for initial data $(\boldsymbol{E}^0, \boldsymbol{H}^0, \Phi^0) \in L^2(Q)^7$. There are numbers K and $\omega > 0$ with*

$$\left\| \begin{pmatrix} \boldsymbol{E}^n \\ \boldsymbol{H}^n \\ \Phi^n \end{pmatrix} \right\|^2 \leq K\mathrm{e}^{-\omega\tau n} \left\| \begin{pmatrix} \boldsymbol{E}^0 \\ \boldsymbol{H}^0 \\ \Phi^0 \end{pmatrix} \right\|^2, \qquad n \in \mathbb{N},$$

for all step sizes $\tau \in (0, \zeta \cdot \min\{\frac{\sqrt{2}}{\kappa_Y}, \frac{1}{2}\}]$. The numbers K and ω depend only on $\varepsilon, \mu, \tilde\sigma, \eta, \zeta$, and Q.

Theorem 3.10 is proved in Chapter 5. The main ingredient of the demonstration is an observability inequality for the conserving scheme (3.23). The observability estimate is derived in Chapter 4. Concerning the statement of Theorem 3.10, two remarks are in order.

Remark 3.11. 1) Let the starting value $(\mathbf{E}^0, \mathbf{H}^0, \Phi^0)$ be contained in $X_{\text{ext},1}$. The error result in Theorem 6.5 yields that the iterates $(\mathbf{E}^n, \mathbf{H}^n, \Phi^n)$ of scheme (3.24) approximate the solution $(\mathbf{E}, \mathbf{H}, 0)$ of the extended Maxwell system (3.1) with initial data $(\mathbf{E}_0, \mathbf{H}_0, \Phi_0) = (\mathbf{E}^0, \mathbf{H}^0, \Phi^0)$. Additionally, Remark 3.6 implies that the first two components $(\mathbf{E}, \mathbf{H})$ solve the original Maxwell system (2.1). Theorem 3.10 moreover yields in this situation that the energy of the iterates $(\mathbf{E}^n, \mathbf{H}^n)$ decays in a uniform exponential way. As a result, Theorem 3.10 represents the time-discrete counterpart to the exponential stability result for (2.1), see [NiPi05, Phun00, Elle19].

2) The upper restriction on the time step size in Theorem 3.10 is due to technical difficulties. The first condition $\tau < \frac{\sqrt{2}}{\kappa_Y}$ arises when we consider the splitting scheme in a subspace Y of $H^1(Q)^7$, see Section 3.4. The second upper bound on τ is used to control boundary terms in a discrete analogue of integration by parts. This technique is applied in Chapter 4 during the proof of the observability estimate for the energy conserving scheme (3.23). ◊

3.4. Regularity theory for the splitting operators

One of the main steps in the demonstration of Theorem 3.10 is an observability estimate for the energy conserving scheme (3.23). The corresponding proof requires H^1-regularity of the iterates of scheme (3.23). To that end, we analyze the splitting operators A, B, D_1, D_2, and D_3 in a subspace of H^1-regular functions. The latter functionspace is used as a state space for (3.23).

Following [EiSc18], the space

$$Y := \{(\mathbf{E}, \mathbf{H}, \Phi) \in H^1(Q)^7 \mid \mathbf{E} \times \nu = 0,\ \mathbf{H} \cdot \nu = 0 \text{ on } \partial Q\} \tag{3.26}$$

is introduced, together with the weighted inner product

$$\left(\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \\ \tilde{\Phi} \end{pmatrix} \right)_Y := \left(\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \\ \tilde{\Phi} \end{pmatrix} \right) + \sum_{j=1}^{3} \int_Q \Big(\varepsilon(\partial_j \mathbf{E}) \cdot (\partial_j \tilde{\mathbf{E}}) + \mu(\partial_j \mathbf{H}) \cdot (\partial_j \tilde{\mathbf{H}}) + \mu(\partial_j \Phi)(\partial_j \tilde{\Phi}) \Big) \, \mathrm{d}x.$$

The latter bilinear form induces the norm $\|\cdot\|_Y$ on Y. The space Y incorporates the perfectly conducting boundary conditions from systems (2.1) and (3.1). It is important that $X_{\mathrm{ext},1}$ is a subspace of Y because Proposition 3.5 then provides solutions of the extended Maxwell system (3.1), which remain in Y. To use Y also as a state space for the numerical schemes (3.23) and (3.24), it is moreover essential that Y is contained in the domains of all splitting operators, see (3.19) and (3.20).

In order to show that the numerical schemes (3.23) and (3.24) indeed provide iterates in Y, we study the parts of the splitting operators in Y. The latter are denoted by A_Y, B_Y, and $D_{i,Y}$ for $i \in \{1,2,3\}$. For the definition of the part of an operator, see Section 2.2. Combining (3.19) and (3.20), the identities

$$\begin{aligned}
\mathcal{D}(A_Y) &:= \{(\mathbf{E},\mathbf{H},\Phi) \in Y \mid \mathscr{C}_1\mathbf{H}, \mathscr{C}_2\mathbf{E} \in H^1(Q)^3,\ (\mathscr{C}_1\mathbf{H}) \times \nu = 0, \\
&\qquad\qquad\qquad\qquad\qquad (\mathscr{C}_2\mathbf{E}) \cdot \nu = 0 \text{ on } \partial Q\}, \\
\mathcal{D}(B_Y) &:= \{(\mathbf{E},\mathbf{H},\Phi) \in Y \mid \mathscr{C}_2\mathbf{H}, \mathscr{C}_1\mathbf{E} \in H^1(Q)^3,\ (\mathscr{C}_2\mathbf{H}) \times \nu = 0, \\
&\qquad\qquad\qquad\qquad\qquad (\mathscr{C}_1\mathbf{E}) \cdot \nu = 0 \text{ on } \partial Q\}, \\
\mathcal{D}(D_{i,Y}) &:= \{(\mathbf{E},\mathbf{H},\Phi) \in Y \mid \partial_i\mathbf{H}_i \in H^1(Q),\ \partial_i\Phi \in H^1(Q), \\
&\qquad\qquad\qquad\qquad\qquad \partial_i\Phi = 0 \text{ on } \Gamma_i\}
\end{aligned} \tag{3.27}$$

immediately follow. Note that we can neglect ε and μ in the above representations, as the parameters satisfy the regularity and positivity assumptions (2.2).

Recall that the operators A and B are obtained by adding zeros into the operator matrix of the respective mappings in [EiSc18]. A similar statement is true for the space Y and the domains $\mathcal{D}(A_Y)$ and $\mathcal{D}(B_Y)$. Consequently, Proposition 3.6 from [EiSc18] provides the following useful result. The statement uses the number κ_Y from (3.25).

Lemma 3.12. *Let ε and μ satisfy (2.2), and let $L \in \{\pm A, \pm B\}$. The following statements are valid for the part L_Y of L in Y.*

a) The interval (κ_Y, ∞) is contained in the resolvent set of L_Y, and the restricted resolvent operator $(I - \tau L)^{-1}|_Y$ coincides with $(I - \tau L_Y)^{-1}$. Furthermore, the inequality

$$\left\|(I - \tau L_Y)^{-1}\right\|_{\mathscr{B}(Y)} \le \frac{1}{1 - \tau\kappa_Y}, \qquad \tau \in (0, 1/\kappa_Y),$$

is valid.

b) The operator L_Y is the generator of a strongly continuous semigroup on Y.

c) There is a constant $\tau_0 \in (0, \frac{1}{2\kappa_Y})$, depending only on κ_Y, with

$$\|S_\tau(L_Y)\|_{\mathscr{B}(Y)} \le \mathrm{e}^{3\kappa_Y\tau}, \qquad \tau \in (0, \tau_0].$$

Here, $S_\tau(L_Y) = (I + \frac{\tau}{2}L_Y)(I - \frac{\tau}{2}L_Y)^{-1}$ denotes the Cayley-Transform of L_Y.

As the schemes (3.23) and (3.24) also involve the splitting operators D_1, D_2, and D_3, it is important for our purposes to have an analogous result for the remaining three operators. To that end, we transfer in the next three lemmas the proofs of Lemmas 3.3–3.5 in [EiSc18] to our operators D_i.

Lemma 3.13. *Let ε and μ satisfy (2.2), and let $i \in \{1,2,3\}$. The operator $D_{i,Y}$ is closed and densely defined in Y.*

Proof. Because $D_{i,Y}$ is the part of the closed operator D_i on Y, it is closed. To show that the domain $\mathcal{D}(D_{i,Y})$ is a dense subset of Y, it remains to approximate a fixed vector $(\mathbf{E}, \mathbf{H}, \Phi) \in Y$ by a sequence in $\mathcal{D}(D_{i,Y})$. We first look for functions $\mathbf{H}_i^n$ in $H^1(Q)$ with $\partial_i \mathbf{H}_i^n \in H^1(Q)$ and $\mathbf{H}_i^n = 0$ on Γ_i, that converge to $\mathbf{H}_i$ in $H^1(Q)$. The existence of such functions follows from the reasoning in part 2 of the proof for Lemma 3.3 in [EiSc18]. Adapting part 3 of the same proof, we receive mappings Φ^n in $H^1(Q)$ with $\partial_i \Phi^n \in H^1(Q)$, $\partial_i \Phi^n = 0$ on Γ_i, and $\Phi^n \to \Phi$ in $H^1(Q)$. In view of formula (3.27), we finally choose $\mathbf{H}_j^n := \mathbf{H}_j$ and $\mathbf{E}^n := \mathbf{E}$ for $n \in \mathbb{N}$ and $j \in \{1,2,3\} \setminus \{i\}$. Taking (2.2) into account, $(\mathbf{E}^n, \mathbf{H}^n, \Phi^n)_n$ is the desired approximating sequence in $\mathcal{D}(D_{i,Y})$. $\square$

The assumption (2.2) for μ is also crucial for our reasoning in the next lemma. The precondition (2.2) here enables us to bound terms involving second derivatives of μ, so that we can derive dissipativity of a perturbation of $D_{i,Y}$. Recall for the statement the number κ_Y from (3.25).

Lemma 3.14. *Let ε and μ satisfy (2.2), and let $i \in \{1,2,3\}$. The operator $\pm D_{i,Y} - \kappa_Y I$ is dissipative in Y.*

Proof. We only consider the operator $D_{i,Y}$ (the mapping $-D_{i,Y}$ can be treated similarly). Let $(\mathbf{E}, \mathbf{H}, \Phi) \in \mathcal{D}(D_{i,Y})$. We first derive an auxiliary equation. Combining the boundary condition $\mathbf{H}_i = 0$ on Γ_i with Lemma 2.1 in [EiSc18], the fact $\partial_j \mathbf{H}_i = 0$ on Γ_i for $j \neq i$ follows. Taking now additionally the boundary condition $\partial_i \Phi = 0$ on Γ_i in an integration by parts into account, we obtain the identities

$$\begin{aligned}\int_Q \Big((\partial_j \partial_i \Phi)(\partial_j \mathbf{H}_i) + (\partial_j \partial_i \mathbf{H}_i)(\partial_j \Phi)\Big)\,\mathrm{d}x &= \int_Q \Big(- (\partial_j \Phi)(\partial_j \partial_i \mathbf{H}_i) + (\partial_j \partial_i \mathbf{H}_i)\partial_j \Phi\Big)\,\mathrm{d}x \\ &= 0 \end{aligned} \tag{3.28}$$

for $j \in \{1,2,3\}$. Recall now that D_i is skewadjoint on X_{ext}, see Lemma 3.8. This gives rise to the formula

$$\left(D_i \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix}, \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix} \right)_Y = -\sum_{j=1}^{3} \int_Q \Big(\mu(\partial_j \partial_i \tfrac{1}{\mu} \Phi) \partial_j \mathbf{H}_i + \mu(\partial_j \tfrac{1}{\mu^2} \partial_i \mu \mathbf{H}_i) \partial_j \Phi \Big)\,\mathrm{d}x$$

$$\begin{aligned} &= -\sum_{j=1}^{3} \int_Q \Big((\partial_j \partial_i \Phi) \partial_j \mathbf{H}_i + (\partial_j \partial_i \mathbf{H}_i) \partial_j \Phi \Big) \, \mathrm{d}x \\ &\quad + \sum_{j=1}^{3} \int_Q \Big(\tfrac{\partial_j \mu}{\mu} (\partial_i \Phi) \partial_j \mathbf{H}_i + \tfrac{\partial_j \mu}{\mu} (\partial_i \mathbf{H}_i) \partial_j \Phi \Big) \, \mathrm{d}x \\ &\quad - \sum_{j=1}^{3} \int_Q \Big(\mu (\partial_j \partial_i \tfrac{1}{\mu}) \Phi \partial_j \mathbf{H}_i + \mu (\partial_j \tfrac{\partial_i \mu}{\mu^2}) \mathbf{H}_i \partial_j \Phi - 2 (\partial_i \mu) \tfrac{\partial_j \mu}{\mu^2} \mathbf{H}_i \partial_j \Phi \Big) \, \mathrm{d}x. \end{aligned} \tag{3.29}$$

Identity (3.28) implies that the first integral on the right hand side of (3.29) is zero. Assumption (2.2) on μ moreover enables us to bound the second integral and the last expression in the third integral by the norm of $(\mathbf{E}, \mathbf{H}, \Phi)$ in Y. We next focus on the two remaining terms in the third integral. Here, Sobolev's embedding $H^1(Q) \hookrightarrow L^6(Q)$, and Hölder's inequality are useful. They lead to the relations

$$\begin{aligned} \Big| \sum_{j=1}^{3} \int_Q \mu (\partial_j \partial_i \tfrac{1}{\mu}) \Phi \partial_j \mathbf{H}_i \, \mathrm{d}x \Big| &\le \Big(\tfrac{1}{\delta^2} \left\| \partial^2 \mu \right\|_{L^3} + \tfrac{2}{\delta^3} \left\| \nabla \mu \right\|_\infty^2 \Big) \left\| \mu \right\|_\infty^{1/2} \left\| \sqrt{\mu} \nabla \mathbf{H}_i \right\|_{L^2} \left\| \Phi \right\|_{L^6} \\ &\le C_S \frac{1+\delta}{2\delta^3} \Big(\left\| \partial^2 \mu \right\|_{L^3} + \tfrac{2}{\delta} \left\| \nabla \mu \right\|_\infty^2 \Big) \left\| \mu \right\|_\infty^{1/2} \left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix} \right\|_Y^2 . \end{aligned}$$

Here, $C_S > 0$ denotes the constant from the applied Sobolev embedding. Recall now that $\partial^2 \mu$ denotes the Hessian of μ. Due to the structural similarity, the integral expression $\sum_{j=1}^{3} \int_Q \mu (\partial_j \frac{\partial_i \mu}{\mu^2}) \mathbf{H}_i \partial_j \Phi \, \mathrm{d}x$ can be handled with the same tools. In view of the choice of κ_Y in (3.25), we have derived the desired relation

$$\left(D_i \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix}, \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix} \right)_Y \le \kappa_Y \left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \Phi \end{pmatrix} \right\|_Y^2 .$$

This means that the difference $D_{i,Y} - \kappa_Y I$ is dissipative on Y. $\square$

Lemma 3.15. *Let ε and μ satisfy (2.2), and let $i \in \{1, 2, 3\}$. The range of the operator $(1 + \kappa_Y) I \pm D_{i,Y}$ is dense in Y.*

Proof. 1) Due to symmetry, it suffices to focus on the operator $(1+\kappa_Y)I - D_{1,Y}$. By Lemma 3.13, it is furthermore enough to demonstrate that the domain $\mathcal{D}(D_{1,Y})$ is contained in the range of the latter mapping. In the following, we use the representation (3.27) for $\mathcal{D}(D_{1,Y})$. Let $(\check{\mathbf{E}}, \check{\mathbf{H}}, \check{\Phi}) \in \mathcal{D}(D_{1,Y})$. We first assume that there is a function $(\mathbf{E}, \mathbf{H}, \Phi) \in \mathcal{D}(D_{1,Y})$ with $((1 + \kappa_Y) I - D_{1,Y})(\mathbf{E}, \mathbf{H}, \Phi) =$

$(\check{\mathbf{E}}, \check{\mathbf{H}}, \check{\Phi})$. It is convenient to write the last relation in detail as the system

$$\begin{aligned} (1+\kappa_Y)\mathbf{E} &= \check{\mathbf{E}}, \\ (1+\kappa_Y)\mathbf{H}_j &= \check{\mathbf{H}}_j \qquad \text{for } j \in \{2,3\}, \\ (1+\kappa_Y)\mathbf{H}_1 + \partial_1(\tfrac{1}{\mu}\Phi) &= \check{\mathbf{H}}_1, \\ (1+\kappa_Y)\Phi + \frac{1}{\mu^2}\partial_1(\mu\mathbf{H}_1) &= \check{\Phi}, \end{aligned} \tag{3.30}$$

see (3.20). The remainder of the proof is concerned with the analysis of (3.30). We derive the existence of a solution, and study its regularity.

To obtain a better formulation, we formally plug the fourth line into the third, obtaining the formula

$$\mathbf{H}_1 - \frac{1}{(1+\kappa_Y)^2}\partial_1(\tfrac{1}{\mu^3}\partial_1\mu\mathbf{H}_1) = \frac{1}{1+\kappa_Y}\check{\mathbf{H}}_1 - \frac{1}{(1+\kappa_Y)^2}\partial_1(\tfrac{1}{\mu}\check{\Phi}) =: h_1, \tag{3.31}$$

being equivalent to the relation

$$\frac{1}{\mu}\mathring{\mathbf{H}}_1 - \frac{1}{(1+\kappa_Y)^2}\partial^2_{1,\mu}\mathring{\mathbf{H}}_1 = h_1. \tag{3.32}$$

Here, the vector $\mathring{\mathbf{H}}_1 := \mu\mathbf{H}_1$, and the operator $\partial^2_{1,\mu} := \partial_1\frac{1}{\mu^3}\partial_1$ are employed. The latter is considered on the domain

$$\mathcal{D}(\partial^2_{1,\mu}) := \{u \in L^2(Q) \mid \partial_1 u, \partial_1^2 u \in L^2(Q),\ u = 0 \text{ on } \Gamma_1\}. \tag{3.33}$$

In view of the definition of h_1 in (3.31), the precondition $(\check{\mathbf{E}}, \check{\mathbf{H}}, \check{\Phi}) \in \mathcal{D}(D_{1,Y})$, and the regularity assumption (2.2) on μ, the function h_1 is an element of $H^1(Q)$. The domain

$$\mathcal{D}(\partial_1) := \{u \in L^2(Q) \mid \partial_1 u \in L^2(Q),\ u = 0 \text{ on } \Gamma_1\}$$

will also be employed.

2) Starting from (3.32), we next determine the solution $(\mathbf{E}, \mathbf{H}, \Phi)$ of (3.30). We therefore associate the left hand side of (3.32) with the operator

$$L_1 w := \frac{1}{\mu}w - \frac{1}{(1+\kappa_Y)^2}\partial^2_{1,\mu}w, \qquad w \in \mathcal{D}(\partial^2_{1,\mu}) =: \mathcal{D}(L_1). \tag{3.34}$$

Following the proof of Lemma 4.3 in [HoJS15], the Lax-Milgram Lemma provides a unique function $w \in \mathcal{D}(L_1)$ with $L_1 w = h_1$. Define then

$$\mathbf{E} = \frac{1}{1+\kappa_Y}\check{\mathbf{E}}, \qquad \mathbf{H}_1 = \frac{1}{\mu}\mathring{\mathbf{H}}_1 := \frac{1}{\mu}w, \qquad \mathbf{H}_j = \frac{1}{1+\kappa_Y}\check{\mathbf{H}}_j \text{ for } j \in \{2,3\},$$

$$\Phi := \frac{1}{1+\kappa_Y}(\check{\Phi} - \tfrac{1}{\mu^2}\partial_1(\mu \mathbf{H}_1)). \tag{3.35}$$

In parts 3)– 6) we demonstrate that the vector $(\mathbf{E}, \mathbf{H}, \Phi)$ indeed belongs to $\mathcal{D}(D_{1,Y})$. This means that $\partial_1 \mathbf{H}_1$ and $\partial_1 \Phi$ have to be elements of $H^1(Q)$, and that $\partial_1 \Phi$ has to be zero on Γ_1. The validity of the third line in (3.30) is finally concluded in part 7). The main idea is to differentiate the formula $L_1 w = h_1$ with respect to the x_2 and x_3 variable, to deduce regularity statements for w. This will then lead to the desired results for $\mathbf{H}_1$ and Φ.

3) Let $k \in \{2,3\}$, and let $\varphi \in H_0^2(Q) = \overline{C_c^\infty(Q)}^{\|\cdot\|_{H^2}}$. For the following reasoning, we note the next facts. The function $\partial_1^2 w$ belongs to $L^2(Q)$ by assumption (2.2). As a result, $\partial_{1,\mu}^2 \partial_k w = \partial_k \partial_{1,\mu}^2 w - \partial_1(\partial_k \frac{1}{\mu^3})\partial_1 w$ is an element of $H^{-1}(Q)$. Because w is contained in $\mathcal{D}(L)$, the distribution $\partial_1 \partial_k w$ is also an element of $H^{-1}(Q)$. The relation $L_1 w = h_1$ and an integration by parts then lead to the equations

$$\begin{aligned}
\langle \tfrac{1}{\mu}\partial_k w - \tfrac{1}{(1+\kappa_Y)^2}\partial_{1,\mu}^2 \partial_k w, \varphi\rangle_{H^{-2}\times H_0^2} &= -\int_Q \Big(w \partial_k(\tfrac{1}{\mu}\varphi) + \tfrac{1}{(1+\kappa_Y)^2}(\partial_1 w)\partial_k(\tfrac{1}{\mu^3}\partial_1\varphi)\Big)\,\mathrm{d}x \\
&= -\int_Q (h_1 \partial_k \varphi + \tfrac{1}{(1+\kappa_Y)^2}(\partial_{1,\mu}^2 w)\partial_k\varphi + w\varphi \partial_k \tfrac{1}{\mu} + \tfrac{1}{(1+\kappa_Y)^2}(\partial_1 w)\partial_k(\tfrac{1}{\mu^3}\partial_1\varphi)\Big)\,\mathrm{d}x \\
&= -\int_Q \Big(h_1 \partial_k\varphi + \tfrac{1}{(1+\kappa_Y)^2}(\partial_1 w)(\partial_1\varphi)\partial_k \tfrac{1}{\mu^3} + w\varphi\partial_k\tfrac{1}{\mu}\Big)\,\mathrm{d}x \\
&= \int_Q (\partial_k h_1)\varphi - (\partial_k \tfrac{1}{\mu}) w\varphi \,\mathrm{d}x + \tfrac{1}{(1+\kappa_Y)^2}\langle \partial_1((\partial_k \tfrac{1}{\mu^3})\partial_1 w), \varphi\rangle_{\mathcal{D}(\partial_1)^*\times\mathcal{D}(\partial_1)}.
\end{aligned}$$

To conclude the same identity in the dual space $\mathcal{D}(\partial_1)^*$ of $\mathcal{D}(\partial_1)$, we use that the space $H_0^2(Q)$ is dense in $\mathcal{D}(\partial_1)$. (Indeed, the space $H_0^2(Q)$ contains the space of test functions $C_c^\infty(Q)$, and the domain $\mathcal{D}(\partial_1)$ is the closure of the same space of test functions with respect to the graph norm of the derivative ∂_1.) By density, we arrive at the formula

$$\begin{aligned}
\frac{1}{\mu}\partial_k w - \frac{1}{(1+\kappa_Y)^2}\partial_{1,\mu}^2\partial_k w &= \partial_k h_1 - (\partial_k \tfrac{1}{\mu})w + \frac{1}{(1+\kappa_Y)^2}\partial_1(\partial_k \tfrac{1}{\mu^3})\partial_1 w \\
&=: \chi(h_1)
\end{aligned} \tag{3.36}$$

in $\mathcal{D}(\partial_1)^*$.

4) To show that the distribution $\partial_1 \partial_k w$ belongs to $L^2(Q)$, we approximate w by regularized functions in part 5).[1] To apply mollifier arguments with respect to the x_2 and x_3 variables, we extend the arising functions in the following to the set

$$\check{Q} := [a_1^-, a_1^+] \times \mathbb{R}^2.$$

[1]The following arguments close a gap in the proof of Lemma 3.5 in [EiSc18].

The mappings h_1 and μ are extended to functions $\tilde{h}_1 \in H^1(\check{Q})$ and $\tilde{\mu} \in W^{1,\infty}(\check{Q})$ by means of Stein's extension operator. Also the operator L_1 from (3.34) is transferred to functions on $\check{Q}$ by

$$\begin{aligned} \tilde{L}_1 g &:= \frac{1}{\tilde{\mu}} g - \frac{1}{(1+\kappa_Y)^2} \partial_1 \frac{1}{\tilde{\mu}^3} \partial_1 g, \\ g \in \mathcal{D}(\tilde{L}_1) &:= \{u \in L^2(\check{Q}) \mid \partial_1 u, \partial_1^2 u \in L^2(\check{Q}),\ u = 0 \text{ on } \{a_1^\pm\} \times \mathbb{R}^2\}. \end{aligned}$$

The Lax-Milgram Lemma now provides a unique map $\tilde{w} \in \mathcal{D}(\tilde{L}_1)$, satisfying the relation $\tilde{L}_1 \tilde{w} = \tilde{h}_1$ on the extended domain $\check{Q}$. As the restriction of $\tilde{w}$ to Q also belongs to the domain $\mathcal{D}(L_1)$ and fulfills the formula $L_1 \tilde{w}|_Q = h_1$, we conclude that $\tilde{w}|_Q$ and w coincide. The mapping $\tilde{w}$ is an extension of w that is used to construct regularized approximations to w. To obtain an analogue of (3.36) on $\check{Q}$, we define the weak derivatives ∂_1 and ∂_k on the domains

$$\begin{aligned} \mathcal{D}(\tilde{\partial}_1) &:= \{u \in L^2(\check{Q}) \mid \partial_1 u \in L^2(\check{Q}),\ u = 0 \text{ on } \{a_1^\pm\} \times \mathbb{R}^2\}, \\ \mathcal{D}(\tilde{\partial}_k) &:= \{u \in L^2(\check{Q}) \mid \partial_k u \in L^2(\check{Q})\}, \qquad k \in \{2,3\}. \end{aligned}$$

Similar calculations as in part 3) now give rise to the formula

$$\begin{aligned} \frac{1}{\tilde{\mu}} \partial_k \tilde{w} - \frac{1}{(1+\kappa_Y)^2} \partial_1 \tfrac{1}{\tilde{\mu}^3} \partial_1 \partial_k \tilde{w} = \partial_k \tilde{h}_1 - (\partial_k \tfrac{1}{\tilde{\mu}}) \tilde{w} + \frac{1}{(1+\kappa_Y)^2} \partial_1 (\partial_k \tfrac{1}{\tilde{\mu}^3}) \partial_1 \tilde{w} \\ =: \chi(\tilde{h}_1) \end{aligned} \tag{3.37}$$

in $\mathcal{D}(\tilde{\partial}_1)^*$.

5) We next construct regularized functions that approximate $\tilde{w}$. This is done by mollifying $\tilde{w}$ with respect to the x_k-variable. Let $\rho_n^k : \mathbb{R} \to [0,1]$ be the smooth standard mollifier with support in $[-\frac{1}{n}, \frac{1}{n}]$ that acts on x_k. We denote the corresponding convolution operator by M_n^k for $n \in \mathbb{N}$. It is given by $M_n^k f := \rho_n^k * f$ for $f \in L^2(\check{Q})$. To extend M_n^k to the dual space $\mathcal{D}(\tilde{\partial}_1)^*$, we also employ the convolution operator with respect to $\rho_n^k(-\cdot)$, which is called M_{-n}^k. Fubini's Theorem then implies the relation

$$\langle M_n^k f, \varphi \rangle_{\mathcal{D}(\tilde{\partial}_1)^* \times \mathcal{D}(\tilde{\partial}_1)} = \langle f, M_{-n}^k \varphi \rangle_{\mathcal{D}(\tilde{\partial}_1)^* \times \mathcal{D}(\tilde{\partial}_1)}, \qquad f, \varphi \in \mathcal{D}(\tilde{\partial}_1),$$

between M_n^k and M_{-n}^k. As M_{-n}^k is the convolution operator for the kernel $\rho_n^k(-\cdot)$ with respect to the x_k-variable, the inclusion $M_{-n}^k(\mathcal{D}(\tilde{\partial}_1)) \subseteq \mathcal{D}(\tilde{\partial}_1)$ is furthermore valid. This reasoning implies that the operator M_n^k can be extended in a continuous way to the space $\mathcal{D}(\tilde{\partial}_1)^*$ via the definition

$$\langle M_n^k f, \varphi \rangle_{\mathcal{D}(\tilde{\partial}_1)^* \times \mathcal{D}(\tilde{\partial}_1)} := \langle f, M_{-n}^k \varphi \rangle_{\mathcal{D}(\tilde{\partial}_1)^* \times \mathcal{D}(\tilde{\partial}_1)}, \qquad f \in \mathcal{D}(\tilde{\partial}_1)^*, \varphi \in \mathcal{D}(\tilde{\partial}_1).$$

The extension obeys the bound $\|M_n^k f\|_{\mathcal{D}(\tilde{\partial}_1)^*} \le \|f\|_{\mathcal{D}(\tilde{\partial}_1)^*}$ for $f \in \mathcal{D}(\tilde{\partial}_1)^*$, compare the proof of Lemma 4.1 in [Spit18]. Standard mollifier theory further establishes the convergence statement

$$\left|\langle M_n^k f - f, \varphi\rangle_{\mathcal{D}(\tilde{\partial}_1)^* \times \mathcal{D}(\tilde{\partial}_1)}\right| = \left|\langle f, M_{-n}^k \varphi - \varphi\rangle_{\mathcal{D}(\tilde{\partial}_1)^* \times \mathcal{D}(\tilde{\partial}_1)}\right| \to 0, \quad n \to \infty, \quad (3.38)$$

for $\varphi \in \mathcal{D}(\tilde{\partial}_1)$. In other words, $M_n^k f$ converges weakly* in $\mathcal{D}(\tilde{\partial}_1)^*$ to f for $f \in \mathcal{D}(\tilde{\partial}_1)^*$.

Define now $\tilde{w}_n := M_n^k \tilde{w}$ for $n \in \mathbb{N}$. Employing that M_n^k is the convolution operator with respect to the mollifier ρ_n^k for the x_k-variable, the mappings $\tilde{w}_n$ and $\partial_k \tilde{w}_n$ are contained in $\mathcal{D}(\tilde{L}_1)$. Mollifier theory further shows that $\tilde{w}_n$ tends to $\tilde{w}$ in $L^2(\check{Q})$ as $n \to \infty$.

6) We will next show that $(\partial_k \tilde{w}_n)_n$ has a weak limit in $\mathcal{D}(\tilde{\partial}_1)$, that coincides with $\partial_k w$. This will establish that $\partial_1 \partial_k w$ belongs to $L^2(Q)$. We first calculate

$$\begin{aligned}
\tilde{L}_1 \partial_k \tilde{w}_n &= \tfrac{1}{\tilde{\mu}} \partial_k M_n^k \tilde{w} - \tfrac{1}{(1+\kappa_Y)^2} \partial_1 \tfrac{1}{\tilde{\mu}^3} \partial_k M_n^k \partial_1 \tilde{w} \\
&= \Big(\tfrac{1}{\tilde{\mu}} \partial_k M_n^k \tilde{w} - M_n^k (\tfrac{1}{\tilde{\mu}} \partial_k \tilde{w})\Big) + M_n^k \Big(\tfrac{1}{\tilde{\mu}} \partial_k \tilde{w} - \tfrac{1}{(1+\kappa_Y)^2} \partial_1 \tfrac{1}{\tilde{\mu}^3} \partial_1 \partial_k \tilde{w}\Big) \\
&\quad + \tfrac{1}{(1+\kappa_Y)^2} \partial_1 \Big(M_n^k (\tfrac{1}{\tilde{\mu}^3} \partial_k \partial_1 \tilde{w}) - \tfrac{1}{\tilde{\mu}^3} \partial_k M_n^k (\partial_1 \tilde{w})\Big) \\
&= e_{1,n} + e_{2,n} + e_{3,n}. \qquad (3.39)
\end{aligned}$$

The summands $e_{1,n}$ and $e_{3,n}$ converge to zero in $\mathcal{D}(\tilde{\partial}_1)^*$ as $n \to \infty$ by Theorem C.14 in [BeSe07]. In consideration of (3.37) and (3.38), the second summand $e_{2,n}$ tends weakly* in $\mathcal{D}(\tilde{\partial}_1)^*$ to $\chi(\tilde{h}_1)$. Altogether, $\tilde{L}_1 \partial_k \tilde{w}_n$ converges weakly* to $\chi(\tilde{h}_1)$ in $\mathcal{D}(\tilde{\partial}_1)^*$.

To deduce a convergence statement for $\partial_k \tilde{w}_n$ from the one for $\tilde{L}_1 \partial_k \tilde{w}_n$, we extrapolate $\tilde{L}_1$. To that end, we use that the operator $\tilde{L}_1$ is selfadjoint and positive definite on $L^2(\check{Q})$. To see this claim, we consider the bilinear form

$$\mathcal{D}(\tilde{\partial}_1)^2 \to \mathbb{R}, \quad (w_1, w_2) \mapsto (\tfrac{1}{\tilde{\mu}} w_1, w_2)_{L^2(\check{Q})} + \frac{1}{(1+\kappa_Y)^2} (\tfrac{1}{\tilde{\mu}^3} \partial_1 w_1, \partial_1 w_2)_{L^2(\check{Q})}.$$

This mapping is closed, symmetric, positive definite and densely defined on $L^2(\check{Q})^2$, so that the claim follows from Theorem VI.2.7 in [Kato95]. Theorem VI.2.23 in [Kato95] also provides the relation $\mathcal{D}(\tilde{\partial}_1) = \mathcal{D}(\tilde{L}_1^{1/2})$.

We then denote the fractional extrapolation space of $L^2(\check{Q})$ with respect to $\tilde{L}_1$ by $L^2(\check{Q})_q$ for $q \in \mathbb{Q} \setminus \{0\}$, see Section 2.2. On $L^2(\check{Q})$, the operator $\tilde{L}_1$ extends to the extrapolation operator $(\tilde{L}_1)_{-1} : L^2(\check{Q}) \to L^2(\check{Q})_{-1}$. The bounded inverse of the operator $(\tilde{L}_1)_{-1}$ is called $(\tilde{L}_1)_{-1}^{-1}$. The relations

$$\mathcal{D}(\tilde{\partial}_1)^* = \mathcal{D}(\tilde{L}_1^{1/2})^* \cong L^2(\check{Q})_{-1/2},$$

see Theorem 1.4.12 in [Aman95], then imply that $(\tilde{L}_1)_{-1}^{-1} : \mathcal{D}(\tilde{\partial}_1)^* \to \mathcal{D}(\tilde{\partial}_1)$ is bounded.

Recall now that the functions $\tilde{L}_1 \partial_k \tilde{w}_n$ tend weakly* to $\chi(\tilde{h}_1)$ in $\mathcal{D}(\tilde{\partial}_1)^*$. As $(\tilde{L}_1)_{-1}^{-1}$ is bounded, the mappings $\partial_k \tilde{w}_n = (\tilde{L}_1)_{-1}^{-1} \tilde{L}_1 \partial_k \tilde{w}_n$ converge weakly in $\mathcal{D}(\tilde{\partial}_1)$ to $(\tilde{L}_1)_{-1}^{-1} \chi(\tilde{h}_1) =: v$. Together with the embedding of $\mathcal{D}(\tilde{\partial}_1)$ into $\mathcal{D}(\tilde{\partial}_k)^*$, this implies weak convergence of $\partial_k \tilde{w}_n$ in $\mathcal{D}(\tilde{\partial}_k)^*$. By definition of $\tilde{w}_n$, however, $(\partial_k \tilde{w}_n)_n$ also has the weak limit $\partial_k \tilde{w}$ in $\mathcal{D}(\tilde{\partial}_k)^*$. By uniqueness, $\partial_k \tilde{w}$ coincides with v and belongs to $\mathcal{D}(\tilde{\partial}_1)$. In other words, $\partial_1 \tilde{w}$ is an element of $H^1(\check{Q})$.

Recalling the choice $\mathbf{H}_1 = \frac{1}{\mu} w = (\frac{1}{\tilde{\mu}} \tilde{w})|_Q$, both functions $\mathbf{H}_1$ and $\partial_1 \mathbf{H}_1$ are contained in $H^1(Q)$, and the boundary condition $\mathbf{H}_1 = 0$ is valid on Γ_1.

7) Note that $\check{\Phi}$ is H^1-regular, as $(\check{\mathbf{E}}, \check{\mathbf{H}}, \check{\Phi})$ belongs to $\mathcal{D}(D_{1,Y})$. The results of step 6) consequently show that the mapping $\Phi = \frac{1}{1+\kappa_Y}(\check{\Phi} - \frac{1}{\mu^2}\partial_1(\mu \mathbf{H}_1))$ belongs to $H^1(Q)$. We next derive the validity of line 3 in system (3.30). Dividing the defining formula for Φ in (3.35) by μ and differentiating with respect to x_1, we conclude the relation

$$\partial_1(\tfrac{1}{\mu}\Phi) = \frac{1}{1+\kappa_Y}(\partial_1(\tfrac{1}{\mu}\check{\Phi}) - \partial_1(\tfrac{1}{\mu^3}\partial_1(\mu \mathbf{H}_1))).$$

Due to the choice of $\mathbf{H}_1$, identity (3.31) is true. We thus arrive at the desired equation

$$\partial_1(\tfrac{1}{\mu}\Phi) = \check{\mathbf{H}}_1 - (1+\kappa_Y)\mathbf{H}_1 \in H^1(Q).$$

As the right hand side vanishes on Γ_1, also the boundary condition $\partial_1(\frac{1}{\mu}\Phi) = 0$ is satisfied on Γ_1. Altogether, the vector $(\mathbf{E}, \mathbf{H}, \Phi)$ is an element of $\mathcal{D}(D_{1,Y})$, and it satisfies the formula $((1+\kappa_Y)I - D_{1,Y})(\mathbf{E}, \mathbf{H}, \Phi) = (\check{\mathbf{E}}, \check{\mathbf{H}}, \check{\Phi})$. □

By means of Lemmas 3.13–3.15, we can now deduce the desired analogue of Lemma 3.12 for the operator $D_{i,Y}$. This statement is used to deduce regularity results for the iterates of the schemes (3.23) and (3.24). It is moreover important for the demonstration of the stability of (3.24) in Y, see Section 6.1. Recall for the statement the number κ_Y from (3.25).

Corollary 3.16. *Let ε and μ satisfy (2.2), and let $i \in \{1,2,3\}$. The following items are true.*

a) The operator $I \pm \tau D_{i,Y} : \mathcal{D}(D_{i,Y}) \to Y$ has a bounded inverse. The latter is equal to the restriction of $(I \pm \tau D_i)^{-1}$ to Y, and it satisfies the estimate

$$\|(I \pm \tau D_{i,Y})^{-1}\|_{\mathscr{B}(Y)} \le \frac{1}{1-\tau\kappa_Y}$$

for $\tau \in (0, \frac{1}{\kappa_Y})$.

b) The Cayley-Transform $S_\tau(D_{i,Y}) = (I + \frac{\tau}{2} D_{i,Y})(I - \frac{\tau}{2} D_{i,Y})^{-1}$ fulfills the inequality

$$\|S_\tau(D_{i,Y})\|_{\mathscr{B}(Y)} \leq \mathrm{e}^{3\kappa_Y \tau}$$

for $\tau \in (0, \tau_0)$. Here, τ_0 is a constant in $(0, \frac{1}{2\kappa_Y})$.

Proof. Combining Lemmas 3.13–3.15 with the reasoning in the proof for Proposition 3.6 in [EiSc18], the asserted statements are obtained. □

To simplify our arguments, we throughout choose the same constant τ_0 for Lemma 3.12 and Corollary 3.16 (this is possible by taking the minimum of the numbers in both statements).

Recall now that we want to deduce that the iterates of (3.24) remain in Y for a starting value in Y. As Y is a subspace of $H^1(Q)^7$, this especially provides H^1-regularity for the numerical approximations. To that end, we next analyze the damping operators from (3.22).

Lemma 3.17. *Let ε and μ satisfy (2.2), let $\tau \in (0, \min\{1, \frac{\sqrt{2}}{\kappa_Y}\})$ for the number κ_Y in (3.25), and let $L \in \{A, B, D_1, D_2, D_3\}$. The operator $V_\tau(L)$ leaves Y invariant.*

Proof. It suffices to consider the case $L = A$, since the other splitting operators can be treated similarly. Combining the identities

$$\begin{aligned} V_\tau(A) &= \Big(I - \tfrac{\tau^3}{4} A^2 (I - \tfrac{\tau^2}{4} A^2)^{-1}\Big)^{-1} = (I - \tfrac{\tau^2}{4} A^2)(I - \tfrac{\tau^2+\tau^3}{4} A^2)^{-1} \\ &= (I + \tfrac{\tau}{2} A)(I + \tfrac{\sqrt{\tau^2+\tau^3}}{2} A)^{-1} (I - \tfrac{\tau}{2} A)(I - \tfrac{\sqrt{\tau^2+\tau^3}}{2} A)^{-1} \end{aligned} \tag{3.40}$$

on X_{ext} with Lemma 3.12, the inclusion $V_\tau(A)(Y) \subseteq Y$ directly follows. □

The above reasoning now leads to the desired regularity statements for the iterates of the schemes (3.23) and (3.24).

Corollary 3.18. *Let ε and μ satisfy (2.2), and let $\tau \in (0, \min\{1, \frac{\sqrt{2}}{\kappa_Y}\})$ be the step size of (3.23) and (3.24). The following statements are valid.*

a) Let the initial data $(\boldsymbol{E}_c^0, \boldsymbol{H}_c^0, \Phi_c^0)$ for (3.23) belong to Y. Then all iterates $(\boldsymbol{E}_c^n, \boldsymbol{H}_c^n, \Phi_c^n)$, $n \in \mathbb{N}$, are elements of Y.

b) Let additionally $\tilde{\sigma}$ and η satisfy (2.2) and (3.3), and let the starting value $(\boldsymbol{E}^0, \boldsymbol{H}^0, \Phi^0)$ for (3.24) be contained in Y. For every $n \in \mathbb{N}$, the iterate $(\boldsymbol{E}^n, \boldsymbol{H}^n, \Phi^n)$ then belongs to Y.

Proof. The statements of Lemma 3.12 and Corollary 3.16 imply a). For b), we note that the operator matrix $\begin{pmatrix} \mathrm{e}^{-\tau\tilde{\sigma}} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathrm{e}^{-\tau\eta} \end{pmatrix}$ leaves Y invariant. (This is a consequence of (2.2) and (3.3).) Using also Lemmas 3.12 and 3.17 as well as Corollary 3.16, we derive b). □

Recall that the implicit parts of other ADI schemes for Maxwell equations require only the solution of one-dimensional elliptic problems, see [Nami00, ZhCZ00, HoJS15, EiSc18, EiJS19, HoKö19]. We next show a similar statement for the scheme (3.24).

Remark 3.19. Let $\varepsilon, \mu, \tilde{\sigma}$, and η satisfy (2.2) and (3.3). We here deduce that the implicit parts of scheme (3.24) can be formulated in a way, in which only one-dimensional elliptic problems have to be solved. Transferring identity (3.40) to all splitting operators, we infer the representation

$$\begin{aligned}\begin{pmatrix}\mathbf{E}^{n+1}\\ \mathbf{H}^{n+1}\\ \Phi^{n+1}\end{pmatrix} &= \begin{pmatrix}\mathrm{e}^{-\tau\tilde{\sigma}} & 0 & 0\\ 0 & I & 0\\ 0 & 0 & \mathrm{e}^{-\tau\eta}\end{pmatrix}\prod_{i=1}^{3}\Big((I+\tfrac{\tau}{2}D_i)^2(I+\tfrac{\sqrt{\tau^2+\tau^3}}{2}D_i)^{-1}(I-\tfrac{\sqrt{\tau^2+\tau^3}}{2}D_i)^{-1}\Big)\\ &\quad\cdot(I+\tfrac{\tau}{2}B)^2(I+\tfrac{\sqrt{\tau^2+\tau^3}}{2}B)^{-1}(I-\tfrac{\sqrt{\tau^2+\tau^3}}{2}B)^{-1}(I+\tfrac{\tau}{2}A)^2(I+\tfrac{\sqrt{\tau^2+\tau^3}}{2}A)^{-1}\\ &\quad\cdot(I-\tfrac{\sqrt{\tau^2+\tau^3}}{2}A)^{-1}\begin{pmatrix}\mathbf{E}^{n}\\ \mathbf{H}^{n}\\ \Phi^{n}\end{pmatrix},\qquad n\in\mathbb{N},\end{aligned}$$

of scheme (3.24). The main effort in the evaluation of (3.24) results from the implicit steps. Since it is well known that the application of the resolvents of A and B corresponds to the solution of essentially one-dimensional elliptic problems, see [Nami00, ZhCZ00, HoJS15, EiSc18], we only deal with the operator $(I+\lambda D_1)^{-1}$ for $\lambda \in (-\frac{1}{\kappa_Y}, \frac{1}{\kappa_Y})$. The resolvent operators for D_2 and D_3 are covered by the same arguments.

Similar to [EiSc18], we choose the initial data $(\mathbf{E}^0, \mathbf{H}^0, \Phi^0)$ in Y to apply the above regularity statements. Indeed, Lemmas 3.12 and 3.17, and Corollaries 3.16 and 3.18 yield the following fact. Every iterate $(\mathbf{E}^n, \mathbf{H}^n, \Phi^n)$ of (3.24) is an element of Y, and it suffices to analyze the case where the resolvent operator $(I+\lambda D_1)^{-1}$ is applied to a vector $(\mathbf{E}, \mathbf{H}, \Phi) \in Y$. Let $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\Phi}) := (I+\lambda D_1)^{-1}(\mathbf{E}, \mathbf{H}, \Phi) \in \mathcal{D}(D_{1,Y})$. By definition of D_1, this relation is equivalent to the system

$$\begin{aligned}\tilde{\mathbf{E}} &= \mathbf{E},\\ \tilde{\mathbf{H}}_j &= \mathbf{H}_j \qquad \text{for } j\in\{2,3\},\\ \tilde{\mathbf{H}}_1 - \lambda\partial_1(\tfrac{1}{\mu}\tilde{\Phi}) &= \mathbf{H}_1,\\ \tilde{\Phi} - \frac{\lambda}{\mu^2}\partial_1(\mu\tilde{\mathbf{H}}_1) &= \Phi.\end{aligned}\tag{3.41}$$

In other words, the application of the resolvent operator $(I+\lambda D_1)^{-1}$ amounts to the task to solve the above system (3.41) of partial differential equations. To insert the last identity in (3.41) into the third, we recall that $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\Phi})$ is contained in $\mathcal{D}(D_{1,Y})$, see (3.27). In this way, the formula

$$\tilde{\mathbf{H}}_1 - \lambda^2 \partial_1(\tfrac{1}{\mu^3} \partial_1(\mu \tilde{\mathbf{H}}_1)) = \mathbf{H}_1 + \lambda \partial_1(\tfrac{1}{\mu} \Phi)$$

follows. As a result, a one-dimensional elliptic problem has to be solved for the application of $(I + \lambda D_1)^{-1}$. All other computations to solve (3.41) can then be done explicit. $\Diamond$

The reasoning in Remark 3.19 also shows that the implicit steps in scheme (3.23) only require the solution of one-dimensional elliptic problems.

4. A uniform observability inequality

In this chapter, we derive an internal observability estimate for the energy conserving scheme (3.23), see Theorem 4.2. The observability estimate is the central tool in the proof of the desired exponential stability result in Theorem 3.10.

To reach this goal, we transfer ideas in [NiPi05] from the continuous setting to the time-discrete one. This is done with a discrete version of the multiplier method, proposed in [Komo94] for boundary controllability of the continuous Maxwell equations.

We start by introducing a substep formalism for (3.23), and we derive useful difference equations. The latter formulas correspond to perturbed discrete versions of the extended Maxwell system (3.1) with $\tilde{\sigma} = \eta = 0$. The proof of the observability estimate is distributed onto Sections 4.2–4.4.

4.1. Difference equations for the conserving scheme

Recall the assumptions (2.2) for ε and μ. Let $n \in \mathbb{N}_0$, $i \in \{1, 2, 3\}$, and fix a time step size $\tau \in (0, \min\{\frac{1}{2}, \frac{\sqrt{2}}{\kappa_Y}\})$ for the energy conserving scheme (3.23). The number κ_Y is given in (3.25). Throughout this chapter, we assume that the initial data $(\mathbf{E}_c^0, \mathbf{H}_c^0, \Phi_c^0)$ for (3.23) belong to the space Y from (3.26). This assumption allows to apply our findings from Section 3.4.

We first introduce a substep formalism for (3.23) by

$$\begin{aligned}
\begin{pmatrix} \mathbf{E}_c^{n,1} \\ \mathbf{H}_c^{n,1} \\ \Phi_c^{n,1} \end{pmatrix} &:= (I - \tfrac{\tau}{2}A)^{-1} \begin{pmatrix} \mathbf{E}_c^{n} \\ \mathbf{H}_c^{n} \\ \Phi_c^{n} \end{pmatrix}; &\qquad \begin{pmatrix} \mathbf{E}_c^{n,2} \\ \mathbf{H}_c^{n,2} \\ \Phi_c^{n,2} \end{pmatrix} &:= (I + \tfrac{\tau}{2}A) \begin{pmatrix} \mathbf{E}_c^{n,1} \\ \mathbf{H}_c^{n,1} \\ \Phi_c^{n,1} \end{pmatrix}; \\
\begin{pmatrix} \mathbf{E}_c^{n,3} \\ \mathbf{H}_c^{n,3} \\ \Phi_c^{n,3} \end{pmatrix} &:= (I - \tfrac{\tau}{2}B)^{-1} \begin{pmatrix} \mathbf{E}_c^{n,2} \\ \mathbf{H}_c^{n,2} \\ \Phi_c^{n,2} \end{pmatrix}; &\qquad \begin{pmatrix} \mathbf{E}_c^{n,4} \\ \mathbf{H}_c^{n,4} \\ \Phi_c^{n,4} \end{pmatrix} &:= (I + \tfrac{\tau}{2}B) \begin{pmatrix} \mathbf{E}_c^{n,3} \\ \mathbf{H}_c^{n,3} \\ \Phi_c^{n,3} \end{pmatrix}; \\
\begin{pmatrix} \mathbf{E}_c^{n,3+2i} \\ \mathbf{H}_c^{n,3+2i} \\ \Phi_c^{n,3+2i} \end{pmatrix} &:= (I - \tfrac{\tau}{2}D_i)^{-1} \begin{pmatrix} \mathbf{E}_c^{n,2+2i} \\ \mathbf{H}_c^{n,2+2i} \\ \Phi_c^{n,2+2i} \end{pmatrix}; &\qquad \begin{pmatrix} \mathbf{E}_c^{n,4+2i} \\ \mathbf{H}_c^{n,4+2i} \\ \Phi_c^{n,4+2i} \end{pmatrix} &:= (I + \tfrac{\tau}{2}D_i) \begin{pmatrix} \mathbf{E}_c^{n,3+2i} \\ \mathbf{H}_c^{n,3+2i} \\ \Phi_c^{n,3+2i} \end{pmatrix}.
\end{aligned} \tag{4.1}$$

We note that the last substep $(\mathbf{E}_c^{n,10}, \mathbf{H}_c^{n,10}, \Phi_c^{n,10})$ coincides with the next iterate $(\mathbf{E}_c^{n+1}, \mathbf{H}_c^{n+1}, \Phi_c^{n+1})$. The substeps are useful for regularity considerations and relations between the iterates $(\mathbf{E}_c^n, \mathbf{H}_c^n, \Phi_c^n)$ and $(\mathbf{E}_c^{n+1}, \mathbf{H}_c^{n+1}, \Phi_c^{n+1})$. The next remark lists important regularity statements for the above substeps of (3.23).

Remark 4.1. Recall that Lemma 3.12 and Corollaries 3.16 and 3.18 imply that all intermediate steps and the next iterate of (3.23) belong to Y. The substeps with odd superscript are even elements of the respective domains $\mathcal{D}(A_Y), \mathcal{D}(B_Y)$, or $\mathcal{D}(D_{i,Y})$ from (3.27) for $i \in \{1,2,3\}$, respectively. $\Diamond$

The facts in the above remark are useful for integration by parts arguments in the proof of the observability inequality for (3.23), see Sections 4.2–4.3. The regularity statements also justify the following derivation of difference equations for (3.23).

In the following, the symbol $H_{c,l}^{n,3+2i}$ stands for the l-th component of the vector $H_c^{n,3+2i}$, while $\mathbf{e}_l$ denotes the l-th standard unit vector. For the next identities, we apply $(I - \frac{\tau}{2}A)$ to the left equation in the first line of (4.1). Taking also the formula on the right hand side of the first line of (4.1) into account, we derive the identities

$$\begin{aligned}
\begin{pmatrix} \mathbf{E}_c^{n,2} + \mathbf{E}_c^n \\ \mathbf{H}_c^{n,2} + \mathbf{H}_c^n \\ \Phi_c^{n,2} + \Phi_c^n \end{pmatrix} &= (I + \tfrac{\tau}{2}A) \begin{pmatrix} \mathbf{E}_c^{n,1} \\ \mathbf{H}_c^{n,1} \\ \Phi_c^{n,1} \end{pmatrix} + (I - \tfrac{\tau}{2}A) \begin{pmatrix} \mathbf{E}_c^{n,1} \\ \mathbf{H}_c^{n,1} \\ \Phi_c^{n,1} \end{pmatrix} = 2 \begin{pmatrix} \mathbf{E}_c^{n,1} \\ \mathbf{H}_c^{n,1} \\ \Phi_c^{n,1} \end{pmatrix}, \\
\begin{pmatrix} \mathbf{E}_c^{n,2} - \mathbf{E}_c^n \\ \mathbf{H}_c^{n,2} - \mathbf{H}_c^n \\ \Phi_c^{n,2} - \Phi_c^n \end{pmatrix} &= (I + \tfrac{\tau}{2}A) \begin{pmatrix} \mathbf{E}_c^{n,1} \\ \mathbf{H}_c^{n,1} \\ \Phi_c^{n,1} \end{pmatrix} - (I - \tfrac{\tau}{2}A) \begin{pmatrix} \mathbf{E}_c^{n,1} \\ \mathbf{H}_c^{n,1} \\ \Phi_c^{n,1} \end{pmatrix} = \tau \begin{pmatrix} \frac{1}{\varepsilon}\mathscr{C}_1 \mathbf{H}_c^{n,1} \\ \frac{1}{\mu}\mathscr{C}_2 \mathbf{E}_c^{n,1} \\ 0 \end{pmatrix}. \qquad (4.2)
\end{aligned}$$

Employing similar arguments for two remaining lines in (4.1), we moreover infer the relations

$$\begin{pmatrix} \mathbf{E}_c^{n,4} + \mathbf{E}_c^{n,2} \\ \mathbf{H}_c^{n,4} + \mathbf{H}_c^{n,2} \\ \Phi_c^{n,4} + \Phi_c^{n,2} \end{pmatrix} = 2 \begin{pmatrix} \mathbf{E}_c^{n,3} \\ \mathbf{H}_c^{n,3} \\ \Phi_c^{n,3} \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{E}_c^{n,4} - \mathbf{E}_c^{n,2} \\ \mathbf{H}_c^{n,4} - \mathbf{H}_c^{n,2} \\ \Phi_c^{n,4} - \Phi_c^{n,2} \end{pmatrix} = -\tau \begin{pmatrix} \frac{1}{\varepsilon}\mathscr{C}_2 \mathbf{H}_c^{n,3} \\ \frac{1}{\mu}\mathscr{C}_1 \mathbf{E}_c^{n,3} \\ 0 \end{pmatrix}, \qquad (4.3)$$

$$\begin{pmatrix} \mathbf{E}_c^{n,4+2i} + \mathbf{E}_c^{n,2+2i} \\ \mathbf{H}_c^{n,4+2i} + \mathbf{H}_c^{n,2+2i} \\ \Phi_c^{n,4+2i} + \Phi_c^{n,2+2i} \end{pmatrix} = 2 \begin{pmatrix} \mathbf{E}_c^{n,3+2i} \\ \mathbf{H}_c^{n,3+2i} \\ \Phi_c^{n,3+2i} \end{pmatrix}, \qquad (4.4)$$

$$\begin{pmatrix} \mathbf{E}_c^{n,4+2i} - \mathbf{E}_c^{n,2+2i} \\ \mathbf{H}_c^{n,4+2i} - \mathbf{H}_c^{n,2+2i} \\ \Phi_c^{n,4+2i} - \Phi_c^{n,2+2i} \end{pmatrix} = -\tau \begin{pmatrix} 0 \\ \partial_i(\frac{1}{\mu}\Phi_c^{n,3+2i})\mathbf{e}_i \\ \frac{1}{\mu^2}\partial_i(\mu \mathbf{H}_{c,i}^{n,3+2i}) \end{pmatrix}. \qquad (4.5)$$

Formulas (4.1)–(4.5) and $\mathbf{E}_c^{n+1} = \mathbf{E}_c^{n,4}$ then give rise to the difference equations

$$\begin{aligned}
\tfrac{1}{\tau}(\mathbf{E}_c^{n+1} - \mathbf{E}_c^n) &= \tfrac{1}{\tau}(\mathbf{E}_c^{n,4} - \mathbf{E}_c^{n,2}) + \tfrac{1}{\tau}(\mathbf{E}_c^{n,2} - \mathbf{E}_c^n) = -\tfrac{1}{\varepsilon}\mathscr{C}_2 \mathbf{H}_c^{n,3} + \tfrac{1}{\varepsilon}\mathscr{C}_1 \mathbf{H}_c^{n,1} \\
&= -\tfrac{1}{\varepsilon}\mathscr{C}_2 \mathbf{H}_c^{n,2} + \tfrac{\tau}{2\varepsilon}\mathscr{C}_2 \tfrac{1}{\mu}\mathscr{C}_1 \mathbf{E}_c^{n,3} + \tfrac{1}{\varepsilon}\mathscr{C}_1 \mathbf{H}_c^{n,2} - \tfrac{\tau}{2\varepsilon}\mathscr{C}_1 \tfrac{1}{\mu}\mathscr{C}_2 \mathbf{E}_c^{n,1}, \qquad (4.6)
\end{aligned}$$

$$
\begin{aligned}
\tfrac{1}{\tau}(\mathbf{H}_c^{n+1} - \mathbf{H}_c^n) &= \tfrac{1}{\tau}(\mathbf{H}_c^{n+1} - \mathbf{H}_c^{n,4}) + \tfrac{1}{\tau}(\mathbf{H}_c^{n,4} - \mathbf{H}_c^{n,2}) + \tfrac{1}{\tau}(\mathbf{H}_c^{n,2} - \mathbf{H}_c^n) \\
&= -\begin{pmatrix} \partial_1 \frac{1}{\mu}\Phi_c^{n,5} \\ \partial_2 \frac{1}{\mu}\Phi_c^{n,7} \\ \partial_3 \frac{1}{\mu}\Phi_c^{n,9} \end{pmatrix} - \tfrac{1}{\mu}\mathscr{C}_1\mathbf{E}_c^{n,3} + \tfrac{1}{\mu}\mathscr{C}_2\mathbf{E}_c^{n,1} \qquad (4.7) \\
&= -\begin{pmatrix} \partial_1 \frac{1}{\mu}\Phi_c^{n,5} \\ \partial_2 \frac{1}{\mu}\Phi_c^{n,7} \\ \partial_3 \frac{1}{\mu}\Phi_c^{n,9} \end{pmatrix} - \tfrac{1}{\mu}\mathscr{C}_1\mathbf{E}_c^{n,2} + \tfrac{\tau}{2\mu}\mathscr{C}_1\tfrac{1}{\varepsilon}\mathscr{C}_2\mathbf{H}_c^{n,3} \\
&\quad + \tfrac{1}{\mu}\mathscr{C}_2\mathbf{E}_c^{n,2} - \tfrac{\tau}{2\mu}\mathscr{C}_2\tfrac{1}{\varepsilon}\mathscr{C}_1\mathbf{H}_c^{n,1}. \qquad (4.8)
\end{aligned}
$$

For the arithmetic mean of two succeeding iterates of (3.23), we infer with (4.1)–(4.5) and $\mathbf{E}_c^{n+1} = \mathbf{E}_c^{n,4}$ the representations

$$
\begin{aligned}
\tfrac{1}{2}(\mathbf{E}_c^{n+1} + \mathbf{E}_c^n) &= \tfrac{1}{2}(\mathbf{E}_c^{n,4} + \mathbf{E}_c^{n,2}) - \tfrac{1}{2}(\mathbf{E}_c^{n,2} - \mathbf{E}_c^n) = \mathbf{E}_c^{n,2} - \tfrac{\tau}{2\varepsilon}\mathscr{C}_2\mathbf{H}_c^{n,3} - \tfrac{\tau}{2\varepsilon}\mathscr{C}_1\mathbf{H}_c^{n,1}, \\
\tfrac{1}{2}(\mathbf{H}_c^{n+1} + \mathbf{H}_c^n) &= \tfrac{1}{2}(\mathbf{H}_c^{n+1} - \mathbf{H}_c^{n,4}) + \tfrac{1}{2}(\mathbf{H}_c^{n,4} + \mathbf{H}_c^{n,2}) - \tfrac{1}{2}(\mathbf{H}_c^{n,2} - \mathbf{H}_c^n) \\
&= -\tfrac{\tau}{2}\begin{pmatrix} \partial_1 \frac{1}{\mu}\Phi_c^{n,5} \\ \partial_2 \frac{1}{\mu}\Phi_c^{n,7} \\ \partial_3 \frac{1}{\mu}\Phi_c^{n,9} \end{pmatrix} + \mathbf{H}_c^{n,2} - \tfrac{\tau}{2\mu}\mathscr{C}_1\mathbf{E}_c^{n,3} - \tfrac{\tau}{2\mu}\mathscr{C}_2\mathbf{E}_c^{n,1}. \qquad (4.9)
\end{aligned}
$$

We next interpret the formulas in (4.9) as representations for $\mathbf{E}_c^{n,2}$ and $\mathbf{H}_c^{n,2}$. In other words, we consider the identity

$$
\mathbf{E}_c^{n,2} = \tfrac{1}{2}(\mathbf{E}_c^{n+1} + \mathbf{E}_c^n) + \tfrac{\tau}{2\varepsilon}\mathscr{C}_2\mathbf{H}_c^{n,3} + \tfrac{\tau}{2\varepsilon}\mathscr{C}_1\mathbf{H}_c^{n,1}
$$

for $\mathbf{E}_c^{n,2}$, and proceed similarly for $\mathbf{H}_c^{n,2}$. These representations are then inserted into (4.6) and (4.8). In view of the splitting relation $\operatorname{curl} = \mathscr{C}_1 - \mathscr{C}_2$, we obtain the fundamental difference equations

$$
\begin{aligned}
\tfrac{1}{\tau}(\mathbf{E}_c^{n+1} - \mathbf{E}_c^n) &= \tfrac{1}{2\varepsilon}\operatorname{curl}(\mathbf{H}_c^{n+1} + \mathbf{H}_c^n) + \tfrac{\tau}{2\varepsilon}\operatorname{curl}\begin{pmatrix} \partial_1 \frac{1}{\mu}\Phi_c^{n,5} \\ \partial_2 \frac{1}{\mu}\Phi_c^{n,7} \\ \partial_3 \frac{1}{\mu}\Phi_c^{n,9} \end{pmatrix} \\
&\quad - \tfrac{\tau}{2\varepsilon}\mathscr{C}_2\tfrac{1}{\mu}\mathscr{C}_2\mathbf{E}_c^{n,1} + \tfrac{\tau}{2\varepsilon}\mathscr{C}_1\tfrac{1}{\mu}\mathscr{C}_1\mathbf{E}_c^{n,3}, \qquad (4.10) \\
\tfrac{1}{\tau}(\mathbf{H}_c^{n+1} - \mathbf{H}_c^n) &= -\tfrac{1}{2\mu}\operatorname{curl}(\mathbf{E}_c^{n+1} + \mathbf{E}_c^n) + \tfrac{\tau}{2\mu}\mathscr{C}_2\tfrac{1}{\varepsilon}\mathscr{C}_2\mathbf{H}_c^{n,3} - \tfrac{\tau}{2\mu}\mathscr{C}_1\tfrac{1}{\varepsilon}\mathscr{C}_1\mathbf{H}_c^{n,1} \\
&\quad - \begin{pmatrix} \partial_1 \frac{1}{\mu}\Phi_c^{n,5} \\ \partial_2 \frac{1}{\mu}\Phi_c^{n,7} \\ \partial_3 \frac{1}{\mu}\Phi_c^{n,9} \end{pmatrix}. \qquad (4.11)
\end{aligned}
$$

Relations (4.10) and (4.11) can be interpreted as perturbed time-discrete versions of the differential equations for $\mathbf{E}$ and $\mathbf{H}$ in (3.1) without damping. In this respect, the last term on the right hand side of (4.11) is the discrete substitute of the gradient of $\frac{1}{\mu}\Phi$ in (3.1). To have a gradient also in the discrete setting, we study the last term in (4.11). The last components of (4.4) and (4.5) provide the auxiliary relations

$$\begin{aligned}
2\Phi_c^{n,5} &= \Phi_c^{n,6} + \Phi_c^{n,4} = 2\Phi_c^{n,6} + \tfrac{\tau}{\mu^2}\partial_1\mu\mathbf{H}_{c,1}^{n,5},\\
2\Phi_c^{n,7} &= \Phi_c^{n,8} + \Phi_c^{n,6} = 2\Phi_c^{n,6} - \tfrac{\tau}{\mu^2}\partial_2\mu\mathbf{H}_{c,2}^{n,7},\\
2\Phi_c^{n,9} &= \Phi_c^{n+1} + \Phi_c^{n,8} = 2\Phi_c^{n,8} - \tfrac{\tau}{\mu^2}\partial_3\mu\mathbf{H}_{c,3}^{n,9} = 2\Phi_c^{n,6} - 2\tfrac{\tau}{\mu^2}\partial_2\mu\mathbf{H}_{c,2}^{n,7} - \tfrac{\tau}{\mu^2}\partial_3\mu\mathbf{H}_{c,3}^{n,9}.
\end{aligned}$$

We can hence conclude the representation

$$\begin{pmatrix}\partial_1\frac{1}{\mu}\Phi_c^{n,5}\\ \partial_2\frac{1}{\mu}\Phi_c^{n,7}\\ \partial_3\frac{1}{\mu}\Phi_c^{n,9}\end{pmatrix} = \nabla\left(\tfrac{1}{\mu}\Phi_c^{n,6}\right) + \frac{\tau}{2}\begin{pmatrix}\partial_1\frac{1}{\mu^3}\partial_1\mu\mathbf{H}_{c,1}^{n,5}\\ -\partial_2\frac{1}{\mu^3}\partial_2\mu\mathbf{H}_{c,2}^{n,7}\\ -2\partial_3\frac{1}{\mu^3}\partial_2\mu\mathbf{H}_{c,2}^{n,7} - \partial_3\frac{1}{\mu^3}\partial_3\mu\mathbf{H}_{c,3}^{n,9}\end{pmatrix}. \tag{4.12}$$

This means that the vector on the left hand side is indeed the gradient of a function, up to a higher order error term. Formula (4.12) is important to derive separate difference equations for the curl- and the divergence-free parts of the magnetic field approximations.

We finally deduce a difference equation for Φ_c^n. To that end, we first note that (4.5) implies the identities

$$\mathbf{H}_{c,1}^{n+1} = \mathbf{H}_{c,1}^{n,6}, \quad \mathbf{H}_{c,2}^{n,6} = \mathbf{H}_{c,2}^{n,4}, \quad \mathbf{H}_{c,3}^{n,8} = \mathbf{H}_{c,3}^{n,4}.$$

From (4.2)–(4.5) we then obtain the supplementary relations

$$\begin{aligned}
2\mathbf{H}_{c,1}^{n,5} &= \mathbf{H}_{c,1}^{n,6} + \mathbf{H}_{c,1}^{n,4} = \mathbf{H}_{c,1}^{n+1} + \mathbf{H}_{c,1}^{n} + (\mathbf{H}_{c,1}^{n,4} - \mathbf{H}_{c,1}^{n,2}) + (\mathbf{H}_{c,1}^{n,2} - \mathbf{H}_{c,1}^{n})\\
&= \mathbf{H}_{c,1}^{n+1} + \mathbf{H}_{c,1}^{n} - \tfrac{\tau}{\mu}(\mathscr{C}_1\mathbf{E}_c^{n,3})_1 + \tfrac{\tau}{\mu}(\mathscr{C}_2\mathbf{E}_c^{n,1})_1,\\
2\mathbf{H}_{c,2}^{n,7} &= \mathbf{H}_{c,2}^{n+1} + \mathbf{H}_{c,2}^{n} - \tfrac{\tau}{\mu}(\mathscr{C}_1\mathbf{E}_c^{n,3})_2 + \tfrac{\tau}{\mu}(\mathscr{C}_2\mathbf{E}_c^{n,1})_2,\\
2\mathbf{H}_{c,3}^{n,9} &= \mathbf{H}_{c,3}^{n+1} + \mathbf{H}_{c,3}^{n} - \tfrac{\tau}{\mu}(\mathscr{C}_1\mathbf{E}_c^{n,3})_3 + \tfrac{\tau}{\mu}(\mathscr{C}_2\mathbf{E}_c^{n,1})_3.
\end{aligned} \tag{4.13}$$

Moreover, formula (4.5) leads to the equation

$$\tfrac{1}{\tau}(\Phi_c^{n+1} - \Phi_c^n) = -\tfrac{1}{\mu^2}\partial_1(\mu\mathbf{H}_{c,1}^{n,5}) - \tfrac{1}{\mu^2}\partial_2(\mu\mathbf{H}_{c,2}^{n,7}) - \tfrac{1}{\mu^2}\partial_3(\mu\mathbf{H}_{c,3}^{n,9}).$$

Plugging (4.13) into this formula, we hence conclude the difference equation

$$\tfrac{1}{\tau}(\Phi_c^{n+1} - \Phi_c^n) = -\tfrac{1}{2\mu^2}\operatorname{div}(\mu(\mathbf{H}_c^{n+1} + \mathbf{H}_c^n)) + \tfrac{\tau}{2\mu^2}\operatorname{div}\mathscr{C}_1\mathbf{E}_c^{n,3} - \tfrac{\tau}{2\mu^2}\operatorname{div}\mathscr{C}_2\mathbf{E}_c^{n,1}.$$

This identity represents a perturbed discrete version of the differential equation for Φ in (3.1). It is further equivalent to the crucial divergence identity

$$\tfrac{1}{2\mu^2}\operatorname{div}(\mu(\mathbf{H}_c^{n+1} + \mathbf{H}_c^n)) = -\tfrac{1}{\tau}(\Phi_c^{n+1} - \Phi_c^n) + \tfrac{\tau}{2\mu^2}\operatorname{div}\mathscr{C}_1\mathbf{E}_c^{n,3}$$

$$- \tfrac{\tau}{2\mu^2} \operatorname{div} \mathscr{C}_2 \mathbf{E}_c^{n,1}. \tag{4.14}$$

We next state the uniform interior observability inequality for (3.23) in terms of the substeps from (4.1). To that end, we recall the corresponding observability inequality from the continuous case, see Section 1.2. Let $T > 0$. There is a constant $C > 0$ with

$$\int_\Omega \left(\varepsilon\, |\mathbf{E}_0|^2 + \mu\, |\mathbf{H}_0|^2\right) \mathrm{d}x \le C \int_0^T \int_\Omega |\mathbf{E}_c|^2 \,\mathrm{d}x\,\mathrm{d}t$$

for all initial data $\left(\begin{smallmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{smallmatrix}\right) \in L^2(\Omega)^6$ with $\operatorname{div}\mathbf{E}_0 = \operatorname{div}\mathbf{H}_0 = 0$ and $\mathbf{H}_0 \cdot \nu = 0$ on $\partial\Omega$, see Lemma 3.1 in [NiPi05]. The field $\left(\begin{smallmatrix}\mathbf{E}_c\\ \mathbf{H}_c\end{smallmatrix}\right)$ denotes here the solution of the undamped Maxwell equations (2.1) on a C^2-domain $\Omega \subseteq \mathbb{R}^3$. It is important that the constants C and T do not depend on the given data.

Since we do not expect the same estimate to hold uniformly in the time discrete setting due to spurious highly oscillating modes, see [InZu99, ZhZZ09, Zuaz05], we add artificial terms on the right hand side of our discrete observability inequality (4.15) below. A similar procedure is also done in [Nica08] for the Maxwell equations on a cube. Here the boundary observability estimate from the continuous case is stabilized, so that it also holds uniformly in the spatial discrete setting.

The next statement also uses the constant κ_Y from (3.25), as well as the space Y from (3.26).

Theorem 4.2. *Let ε and μ satisfy (2.2), and let $\mathring{\tau} \in (0, \min\{\frac{\sqrt{2}}{\kappa_Y}, \frac{1}{2}\})$ be a fixed number. Denote the iterates of scheme (3.23) for initial data $(\boldsymbol{E}^0, \boldsymbol{H}^0, \Phi^0) \in Y$ and step size $\tau \in (0, \mathring{\tau}]$ by $(\boldsymbol{E}_c^k, \boldsymbol{H}_c^k, \Phi_c^k)$. The observability estimate*

$$\begin{aligned}\int_Q (\varepsilon|\boldsymbol{E}^0|^2 + \mu|\boldsymbol{H}^0|^2 + \mu|\Phi^0|^2)\,\mathrm{d}x \le{}& C_o\tau \sum_{k=1}^{N} \int_Q (|\boldsymbol{E}_c^k|^2 + |\Phi_c^k|^2)\,\mathrm{d}x \\ &+ C_o\tau^3 \sum_{k=0}^{N-1} \left(\left\| B \begin{pmatrix} \boldsymbol{E}_c^{k,3} \\ \boldsymbol{H}_c^{k,3} \\ 0 \end{pmatrix} \right\|^2 + \left\| A \begin{pmatrix} \boldsymbol{E}_c^{k,1} \\ \boldsymbol{H}_c^{k,1} \\ 0 \end{pmatrix} \right\|^2 \right) \\ &+ C_o\tau^3 \sum_{k=0}^{N-1} \sum_{i=1}^{3} \left\| D_i \begin{pmatrix} 0 \\ \boldsymbol{H}_c^{k,3+2i} \\ \Phi_c^{k,3+2i} \end{pmatrix} \right\|^2 \end{aligned} \tag{4.15}$$

is valid with a uniform constant $C_o = C_o(\varepsilon, \mu, \mathring{\tau}, Q) > 0$. We also employ here the number $N := \max\{m \in \mathbb{N} \mid m\tau \le 9\mathring{\tau}\}$.

We proceed in three steps for the proof of Theorem 4.2. In Sections 4.2 and 4.3, we derive estimates for the divergence-free and the curl-free parts of the magnetic field approximations. Finally, we put the foregoing steps in Section 4.4 together. Here we use that the scheme (3.23) is energy-conserving to conclude Theorem 4.2.

For the below arguments, it is crucial to have appropriate Helmholtz decompositions for the electric and magnetic fields. According to the decomposition (2.6), the latter can be represented by the formula

$$\mu \mathbf{H}_c^k = \operatorname{curl} \mathbf{J}^k + \nabla q^k, \qquad k \in \mathbb{N}, \tag{4.16}$$

with $q^k \in H^1(Q)$, and $\mathbf{J}^k \in H^1(Q)^3$ satisfying $\operatorname{div} \mathbf{J}^k = 0$, $\mathbf{J}^k \times \nu = 0$ on ∂Q, and $\operatorname{curl} \mathbf{J}^k \in H_0(\operatorname{div}, Q)$. This decomposition of the magnetic field approximations allows to employ Green's rule of integration by parts with a vanishing boundary term for the curl part.

However, we also want to integrate by parts without boundary integral for the gradient part of the electric field. In other words, the first decomposition provides special boundary conditions for the curl part, whereas the desired second decomposition for the electric field should impose boundary conditions on the gradient part. In the next lemma, we establish the latter decomposition. The proof is inspired by the reasoning for Lemma 3.1 in [NiPi05].

Lemma 4.3. *Let ε satisfy (2.2), and let $k \in \mathbb{N}$. There is a unique function $\psi^k \in H_N(\operatorname{curl}, \operatorname{div}, Q)$ with $\operatorname{div} \psi^k \in H_0^1(Q)$, $\operatorname{curl} \psi^k \in H_T(\operatorname{curl}, \operatorname{div}, Q) \subseteq H^1(Q)^3$, and*

$$\varepsilon \boldsymbol{E}_c^k = \operatorname{curl} \operatorname{curl} \psi^k - \nabla \operatorname{div} \psi^k.$$

Proof. First, we consider the bilinear form

$$a(\varphi, \psi) := \int_Q (\operatorname{curl} \varphi) \cdot (\operatorname{curl} \psi) + (\operatorname{div} \varphi)(\operatorname{div} \psi) \, \mathrm{d}x, \quad \varphi, \psi \in H_N(\operatorname{curl}, \operatorname{div}, Q).$$

It is bounded and coercive on $H_N(\operatorname{curl}, \operatorname{div}, Q)$, due to estimate (2.5). The Lax-Milgram Lemma then yields a unique function $\psi^k \in H_N(\operatorname{curl}, \operatorname{div}, Q)$ that solves the equation

$$a(\psi^k, \psi) = \int_Q \varepsilon \mathbf{E}_c^k \cdot \psi \, \mathrm{d}x$$

for all $\psi \in H_N(\operatorname{curl}, \operatorname{div}, Q)$. (The right hand side of this identity defines a bounded linear form on $H_N(\operatorname{curl}, \operatorname{div}, Q)$.)

Theorem 1.1 in [CoDN99] then implies that the function $\operatorname{div} \psi^k$ belongs to $H_0^1(Q)$. Integrating by parts, we derive the identity

$$\int_Q (\operatorname{curl} \psi^k) \cdot (\operatorname{curl} \psi) \, \mathrm{d}x = \int_Q (\varepsilon \mathbf{E}_c^k + \nabla \operatorname{div} \psi^k) \cdot \psi \, \mathrm{d}x.$$

Taking $\psi \in H_0^1(Q)^3$ in the last formula, we obtain that the function $\operatorname{curl}\psi^k$ is contained in $H(\operatorname{curl}, Q)$, and that the asserted formula is true. Since $\operatorname{curl}\psi^k \cdot \nu = 0$ on ∂Q by Remark 2.5 in Section I of [GiRa86], the vector $\operatorname{curl}\psi^k$ belongs to $H_T(\operatorname{curl}, \operatorname{div}, Q) \subseteq H^1(Q)^3$. $\square$

Remark 4.4. The above decomposition of the electric field is again orthogonal in the L^2-sense. Indeed, we employ integration by parts to deduce the identities

$$(\operatorname{curl}\operatorname{curl}\psi^k, \nabla \operatorname{div}\psi^k)_{L^2(Q)^3} = -(\operatorname{div}\operatorname{curl}\operatorname{curl}\psi^k, \operatorname{div}\psi^k)_{L^2(Q)} = 0$$

for all $k \in \mathbb{N}$. Note that we do not have to consider any boundary integrals, since $\operatorname{div}\psi^k$ has zero trace. $\lozenge$

4.2. An estimate for the divergence-free part of the magnetic field approximations

Let ε and μ satisfy (2.2), and let $\mathring{\tau} \in (0, \min\{\frac{\sqrt{2}}{\kappa_Y}, \frac{1}{2}\})$ be fixed with κ_Y from (3.25). We fix for the remaining a smooth function $\alpha : [0, 9\mathring{\tau}] \to [0,1]$, that is supported in $[\frac{9}{4}\mathring{\tau}, \frac{27}{4}\mathring{\tau}]$, and that is equal to 1 on $[3\mathring{\tau}, 6\mathring{\tau}]$. Note that α and in particular its derivative are independent of the discretization parameter $\tau > 0$. Let us also recall the number $N = \max\{m \in \mathbb{N} \mid m\tau \leq 9\mathring{\tau}\}$ from the statement of Theorem 4.2. The assumptions on α then imply the identities

$$\alpha(0) = \alpha((N-1)\tau) = \alpha(N\tau) = 0 \tag{4.17}$$

for all $\tau \in (0, \mathring{\tau}]$.

This section is devoted to the following inequality for the divergence-free part of the magnetic field approximation. It will be complemented by a corresponding estimate for the curl-free part in the next section. In the statement arises the projection p_{curl}, that is associated to the Helmholtz decomposition (2.6). Recall that it maps each function in L^2 onto its divergence-free part. We also use the related vector $\mathbf{J}^k$ from (4.16), as well as the space Y from (3.26).

Lemma 4.5. *Let ε and μ satisfy (2.2), $\tau \in (0, \mathring{\tau}]$, and let $(\boldsymbol{E}^0, \boldsymbol{H}^0, \Phi^0) \in Y$ be the initial data for scheme (3.23). There is a constant $C_c = C_c(\varepsilon, \mu, \mathring{\tau}, Q) > 0$ with*

$$\sum_{k=0}^{N} \alpha(k\tau) \int_Q \boldsymbol{H}_c^k \cdot p_{\mathrm{curl}}(\mu \boldsymbol{H}_c^k)\,\mathrm{d}x \leq \frac{1}{16\,\|\mu\|_\infty} \sum_{k=1}^{N-1} \|\operatorname{curl}\boldsymbol{J}^k\|_{L^2}^2 + C_c \sum_{k=1}^{N-1} \left(\|\boldsymbol{E}_c^k\|_{L^2}^2 + \left\|\Phi_c^k\right\|_{L^2}^2\right)$$

$$+ C_c\tau^2 \sum_{k=0}^{N-1} \left(\left\| A \begin{pmatrix} \boldsymbol{E}_c^{k,1} \\ \boldsymbol{H}_c^{k,1} \\ 0 \end{pmatrix} \right\|^2 + \left\| B \begin{pmatrix} \boldsymbol{E}_c^{k,3} \\ \boldsymbol{H}_c^{k,3} \\ 0 \end{pmatrix} \right\|^2 + \sum_{i=1}^{3} \left\| D_i \begin{pmatrix} 0 \\ \boldsymbol{H}_c^{k,3+2i} \\ \Phi_c^{k,3+2i} \end{pmatrix} \right\|^2 \right).$$

For the sake of a clear presentation, we divide the proof of Lemma 4.5 into two pieces. The first one is given by the next supplementary lemma.

Lemma 4.6. *Let ε and μ satisfy (2.2), $\tau \in (0, \mathring{\tau}]$, and let $(\boldsymbol{E}^0, \boldsymbol{H}^0, \Phi^0) \in Y$ be the initial data for scheme (3.23). Let also $\gamma > 0$ be a fixed number. There is a number $C_1 > 0$ with*

$$\begin{aligned} &\left| \sum_{k=1}^{N-1} \int_Q \varepsilon \boldsymbol{E}_c^k \cdot \tfrac{1}{\tau}(\alpha(k\tau)\boldsymbol{J}^k - \alpha((k-1)\tau)\boldsymbol{J}^{k-1}) \,\mathrm{d}x \right| \\ &\qquad \le \sum_{k=1}^{N-1} \Big(\tfrac{C_T\gamma}{2} \|\operatorname{curl} \boldsymbol{J}^k\|_{L^2}^2 + (\tfrac{1}{2\gamma}\|\alpha'\|_\infty^2 \,\|\varepsilon\|_\infty^2 + C_1)\|\boldsymbol{E}_c^k\|_{L^2}^2 \\ &\qquad\qquad + \tfrac{\tau^2}{8}(\|\mathscr{C}_2 \boldsymbol{H}_c^{k-1,3}\|_{L^2}^2 + \|\mathscr{C}_1 \boldsymbol{H}_c^{k-1,1}\|_{L^2}^2) \\ &\qquad\qquad + C_1\tau^2 \sum_{i=1}^{3} \|\partial_i \mu \boldsymbol{H}_{c,i}^{k,3+2i}\|_{L^2}^2 + C_1\|\Phi_c^k\|_{L^2}^2 \Big). \end{aligned}$$

Note that C_1 depends only on $\varepsilon, \mu, \mathring{\tau}$, and Q.

Proof. A simple algebraic manipulation leads to the formula

$$\begin{aligned} \sum_{k=1}^{N-1} &\int_Q \varepsilon \mathbf{E}_c^k \cdot \tfrac{1}{\tau}(\alpha(k\tau)\mathbf{J}^k - \alpha((k-1)\tau)\mathbf{J}^{k-1}) \,\mathrm{d}x \\ &= \sum_{k=1}^{N-1} \int_Q \varepsilon \mathbf{E}_c^k \cdot \tfrac{1}{\tau}(\alpha(k\tau) - \alpha((k-1)\tau))\mathbf{J}^k \,\mathrm{d}x \\ &\quad + \sum_{k=1}^{N-1} \int_Q \varepsilon \mathbf{E}_c^k \cdot \tfrac{1}{\tau}(\mathbf{J}^k - \mathbf{J}^{k-1})\alpha((k-1)\tau) \,\mathrm{d}x. \end{aligned} \tag{4.18}$$

(i) We start with the first sum on the right hand side of (4.18), and incorporate here the number $\gamma > 0$. Recall that $\mathbf{J}^k$ belongs to $H_N(\operatorname{curl}, \operatorname{div}, Q)$ with $\operatorname{div} \mathbf{J}^k = 0$, see (4.16). As a result, inequality (2.5) is used to bound $\mathbf{J}^k$. Employing also Young's inequality, we infer the estimates

$$\begin{aligned} \left| \sum_{k=1}^{N-1} \int_Q \varepsilon \mathbf{E}_c^k \cdot \frac{\alpha(k\tau) - \alpha((k-1)\tau)}{\tau} \mathbf{J}^k \,\mathrm{d}x \right| &\le \sum_{k=1}^{N-1} \Big(\tfrac{\gamma}{2}\|\mathbf{J}^k\|_{L^2}^2 + \tfrac{1}{2\gamma}\,\|\alpha'\|_\infty^2 \,\|\varepsilon\|_\infty^2 \,\|\mathbf{E}_c^k\|_{L^2}^2 \Big) \\ &\le \sum_{k=1}^{N-1} \Big(\tfrac{C_T\gamma}{2}\|\operatorname{curl} \mathbf{J}^k\|_{L^2}^2 + \tfrac{1}{2\gamma}\,\|\alpha'\|_\infty^2 \,\|\varepsilon\|_\infty^2 \,\|\mathbf{E}_c^k\|_{L^2}^2 \Big). \end{aligned} \tag{4.19}$$

(ii) We next focus on the second sum on the right hand side of (4.18). Here we combine the difference equation (4.11) with the Helmholtz decomposition of $\mu\mathbf{H}_c^k$ in (4.16). In this way, the formula

$$\begin{aligned}
&\tfrac{1}{\tau}\operatorname{curl}(\mathbf{J}^k - \mathbf{J}^{k-1}) + \tfrac{1}{\tau}\nabla(q^k - q^{k-1}) \\
&\quad = -\tfrac{1}{2}\operatorname{curl}(\mathbf{E}_c^k + \mathbf{E}_c^{k-1}) - \mu\begin{pmatrix}\partial_1\frac{1}{\mu}\Phi_c^{k-1,5}\\ \partial_2\frac{1}{\mu}\Phi_c^{k-1,7}\\ \partial_3\frac{1}{\mu}\Phi_c^{k-1,9}\end{pmatrix} - \tfrac{\tau}{2}\mathscr{C}_1\tfrac{1}{\varepsilon}\mathscr{C}_1\mathbf{H}_c^{k-1,1} + \tfrac{\tau}{2}\mathscr{C}_2\tfrac{1}{\varepsilon}\mathscr{C}_2\mathbf{H}_c^{k-1,3}
\end{aligned}$$

follows. To this identity we apply the orthogonal projection p_{curl}, see Remark 2.2. This results in the equation

$$\begin{aligned}
\tfrac{1}{\tau}\operatorname{curl}(\mathbf{J}^k - \mathbf{J}^{k-1}) &= -\tfrac{1}{2}\operatorname{curl}(\mathbf{E}_c^k + \mathbf{E}_c^{k-1}) - p_{\mathrm{curl}}\mu\begin{pmatrix}\partial_1\frac{1}{\mu}\Phi_c^{k-1,5}\\ \partial_2\frac{1}{\mu}\Phi_c^{k-1,7}\\ \partial_3\frac{1}{\mu}\Phi_c^{k-1,9}\end{pmatrix} \\
&\quad - \tfrac{\tau}{2}p_{\mathrm{curl}}(\mathscr{C}_1\tfrac{1}{\varepsilon}\mathscr{C}_1\mathbf{H}_c^{k-1,1} - \mathscr{C}_2\tfrac{1}{\varepsilon}\mathscr{C}_2\mathbf{H}_c^{k-1,3}).
\end{aligned} \tag{4.20}$$

Theorem 2.1 allows us to choose vectors $\check{\varphi}_1^{k-1}$ and $\check{\varphi}_2^{k-1}$ in $H_N(\operatorname{curl},\operatorname{div},Q)$ satisfying

$$\begin{aligned}
\operatorname{curl}\check{\varphi}_1^{k-1} &= p_{\mathrm{curl}}(\mathscr{C}_1\tfrac{1}{\varepsilon}\mathscr{C}_1\mathbf{H}_c^{k-1,1} - \mathscr{C}_2\tfrac{1}{\varepsilon}\mathscr{C}_2\mathbf{H}_c^{k-1,3}), \\
\operatorname{curl}\check{\varphi}_2^{k-1} &= p_{\mathrm{curl}}\mu\begin{pmatrix}\partial_1\frac{1}{\mu}\Phi_c^{k-1,5}\\ \partial_2\frac{1}{\mu}\Phi_c^{k-1,7}\\ \partial_3\frac{1}{\mu}\Phi_c^{k-1,9}\end{pmatrix}.
\end{aligned} \tag{4.21}$$

Inserting the formulas from (4.21) into (4.20) and using Theorem 2.9 in Section I of [GiRa86], there is a function $\eta^{k-1} \in H^1(Q)$ with

$$\tfrac{1}{\tau}(\mathbf{J}^k - \mathbf{J}^{k-1}) = -\tfrac{1}{2}(\mathbf{E}_c^k + \mathbf{E}_c^{k-1}) - \tfrac{\tau}{2}\check{\varphi}_1^{k-1} - \check{\varphi}_2^{k-1} - \nabla\eta^{k-1}. \tag{4.22}$$

Solving this equation for the vector $\nabla\eta^{k-1}$, we conclude that $\nabla\eta^{k-1}$ belongs to $H^1(Q)^3$ with $\nabla\eta^{k-1}\times\nu = 0$ on ∂Q. As a result, $\nabla\eta^{k-1}$ is orthogonal to the space $\operatorname{curl}(H(\operatorname{curl},Q))$.

We next insert the decomposition $\varepsilon\mathbf{E}_c^k = \operatorname{curl}\operatorname{curl}\psi^k - \nabla\operatorname{div}\psi^k$ from Lemma 4.3 into the last sum on the right hand side of (4.18). This leads to the identity

$$\begin{aligned}
&\sum_{k=1}^{N-1}\alpha((k-1)\tau)\int_Q \varepsilon\mathbf{E}_c^k\cdot\tfrac{1}{\tau}(\mathbf{J}^k - \mathbf{J}^{k-1})\,\mathrm{d}x \\
&\qquad\qquad = \sum_{k=1}^{N-1}\alpha((k-1)\tau)\Big[\int_Q \operatorname{curl}\operatorname{curl}\psi^k\cdot\tfrac{1}{\tau}(\mathbf{J}^k - \mathbf{J}^{k-1})\,\mathrm{d}x
\end{aligned}$$

$$- \int_Q \nabla \operatorname{div} \psi^k \cdot \tfrac{1}{\tau}(\mathbf{J}^k - \mathbf{J}^{k-1}) \,\mathrm{d}x \Big].$$

The last integral vanishes, as $\operatorname{div} \psi^k$ belongs to $H_0^1(Q)$ and $\operatorname{div} \psi^k = 0$. Equation (4.22) then implies the formula

$$\begin{aligned}
&\sum_{k=1}^{N-1} \alpha((k-1)\tau) \int_Q \varepsilon \mathbf{E}_c^k \cdot \tfrac{1}{\tau}(\mathbf{J}^k - \mathbf{J}^{k-1}) \,\mathrm{d}x \\
&\qquad = - \sum_{k=1}^{N-1} \alpha((k-1)\tau) \Big(\int_Q (\operatorname{curl} \operatorname{curl} \psi^k) \cdot \tfrac{1}{2}(\mathbf{E}_c^k + \mathbf{E}_c^{k-1}) \,\mathrm{d}x \\
&\qquad\qquad + \tfrac{\tau}{2} \int_Q (\operatorname{curl} \operatorname{curl} \psi^k) \cdot \check{\varphi}_1^{k-1} \,\mathrm{d}x \\
&\qquad\qquad + \int_Q (\operatorname{curl} \operatorname{curl} \psi^k) \cdot \check{\varphi}_2^{k-1} \,\mathrm{d}x \Big).
\end{aligned} \tag{4.23}$$

The three expressions on the right hand side of (4.23) are treated separately in the next three steps.

(ii.a) We consider the first summand on the right hand side of (4.23), using that the decomposition of $\varepsilon \mathbf{E}_c^k$ from Lemma 4.3 is orthogonal. Young's inequality and the bound $0 \le \alpha \le 1$ yield

$$\begin{aligned}
&\left| \sum_{k=1}^{N-1} \alpha((k-1)\tau) \int_Q (\operatorname{curl} \operatorname{curl} \psi^k) \cdot \tfrac{1}{2}(\mathbf{E}_c^k + \mathbf{E}_c^{k-1}) \,\mathrm{d}x \right| \\
&\qquad \le \sum_{k=2}^{N-1} \frac{1}{2} \left\| \operatorname{curl} \operatorname{curl} \psi^k \right\|_{L^2} \left(\left\| \mathbf{E}_c^k \right\|_{L^2} + \left\| \mathbf{E}_c^{k-1} \right\|_{L^2} \right) \\
&\qquad \le \sum_{k=2}^{N-1} \frac{1}{2} \|\varepsilon\|_\infty \left\| \mathbf{E}_c^k \right\|_{L^2} \left(\left\| \mathbf{E}_c^k \right\|_{L^2} + \left\| \mathbf{E}_c^{k-1} \right\|_{L^2} \right) \le \|\varepsilon\|_\infty \sum_{k=1}^{N-1} \left\| \mathbf{E}_c^k \right\|_{L^2}^2 .
\end{aligned} \tag{4.24}$$

(ii.b) The second expression on the right hand side of (4.23) is next integrated by parts. Using the boundary condition $\check{\varphi}_1^{k-1} \times \nu = 0$ on ∂Q and identity (4.21), we calculate

$$\begin{aligned}
&\tfrac{\tau}{2} \sum_{k=1}^{N-1} \alpha((k-1)\tau) \int_Q (\operatorname{curl} \operatorname{curl} \psi^k) \cdot \check{\varphi}_1^{k-1} \,\mathrm{d}x \\
&= \tfrac{\tau}{2} \sum_{k=1}^{N-1} \alpha((k-1)\tau) \int_Q (\operatorname{curl} \psi^k) \cdot (\operatorname{curl} \check{\varphi}_1^{k-1}) \,\mathrm{d}x \\
&= \tfrac{\tau}{2} \sum_{k=1}^{N-1} \alpha((k-1)\tau) \int_Q \operatorname{curl} \psi^k \cdot (\mathscr{C}_1 \tfrac{1}{\varepsilon} \mathscr{C}_1 \mathbf{H}_c^{k-1,1} - \mathscr{C}_2 \tfrac{1}{\varepsilon} \mathscr{C}_2 \mathbf{H}_c^{k-1,3}) \,\mathrm{d}x \\
&\quad - \tfrac{\tau}{2} \sum_{k=1}^{N-1} \alpha((k-1)\tau) \int_Q \operatorname{curl} \psi^k \cdot p_\nabla (\mathscr{C}_1 \tfrac{1}{\varepsilon} \mathscr{C}_1 \mathbf{H}_c^{k-1,1} - \mathscr{C}_2 \tfrac{1}{\varepsilon} \mathscr{C}_2 \mathbf{H}_c^{k-1,3}) \,\mathrm{d}x.
\end{aligned} \tag{4.25}$$

The last term on the right hand side of (4.25) is zero. This can be shown by combining an integration by parts with the boundary condition $\psi^k \times \nu = 0$ on ∂Q, see Lemma 4.3. For the first summand on the right hand side of (4.25), we employ the integration by parts rule (3.18) for $\mathscr{C}_1$ and $\mathscr{C}_2$. To that end, we recall that the vectors $(\frac{1}{\varepsilon}\mathscr{C}_1\mathbf{H}_c^{k-1,1}, \frac{1}{\mu}\mathscr{C}_2\mathbf{E}_c^{k-1,1}, 0)$ and $(\frac{1}{\varepsilon}\mathscr{C}_2\mathbf{H}_c^{k-1,3}, \frac{1}{\mu}\mathscr{C}_1\mathbf{E}_c^{k-1,3}, 0)$ belong to Y by Remark 4.1. As a result,

$$\tfrac{1}{\varepsilon}\mathscr{C}_1\mathbf{H}_c^{k-1,1} \times \nu = \tfrac{1}{\varepsilon}\mathscr{C}_2\mathbf{H}_c^{k-1,3} \times \nu = 0 \text{ on } \partial Q.$$

We then arrive at the identities

$$\begin{aligned}
&\tfrac{\tau}{2}\sum_{k=1}^{N-1} \alpha((k-1)\tau) \int_Q (\operatorname{curl}\operatorname{curl}\psi^k) \cdot \check{\varphi}_1^{k-1} \,\mathrm{d}x \\
&= \tfrac{\tau}{2}\sum_{k=1}^{N-1} \alpha((k-1)\tau) \int_Q (\operatorname{curl}\psi^k) \cdot (\mathscr{C}_1 \tfrac{1}{\varepsilon}\mathscr{C}_1\mathbf{H}_c^{k-1,1} - \mathscr{C}_2 \tfrac{1}{\varepsilon}\mathscr{C}_2\mathbf{H}_c^{k-1,3}) \,\mathrm{d}x \\
&= -\tfrac{\tau}{2}\sum_{k=1}^{N-1} \alpha((k-1)\tau) \int_Q \Big((\mathscr{C}_2 \operatorname{curl}\psi^k) \cdot \tfrac{1}{\varepsilon}\mathscr{C}_1\mathbf{H}_c^{k-1,1} - (\mathscr{C}_1 \operatorname{curl}\psi^k) \cdot \tfrac{1}{\varepsilon}\mathscr{C}_2\mathbf{H}_c^{k-1,3} \Big) \,\mathrm{d}x.
\end{aligned}$$

By Lemma 4.3, the vector $\operatorname{curl}\psi^k$ is H^1-regular on Q, and it satisfies the estimate (2.4). With Remark 4.4, we consequently obtain the estimate

$$\begin{aligned}
&\left| \tfrac{\tau}{2}\sum_{k=1}^{N-1} \alpha((k-1)\tau) \int_Q (\operatorname{curl}\operatorname{curl}\psi^k) \cdot \check{\varphi}_1^{k-1} \,\mathrm{d}x \right| \\
&\qquad \leq \tfrac{\tau}{2\delta}\sum_{k=1}^{N-1} \left\| \operatorname{curl}\psi^k \right\|_{H^1} \Big(\|\mathscr{C}_2\mathbf{H}_c^{k-1,3}\|_{L^2} + \|\mathscr{C}_1\mathbf{H}_c^{k-1,1}\|_{L^2} \Big) \\
&\qquad \leq \sum_{k=2}^{N-1} \Big(2\tfrac{C_T}{\delta^2} \|\varepsilon\|_\infty^2 \|\mathbf{E}_c^k\|_{L^2}^2 + \tfrac{\tau^2}{8} \|\mathscr{C}_2\mathbf{H}_c^{k-1,3}\|_{L^2}^2 + \tfrac{\tau^2}{8} \|\mathscr{C}_1\mathbf{H}_c^{k-1,1}\|_{L^2}^2 \Big). \qquad (4.26)
\end{aligned}$$

(ii.c) The third term on the right hand side of (4.23) is treated similarly, now using the boundary condition $\check{\varphi}_2^{k-1} \times \nu = 0$ on ∂Q, (4.21), and (4.12). In this way, we arrive at the relations

$$\begin{aligned}
&\sum_{k=1}^{N-1} \alpha((k-1)\tau) \int_Q (\operatorname{curl}\operatorname{curl}\psi^k) \cdot \check{\varphi}_2^{k-1} \,\mathrm{d}x \\
&\qquad = \sum_{k=1}^{N-1} \alpha((k-1)\tau) \int_Q (\operatorname{curl}\psi^k) \cdot (\operatorname{curl}\check{\varphi}_2^{k-1}) \,\mathrm{d}x \\
&\qquad = \sum_{k=1}^{N-1} \alpha((k-1)\tau) \Bigg(\int_Q (\operatorname{curl}\psi^k) \cdot (\mu \nabla \tfrac{1}{\mu}\Phi_c^{k-1,6}) \,\mathrm{d}x \qquad (4.27)
\end{aligned}$$

$$+ \tfrac{\tau}{2} \int_Q (\operatorname{curl} \psi^k) \cdot \mu \begin{pmatrix} \partial_1 \frac{1}{\mu^3} \partial_1 \mu \mathbf{H}_{c,1}^{k-1,5} \\ -\partial_2 \frac{1}{\mu^3} \partial_2 \mu \mathbf{H}_{c,2}^{k-1,7} \\ -2\partial_3 \frac{1}{\mu^3} \partial_2 \mu \mathbf{H}_{c,2}^{k-1,7} - \partial_3 \frac{1}{\mu^3} \partial_3 \mu \mathbf{H}_{c,3}^{k-1,9} \end{pmatrix} \mathrm{d}x$$
$$- \int_Q \operatorname{curl} \psi^k \cdot p_\nabla \mu \begin{pmatrix} \partial_1 \frac{1}{\mu} \Phi_c^{k-1,5} \\ \partial_2 \frac{1}{\mu} \Phi_c^{k-1,7} \\ \partial_3 \frac{1}{\mu} \Phi_c^{k-1,9} \end{pmatrix} \mathrm{d}x \Bigg).$$

The last integral on the right hand side of (4.27) vanishes after integrating by parts, since $\psi^k \times \nu = 0$ on ∂Q.

To estimate the remaining two expressions on the right hand side of (4.27), we recall the boundary condition $\operatorname{curl} \psi^k \cdot \nu = 0$ on ∂Q, see Lemma 4.3, as well as the relations $\alpha(0) = 0$ and $\Phi_c^{k-1,6} = \Phi_c^{k-1} - \frac{\tau}{\mu^2} \partial_1 \mu \mathbf{H}_{c,1}^{k-1,5}$ from (4.17) and (4.2)–(4.5). An integration by parts and (2.4) now yield the inequalities

$$\begin{aligned}
&\left| \sum_{k=1}^{N-1} \alpha((k-1)\tau) \int_Q (\operatorname{curl}\operatorname{curl} \psi^k) \cdot \check{\varphi}_2^{k-1} \,\mathrm{d}x \right| \\
&\le \sum_{k=2}^{N-1} \alpha((k-1)\tau) \Bigg(\left| \int_Q (\operatorname{curl} \psi^k) \cdot \tfrac{1}{\mu} (\nabla \mu) \Phi_c^{k-1,6} \,\mathrm{d}x \right| \\
&\qquad + \tfrac{\tau}{2} \left| \int_Q (\operatorname{curl} \psi^k) \cdot \tfrac{1}{\mu^2} \begin{pmatrix} \partial_1^2 \mu \mathbf{H}_{c,1}^{k-1,5} \\ -\partial_2^2 \mu \mathbf{H}_{c,2}^{k-1,7} \\ -2\partial_3 \partial_2 \mu \mathbf{H}_{c,2}^{k-1,7} - \partial_3^2 \mu \mathbf{H}_{c,3}^{k-1,9} \end{pmatrix} \mathrm{d}x \right| \\
&\qquad + \tfrac{3\tau}{2} \left| \int_Q (\operatorname{curl} \psi^k) \cdot \begin{pmatrix} (\frac{\partial_1 \mu}{\mu^3}) \partial_1 \mu \mathbf{H}_{c,1}^{k-1,5} \\ -(\frac{\partial_2 \mu}{\mu^3}) \partial_2 \mu \mathbf{H}_{c,2}^{k-1,7} \\ -2(\frac{\partial_3 \mu}{\mu^3}) \partial_2 \mu \mathbf{H}_{c,2}^{k-1,7} - (\frac{\partial_3 \mu}{\mu^3}) \partial_3 \mu \mathbf{H}_{c,3}^{k-1,9} \end{pmatrix} \mathrm{d}x \right| \Bigg) \\
&\le \sum_{k=2}^{N-1} \Bigg(\frac{\|\nabla \mu\|_\infty \, \|\varepsilon\|_\infty^2}{\delta} C_T \|\mathbf{E}_c^k\|_{L^2}^2 + \frac{\|\nabla \mu\|_\infty}{2\delta} \|\Phi_c^{k-1}\|_{L^2}^2 + \frac{\tau^2 \|\nabla \mu\|_\infty}{2\delta^5} \|\partial_1 \mu \mathbf{H}_c^{k-1,5}\|_{L^2}^2 \\
&\qquad + \frac{\tau}{2} \left| \int_Q (\partial_i \tfrac{1}{\mu^2} (\operatorname{curl} \psi^k)_i)_{i=1}^3 \cdot \begin{pmatrix} \partial_1 \mu \mathbf{H}_{c,1}^{k-1,5} \\ -\partial_2 \mu \mathbf{H}_{c,2}^{k-1,7} \\ -2\partial_2 \mu \mathbf{H}_{c,2}^{k-1,7} - \partial_3 \mu \mathbf{H}_{c,3}^{k-1,9} \end{pmatrix} \mathrm{d}x \right| \\
&\qquad + 9 \|\operatorname{curl} \psi^k\|_{L^2}^2 + \tau^2 \frac{\|\nabla \mu\|_\infty^2}{\delta^6} \sum_{i=1}^3 \|\partial_i \mu \mathbf{H}_{c,i}^{k-1,3+2i}\|_{L^2}^2 \Bigg) \\
&\le \sum_{k=2}^{N-1} \Bigg(\hat{C}_T \|\mathbf{E}_c^k\|_{L^2}^2 + \frac{\|\nabla \mu\|_\infty}{2\delta} \|\Phi_c^{k-1}\|_{L^2}^2 + \hat{C}_\mu \tau^2 \sum_{i=1}^3 \|\partial_i \mu \mathbf{H}_{c,i}^{k-1,3+2i}\|_{L^2}^2 \Bigg),
\end{aligned} \tag{4.28}$$

employing the two positive numbers

$$\hat{C}_T := C_T \|\varepsilon\|_\infty^2 \Big(\frac{\|\nabla\mu\|_\infty}{\delta} + \frac{2}{\delta^4} + 8\frac{\|\nabla\mu\|_\infty^2}{\delta^6} + 9 \Big),$$
$$\hat{C}_\mu := \|\nabla\mu\|_\infty (\frac{1}{\delta^5} + \frac{\|\nabla\mu\|_\infty^2}{\delta^6}) + 1.$$

(iii) Summing up, (4.23)–(4.26) and (4.28) bound the second expression on the right hand side of (4.18) by the inequality

$$\begin{aligned}\Big|\sum_{k=1}^{N-1} \alpha((k-1)\tau) \int_Q \varepsilon \mathbf{E}_c^k \cdot \tfrac{1}{\tau}(\mathbf{J}^k - \mathbf{J}^{k-1})\,\mathrm{d}x\Big| \\ \le \sum_{k=1}^{N-1} \Big(\tilde{C}_T \|\mathbf{E}_c^k\|_{L^2}^2 + \tfrac{\tau^2}{8}(\|\mathscr{C}_2 \mathbf{H}_c^{k-1,3}\|_{L^2}^2 + \|\mathscr{C}_1 \mathbf{H}_c^{k-1,1}\|_{L^2}^2) \\ + \hat{C}_\mu \tau^2 \sum_{i=1}^{3} \|\partial_i \mu \mathbf{H}_{c,i}^{k,3+2i}\|_{L^2}^2 + \tfrac{\|\nabla\mu\|_\infty}{2\delta} \|\Phi_c^k\|_{L^2}^2 \Big),\end{aligned} \tag{4.29}$$

with $\tilde{C}_T := \|\varepsilon\|_\infty + 2\frac{C_T}{\delta^2}\|\varepsilon\|_\infty^2 + \hat{C}_T$. Combining (4.18), (4.19) and (4.29), the asserted estimate

$$\begin{aligned}\Big|\sum_{k=1}^{N-1} \int_Q \varepsilon \mathbf{E}_c^k \cdot \tfrac{1}{\tau}(\alpha(k\tau)\mathbf{J}^k - \alpha((k-1)\tau)\mathbf{J}^{k-1})\,\mathrm{d}x\Big| \\ \le \sum_{k=1}^{N-1} \Big(\tfrac{C_T\gamma}{2} \|\operatorname{curl} \mathbf{J}^k\|_{L^2}^2 + (\tfrac{1}{2\gamma}\|\alpha'\|_\infty^2 \|\varepsilon\|_\infty^2 + \tilde{C}_T) \|\mathbf{E}_c^k\|_{L^2}^2 \\ + \tfrac{\tau^2}{8}(\|\mathscr{C}_2 \mathbf{H}_c^{k-1,3}\|_{L^2}^2 + \|\mathscr{C}_1 \mathbf{H}_c^{k-1,1}\|_{L^2}^2) \\ + \hat{C}_\mu \tau^2 \sum_{i=1}^{3} \|\partial_i \mu \mathbf{H}_{c,i}^{k,3+2i}\|_{L^2}^2 + \tfrac{\|\nabla\mu\|_\infty}{2\delta} \|\Phi_c^k\|_{L^2}^2 \Big)\end{aligned} \tag{4.30}$$

follows. □

By means of Lemma 4.6, we can now derive Lemma 4.5.

Proof of Lemma 4.5. (i) Using the boundary condition $\mathbf{J}^k \times \nu = 0$ on ∂Q and (4.17) in an integration by parts, we obtain the identities

$$\begin{aligned}&\sum_{k=0}^{N} \alpha(k\tau) \int_Q \mathbf{H}_c^k \cdot \operatorname{curl} \mathbf{J}^k \,\mathrm{d}x \\ &\quad = \sum_{k=0}^{N-1} \alpha(k\tau) \Big(\int_Q \tfrac{1}{2} \operatorname{curl}(\mathbf{H}_c^k + \mathbf{H}_c^{k+1}) \cdot \mathbf{J}^k \,\mathrm{d}x - \int_Q \tfrac{1}{2} \operatorname{curl}(\mathbf{H}_c^{k+1} - \mathbf{H}_c^k) \cdot \mathbf{J}^k \,\mathrm{d}x \Big).\end{aligned}$$

Now we plug in (4.10)–(4.11), and use the integration by parts rule (3.18) for $\mathscr{C}_1$ and $\mathscr{C}_2$. The latter is applicable due to the boundary condition $\mathbf{J}^k \times \nu = 0$ on ∂Q. With this reasoning the formula

$$
\begin{aligned}
&\sum_{k=0}^{N} \alpha(k\tau) \int_Q \mathbf{H}_c^k \cdot \operatorname{curl} \mathbf{J}^k \,\mathrm{d}x \\
&\quad = \sum_{k=0}^{N-1} \alpha(k\tau) \int_Q \tfrac{\varepsilon}{\tau}(\mathbf{E}_c^{k+1} - \mathbf{E}_c^k) \cdot \mathbf{J}^k \,\mathrm{d}x - \tfrac{\tau}{2} \sum_{k=0}^{N-1} \alpha(k\tau) \int_Q \operatorname{curl} \begin{pmatrix} \partial_1 \frac{1}{\mu} \Phi_c^{k,5} \\ \partial_2 \frac{1}{\mu} \Phi_c^{k,7} \\ \partial_3 \frac{1}{\mu} \Phi_c^{k,9} \end{pmatrix} \cdot \mathbf{J}^k \,\mathrm{d}x \\
&\qquad + \tfrac{\tau}{2} \sum_{k=0}^{N-1} \alpha(k\tau) \int_Q (\mathscr{C}_2 \tfrac{1}{\mu} \mathscr{C}_2 \mathbf{E}_c^{k,1} - \mathscr{C}_1 \tfrac{1}{\mu} \mathscr{C}_1 \mathbf{E}_c^{k,3}) \cdot \mathbf{J}^k \,\mathrm{d}x \\
&\qquad - \sum_{k=0}^{N-1} \alpha(k\tau) \int_Q \tfrac{1}{2} \operatorname{curl}(\mathbf{H}_c^{k+1} - \mathbf{H}_c^k) \cdot \mathbf{J}^k \,\mathrm{d}x \\
&\quad = \sum_{k=0}^{N-1} \alpha(k\tau) \Big(\int_Q \tfrac{\varepsilon}{\tau}(\mathbf{E}_c^{k+1} - \mathbf{E}_c^k) \cdot \mathbf{J}^k \,\mathrm{d}x - \tfrac{\tau}{2} \int_Q \begin{pmatrix} \partial_1 \frac{1}{\mu} \Phi_c^{k,5} \\ \partial_2 \frac{1}{\mu} \Phi_c^{k,7} \\ \partial_3 \frac{1}{\mu} \Phi_c^{k,9} \end{pmatrix} \cdot \operatorname{curl} \mathbf{J}^k \,\mathrm{d}x \\
&\qquad\qquad - \int_Q \tfrac{1}{2} \operatorname{curl}(\mathbf{H}_c^{k+1} - \mathbf{H}_c^k) \cdot \mathbf{J}^k \,\mathrm{d}x \\
&\qquad\qquad - \tfrac{\tau}{2} \int_Q \tfrac{1}{\mu}(\mathscr{C}_2 \mathbf{E}_c^{k,1}) \cdot (\mathscr{C}_1 \mathbf{J}^k) - \tfrac{1}{\mu}(\mathscr{C}_1 \mathbf{E}_c^{k,3}) \cdot (\mathscr{C}_2 \mathbf{J}^k) \,\mathrm{d}x \Big)
\end{aligned}
$$

follows. Summation by parts and the choice of α yield

$$
\begin{aligned}
&\sum_{k=0}^{N} \alpha(k\tau) \int_Q \mathbf{H}_c^k \cdot \operatorname{curl} \mathbf{J}^k \,\mathrm{d}x \\
&\quad = - \sum_{k=1}^{N-1} \int_Q \varepsilon \mathbf{E}_c^k \cdot \tfrac{1}{\tau} \Big(\alpha(k\tau) \mathbf{J}^k - \alpha((k-1)\tau) \mathbf{J}^{k-1} \Big) \,\mathrm{d}x \\
&\qquad - \sum_{k=0}^{N-1} \alpha(k\tau) \int_Q \tfrac{1}{2} \operatorname{curl}(\mathbf{H}_c^{k+1} - \mathbf{H}_c^k) \cdot \mathbf{J}^k \,\mathrm{d}x \\
&\qquad - \tfrac{\tau}{2} \sum_{k=0}^{N-1} \alpha(k\tau) \int_Q \begin{pmatrix} \partial_1 \frac{1}{\mu} \Phi_c^{k,5} \\ \partial_2 \frac{1}{\mu} \Phi_c^{k,7} \\ \partial_3 \frac{1}{\mu} \Phi_c^{k,9} \end{pmatrix} \cdot \operatorname{curl} \mathbf{J}^k \,\mathrm{d}x \\
&\qquad - \tfrac{\tau}{2} \sum_{k=0}^{N-1} \alpha(k\tau) \int_Q \tfrac{1}{\mu}(\mathscr{C}_2 \mathbf{E}_c^{k,1}) \cdot (\mathscr{C}_1 \mathbf{J}^k) - \tfrac{1}{\mu}(\mathscr{C}_1 \mathbf{E}_c^{k,3}) \cdot (\mathscr{C}_2 \mathbf{J}^k) \,\mathrm{d}x.
\end{aligned} \tag{4.31}
$$

The first sum on the right hand side is already estimated in Lemma 4.6. The remaining three are studied in the next two steps.

(ii) We now deal with the second summand on the right hand side of (4.31). Also here, we incorporate the number $\gamma > 0$, that is used in Lemma 4.6. Recall the difference equation (4.7)

$$\tfrac{1}{\tau}(\mathbf{H}_c^{k+1} - \mathbf{H}_c^k) = -\begin{pmatrix} \partial_1 \frac{1}{\mu}\Phi_c^{k,5} \\ \partial_2 \frac{1}{\mu}\Phi_c^{k,7} \\ \partial_3 \frac{1}{\mu}\Phi_c^{k,9} \end{pmatrix} - \tfrac{1}{\mu}\mathscr{C}_1\mathbf{E}_c^{k,3} + \tfrac{1}{\mu}\mathscr{C}_2\mathbf{E}_c^{k,1}. \tag{4.32}$$

Employing the boundary condition $\mathbf{J}^k \times \nu = 0$ on ∂Q in an integration by parts, as well as formula (4.32), the relations

$$\begin{aligned}
&\left|\sum_{k=1}^{N-1} \alpha(k\tau) \int_Q \tfrac{1}{2}\operatorname{curl}(\mathbf{H}_c^{k+1} - \mathbf{H}_c^k)\cdot \mathbf{J}^k \,\mathrm{d}x\right| \\
&\quad = \tfrac{\tau}{2}\left|\sum_{k=1}^{N-1} \alpha(k\tau) \int_Q \tfrac{1}{\tau}(\mathbf{H}_c^{k+1} - \mathbf{H}_c^k)\cdot \operatorname{curl}\mathbf{J}^k \,\mathrm{d}x\right| \\
&\quad = \tfrac{\tau}{2}\left|\sum_{k=1}^{N-1} \alpha(k\tau) \int_Q \left[-\begin{pmatrix} \partial_1 \frac{1}{\mu}\Phi_c^{k,5} \\ \partial_2 \frac{1}{\mu}\Phi_c^{k,7} \\ \partial_3 \frac{1}{\mu}\Phi_c^{k,9} \end{pmatrix} + \tfrac{1}{\mu}\mathscr{C}_2\mathbf{E}_c^{k,1} - \tfrac{1}{\mu}\mathscr{C}_1\mathbf{E}_c^{k,3}\right]\cdot \operatorname{curl}\mathbf{J}^k \,\mathrm{d}x\right| \\
&\quad \le \sum_{k=1}^{N-1}\Big(3\gamma\|\operatorname{curl}\mathbf{J}^k\|_{L^2}^2 + \tfrac{\tau^2}{16\gamma}\sum_{i=1}^{3}\|\partial_i \tfrac{1}{\mu}\Phi_c^{k,3+2i}\|_{L^2}^2 \\
&\qquad\qquad + \tfrac{\tau^2}{16\gamma\delta^2}(\|\mathscr{C}_2\mathbf{E}_c^{k,1}\|_{L^2}^2 + \|\mathscr{C}_1\mathbf{E}_c^{k,3}\|_{L^2}^2)\Big)
\end{aligned} \tag{4.33}$$

are derived.

(iii) To bound the two remaining terms on the right hand side of (4.31), we apply (2.5) for $\mathbf{J}^k$. (Here we employ that $\mathbf{J}^k$ is contained in $H_N(\operatorname{curl}, \operatorname{div}, Q)$.) We consequently arrive at the inequality

$$\begin{aligned}
&\tfrac{\tau}{2}\left|\sum_{k=1}^{N-1} \alpha(k\tau) \int_Q \begin{pmatrix} \partial_1 \frac{1}{\mu}\Phi_c^{k,5} \\ \partial_2 \frac{1}{\mu}\Phi_c^{k,7} \\ \partial_3 \frac{1}{\mu}\Phi_c^{k,9} \end{pmatrix}\cdot \operatorname{curl}\mathbf{J}^k \,\mathrm{d}x\right| \\
&\quad + \frac{\tau}{2}\left|\sum_{k=1}^{N-1} \alpha(k\tau) \int_Q \tfrac{1}{\mu}(\mathscr{C}_2\mathbf{E}_c^{k,1})\cdot(\mathscr{C}_1\mathbf{J}^k) - \tfrac{1}{\mu}(\mathscr{C}_1\mathbf{E}_c^{k,3})\cdot(\mathscr{C}_2\mathbf{J}^k)\,\mathrm{d}x\right| \\
&\qquad \le \sum_{k=1}^{N-1}\Big(\tfrac{\tau^2}{16\gamma}\sum_{i=1}^{3}\|\partial_i \tfrac{1}{\mu}\Phi_c^{k,3+2i}\|_{L^2}^2 + \gamma(1+\tfrac{C_T}{4})\|\operatorname{curl}\mathbf{J}^k\|_{L^2}^2 \\
&\qquad\qquad + \tfrac{\tau^2}{2\gamma\delta^2}(\|\mathscr{C}_2\mathbf{E}_c^{k,1}\|_{L^2}^2 + \|\mathscr{C}_1\mathbf{E}_c^{k,3}\|_{L^2}^2)\Big).
\end{aligned} \tag{4.34}$$

The desired estimate is now a consequence of (4.31), Lemma 4.6, (4.33)–(4.34), and the choice $\gamma \le (16(\frac{3}{4}C_T + 4)\|\mu\|_\infty)^{-1}$. $\square$

4.3. An inequality concerning the gradient part of the magnetic field approximations

In this Section we establish a bound for the artificial gradient parts of the magnetic field approximations from (3.23). In the related papers [Phun00, NiPi05, Elle19], these components are not present as the magnetic field is divergence-free in the continuous setting. In our case, we exploit the presence of the new variable Φ in the extended Maxwell system (3.1).

As a preparation, we first prove a slight modification of a result in [Gris85], which is of auxiliary character for our purposes. The lemma is used here for a representation of the gradient part of the magnetic field approximations. Note that the statement is well-known to experts in the field.

Lemma 4.7. *Let $q \in L^2(Q)$ with $\int_Q q \,\mathrm{d}x = 0$. The constrained boundary value problem*

$$\begin{aligned} \Delta w &= q && \text{in } Q, \\ \frac{\partial w}{\partial \nu} &= 0 && \text{on } \partial Q, \\ \int_Q w \,\mathrm{d}x &= 0, \end{aligned} \tag{4.35}$$

has a unique solution $w \in H^2(Q)$ with $\|w\|_{H^2} \leq C_G \|q\|_{L^2}$. Here, $C_G > 0$ is a constant depending only on Q.

Proof. The mean of a map $v \in H^1(Q)$ over Q is denoted by $[v]$. We use the Hilbert space $V := \{w \in H^1(Q) \mid [w] = 0\}$, equipped with the standard H^1-norm. The Lax-Milgram Lemma provides us with a unique function $w \in V$ satisfying

$$\int_Q (\nabla w) \cdot (\nabla v) \,\mathrm{d}x = -\int_Q qv \,\mathrm{d}x$$

for all $v \in V$. (By the generalized Poincaré inequality, the left hand side of this identity defines a coercive bilinear form on V.) Since q has by assumption vanishing mean over Q, we can compute

$$\int_Q (\nabla w) \cdot (\nabla v) \,\mathrm{d}x = \int_Q (\nabla w) \cdot \nabla(v - [v]) \,\mathrm{d}x = -\int_Q q(v - [v]) \,\mathrm{d}x = -\int_Q qv \,\mathrm{d}x$$

for every $v \in H^1(Q)$. As a result, w is the unique solution of (4.35). Theorem 3.2.1.3 in [Gris85] then shows that w belongs to $H^2(Q)$.

We next prove the asserted estimate. Hereby we use arguments from the proofs for Theorems 3.1.2.1 and 3.2.1.3 in [Gris85]. Let $m \in \mathbb{N}$. Let $Q_m \subseteq \mathbb{R}^3$ be a convex set with a C^2-boundary ∂Q_m. The set Q_m is supposed to contain Q, and to satisfy $\operatorname{dist}(\partial Q, \partial Q_m) \le \frac{1}{m}$. Such approximating sets exist, see Lemma 2.3.2 in [Hoer94] for instance.

The function $v := w$ also solves the boundary value problem

$$-\Delta v + v = -q + w =: f \quad \text{in } Q, \qquad\qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial Q.$$

This problem is again uniquely solvable due to Theorem 3.2.1.3 in [Gris85]. Let further $\tilde{f}$ denote the trivial extension of f to $\mathbb{R}^3$. We consider for $m \in \mathbb{N}$ the problem

$$-\Delta v_m + v_m = \tilde{f} \quad \text{in } Q_m, \qquad\qquad \frac{\partial v_m}{\partial \nu} = 0 \quad \text{on } \partial Q_m,$$

possessing a unique solution $v_m \in H^2(Q_m)$ with

$$\|v_m\|_{H^2(Q_m)} \le \sqrt{6}\|\tilde{f}\|_{L^2(Q_m)} = \sqrt{6}\|f\|_{L^2(Q)}, \tag{4.36}$$

see Theorem 3.1.2.3 in [Gris85]. The proof of Theorem 3.2.1.3 in [Gris85] moreover yields a subsequence (we denote it by $(v_m)_m$ again) satisfying $v_m|_Q \to v = w$ weakly in $H^2(Q)$ as $m \to \infty$. With (4.36), we now derive the estimates

$$\sqrt{6}\,\|f\|_{L^2(Q)} \ge \liminf_{m\to\infty} \|v_m|_Q\|_{H^2(Q)} \ge \|w\|_{H^2(Q)}\,. \tag{4.37}$$

Using the properties of w in an integration by parts, we also obtain the relations

$$\int_Q |\nabla w|^2 \,\mathrm{d}x = -\int_Q (\Delta w) w \,\mathrm{d}x = -\int_Q q w \,\mathrm{d}x \le \|q\|_{L^2} \|w\|_{L^2}\,. \tag{4.38}$$

The asserted estimate is now a consequence of (4.37), (4.38) and the generalized Poincaré inequality. □

We now estimate the curl-free part of the magnetic field approximations from (3.23). As introduced in Remark 2.2, we denote by p_∇ the projection to the curl-free part in the Helmholtz-decomposition from Theorem 2.1. The statement also uses the fixed maximal time step size $\mathring{\tau} \in (0, \min\{\frac{\sqrt{2}}{\kappa_Y}, \frac{1}{2}\})$ for (3.23) (the number κ_Y is defined in (3.25)). Important is also the cut-off function α from Section 4.2. The initial data $(\mathbf{E}_c^0, \mathbf{H}_c^0, \Phi_c^0)$ are chosen within the space Y from (3.26).

Lemma 4.8. *Let ε and μ satisfy (2.2), and let the initial data $(\boldsymbol{E}^0, \boldsymbol{H}^0, \Phi^0)$ for (3.23) belong to Y. The estimate*

$$\left|\sum_{k=0}^{N} \alpha(k\tau) \int_Q \boldsymbol{H}_c^k \cdot p_\nabla(\mu \boldsymbol{H}_c^k) \,\mathrm{d}x\right|$$

$$
\begin{aligned}
&\le \frac{\delta}{16}\sum_{k=1}^{N-1}\|p_\nabla(\boldsymbol{H}_c^k)\|_{L^2}^2 + C_\nabla\sum_{k=1}^{N-1}(\|\Phi_c^k\|_{L^2}^2+\|\boldsymbol{E}_c^k\|_{L^2}^2)\\
&\quad + C_\nabla\tau^2\sum_{k=1}^{N-1}\left(\left\|A\begin{pmatrix}\boldsymbol{E}_c^{k,1}\\ \boldsymbol{H}_c^{k,1}\\ 0\end{pmatrix}\right\|^2+\left\|B\begin{pmatrix}\boldsymbol{E}_c^{k,3}\\ \boldsymbol{H}_c^{k,3}\\ 0\end{pmatrix}\right\|^2+\sum_{i=1}^{3}\left\|D_i\begin{pmatrix}0\\ \boldsymbol{H}_c^{k,3+2i}\\ \Phi_c^{k,3+2i}\end{pmatrix}\right\|^2\right)
\end{aligned}
$$

is valid for all $\tau\in(0,\mathring{\tau}]$ *with a uniform constant* $C_\nabla = C_\nabla(\varepsilon,\mu,\mathring{\tau},Q)>0$.

Proof. (i) Relation (4.17) for α will again be employed several times without further notice. We apply the Helmholtz decomposition from Theorem 2.1 to have the representation

$$\mathbf{H}_c^k = \operatorname{curl}\tilde{\mathbf{J}}^k + \nabla\tilde{q}^k, \tag{4.39}$$

of the magnetic field approximations for $k\in\mathbb{N}$. Without loss of generality, $\tilde{q}^k\in H^1(Q)$ is chosen to have vanishing mean $\int_Q \tilde{q}^k\,\mathrm{d}x=0$. Lemma 4.7 then provides us with a function $w^k\in H^2(Q)$, satisfying

$$
\begin{aligned}
\Delta w^k &= \tilde{q}^k && \text{in } Q,\\
\frac{\partial w^k}{\partial\nu} &= 0 && \text{on } \partial Q,\\
\int_Q w^k\,\mathrm{d}x &= 0.
\end{aligned}
\tag{4.40}
$$

Recall from Theorem 2.1, that the vector $\tilde{\mathbf{J}}^k$ belongs to $H^1(Q)^3$ with $\operatorname{div}\tilde{\mathbf{J}}^k=0$, $\operatorname{curl}\tilde{\mathbf{J}}^k\in H_0(\operatorname{div},Q)$ and $\tilde{\mathbf{J}}^k\times\nu=0$ on ∂Q.

(ii) The orthogonality of the Helmholtz decomposition from Theorem 2.1 first implies the identities

$$\int_Q \mathbf{H}_c^k\cdot p_\nabla(\mu\mathbf{H}_c^k)\,\mathrm{d}x = \int_Q p_\nabla(\mathbf{H}_c^k)\cdot p_\nabla(\mu\mathbf{H}_c^k)\,\mathrm{d}x = \int_Q \mu p_\nabla(\mathbf{H}_c^k)\cdot\mathbf{H}_c^k\,\mathrm{d}x.$$

We then use the boundary condition $\mathbf{H}_c^k\cdot\nu=0$ on ∂Q in an integration by parts. In this way we calculate

$$
\begin{aligned}
\sum_{k=0}^{N}\alpha(k\tau)\int_Q p_\nabla(\mu\mathbf{H}_c^k)\cdot\mathbf{H}_c^k\,\mathrm{d}x &= \sum_{k=0}^{N}\alpha(k\tau)\int_Q \mu\mathbf{H}_c^k\cdot p_\nabla(\mathbf{H}_c^k)\,\mathrm{d}x\\
&= \sum_{k=0}^{N-1}\alpha(k\tau)\int_Q \tfrac{\mu}{2}(\mathbf{H}_c^k+\mathbf{H}_c^{k+1})\cdot\nabla\tilde{q}^k\,\mathrm{d}x\\
&\quad -\sum_{k=0}^{N-1}\alpha(k\tau)\int_Q \tfrac{\mu}{2}(\mathbf{H}_c^{k+1}-\mathbf{H}_c^k)\cdot\nabla\tilde{q}^k\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha(N\tau) \int_Q \mu \mathbf{H}_c^N \cdot \nabla \tilde{q}^N \,\mathrm{d}x \\
&= -\sum_{k=0}^{N-1} \Big(\alpha(k\tau) \int_Q \tfrac{1}{2} \operatorname{div}(\mu(\mathbf{H}_c^k + \mathbf{H}_c^{k+1}))\tilde{q}^k \,\mathrm{d}x \\
&\qquad + \alpha(k\tau) \int_Q \tfrac{\mu}{2} (\mathbf{H}_c^{k+1} - \mathbf{H}_c^k) \cdot \nabla \tilde{q}^k \,\mathrm{d}x \Big).
\end{aligned}
$$

The important divergence identity (4.14) and the difference equation (4.7) further yield the representation

$$
\begin{aligned}
&\sum_{k=0}^{N} \alpha(k\tau) \int_Q \mathbf{H}_c^k \cdot p_\nabla(\mu \mathbf{H}_c^k) \,\mathrm{d}x \\
&\quad = \sum_{k=0}^{N-1} \Big(\alpha(k\tau) \int_Q \tfrac{\mu^2}{\tau} (\Phi_c^{k+1} - \Phi_c^k) \tilde{q}^k \,\mathrm{d}x - \tfrac{\tau}{2} \alpha(k\tau) \int_Q \operatorname{div}(\mathscr{C}_1 \mathbf{E}_c^{k,3} - \mathscr{C}_2 \mathbf{E}_c^{k,1}) \tilde{q}^k \,\mathrm{d}x \\
&\qquad + \tfrac{\tau}{2} \alpha(k\tau) \int_Q \mu \begin{pmatrix} \partial_1 \frac{1}{\mu} \Phi_c^{k,5} \\ \partial_2 \frac{1}{\mu} \Phi_c^{k,7} \\ \partial_3 \frac{1}{\mu} \Phi_c^{k,9} \end{pmatrix} \cdot \nabla \tilde{q}^k \,\mathrm{d}x + \tfrac{\tau}{2} \alpha(k\tau) \int_Q (\mathscr{C}_1 \mathbf{E}_c^{k,3} - \mathscr{C}_2 \mathbf{E}_c^{k,1}) \cdot \nabla \tilde{q}^k \,\mathrm{d}x \Big).
\end{aligned}
$$

To integrate the second term on the right hand side by parts, we note that the vectors $(\frac{1}{\varepsilon} \mathscr{C}_1 \mathbf{H}_c^{k,1}, \frac{1}{\mu} \mathscr{C}_2 \mathbf{E}_c^{k,1}, 0)$ and $(\frac{1}{\varepsilon} \mathscr{C}_2 \mathbf{H}_c^{k,3}, \frac{1}{\mu} \mathscr{C}_1 \mathbf{E}_c^{k,3}, 0)$ belong to Y, see Remark 4.1. In particular, the boundary conditions $\mathscr{C}_2 \mathbf{E}_c^{k,1} \cdot \nu = \mathscr{C}_1 \mathbf{E}_c^{k,3} \cdot \nu = 0$ on ∂Q are valid. This reasoning consequently leads to the equation

$$
\begin{aligned}
&\sum_{k=0}^{N} \alpha(k\tau) \int_Q \mathbf{H}_c^k \cdot p_\nabla(\mu \mathbf{H}_c^k) \,\mathrm{d}x \\
&\quad = \sum_{k=0}^{N-1} \Big(\alpha(k\tau) \int_Q \tfrac{\mu^2}{\tau} (\Phi_c^{k+1} - \Phi_c^k) \tilde{q}^k \,\mathrm{d}x + \tfrac{\tau}{2} \alpha(k\tau) \int_Q \mu \begin{pmatrix} \partial_1 \frac{1}{\mu} \Phi_c^{k,5} \\ \partial_2 \frac{1}{\mu} \Phi_c^{k,7} \\ \partial_3 \frac{1}{\mu} \Phi_c^{k,9} \end{pmatrix} \cdot \nabla \tilde{q}^k \,\mathrm{d}x \\
&\qquad + \tau \alpha(k\tau) \int_Q (\mathscr{C}_1 \mathbf{E}_c^{k,3} - \mathscr{C}_2 \mathbf{E}_c^{k,1}) \cdot \nabla \tilde{q}^k \,\mathrm{d}x \Big).
\end{aligned} \tag{4.41}
$$

In the next two steps, we deal with the terms on the right hand side of (4.41).

(iii) For the first term on the right hand side of (4.41), summation by parts and (4.17) give rise to the formula

$$
\begin{aligned}
&\sum_{k=0}^{N-1} \alpha(k\tau) \int_Q \tfrac{\mu^2}{\tau} (\Phi_c^{k+1} - \Phi_c^k) \tilde{q}^k \,\mathrm{d}x \\
&\quad = -\sum_{k=1}^{N-1} \int_Q \mu^2 \Phi_c^k \Big[\frac{\alpha(k\tau) - \alpha((k-1)\tau)}{\tau} \tilde{q}^k + \alpha((k-1)\tau) \frac{\tilde{q}^k - \tilde{q}^{k-1}}{\tau} \Big] \,\mathrm{d}x.
\end{aligned} \tag{4.42}
$$

(iii.a) As in the proof of Lemma 4.5, we fix a number $\gamma > 0$ that we determine later. To bound the first term on the right hand side of (4.42), we combine Young's inequality with the generalized Poincaré inequality for $\tilde{q}^k$ with a Poincaré constant $C_P > 0$ (the function $\tilde{q}^k$ is assumed to have zero mean). In this way, we derive the estimates

$$\begin{aligned}
&\left|\sum_{k=1}^{N-1} \int_Q \mu^2 \Phi_c^k \frac{\alpha(k\tau) - \alpha((k-1)\tau)}{\tau} \tilde{q}^k \,\mathrm{d}x\right| \\
&\qquad\le \tfrac{\gamma}{2} \sum_{k=1}^{N-1} \int_Q \mu^4 \left|\tilde{q}^k\right|^2 \,\mathrm{d}x + \tfrac{1}{2\gamma} \sum_{k=1}^{N-1} \int_Q \frac{|\alpha(k\tau) - \alpha((k-1)\tau)|^2}{\tau^2} \left|\Phi_c^k\right|^2 \mathrm{d}x \\
&\qquad\le \sum_{k=1}^{N-1} \Big(\tfrac{\gamma}{2} C_P \|\mu\|_\infty^4 \|\nabla \tilde{q}^k\|_{L^2}^2 + \tfrac{1}{2\gamma} \|\alpha'\|_\infty^2 \|\Phi_c^k\|_{L^2}^2 \Big).
\end{aligned} \tag{4.43}$$

(iii.b) We next deal with the second summand on the right hand side of (4.42). First, we need a convenient representation of the difference quotient in the last expression. To that end, we recall the crucial difference equation (4.11)

$$\begin{aligned}
\tfrac{1}{\tau}(\mathbf{H}_c^{n+1} - \mathbf{H}_c^n) = &-\tfrac{1}{2\mu} \operatorname{curl}(\mathbf{E}_c^{n+1} + \mathbf{E}_c^n) + \tfrac{\tau}{2\mu} \mathscr{C}_2 \tfrac{1}{\varepsilon} \mathscr{C}_2 \mathbf{H}_c^{n,3} - \tfrac{\tau}{2\mu} \mathscr{C}_1 \tfrac{1}{\varepsilon} \mathscr{C}_1 \mathbf{H}_c^{n,1} \\
&- \begin{pmatrix} \partial_1 \frac{1}{\mu} \Phi_c^{n,5} \\ \partial_2 \frac{1}{\mu} \Phi_c^{n,7} \\ \partial_3 \frac{1}{\mu} \Phi_c^{n,9} \end{pmatrix}.
\end{aligned}$$

We then apply the orthogonal projection p_∇ from Remark 2.2 to this identity. By (4.39), the relation $p_\nabla \mathbf{H}_c^k = \nabla \tilde{q}^k$ is valid. Inserting also (4.12), we obtain the formula

$$\begin{aligned}
\tfrac{1}{\tau} \nabla(\tilde{q}^{k+1} - \tilde{q}^k) &= -p_\nabla \begin{pmatrix} \partial_1 \frac{1}{\mu} \Phi_c^{k,5} \\ \partial_2 \frac{1}{\mu} \Phi_c^{k,7} \\ \partial_3 \frac{1}{\mu} \Phi_c^{k,9} \end{pmatrix} - p_\nabla (\tfrac{1}{2\mu} \operatorname{curl}(\mathbf{E}_c^{k+1} + \mathbf{E}_c^k)) \\
&\quad + \tfrac{\tau}{2} p_\nabla \tfrac{1}{\mu} (\mathscr{C}_2 \tfrac{1}{\varepsilon} \mathscr{C}_2 \mathbf{H}_c^{k,3} - \mathscr{C}_1 \tfrac{1}{\varepsilon} \mathscr{C}_1 \mathbf{H}_c^{k,1}) \\
&= -\nabla(\tfrac{1}{\mu} \Phi_c^{k,6}) - \tfrac{\tau}{2} p_\nabla \begin{pmatrix} \partial_1 \frac{1}{\mu^3} \partial_1 \mu \mathbf{H}_{c,1}^{k,5} \\ -\partial_2 \frac{1}{\mu^3} \partial_2 \mu \mathbf{H}_{c,2}^{k,7} \\ -2\partial_3 \frac{1}{\mu^3} \partial_2 \mu \mathbf{H}_{c,2}^{k,7} - \partial_3 \frac{1}{\mu^3} \partial_3 \mu \mathbf{H}_{c,3}^{k,9} \end{pmatrix} \\
&\quad - p_\nabla (\tfrac{1}{2\mu} \operatorname{curl}(\mathbf{E}_c^{k+1} + \mathbf{E}_c^k)) + \tfrac{\tau}{2} p_\nabla (\tfrac{1}{\mu} (\mathscr{C}_2 \tfrac{1}{\varepsilon} \mathscr{C}_2 \mathbf{H}_c^{k,3} - \mathscr{C}_1 \tfrac{1}{\varepsilon} \mathscr{C}_1 \mathbf{H}_c^{k,1})).
\end{aligned} \tag{4.44}$$

We next use Lemma 4.7 to obtain a unique function $\eta^k \in H^2(Q)$ satisfying (4.35) with right hand side $q := \mu^2 \Phi_c^k - \frac{1}{|Q|} \int_Q \mu^2 \Phi_c^k \,\mathrm{d}x$. To reformulate the second summand on the right hand side of (4.42), we use (4.40) and the function η^k. In this way, we arrive at the equations

$$
\begin{aligned}
&\sum_{k=1}^{N-1}\alpha((k-1)\tau)\int_Q \mu^2\Phi_c^k\frac{\tilde{q}^k-\tilde{q}^{k-1}}{\tau}\,\mathrm{d}x\\
&\quad=\sum_{k=1}^{N-1}\alpha((k-1)\tau)\int_Q \mu^2\Phi_c^k\left(\Delta w^k-\Delta w^{k-1}\right)\mathrm{d}x\\
&\quad=\sum_{k=1}^{N-1}\alpha((k-1)\tau)\int_Q \left(\mu^2\Phi_c^k-\tfrac{1}{|Q|}\int_Q\mu^2\Phi_c^k\,\mathrm{d}y\right)\tfrac{1}{\tau}\left(\Delta w^k-\Delta w^{k-1}\right)\mathrm{d}x\\
&\quad=\sum_{k=1}^{N-1}\alpha((k-1)\tau)\int_Q (\Delta\eta^k)\frac{\tilde{q}^k-\tilde{q}^{k-1}}{\tau}\,\mathrm{d}x. \qquad (4.45)
\end{aligned}
$$

We next integrate the right hand side of (4.45) by parts, and we hereby use the boundary condition $\frac{\partial\eta^k}{\partial\nu}=0$ on ∂Q. Inserting furthermore the formula (4.44), the equations

$$
\begin{aligned}
&\sum_{k=1}^{N-1}\alpha((k-1)\tau)\int_Q \mu^2\Phi_c^k\frac{\tilde{q}^k-\tilde{q}^{k-1}}{\tau}\,\mathrm{d}x\\
&\quad=-\sum_{k=1}^{N-1}\alpha((k-1)\tau)\int_Q \nabla\eta^k\cdot\tfrac{1}{\tau}(\nabla\tilde{q}^k-\nabla\tilde{q}^{k-1})\,\mathrm{d}x\\
&\quad=\sum_{k=1}^{N-1}\Bigg(\alpha((k-1)\tau)\int_Q(\nabla\eta^k)\cdot(\nabla\tfrac{1}{\mu}\Phi_c^{k-1,6})\,\mathrm{d}x\\
&\qquad+\tfrac{\tau}{2}\alpha((k-1)\tau)\int_Q(\nabla\eta^k)\cdot\begin{pmatrix}\partial_1\frac{1}{\mu^3}\partial_1\mu\mathbf{H}_{c,1}^{k-1,5}\\ -\partial_2\frac{1}{\mu^3}\partial_2\mu\mathbf{H}_{c,2}^{k-1,7}\\ -2\partial_3\frac{1}{\mu^3}\partial_2\mu\mathbf{H}_{c,2}^{k-1,7}-\partial_3\frac{1}{\mu^3}\partial_3\mu\mathbf{H}_{c,3}^{k-1,9}\end{pmatrix}\mathrm{d}x\\
&\qquad+\alpha((k-1)\tau)\int_Q(\nabla\eta^k)\cdot\tfrac{1}{2\mu}\operatorname{curl}(\mathbf{E}_c^k+\mathbf{E}_c^{k-1})\,\mathrm{d}x\\
&\qquad-\tfrac{\tau}{2}\alpha((k-1)\tau)\int_Q\tfrac{1}{\mu}(\nabla\eta^k)\cdot(\mathscr{C}_2\tfrac{1}{\varepsilon}\mathscr{C}_2\mathbf{H}_c^{k-1,3}-\mathscr{C}_1\tfrac{1}{\varepsilon}\mathscr{C}_1\mathbf{H}_c^{k-1,1})\,\mathrm{d}x\Bigg) \qquad (4.46)
\end{aligned}
$$

follow. All terms on the right hand side are next integrated by parts. For the first three integrals, we hereby employ the boundary conditions $\frac{\partial\eta^k}{\partial\nu}=0$ and $\mathbf{E}_c^k\times\nu=0$ on ∂Q. Formula (3.18) for the operators $\mathscr{C}_1$ and $\mathscr{C}_2$ is applied to the fourth integral. (The function $(\frac{1}{\varepsilon}\mathscr{C}_1\mathbf{H}_c^{k,1})\times\nu$ vanishes on ∂Q, as the vector $(\frac{1}{\varepsilon}\mathscr{C}_1\mathbf{H}_c^{k,1},\frac{1}{\mu}\mathscr{C}_2\mathbf{E}_c^{k,1},0)=A(\mathbf{E}_c^{k,1},\mathbf{H}_c^{k,1},\Phi_c^{k,1})$ belongs to Y, see Remark 4.1. Similarly, the relation $(\frac{1}{\varepsilon}\mathscr{C}_2\mathbf{H}_c^{k,3})\times\nu=0$ on ∂Q is valid.) Using the identity $\Phi_c^{k-1,6}=\Phi_c^{k-1}-\frac{\tau}{\mu^2}\partial_1(\mu\mathbf{H}_{c,1}^{k-1,5})$ from (4.5), we eventually arrive at the inequality

$$
\left|\sum_{k=1}^{N-1}\alpha((k-1)\tau)\int_Q\mu^2\Phi_c^k\frac{\tilde{q}^k-\tilde{q}^{k-1}}{\tau}\,\mathrm{d}x\right|
$$

$$\leq \tfrac{1}{2}\sum_{k=1}^{N}\|\mathbf{E}_c^k\|_{L^2}^2 + \sum_{k=1}^{N-1} C_\mu(\|\Phi_c^k\|_{L^2}^2 + \|\eta^k\|_{H^2}^2) \tag{4.47}$$
$$+ \sum_{k=1}^{N-1}\Big(C_\mu\tau^2\sum_{i=1}^{3}\|\partial_i\mu\mathbf{H}_{c,i}^{k,3+2i}\|_{L^2}^2 + \tfrac{\tau^2}{4\delta^2}(\|\mathscr{C}_2\mathbf{H}_c^{k,3}\|_{L^2}^2 + \|\mathscr{C}_1\mathbf{H}_c^{k,1}\|_{L^2}^2)\Big),$$

where $C_\mu = C_\mu(\mu) > 0$ is a constant. For (4.47) we also apply the inequality

$$\|\eta^k\|_{H^2} \leq C_G\left\|\mu^2\Phi_c^k - \tfrac{1}{|Q|}\int_Q \mu^2\Phi_c^k\,\mathrm{d}y\right\|_{L^2} \leq 2C_G\|\mu\|_\infty^2\|\Phi_c^k\|_{L^2},$$

see Lemma 4.7. Taking then also (4.42) and (4.43) into account, the estimate

$$\left|\sum_{k=0}^{N-1}\alpha(k\tau)\int_Q \tfrac{\mu^2}{\tau}(\Phi_c^{k+1}-\Phi_c^k)\tilde{q}^k\,\mathrm{d}x\right|$$
$$\leq \sum_{k=1}^{N-1}\Big(\tfrac{\gamma}{2}C_P\|\mu\|_\infty^4\|\nabla\tilde{q}^k\|_{L^2}^2 + (\tfrac{1}{2\gamma}\|\alpha'\|_\infty^2 + \tilde{C}_\mu)\|\Phi_c^k\|_{L^2}^2\Big) + \tfrac{1}{2}\sum_{k=1}^{N}\|\mathbf{E}_c^k\|_{L^2}^2$$
$$+ \sum_{k=1}^{N-1}\Big(C_\mu\tau^2\sum_{i=1}^{3}\|\partial_i\mu\mathbf{H}_{c,i}^{k,3+2i}\|_{L^2}^2 + \tfrac{\tau^2}{4\delta^2}(\|\mathscr{C}_2\mathbf{H}_c^{k,3}\|_{L^2}^2 + \|\mathscr{C}_1\mathbf{H}_c^{k,1}\|_{L^2}^2)\Big) \tag{4.48}$$

is obtained. Here, $\tilde{C}_\mu = \tilde{C}_\mu(\mu, Q)$ is a positive number. We have thus estimated the first term on the right hand side of (4.41).

(iv) We now deal with the two remaining expressions in (4.41). To that end, we also incorporate the positive number $\gamma > 0$ from part (iii.a). Applying then Young's inequality as well as the relation $\alpha(0) = 0$, see (4.17), the inequality

$$\left|\sum_{k=0}^{N-1}\frac{\tau}{2}\alpha(k\tau)\Big(\int_Q \mu\begin{pmatrix}\partial_1\frac{1}{\mu}\Phi_c^{k,5}\\ \partial_2\frac{1}{\mu}\Phi_c^{k,7}\\ \partial_3\frac{1}{\mu}\Phi_c^{k,9}\end{pmatrix}\cdot\nabla\tilde{q}^k\,\mathrm{d}x + 2\int_Q(\mathscr{C}_1\mathbf{E}_c^{k,3} - \mathscr{C}_2\mathbf{E}_c^{k,1})\cdot\nabla\tilde{q}^k\,\mathrm{d}x\Big)\right|$$
$$\leq \sum_{k=1}^{N-1}\Big(5\gamma\|\nabla\tilde{q}^k\|_{L^2}^2 + \frac{\tau^2\|\mu\|_\infty^2}{16\gamma}\sum_{i=1}^{3}\|\partial_i\tfrac{1}{\mu}\Phi_c^{k,3+2i}\|_{L^2}^2$$
$$+ \frac{\tau^2}{\gamma}(\|\mathscr{C}_1\mathbf{E}_c^{k,3}\|_{L^2}^2 + \|\mathscr{C}_2\mathbf{E}_c^{k,1}\|_{L^2}^2)\Big) \tag{4.49}$$

is derived. We choose now $\gamma \leq \delta(16(\frac{C_P}{2}\|\mu\|_\infty^4 + 5))^{-1}$, and combine identity (4.41) with our estimates (4.48) and (4.49). In this way, we conclude the asserted inequality

$$\left|\sum_{k=0}^{N}\alpha(k\tau)\int_Q \mathbf{H}_c^k\cdot p_\nabla(\mu\mathbf{H}_c^k)\,\mathrm{d}x\right| \leq \tfrac{\delta}{16}\sum_{k=1}^{N-1}\left\|p_\nabla(\mathbf{H}_c^k)\right\|_{L^2}^2 + C_\nabla\sum_{k=1}^{N-1}(\left\|\Phi_c^k\right\|_{L^2}^2 + \left\|\mathbf{E}_c^k\right\|_{L^2}^2)$$

$$+ C_\nabla \tau^2 \sum_{k=0}^{N-1} \left(\left\| \mathscr{C}_1 \mathbf{H}_c^{k,1} \right\|_{L^2}^2 + \left\| \mathscr{C}_2 \mathbf{H}_c^{k,3} \right\|_{L^2}^2 \right)$$
$$+ C_\nabla \tau^2 \sum_{k=0}^{N-1} \left(\left\| \mathscr{C}_1 \mathbf{E}_c^{k,3} \right\|_{L^2}^2 + \left\| \mathscr{C}_2 \mathbf{E}_c^{k,1} \right\|_{L^2}^2 \right)$$
$$+ C_\nabla \tau^2 \sum_{k=1}^{N-1} \sum_{i=1}^{3} \left(\left\| \partial_i \mu \mathbf{H}_{c,i}^{k,3+2i} \right\|_{L^2}^2 + \left\| \partial_i \tfrac{1}{\mu} \Phi_c^{k,3+2i} \right\|_{L^2}^2 \right),$$

with a constant $C_\nabla > 0$ that is independent of the step size τ. □

4.4. Demonstration of the uniform observability inequality

The results from Lemmas 4.5 and 4.8 at hand, we can now conclude the desired uniform interior observability estimate in Theorem 4.2. The important property of scheme (3.23) is the conservation of energy. This will be exploited in the following proof.

Proof of Theorem 4.2. We first take the sum of the estimates from Lemmas 4.5 and 4.8. The positivity assumption $\mu \geq \delta$ in (2.2) then implies the relation

$$\begin{aligned} &\sum_{k=0}^{N} \alpha(k\tau) \int_Q \mu |\mathbf{H}_c^k|^2 \,\mathrm{d}x \\ &\quad \leq \frac{1}{8} \sum_{k=1}^{N-1} \int_Q \mu |\mathbf{H}_c^k|^2 \,\mathrm{d}x + (C_c + C_\nabla) \Bigg(\sum_{k=1}^{N-1} (\|\mathbf{E}_c^k\|_{L^2}^2 + \|\Phi_c^k\|_{L^2}^2) \\ &\quad + \tau^2 \sum_{k=0}^{N-1} \Bigg(\left\| A \begin{pmatrix} \mathbf{E}_c^{k,1} \\ \mathbf{H}_c^{k,1} \\ 0 \end{pmatrix} \right\|^2 + \left\| B \begin{pmatrix} \mathbf{E}_c^{k,3} \\ \mathbf{H}_c^{k,3} \\ 0 \end{pmatrix} \right\|^2 + \sum_{i=1}^{3} \left\| D_i \begin{pmatrix} 0 \\ \mathbf{H}_c^{k,3+2i} \\ \Phi_c^{k,3+2i} \end{pmatrix} \right\|^2 \Bigg) \Bigg). \end{aligned} \tag{4.50}$$

We now employ that α is equal to 1 on $[3\mathring{\tau}, 6\mathring{\tau}]$, see the choice of α in Section 4.2. Taking also into account that the scheme (3.23) is energy conserving, we deduce the relations

$$\begin{aligned} &\sum_{k=1}^{N} \int_Q (\mu |\mathbf{H}_c^k|^2 + \varepsilon |\mathbf{E}_c^k|^2 + \mu |\Phi_c^k|^2) \,\mathrm{d}x \\ &\quad \leq 4 \sum_{k=\left\lceil \frac{3\mathring{\tau}}{\tau} \right\rceil}^{\left\lfloor \frac{6\mathring{\tau}}{\tau} \right\rfloor} \int_Q (\mu |\mathbf{H}_c^k|^2 + \varepsilon |\mathbf{E}_c^k|^2 + \mu |\Phi_c^k|^2) \,\mathrm{d}x \end{aligned}$$

$$\leq 4\sum_{k=1}^{N}\Big(\alpha(k\tau)\int_Q \mu|\mathbf{H}_c^k|^2\,\mathrm{d}x + \int_Q \varepsilon|\mathbf{E}_c^k|^2 + \mu|\Phi_c^k|^2\,\mathrm{d}x\Big). \tag{4.51}$$

Inserting (4.50) into (4.51) and subtracting the term $\frac{1}{2}\sum_{k=1}^{N-1}\int_Q \mu|\mathbf{H}_c^k|^2\,\mathrm{d}x$, we obtain the result

$$\begin{aligned}
&\sum_{k=1}^{N}\int_Q(\mu|\mathbf{H}_c^k|^2+\varepsilon|\mathbf{E}_c^k|^2+\mu|\Phi_c^k|^2)\,\mathrm{d}x\\
&\quad\leq 8(C_c+C_\nabla+\|\varepsilon\|_\infty)\sum_{k=1}^{N}\|\mathbf{E}_c^k\|_{L^2}^2+8(C_c+C_\nabla+\|\mu\|_\infty)\sum_{k=1}^{N}\|\Phi_c^k\|_{L^2}^2 \qquad (4.52)\\
&\quad\quad+8(C_c+C_\nabla)\tau^2\sum_{k=0}^{N-1}\left(\left\|A\begin{pmatrix}\mathbf{E}_c^{k,1}\\ \mathbf{H}_c^{k,1}\\ 0\end{pmatrix}\right\|^2+\left\|B\begin{pmatrix}\mathbf{E}_c^{k,3}\\ \mathbf{H}_c^{k,3}\\ 0\end{pmatrix}\right\|^2+\sum_{i=1}^{3}\left\|D_i\begin{pmatrix}0\\ \mathbf{H}_c^{k,3+2i}\\ \Phi_c^{k,3+2i}\end{pmatrix}\right\|^2\right).
\end{aligned}$$

Since scheme (3.23) is energy conserving, the identity

$$\int_Q(\mu|\mathbf{H}^0|^2+\varepsilon|\mathbf{E}^0|^2+\mu|\Phi^0|^2)\,\mathrm{d}x=\frac{1}{N}\sum_{k=1}^{N}\int_Q(\mu|\mathbf{H}_c^k|^2+\varepsilon|\mathbf{E}_c^k|^2+\mu|\Phi_c^k|^2)\,\mathrm{d}x$$

is valid. By definition of N in the statement of Theorem 4.2, we have $N\tau > 8\mathring{\tau}$, and we deduce from last equation and (4.52) the desired interior observability estimate (4.15). □

5. Exponential stability of the damped scheme

In this chapter, we prove the uniform exponential stability of the damped scheme (3.24), see Theorem 3.10. We hereby proceed in three steps. First, an energy identity is derived for the iterates of the scheme (3.24). This equation describes the decay of the energy between two subsequent iterates of (3.24). To apply our findings from Chapter 4 also for scheme (3.24), we afterwards compare appropriate substeps of the methods (3.23) and (3.24) in Section 5.2. Finally, the desired exponential stability result is concluded in Section 5.3. We here combine the observability inequality for the undamped scheme (3.23) with the results in Sections 5.1 and 5.2. Our reasoning is here inspired by the strategy in [TeZu03]. This paper deals with discretizations of a one-dimensional wave equation.

5.1. An energy identity for the damped ADI scheme

We start by introducing a substep formalism for the damped scheme (3.24). Similar reasoning is used in (4.1). We recall our permanent assumption that the parameters $\varepsilon, \mu, \tilde{\sigma}$, and η satisfy (2.2) and (3.3). Also important is here the number κ_Y from (3.25), and the space Y from (3.26). The arising splitting operators A, B, D_1, D_2, and D_3 are introduced in (3.19) and (3.20). The associated damping operators $V_\tau(\cdot)$ are defined in (3.22).

Let $n \in \mathbb{N}_0$, $i \in \{1,2,3\}$, and fix a number $\mathring{\tau} \in (0, \min\{\frac{1}{2}, \frac{\sqrt{2}}{\kappa_Y}\})$. Choose the time step size $\tau \in (0, \mathring{\tau}]$ for the damped scheme (3.24), as well as initial data $(\mathbf{E}^0, \mathbf{H}^0, \Phi^0) \in Y$. Define then the substeps

$$\begin{pmatrix}\mathbf{E}^{n,1}\\ \mathbf{H}^{n,1}\\ \Phi^{n,1}\end{pmatrix} := V_\tau(A)\begin{pmatrix}\mathbf{E}^{n}\\ \mathbf{H}^{n}\\ \Phi^{n}\end{pmatrix}; \qquad \begin{pmatrix}\mathbf{E}^{n,2}\\ \mathbf{H}^{n,2}\\ \Phi^{n,2}\end{pmatrix} := (I - \tfrac{\tau}{2}A)^{-1}\begin{pmatrix}\mathbf{E}^{n,1}\\ \mathbf{H}^{n,1}\\ \Phi^{n,1}\end{pmatrix};$$
$$\begin{pmatrix}\mathbf{E}^{n,3}\\ \mathbf{H}^{n,3}\\ \Phi^{n,3}\end{pmatrix} := (I + \tfrac{\tau}{2}A)\begin{pmatrix}\mathbf{E}^{n,2}\\ \mathbf{H}^{n,2}\\ \Phi^{n,2}\end{pmatrix}; \qquad \begin{pmatrix}\mathbf{E}^{n,4}\\ \mathbf{H}^{n,4}\\ \Phi^{n,4}\end{pmatrix} := V_\tau(B)\begin{pmatrix}\mathbf{E}^{n,3}\\ \mathbf{H}^{n,3}\\ \Phi^{n,3}\end{pmatrix};$$

$$\begin{pmatrix}\mathbf{E}^{n,5}\\ \mathbf{H}^{n,5}\\ \Phi^{n,5}\end{pmatrix} := (I - \tfrac{\tau}{2}B)^{-1}\begin{pmatrix}\mathbf{E}^{n,4}\\ \mathbf{H}^{n,4}\\ \Phi^{n,4}\end{pmatrix}; \qquad \begin{pmatrix}\mathbf{E}^{n,6}\\ \mathbf{H}^{n,6}\\ \Phi^{n,6}\end{pmatrix} := (I + \tfrac{\tau}{2}B)\begin{pmatrix}\mathbf{E}^{n,5}\\ \mathbf{H}^{n,5}\\ \Phi^{n,5}\end{pmatrix};$$
$$\begin{pmatrix}\mathbf{E}^{n,4+3i}\\ \mathbf{H}^{n,4+3i}\\ \Phi^{n,4+3i}\end{pmatrix} := V_\tau(D_i)\begin{pmatrix}\mathbf{E}^{n,3+3i}\\ \mathbf{H}^{n,3+3i}\\ \Phi^{n,3+3i}\end{pmatrix}; \qquad \begin{pmatrix}\mathbf{E}^{n,5+3i}\\ \mathbf{H}^{n,5+3i}\\ \Phi^{n,5+3i}\end{pmatrix} := (I - \tfrac{\tau}{2}D_i)^{-1}\begin{pmatrix}\mathbf{E}^{n,4+3i}\\ \mathbf{H}^{n,4+3i}\\ \Phi^{n,4+3i}\end{pmatrix};$$
$$\begin{pmatrix}\mathbf{E}^{n,6+3i}\\ \mathbf{H}^{n,6+3i}\\ \Phi^{n,6+3i}\end{pmatrix} := (I + \tfrac{\tau}{2}D_i)\begin{pmatrix}\mathbf{E}^{n,5+3i}\\ \mathbf{H}^{n,5+3i}\\ \Phi^{n,5+3i}\end{pmatrix}. \tag{5.1}$$

Note also the relation

$$\begin{pmatrix}\mathbf{E}^{n+1}\\ \mathbf{H}^{n+1}\\ \Phi^{n+1}\end{pmatrix} = \begin{pmatrix}\mathrm{e}^{-\tau\tilde{\sigma}}\mathbf{E}^{n,15}\\ \mathbf{H}^{n,15}\\ \mathrm{e}^{-\tau\eta}\Phi^{n,15}\end{pmatrix}.$$

The regularity results in Lemmas 3.12 and 3.17, as well as Corollary 3.16 imply that the above substeps and the next iterate of (3.24) remain in Y. This observation is crucial at the end of the proof for the exponential stability result in Theorem 3.10.

The desired energy identity (5.4) is derived by means of relations between the substeps in (5.1). The last substep satisfies the formula

$$\left\|\begin{pmatrix}\mathrm{e}^{\tau\tilde{\sigma}}\mathbf{E}^{n+1}\\ \mathbf{H}^{n+1}\\ \mathrm{e}^{\tau\eta}\Phi^{n+1}\end{pmatrix}\right\| = \left\|\begin{pmatrix}\mathbf{E}^{n,15}\\ \mathbf{H}^{n,15}\\ \Phi^{n,15}\end{pmatrix}\right\|.$$

This equation represents the damping effect of the terms $-\tilde{\sigma}\mathbf{E}$ and $-\eta\Phi$ in the extended Maxwell system (3.1). As the Cayley-Transform $S_\tau(L) = (I + \frac{\tau}{2}L)(I - \frac{\tau}{2}L)^{-1}$ is an isometry for $L \in \{A, B, D_1, D_2, D_3\}$, we furthermore infer the identity

$$\left\|\begin{pmatrix}\mathbf{E}^{n,3k}\\ \mathbf{H}^{n,3k}\\ \Phi^{n,3k}\end{pmatrix}\right\| = \left\|\begin{pmatrix}\mathbf{E}^{n,3k-2}\\ \mathbf{H}^{n,3k-2}\\ \Phi^{n,3k-2}\end{pmatrix}\right\|, \qquad k \in \{1,2,3,4,5\}.$$

We next employ the supplementary vectors

$$\begin{pmatrix}\check{\mathbf{E}}^{n,1}\\ \check{\mathbf{H}}^{n,1}\\ \check{\Phi}^{n,1}\end{pmatrix} := (I - \tfrac{\tau^2}{4}A^2)^{-1}\begin{pmatrix}\mathbf{E}^{n,1}\\ \mathbf{H}^{n,1}\\ \Phi^{n,1}\end{pmatrix}, \qquad \begin{pmatrix}\check{\mathbf{E}}^{n,4}\\ \check{\mathbf{H}}^{n,4}\\ \check{\Phi}^{n,4}\end{pmatrix} := (I - \tfrac{\tau^2}{4}B^2)^{-1}\begin{pmatrix}\mathbf{E}^{n,4}\\ \mathbf{H}^{n,4}\\ \Phi^{n,4}\end{pmatrix},$$
$$\begin{pmatrix}\check{\mathbf{E}}^{n,4+3i}\\ \check{\mathbf{H}}^{n,4+3i}\\ \check{\Phi}^{n,4+3i}\end{pmatrix} := (I - \tfrac{\tau^2}{4}D_i^2)^{-1}\begin{pmatrix}\mathbf{E}^{n,4+3i}\\ \mathbf{H}^{n,4+3i}\\ \Phi^{n,4+3i}\end{pmatrix}. \tag{5.2}$$

Now the skewadjointness of the operator D_i comes into play, see Lemma 3.8. Also the identity

$$D_i^2(I - \tfrac{\tau^2}{4}D_i^2)^{-1} = D_i(I + \tfrac{\tau}{2}D_i)^{-1}D_i(I - \tfrac{\tau}{2}D_i)^{-1}$$

is applied. We then obtain the relations

$$\begin{aligned}
\left\| \begin{pmatrix} \mathbf{E}^{n,3+3i} \\ \mathbf{H}^{n,3+3i} \\ \Phi^{n,3+3i} \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} \mathbf{E}^{n,4+3i} \\ \mathbf{H}^{n,4+3i} \\ \Phi^{n,4+3i} \end{pmatrix} \right\|^2 - 2\frac{\tau^3}{4}\left(\begin{pmatrix} \mathbf{E}^{n,4+3i} \\ \mathbf{H}^{n,4+3i} \\ \Phi^{n,4+3i} \end{pmatrix}, D_i^2(I - \tfrac{\tau^2}{4}D_i^2)^{-1} \begin{pmatrix} \mathbf{E}^{n,4+3i} \\ \mathbf{H}^{n,4+3i} \\ \Phi^{n,4+3i} \end{pmatrix} \right) \\
&\quad + \frac{\tau^6}{16}\left\| D_i^2 \begin{pmatrix} \check{\mathbf{E}}^{n,4+3i} \\ \check{\mathbf{H}}^{n,4+3i} \\ \check{\Phi}^{n,4+3i} \end{pmatrix} \right\|^2 \qquad (5.3) \\
&= \left\| \begin{pmatrix} \mathbf{E}^{n,4+3i} \\ \mathbf{H}^{n,4+3i} \\ \Phi^{n,4+3i} \end{pmatrix} \right\|^2 + \frac{\tau^3}{2}\left\| D_i \begin{pmatrix} \mathbf{E}^{n,5+3i} \\ \mathbf{H}^{n,5+3i} \\ \Phi^{n,5+3i} \end{pmatrix} \right\|^2 + \frac{\tau^6}{16}\left\| D_i^2 \begin{pmatrix} \check{\mathbf{E}}^{n,4+3i} \\ \check{\mathbf{H}}^{n,4+3i} \\ \check{\Phi}^{n,4+3i} \end{pmatrix} \right\|^2 .
\end{aligned}$$

This identity demonstrates the damping effect of the operator $V_\tau(D_i)$. The same arguments also provide the formulas

$$\left\| \begin{pmatrix} \mathbf{E}^{n,3} \\ \mathbf{H}^{n,3} \\ \Phi^{n,3} \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} \mathbf{E}^{n,4} \\ \mathbf{H}^{n,4} \\ \Phi^{n,4} \end{pmatrix} \right\|^2 + \frac{\tau^3}{2}\left\| B \begin{pmatrix} \mathbf{E}^{n,5} \\ \mathbf{H}^{n,5} \\ \Phi^{n,5} \end{pmatrix} \right\|^2 + \frac{\tau^6}{16}\left\| B^2 \begin{pmatrix} \check{\mathbf{E}}^{n,4} \\ \check{\mathbf{H}}^{n,4} \\ \check{\Phi}^{n,4} \end{pmatrix} \right\|^2 ,$$

$$\left\| \begin{pmatrix} \mathbf{E}^{n} \\ \mathbf{H}^{n} \\ \Phi^{n} \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} \mathbf{E}^{n,1} \\ \mathbf{H}^{n,1} \\ \Phi^{n,1} \end{pmatrix} \right\|^2 + \frac{\tau^3}{2}\left\| A \begin{pmatrix} \mathbf{E}^{n,2} \\ \mathbf{H}^{n,2} \\ \Phi^{n,2} \end{pmatrix} \right\|^2 + \frac{\tau^6}{16}\left\| A^2 \begin{pmatrix} \check{\mathbf{E}}^{n,1} \\ \check{\mathbf{H}}^{n,1} \\ \check{\Phi}^{n,1} \end{pmatrix} \right\|^2 .$$

Altogether, we then conclude the important energy identity

$$\begin{aligned}
&\left\| \begin{pmatrix} \mathbf{E}^{n+1} \\ \mathbf{H}^{n+1} \\ \Phi^{n+1} \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} \mathbf{E}^{n} \\ \mathbf{H}^{n} \\ \Phi^{n} \end{pmatrix} \right\|^2 \\
&= -(\mathrm{e}^{2\tau\tilde{\sigma}} - 1)\|\sqrt{\varepsilon}\mathbf{E}^{n+1}\|_{L^2}^2 - (\mathrm{e}^{2\tau\eta} - 1)\|\sqrt{\eta}\Phi^{n+1}\|_{L^2}^2 \qquad (5.4) \\
&\quad - \sum_{i=1}^{3}\left(\frac{\tau^3}{2}\left\| D_i \begin{pmatrix} \mathbf{E}^{n,5+3i} \\ \mathbf{H}^{n,5+3i} \\ \Phi^{n,5+3i} \end{pmatrix} \right\|^2 + \frac{\tau^6}{16}\left\| D_i^2 \begin{pmatrix} \check{\mathbf{E}}^{n,4+3i} \\ \check{\mathbf{H}}^{n,4+3i} \\ \check{\Phi}^{n,4+3i} \end{pmatrix} \right\|^2 \right) \\
&\quad - \frac{\tau^3}{2}\left(\left\| B \begin{pmatrix} \mathbf{E}^{n,5} \\ \mathbf{H}^{n,5} \\ \Phi^{n,5} \end{pmatrix} \right\|^2 + \left\| A \begin{pmatrix} \mathbf{E}^{n,2} \\ \mathbf{H}^{n,2} \\ \Phi^{n,2} \end{pmatrix} \right\|^2 \right) - \frac{\tau^6}{16}\left(\left\| B^2 \begin{pmatrix} \check{\mathbf{E}}^{n,4} \\ \check{\mathbf{H}}^{n,4} \\ \check{\Phi}^{n,4} \end{pmatrix} \right\|^2 + \left\| A^2 \begin{pmatrix} \check{\mathbf{E}}^{n,1} \\ \check{\mathbf{H}}^{n,1} \\ \check{\Phi}^{n,1} \end{pmatrix} \right\|^2 \right)
\end{aligned}$$

for the damped scheme (3.24). Dividing by τ, this formula is the discrete counterpart to the time derivative of the energy of the extended Maxwell system (3.1). For the remaining reasoning in this chapter, it is essential that the expressions on the right hand side of (5.4) and of the observability inequality (4.15) have a strong similarity.

5.2. Comparison of the damped and undamped schemes

Recall that we want to use the interior observability estimate from Theorem 4.2 to prove Theorem 3.10. The observability estimate, however, depends on the iterates and intermediate steps of the energy-conserving scheme (3.23). We thus want to replace the right hand side of the observability estimate by means of terms that only rely on the iterates of the damped scheme (3.24). Thereby, we follow the strategy of the proof of Theorem 1.1 in [TeZu03].

Throughout, we assume that $\varepsilon, \mu, \tilde{\sigma}$, and η satisfy (2.2) and (3.3). We employ the number κ_Y from (3.25), and fix a number $\mathring{\tau} \in (0, \min\{\frac{1}{2}, \frac{\sqrt{2}}{\kappa_Y}\})$. Moreover, we choose the same fixed time step size $\tau \in (0, \mathring{\tau}]$, as well as the same initial data $(\mathbf{E}^0, \mathbf{H}^0, \Phi^0) \in Y$ for both schemes (3.23) and (3.24).

Combining the observability estimate (4.15) for the undamped scheme (3.23) with the triangle inequality and Young inequality, we first derive the relations

$$
\begin{aligned}
&\int_Q (\mu|\mathbf{H}^0|^2 + \varepsilon|\mathbf{E}^0|^2 + \mu|\Phi^0|^2)\,\mathrm{d}x \\
&\quad \le 2C_o\tau \sum_{k=1}^{N} \Big(\|\mathbf{E}^k\|_{L^2}^2 + \|\Phi^k\|_{L^2}^2 + \|\mathbf{E}^k - \mathbf{E}_c^k\|_{L^2}^2 + \|\Phi^k - \Phi_c^k\|_{L^2}^2 \Big) \\
&\qquad + 2C_o\tau^3 \sum_{k=0}^{N-1} \Bigg(\left\| A \begin{pmatrix} \mathbf{E}^{k,2} \\ \mathbf{H}^{k,2} \\ \Phi^{k,2} \end{pmatrix} \right\|^2 + \left\| A \begin{pmatrix} \mathbf{E}^{k,2} - \mathbf{E}_c^{k,1} \\ \mathbf{H}^{k,2} - \mathbf{H}_c^{k,1} \\ \Phi^{k,2} - \Phi_c^{k,1} \end{pmatrix} \right\|^2 + \left\| B \begin{pmatrix} \mathbf{E}^{k,5} \\ \mathbf{H}^{k,5} \\ \Phi^{k,5} \end{pmatrix} \right\|^2 \\
&\qquad\qquad + \left\| B \begin{pmatrix} \mathbf{E}^{k,5} - \mathbf{E}_c^{k,3} \\ \mathbf{H}^{k,5} - \mathbf{H}_c^{k,3} \\ \Phi^{k,5} - \Phi_c^{k,3} \end{pmatrix} \right\|^2 \Bigg) \\
&\qquad + 2C_o\tau^3 \sum_{k=0}^{N-1} \sum_{i=1}^{3} \Bigg(\left\| D_i \begin{pmatrix} \mathbf{E}^{k,5+3i} \\ \mathbf{H}^{k,5+3i} \\ \Phi^{k,5+3i} \end{pmatrix} \right\|^2 + \left\| D_i \begin{pmatrix} \mathbf{E}^{k,5+3i} - \mathbf{E}_c^{k,3+2i} \\ \mathbf{H}^{k,5+3i} - \mathbf{H}_c^{k,3+2i} \\ \Phi^{k,5+3i} - \Phi_c^{k,3+2i} \end{pmatrix} \right\|^2 \Bigg), \qquad (5.5)
\end{aligned}
$$

where $N = \max\{k \in \mathbb{N} \mid N\tau \le 9\mathring{\tau}\}$ is the number from the statement of Theorem 3.10. The goal of this Section is to control the arising difference expressions on the right hand side by means of terms in the energy identity (5.4).

We abbreviate

$$\begin{pmatrix} \mathbf{E}^{k,\Delta} \\ \mathbf{H}^{k,\Delta} \\ \Phi^{k,\Delta} \end{pmatrix} := \begin{pmatrix} \mathbf{E}^{k} - \mathbf{E}_c^{k} \\ \mathbf{H}^{k} - \mathbf{H}_c^{k} \\ \Phi^{k} - \Phi_c^{k} \end{pmatrix}$$

for $k \in \mathbb{N}$, and proceed in the following manner. The difference terms in (5.5), that involve only substeps of (3.23) and (3.24) (but not the iterates), are first estimated by means of expressions from (5.4) and the energy of $(\mathbf{E}^{k,\Delta}, \mathbf{H}^{k,\Delta}, \Phi^{k,\Delta})$. Next, also the energy of the latter mentioned difference vector is bounded by terms from (5.4). This is achieved with a discrete Gronwall argument.

The following statement uses the supplementary vectors from (5.2). For a compact notation, we also put $D_{-1} := A$ and $D_0 := B$.

Lemma 5.1. *Let ε and μ satisfy the assumptions (2.2). Let $k \in \mathbb{N}_0$, $i \in \{0, \ldots, 3\}$, and $\tau \in (0, \mathring{\tau}]$. For the intermediate difference terms between the damped and undamped schemes (3.23) and (3.24), the estimates*

$$\left\| (I + \tfrac{\tau}{2} A) \begin{pmatrix} \boldsymbol{E}^{k,2} - \boldsymbol{E}_c^{k,1} \\ \boldsymbol{H}^{k,2} - \boldsymbol{H}_c^{k,1} \\ \Phi^{k,2} - \Phi_c^{k,1} \end{pmatrix} \right\|^2 \leq \frac{\tau^6}{16} \left\| A^2 \begin{pmatrix} \check{\boldsymbol{E}}^{k,1} \\ \check{\boldsymbol{H}}^{k,1} \\ \check{\Phi}^{k,1} \end{pmatrix} \right\|^2 + \tau^3 \left\| A \begin{pmatrix} \boldsymbol{E}^{k,2} \\ \boldsymbol{H}^{k,2} \\ \Phi^{k,2} \end{pmatrix} \right\|^2 + (1+\tau) \left\| \begin{pmatrix} \boldsymbol{E}^{k,\Delta} \\ \boldsymbol{H}^{k,\Delta} \\ \Phi^{k,\Delta} \end{pmatrix} \right\|^2, \tag{5.6}$$

$$\left\| (I + \tfrac{\tau}{2} D_i) \begin{pmatrix} \boldsymbol{E}^{k,5+3i} - \boldsymbol{E}_c^{k,3+2i} \\ \boldsymbol{H}^{k,5+3i} - \boldsymbol{H}_c^{k,3+2i} \\ \Phi^{k,5+3i} - \Phi_c^{k,3+2i} \end{pmatrix} \right\|^2 \leq \frac{\tau^6}{16} \left\| D_i^2 \begin{pmatrix} \check{\boldsymbol{E}}^{k,4+3i} \\ \check{\boldsymbol{H}}^{k,4+3i} \\ \check{\Phi}^{k,4+3i} \end{pmatrix} \right\|^2 + \tau^3 \left\| D_i \begin{pmatrix} \boldsymbol{E}^{k,5+3i} \\ \boldsymbol{H}^{k,5+3i} \\ \Phi^{k,5+3i} \end{pmatrix} \right\|^2 + (1+\tau) \left\| (I + \tfrac{\tau}{2} D_{i-1}) \begin{pmatrix} \boldsymbol{E}^{k,2+3i} - \boldsymbol{E}_c^{k,1+2i} \\ \boldsymbol{H}^{k,2+3i} - \boldsymbol{H}_c^{k,1+2i} \\ \Phi^{k,2+3i} - \Phi_c^{k,1+2i} \end{pmatrix} \right\|^2 \tag{5.7}$$

are valid.

Proof. Since the proofs of both inequalities follow essentially the same lines, we only prove the first one. Taking the difference between definitions (5.1) and (4.1), we on the one hand obtain the formula

$$\begin{pmatrix} \mathbf{E}^{k,2} - \mathbf{E}_c^{k,1} \\ \mathbf{H}^{k,2} - \mathbf{H}_c^{k,1} \\ \Phi^{k,2} - \Phi_c^{k,1} \end{pmatrix} = (I - \tfrac{\tau}{2} A)^{-1} (I - \tfrac{\tau^3}{4} A^2 (I - \tfrac{\tau^2}{4} A^2)^{-1})^{-1} \begin{pmatrix} \mathbf{E}^{k} \\ \mathbf{H}^{k} \\ \Phi^{k} \end{pmatrix} - (I - \tfrac{\tau}{2} A)^{-1} \begin{pmatrix} \mathbf{E}_c^{k} \\ \mathbf{H}_c^{k} \\ \Phi_c^{k} \end{pmatrix}$$

$$= (I - \tfrac{\tau}{2}A)^{-1}\tfrac{\tau^3}{4}A^2(I - \tfrac{\tau^2}{4}A^2)^{-1}\begin{pmatrix}\mathbf{E}^{k,1}\\ \mathbf{H}^{k,1}\\ \Phi^{k,1}\end{pmatrix}$$

$$+ (I - \tfrac{\tau}{2}A)^{-1}\begin{pmatrix}\mathbf{E}^{k,\Delta}\\ \mathbf{H}^{k,\Delta}\\ \Phi^{k,\Delta}\end{pmatrix}. \tag{5.8}$$

On the other hand, the skewadjointness of A, see Lemma 3.7, gives rise to the relation

$$\left(\tfrac{\tau^3}{4}A^2(I - \tfrac{\tau^2}{4}A^2)^{-1}\begin{pmatrix}\mathbf{E}^{k,1}\\ \mathbf{H}^{k,1}\\ \Phi^{k,1}\end{pmatrix}, \begin{pmatrix}\mathbf{E}^{k,\Delta}\\ \mathbf{H}^{k,\Delta}\\ \Phi^{k,\Delta}\end{pmatrix}\right)$$

$$= -\left(\tau^{3/2}A(I - \tfrac{\tau}{2}A)^{-1}\begin{pmatrix}\mathbf{E}^{k,1}\\ \mathbf{H}^{k,1}\\ \Phi^{k,1}\end{pmatrix}, \tfrac{\tau^{3/2}}{4}A(I - \tfrac{\tau}{2}A)^{-1}\begin{pmatrix}\mathbf{E}^{k,\Delta}\\ \mathbf{H}^{k,\Delta}\\ \Phi^{k,\Delta}\end{pmatrix}\right). \tag{5.9}$$

We now multiply equation (5.8) from the left by $(I + \frac{\tau}{2}A)$. Furthermore, the isometry of the Cayley-Transform of A, and (5.9) come into play. Using the supplementary vectors from (5.2), we arrive at the estimates

$$\left\|(I + \tfrac{\tau}{2}A)\begin{pmatrix}\mathbf{E}^{k,2} - \mathbf{E}_c^{k,1}\\ \mathbf{H}^{k,2} - \mathbf{H}_c^{k,1}\\ \Phi^{k,2} - \Phi_c^{k,1}\end{pmatrix}\right\|^2 \le \frac{\tau^6}{16}\left\|A^2\begin{pmatrix}\check{\mathbf{E}}^{k,1}\\ \check{\mathbf{H}}^{k,1}\\ \check{\Phi}^{k,1}\end{pmatrix}\right\|^2 + \tau^3\left\|A\begin{pmatrix}\mathbf{E}^{k,2}\\ \mathbf{H}^{k,2}\\ \Phi^{k,2}\end{pmatrix}\right\|^2$$

$$+ \frac{\tau}{16}\left\|\tau A(I - \tfrac{\tau}{2}A)^{-1}\begin{pmatrix}\mathbf{E}^{k,\Delta}\\ \mathbf{H}^{k,\Delta}\\ \Phi^{k,\Delta}\end{pmatrix}\right\|^2 + \left\|\begin{pmatrix}\mathbf{E}^{k,\Delta}\\ \mathbf{H}^{k,\Delta}\\ \Phi^{k,\Delta}\end{pmatrix}\right\|^2.$$

For the third term on the right hand side, we additionally take the identity

$$\tau A(I - \tfrac{\tau}{2}A)^{-1} = 2((I - \tfrac{\tau}{2}A)^{-1} - I),$$

as well as the contractivity of $(I - \frac{\tau}{2}A)^{-1}$ into account. This in particular implies the relation

$$\frac{\tau}{16}\left\|\tau A(I - \tfrac{\tau}{2}A)^{-1}\begin{pmatrix}\mathbf{E}^{k,\Delta}\\ \mathbf{H}^{k,\Delta}\\ \Phi^{k,\Delta}\end{pmatrix}\right\|^2 \le \tau\left\|\begin{pmatrix}\mathbf{E}^{k,\Delta}\\ \mathbf{H}^{k,\Delta}\\ \Phi^{k,\Delta}\end{pmatrix}\right\|^2,$$

and we conclude the desired estimate. □

It will be useful to have a slightly weaker version of the inequalities from Lemma 5.1.

Remark 5.2. By Lemmas 3.7 and 3.8, the splitting operators A, B and D_i are skewadjoint on X_{ext}. Consequently, we can weaken the first estimate in Lemma 5.1 to the form

$$\frac{\tau^2}{4}\left\| A \begin{pmatrix} \mathbf{E}^{k,2} - \mathbf{E}_c^{k,1} \\ \mathbf{H}^{k,2} - \mathbf{H}_c^{k,1} \\ \Phi^{k,2} - \Phi_c^{k,1} \end{pmatrix} \right\|^2 \\ \leq \frac{\tau^6}{16}\left\| A^2 \begin{pmatrix} \check{\mathbf{E}}^{k,1} \\ \check{\mathbf{H}}^{k,1} \\ 0 \end{pmatrix} \right\|^2 + \tau^3 \left\| A \begin{pmatrix} \mathbf{E}^{k,2} \\ \mathbf{H}^{k,2} \\ \Phi^{k,2} \end{pmatrix} \right\|^2 + (1+\tau) \left\| \begin{pmatrix} \mathbf{E}^{k,\Delta} \\ \mathbf{H}^{k,\Delta} \\ \Phi^{k,\Delta} \end{pmatrix} \right\|^2 .$$

Analogous modifications are true for the second estimate in Lemma 5.1. We will use these modifications for the proof of Theorem 3.10. ♢

We note that the upper bounds in Lemma 5.1 still involve the difference vector $(\mathbf{E}^{k,\Delta}, \mathbf{H}^{k,\Delta}, \Phi^{k,\Delta})$. In a next step, we estimate the energy of the latter vectors in terms of a discrete integral over the difference equation (5.4).

Lemma 5.3. *Let $\varepsilon, \mu, \tilde{\sigma}$, and η satisfy (2.2) and (3.3). There is a constant $C_\Delta > 0$ with*

$$\left\| \begin{pmatrix} \boldsymbol{E}^{k+1,\Delta} \\ \boldsymbol{H}^{k+1,\Delta} \\ \Phi^{k+1,\Delta} \end{pmatrix} \right\|^2 \leq C_\Delta \mathrm{e}^{6(k+1)\tau} \left(\left\| \begin{pmatrix} \boldsymbol{E}^{0} \\ \boldsymbol{H}^{0} \\ \Phi^{0} \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} \boldsymbol{E}^{k+1} \\ \boldsymbol{H}^{k+1} \\ \Phi^{k+1} \end{pmatrix} \right\|^2 \right)$$

for all $-1 \leq k \leq N-1$ and $\tau \in (0, \mathring{\tau}]$. The number C_Δ depends only on $\varepsilon, \mu, \tilde{\sigma}$, and η.

Proof. Recall our assumption, that both schemes (3.23) and (3.24) have the same starting value. This implies that the asserted statement is true for $k = -1$. Assume hence that $k \geq 0$. Using the definition of the substeps in (4.1) and (5.1), we derive the equations

$$\begin{pmatrix} \mathbf{E}^{k+1,\Delta} \\ \mathbf{H}^{k+1,\Delta} \\ \Phi^{k+1,\Delta} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\tau\tilde{\sigma}} - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-\tau\eta} - 1 \end{pmatrix} (I + \tfrac{\tau}{2} D_3) \begin{pmatrix} \mathbf{E}^{k,14} \\ \mathbf{H}^{k,14} \\ \Phi^{k,14} \end{pmatrix} \\ + (I + \tfrac{\tau}{2} D_3) \begin{pmatrix} \mathbf{E}^{k,14} - \mathbf{E}_c^{k,9} \\ \mathbf{H}^{k,14} - \mathbf{H}_c^{k,9} \\ \Phi^{k,14} - \Phi_c^{k,9} \end{pmatrix} \\ = \begin{pmatrix} \mathrm{e}^{-\tau\tilde{\sigma}} - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-\tau\eta} - 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\tau\tilde{\sigma}} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathrm{e}^{\tau\eta} \end{pmatrix} \begin{pmatrix} \mathbf{E}^{k+1} \\ \mathbf{H}^{k+1} \\ \Phi^{k+1} \end{pmatrix}$$

$$+ (I + \tfrac{\tau}{2} D_3) \begin{pmatrix} \mathbf{E}^{k,14} - \mathbf{E}_c^{k,9} \\ \mathbf{H}^{k,14} - \mathbf{H}_c^{k,9} \\ \Phi^{k,14} - \Phi_c^{k,9} \end{pmatrix}.$$

Set $C_{\tilde\sigma\eta} := \max\{\|\tilde\sigma\|_\infty, \|\eta\|_\infty\}$. We next combine the inequality $\|\mathrm{e}^{-\tau\xi} - 1\|_\infty \leq \tau\|\xi\|_\infty$, $\xi \in \{\tilde\sigma, \eta\}$, with the Cauchy-Schwarz and the Young inequalities. In this way, the relations

$$\begin{aligned}
\left\| \begin{pmatrix} \mathbf{E}^{k+1,\Delta} \\ \mathbf{H}^{k+1,\Delta} \\ \Phi^{k+1,\Delta} \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} (\mathrm{e}^{-\tau\tilde\sigma} - 1)\mathrm{e}^{\tau\tilde\sigma}\mathbf{E}^{k+1} \\ 0 \\ (\mathrm{e}^{-\tau\eta} - 1)\mathrm{e}^{\tau\eta}\Phi^{k+1} \end{pmatrix} \right\|^2 \\
&\quad + 2 \left(\begin{pmatrix} (\mathrm{e}^{-\tau\tilde\sigma} - 1)\mathrm{e}^{\tau\tilde\sigma}\mathbf{E}^{k+1} \\ 0 \\ (\mathrm{e}^{-\tau\eta} - 1)\mathrm{e}^{\tau\eta}\Phi^{k+1} \end{pmatrix}, (I + \tfrac{\tau}{2} D_3) \begin{pmatrix} \mathbf{E}^{k,14} - \mathbf{E}_c^{k,9} \\ \mathbf{H}^{k,14} - \mathbf{H}_c^{k,9} \\ \Phi^{k,14} - \Phi_c^{k,9} \end{pmatrix} \right) \\
&\quad + \left\| (I + \tfrac{\tau}{2} D_3) \begin{pmatrix} \mathbf{E}^{k,14} - \mathbf{E}_c^{k,9} \\ \mathbf{H}^{k,14} - \mathbf{H}_c^{k,9} \\ \Phi^{k,14} - \Phi_c^{k,9} \end{pmatrix} \right\|^2 \\
&\leq 2\tau C_{\tilde\sigma\eta}^2 \mathrm{e}^{2C_{\tilde\sigma\eta}} \left\| \begin{pmatrix} \mathbf{E}^{k+1} \\ 0 \\ \Phi^{k+1} \end{pmatrix} \right\|^2 + (1+\tau) \left\| (I + \tfrac{\tau}{2} D_3) \begin{pmatrix} \mathbf{E}^{k,14} - \mathbf{E}_c^{k,9} \\ \mathbf{H}^{k,14} - \mathbf{H}_c^{k,9} \\ \Phi^{k,14} - \Phi_c^{k,9} \end{pmatrix} \right\|^2
\end{aligned}$$

are obtained. With Lemma 5.1 and the assumption $\tau \leq 1$, we then infer the estimates

$$\begin{aligned}
\left\| \begin{pmatrix} \mathbf{E}^{k+1,\Delta} \\ \mathbf{H}^{k+1,\Delta} \\ \Phi^{k+1,\Delta} \end{pmatrix} \right\|^2 &\leq 2\tau C_{\tilde\sigma\eta}^2 \mathrm{e}^{2C_{\tilde\sigma\eta}} \left\| \begin{pmatrix} \mathbf{E}^{k+1} \\ 0 \\ \Phi^{k+1} \end{pmatrix} \right\|^2 \\
&\quad + \sum_{i=1}^{3} \left(\frac{2^{4-i}\tau^6}{16} \left\| D_i^2 \begin{pmatrix} \check{\mathbf{E}}^{k,4+3i} \\ \check{\mathbf{H}}^{k,4+3i} \\ \check{\Phi}^{k,4+3i} \end{pmatrix} \right\|^2 + 2^{4-i}\tau^3 \left\| D_i \begin{pmatrix} \mathbf{E}^{k,5+3i} \\ \mathbf{H}^{k,5+3i} \\ \Phi^{k,5+3i} \end{pmatrix} \right\|^2 \right) \\
&\quad + \tau^6 \left\| B^2 \begin{pmatrix} \check{\mathbf{E}}^{k,4} \\ \check{\mathbf{H}}^{k,4} \\ \check{\Phi}^{k,4} \end{pmatrix} \right\|^2 + 16\tau^3 \left\| B \begin{pmatrix} \mathbf{E}^{k,5} \\ \mathbf{H}^{k,5} \\ \Phi^{k,5} \end{pmatrix} \right\|^2 + 2\tau^6 \left\| A^2 \begin{pmatrix} \check{\mathbf{E}}^{k,1} \\ \check{\mathbf{H}}^{k,1} \\ \check{\Phi}^{k,1} \end{pmatrix} \right\|^2 \\
&\quad + 32\tau^3 \left\| A \begin{pmatrix} \mathbf{E}^{k,2} \\ \mathbf{H}^{k,2} \\ \Phi^{k,2} \end{pmatrix} \right\|^2 + (1+\tau)^6 \left\| \begin{pmatrix} \mathbf{E}^{k,\Delta} \\ \mathbf{H}^{k,\Delta} \\ \Phi^{k,\Delta} \end{pmatrix} \right\|^2 .
\end{aligned}$$

After comparing the right hand side with the energy identity (5.4), we conclude

$$\left\|\begin{pmatrix}\mathbf{E}^{k+1,\Delta}\\ \mathbf{H}^{k+1,\Delta}\\ \Phi^{k+1,\Delta}\end{pmatrix}\right\|^2 \le C_\Delta\left(\left\|\begin{pmatrix}\mathbf{E}^{k}\\ \mathbf{H}^{k}\\ \Phi^{k}\end{pmatrix}\right\|^2 - \left\|\begin{pmatrix}\mathbf{E}^{k+1}\\ \mathbf{H}^{k+1}\\ \Phi^{k+1}\end{pmatrix}\right\|^2\right) + \mathrm{e}^{6\tau}\left\|\begin{pmatrix}\mathbf{E}^{k,\Delta}\\ \mathbf{H}^{k,\Delta}\\ \Phi^{k,\Delta}\end{pmatrix}\right\|^2,$$

with a constant $C_\Delta = C_\Delta(\varepsilon,\mu,\tilde{\sigma},\eta) > 0$ being independent of $k \le N-1$ and $\tau > 0$. In presence of the initial choice $(\mathbf{E}^{0,\Delta},\mathbf{H}^{0,\Delta},\Phi^{0,\Delta}) = 0$, we derive by induction the inequality

$$\left\|\begin{pmatrix}\mathbf{E}^{k+1,\Delta}\\ \mathbf{H}^{k+1,\Delta}\\ \Phi^{k+1,\Delta}\end{pmatrix}\right\|^2 \le C_\Delta \mathrm{e}^{6(k+1)\tau}\sum_{j=0}^{k}\left(\left\|\begin{pmatrix}\mathbf{E}^{j}\\ \mathbf{H}^{j}\\ \Phi^{j}\end{pmatrix}\right\|^2 - \left\|\begin{pmatrix}\mathbf{E}^{j+1}\\ \mathbf{H}^{j+1}\\ \Phi^{j+1}\end{pmatrix}\right\|^2\right).$$

This leads to the asserted statement. □

5.3. Demonstration of the exponential stability result for the damped scheme

In this Section, we prove the desired uniform exponential decay of the iterates of the damped scheme (3.24). To that end, we combine the internal observability estimate from Theorem 4.2, the energy identity (5.4) and the estimates from Lemmas 5.1 and 5.3.

Proof of Theorem 3.10. Set

$$\mathring{\tau} = \zeta \cdot \max\{\tfrac{1}{2}, \tfrac{\sqrt{2}}{\kappa_Y}\}, \quad \text{and} \quad N := \max\{k \in \mathbb{N} \mid k\tau \le 9\mathring{\tau}\},$$

involving the fixed number ζ from the statement of Theorem 3.10. The step size τ is then an element of $(0,\mathring{\tau}]$. We first assume the starting value $(\mathbf{E}^0,\mathbf{H}^0,\Phi^0)$ for scheme (3.24) to belong to Y.

The proof mainly consists in estimating all terms on the right hand side of (5.5). It is important that our results from Sections 4.1 and 4.2 only require, that the initial data for schemes (3.23) and (3.24) have to be chosen within Y, and that they have to coincide. The regularity results in Lemmas 3.12 and 3.17, as well as Corollary 3.16 then imply that all iterates of the damped and undamped schemes stay within Y. This is essential, when we want to iterate our argument. (This means, that we want to take the N-th iterate $(\mathbf{E}^N,\mathbf{H}^N,\Phi^N) \in Y$ as a new initial value.)

In the following, $\tilde{C} > 0$ denotes a constant that is allowed to change from line to line. It depends, however, solely on $\varepsilon,\mu,\tilde{\sigma},\eta$, and Q. Only the last summand on the right hand side of (5.5) is considered. All other difference expressions can be handled similar, but with less effort. We first modify inequality (5.7) for $i = 3$ in the spirit of Remark 5.2, and insert then recursively all other estimates from Lemma 5.1 into each other. As $\tau < 1$, we infer the inequalities

$$
\begin{aligned}
&\tau^3 \left\| D_3 \begin{pmatrix} \mathbf{E}^{k,14} - \mathbf{E}_c^{k,9} \\ \mathbf{H}^{k,14} - \mathbf{H}_c^{k,9} \\ \Phi^{k,14} - \Phi_c^{k,9} \end{pmatrix} \right\|^2 \qquad (5.10)\\
&\leq \tilde{C} \left(\tau^7 \left\| D_3^2 \begin{pmatrix} \check{\mathbf{E}}^{k,13} \\ \check{\mathbf{H}}^{k,13} \\ \check{\Phi}^{k,13} \end{pmatrix} \right\|^2 + \tau^4 \left\| D_3 \begin{pmatrix} \mathbf{E}^{k,14} \\ \mathbf{H}^{k,14} \\ \Phi^{k,14} \end{pmatrix} \right\|^2 + \tau \left\| (I + \tfrac{\tau}{2} D_2) \begin{pmatrix} \mathbf{E}^{k,11} - \mathbf{E}_c^{k,7} \\ \mathbf{H}^{k,11} - \mathbf{H}_c^{k,7} \\ \Phi^{k,11} - \Phi_c^{k,7} \end{pmatrix} \right\|^2 \right) \\
&\leq \tilde{C} \Bigg[\sum_{i=1}^{3} \left(\tau^7 \left\| D_i^2 \begin{pmatrix} \check{\mathbf{E}}^{k,4+3i} \\ \check{\mathbf{H}}^{k,4+3i} \\ \check{\Phi}^{k,4+3i} \end{pmatrix} \right\|^2 + \tau^4 \left\| D_i \begin{pmatrix} \mathbf{E}^{k,5+3i} \\ \mathbf{H}^{k,5+3i} \\ \Phi^{k,5+3i} \end{pmatrix} \right\|^2 \right) + \tau^7 \left\| B^2 \begin{pmatrix} \check{\mathbf{E}}^{k,4} \\ \check{\mathbf{H}}^{k,4} \\ \check{\Phi}^{k,4} \end{pmatrix} \right\|^2 \\
&\qquad + \tau^4 \left\| B \begin{pmatrix} \mathbf{E}^{k,5} \\ \mathbf{H}^{k,5} \\ \Phi^{k,5} \end{pmatrix} \right\|^2 + \tau^7 \left\| A^2 \begin{pmatrix} \check{\mathbf{E}}^{k,1} \\ \check{\mathbf{H}}^{k,1} \\ \check{\Phi}^{k,1} \end{pmatrix} \right\|^2 + \tau^4 \left\| A \begin{pmatrix} \mathbf{E}^{k,2} \\ \mathbf{H}^{k,2} \\ \Phi^{k,2} \end{pmatrix} \right\|^2 + \tau \left\| \begin{pmatrix} \mathbf{E}^{k,\Delta} \\ \mathbf{H}^{k,\Delta} \\ \Phi^{k,\Delta} \end{pmatrix} \right\|^2 \Bigg].
\end{aligned}
$$

A similar argument also bounds the remaining difference terms on the right hand side of (5.5) by means of the right hand side of (5.10) (after appropriately modifying the number $\tilde{C}$). Then, all expressions on the right hand side of (5.10) appear also in the difference equation (5.4), except the last one. Lemma 5.3, however, also bounds the last summand. With $N\tau \leq 9\mathring{\tau}$, we consequently obtain the relations

$$
\begin{aligned}
&\tau \sum_{k=1}^{N} \left\| \begin{pmatrix} \mathbf{E}^{k,\Delta} \\ \mathbf{H}^{k,\Delta} \\ \Phi^{k,\Delta} \end{pmatrix} \right\|^2 + \tau^3 \sum_{k=0}^{N-1} \left(\left\| A \begin{pmatrix} \mathbf{E}^{k,2} - \mathbf{E}_c^{k,1} \\ \mathbf{H}^{k,2} - \mathbf{H}_c^{k,1} \\ \Phi^{k,2} - \Phi_c^{k,1} \end{pmatrix} \right\|^2 + \left\| B \begin{pmatrix} \mathbf{E}^{k,5} - \mathbf{E}_c^{k,3} \\ \mathbf{H}^{k,5} - \mathbf{H}_c^{k,3} \\ \Phi^{k,5} - \Phi_c^{k,3} \end{pmatrix} \right\|^2 \right. \\
&\qquad\qquad \left. + \left\| D_i \begin{pmatrix} \mathbf{E}^{k,5+3i} - \mathbf{E}_c^{k,3+2i} \\ \mathbf{H}^{k,5+3i} - \mathbf{H}_c^{k,3+2i} \\ \Phi^{k,5+3i} - \Phi_c^{k,3+2i} \end{pmatrix} \right\|^2 \right) \\
&\leq \tilde{C}\tau \sum_{k=0}^{N-1} \left(\left\| \begin{pmatrix} \mathbf{E}^{k} \\ \mathbf{H}^{k} \\ \Phi^{k} \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} \mathbf{E}^{k+1} \\ \mathbf{H}^{k+1} \\ \Phi^{k+1} \end{pmatrix} \right\|^2 \right) + \tilde{C}\tau \mathrm{e}^{54\mathring{\tau}} \sum_{k=1}^{N-1} \left(\left\| \begin{pmatrix} \mathbf{E}^{0} \\ \mathbf{H}^{0} \\ \Phi^{0} \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} \mathbf{E}^{k+1} \\ \mathbf{H}^{k+1} \\ \Phi^{k+1} \end{pmatrix} \right\|^2 \right) \\
&\leq \tilde{C}\mathring{\tau} \mathrm{e}^{54\mathring{\tau}} \left(\left\| \begin{pmatrix} \mathbf{E}^{0} \\ \mathbf{H}^{0} \\ \Phi^{0} \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} \mathbf{E}^{N} \\ \mathbf{H}^{N} \\ \Phi^{N} \end{pmatrix} \right\|^2 \right). \qquad (5.11)
\end{aligned}
$$

We now use estimate (5.11) for the difference terms on the right hand side of (5.5). For the remaining terms in (5.5), we proceed as above. In this way, we arrive at the inequalities

$$\begin{aligned}
\left\| \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{H}^0 \\ \Phi^0 \end{pmatrix} \right\|^2 &\le \tilde{C} \sum_{k=0}^{N-1} \left(\left\| \begin{pmatrix} \mathbf{E}^k \\ \mathbf{H}^k \\ \Phi^k \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} \mathbf{E}^{k+1} \\ \mathbf{H}^{k+1} \\ \Phi^{k+1} \end{pmatrix} \right\|^2 \right) \\
&\quad + \tilde{C}\mathring{\tau}\mathrm{e}^{54\mathring{\tau}} \left(\left\| \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{H}^0 \\ \Phi^0 \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} \mathbf{E}^N \\ \mathbf{H}^N \\ \Phi^N \end{pmatrix} \right\|^2 \right) \\
&\le \tilde{C}\mathring{\tau}\mathrm{e}^{54\mathring{\tau}} \left(\left\| \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{H}^0 \\ \Phi^0 \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} \mathbf{E}^N \\ \mathbf{H}^N \\ \Phi^N \end{pmatrix} \right\|^2 \right).
\end{aligned}$$

This is equivalent to

$$\left\| \begin{pmatrix} \mathbf{E}^N \\ \mathbf{H}^N \\ \Phi^N \end{pmatrix} \right\|^2 \le \left(1 - \frac{1}{\tilde{C}\mathring{\tau}\mathrm{e}^{54\mathring{\tau}}} \right) \left\| \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{H}^0 \\ \Phi^0 \end{pmatrix} \right\|^2. \tag{5.12}$$

As mentioned above, we can now iterate our argument. As a result, we infer with the same constant $\tilde{C}$ as in (5.12) the estimate

$$\left\| \begin{pmatrix} \mathbf{E}^{mN} \\ \mathbf{H}^{mN} \\ \Phi^{mN} \end{pmatrix} \right\|^2 \le \left(1 - \frac{1}{\tilde{C}\mathring{\tau}\mathrm{e}^{54\mathring{\tau}}} \right)^m \left\| \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{H}^0 \\ \Phi^0 \end{pmatrix} \right\|^2 = \mathrm{e}^{-\omega m\mathring{\tau}} \left\| \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{H}^0 \\ \Phi^0 \end{pmatrix} \right\|^2,$$

with $\omega := \frac{1}{\mathring{\tau}} \ln(\frac{\tilde{C}\mathring{\tau}\mathrm{e}^{54\mathring{\tau}}}{\tilde{C}\mathring{\tau}\mathrm{e}^{54\mathring{\tau}}-1}) > 0$ for all $m \in \mathbb{N}$, compare the proof for Theorem 3.3 in [Nica03]. In particular, ω is independent of τ and the initial data. For starting values in Y, the asserted decay estimate can now be concluded.

Let $k \in \mathbb{N}$ be fixed. We choose $m \in \mathbb{N}_0$ and $r \in \{0, \dots, N-1\}$ with $k = mN + r$. As the energy of the iterates $(\mathbf{E}^n, \mathbf{H}^n, \Phi^n)$ is decreasing, we conclude the relations

$$\begin{aligned}
\left\| \begin{pmatrix} \mathbf{E}^k \\ \mathbf{H}^k \\ \Phi^k \end{pmatrix} \right\|^2 \le \left\| \begin{pmatrix} \mathbf{E}^{mN} \\ \mathbf{H}^{mN} \\ \Phi^{mN} \end{pmatrix} \right\|^2 \le \mathrm{e}^{-\omega m\mathring{\tau}} \left\| \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{H}^0 \\ \Phi^0 \end{pmatrix} \right\|^2 &\le \mathrm{e}^{\omega\mathring{\tau}} \mathrm{e}^{-\omega k\tau} \left\| \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{H}^0 \\ \Phi^0 \end{pmatrix} \right\|^2 \\
&= \frac{\tilde{C}\mathring{\tau}\mathrm{e}^{54\mathring{\tau}}}{\tilde{C}\mathring{\tau}\mathrm{e}^{54\mathring{\tau}} - 1} \mathrm{e}^{-\omega k\tau} \left\| \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{H}^0 \\ \Phi^0 \end{pmatrix} \right\|^2.
\end{aligned}$$

Choosing $M := \frac{\tilde{C}\mathring{\tau}\mathrm{e}^{54\mathring{\tau}}}{\tilde{C}\mathring{\tau}\mathrm{e}^{54\mathring{\tau}}-1} > 0$, we have derived the desired exponential stability estimate.

As the space Y is dense in $X_{\mathrm{ext}} = L^2(Q)^7$ and the damped scheme defines a bounded mapping on X_{ext}, the same inequality is valid for all initial data in X_{ext}. $\square$

6. Error analysis for the damped scheme

We show in this Chapter, that the iterates of the damped scheme (3.24) converge in the dual space Y^* of Y with order one to the solution of the original Maxwell system (2.1), see Theorem 6.5. The space Y is defined in (3.26). For this statement, we need to assume that the initial data of the scheme and of the original Maxwell system are chosen sufficiently regular and compatible, roughly speaking. To demonstrate Theorem 6.5, we furthermore modify arguments from Section 4 of [EiJS19]. Our analysis proceeds in the following way.

We first show that the damped scheme (3.24) is stable in Y. Here, we apply the regularity statements from Chapter 3. In Section 6.2, we define supplementary operators. Among other, the Λ-operators from [HaOs08, HoJS15, EiSc18, EiJS19] are introduced here. The final error result is then obtained in Section 6.3 by estimating the local error in Lemma 6.4 and controlling the error propagation in the proof of Theorem 6.5.

6.1. Stability of the damped scheme

To control the error propagation, we need the stability of the damped scheme (3.24) in Y. Therefore, each operator is estimated separately, that corresponds to one substep in (5.1).

Throughout this Chapter, we assume that $\varepsilon, \mu, \tilde{\sigma}$, and η satisfy (2.2) and (3.3). In the following, we use the parts of the splitting operators from (3.27), as well as the number κ_Y from (3.25). Let $L \in \{A_Y, B_Y, D_{1,Y}, D_{2,Y}, D_{3,Y}\}$. Lemma 3.12 and Corollary 3.16 already bound the Cayley-Transform $S_\tau(L) = (I + \frac{\tau}{2}L)(I - \frac{\tau}{2}L)^{-1}$ by

$$\|S_\tau(L)\|_{\mathscr{B}(Y)} \leq \mathrm{e}^{3\kappa_Y \tau}, \qquad \tau \in (0, \tau_0], \tag{6.1}$$

where $\tau_0 > 0$ is a constant depending only on κ_Y. The regularity assumption on $\tilde{\sigma}$ and η implies the relation

$$\left\| \begin{pmatrix} \mathrm{e}^{-\tau\tilde{\sigma}} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathrm{e}^{-\tau\eta} \end{pmatrix} \right\|_{\mathscr{B}(Y)} \leq \mathrm{e}^{\tau \tilde{C}_S}, \qquad \tau \geq 0, \tag{6.2}$$

with a uniform constant $\tilde{C}_S = \tilde{C}_S(\tilde{\sigma}, \eta) > 0$. This operator matrix is associated with the last intermediate step in (3.24).

The next lemma also bounds the operator $V_\tau(L)$ from (3.22) in Y. The statements will furthermore be used to estimate the local error of (3.24).

Lemma 6.1. *Let $\varepsilon, \mu, \tilde{\sigma}$, and η satisfy (2.2) and (3.3). Let also $L \in \{A_Y, B_Y, D_{1,Y}, D_{2,Y}, D_{3,Y}\}$. The operator $V_\tau(L)$ is well-defined in Y for all $\tau \in (0, \frac{1}{\kappa_Y})$. The estimates*

$$\left\| \tfrac{\tau^2}{4} L^2 (I - \tfrac{\tau^2}{4} L^2)^{-1} \right\|_{\mathscr{B}(Y)} \leq 3, \qquad \|V_\tau(L)\|_{\mathscr{B}(Y)} \leq \frac{1}{1 - 3\tau}$$

are valid for all $\tau \in (0, \tilde{\tau}_0)$ with a uniform constant $\tilde{\tau}_0 \in (0, \min\{\frac{1}{6}, \frac{1}{\kappa_Y}\})$.

Proof. Lemma 3.12 respectively Corollary 3.16 imply that the inverse $(I - \tau^2 L^2)^{-1} = (I - \tau L)^{-1}(I + \tau L)^{-1}$ is well-defined for $\tau \in (0, 1/\kappa_Y)$. The results also provide the bound

$$\left\| (I - \tau^2 L^2)^{-1} \right\|_{\mathscr{B}(Y)} \leq \left\| (I - \tau L)^{-1} \right\|_{\mathscr{B}(Y)} \left\| (I + \tau L)^{-1} \right\|_{\mathscr{B}(Y)} \leq \frac{1}{(1 - \tau \kappa_Y)^2}$$

for $\tau \in (0, \frac{1}{\kappa_Y})$. As a result, there is a number $\tilde{\tau}_0 = \tilde{\tau}_0 \in (0, \min\{\frac{1}{6}, \frac{1}{\kappa_Y}\})$ with $\left\| (I - \frac{\tau^2}{4} L^2)^{-1} \right\|_{\mathscr{B}(Y)} \leq 2$ for all $\tau \in (0, \tilde{\tau}_0)$.

With the formula $\frac{\tau^2}{4} L^2 (I - \frac{\tau^2}{4} L^2)^{-1} = -I + (I - \frac{\tau^2}{4} L^2)^{-1}$, we further infer the estimate $\left\| \frac{\tau^2}{4} L^2 (I - \frac{\tau^2}{4} L^2)^{-1} \right\|_{\mathscr{B}(Y)} \leq 3$. Since $\tilde{\tau}_0 < \frac{1}{6}$, our arguments show that the operator $V_\tau(L)$ is well-defined on Y. It is moreover bounded, according to the relations

$$\begin{aligned} \|V_\tau(L)\|_{\mathscr{B}(Y)} = \left\| (I - \tfrac{\tau^3}{4} L^2 (I - \tfrac{\tau^2}{4} L^2)^{-1})^{-1} \right\|_{\mathscr{B}(Y)} &\leq \frac{1}{1 - \tau \left\| \frac{\tau^2}{4} L^2 (I - \frac{\tau^2}{4} L^2)^{-1} \right\|_{\mathscr{B}(Y)}} \\ &\leq \frac{1}{1 - 3\tau} \end{aligned}$$

for all $\tau \in (0, \tilde{\tau}_0)$. □

By means of the above considerations, we can now provide the desired stability result for the damped scheme (3.24) in Y.

Proposition 6.2. *Let $\varepsilon, \mu, \tilde{\sigma}$, and η satisfy (2.2) and (3.3), and let $T > 0$. The stability estimate*

$$\left\| \begin{pmatrix} \boldsymbol{E}^n \\ \boldsymbol{H}^n \\ \Phi^n \end{pmatrix} \right\|_Y \leq \mathrm{e}^{C_{\mathrm{stab}} T} \left\| \begin{pmatrix} \boldsymbol{E}^0 \\ \boldsymbol{H}^0 \\ \Phi^0 \end{pmatrix} \right\|_Y$$

is valid for all $(\boldsymbol{E}^0, \boldsymbol{H}^0, \Phi^0) \in Y$, $\tau \in (0, \check{\tau}_0)$, and $n \in \mathbb{N}$ with $n\tau \leq T$. Here, the numbers C_{stab}, and $\check{\tau}_0$ are positive, and depend only on $\varepsilon, \mu, \tilde{\sigma}, \eta$, and Q.

Proof. Set $\check{\tau}_0 := \min\{\tau_0, \tilde{\tau}_0\} \in (0, \frac{1}{6})$ with τ_0 from Lemma 3.12 and Corollary 3.16, and $\tilde{\tau}_0$ from Lemma 6.1. The number $\check{\tau}_0$ then only depends on the coefficients $\varepsilon, \mu, \tilde{\sigma}$, and η from the extended Maxwell system (3.1). We also use the number $N := \max\{k \in \mathbb{N} \mid k\tau \leq T\}$. The arguments before this proposition now imply the bound

$$\left\| \begin{pmatrix} \mathbf{E}^n \\ \mathbf{H}^n \\ \Phi^n \end{pmatrix} \right\|_Y \leq \left(\mathrm{e}^{\tau \tilde{C}_S} \frac{1}{(1-3\tau)^5} \mathrm{e}^{15\kappa_Y \tau} \right)^N \left\| \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{H}^0 \\ \Phi^0 \end{pmatrix} \right\|_Y, \qquad \tau \in (0, \check{\tau}_0).$$

Combining the estimate $\frac{1}{1-3\tau} \leq \mathrm{e}^{C\tau}$, $C = \frac{3}{1-3\check{\tau}_0} > 0$, $\tau \in (0, \check{\tau}_0)$, with the relation $N\tau \leq T$, the asserted inequality follows with $C_{\mathrm{stab}} := \tilde{C}_S + 5C + 15\kappa_Y$. □

6.2. Supplementary framework

In this Section, we introduce some auxiliary operators for the analysis of the local error. They come into play, when we expand the iterates of scheme (3.24). Let $L \in \{A_Y, B_Y, D_{1,Y}, D_{2,Y}, D_{3,Y}\}$, and $\tau \in (0, \check{\tau}_0)$ with $\check{\tau}_0$ from Proposition 6.2. We first deal with the operator $V_\tau(L)$ from (3.22). We have the representation

$$\begin{aligned} V_\tau(L) &= (I - \tfrac{\tau^3}{4} L^2 (I - \tfrac{\tau^2}{4} L^2)^{-1})^{-1} = \sum_{n=0}^{\infty} (\tfrac{\tau^3}{4} L^2 (I - \tfrac{\tau^2}{4} L^2)^{-1})^n \\ &= I + V_\tau^{(1)}(L) = I + \tfrac{\tau^3}{4} L^2 (I - \tfrac{\tau^2}{4} L^2)^{-1} + V_\tau^{(2)}(L). \end{aligned} \tag{6.3}$$

Here, we use the operators

$$V_\tau^{(i)}(L) := \sum_{n=i}^{\infty} (\tfrac{\tau^3}{4} L^2 (I - \tfrac{\tau^2}{4} L^2)^{-1})^n, \qquad i \in \{1, 2\},$$

which can be estimated by means of Lemma 6.1 and $\tau < 1/6$ to

$$\begin{aligned} \|V_\tau^{(1)}(L)\|_{\mathscr{B}(Y)} &= \left\| \sum_{n=1}^{\infty} (\tfrac{\tau^3}{4} J^2 (I - \tfrac{\tau^2}{4} J^2)^{-1})^n \right\|_{\mathscr{B}(Y)} \leq \frac{1}{1-3\tau} - 1 \leq 6\tau, \\ \|V_\tau^{(2)}(L)\|_{\mathscr{B}(Y)} &= \left\| \sum_{n=2}^{\infty} (\tfrac{\tau^3}{4} J^2 (I - \tfrac{\tau^2}{4} J^2)^{-1})^n \right\|_{\mathscr{B}(Y)} \leq \frac{1}{1-3\tau} - 1 - 3\tau \leq 18\tau^2. \end{aligned} \tag{6.4}$$

To write the expansions in (6.3) in a convenient form, we define the mappings

$$F^V(j, k, L) := \begin{cases} V_\tau(L) & \text{if } j = k = 0, \\ V_\tau^{(k)}(L) & \text{if } j = k > 0, \\ \frac{\tau^{3j}}{4^j} L^{2j} (I - \frac{\tau^2}{4} L^2)^{-j} & \text{if } j < k, \end{cases} \tag{6.5}$$

for $j \leq k \in \{0,1,2\}$. We then obtain the formula

$$F^V(0,0,L) = \sum_{j=0}^{k} F^V(j,k,L), \qquad k \in \{0,1,2\}.$$

We follow now the preparatory concepts from Section 4.1 in [EiJS19] to derive analogous representations for the Cayley-Transforms $S_\tau(L) = (I + \frac{\tau}{2}L)(I - \frac{\tau}{2}L)^{-1}$, as well as for the semigroup $(\mathrm{e}^{tM_{\mathrm{ext},1}})_{t\geq 0}$. The latter is introduced in Proposition 3.5.

With the relation $(I - \frac{\tau}{2}L)^{-1} = I + \frac{\tau}{2}L(I - \frac{\tau}{2}L)^{-1}$, the identities

$$\begin{aligned} S_\tau(L) &= (I + \tfrac{\tau}{2}L)(I + \tfrac{\tau}{2}L(I - \tfrac{\tau}{2}L)^{-1}) \\ &= I + \tfrac{\tau}{2}L(I + S_\tau(L)), && (6.6) \\ &= I + \tau L + \tfrac{\tau^2}{4}L^2(I + S_\tau(L)) && (6.7) \end{aligned}$$

follow. While the first two equations are true on $\mathcal{D}(L)$, the last one holds on $\mathcal{D}(L^2)$. To obtain the third identity, we recursively insert the second equation for $S_\tau(L)$. As above, we aim for a compact representation of (6.6) and (6.7). To that end, we put

$$F(j,k,L) := \begin{cases} S_\tau(L) & \text{if } j = k = 0, \\ \frac{\tau^k}{2^k}(I + S_\tau(L))L^k & \text{if } j = k > 0, \\ \frac{\tau^j}{j!}L^j & \text{if } j < k, \end{cases} \tag{6.8}$$

for $j \leq k \in \{0,1,2\}$. We then arrive at the formula

$$F(0,0,L) = \sum_{j=0}^{k} F(j,k,L).$$

For the semigroup $(\mathrm{e}^{tM_{\mathrm{ext},1}})_{t\geq 0}$ from Proposition 3.5, we employ operators that are also used in [HaOs08, HoJS15, EiSc18, EiSc17, EiJS19]. These operators will again play a crucial role in the error analysis in the second part of this thesis, see Chapter 10.

Let $\tilde{L}$ be the generator of a strongly continuous semigroup $(\mathrm{e}^{t\tilde{L}})_{t\geq 0}$ on the space $X_{\mathrm{ext},1}$ from Section 3.1. We define the mappings

$$\Lambda_{j,\tilde{L}}(\tau) := \frac{1}{\tau^j (j-1)!} \int_0^\tau (\tau - s)^{j-1} \mathrm{e}^{s\tau}\,\mathrm{d}s, \quad j \in \mathbb{N}, \qquad \Lambda_{0,\tilde{L}}(\tau) := \mathrm{e}^{\tau\tilde{L}}.$$

By construction, these operators are bounded on $X_{\mathrm{ext},1}$, and the vector $\Lambda_{j,\tilde{L}}(\tau)z$ belongs to $\mathcal{D}(\tilde{L})$ for $j \in \mathbb{N}$, $z \in X_{\mathrm{ext},1}$. Furthermore, the recursion formula

$$\tau\tilde{L}\Lambda_{j+1,\tilde{L}}(\tau) = \Lambda_{j,\tilde{L}}(\tau) - \frac{1}{j!}I, \qquad j \in \mathbb{N}_0, \tag{6.9}$$

follows with integration by parts. Choose now $\tilde{L} = M_{\mathrm{ext},1}$, where $M_{\mathrm{ext},1}$ is the part of the extended Maxwell operator M_{ext} in $X_{\mathrm{ext},1}$, see Section 3.1. The linear growth bound from Proposition 3.5 gives rise to the estimate

$$\left\| \Lambda_{j,M_{\mathrm{ext},1}}(\tau) \right\|_{\mathscr{B}(X_{\mathrm{ext},1})} \leq \frac{2C_{\mathrm{stab},1}}{j!}. \tag{6.10}$$

We finally choose

$$\tilde{L} = K_d := \begin{pmatrix} -\tilde{\sigma} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\eta \end{pmatrix}.$$

Applying identity (6.9) twice, we here infer the representation

$$\mathrm{e}^{\tau K_d} = I + \tau K_d \Lambda_{1,K_d}(\tau) = I + \tau K_d + \tau^2 K_d^2 \Lambda_{2,K_d}(\tau). \tag{6.11}$$

Using the supplementary mappings

$$F^{\tilde{\sigma}}(j,k) := \begin{cases} \mathrm{e}^{\tau K_d} & \text{if } j = k = 0, \\ \tau^j K_d^j \Lambda_{j,K_d}(\tau) & \text{if } j = k > 0, \\ \tau^j K_d^j & \text{if } j < k, \end{cases} \tag{6.12}$$

for $j \leq k \in \{0,1,2\}$, the two identities in (6.11) have the form

$$F^{\tilde{\sigma}}(0,0) = \sum_{j=0}^{k} F^{\tilde{\sigma}}(j,k), \qquad k \in \{0,1,2\}.$$

6.3. Convergence result for the damped ADI scheme

The above preparations at hand, we can now show that the damped scheme (3.24) converges with order one in Y^*. To that end, we proceed in two steps. First, we demonstrate that the local error is of order two, see Lemma 6.4. Then, we conclude the global error result with Lady Windermere's fan. In this second step, we apply our stability result and the bound for the semigroup $(\mathrm{e}^{tM_{\mathrm{ext},1}})_{t\geq 0}$. The arguments are here oriented towards the proof of Theorem 4.1 in [EiJS19].

During the proof of the local error bound, we need some facts regarding the extrapolation of operators from the space Y, see (3.26), to the space $X_{\mathrm{ext}} = L^2(Q)^7$. Based on the preliminaries in Section 2.2, we list the important facts in the next remark.

Remark 6.3. 1) Let $X^L_{\mathrm{ext}-1}$ be the extrapolation space of X_{ext} with respect to the operator $L \in \{M_{\mathrm{ext}}, A, B, D_1, D_2, D_3\}$. Proposition 2.10.2 in [TuWe09] then provides the identification $X^L_{\mathrm{ext}-1} \cong \mathcal{D}(L^*)^*$, so that the inclusion of Y in $\mathcal{D}(L) = \mathcal{D}(L^*)$ implies $X^L_{\mathrm{ext}-1} \subseteq Y^*$.

2) It will be useful to have a concrete relation between the extrapolation operator L_{-1} of L, and its bidual $(L^*)^*$. As the operator $L^* : \mathcal{D}(L^*) \to X_{\mathrm{ext}}$ is continuous, we infer that $(L^*)^*$ is continuous from X_{ext} to $\mathcal{D}(L^*)^* \cong X^L_{\mathrm{ext}-1}$. We moreover note that the identity

$$\langle (L^*)^* x, y\rangle_{Y^*\times Y} = (x, L^* y) = (Lx, y)$$

is true for all $y \in Y$, and $x \in \mathcal{D}(L)$. As a result, the operator $(L^*)^*$ is the unique continuous extension of L to an operator on X_{ext}, that now maps into $X^L_{\mathrm{ext}-1}$, see Proposition 2.10.3 in [TuWe09]. These considerations give rise to the important formula

$$\langle L_{-1} x, y\rangle_{Y^*\times Y} = \langle (L^*)^* x, y\rangle_{Y^*\times Y} = (x, L^* y), \qquad x \in X_{\mathrm{ext}},\ y \in Y.$$

3) We also need to extend bounded linear operators from X_{ext} to Y^*. Let P be a bounded linear operator on X_{ext} with an adjoint P^*, leaving Y invariant. The operator P can then be extended in a unique continuous manner to $\tilde{P} \in \mathscr{B}(Y^*)$ via $\tilde{P} := (P^*|_Y)^*$. This argument leads to the identity

$$\langle \tilde{P} z, y\rangle_{Y^*\times Y} = \langle z, P^* y\rangle_{Y^*\times Y}, \qquad z \in Y^*,\ y \in Y,$$

see Proposition 2.9.3 in [TuWe09]. The regularity results from Lemmas 3.12 and 3.17, as well as Corollary 3.16 yield that this extension procedure is reasonable for the arising Cayley-Transforms, as well as the operators $F^V(j,k,L)$ and $F^{\tilde{\sigma}}(j,k)$ for $j \le k \in \{1,2\}$ from (6.5) and (6.12). ♢

The next statement provides a bound of order two for the local error of the damped scheme (3.24) in Y^*. In the proof, we transfer arguments from the proof of Theorem 4.1 in [EiJS19] to our setting. To have a compact notation, we abbreviate the solution of the extended Maxwell system (3.1) with initial datum $(\mathbf{E}_0, \mathbf{H}_0, \Phi_0)$ at time $t \ge 0$ by $v(t)$. The approximation from scheme (3.24) with the same starting value is given by v^n at time $n\tau$. Recall also that $\tau > 0$ is the time step size of (3.24). The statement also uses the space $X_{\mathrm{ext},1}$ from (3.15) for the initial data.

Lemma 6.4. *Let $\varepsilon, \mu, \tilde{\sigma}$, and η satisfy the assumptions (2.2) and (3.3). The local error of scheme (3.24) is bounded by*

$$\left|(v^1 - v(\tau), y)\right| \le C_{\mathrm{loc}} \tau^2 \|v(0)\|_{X_{\mathrm{ext},1}} \|y\|_Y$$

for all $y \in Y$, initial data $v(0) = v^0 \in X_{\mathrm{ext},1}$ and $\tau \in (0, \check{\tau}_0)$. The constants C_{loc} and $\check{\tau}_0$ depend only on $\varepsilon, \mu, \tilde{\sigma}, \eta$, and Q.

Proof. 1) For convenience, we allow the constant C to change from line to line. Let $\tau \in (0, \check{\tau}_0)$ with $\check{\tau}_0$ from Proposition 6.2. We also describe one iteration of scheme (3.24) with step size τ by the application of an operator $S(\tau)$. This gives rise to the formula

$$S(\tau) = \begin{pmatrix} \mathrm{e}^{-\tau\tilde{\sigma}} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathrm{e}^{-\tau\eta} \end{pmatrix} \prod_{i=1}^{3} \Big(S_\tau(D_i) V_\tau(D_i) \Big) \\ \cdot S_\tau(B) V_\tau(B) S_\tau(A) V_\tau(A). \tag{6.13}$$

The local error then possesses the representation

$$v^1 - v(\tau) = (S(\tau) - \mathrm{e}^{\tau M_{\mathrm{ext},1}}) v^0. \tag{6.14}$$

Our next goal is to write the local error in a different form, by means of the expansions from Section 6.2 and the identity

$$\begin{aligned} \mathrm{e}^{\tau M_{\mathrm{ext},1}} v^0 &= v^0 + \tau M_{\mathrm{ext},1} \Lambda_{1, M_{\mathrm{ext},1}}(\tau) v^0 \\ &= v^0 + \tau M_{\mathrm{ext}} v^0 + \tau^2 M_{\mathrm{ext}-1} M_{\mathrm{ext},1} \Lambda_{2, M_{\mathrm{ext},1}}(\tau) v^0. \end{aligned}$$

(Note that the last formula follows by iterating (6.9).) The operator $M_{\mathrm{ext}-1}$ is here the extrapolation of M_{ext} to X_{ext}, see Section 3.1. Inserting this identity for $\mathrm{e}^{\tau M_{\mathrm{ext},1}} v^0$ into (6.14), we obtain the expansion

$$v^1 - v(\tau) = \Big(S(\tau) - I - \tau M_{\mathrm{ext}} - \tau^2 M_{\mathrm{ext}-1} M_{\mathrm{ext},1} \Lambda_{2, M_{\mathrm{ext},1}}(\tau) \Big) v^0. \tag{6.15}$$

As we are only interested in the first two terms on the right hand side of (6.15), we use (6.10) to estimate the remainder term to

$$\begin{aligned} \Big| \langle \tau^2 M_{\mathrm{ext}-1} M_{\mathrm{ext},1} \Lambda_{2, M_{\mathrm{ext},1}}(\tau) v^0, y \rangle_{Y^* \times Y} \Big| &= \Big| (\tau^2 M_{\mathrm{ext}} \Lambda_{2, M_{\mathrm{ext},1}}(\tau) v^0, M_{\mathrm{ext}}^* y) \Big| \\ &\le C \tau^2 \| v^0 \|_{X_{\mathrm{ext},1}} \| y \|_Y. \end{aligned} \tag{6.16}$$

2) It is immediate by construction that the operators $F(j,k,l)$, $F^{\tilde{\sigma}}(j,k)$, and $F^V(j,k,l)$ from Section 6.2 are useful to expand the damped scheme (3.24). This is, however, also possible for the first two terms on the right hand side of (6.15). To be more precise, we arrive at the formula

$$\begin{aligned} &v^0 + \tau M_{\mathrm{ext}} v^0 \\ &= \sum_{k=0}^{1} \sum_{j_1 + \dots + j_6 = k} F^{\tilde{\sigma}}(j_6, 2 - \sum_{i=1}^{5} j_i) \prod_{l=1}^{3} \Big(F(j_{2+l}, 2 - \sum_{i=1}^{l+1} j_i, D_l) F^V(0, 2, D_l) \Big) \\ &\qquad \cdot F(j_2, 2 - j_1, B) F^V(0, 2, B) F(j_1, 2, A) F^V(0, 2, A) v^0. \end{aligned} \tag{6.17}$$

As usual, the product sign means here that the arising operators are concatenated, so that their indices decrease from left to right.

In order to compare the solution v of the extended Maxwell system (3.1) with the approximate numerical solution, we also aim for an analogous representation of the numerical solution by means of the expansions from Section 6.2. To that end, we insert the latter expansions for the splitting steps in (6.13), and obtain the identities

$$\begin{aligned}
S(\tau)v^0 &= F^{\tilde{\sigma}}(0,0)\prod_{l=1}^{3}\Big(F(0,0,D_l)F^V(0,0,D_l)\Big)\,F(0,0,B)F^V(0,0,B)F(0,0,A)\\
&\quad\cdot F^V(0,0,A)v^0\\
&= \sum_{j_1=0}^{2}\sum_{r_1=0}^{2} F^{\tilde{\sigma}}(0,0)\prod_{l=1}^{3}\Big(F(0,0,D_l)F^V(0,0,D_l)\Big)F(0,0,B)F^V(0,0,B)\\
&\quad\cdot F(j_1,2,A)F^V(r_1,2,A)v^0\\
&= \sum_{j_1=0}^{2}\sum_{j_2=0}^{2-j_1}\sum_{r_1=0}^{2}\sum_{r_2=0}^{2-r_1} F^{\tilde{\sigma}}(0,0)\prod_{l=1}^{3}\Big(F(0,0,D_l)F^V(0,0,D_l)\Big)F(j_2,2-j_1,B)\\
&\quad\cdot F^V(r_2,2-r_1,B)F(j_1,2,A)F^V(r_1,2,A)v^0\\
&= \sum_{k=0}^{2}\sum_{j_1+\dots+j_6=k}\sum_{s=0}^{2}\sum_{r_1+\dots+r_5=s} F^{\tilde{\sigma}}(j_6,2-\sum_{i=1}^{5}j_i)\\
&\quad\cdot\prod_{l=1}^{3}\Big(F(j_{2+l},2-\sum_{i=1}^{l+1}j_i,D_l)F^V(r_{2+l},2-\sum_{i=1}^{1+l}r_i,D_l)\Big)F(j_2,2-j_1,B)\\
&\quad\cdot F^V(r_2,2-r_1,B)F(j_1,2,A)F^V(r_1,2,A)v^0
\end{aligned}$$

in Y^*. For summands with $k=2$ in the last equation, we implicitly assume that one of the splitting operators is extrapolated to X_{ext}, if necessary. The succeeding operators in the concatenation (which are automatically bounded on X_{ext}) are then extrapolated to Y^*. A comparison between the last expansion and (6.17) now leads to the formula

$$\begin{aligned}
&(S(\tau)-I-\tau M_{\text{ext}})v^0\\
&\quad=\Bigg(\sum_{\substack{j_1+\dots+j_6=2\\ r_1+\dots+r_5=0}}+\sum_{k=0}^{2}\sum_{j_1+\dots+j_6=k}\sum_{s=1}^{2}\sum_{r_1+\dots+r_5=s}\Bigg)F^{\tilde{\sigma}}(j_6,2-\sum_{i=1}^{5}j_i)\\
&\qquad\cdot\prod_{l=1}^{3}\Big(F(j_{2+l},2-\sum_{i=1}^{l+1}j_i,D_l)F^V(r_{2+l},2-\sum_{i=1}^{l+1}r_i,D_l)\Big)\\
&\qquad\cdot F(j_2,2-j_1,B)F^V(r_2,2-r_1,B)F(j_1,2,A)F^V(r_1,2,A)v^0
\end{aligned}\tag{6.18}$$

on Y^*. The expression consisting of five summation symbols in (6.18) indicates that both summation procedures are done separately and that the results are added afterwards. The desired bound on the local error will be concluded by estimating all summands on the right hand side of (6.18) in Y^*. We categorize the summands in the following eight groups, according to their index tuple $(j_1,\dots,j_6,r_1,\dots,r_5)$.

(i) Let $\sum_{m=1}^6 j_m = 2$, and $\sum_{m=1}^5 r_m = 0$. Prescribe additionally that exactly one of the numbers $j_1,\dots,j_6$ is different from zero. We first consider the case $j_6 = 2$. The associated summand in (6.18) is $F^{\tilde\sigma}(2,2)v^0 = \tau^2 K_d^2 \Lambda_{2,K_d}(\tau)v^0$, because $F(0,2,L) = F^V(0,2,L) = I$ by (6.5) and (6.8). Combining definition (6.12) with the assumptions (2.2) and (3.3) for $\tilde\sigma$ and η, the relations

$$\begin{aligned}\left|\langle F^{\tilde\sigma}(2,2)v^0, y\rangle_{Y^*\times Y}\right| = \left|(F^{\tilde\sigma}(2,2)v^0, y)\right| &\le \tau^2\|K_d^2\Lambda_{2,K_d}(\tau)v^0\|\|y\| \le C\tau^2\|v^0\|\|y\| \\ &\le C\tau^2\left\|v^0\right\|_{X_{\mathrm{ext},1}}\|y\|_Y\end{aligned}$$

follow. Consider now the case $j_5 = 2$. The corresponding summand is

$$F^{\tilde\sigma}(0,0)F(2,2,D_3)v^0 = \tfrac{\tau^2}{4}\mathrm{e}^{\tau K_d}(D_3)_{-1}D_3(I + S_\tau(D_3))v^0,$$

as all other operators, appearing in the product, are equal to the identity, see (6.5) and (6.8). (Recall that $(D_3)_{-1}$ denotes the extrapolation of D_3 to X_{ext}.) Since the Cayley-Transform $S_\tau(D_3)$ is isometric on X_{ext} and $X_{\mathrm{ext},1}$ embeds into $H^1(Q)^7$, we conclude the inequality

$$\begin{aligned}\left|\langle F^{\tilde\sigma}(0,0)F(2,2,D_3)v^0, y\rangle_{Y^*\times Y}\right| &= \frac{\tau^2}{4}\left|((I + S_\tau(D_3))D_3v^0, D_3\mathrm{e}^{\tau K_d}y)\right| \\ &\le C\tau^2\|D_3v^0\|\|D_3\mathrm{e}^{\tau K_d}y\| \\ &\le C\tau^2\|v^0\|_{X_{\mathrm{ext},1}}\|\mathrm{e}^{\tau K_d}y\|_Y \\ &\le C\tau^2\|v^0\|_{X_{\mathrm{ext},1}}\|y\|_Y\,.\end{aligned}$$

The remaining summands on the right hand side of (6.18), that match this category, can be treated in a similar way. Here, one uses that the Cayley-Transforms of the other splitting operators are also isometric on X_{ext}.

(ii) Let $\sum_{m=1}^6 j_m = 2$ and $\sum_{m=1}^5 r_m = 0$, such that exactly two indices j_{i_1} and j_{i_2} are equal to one (all others are then zero). Let us at first consider the case $j_5 = j_6 = 1$, resulting in the term $F^{\tilde\sigma}(1,1)F(1,2,D_3)v^0$ (as above, all other operators in this product are equal to the identity operator). Definitions (6.8) and (6.12) then lead to the formula

$$F^{\tilde\sigma}(1,1)F(1,2,D_3)v^0 = \tau^2 K_d\Lambda_{1,K_d}(\tau)D_3v^0,$$

so that we derive the inequality

$$\left|\langle F^{\tilde\sigma}(1,1)F(1,2,D_3)v^0, y\rangle_{Y^*\times Y}\right| = \tau^2\left|(K_d\Lambda_{1,K_d}(\tau)D_3v^0, y)\right|$$

$$\begin{aligned}&\le C\tau^2\|\Lambda_{1,K_d}(\tau)D_3v^0\|\,\|y\|\\&\le C\tau^2\|D_3v^0\|\,\|y\|\le C\tau^2\|v^0\|_{X_{\mathrm{ext},1}}\,\|y\|_Y\,.\end{aligned}$$

We also study the second option $j_4 = j_5 = 1$. This choice gives rise to the vector $F^{\tilde\sigma}(0,0)F(1,1,D_3)F(1,2,D_2)v^0$. Definitions (6.8) and (6.12) now imply the equation

$$F^{\tilde\sigma}(0,0)F(1,1,D_3)F(1,2,D_2)v^0=\tfrac{\tau^2}{2}\mathrm{e}^{\tau K_d}(D_3)_{-1}(I+S_\tau(D_3))D_2v^0.$$

We now use that the Cayley-Transform $S_\tau(D_3)$ is isometric on X_{ext}, that $X_{\mathrm{ext},1}$ embeds into $H^1(Q)^7$, and that $\tilde\sigma$ and η satisfy (2.2) and (3.3). In this way, we conclude the estimates

$$\begin{aligned}\left|\langle F^{\tilde\sigma}(0,0)F(1,1,D_3)F(1,2,D_2)v^0,y\rangle_{Y^*\times Y}\right|&=\frac{\tau^2}{2}\left|((I+S_\tau(D_3))D_2v^0,D_3\mathrm{e}^{\tau K_d}y)\right|\\&\le C\tau^2\|v^0\|_{X_{\mathrm{ext},1}}\,\|y\|_Y\,.\end{aligned}$$

All other index configurations in this group are tackled in the same way, using now that arising Cayley-Transforms are bounded on Y, see Lemma 3.12 and Corollary 3.16.

(iii) Assume that $\sum_{m=1}^6 j_m=0$ and $\sum_{m=1}^5 r_m=1$. Due to symmetry, we only consider the representative summand $F^V(1,2,D_3)v^0$, being associated to the case $r_5=1$. Definition (6.5) yields here the relation

$$F^V(1,2,D_3)v^0=\tfrac{\tau^3}{4}D_3^2(I-\tfrac{\tau^2}{4}D_3^2)^{-1}v^0.$$

Using now identity

$$\tfrac{\tau}{2}D_3(I-\tfrac{\tau}{2}D_3)^{-1}=-I+(I-\tfrac{\tau}{2}D_3)^{-1}\tag{6.19}$$

together with the skewadjointness of D_3, see Lemma 3.8, the inequalities

$$\begin{aligned}\left|\langle F^V(1,2,D_3)v^0,y\rangle_{Y^*\times Y}\right|&=\tfrac{\tau^2}{2}\left|(\tfrac{\tau}{2}D_3(I-\tfrac{\tau}{2}D_3)^{-1}v^0,(I-\tfrac{\tau}{2}D_3)^{-1}D_3y)\right|\\&\le\tau^2\|v^0\|\|(I-\tfrac{\tau}{2}D_3)^{-1}D_3y\|\le C\tau^2\|v^0\|_{X_{\mathrm{ext},1}}\|D_3y\|\\&\le C\tau^2\|v^0\|_{X_{\mathrm{ext},1}}\|y\|_Y\end{aligned}$$

are derived.

(iv) Let $\sum_{m=1}^6 j_m=0$ and $\sum_{m=1}^5 r_m=2$. Here, only the supplementary operators $F^V(\cdot,\cdot,L)$, $L\in\{A,B,D_1,D_2,D_3\}$ are to be considered. Combining Lemma 6.1 with definition (6.5), it is sufficient to deal with the two configurations $r_4=r_5=1$, and $r_5=2$. The first one leads to the summand

$$F^V(1,1,D_3)F^V(1,2,D_2)v^0=\tfrac{\tau^3}{4}V_\tau^{(1)}(D_3)D_2^2(I-\tfrac{\tau^2}{4}D_2^2)^{-1}v^0.$$

We now use (6.19), as well as Lemma 6.1. Because the resolvents of the splitting operators are contractive on X_{ext} and uniformly bounded on Y, see Lemma 3.8 and Corollary 3.16, the estimate

$$\begin{aligned}
&\left|\langle F^V(1,1,D_3)F^V(1,2,D_2)v^0, y\rangle_{Y^*\times Y}\right| \\
&\qquad = \tfrac{\tau^2}{2}\left|\left(\tfrac{\tau}{2}D_2(I-\tfrac{\tau}{2}D_2)^{-1}v^0, D_2(I-\tfrac{\tau}{2}D_2)^{-1}V_\tau^{(1)}(D_3)^*y\right)\right| \\
&\qquad \le \tau^2\|v^0\|\|D_2(I-\tfrac{\tau}{2}D_2)^{-1}V_\tau^{(1)}(D_3)^*y\| \\
&\qquad \le C\tau^2\|v^0\|_{X_{\mathrm{ext},1}}\|(I-\tfrac{\tau}{2}D_2)^{-1}V_\tau^{(1)}(D_3)^*y\|_Y \\
&\qquad \le C\tau^2\|v^0\|_{X_{\mathrm{ext},1}}\|V_\tau^{(1)}(D_3)^*y\|_Y \\
&\qquad \le C\tau^2\|v^0\|_{X_{\mathrm{ext},1}}\,\|y\|_Y
\end{aligned}$$

is valid. If $r_5 = 2$, the corresponding summand is given by the vector $F^V(2,2,D_3)v^0 = V_\tau^{(2)}(D_3)v^0$. Inequality (6.4) yields the relations

$$\begin{aligned}
\left|\langle F^V(2,2,D_3)v^0, y\rangle_{Y^*\times Y}\right| &= \left|(V_\tau^{(2)}(D_3)v^0, y)\right| \le C\|V_\tau^{(2)}(D_3)v^0\|_Y\|y\|_Y \\
&\le C\tau^2\|v^0\|_{X_{\mathrm{ext},1}}\|y\|_Y \le C\tau^2\|v^0\|_{X_{\mathrm{ext},1}}\,\|y\|_Y\,.
\end{aligned}$$

(v) The case $\sum_{m=1}^6 j_m = 1 = \sum_{m=1}^5 r_m$ can be treated like the first part of (iv). The major tools are again Lemmas 3.7, 3.8, 3.12 and 6.1, as well as Corollary 3.16. The resulting bound has the same form as in parts (i)–(iv).

(vi) Let $\sum_{m=1}^6 j_m = 1$ and $\sum_{m=1}^5 r_m = 2$. Due to structural similarities, we only need to deal with the choice $j_5 = 1$ and $r_5 = 2$, or the case $j_5 = 1$ and $r_4 = r_5 = 1$. The first configuration encodes the expression $F(1,2,D_3)F^V(2,2,D_3)v^0 = \tau D_3 V_\tau^{(2)}(D_3)v^0$. Inequality (6.4) leads here to the estimate

$$\begin{aligned}
\left|\langle F(1,2,D_3)F^V(2,2,D_3)v^0, y\rangle_{Y^*\times Y}\right| &= \tau\left|(D_3V_\tau^{(2)}(D_3)v^0, y)\right| \\
&\le \tau\|D_3V_\tau^{(2)}(D_3)v^0\|\,\|y\| \\
&\le C\tau\|V_\tau^{(2)}(D_3)v^0\|\,\|y\|_Y \\
&\le C\tau^3\|v^0\|_{X_{\mathrm{ext},1}}\,\|y\|_Y\,.
\end{aligned}$$

Second, let $r_4 = r_5 = 1$. With definitions (6.5) and (6.8), we arrive here at the summand

$$F(1,2,D_3)F^V(1,1,D_3)F^V(1,2,D_2)v^0 = \tfrac{\tau^4}{4}D_3V_\tau^{(1)}(D_3)D_2^2(I-\tfrac{\tau^2}{4}D_2^2)^{-1}v^0.$$

With Lemma 6.1 and estimate (6.4), we arrive here at the estimate

$$\begin{aligned}
&\tau^2\left|\langle D_3V_\tau^{(1)}(D_3)\tfrac{\tau^2}{4}D_2^2(I-\tfrac{\tau^2}{4}D_2^2)^{-1}v^0, y\rangle_{Y^*\times Y}\right| \\
&\quad \le C\tau^2\|D_3V_\tau^{(1)}(D_3)\tfrac{\tau^2}{4}D_2^2(I-\tfrac{\tau^2}{4}D_2^2)^{-1}v^0\|\,\|y\|_Y
\end{aligned}$$

$$\begin{aligned}
&\leq C\tau^2 \|V_\tau^{(1)}(D_3)\tfrac{\tau^2}{4}D_2^2(I-\tfrac{\tau^2}{4}D_2^2)^{-1}v^0\|_Y\,\|y\|_Y\\
&\leq C\tau^3 \|\tfrac{\tau^2}{4}D_2^2(I-\tfrac{\tau^2}{4}D_2^2)^{-1}v^0\|_Y\,\|y\|_Y\\
&\leq C\tau^3 \|v^0\|_{X_{\text{ext},1}}\,\|y\|_Y\,.
\end{aligned}$$

(vii) For the case $\sum_{m=1}^{6} j_m = 2$ and $\sum_{m=1}^{5} r_m = 1$, we distinguish between the subclass of summands where exactly one summation index j_i is equal to 2, and the subclass where two indices j_{i_1} and j_{i_2} are equal to 1. The two configurations $(j_6 = 2, r_5 = 1)$ and $(j_5 = 2, r_5 = 1)$ are representative for the first subclass, and correspond to the expressions

$$\begin{aligned}
&F^{\tilde\sigma}(2,2)F^V(1,2,D_3)v^0 = \tau^3 \begin{pmatrix} \tilde\sigma^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \eta^2\end{pmatrix} \Lambda_{2,K_d}(\tau)\tfrac{\tau^2}{4}D_3^2(I-\tfrac{\tau^2}{4}D_3^2)^{-1}v^0,\\
&F^{\tilde\sigma}(0,0)F(2,2,D_3)F^V(1,2,D_3)v^0\\
&\quad = \tfrac{\tau^3}{4}\begin{pmatrix} \mathrm{e}^{-\tau\tilde\sigma} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \mathrm{e}^{-\tau\eta}\end{pmatrix} (D_3)_{-1}(I+S_\tau(D_3))D_3\tfrac{\tau^2}{4}D_2^2(I-\tfrac{\tau^2}{4}D_2^2)^{-1}v^0.
\end{aligned}$$

The first one can be handled by combining the reasoning for the term $F^{\tilde\sigma}(2,2)v^0$ in part (i) with Lemma 6.1. For the second term, we apply the arguments for $F^{\tilde\sigma}(0,0)F(2,2,D_3)v^0$ from (i), as well as Lemma 6.1. This leads to inequalities that have the same form as in parts (i)–(iv).

If exactly tow indices j_{i1} and j_{i2} are equal to 1, we obtain with Lemma 3.12 and Corollary 3.16 that it is enough to deal with the two combinations $j_5 = j_6 = 1 = r_5$, and $j_4 = j_5 = 1 = r_5$. The associated summand for the first choice is the vector $F^{\tilde\sigma}(1,1)F(1,2,D_3)F^V(1,2,D_3)v^0$. Lemma 6.1 implies that the operator $F^V(1,2,D_3)$ is uniformly bounded on Y. Taking now also the reasoning in part (ii) into account, we obtain a uniform estimate of order τ^2 in Y^* for this vector.

The second choice $j_4 = j_5 = 1 = r_5$ leads to the summand

$$F^{\tilde\sigma}(0,0)F(1,1,D_3)F^V(1,2,D_3)F(1,2,D_2)v^0.$$

This term can be handled by combining the reasoning in part (ii) with the boundedness result from Lemma 6.1. We again arrive at a uniform estimate of order two in τ.

(viii) Let $\sum_{m=1}^{6} j_m = 2 = \sum_{m=1}^{5} r_m$. The treatise of the summands in this category reduces to the parts (i) and (ii), by applying Lemma 6.1 and (6.4). We thus derive also here an estimate of order τ^2 in Y^*.

3) The case distinction (i)–(viii) and the assumption $\tau < 1$ lead to the bound

$$\Big|((S(\tau)-I-\tau M_{\text{ext}})v^0, y)\Big| \leq C\tau^2\|v^0\|_{X_{\text{ext},1}}\,\|y\|_Y\,. \tag{6.20}$$

We finally conclude that the local error is of order τ^2 in Y^* by combining identity (6.15) with the estimates (6.16) and (6.20). □

We now combine the above bound on the local error of the damped ADI scheme (3.24) with the stability results for scheme (3.24) and for the extended Maxwell system (3.1). This enables us to derive the desired global error result. To control the error propagation, we use the principle of Lady Windermere's fan. This standard technique has also been used in [EiJS19]. The formulation of the statement incorporates the spaces $X_{\mathrm{ext},1}$ and Y from (3.15) and (3.26).

Theorem 6.5. *Let $\varepsilon, \mu, \tilde{\sigma}$, and η satisfy (2.2) and (3.3). Let further $T > 0$. There are constants C, C_{stab}, and $\check{\tau}$ such that the error estimate*

$$|(v^n - v(n\tau), y)| \leq C\tau(1+T)T\mathrm{e}^{C_{\mathrm{stab}}T} \left\|v(0)\right\|_{X_{\mathrm{ext},1}} \left\|y\right\|_Y, \qquad y \in Y,$$

is valid for the iterates v^n of (3.24) with initial data $v(0) = v^0 \in X_{\mathrm{ext},1}$, $\tau \in (0, \check{\tau}_0)$, and $n \in \mathbb{N}$ with $n\tau \leq T$. The numbers $C, C_{\mathrm{stab}}, \check{\tau}_0 > 0$ depend only on $\varepsilon, \mu, \tilde{\sigma}, \eta$, and Q.

Proof. Let $\tau \in (0, \check{\tau}_0)$ with $\check{\tau}_0$ from Proposition 6.2. Let also $n \in \mathbb{N}$ with $n\tau \leq T$. Recall the operator $S(\tau)$ from (6.13). Its application describes one iteration of the damped scheme (3.24). We denote the error at time $n\tau$ by

$$e_n := v^n - v(n\tau) = S(\tau)^n v^0 - \mathrm{e}^{n\tau M_{\mathrm{ext},1}} v^0.$$

The principle of Lady Windermere's fan yields the formula

$$e_n = \sum_{m=0}^{n-1} S(\tau)^m (S(\tau) - \mathrm{e}^{\tau M_{\mathrm{ext},1}}) \mathrm{e}^{(n-1-m)\tau M_{\mathrm{ext},1}} v^0.$$

The stability result from Proposition 6.2 ensures that the operator $S(\tau)$ is bounded on X_{ext}. From the above representation of the error, we then infer the identity

$$(e_n, y) = \sum_{m=0}^{n-1} \Big((S(\tau) - \mathrm{e}^{\tau M_{\mathrm{ext},1}}) \mathrm{e}^{(n-1-m)\tau M_{\mathrm{ext},1}} v^0, (S(\tau)^m)^* y \Big).$$

The regularity results from Section 3.4 apply also to the adjoint $S(\tau)^*$ of $S(\tau)$, since all arising splitting operators are skewadjoint on X_{ext}. As a result, also $S(\tau)^*$ leaves Y invariant. This means that $(S(\tau)^m)^* y$ belongs to Y for every $m \in \{0, \ldots, n-1\}$. Taking also into account that the family $(\mathrm{e}^{tM_{\mathrm{ext},1}})_{t \geq 0}$ is a strongly continuous semigroup on $X_{\mathrm{ext},1}$, see Proposition 3.5, we furthermore deduce that the vector $\mathrm{e}^{(n-1-m)\tau M_{\mathrm{ext},1}} v^0$ is contained in $X_{\mathrm{ext},1}$. We can hence apply our local error result from Lemma 6.4, to conclude the inequality

$$|(e_n, y)| \leq C_{\mathrm{loc}} \tau^2 \sum_{m=0}^{n-1} \left\| \mathrm{e}^{(n-1-m)\tau M_{\mathrm{ext},1}} v^0 \right\|_{X_{\mathrm{ext},1}} \|(S(\tau)^m)^* y\|_Y,$$

with a constant $C_{\mathrm{loc}} = C_{\mathrm{loc}}(\varepsilon, \mu, \tilde{\sigma}, \eta, Q) > 0$. We finally employ the stability results from Propositions 3.5 and 6.2, as well as the relation $n\tau \leq T$. In this way, we arrive at the desired estimates

$$\begin{aligned} |(e_n, y)| &\leq C_{\mathrm{stab},1} C_{\mathrm{loc}} \tau^2 \mathrm{e}^{C_{\mathrm{stab}} T} (1+T) \sum_{m=0}^{n-1} \|v^0\|_{X_{\mathrm{ext},1}} \|y\|_Y \\ &\leq C_{\mathrm{stab},1} C_{\mathrm{loc}} \tau T (1+T) \mathrm{e}^{C_{\mathrm{stab}} T} \|v^0\|_{X_{\mathrm{ext},1}} \|y\|_Y , \end{aligned}$$

with $C_{\mathrm{stab},1}, C_{\mathrm{stab}}$ being the stability constants from Propositions 3.5 and 6.2. □

Let $v = (\mathbf{E}, \mathbf{H}, \Phi)$ be a solution of the extended Maxwell system (3.1) for initial data $v(0) = v^0 = (\mathbf{E}_0, \mathbf{H}_0, \Phi_0) \in X_{\mathrm{ext},1}$. Although v is a solution of the extended Maxwell system (3.1), Theorem 6.5 still measures the difference between the iterates $(\mathbf{E}^n, \mathbf{H}^n, \Phi^n)$ of (3.24) and the solution of the original Maxwell system (2.1). Indeed, the choice of the initial data in Theorem 6.5 ensures that the tuple $(\mathbf{E}, \mathbf{H})$ is the unique solution of (2.1) for initial data $(\mathbf{E}_0, \mathbf{H}_0)$, see Remark 3.6. We now take functions y within the subspace $\tilde{Y} := \{(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\Phi}) \in Y \mid \tilde{\Phi} = 0\}$ of Y as test functions for the error estimate. In this way, we arrive at the desired estimate of order one for the difference between the iterate $(\mathbf{E}^n, \mathbf{H}^n)$ and the unique solution $(\mathbf{E}, \mathbf{H})$ of the original Maxwell system (2.1). This error is measured in the dual space of $\tilde{Y}$.

Part II.

Error analysis of the Peaceman-Rachford ADI scheme for inhomogeneous Maxwell equations in heterogeneous media

7. Maxwell equations in heterogeneous media and refined framework

We introduce here the Maxwell system analyzed in this part of the thesis, together with some notation concerning jumps at interfaces. Moreover, certain auxiliary results are cited or proved here, that will frequently be used throughout without further notice. The functional analytical framework is further specialized to our purposes in Section 7.3.

7.1. Maxwell system with discontinuous coefficients

We consider the Maxwell equations on a cuboid

$$Q = (a_1^-, a_1^+) \times (a_2^-, a_2^+) \times (a_3^-, a_3^+) \subseteq \mathbb{R}^3,$$

which is divided into two smaller subcuboids $\overline{Q} = \overline{Q_1} \cup \overline{Q_2}$, corresponding to two different media. For convenience, we assume $a_1^- < 0 < a_1^+$, and set

$$\begin{aligned} Q_1 &:= (a_1^-, 0) \times (a_2^-, a_2^+) \times (a_3^-, a_3^+), \\ Q_2 &:= (0, a_1^+) \times (a_2^-, a_2^+) \times (a_3^-, a_3^+). \end{aligned}$$

We investigate only this partition of Q. Other partitions into two subcuboids may be handled with analogous arguments, using appropriate coordinate transformations. The interface between Q_1 and Q_2 is denoted by $\mathscr{F}_{\mathrm{int}}$, and the unit normal vector $\nu_{\mathscr{F}_{\mathrm{int}}}$ at $\mathscr{F}_{\mathrm{int}}$ is chosen to point from Q_1 to Q_2. We furthermore use the symbol $\mathrm{tr}_{\mathscr{F}_{\mathrm{int}}}$ for the trace operator on $\mathscr{F}_{\mathrm{int}}$.

Since we deal with discontinuous material parameters, restrictions to the cuboids Q_i, as well as jumps at the interface $\mathscr{F}_{\mathrm{int}}$ play a crucial role throughout our arguments. Besides the cuboid Q, we also consider other partitioned domains like spheres or discs, whence we make the following definitions more general. Let $O \subseteq \mathbb{R}^n$, $n \in \{2, 3\}$, be an open domain, partitioned into two open subdomains O_1 and O_2, and an interface $O_{\mathrm{int}} \subseteq O$. The restriction of a function $f \in L^2(O)$ to

O_i will be denoted by $f^{(i)}$ for $i \in \{1,2\}$. Moreover, we define the jump

$$[\![f]\!]_{O_{\text{int}}} := \operatorname{tr}_{O_{\text{int}}} f^{(2)} - \operatorname{tr}_{O_{\text{int}}} f^{(1)}$$

for every function $f \in L^2(O)$, whose restrictions $f^{(i)}$ have well defined traces $\operatorname{tr}_{O_{\text{int}}}$ on O_{int} for $i \in \{1,2\}$.

On Q we then consider the inhomogeneous, linear and isotropic Maxwell equations

$$\begin{aligned}
\partial_t \mathbf{E} &= \frac{1}{\varepsilon} \operatorname{curl} \mathbf{H} - \frac{1}{\varepsilon}(\sigma \mathbf{E} + \mathbf{J}) && \text{in } Q \times [0,\infty),\\
\partial_t \mathbf{H} &= -\frac{1}{\mu} \operatorname{curl} \mathbf{E} && \text{in } Q \times [0,\infty),\\
\operatorname{div}(\varepsilon^{(i)} \mathbf{E}^{(i)}) &= \rho^{(i)} && \text{in } Q_i \times [0,\infty),\ i \in \{1,2\},\\
\operatorname{div}(\mu \mathbf{H}) &= 0 && \text{in } Q \times [0,\infty),\\
\varepsilon \mathbf{E} \times \nu &= 0, \qquad \mu \mathbf{H} \cdot \nu = 0 && \text{on } \partial Q \times [0,\infty),\\
\mathbf{E}(0) &= \mathbf{E}_0, \qquad \mathbf{H}(0) = \mathbf{H}_0 && \text{in } Q,
\end{aligned} \tag{7.1}$$

with a perfectly conducting boundary, see Section 1.1 for an introduction. Here, $\mathbf{E}(x,t) \in \mathbb{R}^3$ is the electric field, $\mathbf{H}(x,t) \in \mathbb{R}^3$ the magnetic field, $\varepsilon(x) > 0$ denotes the electric permittivity, $\mu(x) > 0$ the magnetic permeability, $\sigma(x) \geq 0$ the conductivity, and $\nu \in \mathbb{R}^3$ the outer unit normal vector at the boundary ∂Q. Furthermore, $\mathbf{J}(x,t) \in \mathbb{R}^3$ represents the current density, and $\rho(x,t) \in \mathbb{R}$ is the charge density.

The material parameters ε, μ and σ are supposed to be constant on each subcuboid Q_i, and the first two parameters should additionally be strictly positive. This means

$$\varepsilon^{(i)},\ \mu^{(i)} \in (\delta, \infty), \quad \sigma^{(i)} \in [0,\infty) \tag{7.2}$$

on Q_i, $i \in \{1,2\}$, with a positive number δ.

In our setting, we furthermore include a charge density $\rho_{\mathscr{F}_{\text{int}}}$ on the interface $\mathscr{F}_{\text{int}}$, but no free surface current on the interface. The charge density $\rho_{\mathscr{F}_{\text{int}}}$ depends on an initial charge density $\rho_{\mathscr{F}_{\text{int}}}(0)$, as well as on the evolution of the sum $\sigma\mathbf{E}+\mathbf{J}$, see Corollary 9.24. A consequence of the Maxwell equations are the transmission relations

$$\begin{aligned}
&[\![\mathbf{E} \times \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = 0 = [\![\mathbf{H} \times \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}},\\
&[\![\varepsilon\mathbf{E} \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = \rho_{\mathscr{F}_{\text{int}}}, \quad [\![\mu\mathbf{H} \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = 0,
\end{aligned} \tag{7.3}$$

see Section I.4.2.2.3 in [DaLi90] and Sections 7.3.6, 9.4.2 in [Grif13].

The open faces of Q are denoted by

$$\Gamma_j^{\pm} := \{x \in \partial Q \mid x_j \in \{a_j^{\pm}\}, x_l \in (a_l^-, a_l^+) \text{ for } l \neq j\}, \quad \Gamma_j := \Gamma_j^+ \cup \Gamma_j^-, \ \ j \in \{1,2,3\},$$

and the open exterior faces of the smaller cuboids by

$$\Gamma_1^{-,(1)} := \Gamma_1 \cap \partial Q_1, \quad \Gamma_1^{+,(1)} := \mathscr{F}_{\mathrm{int}} =: \Gamma_1^{-,(2)}, \quad \Gamma_1^{+,(2)} := \Gamma_1 \cap \partial Q_2,$$
$$\Gamma_j^{\pm,(i)} := \Gamma_j^{\pm} \cap \partial Q_i, \quad \Gamma_j^{(i)} := \Gamma_j \cap \partial Q_i, \qquad j \in \{2,3\},\ i \in \{1,2\}.$$

For convenience, we also introduce the function $\partial_j g$ on the set $Q \setminus \mathscr{F}_{\mathrm{int}}$ by

$$(\partial_j g)|_{Q_i} := \partial_j (g^{(i)}), \quad i \in \{1,2\}, j \in \{1,2,3\},$$

for every function $g \in L^2(Q)$ such that $\partial_j(g^{(i)})$ exists in $L^2(Q_i)$. Note that this definition coincides with the usual one for any function g, possessing a weak derivative $\partial_j g$ in $L^2(Q)$. Similarly, we extend other differential operators like the Laplace operator Δ to piecewise sufficiently regular functions.

Closely related are the spaces of partial regularity. Let $q \in \mathbb{N}$, and let Γ^* be a union of some faces of Q. The space of piecewise H^q functions, relative to the partition $\overline{Q} = \overline{Q_1} \cup \overline{Q_2}$, is given by

$$PH^q(Q) := \{f \in L^2(Q) \mid f^{(i)} \in H^q(Q_i),\ i \in \{1,2\}\}.$$

Particularly important is also the subspace

$$PH^q_{\Gamma^*}(Q) := \{f \in PH^q(Q) \mid f^{(i)} = 0 \text{ on } \partial Q_i \cap \Gamma^* \text{ for } i \in \{1,2\}\}$$

of functions with zero trace on Γ^*. Finally, the natural norm on $PH^q(Q)$ is defined via

$$\|f\|^2_{PH^q(Q)} := \sum_{i=1}^{2} \left\|f^{(i)}\right\|^2_{H^q(Q_i)}, \qquad f \in PH^q(Q).$$

Note that the spaces $PH^q(Q)$ and $PH^q_{\Gamma^*}(Q)$ are complete with respect to the norm $\|\cdot\|_{PH^q(Q)}$.

7.2. Analytical preparations

In our analysis, we often deal with jumps (or discontinuities) at the interface $\mathscr{F}_{\mathrm{int}}$. The following extension of Lemma 2.1 in [EiSc18] is very useful in this context.

Lemma 7.1. *Let $f \in H^1(Q_1)$ and $g \in H^1(Q_2)$ be two functions with the following properties. The mapping $\partial_1 f$ belongs to $H^1(Q_1)$, $\partial_1 g$ is an element of $H^1(Q_2)$, and $\mathrm{tr}_{\mathscr{F}_{\mathrm{int}}} f - \mathrm{tr}_{\mathscr{F}_{\mathrm{int}}} g = 0$. Then $\mathrm{tr}_{\mathscr{F}_{\mathrm{int}}} \partial_k f - \mathrm{tr}_{\mathscr{F}_{\mathrm{int}}} \partial_k g = 0$ for $k \in \{2,3\}$.*

Proof. Let $k \in \{2,3\}$. In view of the reasoning in Section 2.2, a well-defined trace can be assigned to $\partial_k f$ on the interface $\mathscr{F}_{\text{int}}$, as the function $\partial_1 \partial_k f = \partial_k \partial_1 f$ belongs to $L^2(Q_1)$. An analogous statement is true for $\partial_k g$.

We next employ next a smooth cut-off function $\chi_m : [a_1^-, 0] \to [0,1]$ that is equal to one on $[a_1^-, -\frac{1}{m}]$, that has its support within $[a_1^-, -\frac{1}{2m}]$, and that satisfies $\|\chi_m'\|_\infty \leq Cm$ for $m \in \mathbb{N}$ with $m \geq m_0$. Here, C is a uniform positive constant. We then put

$$h(x) := f(x) - g(\tfrac{a_1^+}{a_1^-} x_1, x_2, x_3), \quad f_m(x) := \chi_m(x_1) h(x) + g(\tfrac{a_1^+}{a_1^-} x_1, x_2, x_3)$$

for $x = (x_1, x_2, x_3) \in Q_1$ and $m \geq m_0$. By construction, f_m and $\partial_1 f_m$ are contained in $H^1(Q_1)$, and $f_m(x) = g(\frac{a_1^+}{a_1^-} x_1, x_2, x_3)$ for $x \in [-\frac{1}{2m}, 0] \times [a_2^-, a_2^+] \times [a_3^-, a_3^+]$. As a result, the traces of $\partial_k f_m$ and $\partial_k g$ on $\mathscr{F}_{\text{int}}$ coincide for $m \geq m_0$. The precondition $\operatorname{tr}_{\mathscr{F}_{\text{int}}} f = \operatorname{tr}_{\mathscr{F}_{\text{int}}} g$ implies that h has a vanishing trace on $\mathscr{F}_{\text{int}}$. Hence, we can conclude as in the proof of Lemma 2.1 in [EiSc18], that the sequence $(\chi_m' \partial_k h)_m$ converges to zero in $L^2(Q_1)$. By means of Lebesgue's dominated convergence theorem, we thus infer the statements $\partial_k f_m \to \partial_k f$ and $\partial_1 \partial_k f_m \to \partial_1 \partial_k f$ in $L^2(Q_1)$ as $m \to \infty$. Employing the continuity of the restricted trace operator $\operatorname{tr}_{\mathscr{F}_{\text{int}}}$ with respect to the graph norm of ∂_1 on Q_1 and Q_2, the assertion follows. $\square$

Remark 7.2. The procedure of reflecting a function at the interface $\mathscr{F}_{\text{int}}$ relies strongly on the special geometric structure of our problem. It is used again in our regularity analysis. $\lozenge$

In the setting of discontinuous material parameters, it is useful to have a representation of the spaces $H(\operatorname{curl}, Q)$ and $H(\operatorname{div}, Q)$ from Section 2.2, that involves continuity requirements at the interface $\mathscr{F}_{\text{int}}$. Using the density of smooth functions in $H(\operatorname{curl}, Q)$ and $H(\operatorname{div}, Q)$, see Theorems 2.4 and 2.10 in Chapter I of [GiRa86], one can deduce the identities

$$\begin{aligned} H(\operatorname{curl}, Q) &= \{\varphi \in L^2(Q)^3 \mid \operatorname{curl} \varphi^{(i)} \in L^2(Q_i)^3,\ i \in \{1,2\}, \\ &\qquad\qquad [\![\varphi \times \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = 0\}, \\ H(\operatorname{div}, Q) &= \{\varphi \in L^2(Q)^3 \mid \operatorname{div} \varphi^{(i)} \in L^2(Q_i),\ i \in \{1,2\}, \\ &\qquad\qquad [\![\varphi \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = 0\}, \end{aligned} \tag{7.4}$$

compare (1.3) and (1.4) in [CoDN99]. These relations play an important role throughout our arguments concerning regularity theory.

One of the most important tools in this part of the thesis is interpolation theory. It is essential both for the regularity analysis, and the error analysis. Throughout, we only employ real interpolation on Hilbert spaces. The resulting interpolation spaces can be constructed by several (equivalent) techniques, such as the

K-method, a trace method, and via fractional powers of selfadjoint positive operators. We sketch the latter method, as it fits best to many of our arguments. The K-method is presented in Section 1.1 in [Luna18], and the trace approach is treated in Section 1.2 in [Luna18] as well as Sections 3–4 in Chapter 1 of [LiMa72].

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two separable Hilbert spaces, where the space Y is a dense subspace of X with a continuous embedding. Section 124 in [RiNa73], and Section 2.1 in Chapter 1 of [LiMa72] show that there is a positive selfadjoint operator Λ on X, whose domain coincides with Y, and whose graph norm is equivalent to the norm in Y. We can then define fractional powers of Λ. In terms of the powers, we introduce the interpolation space

$$(X, Y)_{\theta,2} := \mathcal{D}(\Lambda^\theta), \qquad \theta \in [0, 1],$$

which is equipped with the graph norm in $\mathcal{D}(\Lambda^\theta)$, i.e.,

$$\|x\|_{\theta,2}^2 := \|x\|_X^2 + \|\Lambda^\theta\|_X^2, \qquad x \in (X, Y)_{\theta,2},$$

compare with Definition 2.1 in Chapter 1 of [LiMa72]. Remark 2.3 in Chapter 1 of [LiMa72] further states that the definition of the space $(X, Y)_{\theta,2}$ is independent of the particular choice of Λ.

Attention should also be paid to the order of the spaces X and Y. Lions and Magenes denote the above interpolation spaces with reversed order of X and Y, see Section 2.1 in Chapter 1 of [LiMa72].

In order to treat the non-vanishing charge density ρ as well as the jump of the electric field at $\mathscr{F}_{\mathrm{int}}$, see (7.3), we generalize certain spaces from [EiSc17]. Let $i \in \{1, 2\}$, Γ' be a face of Q_i, $j \in \{2, 3\}$, and $H^1_{\Gamma_j}(\mathscr{F}_{\mathrm{int}}) := \{v \in H^1(\mathscr{F}_{\mathrm{int}}) \mid v = 0 \text{ on } \overline{\Gamma_j} \cap \overline{\mathscr{F}_{\mathrm{int}}}\}$. We then introduce the interpolation spaces

$$\begin{aligned} H^{1/2}_{\Gamma_j}(\mathscr{F}_{\mathrm{int}}) &:= \Big(L^2(\mathscr{F}_{\mathrm{int}}), H^1_{\Gamma_j}(\mathscr{F}_{\mathrm{int}})\Big)_{1/2,2}, \\ H^{1/2}_0(\Gamma') &:= \Big(L^2(\Gamma'), H^1_0(\Gamma')\Big)_{1/2,2}. \end{aligned} \tag{7.5}$$

The first space consists of functions on $\mathscr{F}_{\mathrm{int}}$, that vanish on $\overline{\mathscr{F}_{\mathrm{int}}} \cap \overline{\Gamma_j}$ in a generalized sense, and the second one is the set of functions on Γ' with generalized zero trace on the boundary of Γ' within ∂Q, see Theorem 11.7 in Chapter 1 of [LiMa72]. Note that one cannot assign well-defined traces to functions in $H^{1/2}(\mathscr{F}_{\mathrm{int}}) = (L^2(\mathscr{F}_{\mathrm{int}}), H^1(\mathscr{F}_{\mathrm{int}}))_{1/2,2}$, as the space of test functions $C_c^\infty(\mathscr{F}_{\mathrm{int}})$ is dense in the latter interpolation space, see Theorem 1.4.2.4 in [Gris85]. In particular, $H^{1/2}_0(\mathscr{F}_{\mathrm{int}})$ is not the closure of $C_c^\infty(\mathscr{F}_{\mathrm{int}})$ in $H^{1/2}(\mathscr{F}_{\mathrm{int}})$.

We further define $H^1_{00}(Q_i)$ as the space of all functions $g \in H^1(Q_i)$, whose trace $\mathrm{tr}_{\tilde\Gamma}\, g$ belongs to the interpolation space $H^{1/2}_0(\tilde\Gamma)$ for all faces $\tilde\Gamma$ of Q_i and $i \in \{1, 2\}$.

In other words, functions in $H^1_{00}(Q_i)$ vanish on all edges of Q_i in a generalized sense. Finally, we also employ an interpolation space of higher order, namely

$$H_0^{3/2}(\mathscr{F}_{\text{int}}) := \Big(L^2(\mathscr{F}_{\text{int}}), H_0^1(\mathscr{F}_{\text{int}}) \cap H^2(\mathscr{F}_{\text{int}})\Big)_{3/4,2}. \tag{7.6}$$

There is a useful interpretation of this interpolation space, resulting from the above construction of interpolation spaces. Consider the Dirichlet Laplacian $-\Delta_D$ on the rectangle $\mathscr{F}_{\text{int}}$ with domain $\mathcal{D}(-\Delta_D) = H^2(\mathscr{F}_{\text{int}}) \cap H_0^1(\mathscr{F}_{\text{int}})$. This operator is positive definite and selfadjoint on $L^2(\mathscr{F}_{\text{int}})$, so that one can define fractional powers $(-\Delta_D)^\gamma$ for $\gamma > 0$. The above interpolation space $H_0^{3/2}(\mathscr{F}_{\text{int}})$ then coincides with the domain of the fractional power $(-\Delta_D)^{3/4}$.

The interpolation spaces on $\mathscr{F}_{\text{int}}$ are crucial for several extension arguments. In particular, the space from (7.5) are used to extend traces of Neumann derivatives, while (7.6) is helpful to deal with Dirichlet traces.

We start with two properties of these spaces. The first one will be employed several times to deduce that appropriate functions have a trace on $\mathscr{F}_{\text{int}}$ in one of the above interpolation spaces.

Lemma 7.3. *Let $j \in \{2,3\}$ and $i \in \{1,2\}$. The space $\operatorname{tr}_{\mathscr{F}_{\text{int}}}(H^1_{\Gamma_j}(Q_i))$ is contained in $H^{1/2}_{\Gamma_j}(\mathscr{F}_{\text{int}})$, and $\operatorname{tr}_{\mathscr{F}_{\text{int}}}(H^1_{\Gamma^*}(Q_i))$ with $\Gamma^* = \Gamma_2 \cup \Gamma_3$ is a subspace of $H_0^{1/2}(\mathscr{F}_{\text{int}})$. A similar statement holds true for $V = H^2(Q_i) \cap H^1_{\Gamma^*}(Q_i)$, i.e., the operator $\operatorname{tr}_{\mathscr{F}_{\text{int}}}$ maps V into $H_0^{3/2}(\mathscr{F}_{\text{int}})$.*

Proof. 1) Let $j = 3$, $i = 2$, and $g \in H^1_{\Gamma_3}(Q_2)$. All other choices for j and i can be treated with similar arguments, due to the symmetry of the cuboids. Moreover, we write for convenience $\tilde{g} = g^{(2)}$, use the rectangle $R := (a_2^-, a_2^+) \times (a_3^-, a_3^+)$, and put $I := (0, \infty)$.

First, we extend $\tilde{g}$ by means of Stein's extension operator to a function in $H^1(\mathbb{R}^3)$, and restrict the extended function to $I \times R$. We denote the extension to $\mathbb{R}^3$ and the latter restriction again by $\tilde{g}$, and obtain

$$\tilde{g}, \partial_1 \tilde{g} \in L^2(I \times R) \cong L^2\Big(I, L^2(R)\Big).$$

As a result, $\tilde{g}$ belongs to $H^1(I, L^2(R))$. We employ also a smooth cut-off function $\tilde{\chi} : \overline{I} \to [0,1]$ that is equal to 1 on $[0, \frac{3}{4}a_1^+)$, and that is supported within $[0, a_1^+)$. The choice of $\tilde{\chi}$ then implies that $\tilde{\chi}\tilde{g}$ is an element of $H^1(I, L^2(R))$.

2) Analogously to the interface $\mathscr{F}_{\text{int}}$, the space $H^1_{\Gamma_3}(R)$ consists of all H^1-regular functions on R, that vanish on the intersection of Γ_3 with ∂R. The next goal is to show that the product $\tilde{\chi}\tilde{g}$ also belongs to $L^2(I, H^1_{\Gamma_3}(R))$. This is achieved with an approximation argument. Let $\chi_n : \mathbb{R} \to [0,1]$ be a smooth cut-off function that is

supported on $(a_3^- + \frac{1}{n}, a_3^+ - \frac{1}{n})$, that is equal to 1 on $(a_3^- + \frac{2}{n}, a_3^+ - \frac{2}{n})$, and that satisfies $\|\chi_n'\|_\infty \le Cn$ for $n \ge n_0$ in $\mathbb{N}$ with a uniform constant $C > 0$. We put

$$v_n(x) := \chi_n(x_3)\tilde{\chi}(x_1)\tilde{g}(x), \qquad x = (x_1, x_2, x_3) \in \mathbb{R}^3,\ n \ge n_0.$$

Since $\tilde{g}|_{Q_2}$ has a vanishing trace on $\Gamma_3^{(2)}$, one can adapt the arguments from the proof of Lemma 2.1 in [EiSc18] to the current setting. An additional application of Lebesgue's theorem of dominated convergence thus yields the convergence

$$r_n := \|v_n - \tilde{\chi}\tilde{g}\|_{L^2(I,H^1(R))} \to 0, \qquad n \to \infty. \tag{7.7}$$

Let $\varepsilon > 0$. In presence of the convergence statement (7.7), there is a number $n_\varepsilon \in \mathbb{N}$ with $r_{n_\varepsilon} < \varepsilon$. Let ρ_m^j be the standard smooth mollifier with respect to the j-th variable for $m \in \mathbb{N}$ and $j \in \{1, 2, 3\}$. We define

$$g_m := \rho_m^1 * \rho_m^2 * \rho_m^3 * v_{n_\varepsilon}, \qquad m \in \mathbb{N}.$$

By construction, g_m is smooth, belongs to $H^1(\mathbb{R}^3)$, and vanishes near $\Gamma_3^{(2)}$ for m sufficiently large. Consequently, the restriction $g_m|_{I\times R}$ is contained in the space $L^2(I, H^1_{\Gamma_3}(R))$. Furthermore, $(g_m)_m$ tends to v_{n_ε} in $H^1(\mathbb{R}^3)$ and thus also in $L^2(I, H^1(R))$. The function v_{n_ε} hence belongs to $L^2(I, H^1_{\Gamma_3}(R))$, because the latter space is closed in $L^2(I, H^1(R))$. As $\varepsilon > 0$ is chosen arbitrary and the limit statement (7.7) is valid, the product $\tilde{\chi}\tilde{g}$ belongs to $L^2(I, H^1_{\Gamma_3}(R))$.

Taking also into account that $\tilde{\chi}\tilde{g}$ belongs to $H^1(I, L^2(R))$, Corollary 1.14 from [Luna18] yields that $\operatorname{tr}_{\mathscr{F}_{\mathrm{int}}}(\tilde{\chi}\tilde{g}) = \operatorname{tr}_{\mathscr{F}_{\mathrm{int}}} \tilde{g}$ is contained in the desired interpolation space $(L^2(R), H^1_{\Gamma_3}(R))_{1/2,2}$.

3) The addendum for $H^1_{\Gamma^*}(Q_2)$ is proved by similar methods, employing now also cut-off functions with respect to the second variable x_2. The arguments, however, remain essentially the same.

4) Let $\tilde{g}$ now belong to $H^2(Q_2) \cap H^1_{\Gamma^*}(Q_2)$. We repeat the extension procedure from part 1), obtaining that $\tilde{g}$ is contained in $H^2(I, L^2(R))$. Moreover, as in part 3), we infer that $\tilde{\chi}\tilde{g}$ belongs to $L^2(I, H^1_0(R))$. Employing again the mollifier technique from part 2), we define

$$\check{g}_m := \rho_m^1 * \rho_m^2 * \rho_m^3 * (\tilde{\chi}\tilde{g}), \qquad m \in \mathbb{N}.$$

The function $\check{g}_m$ is smooth, belongs to $H^2(\mathbb{R}^3)$, and its restriction $\check{g}_m|_{I\times R}$ is thus an element of $L^2(I, H^2(R))$. Due to classical mollifier theory, $(\check{g}_m)_m$ converges to $\tilde{\chi}\tilde{g}$ in $H^2(\mathbb{R}^3)$, implying convergence in $L^2(I, H^2(R))$. As a result, $\tilde{\chi}\tilde{g}$ is contained in $L^2(I, H^2(R))$. Altogether, $\tilde{\chi}\tilde{g}|_{I\times R}$ is an element of $L^2(I, H^2(R) \cap H^1_0(R)) \cap H^2(I, L^2(R))$. Theorem 3.2 in Chapter 1 of [LiMa72] now yields that $\operatorname{tr}_{\mathscr{F}_{\mathrm{int}}}(\tilde{\chi}\tilde{g}) = \operatorname{tr}_{\mathscr{F}_{\mathrm{int}}} \tilde{g}$ belongs to $(L^2(R), H^2(R) \cap H^1_0(R))_{3/4,2} = H^{3/2}_0(\mathscr{F}_{\mathrm{int}})$. $\square$

We deduce next a trace inequality for functions in $PH^1_{\Gamma_j}(Q)$, $j \in \{2,3\}$, and for functions in $PH^2(Q) \cap PH^1_{\Gamma_2 \cup \Gamma_3}(Q)$. The estimates become crucial, when we prove piecewise H^2-regularity for the solutions of the Maxwell system.

Lemma 7.4. *Let $i \in \{1,2\}$, $g \in PH^1_{\Gamma_j}(Q)$ with $j \in \{2,3\}$, and let $f \in PH^2(Q) \cap PH^1_{\Gamma_2 \cup \Gamma_3}(Q)$. There is a constant $C_{\text{int}} > 0$, depending only on Q, with*

$$\left\| \operatorname{tr}_{\mathscr{F}_{\text{int}}} g^{(i)} \right\|_{H^{1/2}_{\Gamma_j}(\mathscr{F}_{\text{int}})} \leq C_{\text{int}} \left\| g^{(i)} \right\|_{H^1(Q_i)}, \tag{7.8}$$

$$\left\| \operatorname{tr}_{\mathscr{F}_{\text{int}}} f^{(i)} \right\|_{H^{3/2}_0(\mathscr{F}_{\text{int}})} \leq C_{\text{int}} \left\| f^{(i)} \right\|_{H^2(Q_i)}. \tag{7.9}$$

Proof. 1) We first focus on (7.8). Due to symmetry, it suffices to consider only the restriction $g^{(2)}$ on Q_2 and the case $j = 3$. The rectangle $R = (a_2^-, a_2^+) \times (a_3^-, a_3^+)$ from the previous proof is again employed.

As in the proof of Lemma 7.3, we extend $g^{(2)}$ to a function in $H^1(\mathbb{R}^3)$ by means of Stein's extension operator, and we denote the extended function again by $g^{(2)}$. Since the extension operator is bounded, there is a uniform constant $\tilde{C}_{\text{int}} > 0$ with

$$\left\| g^{(2)} \right\|_{H^1(\mathbb{R}^3)} \leq \tilde{C}_{\text{int}} \left\| g^{(2)}|_{Q_2} \right\|_{H^1(Q_2)}. \tag{7.10}$$

We also employ the smooth cut-off function $\tilde{\chi}$ from part 1) of the proof for Lemma 7.3. By the reasoning of this proof, the function $\tilde{\chi} g^{(2)}|_{(0,\infty)\times R}$ is contained in the space $H^1((0,\infty), L^2(R)) \cap L^2((0,\infty), H^1_{\Gamma_j}(R))$.

Theorem 3.1 in Chapter 1 of [LiMa72] shows the continuity of the trace mapping

$$H^1((0,\infty), L^2(R)) \cap L^2((0,\infty), H^1_{\Gamma_j}(R)) \to H^{1/2}_{\Gamma_j}(R), \quad u \mapsto \operatorname{tr}_{\mathscr{F}_{\text{int}}} u,$$

implying the estimates

$$\begin{aligned} \left\| \operatorname{tr}_{\mathscr{F}_{\text{int}}} (\tilde{\chi} g^{(2)}) \right\|_{H^{1/2}_{\Gamma_j}(R)} &\leq C \left(\left\| \tilde{\chi} g^{(2)} \right\|_{L^2((0,\infty), H^1_{\Gamma_j}(R))} + \left\| \partial_1 (\tilde{\chi} g^{(2)}) \right\|_{L^2((0,\infty)\times R)} \right) \\ &\leq 2C \left\| \tilde{\chi} g^{(2)} \right\|_{H^1((0,\infty)\times R)} \leq 2C \left\| \tilde{\chi} \right\|_{W^{1,\infty}(0,\infty)} \left\| g^{(2)} \right\|_{H^1(Q_2)} \end{aligned}$$

with a uniform constant $C > 0$. Since $\operatorname{tr}_{\mathscr{F}_{\text{int}}} \tilde{\chi} g^{(2)} = \operatorname{tr}_{\mathscr{F}_{\text{int}}} g^{(2)}$, we have thus arrived at (7.8).

2) In order to show the remaining estimate (7.9), we argue in a similar way. The arguments from part 4) of the proof of Lemma 7.3 yield that the product $\tilde{\chi} f^{(2)}$ is an element of $L^2((0,\infty), H^2(R) \cap H^1_0(R)) \cap H^2((0,\infty), L^2(R))$. Employing again Theorem 3.1 in Chapter 1 of [LiMa72], we infer in an analogous way the relations

$$\left\| \operatorname{tr}_{\mathscr{F}_{\text{int}}} (\tilde{\chi} f^{(2)}) \right\|_{H^{3/2}_0(R)} \leq C \left\| \tilde{\chi} f^{(2)} \right\|_{H^2((0,\infty)\times R)} \leq C \left\| \tilde{\chi} \right\|_{W^{2,\infty}(0,\infty)} \left\| f^{(2)} \right\|_{H^2(Q_2)}$$

with a uniform constant $C > 0$. □

Remark 7.5. The statement of the previous lemma is stronger than the trace inequality for functions in $H^1(Q)$ with respect to $H^{1/2}(\mathscr{F}_{\text{int}})$. This is because the topology in $H^{1/2}_{\Gamma_j}(\mathscr{F}_{\text{int}})$ is strictly finer than the one in $H^{1/2}(\mathscr{F}_{\text{int}})$ for $j \in \{2,3\}$, see for instance Theorem 11.7 in Chapter 1 of [LiMa72].

Furthermore, similar arguments as in part 1) of the proof for Lemma 7.4 show that the analogous estimate

$$\left\| \operatorname{tr}_{\mathscr{F}_{\text{int}}} g^{(i)} \right\|_{H_0^{1/2}(\mathscr{F}_{\text{int}})} \leq C_{\text{int}} \left\| g^{(i)} \right\|_{H^1(Q_i)}$$

is valid for all functions $g \in PH^1_{\Gamma_2 \cup \Gamma_3}(Q)$, after a possible modification of the constant $C_{\text{int}} > 0$. $\Diamond$

We close this section by recalling that the Sobolev space $W^{1,1}((a,b),H)$ embeds continuously into $C([a,b],H)$ for real numbers $a < b$, and a Banach space H. Although this result is well known, we prove the embedding statement again, since we are interested in the precise form of the embedding constant. The lemma will be employed to control the error propagation during the error analysis for the Peaceman-Rachford ADI scheme in Chapter 10.

Lemma 7.6. *Let H be a Banach space, $a < b$ be real numbers, and let $f \in W^{1,1}((a,b),H)$. The function f has a continuous representative on $[a,b]$, satisfying*

$$\|f\|_{C([a,b],H)} \leq \max\{1, \tfrac{1}{b-a}\} \, \|f\|_{W^{1,1}((a,b),H)} \,.$$

Proof. The argument is essentially contained in the proof of Lemma 4.24 in [AdFo03]. Let $t \in [a,b]$, and denote the norm on H by $\|\cdot\|_H$. The fundamental theorem of calculus for Sobolev functions already implies that f has a continuous representative on $[a,b]$, fulfilling the identity

$$f(t) = f(r) + \int_r^t f'(s) \, \mathrm{d}s, \qquad r \in [a,b].$$

We can thus estimate

$$\|f(t)\|_H \leq \|f(r)\|_H + \int_a^b \|f'(s)\|_H \, \mathrm{d}s.$$

Integrating both sides of the inequality with respect to r from a to b, we infer

$$\begin{aligned} \|f(t)\|_H &\leq \tfrac{1}{b-a} \int_a^b \|f(r)\|_H \, \mathrm{d}r + \int_a^b \|f'(s)\|_H \, \mathrm{d}s \\ &\leq \max\{1, \tfrac{1}{b-a}\} \, \|f\|_{W^{1,1}((a,b),H)} \,. \end{aligned}$$

$\square$

7.3. Refined framework for the Maxwell system

Recall our assumption (7.2) on the parameters ε, μ, and σ. We consider the Maxwell equations (7.1) as an evolution equation on the space $X := L^2(Q)^6$. For our problem, it is convenient to equip this space with the weighted inner product

$$\left(\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{pmatrix} \right) := \int_Q (\varepsilon \mathbf{E} \cdot \tilde{\mathbf{E}} + \mu \mathbf{H} \cdot \tilde{\mathbf{H}}) \, \mathrm{d}x, \quad \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{pmatrix} \in X, \tag{7.11}$$

inducing the norm $\|\cdot\|$ on X. Note that the assumptions on ε and μ imply that $\|\cdot\|$ is equivalent to the usual L^2-norm on Q. On X we consider the Maxwell operator

$$M := \begin{pmatrix} -\frac{\sigma}{\varepsilon} I & \frac{1}{\varepsilon} \operatorname{curl} \\ -\frac{1}{\mu} \operatorname{curl} & 0 \end{pmatrix} \tag{7.12}$$

with domain

$$\mathcal{D}(M) := H_0(\operatorname{curl}, Q) \times H(\operatorname{curl}, Q).$$

The identities (7.4) imply here the useful representation

$$\begin{aligned} \mathcal{D}(M) = \{ (\mathbf{E}, \mathbf{H}) \in L^2(Q)^6 \mid \operatorname{curl} \mathbf{E}^{(i)}, \operatorname{curl} \mathbf{H}^{(i)} \in L^2(Q_i)^3, \ [\![\mathbf{E} \times \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = 0, \\ [\![\mathbf{H} \times \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = 0, \ \mathbf{E} \times \nu = 0 \text{ on } \partial Q, \ i \in \{1, 2\} \}, \end{aligned} \tag{7.13}$$

involving transmission conditions in tangential direction at the interface $\mathscr{F}_{\text{int}}$. The latter conditions are given by the formulas

$$\mathbf{E}_j^{(1)} - \mathbf{E}_j^{(2)} = 0, \quad \mathbf{H}_j^{(1)} - \mathbf{H}_j^{(2)} = 0, \qquad j \in \{2, 3\}, \tag{7.14}$$

on $\mathscr{F}_{\text{int}}$.

Employing the interpolation spaces from Section 7.2, we can now also incorporate the boundary conditions for the magnetic field, as well as the divergence and the remaining transmission conditions in (7.3). This is done by studying the space

$$\begin{aligned} X_0 := \{ (\mathbf{E}, \mathbf{H}) \in L^2(Q)^6 \mid \operatorname{div}(\varepsilon^{(i)} \mathbf{E}^{(i)}) \in L^2(Q_i), \ \operatorname{div}(\mu \mathbf{H}) = 0, \mu \mathbf{H} \cdot \nu = 0 \text{ on } \partial Q, \\ [\![\varepsilon \mathbf{E} \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} \in H_0^{1/2}(\mathscr{F}_{\text{int}}) \}, \end{aligned}$$

which generalizes the space X_{div} from Section 2 in [EiSc18]. Again, we employ (7.4) to deduce the identity

$$\begin{aligned} X_0 = \{ (\mathbf{E}, \mathbf{H}) \in L^2(Q)^6 \mid \operatorname{div}(\varepsilon^{(i)} \mathbf{E}^{(i)}) \in L^2(Q_i), \ \operatorname{div}(\mu^{(i)} \mathbf{H}^{(i)}) = 0 \text{ on } Q_i, \\ [\![\varepsilon \mathbf{E} \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} \in H_0^{1/2}(\mathscr{F}_{\text{int}}), \ [\![\mu \mathbf{H} \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = 0, \\ \mu \mathbf{H} \cdot \nu = 0 \text{ on } \partial Q, \ i \in \{1, 2\} \}. \end{aligned} \tag{7.15}$$

The new assumption on the normal component of functions in X_0 at $\mathscr{F}_{\text{int}}$ is given by

$$\begin{aligned}&\operatorname{tr}_{\mathscr{F}_{\text{int}}}(\varepsilon^{(1)}\mathbf{E}_1^{(1)}) - \operatorname{tr}_{\mathscr{F}_{\text{int}}}(\varepsilon^{(2)}\mathbf{E}_1^{(2)}) \in H_0^{1/2}(\mathscr{F}),\\ &\operatorname{tr}_{\mathscr{F}_{\text{int}}}(\mu^{(1)}\mathbf{H}_1^{(1)}) - \operatorname{tr}_{\mathscr{F}_{\text{int}}}(\mu^{(2)}\mathbf{H}_1^{(2)}) = 0.\end{aligned}$$

The space X_0 is complete with respect to the norm

$$\begin{aligned}\|(\mathbf{E},\mathbf{H})\|_{X_0} := \|(\mathbf{E},\mathbf{H})\| + \sum_{i=1}^{2} \left\|\operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)})\right\|_{L^2(Q_i)}\\ + \|[\![\varepsilon\mathbf{E}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}\|_{H_0^{1/2}(\mathscr{F}_{\text{int}})}.\end{aligned} \tag{7.16}$$

To show this claim, let $(\mathbf{E}^n,\mathbf{H}^n)_n$ be a Cauchy-sequence in X_0. There is an element $(\mathbf{E},\mathbf{H})$ in $X = L^2(Q)^6$, such that $(\mathbf{E}^n,\mathbf{H}^n)_n$ converges to $(\mathbf{E},\mathbf{H})$ with respect to the norm $\|\cdot\|$. Since also the sequence $(\operatorname{div}(\mu\mathbf{H}^n))_n = (0)_n$ converges in $L^2(Q)$, we infer $\operatorname{div}(\mu\mathbf{H}) = 0$ and $\mu\mathbf{H}\cdot\nu = 0$ on ∂Q by using the continuity of the normal trace operator, see Section 2.2. Regarding the electric field, we observe that also the sequence $(\varepsilon^{(i)}\mathbf{E}^n|_{Q_i})_n$ converges with respect to the graph norm of the divergence operator on Q_i. Consequently, $\operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)})$ is an element of $L^2(Q_i)$. Employing now the continuity of the normal trace operator at $\mathscr{F}_{\text{int}}$, we conclude that $([\![\varepsilon\mathbf{E}^n\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}})_n$ converges in $H^{-1/2}(\mathscr{F}_{\text{int}})$ to $[\![\varepsilon\mathbf{E}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}$. By definition of the norm in X_0 (and the uniqueness of limits), the function $[\![\varepsilon\mathbf{E}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}$ is thus contained in $H_0^{1/2}(\mathscr{F}_{\text{int}})$. Altogether, $(\mathbf{E},\mathbf{H})$ is an element of X_0.

We further denote the restriction of the Maxwell operator to X_0 by M_0, and we consider it on the space

$$X_1 := \mathcal{D}(M_0) = \mathcal{D}(M) \cap X_0, \tag{7.17}$$

which is equipped with the norm

$$\left\|\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\right\|_{X_1} := \left\|\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\right\|_{X_0} + \left\|M\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\right\|, \qquad \begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} \in X_1. \tag{7.18}$$

This space is complete. (The claim is a consequence of the closedness of the Maxwell operator, see Proposition 7.8, and the completeness of X_0.) The part of M in X_1 is denoted by M_1, and it is shown in Proposition 9.22 that M_1 generates a strongly continuous semigroup on X_1. In particular, the Maxwell system (7.1) is then wellposed on X_1. We also prove that X_1 embeds into the space of piecewise H^1-regular functions on $Q_1 \cup Q_2$, see Proposition 9.8. This reasoning then results in an H^1-regularity statement for the solutions of (7.1). The latter result is in particular important for growth bounds on the semigroup during our error analysis in Chapter 10.

While the space X_1 is useful to analyze system (7.1) for piecewise H^1-solutions, we employ still another space to achieve H^2-regularity. We generalize the corresponding construction from Section 3 in [EiSc17] by defining the subspace

$$\begin{aligned} X_2 := \{(\mathbf{E},\mathbf{H}) \in \mathcal{D}(M^2) \cap X_0 \mid\ & \operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)}) \in H^1_{00}(Q_i) \text{ for } i \in \{1,2\}, \\ & [\![\varepsilon \mathbf{E} \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} \in H^{3/2}_0(\mathscr{F}_{\text{int}})\} \end{aligned} \tag{7.19}$$

of the domain $\mathcal{D}(M_0^2)$. This space is equipped with the norm

$$\begin{aligned} \|(\mathbf{E},\mathbf{H})\|_{X_2} := {} & \|(\mathbf{E},\mathbf{H})\|_{\mathcal{D}(M^2)} + \|[\![\varepsilon \mathbf{E} \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}\|_{H^{3/2}_0(\mathscr{F}_{\text{int}})} \\ & + \sum_{i=1}^{2} \Big(\big\|\operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)})\big\|_{H^1(Q_i)} + \sum_{\Gamma' \text{ face of } Q_i} \big\|\operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)})\big\|_{H^{1/2}_0(\Gamma')} \Big). \end{aligned} \tag{7.20}$$

Similar to X_0, the space X_2 is also complete with respect to the norm $\|\cdot\|_{X_2}$. Analogously to the operator M_1, we denote the part of M in X_2 by M_2. We compute the domain of M_2 in Lemma 9.21, and show that it also generates a strongly continuous semigroup on X_2. As a consequence, the Maxwell system (7.1) is also wellposed in X_2, and it possesses solutions of piecewise H^2-regularity.

Remark 7.7. In the definition of the spaces X_0 and X_2, the jump condition for the electric field means essentially, that the flow of information through the interface is regular enough to ensure piecewise regularity of the fields. After establishing that the spaces X_1 and X_2 embed into $PH^1(Q)^6$ respectively $PH^2(Q)^6$, we observe that the definitions of X_1 and X_2 are invariant under appropriate changes of the coefficient function inside the jump condition for the electric field, see Remarks 9.9 and 9.18. This is crucial for the wellposedness of the Maxwell system (7.1) in X_1 and X_2. ◊

The following result states the generator property of the Maxwell operator, corresponding to the Maxwell system (7.1) on X. The statement is essentially contained and proved in Proposition 2.3 of [EiSc18].

Proposition 7.8. *Let ε, μ, and σ satisfy (7.2). The Maxwell operator M generates a contractive C_0-semigroup $(\mathrm{e}^{tM})_{t\geq 0}$ on X.*

Proof. The arguments in part 1) of the proof for Proposition 2.3 in [EiSc18] work also in the current setting of discontinuous coefficients. They yield the generator property, as well as the contractivity. □

8. Elliptic transmission problems

This chapter provides supplementary results for the regularity analysis in Chapter 9. These are namely fundamental regularity results for transmission problems involving the Laplacian on the cuboid Q.

In Section 8.1, we study transmission problems concerning functions with continuous normal derivative across the interface $\mathscr{F}_{\text{int}}$. We apply the results in the study of the first component of the magnetic field. Section 8.2 then allows also functions with a discontinuous Neumann derivative at $\mathscr{F}_{\text{int}}$. The conclusions of this Section are valuable for the analysis of the remaining components of the electric and magnetic fields.

8.1. Elliptic problems with homogeneous transmission conditions

The goal of this section is to transfer parts of Lemma 3.6 in [HoJS15] to our setting of discontinuous coefficients. We first consider elliptic problems with discontinuous coefficients, that are related to the first component of the electric and magnetic fields. Let $\eta > 0$ be a function on Q, that is piecewise constant on Q_1 and Q_2. For later reference, we formulate this assumption in compact form as

$$\eta|_{Q_i} \in (0, \infty) \qquad \text{for } i \in \{1, 2\}. \tag{8.1}$$

The map η is used as a representative for the parameters ε and μ from the Maxwell system (7.1).

Let Γ^* be a union of opposite faces of Q. Throughout, Γ^* represents the part of the boundary, on which homogeneous Dirichlet boundary conditions are imposed. The case $\Gamma^* = \emptyset$ will be referred to as the Neumann case, $\Gamma^* = \partial Q$ as the Dirichlet case, and all remaining ones as the mixed case.

We then define the space

$$\begin{aligned} \mathscr{W} := \{u \in PH^2(Q) \mid [\![\eta u]\!]_{\mathscr{F}_{\text{int}}} = 0 = [\![\partial_1 u]\!]_{\mathscr{F}_{\text{int}}},\ u = 0 \text{ on } \Gamma^*, \\ \partial_\nu u = 0 \text{ on } \partial Q \setminus \Gamma^*\}, \end{aligned} \tag{8.2}$$

that serves as a domain of regular functions for the Laplacian on Q. In view of the boundedness of the trace operator, $\mathscr{W}$ is complete if equipped with the norm in $PH^2(Q)$.

The next proposition states that an appropriately scaled version of the Laplace operator is an isomorphism from $\mathscr{W}$ onto $L^2(Q)$. In the proof, we focus on the more involved cases $\Gamma^* = \emptyset$ or $\Gamma^* = \Gamma_1$, where Neumann boundary conditions are prescribed close to the interface $\mathscr{F}_{\text{int}}$. The case of homogeneous Dirichlet boundary conditions is covered by Theorem 5.3 of [Lemr78] and Theorem 5.1 of [Kell71].

Proposition 8.1. *Let Γ^* be nonempty, let η satisfy (8.1), and let $f \in L^2(Q)$. There is a unique function $u \in V_1 := \{v \in PH^1_{\Gamma^*}(Q) \mid [\![\eta v]\!]_{\mathscr{F}_{\text{int}}} = 0\}$, satisfying the formula*

$$\sum_{i=1}^{2} \int_{Q_i} \eta^{(i)} (\nabla u^{(i)}) \cdot (\nabla \varphi^{(i)}) \,\mathrm{d}x = \int_Q f \varphi \,\mathrm{d}x \tag{8.3}$$

for all functions $\varphi \in V_1$. Furthermore, u belongs to $\mathscr{W}$, and there is a constant $C = C(Q, \eta) > 0$ with $\|u\|_{PH^2(Q)} \leq C \sum_{i=1}^{2} \|\Delta u^{(i)}\|_{L^2(Q_i)}$.

When proving that the magnetic field components of a vector in the space X_1 from (7.17) are piecewise regular of first order, we also consider the Neumann case $\Gamma^* = \emptyset$. The latter is treated in the next proposition.

Proposition 8.2. *Let Γ^* be empty, let η satisfy (8.1), $f \in L^2(Q)$, and define $V_2 = \{v \in PH^1(Q) \mid [\![\eta v]\!]_{\mathscr{F}_{\text{int}}} = 0\}$. The variational problem*

$$\sum_{i=1}^{2} \int_{Q_i} u^{(i)} \varphi^{(i)} + \eta^{(i)} (\nabla u^{(i)}) \cdot (\nabla \varphi^{(i)}) \,\mathrm{d}x = \int_Q f \varphi \,\mathrm{d}x, \qquad \varphi \in V_2,$$

has a unique solution $u \in V_2$. The mapping u belongs to $\mathscr{W}$, and it satisfies the inequality $\|u\|_{PH^2(Q)} \leq C(\|u\|_{L^2(Q)} + \sum_{i=1}^{2} \left\|\Delta u^{(i)}\right\|_{L^2(Q_i)})$ with a uniform constant $C = C(Q, \eta) > 0$.

The remainder of this section is concerned with the derivation of Propositions 8.1 and 8.2. The general structure of the argument is inspired by the papers [Kell71, Lemr78], which treat a Poisson problem with discontinuous coefficients and homogeneous Dirichlet boundary conditions. There are, however, significant differences between our proofs and those papers. The changes are necessary to treat the case of Neumann boundary conditions. In particular, the spectral theory for the considered Laplace-Beltrami operator with discontinuous coefficients on the lower hemisphere is much more involved in our setting: To obtain appropriate lower bounds for the smallest positive eigenvalue, we need to analyze a two-dimensional eigenvalue problem for a precise control on the eigenfunctions, see Lemmas 8.6 and 8.9. In the homogeneous Dirichlet case of [Kell71, Lemr78], however, it is sufficient to deal with an easier one-dimensional eigenvalue problem.

8.1.1. Energy estimates for the Laplacian on the cube

Within the next two lemmas, we derive an energy estimate for the Laplace operator on $\mathscr{W}$ in the spirit of Grisvard, see Section 2 in [Gris75]. We thereby use ideas from the proof of Theorem 2.1 in [Lemr78].

Lemma 8.3. *Let η satisfy (8.1). The identity*

$$\sum_{i=1}^{2} \eta^{(i)} \Big(\big\|\partial_1^2 u^{(i)}\big\|_{L^2(Q_i)}^2 + \big\|\partial_2^2 u^{(i)}\big\|_{L^2(Q_i)}^2 + \big\|\partial_3^2 u^{(i)}\big\|_{L^2(Q_i)}^2 + 2\,\big\|\partial_1\partial_2 u^{(i)}\big\|_{L^2(Q_i)}^2 + 2\,\big\|\partial_1\partial_3 u^{(i)}\big\|_{L^2(Q_i)}^2 + 2\,\big\|\partial_2\partial_3 u^{(i)}\big\|_{L^2(Q_i)}^2 \Big) = \sum_{i=1}^{2} \eta^{(i)}\,\big\|\Delta u^{(i)}\big\|_{L^2(Q_i)}^2$$

is valid for $u \in \mathscr{W}$.

Proof. 1) We only treat the case $\Gamma^* = \Gamma_1$. The remaining cases are proved similarly with appropriate modifications. A simple calculation leads to the equation

$$\begin{aligned} \big\|\Delta u^{(i)}\big\|_{L^2(Q_i)}^2 &= \big\|\partial_1^2 u^{(i)}\big\|_{L^2(Q_i)}^2 + \big\|\partial_2^2 u^{(i)}\big\|_{L^2(Q_i)}^2 + \big\|\partial_3^2 u^{(i)}\big\|_{L^2(Q_i)}^2 \\ &\quad + 2\int_{Q_i} (\partial_1^2 u^{(i)})(\partial_2^2 u^{(i)})\,\mathrm{d}x + 2\int_{Q_i} (\partial_1^2 u^{(i)})(\partial_3^2 u^{(i)})\,\mathrm{d}x + 2\int_{Q_i} (\partial_2^2 u^{(i)})(\partial_3^2 u^{(i)})\,\mathrm{d}x \end{aligned} \tag{8.4}$$

for $i \in \{1,2\}$. To show the desired identity, it remains to consider the three terms on the right hand side.

2) Let $i \in \{1,2\}$. We first employ the boundary condition $\partial_\nu u = 0$ on $\Gamma_2 \cup \Gamma_3$, and apply the reasoning in the proof for Lemma 3.3 in [EiSc18] to the functions $\partial_2 u^{(i)}$ and $\partial_3 u^{(i)}$. Consequently, there are sequences $(\varphi_n)_n$ and $(\psi_n)_n$ in $C^\infty(\overline{Q_i})$ with $\varphi_n \to \partial_3 u^{(i)}$, $\psi_n \to \partial_2 u^{(i)}$ in $H^1(Q_i)$, $n \to \infty$, and $\varphi_n = 0$ on $\Gamma_3^{(i)}$, $\psi_n = 0$ on $\Gamma_2^{(i)}$ for $n \in \mathbb{N}$. Lemma 2.1 in [EiSc18] then implies that $\partial_2 \varphi_n = 0$ on $\Gamma_3^{(i)}$, and $\partial_3 \psi_n = 0$ on $\Gamma_2^{(i)}$. Integrating by parts leads to the identities

$$\int_{Q_i} (\partial_3 \varphi_n)(\partial_2 \psi_n)\,\mathrm{d}x = -\int_{Q_i} (\partial_2\partial_3 \varphi_n)\psi_n\,\mathrm{d}x = \int_{Q_i} (\partial_2 \varphi_n)(\partial_3 \psi_n)\,\mathrm{d}x.$$

Taking limits, we infer the formula

$$\int_{Q_i} (\partial_3^2 u^{(i)})(\partial_2^2 u^{(i)})\,\mathrm{d}x = \int_{Q_i} (\partial_2\partial_3 u^{(i)})^2\,\mathrm{d}x \tag{8.5}$$

for the third term on the right hand side of (8.4).

3) Lemma 7.1 implies that $[\![\eta \partial_2 u]\!]_{\mathscr{F}_{\mathrm{int}}} = 0$, and Lemma 2.1 in [EiSc18] yields $\partial_2 u = 0$ on Γ_1. Introduce then two functions g_1 and g_2 on Q by $g_1^{(i)} := \partial_1 u^{(i)}$ and $g_2^{(i)} := \eta \partial_2 u^{(i)}$. By definition of $\mathscr{W}$ and the above transmission conditions, we

infer that g_1 and g_2 belong to $H^1(Q)$. The reasoning in the proof of Lemma 3.3 in [EiSc18] now provides functions $\tilde{\varphi}_n$ and $\hat{\psi}_n$ in $C^\infty(\overline{Q})$ with $\tilde{\varphi}_n \to g_1$, $\hat{\psi}_n \to g_2$ in $H^1(Q)$, $n \to \infty$, and $\hat{\psi}_n = 0$ on $\Gamma_1 \cup \Gamma_2$. We then put $\tilde{\psi}_n := \frac{1}{\eta}\hat{\psi}_n$. This implies that the transmission conditions $[\![\tilde{\varphi}_n]\!]_{\mathscr{F}_{\mathrm{int}}} = 0 = [\![\eta\tilde{\psi}_n]\!]_{\mathscr{F}_{\mathrm{int}}}$ are true. Lemma 2.1 in [EiSc18] and Lemma 7.1 now further show that $\partial_2\tilde{\psi}_n^{(i)} = 0$ on $\Gamma_1^{(i)}$, $\partial_1\tilde{\psi}_n^{(i)} = 0$ on $\Gamma_2^{(i)}$, and $[\![\partial_2\tilde{\varphi}_n]\!]_{\mathscr{F}_{\mathrm{int}}} = 0$. Integrating by parts twice, we hence arrive at the identities

$$\begin{aligned}\sum_{i=1}^{2}\eta^{(i)}\int_{Q_i}(\partial_1\tilde{\varphi}_n^{(i)})(\partial_2\tilde{\psi}_n^{(i)})\,\mathrm{d}x &= -\sum_{i=1}^{2}\eta^{(i)}\int_{Q_i}(\partial_2\partial_1\tilde{\varphi}_n^{(i)})\tilde{\psi}_n^{(i)}\,\mathrm{d}x\\ &= \sum_{i=1}^{2}\eta^{(i)}\int_{Q_i}(\partial_2\tilde{\varphi}_n^{(i)})(\partial_1\tilde{\psi}_n^{(i)})\,\mathrm{d}x.\end{aligned}$$

As above, the limit $n \to \infty$ gives rise to

$$\sum_{i=1}^{2}\eta^{(i)}\int_{Q_i}(\partial_1^2 u^{(i)})(\partial_2^2 u^{(i)})\,\mathrm{d}x = \sum_{i=1}^{2}\eta^{(i)}\int_{Q_i}(\partial_1\partial_2 u^{(i)})^2\,\mathrm{d}x. \tag{8.6}$$

An analogous reasoning also implies the equation

$$\sum_{i=1}^{2}\eta^{(i)}\int_{Q_i}(\partial_1^2 u^{(i)})(\partial_3^2 u^{(i)})\,\mathrm{d}x = \sum_{i=1}^{2}\eta^{(i)}\int_{Q_i}(\partial_1\partial_3 u^{(i)})^2\,\mathrm{d}x. \tag{8.7}$$

Inserting (8.5)–(8.7) into (8.4), we arrive at the desired formula. $\square$

Lemma 8.4. *Let $u \in \mathscr{W}$, and let η satisfy (8.1).*
a) Let Γ^* be nonempty. Then, the estimate

$$\sum_{i=1}^{2}\eta^{(i)}\left\|u^{(i)}\right\|_{H^2(Q_i)} \le C\sum_{i=1}^{2}\eta^{(i)}\left\|\Delta u^{(i)}\right\|_{L^2(Q_i)}$$

is valid with a uniform constant $C = C(\eta, Q)$.
b) Let Γ^* be empty. There is a constant $C = C(\eta, Q)$ with

$$\sum_{i=1}^{2}\eta^{(i)}\left\|u^{(i)}\right\|_{H^2(Q_i)} \le C\sum_{i=1}^{2}\left\|u^{(i)} - \eta^{(i)}\Delta u^{(i)}\right\|_{L^2(Q_i)}.$$

Proof. a) In view of the interface and boundary conditions in $\mathscr{W}$, see (8.2), an integration by parts leads to the relations

$$\sum_{i=1}^{2}\int_{Q_i}\eta^{(i)}\left|\nabla u^{(i)}\right|^2\mathrm{d}x = -\sum_{i=1}^{2}\int_{Q_i}\eta^{(i)}(\Delta u^{(i)})u^{(i)}\,\mathrm{d}x$$

$$\leq \sum_{i=1}^{2} \left\| u^{(i)} \right\|_{L^2(Q_i)} \eta^{(i)} \left\| \Delta u^{(i)} \right\|_{L^2(Q_i)}.$$

The Poincare inequality in Theorem 13.6.9 of [TuWe09], and Lemma 8.3 now imply the first assertion.

b) If Γ^* is empty, the reasoning from part a) gives rise to the formula

$$\sum_{i=1}^{2} \int_{Q_i} \eta^{(i)} \left| \nabla u^{(i)} \right|^2 \, \mathrm{d}x = - \sum_{i=1}^{2} \int_{Q_i} \eta^{(i)} (\Delta u^{(i)}) u^{(i)} \, \mathrm{d}x,$$

so that part b) now follows from Lemma 8.3. □

8.1.2. Geometric constructions

Lemma 8.4 shows that the ranges of $\eta\Delta$ and $I - \eta\Delta$ from $\mathscr{W}$ into $L^2(Q)$ are closed. To show the bijectivity of these operators, it is thus sufficient to demonstrate that the orthogonal complement

$$\mathscr{N} := \begin{cases} (\eta\Delta(\mathscr{W}))_{\perp} & \text{if } \Gamma^* \neq \emptyset, \\ (I - \eta\Delta)(\mathscr{W})_{\perp} & \text{if } \Gamma^* = \emptyset, \end{cases} \tag{8.8}$$

is empty. Note here that the orthogonal complements are taken in $L^2(Q)$ with respect to the standard unweighted inner product. Employing Weyl's Lemma, see Section IV.4.2 in [Hell60], we conclude the relation

$$\mathscr{N} \subseteq \{ v \in L^2(Q) \mid \Delta v^{(i)} = 0, \ i \in \{1, 2\} \} \cap H^2_{\mathrm{loc}}(Q_1) \cap H^2_{\mathrm{loc}}(Q_2)$$

if Γ^* is nonempty, and

$$\mathscr{N} \subseteq \{ v \in L^2(Q) \mid (I - \Delta) v^{(i)} = 0, \ i \in \{1, 2\} \} \cap H^2_{\mathrm{loc}}(Q_1) \cap H^2_{\mathrm{loc}}(Q_2)$$

if Γ^* is empty. In the following, we investigate the behavior of functions in $\mathscr{N}$ near the boundary of the interface $\mathscr{F}_{\mathrm{int}}$. Indeed, we want to demonstrate that functions in $\mathscr{N}$ are piecewise H^2-regular.

Let $\mathscr{M} \in \partial\mathscr{F}_{\mathrm{int}}$ be for the time being no vertex, and let $B(\mathscr{M}, R)$ be a ball of radius $R > 0$, such that $B(\mathscr{M}, R)$ contains no vertex of $\partial\mathscr{F}_{\mathrm{int}}$. We next introduce spherical domains, representing the regions where η is constant. Set $G_i := \partial B(\mathscr{M}, R) \cap Q_i$ for $i \in \{1, 2\}$. After scaling, shifting and rotating, we can assume the representation

$$\begin{aligned} G_1 &= \{ (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta) \mid \varphi \in I_1, \ \theta \in (\tfrac{\pi}{2}, \pi) \}, \\ G_2 &= \{ (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta) \mid \varphi \in I_2, \ \theta \in (\tfrac{\pi}{2}, \pi) \}, \\ I_1 &= (\tfrac{\pi}{2}, \tfrac{3}{2}\pi), \quad I_2 = (-\tfrac{\pi}{2}, \tfrac{\pi}{2}). \end{aligned} \tag{8.9}$$

We denote the common arc of ∂G_1 and ∂G_2 by S, set $G := G_1 \cup G_2 \cup S$, and define the coefficient function η accordingly. Note that the interface $\mathscr{F}_{\mathrm{int}}$ is now represented by S. Throughout, the tuple (φ, θ) denotes spherical coordinates. We then consider the Laplace-Beltrami operator

$$\begin{aligned}
(L\psi)|_{G_i} &:= \Big(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}\Big)\psi|_{G_i}, \qquad i \in \{1,2\}, \qquad (8.10)\\
\psi \in \mathcal{D}(L) &:= \{\psi \in L^2(G) \mid \psi^{(i)} \in H^2(G_i),\ \partial_\nu\psi^{(i)} = 0 \text{ on } \partial G_i \setminus S \text{ if } \partial G \not\subseteq \Gamma^*,\\
&\qquad\qquad \psi^{(i)} = 0 \text{ on } \partial G_i \setminus S \text{ if } \partial G \subseteq \Gamma^*,\\
&\qquad\qquad [\![\eta\psi]\!]_S = 0 = [\![\partial_{\nu_S}\psi]\!]_S\}
\end{aligned}$$

on the lower half sphere G. To simplify the analysis on G, we next use the stereographic projection with respect to the north pole $(0,0,1)$, see (A.1) in the Appendix. This leads to the identification of G with the unit disc $D := \{(x_1,x_2) \mid x_1^2 + x_2^2 < 1\}$, of G_i with D_i, and of S with $\check{S}$, where

$$\begin{aligned}
&D_1 := \{(x_1,x_2) \in D \mid x_1 < 0\}, \quad D_2 := \{(x_1,x_2) \in D \mid x_1 > 0\},\\
&\check{S} := D \cap \{x_1 = 0\}.
\end{aligned}$$

We next denote the function on D, which results from concatenating η with the inverse stereographic projection, again by η. The assumption (8.1) is then formulated in the following way:

The function η is positive, and piecewise constant on G respectively D.

It is now crucial that the above projection procedure transforms the above Laplace-Beltrami operator on G into the standard 2D-Laplacian (up to a smooth factor), see (A.3) in the Appendix. We hence analyze the 2D-Laplacian

$$\begin{aligned}
\check{L}\psi|_{D_i} &:= \Delta\psi^{(i)}, \qquad i \in \{1,2\}, \qquad (8.11)\\
\psi \in \mathcal{D}(\check{L}) &:= \{\psi \in L^2(D) \mid \psi^{(i)} \in H^2(D_i),\ \partial_\nu\psi^{(i)} = 0 \text{ on } \partial D_i \setminus \check{S} \text{ if } \partial G \not\subseteq \Gamma^*,\\
&\qquad\qquad \psi^{(i)} = 0 \text{ on } \partial D_i \setminus \check{S} \text{ if } \partial G \subseteq \Gamma^*,\\
&\qquad\qquad [\![\eta\psi]\!]_{\check{S}} = 0 = [\![\partial_{\nu_{\check{S}}}\psi]\!]_{\check{S}}\}
\end{aligned}$$

on D. During our arguments, the latter operator is easier to handle than the Laplace-Beltrami operator L.

8.1.3. Analysis of a Laplace operator on the unit disc

By providing an energy estimate, the next lemma yields that the two-dimensional Laplacian $\check{L}$ from (8.11) is closed. Note that a similar estimate is contained in Lemma 2.2 of [Kell71] for the Dirichlet case. As the proof in [Kell71] is, however, not entirely comprehensible to the author, we provide a different one.

Lemma 8.5. *Let η be positive and piecewise constant on $D_1 \cup D_2$. There is a constant $C = C(\eta, D) > 0$ with*

$$\sum_{i=1}^{2} \left\| \psi^{(i)} \right\|_{H^2(D_i)} \leq C(\|\psi\|_{L^2(D)} + \left\| \check{L}\psi \right\|_{L^2(D)})$$

for all $\psi \in \mathcal{D}(\check{L})$.

Proof. 1) We only treat the Neumann case $\partial G \not\subseteq \Gamma^*$. The remaining Dirichlet case can then be obtained in the same way. Let $u \in H^2(D)$ with $\partial_\nu u = 0$ on ∂D. Green's formula, and the Cauchy Schwarz inequality yield the relations

$$\int_D |\nabla u|^2 \,\mathrm{d}x = -\int_D (\Delta u) u \,\mathrm{d}x \leq \frac{1}{2}(\|\Delta u\|_{L^2(D)}^2 + \|u\|_{L^2(D)}^2).$$

In combination with Proposition 7.2 in Chapter 5 of [Tayl11], we hence obtain the bound

$$\|u\|_{H^2(D)}^2 \leq C(\|\Delta u\|_{L^2(D)}^2 + \|u\|_{H^1(D)}^2) \leq 2C(\|\Delta u\|_{L^2(D)}^2 + \|u\|_{L^2(D)}^2) \qquad (8.12)$$

with a uniform constant $C = C(D) > 0$.

2) Let $\psi \in \mathcal{D}(\check{L})$. Denote by $\widehat{\phi^{(i)}}$ the reflection of $\phi^{(i)}$ at the line $\{x_1 = 0\}$ for $\phi \in L^2(D)$, and define functions f and g on $D_1 \cup D_2$ via

$$f^{(1)} := \eta^{(1)}\psi^{(1)} - \eta^{(2)}\widehat{\psi^{(2)}}, \qquad f^{(2)} := \eta^{(2)}\psi^{(2)} - \eta^{(1)}\widehat{\psi^{(1)}},$$
$$g^{(1)} := \psi^{(1)} + \widehat{\psi^{(2)}}, \qquad g^{(2)} := \widehat{\psi^{(1)}} + \psi^{(2)}.$$

In consideration of the construction of f, g and the transmission relations for ψ, we derive the identities

$$f^{(1)} = 0 = f^{(2)}, \quad \partial_1 f^{(1)} = \eta^{(1)}\partial_1\psi^{(1)} + \eta^{(2)}\widehat{\partial_1\psi^{(2)}} = -\eta^{(1)}\partial_1\widehat{\psi^{(1)}} + \eta^{(2)}\partial_1\psi^{(2)} = \partial_1 f^{(2)},$$
$$g^{(1)} = g^{(2)}, \quad \partial_1 g^{(1)} = \partial_1\psi^{(1)} - \widehat{\partial_1\psi^{(2)}} = 0 = -\widehat{\partial_1\psi^{(1)}} + \partial_1\psi^{(2)} = \partial_1 g^{(2)}$$

on the interface S from Section 8.1.2. Lemma 7.1 then implies the equations $\partial_2 f^{(1)} = \partial_2 f^{(2)}$ and $\partial_2 g^{(1)} = \partial_2 g^{(2)}$ on S, since ψ belongs by definition of $\mathcal{D}(\check{L})$ to $PH^2(D)$. We then conclude that f and g are elements of $H^2(D)$. A similar reasoning also shows that f and g satisfy homogeneous Neumann boundary conditions on ∂D. Estimate (8.12) now yields the inequality

$$\|\phi\|_{H^2(D)}^2 \leq 2C(\|\phi\|_{L^2(D)}^2 + \|\Delta\phi\|_{L^2(D)}^2)$$

for $\phi \in \{f, g\}$. Combining the estimates for f and g, we arrive at the relations

$$\left\| \begin{pmatrix} \eta^{(1)} & -\eta^{(2)} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \psi^{(1)} \\ \widehat{\psi^{(2)}} \end{pmatrix} \right\|_{H^2(D_1)}^2$$

$$
\begin{aligned}
&\leq 2C\Big(\|f\|^2_{L^2(D)} + \|g\|^2_{L^2(D)} + \|\Delta f\|^2_{L^2(D)} + \|\Delta g\|^2_{L^2(D)} \Big) \\
&= 4C\Bigg(\left\| \begin{pmatrix} \eta^{(1)} & -\eta^{(2)} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \psi^{(1)} \\ \widehat{\psi^{(2)}} \end{pmatrix} \right\|^2_{L^2(D_1)} + \left\| \begin{pmatrix} \eta^{(1)} & -\eta^{(2)} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \Delta\psi^{(1)} \\ \widehat{\Delta\psi^{(2)}} \end{pmatrix} \right\|^2_{L^2(D_1)} \Bigg).
\end{aligned}
$$

Observe that the matrix $\left(\begin{smallmatrix} \eta^{(1)} & -\eta^{(2)} \\ 1 & 1 \end{smallmatrix}\right)$ is bounded and invertible as an operator on $L^2(D_1)^2$ and on $H^2(D_1)^2$. We thus obtain the asserted estimate. $\square$

To find a spectral decomposition of L in Section 8.1.4, we first deal with its relative $\check{L}$. Recall the intervals $I_1 = (\frac{\pi}{2}, \frac{3\pi}{2})$ and $I_2 = (-\frac{\pi}{2}, \frac{\pi}{2})$ from (8.9). We then use on $I_1 \cup I_2$ our notation for the restriction of a function in the same way as before. Note also that the coefficient function η depends only on the angle φ in polar coordinates, and it can thus be interpreted as a piecewise constant positive function on $I_1 \cup I_2$.

We then consider the discontinuous Sturm-Liouville problem

$$
\begin{aligned}
(\psi^{(i)})''(\varphi) &= -\kappa^2 \psi^{(i)}(\varphi) \qquad \text{for } \varphi \in I_i,\ i \in \{1,2\}, \\
\eta^{(1)}\psi^{(1)}(\tfrac{\pi}{2}) &= \eta^{(2)}\psi^{(2)}(\tfrac{\pi}{2}), \quad \eta^{(1)}\psi^{(1)}(\tfrac{3\pi}{2}) = \eta^{(2)}\psi^{(2)}(-\tfrac{\pi}{2}), \\
(\psi^{(1)})'(\tfrac{\pi}{2}) &= (\psi^{(2)})'(\tfrac{\pi}{2}), \quad (\psi^{(1)})'(\tfrac{3\pi}{2}) = (\psi^{(2)})'(-\tfrac{\pi}{2}).
\end{aligned}
\tag{8.13}
$$

Employing Lemma 4.2 and statement (3.23) in [Kell74], we infer after scaling that (8.13) has a countable set of eigenvalues $0 = \kappa_0^2 < \kappa_1^2 \leq \cdots \to \infty$, and associated piecewise smooth eigenfunctions $\psi_0, \psi_1, \ldots$, forming an orthonormal basis of $L^2(-\frac{\pi}{2}, \frac{3\pi}{2}) \cong L^2(0, 2\pi)$. The latter space is here equipped with the weighted inner product

$$
(f,g)_\eta := \int_0^{2\pi} \eta f g \,\mathrm{d}\varphi.
$$

The first eigenfunction ψ_0 is then piecewise constant, and the eigenvalues κ_1^2 and κ_2^2 are equal to 1. We further denote for $\nu \geq 0$ by J_ν the Bessel function

$$
J_\nu(t) := \sum_{j=0}^{\infty} (-1)^j \frac{(\frac{t}{2})^{\nu+2j}}{j!\Gamma(\nu+j+1)}, \qquad t \geq 0,
$$

which is smooth on $(0,\infty)$, see for example the Theorem in Section 5.5.1 in [Trie92]. Here, $\Gamma(\cdot)$ denotes the Gamma-function. In the following, the positive zeros of the derivative J_ν' are important. These are denoted by $0 < \mu_1^{(\nu)} < \mu_2^{(\nu)} < \cdots \to \infty$.

The above tools at hand, we are now in the position to state the following crucial spectral properties of the 2D-Laplacian $\check{L}$ in the Neumann case, see (8.11). It is essential to have a precise knowledge of the eigenvalues and corresponding eigenfunctions of $\check{L}$ to find a lower bound for eigenvalues of the scaled Laplace Beltrami operator L, see the proof of Lemma 8.9. For the next proof we employ ideas from the proofs of the following statements in [Trie92]. These are the Theorem in Section 5.5.3, Lemma 2 in Section 6.4.2, and Theorem 2 in Section 6.4.2.

Lemma 8.6. *Let η be positive and piecewise constant on $D_1 \cup D_2$, and let $\partial G \not\subseteq \Gamma^*$. The following statements are true.*

a) Let $\lambda_{k,l} := (\mu_k^{(\kappa_l)})^2$ for $l, k \in \mathbb{N}$, $\lambda_{1,0} := 0$, and $\lambda_{k,0} := (\mu_{k-1}^{(0)})^2$ for $k \in \mathbb{N}_{\geq 2}$. The functions

$$\Psi_{k,l}(r, \varphi) := J_{\kappa_l}(\sqrt{\lambda_{k,l}} r) \psi_l(\varphi), \qquad r \in (0,1),\ \varphi \in (-\tfrac{\pi}{2}, \tfrac{3\pi}{2}),\ k \in \mathbb{N},\ l \in \mathbb{N}_0,$$

form a complete orthogonal system in $L^2(D)$ with respect to the weighted inner product

$$(f, g)_{\eta, D} := \int_D \eta f g \,\mathrm{d}x, \qquad f, g \in L^2(D).$$

b) Each basis function $\Psi_{k,l}$ belongs to the domain $\mathcal{D}(\check{L})$, and satisfies the eigenvalue relation

$$\check{L} \Psi_{k,l} = -\lambda_{k,l} \Psi_{k,l}, \qquad k \in \mathbb{N},\ l \in \mathbb{N}_0.$$

c) The operator $\check{L}$ is selfadjoint on $L^2(D)$.

Proof. a) The asserted orthogonality follows from the choice of the functions $\{\psi_l \mid l \in \mathbb{N}_0\}$, and the Theorem in Section 5.5.3 in [Trie92]. The completeness of the system $\{\Psi_{k,l} \mid k \in \mathbb{N},\ l \in \mathbb{N}_0\}$ can be concluded in the same manner as in the proof of Lemma 2 in Section 6.4.2 in [Trie92]. In our case, we employ the completeness of the family $\{\psi_k \mid k \in \mathbb{N}_0\}$ in $L^2(-\frac{\pi}{2}, \frac{3}{2}\pi) \cong L^2(0, 2\pi)$.

b) We show first that $\Psi_{k,l}$ belongs to $\mathcal{D}(\check{L})$. In view of the choice of ψ_l as an eigenfunction of (8.13) and $\sqrt{\lambda_{k,l}} = \mu_k^{(\kappa_l)}$ as a zero of J'_{κ_l}, it suffices to show that $\Psi_{k,l}^{(i)}$ belongs to $H^2(D_i)$. As $\Psi_{1,0}^{(i)}$ is constant on D_i, it is clearly an element of $H^2(D_i)$. We further remark that the function

$$\Psi_{k,0}(r, \varphi) = J_0(\mu_{k-1}^{(0)} r) \psi_0(\varphi) = \sum_{j=0}^{\infty} (-1)^j \left(\frac{\mu_{k-1}^{(0)}}{2} \right)^{2j} \frac{(\frac{r}{2})^{2j}}{j! j!} \psi_0(\varphi),$$

$k \in \mathbb{N}_{\geq 2},\ r \in (0,1),\ \varphi \in (-\frac{\pi}{2}, \frac{3}{2}\pi)$, is smooth on each halfdisc $\overline{D_i}$, and thus an element of $PH^2(D)$.

Let now $l \in \mathbb{N}$ with $\kappa_l = 1$. Since ψ_l solves (8.13), it has the representation

$$\psi_l^{(i)}(\varphi) = a_{i,l} \cos(\varphi) + b_{i,l} \sin(\varphi), \qquad \varphi \in (-\tfrac{\pi}{2}, \tfrac{3}{2}\pi),$$

with real numbers $a_{i,l}$ and $b_{i,l}$. Switching to cartesian coordinates, we then consider the function

$$\begin{aligned}\Psi_{k,l}^{(i)}(r,\varphi) &= \frac{1}{2}r(a_{i,l}\cos(\varphi)+b_{i,l}\sin(\varphi))\sum_{j=0}^{\infty}(-1)^j\Big(\frac{\mu_{k-1}^{(1)}}{2}\Big)^{2j}\frac{(\frac{r}{2})^{2j}}{j!(j+1)!}\\ &= \frac{1}{2}(a_{i,l}x_1+b_{i,l}x_2)\sum_{j=0}^{\infty}(-1)^j\Big(\frac{\mu_{k-1}^{(1)}}{2}\Big)^{2j}\frac{(\frac{x_1^2+x_2^2}{4})^j}{j!(j+1)!} =: \Phi^{(i)}(x_1,x_2),\end{aligned}$$

for $(x_1,x_2)\in D_i$. As a result of the uniform convergence of the series and its derivatives, we conclude that $\Phi^{(i)}$ is smooth on $\overline{D_i}$. This means that Φ, and consequently also $\Psi_{k,l}$, are elements of $PH^2(D)$.

It remains to consider the case $l\in\mathbb{N}$ with $\kappa_l>1$. The map $\Psi_{k,l}$ then has the representation

$$\Psi_{k,l}(r,\varphi) = \frac{1}{2^{\kappa_l}}\psi_l(\varphi)r^{\kappa_l}\sum_{j=0}^{\infty}(-1)^j\frac{(\frac{r}{2})^{2j}}{j!\Gamma(\kappa_l+j+1)}.$$

Employing the uniform convergence of the series and its derivatives, and the piecewise smoothness of ψ_l, we then infer the estimate

$$\begin{aligned}\int_0^1\int_{I_i}\Big(\frac{1}{r}\left|\partial_r\Psi_{k,l}^{(i)}\right|^2+\frac{1}{r^3}\left|\partial_\varphi\Psi_{k,l}^{(i)}\right|^2+\left|\partial_r^2\Psi_{k,l}^{(i)}\right|^2+\frac{1}{r}\left|\partial_r\partial_\varphi\Psi_{k,l}^{(i)}\right|^2&\\ +\frac{1}{r^3}\left|\partial_\varphi^2\Psi_{k,l}^{(i)}\right|^2\Big)\,\mathrm{d}\varphi\,\mathrm{d}r<\infty.&\end{aligned}$$

This shows that $\Psi_{k,l}$ belongs to $PH^2(D)$, see Section A in the Appendix for a representation of partial derivatives of second order in polar coordinates.

Let now $k\in\mathbb{N}$, $l\in\mathbb{N}_0$ and $i\in\{1,2\}$. Applying the Theorem in Section 5.5.3 in [Trie92] together with the choice of ψ_l in (8.13), we arrive at the desired relations

$$\begin{aligned}(\check{L}\Psi_{k,l})^{(i)}(r,\varphi) = \Delta\Psi_{k,l}^{(i)}(r,\varphi) &= \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}J_{\kappa_l}(\sqrt{\lambda_{k,l}}r))\psi_l^{(i)}(\varphi)\\ &\quad+\frac{1}{r^2}(\frac{\partial^2}{\partial\varphi^2}\psi_l^{(i)}(\varphi))J_{\kappa_l}(\sqrt{\lambda_{k,l}}r)\\ &= \Big(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}J_{\kappa_l}(\sqrt{\lambda_{k,l}}r))-\frac{1}{r^2}\kappa_l^2J_{\kappa_l}(\sqrt{\lambda_{k,l}}r)\Big)\psi_l^{(i)}(\varphi)\\ &= -\lambda_{k,l}\psi_l^{(i)}(\varphi)J_{\kappa_l}(\sqrt{\lambda_{k,l}}r) = -\lambda_{k,l}\Psi_{k,l}^{(i)}(r,\varphi).\end{aligned}$$

c) In view of the energy estimate from Lemma 8.5, the operator $\check{L}$ is closed. It remains to show that $\check{L}$ is symmetric. With the statements in a) and b), we can then conclude that $\check{L}$ is selfadjoint, see for instance the Theorem in Section 4.5.4 in

[Trie92]. To that end, let ψ and $\tilde{\psi}$ be two elements of the domain of $\check{L}$. Combining the boundary and transmission conditions from $\mathcal{D}(\check{L})$ in an integration by parts, we infer the desired relations

$$\begin{aligned}(\check{L}\psi, \tilde{\psi})_{\eta,D} &= \sum_{i=1}^{2} \int_{D_i} \eta^{(i)} (\Delta\psi^{(i)}) \tilde{\psi}^{(i)} \,\mathrm{d}x = -\sum_{i=1}^{2} \int_{D_i} \eta^{(i)} \nabla\psi^{(i)} \cdot \nabla\tilde{\psi}^{(i)} \,\mathrm{d}x \\ &= \sum_{i=1}^{2} \int_{D_i} \eta^{(i)} \psi^{(i)} (\Delta\tilde{\psi}^{(i)}) \,\mathrm{d}x = (\psi, \check{L}\tilde{\psi})_{\eta,D}.\end{aligned}$$

Choosing here in particular $\psi = \tilde{\psi}$, we also conclude that $-\check{L}$ is positive. □

8.1.4. Spectral analysis of the Laplace-Beltrami operator

The results of Lemma 8.6 for the 2D-Laplacian $\check{L}$ from (8.11) hand, we now derive spectral properties for its counterpart L on the half sphere within the next two lemmas. Recall that L is defined in (8.10). The statements correspond to Theorem 4.1 and Proposition 4.2 in [Lemr78].

Lemma 8.7. *Let η be positive and piecewise constant on $G_1 \cup G_2$. The operator $I - L : \mathcal{D}(L) \to L^2(G)$ is an isomorphism.*

Proof. 1) We again only consider the Neumann case $\partial G \not\subseteq \Gamma^*$. For the Dirichlet case, consult Theorem 4.1 in [Lemr78]. Let $f \in L^2(G)$. We seek for a function $u \in \mathcal{D}(L)$ with $(I - L)u = f$. Therefore, the lower half sphere G is transformed into the disc D via the stereographic projection with respect to the north pole $(0, 0, 1)$, see (A.1) in the Appendix. For a function $w \in L^2(G)$ the transformed function on D is then denoted by $\tilde{w}$. To derive an appropriate weak formulation of the identity $(I - L)u = f$, we use certain facts about the stereographic projection, see Section A in the Appendix.

Assume first that there is a function $u \in \mathcal{D}(L)$ with $(I - L)u = f$. The stereographic projection being a C^∞-smooth diffeomorphism, the function $\tilde{u}$ is again piecewise H^2-regular on D, and $\eta\tilde{u}$ belongs to $H^1(D)$. This means that $\tilde{u}$ satisfies the zero order transmission condition in $\mathcal{D}(\check{L})$. Similar reasoning also shows that $\tilde{u}$ fulfills the first order transmission conditions. To conclude that $\tilde{u}$ is an element of the domain of $\check{L}$, it hence remains to consider the Neumann boundary conditions. Denoting in the following calculation polar coordinates on D by $(r, \tilde{\varphi})$ and spherical coordinates on G by (θ, φ), the stereographic projection is given by the mapping property

$$r = \frac{\sin\theta}{1 - \cos\theta}, \qquad \tilde{\varphi} = \varphi,$$

see (A.4) in the Appendix. The first identity implies the formula

$$\partial_\theta u(\theta,\varphi) = \frac{\partial \tilde{u}}{\partial r}(r,\tilde{\varphi})\frac{\partial r}{\partial \theta} = \frac{1}{\cos\theta - 1}\partial_r \tilde{u}(r,\tilde{\varphi}).$$

Choosing $\theta = \pi/2$, we conclude that $\tilde{u}$ satisfies homogeneous Neumann boundary conditions on ∂D. Altogether, $\tilde{u}$ is an element of the domain of $\check{L}$, see (8.11).

2) Recall now the volume factor from (A.2) in the Appendix, and the representation of the Laplace-Beltrami operator by means of the 2D Laplacian on the disc, see (A.3) in the Appendix. Combining the boundary and interface conditions in an integration by parts, we first infer the equations

$$\begin{aligned}\int_G \eta((I-L)u)v \,\mathrm{d}\varsigma &= \sum_{i=1}^{2}\int_{D_i} \frac{4\eta^{(i)}}{(1+x_1^2+x_2^2)^2}\tilde{u}^{(i)}\tilde{v}^{(i)} - \eta^{(i)}(\Delta\tilde{u}^{(i)})\tilde{v}^{(i)}\,\mathrm{d}(x_1,x_2)\\ &=\sum_{i=1}^{2}\int_{D_i} \frac{4\eta^{(i)}}{(1+x_1^2+x_2^2)^2}\tilde{u}^{(i)}\tilde{v}^{(i)} + \frac{1}{\eta^{(i)}}\nabla(\eta^{(i)}\tilde{u}^{(i)})\cdot\nabla(\eta^{(i)}\tilde{v}^{(i)})\,\mathrm{d}(x_1,x_2),\\ v\in V :&=\{\psi\in PH^1(G) \mid [\![\eta\psi]\!]_S = 0\}.\end{aligned}$$

Taking the fact $(I-L)u = f$ into account, we hence infer the weak formulation

$$\begin{aligned}\sum_{i=1}^{2}\int_{D_i}\Big(\frac{4\eta^{(i)}}{(1+x_1^2+x_2^2)^2}\tilde{u}^{(i)}w^{(i)} + \frac{1}{\eta^{(i)}}\nabla(\eta^{(i)}\tilde{u}^{(i)})\cdot\nabla(\eta^{(i)}w^{(i)})\Big)\,\mathrm{d}(x_1,x_2)&\\ =\int_D \frac{4\eta}{(1+x_1^2+x_2^2)^2}\tilde{f}w\,\mathrm{d}(x_1,x_2),&\qquad (8.14)\\ w\in\tilde{V} :=\{\psi\in PH^1(D)\mid [\![\eta\psi]\!]_{\check{S}} = 0\}&\end{aligned}$$

of the identity $(I-L)u = f$. We hereby tacitly use that the spaces V and $\tilde{V}$ are transformed into each other by means of the stereographic projection, see our reasoning in part 1).

3) We now prove the existence of the desired function u. Thanks to the Lax-Milgram Lemma, (8.14) has a unique weak solution $\tilde{u}\in\tilde{V}$. It satisfies

$$\begin{aligned}\sum_{i=1}^{2}\int_{D_i}\Big(\eta^{(i)}\tilde{u}^{(i)}v^{(i)} + \frac{1}{\eta^{(i)}}\nabla(\eta^{(i)}\tilde{u}^{(i)})\cdot\nabla(\eta^{(i)}v^{(i)})\Big)\,\mathrm{d}(x_1,x_2)&\\ =\int_D \eta g v\,\mathrm{d}(x_1,x_2),&\qquad (8.15)\end{aligned}$$

for all $v\in\tilde{V}$, with $g(x_1,x_2) := \frac{4}{(1+x_1^2+x_2^2)^2}(\tilde{f}-\tilde{u})(x_1,x_2) + \tilde{u}(x_1,x_2)$. We may consider $\tilde{u}$ as a fixed function on the right hand side, so that (8.15) is the weak formulation of the identity $(I-\check{L})\tilde{u} = g$. By Lemma 8.6, $\tilde{u}$ thus belongs to $\mathcal{D}(\check{L})$. Repeating the reasoning in parts 1) and 2) in reverse order, the resulting function u is an element of $\mathcal{D}(L)$, and it satisfies $(I-L)u = f$.

We finally note that a similar reasoning shows that L is also closed, implying that $I - L$ is invertible. □

Lemma 8.8. *Let η be positive and piecewise constant on $G_1 \cup G_2$. The spectrum of $-L$ consists of a countable set of eigenvalues $0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \to \infty$, and there is a complete associated family of eigenvectors $\{\Phi_k \mid k \in \mathbb{N}_0\}$, satisfying the orthonormality relations*

$$\int_G \eta \Phi_k \Phi_l \, \mathrm{d}\varsigma = \delta_{kl}, \qquad k, l \in \mathbb{N}_0.$$

Proof. We again only focus on the Neumann case $\partial G \nsubseteq \Gamma^*$. The Dirichlet setting $\partial G \subseteq \Gamma^*$ is covered by Proposition 4.2 in [Lemr78]. We show first that L is selfadjoint with respect to the weighted inner product

$$(g, f)_{\eta,G} := \int_G \eta g f \, \mathrm{d}\varsigma, \qquad g, f \in L^2(G).$$

Let $\psi, \phi \in \mathcal{D}(L)$. We employ here the reasoning from parts 1) and 2) of the proof for Lemma 8.7. This means that G is projected onto the unit disc D by means of the stereographic projection. The transformed functions are denoted by $\tilde{\psi}$ and $\tilde{\phi}$. The latter mappings are then elements of the domain of $\check{L}$. Combining the symmetry of $\check{L}$, see Lemma 8.6, with the arguments in part 2) of the proof for Lemma 8.7, we arrive at the crucial formula

$$(L\psi, \phi)_{\eta,G} = \int_D \eta (\check{L}\tilde{\psi}) \tilde{\phi} \, \mathrm{d}x = \int_D \eta \tilde{\psi} \check{L} \tilde{\phi} \, \mathrm{d}x = (\psi, L\phi)_{\eta,G}. \tag{8.16}$$

As a result, L is symmetric with respect to the inner product $(\cdot,\cdot)_{\eta,G}$. By Lemma 8.7, the operator $I - L$ is invertible, implying that L is selfadjoint on $L^2(G)$. Since the domain $\mathcal{D}(L)$ is embedded into $\frac{1}{\eta} \cdot H^1(G)$, the embedding of $\mathcal{D}(L)$ into $L^2(G)$ is compact. As $-L$ is further nonnegative, the spectral theorem for selfadjoint operators with compact resolvent implies the statements about the structure of the spectrum of $-L$, and the associated eigenbasis of $-L$. □

In the Neumann case $\partial G \nsubseteq \Gamma^*$, we choose the first eigenvector of L to be piecewise constant, meaning $\Phi_0 := \frac{\alpha}{\eta}$, with $\alpha \in \mathbb{R}$ a normalizing factor. The next lemma provides us with a lower bound for the first nonzero eigenvalue of $-L$, which turns out to be crucial for the regularity of functions in the space $\mathscr{N}$ from (8.8).

Lemma 8.9. *Let η be positive and piecewise constant on $G_1 \cup G_2$. In case of Neumann boundary conditions $\partial G \nsubseteq \Gamma^*$, the eigenvalue λ_1 of $-L$ satisfies $\lambda_1 > \frac{3}{4}$. For Dirichlet boundary conditions $\partial G \subseteq \Gamma^*$, the estimate $\lambda_0 \geq 1$ is valid.*

Proof. We consider only the Neumann case. The other asserted inequality may be obtained with the same arguments as in the proof of Proposition 4.3 in [Lemr78].

Let V be a two-dimensional subspace of $\mathcal{D}(L)$. For $\psi \in V$, we denote by $\tilde{\psi}$ the function on D, obtained by projecting G onto D via the stereographic projection. The latter projection is introduced in Section A in the Appendix. Then the space

$$\tilde{V} := \{g \mid g = \tilde{\psi} \text{ with } \psi \in V\}$$

is contained in $\mathcal{D}(\check{L})$, and it is also two-dimensional, see part 1) of the proof of Lemma 8.7. We recall the inner products $(\cdot,\cdot)_{\eta,G}$ and $(\cdot,\cdot)_\eta$ on G respectively D, which induce norms $\|\cdot\|_{\eta,G}$ and $\|\cdot\|_\eta$. The latter are equivalent to the standard L^2-norms on G and D, respectively. In view of (A.2) in the Appendix and (8.16), we obtain the inequality

$$\frac{(-L\psi,\psi)_{\eta,G}}{\|\psi\|_{\eta,G}^2} = \frac{(-\check{L}\tilde{\psi},\tilde{\psi})_\eta}{\int_D \frac{4\eta}{(1+x_1^2+x_2^2)^2}|\tilde{\psi}|^2 \,\mathrm{d}(x_1,x_2)} \geq \frac{1}{4}\frac{(-\check{L}\tilde{\psi},\tilde{\psi})_\eta}{\|\tilde{\psi}\|_\eta^2}.$$

Since V is an arbitrary two-dimensional subspace of $\mathcal{D}(L)$, we infer the relations

$$\begin{aligned} \max_{\psi\in V\setminus\{0\}} \frac{(-L\psi,\psi)_{\eta,G}}{\|\psi\|_{\eta,G}^2} &\geq \frac{1}{4}\max_{\tilde{\psi}\in\tilde{V}\setminus\{0\}} \frac{(-\check{L}\tilde{\psi},\tilde{\psi})_\eta}{\|\tilde{\psi}\|_\eta^2}, \\ \min_{\substack{V\leq\mathcal{D}(L),\\ \dim V=2}} \max_{\psi\in V\setminus\{0\}} \frac{(-L\psi,\psi)_{\eta,G}}{\|\psi\|_{\eta,G}^2} &\geq \frac{1}{4}\min_{\substack{\tilde{V}\leq\mathcal{D}(\check{L}),\\ \dim\tilde{V}=2}} \max_{\tilde{\psi}\in\tilde{V}\setminus\{0\}} \frac{(-\check{L}\tilde{\psi},\tilde{\psi})_\eta}{\|\tilde{\psi}\|_\eta^2}. \end{aligned}$$

The Courant-Fischer Theorem now yields the estimate

$$\lambda_1 \geq \frac{1}{4}\min\Big(\{\lambda_{k,0} \mid k\geq 2\} \cup \{\lambda_{k,l} \mid k,l\in\mathbb{N}\}\Big).$$

In consideration of the formula

$$J_0'(t) = \sum_{j=1}^{\infty}(-1)^j j\frac{(\frac{t}{2})^{2j-1}}{j!j!} = -\sum_{j=0}^{\infty}(-1)^j\frac{(\frac{t}{2})^{2j+1}}{j!(j+1)!} = -J_1(t), \qquad t\geq 0,$$

estimate (1) in [Lorc93] provides the lower bound $\lambda_{k,0}\geq 12$ for $k\geq 2$. Being related to zeros of the derivatives of certain Bessel functions in Lemma 8.6, the remaining relevant eigenvalues satisfy $\lambda_{k,l} > 3$ for $k,l\in\mathbb{N}$, see Section 15.3 in [Wats66]. This shows the asserted estimate. □

8.1.5. Conclusion of the elliptic regularity statements

To demonstrate the desired regularity statements in Propositions 8.1 and 8.2, we show that the space $\mathscr{N}$ from (8.8) is trivial. A major step in this direction is the result that all functions in $\mathscr{N}$ are piecewise H^2-regular on Q.

By means of the above preparations, we first deduce H^2-regularity for functions in $\mathscr{N}$ in a neighborhood of the intersection of the interface $\mathscr{F}_{\text{int}}$ with the boundary ∂Q. We denote this intersection by $\partial\mathscr{F}_{\text{int}}$ in the next statement (because it is the boundary of $\mathscr{F}_{\text{int}}$ in $\mathbb{R}^2$). Although the associated proof follows essentially by combining the arguments of the proofs for Theorems 2.1 and 5.1 in [Kell71], and the proof for Proposition 5.2 in [Lemr78] with our results in Lemmas 8.7–8.9, we include the proof for the sake of a self-contained presentation. Note that the ideas of the proof are also employed by Grisvard in Lemma 2.4 of [Gris75].

Lemma 8.10. *Let η satisfy (8.1). Let further $v \in \mathscr{N}$, and $\mathscr{M} \in \partial\mathscr{F}_{\text{int}}$ be no vertex. Then, $v^{(i)}$ belongs to $H^2(Q_i \cap B(\mathscr{M}, \rho))$ for some $\rho > 0$.*

Proof. 1) We again only treat the Neumann case $\mathscr{M} \notin \Gamma^*$, since the Dirichlet case follows in an analogous way. Let $\delta > 0$ be chosen such that the ball $B(\mathscr{M}, \delta)$ contains no vertex of $\partial\mathscr{F}_{\text{int}}$. After shifting and rotating, we can assume the formula

$$B(\mathscr{M}, \delta) \cap Q_i = \{rs \mid r \in (0, \delta),\ s \in G_i\}.$$

We employ here the spherical domains G_i from (8.9). The assumption $\mathscr{M} \notin \Gamma^*$ then corresponds to the Neumann setting $\partial G \not\subseteq \Gamma^*$.

2) In view of Fubini's Theorem, the function $\tilde{v}_r : G \to \mathbb{R}$, $s \mapsto v(rs)$, is L^2-integrable for almost all $r \in (0, \delta)$. Employing the eigenbasis $\{\Phi_k \mid k \in \mathbb{N}\}$ of the Laplace-Beltrami operator L on the lower hemisphere G, see Lemma 8.8, $\tilde{v}_r$ has the representation

$$\tilde{v}_r = \sum_{k=0}^{\infty} \alpha_k(r)\Phi_k, \qquad \alpha_k(r) := \int_G \eta v(rs)\Phi_k(s)\,\mathrm{d}s, \qquad r \in (0, \delta).$$

Since

$$\begin{aligned}\int_0^\delta r \int_G |\eta v(rs)\Phi_k(s)|\,\mathrm{d}s\,\mathrm{d}r &\leq \|\eta\|_\infty^{1/2} \int_0^\delta r\Big(\int_G |v(rs)|^2\,\mathrm{d}s\Big)^{1/2}\Big(\int_G \eta|\Phi_k|^2\,\mathrm{d}s\Big)^{1/2}\mathrm{d}r \\ &\leq \|\eta\|_\infty^{1/2}\,\|v\|_{L^2(Q)},\end{aligned}$$

we conclude by Fubini's Theorem that the function $(0, \delta) \to \mathbb{R}$, $r \mapsto \alpha_k(r)$, is measurable. We further obtain with Parseval's identity the relations

$$\sum_{k=0}^{\infty} \int_0^\delta r^2|\alpha_k(r)|^2\,\mathrm{d}r = \int_0^\delta \sum_{k=0}^{\infty} r^2|\alpha_k|^2\,\mathrm{d}r = \int_0^\delta \int_G r^2\eta|v(rs)|^2\,\mathrm{d}s\,\mathrm{d}r$$

$$\leq \|\eta\|_\infty \|v\|_{L^2(G)}^2 < \infty. \tag{8.17}$$

In particular, α_k is integrable on every subinterval $[a, \delta]$ for $a > 0$.

3) Take a test function $\chi \in C_c^\infty(0, \delta)$, and set

$$u_k(rs) := \begin{cases} \chi(r)\Phi_k(s) & \text{if } r \in (0, \delta), \\ 0 & \text{else,} \end{cases}$$

for $s \in G$. Since $\Phi_k \in \mathcal{D}(L)$, the function u_k belongs to $PH^2(Q)$, and it satisfies the relations $[\![\eta u_k]\!]_{\mathscr{F}_{\text{int}}} = [\![\partial_1 u_k]\!]_{\mathscr{F}_{\text{int}}} = 0$, $\partial_\nu u_k = 0$ on $\Gamma_2 \cup \Gamma_3$, and $u_k = 0$ on Γ_1. This means that u_k is an element of $\mathscr{W}$, see (8.2). By definition of $\mathscr{N}$ in (8.8), the identities

$$0 = (v, \eta \Delta u_k)_{L^2} = \sum_{i=1}^{2} \int_0^\delta \int_{G_i} \eta v^{(i)}(rs) \Delta u_k^{(i)}(rs) r^2 \,\mathrm{d}s \,\mathrm{d}r \tag{8.18}$$

follow. We next rewrite the Laplacian Δ in three-dimensional polar coordinates, see Section A in the Appendix. The definitions of L in (8.10) and u_k, as well as the eigenvector relation for Φ_k from Lemma 8.8 yield

$$\begin{aligned} \Delta u_k^{(i)}(r, \varphi, \theta) &= \frac{1}{r^2} \partial_r (r^2 \partial_r u_k^{(i)}(r, \varphi, \theta)) + \frac{1}{r^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta u_k^{(i)}(r, \varphi, \theta)) \\ &\quad + \frac{1}{r^2 \sin^2 \theta} \partial_\varphi^2 u_k^{(i)}(r, \varphi, \theta) \\ &= \frac{1}{r^2} \partial_r (r^2 \partial_r u_k^{(i)}(r, \varphi, \theta)) + \frac{1}{r^2} \chi(r) L \Phi_k^{(i)}(\varphi, \theta) \\ &= \frac{1}{r^2} (2r\chi'(r) + r^2 \chi''(r) - \lambda_k \chi(r)) \Phi_k^{(i)}(\varphi, \theta). \end{aligned}$$

So (8.18) leads to the formula

$$\begin{aligned} 0 &= \int_0^\delta \int_G \eta (2r\chi'(r) + r^2 \chi''(r) - \lambda_k \chi(r)) \Phi_k(s) v(rs) \,\mathrm{d}s \,\mathrm{d}r \\ &= \int_0^\delta (r^2 \chi''(r) + 2r\chi'(r) - \lambda_k \chi(r)) \alpha_k(r) \,\mathrm{d}r. \end{aligned} \tag{8.19}$$

4) Weyl's Lemma, see Section IV.4.2 in [Hell60], now implies that α_k can be identified with a twice continuously differentiable function, up to a set of measure zero. We can thus assume that α_k belongs to $C^2(0, \delta)$. Integrating by parts, we hence infer from (8.19) the identity

$$(r^2 \alpha_k')' - \lambda_k \alpha_k = 0 \tag{8.20}$$

on $(0,\delta)$, since $\chi \in C_c^\infty(0,\delta)$ is chosen arbitrarily. Interpreting (8.20) as an end-point respectively initial value problem on $(0,\delta/2]$ and $[\delta/2,\delta)$ with final respectively initial values $\alpha(\delta/2), \alpha'(\delta/2) \in \mathbb{R}$, the theory of ordinary differential equations shows that the space of solutions of (8.20) is two-dimensional. We can thus write

$$\alpha_k(r) = a_k r^{\zeta_{1,k}} + b_k r^{\zeta_{2,k}}, \tag{8.21}$$

for $k \in \mathbb{N}$, and for the exponents

$$\zeta_{1,k} = \frac{-1 + (1+4\lambda_k)^{1/2}}{2}, \qquad \zeta_{2,k} = \frac{-1 - (1+4\lambda_k)^{1/2}}{2},$$

where $a_k, b_k \in \mathbb{R}$ are chosen such that $\alpha_k(r) = \int_G \eta v(rs)\Phi_k(s)\,\mathrm{d}s$. Recall also that

$$\int_G \eta\,|v(rs)|^2\,\mathrm{d}s = \sum_{k=0}^\infty |\alpha_k(r)|^2.$$

5) We next deduce that $b_k = 0$ for $k \in \mathbb{N}_0$, and start with $k = 0$. As in the proof of Theorem 2.1 in [Kell71], we choose a smooth function $\check{\chi} \in C^\infty(0,\delta)$ with $\check{\chi} = 1$ on $[0,\delta/4]$, and $\check{\chi} = 0$ on $[3\delta/4,\delta]$. Define then

$$\check{u}_0(rs) := \begin{cases} \check{\chi}(r)\Phi_0(s) & \text{if } r \in (0,\delta), \\ 0 & \text{else,} \end{cases}$$

for $s \in G$. Since Φ_0 is piecewise constant, $\check{u}_0$ belongs to $\mathscr{W}$, see (8.2). As in (8.19), we infer the identity

$$0 = \int_0^\delta (r^2\check{\chi}')'\alpha_0(r)\,\mathrm{d}r = \int_0^\delta (r^2\check{\chi}')'\Big(a_0 + \frac{b_0}{r}\Big)\,\mathrm{d}r,$$

using (8.21). Integrating by parts, we then calculate

$$0 = \int_0^\delta b_0\check{\chi}'\,\mathrm{d}r = b_0(\check{\chi}(\delta) - \check{\chi}(0)) = -b_0.$$

Let now $k \in \mathbb{N}$. On the one hand, Lemma 8.9 implies $\zeta_{2,k} \le -\frac{3}{2}$. On the other hand, (8.17) and (8.21) yield the relations

$$\infty > \sum_{k=1}^\infty \int_0^\delta r^2|\alpha_k(r)|^2\,\mathrm{d}r = \sum_{k=1}^\infty \int_0^\delta \Big(|a_k|^2 r^{2+2\zeta_{1,k}} + 2a_k b_k r^{2+\zeta_{1,k}+\zeta_{2,k}} + |b_k|^2 r^{2+2\zeta_{2,k}}\Big)\,\mathrm{d}r.$$

This shows that $b_k = 0$ for all $k \in \mathbb{N}$, and leads to the estimate

$$\infty > \sum_{k=1}^\infty \int_0^\delta |a_k|^2 r^{2+2\zeta_{1,k}}\,\mathrm{d}r = \sum_{k=1}^\infty |a_k|^2 \frac{\delta^{3+2\zeta_{1,k}}}{3+2\zeta_{1,k}}. \tag{8.22}$$

6) It now suffices to deduce that the series $\sum_{k=1}^{\infty} \alpha_k \Phi_k$ converges on $G_i \times (0,\rho)$ in the H^2-sense in polar coordinates for $\rho = \delta/2$. We first note that the convergence in L^2 follows already from (8.17). Let $l \geq m$ in $\mathbb{N}$. We compute

$$\begin{aligned}\sum_{k=m}^{l} \int_0^\rho \int_{G_i} \eta^{(i)} \Big(|\partial_r^2 \alpha_k(r)|^2 + \frac{1}{r^2} |\partial_r \alpha_k(r)|^2 \Big) |\Phi_k(s)|^2 r^2 \, \mathrm{d}s \, \mathrm{d}r \\ = \sum_{k=m}^{l} \int_0^\rho a_k^2 (\zeta_{1,k}^2 (\zeta_{1,k} - 1)^2 r^{2\zeta_{1,k}-2} + \zeta_{1,k}^2 r^{2\zeta_{1,k}-2}) \, \mathrm{d}r \\ = \sum_{k=m}^{l} a_k^2 \zeta_{1,k}^2 \frac{(\zeta_{1,k}-1)^2 + 1}{2\zeta_{1,k} - 1} \rho^{2\zeta_{1,k}-1}.\end{aligned}$$

Lemma 8.8 and (8.21) imply that $(\zeta_{1,k})_k$ is a monotonically increasing divergent sequence. Consequently, there are uniform positive constants C and K with

$$\zeta_{1,k}^2 \frac{(\zeta_{1,k}-1)^2 + 1}{2\zeta_{1,k} - 1} \leq C \frac{2^{2\zeta_{1,k}-1}}{3 + 2\zeta_{1,k}}, \qquad k > K.$$

Estimate (8.22) consequently yields the inequalities

$$\begin{aligned}\sum_{k=m}^{l} \int_0^\rho \int_{G_i} \eta^{(i)} \Big(|\partial_r^2 \alpha_k(r)|^2 + \frac{1}{r^2} |\partial_r \alpha_k(r)|^2 \Big) |\Phi_k(s)|^2 r^2 \, \mathrm{d}s \, \mathrm{d}r &\leq 2C \sum_{k=m}^{l} a_k^2 \frac{\delta^{2\zeta_{1,k}+3}}{3 + 2\zeta_{1,k}} \\ &\leq \overline{C}\end{aligned}$$

for all $l, m \geq K$ with a uniform constant $\overline{C} > 0$. Denote next by $\tilde{\Phi}_k$ the function being obtained from Φ_k by projecting G onto D via a stereographic projection, see Section A in the Appendix. Applying then Lemmas 8.5 and 8.9, we also derive the estimates

$$\begin{aligned}\|\Phi_k\|_{PH^2(G)} &\leq \tilde{C}_1 \big\|\tilde{\Phi}_k\big\|_{PH^2(D)} \leq \tilde{C}_2 (\big\|\tilde{\Phi}_k\big\|_{L^2(D)} + \big\|\Delta \tilde{\Phi}_k\big\|_{L^2(D)}) \\ &\leq 4\tilde{C}_2 (\|\Phi_k\|_{L^2(G)} + \|L\Phi_k\|_{L^2(G)}) \\ &= 4\tilde{C}_2 (1 + \lambda_k) \leq 12 \tilde{C}_2 \lambda_k, \qquad k \in \mathbb{N},\end{aligned}$$

with uniform constants $\tilde{C}_1, \tilde{C}_2 > 0$. Using also (8.22), we arrive at the remaining inequalities

$$\begin{aligned}\sum_{k=m}^{l} \Big(\int_0^\rho |\partial_r \alpha_k(r)|^2 \big\|\Phi_k^{(i)}\big\|_{H^1(G_i)}^2 + \frac{1}{r^2} |\alpha_k(r)|^2 \|\Phi_k^{(i)}\|_{H^2(G_i)}^2 \, \mathrm{d}r \Big) \\ \leq 144 \tilde{C}_2^2 \sum_{k=m}^{l} \lambda_k^2 a_k^2 (\zeta_{1,k}^2 + 1) \frac{\rho^{2\zeta_{1,k}-1}}{2\zeta_{1,k} - 1}\end{aligned}$$

$$\leq \check{C}_1 \sum_{k=m}^{l} a_k^2 \frac{\delta^{3+2\zeta_{1,k}}}{3+2\zeta_{1,k}} \leq \check{C}_2,$$

for $l, m > K$ with uniform positive constants $\check{C}_1$ and $\check{C}_2$. $\square$

In the next result, we analyze functions in $\mathscr{N}$ in a neighborhood of the corners of $\mathscr{F}_{\text{int}}$. Recall for the proof that Γ^* is the set of faces on Q, on which Dirichlet boundary conditions are prescribed in the definition of the space $\mathscr{W}$, see (8.2).

Lemma 8.11. *Let η satisfy (8.1). Let further $v \in \mathscr{N}$, $i \in \{1,2\}$, and $\mathscr{M} \in \overline{\mathscr{F}_{\text{int}}} \cap \partial Q$. Then, $v^{(i)}$ belongs to $H^2(Q_i \cap B(\mathscr{M}, \rho))$ for a constant $\rho > 0$.*

Proof. In view of Lemma 8.10, it remains to consider the case of $\mathscr{M}$ being a corner of $\mathscr{F}_{\text{int}}$. In the following, we adapt previous constructions from Section 8.1.2 to the current setting. Note also that we consider first only the cases $\Gamma^* = \Gamma_1$ or $\Gamma^* = \emptyset$. Both options correspond to Neumann boundary conditions on the faces of Q that touch $\mathscr{F}_{\text{int}}$.

1) For sufficiently small $R > 0$, the sets $B(\mathscr{M}, R) \cap Q_i$, $i \in \{1,2\}$, can be represented as

$$\begin{aligned} B(\mathscr{M}, R) \cap Q_i &= \{rs \mid r \in (0,R),\ s \in G_{i,c}\}, \\ G_{1,c} &= \{(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta) \mid \varphi \in (\tfrac{\pi}{2}, \pi),\ \theta \in (\tfrac{\pi}{2}, \pi)\}, \\ G_{2,c} &= \{(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta) \mid \varphi \in (0, \tfrac{\pi}{2}),\ \theta \in (\tfrac{\pi}{2}, \pi)\}, \end{aligned}$$

(after rotation and translation). Analogous to the sets G and S in Section 8.1.2, we define the interface and spherical domain

$$S_c := \overline{G_{1,c}} \cap \overline{G_{2,c}}, \quad G_c := G_{1,c} \cup G_{2,c} \cup S_c.$$

The Laplace-Beltrami operator L from (8.10) is then adapted to an operator L_c on G_c. More precisely, we set

$$\begin{aligned} (L_c\psi)|_{G_{i,c}} &:= \Big(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}\Big)\psi|_{G_{i,c}}, \qquad i \in \{1,2\}, \\ \psi \in \mathcal{D}(L_c) &:= \{\psi \in L^2(G_c) \mid \psi^{(i)} \in H^2(G_{i,c}),\ \partial_\nu\psi^{(i)} = 0 \text{ on } \partial G_{i,c} \setminus S_c \\ &\qquad [\![\eta\psi]\!]_{S_c} = 0 = [\![\partial_{\nu_{S_c}}\psi]\!]_{S_c}\}. \end{aligned}$$

2) We next show that the operator $I - L_c : \mathcal{D}(L_c) \to L^2(G_c)$ is an isomorphism. Let $f \in L^2(G_c)$. Denote the reflection of a function $w \in L^2(G_c)$ at the plane $\{x_2 = 0\}$ by $\hat{w}$, and define a function $\tilde{w}$ on G by $\tilde{w}|_{\{x_2<0\}} := \hat{w}$ and $\tilde{w}|_{\{x_2>0\}} := w$. Note that $\tilde{w}$ belongs to $L^2(G)$. Lemma 8.7 then provides a unique function $u \in \mathcal{D}(L)$

with $(I-L)u=\tilde f$ and $\|u\|_{PH^2(G)} \le C\|\tilde f\|_{L^2(G)} = 2C\,\|f\|_{L^2(G_c)}$. Consider now the function

$$\mathring{u}(x_1,x_2,x_3) := \frac{1}{2}(u(x_1,x_2,x_3)+u(x_1,-x_2,x_3)), \qquad (x_1,x_2,x_3)\in G_c,$$

belonging to $\mathcal{D}(L_c)$ by construction. The choice of u and $\tilde f$ then gives rise to the formula $(I-L_c)\mathring{u}=f$, as well as to the estimate $\|\mathring{u}\|_{PH^2(G_c)} \le 2C\,\|f\|_{L^2(G_c)}$. This establishes that $I-L_c$ is an isomorphism.

3) Arguing analogously to the proof of Lemma 8.8, we conclude that the spectrum of $-L_c$ consists of eigenvalues $0=\lambda_0^c \le \lambda_1^c \le \cdots \to \infty$, and that there is an associated orthonormal basis of eigenvectors $\{\Phi_k^c \mid k\in\mathbb{N}_0\}$. The latter family is orthonormal with respect to the L^2-inner product on G_c with weight η.

4) We again use the notation from part 2). In view of the vanishing outer normal derivative at $\partial G_{i,c}\cap\partial Q$, we infer that the system $\{\tilde{\Phi}_k^c \mid k\in\mathbb{N}_0\}$ is contained in $\mathcal{D}(L)$. By construction, the eigenvalue relations $-L\tilde{\Phi}_k^c = \lambda_k^c\tilde{\Phi}_k^c$ are further satisfied. We consequently infer the estimate $\lambda_1^c \ge \lambda_1 \ge \frac{3}{4}$, employing Lemma 8.9.

5) We finally conclude the asserted statement by employing the results of parts 3) and 4), and adapting the arguments in the proof of Lemma 8.10 to the current setting.

6) Consider now the case of different boundary conditions on the faces of Q that touch $\mathscr{F}_{\mathrm{int}}$, meaning $\Gamma^* \notin \{\partial Q, \Gamma_1\}$. We then assume without loss of generality that the Neumann boundary part of G_c coincides with $\partial G_c \cap \{x_2=0\}$. (This can be obtained after rotating.) The operator L_c is now studied on the domain

$$\begin{aligned}\mathcal{D}(L_c) := \{\psi\in L^2(G_c) \mid {}& \psi^{(i)}\in H^2(G_{i,c}),\ \partial_\nu\psi^{(i)}=0 \text{ on } \partial G_{i,c}\cap\{x_2=0\}\setminus S_c,\\ & \psi^{(i)}=0 \text{ on } \partial G_{i,c}\cap\{x_3=0\}\setminus S_c,\\ & [\![\eta\psi]\!]_{S_c} = 0 = [\![\partial_{\nu_{S_c}}\psi]\!]_{S_c}\}.\end{aligned}$$

The reasoning in parts 2)–5) applies again, and we can conclude the asserted statement.

7) It finally remains to deal with the case of homogeneous Dirichlet boundary conditions on the faces next to $\mathscr{F}_{\mathrm{int}}$. Here we argue essentially in the same way. The operator L_c is defined as in part 6), except that we now assume homogeneous Dirichlet boundary conditions on ∂G_c. In this setting, we extend functions $w\in L^2(G_c)$ by

$$\tilde w(x_1,x_2,x_3) := \begin{cases} w(x_1,x_2,x_3) & \text{if } (x_1,x_2,x_3)\in G_c,\\ -w(x_1,-x_2,x_3) & \text{if } (x_1,x_2,x_3)\in G\setminus G_c,\end{cases}$$

to G. For $f\in L^2(G_c)$, Lemma 8.7 provides a unique function $u\in\mathcal{D}(L)$ with $(I-L)u=\tilde f$, and with $\|u\|_{PH^2(G)} \le C\|f\|_{L^2(G_c)}$. Choose now the map

$$\mathring{u}(x_1,x_2,x_3) = \frac{1}{2}\Big(u(x_1,x_2,x_3)-u(x_1,-x_2,x_3)\Big), \qquad (x_1,x_2,x_3)\in G_c.$$

By construction, this mapping is an element of $\mathcal{D}(L_c)$, and it satisfies the identity $(I - L_c)\mathring{u} = f$, as well as the inequality $\|\mathring{u}\|_{PH^2(G_c)} \leq C\|f\|_{L^2(G_c)}$. We conclude that $I - L_c$ is an isomorphism. The arguments in parts 3)–5) yield also in the current setting the asserted statement. $\square$

The above statements at hand, we are now in the position to deduce that the Laplace operator maps the space $\mathscr{W}$ from (8.2) isomorphically onto $L^2(Q)$.

Proposition 8.12. *Let η satisfy (8.1). If Γ^* is nonempty, the operator $\eta\Delta : \mathscr{W} \to L^2(Q)$ is an isomorphism. In the contrary case $\Gamma^* = \emptyset$, the operator $I - \eta\Delta : \mathscr{W} \to L^2(Q)$ is an isomorphism.*

Proof. We only consider the case $\Gamma^* = \Gamma_1$ here, since all other cases can be treated by analogous arguments. In view of Lemma 8.4, it remains to show that the orthogonal complement $\mathscr{N}$ from (8.8) is trivial. Let $v \in \mathscr{N}$.

1a) We first deduce that v is piecewise H^2-regular on Q. Standard elliptic regularity theory shows that $v^{(i)}$ is contained in $H^2_{\mathrm{loc}}(Q_i)$ for $i \in \{1,2\}$, see Weyl's Lemma in Section IV.4.2 of [Hell60]. Employing the compactness of $\partial\mathscr{F}_{\mathrm{int}}$ and Lemma 8.11, there is a union $\mathscr{T}$ of open tubes of inner radius $\delta > 0$ around $\partial\mathscr{F}_{\mathrm{int}}$, such that v is contained in $PH^2(\mathscr{T})$, relative to the partition $\overline{Q} = \overline{Q_1} \cup \overline{Q_2}$.

Let $U \subseteq Q$ be an open superset of

$$(-\delta/2, \delta/2) \times (a_2^- + \tfrac{3}{4}\delta, a_2^+ - \tfrac{3}{4}\delta) \times (a_3^- + \tfrac{3}{4}\delta, a_3^+ - \tfrac{3}{4}\delta)$$

with the following properties: U has a smooth boundary, does not touch ∂Q, and is symmetric with respect to the plane $\{x_1 = 0\}$. We denote $U_i := U \cap Q_i$ for $i \in \{1,2\}$. Let further $\chi_1 : (a_1^-, a_1^+) \to [0,1]$ and $\chi_j : (a_j^-, a_j^+) \to [0,1]$ be smooth cut-off functions with $\chi_1 = 1$ on $(-\delta/8, \delta/8)$, $\operatorname{supp}\chi_1 \subseteq (-\delta/4, \delta/4)$, as well as $\chi_j = 1$ on $(a_j^- + \frac{15}{16}\delta, a_j^+ - \frac{15}{16}\delta)$ and $\operatorname{supp}\chi_j \subseteq (a_j^- + \frac{7}{8}\delta, a_j^+ - \frac{7}{8}\delta)$ for $j \in \{2,3\}$. Define

$$\chi(x_1, x_2, x_3) := \chi_1(x_1)\chi_2(x_2)\chi_3(x_3), \qquad (x_1, x_2, x_3) \in Q.$$

For a function $v \in L^2(U)$, denote in the following the reflection of $v^{(i)}$ at the plane $\{x_1 = 0\}$ by $\widehat{v^{(i)}}$. Similarly to the proof of Lemma 8.5, we construct functions $f, g \in L^2(U)$ via

$$\begin{aligned} f^{(1)} &:= \chi^{(1)}\eta^{(1)}v^{(1)} - \eta^{(2)}\widehat{\chi^{(2)}v^{(2)}}, \qquad & f^{(2)} &:= \chi^{(2)}\eta^{(2)}v^{(2)} - \eta^{(1)}\widehat{\chi^{(1)}v^{(1)}}, \\ g^{(1)} &:= \chi^{(1)}v^{(1)} + \widehat{\chi^{(2)}v^{(2)}}, & g^{(2)} &:= \widehat{\chi^{(1)}v^{(1)}} + \chi^{(2)}v^{(2)}. \end{aligned}$$

Take $w \in H^2(U) \cap H^1_0(U)$. We first note that the function ψ, given by

$$\psi^{(1)}|_{U_1} := \chi^{(1)}(w^{(1)} - \widehat{w^{(2)}}), \quad \psi^{(2)}|_{U_2} := \chi^{(2)}(w^{(2)} - \widehat{w^{(1)}}),$$

$$\psi|_{(Q_1\cup Q_2)\setminus U} := 0,$$

belongs to $\mathscr{W}$, see (8.2). Simple algebraic manipulations then lead to the identities

$$\begin{aligned}
\langle f,\Delta w\rangle_{L^2(U)} &= \langle \chi^{(1)}\eta^{(1)}v^{(1)} - \eta^{(2)}\widehat{\chi^{(2)}v^{(2)}}, \Delta w\rangle_{L^2(U_1)} \\
&\quad + \langle \eta^{(2)}\chi^{(2)}v^{(2)} - \eta^{(1)}\widehat{\chi^{(1)}v^{(1)}}, \Delta w\rangle_{L^2(U_2)} \\
&= \langle \chi^{(1)}v^{(1)}, \eta^{(1)}\Delta(w^{(1)} - \widehat{w^{(2)}})\rangle_{L^2(U_1)} + \langle \chi^{(2)}v^{(2)}, \eta^{(2)}\Delta(w^{(2)} - \widehat{w^{(1)}})\rangle_{L^2(U_2)} \\
&= \langle v^{(1)}, \eta^{(1)}\Delta(\chi^{(1)}(w^{(1)} - \widehat{w^{(2)}}))\rangle_{L^2(U_1)} + \langle v^{(2)}, \eta^{(2)}\Delta(\chi^{(2)}(w^{(2)} - \widehat{w^{(1)}}))\rangle_{L^2(U_2)} \\
&\quad - \langle v^{(1)}, \eta^{(1)}(\Delta\chi^{(1)})(w^{(1)} - \widehat{w^{(2)}})\rangle_{L^2(U_1)} - \langle v^{(2)}, \eta^{(2)}(\Delta\chi^{(2)})(w^{(2)} - \widehat{w^{(1)}})\rangle_{L^2(U_2)} \\
&\quad - 2\langle v^{(1)}, \eta^{(1)}(\nabla\chi^{(1)}) \cdot \nabla(w^{(1)} - \widehat{w^{(2)}})\rangle_{L^2(U_1)} \\
&\quad - 2\langle v^{(2)}, \eta^{(2)}(\nabla\chi^{(2)}) \cdot \nabla(w^{(2)} - \widehat{w^{(1)}})\rangle_{L^2(U_2)}.
\end{aligned}$$

By construction of χ, the support of $\nabla\chi$ is a compact subset of the union $\mathscr{T} \cup U_1 \cup U_2$. Combining the fact that $v^{(i)}$ is contained in $H^2(\mathscr{T} \cap Q_i)$ and in $H^2_{\mathrm{loc}}(Q_i)$ with the smoothness of χ, we conclude that the product $v\nabla\chi$ is an element of $PH^2(\mathscr{T}\cup U_1\cup U_2)^3$. As v is by definition of $\mathscr{N}$ in (8.8) orthogonal to the image $\eta\Delta(\mathscr{W})$, an integration by parts establishes the formulas

$$\begin{aligned}
\langle f,\Delta w\rangle_{L^2(U)} &= -\langle \eta^{(1)}(\Delta\chi^{(1)})v^{(1)}, w^{(1)} - \widehat{w^{(2)}}\rangle_{L^2(U_1)} \\
&\quad - \langle \eta^{(2)}(\Delta\chi^{(2)})v^{(2)}, w^{(2)} - \widehat{w^{(1)}}\rangle_{L^2(U_2)} + 2\langle \operatorname{div}(\eta^{(1)}v^{(1)}\nabla\chi^{(1)}), w^{(1)} - \widehat{w^{(2)}}\rangle_{L^2(U_1)} \\
&\quad + 2\langle \operatorname{div}(\eta^{(2)}v^{(2)}\nabla\chi^{(2)}), w^{(2)} - \widehat{w^{(1)}}\rangle_{L^2(U_2)} \\
&= \langle 2\operatorname{div}(\eta^{(1)}v^{(1)}\nabla\chi^{(1)}) - 2\ (\operatorname{div}(\eta^{(2)}v^{(2)}\nabla\chi^{(2)}))\,\widehat{}\, - \eta^{(1)}(\Delta\chi^{(1)})v^{(1)} \\
&\qquad + (\eta^{(2)}(\Delta\chi^{2})v^{(2)})\,\widehat{}\,, w^{(1)}\rangle_{L^2(U_1)} \\
&\quad + \langle 2\operatorname{div}(\eta^{(2)}v^{(2)}\nabla\chi^{(2)}) - 2(\operatorname{div}(\eta^{(1)}v^{(1)}\nabla\chi^{(1)}))\,\widehat{}\, - \eta^{(2)}(\Delta\chi^{(2)})v^{(2)} \\
&\qquad + (\eta^{(1)}(\Delta\chi^{(1)})v^{(1)})\,\widehat{}\,, w^{(2)}\rangle_{L^2(U_2)} \\
&=: \langle \Phi_1, w\rangle_{L^2(U)}.
\end{aligned}$$

The above reasoning implies that Φ_1 belongs to $L^2(U)$. Employing Proposition 1.1 and Theorem 1.3 in Chapter 5 of [Tayl11], there is a unique function $\tilde{f} \in H^2(U) \cap H^1_0(U)$ with $\Delta\tilde{f} = \Phi_1$. As a result, $f - \tilde{f}$ is orthogonal to the image of the Laplacian Δ on $H^2(U) \cap H^1_0(U)$, being $L^2(U)$. We conclude $f = \tilde{f} \in H^2(U) \cap H^1_0(U)$.

Similar arguments are now applied to g. Consider first the function Ψ on Q, given by

$$\Psi|_{U_1} = \tfrac{\chi^{(1)}}{\eta^{(1)}}(w^{(1)} + \widehat{w^{(2)}}), \quad \Psi|_{U_2} = \tfrac{\chi^{(2)}}{\eta^{(2)}}(\widehat{w^{(1)}} + w^{(2)}),$$
$$\Psi_{Q_1\cup Q_2\setminus U} = 0.$$

Using the definition of the cut-off function χ, the mapping Ψ is an element of $\mathscr{W}$, see (8.2). Analogous calculations as for f lead to the relations

$$\begin{aligned}
&\langle g, \Delta w\rangle_{L^2(U)} = \langle \chi^{(1)}v^{(1)} + \widehat{\chi^{(2)}v^{(2)}}, \Delta w\rangle_{L^2(U_1)} + \langle \widehat{\chi^{(1)}v^{(1)}} + \chi^{(2)}v^{(2)}, \Delta w\rangle_{L^2(U_2)} \\
&= \langle \chi^{(1)}v^{(1)}, \eta^{(1)}\Delta \frac{1}{\eta^{(1)}}(w^{(1)} + \widehat{w^{(2)}})\rangle_{L^2(U_1)} + \langle \chi^{(2)}v^{(2)}, \eta^{(2)}\Delta \frac{1}{\eta^{(2)}}(\widehat{w^{(1)}} + w^{(2)})\rangle_{L^2(U_2)} \\
&= \langle v^{(1)}, \eta^{(1)}\Delta\Big(\frac{\chi^{(1)}}{\eta^{(1)}}(w^{(1)} + \widehat{w^{(2)}})\Big)\rangle_{L^2(U_1)} + \langle v^{(2)}, \eta^{(2)}\Delta\Big(\frac{\chi^{(2)}}{\eta^{(2)}}(\widehat{w^{(1)}} + w^{(2)})\Big)\rangle_{L^2(U_2)} \\
&\quad - \langle v^{(1)}, (\Delta\chi^{(1)})(w^{(1)} + \widehat{w^{(2)}})\rangle_{L^2(U_1)} - \langle v^{(2)}, (\Delta\chi^{(2)})(\widehat{w^{(1)}} + w^{(2)})\rangle_{L^2(U_2)} \qquad (8.23) \\
&\quad - 2\langle v^{(1)}, (\nabla\chi^{(1)}) \cdot \nabla(w^{(1)} + \widehat{w^{(2)}})\rangle_{L^2(U_1)} - 2\langle v^{(2)}, (\nabla\chi^{(2)}) \cdot \nabla(\widehat{w^{(1)}} + w^{(2)})\rangle_{L^2(U_2)}.
\end{aligned}$$

Note that the sum of the first two terms on the right-hand side of (8.23) vanish as v is an element of $\mathscr{N}$. Using again the properties of χ in an integration by parts, we obtain the relations

$$\begin{aligned}
&\langle g, \Delta w\rangle_{L^2(U)} = -\langle (\Delta\chi^{(1)})v^{(1)}, w^{(1)} + \widehat{w^{(2)}}\rangle_{L^2(U_1)} - \langle (\Delta\chi^{(2)})v^{(2)}, \widehat{w^{(1)}} + w^{(2)}\rangle_{L^2(U_2)} \\
&+ 2\langle \operatorname{div}(v^{(1)}\nabla\chi^{(1)}), w^{(1)} + \widehat{w^{(2)}}\rangle_{L^2(U_1)} + 2\langle \operatorname{div}(v^{(2)}\nabla\chi^{(2)}), \widehat{w^{(1)}} + w^{(2)}\rangle_{L^2(U_2)} \\
&= \langle 2\operatorname{div}(v^{(1)}\nabla\chi^{(1)}) + 2(\operatorname{div}(v^{(2)}\nabla\chi^{(2)}))\,\widehat{}\, - (\Delta\chi^{(1)})v^{(1)} - \Big((\Delta\chi^{(2)})v^{(2)}\Big)\widehat{}\,, w^{(1)}\rangle_{L^2(U_1)} \\
&+ \langle 2\operatorname{div}(v^{(2)}\nabla\chi^{(2)}) + 2(\operatorname{div}(v^{(1)}\nabla\chi^{(1)}))\,\widehat{}\, - (\Delta\chi^{(2)})v^{(2)} - \Big((\Delta\chi^{(1)})v^{(1)}\Big)\widehat{}\,, w^{(2)}\rangle_{L^2(U_2)} \\
&=: \langle \Phi_2, w\rangle_{L^2(U)},
\end{aligned}$$

with $\Phi_2 \in L^2(U)$. As above, we conclude that g belongs to $H^2(U) \cap H^1_0(U)$. Since the matrix $\begin{pmatrix}\eta^{(1)} & -\eta^{(2)} \\ 1 & 1\end{pmatrix}$ is an isomorphism on $H^2(U_1)$, we infer that χv is an element of $PH^2(U)$.

1b) Let now $\check{\chi} : [a_1^-, 0] \to [0,1]$ be a smooth cut-off function with $\check{\chi} = 1$ on $[a_1^-, -\frac{\delta}{12}]$, and $\operatorname{supp}\check{\chi} \subseteq [a_1^-, -\frac{\delta}{16}]$. Part 1a) then shows that $\check{\chi}'(x_1)v^{(1)}$ belongs to $H^2(Q_1)$. Let further $u \in H^2(Q_1)$ with $u = 0$ on $\Gamma_1^- \cup \mathscr{F}_{\text{int}}$, and $\partial_\nu u = 0$ on $\Gamma_2^{(1)} \cup \Gamma_3^{(1)}$. Clearly, $\frac{1}{\eta}\check{\chi}(x_1)u$ is an element of $\mathscr{W}$ after trivial extension to Q, see (8.2). Integrating by parts, we thus obtain the identities

$$\begin{aligned}
\int_{Q_1} \check{\chi}(x_1)v(x)\Delta u(x)\,\mathrm{d}x &= \int_{Q_1} v\Delta(\check{\chi}u)\,\mathrm{d}x - \int_{Q_1} \Big(v(\Delta\check{\chi})u + 2v(\nabla\check{\chi}) \cdot (\nabla u)\Big)\,\mathrm{d}x \\
&= \int_Q \eta v\Delta\Big(\frac{1}{\eta}\check{\chi}u\Big)\,\mathrm{d}x - \int_{Q_1} (v(\Delta\check{\chi}) - \partial_1(v\check{\chi}'))u\,\mathrm{d}x \\
&= -\int_{Q_1} (v(\Delta\check{\chi}) - \partial_1(v\check{\chi}'))u\,\mathrm{d}x.
\end{aligned}$$

Setting $\Phi_3 := \check{\chi}v + v\Delta\check{\chi} - \partial_1(v\check{\chi}') \in L^2(Q_1)$, we have derived the identity

$$\langle \check{\chi}v, u - \Delta u\rangle_{L^2(Q_1)} = \langle \Phi_3, u\rangle_{L^2(Q_1)}.$$

By Lemma 3.6 in [HoJS15], the operator

$$I - \Delta : \{u \in H^2(Q_1) \cap H^1_{\Gamma_1^- \cup \mathscr{F}_{\mathrm{int}}}(Q_1) \mid \partial_\nu u = 0 \text{ on } \Gamma_2^{(1)} \cup \Gamma_3^{(1)}\} \to L^2(Q_1)$$

is bijective, so that the same reasoning as in part 1a) implies that $\check{\chi} v$ is an element of $H^2(Q_1)$. Adapting the construction to Q_2, and combining with the results of part 1a), we then deduce that v belongs to $PH^2(Q)$.

2) It now suffices to show that v is a strong solution of the transmission problem

$$\begin{aligned} \Delta v^{(i)} &= 0 && \text{on } Q_i,\ i \in \{1,2\}, \\ v &= 0 && \text{on } \Gamma_1, \\ \partial_\nu v &= 0 && \text{on } \Gamma_2 \cup \Gamma_3, \\ [\![\eta v]\!]_{\mathscr{F}_{\mathrm{int}}} &= [\![\partial_1 v]\!]_{\mathscr{F}_{\mathrm{int}}} = 0. \end{aligned} \tag{8.24}$$

Since this problem has a unique weak solution, we can then conclude $v = 0$.

To that end, we use the fact that v is contained in $PH^2(Q)$, and that v satisfies the identity $0 = (v, \eta \Delta w)_{L^2(Q)}$ for $w \in \mathscr{W}$ by definition of $\mathscr{N}$ in (8.8). Choosing a smooth test function w that has compact support in one of the subcuboids Q_1 and Q_2, we infer that $\Delta v^{(i)} = 0$ on Q_i for $i \in \{1,2\}$.

Since the boundary conditions in (8.24) can be treated in the same way as the transmission conditions, we focus on the latter. Let $\psi \in C_c^\infty(\mathscr{F}_{\mathrm{int}})$, and $\tilde{\chi} : [a_1^-, a_1^+] \to [0,1]$ be a smooth cut-off function with $\tilde{\chi} = 1$ on $[a_1^-/4, a_1^+/4]$, and $\operatorname{supp} \tilde{\chi} \subseteq [3a_1^-/4, 3a_1^+/4]$. We then consider the function

$$w_1(x_1,x_2,x_3) := \begin{cases} \frac{1}{\eta^{(1)}} \tilde{\chi}(x_1)\psi(x_2,x_3) & \text{for } (x_1,x_2,x_3) \in Q_1, \\ \frac{1}{\eta^{(2)}} \tilde{\chi}(x_1)\psi(x_2,x_3) & \text{for } (x_1,x_2,x_3) \in Q_2. \end{cases}$$

By construction, w_1 is an element of $\mathscr{W}$, see (8.2). Integrating by parts, we hence conclude

$$\begin{aligned} 0 = (v, \eta \Delta w_1)_{L^2(Q)} &= -\sum_{i=1}^{2} \int_{Q_i} \eta^{(i)} \nabla v^{(i)} \cdot \nabla w_1^{(i)} \,\mathrm{d}x - \int_{\mathscr{F}_{\mathrm{int}}} [\![v]\!]_{\mathscr{F}_{\mathrm{int}}} (\partial_1 \tilde{\chi}) \psi \,\mathrm{d}\varsigma \\ &= \sum_{i=1}^{2} \int_{Q_i} \eta^{(i)} (\Delta v^{(i)}) w_1^{(i)} \,\mathrm{d}x + \int_{\mathscr{F}_{\mathrm{int}}} [\![\partial_1 v]\!]_{\mathscr{F}_{\mathrm{int}}} \tilde{\chi} \psi \,\mathrm{d}\varsigma \\ &= \int_{\mathscr{F}_{\mathrm{int}}} [\![\partial_1 v]\!]_{\mathscr{F}_{\mathrm{int}}} \psi \,\mathrm{d}\varsigma. \end{aligned}$$

As the space $C_c^\infty(\mathscr{F}_{\mathrm{int}})$ is dense in $L^2(\mathscr{F}_{\mathrm{int}})$, we conclude that v satisfies the first order transmission conditions in (8.24).

To check the zero order transmission conditions, we employ the function

$$w_2(x_1,x_2,x_3) := x_1 \tilde{\chi}(x_1) \psi(x_2,x_3), \qquad (x_1,x_2,x_3) \in Q.$$

This mapping is also an element of $\mathscr{W}$, and as above we infer the relations

$$\begin{aligned} 0 = (v, \eta\Delta w_2)_{L^2(Q)} &= -\sum_{i=1}^{2}\int_{Q_i}\eta^{(i)}\nabla v^{(i)}\cdot\nabla w_2^{(i)}\,\mathrm{d}x - \int_{\mathscr{F}_{\text{int}}}[\![\eta v]\!]_{\mathscr{F}_{\text{int}}}\psi\,\mathrm{d}\varsigma \\ &= \sum_{i=1}^{2}\int_{Q_i}\eta^{(i)}(\Delta v^{(i)})w_2^{(i)}\,\mathrm{d}x - \int_{\mathscr{F}_{\text{int}}}\Big([\![\eta v]\!]_{\mathscr{F}_{\text{int}}}\psi - [\![\partial_1 v]\!]_{\mathscr{F}_{\text{int}}}\eta x_1\chi\psi\Big)\,\mathrm{d}\varsigma \\ &= -\int_{\mathscr{F}_{\text{int}}}[\![\eta v]\!]_{\mathscr{F}_{\text{int}}}\psi\,\mathrm{d}\varsigma. \end{aligned}$$

Using again the density of $C_c^\infty(\mathscr{F}_{\text{int}})$ in $L^2(\mathscr{F}_{\text{int}})$, we conclude that $[\![\eta v]\!]_{\mathscr{F}_{\text{int}}}$ is zero. Altogether, v is the strong solution of (8.24). □

The proofs of Propositions 8.1 and 8.2 are now mere applications of Proposition 8.12.

Proof of Proposition 8.1. We first note that the left hand side of (8.3) defines an inner product on the space V_1, and that V_1 is complete with respect to the induced norm. As the right hand side of (8.3) is a bounded linear form on V_1, the Lax-Milgram Lemma consequently yields that there is a unique function $u \in V_1$, satisfying (8.3) for all $\varphi \in V_1$.

By Proposition 8.12, the Poisson problem

$$\begin{aligned} \eta^{(i)}\Delta v^{(i)} &= -f^{(i)} && \text{on } Q_i,\ i \in \{1,2\}, \\ \partial_\nu v &= 0 && \text{on } \partial Q \setminus \Gamma^*, \\ v &= 0 && \text{on } \Gamma^*, \\ [\![\eta v]\!]_{\mathscr{F}_{\text{int}}} &= 0 = [\![\partial_\nu v]\!]_{\mathscr{F}_{\text{int}}}, \end{aligned}$$

has a unique solution $v \in \mathscr{W}$. In particular, v satisfies the identity

$$\sum_{i=1}^{2}\int_{Q_i}\eta^{(i)}(\nabla v^{(i)})\cdot(\nabla\varphi^{(i)})\,\mathrm{d}x = \int_Q f\varphi\,\mathrm{d}x, \qquad \varphi \in V_1,$$

which means that u and v coincide. The asserted estimate is now a direct consequence of Lemma 8.4. □

Proof of Proposition 8.2. An appropriate modification of the proof for Proposition 8.1 shows the asserted statement. □

8.2. Elliptic transmission problems with prescribed discontinuities

One main step in the proof of the desired piecewise H^2-regularity of solutions to the Maxwell system (7.1) is the embedding of the space X_2 into $PH^2(Q)^6$, see

Chapter 9. To achieve the latter goal, we still need to draw two conclusions from Propositions 8.1 and 8.2.

The first lemma in this section deals with an elliptic transmission problem for functions which are continuous but have a discontinuous normal derivative across the interface $\mathscr{F}_{\text{int}}$. It is employed to analyze the second and third component of the vectors in X_2. To demonstrate the statement, we employ ideas and techniques from the proof of Lemma 3.1 in [EiSc17]. The main idea of our proof is to distribute the jump of the normal derivative in a symmetric way onto the subcuboids Q_1 and Q_2. By means of interpolation theory, we can then separately extend the normal derivatives to both cuboids Q_1 and Q_2. Altogether, we arrive at the desired solution of the transmission problem.

For the statement, recall the faces $\Gamma_l^{\pm}$ and $\Gamma_l^{\pm,(i)}$ of Q and Q_i from Section 7.1 for $i \in \{1,2\}$, and $l \in \{2,3\}$. The l-th component of the exterior unit normal vector for the cuboid Q_i is moreover denoted by $(\nu^{(i)})_l$.

Lemma 8.13. *Let $j, l \in \{2,3\}$, $l \neq j$, $\Gamma^* := \Gamma_j$ or $\Gamma^* := \Gamma_1 \cup \Gamma_j$. Let additionally $\tilde{f} \in L^2(Q)$ and $g \in H^{1/2}_{\Gamma_j}(\mathscr{F}_{\text{int}})$. Let $h \in L^2(\Gamma_l^+ \cup \Gamma_l^-)$ be a function with $h^{(i)} \in H_0^{1/2}(\Gamma_l^{+,(i)} \cup \Gamma_l^{-,(i)})$, $i \in \{1,2\}$. If $\Gamma^* \cap \Gamma_1 = \emptyset$, we set $h = 0$. Then there is a unique function $u \in H^1_{\Gamma^*}(Q)$ with $\Delta u^{(i)} \in L^2(Q_i)$, solving*

$$\int_Q (\nabla u) \cdot (\nabla \varphi)\, \mathrm{d}x = \int_Q \tilde{f} \varphi \, \mathrm{d}x - \int_{\mathscr{F}_{\text{int}}} g\varphi \, \mathrm{d}\varsigma + \sum_{i=1}^{2} \int_{\Gamma_l^{\pm,(i)}} h^{(i)} \varphi^{(i)} (\nu^{(i)})_l \, \mathrm{d}\varsigma \tag{8.25}$$

for all functions $\varphi \in H^1_{\Gamma^*}(Q)$. Moreover, u belongs even to $PH^2(Q)$ with $\partial_l u^{(i)} = h^{(i)}$ on $\Gamma_l^{(i)}$. Finally, the estimate

$$\|u\|_{PH^2(Q)} \leq C \left(\sum_{i=1}^{2} \|\Delta u^{(i)}\|_{L^2(Q_i)} + \|g\|_{H^{1/2}_{\Gamma_j}(\mathscr{F}_{\text{int}})} + \sum_{i=1}^{2} \|h^{(i)}\|_{H_0^{1/2}(\Gamma_l^{+,(i)} \cup \Gamma_l^{-,(i)})} \right)$$

is valid with a constant $C > 0$ being independent of u.

Proof. 1) Let $j = 3$ and $l = 2$. This corresponds to the two settings $\Gamma^* = \Gamma_3$ or $\Gamma^* = \Gamma_1 \cup \Gamma_3$. The remaining cases are treated in the same way, employing the symmetry structure of the domain. We first observe that the Lax-Milgram Lemma yields a unique function $u \in H^1_{\Gamma_1 \cup \Gamma_3}(Q)$ solving (8.25). In order to deduce more about its regularity, we write it as the sum of more regular functions.

Let $R_1 := (a_1^-, 0) \times (a_3^-, a_3^+)$ and $R_2 := (a_2^-, a_2^+) \times (a_3^-, a_3^+)$. On $L^2(R_1)$, we consider the Laplacian Δ_{R_1} with homogeneous Dirichlet boundary conditions. On $L^2(R_2)$, the Laplacian Δ_{R_2} with mixed boundary conditions is studied, meaning

$$\begin{aligned} \mathcal{D}(\Delta_{R_1}) &:= H^2(R_1) \cap H_0^1(R_1), \\ \mathcal{D}(\Delta_{R_2}) &:= \{v \in H^2(R_2) \mid v(\cdot, a_3^-) = v(\cdot, a_3^+) = 0,\ \partial_2 v(a_2^-, \cdot) = \partial_2 v(a_2^+, \cdot) = 0\}. \end{aligned}$$

The operator Δ_{R_1} is also considered with an analogous domain on the faces $\{x_2\} \times R_1$, and Δ_{R_2} is analyzed on $\{x_1\} \times R_2$ for $x_1 \in (a_1^-, a_1^+)$ and $x_2 \in (a_2^-, a_2^+)$. These operators have the same qualitative properties as on R_1 and R_2, respectively.

Let $k \in \{1, 2\}$. Using the closed symmetric bilinear forms

$$\begin{aligned} a_1(\tilde{u}, v) &:= \int_{R_1} (\partial_1 \tilde{u})(\partial_1 v) + (\partial_3 \tilde{u})(\partial_3 v) \,\mathrm{d}x, \\ a_2(\tilde{u}, v) &:= \int_{R_2} (\partial_2 \tilde{u})(\partial_2 v) + (\partial_3 \tilde{u})(\partial_3 v) \,\mathrm{d}x, \end{aligned} \tag{8.26}$$

defined on the spaces

$$\begin{aligned} &\mathcal{D}(a_1) := H_0^1(R_1), \\ &\mathcal{D}(a_2) := H_{\Gamma_3}^1(R_2) = \{\tilde{u} \in H^1(R_2) \mid \tilde{u}(\cdot, a_3^-) = \tilde{u}(\cdot, a_3^+) = 0\}, \end{aligned}$$

one can show that the operator $-\Delta_{R_k}$ is self-adjoint on $L^2(R_k)$, $k \in \{1, 2\}$, thanks to Theorem VI.2.6 in [Kato95]. Moreover, both operators $-\Delta_{R_k}$ are positive definite, what is a consequence of the Poincaré inequality, see Theorem 1.1.1 in [Neča12] in the case $k = 1$, and Theorem 13.6.9 in [TuWe09] for $k = 2$. In particular, one can define positive definite and self-adjoint fractional powers $(-\Delta_{R_k})^\gamma$ for $\gamma > 0$ by means of functional calculus.

The fractional powers $(-\Delta_{R_k})^\gamma$ generate analytic semigroups $(\mathrm{e}^{t(-\Delta_{R_k})^\gamma})_{t \geq 0}$ on $L^2(R_k)$. Theorem VI.2.23 in [Kato95] moreover yields the identity $\mathcal{D}((-\Delta_{R_k})^{1/2}) = \mathcal{D}(a_k)$, so that we obtain the relations

$$\begin{aligned} H_0^{1/2}(R_1) &= \Big(L^2(R_1), \mathcal{D}((-\Delta_{R_1})^{1/2})\Big)_{1/2,2}, \\ H_{\Gamma_3}^{1/2}(\mathscr{F}_{\mathrm{int}}) &= \Big(L^2(\mathscr{F}_{\mathrm{int}}), \mathcal{D}((-\Delta_{R_2})^{1/2})\Big)_{1/2,2}, \end{aligned}$$

see (7.5). We abbreviate these spaces in the following by $V_1 := H_0^{1/2}(R_1)$ and $V_2 := H_{\Gamma_3}^{1/2}(\mathscr{F}_{\mathrm{int}})$.

Let $T > 0$. Proposition 6.2 from [Luna18] implies the estimate

$$\int_0^T \left\| (-\Delta_{R_k})^{1/2} \mathrm{e}^{t(-\Delta_{R_k})^{1/2}} \psi \right\|_{L^2(R_k)}^2 \mathrm{d}t \leq C \left\| \psi \right\|_{V_k}^2 \tag{8.27}$$

for all $\psi \in V_k$, and $k \in \{1, 2\}$, with a uniform constant $C = C(T) > 0$.

2) In order to represent u, we choose a smooth cut-off function $\chi_2 : [a_2^-, a_2^+] \to [0, 1]$, which is equal to 1 on $[a_2^-, \frac{1}{2}a_2^- + \frac{1}{2}a_2^+]$, and which is supported within $[a_2^-, \frac{3}{4}a_2^+ + \frac{1}{4}a_2^-]$. We further note that the maps $h_1^{(i)} := h^{(i)}|_{\Gamma_2^{-,(i)}}$ and $h_2^{(i)} := h^{(i)}|_{\Gamma_2^{+,(i)}}$ belong to V_1 by assumption. By means of these functions, we define the mapping

$$\check{u}_1(x_1, x_2, x_3) := \chi_2(x_2) \left((-\Delta_{R_1})^{-1/2} \mathrm{e}^{(x_2 - a_2^-)(-\Delta_{R_1})^{1/2}} h_1^{(1)} \right) (x_1, x_3)$$

$$
\begin{aligned}
&+ (\chi_2(x_2) - 1) \left((-\Delta_{R_1})^{-1/2} \mathrm{e}^{(a_2^+ - x_2)(-\Delta_{R_1})^{1/2}} h_2^{(1)} \right) (x_1, x_3) \\
&=: \check{u}_{1,1}(x_1, x_2, x_3) + \check{u}_{1,2}(x_1, x_2, x_3)
\end{aligned}
$$

on Q_1. We thus have the boundary conditions

$$
\begin{aligned}
\check{u}_1(\cdot, x_2, a_3^-) &= \check{u}_1(\cdot, x_2, a_3^+) = 0 && \text{on } (a_1^-, 0), \\
\check{u}_1(a_1^-, x_2, \cdot) &= \check{u}_1(0, x_2, \cdot) = 0 && \text{on } (a_3^-, a_3^+).
\end{aligned}
$$

Since the semigroup $(\mathrm{e}^{t(-\Delta_{R_1})^{1/2}})_{t \geq 0}$ is analytic, the function $\check{u}_1(\cdot, x_2, \cdot)$ furthermore belongs to the space $H^2(R_1)$ for all $x_2 \in (a_2^-, a_2^+)$.

Let $(x_1, x_2, x_3) \in Q_1$. We calculate

$$
\begin{aligned}
\partial_2 \check{u}_{1,1}(x_1, x_2, x_3) = {}& \chi_2'(x_2) \left((-\Delta_{R_1})^{-1/2} \mathrm{e}^{(x_2 - a_2^-)(-\Delta_{R_1})^{1/2}} h_1^{(1)} \right) (x_1, x_3) \\
&+ \chi_2(x_2) \left(\mathrm{e}^{(x_2 - a_2^-)(-\Delta_{R_1})^{1/2}} h_1^{(1)} \right) (x_1, x_3), \qquad (8.28) \\
\partial_2^2 \check{u}_{1,1}(x_1, x_2, x_3) = {}& \chi_2''(x_2) \left((-\Delta_{R_1})^{-1/2} \mathrm{e}^{(x_2 - a_2^-)(-\Delta_{R_1})^{1/2}} h_1^{(1)} \right) (x_1, x_3) \\
&+ 2\chi_2'(x_2) \left(\mathrm{e}^{(x_2 - a_2^-)(-\Delta_{R_1})^{1/2}} h_1^{(1)} \right) (x_1, x_3) \\
&+ \chi_2(x_2) \left((-\Delta_{R_1})^{1/2} \mathrm{e}^{(x_2 - a_2^-)(-\Delta_{R_1})^{1/2}} h_1^{(1)} \right) (x_1, x_3). \qquad (8.29)
\end{aligned}
$$

Similar formulas hold for $\check{u}_{1,2}$. Again, the analyticity of $(\mathrm{e}^{t(-\Delta_{R_1})^{1/2}})_{t \geq 0}$ implies that $\partial_2 \check{u}(\cdot, x_2, \cdot)$ belongs to $H^1(R_1)$ for any $x_2 \in (a_2^-, a_2^+)$.

We next employ certain norm equivalences. First, the norms $\|\cdot\|_{H^1(R_1)}$ and $\|\cdot\|_{\mathcal{D}(-\Delta_{R_1})^{1/2}}$ are equivalent on $\mathcal{D}(-\Delta_{R_1})^{1/2}$, since the operator $-\Delta_{R_1}$ is associated to the bilinear form $a_1(\cdot, \cdot)$ from (8.26). Second, the norms $\|\cdot\|_{H^2(R_1)}$ and $\|\cdot\|_{\mathcal{D}(\Delta_{R_1})}$ are equivalent on $\mathcal{D}(\Delta_{R_1})$. (The domain $\mathcal{D}(\Delta_{R_1}) = H^2(R_1) \cap H_0^1(R_1)$ is complete with respect to both norms, and the identity mapping is bounded from $(\mathcal{D}(\Delta_{R_1}), \|\cdot\|_{H_2(R_1)})$ into $(\mathcal{D}(\Delta_{R_1}), \|\cdot\|_{\mathcal{D}(\Delta_{R_1})})$. The equivalence is thus a consequence of the open mapping theorem.) From (8.27)–(8.29), we can hence deduce the inequalities

$$
\begin{aligned}
\|\partial_2 \check{u}_{1,1}\|_{L^2(Q_1)}^2 &\leq C \left\| h_1^{(1)} \right\|_{L^2(R_1)}^2, \\
\|\partial_j \check{u}_{1,1}\|_{L^2(Q_1)}^2 &\leq C \int_{a_2^-}^{a_2^+} \|\check{u}_{1,1}(\cdot, x_2, \cdot)\|_{\mathcal{D}(-\Delta_{R_1})^{1/2}}^2 \, \mathrm{d}x_2 \leq C \left\| h_1^{(1)} \right\|_{L^2(R_1)}^2, \\
\left\|\partial_2^2 \check{u}_{1,1}\right\|_{L^2(Q_1)}^2 &\leq C \left\| h_1^{(1)} \right\|_{H_0^{1/2}(R_1)}^2, \\
\|\partial_j \partial_2 \check{u}_{1,1}\|_{L^2(Q_1)}^2 &\leq C \int_{a_2^-}^{a_2^+} \|\partial_2 \check{u}_{1,1}(\cdot, x_2, \cdot)\|_{\mathcal{D}(-\Delta_{R_1})^{1/2}}^2 \, \mathrm{d}x_2 \leq C \left\| h_1^{(1)} \right\|_{H_0^{1/2}(R_1)}^2, \\
\sum_{j,k \in \{1,3\}} \|\partial_j \partial_k \check{u}_{1,1}\|_{L^2(Q_1)}^2 &\leq \int_{a_2^-}^{a_2^+} \|\check{u}_{1,1}(\cdot, x_2, \cdot)\|_{H^2(R_1)}^2 \, \mathrm{d}x_2
\end{aligned}
$$

$$\begin{aligned}
&\leq C \int_{a_2^-}^{a_2^+} \|\check{u}_{1,1}(\cdot, x_2, \cdot)\|^2_{\mathcal{D}(\Delta_{R_1})} \, \mathrm{d}x_2 \\
&\leq C \left\| h_1^{(1)} \right\|^2_{H_0^{1/2}(R_1)}, \qquad (8.30)
\end{aligned}$$

for $j \in \{1, 3\}$, with a constant $C > 0$ being independent of $h_1^{(1)}$, and thus independent of $\check{u}_{1,1}$. The other summand $\check{u}_{1,2}$ is treated in the same way. As a result, $\check{u}_1$ is contained in $H^2(Q_1)$, and it satisfies the estimate

$$\|\check{u}_1\|_{H^2(Q_1)} \leq C_1 \left\| h^{(1)} \right\|_{H_0^{1/2}(\Gamma_2^{(1)})}$$

with a uniform constant $C_1 > 0$.

3) Identity (8.28) also implies the relations

$$\begin{aligned}
\partial_2 \check{u}_1(\cdot, x_2, \cdot)(\nu_{\Gamma_2^{-,(1)}})_2|_{x_2 = a_2^-} &= -h_1^{(1)}, \\
\partial_2 \check{u}_1(\cdot, x_2, \cdot)(\nu_{\Gamma_2^{+,(1)}})_2|_{x_2 = a_2^+} &= h_2^{(1)},
\end{aligned}$$

on R_1.

Repeating all arguments on the rectangle $(0, a_1^+) \times (a_3^-, a_3^+)$, we obtain also a function $\check{u}_2$ on Q_2 that belongs to $H^2(Q_2)$, and that satisfies the following properties. It vanishes on the faces $\Gamma_1^{\pm,(2)}$ and $\Gamma_3^{\pm,(2)}$, is bounded in norm by $\|\check{u}_2\|_{H^2(Q_2)} \leq C_2 \left\| h^{(2)} \right\|_{H_0^{1/2}(\Gamma_2^{(2)})}$ for some uniform constant $C_2 > 0$, and has the Neumann traces

$$\begin{aligned}
\partial_2 \check{u}_2(\cdot, x_2, \cdot)(\nu_{\Gamma_2^{-,(2)}})_2|_{x_2 = a_2^-} &= -h_1^{(2)}, \\
\partial_2 \check{u}_2(\cdot, x_2, \cdot)(\nu_{\Gamma_2^{+,(2)}})_2|_{x_2 = a_2^+} &= h_2^{(2)},
\end{aligned}$$

on $(0, a_1^+) \times (a_3^-, a_3^+)$. Altogether, the function

$$\check{u} := \begin{cases} \check{u}_1 & \text{on } Q_1, \\ \check{u}_2 & \text{on } Q_2, \end{cases}$$

belongs to $PH^2(Q) \cap H^1_{\Gamma_1 \cup \Gamma_3}(Q)$, vanishes on $\mathscr{F}_{\mathrm{int}}$, and satisfies the estimate

$$\|\check{u}\|_{PH^2(Q)} \leq (C_1 + C_2) \sum_{i=1}^{2} \left\| h^{(i)} \right\|_{H_0^{1/2}(\Gamma_2^{(i)})}. \qquad (8.31)$$

In particular, the jump $v := [\![\nabla \check{u} \cdot \nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}}$ is contained in $H^{1/2}_{\Gamma_3}(\mathscr{F}_{\mathrm{int}})$.

4) Although the function $\check{u}$ extends the Neumann trace of u on Γ_2 in the desired way, its behavior on the interface needs to be improved. For that purpose, we

choose a second smooth cut-off function $\tilde{\chi}_1 : [a_1^-, a_1^+] \to [0,1]$ that is equal to 1 on $[\frac{3}{4}a_1^-, \frac{3}{4}a_1^+]$, and that is supported within $[\frac{5}{6}a_1^-, \frac{5}{6}a_1^+]$. We then define the mapping

$$\tilde{u}(x_1,x_2,x_3) := \begin{cases} \frac{1}{2}\tilde{\chi}_1(x_1)\left((-\Delta_{R_2})^{-1/2}\mathrm{e}^{-x_1(-\Delta_{R_2})^{1/2}}(g-v)\right)(x_2,x_3) & \text{on } Q_1, \\ \frac{1}{2}\tilde{\chi}_1(x_1)\left((-\Delta_{R_2})^{-1/2}\mathrm{e}^{x_1(-\Delta_{R_2})^{1/2}}(g-v)\right)(x_2,x_3) & \text{on } Q_2. \end{cases}$$

In view of the analyticity of $(\mathrm{e}^{t(-\Delta_{R_2})^{1/2}})_{t\geq 0}$, the function $\tilde{u}(x_1,\cdot,\cdot)$ belongs to the space $H^2(R_2)$, and it satisfies the boundary conditions

$$\begin{aligned} \tilde{u}(x_1,\cdot,a_3^-) = \tilde{u}(x_1,\cdot,a_3^+) &= 0 && \text{on } (a_2^-,a_2^+), \\ \partial_2\tilde{u}(x_1,a_2^-,\cdot) = \partial_2\tilde{u}(x_1,a_2^+,\cdot) &= 0 && \text{on } (a_3^-,a_3^+), \end{aligned}$$

for all $x_1 \in (a_1^-,a_1^+)\setminus\{0\}$. Note that $\tilde{u}$ is continuous in x_1 at $\mathscr{F}_{\mathrm{int}}$, and that $\tilde{u}=0$ on Γ_1 by construction. We hence conclude that $\tilde{u}$ is an element of $H^1_{\Gamma_1\cup\Gamma_3}(Q)$. Employing that g and v belong to $H^{1/2}_{\Gamma_3}(\mathscr{F}_{\mathrm{int}})$, similar arguments as in part 2) show that $\tilde{u}$ is also an element of $PH^2(Q)$, and that it can be bounded according to

$$\|\tilde{u}\|_{PH^2(Q)} \leq C_3 \|g-v\|_{H^{1/2}_{\Gamma_3}(\mathscr{F}_{\mathrm{int}})}, \tag{8.32}$$

with some uniform constant $C_3 > 0$. The norm equivalences of $\|\cdot\|_{H^1(R_2)}$ with $\|\cdot\|_{\mathcal{D}(-\Delta_{R_2})^{1/2}}$, and of $\|\cdot\|_{H^2(R_2)}$ with $\|\cdot\|_{\mathcal{D}(\Delta_{R_2})}$, respectively, are analogously verified to part 2).

A straightforward calculation yields the Neumann traces

$$\begin{aligned} \partial_1\tilde{u}^{(1)}(x_1,\cdot,\cdot)(\nu_{\Gamma_1^+,(1)})_1|_{x_1=0} &= -\frac{1}{2}(g-v), \\ \partial_1\tilde{u}^{(2)}(x_1,\cdot,\cdot)(\nu_{\Gamma_1^-,(2)})_1|_{x_1=0} &= -\frac{1}{2}(g-v), \\ \partial_1\tilde{u}^{(1)}(x_1,\cdot,\cdot)|_{x_1=a_1^-} = 0 &= \partial_1\tilde{u}^{(2)}(x_1,\cdot,\cdot)|_{x_1=a_1^+}, \end{aligned}$$

on R_2. The first line denotes here the interface trace with respect to Q_1, the second line with respect to Q_2.

5) Define now two functions ζ and ψ in $L^2(Q)$ via $\psi^{(i)} := \tilde{f}^{(i)} - \zeta^{(i)}$, and $\zeta^{(i)} := -\Delta(\check{u}^{(i)} + \tilde{u}^{(i)})$ for $i \in \{1,2\}$. Proposition 8.1 applies with $\eta = 1$, and provides us with a unique function $\hat{u} \in PH^2(Q) \cap H^1_{\Gamma_1\cup\Gamma_3}(Q)$, satisfying (8.3) for $f = \psi$. Consequently $u := \tilde{u}+\check{u}+\hat{u} \in PH^2(Q)\cap H^1_{\Gamma_1\cup\Gamma_3}(Q)$ is the unique solution of (8.25). Note that the boundary integral $\int_{\Gamma_1}(\nabla\check{u}\cdot\nu)\varphi\,\mathrm{d}\varsigma$ vanishes for $\varphi \in H^1_{\Gamma_1\cup\Gamma_3}(Q)$ due to the assumptions on $h^{(i)}$. It hence remains to show the asserted estimate.

Lemma 7.4 and the definition of v yield the relations

$$\|v\|_{H^{1/2}_{\Gamma_3}(\mathscr{F}_{\mathrm{int}})} = \|[\![\nabla\check{u}\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}}\|_{H^{1/2}_{\Gamma_3}(\mathscr{F}_{\mathrm{int}})} \leq C\,\|\nabla\check{u}\|_{PH^1(Q)} \leq C\,\|\check{u}\|_{PH^2(Q)}$$

with a uniform constant $C > 0$. In view of Proposition 8.1 for $\hat{u}$, (8.31) for $\check{u}$, and (8.32) for $\tilde{u}$, we arrive at the inequalities

$$\begin{aligned}
\|u\|_{PH^2(Q)} &\leq \|\tilde{u}\|_{PH^2(Q)} + \|\check{u}\|_{PH^2(Q)} + \|\hat{u}\|_{PH^2(Q)} \\
&\leq C_4(\|g - v\|_{H^{1/2}_{\Gamma_3}(\mathscr{F}_{\text{int}})} + \sum_{i=1}^{2} \left\|\Delta\hat{u}^{(i)}\right\|_{L^2(Q_i)} + \|\check{u}\|_{PH^2(Q)}) \\
&\leq C_5\Big(\|g\|_{H^{1/2}_{\Gamma_3}(\mathscr{F}_{\text{int}})} + \sum_{i=1}^{2} \left\|h^{(i)}\right\|_{H_0^{1/2}(\Gamma_2^{(i)})} + \sum_{i=1}^{2} \left\|\Delta(\tilde{u}^{(i)} + \check{u}^{(i)} + \hat{u}^{(i)})\right\|_{L^2(Q_i)} \Big),
\end{aligned}$$

where C_4 and C_5 are tow positive constants that depend only on Q. This shows the asserted estimate since $u = \tilde{u} + \check{u} + \hat{u}$. □

To analyze the first component of vectors in the space X_2, we still need another conclusion from Proposition 8.1.

Lemma 8.14. *Let η be a positive function that is piecewise constant on the cuboids Q_1 and Q_2, and let $\Gamma^* := \Gamma_2 \cup \Gamma_3$. Moreover, let $\tilde{f} \in L^2(Q)$, $g \in H_0^{1/2}(\mathscr{F}_{\text{int}})$, and $h \in H_0^{1/2}(\Gamma_1)$. There is a unique function $u \in V := \{v \in PH^1_{\Gamma^*}(Q) \mid [\![\eta v]\!]_{\mathscr{F}_{\text{int}}} = 0\}$, satisfying the formula*

$$\sum_{i=1}^{2} \int_{Q_i} \eta^{(i)}(\nabla u^{(i)}) \cdot (\nabla\varphi^{(i)}) \,\mathrm{d}x = \int_Q \tilde{f}\varphi \,\mathrm{d}x + \int_{\Gamma_1} h\nu_1\varphi \,\mathrm{d}\varsigma - \int_{\mathscr{F}_{\text{int}}} g\varphi \,\mathrm{d}\varsigma \tag{8.33}$$

for all functions $\varphi \in V$. The mapping u belongs to $PH^2(Q)$ with $\frac{\partial u^{(i)}}{\partial \nu} = h$ on $\Gamma_1 \cap \partial Q_i$ for $i \in \{1, 2\}$. Finally, the estimate

$$\|u\|_{PH^2(Q)} \leq C(\sum_{i=1}^{2} \left\|\Delta u^{(i)}\right\|_{L^2(Q_i)} + \|h\|_{H_0^{1/2}(\Gamma_1)} + \|g\|_{H_0^{1/2}(\mathscr{F}_{\text{int}})})$$

is true with some uniform constant $C > 0$, depending only on Q and η.

Proof. The proof mainly modifies arguments from the proof of Lemma 8.13. In particular, concepts from the proof for Lemma 3.1 in [EiSc17] are again employed. Due to the strong similarities with Lemma 8.13, we only focus on the differences.

As before, the Lax-Milgram Lemma shows the existence of a unique function $u \in V$, satisfying (8.33). We again employ the rectangle $R := (a_2^-, a_2^+) \times (a_3^-, a_3^+)$. This time, we consider the Dirichlet Laplacian Δ_R on R, i.e., $\mathcal{D}(\Delta_R) := H_0^1(R) \cap H^2(R)$. Following the arguments for Δ_{R_1} in part 1) of the proof of Lemma 8.13, the operator $-\Delta_R$ is self-adjoint, and positive definite on $L^2(R)$. The same statement is true for the well-defined fractional powers $(-\Delta_R)^\gamma$ for $\gamma > 0$. Consequently,

$(-\Delta_R)^{1/2}$ generates an analytic semigroup $(\mathrm{e}^{(-\Delta_R)^{1/2}})_{t\geq 0}$ on $L^2(R)$. Recall that $\mathcal{D}((-\Delta_R)^{1/2}) = H_0^1(R)$, and

$$H_0^{1/2}(R) = (L^2(R), \mathcal{D}(-\Delta_R)^{1/2})_{1/2,2}.$$

Let $\chi : [a_1^-, a_1^+] \to [0,1]$ be a smooth cut-off function that is equal to 1 on $[\frac{3}{4}a_1^-, \frac{3}{4}a_1^+]$, and that is supported within $[\frac{5}{6}a_1^-, \frac{5}{6}a_1^+]$. Denote $h_1 := h|_{\Gamma_1^{-,(1)}}$ and $h_2 := h|_{\Gamma_1^{+,(2)}}$. In order to extend h and g, we consider the function $\tilde{u}$, which is defined via its restrictions to Q_1 and Q_2 as

$$\begin{aligned}
\tilde{u}^{(1)}(x_1,x_2,x_3) &:= \tfrac{1-\chi(x_1)}{\eta^{(1)}}\left((-\Delta_R)^{-1/2}\mathrm{e}^{(x_1-a_1^-)(-\Delta_R)^{1/2}}h_1\right)(x_2,x_3)\\
&\quad + \tfrac{\chi(x_1)}{2\eta^{(1)}}\left((-\Delta_R)^{-1/2}\mathrm{e}^{-x_1(-\Delta_R)^{1/2}}g\right)(x_2,x_3),\\
\tilde{u}^{(2)}(x_1,x_2,x_3) &:= -\tfrac{1-\chi(x_1)}{\eta^{(2)}}\left((-\Delta_R)^{-1/2}\mathrm{e}^{(a_1^+-x_1)(-\Delta_R)^{1/2}}h_2\right)(x_2,x_3)\\
&\quad + \tfrac{\chi(x_1)}{2\eta^{(2)}}\left((-\Delta_R)^{-1/2}\mathrm{e}^{x_1(-\Delta_R)^{1/2}}g\right)(x_2,x_3).
\end{aligned}$$

By construction, $\eta\tilde{u}$ is continuous in x_1 across $\mathscr{F}_{\mathrm{int}}$. We next use that the functions h_1, h_2, and g belong to $H_0^{1/2}(R)$. As a result, a modification of the arguments in part 2) of the proof of Lemma 8.13 shows that $\tilde{u}$ is contained in $PH^2(Q) \cap PH^1_{\Gamma^*}(Q)$, and that its norm satisfies the estimate

$$\|\tilde{u}\|_{PH^2(Q)} \leq C\left(\|h\|_{H_0^{1/2}(\Gamma_1)} + \|g\|_{H_0^{1/2}(\mathscr{F}_{\mathrm{int}})}\right) \tag{8.34}$$

with a uniform positive constant C, depending only on Q and η. The Neumann traces of $\tilde{u}$ at the faces Γ_1 and $\mathscr{F}_{\mathrm{int}}$ are given as

$$\begin{aligned}
\partial_1\tilde{u}^{(1)}(x_1,\cdot,\cdot)(\nu_{\Gamma_1^{-,(1)}})_1|_{x_1=a_1^-} &= -\tfrac{1}{\eta^{(1)}}h_1,\\
\partial_1\tilde{u}^{(2)}(x_1,\cdot,\cdot)(\nu_{\Gamma_1^{+,(2)}})_1|_{x_1=a_1^+} &= \tfrac{1}{\eta^{(2)}}h_2,\\
\partial_1\tilde{u}^{(1)}(x_1,\cdot,\cdot)(\nu_{\Gamma_1^{+,(1)}})_1|_{x_1=0} &= -\tfrac{1}{2\eta^{(1)}}g,\\
\partial_1\tilde{u}^{(2)}(x_1,\cdot,\cdot)(\nu_{\Gamma_1^{-,(2)}})_1|_{x_1=0} &= -\tfrac{1}{2\eta^{(2)}}g,
\end{aligned}$$

on R.

Finally, we introduce two functions ζ and ψ in $L^2(Q)$ by $\zeta^{(i)} := -\eta^{(i)}\Delta\tilde{u}^{(i)}$ and $\psi^{(i)} := \tilde{f}^{(i)} - \zeta^{(i)}$. Proposition 8.1 then yields a function $\hat{u} \in PH^2(Q) \cap V$ that solves (8.3) for $f = \psi$. As a result, $u := \tilde{u} + \hat{u} \in PH^2(Q) \cap V$ is the unique solution of (8.33). Proposition 8.1 and (8.34) also provide the desired estimate

$$\|u\|_{PH^2(Q)} \leq \tilde{C}\left(\|h\|_{H_0^{1/2}(\Gamma_1)} + \|g\|_{H_0^{1/2}(\mathscr{F}_{\mathrm{int}})} + \sum_{i=1}^{2}\left\|\Delta\hat{u}^{(i)}\right\|_{L^2(Q_i)}\right)$$

$$\leq \tilde{C}\Big(2\,\|h\|_{H_0^{1/2}(\Gamma_1)} + 2\,\|g\|_{H_0^{1/2}(\mathscr{F}_{\mathrm{int}})} + \sum_{i=1}^{2}\Big\|\Delta(\tilde{u}^{(i)} + \hat{u}^{(i)})\Big\|_{L^2(Q_i)}\Big),$$

with a constant $\tilde{C} > 0$ that depends only on η and Q. $\square$

9. Regularity analysis for the Maxwell equations

The main goal of this chapter is to show that the Maxwell equations (7.1) have solutions with piecewise H^1- and H^2-regularity, provided that the initial data are chosen appropriately. These results play a crucial role in our error analysis for the Peaceman-Rachford ADI scheme in Chapter 10. To reach this goal, we proceed in the following steps.

In Sections 9.1 and 9.2, we demonstrate that the spaces X_1 and X_2 from (7.17) and (7.19) embed into the spaces $PH^1(Q)^6$ and $PH^2(Q)^6$, respectively. We then show that the spaces X_1 and X_2 are state spaces for the Maxwell equations, see Section 9.3. In other words, the Maxwell equations are wellposed on X_1 and X_2.

9.1. First order regularity for the space X_1

In this section, we extend the well-known regularity statements for the spaces $H_N(\mathrm{curl}, \mathrm{div}, Q)$ and $H_T(\mathrm{curl}, \mathrm{div}, Q)$, see Section 2.2, to the case of piecewise constant coefficients. Therefore, we transfer arguments from the continuous setting in [GiRa86] to the discontinuous one.

Throughout, we make the assumption (7.2) for the parameters ε, μ, and σ. The corresponding spaces for our problem are

$$\begin{aligned}
H_{N,00}(\mathrm{curl}, \mathrm{div}\,\varepsilon, Q) &:= \{\mathbf{E} \in H_0(\mathrm{curl}, Q) \mid \mathrm{div}(\varepsilon\mathbf{E}) = 0\},\\
H_{N,0}(\mathrm{curl}, \mathrm{div}\,\varepsilon, Q) &:= \{\mathbf{E} \in H_0(\mathrm{curl}, Q) \mid \mathrm{div}(\varepsilon\mathbf{E}) \in L^2(Q)\},\\
H_{T,00}(\mathrm{curl}, \mathrm{div}\,\mu, Q) &:= \{\mathbf{H} \in H(\mathrm{curl}, Q) \mid \mathrm{div}(\mu\mathbf{H}) = 0,\ \mu\mathbf{H}\cdot\nu = 0 \text{ on } \partial Q\}.
\end{aligned}$$

While the first and last spaces are already complete with respect to the norm in $H(\mathrm{curl}, Q)$, the second one is complete with respect to the norm

$$\|\mathbf{E}\|_{H_{N,0}}^2 := \|\mathbf{E}\|_{L^2(Q)}^2 + \|\mathrm{curl}\,\mathbf{E}\|_{L^2(Q)}^2 + \|\mathrm{div}(\varepsilon\mathbf{E})\|_{L^2(Q)}^2, \quad \mathbf{E} \in H_{N,0}(\mathrm{curl}, \mathrm{div}\,\varepsilon, Q).$$

Our first goal are embeddings of the first and third space into the space of piecewise H^1-regular functions. In a next step, we then derive the desired embedding of X_1 into $PH^1(Q)^6$. These embeddings are on the one hand useful for our purposes, since they yield piecewise H^1-regularity for functions in the space X_1. On the other hand, these embeddings are valuable for other applications like certain Helmholtz decompositions, see Section 2.2.

As we could neither detect the statements nor the proofs in the literature, we here deduce the desired embeddings in a sequence of several lemmas and propositions. Our plan is to transfer some of the arguments from the Sections I.3.3–I.3.5 in [GiRa86] to our setting of a transmission problem.

We start with the embedding of $H_{N,00}(\operatorname{curl}, \operatorname{div}\varepsilon, Q)$ into $PH^1(Q)^3$, and we first prove the injectivity of the curl-operator on the former space. The statement generalizes Remark I.3.9 in [GiRa86].

Lemma 9.1. *Let ε satisfy (7.2), and let $\boldsymbol{E} \in H_{N,00}(\operatorname{curl}, \operatorname{div}\varepsilon, Q)$ with $\operatorname{curl}\boldsymbol{E} = 0$. Then $\boldsymbol{E} = 0$.*

Proof. Theorem I.3.4 in [GiRa86] implies the existence of a vector $\Phi \in H^1(Q)^3$ with $\mathbf{E} = \frac{1}{\varepsilon}\operatorname{curl}\Phi$, and $\operatorname{div}\Phi = 0$ on Q. Integrating by parts, we obtain the result

$$\int_Q \varepsilon\,|\mathbf{E}|^2\,\mathrm{d}x = \int_Q (\operatorname{curl}\Phi)\cdot\mathbf{E}\,\mathrm{d}x = \int_Q \Phi\cdot\operatorname{curl}\mathbf{E}\,\mathrm{d}x = 0. \qquad \square$$

In order to determine the image of the curl-operator, we define the space

$$H_\xi := \{\mathbf{E} \in L^2(Q)^3 \mid \operatorname{div}(\xi\mathbf{E}) = 0,\ \xi\mathbf{E}\cdot\nu = 0 \text{ on } \partial Q\}$$

for $\xi \in \{\varepsilon, \mu\}$. Note that H_ξ is a closed subspace of $L^2(Q)^3$. This claim can be verified by combining the closedness of the divergence operator on $H(\operatorname{div}, Q)$ with the boundedness of the normal trace operator from $H(\operatorname{div}, Q)$ into $H^{-1/2}(\partial Q)$, see Section 2.2.

The following statement characterizes the image of the curl-operator on the space $H_{N,00}(\operatorname{curl}, \operatorname{div}\varepsilon, Q)$. It corresponds to Theorem I.3.6 in [GiRa86], and it extends Lemma 6.3 in [BoHL99] to our setting of a cuboid. We use ideas from the proof of the latter statement. But as we cannot follow its arguments, we give more details.

Lemma 9.2. *Let ε satisfy (7.2). Each function $\boldsymbol{E} \in H_\varepsilon$ has the representation*

$$\boldsymbol{E} = \frac{1}{\varepsilon}\operatorname{curl}\Phi$$

for a unique function $\Phi \in H_{N,00}(\operatorname{curl}, \operatorname{div}\varepsilon, Q)$. Moreover, Φ belongs to the space $PH^1(Q)^3$.

Proof. 1) Lemma 9.1 already implies that there is at most one function Φ with the required properties. Consequently, it remains to show the existence of the desired vector Φ, as well as its regularity.

2) Employing Theorem I.3.6 from [GiRa86] to the cuboid Q, we obtain a vector $\tilde{\Phi} \in H_N(\mathrm{curl}, \mathrm{div}, Q) \hookrightarrow H^1(Q)^3$ that satisfies the identities $\frac{1}{\varepsilon} \mathrm{curl}\, \tilde{\Phi} = \mathbf{E}$, and $\mathrm{div}\, \tilde{\Phi} = 0$ on Q. Since the function ε is constant on Q_1 and Q_2, this implies the formula

$$\mathrm{div}(\varepsilon^{(i)} \tilde{\Phi}^{(i)}) = 0 \quad \text{on } Q_i$$

for $i \in \{1, 2\}$. We cannot, however, expect $\tilde{\Phi}$ to satisfy the additional transmission condition $[\![\varepsilon \tilde{\Phi} \cdot \nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}} = 0$. We thus seek for a function $\psi \in PH^2(Q) \cap H_0^1(Q)$, solving the elliptic transmission problem

$$\begin{aligned} \Delta \psi^{(i)} &= 0 && \text{on } Q_i \text{ for } i \in \{1, 2\}, \\ \psi &= 0 && \text{on } \partial Q, \\ [\![\psi]\!]_{\mathscr{F}_{\mathrm{int}}} &= 0, \\ [\![\varepsilon \partial_1 \psi]\!]_{\mathscr{F}_{\mathrm{int}}} &= [\![\varepsilon \tilde{\Phi}_1]\!]_{\mathscr{F}_{\mathrm{int}}}. \end{aligned} \tag{9.1}$$

The weak formulation of (9.1) is given by the formula

$$\int_Q \varepsilon (\nabla \psi) \cdot (\nabla \varphi) \, \mathrm{d}x = - \int_{\mathscr{F}_{\mathrm{int}}} [\![\varepsilon \tilde{\Phi}_1]\!]_{\mathscr{F}_{\mathrm{int}}} \varphi \, \mathrm{d}\varsigma, \qquad \varphi \in H_0^1(Q). \tag{9.2}$$

Since the left hand side of (9.2) defines a bounded and coercive symmetric bilinear form on $H_0^1(Q)$, and the right hand side is a bounded linear form, the Lax-Milgram Lemma provides a unique solution $\psi \in H_0^1(Q)$ of (9.2).

3) We show next that the weak solution ψ belongs indeed to $PH^2(Q)$, and that it satisfies (9.1). To that end, we proceed similar to the proof of Lemma 8.13. Let $R := (a_2^-, a_2^+) \times (a_3^-, a_3^+)$, and consider in $L^2(R)$ the Dirichlet Laplacian Δ_D on its domain

$$\mathcal{D}(\Delta_D) := H^2(R) \cap H_0^1(R).$$

Then, $-\Delta_D$ is self-adjoint and positive definite on $L^2(R)$. Consequently, we can define fractional powers $(-\Delta_D)^\gamma$ for $\gamma > 0$ by means of functional calculus for selfadjoint operators. The powers are self-adjoint and positive definite as well. As a result, the fractional powers $(-\Delta_D)^\gamma$ generate analytic semigroups $(\mathrm{e}^{t(-\Delta_D)^\gamma})_{t \geq 0}$ on $L^2(R)$. Recall that $\mathcal{D}((-\Delta_D)^{1/2}) = H_0^1(R)$.

Next, we define an auxiliary function whose Neumann derivative has the required jump being specified in (9.1). Let $\chi : [a_1^-, a_1^+] \to [0, 1]$ be a smooth cut-off function that equals 1 on $[\frac{3}{4} a_1^-, \frac{3}{4} a_1^+]$, and that is supported within $[\frac{5}{6} a_1^-, \frac{5}{6} a_1^+]$. Abbreviate furthermore $g := [\![\varepsilon \tilde{\Phi}_1]\!]_{\mathscr{F}_{\mathrm{int}}}$. We define the mapping

$$\tilde{\psi}(x_1, x_2, x_3) := \begin{cases} \frac{1}{2} \chi(x_1) \left((-\Delta_D)^{-1/2} \mathrm{e}^{-\frac{x_1}{\varepsilon^{(1)}} (-\Delta_D)^{1/2}} g \right) (x_2, x_3) & \text{for } x \in Q_1, \\ \frac{1}{2} \chi(x_1) \left((-\Delta_D)^{-1/2} \mathrm{e}^{\frac{x_1}{\varepsilon^{(2)}} (-\Delta_D)^{1/2}} g \right) (x_2, x_3) & \text{for } x \in Q_2. \end{cases}$$

Taking into account that $\tilde{\Phi}_1$ is an element of $PH^1_{\Gamma_2\cup\Gamma_3}(Q)$, Lemma 7.3 implies that g belongs to $H_0^{1/2}(R) := (L^2(R), H_0^1(R))_{1/2,2}$. Thus, the arguments from the proof of Lemma 8.13 and the smoothness of χ imply that the function $\tilde{\psi}$ belongs to $PH^2(Q)$, see also the proof for Lemma 3.1 in [EiSc17]. It moreover satisfies the boundary and transmission conditions specified in (9.1).

4) Because the function $\tilde{\psi}$ will in general not satisfy the Poisson equation $\Delta\tilde{\psi}^{(i)} = 0$ on Q_i, we also need to consider another elliptic transmission problem, namely

$$\begin{aligned} \Delta\hat{\psi}^{(i)} &= -\Delta\tilde{\psi}^{(i)} && \text{on } Q_i \text{ for } i \in \{1,2\}, \\ \hat{\psi} &= 0 && \text{on } \partial Q, \\ [\![\hat{\psi}]\!]_{\mathscr{F}_{\text{int}}} = [\![\varepsilon\partial_1\hat{\psi}]\!]_{\mathscr{F}_{\text{int}}} &= 0. \end{aligned} \tag{9.3}$$

Proposition 8.1 provides a function $u \in PH^2(Q) \cap PH_0^1(Q)$, satisfying (8.3) with $\eta = 1/\varepsilon$ and $f^{(i)} = \Delta\tilde{\psi}^{(i)}$ for $i \in \{1,2\}$. Choose then $\hat{\psi} := u/\varepsilon$. Using the interface conditions $[\![u/\varepsilon]\!]_{\mathscr{F}_{\text{int}}} = [\![\partial_1 u]\!]_{\mathscr{F}_{\text{int}}} = 0$, we conclude that $\hat{\psi}$ is the desired solution of (9.3).

Setting $\psi := \tilde{\psi} + \hat{\psi} \in PH^2(Q) \cap H_0^1(Q)$, we have thus constructed the unique solution of (9.1). Altogether, $\Phi := \tilde{\Phi} - \nabla\psi \in PH^1(Q)^3$ is the desired vector field. $\square$

The next proposition summarizes the results of the last two lemmas, and the proof follows the lines of the proofs for Lemma I.3.4 and Theorem I.3.7 in [GiRa86].

Proposition 9.3. *Let ε satisfy (7.2). The space $H_{N,00}(\operatorname{curl}, \operatorname{div}\varepsilon, Q)$ embeds into $PH^1(Q)^3$.*

Proof. Lemma 9.2 yields that the mapping $\frac{1}{\varepsilon}\operatorname{curl} : H_{N,00}(\operatorname{curl}, \operatorname{div}\varepsilon, Q) \to H_\varepsilon$ is bijective. Since it is also bounded and both mentioned spaces are complete, we infer by the open mapping principle that $\frac{1}{\varepsilon}\operatorname{curl}$ is an isomorphism between these spaces. Lemma 9.2 further shows the identities

$$\frac{1}{\varepsilon}\operatorname{curl}\left(H_{N,00}(\operatorname{curl}, \operatorname{div}\varepsilon, Q)\right) = H_\varepsilon = \frac{1}{\varepsilon}\operatorname{curl}\left(H_{N,00}(\operatorname{curl}, \operatorname{div}\varepsilon, Q) \cap PH^1(Q)^3\right).$$

Since $\frac{1}{\varepsilon}\operatorname{curl}$ is an isomorphism, this means that $H_{N,00}(\operatorname{curl}, \operatorname{div}\varepsilon, Q)$ is a subspace of $PH^1(Q)^3$ with a continuous embedding. $\square$

In order to deduce the embedding property of the space X_1 into $PH^1(Q)^6$, we transfer the statement of Lemma I.3.4 in [GiRa86] to our setting. Hereby, we employ ideas from its proof.

Lemma 9.4. *Let ε satisfy (7.2). The estimate*

$$\|\boldsymbol{E}\|_{L^2(Q)} \le C_{N0}(\|\operatorname{curl}\boldsymbol{E}\|_{L^2(Q)} + \|\operatorname{div}(\varepsilon\boldsymbol{E})\|_{L^2(Q)}) \tag{9.4}$$

is valid for all vectors $\boldsymbol{E} \in H_{N,0}(\operatorname{curl}, \operatorname{div}\varepsilon, Q)$ with a uniform constant $C_{N0} > 0$.

Proof. 1) We first deduce the asserted estimate for the space $H_{N,00}(\operatorname{curl}, \operatorname{div} \varepsilon, Q)$. Let $\mathbf{E} \in H_{N,00}(\operatorname{curl}, \operatorname{div} \varepsilon, Q)$. Recall that the mapping $\frac{1}{\varepsilon} \operatorname{curl}$ is an isomorphism from $H_{N,00}(\operatorname{curl}, \operatorname{div} \varepsilon, Q)$ to H_ε. Hence we infer the inequality

$$\|\mathbf{E}\|_{H(\operatorname{curl},Q)} \leq \left\|(\tfrac{1}{\varepsilon} \operatorname{curl})^{-1}\right\| \left\|\tfrac{1}{\varepsilon} \operatorname{curl} \mathbf{E}\right\|_{L^2(Q)},$$

yielding the asserted estimate for $\mathbf{E}$.

2) Let $\mathbf{E} \in H_{N,0}(\operatorname{curl}, \operatorname{div} \varepsilon, Q)$. The main tool is an appropriate decomposition of $\mathbf{E}$ into a field we can apply part 1) to, and a remainder. To that end, we consider the elliptic transmission problem

$$\begin{aligned} \Delta \phi^{(i)} &= \operatorname{div} \mathbf{E}^{(i)} && \text{on } Q_i \text{ for } i \in \{1,2\}, \\ \phi &= 0 && \text{on } \partial Q, \\ [\![\phi]\!]_{\mathscr{F}_{\text{int}}} &= [\![\varepsilon \partial_1 \phi]\!]_{\mathscr{F}_{\text{int}}} = 0. \end{aligned} \tag{9.5}$$

As in part 4) of the proof for Lemma 9.2, this problem has a unique solution $\phi \in PH^2(Q) \cap H_0^1(Q)$. Employing the homogeneous Dirichlet boundary conditions, we obtain $\nabla \phi \times \nu = 0$ on ∂Q, see Lemma 2.1 in [EiSc18]. The first order transmission condition further implies that $\nabla \phi$ is an element of $H_{N,0}(\operatorname{curl}, \operatorname{div} \varepsilon, Q)$. Consequently $\psi := \nabla \phi - \mathbf{E}$ belongs to $H_{N,00}(\operatorname{curl}, \operatorname{div} \varepsilon, Q)$, and it thus satisfies (9.4) without the divergence term, see part 1). We obtain in this way the relations

$$\begin{aligned} \|\mathbf{E}\|_{L^2(Q)} \leq \|\psi\|_{L^2(Q)} + \|\nabla \phi\|_{L^2(Q)} &\leq C_{00} \|\operatorname{curl} \psi\|_{L^2(Q)} + \|\nabla \phi\|_{L^2(Q)} \\ &= C_{00} \|\operatorname{curl} \mathbf{E}\|_{L^2(Q)} + \|\nabla \phi\|_{L^2(Q)}, \end{aligned}$$

where C_{00} is the uniform positive constant from the desired inequality in the space $H_{N,00}(\operatorname{curl}, \operatorname{div} \varepsilon, Q)$. In view of the weak formulation of system (9.5) and Poincaré's inequality, we infer

$$\begin{aligned} \|\varepsilon \nabla \phi\|_{L^2(Q)}^2 &= -\sum_{i=1}^{2} \int_{Q_i} \varepsilon^{(i)} \phi^{(i)} \operatorname{div} \mathbf{E}^{(i)} \, \mathrm{d}x \leq \|\phi\|_{L^2(Q)} \|\operatorname{div}(\varepsilon \mathbf{E})\|_{L^2(Q)} \\ &\leq C_P \|\nabla \phi\|_{L^2(Q)} \|\operatorname{div}(\varepsilon \mathbf{E})\|_{L^2(Q)}, \end{aligned}$$

employing the Poincaré constant $C_P > 0$. This yields the assertion. □

We continue by proving that the space $H_{T,00}(\operatorname{curl}, \operatorname{div} \mu, Q)$ embeds continuously into $PH^1(Q)^3$. This implies that the magnetic field component of a vector in X_1 is piecewise H^1-regular. For that purpose, we first deduce the injectivity of the curl-operator on $H_{T,00}(\operatorname{curl}, \operatorname{div} \mu, Q)$, compare Remark I.2.2 in [GiRa86].

Lemma 9.5. *Let μ satisfy (7.2). The operator* curl *is injective on the space $H_{T,00}(\operatorname{curl}, \operatorname{div} \mu, Q)$.*

Proof. Let $\mathbf{H} \in H_{T,00}(\operatorname{curl}, \operatorname{div}\mu, Q)$ with $\operatorname{curl}\mathbf{H} = 0$. Theorem I.2.9 in [GiRa86] then yields a function $q \in H^1(Q)$ with $\mathbf{H} = \nabla q$. Since $\mu\mathbf{H} \cdot \nu = 0$ on ∂Q, an integration by parts implies the identities

$$\|\sqrt{\mu}\mathbf{H}\|^2_{L^2(Q)} = (\mathbf{H}, \mu\mathbf{H})_{L^2(Q)} = (\nabla q, \mu\mathbf{H})_{L^2(Q)} = (q, \operatorname{div}(\mu\mathbf{H}))_{L^2(Q)} = 0.$$

As μ is positive, we infer that $\mathbf{H} = 0$. □

The next statement transfers Theorem I.3.5 from [GiRa86] to our current setting. It is proved in a similar way as Lemma 9.2. More precisely, we establish again the bijectivity of the curl-operator, being now defined on $H_{T,00}(\operatorname{curl}, \operatorname{div}\mu, Q)$.

Lemma 9.6. *Let μ satisfy (7.2), and let $\boldsymbol{H} \in H_\mu$. There is a unique function $\Phi \in H_{T,00}(\operatorname{curl}, \operatorname{div}\mu, Q)$ with*

$$\boldsymbol{H} = \frac{1}{\mu}\operatorname{curl}\Phi.$$

The mapping Φ belongs to $PH^1(Q)^3$.

Proof. 1) Lemma 9.5 shows that there exists at most one function Φ in the space $H_{T,00}(\operatorname{curl}, \operatorname{div}\mu, Q)$ with $\mathbf{H} = \frac{1}{\mu}\operatorname{curl}\Phi$. In the following, we deduce the existence of such a function.

2) Theorem I.3.5 in [GiRa86] shows that there is a vector $\tilde{\Phi} \in H_T(\operatorname{curl}, \operatorname{div}, Q)$, satisfying

$$\mathbf{H} = \frac{1}{\mu}\operatorname{curl}\tilde{\Phi}, \quad \operatorname{div}\tilde{\Phi} = 0 \quad \text{on } Q.$$

(Note that $\tilde{\Phi}$ belongs also to $H^1(Q)^3$, see Section 2.2.) It follows

$$\operatorname{div}(\mu^{(i)}\tilde{\Phi}^{(i)}) = 0 \quad \text{on } Q_i,$$

for $i \in \{1, 2\}$, because μ is piecewise constant. As above, the vector $\tilde{\Phi}$ does, however, in general not satisfy the transmission condition $[\![\mu\tilde{\Phi} \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = 0$.

Since $\tilde{\Phi} \in H^1(Q)^3$, the function $g := [\![\mu\tilde{\Phi} \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}$ is an element of $H^{1/2}(\mathscr{F}_{\text{int}})$. To extend g to the cuboid Q, we now consider the Neumann Laplacian Δ_N on the rectangle $R := [a_2^-, a_2^+] \times [a_3^-, a_3^+]$ with domain

$$\mathcal{D}(\Delta_N) := \{u \in H^2(R) \mid \partial_2 u(a_2^\pm, \cdot) = 0 \text{ on } (a_3^-, a_3^+),\ \partial_3 u(\cdot, a_3^\pm) = 0 \text{ on } (a_2^-, a_2^+)\}.$$

Theorems 3.2.1.3 and 4.3.1.4 in [Gris85] imply that $-\Delta_N$ is non-negative and self-adjoint on $L^2(R)$. In particular, its fractional powers $(I - \Delta_N)^\gamma$, $\gamma > 0$, are well defined, and positive definite. As the operator $I - \Delta_N$ is given by the symmetric form $\int_R (uv + \nabla u \cdot \nabla v)\,\mathrm{d}x$, we further conclude the identity $\mathcal{D}((I - \Delta_N)^{1/2}) = H^1(R)$.

Let $\chi : [a_1^-, a_1^+] \to [0,1]$ be a smooth cut-off function that is equal to 1 on $[a_1^-/2, a_1^+/2]$, and that is supported within $[3a_1^-/4, 3a_1^+/4]$. Define then the mapping

$$\tilde{\psi}(x_1,x_2,x_3) := \begin{cases} \frac{1}{2}\chi(x_1)\left((I-\Delta_N)^{-1/2}\mathrm{e}^{-\frac{x_1}{\mu^{(1)}}(I-\Delta_N)^{1/2}} g\right)(x_2,x_3) & \text{for } x \in Q_1, \\ \frac{1}{2}\chi(x_1)\left((I-\Delta_N)^{-1/2}\mathrm{e}^{\frac{x_1}{\mu^{(2)}}(I-\Delta_N)^{1/2}} g\right)(x_2,x_3) & \text{for } x \in Q_2. \end{cases}$$

For $\tilde{\psi}$ we employ appropriate modifications of the arguments from the proof for Lemma 8.13, which uses itself ideas from the proof of Lemma 3.1 in [EiSc17]. In this way, we can conclude that $\tilde{\psi}$ belongs to $PH^2(Q)$, and that it satisfies homogeneous Neumann boundary conditions on ∂Q. A straightforward calculation further yields the transmission conditions

$$[\![\tilde{\psi}]\!]_{\mathscr{F}_{\mathrm{int}}} = 0, \qquad [\![\mu\partial_1\tilde{\psi}]\!]_{\mathscr{F}_{\mathrm{int}}} = g. \tag{9.6}$$

It then remains to treat the elliptic transmission problem

$$\begin{aligned} \Delta\hat{\psi}^{(i)} &= -\Delta\tilde{\psi}^{(i)} && \text{on } Q_i \text{ for } i \in \{1,2\}, \\ \partial_\nu\hat{\psi} &= 0 && \text{on } \partial Q, \\ [\![\hat{\psi}]\!]_{\mathscr{F}_{\mathrm{int}}} &= [\![\mu\partial_1\hat{\psi}]\!]_{\mathscr{F}_{\mathrm{int}}} = 0, \end{aligned} \tag{9.7}$$

being equivalent to the system

$$\begin{aligned} \frac{1}{\mu^{(i)}}\Delta u^{(i)} &= -\Delta\tilde{\psi}^{(i)} && \text{on } Q_i \text{ for } i \in \{1,2\}, \\ \partial_\nu u &= 0 && \text{on } \partial Q, \\ [\![\tfrac{1}{\mu}u]\!]_{\mathscr{F}_{\mathrm{int}}} &= [\![\partial_1 u]\!]_{\mathscr{F}_{\mathrm{int}}} = 0, \end{aligned} \tag{9.8}$$

by choosing $\hat{\psi} := \frac{1}{\mu}u$. We analyze in the following (9.8).

We next denote the mean of a function $v \in L^2(Q)$ on Q by $[v]_Q$, and equip the space

$$\tilde{V} := \{w \in PH^1(Q) \mid [\tfrac{1}{\mu}w]_Q = 0,\ [\![\tfrac{1}{\mu}w]\!]_{\mathscr{F}_{\mathrm{int}}} = 0\}$$

with the weighted norm $\|\frac{1}{\mu}\cdot\|_{H^1(Q)}$. Note that $\tilde{V}$ is then complete. Combining the Lax-Milgram Lemma with the generalized Poincaré inequality, there is a unique function $u \in \tilde{V}$ with

$$\int_Q \tfrac{1}{\mu}(\nabla u)\cdot(\nabla w)\,\mathrm{d}x = \sum_{i=1}^{2}\int_{Q_i}(\Delta\tilde{\psi}^{(i)})w\,\mathrm{d}x, \qquad w \in \tilde{V}.$$

For the next argument, we note that the mapping $v - \mu[\frac{1}{\mu}v]_Q$ belongs to $\tilde{V}$ for every function $v \in V := \{\phi \in PH^1(Q) \mid [\![\frac{1}{\mu}\phi]\!]_{\mathscr{F}_{\mathrm{int}}} = 0\}$. We additionally employ the homogeneous Neumann boundary conditions of $\tilde{\psi}$ and the transmission conditions (9.6) in an integration by parts. The formula

$$\begin{aligned}
&\int_Q (uv + \tfrac{1}{\mu}(\nabla u)\cdot(\nabla v))\,\mathrm{d}x \qquad (9.9)\\
&\quad = \sum_{i=1}^{2} \int_{Q_i} (uv + \tfrac{1}{\mu^{(i)}}(\nabla u^{(i)})\cdot \nabla(v^{(i)} - \mu^{(i)}[\tfrac{1}{\mu}v]_Q))\,\mathrm{d}x\\
&\quad = \sum_{i=1}^{2} \int_{Q_i} (uv + (\Delta\tilde{\psi}^{(i)})(v^{(i)} - \mu^{(i)}[\tfrac{1}{\mu}v]_Q))\,\mathrm{d}x = \sum_{i=1}^{2} \int_{Q_i} (u + \Delta\tilde{\psi}^{(i)})v\,\mathrm{d}x
\end{aligned}$$

then follows for $v \in V$. Applying now also Proposition 8.2, we finally conclude that u belongs to $PH^2(Q)$, and that it satisfies the boundary and transmission conditions in (9.8). Choosing test functions in $C_c^\infty(Q_1) \cup C_c^\infty(Q_2)$ for v in (9.9), we additionally conclude that u is the strong solution of (9.8). By defining

$$\Phi := \tilde{\Phi} - \nabla(\tilde{\psi} + \hat{\psi}) \in PH^1(Q)^3 \cap H_\mu,$$

we have altogether constructed the desired function. □

The next result is a counterpart to Proposition 9.3. It summarizes the last two lemmas, and yields that the magnetic field component $\mathbf{H}$ is piecewise H^1-regular for all vectors $(\mathbf{E}, \mathbf{H})$ in X_1.

Proposition 9.7. *Let μ satisfy (7.2). The space $H_{T,00}(\mathrm{curl}, \mathrm{div}\,\mu, Q)$ embeds into $PH^1(Q)^3$.*

Proof. The proof basically follows the lines of the proof for Proposition 9.3 (and employs in particular the arguments from the proofs for Lemma I.3.4 and Theorem I.3.7 in [GiRa86]). Instead of Lemma 9.2, the just established Lemma 9.6 is used, however. Furthermore, the parameter ε is replaced by μ. □

We now deduce the desired piecewise H^1-regularity of functions in the space X_1. Recall in this respect the spaces

$$\begin{aligned}
\mathcal{D}(M) &:= H_0(\mathrm{curl}, Q) \times H(\mathrm{curl}, Q),\\
X_0 &:= \{(\mathbf{E}, \mathbf{H}) \in L^2(Q)^6 \mid \mathrm{div}(\varepsilon^{(i)}\mathbf{E}^{(i)}) \in L^2(Q_i),\ \mathrm{div}(\mu\mathbf{H}) = 0,\\
&\qquad \mu\mathbf{H}\cdot\nu = 0 \text{ on } \partial Q,\ [\![\varepsilon\mathbf{E}\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}} \in H_0^{1/2}(\mathscr{F}_{\mathrm{int}})\},\\
X_1 &:= \mathcal{D}(M) \cap X_0.
\end{aligned}$$

The associated norms are defined in (7.16) and (7.18). In view of Propositions 9.3 and 9.7, it remains to generalize our findings in this section to the case of non-vanishing electric charges.

Proposition 9.8. *Let ε and μ satisfy (7.2). The space X_1 embeds continuously into $PH^1(Q)^6$.*

Proof. 1) Let $(\mathbf{E},\mathbf{H}) \in X_1 = \mathcal{D}(M) \cap X_0$. We start to show the asserted regularity of the vector $(\mathbf{E},\mathbf{H})$, and need to deal only with the electric field $\mathbf{E}$. (The magnetic field component $\mathbf{H}$ is an element of $H_{T,00}(\operatorname{curl}\operatorname{div}\mu, Q)$. So Proposition 9.7 provides the desired regularity statement for $\mathbf{H}$.) Our goal is to reformulate our problem in such a way, that we can apply Proposition 9.3. Our arguments follow here essentially the proof of Lemma 9.2.

We search for a function $\psi \in PH^2(Q) \cap H_0^1(Q)$, that solves the elliptic transmission problem

$$\begin{aligned} \Delta\psi^{(i)} &= \operatorname{div}\mathbf{E}^{(i)} && \text{on } Q_i \text{ for } i \in \{1,2\}, \\ \psi &= 0 && \text{on } \partial Q, \\ [\![\psi]\!]_{\mathscr{F}_{\text{int}}} &= 0, \\ [\![\varepsilon\partial_1\psi]\!]_{\mathscr{F}_{\text{int}}} &= [\![\varepsilon\mathbf{E}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} \in H_0^{1/2}(\mathscr{F}_{\text{int}}). \end{aligned} \tag{9.10}$$

As in part 2) of the proof for Lemma 9.2, system (9.10) has a unique weak solution $\psi \in H_0^1(Q)$. Employing our arguments from part 3) of the same proof, we can construct a function $\tilde{\psi}$ belonging to $PH^2(Q) \cap H_0^1(Q)$ and satisfying the transmission conditions required in (9.10). (Here we use the fact $[\![\varepsilon\mathbf{E}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} \in H_0^{1/2}(\mathscr{F}_{\text{int}})$ for $(\mathbf{E},\mathbf{H}) \in X_0$.) The mapping can be estimated in norm via

$$\|\tilde{\psi}\|_{PH^2(Q)} \le C_1 \left\|[\![\varepsilon\mathbf{E}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}\right\|_{H_0^{1/2}(\mathscr{F}_{\text{int}})}, \tag{9.11}$$

see for instance the arguments in part 2) from the proof of Lemma 8.13. Similar to part 4) of the proof for Lemma 9.2, there is a unique function $\hat{\psi} \in PH^2(Q) \cap H_0^1(Q)$ that fulfills the system

$$\begin{aligned} \Delta\hat{\psi}^{(i)} &= -\Delta\tilde{\psi}^{(i)} + \operatorname{div}\mathbf{E}^{(i)} && \text{on } Q_i \text{ for } i \in \{1,2\}, \\ \hat{\psi} &= 0 && \text{on } \partial Q, \\ [\![\hat{\psi}]\!]_{\mathscr{F}_{\text{int}}} &= [\![\varepsilon\partial_1\hat{\psi}]\!]_{\mathscr{F}_{\text{int}}} = 0. \end{aligned} \tag{9.12}$$

Altogether, the function $\psi = \tilde{\psi} + \hat{\psi} \in PH^2(Q) \cap H_0^1(Q)$ solves (9.10) in strong form. Consequently, $\mathbf{E} - \nabla\psi$ is an element of $H_{N,00}(\operatorname{curl}, \operatorname{div}\varepsilon, Q) \subseteq PH^1(Q)^3$, meaning that $\mathbf{E}$ is contained in $PH^1(Q)^3$.

2) It remains to show the asserted embedding property. To that end, we first treat the electric field component $\mathbf{E}$. Applying Proposition 9.3, the relations

$$\|\mathbf{E}\|_{PH^1(Q)} \le \|\mathbf{E} - \nabla\psi\|_{PH^1(Q)} + \|\nabla\psi\|_{PH^1(Q)} \le C_{00}\,\|\operatorname{curl}\mathbf{E}\|_{L^2(Q)} + \|\nabla\psi\|_{PH^1(Q)}$$

$$\leq C_{00}\left\|\mu\right\|_\infty \left\|M\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\right\| + \left\|\tilde{\psi}\right\|_{PH^2(Q)} + \left\|\nabla\hat{\psi}\right\|_{PH^1(Q)} \tag{9.13}$$

follow, where the uniform constant $C_{00} > 0$ stems from the embedding in Proposition 9.3. The boundary and transmission conditions in (9.12) imply that the potential $\nabla\hat{\psi}$ is also an element of $H_{N,0}(\operatorname{curl}, \operatorname{div}\varepsilon, Q)$, so that we can apply Lemma 9.4 to the last term on the right hand side of (9.13). By means of the identity $\Delta\hat{\psi}^{(i)} = -\Delta\tilde{\psi}^{(i)} + \operatorname{div}\mathbf{E}^{(i)}$ and inequality (9.11), we then conclude the desired estimates

$$\begin{aligned}
\left\|\mathbf{E}\right\|_{PH^1(Q)} &\leq C_{00}\left\|\mu\right\|_\infty \left\|M\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\right\| + \left\|\tilde{\psi}\right\|_{PH^2(Q)} + C_{N0}\sum_{i=1}^{2}\left\|\varepsilon^{(i)}\Delta\hat{\psi}^{(i)}\right\|_{L^2(Q_i)} \\
&\leq C_{00}\left\|\mu\right\|_\infty \left\|M\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\right\| + (1 + C_{N0})(1 + \left\|\varepsilon\right\|_\infty)\left\|\tilde{\psi}\right\|_{PH^2(Q)} \\
&\quad + C_{N0}\sum_{i=1}^{2}\left\|\operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)})\right\|_{L^2(Q_i)} \\
&\leq C_{00}\left\|\mu\right\|_\infty \left\|M\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\right\| + (1 + C_{N0})(1 + \left\|\varepsilon\right\|_\infty)C_1\left\|[\![\varepsilon\mathbf{E}\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}}\right\|_{H_0^{1/2}(\mathscr{F}_{\mathrm{int}})} \\
&\quad + C_{N0}\sum_{i=1}^{2}\left\|\operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)})\right\|_{L^2(Q_i)} \\
&\leq C_3\left\|\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\right\|_{X_1}
\end{aligned}$$

with a uniform constant $C_3 = C_3(\varepsilon, Q) > 0$.

3) Concerning the magnetic field $\mathbf{H}$, we apply Proposition 9.7. Here we arrive at the estimates

$$\begin{aligned}
\left\|\mathbf{H}\right\|_{PH^1(Q)} &\leq C_4(\left\|\mathbf{H}\right\|_{L^2(Q)} + \left\|\operatorname{curl}\mathbf{H}\right\|_{L^2(Q)}) \\
&\leq C_4\Big(\tfrac{1}{\sqrt{\delta}}\left\|\sqrt{\mu}\mathbf{H}\right\|_{L^2(Q)} + \left\|\tfrac{\sigma}{\varepsilon}\right\|_\infty\left\|\varepsilon\right\|_\infty\left\|\sqrt{\varepsilon}\mathbf{E}\right\|_{L^2(Q)} \\
&\qquad + \left\|\varepsilon\right\|_\infty\left\|\sqrt{\varepsilon}(-\tfrac{\sigma}{\varepsilon}\mathbf{E} + \tfrac{1}{\varepsilon}\operatorname{curl}\mathbf{H})\right\|_{L^2(Q)}\Big) \\
&\leq C_4(1 + \tfrac{1}{\sqrt{\delta}} + \left\|\tfrac{\sigma}{\varepsilon}\right\|_\infty\left\|\varepsilon\right\|_\infty + \left\|\varepsilon\right\|_\infty)\left\|\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\right\|_{\mathcal{D}(M)} \\
&\leq C_4(1 + \tfrac{1}{\sqrt{\delta}} + (1 + \left\|\tfrac{\sigma}{\varepsilon}\right\|_\infty)\left\|\varepsilon\right\|_\infty)\left\|\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\right\|_{X_1},
\end{aligned}$$

where $C_4 > 0$ is the uniform constant from Proposition 9.7. □

The following remark makes an observation, which turns out to be crucial for the regularity analysis of the Maxwell system (7.1). Roughly speaking, it says that the definition of X_1 is independent of the actual coefficient function in the jump condition for the electric field.

Remark 9.9. Let $\tilde{\varepsilon}$ be a positive function on Q, that is piecewise constant on the cuboids Q_1 and Q_2. We consider the modification

$$\tilde{X}_0 := \{(\mathbf{E},\mathbf{H}) \in L^2(Q)^6 \mid \operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)}) \in L^2(Q_i),\ \operatorname{div}(\mu\mathbf{H}) = 0, \mu\mathbf{H}\cdot\nu = 0 \text{ on } \partial Q,\\ [\![\tilde{\varepsilon}\mathbf{E}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} \in H_0^{1/2}(\mathscr{F}_{\text{int}})\}$$

of X_0, as well as the space

$$\tilde{W} := \mathcal{D}(M) \cap \tilde{X}_0.$$

We claim the identity $\tilde{W} = \mathcal{D}(M) \cap X_0 = X_1$. To show this statement, let $(\mathbf{E},\mathbf{H}) \in \tilde{W}$. Since ε is piecewise constant on the subcuboids, we infer that $\operatorname{div}(\mathbf{E}^{(i)})$ is also contained in $L^2(Q_i)$ for $i \in \{1,2\}$. In particular, $\operatorname{div}(\tilde{\varepsilon}^{(i)}\mathbf{E}^{(i)})$ is an element of $L^2(Q_i)$. Replacing ε by $\tilde{\varepsilon}$ for Proposition 9.8, we infer that $\mathbf{E}$ is an element of $PH^1(Q)^3$. The boundary conditions for $\mathbf{E}_1$ now imply that $\mathbf{E}_1$ belongs to $PH^1_{\Gamma_2\cup\Gamma_3}(Q)$, and we infer that $[\![\varepsilon\mathbf{E}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}$ is contained in $H_0^{1/2}(\mathscr{F}_{\text{int}})$, see Lemma 7.3. Altogether, $(\mathbf{E},\mathbf{H})$ is an element of $X_1 = \mathcal{D}(M) \cap X_0$. A similar reasoning yields the reverse inclusion. ◊

An analogous statement is true for the space X_2, see Remark 9.18.

9.2. Piecewise H^2-regularity for the space X_2

Based on the results of the above sections, we are now in the position to deduce the desired piecewise H^2-regularity for functions in the space

$$X_2 = \{(\mathbf{E},\mathbf{H}) \in \mathcal{D}(M^2) \cap X_0 \mid \operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)}) \in H^1_{00}(Q_i) \text{ for } i \in \{1,2\},\\ [\![\varepsilon\mathbf{E}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} \in H_0^{3/2}(\mathscr{F}_{\text{int}})\},$$

see (7.19). By interpreting X_2 as a state space for the Maxwell system (7.1) in Section 9.3.

To demonstrate the embedding statement for the space X_2, we first conclude that each vector in X_2 is piecewise H^2-regular in the interior of Q_1 and Q_2.

Lemma 9.10. *Let ε, μ, and σ satisfy (7.2). Let also $(\boldsymbol{E},\boldsymbol{H}) \in X_2$, and $i \in \{1,2\}$. The functions $\Delta\boldsymbol{E}^{(i)}$ and $\Delta\boldsymbol{H}^{(i)}$ belong to $L^2(Q_i)^3$.*

Proof. Since the coefficients ε and μ are piecewise constant, the definition of X_2 implies that the function $\operatorname{div}\mathbf{E}^{(i)}$ belongs to $H^1_{00}(Q_i)$, and that the vector $\operatorname{curl}\operatorname{curl}\mathbf{E}^{(i)}$ is contained in $L^2(Q_i)^3$. We calculate

$$\operatorname{curl}\operatorname{curl}\mathbf{E}^{(i)} = -\Delta\mathbf{E}^{(i)} + \nabla\operatorname{div}\mathbf{E}^{(i)} \tag{9.14}$$

in $H^{-1}(Q_i)$. As a result, $\Delta\mathbf{E}^{(i)}$ belongs to $L^2(Q_i)^3$. For the magnetic field $\mathbf{H}$, one employs that $\operatorname{div}(\mu\mathbf{H}) = 0$, and that the functions $\mathbf{E}^{(i)}$ and $-\frac{\sigma^{(i)}}{2\varepsilon^{(i)}}\mathbf{E}^{(i)} + \frac{1}{\varepsilon^{(i)}}\operatorname{curl}\mathbf{H}^{(i)}$ belong to $H(\operatorname{curl}, Q_i)$. The latter statement implies that also $\operatorname{curl}\operatorname{curl}\mathbf{H}^{(i)} = -\Delta\mathbf{H}^{(i)}$ is an element of $L^2(Q_i)^3$. □

Remark 9.11. The interior regularity inside the cuboids Q_1 and Q_2 follows for vectors $(\mathbf{E}, \mathbf{H}) \in X_2$ from Lemma 9.10 by means of elliptic regularity results, see Theorem 1 in Section 6.3.1 in [Evan15] for instance. Our error analysis in Chapter 10, however, demands for H^2-regularity on each subcuboid up to the boundary. ♢

In the next four lemmas, we apply the general elliptic regularity results from Chapter 8. Hereby we proceed in the following way.

For both the electric $\mathbf{E}$ and the magnetic field $\mathbf{H}$ components of a vector $(\mathbf{E}, \mathbf{H}) \in X_2$, we start with the first components $\mathbf{E}_1$ and $\mathbf{H}_1$, and consider then the two other components. The functions $\mathbf{E}_1$ and $\mathbf{H}_1$ are more convenient to treat, since we can gain here regularity by means of the divergence constraints. We derive the result by means of ideas and techniques from Lemma 3.7 in [HoJS15], and Proposition 3.2 in [EiSc17].

Lemma 9.12. *Let ε, μ, and σ satisfy (7.2), and let $(\boldsymbol{E}, \boldsymbol{H}) \in X_2$. Then $\boldsymbol{E}_1$ belongs to $PH^2(Q)$, and the estimate*

$$\begin{aligned}\|\boldsymbol{E}_1\|_{PH^2(Q)} \le C_{E,1} \sum_{i=1}^{2} \Big(&\big\|\Delta\boldsymbol{E}_1^{(i)}\big\|_{L^2(Q_i)} + \big\|\operatorname{div}(\varepsilon^{(i)}\boldsymbol{E}^{(i)})\big\|_{H_0^{1/2}(\Gamma_1^{(i)})} \\ &+ \big\|\operatorname{div}(\varepsilon^{(i)}\boldsymbol{E}^{(i)})\big\|_{H_0^{1/2}(\mathscr{F}_{\mathrm{int}})}\Big) + \|[\![\varepsilon\boldsymbol{E}_1]\!]_{\mathscr{F}_{\mathrm{int}}}\|_{H_0^{3/2}(\mathscr{F}_{\mathrm{int}})}\end{aligned}$$

is valid with some uniform constant $C_{E,1} > 0$ being independent of $(\boldsymbol{E}, \boldsymbol{H})$.

Proof. 1) We start with a density result, that is inspired by the third part of the proof of Lemma 3.3 in [EiSc18]. Let $\Gamma^* := \Gamma_2 \cup \Gamma_3$, and put

$$\begin{aligned}&V := \{\varphi \in PH^1_{\Gamma^*}(Q) \mid [\![\varepsilon\varphi]\!]_{\mathscr{F}_{\mathrm{int}}} = 0\},\\ &W := \{\varphi \in V \mid \varphi^{(i)} \text{ is smooth on } Q_i,\ \operatorname{supp}(\varphi) \subseteq [a_1^-, a_1^+] \times (a_2^-, a_2^+) \times (a_3^-, a_3^+)\}.\end{aligned}$$

The first goal is to show that W is dense in V with respect to the norm in $PH^1(Q)$. To verify the claim, let $\varphi \in V$. For $j \in \{2, 3\}$, let $\chi_{n,j} : [a_j^-, a_j^+] \to [0, 1]$

be a smooth cut-off function which is equal to 1 on $[a_j^- + \frac{1}{n}, a_j^+ - \frac{1}{n}]$, which is supported within $[a_j^- + \frac{1}{2n}, a_j^+ - \frac{1}{2n}]$, and which satisfies the estimate $\left\|\chi_{n,j}'\right\|_\infty \leq Cn$ with some uniform constant $C > 0$ for all $n \geq n_0$. The product $\varepsilon\varphi$ belongs to $H^1(Q)$ by definition of V, and thus the function

$$g_n(x_1, x_2, x_3) := \chi_{n,2}(x_2)\chi_{n,3}(x_3)\varepsilon(x)\varphi(x), \qquad x = (x_1, x_2, x_3) \in Q,\ n \geq n_0,$$

can be extended by means of Stein's extension operator to a function in $H^1(\mathbb{R}^3)$. The resulting map is again called g_n.

Let $\kappa > 0$. Since φ vanishes on Γ^*, the sequence $(g_n|_Q)_n$ tends to $\varepsilon\varphi$ in $H^1(Q)$, see the proof of Lemma 2.1 in [EiSc18] for instance. Consequently, there is a number $n_\kappa \geq n_0$ with

$$\|g_{n_\kappa} - \varepsilon\varphi\|_{H^1(Q)} \leq \kappa. \tag{9.15}$$

Let $\rho_{m,l}$ be the standard mollifier, acting on the l-th coordinate for $l \in \{1, 2, 3\}$ and $m \geq n_0$. We define the function

$$\varphi_m := \frac{1}{\varepsilon}\left(\rho_{m,1} * \rho_{m,2} * \rho_{m,3} * g_{n_\kappa}\right)|_Q, \qquad m \in \mathbb{N}.$$

It is piecewise smooth, satisfies the transmission condition $[\![\varepsilon\varphi_m]\!]_{\mathscr{F}_{\text{int}}} = 0$, and is supported within $[a_1^-, a_1^+] \times (a_2^-, a_2^+) \times (a_3^-, a_3^+)$ for m sufficiently large. In particular, φ_m is an element of W. Because the restriction $g_{n_\kappa}|_Q$ is contained in $H^1(Q)$, we obtain by standard mollifier theory that the sequence $(\varepsilon\varphi_m)_m$ tends in $H^1(Q)$ to g_{n_κ}. As a result, $(\varphi_m)_m$ converges to $\frac{1}{\varepsilon}g_{n_\kappa}$ in $PH^1(Q)$. The choice of g_{n_κ} in (9.15) implies that W is dense in V.

2) Let $(\mathbf{E}, \mathbf{H}) \in X_2$. We proceed in two steps. First, we extend the normal jump of the electric field component $\mathbf{E}_1$ across the interface $\mathscr{F}_{\text{int}}$ in such a way, that we obtain a piecewise H^2-regular function ψ. In the second step, we apply Lemma 8.14 to the difference $\mathbf{E}_1 - \psi$ to deduce the desired regularity for $\mathbf{E}_1$.

By definition of X_2 in (7.19), the normal jump $\check{g} := [\![\varepsilon\mathbf{E} \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = [\![\varepsilon\mathbf{E}_1]\!]_{\mathscr{F}_{\text{int}}}$ is an element of $H_0^{3/2}(\mathscr{F}_{\text{int}})$. In order to extend $\check{g}$ to Q, we use arguments from the proof of Lemma 8.13, see also the proof of Lemma 3.1 in [EiSc17]. We consider again the rectangle $R = (a_2^-, a_2^+) \times (a_3^-, a_3^+)$, as well as the Dirichlet Laplacian Δ_R on $L^2(R)$ with the domain

$$\mathcal{D}(\Delta_R) := H^2(R) \cap H_0^1(R).$$

As in the proof of Lemma 8.13, we can define positive definite and self-adjoint fractional powers $(-\Delta_R)^\gamma$ for $\gamma > 0$. These generate analytic semigroups on $L^2(R)$. Theorem 4.36 from [Luna18] shows that $\check{g}$ belongs to the domain $\mathcal{D}((-\Delta_R)^{3/4})$, and thus $g := (-\Delta_R)^{1/2}\check{g}$ is an element of the domain

$$\mathcal{D}((-\Delta_R)^{1/4}) = (L^2(R), \mathcal{D}((-\Delta_R)^{1/2}))_{1/2,2} = H_0^{1/2}(R).$$

Finally, we also employ a smooth cut-off function $\chi : [a_1^-, a_1^+] \to [0,1]$ that is supported within $[\frac{5}{6}a_1^-, \frac{5}{6}a_1^+]$, and that is equal to 1 on $[\frac{3}{4}a_1^-, \frac{3}{4}a_1^+]$. We can then define the desired function

$$\psi(x_1,x_2,x_3) := \begin{cases} -\frac{\chi(x_1)}{2\varepsilon^{(1)}}\Big((-\Delta_R)^{-1/2}\mathrm{e}^{-x_1(-\Delta_R)^{1/2}}g\Big)(x_2,x_3) & \text{on } Q_1, \\ \frac{\chi(x_1)}{2\varepsilon^{(2)}}\Big((-\Delta_R)^{-1/2}\mathrm{e}^{x_1(-\Delta_R)^{1/2}}g\Big)(x_2,x_3) & \text{on } Q_2. \end{cases}$$

Modifying the arguments in part 2) of the proof for Lemma 8.13, we infer that ψ belongs to $PH^2(Q) \cap PH^1_{\Gamma^*}(Q)$ with $\Gamma^* = \Gamma_2 \cup \Gamma_3$, and that it satisfies the transmission condition $[\![\varepsilon\psi]\!]_{\mathscr{F}_{\mathrm{int}}} = [\![\varepsilon\mathbf{E}_1]\!]_{\mathscr{F}_{\mathrm{int}}}$. We further obtain the extension property $\mathrm{tr}_{\mathscr{F}_{\mathrm{int}}}\partial_1\psi^{(i)} = \frac{1}{2\varepsilon^{(i)}}g$. Recall that the latter function is contained in $H_0^{1/2}(\mathscr{F}_{\mathrm{int}})$ for $i \in \{1,2\}$. Additionally, the relations

$$\|\psi\|_{PH^2(Q)} \le \tilde{C}_1\,\|g\|_{H_0^{1/2}(\mathscr{F}_{\mathrm{int}})} \le \tilde{C}_2\,\|[\![\varepsilon\mathbf{E}_1]\!]_{\mathscr{F}_{\mathrm{int}}}\|_{H_0^{3/2}(\mathscr{F}_{\mathrm{int}})} \tag{9.16}$$

are valid with uniform positive constants $\tilde{C}_1$ and $\tilde{C}_2$.

3) We consider now the difference $\mathbf{E}_1 - \psi$. It fulfills the transmission condition $[\![\varepsilon(\mathbf{E}_1 - \psi)]\!]_{\mathscr{F}_{\mathrm{int}}} = 0$ by construction of ψ. Consequently, it belongs to the space V. We want to apply Lemma 8.14 to deduce the desired regularity for $\mathbf{E}_1$, and use concepts from the proof of Lemma 3.7 in [HoJS15]. First we take a test function $\varphi \in W$, and define for sufficiently large $n \in \mathbb{N}$ the open subcuboids

$$\begin{aligned} Q_{1,n} &:= (a_1^- + \tfrac{1}{n}, -\tfrac{1}{n}) \times (a_2^- + \tfrac{1}{n}, a_2^+ - \tfrac{1}{n}) \times (a_3^- + \tfrac{1}{n}, a_3^+ - \tfrac{1}{n}), \\ Q_{2,n} &:= (\tfrac{1}{n}, a_1^+ - \tfrac{1}{n}) \times (a_2^- + \tfrac{1}{n}, a_2^+ - \tfrac{1}{n}) \times (a_3^- + \tfrac{1}{n}, a_3^+ - \tfrac{1}{n}). \end{aligned} \tag{9.17}$$

The corresponding faces of these cuboids are denoted analogously to the faces of Q_1 and Q_2, i.e., $Q_{i,n}$ has the faces $\Gamma_{j,n}^{\pm,(i)}$ for $j \in \{1,2,3\}$ and $i \in \{1,2\}$.

Proposition 9.8 shows that $\mathbf{E}$ is an element of $PH^1(Q)^3$. Moreover, Lemma 9.10 and Remark 9.11 yield that $\Delta\mathbf{E}_j^{(i)}$ belongs to $L^2(Q_i)$, and that $\mathbf{E}_j^{(i)}$ is contained in $H^2(Q_{i,n})$ for $j \in \{1,2,3\}$, $i \in \{1,2\}$, and sufficiently large n. By construction of ψ, the function $u := \mathbf{E}_1 - \psi$ satisfies the transmission condition $[\![\varepsilon u]\!]_{\mathscr{F}_{\mathrm{int}}} = 0$. Integrating by parts, we hence obtain

$$\begin{aligned} \sum_{i=1}^{2}\int_{Q_i}\varepsilon^{(i)}(\nabla u^{(i)})\cdot(\nabla\varphi^{(i)})\,\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^{2}\int_{Q_{i,n}}\varepsilon^{(i)}(\nabla u^{(i)})\cdot(\nabla\varphi^{(i)})\,\mathrm{d}x \\ &= \lim_{n\to\infty}\Big(\sum_{i=1}^{2}\int_{Q_{i,n}} -\varepsilon^{(i)}(\Delta u^{(i)})\varphi^{(i)}\,\mathrm{d}x + \sum_{i=1}^{2}\int_{\partial Q_{i,n}}(\varepsilon^{(i)}\nabla u^{(i)}\cdot\nu^{(i)})\varphi^{(i)}\,\mathrm{d}\varsigma\Big). \end{aligned} \tag{9.18}$$

The boundary integrals on the right hand side reduce to integrals over the faces $\Gamma_{1,n}^{\pm,(i)}$ for sufficiently large n, since φ has compact support within $[a_1^-, a_1^+] \times (a_2^-, a_2^+) \times (a_3^-, a_3^+)$. We write in the following $\operatorname{div}\mathbf{E}$ and $\operatorname{div}(\varepsilon\mathbf{E})$ for convenience, meaning the functions that are defined piecewise on Q_1 and Q_2. Plugging in the divergence of $\varepsilon\mathbf{E}$, integration by parts thus implies the identities

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{2}\int_{\partial Q_{i,n}}(\varepsilon^{(i)}\nabla u^{(i)}\cdot\nu^{(i)})\varphi^{(i)}\,\mathrm{d}\varsigma = \lim_{n\to\infty}\sum_{i=1}^{2}\int_{\Gamma_{1,n}^{\pm,(i)}}(\partial_1\varepsilon^{(i)}u^{(i)})(\nu^{(i)})_1\varphi^{(i)}\,\mathrm{d}\varsigma\\
&= \lim_{n\to\infty}\sum_{i=1}^{2}\int_{\Gamma_{1,n}^{\pm,(i)}}(\operatorname{div}(\mathbf{E}^{(i)})-\partial_1\psi^{(i)}-\partial_2\mathbf{E}_2^{(i)}-\partial_3\mathbf{E}_3^{(i)})\varepsilon^{(i)}(\nu^{(i)})_1\varphi^{(i)}\,\mathrm{d}\varsigma\\
&= \lim_{n\to\infty}\Big(\int_{\Gamma_{1,n}^{-,(1)}\cup\Gamma_{1,n}^{+,(2)}}\varepsilon^{(i)}(\nu^{(i)})_1[(\operatorname{div}(\mathbf{E}^{(i)})-\partial_1\psi^{(i)})\varphi^{(i)}+\mathbf{E}_2^{(i)}\partial_2\varphi^{(i)}+\mathbf{E}_3^{(i)}\partial_3\varphi^{(i)}]\,\mathrm{d}\varsigma\\
&\quad+\int_{\Gamma_{1,n}^{+,(1)}\cup\Gamma_{1,n}^{-,(2)}}\varepsilon^{(i)}(\nu^{(i)})_1[(\operatorname{div}(\mathbf{E}^{(i)})-\partial_1\psi^{(i)})\varphi^{(i)}+\mathbf{E}_2^{(i)}\partial_2\varphi^{(i)}+\mathbf{E}_3^{(i)}\partial_3\varphi^{(i)}]\,\mathrm{d}\varsigma\Big),
\end{aligned}$$

where all boundary integrals over $\partial\Gamma_{1,n}^{\pm,(i)}$ vanish for sufficiently large n, due to the location of the support of φ. Combining the facts $[\![\varepsilon\varphi]\!]_{\mathscr{F}_{\text{int}}}=0$ and $\varphi\in PH^2(Q)$ with Lemma 7.1, we infer that $[\![\varepsilon\partial_2\varphi]\!]_{\mathscr{F}_{\text{int}}}=[\![\varepsilon\partial_3\varphi]\!]_{\mathscr{F}_{\text{int}}}=0$. Recalling also the boundary and transmission conditions

$$\mathbf{E}_2=\mathbf{E}_3=\partial_1\psi=0 \text{ on } \Gamma_1, \quad\text{and}\quad [\![\mathbf{E}_2]\!]_{\mathscr{F}_{\text{int}}}=[\![\mathbf{E}_3]\!]_{\mathscr{F}_{\text{int}}}=0,$$

we thus arrive at the formula

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{2}\int_{\partial Q_{i,n}}(\varepsilon^{(i)}\nabla u^{(i)}\cdot\nu^{(i)})\varphi^{(i)}\,\mathrm{d}\varsigma\\
&\qquad=\int_{\Gamma_1}\operatorname{div}(\varepsilon\mathbf{E})\nu_1\varphi\,\mathrm{d}\varsigma+\int_{\mathscr{F}_{\text{int}}}[\![\partial_1\psi-\operatorname{div}(\mathbf{E})]\!]_{\mathscr{F}_{\text{int}}}\varepsilon^{(1)}\varphi^{(1)}\,\mathrm{d}\varsigma.
\end{aligned}$$

Altogether, we infer from (9.18) the identity

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{Q_i}\varepsilon^{(i)}(\nabla u^{(i)})\cdot(\nabla\varphi^{(i)})\,\mathrm{d}x\\
&\qquad=\sum_{i=1}^{2}\int_{Q_i}-\varepsilon^{(i)}(\Delta u^{(i)})\varphi^{(i)}\,\mathrm{d}x+\int_{\Gamma_1}\operatorname{div}(\varepsilon\mathbf{E})\nu_1\varphi\,\mathrm{d}\varsigma\\
&\qquad\quad+\int_{\mathscr{F}_{\text{int}}}[\![\partial_1\psi-\operatorname{div}(\mathbf{E})]\!]_{\mathscr{F}_{\text{int}}}\varepsilon^{(1)}\varphi^{(1)}\,\mathrm{d}\varsigma. \qquad (9.19)
\end{aligned}$$

Since $\varphi\in W$ is chosen arbitrarily, part 1) implies that (9.19) holds in fact for all $\varphi\in V$. Now, Lemma 8.14 applies for $\eta=\varepsilon$. It yields the asserted regularity for $\mathbf{E}_1$ by construction of ψ. Since ψ belongs to $PH^2(Q)\cap PH^1_{\Gamma_2\cup\Gamma_3}(Q)$, its derivative $\partial_1\psi^{(i)}$ is an element of $H^1_{\Gamma_2\cup\Gamma_3}(Q_i)$ for $i\in\{1,2\}$. Remark 7.5 and (9.16) consequently imply the inequalities

$$\left\|\partial_1\varepsilon^{(i)}\psi^{(i)}\right\|_{H_0^{1/2}(\mathscr{F}_{\text{int}})}\le C_{\text{int}}\left\|\varepsilon^{(i)}\psi^{(i)}\right\|_{H^2(Q_i)}\le C_{\text{int}}\tilde{C}_2\varepsilon^{(i)}\left\|[\![\varepsilon\mathbf{E}_1]\!]_{\mathscr{F}_{\text{int}}}\right\|_{H_0^{3/2}(\mathscr{F}_{\text{int}})}.$$

Employing again (9.16) and Lemma 8.14, we thus infer the desired estimates

$$\begin{aligned}\|\mathbf{E}_1\|_{PH^2(Q)} &\le \check{C}_1 \sum_{i=1}^{2} \Big(\big\|\Delta u^{(i)}\big\|_{L^2(Q_i)} + \big\|\operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)})\big\|_{H_0^{1/2}(\Gamma_1^{(i)})} \\ &\qquad + \big\|\partial_1\varepsilon^{(i)}\psi^{(i)} - \operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)})\big\|_{H_0^{1/2}(\mathscr{F}_{\mathrm{int}})}\Big) \\ &\le \check{C}_1 \sum_{i=1}^{2} \Big[\big\|\Delta \mathbf{E}_1^{(i)}\big\|_{L^2(Q_i)} + \big\|\operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)})\big\|_{H_0^{1/2}(\Gamma_1^{(i)})} \\ &\qquad + \big\|\operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)})\big\|_{H_0^{1/2}(\mathscr{F}_{\mathrm{int}})} + (C_{\mathrm{int}}+1)\tilde{C}_2\varepsilon^{(i)}\, \|[\![\varepsilon\mathbf{E}_1]\!]_{\mathscr{F}_{\mathrm{int}}}\|_{H_0^{3/2}(\mathscr{F}_{\mathrm{int}})}\Big],\end{aligned}$$

where $\check{C}_1 > 0$ is a uniform constant from Lemma 8.14. □

The next statement treats the remaining components of the electric field. We here use the already established piecewise regularity of $\operatorname{curl}\mathbf{E}$ and $\mathbf{E}_1$, see Proposition 9.7 and Lemma 9.12. The proof furthermore transfers ideas from the proofs of Lemma 3.7 in [HoJS15], and Proposition 3.2 in [EiSc17]. Among other changes with respect to [HoJS15, EiSc17], an additional integral over the interface $\mathscr{F}_{\mathrm{int}}$ is present in our analysis. To control this term, we combine the above regularity statements for $\operatorname{curl}\mathbf{E}$ and $\mathbf{E}_1$ with our findings in Section 8.2.

Lemma 9.13. *Let ε, μ, and σ satisfy (7.2), and let $(\boldsymbol{E}, \boldsymbol{H}) \in X_2$. The components $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$ belong to $PH^2(Q)$. Furthermore, there is a constant $C_{E,2} > 0$ such that the estimate*

$$\|\boldsymbol{E}_j\|_{PH^2(Q)} \le C_{E,2} \sum_{i=1}^{2} \Big(\big\|\Delta \boldsymbol{E}_j^{(i)}\big\|_{L^2(Q_i)} + \big\|\partial_1 \boldsymbol{E}_j^{(i)}\big\|_{H^1(Q_i)} + \big\|\operatorname{div}(\varepsilon\boldsymbol{E}^{(i)})\big\|_{H_0^{1/2}(\Gamma_j^{\pm,(i)})}\Big)$$

is valid for $j \in \{2,3\}$. The number $C_{E,2}$ depends only on ε, μ, σ, and Q.

Proof. 1) We consider only the function $\mathbf{E}_2$. The remaining component $\mathbf{E}_3$ is treated by similar arguments. Let $\Gamma^* := \Gamma_1 \cup \Gamma_3$, and put

$$\begin{aligned} V &:= \{\varphi \in PH^1_{\Gamma^*}(Q) \mid [\![\varphi]\!]_{\mathscr{F}_{\mathrm{int}}} = 0\}, \\ W &:= \{\varphi \in V \mid \varphi^{(i)} \text{ is smooth on } \overline{Q_i},\ \operatorname{supp}(\varphi) \subseteq (a_1^-, a_1^+) \times [a_2^-, a_2^+] \times (a_3^-, a_3^+)\}. \end{aligned}$$

Similar to part 1) of the proof of Lemma 9.12, one can show that W is dense in V with respect to the norm in $H^1(Q)$.

We employ also the cuboids $Q_{i,n}$ for $i \in \{1,2\}$ and $n \ge n_0$, being defined in (9.17). Their faces are again denoted by $\Gamma_{j,n}^{\pm,(i)}$ for $j \in \{1,2,3\}$.

2) Lemma 9.12 shows that the function $\mathbf{E}_1$ belongs to $PH^2(Q)$, and Proposition 9.7 implies that $\operatorname{curl}\mathbf{E}$ is contained in $PH^1(Q)^3$. Consequently, the function

$\partial_1 \mathbf{E}_2^{(i)} = (\operatorname{curl} \mathbf{E}^{(i)})_3 + \partial_2 \mathbf{E}_1^{(i)}$ is an element of $H^1(Q_i)$ for $i \in \{1,2\}$. Moreover, $\partial_1 \mathbf{E}_2^{(i)}$ vanishes on the exterior face $\Gamma_3^{(i)}$, due to $\mathbf{E}_2 = 0$ on Γ_3 and Lemma 2.1 from [EiSc18]. Lemma 7.3 now shows that $\operatorname{tr}_{\mathscr{F}_{\mathrm{int}}}(\partial_1 \mathbf{E}_2^{(i)})$ is contained in the trace space $H^{1/2}_{\Gamma_3}(\mathscr{F}_{\mathrm{int}})$. Finally, $\mathbf{E}_2^{(i)}$ belongs to $H^1(Q_i) \cap H^2_{\mathrm{loc}}(Q_i)$ for $i \in \{1,2\}$, see Proposition 9.8 and Remark 9.11.

We next transfer our problem into the form of (8.25). Let $\varphi \in W$. Integrating by parts, we obtain the relations

$$\begin{aligned} \sum_{i=1}^{2} \int_{Q_i} (\nabla \mathbf{E}_2^{(i)}) \cdot (\nabla \varphi^{(i)}) \,\mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{2} \int_{Q_{i,n}} (\nabla \mathbf{E}_2^{(i)}) \cdot (\nabla \varphi^{(i)}) \,\mathrm{d}x \\ &= \lim_{n\to\infty} \Big(\sum_{i=1}^{2} \int_{Q_{i,n}} (-\Delta \mathbf{E}_2^{(i)}) \varphi^{(i)} \,\mathrm{d}x + \sum_{i=1}^{2} \int_{\partial Q_{i,n}} (\nabla \mathbf{E}_2^{(i)} \cdot \nu^{(i)}) \varphi^{(i)} \,\mathrm{d}\varsigma \Big). \end{aligned} \tag{9.20}$$

In the following, we deal with the boundary integrals on the right hand side. First, we note that all integrals over the faces $\Gamma_{3,n}^{\pm,(i)}$ vanish for sufficiently large n, due to the location of the support of φ. Employing the divergence of $\mathbf{E}^{(i)}$, we thus obtain

$$\begin{aligned} &\lim_{n\to\infty} \sum_{i=1}^{2} \int_{\partial Q_{i,n}} (\nabla \mathbf{E}_2^{(i)} \cdot \nu^{(i)}) \varphi^{(i)} \,\mathrm{d}\varsigma \\ &\quad = \lim_{n\to\infty} \sum_{i=1}^{2} \int_{\Gamma_{1,n}^{\pm,(i)}} (\partial_1 \mathbf{E}_2^{(i)}) (\nu^{(i)})_1 \varphi^{(i)} \,\mathrm{d}\varsigma + \sum_{i=1}^{2} \int_{\Gamma_{2,n}^{\pm,(i)}} (\partial_2 \mathbf{E}_2^{(i)}) (\nu^{(i)})_2 \varphi^{(i)} \,\mathrm{d}\varsigma \\ &\quad = \lim_{n\to\infty} \Big(\sum_{i=1}^{2} \int_{\Gamma_{1,n}^{\pm,(i)}} (\partial_1 \mathbf{E}_2^{(i)}) (\nu^{(i)})_1 \varphi^{(i)} \,\mathrm{d}\varsigma \\ &\qquad\qquad + \sum_{i=1}^{2} \int_{\Gamma_{2,n}^{\pm,(i)}} [\operatorname{div}(\mathbf{E}^{(i)}) - \partial_1 \mathbf{E}_1^{(i)} - \partial_3 \mathbf{E}_3^{(i)}] (\nu^{(i)})_2 \varphi^{(i)} \,\mathrm{d}\varsigma \Big). \end{aligned}$$

Since φ vanishes in a neighborhood of the exterior faces Γ_1 and Γ_3, the boundary integrals over $\Gamma_{1,n}^{+,(2)}$ and $\Gamma_{1,n}^{-,(1)}$ are zero for sufficiently large n. Due to the same reason, we can integrate the second term on the right hand side by parts in x_1 and x_3 with vanishing boundary integrals over $\partial \Gamma_{2,n}^{\pm,(i)}$. Since $\partial_1 \mathbf{E}_2^{(i)}$ belongs to $H^1(Q_i)$, we arrive at the identities

$$\begin{aligned} &\lim_{n\to\infty} \sum_{i=1}^{2} \int_{\partial Q_{i,n}} (\nabla \mathbf{E}_2^{(i)} \cdot \nu^{(i)}) \varphi^{(i)} \,\mathrm{d}\varsigma \\ &\quad = - \int_{\mathscr{F}_{\mathrm{int}}} [\![(\partial_1 \mathbf{E}_2) \varphi]\!]_{\mathscr{F}_{\mathrm{int}}} \,\mathrm{d}\varsigma \\ &\qquad + \lim_{n\to\infty} \sum_{i=1}^{2} \int_{\Gamma_{2,n}^{\pm,(i)}} (\nu^{(i)})_2 [\operatorname{div}(\mathbf{E}^{(i)}) \varphi^{(i)} + \mathbf{E}_1^{(i)} \partial_1 \varphi^{(i)} + \mathbf{E}_3^{(i)} \partial_3 \varphi^{(i)}] \,\mathrm{d}\varsigma \end{aligned}$$

$$= -\int_{\mathscr{F}_{\text{int}}} [\![\partial_1 \mathbf{E}_2]\!]_{\mathscr{F}_{\text{int}}} \varphi \,\mathrm{d}\varsigma + \sum_{i=1}^{2} \int_{\Gamma_2^{\pm,(i)}} \operatorname{div}(\mathbf{E}^{(i)}) \varphi^{(i)} (\nu^{(i)})_2 \,\mathrm{d}\varsigma,$$

where we employ for the last relation that φ is continuous across $\mathscr{F}_{\text{int}}$, and that $\mathbf{E}_1$ and $\mathbf{E}_3$ vanish on Γ_2. Together with (9.20) we have derived the formula

$$\begin{aligned}
&\sum_{i=1}^{2} \int_{Q_i} (\nabla \mathbf{E}_2^{(i)}) \cdot (\nabla \varphi^{(i)}) \,\mathrm{d}x \\
&\qquad = \sum_{i=1}^{2} \int_{Q_i} (-\Delta \mathbf{E}_2^{(i)}) \varphi^{(i)} \,\mathrm{d}x - \int_{\mathscr{F}_{\text{int}}} [\![\partial_1 \mathbf{E}_2]\!]_{\mathscr{F}_{\text{int}}} \varphi \,\mathrm{d}\varsigma \\
&\qquad\quad + \sum_{i=1}^{2} \int_{\Gamma_2^{\pm,(i)}} \operatorname{div}(\mathbf{E}^{(i)}) \varphi^{(i)} (\nu^{(i)})_2 \,\mathrm{d}\varsigma
\end{aligned} \tag{9.21}$$

for all functions $\varphi \in V$, since W is a dense subspace of V with respect to the norm in $H^1(Q)$. Because the trace of $\partial_1 \mathbf{E}_2^{(i)}$ on $\mathscr{F}_{\text{int}}$ belongs to $H^{1/2}_{\Gamma_3}(\mathscr{F}_{\text{int}})$ for $i \in \{1, 2\}$, Lemmas 7.4 and 8.13 yield the asserted estimate. □

In the next three lemmas, we deal with the magnetic field component $\mathbf{H}$ of a vector $(\mathbf{E}, \mathbf{H}) \in X_2$. To tackle arising face integrals, we first derive an auxiliary result.

Lemma 9.14. *Let μ satisfy (7.2). Let further $\Phi_1, \Phi_2 \in H^1(Q)$ with $\Phi_1 = 0$ on Γ_2 and $\Phi_2 = 0$ on Γ_3. The formula*

$$\sum_{i=1}^{2} \int_{Q_i} \mu^{(i)} (\nabla \Phi_1^{(i)}) \cdot \operatorname{curl} \begin{pmatrix} 0 \\ 0 \\ \varphi^{(i)} \end{pmatrix} \mathrm{d}x = 0 = \sum_{i=1}^{2} \int_{Q_i} \mu^{(i)} (\nabla \Phi_2^{(i)}) \cdot \operatorname{curl} \begin{pmatrix} 0 \\ \varphi^{(i)} \\ 0 \end{pmatrix} \mathrm{d}x$$

is true for all $\varphi \in PH^2_{\Gamma_1}(Q)$ with $[\![\mu\varphi]\!]_{\mathscr{F}_{\text{int}}} = 0$.

Proof. We only treat the function Φ_1. The remaining case can be handled with similar arguments. The vector $\nabla \Phi_1$ is an element of $H(\operatorname{curl}, Q)$, and an integration by parts leads to the identity

$$\begin{aligned}
&\sum_{i=1}^{2} \int_{Q_i} \mu^{(i)} (\nabla \Phi_1^{(i)}) \cdot \operatorname{curl} \begin{pmatrix} 0 \\ 0 \\ \varphi^{(i)} \end{pmatrix} \mathrm{d}x \\
&\qquad = \sum_{i=1}^{2} \langle \nabla \Phi_1^{(i)} \times \nu, \mu^{(i)} \begin{pmatrix} 0 \\ 0 \\ \varphi^{(i)} \end{pmatrix} \rangle_{H^{-1/2}(\partial Q_i) \times H^{1/2}(\partial Q_i)}.
\end{aligned} \tag{9.22}$$

We next proceed similar to the proof of Lemma 7.1. Let $\chi_m : \mathbb{R} \to [0, 1]$ be a smooth cut-off function that is equal to 1 on $[a_2^- + 1/m, a_2^+ - 1/m]$, that has its support within $[a_2^- + 1/(2m), a_2^+ - 1/(2m)]$, and that satisfies $\|\chi_m'\|_\infty \le Cm$ for $m \in \mathbb{N}$ with $m \ge m_0 := \lceil \frac{4}{a_2^+ - a_2^-} \rceil$ and a uniform constant $C > 0$. Let further $\rho_n : [0,1]^3 \to \mathbb{R}$ be the standard smooth mollifier with support in the ball $B(0, 1/n)$ for $n \in \mathbb{N}$.

Since Φ_1 belongs to $H^1(Q)$, it can be extended by means of Stein's extension operator to a function in $H^1(\mathbb{R}^3)$. The extension is again denoted by Φ_1. We then define the maps

$$\begin{aligned} h_m(x_1, x_2, x_3) &:= \chi_m(x_2)\Phi_1(x_1, x_2, x_3), && (x_1, x_2, x_3) \in \mathbb{R}^3, \\ g_{m,n} &:= \rho_n * h_m, && m, n \in \mathbb{N}. \end{aligned}$$

Let $\kappa > 0$. Since $\Phi_1|_Q$ has a vanishing trace on Γ_2, the proof of Lemma 2.1 in [EiSc18] shows that there is a number $m_\kappa \in \mathbb{N}$ with

$$\|h_{m_\kappa} - \Phi_1\|_{H^1(Q)} \le \kappa.$$

By construction of $g_{m_\kappa,n}$, the function is smooth on $\mathbb{R}^3$, vanishes near Γ_2 for sufficiently large n, and the sequence $(g_{m_\kappa,n})_n$ converges to h_{m_κ} in $H^1(Q)$. As a result, $(\nabla g^{(i)}_{m_\kappa,n})_n$ converges in $H(\mathrm{curl}, Q_i)$ to $\nabla h^{(i)}_{m_\kappa}$, and the estimate $\|\nabla h_{m_\kappa} - \nabla\Phi_1\|_{H(\mathrm{curl},Q_i)} \le \kappa$ is valid. The continuity of the tangential trace operator consequently implies the statements

$$\begin{aligned} &\nabla g^{(i)}_{m_\kappa,n} \times \nu \to \nabla h^{(i)}_{m_\kappa} \times \nu \text{ in } H^{-1/2}(\partial Q_i), \quad n \to \infty, \\ &\|\nabla h_{m_\kappa} \times \nu - \nabla\Phi_1 \times \nu\|_{H^{-1/2}(\partial Q_i)} \le C\kappa, \end{aligned} \tag{9.23}$$

with a uniform constant $C > 0$. Employing the smoothness of $g_{m_\kappa,n}$, as well as the transmission and boundary conditions for φ, we next calculate

$$\begin{aligned} &\sum_{i=1}^{2} \langle \nabla g^{(i)}_{m_\kappa,n} \times \nu, \mu^{(i)} \begin{pmatrix} 0 \\ 0 \\ \varphi^{(i)} \end{pmatrix} \rangle_{H^{-1/2}(\partial Q_i) \times H^{1/2}(\partial Q_i)} \\ &\qquad = \sum_{i=1}^{2} \int_{\partial Q_i} (\nabla g^{(i)}_{m_\kappa,n} \times \nu) \cdot \mu^{(i)} \begin{pmatrix} 0 \\ 0 \\ \varphi^{(i)} \end{pmatrix} \mathrm{d}\varsigma \\ &\qquad = \sum_{i=1}^{2} \Big(- \int_{\Gamma_1^{(i)}} \partial_2 g^{(i)}_{m_\kappa,n} \nu_1 \mu^{(i)} \varphi^{(i)} \,\mathrm{d}\varsigma + \int_{\Gamma_2^{(i)}} \partial_1 g^{(i)}_{m_\kappa,n} \nu_2 \mu^{(i)} \varphi^{(i)} \,\mathrm{d}\varsigma \Big) \\ &\qquad = 0. \end{aligned} \tag{9.24}$$

Since $\kappa > 0$ is arbitrary, the results (9.22)–(9.24) imply the desired formula. □

Similar to the considerations for the electric field, we first investigate the regularity of the first component $\mathbf{H}_1$. Thereby we apply techniques from the proof of

Lemma 3.7 in [HoJS15]. To cover our setting of discontinuous material parameters, we combine the divergence constraint $\operatorname{div}(\mu\mathbf{H}) = 0$ with Lemma 9.14. In this way, we can control arising interface integrals.

Lemma 9.15. *Let ε, μ, and σ satisfy (7.2), and let $(\boldsymbol{E}, \boldsymbol{H}) \in X_2$. The function $\boldsymbol{H}_1$ is contained in $PH^2(Q)$, and it satisfies the transmission relation $[\![\partial_1 \boldsymbol{H}_1]\!]_{\mathscr{F}_{\text{int}}} = 0$. Moreover, the estimate*

$$\|\boldsymbol{H}_1\|_{PH^2(Q)} \leq C_{H,1} \sum_{i=1}^{2} \left\|\Delta \boldsymbol{H}_1^{(i)}\right\|_{L^2(Q_i)}$$

is true with a constant $C_{H,1} > 0$, being independent of $(\boldsymbol{E}, \boldsymbol{H})$.

Proof. 1) We proceed similar to the above proofs, and define the spaces

$$\begin{aligned} V &:= \{\varphi \in PH^1_{\Gamma_1}(Q) \mid [\![\mu\varphi]\!]_{\mathscr{F}_{\text{int}}} = 0\}, \\ W &:= \{\varphi \in V \mid \varphi^{(i)} \text{ is smooth on } \overline{Q_i},\ \operatorname{supp}(\varphi) \subseteq (a_1^-, a_1^+) \times [a_2^-, a_2^+] \times [a_3^-, a_3^+]\}. \end{aligned}$$

Arguing analogously to part 1) from the proof of Lemma 9.12, W is dense in V with respect to the norm in $PH^1(Q)$. The smaller subcuboids $Q_{i,n}$ from (9.17) are again employed for sufficiently large $n \in \mathbb{N}$. These have the faces $\Gamma_{j,n}^{\pm,(i)}$ for $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$.

2) Remark 9.11 and Proposition 9.7 state that $\mathbf{H}^{(i)}$ belongs to $H^1(Q_i)^3 \cap H^2_{\text{loc}}(Q_i)^3$. We next deduce that the vector $\operatorname{curl}\mathbf{H}$ is contained in $PH^1(Q)^3$. In fact, the function $\tilde{\mathbf{H}} := \frac{1}{\varepsilon}\operatorname{curl}\mathbf{H} - \frac{\sigma}{\varepsilon}\mathbf{E}$ is contained in $H_0(\operatorname{curl}, Q)$ as $(\mathbf{E}, \mathbf{H}) \in \mathcal{D}(M^2)$. Additionally, $\operatorname{div}(\varepsilon^{(i)}\tilde{\mathbf{H}}^{(i)}) = -\operatorname{div}(\sigma^{(i)}\mathbf{E}^{(i)})$ is an element of $L^2(Q_i)$, and the jump $[\![\varepsilon\tilde{\mathbf{H}} \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = -[\![\sigma\mathbf{E} \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}$ belongs to $H_0^{1/2}(\mathscr{F}_{\text{int}})$ by Lemma 7.3, and Lemmas 9.12–9.13. (For the last identity we use the fact $\operatorname{curl}\mathbf{H} \in H(\operatorname{div}, Q)$, as well as Green's formulas.) Consequently, the vector $(\tilde{\mathbf{H}}, 0)$ is contained in X_1, so that Proposition 9.8 implies the piecewise H^1-regularity of $\tilde{\mathbf{H}}$. Knowing that $\mathbf{E}$ is piecewise H^2-regular, we conclude that $\operatorname{curl}\mathbf{H}$ is an element of $PH^1(Q)^3$.

Since $\tilde{\mathbf{H}}$ and $\mathbf{E}$ are elements of $H_0(\operatorname{curl}, Q)$, also the tangential boundary condition $\operatorname{curl}\mathbf{H}^{(i)} \times \nu = 0$ on $\partial Q_i \setminus \mathscr{F}_{\text{int}}$ is valid.

3) The desired statement will be a consequence of Proposition 8.1. To apply the latter result, we derive a variational problem of the form (8.3). Let $\varphi \in W$. An integration by parts leads to the identities

$$\begin{aligned} \sum_{i=1}^{2} \int_{Q_i} \mu^{(i)}(\nabla\mathbf{H}_1^{(i)}) \cdot (\nabla\varphi^{(i)}) \,\mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{2} \int_{Q_{i,n}} \mu^{(i)}(\nabla\mathbf{H}_1^{(i)}) \cdot (\nabla\varphi^{(i)}) \,\mathrm{d}x \\ &= \lim_{n\to\infty} \sum_{i=1}^{2} \Big(\int_{Q_{i,n}} -\mu^{(i)}(\Delta\mathbf{H}_1^{(i)})\varphi^{(i)} \,\mathrm{d}x + \int_{\partial Q_{i,n}} \mu^{(i)}(\nabla\mathbf{H}_1^{(i)} \cdot \nu^{(i)})\varphi^{(i)} \,\mathrm{d}\varsigma \Big). \end{aligned} \tag{9.25}$$

In the next steps we show that the boundary integral term on the right hand side converges to zero as $n \to \infty$. Note that the integrals over the exterior faces $\Gamma_{1,n}^{-,(1)}$ and $\Gamma_{1,n}^{+,(2)}$ vanish for sufficiently large n, since these faces are disjoint from the support of φ. Inserting $\pm\partial_1 \mathbf{H}_2^{(i)}$, $\pm\partial_1 \mathbf{H}_3^{(i)}$ and the identity $\operatorname{div}(\mu\mathbf{H}) = 0$, we thus obtain the formula

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=1}^{2} \int_{\partial Q_{i,n}} \mu^{(i)} (\nabla \mathbf{H}_1^{(i)} \cdot \nu^{(i)}) \varphi^{(i)} \, \mathrm{d}\varsigma \\
&\quad = \lim_{n\to\infty} \sum_{i=1}^{2} \Big(\int_{\Gamma_{1,n}^{\pm,(i)}} -\mu^{(i)} \nu_1^{(i)} (\partial_2 \mathbf{H}_2^{(i)} + \partial_3 \mathbf{H}_3^{(i)}) \varphi^{(i)} \, \mathrm{d}\varsigma \\
&\quad\quad - \int_{\Gamma_{2,n}^{\pm,(i)}} \mu^{(i)} \nu_2^{(i)} (\operatorname{curl} \mathbf{H})_3 \varphi^{(i)} \, \mathrm{d}\varsigma + \int_{\Gamma_{3,n}^{\pm,(i)}} \mu^{(i)} \nu_3^{(i)} (\operatorname{curl} \mathbf{H}^{(i)})_2 \varphi^{(i)} \, \mathrm{d}\varsigma \\
&\quad\quad + \int_{\Gamma_{2,n}^{\pm,(i)}} \mu^{(i)} \nu_2^{(i)} (\partial_1 \mathbf{H}_2^{(i)}) \varphi^{(i)} \, \mathrm{d}\varsigma + \int_{\Gamma_{3,n}^{\pm,(i)}} \mu^{(i)} \nu_3^{(i)} (\partial_1 \mathbf{H}_3^{(i)}) \varphi^{(i)} \, \mathrm{d}\varsigma \Big). \qquad (9.26)
\end{aligned}$$

Because the trace of the vector $\operatorname{curl} \mathbf{H}$ is tangential to the normal vector on the exterior faces, see part 2), the second and third summand on the right hand side of (9.26) tend to zero. By means of Green's formula for curl, we next calculate

$$\begin{aligned}
&\sum_{i=1}^{2} \Big(\int_{\Gamma_{1,n}^{\pm,(i)}} -\mu^{(i)} \nu_1^{(i)} (\partial_2 \mathbf{H}_2^{(i)} + \partial_3 \mathbf{H}_3^{(i)}) \varphi^{(i)} \, \mathrm{d}\varsigma + \int_{\Gamma_{2,n}^{\pm,(i)}} \mu^{(i)} \nu_2^{(i)} (\partial_1 \mathbf{H}_2^{(i)}) \varphi^{(i)} \, \mathrm{d}\varsigma \\
&\quad + \int_{\Gamma_{3,n}^{\pm,(i)}} \mu^{(i)} \nu_3^{(i)} (\partial_1 \mathbf{H}_3^{(i)}) \varphi^{(i)} \, \mathrm{d}\varsigma \Big) \\
&= \sum_{i=1}^{2} \Big(\int_{\partial Q_{i,n}} (\nabla \mathbf{H}_2^{(i)} \times \nu^{(i)}) \cdot \mu^{(i)} \begin{pmatrix} 0 \\ 0 \\ \varphi^{(i)} \end{pmatrix} \mathrm{d}\varsigma - \int_{\partial Q_{i,n}} (\nabla \mathbf{H}_3^{(i)} \times \nu^{(i)}) \cdot \mu^{(i)} \begin{pmatrix} 0 \\ \varphi^{(i)} \\ 0 \end{pmatrix} \mathrm{d}\varsigma \Big) \\
&= \sum_{i=1}^{2} \int_{Q_{i,n}} \mu^{(i)} (\nabla \mathbf{H}_2^{(i)}) \cdot \operatorname{curl} \begin{pmatrix} 0 \\ 0 \\ \varphi^{(i)} \end{pmatrix} - \mu^{(i)} (\nabla \mathbf{H}_3^{(i)}) \cdot \operatorname{curl} \begin{pmatrix} 0 \\ \varphi^{(i)} \\ 0 \end{pmatrix} \mathrm{d}x.
\end{aligned}$$

Passing to the limit $n \to \infty$ and using Lemma 9.14, we arrive at the identity

$$\begin{aligned}
&\sum_{i=1}^{2} \Big(\int_{\Gamma_{1,n}^{\pm,(i)}} -\mu^{(i)} \nu_1^{(i)} (\partial_2 \mathbf{H}_2^{(i)} + \partial_3 \mathbf{H}_3^{(i)}) \varphi^{(i)} \, \mathrm{d}\varsigma + \int_{\Gamma_{2,n}^{\pm,(i)}} \mu^{(i)} \nu_2^{(i)} (\partial_1 \mathbf{H}_2^{(i)}) \varphi^{(i)} \, \mathrm{d}\varsigma \\
&\quad + \int_{\Gamma_{3,n}^{\pm,(i)}} \mu^{(i)} \nu_3^{(i)} (\partial_1 \mathbf{H}_3^{(i)}) \varphi^{(i)} \, \mathrm{d}\varsigma \Big) \\
&= 0.
\end{aligned}$$

Altogether, (9.25) now has the representation

$$\sum_{i=1}^{2} \int_{Q_i} \mu^{(i)} (\nabla \mathbf{H}_1^{(i)}) \cdot (\nabla \varphi^{(i)}) \, \mathrm{d}x = \sum_{i=1}^{2} \int_{Q_i} -\mu^{(i)} (\Delta \mathbf{H}_1^{(i)}) \varphi^{(i)} \, \mathrm{d}x.$$

By density, the same formula holds also for all functions $\varphi \in V$. Finally, we employ Proposition 8.1 with $\eta = \mu$ and $\Gamma^* = \Gamma_1$ to conclude the asserted statement. □

Similar to our studies of the electric field components $\mathbf{E}_2$ and $\mathbf{E}_3$ in Lemma 9.13, we combine techniques from the proof for Lemma 3.7 in [HoJS15] with the statements of Lemmas 8.13 and 9.15 to treat the remaining two components of the magnetic field.

Lemma 9.16. *Let ε, μ, and σ satisfy (7.2), and let $(\boldsymbol{E}, \boldsymbol{H}) \in X_2$. The components $\boldsymbol{H}_2$ and $\boldsymbol{H}_3$ are then contained in $PH^2(Q)$, and they satisfy the estimates*

$$\|\boldsymbol{H}_2\|_{PH^2(Q)} \leq C_{H,2} \left(\sum_{i=1}^{2} \left\| \Delta \boldsymbol{H}_2^{(i)} \right\|_{L^2(Q_i)} + \|(\operatorname{curl} \boldsymbol{H})_3\|_{PH^1(Q)} + \|\partial_2 \boldsymbol{H}_1\|_{PH^1(Q)} \right),$$
$$\|\boldsymbol{H}_3\|_{PH^2(Q)} \leq C_{H,2} \left(\sum_{i=1}^{2} \left\| \Delta \boldsymbol{H}_3^{(i)} \right\|_{L^2(Q_i)} + \|(\operatorname{curl} \boldsymbol{H})_2\|_{PH^1(Q)} + \|\partial_3 \boldsymbol{H}_1\|_{PH^1(Q)} \right),$$

with a constant $C_{H,2} > 0$, that is independent of $(\boldsymbol{E}, \boldsymbol{H})$.

Proof. 1) Again, we only treat the second component $\mathbf{H}_2$ of the magnetic field. The third one can be handled in a similar way. We define the spaces

$$V := \{\varphi \in PH^1_{\Gamma_2}(Q) \mid [\![\varphi]\!]_{\mathscr{F}_{\text{int}}} = 0\},$$
$$W := \{\varphi \in V \mid \varphi^{(i)} \text{ is smooth on } \overline{Q_i},\ \operatorname{supp}(\varphi) \subseteq [a_1^-, a_1^+] \times (a_2^-, a_2^+) \times [a_3^-, a_3^+]\}.$$

Similar to part 1) from the proof of Lemma 9.12, W is a dense subspace of V with respect to the H^1-norm on Q. As above, we use the subcuboids $Q_{i,n}$ from (9.17), which have the faces $\Gamma_{j,n}^{\pm,(i)}$, for $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$.

2) The interior regularity of $\mathbf{H}_2^{(i)}$ is already established. Remark 9.11 in particular shows that $\mathbf{H}_2^{(i)}$ belongs to $H^2_{\text{loc}}(Q_i)$. Recall also from part 2) of the proof for Lemma 9.15 that the vector $\operatorname{curl} \mathbf{H}$ belongs to $PH^1(Q)^3$, and that it fulfills the boundary condition $\operatorname{curl} \mathbf{H}^{(i)} \times \nu = 0$ on $\partial Q_i \setminus \mathscr{F}_{\text{int}}$. Taking additionally the regularity result for $\mathbf{H}_1$ from Lemma 9.15 into account, we deduce that $\partial_1 \mathbf{H}_2$ is an element of $PH^1(Q)$. The boundary condition $\mathbf{H}_2 = 0$ on Γ_2, and Lemma 2.1 from [EiSc18] further imply that $\partial_1 \mathbf{H}_2 = 0$ on Γ_2. Combining these facts with the boundary condition of $\operatorname{curl} \mathbf{H}$, we conclude that $\partial_2 \mathbf{H}_1 = 0$ on Γ_2. This means that $\operatorname{tr}_{\mathscr{F}_{\text{int}}} \partial_2 \mathbf{H}_1$ is contained in $H^{1/2}_{\Gamma_2}(\mathscr{F}_{\text{int}})$, see Lemma 7.3.

3) Let $\varphi \in W$. Integrating by parts, we obtain the relations

$$\sum_{i=1}^{2} \int_{Q_i} (\nabla \mathbf{H}_2^{(i)}) \cdot (\nabla \varphi^{(i)}) \, \mathrm{d}x = \lim_{n \to \infty} \sum_{i=1}^{2} \int_{Q_{i,n}} (\nabla \mathbf{H}_2^{(i)}) \cdot (\nabla \varphi^{(i)}) \, \mathrm{d}x$$
$$= \lim_{n \to \infty} \sum_{i=1}^{2} \Big(\int_{Q_{i,n}} -(\Delta \mathbf{H}_2^{(i)}) \varphi^{(i)} \, \mathrm{d}x + \int_{\partial Q_{i,n}} (\nabla \mathbf{H}_2^{(i)} \cdot \nu^{(i)}) \varphi^{(i)} \, \mathrm{d}\varsigma \Big). \qquad (9.27)$$

Since the faces $\Gamma_{2,n}^{\pm,(i)}$ are disjoint from the support of φ for sufficiently large n, all boundary integrals on the right hand side of (9.27) with respect to these faces vanish. Employing the vector $\operatorname{curl}\mathbf{H}$, we then derive the formula

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{2}\int_{\partial Q_{i,n}}(\nabla\mathbf{H}_2^{(i)}\cdot\nu^{(i)})\varphi^{(i)}\,\mathrm{d}\varsigma\\
&\quad=\lim_{n\to\infty}\sum_{i=1}^{2}\Big(\int_{\Gamma_{1,n}^{\pm,(i)}}\nu_1^{(i)}(\partial_1\mathbf{H}_2^{(i)})\varphi^{(i)}\,\mathrm{d}\varsigma+\int_{\Gamma_{3,n}^{\pm,(i)}}\nu_3^{(i)}(\partial_3\mathbf{H}_2^{(i)})\varphi^{(i)}\,\mathrm{d}\varsigma\Big)\\
&\quad=\lim_{n\to\infty}\sum_{i=1}^{2}\Big(\int_{\Gamma_{1,n}^{\pm,(i)}}\nu_1^{(i)}(\operatorname{curl}\mathbf{H}^{(i)})_3\varphi^{(i)}\,\mathrm{d}\varsigma+\int_{\Gamma_{1,n}^{\pm,(i)}}\nu_1^{(i)}(\partial_2\mathbf{H}_1^{(i)})\varphi^{(i)}\,\mathrm{d}\varsigma\\
&\qquad-\int_{\Gamma_{3,n}^{\pm,(i)}}\nu_3^{(i)}(\operatorname{curl}\mathbf{H}^{(i)})_1\varphi^{(i)}\,\mathrm{d}\varsigma+\int_{\Gamma_{3,n}^{\pm,(i)}}\nu_3^{(i)}(\partial_2\mathbf{H}_3^{(i)})\varphi^{(i)}\,\mathrm{d}\varsigma\Big).
\end{aligned}\tag{9.28}$$

In a next step, we integrate the fourth summand on the right hand side by parts. Hereby, we notice that the corresponding boundary integrals over $\partial\Gamma_{3,n}^{\pm,(i)}$ vanish, due to the location of the support of φ. Also the boundary conditions $\mathbf{H}_1=0$ on Γ_1, and $\mathbf{H}_3=0$ on Γ_3 come into play. The first one in particular implies $\partial_2\mathbf{H}_1=0$ on Γ_1, see Lemma 2.1 in [EiSc18]. Using additionally the transmission condition $[\![\varphi]\!]_{\mathscr{F}_{\mathrm{int}}}=0$, we thus obtain the identities

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{2}\Big(\int_{\Gamma_{1,n}^{\pm,(i)}}\nu_1^{(i)}(\partial_2\mathbf{H}_1^{(i)})\varphi^{(i)}\,\mathrm{d}\varsigma+\int_{\Gamma_{3,n}^{\pm,(i)}}\nu_3^{(i)}(\partial_2\mathbf{H}_3^{(i)})\varphi^{(i)}\,\mathrm{d}\varsigma\Big)\\
&\quad=\lim_{n\to\infty}\sum_{i=1}^{2}\left(\int_{\Gamma_{1,n}^{\pm,(i)}}\nu_1^{(i)}(\partial_2\mathbf{H}_1^{(i)})\varphi^{(i)}\,\mathrm{d}\varsigma-\int_{\Gamma_{3,n}^{\pm,(i)}}\nu_3^{(i)}\mathbf{H}_3^{(i)}\partial_2\varphi^{(i)}\,\mathrm{d}\varsigma\right)\\
&\quad=-\int_{\mathscr{F}_{\mathrm{int}}}[\![\partial_2\mathbf{H}_1]\!]_{\mathscr{F}_{\mathrm{int}}}\varphi\,\mathrm{d}\varsigma.
\end{aligned}$$

The boundary conditions for the vector $\operatorname{curl}\mathbf{H}$ further imply that the third summand in (9.28) tends to zero as $n\to\infty$. The same is true for the boundary integrals in the first summand in (9.28) with respect to $\Gamma_{1,n}^{-,(1)}$ and $\Gamma_{1,n}^{+,(2)}$. From (9.27) we altogether derive the formula

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{Q_i}(\nabla\mathbf{H}_2^{(i)})\cdot(\nabla\varphi^{(i)})\,\mathrm{d}x\\
&\quad=-\sum_{i=1}^{2}\int_{Q_i}(\Delta\mathbf{H}_2^{(i)})\varphi^{(i)}\,\mathrm{d}x-\int_{\mathscr{F}_{\mathrm{int}}}([\![(\operatorname{curl}\mathbf{H})_3]\!]_{\mathscr{F}_{\mathrm{int}}}+[\![\partial_2\mathbf{H}_1]\!]_{\mathscr{F}_{\mathrm{int}}})\varphi\,\mathrm{d}\varsigma.
\end{aligned}\tag{9.29}$$

Since W is dense in V, (9.29) holds also for all functions $\varphi\in V$. The asserted statement is now concluded by combining Lemmas 7.4 and 8.13. □

The next theorem summarizes the last results. It states the desired piecewise H^2-regularity of functions in the space X_2 from (7.19).

Theorem 9.17. *Let ε, μ, and σ satisfy (7.2). The space X_2 embeds continuously into $PH^2(Q)^6$.*

Proof. Lemmas 9.12–9.13, and 9.15–9.16 already show that X_2 is a subspace of $PH^2(Q)^6$. It consequently only remains to show the embedding property.

1) Let $(\mathbf{E}, \mathbf{H}) \in X_2$. We consider first the electric field. Lemmas 9.12–9.13 yield the estimates

$$\begin{aligned}
\|\mathbf{E}\|_{PH^2(Q)} &\le C_{E,1,2}\Big(\sum_{i=1}^{2} \Big(\big\|\Delta\mathbf{E}^{(i)}\big\|_{L^2(Q_i)} + \big\|\partial_1 \mathbf{E}_2^{(i)}\big\|_{H^1(Q_i)} + \big\|\partial_1 \mathbf{E}_3^{(i)}\big\|_{H^1(Q_i)} \\
&\qquad + \big\|\operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)})\big\|_{H_0^{1/2}(\mathscr{F}_{\mathrm{int}})} + \sum_{j=1}^{3} \big\|\operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)})\big\|_{H_0^{1/2}(\Gamma_j^{\pm,(i)})}\Big) \\
&\qquad + \|[\![\varepsilon\mathbf{E}\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}}\|_{H_0^{3/2}(\mathscr{F}_{\mathrm{int}})}\Big) \\
&\le (C_{E,1}+1)C_{E,1,2}\Big(\sum_{i=1}^{2} \Big(\big\|\Delta\mathbf{E}^{(i)}\big\|_{L^2(Q_i)} + \big\|\operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)})\big\|_{H_0^{1/2}(\mathscr{F}_{\mathrm{int}})} \\
&\qquad + \sum_{j=1}^{3} \big\|\operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)})\big\|_{H_0^{1/2}(\Gamma_j^{\pm,(i)})}\Big) \\
&\qquad + \|\operatorname{curl}\mathbf{E}\|_{PH^1(Q)} + \|[\![\varepsilon\mathbf{E}\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}}\|_{H_0^{3/2}(\mathscr{F}_{\mathrm{int}})}\Big),
\end{aligned} \tag{9.30}$$

with a constant $C_{E,1,2} > 0$, depending only on the constants $C_{E,1}$ and $C_{E,2}$ from Lemmas 9.12–9.13. As a result, it is enough to estimate the norm of $\operatorname{curl}\mathbf{E}$ in $PH^1(Q)^3$, as well as the norm of $\Delta\mathbf{E}^{(i)}$ in $L^2(Q_i)$, $i \in \{1,2\}$.

2) We put $f := \operatorname{curl}\mathbf{E}$, and define

$$g(x_1,x_2,x_3) := \begin{cases} f(x_1,x_2,x_3) & \text{if } (x_1,x_2,x_3) \in Q_2, \\ f(-x_1,x_2,x_3) & \text{if } (x_1,x_2,x_3) \in Q_1. \end{cases}$$

As $\mathbf{E}$ is piecewise H^2-regular, g belongs to $H^1(Q)^3$. Taking also the fact $(\mathbf{E}, \mathbf{H}) \in \mathcal{D}(M) \cap X_0$ into account, we conclude that $g \cdot \nu = 0$ on ∂Q, see Remark 2.5 in Chapter I of [GiRa86]. Altogether, g is also an element of $H(\operatorname{curl}, Q) \cap H_0(\operatorname{div}, Q)$. Doing a similar procedure for $f^{(1)}$ instead of $f^{(2)}$, (2.4) and the properties of μ in (7.2) imply the relations

$$\|\operatorname{curl}\mathbf{E}\|_{PH^1(Q)} \le C_T \sum_{i=1}^{2} \big\|\operatorname{curl}\operatorname{curl}\mathbf{E}^{(i)}\big\|_{L^2(Q_i)}$$

$$\leq C_T \|\mu\|_\infty \sum_{i=1}^{2} \left\| \operatorname{curl} \tfrac{1}{\mu^{(i)}} \operatorname{curl} \mathbf{E}^{(i)} \right\|_{L^2(Q_i)}. \tag{9.31}$$

3) We continue by estimating the L^2-norm of the vector $\operatorname{curl} \frac{1}{\mu^{(i)}} \operatorname{curl} \mathbf{E}^{(i)}$ on Q_i. Since the injection $\mathcal{D}(M^2) \hookrightarrow \mathcal{D}(M)$ is bounded (this can be seen with the closed graph theorem), the inequalities

$$\begin{aligned}
\left\| \begin{pmatrix} \operatorname{curl} \frac{1}{\mu^{(i)}} \operatorname{curl} \mathbf{E}^{(i)} \\ \operatorname{curl} \frac{1}{\varepsilon^{(i)}} \operatorname{curl} \mathbf{H}^{(i)} \end{pmatrix} \right\|_{L^2(Q_i)} &\leq (\|\varepsilon\|_\infty + \|\mu\|_\infty) \left\| \begin{pmatrix} \frac{1}{\varepsilon^{(i)}} \operatorname{curl} \frac{1}{\mu^{(i)}} \operatorname{curl} \mathbf{E}^{(i)} \\ \frac{1}{\mu^{(i)}} \operatorname{curl} \frac{1}{\varepsilon^{(i)}} \operatorname{curl} \mathbf{H}^{(i)} \end{pmatrix} \right\|_{L^2(Q_i)} \\
&\leq (\|\varepsilon\|_\infty + \|\mu\|_\infty) \Big(\left\| \tfrac{\sigma}{\varepsilon} \right\|_\infty \left\| M \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{L^2(Q)} \\
&\qquad + \left\| \begin{pmatrix} \frac{\sigma^2}{\varepsilon^2} \mathbf{E}^{(i)} - \frac{\sigma}{\varepsilon^2} \operatorname{curl} \mathbf{H}^{(i)} - \frac{1}{\varepsilon} \operatorname{curl} \frac{1}{\mu^{(i)}} \operatorname{curl} \mathbf{E}^{(i)} \\ -\frac{1}{\mu} \operatorname{curl}(-\frac{\sigma^{(i)}}{\varepsilon^{(i)}} \mathbf{E}^{(i)} + \frac{1}{\varepsilon^{(i)}} \operatorname{curl} \mathbf{H}^{(i)}) \end{pmatrix} \right\|_{L^2(Q_i)} \Big) \\
&\leq (\|\varepsilon\|_\infty + \|\mu\|_\infty)(1 + \left\| \tfrac{\sigma}{\varepsilon} \right\|_\infty) \Big(\left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{\mathcal{D}(M)} + \left\| M^2 \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\| \Big) \\
&\leq C_3 \left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{\mathcal{D}(M^2)}
\end{aligned} \tag{9.32}$$

follow for $i \in \{1, 2\}$ with a uniform constant $C_3 = C_3(\varepsilon, \mu, \sigma, Q) > 0$. Furthermore, the formula $\operatorname{curl} \operatorname{curl} \mathbf{E}^{(i)} = -\Delta \mathbf{E}^{(i)} + \nabla \operatorname{div} \mathbf{E}^{(i)}$ implies the estimate

$$\left\| \Delta \mathbf{E}^{(i)} \right\|_{L^2(Q_i)} \leq \|\mu\|_\infty \left\| \operatorname{curl} \tfrac{1}{\mu^{(i)}} \operatorname{curl} \mathbf{E}^{(i)} \right\|_{L^2(Q_i)} + \tfrac{1}{\delta} \left\| \operatorname{div}(\varepsilon^{(i)} \mathbf{E}^{(i)}) \right\|_{H^1(Q_i)}. \tag{9.33}$$

In view of (9.30)–(9.33), we arrive at the desired inequality

$$\|\mathbf{E}\|_{PH^2(Q)} \leq C_4 \left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{X_2}$$

with a uniform constant $C_4 = C_4(\varepsilon, \mu, \sigma, Q) > 0$, see (7.20) for the definition of the norm in X_2.

4) Concerning the magnetic field, Lemmas 9.15 and 9.16 imply the relation

$$\|\mathbf{H}\|_{PH^2(Q)} \leq C_{H,1,2} \left(\sum_{i=1}^{2} \left\| \Delta \mathbf{H}^{(i)} \right\|_{L^2(Q_i)} + \|\operatorname{curl} \mathbf{H}\|_{PH^1(Q)} \right),$$

with a uniform constant $C_{H,1,2}$, that depends only on the constants $C_{H,1}$ and $C_{H,2}$ from Lemmas 9.15 and 9.16. Employing the constraint $\operatorname{div}(\mu \mathbf{H}) = 0$, similar arguments as in parts 2)–3) yield the corresponding uniform estimate

$$\|\mathbf{H}\|_{PH^2(Q)} \leq C_4 \left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{X_2}, \tag{9.34}$$

after a uniform modification of the constant C_4. $\square$

Similar to the space $X_1 = \mathcal{D}(M) \cap X_0$, we observe in the next remark that the definition of X_2 is in a certain sense independent of the coefficient function arising in the jump condition for the electric field. This means that only qualitatively regular information flow through the interface is needed to ensure regularity. This remark becomes crucial for the wellposedness of the Maxwell system (7.1) in X_2.

Remark 9.18. Let $\tilde{\varepsilon}$ and ε be two positive functions on Q, that are piecewise constant on the cuboids Q_1 and Q_2. Define the space

$$\begin{aligned}\tilde{X}_2 := \{(\mathbf{E},\mathbf{H}) \in \mathcal{D}(M^2) \cap X_0 \mid \ & \operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)}) \in H^1_{00}(Q_i) \text{ for } i \in \{1,2\}, \\ & [\![\tilde{\varepsilon}\mathbf{E} \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} \in H^{3/2}_0(\mathscr{F}_{\text{int}})\},\end{aligned}$$

compare the definition of X_2 in (7.19). We claim that $\tilde{X}_2 = X_2$. The proof of this identity is analogous to the one in Remark 9.9.

Let $(\mathbf{E},\mathbf{H}) \in \tilde{X}_2$. Since ε and $\tilde{\varepsilon}$ are piecewise positive constants, we infer that the function $\operatorname{div}(\tilde{\varepsilon}^{(i)}\mathbf{E}^{(i)})$ belongs to $H^1_{00}(Q_i)$ for $i \in \{1,2\}$. In view of Remark 9.9 and Proposition 9.7, the vectors $\mathbf{E}$ and $\operatorname{curl}\mathbf{E}$ then belong to $PH^1(Q)^3$. Since both coefficient functions satisfy the same assumptions, the proofs of Lemmas 9.12–9.13 still work. This means that the vector $\mathbf{E}$ is an element of $PH^2(Q)^3$. The boundary condition for $\mathbf{E}$ then implies that $\mathbf{E}_1$ belongs to $PH^2(Q) \cap PH^1_{\Gamma_2 \cup \Gamma_3}(Q)$. Employing additionally Lemma 7.3, we deduce that the jump $[\![\varepsilon\mathbf{E}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}$ belongs to $H^{3/2}_0(\mathscr{F}_{\text{int}})$. Altogether, $(\mathbf{E},\mathbf{H})$ is an element of X_2. Similar arguments show the reverse inclusion $X_2 \subseteq \tilde{X}_2$. ♢

9.3. Wellposedness of the Maxwell equations in piecewise regular spaces

The main result of Section 9.2 shows piecewise H^2-regularity for the space X_2, see Theorem 9.17. So far, however, we have not deduced whether the solutions of the Maxwell system (7.1) stay within X_2 for sufficiently regular initial data. In other words, it is not clear yet whether the Maxwell equations are wellposed on X_2. In fact, we deduce the wellposedness of (7.1) in X_2 by means of semigroup theory in this section.

For the error analysis in Chapter 10, the wellposedness of (7.1) in a subspace of $PH^1(Q)^6$, namely X_1, is also an important issue. In fact, we employ this property to control the error propagation. It moreover turns out that the arguments for the H^1- and the H^2-regularity are similar.

As a starting point, we show that the operator M_0 is not only the restriction of M to X_0, but also its part in this space. The statement corresponds to relation (2.5) in [EiSc18]. Recall here the spaces X_0 and X_1 from (7.15) and (7.17).

Lemma 9.19. *Let ε, μ, and σ satisfy (7.2). The following items are true.*

a) The identity $\mathcal{D}(M_0^k) = \mathcal{D}(M^k) \cap X_0$ is valid for all $k \in \mathbb{N}$, and $M_0(\mathcal{D}(M_0))$ is a subset of X_0. In particular, M_0 is the part of M in X_0.

b) The graph norm of M_0 defines an equivalent norm on X_1.

Proof. a) Employing our H^1-regularity results, we transfer the arguments from the proof of (2.5) in [EiSc18] to our current setting.

We first note that the identity $\mathcal{D}(M_0) = \mathcal{D}(M) \cap X_0$ is true by definition (7.17). To show that M_0 is the part of M in X_0, we demonstrate that $M(\mathcal{D}(M_0))$ is a subspace of X_0. Recall to that end (7.15) for the definition of X_0. Let $(\mathbf{E}, \mathbf{H}) \in \mathcal{D}(M) \cap X_0$. Then, $(M(\mathbf{E}, \mathbf{H}))_1 = -\frac{\sigma}{\varepsilon}\mathbf{E} + \frac{1}{\varepsilon}\operatorname{curl}\mathbf{H}$, and thus

$$\operatorname{div}(\varepsilon^{(i)} M(\mathbf{E}, \mathbf{H})_1^{(i)}) = -\tfrac{\sigma^{(i)}}{\varepsilon^{(i)}} \operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)}) \in L^2(Q_i)$$

for $i \in \{1, 2\}$. Combining the boundary condition $\mathbf{E}_1 = 0$ on $\Gamma_2 \cup \Gamma_3$ with Proposition 9.8 and Lemma 7.3, we infer that the jump $[\![\sigma\mathbf{E} \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}$ belongs to the trace space $H_0^{1/2}(\mathscr{F}_{\text{int}})$. As the vector $\operatorname{curl}\mathbf{H}$ is contained in the divergence space $H(\operatorname{div}, Q)$, we furthermore obtain the identity $[\![\operatorname{curl}\mathbf{H} \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = 0$ by means of the divergence formula. Consequently, the divergence and the transmission conditions in X_0 are satisfied for $M(\mathbf{E}, \mathbf{H})_1$. The remaining magnetic conditions in X_0 are also satisfied for $M(\mathbf{E}, \mathbf{H})$, because the curl-operator maps $H_0(\operatorname{curl}, Q)$ into $H_0(\operatorname{div}, Q)$, see Remark 2.5 in Section I.2 of [GiRa86]. Altogether, the space $M(\mathcal{D}(M) \cap X_0) = M(\mathcal{D}(M_0))$ is contained in X_0. Induction over $k \in \mathbb{N}$ now implies the identity $\mathcal{D}(M_0^k) = \mathcal{D}(M^k) \cap X_0$. (In fact, the inclusion $\mathcal{D}(M_0^{k+1}) \subseteq \mathcal{D}(M^{k+1}) \cap X_0$ is clear, as M_0 is the restriction of M to X_0. To show the reverse relation, let $u \in \mathcal{D}(M^{k+1}) \cap X_0$. Then, $y := M^{k-1}u$ is by induction hypothesis an element of X_0. Our above reasoning now shows that $My = M^k u$ is contained in X_0. As $M^k u$ belongs also to $\mathcal{D}(M)$, we conclude $u \in \mathcal{D}(M_0^{k+1})$.)

b) Let $(u, v) \in X_1 = \mathcal{D}(M) \cap X_0$. We estimate

$$\begin{aligned}\sum_{i=1}^{2} \left\|\operatorname{div}(\varepsilon^{(i)} M \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)_1^{(i)})\right\|_{L^2(Q_i)} &= \sum_{i=1}^{2} \left\|\operatorname{div}(\sigma^{(i)} u^{(i)})\right\|_{L^2(Q_i)} \\ &\leq \tfrac{\|\sigma\|_\infty}{\delta} \sum_{i=1}^{2} \left\|\operatorname{div}(\varepsilon^{(i)} u^{(i)})\right\|_{L^2(Q_i)}.\end{aligned}$$

Employing that $\operatorname{curl} v$ is contained in $H(\operatorname{div}, Q)$, we infer $[\![\operatorname{curl} v \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = 0$, and with Lemma 7.4 and Proposition 9.8 we derive the relations

$$\begin{aligned}\|[\![\varepsilon(M \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right))_1 \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}\|_{H_0^{1/2}(\mathscr{F}_{\text{int}})} &= \|[\![\sigma u \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}\|_{H_0^{1/2}(\mathscr{F}_{\text{int}})} \\ &\leq C_{\text{int}} \|\sigma\|_\infty \|u\|_{PH^1(Q)} \\ &\leq C_e C_{\text{int}} \|\sigma\|_\infty \|\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\|_{X_1},\end{aligned}$$

where C_{int} and C_e denote uniform constants from Lemma 7.4 and Proposition 9.8. The definition of the norms $\|\cdot\|_{X_0}$ and $\|\cdot\|_{X_1}$ in (7.16) and (7.18) now shows the desired inequality

$$\|\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\|_{\mathcal{D}(M_0)} = \|\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\|_{X_0} + \|M\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\|_{X_0} \leq \|\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\|_{X_1} + \left(\tfrac{\|\sigma\|_\infty}{\delta} + C_e C_{\text{int}} \|\sigma\|_\infty\right) \|\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\|_{X_1}.$$

The reverse inequality is immediate. Altogether, the desired norm equivalence is established. □

Remark 9.20. If X_1 is equipped with the norm

$$\left|\!\left|\!\left|\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}\right|\!\right|\!\right|_{X_1}^2 := \left\|\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}\right\|_{X_0}^2 + \left\|M\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}\right\|_{X_0}^2, \qquad \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in X_1,$$

which is equivalent to the norm $\|\cdot\|_{X_1}$ due to Lemma 9.19, it becomes a Hilbert space. ♢

The next lemma transfers a part of Proposition 3.2 in [EiSc17] to our setting. Similar to the proof of Lemma 9.19, the regularity result for the space X_2 plays an important role.

Lemma 9.21. *Let ε, μ, and σ satisfy (7.2). The operator M_2 has the domain $\mathcal{D}(M_2) = \mathcal{D}(M^3) \cap X_2 = \mathcal{D}(M_0^3) \cap X_2$.*

Proof. Having Lemma 9.19 in mind, it suffices to check the first identity. Furthermore, the definition of X_2 immediately implies that $\mathcal{D}(M_2)$ is a subset of $\mathcal{D}(M^3) \cap X_2$. Consequently, it is enough to show the inclusion $\mathcal{D}(M^3) \cap X_2 \subseteq \mathcal{D}(M_2)$.

Let $(\mathbf{E}, \mathbf{H}) \in \mathcal{D}(M^3) \cap X_2$. We set

$$\begin{pmatrix} u \\ v \end{pmatrix} := M\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\frac{\sigma}{\varepsilon}\mathbf{E} + \frac{1}{\varepsilon}\operatorname{curl}\mathbf{H} \\ -\frac{1}{\mu}\operatorname{curl}\mathbf{E} \end{pmatrix}.$$

By definition, the vector (u, v) is an element of the domain $\mathcal{D}(M^2)$. We compute

$$\operatorname{div}(\varepsilon^{(i)} u^{(i)}) = -\sigma^{(i)} \operatorname{div}(\mathbf{E}^{(i)}) \in H^1_{00}(Q_i), \qquad \operatorname{div}(\mu^{(i)} v^{(i)}) = 0,$$

for $i \in \{1, 2\}$. By definition of $\mathcal{D}(M)$, $\mathbf{E}$ is continuous in tangential direction across $\mathscr{F}_{\text{int}}$. Lemma 7.1 then implies the identities

$$[\![\mu v \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = -(\partial_2 \mathbf{E}_3^{(2)} - \partial_3 \mathbf{E}_2^{(2)})|_{\mathscr{F}_{\text{int}}} + (\partial_2 \mathbf{E}_3^{(1)} - \partial_3 \mathbf{E}_2^{(1)})|_{\mathscr{F}_{\text{int}}} = 0. \tag{9.35}$$

Similarly, the boundary conditions of $\mathbf{E}$ yield $\mu v \cdot \nu = -\operatorname{curl}\mathbf{E} \cdot \nu = 0$ on ∂Q. As a result, v satisfies all required magnetic conditions.

Concerning the remaining transmission conditions for u, we note that the same arguments as in (9.35) show that also the field $\operatorname{curl}\mathbf{H}$ is continuous in normal direction at the interface $\mathscr{F}_{\text{int}}$. Employing that $\mathbf{E}_1$ belongs to the space $PH^2(Q) \cap PH^1_{\Gamma_2\cup\Gamma_3}(Q)$, we can thus conclude that $[\![\varepsilon u \cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = [\![-\sigma\mathbf{E}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}$ is an element of $H_0^{3/2}(\mathscr{F}_{\text{int}})$, see Lemma 7.3. Altogether, (u, v) belongs to X_2, implying that $(\mathbf{E}, \mathbf{H})$ is contained in $\mathcal{D}(M_2)$. $\square$

The next proposition yields the classical wellposedness of the Maxwell system (7.1) in X_1. The main tool is here semigroup theory. During the proof, we transfer parts of the proof for Proposition 2.3 from [EiSc18] to our current setting of discontinuous coefficients. Due to the discontinuous behavior of the conductivity σ, our proof is, however, more involved. We in particular have to control the jump of the first component of the electric field in normal direction across the interface. Among other things, we therefore use the Yosida approximation techniques from the proof of the Hille-Yosida Theorem II.3.5 in [EnNa00]. To show the crucial bounds for the resolvent of the Maxwell operator M, we then apply a scaling technique, which eventually leads to the exponential growth factor in the final bound of the semigroup $(\mathrm{e}^{tM_1})_{t\geq 0}$.

Proposition 9.22. *Let ε, μ, and σ satisfy (7.2). The part M_1 of M generates a C_0-semigroup $(\mathrm{e}^{tM_1})_{t\geq 0}$ on X_1. The family $(\mathrm{e}^{tM_1})_{t\geq 0}$ is the restriction of $(\mathrm{e}^{tM})_{t\geq 0}$ to X_1, and it satisfies the growth bound*

$$\left\|\mathrm{e}^{tM_1}\right\|_{\mathcal{L}(X_1)} \leq \mathrm{e}^{C_{g,1}t}, \qquad t \geq 0,$$

with a constant $C_{g,1} > 0$ that depends only on ε, μ, σ, and Q.

Proof. 1) Employing the theory of subspace semigroups, see Paragraph II.2.3 in [EnNa00] for instance, all desired statements (except the estimate) follow by showing that the family $(\mathrm{e}^{tM})_{t\geq 0}$ restricts to a strongly continuous semigroup on X_1. This is here concluded by considering the scaled family $(\mathrm{e}^{t(M-\omega)})_{t\geq 0}$ for a fixed number $\omega \geq 0$, that is determined later. While we consider $\omega > 0$ to obtain the desired generator property for M_1, the asserted estimate follows by considering the special choice $\omega = 0$, see part 6).

We show first that the semigroup $(\mathrm{e}^{t(M-\omega)}|_{X_1})_{t\geq 0}$ restricts to a family of operators on X_1. As a consequence of semigroup theory, the inclusion $\mathrm{e}^{t(M-\omega)}(\mathcal{D}(M)) \subseteq \mathcal{D}(M)$ is valid for $t \geq 0$. Concerning the conditions for the magnetic field, the arguments in the proof of Proposition 2.3 in [EiSc18] further show that the space

$$X_{\text{mag}} := \{(u, v) \in X \mid \operatorname{div}(\mu v) = 0,\ \mu v \cdot \nu = 0 \text{ on } \partial Q\}$$

is invariant under the resolvent maps $R(\lambda, M - \omega)$ for $\lambda > 0$. The same is true for the family $(\mathrm{e}^{t(M-\omega)})_{t\geq 0}$.

2) Next, we treat the remaining conditions for the electric field. For that purpose, we employ that the semigroup $(\mathrm{e}^{t(M-\omega)})_{t\geq 0}$ can be approximated by means of the resolvents of $M-\omega$. Consequently, we show first that the resolvent operator $R(\lambda, M-\omega)$ leaves X_1 for $\lambda > 0$ invariant.

Let $(\tilde{u}, \tilde{v}) \in X_1$, $\lambda > 0$, and put $(u, v) := R(\lambda, M-\omega)(\tilde{u}, \tilde{v})$. The definition of M in (7.12) then implies the relation

$$\tilde{u} = (\lambda + \omega + \tfrac{\sigma}{\varepsilon})u - \tfrac{1}{\varepsilon} \operatorname{curl} v \tag{9.36}$$

on Q. Taking first the divergence of this identity, we infer the formula

$$\operatorname{div}(\varepsilon^{(i)} \tilde{u}^{(i)}) = \Big((\lambda + \omega)\varepsilon^{(i)} + \sigma^{(i)}\Big) \operatorname{div}(u^{(i)}) \tag{9.37}$$

in $H^{-1}(Q_i)$. This means that the function $\operatorname{div}(\varepsilon^{(i)} u^{(i)})$ is an element of $L^2(Q_i)$, compare to the proof of Proposition 2.3 in [EiSc18]. Using (9.36), we also obtain the identity

$$[\![\varepsilon \tilde{u} \cdot \nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}} = [\![((\lambda + \omega)\varepsilon + \sigma)u \cdot \nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}} - [\![\operatorname{curl} v \cdot \nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}} \tag{9.38}$$

on the interface $\mathscr{F}_{\mathrm{int}}$. Since the vector $\operatorname{curl} v$ is contained in $H(\operatorname{div}, Q)$, the divergence-theorem shows that $[\![\operatorname{curl} v \cdot \nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}} = 0$. As a result, we have derived that the jump $[\![((\lambda + \omega)\varepsilon + \sigma)u \cdot \nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}}$ is contained in the trace space $H_0^{1/2}(\mathscr{F}_{\mathrm{int}})$. Since λ is positive and (u, v) belongs to $\mathcal{D}(M)$, Remark 9.9 yields that the function $[\![\varepsilon u \cdot \nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}}$ belongs to $H_0^{1/2}(\mathscr{F}_{\mathrm{int}})$.

Altogether, (u, v) is an element of $\mathcal{D}(M^2) \cap X_0 = \mathcal{D}(M_0^2)$. In particular, the resolvent $R(\lambda, M-\omega)$ leaves X_1 invariant.

3) Let $t > 0$ be fixed. To approximate the semigroup $(\mathrm{e}^{\tilde{t}(M-\omega)})_{\tilde{t}\geq 0}$, we show that the family of operators $\{(\frac{n}{t} R(\frac{n}{t}, M-\omega))^n \mid n \in \mathbb{N}\}$ is uniformly bounded on X_1. Recall to this end that $(\mathrm{e}^{\tilde{t}M})_{\tilde{t}\geq 0}$ is contractive on X by Proposition 7.8.

Let again $(\tilde{u}, \tilde{v}) \in X_1$, and put $(u, v) := \frac{n}{t} R(\frac{n}{t}, M-\omega)(\tilde{u}, \tilde{v})$. We have just seen that (u, v) belongs to $\mathcal{D}(M_0^2)$. Employing the resolvent bounds for generators of rescaled semigroups, we infer the relations

$$\begin{aligned} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\mathcal{D}(M)} &= \left\| \tfrac{n}{t} R(\tfrac{n}{t}, M-\omega) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\| + \left\| \tfrac{n}{t} R(\tfrac{n}{t}, M-\omega) M \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\| \\ &\leq \frac{\frac{n}{t}}{\omega + \frac{n}{t}} \left(\left\| \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\| + \left\| M \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\| \right) = \frac{1}{1 + \frac{t}{n}\omega} \left\| \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\|_{\mathcal{D}(M)}. \end{aligned} \tag{9.39}$$

Similar to (9.37), we obtain the equation

$$\tfrac{n}{t} \operatorname{div}(\varepsilon^{(i)} \tilde{u}^{(i)}) = \Big(\tfrac{n}{t} + \omega + \tfrac{\sigma^{(i)}}{\varepsilon^{(i)}}\Big) \operatorname{div}(\varepsilon^{(i)} u^{(i)})$$

on Q_i for $i \in \{1,2\}$, resulting in the relations

$$\begin{aligned}\left\|\operatorname{div}(\varepsilon^{(i)}u^{(i)})\right\|_{L^2(Q_i)} &= \frac{\frac{n}{t}}{\frac{n}{t}+\frac{\sigma^{(i)}}{\varepsilon^{(i)}}+\omega}\left\|\operatorname{div}(\varepsilon^{(i)}\tilde{u}^{(i)})\right\|_{L^2(Q_i)}\\ &\leq \frac{1}{1+\frac{t}{n}\omega}\left\|\operatorname{div}(\varepsilon^{(i)}\tilde{u}^{(i)})\right\|_{L^2(Q_i)}.\end{aligned} \tag{9.40}$$

Arguing as in part 2), we also derive the formula

$$[\![\tfrac{n}{t}\varepsilon\tilde{u}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = [\![((\tfrac{n}{t}+\omega)\varepsilon+\sigma)u\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}},$$

being equivalent to

$$[\![\varepsilon\tilde{u}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}} = [\![(1+\omega\tfrac{t}{n}+\tfrac{\sigma}{\frac{n}{t}\varepsilon})\varepsilon u\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}. \tag{9.41}$$

By means of the triangle inequality and the trace estimate from Lemma 7.4, we infer the relations

$$\begin{aligned}(1+\tfrac{\omega t}{n})\left\|[\![\varepsilon u\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}\right\|_{H_0^{1/2}(\mathscr{F}_{\text{int}})} &\leq \left\|[\![\varepsilon\tilde{u}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}\right\|_{H_0^{1/2}(\mathscr{F}_{\text{int}})}\\ &\quad+\tfrac{\|\sigma\|_\infty t}{n}\sum_{i=1}^{2}\left\|u^{(i)}\cdot\nu_{\mathscr{F}_{\text{int}}}\right\|_{H_0^{1/2}(\mathscr{F}_{\text{int}})}\\ &\leq \left\|[\![\varepsilon\tilde{u}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}\right\|_{H_0^{1/2}(\mathscr{F}_{\text{int}})}+\tfrac{\|\sigma\|_\infty t}{n}C_{\text{int}}\left\|u_1\right\|_{PH^1(Q)}.\end{aligned}$$

The embedding of $X_1 = \mathcal{D}(M_0)$ into $PH^1(Q)^6$ from Proposition 9.8 yields now the inequalities

$$\begin{aligned}(1+\tfrac{\omega t}{n})\left\|[\![\varepsilon u\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}\right\|_{H_0^{1/2}(\mathscr{F}_{\text{int}})} &\leq \left\|[\![\varepsilon\tilde{u}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}\right\|_{H_0^{1/2}(\mathscr{F}_{\text{int}})}\\ &\quad+\tfrac{\|\sigma\|_\infty t}{n}C_eC_{\text{int}}\left\|\begin{pmatrix}u\\v\end{pmatrix}\right\|_{X_1},\end{aligned} \tag{9.42}$$

where C_e is a uniform constant from the embedding in Proposition 9.8. Altogether, estimates (9.39), (9.40), and (9.42) imply the relation

$$\left\|\tfrac{n}{t}R(\tfrac{n}{t},M-\omega)\begin{pmatrix}\tilde{u}\\\tilde{v}\end{pmatrix}\right\|_{X_1} \leq \frac{1}{1+\frac{\omega t}{n}}\left\|\begin{pmatrix}\tilde{u}\\\tilde{v}\end{pmatrix}\right\|_{X_1}+\tfrac{C_0t}{n}\left\|\tfrac{n}{t}R(\tfrac{n}{t},M-\omega)\begin{pmatrix}\tilde{u}\\\tilde{v}\end{pmatrix}\right\|_{X_1},$$

being equivalent to

$$\left\|\tfrac{n}{t}R(\tfrac{n}{t},M-\omega)\begin{pmatrix}\tilde{u}\\\tilde{v}\end{pmatrix}\right\|_{X_1} \leq \frac{1}{(1-\frac{C_0t}{n})(1+\frac{\omega t}{n})}\left\|\begin{pmatrix}\tilde{u}\\\tilde{v}\end{pmatrix}\right\|_{X_1}, \tag{9.43}$$

with the uniform constant $C_0 := \|\sigma\|_\infty C_e C_{\text{int}} \geq 0$. The latter number does in particular not depend on ω. As a result, we arrive at the uniform bound

$$\left\|\left(\tfrac{n}{t}R(\tfrac{n}{t}, M-\omega)\right)^n\right\|_{\mathcal{L}(X_1)} \leq \frac{1}{(1-\frac{C_0 t}{n})^n} \leq e^{C_0 t} \tag{9.44}$$

for all $n \in \mathbb{N}$, $t > 0$ with $n/t > C_0$. Taking $\omega := 2C_0$ and $t = 1$ in (9.43), we moreover derive the estimate

$$\left\| nR(n, M-\omega)\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}\right\|_{X_1} \leq \left\|\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}\right\|_{X_1} \tag{9.45}$$

for $n \geq \omega$.

4) We deduce now by means of the results from parts 2) and 3), that the family $(\mathrm{e}^{t(M-\omega)})_{t\geq 0}$ leaves X_1 invariant. Recall that X_1 is a Hilbert space with respect to the equivalent norm in Remark 9.20. Since the sequence $\left((\frac{n}{t}R(\frac{n}{t}, M-\omega))^n \left(\begin{smallmatrix}\tilde{u}\\ \tilde{v}\end{smallmatrix}\right)\right)_n$ is bounded due to part 3), the Theorem of Banach-Alaoglu provides us with a weakly converging subsequence. Let $(\check{u}, \check{v})$ be its limit in X_1. Approximation theory for semigroups implies also that the same subsequence tends to $\mathrm{e}^{t(M-\omega)}\left(\begin{smallmatrix}\tilde{u}\\ \tilde{v}\end{smallmatrix}\right)$ in X, see Corollary III.5.5 in [EnNa00]. Since the embedding of X_1 into X is bounded, we infer hat $\left(\begin{smallmatrix}\check{u}\\ \check{v}\end{smallmatrix}\right) = \mathrm{e}^{t(M-\omega)}\left(\begin{smallmatrix}\tilde{u}\\ \tilde{v}\end{smallmatrix}\right)$ is an element of X_1, and that $\mathrm{e}^{t(M-\omega)}$ leaves X_1 invariant.

5) It remains to prove that the semigroup $(\mathrm{e}^{t(M-\omega)}|_{X_1})_{t\geq 0}$ is strongly continuous on X_1. Let $\left(\begin{smallmatrix}u(t)\\ v(t)\end{smallmatrix}\right) := \mathrm{e}^{t(M-\omega)}\left(\begin{smallmatrix}\tilde{u}\\ \tilde{v}\end{smallmatrix}\right)$ for $t \geq 0$. Since the vector $(\tilde{u}, \tilde{v})$ belongs to $X_1 \subseteq \mathcal{D}(M)$, we infer that the mapping $(u, v) : [0, \infty) \to \mathcal{D}(M)$ is continuous. For the divergence of u, we argue similar to part 2) and the proof of Proposition 2.3 in [EiSc18]. Employing that (u, v) solves the Maxwell equations with $\mathbf{J} = 0$ and perturbation $-\omega I$, we obtain the formula

$$\partial_t u(t) = \tfrac{1}{\varepsilon}\operatorname{curl}(v(t)) - (\tfrac{\sigma}{\varepsilon} + \omega)u(t), \qquad t \geq 0. \tag{9.46}$$

Taking the divergence of this equation, we arrive at the relation $\partial_t \operatorname{div}(\varepsilon^{(i)}u^{(i)}(t)) = -(\frac{\sigma^{(i)}}{\varepsilon^{(i)}} + \omega)\operatorname{div}(\varepsilon^{(i)}u^{(i)}(t))$, and thus

$$\partial_t\Big(\mathrm{e}^{(\frac{\sigma^{(i)}}{\varepsilon^{(i)}}+\omega)t}\operatorname{div}(\varepsilon^{(i)}u^{(i)}(t))\Big) = 0$$

in $L^2(Q_i)$. Integrating this formula yields the identity

$$\operatorname{div}(\varepsilon^{(i)}u^{(i)}(t)) = \mathrm{e}^{-(\frac{\sigma^{(i)}}{\varepsilon^{(i)}}+\omega)t}\operatorname{div}(\varepsilon^{(i)}\tilde{u}^{(i)}), \qquad t \geq 0, \tag{9.47}$$

on Q_i. As a result, the mapping $[0, \infty) \to L^2(Q_i)$, $t \mapsto \operatorname{div}(\varepsilon^{(i)}u^{(i)}(t))$, is smooth for $i \in \{1, 2\}$. We further conclude that the function $[0, \infty) \to H(\operatorname{div}, Q_i)$, $t \mapsto$

$\varepsilon^{(i)}u^{(i)}(t)$, is continuously differentiable. Due to the continuity of the normal trace operator, we deduce from (9.46) the relation

$$\partial_t[\![\varepsilon u(t)\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}} = [\![\mathrm{curl}(v(t))\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}} - [\![(\sigma+\varepsilon\omega)u(t)\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}}, \qquad t\ge 0,$$

in $H^{-1/2}(\mathscr{F}_{\mathrm{int}})$. As before, the first summand on the right hand side vanishes, and we obtain by integration

$$[\![\varepsilon u(t)\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}} = [\![\varepsilon\tilde{u}\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}} - \int_0^t [\![(\sigma+\omega\varepsilon)u(s)\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}}\,\mathrm{d}s \tag{9.48}$$

in $H^{-1/2}(\mathscr{F}_{\mathrm{int}})$.

6) Let $T>0$. It suffices now to show that the function $s\mapsto[\![(\sigma+\omega\varepsilon)u(s)\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}}$ belongs to $L^2([0,T],H_0^{1/2}(\mathscr{F}_{\mathrm{int}}))$. Formula (9.48) will then imply that the mapping $[0,T]\to H_0^{1/2}(\mathscr{F}_{\mathrm{int}})$, $t\mapsto[\![\varepsilon u(t)\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}}$, is continuous. Our results from above will furthermore show that the restriction of the family $(\mathrm{e}^{t(M-\omega)}|_{X_1})_{t\ge0}$ is strongly continuous on X_1.

Define the linear operators

$$V(k) := kR(k,M-\omega), \quad M_k := k(V(k)-I), \qquad k\in\mathbb{N}, \tag{9.49}$$

which are both bounded on X and on X_1. Note that the sequence $(M_k)_k$ forms the classical Yosida approximations of $M-\omega I$ that converge on $\mathcal{D}(M)$ pointwise to $M-\omega I$ in the norm of X. Moreover, the single sequence elements M_k generate contractive semigroups $(\mathrm{e}^{tM_k})_{t\ge0}$, denoted by $\mathscr{T}_k$. The latter tend to $(\mathrm{e}^{t(M-\omega)})_{t\ge0}$ uniformly on compact time intervals, see the proof of the Hille-Yosida Theorem II.3.5 in [EnNa00]. In particular, the sequence $(\mathscr{T}_k\left(\begin{smallmatrix}\tilde{u}\\ \tilde{v}\end{smallmatrix}\right))_k$ converges in $L^2([0,T],X)$ to $(\mathrm{e}^{t(M-\omega)}\left(\begin{smallmatrix}\tilde{u}\\ \tilde{v}\end{smallmatrix}\right))_{t\ge0}$.

Since each operator M_k is also bounded on X_1, it generates a strongly continuous semigroup on X_1, which coincides with $\mathscr{T}_k|_{X_1}$. (This can be seen by means of the exponential series representation for a semigroup generated by a bounded operator, for instance.)

Let $k\in\mathbb{N}$ with $k\ge\omega=2C_0$. Employing (9.45), we estimate

$$\begin{aligned}\left\|\mathrm{e}^{tM_k}\right\|_{\mathcal{L}(X_1)} &\le \mathrm{e}^{-tk}\left\|\mathrm{e}^{tkV(k)}\right\|_{\mathcal{L}(X_1)} \le \mathrm{e}^{-tk}\sum_{n=0}^{\infty}\tfrac{(tk)^n\|(kR(k,M-\omega))^n\|_{\mathcal{L}(X_1)}}{n!}\\ &\le \mathrm{e}^{-tk}\sum_{n=0}^{\infty}\tfrac{(tk)^n}{n!} = 1\end{aligned} \tag{9.50}$$

for $t\ge0$. As a result, the sequence $(\mathscr{T}_k\left(\begin{smallmatrix}\tilde{u}\\ \tilde{v}\end{smallmatrix}\right))_k$ is uniformly bounded in $L^2([0,T],X_1)$, and the Theorem of Banach-Alaoglu yields a subsequence $(\mathscr{T}_{k_l}\left(\begin{smallmatrix}\tilde{u}\\ \tilde{v}\end{smallmatrix}\right))_l$, converging weakly to a mapping $\mathscr{T}\left(\begin{smallmatrix}\tilde{u}\\ \tilde{v}\end{smallmatrix}\right)$ in $L^2([0,T],X_1)$. Due to the continuous embedding of X_1 into X, the same weak convergence statement is valid in $L^2([0,T],X)$. The above arguments, however, show that $(\mathscr{T}_{k_l}\left(\begin{smallmatrix}\tilde u\\ \tilde v\end{smallmatrix}\right))_l$ converges already in norm to $\left(\begin{smallmatrix}u\\ v\end{smallmatrix}\right)$ in $L^2([0,T],X)$. Consequently, $\mathscr{T}\left(\begin{smallmatrix}\tilde u\\ \tilde v\end{smallmatrix}\right) = \left(\begin{smallmatrix}u\\ v\end{smallmatrix}\right)$ is a function in $L^2([0,T],X_1)$. Lemma 7.4 and Proposition 9.8 further imply that the trace-jump-mapping $X_1 \to H_0^{1/2}(\mathscr{F}_{\text{int}})$, $(\mathbf{E},\mathbf{H}) \mapsto [\![(\sigma+\omega\varepsilon)\mathbf{E}\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}$, is continuous. The function $[\![(\sigma+\omega\varepsilon)u\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}$ thus belongs to $L^2([0,T],H_0^{1/2}(\mathscr{F}_{\text{int}}))$.

Altogether, we have derived that the family $(\mathrm{e}^{t(M-\omega)}|_{X_1})_{t\geq 0}$ is also a strongly continuous semigroup on X_1, being generated by M_1. The asserted growth bound follows from (9.43) by choosing $\omega = 0$, $\lambda = \frac{n}{t} > 0$, and employing standard generation theorems for semigroups, see Theorem II.3.8 in [EnNa00] for instance. □

In almost the same way, we obtain an associated result in the space X_2 from (7.19).

Proposition 9.23. *Let ε,μ, and σ satisfy (7.2). The restricted family $(\mathrm{e}^{tM}|_{X_2})_{t\geq 0}$ forms a strongly continuous semigroup on X_2 with generator M_2. We denote it by $(\mathrm{e}^{tM_2})_{t\geq 0}$. The semigroup can be bounded in operatornorm by*

$$\left\|\mathrm{e}^{tM_2}\right\|_{\mathcal{L}(X_2)} \leq \mathrm{e}^{C_{g,2}t}, \qquad t \geq 0,$$

with a positive constant $C_{g,2}$, that depends only on ε,μ,σ, and Q.

Proof. Due to the strong similarity with the proof of Proposition 9.22, we only sketch the common parts. The first two steps of this proof use again arguments from the proof of Proposition 2.3 in [EiSc18]. As above, it suffices to show that the family $(\mathrm{e}^{tM}|_{X_2})_{t\geq 0}$ leaves X_2 invariant, and that it is strongly continuous on it. To that end, we consider another time the scaled family $(\mathrm{e}^{t(M-\omega)})_{t\geq 0}$ for some fixed $\omega \geq 0$.

1) The arguments from part 1) of the proof for Proposition 9.22 imply here that the space $\mathcal{D}(M^2)\cap X_{\text{mag}}$ is invariant under $(\mathrm{e}^{t(M-\omega)})_{t\geq 0}$ and $R(\lambda, M-\omega)$ for $\lambda > 0$.

2) We show first that the resolvent operator $R(\lambda, M-\omega)$ leaves X_2 for $\lambda > 0$ invariant. Let $\left(\begin{smallmatrix}\tilde u\\ \tilde v\end{smallmatrix}\right) \in X_2$ and put $\left(\begin{smallmatrix}u\\ v\end{smallmatrix}\right) := R(\lambda, M-\omega)\left(\begin{smallmatrix}\tilde u\\ \tilde v\end{smallmatrix}\right)$. Part 2) of the proof for Proposition 9.22 states that (u,v) is contained in X_1. In particular, (u,v) is an element of X_0. Relation (9.37) now implies that the function $\operatorname{div}(\varepsilon^{(i)}u^{(i)})$ belongs to $H^1_{00}(Q_i)$, and identity (9.38) means that the jump $[\![((\lambda+\omega)\varepsilon+\sigma)u\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}$ is an element of $H_0^{3/2}(\mathscr{F}_{\text{int}})$. As a consequence of the arguments in Remark 9.18, also the jump $[\![\varepsilon u\cdot\nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}$ belongs to $H_0^{3/2}(\mathscr{F}_{\text{int}})$, and the vector (u,v) is contained in X_2. Thus, the resolvent operator $R(\lambda, M-\omega)$ leaves X_2 invariant.

3) Let $t > 0$, $n \in \mathbb{N}$, and put $\left(\begin{smallmatrix}u\\ v\end{smallmatrix}\right) := \frac{n}{t}R(\frac{n}{t}, M-\omega)\left(\begin{smallmatrix}\tilde u\\ \tilde v\end{smallmatrix}\right)$. Analogously to relations (9.39) and (9.40), we conclude the estimates

$$\left\|\begin{pmatrix}u\\ v\end{pmatrix}\right\|_{\mathcal{D}(M^2)} = \left\|\tfrac{n}{t}R(\tfrac{n}{t}, M-\omega)\begin{pmatrix}\tilde u\\ \tilde v\end{pmatrix}\right\| + \left\|\tfrac{n}{t}R(\tfrac{n}{t}, M-\omega)M^2\begin{pmatrix}\tilde u\\ \tilde v\end{pmatrix}\right\|$$

$$\leq \frac{1}{1+\frac{t}{n}\omega} \left\| \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\|_{\mathcal{D}(M^2)}, \tag{9.51}$$

$$\left\| \operatorname{div}(\varepsilon^{(i)} u^{(i)}) \right\|_{H^1(Q_i)} \leq \frac{1}{1+\frac{t}{n}\omega} \left\| \operatorname{div}(\varepsilon^{(i)} \tilde{u}^{(i)}) \right\|_{H^1(Q_i)}, \tag{9.52}$$

$$\left\| \operatorname{div}(\varepsilon^{(i)} u^{(i)}) \right\|_{H_0^{1/2}(\Gamma')} \leq \frac{1}{1+\frac{t}{n}\omega} \left\| \operatorname{div}(\varepsilon^{(i)} \tilde{u}^{(i)}) \right\|_{H_0^{1/2}(\Gamma')}, \tag{9.53}$$

where Γ' is an arbitrary face of Q_i. Formula (9.41) is again valid, and together with Lemma 7.4 and Theorem 9.17 it leads to the inequalities

$$\begin{aligned}(1+\tfrac{\omega}{n}t)\,\|[\![\varepsilon u \cdot \nu_{\mathscr{F}_{\rm int}}]\!]_{\mathscr{F}_{\rm int}}\|_{H_0^{3/2}(\mathscr{F}_{\rm int})} &\leq \|[\![\varepsilon \tilde{u} \cdot \nu_{\mathscr{F}_{\rm int}}]\!]_{\mathscr{F}_{\rm int}}\|_{H_0^{3/2}(\mathscr{F}_{\rm int})} \\ &\quad + \tfrac{\|\sigma\|_\infty t}{n} C_{\rm int}\, \|u_1\|_{PH^2(Q)} \\ &\leq \|[\![\varepsilon \tilde{u} \cdot \nu_{\mathscr{F}_{\rm int}}]\!]_{\mathscr{F}_{\rm int}}\|_{H_0^{3/2}(\mathscr{F}_{\rm int})} \\ &\quad + \tfrac{\|\sigma\|_\infty t}{n} C_{\rm int} C_e \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{X_2}, \end{aligned} \tag{9.54}$$

where C_e denotes the uniform embedding constant from Theorem 9.17. Analogously to (9.43)–(9.45), the relations (9.51)–(9.54) give rise to the estimates

$$\left\| \tfrac{n}{t} R(\tfrac{n}{t}, M-\omega) \right\|_{\mathcal{L}(X_2)} \leq \frac{1}{(1-\frac{C_0 t}{n})(1+\frac{\omega t}{n})}, \tag{9.55}$$

$$\left\| \tfrac{n}{t} R(\tfrac{n}{t}, M-\omega) \right\|_{\mathcal{L}(X_2)} \leq \mathrm{e}^{C_0 t}, \tag{9.56}$$

for all $n \in \mathbb{N}$, and $t > 0$ with $n/t > C_0$. They also lead to the bound

$$\|nR(n, M-2C_0)\|_{\mathcal{L}(X_2)} \leq 1 \tag{9.57}$$

for $n \geq 2C_0$. Here $C_0 := \|\sigma\|_\infty C_e C_{\rm int} \geq 0$ is a uniform constant, that is independent of ω. We choose in the following $\omega = 2C_0$. In view of the above reasoning, the arguments in part 4) from the proof of Proposition 9.22 remain for X_2 essentially the same, and imply that $\mathrm{e}^{t(M-\omega)}$ leaves X_2 invariant.

4) Let $\binom{u(t)}{v(t)} := \mathrm{e}^{t(M-\omega)} \binom{\tilde{u}}{\tilde{v}}$ for $t \geq 0$. We only have to show that the mapping (u, v) is continuous on X_2.

First, $(u, v) : [0, \infty) \to \mathcal{D}(M^2)$ is continuous, employing that $(\tilde{u}, \tilde{v})$ is an element of $\mathcal{D}(M_0^2)$. Formula (9.47) is also true in our current setting. It shows that the mapping $t \mapsto \operatorname{div}(\varepsilon^{(i)} u^{(i)}(t))$ is continuous with respect to the toplogies in $H^1(Q_i)$ and $H_0^{1/2}(\Gamma')$, where Γ' is an arbitrary face of Q_i.

Let $T > 0$. As above, we can deduce identity (9.48), and we aim this time at proving that the function $s \mapsto [\![(\sigma + \omega\varepsilon) u(s) \cdot \nu_{\mathscr{F}_{\rm int}}]\!]_{\mathscr{F}_{\rm int}}$ belongs to the space

$L^2([0,T],H_0^{3/2}(\mathscr{F}_{\text{int}}))$. This results then in the continuity of the mapping $[0,T]\to H_0^{3/2}(\mathscr{F}_{\text{int}})$, $t\mapsto [\![\varepsilon u(t)\cdot \nu_{\mathscr{F}_{\text{int}}}]\!]_{\mathscr{F}_{\text{int}}}$, and in the strong continuity of the family $(\mathrm{e}^{t(M-\omega)}|_{X_2})_{t\geq 0}$ on X_2.

We employ again the linear operators $V(k)$ and M_k for $k\in\mathbb{N}$ from (9.49), which are here bounded on X_2 and X. Furthermore, we use the same notation as in part 6) of the proof for Proposition 9.22. The operator M_k then generates a strongly continuous semigroup on X_2, coinciding with the restriction of the family $\mathscr{T}_k$ to X_2. In view of (9.57), the estimate

$$\left\|e^{tM_k}\right\|_{\mathcal{L}(X_2)}\leq 1$$

follows for $t\geq 0$ and $k\geq\omega$, see (9.50). The remaining arguments in the proof of Proposition 9.22 now transfer immediately to X_2, and we conclude the assertion. The growth bound of $(\mathrm{e}^{tM_2})_{t\geq 0}$ is obtained from (9.55), after choosing $\omega=0$ and $\lambda=\frac{n}{t}>0$. □

By means of classical semigroup theory, we can deduce the wellposedness of the Maxwell system (7.1) in the space X_2. The statement transfers parts of Proposition 3.3 from [EiSc17] to our setting of discontinuous coefficients. The formula for the charge density $\rho_{\mathscr{F}_{\text{int}}}$ on the interface is also contained in [ScSp18]. For the statement, the inhomogeneity of (7.1) is supposed to be contained in the space

$$\begin{aligned} W_{1,T} &:= L^1([0,T],\mathcal{D}(M_2))+W^{1,1}([0,T],X_2),\\ \|g\|_{W_{1,T}} &:= \inf_{\substack{g=g_1+g_2,\\ g_1\in L^1([0,T],\mathcal{D}(M_2)),\\ g_2\in W^{1,1}([0,T],X_2)}}\left(\|g_1\|_{L^1([0,T],\mathcal{D}(M_2))}+\|g_2\|_{W^{1,1}([0,T],X_2)}\right),\quad g\in W_{1,T}, \end{aligned}$$

for a fixed number $T>0$. Note that $W_{1,T}$ is a Banach space.

Corollary 9.24. *Let ε,μ, and σ satisfy (7.2). Let moreover $T>0$, and $w_0=(\boldsymbol{E}_0,\boldsymbol{H}_0)$ be initial data for (7.1) from the space $\mathcal{D}(M_2)=\mathcal{D}(M^3)\cap X_2$. Let also $g:=(\frac{1}{\varepsilon}\boldsymbol{J},0):[0,T]\to X_2$ be a weighted current density that is continuous, and an element of $W_{1,T}$. The following statements are true.*

a) The Maxwell system (7.1) possesses a unique classical solution $w=(\boldsymbol{E},\boldsymbol{H})$ in $C([0,T],\mathcal{D}(M_2))\cap C^1([0,T],X_2)$. It satisfies the bounds

$$\begin{aligned} \|w(t)\|_{X_2} &\leq \mathrm{e}^{C_{g,2}t}(\|w_0\|_{X_2}+\|g\|_{L^1([0,t],X_2)}),\\ \|M_2w(t)\|_{X_2} &\leq \mathrm{e}^{C_{g,2}t}(\|w_0\|_{\mathcal{D}(M_2)}+(\tfrac{2}{T}+3)\,\|g\|_{W_{1,T}}), \end{aligned}$$

for $t\in[0,T]$. The constant $C_{g,2}$ is taken from Proposition 9.23, and depends only on ε,μ,σ, and Q.

b) The charge densities $\rho^{(i)}$ on Q_i, and $\rho_{\mathscr{F}_{\mathrm{int}}}$ on the interface $\mathscr{F}_{\mathrm{int}}$ are given via the formulas

$$\rho^{(i)}(t) = \operatorname{div}(\varepsilon^{(i)}\boldsymbol{E}^{(i)}(t)) = \operatorname{div}(\varepsilon^{(i)}\boldsymbol{E}_0^{(i)}) - \int_0^t \operatorname{div}(\sigma^{(i)}\boldsymbol{E}^{(i)}(s) + \boldsymbol{J}^{(i)}(s))\,\mathrm{d}s,$$

$$\rho_{\mathscr{F}_{\mathrm{int}}}(t) = [\![\varepsilon\boldsymbol{E}(t)\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}} = [\![\varepsilon\boldsymbol{E}_0\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}} - \int_0^t [\![(\sigma\boldsymbol{E}(s) + \boldsymbol{J}(s))\cdot\nu_{\mathscr{F}_{\mathrm{int}}}]\!]_{\mathscr{F}_{\mathrm{int}}}\,\mathrm{d}s,$$

for $t \in [0,T]$, and $i \in \{1,2\}$.

Proof. a) Proposition 9.23 shows that M_2 generates a strongly continuous semigroup $(\mathrm{e}^{tM_2})_{t\geq 0}$ on X_2. The classical wellposedness is thus a standard consequence of semigroup theory, see Theorem 8.1.4 in [Vrab03] for instance. The corresponding solution is given via Duhamel's formula

$$w(t) = \mathrm{e}^{tM_2}w_0 + \int_0^t \mathrm{e}^{(t-s)M_2}g(s)\,\mathrm{d}s = \mathrm{e}^{tM_2}w_0 + \tfrac{1}{\varepsilon}\int_0^t \mathrm{e}^{(t-s)M_2}(\mathbf{J}(s),0)\,\mathrm{d}s.$$

Employing the growth bound for $(\mathrm{e}^{tM_2})_{t\geq 0}$ from Proposition 9.23 in this identity, we infer the relations

$$\begin{aligned}\|w(t)\|_{X_2} &\leq \mathrm{e}^{C_{g,2}t}\,\|w_0\|_{X_2} + \int_0^t \mathrm{e}^{C_{g,2}(t-s)}\left\|(\tfrac{1}{\varepsilon}\mathbf{J}(s),0)\right\|_{X_2}\,\mathrm{d}s\\ &\leq \mathrm{e}^{C_{g,2}t}(\|w_0\|_{X_2} + \left\|(\tfrac{1}{\varepsilon}\mathbf{J},0)\right\|_{L^1([0,T],X_2)}).\end{aligned}$$

Let $(\frac{1}{\varepsilon}\mathbf{J},0) \in W_{1,T}$. Let additionally $\zeta > 0$, and $\mathbf{J}_1 \in L^1([0,T],\mathcal{D}(M_2))$, $\mathbf{J}_2 \in W^{1,1}([0,T],X_2)$ with

$$\begin{aligned}(\tfrac{1}{\varepsilon}\mathbf{J},0) &= \mathbf{J}_1 + \mathbf{J}_2,\\ \|(\tfrac{1}{\varepsilon}\mathbf{J},0)\|_{W_{1,T}} &\geq \|\mathbf{J}_1\|_{L^1([0,T],\mathcal{D}(M_2))} + \|\mathbf{J}_2\|_{W^{1,1}([0,T],X_2)} - \zeta.\end{aligned}$$

Integrating by parts, we conclude from Duhamel's formula the identities

$$\begin{aligned}M_2w(t) &= \mathrm{e}^{tM_2}M_2w_0 + \int_0^t M_2\mathrm{e}^{(t-s)M_2}(\mathbf{J}_1(s) + \mathbf{J}_2(s))\,\mathrm{d}s\\ &= \mathrm{e}^{tM_2}M_2w_0 + \int_0^t \mathrm{e}^{(t-s)M_2}M_2\mathbf{J}_1(s)\,\mathrm{d}s - \int_0^t (\frac{d}{ds}\mathrm{e}^{(t-s)M_2})\mathbf{J}_2(s)\,\mathrm{d}s\\ &= \mathrm{e}^{tM_2}M_2w_0 + \int_0^t \mathrm{e}^{(t-s)M_2}M_2\mathbf{J}_1(s)\,\mathrm{d}s - \mathbf{J}_2(t) + \mathrm{e}^{tM_2}\mathbf{J}_2(0)\\ &\quad + \int_0^t \mathrm{e}^{(t-s)M_2}\mathbf{J}_2'(s)\,\mathrm{d}s.\end{aligned}$$

By means of Lemma 7.6 and Proposition 9.23, we obtain the estimate

$$\|M_2w(t)\|_{X_2} \leq \mathrm{e}^{C_{g,2}t}\Big(\,\|w_0\|_{\mathcal{D}(M_2)} + \|\mathbf{J}_1\|_{L^1([0,T],\mathcal{D}(M_2))} + (\tfrac{2}{T}+3)\,\|\mathbf{J}_2\|_{W^{1,1}([0,T],X_2)}\Big)$$

$$\leq \mathrm{e}^{C_{g,2}t}\Big(\|w_0\|_{\mathcal{D}(M_2)} + (\tfrac{2}{T}+3)(\|(\tfrac{1}{\varepsilon}\mathbf{J},0)\|_{W_{1,T}} + \zeta)\Big).$$

As the solution w and all arising constants (except ζ) are independent of the partition $(\frac{1}{\varepsilon}\mathbf{J},0) = \mathbf{J}_1 + \mathbf{J}_2$, we let ζ tend to zero, and infer the second asserted inequality.

b) The representation for the current density on Q_i may be obtained by appropriately modifying the arguments from Proposition 2.3 in [EiSc18], and part 5) from the proof of Proposition 9.22. Since the mapping $X_2 \to H(\operatorname{div}, Q_i)$, $(u,v) \mapsto \operatorname{div}(u^{(i)})$, is bounded for $i \in \{1,2\}$, the regularity of w implies that the charge density $\rho^{(i)} : [0,T] \to L^2(Q_i)$ is continuously differentiable. Due to the same reasoning, the function $[0,T] \to L^2(Q_i)$, $s \mapsto \operatorname{div}(\mathbf{J}^{(i)}(s))$, is L^1-integrable. By taking the divergence in (7.1), we consequently infer the equation

$$\partial_t \operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)}(t)) = -\sigma^{(i)} \operatorname{div}(\mathbf{E}^{(i)}(t)) - \operatorname{div}(\mathbf{J}^{(i)}(t))$$

in $L^2(Q_i)$. Integrating with respect to t yields the first asserted formula. In a similar way, we obtain from our arguments in part 5) of the proof for Proposition 9.22 also the asserted formula for $\rho_{\mathscr{F}_{\text{int}}}$ in $H_0^{3/2}(\mathscr{F}_{\text{int}})$. $\square$

During the error analysis in Chapter 10, the following consequence of Corollary 9.24 is crucial.

Remark 9.25. Let the assumptions of Corollary 9.24 be true. If $g = (\frac{1}{\varepsilon}\mathbf{J},0)$ is additionally an element of the space $W^{1,1}([0,T],X_2)$, Corollary 9.24 implies the estimate

$$\|M_2 w(t)\|_{X_2} \leq \mathrm{e}^{C_{g,2}t}\Big(\|w_0\|_{\mathcal{D}(M_2)} + (\tfrac{2}{T}+3)\, \|g\|_{W^{1,1}([0,T],X_2)}\Big)$$

for $t \in [0,T]$. $\Diamond$

10. Error analysis for the Peaceman-Rachford ADI scheme

The goal of this chapter is an error result in L^2 for time-discrete approximations to the Maxwell system (7.1), that are obtained by means of the Peaceman-Rachford ADI scheme. The most important ingredients of our analysis are the regularity results from Chapter 9. They enable us to estimate arising interface integrals during the study of the local error, so that we lose only half an order in the convergence result, compared to the continuous setting in [HoJS15, EiSc17, Eili17, Köhl18].

We start by recalling the Peaceman-Rachford ADI splitting, and we state certain useful properties of the splitting operators. The definition of the splitting operators follows Section 2.2 of [HoJS15], and Section 3 of [EiSc18]. Note also that some of the below operators already arise in the error analysis in Chapter 6. To have a self-contained presentation in this chapter, we however repeat the common parts.

Recall our permanent assumption (7.2) for the parameters ε, μ, and σ. The curl operator is splitted into the difference

$$\operatorname{curl} = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} = \mathscr{C}_1 - \mathscr{C}_2,$$

employing the two operators

$$\mathscr{C}_1 = \begin{pmatrix} 0 & 0 & \partial_2 \\ \partial_3 & 0 & 0 \\ 0 & \partial_1 & 0 \end{pmatrix} \quad \text{and} \quad \mathscr{C}_2 = \begin{pmatrix} 0 & \partial_3 & 0 \\ 0 & 0 & \partial_1 \\ \partial_2 & 0 & 0 \end{pmatrix},$$

endowed with their maximal domains

$$\begin{aligned}\mathcal{D}(\mathscr{C}_j) =& \{u \in L^2(Q)^3 \mid \mathscr{C}_j u \in L^2(Q)^3\} \\ =& \{u \in L^2(Q)^3 \mid \mathscr{C}_j u^{(i)} \in L^2(Q_i)^3 \text{ for } i \in \{1,2\},\ [\![u_2]\!]_{\mathscr{F}_{\text{int}}} = 0 \text{ if } j = 1, \\ & \qquad \text{or } [\![u_3]\!]_{\mathscr{F}_{\text{int}}} = 0 \text{ if } j = 2\}\end{aligned}$$

for $j \in \{1,2\}$. By means of these operators, we can now split the Maxwell operator M from (7.12) into the sum $M = A + B$ with the operators

$$A := \begin{pmatrix} -\frac{\sigma}{2\varepsilon} I & \frac{1}{\varepsilon}\mathscr{C}_1 \\ \frac{1}{\mu}\mathscr{C}_2 & 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} -\frac{\sigma}{2\varepsilon} I & -\frac{1}{\varepsilon}\mathscr{C}_2 \\ -\frac{1}{\mu}\mathscr{C}_1 & 0 \end{pmatrix}.$$

We consider both operators on their corresponding domains

$$\begin{aligned} \mathcal{D}(A) :=& \{(\mathbf{E},\mathbf{H}) \in L^2(Q)^6 \mid (\mathscr{C}_1\mathbf{H}^{(i)}, \mathscr{C}_2\mathbf{E}^{(i)}) \in L^2(Q_i)^6,\ [\![\mathbf{E}_3]\!]_{\mathscr{F}_{\mathrm{int}}} = 0, \\ & \qquad [\![\mathbf{H}_2]\!]_{\mathscr{F}_{\mathrm{int}}} = 0,\ \mathbf{E}_1^{(i)} = 0 \text{ on } \Gamma_2^{(i)},\ \mathbf{E}_2^{(i)} = 0 \text{ on } \Gamma_3^{(i)}, \\ & \qquad \mathbf{E}_3^{(i)} = 0 \text{ on } \Gamma_1^{(i)} \text{ for } i \in \{1,2\}\}, \\ =& \{(\mathbf{E},\mathbf{H}) \in L^2(Q)^6 \mid (\mathscr{C}_1\mathbf{H}, \mathscr{C}_2\mathbf{E}) \in L^2(Q)^6,\ \mathbf{E}_1 = 0 \text{ on } \Gamma_2,\ \mathbf{E}_2 = 0 \text{ on } \Gamma_3, \\ & \qquad \mathbf{E}_3 = 0 \text{ on } \Gamma_1\}, \\ \mathcal{D}(B) :=& \{(\mathbf{E},\mathbf{H}) \in L^2(Q)^6 \mid (\mathscr{C}_2\mathbf{H}^{(i)}, \mathscr{C}_1\mathbf{E}^{(i)}) \in L^2(Q_i)^6,\ [\![\mathbf{E}_2]\!]_{\mathscr{F}_{\mathrm{int}}} = 0, \\ & \qquad [\![\mathbf{H}_3]\!]_{\mathscr{F}_{\mathrm{int}}} = 0,\ \mathbf{E}_1^{(i)} = 0 \text{ on } \Gamma_3^{(i)},\ \mathbf{E}_2^{(i)} = 0 \text{ on } \Gamma_1^{(i)}, \\ & \qquad \mathbf{E}_3^{(i)} = 0 \text{ on } \Gamma_2^{(i)} \text{ for } i \in \{1,2\}\} \\ =& \{(\mathbf{E},\mathbf{H}) \in L^2(Q)^6 \mid (\mathscr{C}_2\mathbf{H}, \mathscr{C}_1\mathbf{E}) \in L^2(Q)^6,\ \mathbf{E}_1 = 0 \text{ on } \Gamma_3,\ \mathbf{E}_2 = 0 \text{ on } \Gamma_1, \\ & \qquad \mathbf{E}_3 = 0 \text{ on } \Gamma_2\}, \end{aligned} \tag{10.1}$$

We collect some observations in the next remark, concerning the domains of A and B.

Remark 10.1. 1) The boundary condition for the electric field is distributed onto the domains of both splitting operators. Note that all traces and interface conditions are well-defined due to the imposed partial regularity, see Section 2.2. The boundary conditions for the magnetic field are not included in the domains, but will be incorporated by restricting the setting to the subspaces X_1 and X_2. This reasoning is inspired by the arguments in [EiSc18, EiSc17].

2) Although we deal with piecewise regularity in Chapter 9 and consider discontinuous coefficients, we define both splitting operators in such a way that we can apply the corresponding differential operator on the whole domain Q. In other words, we impose interface conditions in the domains of A and B. This is crucial, since the current definition ensures that the intersection $\mathcal{D}(A) \cap \mathcal{D}(B)$ is contained in $\mathcal{D}(M)$, that A and B are skew-adjoint on X, and that the inverses $(I - \tau A)^{-1}$ and $(I - \tau B)^{-1}$ exist. The latter resolvents are needed to formulate the ADI scheme. Note that $I - \tau A$ and $I - \tau B$ would loose their injectivity if being extended to domains without interface conditions. $\Diamond$

In terms of the operators A and B, we now formulate the considered Peaceman-Rachford ADI scheme for the approximation of the inhomogeneous Maxwell system

(7.1), see [ZhCZ00, HoJS15, EiSc18, EiSc17, Eili17, Köhl18, HoKö20]. Fix a step size $\tau > 0$, and take initial data $(\mathbf{E}^0, \mathbf{H}^0) \in \mathcal{D}(B)$. Then we approximate the solution of (7.1) at time $t = n\tau$ by

$$\begin{aligned}\begin{pmatrix}\mathbf{E}^n\\ \mathbf{H}^n\end{pmatrix} &= \mathscr{T}_\tau\left(\begin{pmatrix}\mathbf{E}^{n-1}\\ \mathbf{H}^{n-1}\end{pmatrix}\right)\\ &:= (I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A)\Big[(I - \tfrac{\tau}{2}A)^{-1}(I + \tfrac{\tau}{2}B)\begin{pmatrix}\mathbf{E}^{n-1}\\ \mathbf{H}^{n-1}\end{pmatrix}\\ &\qquad\qquad - \tfrac{\tau}{2\varepsilon}(\mathbf{J}((n-1)\tau) + \mathbf{J}(n\tau), 0)\Big] \qquad (10.2)\end{aligned}$$

for $n \in \mathbb{N}$.

The following statement corresponds to Proposition 3.1 in [EiSc18]. It is crucial for our analysis, as it shows that the scheme (10.2) is well-defined. The lemma is furthermore essential for the subsequent unconditional stability of the scheme.

Lemma 10.2. *Let ε, μ, and σ satisfy (7.2). The following items are valid.*

a) The adjoint operators of A and B on X are given as

$$A^* = \begin{pmatrix} -\frac{\sigma}{2\varepsilon}I & -\frac{1}{\varepsilon}\mathscr{C}_1 \\ -\frac{1}{\mu}\mathscr{C}_2 & 0\end{pmatrix} \qquad \textit{and} \qquad B^* = \begin{pmatrix} -\frac{\sigma}{2\varepsilon}I & \frac{1}{\varepsilon}\mathscr{C}_2 \\ \frac{1}{\mu}\mathscr{C}_1 & 0\end{pmatrix},$$

and their domains are $\mathcal{D}(A) = \mathcal{D}(A^*)$ and $\mathcal{D}(B) = \mathcal{D}(B^*)$.

b) The operators A, A^*, B, and B^* are generators of contractive C_0-semigroups on X. In particular, the operators $(I - \tau L)^{-1}$ and $S_\tau(L) := (I + \tau L)(I - \tau L)^{-1}$ are contractive on X for $L \in \{A, A^*, B, B^*\}$ and $\tau > 0$.

Proof. The arguments from the proof of Proposition 3.1 in [EiSc18] yield also here the assertions. □

The above lemma leads to the unconditional stability of the scheme (10.2). The statement of this fact is formulated in Corollary 10.3. Note that the result is already contained and proved in Corollary 4.12 of [Köhl18]. The same result is also established in Theorem 4.2 of [EiSc18] for the case of regular coefficients.

Corollary 10.3. *Let ε, μ, and σ satisfy (7.2). Let further $T > 0$, $\tau \in (0, 1)$, and $(\boldsymbol{E}^0, \boldsymbol{H}^0) \in \mathcal{D}(B)$. Let the inhomogeneity $(\frac{1}{\varepsilon}\boldsymbol{J}, 0)$ be contained in $C([0,T], \mathcal{D}(A))$, and $n \in \mathbb{N}$ with $n\tau \leq T$. The stability estimate*

$$\|(\boldsymbol{E}^n, \boldsymbol{H}^n)\| \leq C\Big(\|(\boldsymbol{E}^0, \boldsymbol{H}^0)\|_{\mathcal{D}(B)} + T\|(\tfrac{1}{\varepsilon}\boldsymbol{J}, 0)\|_{C([0,T],\mathcal{D}(A))}\Big)$$

is valid with a uniform constant $C > 0$.

The error analysis in [HaOs08, HoJS15, EiSc17] depends in a crucial way on the embedding of the domain $\mathcal{D}(M_2)$ into $\mathcal{D}(AB)$. Recall here that M_2 denotes the part of the Maxwell operator M in the space X_2 from (7.19). The useful embedding is valid if the coefficients are sufficiently regular, i.e., if the parameters belong at least to $W^{1,\infty}(Q)$. For discontinous coefficients ε, μ, and σ, however, this embedding is in general not valid anymore, see Remark 10.5. The failure of the embedding is the main reason, why our error analysis suffers from a loss of convergence order. Nevertheless, we can at least state the following weaker result.

Lemma 10.4. *Let ε, μ, and σ satisfy (7.2). The embedding $X_1 \hookrightarrow PH^1(Q)^6 \cap \mathcal{D}(A) \cap \mathcal{D}(B)$ is valid.*

Proof. The embedding of X_1 into $PH^1(Q)^6$ is already shown in Proposition 9.8. As a result, it remains to check that all vectors in X_1 satisfy the boundary and interface conditions, imposed in the intersection $\mathcal{D}(A) \cap \mathcal{D}(B)$. Taking the definition of the domain of M in (7.12) into account, we can also conclude that the desired boundary and interface conditions in $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are valid for every vector in X_1. $\square$

Remark 10.5. Note that the result of Lemma 10.4 can in general not be improved to yield the embedding of $\mathcal{D}(M_2)$ into $\mathcal{D}(AB)$. To see this claim, let ε, and μ satisfy (7.2), and let $\sigma = 0$. Let also $(\mathbf{E}, \mathbf{H}) \in \mathcal{D}(M_2) \subseteq \mathcal{D}(B)$. (The latter inclusion is true, as $\mathcal{D}(M_2)$ is a subset of the space X_2 from (7.19), which itself is a subspace of X_1.) The definition of B then implies the identity

$$\left(B \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right)_3 = -\tfrac{1}{\varepsilon} \partial_2 \mathbf{H}_1.$$

Since $\mu \mathbf{H}_1$ is continuous across $\mathscr{F}_{\text{int}}$ by definition of X_2 in (7.19), we infer from Lemma 7.1 and Theorem 9.17 the relation $[\![\mu \partial_2 \mathbf{H}_1]\!]_{\mathscr{F}_{\text{int}}} = 0$. Unless the refractive index $\varepsilon\mu$ is continuous on Q, the function $\frac{1}{\varepsilon} \partial_2 \mathbf{H}_1 = \frac{\mu}{\varepsilon\mu} \partial_2 \mathbf{H}_1$ will thus have a discontinuous trace at $\mathscr{F}_{\text{int}}$. We consequently obtain that $B \left(\begin{smallmatrix} \mathbf{E} \\ \mathbf{H} \end{smallmatrix} \right)$ does not belong to $\mathcal{D}(A)$. One can argue in a similar way to show that $\mathcal{D}(M_2)$ does neither embed into $\mathcal{D}(BA)$. Note that the continuity of the refractive index is an assumption, that is too restrictive for composite materials. $\Diamond$

Let $l \in \{1, 2\}$. In order to expand the semigroup $(\mathrm{e}^{tM_l})_{t \geq 0}$ for positive times, we employ the operators

$$\Lambda_{j,l}(t) w := \frac{1}{t^j (j-1)!} \int_0^t (t-s)^{j-1} \mathrm{e}^{sM_l} w \, \mathrm{d}s, \qquad \Lambda_0(t) := \mathrm{e}^{tM_l}, \tag{10.3}$$

for $w \in X_l$, $t \geq 0$, and $j \in \mathbb{N}$, see [HaOs08, HoJS15]. Note that the semigroup $(\mathrm{e}^{tM_l})_{t \geq 0}$ is introduced in Propositions 9.22 and 9.23, respectively. One can define the functions $\Lambda_j(t)$ in the same way on X, using the semigroup $(\mathrm{e}^{tM})_{t\geq 0}$. Propositions 9.22 and 9.23 also imply that $\Lambda_{j,l}(t)$ and $\Lambda_j(t)|_{X_l}$ coincide on X_l for all $j \in \mathbb{N}_0$ and $t \geq 0$. For notational simplicity, we shall thus write $\Lambda_j(t)$ instead of $\Lambda_{j,l}(t)$.

By means of standard semigroup theory and the growth bounds from Propositions 9.22 and 9.23, one can show the relations

$$\|\Lambda_j(t)\|_{\mathcal{L}(X)} \leq \frac{1}{j!}, \quad \|\Lambda_j(t)\|_{\mathcal{L}(X_k)} \leq \tfrac{1}{j!}\mathrm{e}^{C_{g,k}t}, \qquad k \in \{1,2\}, \tag{10.4}$$

$$tM_l\Lambda_{j+1}(t) = \Lambda_j(t) - \frac{1}{j!}I \quad \text{on } \mathcal{D}(M_l), \qquad j \in \mathbb{N}_0, \tag{10.5}$$

$$\Lambda_0(t) = I + tM_l\Lambda_1(t) = I + tM_l + \tfrac{1}{2}t^2M_l^2 + t^3M_l^3\Lambda_3(t) \quad \text{on } \mathcal{D}(M_l^3), \tag{10.6}$$

for $t \geq 0$, see Section 4 in [HoJS15]. Furthermore, $\Lambda_j(t)$ maps $\mathcal{D}(M_l^k)$ into $\mathcal{D}(M_l^k)$ for all $j,k \in \mathbb{N}$ and $t \geq 0$. The statements remain true if we replace M_l by M.

Those operators are also involved in the next statement. It deals with a term, that is critical within our error analysis. This is the moment, where our regularity results come into play, and we can gain half an order in the error result by estimating arising interface integrals. As introduced in Section 2.2, we denote the extrapolation of A onto $L^2(Q)^6$ by A_{-1}. Moreover, we often abbreviate the electromagnetic field $(\mathbf{E},\mathbf{H})$ by w.

Lemma 10.6. *Let ε, μ, and σ satisfy (7.2), let $w \in \mathcal{D}(M_2)$, and let $\tau \in (0,\frac{1}{2}]$. The estimate*

$$\left\|(I - \tfrac{\tau}{2}A_{-1})^{-1}A_{-1}BM_2\Lambda_1(\tau)w\right\|_{L^2(Q)} \leq \tfrac{C_b}{\tau^{1/2}}\, \|w\|_{\mathcal{D}(M_2)}$$

holds with a uniform constant $C_b > 0$, being independent of w and τ.

Proof. 1) Let $v \in X = L^2(Q)^6$, and define the vectors

$$\begin{pmatrix}\hat{\mathbf{E}}\\ \hat{\mathbf{H}}\end{pmatrix} := \Lambda_1(\tau)M_2w, \qquad \begin{pmatrix}\check{\mathbf{E}}\\ \check{\mathbf{H}}\end{pmatrix} := B\begin{pmatrix}\hat{\mathbf{E}}\\ \hat{\mathbf{H}}\end{pmatrix}.$$

Theorem 9.17 and (10.4) imply that $\left(\begin{smallmatrix}\hat{\mathbf{E}}\\ \hat{\mathbf{H}}\end{smallmatrix}\right)$ belongs to the space $X_2 \subseteq PH^2(Q)^6$, and that the estimate

$$\left\|\begin{pmatrix}\hat{\mathbf{E}}\\ \hat{\mathbf{H}}\end{pmatrix}\right\|_{PH^2(Q)} \leq k_1\, \|w\|_{\mathcal{D}(M_2)} \tag{10.7}$$

is valid with a uniform constant $k_1 > 0$. Lemma 10.4 and Theorem 9.17 further show that the vector $\left(\begin{smallmatrix}\check{\mathbf{E}}\\ \check{\mathbf{H}}\end{smallmatrix}\right)$ is an element of $PH^1(Q)^6$.

We recall for the following calculations that $(\cdot,\cdot)$ denotes the weighted L^2-inner product from (7.11). On X we also note the identities

$$(I - \tfrac{\tau}{2}A_{-1})^{-1}A_{-1} = A(I - \tfrac{\tau}{2}A)^{-1} = -\tfrac{2}{\tau}I + \tfrac{2}{\tau}(I - \tfrac{\tau}{2}A)^{-1}. \tag{10.8}$$

By means of Lemma 10.2, we then obtain the relations

$$\begin{aligned}\left((I - \tfrac{\tau}{2}A_{-1})^{-1}A_{-1}\begin{pmatrix}\check{\mathbf{E}}\\ \check{\mathbf{H}}\end{pmatrix}, v\right) &= \left(\begin{pmatrix}\check{\mathbf{E}}\\ \check{\mathbf{H}}\end{pmatrix}, (-\tfrac{2}{\tau}I + \tfrac{2}{\tau}(I - \tfrac{\tau}{2}A^*)^{-1})v\right)\\ &= \left(\begin{pmatrix}\check{\mathbf{E}}\\ \check{\mathbf{H}}\end{pmatrix}, A^*(I - \tfrac{\tau}{2}A^*)^{-1}v\right).\end{aligned} \tag{10.9}$$

Denoting $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) := (I - \tfrac{\tau}{2}A^*)^{-1}v$, we compute with Green's formula

$$\begin{aligned}\left(\begin{pmatrix}\check{\mathbf{E}}\\ \check{\mathbf{H}}\end{pmatrix}, A^*\begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{H}}\end{pmatrix}\right) &= -\int_Q \Big(\check{\mathbf{E}}\cdot\mathscr{C}_1\tilde{\mathbf{H}} + \check{\mathbf{H}}\cdot\mathscr{C}_2\tilde{\mathbf{E}} + \tfrac{\sigma}{2}\check{\mathbf{E}}\cdot\tilde{\mathbf{E}}\Big)\,\mathrm{d}x\\ &= \sum_{i=1}^{2}\int_{Q_i}\Big((\mathscr{C}_2\check{\mathbf{E}}^{(i)})\cdot\tilde{\mathbf{H}}^{(i)} + (\mathscr{C}_1\check{\mathbf{H}}^{(i)})\cdot\tilde{\mathbf{E}}^{(i)} - \tfrac{\sigma^{(i)}}{2}\check{\mathbf{E}}^{(i)}\cdot\tilde{\mathbf{E}}^{(i)}\Big)\,\mathrm{d}x\\ &\quad + \int_{\mathscr{F}_{\mathrm{int}}}\Big([\![\check{\mathbf{E}}_3]\!]_{\mathscr{F}_{\mathrm{int}}}\tilde{\mathbf{H}}_2 + [\![\check{\mathbf{H}}_2]\!]_{\mathscr{F}_{\mathrm{int}}}\tilde{\mathbf{E}}_3\Big)\,\mathrm{d}\varsigma.\end{aligned} \tag{10.10}$$

Note that all other boundary integrals from Green's formula vanish, due to the boundary conditions for $(\hat{\mathbf{E}}, \hat{\mathbf{H}}) \in X_2$ and $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) \in \mathcal{D}(A)$, see (7.19) and (10.1). For the interface integral, we exploit that $\tilde{\mathbf{H}}_2$ and $\tilde{\mathbf{E}}_3$ are continuous across $\mathscr{F}_{\mathrm{int}}$, since $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$ is contained in $\mathcal{D}(A)$.

2) We next deduce a trace inequality for the space $H^{1/2}(\mathscr{F}_{\mathrm{int}})^*$, which is the dual space of

$$H^{1/2}(\mathscr{F}_{\mathrm{int}}) = (L^2(\mathscr{F}_{\mathrm{int}}), H^1(\mathscr{F}_{\mathrm{int}}))_{1/2,2}$$

with respect to the pivot space $L^2(\mathscr{F}_{\mathrm{int}})$. In this way, we want to estimate the interface integral on the right hand side of (10.10). As usual, we identify $\mathscr{F}_{\mathrm{int}}$ with the rectangle $R := (a_2^-, a_2^+) \times (a_3^-, a_3^+)$.

By means of Theorem 6.2 in Chapter 1 of [LiMa72], we can express $H^{1/2}(\mathscr{F}_{\mathrm{int}})^*$ in terms of the formula

$$H^{1/2}(\mathscr{F}_{\mathrm{int}})^* = \Big(L^2(\mathscr{F}_{\mathrm{int}}), H^1(\mathscr{F}_{\mathrm{int}})\Big)^*_{1/2,2} = \Big(H^1(\mathscr{F}_{\mathrm{int}})^*, L^2(\mathscr{F}_{\mathrm{int}})\Big)_{1/2,2},$$

employing the dual $H^1(\mathscr{F}_{\mathrm{int}})^*$ of $H^1(\mathscr{F}_{\mathrm{int}})$ with respect to the space $L^2(\mathscr{F}_{\mathrm{int}})$. Remark 3.5 in Chapter 1 of [LiMa72] now yields the trace inequality

$$\|\mathrm{tr}_{\mathscr{F}_{\mathrm{int}}} f\|_{H^{1/2}(\mathscr{F}_{\mathrm{int}})^*} \le k_2\,\|f\|^{1/2}_{L^2((0,\infty),L^2(R))}\,\|\partial_1 f\|^{1/2}_{L^2((0,\infty),H^1(R)^*)} \tag{10.11}$$

for all functions $f \in H^1((0,\infty), H^1(R)^*) \cap L^2((0,\infty), L^2(R))$. Here, k_2 denotes a positive uniform constant.

In order to apply (10.11) to our particular case, we employ a cut-off argument. Let $g \in H^1((0,a_1^+), L^2(R))$, and let $\chi : [0,a_1^+] \to [0,1]$ be a smooth function that is equal to 1 on $[0,\frac{1}{2}a_1^+]$, and that is supported within $[0,\frac{3}{4}a_1^+]$. The function

$$\tilde{g}(x_1,x_2,x_3) := \chi(x_1) g(x_1,x_2,x_3), \qquad (x_1,x_2,x_3) \in Q_2,$$

is then still contained in $H^1((0,a_1^+), L^2(R))$. By defining $\tilde{g}(x_1,\cdot) := 0$ for $x_1 > a_1^+$, we obtain a function in $H^1((0,\infty), L^2(R))$. Consequently, estimate (10.11) is applicable to $\tilde{g}$. Taking also the properties of χ as well as the embedding $L^2(R) \hookrightarrow H^1(R)^*$ into account, we infer the relations

$$\begin{aligned}
\|\mathrm{tr}_{\mathscr{F}_{\mathrm{int}}} g\|_{H^{1/2}(\mathscr{F}_{\mathrm{int}})^*} &= \|\mathrm{tr}_{\mathscr{F}_{\mathrm{int}}} \tilde{g}\|_{H^{1/2}(\mathscr{F}_{\mathrm{int}})^*} \le k_2 \,\|\tilde{g}\|_{L^2(Q_2)}^{1/2} \,\|\partial_1 \tilde{g}\|_{L^2(Q_2)}^{1/2} \\
&\le k_2 \,\|g\|_{L^2(Q_2)}^{1/2} \Big(\|\partial_1 g\|_{L^2(Q_2)} + \|\chi'\|_\infty \,\|g\|_{L^2(Q_2)}\Big)^{1/2} \\
&\le k_3 \,\|g\|_{L^2(Q_2)}^{1/2} \,\|g\|_{H^1((0,a_1^+),L^2(R))}^{1/2}
\end{aligned} \tag{10.12}$$

with the uniform constant $k_3 := k_2(1 + \|\chi'\|_\infty)^{1/2} > 0$.

3) By means of the trace inequality (10.12), we are now in the position to estimate the interface integral in (10.10). To that end, we employ that $\tilde{\mathbf{H}}_2$ and $\tilde{\mathbf{E}}_3$ belong to $H^1((0,a_1^+), L^2(R))$, and that $\check{\mathbf{H}}_2$ and $\check{\mathbf{E}}_3$ are contained in $PH^1(Q)$. Altogether, we arrive at the estimates

$$\begin{aligned}
&\Big|\int_{\mathscr{F}_{\mathrm{int}}} \Big([\![\check{\mathbf{E}}_3]\!]_{\mathscr{F}_{\mathrm{int}}} \tilde{\mathbf{H}}_2 + [\![\check{\mathbf{H}}_2]\!]_{\mathscr{F}_{\mathrm{int}}} \tilde{\mathbf{E}}_3 \Big) \,\mathrm{d}\varsigma \Big| \\
&\quad\le \Big\| \tfrac{1}{\sqrt{\mu^{(2)}}} [\![\check{\mathbf{E}}_3]\!]_{\mathscr{F}_{\mathrm{int}}} \Big\|_{H^{1/2}(\mathscr{F}_{\mathrm{int}})} \Big\| \sqrt{\mu^{(2)}}\, \tilde{\mathbf{H}}_2 \Big\|_{H^{1/2}(\mathscr{F}_{\mathrm{int}})^*} \\
&\qquad + \Big\| \tfrac{1}{\sqrt{\varepsilon^{(2)}}} [\![\check{\mathbf{H}}_2]\!]_{\mathscr{F}_{\mathrm{int}}} \Big\|_{H^{1/2}(\mathscr{F}_{\mathrm{int}})} \Big\| \sqrt{\varepsilon^{(2)}}\, \tilde{\mathbf{E}}_3 \Big\|_{H^{1/2}(\mathscr{F}_{\mathrm{int}})^*} \\
&\quad\le k_3 \Bigg[\Big\| \tfrac{1}{\sqrt{\mu^{(2)}}} [\![\check{\mathbf{E}}_3]\!]_{\mathscr{F}_{\mathrm{int}}} \Big\|_{H^{1/2}(\mathscr{F}_{\mathrm{int}})} \Big\| \sqrt{\mu^{(2)}}\, \tilde{\mathbf{H}}_2 \Big\|_{L^2(Q_2)}^{1/2} \\
&\qquad\qquad \cdot \bigg(\Big\| \sqrt{\mu^{(2)}}\, \tilde{\mathbf{H}}_2 \Big\|_{L^2(Q_2)} + \Big\| \sqrt{\mu^{(2)}}\, \partial_1 \tilde{\mathbf{H}}_2 \Big\|_{L^2(Q_2)} \bigg)^{1/2} \\
&\qquad\quad + \Big\| \tfrac{1}{\sqrt{\varepsilon^{(2)}}} [\![\check{\mathbf{H}}_2]\!]_{\mathscr{F}_{\mathrm{int}}} \Big\|_{H^{1/2}(\mathscr{F}_{\mathrm{int}})} \Big\| \sqrt{\varepsilon^{(2)}}\, \tilde{\mathbf{E}}_3 \Big\|_{L^2(Q_2)}^{1/2} \\
&\qquad\qquad \cdot \bigg(\Big\| \sqrt{\varepsilon^{(2)}}\, \tilde{\mathbf{E}}_3 \Big\|_{L^2(Q_2)} + \Big\| \sqrt{\varepsilon^{(2)}}\, \partial_1 \tilde{\mathbf{E}}_3 \Big\|_{L^2(Q_2)} \bigg)^{1/2} \Bigg].
\end{aligned} \tag{10.13}$$

The definition of $(\check{\mathbf{E}}, \check{\mathbf{H}})$ yields the formulas

$$\mathrm{tr}_{\mathscr{F}_{\mathrm{int}}}(\check{\mathbf{E}}_3^{(2)} - \check{\mathbf{E}}_3^{(1)}) = -\,\mathrm{tr}_{\mathscr{F}_{\mathrm{int}}}\big(\tfrac{1}{\varepsilon^{(2)}} \partial_2 \hat{\mathbf{H}}_1^{(2)} - \tfrac{1}{\varepsilon^{(1)}} \partial_2 \hat{\mathbf{H}}_1^{(1)} + \tfrac{\sigma^{(2)}}{2\varepsilon^{(2)}} \hat{\mathbf{E}}_3^{(2)} - \tfrac{\sigma^{(1)}}{2\varepsilon^{(1)}} \hat{\mathbf{E}}_3^{(1)}\big),$$

$$\operatorname{tr}_{\mathscr{F}_{\mathrm{int}}}(\check{\mathbf{H}}_2^{(2)} - \check{\mathbf{H}}_2^{(1)}) = -\operatorname{tr}_{\mathscr{F}_{\mathrm{int}}}(\tfrac{1}{\mu^{(2)}}\partial_3\hat{\mathbf{E}}_1^{(2)} - \tfrac{1}{\mu^{(1)}}\partial_3\hat{\mathbf{E}}_1^{(1)}).$$

We now combine these identities, the trace inequality for $PH^1(Q)$, (10.9), (10.10), and (10.13). It follows

$$\begin{aligned}
&\left|\left((I - \tfrac{\tau}{2}A_{-1})^{-1}A_{-1}\begin{pmatrix}\check{\mathbf{E}}\\ \check{\mathbf{H}}\end{pmatrix}, v\right)\right| \\
&\le \sum_{i=1}^{2}\left[\left\|\begin{pmatrix}C_2\check{\mathbf{E}}^{(i)}\\ C_1\check{\mathbf{H}}^{(i)}\end{pmatrix}\right\|_{L^2(Q_i)} + \tfrac{\|\sigma\|_\infty}{2}\left\|\check{\mathbf{E}}^{(i)}\right\|_{L^2(Q_i)}\right]\left\|(I - \tfrac{\tau}{2}A^*)^{-1}v\right\|_{L^2(Q)} \\
&\quad + k_3(1+\tilde{C}_{\mathrm{int}})\left(\left\|\tfrac{1}{\sqrt{\varepsilon^{(2)}}}\tfrac{1}{\mu}\partial_3\hat{\mathbf{E}}_1\right\|_{PH^1(Q)} + \left\|\tfrac{1}{\sqrt{\mu^{(2)}}}\tfrac{1}{\varepsilon}\partial_2\hat{\mathbf{H}}_1\right\|_{PH^1(Q)} + \left\|\tfrac{\sigma}{2\varepsilon}\tfrac{1}{\sqrt{\mu^{(2)}}}\hat{\mathbf{E}}_3\right\|_{PH^1(Q)}\right) \\
&\quad \cdot \left\|\begin{pmatrix}\sqrt{\varepsilon}\tilde{\mathbf{E}}_3\\ \sqrt{\mu}\tilde{\mathbf{H}}_2\end{pmatrix}\right\|_{L^2(Q)}^{1/2}\left(\left\|\begin{pmatrix}\sqrt{\varepsilon}\tilde{\mathbf{E}}_3\\ \sqrt{\mu}\tilde{\mathbf{H}}_2\end{pmatrix}\right\|_{L^2(Q_2)} + \left\|\begin{pmatrix}\sqrt{\varepsilon}\partial_1\tilde{\mathbf{E}}_3\\ \sqrt{\mu}\partial_1\tilde{\mathbf{H}}_2\end{pmatrix}\right\|_{L^2(Q_2)}\right)^{1/2}.
\end{aligned}$$

Here $\tilde{C}_{\mathrm{int}} > 0$ denotes the constant, resulting from the boundedness of the trace mapping $\operatorname{tr}_{\mathscr{F}_{\mathrm{int}}} : PH^1(Q) \to H^{1/2}(\mathscr{F}_{\mathrm{int}})$. The definition of A, the assumptions on ε, μ and σ in (7.2), Lemma 10.2 and (10.8) now imply the relations

$$\begin{aligned}
\left|\left((I - \tfrac{\tau}{2}A_{-1})^{-1}A_{-1}\begin{pmatrix}\check{\mathbf{E}}\\ \check{\mathbf{H}}\end{pmatrix}, v\right)\right| &\le \tfrac{1+\frac{\|\sigma\|_\infty}{2}}{\delta^{1/2}}\left\|\begin{pmatrix}\check{\mathbf{E}}\\ \check{\mathbf{H}}\end{pmatrix}\right\|_{PH^1(Q)}\|v\| \\
&\quad + \tfrac{k_3\sqrt{2}}{\delta^{3/2}}(1+\tilde{C}_{\mathrm{int}})\left(\left\|\begin{pmatrix}\partial_3\hat{\mathbf{E}}_1\\ \partial_2\hat{\mathbf{H}}_1\end{pmatrix}\right\|_{PH^1(Q)} + \tfrac{\|\sigma\|_\infty}{2}\left\|\hat{\mathbf{E}}_3\right\|_{PH^1(Q)}\right)\|v\|^{1/2} \\
&\quad \cdot\left(\|v\| + \sqrt{\|\varepsilon\|_\infty^2 + \|\mu\|_\infty^2}\left(\left\|A\begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{H}}\end{pmatrix}\right\| + \|\sigma\|_\infty\left\|\tfrac{1}{\sqrt{\varepsilon}}\tilde{\mathbf{E}}_3\right\|_{L^2(Q)}\right)\right)^{1/2} \\
&\le k_4\left\|\begin{pmatrix}\hat{\mathbf{E}}\\ \hat{\mathbf{H}}\end{pmatrix}\right\|_{PH^2(Q)}\|v\| + \tfrac{k_3\sqrt{2}}{\delta^{3/2}}(1+\tilde{C}_{\mathrm{int}})(1+\tfrac{\|\sigma\|_\infty}{2})\left\|\begin{pmatrix}\hat{\mathbf{E}}\\ \hat{\mathbf{H}}\end{pmatrix}\right\|_{PH^2(Q)}\|v\|^{1/2} \\
&\quad \cdot\Big(\|v\| + (\|\varepsilon\|_\infty + \|\mu\|_\infty)\big[(\tfrac{2}{\tau} + \tfrac{\|\sigma\|_\infty}{\delta})\|v\| + \tfrac{2}{\tau}\left\|(I + \tfrac{\tau}{2}A)^{-1}v\right\|\big]\Big)^{1/2} \\
&\le \tfrac{k_5}{\tau^{1/2}}\left\|\begin{pmatrix}\hat{\mathbf{E}}\\ \hat{\mathbf{H}}\end{pmatrix}\right\|_{PH^2(Q)}\|v\|,
\end{aligned}$$

with positive uniform constants k_4, and k_5, since $\tau < 1/2$. We finally conclude the asserted estimate by means of inequality (10.7). $\square$

We are now in the position to state, and to prove our error result for the Peaceman-Rachford ADI scheme (10.2). Hereby we employ arguments from the proof of Theorem 4.2 in [HoJS15], which uses itself an error formula from [HaOs08].

Arguments from the proof of Theorem 5.1 in [EiSc18] are also applied. As a first main difference to the mentioned literature, we cannot use an embedding of $\mathcal{D}(M_2)$ into $\mathcal{D}(AB)$. We thus extrapolate A. Second, we can estimate the expression from the statement of Lemma 10.6 only with a loss of convergence order.

For notational convenience, the solution of the Maxwell system (7.1) with initial data $(\mathbf{E}_0, \mathbf{H}_0)$ is denoted by $w = (\mathbf{E}, \mathbf{H})$. The approximate solution at time $t_n = n\tau$ from scheme (10.2) with starting value $w_0 = (\mathbf{E}_0, \mathbf{H}_0)$ is called w_n. In particular, the scheme (10.2) is assumed to start with exact initial data. Let $T > 0$. For the statement, the current density is supposed to belong to the space

$$\begin{aligned} W_{2,T} &:= W^{1,1}([0,T], X_2) \cap W^{2,1}([0,T], X_1), \\ \|f\|_{W_{2,T}} &:= \|f\|_{W^{1,1}([0,T],X_2)} + \|f\|_{W^{2,1}([0,T],X_1)}, \qquad f \in W_{2,T}, \end{aligned}$$

which is a Banach space.

Theorem 10.7. *Let ε, μ, and σ satisfy (7.2), and let $T \geq 1$. Let $\tau \in (0, 1/2)$ be a fixed step size, and let $w_0 = (\boldsymbol{E}_0, \boldsymbol{H}_0) \in \mathcal{D}(M_2)$ be the initial data for the Maxwell system (7.1) and the scheme (10.2). Moreover, let $(\frac{1}{\varepsilon}\boldsymbol{J}, 0)$ be contained in $W_{2,T}$. There is a uniform constant $C_{\mathrm{err}} > 0$ with*

$$\|w_n - w(n\tau)\|_{L^2} \leq C_{\mathrm{err}} T \mathrm{e}^{C_{\mathrm{err}} T} \tau^{3/2} (\|w_0\|_{\mathcal{D}(M_2)} + \|(\tfrac{1}{\varepsilon}\boldsymbol{J}, 0)\|_{W_{2,T}}) \tag{10.14}$$

for all $n \in \mathbb{N}_0$ with $n\tau \leq T$. The number C_{err} depends only on ε, μ, σ, and Q.

Proof. 1) We start by estimating the local error of the ADI scheme (10.2). Combining the definition of X_2 in (7.19) with Lemma 9.19, we infer that the operator M^2 maps $\mathcal{D}(M_2)$ into $\mathcal{D}(M) \cap X_0 = X_1$. Lemma 10.4 consequently implies that the function $M^l \Lambda_j(\tau) w_0$ is contained in the intersection $\mathcal{D}(A_{-1}B) \cap \mathcal{D}(A) \cap \mathcal{D}(B)$ for $l \in \{0, 1, 2\}$, and $j \in \mathbb{N}_0$.

The first goal is a convenient representation formula for the local error. We thus expand the current density $\mathbf{J}$ in the formula

$$\tfrac{1}{\varepsilon}\mathbf{J}(k\tau + s) = \tfrac{1}{\varepsilon}\mathbf{J}(k\tau) + \tfrac{s}{\varepsilon}\mathbf{J}'(k\tau) + \int_{k\tau}^{k\tau+s} (k\tau + s - r) \tfrac{1}{\varepsilon}\mathbf{J}''(r) \,\mathrm{d}r \tag{10.15}$$

for $s \in [0, \tau]$. Employing the Λ-operators from (10.3) in Duhamel's formula, we can now write the solution $w((k+1)\tau)$ of the Maxwell system (7.1) in the way

$$\begin{aligned} w((k+1)\tau) &= \mathrm{e}^{\tau M} w(k\tau) + \int_0^\tau \mathrm{e}^{(\tau - s)M} (-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau + s), 0) \,\mathrm{d}s \\ &= \mathrm{e}^{\tau M} w(k\tau) + \int_0^\tau \mathrm{e}^{(\tau - s)M} (-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau) - \tfrac{s}{\varepsilon}\mathbf{J}'(k\tau), 0) \,\mathrm{d}s \\ &\quad + \int_0^\tau \mathrm{e}^{(\tau - s)M} \Big(\int_{k\tau}^{k\tau+s} (k\tau + s - r)(-\tfrac{1}{\varepsilon}\mathbf{J}''(r), 0) \,\mathrm{d}r \Big) \,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= \Lambda_0(\tau)w(k\tau) + \tau\Lambda_1(\tau)(-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0) + \tau^2\Lambda_2(\tau)(-\tfrac{1}{\varepsilon}\mathbf{J}'(k\tau), 0) \\
&\quad + R_k(\tau),
\end{aligned} \tag{10.16}
$$

with the remainder

$$
R_k(\tau) = \int_0^\tau \mathrm{e}^{(\tau-s)M}\Big(\int_{k\tau}^{k\tau+s} (k\tau + s - r)(-\tfrac{1}{\varepsilon}\mathbf{J}''(r), 0)\,\mathrm{d}r\Big)\,\mathrm{d}s.
$$

(Compare the proof of Theorem 5.1 in [EiSc18].) Applying (10.15) for the ADI scheme (10.2), we obtain the analogous formula

$$
\begin{aligned}
\mathscr{T}_\tau(w(k\tau)) &= (I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A)\Big[(I - \tfrac{\tau}{2}A)^{-1}(I + \tfrac{\tau}{2}B)w(k\tau) + \tau(-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0) \\
&\qquad + \tfrac{\tau^2}{2}(-\tfrac{1}{\varepsilon}\mathbf{J}'(k\tau), 0) + \tfrac{\tau}{2}\int_{k\tau}^{(k+1)\tau} ((k+1)\tau - r)(-\tfrac{1}{\varepsilon}\mathbf{J}''(r), 0)\,\mathrm{d}r\Big] \\
&= (I - \tfrac{\tau}{2}B)^{-1}\Big[(I - \tfrac{\tau}{2}A_{-1})^{-1}(I + \tfrac{\tau}{2}A_{-1})(I + \tfrac{\tau}{2}B)w(k\tau) \\
&\qquad + (I + \tfrac{\tau}{2}A)\Big(\tau(-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0) + \tfrac{\tau^2}{2}(-\tfrac{1}{\varepsilon}\mathbf{J}'(k\tau), 0) \\
&\qquad + \tfrac{\tau}{2}\int_{k\tau}^{(k+1)\tau} ((k+1)\tau - r)(-\tfrac{1}{\varepsilon}\mathbf{J}''(r), 0)\,\mathrm{d}r\Big)\Big].
\end{aligned} \tag{10.17}
$$

Here we have to extrapolate A in the last identity, since $Bw(k\tau)$ is in general not contained in $\mathcal{D}(A)$. Subtracting both representations (10.16) and (10.17), we arrive at the basic equation for the local error

$$
\begin{aligned}
&\mathscr{T}_\tau(w(k\tau)) - w((k+1)\tau) \\
&= (I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A_{-1})^{-1}\Big[(I + \tfrac{\tau}{2}A_{-1})(I + \tfrac{\tau}{2}B) - (I - \tfrac{\tau}{2}A_{-1})(I - \tfrac{\tau}{2}B)\mathrm{e}^{\tau M}\Big]w(k\tau) \\
&\quad + (I - \tfrac{\tau}{2}B)^{-1}\Big[\tau(I + \tfrac{\tau}{2}A) - \tau(I - \tfrac{\tau}{2}B)\Lambda_1(\tau)\Big](-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0) \\
&\quad + (I - \tfrac{\tau}{2}B)^{-1}\Big[\tfrac{\tau^2}{2}(I + \tfrac{\tau}{2}A) - \tau^2(I - \tfrac{\tau}{2}B)\Lambda_2(\tau)\Big](-\tfrac{1}{\varepsilon}\mathbf{J}'(k\tau), 0) \\
&\quad + \tfrac{\tau}{2}(I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A)\int_{k\tau}^{(k+1)\tau} ((k+1)\tau - r)(-\tfrac{1}{\varepsilon}\mathbf{J}''(r), 0)\,\mathrm{d}r - R_k(\tau) \\
&=: e_{1,k}(\tau) + e_{2,k}(\tau) + e_{3,k}(\tau) + e_{4,k}(\tau) - R_k(\tau).
\end{aligned} \tag{10.18}
$$

Note that all arising expressions in (10.18) are well-defined, due to the inclusion of $\Lambda_2(\tau)(X_1)$ and X_1 in $\mathcal{D}(A) \cap \mathcal{D}(B)$, as well as the assumed regularity of $\mathbf{J}$. We estimate in the sequel each summand on the right hand side of (10.18) separately.

1.a) We start by rewriting $e_{1,k}(\tau)$ as

$$
\begin{aligned}
e_{1,k}(\tau) =&(I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A_{-1})^{-1} \\
&\cdot [I + \tfrac{\tau}{2}M_2 + \tfrac{\tau^2}{4}A_{-1}B - (I - \tfrac{\tau}{2}M_2 + \tfrac{\tau^2}{4}A_{-1}B)\Lambda_0(\tau)]w(k\tau).
\end{aligned} \tag{10.19}
$$

Employing (10.5) twice, the formulas

$$
\begin{aligned}
&(I+\Lambda_0(\tau))w(k\tau) = (2I+\tau M\Lambda_1(\tau))w(k\tau) = (2I+\tau M+\tau^2M^2\Lambda_2(\tau))w(k\tau),\\
&(I-\Lambda_0(\tau))w(k\tau) = -\tau M\Lambda_1(\tau)w(k\tau),
\end{aligned}
$$

follow, while relation (10.6) yields

$$(I-\Lambda_0(\tau))w(k\tau) = (-\tau M - \tfrac{\tau^2}{2}M^2 - \tau^3M^3\Lambda_3(\tau))w(k\tau).$$

Altogether, we deduce from (10.19) the identities

$$
\begin{aligned}
e_{1,k}(\tau) &= (I-\tfrac{\tau}{2}B)^{-1}(I-\tfrac{\tau}{2}A_{-1})^{-1}\Big[I-\Lambda_0(\tau)+\tfrac{\tau}{2}M_2(I+\Lambda_0(\tau))\\
&\qquad\qquad + \tfrac{\tau^2}{4}A_{-1}B(I-\Lambda_0(\tau))\Big]w(k\tau)\\
&= (I-\tfrac{\tau}{2}B)^{-1}(I-\tfrac{\tau}{2}A_{-1})^{-1}\Big[-\tau M-\tfrac{\tau^2}{2}M^2-\tau^3M^3\Lambda_3(\tau)+\tau M+\tfrac{\tau^2}{2}M^2\\
&\qquad\qquad + \tfrac{\tau^3}{2}M^3\Lambda_2(\tau)-\tfrac{\tau^3}{4}A_{-1}BM\Lambda_1(\tau)\Big]w(k\tau)\\
&= (I-\tfrac{\tau}{2}B)^{-1}(I-\tfrac{\tau}{2}A_{-1})^{-1}\Big[-\tau^3M^3\Lambda_3(\tau)+\tfrac{\tau^3}{2}M^3\Lambda_2(\tau)\\
&\qquad\qquad -\tfrac{\tau^3}{4}A_{-1}BM\Lambda_1(\tau)\Big]w(k\tau).
\end{aligned}
$$

To estimate $e_{1,k}(\tau)$, we recall that the definition of the norm $\|\cdot\|_{X_2}$ in (7.20) yields the relations

$$\|M^3w(k\tau)\| \le \|Mw(k\tau)\|_{\mathcal{D}(M^2)} \le \|Mw(k\tau)\|_{X_2},$$

and that the resolvent of the extrapolated operator A_{-1} coincides with the one of A on X. Formula (10.4) and Lemma 10.6 thus yield the bounds

$$
\begin{aligned}
\|e_{1,k}(\tau)\| &\le \tau^3\left\|(I-\tfrac{\tau}{2}B)^{-1}\right\|\Big(\left\|(I-\tfrac{\tau}{2}A)^{-1}\right\|\left\|\tfrac{1}{2}\Lambda_2(\tau)-\Lambda_3(\tau)\right\|\|Mw(k\tau)\|_{X_2}\\
&\qquad\qquad + \tfrac{1}{4}\left\|(I-\tfrac{\tau}{2}A_{-1})^{-1}A_{-1}BM\Lambda_1(\tau)w(k\tau)\right\|\Big)\\
&\le \tau^3(\tfrac{5}{12}+\tfrac{C_b}{4\tau^{1/2}})\,\|w(k\tau)\|_{\mathcal{D}(M_2)},
\end{aligned}
\tag{10.20}
$$

where we also employ Lemma 10.2. Similarly, we obtain

$$\left\|(I+\tfrac{\tau}{2}B)e_{1,k}(\tau)\right\| \le \tau^3(\tfrac{5}{12}+\tfrac{C_b}{4\tau^{1/2}})\,\|w(k\tau)\|_{\mathcal{D}(M_2)}. \tag{10.21}$$

1.b) We next deal with the second summand on the right hand side of (10.18). Due to the regularity of $\mathbf{J}$, simple rearranging of operators immediately yields the identities

$$(I-\tfrac{\tau}{2}A)^{-1}(I-\tfrac{\tau}{2}B)\Lambda_1(\tau)(-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau),0)$$

$$
\begin{aligned}
&= (I - \tfrac{\tau}{2}A)^{-1}(I - \tfrac{\tau}{2}M + \tfrac{\tau}{2}A)\Lambda_1(\tau)(-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0) \\
&= \Big[(I - \tfrac{\tau}{2}A)^{-1}(I - \tfrac{\tau}{2}M) - I + (I - \tfrac{\tau}{2}A)^{-1}\Big]\Lambda_1(\tau)(-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0).
\end{aligned}
$$

Inserting the last formula into the definition of $e_{2,k}(\tau)$ and employing the formula $(I - \frac{\tau}{2}A)^{-1}(I + \frac{\tau}{2}A) = -I + 2(I - \frac{\tau}{2}A)^{-1}$, we thus arrive at the relations

$$
\begin{aligned}
e_{2,k}(\tau) &= \tau(I - \tfrac{\tau}{2}B)^{-1}\Big[(I + \tfrac{\tau}{2}A) - (I - \tfrac{\tau}{2}B)\Lambda_1(\tau)\Big](-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0) \\
&= \tau(I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A)\Big[(I - \tfrac{\tau}{2}A)^{-1}(I + \tfrac{\tau}{2}A) \\
&\qquad\qquad - (I - \tfrac{\tau}{2}A)^{-1}(I - \tfrac{\tau}{2}B)\Lambda_1(\tau)\Big](-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0) \\
&= \tau(I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A)\Big[- I + \Lambda_1(\tau) + 2(I - \tfrac{\tau}{2}A)^{-1}(I - \Lambda_1(\tau)) \\
&\qquad\qquad + \tfrac{\tau}{2}(I - \tfrac{\tau}{2}A)^{-1}M\Lambda_1(\tau)](-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0).
\end{aligned}
$$

Applying now (10.5) three times, the formula

$$
\begin{aligned}
e_{2,k}(\tau) = \tau(I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A)\Big[&\tau M\Lambda_2(\tau) - 2\tau(I - \tfrac{\tau}{2}A)^{-1}M\Lambda_2(\tau) \\
&+ \tfrac{\tau}{2}(I - \tfrac{\tau}{2}A)^{-1}M + \tfrac{\tau^2}{2}(I - \tfrac{\tau}{2}A)^{-1}M^2\Lambda_2(\tau)\Big](-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0)
\end{aligned}
$$

follows. In view of the relations

$$
\begin{aligned}
\tau M\Lambda_2(\tau) &= \tau(I - \tfrac{\tau}{2}A)^{-1}(I - \tfrac{\tau}{2}A)M\Lambda_2(\tau) \\
&= \tau(I - \tfrac{\tau}{2}A)^{-1}M\Lambda_2(\tau) - \tfrac{\tau^2}{2}(I - \tfrac{\tau}{2}A)^{-1}AM\Lambda_2(\tau)
\end{aligned}
$$

on X_2 and (10.5), we have thus derived the useful representation

$$
\begin{aligned}
e_{2,k}(\tau) &= \tau(I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A)\Big[- \tau(I - \tfrac{\tau}{2}A)^{-1}M(\Lambda_2(\tau) - \tfrac{1}{2}I) \\
&\qquad\qquad + \tfrac{\tau^2}{2}(I - \tfrac{\tau}{2}A)^{-1}BM\Lambda_2(\tau)\Big](-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0) \\
&= \tau(I - \tfrac{\tau}{2}B)^{-1}\Big[- \tau^2M^2\Lambda_3(\tau) + \tfrac{\tau^2}{2}BM\Lambda_2(\tau)\Big](-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0).
\end{aligned}
$$

Lemma 10.2, (10.4), as well as Theorem 9.17 yield the estimates

$$
\begin{aligned}
\|e_{2,k}(\tau)\| &\le \tau^3 \left\|(I - \tfrac{\tau}{2}B)^{-1}\right\| \Big(\|\Lambda_3(\tau)\| \left\|M^2(-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0)\right\| \\
&\qquad\qquad + C_0 \left\|\Lambda_2(\tau)(-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0)\right\|_{PH^2(Q)} \Big) \\
&\le \tau^3(\tfrac{1}{6} \left\|(-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0)\right\|_{X_2} + \tfrac{C_1}{2}\mathrm{e}^{C_{g,2}\tau} \left\|(-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0)\right\|_{X_2}) \\
&= \tau^3(\tfrac{1}{6} + \tfrac{C_1}{2}\mathrm{e}^{C_{g,2}\tau}) \left\|(-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0)\right\|_{X_2}, && (10.22) \\
\left\|(I + \tfrac{\tau}{2}B)e_{2,k}(\tau)\right\| &\le \tau^3(\tfrac{1}{6} + \tfrac{C_1}{2}\mathrm{e}^{C_{g,2}\tau}) \left\|(-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0)\right\|_{X_2}, && (10.23)
\end{aligned}
$$

where C_0 and C_1 are two positive uniform constants, depending only on ε, μ, σ, and Q.

1.c) Using (10.5), we can rewrite the third error term on the right hand side of (10.18), obtaining

$$\begin{aligned} e_{3,k}(\tau) &= \tau^2 (I - \tfrac{\tau}{2}B)^{-1} \Big[\tfrac{1}{2}(I + \tfrac{\tau}{2}A) - (I - \tfrac{\tau}{2}B)\Lambda_2(\tau) \Big] (-\tfrac{1}{\varepsilon}\mathbf{J}'(k\tau), 0) \\ &= \tau^2 (I - \tfrac{\tau}{2}B)^{-1} \Big[\tfrac{\tau}{4}A - \tau M \Lambda_3(\tau) + \tfrac{\tau}{2} B \Lambda_2(\tau) \Big] (-\tfrac{1}{\varepsilon}\mathbf{J}'(k\tau), 0). \end{aligned}$$

Lemma 10.2, Proposition 9.8, and (10.4) now imply the bounds on the third error term

$$\begin{aligned} \|e_{3,k}(\tau)\| &\le \tilde{C}_2 \tau^3 \Big(\tfrac{1}{4} \left\| (-\tfrac{1}{\varepsilon}\mathbf{J}'(k\tau), 0) \right\|_{PH^1(Q)} + \left\| \Lambda_3(\tau)(-\tfrac{1}{\varepsilon}\mathbf{J}'(k\tau), 0) \right\|_{PH^1(Q)} \\ &\qquad + \tfrac{1}{2} \left\| \Lambda_2(\tau)(-\tfrac{1}{\varepsilon}\mathbf{J}'(k\tau), 0) \right\|_{PH^1(Q)} \Big) \\ &\le C_2 \tau^3 (\tfrac{1}{4} + (\tfrac{1}{6} + \tfrac{1}{4}) \mathrm{e}^{C_{g,1}\tau}) \left\| (-\tfrac{1}{\varepsilon}\mathbf{J}'(k\tau), 0) \right\|_{X_1}, \end{aligned} \tag{10.24}$$

$$\left\| (I + \tfrac{\tau}{2}B) e_{3,k}(\tau) \right\| \le C_2 \tau^3 (\tfrac{1}{4} + \tfrac{5}{12} \mathrm{e}^{C_{g,1}\tau}) \left\| (-\tfrac{1}{\varepsilon}\mathbf{J}'(k\tau), 0) \right\|_{X_1}, \tag{10.25}$$

with uniform constants $\tilde{C}_2$ and C_2 that depend only on ε, μ, σ, and Q.

1.d) The two remaining summands in (10.18) are directly estimated by means of Lemma 10.2, Proposition 7.8, and Proposition 9.8, concluding the statements

$$\begin{aligned} \|e_{4,k}(\tau)\| + \|R_k(\tau)\| &\le \tilde{C}_3 \Big(\tau \int_{k\tau}^{(k+1)\tau} ((k+1)\tau - r) \left\| (-\tfrac{1}{\varepsilon}\mathbf{J}''(r), 0) \right\|_{PH^1(Q)} \,\mathrm{d}r \\ &\qquad + \int_0^\tau \int_{k\tau}^{k\tau + s} (k\tau + s - r) \left\| (-\tfrac{1}{\varepsilon}\mathbf{J}''(r), 0) \right\| \,\mathrm{d}r \,\mathrm{d}s \Big) \\ &\le C_3 \tau^2 \left\| (-\tfrac{1}{\varepsilon}\mathbf{J}, 0) \right\|_{W^{2,1}([k\tau,(k+1)\tau], X_1)}, \end{aligned} \tag{10.26}$$

$$\left\| (I + \tfrac{\tau}{2}B) e_{4,k}(\tau) \right\| + \left\| (I + \tfrac{\tau}{2}B) R_k(\tau) \right\| \le C_3 \tau^2 \left\| (-\tfrac{1}{\varepsilon}\mathbf{J}, 0) \right\|_{W^{2,1}([k\tau,(k+1)\tau], X_1)} \tag{10.27}$$

with constants $\tilde{C}_3$ and C_3 that depend only on ε, μ, σ, and Q.

Altogether, we have estimated the local error $\mathscr{T}_\tau(w(k\tau)) - w((k+1)\tau)$, as well as the difference $(I + \frac{\tau}{2}B)(\mathscr{T}_\tau(w(k\tau)) - w((k+1)\tau))$. The latter term is crucial when controlling the error propagation.

2) To bound the global error, we now combine the unconditional stability of the ADI sheme, see Corollary 10.3, with the bounds on the local error from part 1). Using (10.16) and (10.18), we first derive the more convenient representation of the global error

$$\begin{aligned} w_n - w(n\tau) = (I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A_{-1})^{-1} \Big[&(I + \tfrac{\tau}{2}A_{-1})(I + \tfrac{\tau}{2}B) w_{n-1} \\ &- (I - \tfrac{\tau}{2}A_{-1})(I - \tfrac{\tau}{2}B) \mathrm{e}^{\tau M} w((n-1)\tau) \Big] \end{aligned}$$

$$
\begin{aligned}
&+ (I - \tfrac{\tau}{2}B)^{-1}\Big[\tau(I + \tfrac{\tau}{2}A) - \tau(I - \tfrac{\tau}{2}B)\Lambda_1(\tau)\Big](-\tfrac{1}{\varepsilon}\mathbf{J}((n-1)\tau), 0) \\
&+ (I - \tfrac{\tau}{2}B)^{-1}\Big[\tfrac{\tau^2}{2}(I + \tfrac{\tau}{2}A) - \tau^2(I - \tfrac{\tau}{2}B)\Lambda_2(\tau)\Big](-\tfrac{1}{\varepsilon}\mathbf{J}'((n-1)\tau), 0) \\
&+ \tfrac{\tau}{2}(I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A)\int_{(n-1)\tau}^{n\tau}(n\tau - r)(-\tfrac{1}{\varepsilon}\mathbf{J}''(r), 0)\,\mathrm{d}r - R_{n-1}(\tau) \\
= {}& (I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A_{-1})^{-1}\Big[(I + \tfrac{\tau}{2}A_{-1})(I + \tfrac{\tau}{2}B)(w_{n-1} - w((n-1)\tau)) \\
&\quad + ((I + \tfrac{\tau}{2}A_{-1})(I + \tfrac{\tau}{2}B) - (I - \tfrac{\tau}{2}A_{-1})(I - \tfrac{\tau}{2}B)\mathrm{e}^{\tau M})w((n-1)\tau)\Big] \\
&+ \sum_{l=2}^{4} e_{l,n-1}(\tau) - R_{n-1}(\tau).
\end{aligned}
$$

This recursive formula can also be written in the explicit form

$$
\begin{aligned}
w_n - w(n\tau) &= \sum_{k=0}^{n-1}\Big[(I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A)(I - \tfrac{\tau}{2}A)^{-1}(I + \tfrac{\tau}{2}B)\Big]^{n-1-k} \\
&\qquad \cdot \Big((I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A_{-1})^{-1} \\
&\qquad\quad \cdot \Big((I + \tfrac{\tau}{2}A_{-1})(I + \tfrac{\tau}{2}B) - (I - \tfrac{\tau}{2}A_{-1})(I - \tfrac{\tau}{2}B)\mathrm{e}^{\tau M}\Big)w(k\tau) \\
&\qquad\quad + \sum_{l=2}^{4} e_{l,k}(\tau) - R_k(\tau)\Big) \\
&= \sum_{k=0}^{n-1}\Big[(I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A)(I - \tfrac{\tau}{2}A)^{-1}(I + \tfrac{\tau}{2}B)\Big]^{n-1-k} \\
&\qquad \cdot \Big(\sum_{l=1}^{4} e_{l,k}(\tau) - R_k(\tau)\Big).
\end{aligned}
$$

Similar arguments are also employed in the proof of Theorem 9.3 in [Eili17]. Recall that $S_\tau(L) = (I + \frac{\tau}{2}L)(I - \frac{\tau}{2}L)^{-1}$ denotes the Cayley-Transform of $L \in \{A, B\}$. Estimates (10.20)–(10.27), Lemma 10.2, and the assumption $\tau < 1$ now imply the relations

$$
\begin{aligned}
\|w_n - w(n\tau)\| &\le \sum_{k=0}^{n-2}\Big\|(I - \tfrac{\tau}{2}B)^{-1}\Big\|\,\Big\|(S_\tau(A)S_\tau(B))^{n-2-k}S_\tau(A)\Big\| \\
&\qquad \cdot \Big\|(I + \tfrac{\tau}{2}B)\Big(\sum_{l=1}^{4} e_{l,k}(\tau) - R_k(\tau)\Big)\Big\| + \Big\|\sum_{l=1}^{4} e_{l,n-1}(\tau) - R_{n-1}(\tau)\Big\| \\
&\le \tau^3\sum_{k=0}^{n-1}\Big((\tfrac{5}{12} + \tfrac{C_b}{4\tau^{1/2}})\,\|w(k\tau)\|_{\mathcal{D}(M_2)} + (\tfrac{1}{6} + \tfrac{C_1}{2}\mathrm{e}^{C_{g,2}})\,\Big\|(-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0)\Big\|_{X_2} \\
&\qquad + C_2(\tfrac{1}{4} + \tfrac{5}{12}\mathrm{e}^{C_{g,1}})\,\Big\|(-\tfrac{1}{\varepsilon}\mathbf{J}'(k\tau), 0)\Big\|_{X_1} \\
&\qquad + \tfrac{C_3}{\tau}\,\Big\|(-\tfrac{1}{\varepsilon}\mathbf{J}, 0)\Big\|_{W^{2,1}([k\tau,(k+1)\tau], X_1)}\Big).
\end{aligned}
$$

In view of Lemma 7.6 and Remark 9.25, we finally arrive at the desired estimate

$$\|w_n - w(n\tau)\| \leq C_5 T \mathrm{e}^{C_g T} \tau^{3/2} \left(\|w_0\|_{\mathcal{D}(M_2)} + \left\| \left(\tfrac{1}{\varepsilon}\mathbf{J}, 0\right) \right\|_W \right).$$

Here C_5 is a positive uniform constant, depending only on ε, μ, σ, and Q, while $C_g := \max\{C_{g,1}, C_{g,2}\}$. Recall that $C_{g,1}$ and $C_{g,2}$ are the constants from Propositions 9.22 and 9.23. □

Appendices

A. Differential expressions in different coordinate systems

Here we list useful concepts from geometric analysis, that are employed in Part II of this thesis. First, we recall plane polar coordinates (r, φ). The representation of cartesian coordinates by means of polar coordinates is given by

$$(x, y) = (r\cos\varphi, r\sin\varphi), \qquad r \geq 0, \varphi \in [0, 2\pi).$$

In the following, we tacitly assume that all arising expressions are well-defined. To avoid an overloaded notation, we denote a function in different coordinate systems by the same symbol. In other words, we write $u(r, \varphi) = u(r\cos\varphi, r\sin\varphi)$ for a function u on $\mathbb{R}^2$. (To be precise, the mapping on the left is a mapping in the polar coordinate system, while the one on the right is a mapping in the cartesian coordinate system.) The chain rule then gives rise to the formulas

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\cos\varphi - \frac{\partial u}{\partial \varphi}\frac{\sin\varphi}{r}, \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\sin\varphi + \frac{\partial u}{\partial \varphi}\frac{\cos\varphi}{r}, \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial r^2}\cos^2\varphi - 2\frac{\partial^2 u}{\partial r\partial\varphi}\frac{\sin\varphi\cos\varphi}{r} + 2\frac{\partial u}{\partial\varphi}\frac{\sin\varphi\cos\varphi}{r^2} + \frac{\partial u}{\partial r}\frac{\sin^2\varphi}{r} + \frac{\partial^2 u}{\partial\varphi^2}\frac{\sin^2\varphi}{r^2}, \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial r^2}\sin^2\varphi + 2\frac{\partial^2 u}{\partial r\partial\varphi}\frac{\sin\varphi\cos\varphi}{r} - 2\frac{\partial u}{\partial\varphi}\frac{\sin\varphi\cos\varphi}{r^2} + \frac{\partial u}{\partial r}\frac{\cos^2\varphi}{r} + \frac{\partial^2 u}{\partial\varphi^2}\frac{\cos^2\varphi}{r^2}, \\
\frac{\partial^2 u}{\partial x\partial y} &= \frac{\partial^2 u}{\partial r^2}\sin\varphi\cos\varphi + \frac{\partial^2 u}{\partial r\partial\varphi}\frac{\cos^2\varphi - \sin^2\varphi}{r} - \frac{\partial u}{\partial\varphi}\frac{\cos^2\varphi - \sin^2\varphi}{r^2} \\
&\quad - \frac{\partial u}{\partial r}\frac{\sin\varphi\cos\varphi}{r} - \frac{\partial^2 u}{\partial\varphi^2}\frac{\sin\varphi\cos\varphi}{r^2},
\end{aligned}$$

see Section 1.5.4 in [Zeid13] for instance. The two-dimensional Laplacian has in this coordinate system the representation

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}\cdot) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}.$$

We next proceed to three-dimensional polar coordinates (r, θ, φ). Note that we also employ spherical coordinates to parametrize the unit sphere. The latter coordinates are covered by setting $r = 1$, and by omitting the derivatives with respect to r in the below formulas.

Cartesian coordinates (x, y, z) are represented in three-dimensional polar coordinates by means of the identities

$$(x, y, z) = (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta), \quad r \geq 0, \varphi \in [0, 2\pi), \theta \in [0, \pi].$$

Let u be now a H^1-regular function on $\mathbb{R}^3$. By means of the chain rule we then infer the identities

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\sin\theta\cos\varphi + \frac{\partial u}{\partial \theta}\frac{1}{r}\cos\theta\cos\varphi - \frac{\partial u}{\partial \varphi}\frac{1}{r}\frac{\sin\varphi}{\sin\theta}, \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\sin\theta\sin\varphi + \frac{\partial u}{\partial \theta}\frac{1}{r}\cos\theta\sin\varphi + \frac{\partial u}{\partial \varphi}\frac{1}{r}\frac{\cos\varphi}{\sin\theta}, \\
\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{1}{r}\sin\theta.
\end{aligned}$$

The three-dimensional Laplacian is in polar coordinates given by the formula

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^2},$$

employing the Laplace-Beltrami operator

$$\Delta_{S^2} = \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \varphi^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \theta^2}$$

on the unit sphere $S^2 \subseteq \mathbb{R}^3$, see Examples 6.10 d) and e) in Section XI.6 of [AmEs09] for instance.

To transfer the analysis from the lower hemisphere $S^2_{\text{low}} := S^2 \cap \{x_3 \leq 0\}$ to the unit disc D, the stereographic projection is a very useful tool for us. The stereographic projection with respect to the north pole $(0, 0, 1)$ maps S^2_{low} onto the unit disc D. It is given by the formula

$$f(x, y, z) = \Big(\frac{x}{1-z}, \frac{y}{1-z}\Big) =: (w_1, w_2), \qquad (x, y, z) \in S^2_{\text{low}}, \tag{A.1}$$

see Example 1 in Section 1.1 of [Jost17]. In particular, this transformation is a C^∞-diffeomorphism. In the following, we denote the cartesian coordinates on D by (w_1, w_2). The associated metric (g_{ij}) is then given by the matrix with entries

$$g_{ij}(w_1, w_2) = \frac{4}{(1 + w_1^2 + w_2^2)^2}\delta_{ij}, \qquad i, j \in \{1, 2\},$$

employing the Kronecker delta δ_{ij}. This gives rise to the valume factor

$$\sqrt{g(w_1, w_2)} := \sqrt{\det(g_{ij})} = \frac{4}{(1 + w_1^2 + w_2^2)^2}, \tag{A.2}$$

see Section 1.4 in [Jost17], in particular pages 32–33 therein. The inverse matrix of (g_{ij}) is next denoted by (g^{ij}). We can then represent the Laplace-Beltrami operator on the lower hemisphere S^2_{low} in terms of the coordinates (w_1, w_2) on the disc D. The corresponding formula is

$$\Delta_{S^2} = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{2} \frac{\partial}{\partial w_i} \Big(\sqrt{g} g^{ij} \frac{\partial}{\partial w_j} \Big) = \frac{(1 + w_1^2 + w_2^2)^2}{4} \sum_{i=1}^{2} \frac{\partial^2}{\partial w_i^2}, \tag{A.3}$$

see Section 3.1 in [Jost17], for instance.

As many of our computations are done in polar and spherical coordinates, it is also convenient to have a representation of the stereographic projection in these coordinates. Denoting polar coordinates on D by $(r, \tilde{\varphi})$, and spherical coordinates on S^2_{low} by (θ, φ), (A.1) takes the form

$$r \cos \tilde{\varphi} = \frac{\cos \varphi \sin \theta}{1 - \cos \theta}, \quad r \sin \tilde{\varphi} = \frac{\sin \varphi \sin \theta}{1 - \cos \theta}.$$

This in particular implies the formulas

$$\begin{aligned} r &= \sqrt{\tfrac{\cos^2 \varphi \sin^2 \theta}{(1-\cos \theta)^2} + \tfrac{\sin^2 \varphi \sin^2 \theta}{(1-\cos \theta)^2}} = \frac{\sin \theta}{1 - \cos \theta}, \\ \tilde{\varphi} &= \varphi. \end{aligned} \tag{A.4}$$

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I continue in German.

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Glossary: Part I

Φ artificial variable in extended Maxwell system 25
η artificial damping parameter in extended Maxwell system 24
$\mathbf{E}^n$ iterate of damped ADI scheme 40
$\mathbf{E}^{n,i}$ substep of damped ADI scheme 79
$\mathbf{E}_c^n$ iterate of conserving ADI scheme 40
$\mathbf{E}_c^{n,i}$ substep of conserving ADI scheme 54
$\mathbf{E}$ electric field 18
$\mathbf{H}$ magnetic field 18
μ magnetic permeability 18
$\tilde{\sigma}$ scaled conductivity 18
ε electric permittivity 18

Function spaces
$H(\mathrm{curl}, Q)$ domain of curl-operator 20
$H(\mathrm{div}, Q)$ domain of div-operator 20
$H_0(\mathrm{curl}, Q)$ domain of curl-operator with zero normal boundary conditions 20
$H_0(\mathrm{div}, Q)$ domain of div-operator with zero tangential boundary conditions 20
$H_N(\mathrm{curl}, \mathrm{div}, Q)$ intersection of $H_0(\mathrm{curl}, Q)$ and $H(\mathrm{div}, Q)$ 21
$H_T(\mathrm{curl}, \mathrm{div}, Q)$ intersection of $H_0(\mathrm{div}, Q)$ and $H(\mathrm{curl}, Q)$ 21
$X_{\mathbf{ext},1}$ H^1-regular state space for extended Maxwell system 31
Y H^1-regular state space for numerical schemes 42
X_{ext} basic space for extended Maxwell system 26

Geometric domains
Q cuboid 18
Γ_j boundary faces of Q 20

Inner products
$(\cdot, \cdot)_Y$ inner product on Y 42
$(\cdot, \cdot)$ inner product on X_{ext} 26

Norms
$\|\cdot\|_{H_N}$ norm in $H_N(\mathrm{curl}, \mathrm{div}, Q)$ 21
$\|\cdot\|_{H_T}$ norm in $H_T(\mathrm{curl}, \mathrm{div}, Q)$ 21
$\|\cdot\|_{\mathrm{curl}}$ operator norm of curl-operator 20

$\|\cdot\|_{\text{div}}$ operator norm of div-operator 20
$\|\cdot\|$ norm on X_{ext} 26

Operators

A, B, D_i splitting operators for extended Maxwell system 36, 37
$A_Y, B_Y, D_{i,Y}$ parts of the splitting operators in Y 43
$S_\tau(L)$ Cayley-Transform w.r.t. an operator L 36
$V_\tau(L)$ artificial damping operator 39
M_{ext} Maxwell operator for extended system 26
$M_{\text{ext},1}$ part of M_{ext} on $X_{\text{ext},1}$ 33
$\mathscr{C}_1$ splitting operator for curl 35
p_{curl} projection on curl part 22
p_∇ projection on gradient part 22

Glossary: Part II

$\mathscr{M}$ point on boundary of interface 121
$\mathscr{T}_\tau$ abbreviation for PR-ADI scheme 190
η placeholder for ε and μ 117
$\mathbf{E}^n$ iterate of PR-ADI scheme 190
$\mathbf{E}$ electric field 106
$\mathbf{H}$ magnetic field 106
$\mathbf{J}$ external electric current 106
$\mu_1^{(\nu)}$ zero of the derivative of the Bessel function J_ν 124
μ magnetic permeability 106
$\nu_{\mathscr{F}_{\text{int}}}$ unit normal vector of interface $\mathscr{F}_{\text{int}}$ 105
$\rho_{\mathscr{F}_{\text{int}}}$ electric surface charge at $\mathscr{F}_{\text{int}}$ 106
ρ electric charge density 106
σ electric conductivity 106
ε electric permittivity 106

Eigenfunctions
- Φ_l eigenfunction of Laplace-Beltrami operator on lower hemisphere 129
- $\Psi_{k,l}$ eigenfunction of Laplacian on disc 125
- ψ_l eigenfunction of one-dimensional eigenvalue problem (8.13) 124

Eigenvalues
- κ_l^2 eigenvalue of one-dimensional eigenvalue problem (8.13) 124
- $\lambda_{k,l}$ eigenvalue of Laplacian on disc 125
- λ_l eigenvalue of Laplace-Beltrami operator on lower hemisphere 129

Function spaces
- $(X, Y)_{\theta,2}$ interpolation space between the spaces X, Y with parameter θ 109
- $PH^q(Q)$ piecewise Sobolev space 107
- X_0 subspace of X with normal transmission and divergence conditions 114
- X_1 H^1-regular state space for Maxwell system 115
- X_2 H^2-regular state space for the Maxwell equations 116
- X basic space X, coincides with $L^2(Q)^6$ 114
- $\mathscr{N}$ orthogonal complement of the image of the Laplacian on Q 121
- $\mathscr{W}$ H^2-regular space for elliptic transmission problem 117

Functions

J_ν Bessel function with parameter ν 124
$[\![f]\!]_F$ jump of a function f at an interface F 106
$f^{(i)}$ restriction of a function f to the i-th subdomain of a partition 106

Geometric domains
D_i part of unit disc, where η is constant 122
D unit disc 122
G_i part of unit sphere, where η is constant 121
G lower hemisphere 122
I_i subinterval of $(0, 2\pi)$ to parametrize G_i 121
Q_i subcuboid 105
Q cuboid 105
S common arc of G_1 and G_2 122
$\mathscr{F}_{\text{int}}$ interface between Q_1 and Q_2 105
Γ^* union of some of the boundary faces of Q with zero Dirichlet b.c. 117
$\Gamma_j^\pm, \Gamma_j^{\pm,(i)}$ boundary faces of Q or Q_i 107
$\check{S}$ interface on unit disc 122

Inner products
$(\cdot,\cdot)_{\eta,D}$ L^2-inner product on disc D with weight η 125
$(\cdot,\cdot)_\eta$ L^2-inner product on $[0, 2\pi]$ with weight η 124
$(\cdot,\cdot)$ inner product on X 114

Norms
$\|\cdot\|$ norm on X 114

Operators
A, B splitting operators for Maxwell system 189
L Laplace-Beltrami operator on G 122
M_0 restriction of the Maxwell operator to X_0 115
M_1 part of M in X_1 115
M_2 part of M in X_2 116
M Maxwell operator 114
$\mathscr{C}_1$ splitting operator for curl 188
$\check{L}$ Laplace operator on disc 122

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