Karlsruhe Institute of Technology

Wave phenomena
analysis and numerics

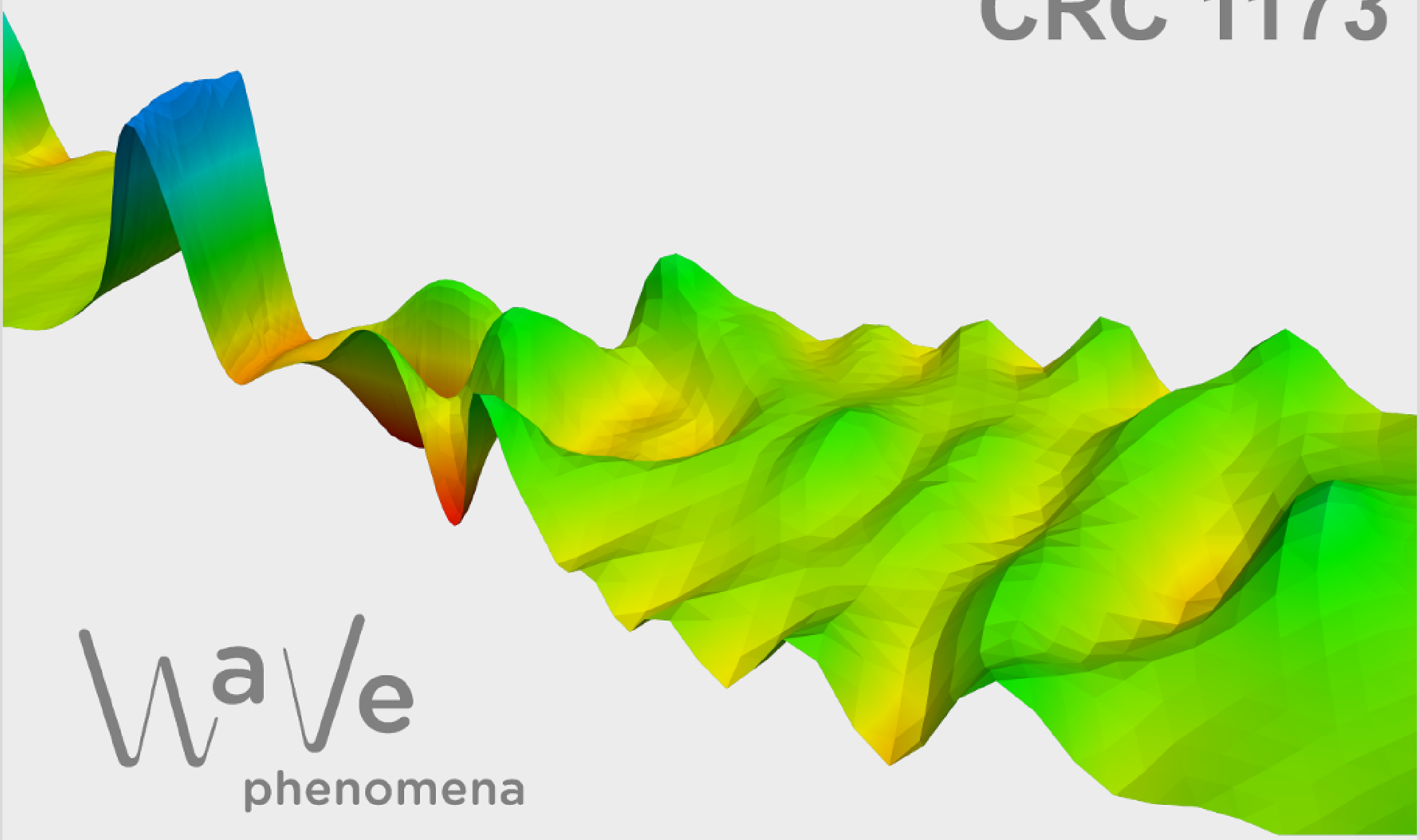
Error analysis for space discretizations of quasilinear wave-type equations

Marlis Hochbruck, Bernhard Maier

CRC Preprint 2021/2, January 2021

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173



Participating universities

Universität Stuttgart

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

TU WIEN

Funded by

DFG

ISSN 2365-662X

ERROR ANALYSIS FOR SPACE DISCRETIZATIONS OF QUASILINEAR WAVE-TYPE EQUATIONS

MARLIS HOCHBRUCK AND BERNHARD MAIER*

Institute for Applied and Numerical Mathematics, Karlsruhe Institute of Technology, Englerstr. 2, 76131 Karlsruhe, Germany

ABSTRACT. In this paper we study space discretizations of a general class of first- and second-order quasilinear wave-type problems. We present a rigorous error analysis based on a combination of inverse estimates with semigroup theory for nonautonomous linear Cauchy problems. Moreover, we provide refined results for the special case that the nonlinearities are local in space. As applications of these general results we derive novel error estimates for two prominent examples from nonlinear physics: the Westervelt equation and the Maxwell equations with Kerr nonlinearity. We conclude with a numerical example to illustrate our theoretical findings.

1. INTRODUCTION

Wave-type problems cover a wide range of applications in physics. This includes for instance the Maxwell equations and the acoustic wave equation to describe the propagation of light and sound waves, respectively. To take the surrounding media into account, these equations are equipped with material models. Due to their simplicity, the consideration of linear wave-type problems based on linearized material models is very appealing. However, this is not reasonable if waves with high frequency or intensity occur. Thus, nonlinear material models have to be considered, which yield quasilinear wave-type problems. With respect to the previous examples, this includes the Maxwell equations with Kerr nonlinearity and the Westervelt equation as a basic model for the propagation of ultrasound.

Concerning the analysis of these problems, Kato (1975) proved wellposedness for a general class of quasilinear wave-type problems on the full space using semigroup theory. In subsequent papers these ideas were also applied to quasilinear wave-type problems on bounded domains with sufficiently regular boundary. Additionally, as Kato's framework is restrictive with respect to boundary conditions, also alternative techniques have been developed. For instance, based on energy techniques, Spitz (2019) proves wellposedness of the quasilinear Maxwell equations with perfectly conducting boundary conditions.

In contrast, the analysis of numerical approximations for the space discretizations of quasilinear wave-type problems is much less developed. Up to our knowledge, the discretization of quasilinear first-order wave-type problems was not analyzed before. However, at least for second-order problems there are a few results: For the conforming space discretization of quasilinear elastic wave equations with finite elements, Makridakis (1993) proves an error estimate using Banach's fixed-point theorem and inverse estimates. Based on this result, Ortner & Süli (2007) provide a corresponding result for the

E-mail address: marlis.hochbruck@kit.edu, bernhard.maier@kit.edu.

Key words and phrases. quasilinear wave-type equations; abstract error analysis; a priori error estimates; Westervelt equation; Maxwell equations; Kerr nonlinearity.

nonconforming discretization of these equations with discontinuous Galerkin finite elements. For the Westervelt equation with strong damping, error estimates for the space discretization with continuous and discontinuous Galerkin finite elements were derived by Nikolić & Wohlmuth (2019) and Antonietti *et al.* (2020), respectively.

Moreover, there are some results considering the full discretization of quasilinear second-order wave-type problems: Based on conforming space discretizations, Ewing (1980), Bales (1986, 1988), Bales & Dougalis (1989), and Makridakis (1993) analyzed the full discretization of quasilinear wave equations with various time integration schemes. More recently, the full discretization of specific quasilinear wave equations in 1D was considered by Gerner (2013) and Gauckler *et al.* (2019) using a continuous Galerkin discretization and the Fourier spectral method, respectively.

In the present paper we analyze space discretizations of quasilinear wave-type problems in a very general abstract framework, i.e., we consider

$$\begin{cases} \Lambda(y(t))\partial_t y(t) = Ay(t) + F(t, y(t)), \quad t \in [0, T], \\ \qquad\qquad y(0) = y_0, \end{cases} \tag{1.1}$$

in a suitable Hilbert space X. Here, Λ is a nonlinear operator, which is locally Lipschitz continuous and bounded in X, whereas A is linear but unbounded in X. The nonlinear right-hand side F is assumed to be sufficiently regular. Note that (1.1) covers both first- and second-order quasilinear wave-type problems and hence includes both the Maxwell equations with Kerr nonlinearity and the Westervelt equation.

Up to our knowledge, all previously obtained results concerning the analysis for space discretizations of quasilinear wave-type equations rely on fixed-point arguments, where wellposedness and the error estimate are proven simultaneously. In contrast, we provide an alternative approach based on semigroup theory. More precisely, our analysis consists of the following three steps:

(1) We exploit that the discrete quasilinear wave-type problem is finite dimensional to prove wellposedness. This only yields a non-optimal lower bound for the maximal time of existence, which may degenerate if the mesh parameter associated to the spatial grid tends to zero.
(2) By using semigroup theory for linear, nonautonomous Cauchy problems, we derive a rigorous error estimate for the quasilinear wave-type problem.
(3) Based on a consistency assumption and inverse estimates, we show that the maximal time of existence of the continuous problem is a lower bound for its discrete counterpart.

Thus, compared to the other results, which are based on fixed-point arguments, our approach allows for a better insight into the various contributions to the error.

Hipp *et al.* (2019) and Hochbruck & Leibold (2020) provided a unified error analysis for linear and semilinear wave-type equations, respectively. In this paper, we extend it to quasilinear problems. Note that the original framework allows for very general nonconforming space discretizations including domain approximations. This is relevant in the quasilinear setting, where wellposedness results in general rely on severe regularity assumptions on the spatial domain. However, for the sake of readability we only consider discretizations where the discrete function space is a subset of X here. For the analysis of general nonconforming space discretizations, we refer to Maier (2020), where all details are given.

Outline. In Section 2 we present the abstract framework including basic assumptions. Correspondingly, we introduce the spatially discrete framework in Section 3 and discuss properties of the discrete operators. In Section 4 we prove the main results of this paper: wellposedness of the discrete problem and a rigorous error estimate within the abstract framework. For the special case of nonlinearities which are local in space, we refine these results in Section 5. In Section 6 we apply the abstract results to specific examples. First, we analyze the Maxwell equations with Kerr nonlinearity in Section 6.1. We further derive an error estimate for the Westervelt equation in Section 6.2. Finally, we validate the theoretical results by a numerical experiment for the Westervelt equation in Section 7.

Notation. For normed spaces X and Y we denote the space of bounded linear operators from X to Y by $\mathcal{L}(X,Y)$. The corresponding norm is given by

$$\|A\|_{\mathcal{L}(X,Y)} \coloneqq \sup_{x\in X\setminus\{0\}} \frac{\|Ax\|_Y}{\|x\|_X}, \qquad A \in \mathcal{L}(X,Y).$$

We denote by $B_X(R)$ the open ball of radius $R > 0$ in X centered around 0. Finally, we use $C > 0$ as a generic constant, which may have different values on any occurrence.

2. Analytical setting

In this section, we collect the basic assumptions on the quasilinear wave-type problem (1.1). Moreover, since there is no wellposedness result which applies to this quite general problem, we also assume wellposedness and justify this assumption in Section 6 for specific examples.

Let $(X, (\cdot \mid \cdot)_X)$, $(Y, (\cdot \mid \cdot)_Y)$, $(Z_\partial, (\cdot \mid \cdot)_{Z_\partial})$, and $(Z, (\cdot \mid \cdot)_Z)$ be Hilbert spaces with dense and continuous embeddings $Z \hookrightarrow Z_\partial \hookrightarrow Y \hookrightarrow X$. We denote the induced norms by $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_{Z_\partial}$, and $\|\cdot\|_Z$, respectively. Moreover, $|\cdot|_Y$ denotes a seminorm on Y with

$$|\xi|_Y \le C_Y \|\xi\|_Y, \qquad \xi \in Y,$$

for a constant $C_Y > 0$.

Assumption 2.1 With a radius $R_Y > 0$ the operators appearing in (1.1) satisfy the following properties.

(Λ) $\{\Lambda(\xi) \mid \xi \in B_Y(R_Y)\} \subset \mathcal{L}(X)$ is a family of symmetric operators, which are uniformly positive definite and bounded, i.e., there are constants $c_\Lambda, C_\Lambda > 0$ such that

$$c_\Lambda \|\varphi\|_X^2 \le (\Lambda(\xi)\varphi \mid \varphi)_X, \qquad \|\Lambda(\xi)\|_{\mathcal{L}(X)} \le C_\Lambda, \qquad \varphi \in X,\ \xi \in B_Y(R_Y) \tag{2.1}$$

holds.

(A) Let $A \in \mathcal{L}(D(A), X)$ with $Y \subset D(A) \subset X$, where $D(A)$ denotes the domain of A.

(F) $F\colon [0,T] \times B_Y(R_Y) \to X$ is continuous in time and bounded, i.e., there is a constant $C_F > 0$ such that

$$\|F(t,\xi)\|_X \le C_F, \qquad t \in [0,T],\ \xi \in B_Y(R_Y).$$

We emphasize that all our results can be generalized to bounded domains instead of spheres $B_Y(R_Y)$. Moreover, they remain valid if the nonlinear operators in (Λ) additionally depend on time, i.e., for an operator family $\{\Lambda(t,\xi) \mid t \in [0,T],\ \xi \in B_Y(R_Y)\} \subset \mathcal{L}(X)$. However, we refrain from considering the most general case here for the sake of readability.

Due to (2.1), the family of inverse operators $\{\Lambda(\xi)^{-1} \mid \xi \in B_Y(R_Y)\} \subset \mathcal{L}(X)$ exists. Thus, (1.1) implies

$$\begin{cases} \partial_t y(t) = \mathcal{A}(y(t))y(t) + \mathcal{F}(t, y(t)), & t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{2.2}$$

with

$$\mathcal{A}(\xi) \coloneqq \Lambda(\xi)^{-1}A, \qquad \mathcal{F}(t,\xi) \coloneqq \Lambda(\xi)^{-1}F(t,\xi), \qquad t \in [0,T],\ \xi \in B_Y(R_Y).$$

Our analysis then relies on the following assumption.

ASSUMPTION 2.2 Let $R_Y > 0$ be a given radius such that Assumption 2.1 holds. The quasilinear Cauchy problem (2.2) has a unique solution y with maximal time of existence $t^*(y_0) > 0$, which satisfies

$$y \in C^1([0,T], Z_\partial) \cap C([0,T], Z \cap B_Y(R_Y))$$

for $T < t^*(y_0)$. Moreover, there exist radii $R_{\partial_t}, R_{\mathcal{A}} > 0$ such that

$$\|\partial_t y(t)\|_Y < R_{\partial_t}, \qquad |\mathcal{A}(y(t))y(t)|_Y < R_{\mathcal{A}}$$

hold uniformly in $[0,T]$.

We further introduce the state-dependent inner product

$$(\varphi \mid \psi)_{\Lambda(\xi)} \coloneqq (\Lambda(\xi)\varphi \mid \psi)_X, \qquad \varphi, \psi \in X,\, \xi \in B_Y(R_Y),$$

and the state-dependent norm

$$\|\varphi\|^2_{\Lambda(\xi)} \coloneqq (\varphi \mid \varphi)_{\Lambda(\xi)}, \qquad \varphi \in X,\, \xi \in B_Y(R_Y),$$

which is equivalent to the norm in X due to (2.1).

If Assumption 2.2 is satisfied, then the weak formulation of (1.1) considered on $(X, (\cdot \mid \cdot)_X)$ is equivalent to the weak formulation of (2.2) considered on $(X, (\cdot \mid \cdot)_{\Lambda(y)})$. Hence, we only focus on (2.2).

3. SPACE DISCRETIZATION

The space discretization of (2.2) is based on spaces X_h and Y_h with

$$X_h = (V_h, (\cdot \mid \cdot)_X) \subset X, \qquad Y_h = (V_h, \|\cdot\|_{Y_h}), \tag{3.1}$$

where V_h is a finite-dimensional function space and $\|\cdot\|_{Y_h}$ corresponds to the norm induced by the inner product of Y. Furthermore, we also introduce a seminorm $|\cdot|_{Y_h}$ corresponding to $|\cdot|_Y$, which satisfies for a constant $C_{Y_h} > 0$ the bound

$$|\xi_h|_{Y_h} \le C_{Y_h} \|\xi_h\|_{Y_h}, \qquad \xi_h \in Y_h. \tag{3.2}$$

The subscript $h > 0$ denotes the space discretization parameter; e.g., for the discretization with finite elements, h represents the diameter of the mesh elements.

We emphasize that we only consider space discretizations with $X_h \subset X$ for the sake of simplicity of presentation. Nevertheless, all our results can be generalized to more general nonconforming space discretizations, in particular, where $X_h \not\subset X$. This is of interest, as wellposedness of quasilinear wave-type equations in many cases depends on smoothness of the boundary of the domain, which will not necessarily be available in the framework considered here. The detailed analysis for the nonconforming discretization can be found in (Maier, 2020).

Since X_h and Y_h are finite dimensional, the norms of these spaces are equivalent for $h > 0$ sufficiently small, i.e.,

$$\frac{1}{C_{X_h,Y_h}(h)} \|\xi_h\|_{X_h} \le \|\xi_h\|_{Y_h} \le C_{Y_h,X_h}(h) \|\xi_h\|_{X_h}, \qquad \xi_h \in Y_h, \tag{3.3}$$

for constants $C_{Y_h,X_h}(h), C_{X_h,Y_h}(h) > 0$ which may depend on h. For the specific examples in Section 6, we have $C_{Y_h,X_h}(h) \sim h^{-\frac{d}{2}}$, where $d \in \mathbb{N}$ is the spatial dimension. Moreover, we have $C_{X_h,Y_h}(h) \sim h^{-1}$ for the Westervelt equation, whereas $C_{X_h,Y_h}(h)$ is independent of h for the Maxwell equations.

With discretizations Λ_h, A_h, and F_h of Λ, A, and F, respectively, a variational formulation leads to the discrete system

$$\begin{cases} \Lambda_h(y_h(t))\partial_t y_h(t) = A_h y_h(t) + F_h(t, y_h(t)), \quad t \in [0,T], \\ \qquad\qquad y_h(0) = y_{h,0}, \end{cases} \tag{3.4}$$

where $y_{h,0} \in X_h$ is the discrete initial value.

ASSUMPTION 3.1 With a radius $R_{Y_h} > 0$ the discrete operators appearing in (3.4) satisfy the following properties uniformly in $h > 0$.

(Λ_h) $\{\Lambda_h(\xi_h) \mid \xi_h \in B_{Y_h}(R_{Y_h})\} \subset \mathcal{L}(X_h)$ is a family of symmetric operators, which are uniformly positive definite and bounded, i.e., there are constants $c_{\Lambda_h}, C_{\Lambda_h} > 0$ such that

$$c_{\Lambda_h} \|\varphi_h\|_{X_h}^2 \le (\Lambda_h(\xi_h)\varphi_h \mid \varphi_h)_{X_h}, \quad \|\Lambda_h(\xi_h)\|_{\mathcal{L}(X_h)} \le C_{\Lambda_h}, \quad \varphi_h \in X_h,\ \xi_h \in B_{Y_h}(R_{Y_h}) \tag{3.5}$$

holds. Moreover, there are constants $L_{\Lambda_h}^{X_h}, L_{\Lambda_h}^{Y_h} > 0$ such that

$$\|\Lambda_h(\varphi_h) - \Lambda_h(\psi_h)\|_{\mathcal{L}(X_h)} \le L_{\Lambda_h}^{X_h} \|\varphi_h - \psi_h\|_{Y_h}, \qquad \varphi_h, \psi_h \in B_{Y_h}(R_{Y_h}), \tag{3.6a}$$

$$\|\big(\Lambda_h(\varphi_h) - \Lambda_h(\psi_h)\big)\xi_h\|_X \le L_{\Lambda_h}^{Y_h} \|\varphi_h - \psi_h\|_X |\xi_h|_{Y_h}, \quad \varphi_h, \psi_h \in B_{Y_h}(R_{Y_h}),\ \xi_h \in Y_h \tag{3.6b}$$

hold.

(A_h) $A_h \colon X_h \to X_h$ is dissipative in X_h, i.e.,

$$(A_h \xi_h \mid \xi_h)_{X_h} \le 0, \qquad \xi_h \in X_h$$

holds.

(F_h) We have $F_h \colon [0,T] \times B_{Y_h}(R_{Y_h}) \to X_h$, which is continuous in time and bounded in Y_h, i.e., there is a constant $C_{F_h} > 0$ such that

$$|F_h(t, \xi_h)|_{Y_h} \le C_{F_h}, \qquad t \in [0,T],\ \xi_h \in B_{Y_h}(R_{Y_h})$$

holds. Furthermore, F_h is Lipschitz continuous in the second argument, i.e., there is a constant $L_{F_h} > 0$ such that

$$\|F_h(t, \varphi_h) - F_h(t, \psi_h)\|_{X_h} \le L_{F_h} \|\varphi_h - \psi_h\|_{X_h}, \qquad t \in [0,T],\ \varphi_h, \psi_h \in B_{Y_h}(R_{Y_h}) \tag{3.7}$$

holds.

Due to (3.5), the family of discrete inverse operators $\{\Lambda_h(\xi_h)^{-1} \mid \xi_h \in B_{Y_h}(R_{Y_h})\} \subset \mathcal{L}(X_h)$ is well defined. Thus, (3.4) yields

$$\begin{cases} \partial_t y_h(t) = \mathcal{A}_h(y_h(t)) y_h(t) + \mathcal{F}_h(t, y_h(t)), \quad t \in [0,T], \\ \quad y_h(0) = y_{h,0}, \end{cases} \tag{3.8}$$

with

$$\mathcal{A}_h(\xi_h) \coloneqq \Lambda_h(\xi_h)^{-1} A_h, \quad \mathcal{F}_h(t, \xi_h) \coloneqq \Lambda_h(\xi_h)^{-1} F_h(t, \xi_h), \quad t \in [0,T],\ \xi_h \in B_{Y_h}(R_{Y_h}), \tag{3.9}$$

cf. (2.2). We also introduce for $\xi_h \in B_{Y_h}(R_{Y_h})$ the discrete state-dependent inner product

$$(\varphi_h \mid \psi_h)_{\Lambda_h(\xi_h)} \coloneqq (\Lambda_h(\xi_h)\varphi_h \mid \psi_h)_{X_h}, \qquad \varphi_h, \psi_h \in X_h,$$

and the state-dependent norm

$$\|\varphi_h\|_{\Lambda_h(\xi_h)}^2 \coloneqq (\varphi_h \mid \varphi_h)_{\Lambda_h(\xi_h)}, \qquad \varphi_h \in X_h,$$

which is equivalent to the norm of X_h due to (3.5), i.e., we have

$$c_{\Lambda_h}\|\xi_h\|_{X_h}^2 \le \|\xi_h\|_{\Lambda_h(\zeta_h)}^2 \le C_{\Lambda_h}\|\xi_h\|_{X_h}^2, \qquad \xi_h \in X_h. \tag{3.10}$$

Again, if the solution y_h of (3.8) satisfies

$$y_h \in C^1([0,T],X_h) \cap C([0,T],B_{Y_h}(R_{Y_h})),$$

then the weak form of (3.4) considered on $X_h = (V_h, (\cdot \mid \cdot)_X)$ is equivalent to the weak form of (3.8) considered on $(V_h, (\cdot \mid \cdot)_{\Lambda_h(y_h)})$.

In the next lemma, we prove that the state-dependent norm depends continuously on time.

LEMMA 3.2 Let $R_{Y_h}, \widehat{R}_{\partial_t} > 0$ be given radii such that Assumption 3.1 is satisfied and

$$z_h \in C^1([0,T],Y_h) \cap C([0,T],B_{Y_h}(R_{Y_h}))$$

holds with $\|\partial_t z_h\|_{Y_h} < \widehat{R}_{\partial_t}$. Then, we have

$$\|\xi_h\|_{\Lambda_h(z_h(t))} \le \mathrm{e}^{C'|t-s|}\|\xi_h\|_{\Lambda_h(z_h(s))}, \qquad s,t \in [0,T],\ \xi_h \in X_h \tag{3.11}$$

with $C' = \frac{1}{2}L_{\Lambda_h}^{X_h} c_{\Lambda_h}^{-1}\widehat{R}_{\partial_t}$.

Proof. Let $s,t \in [0,T]$ and $\xi_h \in X_h$. The Lipschitz continuity (3.6a) of Λ_h, the norm equivalence (3.10), and the fundamental theorem of calculus yield

$$\begin{aligned}
\|\xi_h\|_{\Lambda_h(z_h(t))}^2 &= (\Lambda_h(z_h(s))\xi_h \mid \xi_h)_{X_h} + \big((\Lambda_h(z_h(t)) - \Lambda_h(z_h(s)))\xi_h \mid \xi_h\big)_{X_h} \\
&\le \|\xi_h\|_{\Lambda_h(z_h(s))}^2 + L_{\Lambda_h}^{X_h}\|z_h(t) - z_h(s)\|_{Y_h}\|\xi_h\|_{X_h}^2 \\
&\le \mathrm{e}^{2C'|t-s|}\|\xi_h\|_{\Lambda_h(z_h(s))}^2.
\end{aligned}$$

□

Based on assumptions (Λ_h), (A_h), and (F_h), we show in the following lemma that the operators appearing in (3.8) are Lipschitz continuous.

LEMMA 3.3 Let $R_{Y_h} > 0$ be a given radius such that Assumption 3.1 is satisfied. There are constants $L_{\mathcal{A}_h}, L_{\mathcal{F}_h} > 0$ such that

$$\|\big(\mathcal{A}_h(\varphi_h) - \mathcal{A}_h(\psi_h)\big)\xi_h\|_{X_h} \le L_{\mathcal{A}_h}|\mathcal{A}_h(\varphi_h)\xi_h|_{Y_h}\|\varphi_h - \psi_h\|_{X_h}, \qquad \xi_h \in X_h, \tag{3.12}$$

$$\|\mathcal{F}_h(t,\varphi_h) - \mathcal{F}_h(t,\psi_h)\|_{X_h} \le L_{\mathcal{F}_h}\|\varphi_h - \psi_h\|_{X_h}, \qquad t \in [0,T], \tag{3.13}$$

is satisfied for all $\varphi_h, \psi_h \in B_{Y_h}(R_{Y_h})$.

Proof. Let $t \in [0,T]$, $\xi_h \in X_h$ and $\varphi_h, \psi_h \in B_{Y_h}(R_{Y_h})$ be chosen arbitrarily. The first estimate in (3.5) implies

$$\|\Lambda_h(\xi)^{-1}\|_{\mathcal{L}(X_h)} \le c_{\Lambda_h}^{-1}. \tag{3.14}$$

Hence, we obtain from the definition (3.9) of the discrete operators

$$\begin{aligned}
\|\big(\mathcal{A}_h(\varphi_h) - \mathcal{A}_h(\psi_h)\big)\xi_h\|_{X_h} &\le c_{\Lambda_h}^{-1}\|\big(\Lambda_h(\psi_h)\mathcal{A}_h(\varphi_h) - A_h\big)\xi_h\|_{X_h} \\
&\le c_{\Lambda_h}^{-1}\|\big(\Lambda_h(\psi_h) - \Lambda_h(\varphi_h)\big)\mathcal{A}_h(\varphi_h)\xi_h\|_{X_h}.
\end{aligned}$$

Thus, the Lipschitz continuity (3.6b) of Λ_h yields (3.12).

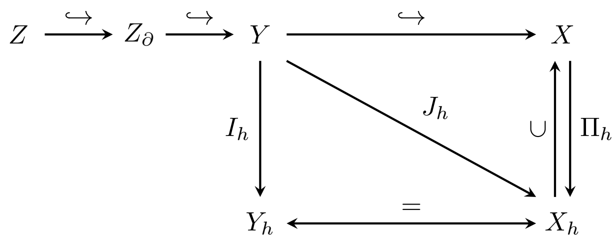


FIGURE 1. Overview of discrete and continuous spaces and operators. Note that each vector space is equipped with the corresponding norm specified above.

For (3.13), we deduce from (3.9), (3.6b), and the Lipschitz continuity (3.7) of F_h

$$\begin{aligned}\|\mathcal{F}_h(t,\varphi_h) - \mathcal{F}_h(t,\psi_h)\|_{X_h} &\le c_{\Lambda_h}^{-1} \|\Lambda_h(\psi_h)\big(\mathcal{F}_h(t,\varphi_h) - \mathcal{F}_h(t,\psi_h)\big)\|_{X_h} \\ &\le c_{\Lambda_h}^{-1} \|\big(\Lambda_h(\psi_h) - \Lambda_h(\varphi_h)\big)\mathcal{F}_h(t,\varphi_h)\|_{X_h} + \|F_h(t,\varphi_h) - F_h(t,\psi_h)\|_{X_h} \\ &\le c_{\Lambda_h}^{-1} (L_{\Lambda_h}^{Y_h} C_{F_h} + L_{F_h}) \|\varphi_h - \psi_h\|_{X_h}.\end{aligned}$$

This completes the proof. □

Finally, we introduce operators relating the continuous to the discrete setting. These relations are illustrated in Figure 1.

(J_h) Let $J_h : Y \to X_h$ be a bounded linear operator with

$$\|J_h \xi\|_{X_h} \le C_{J_h} \|\xi\|_Y, \qquad \xi \in Y. \tag{3.15}$$

(I_h) Let $I_h : Y \to Y_h$ be a bounded operator with

$$\|I_h \xi\|_{Y_h} \le C_{I_h} \|\xi\|_Y, \qquad |I_h \xi|_{Y_h} \le C_{I_h} |\xi|_Y, \qquad \xi \in Y. \tag{3.16}$$

Note that this condition is stronger than (3.15), as the norm of Y_h is in general stronger than the one of X_h.

(Π_h) Let $\Pi_h : X \to X_h$ be the projection with respect to the standard inner product of X, i.e.,

$$(\varphi_h \mid \psi)_X = (\varphi_h \mid \Pi_h \psi)_X, \qquad \varphi_h \in X_h,\ \psi \in X. \tag{3.17}$$

In our specific examples, I_h is a nodal interpolation operator and the reference operator J_h relates the continuous solution to the discrete framework. For first-order wave-type equations, we simply choose $J_h = I_h$. However, for second-order wave-type equations, one has to incorporate the projection Π_h in order to obtain the expected order of convergence. For further details, we refer to Section 6.2.

4. ERROR ANALYSIS

We now analyze the error of the solution of the discrete quasilinear Cauchy problem (3.8). Usually, the first step would be to show wellposedness, followed by a rigorous error estimate. However, this approach is not suitable here, as wellposedness and convergence are intertwined: On the one hand, we rely on Assumption 3.1 to prove the existence of a unique solution of (3.8) in X_h. On the other hand, Assumption 3.1 is only valid if we also ensure that this solution stays bounded in Y_h, which is obtained as a consequence of the error estimate. To resolve this dilemma, we proceed as follows.

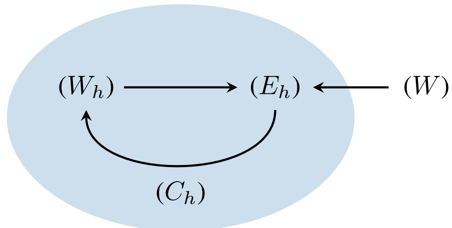


FIGURE 2. Relations between the main steps for the analysis of the abstract space discretization.

Roadmap for the analysis of the abstract space discretization.

- (W) Assumption 2.2 yields wellposedness of the continuous quasilinear Cauchy problem (2.2). In particular, for all $T < t^*(y_0)$ we have $\|y\|_Y < R_Y$ uniformly on $[0, T]$.
- (W_h) In Lemma 4.2 we prove wellposedness of the discrete quasilinear Cauchy problem (3.8) based on Assumption 3.1. More precisely, we show the existence of maximal time $t_h^*(y_{h,0})$ such that for all $T_h < t_h^*(y_{h,0})$ the solution satisfies $\|y_h\|_{Y_h} < R_{Y_h}$ uniformly on $[0, T_h]$. However, at this point, $t_h^*(y_{h,0})$ might deteriorate for $h \to 0$.
- (E_h) Based on (W), (W_h), and semigroup theory, we derive a rigorous estimate for the error $\|y - y_h\|_X$ in Theorem 4.3 on a common time interval $\widetilde{J} = [0, \min\{T, T_h\}]$.
- (C_h) Using an inverse estimate and the consistency from Assumption 4.5, we prove in Theorem 4.7 that $\|y_h - I_h y\|_{Y_h} \to 0$ holds uniformly on $\widetilde{J}$ for $h \to 0$. Thus, we conclude $t_h^*(y_{h,0}) > T$ for h sufficiently small, which closes the analysis.

These steps are illustrated in Figure 2. The results which are proven in this section are highlighted by the blue ellipse.

Preliminary to the analysis, we fix the radii $R_Y, R_{\partial_t}, R_{\mathcal{A}}, R_{Y_h}$ introduced in the previous sections for the rest of this paper.

ASSUMPTION 4.1 First, let $R_{Y_h} > 0$ be a given radius such that Assumption 3.1 and $y_{h,0} \in B_{Y_h}(R_{Y_h})$ hold. Moreover, the radii $R_Y, R_{\partial_t}, R_{\mathcal{A}} > 0$ are chosen such that Assumption 2.2 and

$$C_{I_h} R_Y < R_{Y_h} \tag{4.1}$$

are satisfied. Finally, we define $R_{\mathcal{A}_h} > 0$ with

$$\max\Big\{ C_{I_h} R_{\mathcal{A}},\ |\mathcal{A}_h(y_{h,0}) y_{h,0}|_{Y_h} \Big\} < R_{\mathcal{A}_h}. \tag{4.2}$$

We now start the analysis of the discrete quasilinear Cauchy problem (3.8) by proving (W_h), i.e., wellposedness of the discrete quasilinear Cauchy problem (3.8). To do so, we use that X_h is a finite-dimensional space and hence there is a constant $C_{A_h}(h) > 0$ such that

$$\|A_h\|_{\mathcal{L}(X_h)} \le C_{A_h}(h) \tag{4.3}$$

holds. We emphasize that $C_{A_h}(h)$ might deteriorate for $h \to 0$.

LEMMA 4.2 If Assumption 4.1 holds, then there exists a maximal time of existence $t_h^*(y_{h,0}) > 0$ such that for all $T_h < t_h^*(y_{h,0})$, the discrete quasilinear Cauchy problem (3.8) has a unique solution y_h with

$$y_h \in C^1([0,T_h], X_h) \cap C([0,T_h], Y_h)$$

and

$$\|y_h(t)\|_{Y_h} < R_{Y_h}, \qquad |\mathcal{A}_h(y_h(t))y_h(t)|_{Y_h} < R_{\mathcal{A}_h}, \qquad t \in [0,T_h].$$

Proof. Let $\varphi_h, \psi_h \in Y_h$ such that for $\xi_h \in \{\varphi_h, \psi_h\}$

$$\|\xi_h\|_{Y_h} < R_{Y_h}, \qquad |\mathcal{A}_h(\xi_h)\xi_h|_{Y_h} < R_{\mathcal{A}_h}$$

holds. The triangle inequality yields

$$\|\mathcal{A}_h(\varphi_h)\varphi_h - \mathcal{A}_h(\psi_h)\psi_h\|_{X_h} \le \|(\mathcal{A}_h(\varphi_h) - \mathcal{A}_h(\psi_h))\varphi_h\|_{X_h} + \|\mathcal{A}_h(\psi_h)(\varphi_h - \psi_h)\|_{X_h}.$$

Thus, the Lipschitz continuity (3.12) yields for the first term

$$\|\big(\mathcal{A}_h(\varphi_h) - \mathcal{A}_h(\psi_h)\big)\psi_h\|_{X_h} \le L_{\mathcal{A}_h} R_{\mathcal{A}_h} \|\varphi_h - \psi_h\|_{X_h}.$$

For the second term, (3.9) and the bounds (3.14) and (4.3) for $\Lambda_h(\varphi_h)^{-1}$ and A_h, respectively, imply

$$\|\mathcal{A}_h(\varphi_h)(\varphi_h - \psi_h)\|_{X_h} \le c_{\Lambda_h}^{-1} C_{A_h}(h) \|\varphi_h - \psi_h\|_{X_h}.$$

Morover, we get from (3.13)

$$\|\mathcal{F}_h(t,\varphi_h) - \mathcal{F}_h(t,\psi_h)\|_{X_h} \le L_{\mathcal{F}_h} \|\varphi_h - \psi_h\|_{X_h}, \qquad t \in [0,T].$$

Collecting these results and due to the inverse estimates (3.3) we obtain

$$\begin{aligned}
&\|\mathcal{A}_h(\varphi_h)\varphi_h + \mathcal{F}_h(t,\varphi_h) - \mathcal{A}_h(\psi_h)\psi_h - \mathcal{F}_h(t,\psi_h)\|_{Y_h} \\
&\qquad \le C C_{Y_h,X_h}(h)(1 + C_{A_h}(h)) C_{X_h,Y_h}(h) \|\varphi_h - \psi_h\|_{Y_h},
\end{aligned}$$

with a constant $C > 0$ depending on $R_{\mathcal{A}_h}$ and the constants from Assumption 3.1. Finally, the local version of the Picard–Lindelöf theorem yields the result. □

We emphasize that the previous proof only yields a lower bound for $t_h^*(y_{h,0})$, which deteriorates for $h \to 0$. Nevertheless, this turns out to be sufficient for the derivation of an error estimate.

Motivated by the unified error analysis proposed by Hipp *et al.* (2019) and Hochbruck & Leibold (2020) for linear and semi-linear wave-type problems, respectively, we now derive an estimate for the error between the solution y of the continuous problem (2.2) and the solution y_h of the discrete problem (3.8). To do so, we define the following remainder terms.

$$\mathcal{R}_\Lambda(\xi) \coloneqq \Lambda_h(I_h\xi)J_h - \Pi_h\Lambda(\xi), \qquad \xi \in B_Y(R_Y), \tag{4.4a}$$

$$\mathcal{R}_A \coloneqq A_h J_h - \Pi_h A, \tag{4.4b}$$

$$\mathcal{R}_F(t,\xi) \coloneqq F_h(t, I_h\xi) - \Pi_h F(t,\xi), \qquad t \in [0,T],\ \xi \in B_Y(R_Y). \tag{4.4c}$$

Note that due to the boundedness (3.16) of I_h and the relation (4.1) of R_Y and R_{Y_h}, the remainder terms $\mathcal{R}_\Lambda(\xi)$ and $\mathcal{R}_F(t,\xi)$ are well defined.

THEOREM 4.3 Let Assumption 4.1 be true and $\widetilde{T} < \min\{t^*(y_0), t_h^*(y_{h,0})\}$. Then, for $t \in [0,\widetilde{T}]$ we have

$$\begin{aligned}
\|y(t) - y_h(t)\|_X \le \|(\mathrm{Id} - J_h)y(t)\|_X + C(1+t)\mathrm{e}^{Ct}\Big(&\|J_h y_0 - y_{h,0}\|_X + \sup_{[0,t]}\|(I_h - J_h)y\|_X \\
&+ \sup_{[0,t]}\|\mathcal{R}_\Lambda(y)\partial_t y\|_X + \sup_{[0,t]}\|\mathcal{R}_A y\|_X + \sup_{[0,t]}\|\mathcal{R}_F(\cdot,y)\|_X\Big).
\end{aligned} \tag{4.5}$$

The constant $C > 0$ depends on $t^*(y_0)$, R_{∂_t}, and $R_{\mathcal{A}_h}$, but is independent of h and $\widetilde{T}$.

Proof. Let $t < \min\{t^*(y_0), t_h^*(y_{h,0})\}$. We first split the error into

$$\|y(t) - y_h(t)\|_X \le \|(\mathrm{Id} - J_h) y(t)\|_X + \|e_h(t)\|_X,$$

with the discretization error

$$e_h(t) = J_h y(t) - y_h(t).$$

As the first term appears in the right-hand side of the estimate (4.5), we focus on the second term. By (1.1) and (3.8), e_h satisfies for $\widetilde{T} < \min\{t^*(y_0), t_h^*(y_{h,0})\}$ the linear, but nonautonomous Cauchy problem

$$\begin{cases} \partial_t e_h(t) = \mathcal{A}_h(I_h y(t)) e_h(t) + g_h(t), \quad t \in [0, \widetilde{T}], \\ \quad e_h(0) = J_h y_0 - y_{h,0}, \end{cases}$$

where

$$g_h = \big(\mathcal{A}_h(I_h y) - \mathcal{A}_h(y_h)\big) y_h + \mathcal{F}_h(I_h y) - \mathcal{F}_h(y_h) + \Lambda_h(I_h y)^{-1} \Big(\mathcal{R}_\Lambda(y)\partial_t y - \mathcal{R}_A y - \mathcal{R}_F(y)\Big).$$

Note that we omit the dependency on time whenever possible for the ease of presentation. The bound (3.14) for ${\Lambda_h}^{-1}$ and the Lipschitz continuity (3.12) and (3.13) of $\mathcal{A}_h$ and $\mathcal{F}_h$, respectively, yield

$$\begin{aligned} \|g_h\|_X \le{}& \big(L_{\mathcal{A}_h} R_{\mathcal{A}_h} + L_{\mathcal{F}_h}\big)\big(\|(I_h - J_h)y\|_X + \|e_h\|_X\big) \\ &+ c_{\Lambda_h}^{-1} \Big(\|\mathcal{R}_\Lambda(y)\partial_t y\|_X + \|\mathcal{R}_A y\|_X + \|\mathcal{R}_F(y)\|_X\Big). \end{aligned} \tag{4.6}$$

As discussed in detail in the proof of (Maier, 2020, Thm. 4.20), we apply (Kato, 1970, Thm. 6.1) to prove existence of a unique evolution family $\big(U_h(t,s)\big)_{\widetilde{T} \ge t \ge s \ge 0}$ such that the discrete error is given by

$$e_h(t) = U_h(t,0) e_h(0) + \int_0^t U_h(t,s) g_h(s) \,\mathrm{d}s.$$

Moreover, we have

$$\|U_h(t,s)\|_{\mathcal{L}(X_h)} \le \left(\frac{C_{\Lambda_h}}{c_{\Lambda_h}}\right)^{\frac{1}{2}} \mathrm{e}^{2C'(t-s)},$$

where the constant C' is given in (3.11). Thus, we obtain

$$\|e_h(t)\|_X \le \left(\frac{C_{\Lambda_h}}{c_{\Lambda_h}}\right)^{\frac{1}{2}} \mathrm{e}^{2C't} \|J_h y_0 - y_{h,0}\|_X + \left(\frac{C_{\Lambda_h}}{c_{\Lambda_h}}\right)^{\frac{1}{2}} \int_0^t \mathrm{e}^{2C'(t-s)} \|g_h(s)\|_X \,\mathrm{d}s.$$

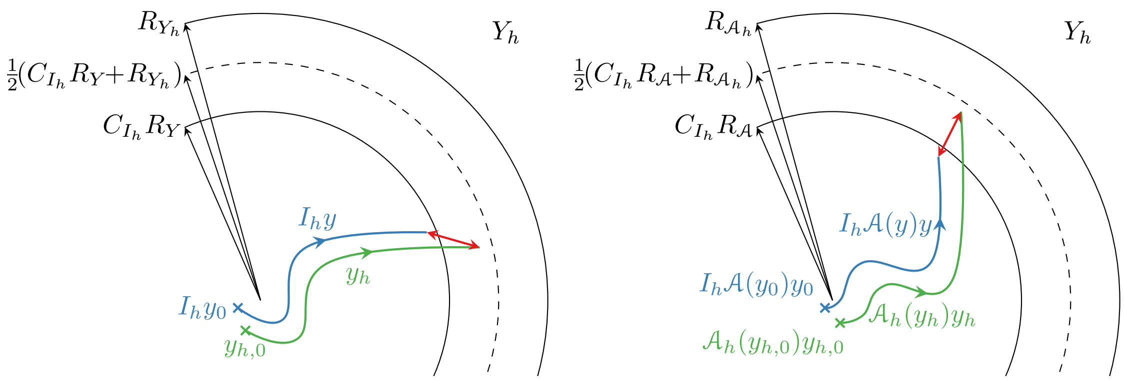
Hence, we get with (4.6)

$$\begin{aligned} \mathrm{e}^{-Ct}\|e_h(t)\|_X \le{}& C\|J_h y_0 - y_{h,0}\|_X + C\int_0^t \mathrm{e}^{-Cs}\|e_h(s)\|_X \,\mathrm{d}s + tC \sup_{[0,t]} \|(I_h - J_h)y\|_X \\ &+ tC\big(\sup_{[0,t]}\|\mathcal{R}_\Lambda(y)\partial_t y\|_X + \sup_{[0,t]}\|\mathcal{R}_A y\|_X + \sup_{[0,t]}\|\mathcal{R}_F(\cdot, y)\|_X\big), \end{aligned}$$

with a constant $C > 0$, which is independent of h. With the Gronwall inequality, we finally obtain

$$\begin{aligned} \|e_h(t)\|_X \le{}& C(1+t)\mathrm{e}^{Ct}\Big(\|J_h y_0 - y_{h,0}\|_X + \sup_{[0,t]}\|(I_h - J_h)y\|_X \\ &+ \sup_{[0,t]}\|\mathcal{R}_\Lambda(y)\partial_t y\|_X + \sup_{[0,t]}\|\mathcal{R}_A y\|_X + \sup_{[0,t]}\|\mathcal{R}_F(\cdot, y)\|_X\Big), \end{aligned}$$

which concludes the proof. □



(A) Balls for the solutions y and y_h.

(B) Balls for the differential operators $\mathcal{A}$ and $\mathcal{A}_h$ applied to the respective solution y and y_h.

FIGURE 3. Illustration of the different balls centered at the origin with the radii specified in Assumption 4.1. Additionally, the concept of the proof of Theorem 4.7 is depicted.

In the rest of this section, we use the error estimate to improve the wellposedness result from Lemma 4.2. More precisely, for $T < t^*(y_0)$ and h sufficiently small we show $t_h^*(y_h) > T$, i.e., we prove that the discrete approximation exists at least as long as the continuous solution.

In Figure 3 the basic idea for the analysis is illustrated. The interpolation $I_h y$ (blue) of the solution of (2.2) and the discrete solution y_h (green) of (3.8) are shown in Figure 3a for $t \in [0,T]$. Correspondingly, the application of the differential operators $\mathcal{A}$ and $\mathcal{A}_h$ to the respective solution are depicted in Figure 3b. Lemma 4.2 yields that for $t \in [0, t_h^*(y_{h,0}))$ the green curves stay in balls centered in the origin with radii R_{Y_h} and $R_{\mathcal{A}_h}$, respectively. As we see in the following, these bounds are not sharp. More precisely, we show that for $t \in [0,T]$ the green curves are even bounded by the intermediate radii indicated by the dashed lines. To this end, we use the error estimate from Theorem 4.3 together with a consistency argument based on Assumption 4.5 below to bound the differences illustrated by the red arrows. This finally implies $t_h^*(y_{h,0}) > T$.

As the first step, we define the constant $C_{\mathcal{A}_h,Y_h,X_h}(h) > 0$ such that

$$|\mathcal{A}_h(\xi_h)\zeta_h|_{Y_h} \le C_{\mathcal{A}_h,Y_h,X_h}(h)\|\zeta_h\|_X, \qquad \xi_h \in B_{Y_h}(R_{Y_h}),\ \zeta_h \in X_h, \tag{4.7}$$

holds for $h > 0$ sufficiently small. Moreover, we introduce

$$C_{\max}(h) = \max\{1, C_{Y_h,X_h}(h), C_{\mathcal{A}_h,Y_h,X_h}(h)\}. \tag{4.8}$$

In general, this constant deteriorates for $h \to 0$.

EXAMPLE 4.4 For the Maxwell equations discussed in Section 6.1, we get

$$C_{Y_h,X_h}(h) = Ch^{-\frac{3}{2}}, \qquad C_{\mathcal{A}_h,Y_h,X_h}(h) = Ch^{-\frac{5}{2}}.$$

For the Westervelt equation studied in Section 6.2, we have

$$C_{Y_h,X_h}(h) = Ch^{-\frac{d}{2}}, \qquad C_{\mathcal{A}_h,Y_h,X_h}(h) = Ch^{-1-\frac{d}{6}},$$

where $d = 2,3$ is the dimension of the spatial domain Ω. In both cases, $C > 0$ is a constant independent of h.

We now assume consistency of the space discretization in terms of the constant defined in (4.8).

ASSUMPTION 4.5 Let Assumption 4.1 be satisfied. For $h \to 0$ it holds

(A_1) $\|(\mathrm{Id} - J_h)\zeta\|_X \to 0$, (A_2) $C_{\max}(h)\|J_h y_0 - y_{h,0}\|_X \to 0$,
(A_3) $C_{\max}(h)\|(I_h - J_h)\zeta\|_X \to 0$, (A_4) $C_{\max}(h)\|\mathcal{R}_A\zeta\|_X \to 0$,
(A_5) $C_{\max}(h)\|\mathcal{R}_\Lambda(\xi)\zeta_\partial\|_X \to 0$, (A_6) $C_{\max}(h)\sup_{[0,T]}\|\mathcal{R}_F(\cdot,\xi)\|_X \to 0$,

uniformly for $\zeta_\partial \in Z_\partial$, $\zeta \in Z$, and $\xi \in Z \cap B_Y(R_Y)$.

Preliminary to the final theorem of this section, we prove the following lemma. Since it is also relevant for the analysis of fully discrete schemes, which will be studied in a follow up paper, we provide a more general version than necessary for the framework considered here.

LEMMA 4.6 Let Assumptions 4.1 and 4.5 be satisfied and $\xi \in Z \cap B_Y(R_Y)$ with $\mathcal{A}(\xi)\xi \in B_Y(R_\mathcal{A})$. Furthermore, let $\xi_{h,1}, \xi_{h,2} \in B_{Y_h}(R_{Y_h})$ such that

$$C_{\max}(h)\|J_h\xi - \xi_{h,i}\|_X \le C_{\text{conv}}(h), \qquad i = 1, 2, \tag{4.9}$$

where $C_{\text{conv}}(h)$ may depend on R_Y, but is independent of ξ. If $C_{\text{conv}}(h) \to 0$ holds for $h \to 0$, then there exists an $h_0 > 0$ with

$$\|\xi_{h,1}\|_{Y_h} < \tfrac{1}{2}(R_{Y_h} + C_{I_h}R_Y), \qquad |\mathcal{A}_h(\xi_{h,2})\xi_{h,1}|_{Y_h} < \tfrac{1}{2}(R_{\mathcal{A}_h} + C_{I_h}R_\mathcal{A}), \qquad h < h_0. \tag{4.10}$$

Proof. First, the inverse estimate (3.3) and (3.16) imply

$$\begin{aligned}\|\xi_{h,1}\|_{Y_h} &\le \|\xi_{h,1} - J_h\xi\|_{Y_h} + \|(J_h - I_h)\xi\|_{Y_h} + \|I_h\xi\|_{Y_h} \\ &\le C_{Y_h,X_h}(h)\|\xi_{h,1} - J_h\xi\|_X + C_{Y_h,X_h}(h)\|(J_h - I_h)\xi\|_X + C_{I_h}R_Y.\end{aligned}$$

Due to (4.9) and Assumption 4.5, we obtain for $h \to 0$

$$C_{Y_h,X_h}(h)\|\xi_{h,1} - J_h\xi\|_X + C_{Y_h,X_h}(h)\|(J_h - I_h)\xi\|_X \to 0.$$

Hence, (4.1) yields the existence of an $h_1 > 0$ such that the first bound in (4.10) holds for all $h < h_1$.

For the second bound in (4.10), we have

$$\mathcal{A}_h(\xi_{h,2})\xi_{h,1} = \mathcal{A}_h(\xi_{h,2})(\xi_{h,1} - J_h\xi) + (\mathcal{A}_h(\xi_{h,2}) - \mathcal{A}_h(I_h\xi))J_h\xi + \mathcal{A}_h(I_h\xi)J_h\xi. \tag{4.11}$$

For the first term, we further get with (4.7) and (4.9)

$$|\mathcal{A}_h(\xi_{h,2})(\xi_{h,1} - J_h\xi)|_{Y_h} \le C_{\mathcal{A}_h,Y_h,X_h}(h)\|\xi_{h,1} - J_h\xi\|_X \le C_{\text{conv}}(h).$$

Due to (3.2), (3.3), and the Lipschitz continuity (3.12) of $\mathcal{A}_h$, the second term is bounded by

$$\begin{aligned}&|\big(\mathcal{A}_h(\xi_{h,2}) - \mathcal{A}_h(I_h\xi)\big)J_h\xi|_{Y_h} \\ &\qquad \le C_{Y_h}C_{Y_h,X_h}(h)L_{\mathcal{A}_h}|\mathcal{A}_h(I_h\xi)J_h\xi|_{Y_h}\Big(\|\xi_{h,2} - J_h\xi\|_X + \|(J_h - I_h)\xi\|_X\Big).\end{aligned}$$

Thus, (4.9) and Assumption 4.5 imply the existence of a constant $C_0(h) > 0$ with $C_0(h) \to 0$ for $h \to 0$ such that

$$|\big(\mathcal{A}_h(\xi_{h,2}) - \mathcal{A}_h(I_h\xi)\big)J_h\xi|_{Y_h} \le C_0(h)|\mathcal{A}_h(I_h\xi)J_h\xi|_{Y_h}.$$

Using these estimates in (4.11), we obtain

$$|\mathcal{A}_h(\xi_{h,2})\xi_{h,1}|_{Y_h} \le C_{\text{conv}}(h) + \big(1 + C_0(h)\big)|\mathcal{A}_h(I_h\xi)J_h\xi|_{Y_h}. \tag{4.12}$$

Further, we get from the definitions (4.4a) and (4.4b) of the remainder terms

$$\mathcal{A}_h(I_h\xi)J_h\xi = \Lambda_h(I_h\xi)^{-1}\mathcal{R}_A\xi - \Lambda_h(I_h\xi)^{-1}\mathcal{R}_\Lambda(\xi)\mathcal{A}(\xi)\xi + (J_h - I_h)\mathcal{A}(\xi)\xi + I_h\mathcal{A}(\xi)\xi, \tag{4.13}$$

where we again consider each term separately. For the first term, we obtain with (3.2), (3.3), and (3.14)

$$|\Lambda_h(I_h\xi)^{-1}\mathcal{R}_A\xi|_{Y_h} \le C_{Y_h}C_{Y_h,X_h}(h)c_{\Lambda_h}^{-1}\|\mathcal{R}_A\xi\|_X.$$

With the same arguments, we get for the second term

$$|\Lambda_h(I_h\xi)^{-1}\mathcal{R}_\Lambda(\xi)\mathcal{A}(\xi)\xi|_{Y_h} \le C_{Y_h}C_{Y_h,X_h}(h)c_{\Lambda_h}^{-1}\|\mathcal{R}_\Lambda(\xi)\mathcal{A}(\xi)\xi\|_X.$$

Finally, (3.3) yields

$$|(J_h - I_h)\mathcal{A}(\xi)\xi|_{Y_h} \le C_{Y_h}C_{Y_h,X_h}(h)\|(J_h - I_h)\mathcal{A}(\xi)\xi\|_X.$$

Using these bounds in (4.12) and (4.13), we hence obtain

$$\begin{aligned}|\mathcal{A}_h(\xi_{h,2})\xi_{h,1}|_{Y_h} \le C_{\mathrm{conv}}(h) + CC_{\max}(h)\big(\|\mathcal{R}_A\xi\|_X + \|\mathcal{R}_\Lambda(\xi)\mathcal{A}(\xi)\xi\|_X + \|(J_h - I_h)\mathcal{A}(\xi)\xi\|_X\big)\\ + \big(1 + C_0(h)\big)|I_h\mathcal{A}(\xi)\xi|_{Y_h},\end{aligned}$$

with a constant $C > 0$ independent of h. Thus, Assumption 4.5 and (4.2) imply the existence of an $h_2 > 0$ such that the second bound in (4.10) holds for all $h < h_2$. Finally, we set $h_0 = \min\{h_1, h_2\}$. □

We conclude the analysis of the abstract space discretization with the following theorem, where we provide wellposedness of (3.8) on the time interval $[0, T]$ from Assumption 2.2. In addition, we show convergence of the discrete solution y_h to the solution y of the continuous problem (2.2).

THEOREM 4.7 Let Assumptions 4.1 and 4.5 be satisfied. For all $T < t^*(y_0)$, there is an $h_0 > 0$ such that Lemma 4.2 and Theorem 4.3 hold with $t_h^*(y_{h,0}) > T$ for all $h < h_0$. Moreover, we have for $h \to 0$

$$\|y(t) - y_h(t)\|_X \to 0, \qquad t \in [0, T]. \tag{4.14}$$

Proof. We prove the statement by contradiction, i.e., we assume that $t_h^*(y_{h,0}) \le T$ holds for $h > 0$ arbitrary small. In particular, this implies $t_h^*(y_{h,0}) < \infty$. Since $t_h^*(y_{h,0})$ is the maximal time of existence of the solution y_h of (3.8), we have

$$\lim_{t\to t_h^*(y_{h,0})^-}\|y_h(t)\|_{Y_h} = R_{Y_h} \qquad \text{or} \qquad \lim_{t\to t_h^*(y_{h,0})^-}|\mathcal{A}_h(y_h(t))y_h(t)|_{Y_h} = R_{\mathcal{A}_h}. \tag{4.15}$$

Then, Theorem 4.3 together with Assumption 4.5 yield that (4.9) is satisfied with $\xi = y(t)$ and $\xi_{h,i} = y_h(t)$, $i = 1, 2$. Hence, due to Lemma 4.6 there is $h_0 > 0$ such that

$$\sup_{[0,T]}\|y_h(t)\|_{Y_h} < \tfrac{1}{2}(R_{Y_h} + C_{I_h}R_Y), \qquad \sup_{[0,T]}|\mathcal{A}_h(y_h)y_h|_{Y_h} < \tfrac{1}{2}(R_{\mathcal{A}_h} + C_{I_h}R_{\mathcal{A}}),$$

for all $h < h_0$. However, this yields

$$\lim_{t\to t_h^*(y_{h,0})}\|y_h(t)\|_{Y_h} \le \tfrac{1}{2}(R_{Y_h} + C_{I_h}R_Y), \quad \lim_{t\to t_h^*(y_{h,0})}|\mathcal{A}_h(y_h(t))y_h(t)|_{Y_h} \le \tfrac{1}{2}(R_{\mathcal{A}_h} + C_{I_h}R_{\mathcal{A}})$$

for all $h < h_0$, which is a contradiction to (4.15) due to $C_{I_h}R_Y < R_{Y_h}$ and $C_{I_h}R_{\mathcal{A}} < R_{\mathcal{A}_h}$.

Finally, (4.14) is a direct consequence of Assumption 4.5 and the error estimate (4.5), as all terms on the right-hand side tend to zero. □

5. Discretization of local nonlinearities

We investigate the remainder terms $\mathcal{R}_\Lambda$ and $\mathcal{R}_F$ defined in (4.4a) and (4.4c), respectively, under the additional assumption that Λ and F are local in space. More precisely, we provide bounds in terms of the operators introduced at the end of Section 3.

To this end, we narrow down the abstract framework to the space discretization of partial differential equations: For some $d, d_r \in \mathbb{N}$ let X, Y and Z be function spaces consisting of functions defined on a bounded domain $\Omega \subset \mathbb{R}^d$ with values in $\mathbb{R}^{d_r}$. Correspondingly, X_h and Y_h are function spaces from a bounded domain $\Omega_h \subset \mathbb{R}^d$ to $\mathbb{R}^{d_r}$.

Assumption 5.1 The interpolation operator I_h and the nonlinearities Λ and F satisfy the following properties.

(λ) For $\xi \in B_Y(R_Y)$ let $\Lambda(\xi) \in \mathcal{L}(Y)$. Further, Λ is local in space, i.e., there is $\lambda : \Omega \times \mathbb{R}^{d_r} \to \mathbb{R}^{d_r \times d_r}$ such that for every $\xi \in B_Y(R_Y)$ and $\varphi \in X$ the identity

$$\big(\Lambda(\xi)\varphi\big)(x) = \lambda(x, \xi(x))\varphi(x), \qquad x \in \Omega,$$

holds in X.

(f) The nonlinearity F is local in space, i.e., there exists $f : [0,T] \times \Omega \times \mathbb{R}^{d_r} \to \mathbb{R}^{d_r}$ such that for every $\xi \in B_Y(R_Y)$ and $t \in [0,T]$ we have

$$\big(F(t,\xi)\big)(x) = f(t, x, \xi(x)), \qquad x \in \Omega,$$

in X.

(I_h) The operator I_h is a nodal interpolation operator, i.e., for some $M \in \mathbb{N}$ there are interpolation points $\Omega_{I_h} = \{\underline{x}^0, \dots, \underline{x}^M\} \subset \Omega \cap \Omega_h$ and basis functions $\{\phi_h^0, \dots, \phi_h^M\} \subset Y_h$ with

$$I_h\xi = \sum_{m=0}^{M} \xi(\underline{x}^m)\, \phi_h^m, \qquad I_h\xi(\underline{x}) = \xi(\underline{x}), \qquad \xi \in Y,\ \underline{x} \in \Omega_{I_h}.$$

We define the discrete operator Λ_h corresponding to Λ by

$$\Lambda_h(\xi_h)\varphi_h = I_h\Lambda(\xi_h)\varphi_h \tag{5.1a}$$

$$= \sum_{m=0}^{M} \lambda(\underline{x}^m, \xi_h(\underline{x}^m))\varphi_h(\underline{x}^m)\, \phi_h^m, \tag{5.1b}$$

for $\xi_h \in B_{Y_h}(R_{Y_h})$ and $\psi_h \in X_h$. We emphasize that (5.1a) has to be interpreted as a short notation for (5.1b), but does not yield a well-defined operator on its own, since in general $\xi_h \notin B_Y(R_Y)$ and $\Lambda(\xi_h)\psi_h \notin Y$. Nevertheless, we use (5.1a) for the sake of readability, keeping (5.1b) in mind.

Similarly, we define F_h via

$$F_h(t,\xi_h) = I_hF(t,\xi_h) \tag{5.2a}$$

$$= \sum_{m=0}^{M} f(t, \underline{x}^m, \xi_h(\underline{x}^m))\, \phi_h^m, \tag{5.2b}$$

for $t \in [0,T]$ and $\xi_h \in B_{Y_h}(R_{Y_h})$. Again, (5.2a) is only well defined in the sense of (5.2b).

We now bound the remainder terms.

LEMMA 5.2 Under Assumption 3.1 and Assumption 5.1, we have for $t \in [0,T]$, $\xi \in Y$, and $\zeta \in B_Y(R_Y)$

$$\|\mathcal{R}_\Lambda(\zeta)\xi\|_X \leq \|(\mathrm{Id} - I_h)\Lambda(\zeta)\xi\|_X + C_{\Lambda_h}\|(I_h - J_h)\xi\|_X, \tag{5.3}$$

$$\|\mathcal{R}_F(t,\zeta)\|_X \leq \|(\mathrm{Id} - I_h)F(t,\zeta)\|_X. \tag{5.4}$$

Proof. Let $t \in [0,T]$, $\xi \in Y$, and $\zeta \in B_Y(R_Y)$ arbitrary. Using the definition (4.4a) of $\mathcal{R}_\Lambda$ and the short notation (5.1a) for Λ_h we obtain

$$\begin{aligned}\mathcal{R}_\Lambda(\zeta) &= I_h\Lambda(I_h\zeta)J_h - \Pi_h\Lambda(\zeta)\\ &= I_h\Lambda(I_h\zeta)(J_h - I_h) + I_h(\Lambda(I_h\zeta)I_h - \Lambda(\zeta)) + (I_h - \Pi_h)\Lambda(\zeta).\end{aligned}$$

We further get from (3.5) the bound

$$\|I_h\Lambda(I_h\zeta)(J_h - I_h)\xi\|_X = \|\Lambda_h(I_h\zeta)(J_h - I_h)\xi\|_X \leq C_{\Lambda_h}\|(J_h - I_h)\xi\|_X.$$

Due to the definition (5.1b) of Λ_h, we have

$$I_h\big(\Lambda(I_h\zeta)I_h\xi - \Lambda(\zeta)\xi\big) = \sum_{m=0}^{M}\big(\lambda(\underline{x}^m,\zeta(\underline{x}^m))\xi(\underline{x}^m) - \lambda(\underline{x}^m,\zeta(\underline{x}^m))\xi(\underline{x}^m)\big)\,\phi_h^m = 0.$$

Finally, the definition (3.17) of Π_h yields

$$\|(I_h - \Pi_h)\Lambda(\zeta)\xi\|_X = \sup_{\|\xi_h\|_{X_h}=1}((I_h - \mathrm{Id})\Lambda(\zeta)\xi \mid \xi_h)_X = \|(I_h - \mathrm{Id})\Lambda(\zeta)\xi\|_X, \tag{5.5}$$

which proves (5.3).

Similarly, we obtain with the definition (4.4c) of $\mathcal{R}_F$ and the short notation (5.2a) for F_h

$$\begin{aligned}\mathcal{R}_F(t,\zeta) &= I_hF(t,I_h\zeta) - \Pi_hF(t,\zeta)\\ &= I_h\big(F(t,I_h\zeta) - F(t,\zeta)\big) + \big(I_h - \Pi_h\big)F(t,\zeta).\end{aligned}$$

Again, due to (5.2b) the first term vanishes, i.e., we have

$$I_h\big(F(t,I_h\zeta) - F(t,\zeta)\big) = \sum_{m=0}^{M}\big(f(t,\underline{x}^m,\zeta(\underline{x}^m)) - f(t,\underline{x}^m,\zeta(\underline{x}^m))\big)\xi(\underline{x}^m)\,\phi_h^m = 0.$$

Since the second term can be bounded as in (5.5), this proves (5.4). □

To conclude this section, we state the following refined version of Theorem 4.7 for nonlinearities which are local in space.

COROLLARY 5.1 Let (A_1)–(A_4) of Assumption 4.5 as well as Assumption 4.1 and Assumption 5.1 be satisfied. Then, Theorem 4.7 holds and for $T < t^*(y_0)$ the error satisfies

$$\begin{aligned}\|y(t) - y_h(t)\|_X \leq \|(\mathrm{Id} - J_h)y(t)\|_X + C(1+t)\mathrm{e}^{Ct}\Big(&\|J_hy_0 - y_{h,0}\|_X + \sup_{[0,t]}\|(I_h - J_h)y\|_X\\ &+ \sup_{[0,t]}\|(I_h - J_h)\partial_t y\|_X + \sup_{[0,t]}\|(\mathrm{Id} - I_h)\Lambda(y)\partial_t y\|_X\\ &+ \sup_{[0,t]}\|\mathcal{R}_A y\|_X + \sup_{[0,t]}\|(\mathrm{Id} - I_h)F(\cdot,y)\|_X\Big),\end{aligned}$$

with a constant $C > 0$ independent of h.

6. Application to specific examples

We conclude the theoretical part of this paper by showing that important classes of applications fit into our abstract framework and by providing more specific bounds for them. In particular, we investigate two specific examples from physics: the Maxwell equations with Kerr nonlinearity and the Westervelt equation. We emphasize that these examples are discussed in detail in Maier (2020), where Sections 3.3 and 5.2 are devoted to the Maxwell equations and Sections 3.2 and 5.1.2 to the Westervelt equation, respectively.

6.1. **Maxwell equations.** The Maxwell equations with Kerr nonlinearity are for a final time $T > 0$ and a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ given as

$$\begin{cases} \partial_t \mathcal{H} = -\nabla\times \mathcal{E}, & \text{on } [0,T]\times\Omega, \\ \big((1+|\mathcal{E}|^2\chi)\,\mathrm{Id} + 2(\mathcal{E}\otimes\mathcal{E})\chi\big)\partial_t \mathcal{E} = \nabla\times\mathcal{H}, & \text{on } [0,T]\times\Omega, \\ \mathcal{H}(0) = \mathcal{H}_0, \qquad \mathcal{E}(0) = \mathcal{E}_0 & \text{on } \Omega. \end{cases} \tag{6.1}$$

Here, $\nabla\times$ is the curl operator, $\otimes$ denotes the Kronecker product, and $\chi \in L^\infty(\Omega)$ is the nonlinear susceptibility. We consider these equations with given initial values $\mathcal{H}_0, \mathcal{E}_0 \colon \Omega \to \mathbb{R}^3$ and subject to homogeneous perfectly conducting boundary conditions.

Introducing for $y = (\mathcal{H}, \mathcal{E})$ the operators

$$\Lambda(y) = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & (1+\chi|\mathcal{E}|^2)\,\mathrm{Id} + 2\chi(\mathcal{E}\otimes\mathcal{E}) \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & -\nabla\times \\ \nabla\times & 0 \end{pmatrix}, \qquad F(y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

we observe that the Maxwell equations with Kerr nonlinearity (6.1) fit into the abstract framework (2.2). Then, Assumption 2.1 is satisfied for spaces

$$\begin{aligned} X &= L^2(\Omega)^3 \times L^2(\Omega)^3, \\ Y &= H^2(\Omega)^3 \times \{\varphi \in H^2(\Omega)^3 \mid \varphi\times\nu = 0\}, \\ Z_\partial &= H^p(\Omega)^3 \times H^p(\Omega)^3, \\ Z &= H^{p+1}(\Omega)^3 \times H^{p+1}(\Omega)^3, \end{aligned}$$

equipped with the standard inner products and $|\cdot|_Y = \|\cdot\|_Y$. Here, $\times$ denotes the cross product and ν is the outer unit normal of Ω. Moreover, Assumption 2.2 follows from (Spitz, 2019, Thm. 5.3) if $\partial\Omega$ is sufficiently smooth. However, since we only consider $X_h \subset X$ and thus particularly that both spaces are based on the same spatial domain Ω here, Ω is assumed to be a polyhedron. Thus, we require that Assumption 2.2 is also valid in this case.

We use the discontinuous Galerkin finite element method to discretize in space. More precisely, we introduce the discrete function space $V_h \subset L^2(\Omega)^6$ consisting of piecewise polynomials of degree at most $p \in \mathbb{N}$ and define the discrete spaces (3.1) with

$$\|\cdot\|_{Y_h} = \|\cdot\|_{L^\infty(\Omega)^3\times L^\infty(\Omega)^3}, \qquad\qquad |\cdot|_{Y_h} = \|\cdot\|_{Y_h}.$$

Then, Assumption 3.1 is satisfied for $R_{Y_h} < (9\|\chi\|_{L^\infty(\Omega)})^{-\frac{1}{2}}$. Moreover, the inverse estimate (3.3) and the bound (4.7) hold with

$$C_{X_h,Y_h}(h) = C, \qquad C_{Y_h,X_h}(h) = Ch^{-\frac{3}{2}}, \qquad C_{\mathcal{A}_h,Y_h,X_h}(h) = Ch^{-\frac{5}{2}}.$$

Finally, we choose I_h to be the Lagrange interpolation operator and $J_h = I_h$. Up to our knowledge, Corollary 5.1 then yields the first rigorous error estimate for quasilinear Maxwell equations.

THEOREM 6.1 Let Assumption 4.1 be true, χ sufficiently smooth, and $p \geq 3$. For $T > 0$, let the solution $y = (\mathcal{H}, \mathcal{E})$ of (6.1) satisfy

$$y \in C^1([0,T], Z_\partial) \cap C([0,T], Z).$$

Then, there is $h_0 > 0$ such that for all $h < h_0$ the discrete approximation $y_h = (\mathcal{H}_h, \mathcal{E}_h)$ satisfies

$$y_h \in C^1([0,T], X_h) \cap C([0,T], B_{Y_h}(R_{Y_h})).$$

Further, we have for $t \in [0,T]$

$$\|\mathcal{H}(t) - \mathcal{H}_h(t)\|_{L^2(\Omega)^3} + \|\mathcal{E}(t) - \mathcal{E}_h(t)\|_{L^2(\Omega)^3} \leq C_{\mathcal{H},\mathcal{E},\chi}(1+t)\mathrm{e}^{Ct}h^p,$$

where $C_{\mathcal{H},\mathcal{E},\chi}, C > 0$ are constants independent of h, t, and T, but $C_{\mathcal{H},\mathcal{E},\chi}$ depends on $\mathcal{H}$, $\mathcal{E}$, and χ, including their derivatives.

To conclude, we emphasize that Maier (2020) also considers more general nonconforming space discretizations, including the case $X_h \not\subset X$. In particular, this allows for an approximation of the spatial domain Ω using discontinuous isoparametric finite elements. Thus, (Spitz, 2019, Thm. 5.3) is applicable to prove Assumption 2.2. However, note that in this case we have to use piecewise polynomials of degree $p+1$ for the discrete space for the electric field $\mathcal{E}_h$. This is due to the fact that the boundary condition for $\mathcal{E}$ is not exactly true but has to be approximated with the corresponding order of convergence in h.

6.2. **Westervelt equation.** In nonlinear acoustics, the Westervelt equation (Westervelt, 1963) is a basic model for the propagation of ultrasound. For a finite time interval $[0,T]$ and a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, the pressure $u\colon [0,T] \times \Omega \to \mathbb{R}$ satisfies

$$\begin{cases} (1-\kappa u)\partial_t^2 u = \Delta\, u + \kappa(\partial_t u)^2 & \text{on } [0,T] \times \Omega, \\ \qquad u(0) = u_0, \qquad \partial_t u(0) = v_0 & \text{on } \Omega, \end{cases} \tag{6.2}$$

with given initial values $u_0, v_0 : \Omega \to \mathbb{R}$ and subject to homogeneous Dirichlet boundary conditions. The parameter $\kappa \in \mathbb{R}$ models the nonlinearity of the medium.

With the operators

$$\Lambda(y) = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 1-\kappa u \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta & 0 \end{pmatrix}, \qquad F(y) = \begin{pmatrix} 0 \\ \kappa(\partial_t u)^2 \end{pmatrix},$$

for $y = (u, \partial_t u)$, the Westervelt equation (6.2) fits into the abstract framework (2.2). Moreover, Assumption 2.1 is satisfied for

$$\begin{aligned} X &= H_0^1(\Omega) \times L^2(\Omega), \\ Y &= \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times \big(H^2(\Omega) \cap H_0^1(\Omega)\big), \\ Z_\partial &= \big(H^p(\Omega) \cap H_0^1(\Omega)\big) \times \big(H^{p-1}(\Omega) \cap H_0^1(\Omega)\big), \\ Z &= \big(H^{p+1}(\Omega) \cap H_0^1(\Omega)\big) \times \big(H^p(\Omega) \cap H_0^1(\Omega)\big), \end{aligned}$$

equipped with the standard inner products and

$$|\xi|_Y = \|\xi^v\|_{H^2(\Omega) \cap H_0^1(\Omega)}, \qquad \xi = (\xi^u, \xi^v) \in Y.$$

Then, (Dörfler *et al.*, 2016, Thm. 4.1) yields existence of a unique solution of (6.2) for $\partial\Omega$ being sufficiently smooth. Thus, Assumption 2.2 is justified.

For the discretization in space we use finite elements of order $p \in \mathbb{N}$ to define the approximation space $V_h \subset C(\Omega)^2$. We then define the discrete spaces as in (3.1) with

$$\|\xi_h\|_{Y_h} = \|\xi_h\|_{L^\infty(\Omega)\times L^\infty(\Omega)}, \qquad |\xi_h|_{Y_h} = \|\psi_h\|_{L^3(\Omega)}, \qquad \xi_h = (\varphi_h, \psi_h) \in V_h.$$

Then, Assumption 3.1 is satisfied for $R_{Y_h} < |\kappa|^{-1}$, where we follow the arguments of (Maier, 2020, Lem. 5.1), but now present an improved estimate for the Lipschitz continuity (3.6b) of Λ_h using the seminorm in Y_h, Hölder's inequality, and the Sobolev inequality, i.e., we have

$$\begin{aligned}\|\big(\Lambda_h(\varphi_h) - \Lambda_h(\psi_h)\big)\xi_h\|_X &= \|\kappa(\varphi_h^u - \psi_h^u)\xi_h^v\|_{L^2(\Omega)}\\ &\le |\kappa|\|\varphi_h^u - \psi_h^u\|_{L^6(\Omega)}\|\xi_h^v\|_{L^3(\Omega)} \le C\|\varphi_h - \psi_h\|_X|\xi_h|_{Y_h},\end{aligned}$$

for $\varphi_h = (\varphi_h^u, \varphi_h^v), \psi_h = (\psi_h^u, \psi_h^v) \in B_{Y_h}(R_{Y_h})$ and $\xi_h = (\xi_h^u, \xi_h^v) \in Y_h$. Moreover, for (3.3) we obtain with inverse estimates

$$C_{X_h,Y_h}(h) = Ch^{-1}, \qquad\qquad C_{Y_h,X_h}(h) = Ch^{-\frac{d}{2}}.$$

With the discrete Laplace operator $\Delta_h \colon V_h \to V_h$ defined by

$$(\Delta_h \varphi_h \mid \psi_h)_{L^2(\Omega)} = (\varphi_h \mid \psi_h)_{H_0^1(\Omega)}, \qquad\qquad \varphi_h, \psi_h \in V_h,$$

we further obtain the bound

$$\begin{aligned}|\mathcal{A}_h(\xi_h)\zeta_h|_{Y_h} &= \|(1-\kappa\xi_h^u)\,\Delta_h\,\zeta_h^v\|_{L^3(\Omega)}\\ &\le C\|1-\kappa\xi_h^u\|_{L^\infty(\Omega)}h^{-\frac{d}{6}}\|\Delta_h\,\zeta_h^v\|_{L^2(\Omega)}\\ &\le Ch^{-1-\frac{d}{6}}\|\zeta_h\|_X,\end{aligned}$$

for $\zeta_h = (\zeta_h^u, \zeta_h^v) \in X_h$ and $\xi_h = (\xi_h^u, \xi_h^v) \in B_{Y_h}(R_{Y_h})$ with $R_{Y_h} < |\kappa|^{-1}$. This yields (4.7).

Finally, we choose I_h as the Lagrange interpolation operator. With the representations

$$I_h = \begin{pmatrix} I_h^u & 0 \\ 0 & I_h^v \end{pmatrix}, \qquad\qquad \Pi_h = \begin{pmatrix} \Pi_h^u & 0 \\ 0 & \Pi_h^v \end{pmatrix},$$

we further define the reference operator by

$$J_h = \begin{pmatrix} \Pi_h^u & 0 \\ 0 & I_h^v \end{pmatrix}.$$

The abstract result from Corollary 5.1 then yields the following.

THEOREM 6.2 Let Assumption 4.1 be true, $p \ge 2$, and $T > 0$. If

$$(u, \partial_t u) = y \in C^1([0,T], Z_\partial) \cap C([0,T], Z)$$

is the solution of (6.2), then there exists $h_0 > 0$ such that for all $h < h_0$ the discrete approximation satisfies

$$(u_h, v_h) = y_h \in C^1([0,T], X_h) \cap C([0,T], B_{Y_h}(R_{Y_h})).$$

Further, we have for all $t \in [0,T]$ the estimate

$$\|u(t) - u_h(t)\|_{H_0^1(\Omega)} + \|\partial_t u(t) - v_h(t)\|_{L^2(\Omega)} \le C_u(1+t)\mathrm{e}^{Ct}h^p, \tag{6.3}$$

where $C_u, C > 0$ are constants independent of h, t, and T, but C_u depends on u including derivatives.

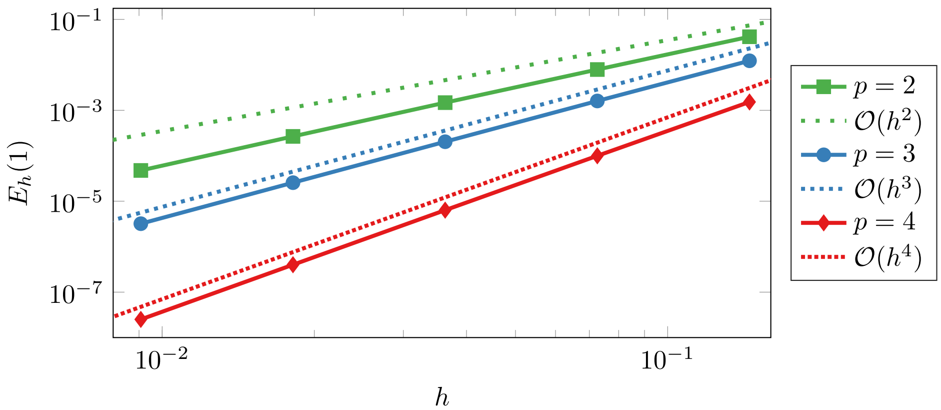


FIGURE 4. Space discretization error $E_h(1)$ defined in (7.2) for different polynomial degrees $p = 2, 3, 4$ and various space discretization parameters h.

We emphasize that due to the introduction of the seminorms $|\cdot|_Y$ and $|\cdot|_{Y_h}$ we obtain a more relaxed constant $C_{\max}(h)$ here. Eventually, this yields that the result stated above is also valid for $p = 2$, whereas only $p \geq 3$ is allowed in (Maier, 2020, Thm. 5.9).

Note that in (Maier, 2020, Sec. 5.1) also the one-dimensional case ($d = 1$) as well as nonconforming space discretizations including domain approximation with isoparametric finite elements are considered. As for the Maxwell equations, this allows to close the gap with respect to the wellposedness result, which is only applicable for spatial domains with smooth boundaries.

In the literature, the term "Westervelt equation" also refers to a refined version of (6.2) with strong damping, i.e.,

$$\begin{cases} (1-\kappa u)\partial_t^2 u = \Delta u + b\,\Delta\,\partial_t u + \kappa(\partial_t u)^2 & \text{on } [0,T]\times\Omega, \\ \qquad u(0) = u_0, \qquad \partial_t u(0) = v_0 & \text{on } \Omega, \end{cases}$$

with the sound diffusivity $b > 0$. Due to this additional damping term, the refined problem behaves rather parabolically than hyperbolically. The space discretization of this model was considered by Nikolić & Wohlmuth (2019) and Antonietti *et al.* (2020) using continuous and discontinuous Galerkin finite elements, respectively. Based on a rigorous error analysis, which is tailored for the presence of strong damping, they obtain stronger convergence rates than in Theorem 6.2. However, these results deteriorate for $b \to 0$.

7. NUMERICAL EXPERIMENT

We finally present numerical results based on an implemenation with the C++ finite element library MFEM (2018). The code to reproduce the computational results of this experiment is available on `https://doi.org/10.5445/IR/1000128712`.

In order to validate the error estimate (6.3), we consider a modified variant of the Westervelt equation (6.2) with a right-hand side $f\colon [0,T]\times\Omega\to\mathbb{R}$, i.e.,

$$\begin{cases} (1-\kappa u)\partial_t^2 u = \Delta u + \kappa(\partial_t u)^2 + f & \text{on } [0,T]\times\Omega, \\ \qquad u(0) = u_0, \qquad \partial_t u(0) = v_0 & \text{on } \Omega. \end{cases} \tag{7.1}$$

Note that contrary to the original equation (6.2) we can easily construct solutions of the modified variant by choosing f accordingly. We further emphasize that for f sufficiently regular, Theorem 6.2 is also valid under this modification.

For the numerical example, we fix $\kappa = 1$ and choose the initial values u_0, v_0 as well as the right-hand side f such that

$$u(t,x) = x_1 x_2 \sin\bigl(2\pi x_1\bigr) \sin\bigl(3\pi x_2\bigr) \cos\bigl(\tfrac{\pi}{2} t\bigr), \qquad t \in [0,1],\ x = (x_1, x_2) \in [0,1]^2,$$

solves (7.1) with $T = 1$ and $\Omega = [0,1]^2$. For the space discretization, we use continuous finite elements of order $p = 2, 3, 4$ on an unstructured triangular grid. For the discretization in time, we apply a linearized variant of the leapfrog scheme which is discussed in detail in (Maier, 2020, Sec. 7.2) including a rigorous error analysis. The time-step size $\tau = 10^{-4}$ is chosen sufficiently small such that the space discretization error dominates.

In Figure 4, the error

$$E_h(T) = \max_{t \in [0,T]} \Bigl\{ \|u(t) - u_h(t)\|_{H_0^1(\Omega)} + \|\partial_t u(t) - v_h(t)\|_{L^2(\Omega)} \Bigr\} \tag{7.2}$$

is depicted for various space discretization parameters h. Corresponding to Theorem 6.2 we observe convergence of order at least p for all computations.

Acknowledgment

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

References

ANTONIETTI, P. F., MAZZIERI, I., MUHR, M., NIKOLIĆ, V. & WOHLMUTH, B. (2020) A high-order discontinuous Galerkin method for nonlinear sound waves. *Journal of Computational Physics*, **415**, 109484.

BALES, L. A. (1986) Higher-order single-step fully discrete approximations for nonlinear second-order hyperbolic equations. *Computers & Mathematics with Applications*, **12**, 581–604.

BALES, L. A. (1988) Error estimates for single step fully discrete approximations for nonlinear second order hyperbolic equations. *Computers & Mathematics with Applications*, **15**, 535–553.

BALES, L. A. & DOUGALIS, V. A. (1989) Cosine methods for nonlinear second-order hyperbolic equations. *Math. Comp.*, **52**, 299–319, S15–S33.

DÖRFLER, W., GERNER, H. & SCHNAUBELT, R. (2016) Local well-posedness of a quasilinear wave equation. *Appl. Anal.*, **95**, 2110–2123.

EWING, R. E. (1980) On efficient time-stepping methods for nonlinear partial differential equations. *Comput. Math. Appl.*, **6**, 1–13.

GAUCKLER, L., LU, J., MARZUOLA, J. L., ROUSSET, F. & SCHRATZ, K. (2019) Trigonometric integrators for quasilinear wave equations. *Math. Comp.*, **88**, 717–749.

GERNER, H. (2013) Stetige Galerkinverfahren für zeitabhängige Maxwellgleichungen mit Kerr-Nichtlinearität. *Ph.D. thesis*, Karlsruhe Institute for Technology (KIT).

HIPP, D., HOCHBRUCK, M. & STOHRER, C. (2019) Unified error analysis for nonconforming space discretizations of wave-type equations. *IMA J. Numer. Anal.*, **39**, 1206–1245.

HOCHBRUCK, M. & LEIBOLD, J. (2020) Finite element discretization of semilinear acoustic wave equations with kinetic boundary conditions. *Electron. Trans. Numer. Anal.*, **53**, 522–540.

KATO, T. (1970) Linear evolution equations of "hyperbolic" type. *J. Fac. Sci. Univ. Tokyo Sect. I*, **17**, 241–258.

KATO, T. (1975) Quasi-linear equations of evolution, with applications to partial differential equations. *Spectral theory and differential equations (Proc. Sympos., Dundee, 1974; dedicated to Konrad Jörgens)*. Springer Berlin Heidelberg, pp. 25–70. Lecture Notes in Math., Vol. 448.

MAIER, B. (2020) Error analysis for space and time discretizations of quasilinear wave-type equations. *Ph.D. thesis*, Karlsruhe Institute for Technology (KIT), Germany.

MAKRIDAKIS, C. G. (1993) Finite element approximations of nonlinear elastic waves. *Math. Comp.*, **61**, 569–594.

MFEM (2018) MFEM: Modular Finite Element Methods. `http://mfem.org`. [Software].

NIKOLIĆ, V. & WOHLMUTH, B. (2019) A priori error estimates for the finite element approximation of Westervelt's quasi-linear acoustic wave equation. *SIAM J. Numer. Anal.*, **57**, 1897–1918.

ORTNER, C. & SÜLI, E. (2007) Discontinuous Galerkin finite element approximation of nonlinear second-order elliptic and hyperbolic systems. *SIAM J. Numer. Anal.*, **45**, 1370–1397.

SPITZ, M. (2019) Local wellposedness of nonlinear Maxwell equations with perfectly conducting boundary conditions. *J. Differential Equations*, **266**, 5012–5063.

WESTERVELT, P. J. (1963) Parametric Acoustic Array. *The Journal of the Acoustical Society of America*, **35**, 535–537.