Contracting boundaries of amalgamated free products of CAT(0) groups with applications for right-angled Coxeter groups

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Annette Karrer

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1. Referentin:	Prof. Dr. Petra Schwer
2. Referent:	Prof. Dr. Tobias Hartnick

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Contents

1 Introduction

In this thesis, we explore topological spaces associated to CAT(0) groups that are called contracting boundaries. They generalize Gromov boundaries and were introduced by Charney and Sultan in [CS15]. Cordes generalized contracting boundaries to Morse boundaries [Cor17]. Thus, contracting boundaries are also known as Morse boundaries. We study contracting boundaries of amalgamated free products of CAT(0) groups and focus on situations where totally disconnected contracting boundaries are involved. We use our insights for examining which right-angled Coxeter groups have totally disconnected contracting boundaries.

We motivate the interest in totally disconnected contracting boundaries with two examples of right-angled Coxeter groups. Let Λ be a finite simplicial graph with vertex set S of size n and edge set E. The *right-angled Coxeter group* associated to Λ is the group

$$W_\Lambda = \langle S \mid s^2 = \mathrm{id} \text{ for all } s \in S, ss' = s's \text{ for all } \{s, s'\} \in E \rangle. \tag{1.0.1}$$

We say that Λ is the *defining graph* of W_Λ. Every right-angled Coxeter group W_Λ has an associated cube complex, called *Davis complex* [Dav08]. The Davis complex can be constructed as follows. Whenever we find the 1-skeleton of an n-cube in the Cayley graph associated to the presentation in Equation (1.0.1), we attache a filled Euclidean n-cube. This way, we obtain a cube complex consisting of Euclidean cubes. For instance, Figure 1.1 shows the Davis complex of a right-angled Coxeter group whose defining graph is a 5-cycle.

This complex is an example of a *hyperbolic space*. A *hyperbolic space* is a geodesic metric space in which large geodesic triangles are *thin*, i.e. there exists $\delta > 0$ such that the δ-neighborhood of any two sides of the triangle contains the third side. The *Gromov boundary* of a hyperbolic space is a topological space that can be described as the space of directions in which we are able to go if we start at a fixed base point. It consists of equivalence classes of geodesic rays. Two geodesic rays are equivalent if their Hausdorff distance is bounded. In the topology of this space, two equivalence classes of geodesic rays are close to each other if they stay close for a long time. The Gromov boundary of a hyperbolic group is the Gromov boundary of a hyperbolic space on which it acts geometrically. For instance, if W_Λ is a right-angled Coxeter group, it acts geometrically on its Davis complex. The Gromov boundary of the Davis complex pictured in Figure 1.1 is a 1-sphere. Thus, the contracting boundary of the associated right-angled Coxeter group is a 1-sphere.

We study the Davis complex in Figure 1.1 more thoroughly. Let C be the cycle pictured on the left. The vertices b, c and d build a 2-path P_2 in C. Every pink strip is

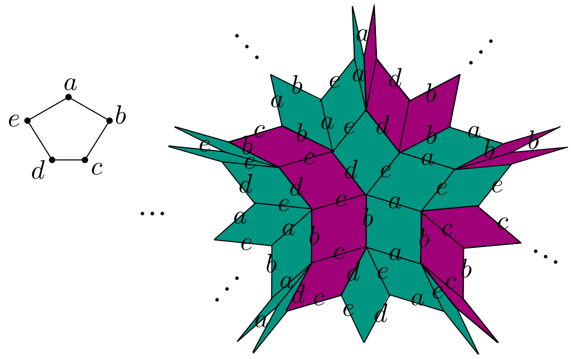

Figure 1.1 Left: a 5-cycle C. Right: The Davis complex associated to the right-angled Coxeter group with defining graph C. The pink strips are subcomplexes isometric to the Davis complex of the right-angled Coxeter group that has the 2-path b, c, d as defining graph.

isometric to the Davis complex of a right-angled Coxeter group that has P_2 as defining graph. Suppose that we delete all pink strips. Then the Davis complex decomposes into uncountably many green subcomplexes. Each such green subcomplex is isometric to the Davis complex of the right-angled Coxeter group whose defining graph is the path $P_3 = b, a, e, d$. If we look at such a complex from far afar, it looks like an infinite tree whose vertices have degree 4, i.e., it is quasi-isometric to a 4-valent tree. See Figure 1.2. Every pink strip is glued to two green subcomplexes along its sides. The side of every

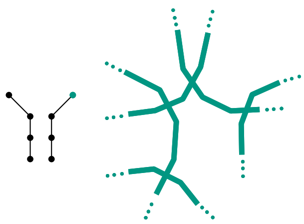

Figure 1.2 Left: A 3-path. Right: The Davis complex associated to the right-angled Coxeter group with defining graph a 3-path. It looks like a 4-valent tree if we look at it from far afar.

pink strip is isometric to the Davis complex of the right-angled Coxeter group whose defining Graph C' consists of the vertices b and d.

These observations have interesting consequences. For instance, the group W_C can be written as the amalgamated free product $W_C = W_{P_2} *_{C'} W_{P_3}$. The group $W_{C'}$ is quasi-isometric to $\mathbb{Z}$, i.e., W_C *splits* over a group quasi-isometric to $\mathbb{Z}$. We look at the corresponding vertices b and d in C. If we delete these two vertices form the graph, the graph decomposes into two components. Similarly, the Davis complex decomposes into two complexes if we delete a side of one pink strip. Every side of a pink strip corresponds to two 'directions', i.e. to two points in the boundary. Recall that the boundary is a

1-sphere. If we delete two points from a 1-sphere, the 1-sphere decomposes into more than one component. We say that two points whose deletion decomposes the space build a *cut pair*. We see, there is an interesting relation between cut pairs in the boundary, cut sets in the Davis complex and splittings over subgroups that are quasi-isometric to $\mathbb{Z}$. Bowditch studied this interaction in general and found out that cut pairs in the Gromov boundary of a hyperbolic group allow conclusions about splittings of the group. He proved the following (See p. 21 for a definition of Fuchsian groups):

Theorem 1.1 ([Bow98a]). *The boundary of a (non-Fuchsian) hyperbolic group has a cut pair if and only if the group splits over a subgroup quasi-isometric to $\mathbb{Z}$.*

We consider now the boundaries of the pink and the green subcomplexes. The boundary of each pink strip corresponds to two directions. Thus, the Gromov boundary of a pink strip consists of two single points. Next, we consider the green subcomplexes. Every 'direction' is given by the equivalent class of an infinite geodesic ray in a 4-valent tree. In this case, the Gromov boundary is a Cantor set. The boundaries of both types of subcomplexes are totally disconnected. Surprisingly, the boundary of the whole complex is a 1-sphere.

On the other extreme, we consider the example pictured in Figure 1.3.

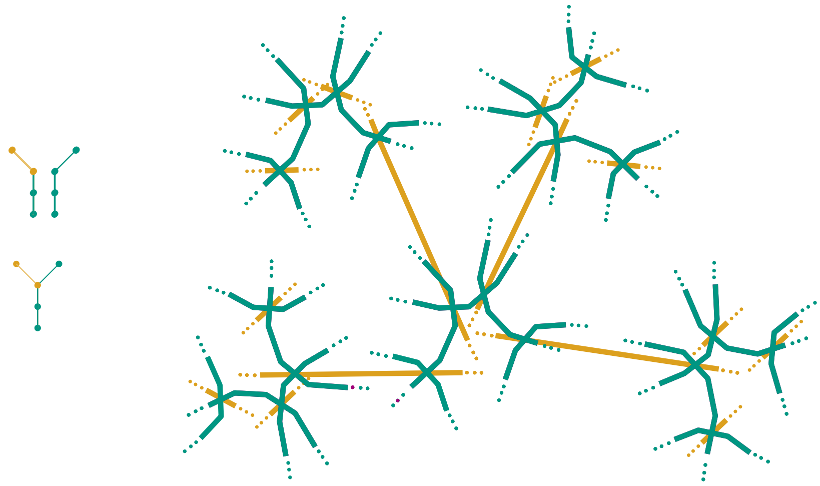

Figure 1.3 Left: The graph at the bottom contains the two graphs above as induced subgraphs. Right: The Davis complex associated to the right-angled Coxeter group with defining graph pictured left at the bottom. The picture is skewed for highlighting the structure of the Davis complex: it looks like a 6-valent tree. The contracting boundary of the pictured Davis complex is totally disconnected.

Left above, we see two 3-paths that can be glued such that we obtain the graph pictured left at the bottom. We denote it by Λ. The Davis complex associated to a right-angled Coxeter group with defining graph a 3-path is pictured in Figure 1.2. The Davis complex associated to W_Λ looks like a 6-valent tree if we look at it form far afar.

As before, the group splits over a group quasi-isometric to $\mathbb{Z}$. But in this case, the Davis complex has totally disconnected Gromov boundary. Why do we obtain a whole sphere in the first case but a totally disconnected boundary in the second case? In contrast to the second case, the Davis complex in the first case is quasi-isometric to the hyperbolic plane. Might it be that the Davis complex of any hyperbolic right-angled Coxeter group with a 1-sphere as Gromov boundary is quasi-isometric to the hyperbolic plane? The answer to this question is positive. Furthermore, we obtain a positive answer too if we probe an analogous question for 2-dimensional spheres. Indeed, hyperbolic right-angled Coxeter groups satisfy Cannon's Conjecture [Haï15].

Conjecture 1.2 (Cannon's Conjecture [Can91])**.** *A hyperbolic group whose Gromov boundary is the* 2*-sphere* S^2 *is virtually Kleinian, i.e., it contains a subgroup of finite index that is isomorphic to a discrete subgroup of* $PSL(2,\mathbb{C})$*. In particular, if the group is torsion-free, then it is isomorphic to the fundamental group of a closed hyperbolic 3-manifold.*

We studied boundaries of hyperbolic groups so far. But not all right-angled Coxeter groups are hyperbolic. The Davis complex of a right-angled Coxeter group with a 4-cycle as defining graph is isometric to the Euclidean plane, and the Euclidean plane is the standard example of a CAT(0) space that is not hyperbolic. See Figure 1.4. Now,

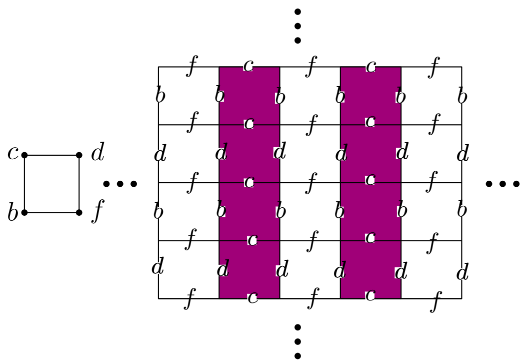

Figure 1.4 Left: a 4-cycle. Right: The Davis complex associated to the right-angled Coxeter group with defining graph a 4-cycle.

Charney and Sultan introduced a boundary for CAT(0) groups. This enables us to examine the phenomena shown in the examples above for contracting boundaries of CAT(0) groups. Since every Davis complex is a CAT(0) cube complex [Gro87], we can examine the contracting boundary of every right-angled Coxeter group. The contracting boundary of a complete CAT(0) space is a topological space that can be described as the space of directions in which the space looks "hyperbolic-like". In this sense, the contracting boundary of a CAT(0) space is a generalization of the Gromov boundary and measures how hyperbolic-like a group behaves. For instance, the contracting boundary of the complex pictured in Figure 1.1 coincides with its Gromov boundary and is a 1-sphere. The Davis complex pictured in Figure 1.4 has no hyperbolic-like behavior. Its contracting boundary is empty. Analogously to the Gromov boundary, the contracting boundary can be defined for CAT(0) groups, i.e. groups that act geometrically on CAT(0) spaces. The contracting boundary of a CAT(0) group G is defined as the contracting boundary of a CAT(0) space on which G acts geometrically.

Motivated by the made considerations, we pursue the target to find out which right-angled Coxeter groups have totally disconnected contracting boundaries. The starting point for our research is a conjecture formulated by Tran in [Tra19, Conj. 1.14]. This conjecture is related to an example of a right-angled Coxeter group with totally disconnected contracting boundary that was investigated by Charney and Sultan in Section 4.2 in [CS15]. In their calculation of the contracting boundary of this right-angled Coxeter group, Charney and Sultan used a certain decomposition of the defining graph. Inspired by this, we study the following question:

Question 1. Suppose that Λ is the union of two induced subgraphs Λ_0 and Λ_1, i.e. every edge in Λ connecting two vertices of Λ_i is contained in Λ_i, $i \in \{0, 1\}$. Assume that the contracting boundaries of W_{Λ_0} and W_{Λ_1} are known. When is the contracting boundary of W_Λ totally disconnected?

Suppose that Λ is the union of two induced subgraphs Λ_0 and Λ_1 with intersection graph Λ_*. Then the group W_Λ can be written as the amalgamated free product $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$. Thereby, the four groups $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$, W_{Λ_0}, W_{Λ_1} and W_{Λ_*} each act on an associated Davis complex. Recall that every Davis complex is a CAT(0) space. So, $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$ is an example of a CAT(0) group that is an amalgamated free product of two CAT(0) groups along a CAT(0) group. Suppose that G_0, G_1 and H are CAT(0) groups. The Equivariant Gluing Theorem of Bridson and Haefliger [BH99, Thm 11.18 in II] formulates conditions under which the amalgamated free product $G = G_0 *_H G_1$ is a CAT(0) group. The group $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$ is an example satisfying these conditions. Thus, we can solve Question 1 by examining amalgamated free products of CAT(0) groups that satisfy the conditions of the Equivariant Gluing Theorem of Bridson and Haefliger [BH99, Thm 11.18]. For such groups, we study the following question.

Question 2. Suppose that the contracting boundaries of G_0 and G_1 are known. When is the contracting boundary of $G = G_0 *_H G_1$ totally disconnected?

To examine this question, we study visual boundaries of spaces on which such groups act geometrically and deduce consequences for contracting boundaries. First, we work in the general setting of amalgamated free products of CAT(0) groups. Afterwards, we apply our results to right-angled Coxeter groups.

1.1 History of research

Two important theorems can be seen as the roots of this thesis. The first is the Morse Lemma about stability of quasi-geodesics in hyperbolic spaces. The origin of this theorem and of the notion of Morse geodesic rays lies in [Mor24] and [Mor21]. The Morse Lemma is a key for the pleasant behavior of hyperbolicity and can be seen as an important pillar in geometric group theory. One object arising from this pillar is the Gromov boundary. The contracting boundary is a generalization of the Gromov boundary. Thus, it comes as no surprise that contracting boundaries are also known as *Morse boundaries*.

The second classic theorem at the roots of this thesis is the Seifert–van Kampen Theorem [Sei31; VK33]. Suppose that we would like to understand the fundamental group of a space that is covered by two open, path-connected subspaces. The Seifert–van Kampen Theorem says that the fundamental group is the amalgamated free product of two fundamental groups in that case. This observation has wide-ranging consequences. It can be seen as the starting point of the Bass-Serre theory. As this thesis concerns amalgamated free products, it is deeply connected to that area. We summarize the history of research on the two described perspectives in the following two subsections.

1.1.1 History of research on contracting boundaries

In this subsection, we summarize the research on contracting boundaries in general. We consider the research on contracting boundaries of right-angled Coxeter groups in an own section and refer the reader to Section 5.1.

As described above, the root of contracting boundaries can be seen in the Morse Lemma about stability of quasi-geodesics in hyperbolic spaces that has its origin in [Mor24] and [Mor21]. The Morse lemma says that quasi-geodesics stay close to geodesics: if a quasi-geodesic connects two points of a geodesic, then the distance of any point on the quasi-geodesic to the geodesic is bounded by a constant that depends only on the quasi-constants. Using the Morse Lemma, one can prove that the Gromov boundary is a quasi-isometry invariant, i.e., two quasi-isometric hyperbolic spaces have homeomorphic Gromov boundaries. This crucial property enables us to define Gromov boundaries not only for hyperbolic spaces but also for hyperbolic groups, i.e., groups that act *geometrically* on hyperbolic spaces. A group acts *geometrically* on a metric space if it acts on it properly and cocompactly by isometries. If X is a metric space, we denote its isometry group by $Isom(X)$. A metric space X is *cocompact* if there exists a compact set $K \subset X$ such that $X = \cup_{g \in Isom(X)} gK$.

As a hyperbolic space, a complete CAT(0) space has a boundary, called the *visual boundary*, defined analogously to the Gromov boundary. In the case that a CAT(0) space is hyperbolic, its visual boundary coincides with its Gromov boundary. Recall, a group is a CAT(0) group if it acts geometrically on a CAT(0) space. One may hope that it is possible to define the visual boundary of a CAT(0) group as the visual boundary of a CAT(0) space on which it acts geometrically. Sadly, this is not possible. Croke and

Kleiner [CK00] found an example of a CAT(0) group that acts geometrically on two different CAT(0) spaces whose visual boundaries are not homeomorphic. Why do rays in CAT(0) boundaries behave differently to Gromov boundaries? And which properties of the CAT(0) boundary are CAT(0) group invariant? Charney and Sultan examined these questions in [CS15]. They used the notion of *contracting geodesic rays* of Bestvina and Fujiwara [BF09] for defining a new boundary for CAT(0) groups. This notion is based on a more general variant of contracting geodesic rays in [MM99]. Charney and Sultan observed that contracting geodesic rays behave like geodesic rays in hyperbolic spaces, i.e., they satisfy the properties of geodesic rays in the Morse Lemma. That evokes the idea to equip this subset of the visual boundary with a topology and to examine the matter if the resulting topological space is a quasi-isometry invariant. A first possibility would be to equip this set with the subspace topology of the visual boundary. However, this turns out not to be a good choice. Cashen [Cas16] has proven that the resulting topological space is not a quasi-isometry invariant. Accordingly, Charney and Sultan equipped the set of equivalence classes of contracting geodesic rays with another topology - namely with a direct limit topology. The resulting topological space is the contracting boundary of the underlying space. If the underlying space is hyperbolic, the contracting boundary coincides with the Gromov boundary. In this sense, the contracting boundary is a generalization of the Gromov boundary. It is a quasi-isometry invariant like the Gromov boundary. This way it is possible to define the contracting boundary of a CAT(0) group to be the contracting boundary of a CAT(0) space on which the group acts geometrically. The key-property of the contracting boundary is that its elements are equivalence classes of geodesic rays behaving as in hyperbolic spaces, i.e., they satisfy the conditions of the Morse Lemma. Might it be possible to concentrate on this property and to define a boundary for proper metric spaces that behaves similarly to the contracting boundary? Indeed, this is possible. This generalization was done by Cordes. He generalized in [Cor17] contracting boundaries to *Morse boundaries* of proper metric spaces. As the contracting boundary, the Morse boundary is a quasi-isometry invariant. The Morse boundary of a finitely generated group is the Morse boundary of a proper metric space on which the group acts geometrically. In the case of CAT(0) spaces, it coincides with the contracting boundary. Thus, contracting boundaries of complete CAT(0) spaces are also known as *Morse boundaries*. Furthermore, Cordes defined Morse boundaries for not-necessarily-proper geodesic spaces. This can be found in the survey of Cordes about Morse boundaries [Cor19]. In [Cor19], Cordes survives known results concerning Morse boundaries. We highlight a few results explained in this survey and expand it with new results. For more details, we refer to [Cor19].

In CAT(0) spaces, a geodesic ray is Morse if and only if it is contracting. In general, being contracting is a stronger property than being Morse. Though, the 'contracting property' is a useful tool to study Morse boundaries. Morse geodesics in geodesic metric spaces can be examined by studying their contracting behavior and their divergence. For CAT(0) spaces, we refer to Theorem 2.14 in [CS15]. For characterizations of Morse geodesics by divergence, see [Arz+17].

It is important to notice that many groups have empty Morse boundaries ([DS05]). The only information we can read off an empty Morse boundary is that the group does not have any geodesic ray that behaves like a geodesic ray in a hyperbolic space. If the Morse boundary of a finitely generated group is not empty, it carries interesting information about the large-scale geometry of the group. As the Morse boundary of a proper metric space measures how hyperbolic-like the space behaves it is interesting to study Morse boundaries of groups that are itself not hyperbolic but have some hyperbolic-like behavior. One important class of groups with nonempty Morse boundary are acylindrically hyperbolic groups. Acylindrically hyperbolic groups occurred parallel to diverse other classes of groups in literature which turned all to be the same class of groups [Osi16]. Many groups of great interest are acylindrically hyperbolic. Cordes [Cor19] mentioned the following acylindrically hyperbolic groups in his survey: hyperbolic groups, relative hyperbolic groups, non-directly decomposable right-angled Artin groups, mapping class groups, and $Out(F_n)$. Sisto [Sis16] showed that every acylindrically hyperbolic group has a bi-infinite Morse geodesic. In particular, their Morse boundaries are not empty. Cordes remarked that acylindrical hyperbolic groups are not the only groups with nonempty Morse boundaries and refereed to [OsOS09].

Some properties of Gromov boundaries can be transferred to Morse boundaries. The Gromov boundary and the Morse boundary of proper metric spaces are visibility spaces. This was proven in [CS15] for contracting boundaries. It was proven in [Cor17] for proper metric spaces this. Further phenomena were mentioned in [CCS19]: Charney, Cordes and Murray [CCM19] proved that a homeomorphism between the Morse boundaries of two proper, cocompact spaces is induced by a quasi-isometry if and only if the homeomorphism is quasi-mobius and 2-stable. This is a counterpart to a classical theorem by Paulin [Pau96]. In [CH17], Cordes and Hume showed that the Morse boundary is recovered as a direct limit of the usual Gromov boundaries of certain hyperbolic subspaces that are *N-stable* for some $N \in \mathbb{N}$.

Another interesting property of Gromov boundaries of hyperbolic groups is that its axial isometries act by North-South Dynamics on its boundary. North-South Dynamics has many interesting consequences; we study this property more closely in Section 4.3. In [Ham09], Hamenstädt examined the action of axial rank-one isometries on visual boundaries of proper CAT(0) spaces. She proved that an axial isometry of a proper CAT(0) space is rank-one if and only if it acts with North-South Dynamics on the visual boundary. In this thesis, we work with CAT(0) cube complexes. For CAT(0) cube complexes, Caprace and Sageev [CS11] proved existence theorems of rank-one isometries. Murray showed in Corollary 2.16 of [Mur19], that every CAT(0) group with nonempty Morse boundary has a rank-one isometry and that rank-one isometries act with *weak North-South Dynamics.*

Another important tool for this thesis are Hamenstädt's and Murray's results concerning the denseness of orbit of boundary points. Murray showed that the orbit of a contracting boundary point is dense in the contracting boundary of a proper CAT(0) space if it admits a cocompact action of a group that does not globally fix a point of the contracting boundary. For visual boundaries, Hamenstädt achieved an analogous result for boundary points associated to rank-one isometries. Recently, Liu [Liu19] generalized Hamenstädt's and Murray's considerations and achieved analogous results for Morse boundaries of proper metric spaces.

We concentrated on good-natured properties of Morse boundaries unit know. But disadvantages are raised by the matter that the topology of the contracting boundary is not the subspace topology of the visual boundary. The contracting boundary does not behave as nice as the Gromov boundary. In [Mur19], Murray proved that contracting boundaries are in general not first-countable and hence not metrizable. Furthermore, the contracting boundary of a CAT(0) group is compact if and only if the group is hyperbolic.

Diverse efforts were done for finding definitions or generalizations of Morse boundaries that behave better. For example, Cashen and Mackay [CM19] studied a set of equivalence classes of 'contracting quasi-geodesics' and equipped this set with a topology that is second-countable and thus metrizable. They used the definition 'contracting' by Arzhantseva, Cashen, Gruber and Hume [Arz+17]. This notion is weaker than that of Charney and Sultan. Recently, Qing, Rafi and Tiozzo introduced a so-called k-Morse boundary of CAT(0) groups [QRT19]. This boundary is metrizable and a quasi-isometry invariant and its underlying set is larger than the Morse boundary defined by Cordes. Qing and Zalloum [QZ19] proved that this k-Morse boundary is a strong visibility space. Furthermore, they proved that every group with nonempty k-Morse boundary contains a rank-one isometry. The set of strongly contracting rays is a dense subspace of the ambient k-Morse boundary. As a consequence, any CAT(0) group with nonempty k-Morse boundary is an acylindrically hyperbolic group.

In the following, we list concrete classes of groups that have interesting Morse boundaries and summarize what is known about them: Small cancellation groups are one such class. In [Arz+19], Arzhantseva, Cashen, Gruber and Hume examined contracting geodesic rays in graphical small cancellation groups that are infinitely presented. They determined the contracting property of a geodesic ray in their Cayley graphs by means of the defining graph of the group. Not all small cancellation groups are CAT(0). Hence, there might be interesting properties of Morse boundaries of small cancellation groups that might not occur in case the of CAT(0) groups. Recently, Charney, Cordes and Sisto [CCS19] mentioned that one such property might be *σ-compactness*. A metric space is *σ-compact* if it is the union of countably many compact subspaces. By the main theorem of Charney and Sultan in [CS15], the contracting boundary of every proper CAT(0) space is σ-compact. Charney, Cordes and Sisto remarked in [CCS19] that to the best of their knowledge there is no known example of a group whose Morse boundary is not σ-compact. They think, nevertheless, that it might be possible that every infinitely presented $c'(\frac{1}{6})$-small cancellation group is non-σ-compact.

In [Fin17], Fink investigated Morse geodesics in infinite unbounded torsion groups. This led to a first example of a group that contains Morse geodesics but no *Morse elements*. A group element g of infinite order is a *Morse element* if we obtain a Morse quasi-geodesic by embedding the cyclic subgroup generated by g in the Cayley graph and connecting consecutive elements by a geodesic.

Charney, Cordes and Sisto observed in [CCS19], that many motivating examples of groups with nonempty Morse boundaries contain a stable free subgroup, i.e., the corresponding Morse boundaries contain a Cantor subspace. For instance, the non-compact contracting boundaries we consider in this thesis, contain a Cantor subspace by construction. Charney, Cordes and Sisto point out that there is no known example of a group with non-compact Morse boundary that does not contain a Cantor subspace. Even the examples of Fink mentioned above contain a Cantor subspace most probably. This is interesting in view of Theorem 1.2 in [CCS19]. It says the following: Suppose that a group is finitely generated and that its Morse boundary contains a Cantor subspace. If the Morse boundary is totally disconnected and σ-compact, then the Morse boundary of the group is either a Cantor space or an ω-*Cantor* subspace. An ω-*Cantor subspace* is defined as the direct limit of a sequence of Cantor spaces $X_1 \subseteq X_2 \subseteq X_3 \dots$ such that X_i has an empty interior in X_{i+1}. In this thesis, we study the question of which right-angled Coxeter groups have totally disconnected contracting boundaries. Right-angled Coxeter groups are CAT(0) groups and by the main theorem of Charney and Sultan in [CS15], they are σ-compact. As mentioned above, the non-compact contracting boundaries of groups we consider in this thesis contain a Cantor subspace. These boundaries are either Cantor spaces or ω-Cantor spaces [CCS19].

One consequence of Theorem 1.2 of Charney, Cordes and Sisto is that the Morse-boundaries of right-angled Artin groups can be classified. In Theorem 1.1 of [CCS19], Charney, Cordes and Sisto showed that the contracting boundary $\vec{\partial}_c A_\Gamma$ of any right-angled Artin group A_Γ with defining graph Γ satisfies exactly one of the following:

- $\vec{\partial}_c A_\Gamma$ is empty,
- $\vec{\partial}_c A_\Gamma$ consists of two points,
- $\vec{\partial}_c A_\Gamma$ is a Cantor space, or
- $\vec{\partial}_c A_\Gamma$ is an ω-Cantor space.

In addition, they gave precise conditions on the defining graph Λ for realizing each of the four cases. Furthermore, they proved that any two ω-Cantor spaces are homeomorphic.

Furthermore, the authors proved that the Morse boundary of a fundamental group of a finite-volume hyperbolic 3-manifold with at least one cusp is an ω-*Sierpiński curve.* An ω-*Sierpiński curve* is a direct limit of a sequence $S_1 \subseteq S_1 \subseteq S_3 \dots$ of Sierpiński curves such that the peripheral curves of S_i are disjoint from those of S_{i+1}. The authors proved that any two such ω-Sierpiński curves are homeomorphic.

Further non-hyperbolic groups with non-empty Morse boundaries might be found among systolic groups; see Januszkiewicz's and Świątkowski's work [JS06]. Interesting examples of systolic groups can for instance be found in Przytycki's and Schwer's work [PS16].

In this thesis, we are interested in the question of which right-angled Coxeter groups have totally disconnected contracting boundary. We examine their contracting boundaries by the study of their Davis complexes introduced by Davis [Dav08]. Davis complexes are CAT(0) cube complexes. The origin of CAT(0) cube complexes is Gromov's work [Gro87]. We refer to Haglund's and Wise's work [HW08], to Sageev's works [Sag95] and to Caprace's and Sageev's work [CS11] as important contributions to this field. We refer to the notes [Sag14] for a summary of important results.

We come back to right-angled Coxeter groups. Coxeter groups have lots of interesting properties like reflecting lengths for instance (see [Lew+18]). Like the contracting boundary is a quasi-isometry invariant, contracting boundaries can be used to distinguish groups according to their large-scale geometry. We refer to Dani's survey [Dan18] for general information concerning the large-scale geometry of right-angled Coxeter groups. At this point, we report only three results explained in [Dan18]. The first two results show that visual boundaries of spaces admitting a geometric action of a right-angled Coxeter group don't behave so nicely as one would like to believe. First, we recap Theorem 4.4 of [Dan18]. This theorem was proven by Yamagata [Yam09] by means of [BR96]. It says that there exists a right-angled Coxeter group which admits two different geometric actions on a space X such that the natural quasi-isometry of X induced by the two group actions does not extend to a continuous map $\hat{\partial} X \to \hat{\partial} X$. Secondly, Stark [Sta18] found a quotient of a Davis complex by a one-ended right-angled Coxeter group which has two non-homeomorphic finite covers that are homotopy equivalent. We finish with Qing's [Qin16b] study of geometric actions of right-angled Coxeter groups on the spaces occurring in the example of Croke and Kleiner and extensions of this example in [Wil05]. Qing [Qin16a] proved that there exists a right-angled Coxeter group which acts geometrically on all Croke-Kleiner spaces obtained by a certain variation of the construction, but not all of the boundaries of these spaces are equivalently homeomorphic. Dani remarked that it is still possible that the considered spaces are homeomorphic. Accordingly, it is unknown whether visual boundaries are well-defined for right-angled Coxeter groups. In other words, the following question is still open:

Question 3 (Question 4.6 in [Dan18])**.** Is there a right-angled Coxeter group acting geometrically on two spaces whose boundaries are not homeomorphic?

1.1.2 History of research on amalgamated free products of CAT(0) groups

Amalgamated free products of groups are of interest in Bass-Serre theory; we refer to Serre's book [Ser03] and Wall's and Scott's work [SW79] as fundamental contributions. This theory analyzes actions of groups on trees and how they are related to fundamental groups on associated spaces. The Seifert-van Kampen Theorem can be seen as the simplest example: Suppose that we would like to understand the fundamental group of a space that is covered by two open, path-connected subspaces. Then we can write this fundamental group as an amalgamated free product of two fundamental groups along a fundamental group. On the other hand, suppose that we have given some amalgamated free product $G = G_0 *_H G_1$. Then we can construct a space, a so-called *total space* X, that has the amalgamated free product as fundamental group. This total space X has two subspaces X_0 and X_1. These two subspaces are glued along a thickened copy of a space Y. Thereby, G_i is the fundamental group of X_i, $i \in \{0, 1\}$ and H is the fundamental group of Y. The universal cover of the total space consists of copies of X_0, X_1 and Y. If we project all copies of X_0 and X_1 to points and project the thickened copies of Y to edges, we obtain a tree, namely the *Bass-Serre tree*. The action of the amalgamated free product on the universal cover of X induces an action of the amalgamated free product on the Bass-Serre tree.

On the other hand, if an action of a group on a tree is suitable, this group is a fundamental group of a space that is obtained by iterating constructions as above. This leads to the study of *trees of groups* that can be described as iterated amalgamated free products. If one generalizes the theory of trees of groups, one ends up at the theory of *graphs of groups*. In this thesis, we study amalgamated free products and iterations of amalgamated free products. We remain in the world of trees of groups. Furthermore, we are only interested in amalgamated free products of CAT(0) groups that are itself CAT(0) groups, i.e. we are interested in amalgamated free products that act geometrically on CAT(0) spaces.

Suppose that $G = G_0 *_H G_1$ is an amalgamated free product of two groups G_0 and G_1 along a group H that acts geometrically on CAT(0) spaces X_0, X_1 and Y respectively. The question arises if we can construct a CAT(0) space, on which the amalgamated free product $G = G_0 *_H G_1$ acts geometrically. An idea for constructing such a space could mimic the construction explained above. In the construction above, we glued copies of spaces together whose fundamental groups coincided with the factors of $G = G_0 *_H G_1$. An imitation would glue copies of X_0 and X_1 along copies of Y instead of copies of space whose fundamental groups coincide with G_0, G_1 and H. Sadly, the resulting space is not always a CAT(0) space on which G acts geometrically. Bridson and Haefliger proved in Proposition 6.10 of Γ.6 in part III of [BH99] that there is an amalgamated free product of two CAT(0) groups that is not CAT(0). On the other hand, Bridson and Haefliger formulated in Theorem 11.18 of Chapter II in [BH99] conditions under which the described construction can be done. If these conditions are satisfied, the amalgamated free product itself is a CAT(0) group.

In this thesis, we concentrate on amalgamated free products of CAT(0) groups that allow the described construction, i.e. we focus on amalgamated free products of CAT(0) groups satisfying the Equivariant Gluing Theorem of Bridson and Haefliger. There are many reasons for studying such amalgamated free products. One reason is that they are a source of interesting examples of groups. One such example is described by Croke and Kleiner in [CK00]. This example is crucial for this thesis because it proves that the visual boundary of CAT(0) groups is not a quasi-isometry invariant. It is an amalgamated free product of two CAT(0) groups along a CAT(0) group. It is obtained by gluing three tori along nontrivial loops. We refer to this space as the Croke-Kleiner complex. The fundamental group of this complex is an amalgamated free product of the form $\mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}^2$. In [Wil05], Wilson proved that this group acts geometrically on uncountably many CAT(0) spaces with pairwise non-homeomorphic visual boundaries. In [Moo10], Mooney extended this example to groups of the form $(\Gamma_- \times \mathbb{Z}^m) *_{\mathbb{Z}^m} (\mathbb{Z}^m \times \mathbb{Z}^n) *_{\mathbb{Z}^n} (\mathbb{Z}^n \times \Gamma_+)$ where Γ_- and Γ_+ denote infinite CAT(0) groups. For studying these examples, Mooney introduced a notion for spaces on which such groups act geometrically, namely *CAT(0) spaces with block structure*. Ben-Zvi [BZ19] and Ben-Zvi and Kropholler [BZK19] worked with such spaces as well and called them *CAT(0) spaces with block decomposition.* Among other things, Ben-Zvi [BZK19] considers block decompositions of the spaces arising from the Equivariant Gluing Theorem of Bridson and Haefliger. In this thesis, we call such decompositions *block decompositions with thin wall.* It is an important tool for us.

Though the visual boundary is not a quasi-isometry invariant, it is a question of great interest in how the topology of the visual boundary corresponds to the geometry and algebraic structure of the group. For instance, the *ends* of a CAT(0) or hyperbolic space are deeply connected to the corresponding boundary. An *end* of a topological space X is an equivalence class of an equivalence relation on continuous proper rays in X. Two such rays $r_1 : [0, \infty) \to X$ and $r_2 : [0, \infty] \to X$ are equivalent if for every compact $C \subseteq X$ there exists $n \in \mathbb{N}$ such that $r_1[n, \infty)$ and $r_2[n, \infty)$ are contained in the same path component of $X \setminus C$. The *ends* of a finitely generated group are the ends of an associated Cayley graph. It is a classical theorem shown by Hopf [Hop44] that each finitely generated group has either 0, 1, 2, or infinitely many ends. The group has no end if and only if it is finite. It has two ends if and only if it contains $\mathbb{Z}$ as a subgroup of finite index, i.e. it is quasi-isometric to $\mathbb{Z}$. It has infinitely many ends if and only if it can be expressed as an amalgamated free product or HNN extension along a finite group. See [BH99, Thm.8.32 in Ch.I]. This characterization was shown by Stalling [Sta68].

Suppose that a hyperbolic or CAT(0) group acts geometrically on a hyperbolic or CAT(0) space. Then the visual boundary of the space is connected if and only if the group is one-ended. The question arises of whether connectedness-properties reflect the behavior of the group. Figure 1.5 summarizes the connectedness-properties which the visual boundary of a CAT(0) space might have. We mention that a connected visual or Gromov boundary that is locally connected is path connected. In general, there are connected topological spaces that are locally connected but not path connected.

Much research has been done on locally connectedness and path connectedness of visual boundaries. It was proven that the Gromov boundary of every one-ended hyperbolic group is connected and locally connected. More precisely, Bestvina and Mess [BM91] proved that if the Gromov boundary of a one-ended hyperbolic group does not have cut

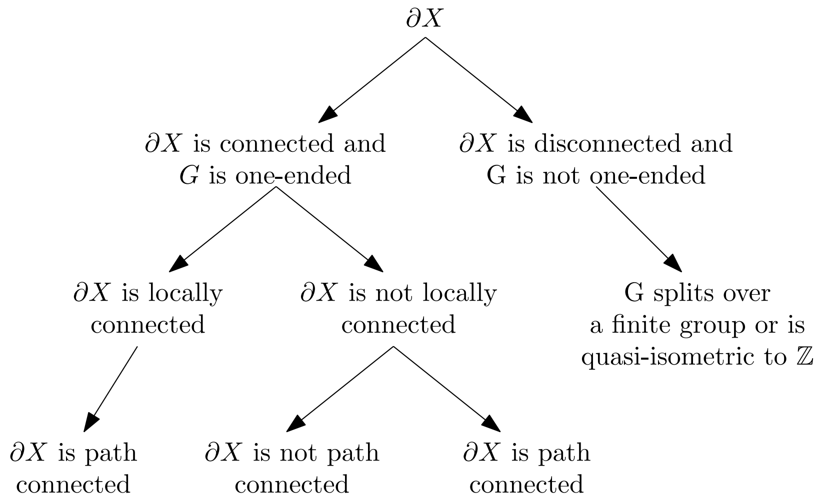

Figure 1.5 The space X denotes a CAT(0) space and G denotes a group acting geometrically on X. Suppose that A denotes the property of a vertex of the pictured tree. Then the properties at the peaks of the outgoing arrows of A denote all possibilities that might occur if all the properties on the path from A to the root of the tree are satisfied.

points then it is locally connected. Afterwards, Swarup [Swa96] showed that the Gromov boundary of a one-ended hyperbolic group does not have a cut point using results of Levitt [Lev98] and Bowditch [Bow98a; Bow98b; Bow99].

The situation is different in the case of CAT(0) spaces. The standard example is the group $\mathbb{F}_2 \times \mathbb{Z}$ where $\mathbb{F}_2$ denotes the free group of rank two. This group acts on the direct product of $\mathbb{R}$ and the infinite 4-valent tree and the visual boundary of this space is a suspension of a Cantor set. Accordingly, it is connected but not locally connected. This example was for instance mentioned in [PS09] and [Dan18]. The visual boundaries arising from the examples of Croke and Kleiner are also non-locally connected. See [CM14]. Another interesting example arises from the research of Schreve and Stark [SS20]. They examined the Croke-Kleiner complex and proved for any locally CAT(0) metric: if a finite graph is embedded in the visual boundary of one universal cover, then the graph is embedded in the other. On the other hand, they created two locally CAT(0) complexes whose universal covers have homeomorphic visual boundaries, but only one of both visual boundaries contains a non-planar graph. The visual boundaries of these spaces are not locally connected. Moreover, the studied spaces are not quasi-isometric to 3-manifold groups. As a consequence, Dani pointed out that the following question is still open:

Question 4 (Question 4.7 in [Dan18])**.** Is there a CAT(0) group that acts on two different spaces, such that one has locally connected boundary and the other has non-locally connected boundary?

We extend Dani's survey on research that was done concerning this question by the new results of Ben-Zvi and Kropholler. They concentrated on path connectedness. Ben-Zvi's work [BZ19] is based on the studies of Hruska and Ruane [HR20] who examined one-ended CAT(0) groups with isolated flats. According to [Hru05], the visual boundary for such groups is an invariant, i.e. the visual boundary of a CAT(0) group with isolated flats can be defined as the visual boundary on which it acts geometrically. Hruska and Ruane found necessary and sufficient conditions for determining when a CAT(0) group with isolated flats has a locally connected visual boundary. Many visual boundaries of CAT(0) groups with isolated flats are non-locally connected. A few such examples can be found in Section 2 of [BZ19]. If a visual boundary is connected and not locally connected it is still possible that it is path connected. In [BZ19], Ben-Zvi proved that if a one-ended CAT(0) group with isolated flats acts geometrically on a CAT(0) space, then the visual boundary of the space is path connected.

In [BZK19], Ben-Zvi and Kropholler found examples of right-angled Artin groups that act on a space that is not path-connected. This space is constructed by means of Croke and Kleiner spaces. They proved that neither path-connectedness nor n-connectedness are invariants for CAT(0) groups. They showed that for each n, there is a group G_n and a CAT(0) space X_n and Y_n admitting geometric group actions by G_n with the following properties:

- the visual boundaries of X_n and Y_n are n-connected
- the visual boundaries of X_n and Y_n are not homeomorphic.

They asked the following question:

Question 5. Does there exist a group G which acts on two CAT(0) spaces X, Y geometrically and the visual boundary of X is n-connected but the visual boundary of Y not?

Another important question is: how are cut points in the boundary related to the structure of the group? Bowditch examined this question in [Bow98a] for hyperbolic groups. He proved that local cut points in the boundary are associated to splittings along 2-ended groups. More precisely, suppose that G is a one-ended hyperbolic group which is not a cocompact *Fuchsian group*. A finitely generated group is *Fuchsian* if it is *non-elementary* and acts properly discontinuously on the hyperbolic plane. A group is *elementary* if its Gromov boundary consists of at most two points or the group fixes a point in the boundary. Bowditch proved in Theorem 0.1 that there is a canonical splitting of G as a finite graph of groups such that each edge group is two-ended and each vertex group is of one of three types. This canonical splitting is the *JSJ splitting* or *JSJ decomposition* of G. The described graph of groups is the quotient of an action of G on an associated tree, the JSJ tree. This JSJ tree is a quasi-isometry invariant and is defined by means of the structure of local cut points of the boundary. It follows from Bowditch's considerations that the boundary of a one-ended hyperbolic group has a local cut point if and only if either the group is a cocompact Fuchsian group or the group splits over a 2-ended group. See Theorem 6.2. In addition, Bowditch concluded Theorem 1.1, the theorem stated at the beginning of this thesis, on page 147 in [Bow98a].

JSJ splittings were introduced by Sela [Sel97]. The name is motivated by analogies to the work of Jaco and Shalen [JS79] and Johannson [Joh79].

Bowditch's result motivates that JSJ decompositions are an interesting tool for examining the interaction of the Gromov boundary of a hyperbolic group and a space on which the group acts geometrically. The question arises of whether such JSJ decompositions can be defined not only for hyperbolic groups. Swarup and Scott [SS03] constructed a canonical JSJ decomposition for finitely presented groups.

In this thesis, we are interested in CAT(0) groups. For the case of CAT(0) groups, another, similar decomposition as in [SS03] was exhibited in [PS09]. In [PS09], Papasoglu and Swenson study how to find a canonical JSJ decomposition for CAT(0) groups by means of their visual boundaries. This canonical JSJ decomposition is defined by means of cut pairs in the boundary., i.e. pairs of points whose deletion decomposes the boundary in at least two connected components. By means of a construction of an $\mathbb{R}$-tree in [PS06] they defined a certain tree on which G acts. The quotient of this group action gives a canonical JSJ decomposition of G over 2-ended groups. See Theorem 3 in [PS09]. In the case that G is a one-ended group acting geometrically on a proper CAT(0) space X, Papasoglu and Swenson showed the following theorems:

Theorem 1.3 (Theorem 1 in [PS09])**.** *Let G be a one-ended group that acts geometrically on a CAT(0) space X. Then the visual boundary of X has no cut points.*

Theorem 1.4 (Theorem 2 in [PS09])**.** *Let G be a one-ended group that acts geometrically on a CAT(0) space X. Suppose that the visual boundary of X has a cut pair. If G does not split over a 2-ended group, then G is virtually a surface group.*

Papasoglu and Swenson remarked that the last statement follows from [Pap05] if the visual boundary of X is assumed to be locally connected. In [Pap05], Papasoglu studied 1-ended, finitely presented groups that are not commensurable to surface groups. Papasoglu proved that two such groups split over a 2-ended subgroup if they are quasi-isometric, i.e. to split over a 2-ended subgroup is a quasi-isometry invariant.

Guirardel and Levitt [GL17] give a simple general definition of JSJ decompositions and unite results concerning JSJ decompositions. We refer to [GL17] for more information concerning JSJ decompositions.

In the following, we summarize the consequences of the research above for right-angled Coxeter groups that are 2-dimensional, i.e. right-angled Coxeter groups whose defining graphs don't contain triangles. We start with a summary of the most important statements considered in Dani's survey [Dan18]. First, Dani unified known results of [DT17], [Tuk88], [Gab92] and [CJ94] as follows:

Theorem 1.5 (Theorem 5.10 in [Dan18])**.** *For a 2-dimensional right-angled Coxeter group W, the following statements are equivalent:*

- *W is cocompact Fuchsian.*
- *W is quasi-isometric to a cocompact Fuchsian group.*

- *W has an n-cycle with $n \geq 5$ as defining graph.*

We like to apply Bowditch's theorem to hyperbolic two-dimensional right-angled Coxeter groups. Suppose that Λ is triangle-free. In this case, the structure of Λ is related to splittings of W_Λ over 2-ended groups. Indeed, it follows from [MT09] that W_Λ splits over a 2-ended group if and only if Λ contains two vertices whose deletion decomposes the graph in more than one connected component. Dani combined again known results and observed that the conditions of Bowditch's theorem can be expressed in terms of the defining graph. In 'Assumptions 5.11' in [Dan18], Dani listed these conditions as follows: Suppose that Λ is the defining graph of a right-angled Coxeter group that does not contain triangles. A subgraph Λ' of a graph Λ is *induced* if every edge of Λ with endvertices in $V(\Lambda')$ is an edge of Λ'. The conditions of Bowditch's theorem are satisfied if Λ satisfies the following:

a) Λ is connected and has no separating vertices or edges (W_Λ is one-ended);

b) Λ does not contain any induced 4-cycle (W_Λ is hyperbolic);

c) Λ is not a cycle of length at least 5 (W_Λ is not cocompact Fuchsian); and

d) Λ has a separating pair of non-adjacent vertices (W_Λ splits over a 2-ended subgroup).

Dani and Thomas [DT17] studied two-dimensional right-angled Coxeter groups whose defining graphs satisfy the listed conditions and described associated JSJ trees in terms of Λ. They concluded that the quasi-isometry problem is decidable for all right-angled Coxeter groups whose defining graph is triangle-free, satisfies the listed properties, and does not contain a K_4 minor. Cashen and Dani proved that it is not possible to remove the K_4-condition (see Appendix B in [DT17]).

One application of the JSJ decompositions above can be found in [DST18]. Dani, Stark and Thomas studied hyperbolic right-angled Coxeter groups whose defining graphs are so-called *θ-graphs*. They found necessary and sufficient conditions for figuring out whether two such right-angled Coxeter groups are commensurable. One such θ-graph occurs as a subgraph in a counterexample studied in Section 5.5 of this thesis and plays a crucial role. The paths in Definition 1.23 build sometimes a θ-graph. Definition 1.23 is related to a conjecture with which we complete this thesis (Conjecture 1.24).

Dani remarked in [Dan18] that it might be possible to transfer the JSJ decomposition in [DT17] to the non-triangle-free case. But if the hyperbolic group does not split over a 2-ended group, the JSJ tree is trivial and the considerations in [Dan18] cannot be used. Dani remarked that there are many such hyperbolic right-angled Coxeter groups. At this point, we cite a result of Kapovich and Kleiner stated as Theorem 4.2 in Dani's survey [Dan18] that concerns such groups.

Theorem 1.6. *Let G be a hyperbolic group with one-dimensional Gromov boundary which does not split over a finite or 2-ended group. Then the Gromov boundary is either a circle, a Sierpiński carpet or a Menger curve.*

Dani explained that this result follows from the research on topological characterizations of Sierpiński carpets and Menger curves. We refer to Theorem 4.3 in [Dan18] for more details. The question arises of which hyperbolic groups have circles, Sierpiński carpets and Menger curves in their boundaries. We refer to Dani's survey for more information concerning this topic and cite just one important result. We remark that if a hyperbolic group acts geometrically on a hyperbolic space, then the group acts as convergence group on its Gromov boundary. The following follows from [Tuk88], [Gab92] and [CJ94]:

Theorem 1.7. *If G is a hyperbolic group whose Gromov boundary is a 1-sphere such that G acts as a convergence group on $\vec{\partial}_c G$, then G is cocompact Fuchsian.*

This, combined with Theorem 1.5, implies that the 1-dimensional analogy to Cannon's Conjecture is true that was mentioned at the beginning of this thesis. What about Cannon's Conjecture itself? Haïssinsky [Haï15] proved that Cannon's Conjecture is true for hyperbolic groups that act geometrically on a CAT(0) cube complex. As right-angled Coxeter groups act on CAT(0) cube complexes, hyperbolic right-angled Coxeter groups satisfy Cannon's conjecture.

More precisely, Haïssinsky studied a more sophisticated conjecture concerning *planarity* of visual boundaries. We say that a topological space is *planar* if it can be embedded in a 2-sphere. The following conjecture can be found in [DHW19]. It was asked as a question in [DST18].

Conjecture 1.8. *Let G be a CAT(0) group with a planar visual boundary. Then every visual boundary of G is planar, and furthermore, G is virtually the fundamental group of a compact 3-manifold.*

Haïssinsky [Haï15] proved that every hyperbolic group acting geometrically on a CAT(0) cube complex satisfies Conjecture 1.8.

Recently, Dani, Haulmark and Walsh [DHW19] examined Conjecture 1.8 for cases in which the studied group is not assumed to be hyperbolic. They ascertained that it is more difficult to handle Conjecture 1.8 for non-hyperbolic CAT(0) groups. First, a CAT(0) group can have uncountably many visual boundaries. Secondly, the visual boundary of a CAT(0) space admitting an action of a one-ended CAT(0) group is not always locally connected. Suppose that a one-ended group G acts geometrically on a CAT(0) space X such that its visual boundary is locally connected. If G does not contain an infinite torsion group, then the visual boundary of X is non-planar if and only if it contains a non-planar graph. This follows from [Cla34], as Schreve and Stark observed in [SS20]. If the visual boundary of X is not locally connected, this equivalence is false in general: recall that Schreve and Stark [SS20] created two locally CAT(0) complexes whose universal covers have homeomorphic visual boundaries, but only one of both visual boundaries contains a non-planar graph. The associated groups are torsion-free, and the studied visual boundaries are both not locally connected and non-planar.

Suppose that G is quasi-isometric to a 3-manifold group and that G acts geometrically on a CAT(0) space X. Then the visual boundary of X does not contain a non-planar graph. This follows from a very special case of a theorem in [BKK02] as Schreve and

Stark observed in [SS20]. So, if the visual boundary of a CAT(0) space X contains a non-planar graph, then there is no 3-manifold group acting on X geometrically. It follows that the boundaries examined by Schreve and Stark, don't belong to 3-manifolds.

In [DHW19], Dani, Haulmark and Walsh proved that Conjecture 1.8 is true for a class of right-angled Coxeter groups. They showed the following:

Corollary 1.9 (Corollary 1.4 in [DHW19]). *Let Λ be a triangle-free graph and X a CAT(0) space on which W_Λ acts geometrically. If the visual boundary of X is a Sierpiński carpet, W_Λ is virtually a* 3*-manifold group.*

In addition, they showed the following.

Corollary 1.10 (Corollary 1.5 in [DHW19]). *Let Λ be a triangle-free graph. Suppose that Λ is connected, has no separating complete subgraphs, no cut pair and no separating complete subgraph suspension. Suppose that W_Λ is hyperbolic or CAT(0) with isolated flats. Then the following statements are equivalent.*

a) Λ is planar.

b) Every visual boundary of a CAT(0) space admitting a geometric action of W_Λ is a Menger curve.

c) W_Λ acts geometrically on a CAT(0) space whose visual boundary is a Menger curve.

In this thesis, we study contracting boundaries. The advantage of the contracting boundaries is that they are well-defined for CAT(0) groups. It is interesting to study the topics above for contracting boundaries instead of visual boundaries.

Let us consider Cannon's conjecture in the setting of contracting boundaries of CAT(0) groups. Murray [Mur19] proved that the contracting boundary of a complete CAT(0) space is compact if and only if the space is hyperbolic. In particular, the contracting boundary of a CAT(0) group is a 2-sphere if and only if the group is hyperbolic and its Gromov boundary is a two-sphere. Thus, if we transfer Cannon's Conjecture word-for-word to contacting boundaries, the meaning of the conjecture stays the same as before. However, it is very interesting to examine related questions: When do arise spheres in the contracting boundary? And if there arise spheres in the contracting boundary, where do they come from?

In this thesis, we study how cutsets in a CAT(0) space are related to the contracting boundary of a CAT(0) group that acts geometrically on it. For this purpose, we study how contracting boundaries of two CAT(0) groups change if we amalgamate these two groups, i.e., we study contracting boundaries of amalgamated free products of CAT(0) groups. We concentrate on the cases where totally disconnected boundaries are involved. This thesis provides examples where the contracting boundaries either have totally disconnected contracting boundaries or they contain large connected components. It is not clear how large these connected components are. Adjoining questions are: how are these large connected components related to a hyperbolic-lice structure in the studied group and when are embedded in spheres?

1.2 Outline of the thesis and scientific contribution

Initial question

The Equivariant Gluing Theorem of Bridson and Haefliger [BH99, Thm 11.18 in II] formulates conditions under which an amalgamated free product G of two CAT(0) groups G_0 and G_1 along a CAT(0) group H is itself a CAT(0) group. We examine when such a group G has totally disconnected contracting boundary and apply our insights to right-angled Coxeter groups.

Methods of the research on the initial question

If an amalgamated free product of two CAT(0) groups satisfies the Equivariant Gluing Theorem of Bridson and Haefliger [BH99, Thm 11.18 in II], then it acts on a CAT(0) space with block decomposition (see Chapter 3). We examine contracting boundaries of amalgamated free products of CAT(0) groups by studying CAT(0) spaces with block decomposition on which they act geometrically. All results in this thesis arise from exchanging the topology of the contracting boundary with the subspace topology of the visual boundary. The former topology is finer than the latter topology. Thus, the study of the latter topology implies insights about the former topology.

Motivation

Our research is firstly inspired by an example of a right-angled Coxeter group with totally disconnected contracting boundary studied by Charney and Sultan in Section 4.2 of [CS15] (see Figure 1.13). We refer to it as the *Cycle-Join-Example*. Secondly, it is motivated by a conjecture of Tran [Tra19, Conjecture 1.14] about right-angled Coxeter groups with totally disconnected contracting boundaries. We refer to this conjecture as the *Burst-Cycle-Conjecture* (see Conjecture 1.16).

Content of the chapters

In Chapter 2, we recall basic concepts. As a preparation for our main theorems, we study CAT(0) spaces with block decomposition in Chapter 3. Chapter 4 and Chapter 5 contain our main theorems about contracting boundaries of amalgamated free products of CAT(0) groups and right-angled Coxeter groups respectively.

Main results

In Chapter 4, we prove three main results about contracting boundaries of amalgamated free products of CAT(0) groups. Firstly, we generalize the Cycle-Join-Example of Charney and Sultan in Theorem 4.10. Secondly, we find an interesting property of connected components associated to axes for axial rank-one isometries in Theorem 4.24. Thirdly, we analyze contracting boundaries of amalgamated free products along groups quasi-isometric to $\mathbb{Z}$ in Theorem 4.50.

In Chapter 5, we apply the results of Chapter 4 for studying which right-angled Coxeter groups have totally disconnected contracting boundaries. We obtain two main results. In Theorem 5.32, we show a variant of Theorem 4.10 for right-angled Coxeter groups. Using this theorem, we define a class of graphs $\mathcal{J}$ (see Definition 5.37) and prove that every right-angled Coxeter group whose defining graph is contained in $\mathcal{J}$, has totally disconnected contracting boundary (see Corollary 5.38). In particular, $\mathcal{J}$ satisfies the Burst-Cycle-Conjecture (see Corollary 5.39). The second main result of Chapter 5 concerns the question of how the contracting boundary of a right-angled Coxeter group changes when we glue a path of length at least two on its defining graph. We prove a Dichotomy in Theorem 5.58.

We finish this thesis with an outlook on a joint work with Graeber, Lazarovich and Stark. We sketch counterexamples proving that the Burst-Cycle-Conjecture is wrong in general. We examine these examples and formulate a new conjecture.

Notation

Before we start with the outline of this thesis, we recap the different boundaries we work with and fix notation. Suppose that X is a complete CAT(0) space. We say that two geodesic rays in X are *asymptotic* if their Hausdorff distance is bounded. Being asymptotic is an equivalence relation. A *boundary point* of X is an equivalence class of a geodesic ray γ, denoted by $\gamma(\infty)$. We denote the set of boundary points by ∂X. If we equip this set with the cone topology, we obtain the *visual boundary* of X, denoted by $\hat{\partial} X$. In this topological space, two boundary points are close to each other if their representatives stay close for a long time. If X is hyperbolic, the visual boundary of X coincides with the Gromov boundary. A geodesic ray γ is *contracting*, if there exists $D > 0$ such that the closest point projection sends all balls that don't intersect γ onto a subsegment of γ that has length at most D. This way, large balls are "contracted" to a "short" segment. We denote the set of equivalence classes of contracting geodesic rays by trX. If we equip this set with the subspace topology of the visual boundary of X, we obtain a topological space that we denote by $\hat{\partial}_c X$. Cashen [Cas16] proved that this topological space is not a quasi-isometry invariant. Thus, we equip $\partial_c X$ with another topology, the direct limit topology. This way we obtain the *contracting boundary* of X, denoted by $\vec{\partial}_c X$. If X is hyperbolic, the Gromov boundary of X coincides with the contracting boundary of X. The contracting boundary of a CAT(0) group G, denoted by $\vec{\partial}_c G$, is defined to be the contracting boundary of a complete CAT(0) space X on which G acts geometrically. If G is hyperbolic, its Gromov boundary coincides with its contracting boundary. We summarize for the benefit of the reader our notation.

Notation 1.1. In the following tabular, X denotes a CAT(0) space and Y a convex subspace of X. Furthermore, G denotes a CAT(0) group.

Object	Notation
∂X	The set of boundary points of X
$\partial_c X$	The set of contracting boundary points of X
$\hat{\partial} X$	The visual boundary of X
$\hat{\partial}_c X$	The set $\partial_c X$ equipped with the subspace topology of $\hat{\partial} X$
$\vec{\partial}_c X$	The contracting boundary of X
$\vec{\partial}_c G$	The contracting boundary of G.
$\partial_{c,Y}$	The set $\{\gamma(\infty) \in \partial_c X \mid \gamma \subseteq Y\}$
$\hat{\partial}_{c,Y}$	The set $\partial_{c,Y}$ equipped with the subspace topology of $\hat{\partial}_c X$.
$\vec{\partial}_{c,Y}$	The set $\partial_{c,Y}$ equipped with the subspace topology of $\vec{\partial}_c X$.

We focus on the topological space $\hat{\partial}_c X$. We use the fact that the topology of the contracting boundary is finer than the topology of $\hat{\partial}_c X$. Hence, every connected component in the contracting boundary $\vec{\partial}_c X$ of X is contained in a connected component of $\hat{\partial}_c X$. In particular, if $\hat{\partial}_c X$ is totally disconnected, then $\vec{\partial}_c X$ is totally disconnected. We summarize the content of Chapter 3, Chapter 4 and Chapter 5 in the following and explain the main results of this thesis.

1.2.1 Chapter 3: boundaries of CAT(0) spaces with block decompositions

In Chapter 3, we study boundaries of *CAT(0) spaces with block decomposition.* We are interested in such spaces as there arise naturally as spaces on which interesting examples of amalgamated free products of CAT(0) groups act geometrically. An important example is the example of Croke and Kleiner in [CK00] that shows that the visual boundaries of two spaces on which a group acts geometrically might have non-homeomorphic visual boundaries. Inspired by this example, Mooney [Moo10] introduced *CAT(0) spaces with block structure.* Such spaces were further studied by Ben-Zvi [BZ19], and Ben-Zvi and Kropholler [BZK19] as *CAT(0) spaces with block decomposition.* Here, we compare two different kinds of block decompositions with thin and thick walls respectively. The first kind coincides with the decomposition studied by the authors above. The second decomposition is a variant of the first one. We introduce a common notion for both types of decompositions and examine properties of them.

Section 3.1: Block decompositions with thin walls

Roughly speaking, a block decomposition with thin walls of a CAT(0) space X consists of a collection $\mathcal{B}$ of closed convex subspaces, called blocks whose union covers X. Each block has a parity $(+)$ or $(-)$ such that we obtain a bipartite tree when we take the blocks as vertex set and connect two blocks by an edge if and only if they have a nonempty intersection. Such a nonempty intersection is a *wall.* It is helpful for our considerations to insert vertices corresponding to walls in such a tree. If we speak of a tree associated to a block decomposition, we mean the barycentric subdivisions of the tree described above. We metrize the obtained tree by considering each edge as isometric to an interval of length $\frac{1}{2}$. Later on, we will study block decompositions whose associated trees are isometric to the barycentric subdivision of Bass-Serre trees. If we speak of an *extended Bass-Serre tree*, we mean its barycentric subdivision and denote it by $\mathcal{T}_{\text{ext}}$. With help of this tree, it is possible to define *itineraries of geodesic rays.* The definition of itineraries has its origin in [CK00]. The itinerary of a geodesic ray γ, denoted by $I(\gamma)$, is a path in the tree associated to the block decomposition. This path describes how the geodesic ray runs through the blocks and the walls of the space. The itinerary of a boundary point is the itinerary of its representative that starts in a chosen fixed base point of X.

Section 3.2: Block decompositions with thick walls

For amalgamated free products of CAT(0) groups that satisfy the conditions of the Equivariant Gluing Theorem, Bridson and Haefliger construct CAT(0) spaces on which the group acts geometrically. These CAT(0) spaces have similar decompositions as described above. But in these decompositions, the walls are thick, i.e. they are isometric to $C \times [0, 1]$ where C denotes a convex set in X. We, therefore, define *block decompositions with thick walls* in the second section of Chapter 3. We define itineraries of geodesic rays in such spaces. We observe that itineraries behave more naturally in block decompositions with thick walls than in block decompositions with thin walls.

Section 3.3: Itineraries of geodesic rays in CAT(0) spaces with block decomposition

In Section 3.3, we introduce a common language for CAT(0) spaces with thick or thin walls. We say that such a space has a *block decomposition*. We summarize properties of itineraries of geodesic rays in such spaces.

Section 3.4: The boundary points of every wall behave like a cutset

The most important aspect of our matter is the study of boundary points of walls in Section 3.4. Inspired by the study of cutpoints of Bowditch [Bow98a] and others as described in Section 1.1.2, Lemma 7 in Section 1.7 of [CK00] and the Cycle-Join-Example of Charney and Sultan, we observe that the set of boundary points of a wall in a CAT(0) space with block decomposition behaves like a cutset in the visual and contracting boundary (See Corollary 3.45). A similar observation was recently and independently made by Ben-Zvi and Kropholler in Lemma 3.1 of [BZK19]. Ben-Zvi and Kropholler study path connectedness of visual boundaries and prove that the set of boundary points of any wall behaves like a cutset of a path-component. Differently to Lemma 3.1 in [BZK19], Corollary 3.45 can be applied to contracting boundaries.

Definition 1.2. Let X be a CAT(0) space with block decomposition $(\mathcal{B}, \mathcal{A})$. Let I_0 and I_1 be two paths in the associated tree $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ starting with v_{base}. Let I' be the subgraph of $I_0 \cup I_1$ consisting of all edges that lie in I_0 or I_1 but not in I_0 and I_1 simultaneously. We say that a vertex v *is between* I_0 and I_1 if it is contained in I' and say that I' is the *path between* I_0 *and* I_1.

For our considerations, the following consequence of the cutset property, which we prove in 3.50, is an important tool:

Lemma 1.3 (Key-lemma)**.** *Let X be a complete CAT(0) space with block decomposition $(\mathcal{B}, \mathcal{A})$. Let κ be a connected component of a subspace of $\hat{\partial} X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) containing two points with different itineraries. For every vertex between their itineraries corresponding to a wall A there exists a point $\xi \in \partial A$ such that $\xi \in \kappa$.*

Section 3.5: Types of connected components

In Section 3.5, we classify connected components of boundaries of CAT(0) spaces with block decomposition; we classify them into two types, 1 and 2, as in the Cycle-Join-Example of Charney and Sultan. A connected component is of *type* 1 if all its elements have the same itinerary. If this itinerary is finite, the connected component is of *type* 1_f. If the itinerary is infinite, it is of *type* 1_∞. Otherwise, if a connected component contains elements with distinct itineraries, it is of *type* 2. If a connected component is of type 1_f, it comes from a block, i.e., it is topologically embedded in the boundary of a block. In general, it is difficult to understand connected components of type 1_∞. If we look, for example, at the situations studied in [CK02, Cor. 5.29], there occur connected components in the Tits boundary that are of type 1_∞ and isometric to an interval of length at most π. However, in special situations, connected components of type 1_∞ consist of single points. We will prove that this is the case for the spaces which we study in this thesis. This is an important part of two main results in Chapter 4. It remains to consider connected components of type 2. The Key-lemma above implies that every connected component of type 2 contains a boundary point corresponding to a geodesic ray in a wall. We summarize the classification in Figure 1.6 and Figure 1.7. In Figure 1.7, we use the fact that the topology of the contracting boundary $\vec{\partial}_c X$ is finer than the topology of $\hat{\partial}_c x$.

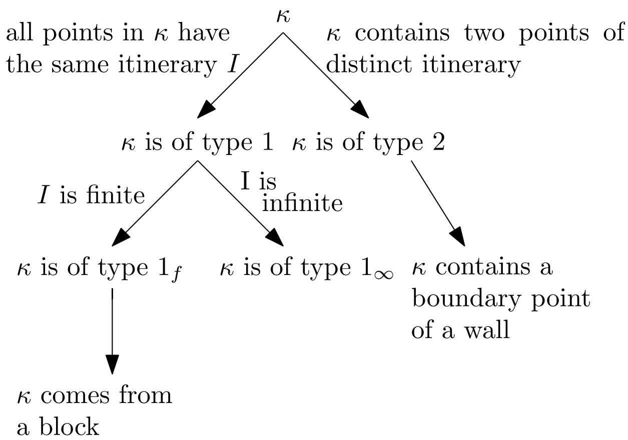

Figure 1.6 Possible types of a connected component. Suppose that X is a complete CAT(0) space with block decomposition and $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is its associated tree. The letter κ denotes a connected component in $\hat{\partial} X$, $\vec{\partial}_c X$ or $\hat{\partial}_c X$. The arrows denote implications valid under the conditions of the attached labels.

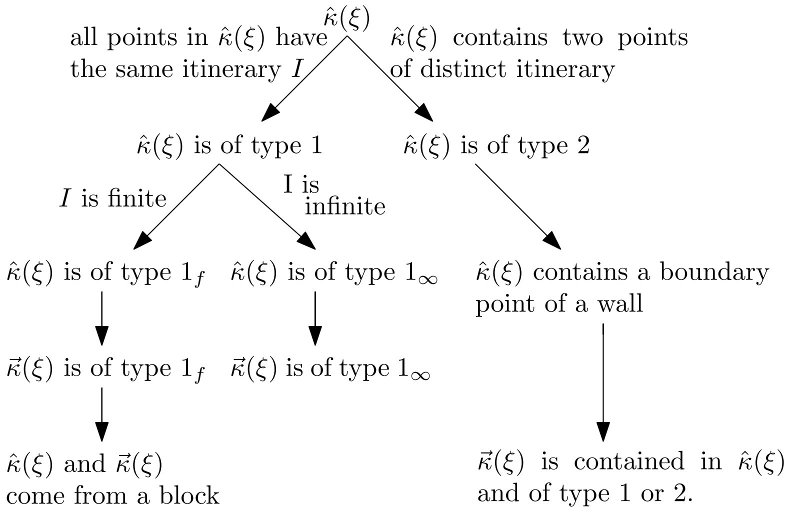

Figure 1.7 Possible types of a connected component of an element ξ in $\partial_c X$ where X is a CAT(0) space with block decomposition and $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is its associated tree. The connected component of ξ in $\hat{\partial}_c X$ is denoted by $\hat{\kappa}(\xi)$ and the connected component of ξ in $\vec{\partial}_c X$ is denoted by $\vec{\kappa}(\xi)$. The arrows denote implications valid under the conditions of the labels at the arrows.

Section 3.6: Contracting boundaries of amalgamated free products of CAT(0) groups

In Section 3.6 we study geometric group actions on CAT(0) spaces with block decomposition. In their Equivariant Gluing Theorem 11.18 of Chapter II in [BH99], Bridson and Haefliger formulate conditions under which it is possible to construct a proper CAT(0) space on which an amalgamated free product of CAT(0) groups acts geometrically. We recall their construction and observe that the obtained space has a block decomposition with thick walls. We examine the construction and formulate conditions under which it is possible to shrink the thick walls to thin walls such that the action of G on the resulting space is still geometric. The goal of the next chapter is to examine contracting boundaries of spaces admitting a geometric action of an amalgamated free product of CAT(0) groups that arise from the construction of Bridson and Haefliger with or without shrunken walls. Thus, we assume that the spaces we work with have all crucial properties of such spaces. These properties are listed in the following convention.

Convention 1.4. Let G_0, G_1 and H be groups acting geometrically on proper CAT(0) spaces X_0, X_1 and Y respectively. Suppose that $G = G_0 *_H G_1$ acts geometrically on a proper CAT(0) space $\mathbb{X} = \mathbb{X}(G_0, X_0, G_1, X_1, H, Y)$ with block decomposition $(\mathcal{B}, \mathcal{A})$ satisfying the following conditions.

a) For every coset gG_0 of G_0 in G, $\mathcal{B}$ contains a block $B^{(gG_0)}$ that is isometric to X_0 and has parity $(-)$.

b) For every coset gG_1 of G_1 in G, $\mathcal{B}$ contains a block $B^{(gG_1)}$ that is isometric to X_1 and has parity $(+)$.

c) For every coset gH of H in G, $\mathcal{A}$ contains a wall $A^{(gH)}$. If $(\mathcal{B}, \mathcal{A})$ is a block decomposition with thin walls, $A^{(gH)}$ is isometric to Y. Otherwise, $A^{(gH)}$ is isometric to $[0, 1] \times Y$.

d) Any wall $A^{(gH)}$ in $\mathcal{A}$ is adjacent to the blocks $B^{(gG_0)}$ and $B^{(gG_1)}$.

e) The tree $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ associated to $(\mathcal{B}, \mathcal{A})$ is isometric to the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$ associated to $G = G_0 *_H G_1$. We identify $\mathcal{T}_{\text{ext}}$ with $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ and say that a vertex with label gH in $\mathcal{T}_{\text{ext}}$ corresponds to the wall $A^{(gH)}$. Analogously a vertex with label gG_i corresponds to the block $B^{(gG_i)}$.

f) The stabilizer of $B^{(gG_i)}$ in G is gG_ig^{-1} for all $g \in G$, $i \in \{0, 1\}$. The stabilizer of every side of $A^{(gH)}$ in G is gHg^{-1} for all $g \in G$. The action of the stabilizer G_i on $B^{(\mathrm{id}\, G_i)}$ is given by the action of G_i on X_i, $i \in \{0, 1\}$. The action of the stabilizer H on every side of $A^{(\mathrm{id}\, H)}$ is given by the action of H on Y.

We denote the set of all blocks of parity $(-)$ by $\mathcal{B}^-$ and the set of all blocks of parity $(+)$ by $\mathcal{B}^+$.

Now we are well-prepared for studying contracting boundaries of amalgamated free products of CAT(0) groups as done in Chapter 4.

1.2.2 Chapter 4: contracting boundaries of amalgamated free products of CAT(0) groups

In Chapter 4, we study contracting boundaries of CAT(0) spaces with block decomposition on which amalgamated free products of CAT(0) groups act geometrically.

Section 4.1: A variant of a theorem of Ben-Zvi and Kropholler

Before we arrive at our first main result, we analyze a theorem that was recently proven independently by Ben-Zvi and Kropholler in the second section of Chapter 4 and compare this theorem with the focus of this thesis. The theorem of Ben-Zvi and Kropholler provides examples for visual boundaries that are not path connected but contain a big path-component and belong to CAT(0) spaces admitting a geometric action of a free amalgamated product of CAT(0) groups. For showing their theorem, Ben-Zvi and Kropholler use a cutset property similar to the one we examine in Chapter 3. Our variant of the cutset property enables us to formulate an analogue to the theorem of Ben-Zvi and Kropholler for contracting boundaries of CAT(0) spaces with block decomposition. For completeness, we formulate this variant for visual- and contracting boundaries. In the case of visual boundaries, the following theorem follows from Theorem 3.2 in [BZK19]. We use notation as in Notation 1.1. The *limit set* $\Lambda(H)$ of a subgroup H of $\mathrm{Iso}(X)$ is the set of accumulation points in $\hat{\partial}X$ $(\hat{\partial}_c X, \vec{\partial}_c X)$ of an orbit of the action of G on X. The following statement is Theorem 4.2.

Theorem 1.5. *(Variant of Theorem 3.2 in [BZK19]) Let $G = G_0 *_H G_1$ be a CAT(0) group acting geometrically on a proper CAT(0) space X with block decomposition. Suppose that G_0 and H act geometrically on a block $\mathbf{B}$ and a wall A of X respectively. Furthermore, suppose A and its translates to separate $\mathbf{B}$ from the rest of X. Lastly, suppose that $\mathbf{B}$ satisfies the following*

a) $\mathbf{B}$ *has a block decomposition $(\mathcal{B}, \mathcal{A})$ such that $\bigcup_{B\in\mathcal{B}} \hat{\partial}B$ $(\bigcup_{B\in\mathcal{B}} \hat{\partial}_{c,X}B$, $\bigcup_{B\in\mathcal{B}} \vec{\partial}_{c,X}B)$ is nonempty and path connected,*

b) $\hat{\partial}\mathbf{B}$ $(\hat{\partial}_{c,X}\mathbf{B}, \vec{\partial}_{c,X}\mathbf{B})$ *is not path connected and*

c) $\Lambda(H)$ *is contained in the path component of* $\hat{\partial}\mathbf{B}$ $(\hat{\partial}_{c,X}\mathbf{B}, \vec{\partial}_{c,X}\mathbf{B})$ *that contains* $(\bigcup_{B\in\mathcal{B}} \hat{\partial}_{c,X}B$, $\bigcup_{B\in\mathcal{B}} \vec{\partial}_{c,X}B)$

Then $\hat{\partial}X$ $(\hat{\partial}_c X\ \vec{\partial}_c X)$ *is not path connected.*

Ben-Zvi and Kropholler examine examples of spaces whose visual boundaries satisfy the conditions of the theorem reported above. The contracting boundaries of these examples don't satisfy the properties of the theorem anymore. It is an interesting question if there are, nevertheless, contracting boundaries that satisfy the conditions of the theorem above. If there exist examples of such contracting boundaries, they are not path connected but contain a large connected component. This large connected component arises from a path-component that is assumed to exist in the contracting boundary of one of the blocks of the space. The focus of this thesis is differently to the focus of the variant of the theorem of Ben-Zvi and Kropholler above. Indeed, we are mainly interested in the question of which contracting boundaries of amalgamated free products have totally disconnected contracting boundaries. In particular, we are interested to understand contracting boundaries of CAT(0) space with block decomposition whose blocks have totally disconnected contracting boundaries.

Section 4.2: Generalization of an example of Charney and Sultan

In Section 4.2, we prove our first main result. We generalize the Cycle-Join-Example of Charney and Sultan [CS15, Section 4.2]. In their example, Charney and Sultan calculate the contracting boundary of a right-angled Coxeter group W. For that purpose, they examine the contracting boundary of its Davis complex X. This Davis complex X is a CAT(0) space with block decomposition. All blocks of one parity $(+)$ of X have an empty contracting boundary and all blocks of the other parity $(-)$ have a 1-sphere S^1 as contracting boundary. Thereby, in each such sphere a dense set of points corresponds to geodesic rays that are not contracting in the ambient space. Thus, every block of parity $(-)$ contributes a totally disconnected subset of a 1-sphere to the contracting boundary of X. Such a set is totally disconnected. Using this observation, Charney and Sultan prove that the contracting boundary of X is totally disconnected. The crucial point in their proof is that no wall contains any geodesic ray that is contracting in the ambient space.

We transfer their considerations to the setting of amalgamated free products of CAT(0) groups that act on a CAT(0) space with block decomposition. We study the case that no wall contains any geodesic ray that is contracting in the ambient space. Differently to the Cycle-Join-Example of Charney and Sultan, we allow each block to have a nonempty contracting boundary. By means of our preparation in Chapter 3, we observe that every connected component is of type 1 if no wall contains any geodesic ray that is contracting in the ambient space. Because connected components of type 1_f are well-understood, we concentrate on understanding connected components of type 1_∞. We assume that a technical property (QG) is satisfied. This property ensures that certain curves connecting particular orbit points behave well enough. This condition is defined in Definition 4.8. Using the methods of the Cycle-Join-Example, we prove the following theorem by means of the Stability Lemma 2.25 of Bestvina and Fujiwara.

We use notation as in Notation 1.1. We study the following result in Theorem 4.10.

Theorem 1.6 (Generalization of the example of Charney and Sultan)**.** *Let G_0, G_1 and H be groups acting geometrically on proper CAT(0) spaces X_0, X_1 and Y respectively. Suppose that $G_0 *_H G_1$ acts geometrically on a proper CAT(0) space $\mathbb{X} = \mathbb{X}(G_0, X_0, G_1, X_1, H, Y)$ with block decomposition as in Convention 1.4. Assume that*

1. *one side of a wall in $\mathbb{X}$ does not contain any geodesic ray that is contracting in $\mathbb{X}$ and that*
2. *$\mathbb{X}$ satisfies property (QG) as defined in Definition 4.8.*

Suppose that κ is a connected component of $\vec{\partial}_c\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$). Then

1. *κ consists of a single point or*
2. *for all $B \in \mathcal{B}^-$, κ is homeomorphic to a connected component of $\vec{\partial}_{c,\mathbb{X}}B$ ($\hat{\partial}_{c,\mathbb{X}}B$) or*
3. *for all $B \in \mathcal{B}^+$, κ is homeomorphic to a connected component of $\vec{\partial}_{c,\mathbb{X}}B$ ($\hat{\partial}_{c,\mathbb{X}}B$).*

The following corollary, that we study in Corollary 4.11, is a direct consequence.

Corollary 1.7. *Let G, G_0, G_1 H, X_0, X_1, Y and $\mathbb{X}$ be as in Convention 1.4. If all assumptions of Theorem 1.6 are satisfied and $\hat{\partial}_{c,\mathbb{X}}X_0$ and $\hat{\partial}_{c,\mathbb{X}}X_1$ ($\vec{\partial}_{c,\mathbb{X}}X_0$ and $\vec{\partial}_{c,\mathbb{X}}X_1$) each are totally disconnected, then $\hat{\partial}_c\mathbb{X}$ ($\vec{\partial}_c\mathbb{X}$) is totally disconnected.*

The question arises of when the property (QG) is satisfied. We remark that the property (QG) is satisfied if the behavior of the shortest point projections of points to walls is good enough. Further work may be done for examining how this is related to the behavior of the shortest point projections in spaces that admit a geometric action of a relative hyperbolic group. We concentrate on the case that G is a Coxeter group and prove the following in Corollary 4.15.

Corollary 1.8. *Let G, G_0, G_1 H, X_0, X_1, Y and $\mathbb{X}$ be as in Convention 1.4. If G is a Coxeter group, the space $\mathbb{X}$ satisfies the property (QG).*

For showing this, we take advantage of the fact that elements of the amalgamated free product $G = G_0 *_H G_1$ can uniquely be represented as words in $G_1 \cup G_2$ whose letters (except for the last letter) are fixed representatives for G_0/H and G_1/H. By means of the Deletion Condition of Coxeter groups and the Švarc-Milnor Lemma, we conclude that $\mathbb{X}$ satisfies property (QG).

Corollary 1.8 can be seen as a preparation for Chapter 5 of this thesis. In Chapter 5, we apply Theorem 1.6 to a class of right-angled Coxeter groups; see Theorem 1.18. This leads to a class of right-angled Coxeter groups that have empty or totally disconnected contracting boundaries; see Corollary 1.19.

Overall, this section provides a good understanding of how contracting boundaries of CAT(0) spaces with block decomposition behave if walls don't contain contracting geodesic rays. The question remains what happens if walls contain contracting geodesic rays. The easiest case for studying this question is when walls are quasi-isometric to $\mathbb{Z}$. In this situation, they contain an axis for an axial isometry that is *rank-one*, i.e., it is not bounded by a Euclidean half-plane. This is our motivation for studying axial rank-one isometries in the next section.

Section 4.3: Boundary points of axes for rank-one isometries

In Section 4.3, we study axes for axial rank-one isometries in proper CAT(0) spaces. Suppose that X is an arbitrary proper CAT(0) space on which an axial rank-one isometry g acts. Let γ be an axis for g, i.e., γ is a bi-infinite geodesic ray and g acts on γ by translations and no Euclidean half-plane bounds γ. Let $\gamma^+(\infty)$ and $\gamma^-(\infty)$ be the associated boundary points. Assume further that g acts as a homeomorphism on a subspace Z of $\hat{\partial}X$. Based on the results concerning North-South Dynamics of Hamenstädt in [Ham09] we examine connected components of Z that contain $\gamma^+(\infty)$ or $\gamma^-(\infty)$. We prove the following theorem in Theorem 4.24.

Theorem 1.9. *Let g be an axial rank-one isometry of a proper CAT(0) space X and γ an axis for g. Suppose that Z is a subspace of the visual boundary of X containing $\gamma^+(\infty)$ and $\gamma^-(\infty)$ such that g acts on Z as a homeomorphism. Let $\kappa(\gamma^+(\infty))$ and $\kappa(\gamma^-(\infty))$ be the connected components of $\gamma^+(\infty)$ and $\gamma^-(\infty)$ in Z respectively. Then, either*

a) $|\kappa(\gamma^+(\infty))| = |\kappa(\gamma^-(\infty))| = 1$ *or*

b) $\kappa(\gamma^+(\infty)) = \kappa(\gamma^-(\infty))$.

If Z is not connected, then every open neighborhood of $\gamma^+(\infty)$ ($\gamma^-(\infty)$) contains a connected component. If Z is not connected and contains more than two points, then every open neighborhood of $\gamma^+(\infty)$ ($\gamma^-(\infty)$) contains a connected component that does not contain $\gamma^+(\infty)$ ($\gamma^-(\infty)$).

Murray proves a weak North-South Dynamics of rank-one isometries in contracting boundaries in [Mur19]. It follows that Theorem 1.9 holds for contracting boundaries if we add the condition that $\kappa(\gamma^+(\infty))$ and $\kappa(\gamma^-(\infty))$ are contained in a compact subset of the contracting boundary of X.

The following is a direct consequence of Theorem 1.9. We study it in Corollary 4.25.

Corollary 1.10. *Let g be an axial rank-one isometry of a proper CAT(0) space X and γ an axis for g. Either $\hat{\partial}_c X$ has a connected component containing $\gamma^+(\infty)$ and $\gamma^-(\infty)$ simultaneously or the connected components of $\gamma^+(\infty)$ and $\gamma^-(\infty)$ in $\hat{\partial}_c X$ and $\vec{\partial}_c X$ each consist of a single point.*

Suppose that γ is an axis for an axial rank-one isometry. Suppose further that the contracting boundary has a connected component that consists of at least two points and contains one boundary point associated to γ. Then the topological space $\hat{\partial}_c X$ has a connected component that contains both boundary points associated to γ. Intuitively, the boundary points associated to γ are far away from each other and a connected component has to be large if it contains two such points. Suppose that the itinerary of γ is infinite. If both associated boundary points are contained in a common connected component, we can apply the Key-lemma of Chapter 3 (Lemma 1.3). It says that the connected component of $\gamma^+(\infty)$ and $\gamma^-(\infty)$ contains one boundary point for every wall associated to a vertex in the itinerary of γ. See Lemma 4.32. Because of this strong property, we call γ *essential* if it is an axis for an axial rank-one isometry whose itinerary is infinite.

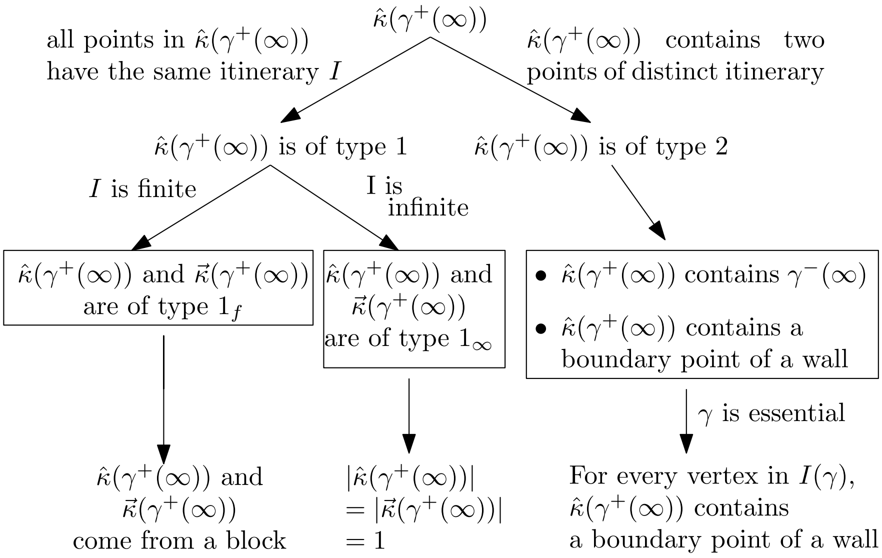

Figure 1.8 The letter γ denotes an axis for a rank-one isometry in a CAT(0) space $\mathbb{X}$ as in Convention 1.4. The terms $\hat{\kappa}(\gamma^+(\infty))$ and $\vec{\kappa}(\gamma^+(\infty))$ denote the connected component of $\gamma^+(\infty)$ in $\hat{\partial}_c\mathbb{X}$ and $\vec{\partial}_c\mathbb{X}$ respectively. The arrows denote implications. The property at a peak follows if the conditions at the arrows are satisfied.

Figure 1.8 summarizes the properties of a connected component of an equivalence class of an oriented axis for a rank-one isometry in $\hat{\partial}_c\mathbb{X}$.

We will consider essential axes for rank-one isometry at the end of the next section.

Section 4.4: Amalgamated free products along groups quasi-isometric to $\mathbb{Z}$

In Section 4.4, we study contracting boundaries of amalgamated free products $G = G_0 *_H G_1$ of CAT(0) groups along groups that are quasi-isometric to $\mathbb{Z}$. Corollary 11.19 in part II in [BH99] of Bridson and Haefliger implies that such groups each act geometrically on a CAT(0) space with block decomposition $\mathbb{X}$. Recall that we studied the case where no wall contains a contracting geodesic ray in Section 4.2. Hence, we concentrate now on the remaining case where every wall of $\mathbb{X}$ contains a geodesic ray that is contracting in the ambient space. For simplicity, we work with the following convention. We remark that it is always possible to construct the space $\mathbb{X}$ so that the conditions of the following convention are satisfied. If g is a rank-one isometry, we denote by $\mathrm{Min}(g) \subseteq \mathbb{X}$ the set in $\mathbb{X}$ on which the displacement function of g is minimal.

Convention 1.11. Let G_0 and G_1 and H be groups acting geometrically on CAT(0) spaces X_0, X_1 and Y respectively. Let Y be quasi-isometric to $\mathbb{Z}$. Then H contains an axial isometry h_α. We assume that h_α is rank-one and that $Min(h_\alpha) = Y$. Let $\mathbb{X}$ be a CAT(0) space with block decomposition associated to G, X_0, X_1 and Y as in Convention 1.4. Let α be an axis for h_α which is contained in $A^{(\mathrm{id}\, H)}$. We choose a base point x_{base} that is contained in α. Then both, the itinerary of $\alpha^+(\infty)$ and $\alpha^-(\infty)$ consist of the vertex v_{base} in the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$ associated to $G = G_0 *_H G_1$. Without loss of generality we suppose that $\alpha(0) = x_{\text{base}}$.

In this situation, two types of connected components can occur. We characterize these two types. In Corollary 4.39, we prove that connected components of type 1 in $\hat{\partial}_c\mathbb{X}$ and $\vec{\partial}_c\mathbb{X}$ either consist of single points or come from the boundaries of blocks. For that purpose, we adapt a proof of Murray in [Mur19]. Afterwards, we study connected components of type 2 in the topological space $\hat{\partial}_c\mathbb{X}$. For that purpose, we consider the following subgraphs of the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$ of $G = G_0 *_H G_1$.

Definition 1.12. Let $\Xi \in \{\hat{\partial}\mathbb{X}, \hat{\partial}_c\mathbb{X}\}$. Let $g \in G$. Let $T_{g\cdot\alpha} = T_{g\cdot\alpha}(\Xi)$ be the subgraph of $\mathcal{T}_{\text{ext}}$ induced by all vertices whose corresponding wall or block in $\mathbb{X}$ contains a geodesic ray γ such that $\gamma(\infty) \in \kappa(g \cdot \alpha^+(\infty))$ in Ξ.

We show that these subgraphs are trees. We are mainly interested in $\hat{\partial}_c\mathbb{X}$. For completeness, we study $\hat{\partial}\mathbb{X}$ as well. In the following, let $\Xi \in \{\hat{\partial}\mathbb{X}, \hat{\partial}_c\mathbb{X}\}$. If we write $T_{g\cdot\alpha}$, we always mean the associated tree $T_{g\cdot\alpha}(\Xi)$ as defined above. By means of the results in Section 4.3, we show in Lemma 4.49 the in following:

Lemma 1.13. *Suppose that Ξ contains a connected component of type 2. Then the trees in the set $\{T_{g\cdot\alpha} \mid g \in M\}$ are pairwise edge-disjoint, isometric, and cover $\mathcal{T}_{ext}$, i.e., every edge of $\mathcal{T}_{ext}$ is contained in an edge of a tree in $\{T_{g\cdot\alpha} \mid g \in M\}$. Furthermore, G acts on the set $\{T_{g\cdot\alpha} \mid g \in M\}$ transitively.*

This leads to the following theorem which we study in Theorem 4.50

Theorem 1.14. *Let G, G_0, X_0, G_1, X_1, H, Y and $\mathbb{X}$ be as in Convention 1.11. Suppose that $\hat{\partial}_c\mathbb{X}$ contains a connected component of type 2. Then the set of connected components of type 2 is bijective to the set of edge-disjoint subtrees $\{T_{g\cdot\alpha} \mid g \in M\}$ of $\mathcal{T}_{ext}$ covering $\mathcal{T}_{ext}$.*

Figure 1.9 summarizes the classification of connected components of $\hat{\partial}_c\mathbb{X}$ resulting from this section. Because $\vec{\partial}_c\mathbb{X}$ is finer than $\hat{\partial}_c\mathbb{X}$, every connected component of an element ξ in $\vec{\partial}_c\mathbb{X}$ is contained in a connected component of ξ in $\hat{\partial}_c\mathbb{X}$. If the connected component of ξ is of type 1 in $\hat{\partial}_c\mathbb{X}$, then it is also of type 1 in $\vec{\partial}_c\mathbb{X}$. If the connected component of ξ is of type 2 in $\hat{\partial}_c\mathbb{X}$, then it cannot be larger in $\vec{\partial}_c\mathbb{X}$. The arising consequences are pictured in Figure 1.10. The question arises: what do these statements imply if G_0 and G_1 have totally disconnected contracting boundaries? We observe that $G_0 *_H G_1$ is totally disconnected if the connected component of the two boundary points associated to α each consist of a single point. Suppose that the contracting boundary of $G_0 *_H G_1$ is not totally disconnected. We conclude that then $\hat{\partial}_c\mathbb{X}$ has a connected component containing both boundary points associated to α. The question arises: when does such a connected component in $\hat{\partial}_c\mathbb{X}$ exist? Recall that we call an axis *essential* if it is an axis for an axial rank-one isometry of infinite itinerary. It might be possible that the existence of a connected component of size at least two is highly related to the existence of essential axes for axial rank-one isometries, whose associated boundary points lie in the a common connected component. We finish Section 4.4 with the following question.

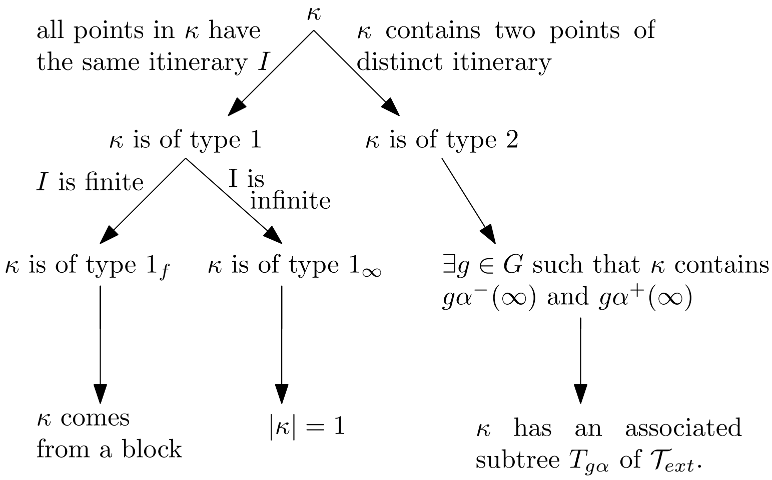

Figure 1.9 Possible types of a connected component κ in $\hat{\partial}_c\mathbb{X}$ where $\mathbb{X}$ is as in Convention 1.11. The arrows denote implications under the conditions of the labels of the arrows.

Question 6. Let G_0 and G_1 CAT(0) groups and H a group quasi-isometric to $\mathbb{Z}$. Suppose that $\vec{\partial}_c G_0$ and $\vec{\partial}_c G_1$ are totally disconnected. Are the following statements equivalent?

a) The contracting boundary of $G = G_0 *_H G_1$ is totally disconnected or empty.

b) G acts geometrically on a CAT(0) space X such that the connected component of every equivalence class of an oriented essential axis in $\hat{\partial}_c X$ consists of a single point.

In summary, we studied contracting boundaries of amalgamated free products by the examination of subspaces of visual boundaries up to Chapter 4. In the remaining chapter (Chapter 5) of this thesis, we apply our results to right-angled Coxeter groups.

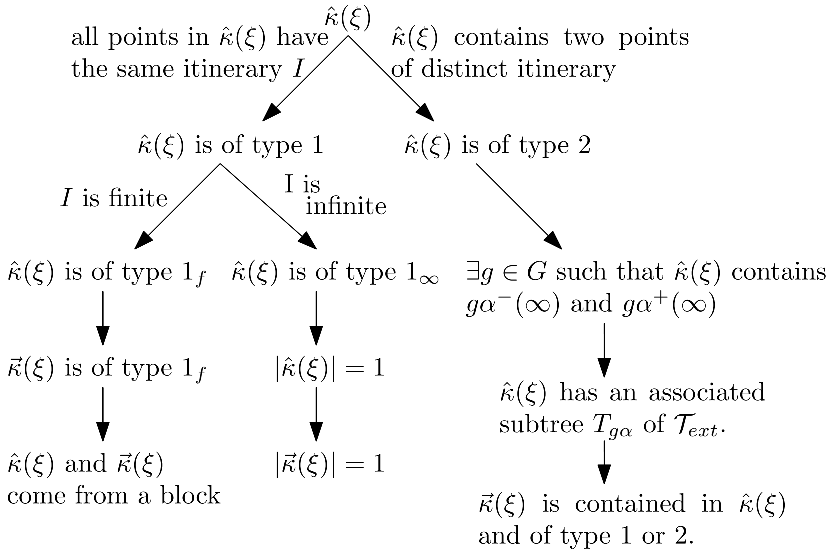

Figure 1.10 Possible types of a connected component of an element ξ in $\partial_c\mathbb{X}$ where $\mathbb{X}$ is as in Convention 1.11. The connected component of ξ in $\hat{\partial}_c\mathbb{X}$ is denoted by $\hat{\kappa}(\xi)$ and the connected component of ξ in $\vec{\partial}_c\mathbb{X}$ is denoted by $\vec{\kappa}(\xi)$. The arrows denote implications under the conditions of the labels of the arrows.

1.2.3 Chapter 5: Contracting boundaries of right-angled Coxeter groups

In Chapter 5, we apply our results of Chapter 4 for studying the question of when the contracting boundary of a right-angled Coxeter group with defining graph Λ is totally disconnected. For examining the contracting boundary of W_Λ, we have to study the contracting boundary of a space on which W_Λ acts geometrically. Such a space is the Davis complex Σ_Λ of W_Λ. We use notation as in Notation 1.1. We examine the contracting boundary $\vec{\partial}_c\Sigma_\Lambda$ of Σ_Λ by investigating the topological space $\hat{\partial}_c\Sigma_\Lambda$. We define the Davis complex of a graph to be the Davis complex of its associated right-angled Coxeter group. Throughout Chapter 5, we assume every graph to be simplicial. If Λ is a graph, $V(\Lambda)$ denote its vertex set and $E(\Lambda)$ denotes its edge set.

Section 5.1: A conjecture about contracting boundaries of right-angled Coxeter groups

Section 5.1 concerns Conjecture 1.14 in [Tra19] formulated by Tran. We refer to it as the Burst-Cycle-Conjecture in Conjecture 5.5 and summarize what is known about this conjecture. In particular, we explain how the Burst-Cycle-Conjecture is related to an example of Charney and Sultan in section 4.2 of [CS15], to which we refer as the Cycle-Join-Example. We summarize this example in Section 5.1.1.

In the following, we give a short overview of Section 5.1. We say that an edge of Λ is a *diagonal* of a cycle C if it connects two non-consecutive vertices of C. A cycle is *induced* if it does not have diagonals.

Definition 1.15 (burst cycles)**.** We say that a cycle in a graph Λ is *burst* in Λ if one of the following three conditions is satisfied:

- C has length 3 or 4,
- C has a diagonal, i.e., two non-consecutive vertices of C are connected by an edge,
- the vertex set of C contains a pair of non-adjacent vertices of an induced 4-cycle.

A cycle in a graph is *intact* if it is not burst in Λ.

See Figure 1.11 for some examples of burst cycles. The following is Conjecture 5.5.

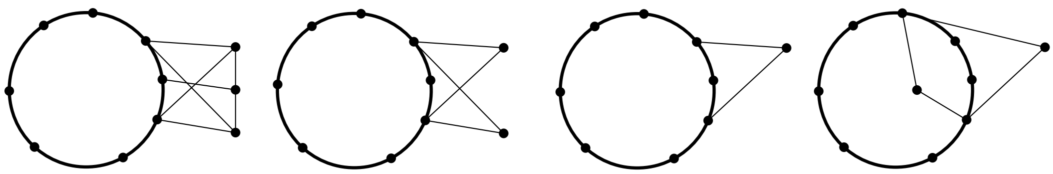

Figure 1.11 The thickened cycles are burst.

Conjecture 1.16 (The Burst-Cycle-Conjecture in [Tra19] (Conjecture 1.14).)**.** *Every cycle in the defining graph Λ of a right-angled Coxeter group W_Λ is burst if and only if the contracting boundary of W_Λ is totally disconnected.*

It was proven in the last years that one part of the Burst-Cycle-Conjecture is true: If the defining graph of a right-angled Coxeter group contains an intact cycle, the contracting boundary of the right-angled Coxeter group is not totally disconnected. Indeed, every intact cycle leads to the existence of a 1-sphere in the contracting boundary of the corresponding right-angled Coxeter group. This was proven for the case of triangle-free graphs by Corollary 7.12 of [Tra19]. For general graphs, it follows from Proposition 4.9 of Genevois [Gen20]. Russell, Spriano and Tran formulated another proof of Genevois' statement in Theorem 7.5 of [RST18]. At the end of Section 5.1, we add to these proofs another one of Lazarovich, presented to me in a discussion we had (see Proof 5.23).

The question remains: how does the contracting boundary of a right-angled Coxeter group looks like whose defining graph does not contain any intact cycle? The easiest examples of such graphs are cliques and nontrivial joins. A *clique* is a graph whose vertices are pairwise connected by an edge. A *join* is a graph that is obtained by two vertex-disjoint graphs Λ_0 and Λ_1 by connecting each vertex of Λ_0 with each vertex of Λ_1. A join is *nontrivial* if neither Λ_0 nor Λ_1 is a clique. See Figure 1.12 for examples of nontrivial joins. If a graph is a clique, the corresponding right-angled Coxeter group is finite. If a graph is a nontrivial join, the corresponding right-angled Coxeter group

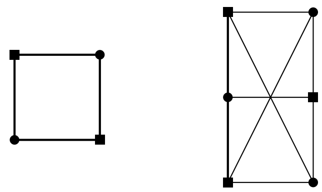

Figure 1.12 The pictured graphs both are joints of two graphs. vertices of different shapes belong to different graphs.

is the direct product of two infinite right-angled Coxeter groups. In both cases, the corresponding right-angled Coxeter groups have empty contracting boundaries. If a graph is neither a clique nor a nontrivial join, its contracting boundary contains at least one element. This follows from Caprace's and Sageev's observation in [CS11, Cor. B]. Thus it is an interesting question of how contracting boundaries of right-angled Coxeter groups look like whose defining graphs don't contain intact cycles and are neither cliques nor nontrivial joins. The Cycle-Join-Example of Charney and Sultan in Section 4.2 of [CS15] is such an example. The defining graph of the Cycle-Join-Example is pictured in Figure 1.13. Charney and Sultan prove that the corresponding right-angled

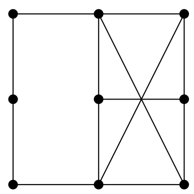

Figure 1.13 The defining graph of the Cycle-Join-Example studied by Charney and Sultan in Section 4.2 in [CS15].

Coxeter group has totally disconnected contracting boundary. In particular, it satisfies the Burst-Cycle-Conjecture. The defining graph consists of a 6-cycle and a nontrivial join; see Figure 1.14. The *union of two graphs* Λ_0 *and* Λ_1 is the graph whose vertex set is the union of the (possibly non-disjoint) vertex sets $V(\Lambda_0)$ and $V(\Lambda_1)$ and whose edge set is the union of the (possibly non-disjoint) edge sets $E(\Lambda_0)$ and $E(\Lambda_1)$. If a graph is the union of a cycle of length at least 5 and a nontrivial join, it can be proven analogously that the corresponding right-angled Coxeter group has totally disconnected contracting boundary. In Section 5.3, these graphs play an important role and we call them *Charney-Sultan-graph*s (see Definition 5.35). The examples of bad cycles in Figure 1.11 are Charney-Sultan-graphs, i.e., the associated right-angled Coxeter groups have all totally disconnected contracting boundaries.

Furthermore, it can be shown like in the Cycle-Join-Example that the contracting boundary of a right-angled Coxeter group with totally disconnected contracting boundary stays totally disconnected if we glue a non-trivial join on its defining graph. Nguyen and Tran used this observation to show that each graph in the graph class $\mathcal{G}$ defined in Definition 5.14 corresponds to a right-angled Coxeter group with totally disconnected contracting boundary. In particular, $\mathcal{G}$ satisfies the Burst-Cycle-Conjecture. Our goal is to find a larger graph class satisfying Conjecture 1.16.

Section 5.2: Block decompositions of Davis complexes

In Section 5.2, we prove that every Davis complex of an infinite right-angled Coxeter group has a non-trivial block decomposition. For proving this, we study *proper separations* of graphs. Recall, a subgraph Λ' of a graph Λ is *induced* if every edge of Λ with endvertices in $V(\Lambda')$ is an edge of Λ'. The intersection of two graphs Λ_0 and Λ_1 is the graph whose vertex set is $V(\Lambda_0) \cap V(\Lambda_1)$ and whose edge set is $E(\Lambda_0) \cap E(\Lambda_1)$. A *separation* of a graph Λ is an unordered pair of two induced subgraphs Λ_0 and Λ_1 such that $\Lambda = \Lambda_0 \cup \Lambda_1$. A separation $\{\Lambda_0, \Lambda_1\}$ is *proper* if both Λ_0 and Λ_1 have at least one vertex that is not contained in the *separating subgraph* $\Lambda_* = \Lambda_0 \cap \Lambda_1$. Suppose that a graph Λ has a proper separation into two subgraphs Λ_0 and Λ_1 with separating subgraph Λ_*. Then W_Λ is isomorphic to $W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$. For example, the defining graph of the Cycle-Join-Example has a proper separation into the two induced subgraphs pictured in Figure 1.14.

If Λ' is an induced subgraph of a graph Λ, then the Davis complex of Λ' can be canonically embedded in the Davis complex of Λ, i.e., there is an isometric embedding of $\Sigma_{\Lambda'}$ in Σ_Λ such that the embedded Davis complex contains the identity vertex of Σ_Λ. This observation helps in proving the following proposition. That proposition says that every proper separation of the defining graph Λ of a right-angled Coxeter group corresponds to a block decomposition with thin walls of Σ_Λ such that every block is isometric to a Davis complex of one of the induced subgraphs in the proper separation. We remark that we allow the separating subgraph Λ_* of a proper separation to be the trivial graph $(\emptyset, \emptyset)$. Such a graph is a clique on 0 vertices. The Davis complex of $(\emptyset, \emptyset)$ consists of a vertex. If we embed this vertex canonically in Σ_Λ, we identify this vertex with the vertex corresponding to the identity vertex in Σ_Λ.

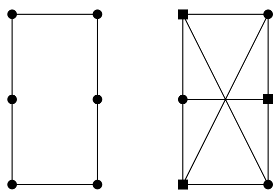

Figure 1.14 Decomposition of the graph in Figure 1.13 into two induced subgraphs Λ_0 (left) and Λ_1 (right).

In Proposition 5.28, we consider the following proposition.

Proposition 1.17. *Let $\{\Lambda_0, \Lambda_1\}$ be a proper separation of a graph Λ into two induced subgraphs Λ_0 and Λ_1 with separating subgraph Λ_*. Let Σ_{Λ_0}, Σ_{Λ_1} and Σ_{Λ_*} be the canonically embedded Davis complexes of Λ_0, Λ_1 and Λ_* in the Davis complex Σ_Λ of Λ. Then*

$$(\{g\Sigma_{\Lambda_0} \mid g \in W_\Lambda\} \cup \{g\Sigma_{\Lambda_1} \mid g \in W_\Lambda\},\ \{g\Sigma_{\Lambda_*} \mid g \in W_\Lambda\})$$

is a block decomposition with thin walls of Σ_Λ. All blocks of parity $(-)$ and $(+)$ are of the form $g\Sigma_{\Lambda_0}$ and $g\Sigma_{\Lambda_1}$, $g \in W_\Lambda$, respectively. The action of $W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$ on Σ_Λ with this block decomposition satisfies all properties of Convention 1.4.

This proposition enables us to apply our results in Chapter 4 to right-angled Coxeter groups.

Section 5.3: Right-angled Coxeter groups satisfying the conjecture

In Section 5.3, we prove our first main result of Chapter 5. We generalize the Cycle-Join-Example. The defining graph of the Cycle-Join-Example has a proper separation where the separating subgraph is contained in a nontrivial join. In the following theorem, we suppose that a graph has a similar behavior. We assume that it has a proper separation such that the separating subgraph Λ_* is either empty or contained in a clique or in a nontrivial join. Recall that W_{Λ_*} has an empty contracting boundary if and only if Λ_* is a clique or a nontrivial join. In this situation, no wall of the associated block decomposition of the Davis complex contains a geodesic ray that is contracting in the ambient Davis complex. Thus, all conditions of Theorem 1.6 in Section 4.2 are satisfied. This way, we obtain the following variant of Theorem 1.6 for right-angled Coxeter groups. It is Theorem 5.32.

Theorem 1.18 (Variant of Theorem 1.6 for right-angled Coxeter groups)**.** *Let Λ be a graph with a proper separation $\{\Lambda_0, \Lambda_1\}$ with separating subgraph Λ_*. Suppose that Λ_* satisfies one of the following two conditions.*

a) Λ_* is contained in a clique

b) Λ_* is contained in a nontrivial join of two induced subgraphs of Λ.

Let Σ_{Λ_0} and Σ_{Λ_1} be the canonically embedded Davis complexes of Λ_0 and Λ_1 in Σ_Λ. Then every connected component of $\vec{\partial}_c \Sigma_\Lambda$ $(\hat{\partial}_c \Sigma_\Lambda)$

a) consists of a single point or

b) is homeomorphic to a connected component of $\vec{\partial}_{c,\Sigma_\Lambda} \Sigma_{\Lambda_0}$ $(\hat{\partial}_{c,\Sigma_\Lambda} \Sigma_{\Lambda_0})$ or

c) is homeomorphic to a connected component of $\vec{\partial}_{c,\Sigma_\Lambda} \Sigma_{\Lambda_1}$ $(\hat{\partial}_{c,\Sigma_\Lambda} \Sigma_{\Lambda_1})$.

Using Theorem 1.18, we generalize the Cycle-Join-Example in the following manner. We define a graph class $\mathcal{J}$ of so-called *join-decomposable graphs* in Definition 5.37. This graph class is defined recursively. It is the largest graph class that can be obtained as follows: At first, we add cliques, trees, empty graphs and Charney-Sultan-graphs. In the next step, we add unions of graphs Λ_0, Λ_1 in $\mathcal{J}$ such that their intersection is empty, a clique or contained in a nontrivial join, where the union of two graphs Λ_0, Λ_1 is the graph with vertex set $V(\Lambda_0) \cup V(\Lambda_1)$ and edge set $E(\Lambda_1) \cup E(\Lambda_2)$. Recall, [CS11] implies that the contracting boundary of a right-angled Coxeter group is empty if and only if its defining graph is neither a nontrivial join nor a clique. We obtain the following corollary that we study in Corollary 5.38.

Corollary 1.19. *Let Λ be a join-decomposable graph. If Λ is a clique or a nontrivial join, the contracting boundary of W_Λ is empty. In the remaining case, the contracting boundary of W_Λ is nonempty and totally disconnected.*

We conclude in Corollary 5.39 that the graph class $\mathcal{J}$ satisfies Conjecture 1.16.

Corollary 1.20. *The Burst-Cycle-Conjecture 1.16 is true for every right-angled Coxeter group whose defining graph is join-decomposable.*

Recall that Nguyen and Tran observed in [NT19] that the graph class $\mathcal{G}$ defined in Definition 5.14 satisfies the Burst-Cycle-Conjecture 1.16. The graph class $\mathcal{J}$ contains $\mathcal{G}$ and is larger than $\mathcal{G}$. For instance, $\mathcal{J}$ contains the defining graph of an example of a right-angled Coxeter group that was examined by Russell, Spriano and Tran [RST18, Example 7.7]. The defining graph Λ of the example is pictured in Figure 1.15. The

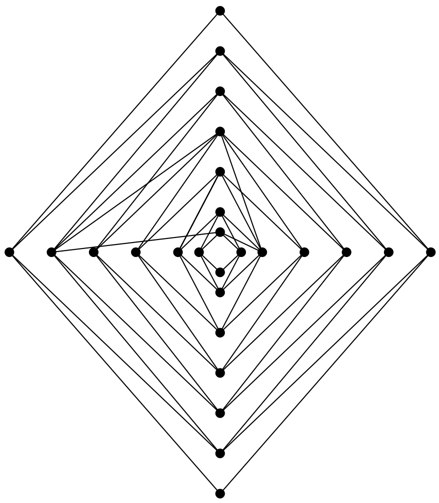

Figure 1.15 Defining graph of a right-angled Coxeter group studied in [RST18, Example 7.7]

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contracting boundary of this group was unknown (see the tabular in Example 7.7 of

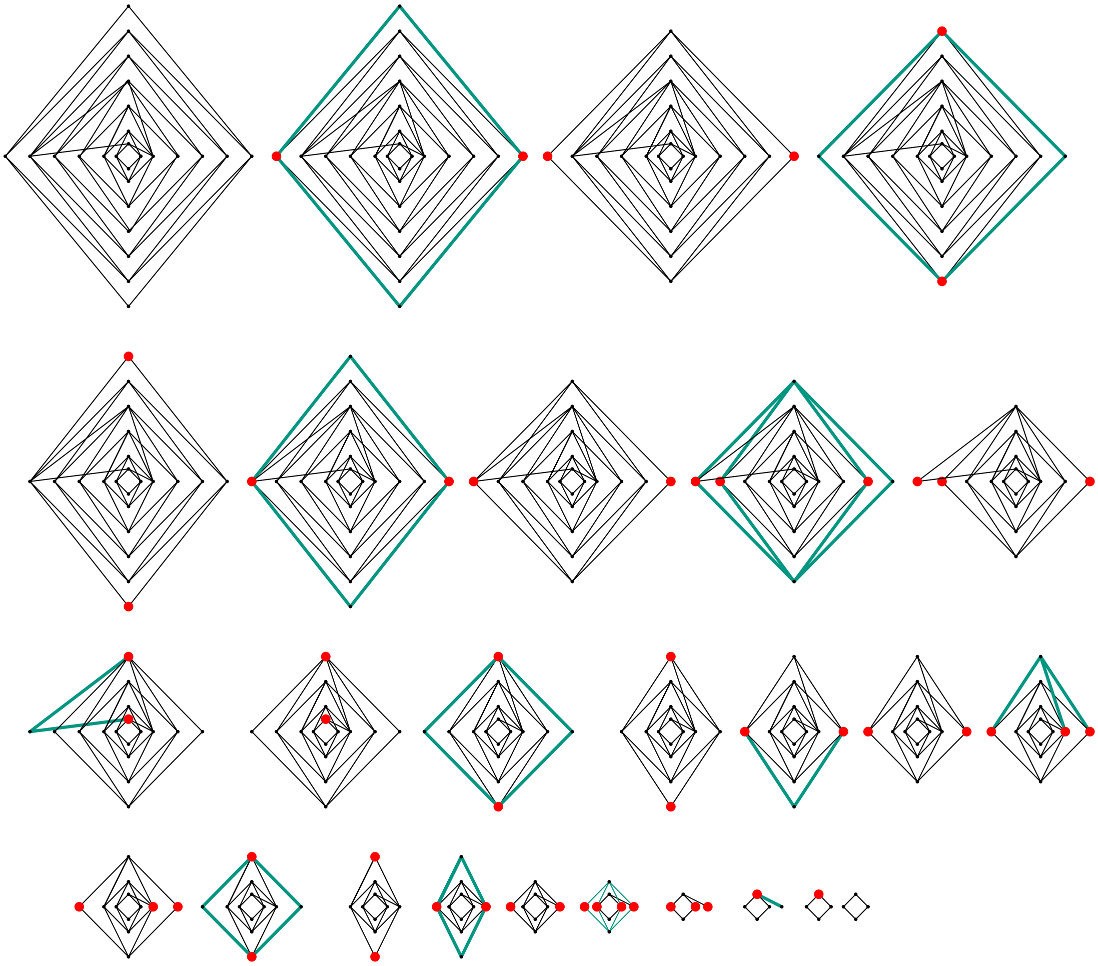

Figure 1.16 Decomposition of the graph in the left upper corner. This decomposition shows that the graph in the left upper corner is join-decomposable.

[RST18]), but it can be said more now. The decomposition pictured in Figure 1.16 shows that Λ is join-decomposable. The graph Λ is pictured in the left upper corner. We decompose Λ from left to right and above to bottom. In the first step (second graph in the first row), we decompose Λ into a green and a black subgraph. The intersection graph consists of two red-colored vertices. The red vertices are contained in a 4-cycle, namely the green colored one. We delete the green 4-cycle and obtain the third graph in the first row. We continue in this manner. In every second step, we decompose the graph into a green and a black graph. The intersection of these two graphs consists always of the thick red vertices. These red vertices are either contained in an induced 4-cycle or in another nontrivial join or in a clique. In every second step, we delete the green subgraph and continue to decompose the obtained graph in the next step. Finally, we end up with a 4-cycle. By definition, a 4-cycle is join-decomposable. We conclude that

Λ is join-decomposableand that W_Λ has totally disconnected contracting boundary. In summary, Theorem 1.18 and Corollary 1.19 build the first main result of this chapter.

Section 5.4: Gluing paths on graphs

In Section 5.4, we study how the contracting boundary of a right-angled Coxeter group changes when we glue a path P of length at least two on its defining graph. In this case, P is an *independent path* in the resulting graph Λ. Let $\bar{P}$ be the graph we obtain by deleting all inner vertices of P and all edges incident to inner vertices of P. Then $\Lambda = \bar{P} \cup P$. We consider the interesting case where the endvertices of P are glued to two non-adjacent vertices. Then the corresponding right-angled Coxeter group can be written as an amalgamated free product along a group that is quasi-isometric to $\mathbb{Z}$. Thus, we can apply the results of Section 4.4. Furthermore, the considerations of Section 4.3 about axes for axial rank-one isometries are very useful. In particular, we use our considerations on essential axes for axial rank-one isometries in splittings over groups quasi-isometric to $\mathbb{Z}$ (Section 4.4). With these ingredients and with help of Hamenstädt's and Murray's theorems about denseness of orbits of contracting boundary points in [Ham09] and [Mur19, Prop. 4.5, Cor. 4.7], we prove the following second main result of this chapter. In the proof of this theorem, the behavior of essential axes for axial rank-one isometries in different block decompositions of Σ_Λ plays a crucial role. The properties of the essential axes cause that there occur just two extreme cases if we glue P on $\bar{P}$. In the first case, all "new connected components" are single points, i.e., if a connected component of the contracting boundary of W_Λ consists of more than one point, then it is topologically embedded in the contracting boundary of $W_{\bar{P}}$. In the case that the contracting boundary of $W_{\bar{P}}$ is totally disconnected, this implies that the contracting boundary of W_Λ is totally disconnected. In the second case, there arises a large connected component in the subspace $\hat{\partial}_c\Sigma_\Lambda$ of the visual boundary of Σ_Λ. Indeed, the visual boundary of the Davis complex of P is topologically embedded in this connected component. We use notation as in Notation 1.1. The following statement is Theorem 5.58.

Theorem 1.21 (Gluing paths on graphs)**.** *Let Λ be a graph that contains an independent path P with distinct endvertices s and t that are not adjacent. Let $\bar{P}$ be the graph obtained from Λ by deleting all inner vertices of P. Let Σ_P and $\Sigma_{\bar{P}}$ be the canonically embedded Davis complexes of P and $\bar{P}$ in Σ_Λ respectively. Let $\alpha_{s,t}$ be the axis for the axial isometry st that intersects the identity-vertex of Σ_Λ. One of the following statements holds.*

a) Every geodesic ray in Σ_P is contracting in Σ_Λ and for each $g \in W_\Lambda$ there exists a connected component in $\hat{\partial}_c\Sigma_\Lambda$ containing $g \cdot \partial\Sigma_P$.

b) For all $\xi \in W_\Lambda \cdot \partial_{c,\Sigma_\Lambda}\Sigma_P \setminus W_\Lambda \cdot \alpha^+_{s,t}(\infty)$, the connected component of ξ in $\hat{\partial}_c\Sigma_\Lambda$ and $\vec{\partial}_c\Sigma_\Lambda$ consists of a single point.

Suppose that Item b) is satisfied. Then every connected component of $\vec{\partial}_c\Sigma_\Lambda$ consists of a single point or is homeomorphic to a connected component of $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$. Analogously, every connected component of $\hat{\partial}_c\Sigma_\Lambda$ consists of a single point or is homeomorphic to a connected component of $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$.

The following corollary, which we study in Corollary 5.59, is a direct consequence.

Corollary 1.22. *Let Λ be a graph that contains an independent path P whose endvertices are not adjacent. Let $\bar{P}$ be the graph obtained from Λ by deleting all inner vertices of P. Suppose that the contracting boundaries of $W_{\bar{P}}$ and W_P are totally disconnected. Then exactly one of the following is true*

a) The contracting boundary of W_Λ is totally disconnected or empty and the topological space $\hat{\partial}_c\Sigma_\Lambda$ is totally disconnected or empty.

b) The topological space $\hat{\partial}_c\Sigma_\Lambda$ has a connected component that contains a set bijective to the visual boundary of Σ_P.

We compare the last corollary with the motivating examples explained at the beginning of this thesis. The 5-cycle C in Figure 1.1 can be obtained by gluing the endvertices of a path P_2 of length two to the endvertices of a path P_3 of length three. If we apply Corollary 1.22, we are in the second case: The contracting boundary of W_C has a connected component κ that contains a set bijective to the Davis complex of W_{P_2}. In this special example, κ coincides with the contracting boundary of the whole complex, i.e. it is homeomorphic to a 1-sphere.

Recall, we asked at the beginning of this thesis why the contracting boundary of W_C is not totally disconnected but the contracting boundary of the example with defining graph as in Figure 1.2 is. We observed that a space quasi-isometric to the hyperbolic plane occurs in one of both examples but not in both. The considerations of Section 5.4 and Section 4.4 provide a further answer to why one of the contracting boundaries is connected but the other is totally disconnected: the behavior of essential axes of rank-one isometries in the two examples is different. In the example pictured in Figure 1.2, the connected component of each boundary point associated to an axis of a rank-one isometry consists of a single point. In the second example pictured in Figure 1.1, the connected component of each boundary point associated to a rank-one isometry g contains both boundary points associated to g. This observation is related to Question 6. It would be very interesting to examine this question for the situation in Corollary 1.22.

We consider Corollary 1.22 more generally. Suppose that we are in the situation of Corollary 1.22. Then either all connected components in $\vec{\partial}_c W_\Lambda$ and $\hat{\partial}_c\Sigma_\Lambda$ are single points or $\hat{\partial}_c\Sigma_\Lambda$ contains a large connected component κ. This connected component contains all equivalence classes of geodesic rays that are contained in the canonically embedded Davis complex Σ_P in Σ_Λ. In the example above, κ was a sphere. It is an interesting question how large this connected component κ is in this general setting. By our considerations in Section 4.4, it is a connected component of type 2. Associated to this connected component is a subtree $T_{\alpha_{s,t}}$ of the Bass-Serre tree $\mathcal{T}_{\text{ext}}$ associated to $W_\Lambda = W_{\bar{P}} *_{W_{\Lambda_{s,t}}} W_P$ (see Definition 1.12). The larger this tree $T_{\alpha_{s,t}}$ is, the larger is the connected component κ. It would be interesting to understand how large this tree is.

Another important observation is the following. Suppose that the connected component κ in $\hat{\partial}_c\Sigma_\Lambda$ described above is also connected in the contracting boundary of Σ_Λ. Assume further that Λ does not contain any intact cycle. Then Λ is a counterexample to Conjecture 1.16. Such examples occur in the next section. They imply that Conjecture 1.16 is wrong in general.

Section 5.5: Counterexamples to the conjecture

Section 5.5 is an outlook. The content is joint work with Graeber, Lazarovich and Stark. We consider three counterexamples proving that Conjecture 1.16 is wrong in general, even for triangle-free graphs. The first example was found by Graeber (Section 5.5.1). The two other examples (described in Section 5.5.3 and Section 5.5.3) are inspired by this first example. The contracting boundaries of all three examples contain a 1-sphere although their defining graphs don't contain any intact cycle. The first two examples (in Section 5.5.1 and Section 5.5.2) can be obtained by gluing a path on a graph as described above. The first example contains triangles. The second example is triangle-free. Both examples are *path-decomposable*, i.e., they can be obtained as follows. We start with a clique and glue successively on paths of length at least two. See Definition 5.60 for a formal definition of path-decomposable graphs. In the first two counterexamples, the spheres in the corresponding contracting boundaries come from three paths in the defining graphs that satisfy certain properties. Inspired by this discovery, we study graphs without intact cycles that don't contain such paths. We call such graphs *totally burst*. The following is Definition 5.77.

Definition 1.23. A graph Λ is *totally burst*, if all its cycles are burst and Λ does not contain a pair of non-adjacent vertices u and v that are linked by three paths P_0, P_1 such that

a) P_0, P_1 and P_2 are independent to each other,

b) for all $i \in \{1, 2, 3\}$, no pair of non-adjacent vertices in P_i are contained in an induced 4-cycle of Λ,

c) two of the three pairs of the three paths build an induced cycle of length at least 5, i.e., there exists $i \in \{1, 2, 3\}$ such that for $j \in \{1, 2, 3\} \setminus \{i\}$, $P_i \cup P_j$ is an induced cycle of length at least 5.

A graph that is not totally burst is *pretty intact*.

We study the question if Conjecture 1.16 might become true if we formulate it in terms of totally burst graphs. The third counterexample in Section 5.5 shows that this is not the case. Indeed, the defining graph is totally burst but the contracting boundary of the corresponding right-angled Coxeter group contains a 1-sphere. The defining graph is triangle-free. So, not all triangle-free graphs satisfy the reformulated conjecture. But the defining graph is not path-decomposable. So, the reformulated conjecture might be true for path-decomposable graphs.

Section 5.6: Summary of the results of this chapter and a new conjecture

In summary, our two main results in this chapter together with the study of the three counterexamples in Section 5.5 provide hints that the following conjecture, which we consider in Conjecture 5.85 might be true for path-decomposable graphs.

Conjecture 1.24. *Let Λ be a path-decomposable graph. The following are equivalent.*

- *The contracting boundary of W_Λ is empty or totally disconnected.*
- *Λ is totally burst.*
- *Λ is join-decomposable.*
- *The contracting boundary of W_Λ does not contain a 1-sphere.*

This conjecture concerns only path-decomposable graphs. For general graphs, we ask the following question.

Question 7. Suppose that Λ is a graph that is not join-decomposable. When does it contain a 1-sphere?

A promising approach for examining this question is to study the third counterexample in Section 5.5.

2 Basic concepts

2.1 Simplicial complexes and graphs

In this section, we define simplicial complexes and graphs. The definitions concerning simplicial complexes comply with the assignments in Appendix A.2 in [Dav08]. The definitions and facts concerning graphs are based on West's book about graph theory [Wes01]. Another reference is [Die17].

We define simplicial complexes as Davis in Definition A.2.6 of [Dav08].

Definition 2.1. An *abstract simplicial complex* Λ consists of a set $\mathcal{V}$ and a collection $\mathcal{E}$ of finite subsets of $\mathcal{V}$, such that

a) for each $v \in \mathcal{V}$, $\{v\} \in \mathcal{E}$ and

b) if $A \in \mathcal{E}$ and if $A' \subseteq A$, then $A' \in \mathcal{E}$.

An abstract simplicial complex is a set partially ordered by inclusion.

If Λ is a simplicial complex, we denote its vertex set by $\mathcal{V}(\Lambda)$ and its edge set by $\mathcal{E}(\Lambda)$. Let $\Lambda = (\mathcal{V}, \mathcal{E})$ be a simplicial complex. It is *finite* if its vertex set is finite. The elements of $\mathcal{V}$ are called *vertices* and the elements of $\mathcal{E}$ are called *simplices*. The *rank* of a simplex A is the number of elements $|A|$ contained in it. The *dimension* of a simplex A is $|A| - 1$. A simplex A' that is contained in a simplex A is a *face* of A. A k-simplex is a simplex of dimension k. A 0-simplex is a set that contains one vertex. For simplicity, we refer to a 0-simplex as a vertex. A 1-simplex is an *edge*. Two vertices are *adjacent* if they are contained in an edge. The dimension of Λ is the supremum of the dimensions of all simplices in $\mathcal{E}$. A simplicial complex $\Lambda' = (\mathcal{V}', \mathcal{E}')$ is a *subcomplex* of Λ if each of its simplices is a simplex of Λ. The k-skeleton of Λ is the k-dimensional subcomplex $\Lambda^{(k)}$ consisting of all simplices in $\mathcal{E}$ of dimension k or less than k. A simplicial complex $\Lambda' = (\mathcal{V}', \mathcal{E}')$ is an *induced subcomplex* of Λ if every simplex $A \in \Lambda$ with $A \subseteq \mathcal{V}'$ is contained in Λ'. Then, Λ' is *spanned* by the vertex set $\mathcal{V}'$. Such subcomplexes are also called *full*. Suppose that every vertex set of pairwise adjacent vertices in $\mathcal{V}$ spans a simplex of Λ; then Λ is a *flag* simplicial complex. The *link* of a vertex v is the subcomplex of Λ spanned by all vertices that are adjacent to v.

Definition 2.2. A *simplicial graph* or *simple graph* is a 2-dimensional simplicial complex.

Every simplicial graph is a *graph*. A *graph* Λ consists of a vertex set $V(G)$ (for short V), an edge set $E(G)$ (for short E) and a relation that associates with each edge two vertices, called its *endpoints* or its *endvertices*. In contrast to simplicial graphs, it is possible that the endpoints of an edge coincide in a graph (such an edge is a *loop*) and that two vertices are connected by more than one edge (called multiple edges). A vertex and an edge are *incident* if the vertex is contained in the edge. Two vertices are *adjacent* if they are both contained in an edge. We also say that the two vertices are *connected by an edge*. The *degree* or *valence* of a vertex v is the number of edges that contain v. The trivial graph is the graph $(\emptyset, \emptyset)$. A graph is *complete* or a *clique*, if it is a simple graph whose vertices are pairwise adjacent. The (unlabeled) complete graph with k vertices is denoted by K_k. If a graph contains k vertices and no edges, it is an *empty* graph on k vertices. The trivial graph is a complete graph on 0 vertices. The (unlabeled) empty graph on k vertices is denoted by $\bar{K}_k$. The *barycentric subdivision* of a graph is the graph we obtain by adding a vertex to every edge. More precisely, if Λ is a graph with vertex set V and edge set E, then the barycentric subdivision $\bar{\Lambda}$ is the following graph. Its vertex set is the set $\{v_m \mid m \in E \cup V\}$. Two vertices v_m and v_n are adjacent in $\bar{\Lambda}$ if the set $\{m, n\}$ consists of an edge e and a vertex v in Λ such that e is incident to v in Λ. If we *delete a vertex v from a graph Λ* , we delete v from the vertex set $V(\Lambda)$ and all edges in the edge set $E(\Lambda)$ that are incident to v. This way we obtain a graph with less vertices than before. If we *delete an edge e from a graph Λ* , we delete e from the edge set $E(\Lambda)$. This way, we obtain a graph with less edges than before.

The union of two graphs Λ_0 and Λ_1, denoted by $\Lambda_0 \cup \Lambda_1$, is the graph with vertex set $V(\Lambda_0) \cup V(\Lambda_1)$ and edge set $E(\Lambda_0) \cup (\Lambda_1)$. If Λ_0 and Λ_1 don't have any vertices and edges in common, $G_0 \cup G_1$ is a graph that consists of the two disjoint graphs G_0 and G_1. Otherwise, the vertex sets of G_0 and G_1 have non-empty intersection. The *intersection* of two graphs Λ_0 and Λ_1, denoted by $\Lambda_0 \cap \Lambda_1$, is the graph with vertex set $V(\Lambda_0) \cap V(\Lambda_1)$ and edge set $E(\Lambda_0) \cap E(\Lambda_1)$. The graph $\Lambda_0 \setminus \Lambda_1 = \Lambda_0 \setminus V(\Lambda_1)$ is the graph which we obtain from Λ_0 by deleting all vertices that are contained in Λ_1 and all edges that are incident to a vertex of Λ_1. We say that $\Lambda_0 \setminus \Lambda_1 = \Lambda_0 \setminus V(\Lambda_1)$ is the graph obtained by deleting Λ_1 from Λ_0. We allow to delete the trivial graph $(\emptyset, \emptyset)$ from a graph. The graph obtained by deleting $(\emptyset, \emptyset)$ is the same graph as before. The graph $\Lambda_0 \setminus E(\Lambda_1)$ is the graph we obtain from Λ_0 by deleting all edges that are contained in Λ_1. A graph Λ' is a *subgraph* of a graph Λ if $V(\Lambda') \subseteq V(\Lambda)$ and $E(\Lambda') \subseteq E(\Lambda)$. In this thesis, *induced subgraphs* play an important role. A subgraph Λ' of Λ is an *induced subgraph* of a graph Λ if every edge of Λ with endvertices in $V(\Lambda')$ is an edge of Λ'. A graph Λ' is induced by a vertex set $V' \subset V(\Lambda)$ if it is an induced subgraph of Λ with vertex set V'. Then Λ' *is spanned* by V'. A *walk* in a graph Λ is a nonempty alternating sequence $v_0, e_0, \dots, e_k, v_k$ of vertices and edges in Λ such that $e_i = \{v_i, v_{i+1}\}$, $i \in \{0, \dots, k-1\}$. If Λ is simple, $v_0, e_0, \dots, e_k, v_k$ is determined by its associated sequence of vertices $v_0, v_1, \dots, v_{k-1}, v_k$. Because almost all graphs in this thesis are simplicial, we often refer to a walk as a sequence of vertices. A u, v-walk has the first vertex u and the last vertex v; these are its *endpoints* or *endvertices*.

A *path* P is a simple graph whose vertices can be labeled with natural numbers so that the labels of two distinct vertices are distinct and so that two vertices are adjacent if and only if their labels are consecutive natural numbers. A path is an *infinite path* if it contains infinitely many vertices. A bi-infinite path P is a simple graph whose vertices can be labeled with integers so that the labeling induces a bijective map between the vertex set of P and $\mathcal{Z}$ and so that two vertices are adjacent if and only if their labels are consecutive integers. If Λ is simple, P is a path with a sequence of its vertices that defines a walk with pairwise distinct vertices and edges. In this case, if P is finite, we often refer to P as the finite sequence of vertices $v_0, \ldots, v_k$ and say that it is a *path from* v_0 *to* v_k. Then, v_0 is the first or start vertex of P and v_k is the last vertex of P. A $u, v-$path is a path whose vertices of degree 1 (its endpoints or endvertices) are u and v. We say that u and v are *linked* by P. The other vertices are *internal* or *inner* vertices of P.The *length* of a walk or a path is the number of its edges. A walk is closed if its endpoints are the same. If a path consists of one vertex, it is *trivial.* Two or more paths in a graph are *independent* if none of them contains an inner vertex of another one.

A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a cycle so that two vertices are adjacent if and only if they appear consecutively along the cycle. We often refer to a cycle as such a sequence $v_0, \ldots v_k$ of its vertices. In this case, the vertex set of the cycle is $\{v_0, \ldots v_k\}$ and the edge set of the cycle is $\{\{v_0, v_1\}, \ldots \{v_{k-1}, v_k\}, \{v_k, v_0\}\}$. A cycle has a *diagonal* if there are two non-consecutive vertices in $v_0, \ldots v_k$ that are adjacent in Λ. In other words, a cycle has a diagonal, if it is not induced. A graph is *connected* if every pair of vertices is linked by a finite path. Otherwise it is *disconnected.* A *connected component* of a graph is a maximal connected subgraph of Λ. The *connectivity* of a graph is the minimum size of a vertex set V' such that $G \setminus V$ is disconnected or has only one vertex. A graph is k-connected if its connectivity is at least k. By Menger's theorem [Men27], a graph with at least two vertices is k-connected if and only if every two of its vertices are linked by k independent paths. In this thesis, almost all graphs are simplicial. Thus, from now on, we always mean by a *graph* a simplicial graph if we don't highlight that the considered graph might have a loop or a multiple edge.

2.2 Geometric actions, Cayley graphs and the Švarc-Milnor Lemma

The definitions and facts of this section refer to [BH99]. Some of the following notation is based on [Dav08]. Let (X,d) be a metric space. A *curve* or *path* in X is a continuous map from a compact interval in $\mathbb{R}$ into X. A *generalized curve* is a continuous map $D \to X$ where $D = [0,R]$, $R \geq 0$ or $D = [0,\infty) \to X$. Let $c : [a,b] \to X$ be a curve. For the definition of textitconcatenations of curves and the *length* of a curve c, we refer to [BH99, p.12]. We denote the length of c by $l(c)$ and say that c is *rectifiable* if its length is finite. If we speak of c, we often mean the image of c. If it is not clear from the context, if the curve c or its image is meant, we highlight what we mean. We say that the image of c is a *geodesic segment* if c is an isometric embedding. A *geodesic ray* γ is an isometric embedding $\gamma : [0,\infty) \to X$. A *geodesic line* (for short *line*) or *bi-infinite geodesic ray* γ is an isometric embedding $\gamma : \mathbb{R} \to X$. A *generalized geodesic ray* is an isometric embedding $D \to X$ where $D = [0,R]$, $R \geq 0$ or $D = [0,\infty) \to X$. A *geodesic* is an isometric embedding of a possibly infinite interval in $\mathbb{R}$ to X. Let $[a,b]$ be an interval in $\mathbb{R}$. A *subgeodesic* γ' of a geodesic γ is a geodesic whose image is contained in γ. A subgeodesic γ' is a *subgeodesic ray* (*subgeodesic segment*) of γ if the image of γ' is isometric to $[0,\infty)$ (a compact interval in $\mathbb{R}$). A map $\gamma : I \to X$ is a *linearly reparametrized geodesic* if there is a constant D such that $d(\gamma(s),\gamma(t)) = D|s-t|$ for all $s,t \in I$. In this case the image of γ is *parametrized proportionally to arc length.* Like in the case of curves, we use the letter γ to refer to a (linearly reparamized) geodesic γ or to its image.

The metric d of X is a *length metric* or an *inner metric* if the distance between every pair of points x, $y \in X$ is equal to the infimum of the length of rectifiable curves joining them. In this case, (X,d) is a *length space.* The space (X,d) is a *geodesic metric space* if every two points in X are joint by a geodesic segment. An action of a group G on X is *transitive*, if there exists $x \in X$ such that $G \cdot x = X$. The *isotropy subgroup* of $x \in X$ is the *stabilizer* of x, i.e. the subgroup $G_x = \{g \in G \mid g \cdot x = x\}$. Let $x \in X$ and $r \in [0,\infty]$. We denote the open ball of radius r about x by $B(x,r)$. The space X is proper if for every $x \in X$ and every $r > 0$, the closed ball $\bar{B}(x,r)$ is compact. We say like Bridson and Haefliger in [BH99, p. 132] that an action is *proper* (i.e. G acts properly on X) if for each $x \in X$ there exists $r > 0$ such that the set $\{g \in G \mid gB(x,r) \cap B(x,r) \neq \emptyset\}$ is finite. The action is *cocompact* (i.e. G acts cocompactly on X) if there exists a compact set $K \subseteq X$ such that $X = G \cdot K$. A metric space X is *cocompact* if there exists a compact set $K \subset X$ such that $X = \cup_{g \in Isom(X)} gK$. We denote the Isometry group of X by $\mathrm{Isom}(X)$. A group G acts *geometrically* on a metric space X if it acts properly and cocompactly by isometries on X. Suppose that X is a length space and that a group acts geometrically on X. By Exercise 8.4 (1) in [BH99], X is complete and locally compact. By the Hopf-Rinow Theorem, a length space is complete and locally compact if and only if it is proper (see Corollary 3.8 in part I of [BH99]). So, a space on which a group acts geometrically behaves nicely.

Remark 2.3. We remark that one usually defines an action of a group G on X to be proper, if for every compact set $K \subset X$ the set of elements $\{g \in G \mid g \cdot K \cap K \neq \emptyset\}$ is finite. If G acts on X by isometries and X is a proper metric space, this is equivalent to the definition above used by Bridson and Haefliger in [BH99, p. 132]. We study only actions on proper metric spaces. So, in the situations we study, the two definitions of a proper action are equivalent.

Let G be a group generated by a finite set S. Suppose that S does not contain the identity element id of G. In this thesis, we often study cases where $S = S^{-1}$. A *word in* $S \cup S^{-1}$ is a finite sequence of letters $s_i \in S \cup S^{-1}$. An *infinite word* is an infinite sequence of letters in $S \cup S^{-1}$. A bi-infinite word in S is a sequence $(s_i)_{i \in \mathbb{Z}}$, $s_i \in S \cup S^{-1}$. A subword of (an infinite) word $\vec{w}'$ is a subsequence of $\vec{w}'$ in which every two consecutive letters are consecutive in $\vec{w}'$. It is an *initial subword*, if it starts with the first letter of $\vec{w}'$. Let $s_0, \ldots, s_k$ be a word in $S \cup S^{-1}$. If we delete a letter s_i, $i \in \{0, \ldots, k\}$ we denote the obtained word by $s_0, \ldots \hat{s}_i, \ldots, s_k$. We proceed Analogously if we delete more than one letter of a word. The *value* of a word $s_0, \ldots, s_k$ in $S \cup S^{-1}$ is the group element $s_0 \cdots s_k$. Every group element $g \in G$ is a value of a word in $S \cup S^{-1}$. A word $\vec{g}$ in G *representing* $g \in G$ is a finite sequence $s_0, \ldots, s_k$ of elements $s_i \in S \cup S^{-1}$, $i \in \{0, \ldots, k\}$ such that $g = s_0 \cdots s_k$. We say that $s_0, \ldots, s_k$ is an *expression for* g. The word $\vec{g}$ is S-reduced (for short reduced), if k is chosen minimally, i.e. there is no word of length less than k that represents g. In this case, k is the *word length* of g, denoted by $l(g)$. An infinite or bi-infinite word is reduced if every of its finite initial subwords is reduced. The *word metric* $d^1 : G \times G \to \mathbb{N}_0$ on G is the following metric on G. The distance of two group elements g and h in G is the word length of gh^{-1}, i.e. the smallest natural number k such that $g = ha$ and $l(a) = k$. The word metric can also be defined with help of the *Cayley graph*.

Definition 2.4. Let G be a group generated by a finite set S. The *Cayley graph* $Cay(G, S)$ is a graph defined as follows. Its vertex set is the group G. If G is the trivial group consisting of the identity element id, it does not contain any edges and is the graph that consists of the identity vertex id. Suppose otherwise that S does not contain the identity id. Two elements of G are contained in an edge if and only if it is of the form $\{g, gs\}$ for some $g \in G$, $s \in S$. The label of $\{g, gs\}$ is the generator s. If $s \neq s^{-1}$, the edge has a direction from g to gs. The *initial vertex* of $\{g, gs\}$ is g and the *terminal vertex* is gs. Otherwise, if $s = s^{-1}$, the edge is *undirected*.

Let $\sigma^1 = g_0, \ldots, g_k$ be a walk in $Cay(G, S)$. Recall that the edges of σ^1 are labeled by elements of S. We read off these labels and obtain an associated word in $S \cup S^{-1}$ as follows. For all $i \in \{0, \ldots, k\}$, we choose the label s_i and $\epsilon_i \in \{-1, +1\}$ such that $g_i = g_{i-1} s^{\epsilon_{i-1}}$ for all $i \in \{0, \ldots, k\}$. Then $\vec{\sigma}^1 = s_0^{\epsilon_0}, \ldots, s_k^{\epsilon_k}$ is the word in $S \cup S^{-1}$ associated to σ^1. On the other hand, if a group element g and a word $\vec{a}$ are given, they define a corresponding walk σ^1 in $Cay(G, S)$ that starts in v_g and has $\vec{a}$ as associated word. We call this walk $\sigma^1(g, \vec{a})$. It is $\vec{a} = \overrightarrow{(\sigma^1(\vec{a}, g))}$. Accordingly, for each walk from g_0 to g_k there is an associated word $\vec{a}$ with $g_k = g_0 a$ and vice versa. We define a metric $d : Cay(G, S) \times Cay(G, S) \to [0, \infty)$ as follows. We define each edge to be isometric to

$[0,1]$. This way, every edge has length 1. The distance between two points $d(x,y)$ is the length of the shortest curve c from x to y. For details substantiating why this is a well-defined metric, see [BH99]. The metric we obtain by restricting d to the vertex set of $Cay(G,S)$ is the *word metric* $d^1 : G \times G \to \mathbb{N}_0$ as defined above. The group G acts by left multiplication properly and cocompactly by isometries on $Cay(G,S)$. This observation leads to the lemma of Švarc-Milnor. It has is origin in [Efr53] and[Š55]. It was rediscovered by John Milnor in Lemma 2 of [Mil68]. The following definition and formulation of the Švarc-Milnor Lemma are Definition 8.14 and Proposition 8.19 in Chapter I in [BH99].

Definition 2.5. Let (X,d_0), (X,d_1) be metric spaces. A map $f : X_0 \to X_1$ is a (K,L)-quasi-isometric embedding if there exists $K \geq 1$ and $L \geq 0$ such that for all x, $y \in X_0$ we have

$$\frac{1}{K}d_0(x,y) - L \leq d_1(f(x),f(y)) \leq Kd_0(x,y) + L.$$

If, there exists in addition a constant $C \geq 0$ such that every point of X_1 lies in the C-neighborhood of the image of f, then f is a (K,L)-*quasi-isometry.* When such a map exists, X_0 and X_1 are *quasi-isometric.*

Theorem 2.6 (The Švarc-Milnor Lemma)**.** *Let X be a length space. If a group G acts properly and cocompactly by isometries on X, then G is finitely generated and for any choice of basepoint $x_0 \in X$, the map $g \to g \cdot x_0$ is a quasi-isometry.*

Another important tool we need is the following consequence of the Theorem of Arzelà-Ascoli (see for instance [BH99, Lem 3.10 in I.3]) that can be found as Corollary 1.4 in [Cor17].

Lemma 2.7. *Let X be a proper metric space and $p \in X$. Then any sequence of geodesic rays $\gamma_n : [0,L_n] \to X$ with $\gamma_n(0) = p$ and $L_n \to \infty$ has a subsequence that converges uniformly on compact sets to a geodesic ray $\gamma : [0,\infty) \to X$.*

2.3 Hyperbolic and CAT(0) spaces and groups and their boundaries

In this section, we introduce hyperbolic and CAT(0) spaces and groups and their boundaries. There are diverse definitions of boundaries. We repeat here only the definitions that are based on geodesic rays. The following definitions and facts concerning CAT(0) and hyperbolic spaces and groups and their boundaries refer to Chapter II.8 and Chapter III.H in [BH99]. Another reference is [Bal95].

Let X be a metric space and $\epsilon \geq 0$. If A is a subset of X, the closed ϵ-neighborhood of A in X, denoted by $N_\epsilon(A)$ is the set $\bigcup_{a\in A}\{x \in X \mid d(a,x) \leq \epsilon\}$. If $\epsilon > 0$ and if A is contained in the closed ϵ-neighborhood of B and if B is contained in the closed ϵ-neighborhood of A, then A and B have *Hausdorff distance at most* ϵ. The *Hausdorff distance* $d_H(A,B)$ between A and B is the infimum over all such ϵ. Two geodesics are *asymptotic* if their images have bounded Hausdorff distance. If $\gamma : [0,\infty) \to X$ and $\gamma' : [0,\infty) \to X$ are two geodesic rays, they have bounded Hausdorff distance if and only if there exists $D > 0$ such that $d(\gamma(t), \gamma'(t)) \leq D$ for all $t \geq 0$. Being asymptotic is an equivalence relation on the set of geodesic rays. The set of equivalence classes of geodesic rays is denoted by ∂X. We call the elements of $\partial(X)$ *boundary points* or *points at infinity.* If γ is a geodesic ray, it is a representative of its equivalence class that is denoted by $\gamma(\infty)$. If we refer to an element of ∂X and are not interested in its representatives, we denote it often by ξ. Let ∂X be the set of all equivalence classes of geodesic rays in X. Let x_{base} be a base point of X. If the choice of x_{base} is important for our considerations, we write $X_{x_{\text{base}}}$ for the space X. Recall that a generalized geodesic ray is a geodesic $\gamma : D \to X$ where $D = [0,R]$ for some $R > 0$ or $D = [0,\infty)$. If $D = [0,R]$, we define $\gamma(t) = \gamma(R)$ for all $t \geq t$. This way, we can interpret every point in X as a generalized ray and identify $X \cup \partial X$ with the set $\{c(\infty) \mid c \text{ is a generalized ray}\}$.

In the following, we will see that boundaries of hyperbolic and CAT(0) spaces can be equipped with certain typologies. If X is a hyperbolic or CAT(0) space, these topological spaces are usually denoted by ∂X. We work with different typologies on boundary points. For distinguishing these topologies, we introduce another notation. If we write ∂X, we mean always the set of boundary points without a certain topology.

2.3.1 Hyperbolic spaces and groups and the Gromov boundary

The following definition complies with [BH99].

Definition 2.8. Let $\delta > 0$. A geodesic triangle in a metric space is said to be δ-slim if each of its sides is contained in the δ-neighborhood of the union of the other two sides. A geodesic space X is δ*-hyperbolic* if every geodesic triangle in X is δ-slim. If there exists δ such that X is δ-hyperbolic, one says that X is *hyperbolic.* A group is *hyperbolic* if it acts geometrically on a hyperbolic space.

Suppose that X is a proper hyperbolic space. It is possible to equip $X \cup \partial X$ with a certain topology, such that we obtain a topological space $X \cup \vec{\partial}_c X$ with the following properties

- The topology on $X \cup \vec{\partial}_c X$ is independent of the choice of basepoint.
- $X \hookrightarrow X \cup \vec{\partial}_c X$ is a homeomorphism onto its image and ∂X is closed in $X \cup \vec{\partial}_c X$.
- $X \cup \vec{\partial}_c X$ is compact.
- If X is a proper, geodesic hyperbolic space, then $\vec{\partial}_c X$ is *visible*, i.e. for each pair of distinct points $\xi_1, \xi_2 \in \partial X$ there exists a geodesic line $\gamma : \mathbb{R} \to X$ with $\gamma(\infty) = \xi_1$ and $\gamma(-\infty) = \xi_2$.

This topological space is the *Gromov boundary* of X. See [BH99, p.429] for a definition of the Gromov boundary, and see Proposition 3.7 and Lemma 3.2 in Part III.H in [BH99] for proofs of the listed properties. The Gromov boundary is a quasi-isometry invariant, i.e. if two hyperbolic metric spaces are quasi-isometric, then their Gromov boundaries coincide. This is the content of Theorem 3.9 in Chapter III.H [BH99].

Theorem 2.9 (Theorem 3.9 in Chapter III.H [BH99])**.** *Let X, X' be proper δ-hyperbolic geodesic spaces. If $f : X \to X'$ is a quasi-isometric embedding, then the map sending an equivalence class of a geodesic ray $\gamma : [0, \infty) \to X$ to the equivalence class of $\gamma \circ f$ induces a topological embedding $f_\partial : \vec{\partial}_c X \to \vec{\partial}_c X'$. If f is a quasi-isometry, then f_∂ is a homeomorphism.*

This implies that the Gromov boundary can be defined for hyperbolic groups according to the Lemma of Švarc-Milnor.

Definition 2.10. The Gromov boundary of a hyperbolic group G is the Gromov boundary of a metric space on which G acts geometrically.

That the Gromov boundary is a quasi-isometry, results from the stability of *quasi-geodesics* in hyperbolic spaces. A *(quasi-)geodesic* γ is a (quasi-)isometric embedding of a possibly infinite interval to X. Roughly speaking, the stability of quasi-geodesics means that every *quasi-geodesic* with endpoints on a geodesic γ stays close to γ. Such geodesic rays are called *Morse.* Formally, Morse-geodesics are defined as follows.

Definition 2.11. A (quasi-) geodesic γ is *M-Morse* if, for any $K \geq 1$, $L \geq 0$ there is a constant $M = M(K, L)$ such that, for every (K, L)-quasi-geodesic σ with endpoints on γ, we have that σ is contained in the M-neighborhood of γ. We say that γ is *Morse* if it is M-Morse for some M. We call the function $M(K, L)$ *Morse gauge.*

The following theorem says that quasi-geodesics in hyperbolic spaces are stable. It can be found as Theorem 1.7 in Chapter III.H of [BH99]. The roots of this theorem and the notion of Morse geodesic rays lie in [Mor24] and [Mor21].

Theorem 2.12 (Stability of Quasi-geodesics)**.** *For all $\delta > 0$ there is a Morse gauge $M : [1, \infty), [0, \infty) \to \mathbb{R}$ with the following property: If X is a δ-hyperbolic geodesic space, γ is a (K, L)-quasi-geodesic in X and $[p, q]$ is a geodesic segment joining the endpoints of γ, then the Hausdorff distance between $[p, q]$ and the image of γ is less than $M(K, L)$.*

Next, we define Morse subsets.

Definition 2.13 (Morse subsets)**.** A subset A of a geodesic metric space X is *Morse* if for every $K \geq 1$, and $L \geq 0$ there is some $M = M(K, L)$ such that every (K, L)-quasigeodesic with endpoints in A is contained in the closed M-neighborhood of A.

We finish this section with the notion of *stability* introduced by Durham and Taylor in [DT15b]. The concept of stability is motivated by *quasiconvexity.* A subset X of a geodesic metric space X is *D-quasiconvex* if for any x_0, x_1 in A and any geodesic segment σ connecting x_0 and x_0 we have that σ is contained in the closed D-neighborhood of X. If X is hyperbolic, the preimage of a quasiconvex subspace through a quasi-isometric embedding quasiconvex is itself. In general, this property fails if X is not hyperbolic. *Stability* is a strong notion of quasiconvexity that is preserved under quasi-isometry. A quasi-isometric embedding $f : X \to Y$ between two geodesic metric spaces X and Y is *stable* if for any $K \geq 1$, $L \geq 0$ there exists $M = M(K, L) \geq 0$ such that for all (K, L)-quasi-geodesics α and β with the same endpoints in $f(X)$ we have that the Hausdorff distance between α and β is less than M. Suppose that X is stable in Y. By Proposition 3.2 of [DT15b], stability is preserved under quasi-isometric embeddings. Let G be a finitely generated group. A finitely generated subgroup H of G is *undistorted* if the inclusion of H in G is a quasi-isometric embedding for some word metrics on H and G. If H is undistorted in G, the inclusion of H in G is a quasi-isometric embedding for any word metrics on H and G because of the Švarc-Milnor Lemma. The following is Definition 3 in [DT15b].

Definition 2.14 (Stability)**.** Let G be a finitely generated group equipped with the word metric associated to a finitely generating set S of G. A finitely generated subgroup H of G is *stable*, if H is undistorted in G and the inclusion of H in G is stable for any choice of word metric on H.

By Lemma 3.4 in [DT15b], the definition of stability does not depend on the choice of the generating set S for G. For hyperbolic groups, stability agrees with quasiconvexity. Motivated by this observation, so-called *strongly quasiconvex subgroups* of finitely generated groups were introduced by Tran in [Tra19] and independently by Genevois in [Gen20] under the name *Morse subgroups.* An analogous notion are *N-stable* subsets of geodesic metric spaces as studied by Cordes and Hume in [CH17]. The following is Definition 4.5 in [Tra19].

Definition 2.15 (strongly quasiconvex)**.** A subgroup H of a finitely generated group G is *strongly quasiconvex* in G if for some (any) finite generating set S of G we have that H is a Morse subset in the Cayley graph $Cay(G, S)$.

The following characterization of stable subgroups is Theorem 4.8 in [Tra19]. It uses the notion of *lower divergence in spaces* that was originally introduced by Tran in [Tra15]. See[Tra15] for a Definition.

Theorem 2.16. *Let G be a finitely generated group and H an infinite subgroup of G. Then the following statements are equivalent:*

a) H is stable in G.

b) H is hyperbolic and strongly quasiconvex in G.

c) H is hyperbolic and the lower relative divergence of G with respect to H is completely superlinear.

2.3.2 CAT(0) spaces and groups and the visual boundary

Like Bridson and Haefliger, we denote the Euclidean space $\mathbb{R}^n$ by M_0^n ; Let $\kappa \leq 0$. Let M_κ^n be the metric space obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by $\frac{1}{\kappa}$.

Definition 2.17. Let X be a metric space, $\kappa \leq 0$ and Δ a geodesic triangle in X. Let $\bar{\Delta}$ be a comparison triangle for Δ in M_κ^2. The triangle Δ satisfies the *CAT(κ) inequality* if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$, $d(x, y) \leq d(\bar{x}, \bar{y})$. If X is a geodesic space such that every geodesic triangle satisfies the CAT(κ) inequality, X is a *CAT(κ) space.* A group is CAT(κ) if it acts geometrically on a CAT(κ) space.

Let $\kappa < 0$. By Proposition 1.2 in Chapter III in [BH99], every CAT(κ) space is δ-hyperbolic where δ depends only on κ. By the Flat Plane Theorem 1.5 in Chapter III in [BH99], a proper, cocompact CAT(0) space X is hyperbolic if and only if it does not contain an isometrically embedded copy of the Euclidean plane. If a group is CAT(0) it might not be hyperbolic. For example, $\mathbb{Z}^2$ is a CAT(0) group but not hyperbolic. Suppose that G is a hyperbolic group. It is an open problem if G is a CAT(0) group.

Let X be a CAT(0) space with metric d. An important property of X is that d is convex, i.e. if $\gamma : [0, 1] \to X$ and $\gamma' : [0, 1] \to X$ are geodesics parametrizied proportional to arc length, then

$$d(\gamma(t), \gamma'(t)) \leq (1 - t)d(\gamma(0), \gamma'(0)) + t(d(\gamma(1), \gamma'(1)) \tag{2.17.1}$$

for all $t \in [0, 1]$. The Convexity of the metric has a lot of consequences. For example, every two points in X are connected by a unique geodesic ray. If x and y are two points in a CAT(0) space we denote the shortest geodesic connecting x and y by $[x, y]$. Furthermore, there is a nearest-point-projection map for every convex set in X that sends every point of X onto C such that distances don't increase. More precisely, let C be a convex subset of X that is complete in the induced metric. Then there is a well-defined nearest point projection map $\pi_C : X \to C$. This projection map is continuous and does not increase distances. For more details see Proposition 2.4 in Chapter II in [BH99].

If X is a complete CAT(0) space, it is possible to equip ∂X with a topology similar to the topology of the Gromov boundary of a hyperbolic space. The obtained boundary is called *visual boundary* of X. Let X be a complete CAT(0) space and x_{base} a fixed chosen base point of X. For defining the visual boundary of X we consider

$$\partial X_{x_{\text{base}}} := \{\alpha(\infty) \in \partial X \mid \alpha \text{ is a geodesic ray with } \alpha(0) = x_{\text{base}}\} \tag{2.17.2}$$

It is a basic fact of complete CAT(0) spaces that every equivalence class in ∂X is represented by a unique geodesic ray starting in x_{base}. See Proposition 8.2 in Chapter II of [BH99]. Accordingly, $\partial X_{x_{\text{base}}}$ can be identified with the set of geodesic rays in X that start in x_{base}.

Definition 2.18. Let α be a geodesic ray in a complete CAT(0) space X. We say that a geodesic ray is (ϵ, r)-*close* to α if it is based at $\alpha(0)$ and represents an element of the set

$$U(\alpha(\infty), r, \epsilon) = \{\gamma(\infty) \in \partial X \mid \gamma \text{ is a geodesic ray at } \alpha(0),\ d(\alpha(t), \gamma(t)) < \epsilon\ \forall\ t \leq r\}.$$

Suppose that $\alpha(0) = x_{x_{\text{base}}}$. We equip $\partial X_{x_{\text{base}}}$ with the so-called *cone topology.* In this topology, the open sets of the form as in Definition 2.18 define a neighborhood basis for $\alpha(\infty)$ in $\partial X_{x_{\text{base}}}$. We denote the corresponding topological space by $\hat{\partial} X_{x_{\text{base}}}$. It is independent of the choice of the base point. Hence, it induces a topology on ∂X, the *visual boundary* of X. We denote it by $\hat{\partial} X$. The visual boundary of X has the following properties. For more details see [Def.8.6 in part II][BH99].

- If X is a proper hyperbolic CAT(0) space, the visual and Gromov boundary of X coincide;
- the topology on $X \cup \hat{\partial} X$ is independent of the choice of the base point;
- $X \hookrightarrow X \cup \hat{\partial} X$ is a homeomorphism onto its image and ∂X is closed in $\subset X \cup \hat{\partial} X$.
- if X is proper, $X \cup \hat{\partial} X$ is compact.

The following theorem is Corollary 8.9 in Chapter II in [BH99]. It shows that every isometry on a space X induces a homeomorphism on its visual boundary.

Theorem 2.19 (Corollary 8.9 in Chapter II in [BH99])**.** *Let f be an isometry of a complete CAT(0) space X. The natural extension of f to $X \cup \hat{\partial} X$ equipped with the cone topology is a homeomorphism.*

Differently to the Gromov boundary, the visual boundary is not a quasi-isometry invariant. Croke and Kleiner have proven that there is a CAT(0) group that acts on two CAT(0) space X and Y whose visual boundaries are not homeomorphic [CK00]. In [CS15], Charney and Sultan introduced the *contracting boundary* of complete CAT(0) spaces. It is a topological space that is a quasi-isometry invariant and can be seen as a generalization of the Gromov boundary to the setting of CAT(0) spaces. We define contracting boundaries in the next section.

2.4 The contracting boundary

In this section, we define contracting geodesic rays, consider properties of them, and define the contracting boundary that was introduced by Charney and Sultan in [CS15]. There are many generalizations concerning contracting boundaries. Cordes generalized contracting boundaries to *Morse boundaries* of proper metric spaces [Cor17] and gives a more general definition of Morse boundaries of not-necessarily proper geodesic spaces in [Cor19]. For more details see [Cor19]. Recently, Qing and Rafi introduced a so-called k-Morse boundary in [QRT19]. This boundary is a quasi-isometry invariant and its underlying set is larger than the Morse boundary defined by Cordes.

2.4.1 Contracting geodesic rays

Let X be a complete CAT(0) space. For introducing the contracting boundary, Charney and Sultan use the definition of contracting geodesic rays of Bestvina and Fujiwara given in [BF09]. A more general variant of contracting geodesic rays was originally used in [MM99]. The following definition is Definition 2.3 in [CS15].

Definition 2.20 (contracting geodesics)**.** Given a fixed constant D, a geodesic ray or geodesic segment or line γ in X is said to be *D-contracting* if for all x, $y \in X$,

$$d_X(x, y) < d_X(x, \pi_\gamma(x)) \Rightarrow d_X(\pi_\gamma(x), \pi_\gamma(y)) < D. \tag{2.20.1}$$

We say that γ is *contracting* if it is D-contracting for some D. Equivalently, any metric ball B that does not intersect γ projects to a segment of length less than $2D$ on γ.

There are many ways to characterize contracting geodesic rays. The most important one is that a geodesic ray is contracting if and only if it is Morse.

A proof, that every contracting geodesic is Morse can for instance be found in Sultan's paper [Sul14] and Algom-Kfir's paper [AK11]. Theorem 3.4 of [Sul14] includes also the other direction, i.e. that every Morse geodesic is contracting but without explicit bounds for the constants. In Theorem 2.9 of [CS15], Charney and Sultan reprove this fact with explicit bounds. That a geodesic ray is contracting if and only if it is Morse means that contracting geodesics behave like geodesics in hyperbolic spaces. This is the basic idea of the contracting boundary. The special behavior of Morse-geodesic rays allows us to equip the set of equivalence classes of contracting geodesic rays with a topology such that the obtained topological space is a quasi-isometry invariant. The obtained topological space is called *contracting boundary.* For hyperbolic spaces, it coincides with the Gromov boundary. Thus, it can be seen as a generalization of the Gromov boundary of hyperbolic spaces.

Another important characterization of contracting geodesics proven by Charney and Sultan [CS15, Theorem 2.14] is that a contracting geodesic ray is contracting if and only if it is *slim.* If $x, y \in X$, we denote the shortest geodesic connecting x and y by $[x, y]$.

Definition 2.21. A geodesic ray γ is *δ-slim* if for all $x \in X$, $y \in \gamma$, $d(\pi_\gamma(x), [x, y]) < \delta$.

We will use the following lemma of Murray in [Mur19] concerning slimness.

Lemma 2.22 (Lemma 2.10 of [Mur19])**.** *If X is a proper CAT(0) space and γ is a δ-slim geodesic ray in X then for any $x \in X$ the distance $d(\pi_i\gamma(x), [x, \alpha(\infty)])$ is less or equal to δ.*

We will also use the following lemma of Murray.

Lemma 2.23 (Lemma 2.17 of [Mur19])**.** *Let γ be a geodesic ray in a metric space X and let x be a point in X such that $d(x, \gamma(0)) = t_0$. If the distance $d(x, \gamma) \leq D$, then $d(x, \gamma(t_0)) \leq 2D$.*

Another characterization uses a variant of the classical notion of *divergence* (see for example [DMS10]), namely the so-called *lower divergence* of geodesic rays. Lower divergence was introduced by Charney and Sultan in [CS15]. We summarize characterizations of contracting geodesic rays by citing Theorem 2.4 in [CS15]. Parts of this theorem were proven in [AK11], [BF09] and [Sul14].

Theorem 2.24 (Theorem 2.4 of [CS15])**.** *Let X be a CAT(0) space and let $\alpha \subseteq X$ be a geodesic ray or line. Then the following statements are equivalent:*

a) γ is contracting;

b) γ is Morse;

c) γ is slim;

d) γ has superlinear lower divergence;

e) γ has at least quadratic lower divergence.

Another important statement is the stability Lemma 3.8 of Bestvina and Fujiwara in [BF09].

Lemma 2.25 (Stability Lemma 3.8 of Bestvina and Fujiwara [BF09])**.** *Suppose that X is a CAT(0) space and a, a', b, b' are points in X such that $[a, b]$ is D-contracting, $d(a, a') \leq C$ and $d(b, b') \leq C$. Then there exists a constant D' depending only on D and C such that $[a', b']$ is D'-contracting.*

From this stability lemma follows the next important corollary.

Corollary 2.26. *If α and β are asymptotic geodesics, then α is contracting if and only if β is contracting.*

Because of this corollary, the following definition is well-defined.

Definition 2.27. Let X be a complete CAT(0) space. A boundary point ξ is *contracting* if some (or any) representative geodesic ray of ξ is contracting.

For more details about contracting and Morse geodesic rays see for instance [CS15], [BF09], [Mur19] and [Cor19].

2.4.2 Definition of the contracting boundary

In this subsection, we define contracting boundaries of complete CAT(0) spaces introduced by Charney and Sultan in [CS15] and summarize basic facts we need in this thesis. Cordes generalized in [Cor17] contracting boundaries to *Morse boundaries* of proper metric spaces. Thus, we can also speak of Morse boundaries instead of contracting boundaries.

Let X be a complete CAT(0) space with basepoint x_{base}. Recall that ∂X denotes the set of boundary points of X and that every boundary point in ∂X is represented by a unique geodesic ray starting in x_{base}. We recap that $\hat{\partial} X$ denotes the visual boundary of X, i.e. ∂X equipped with the cone topology. Let $\partial_c X$ be the set of all equivalence classes of contracting geodesic rays in X. First, we equip $\partial_c X$ with the subspace topology of the visual boundary of X. We denote this topological space by $\hat{\partial}_c X$. By [Cas16], a quasi-isometry need not induce homeomorphisms of contracting boundaries equipped with the cone topology. Thus, we equip $\partial_c X$ with another topology. For that purpose we choose a fixed base point x_{base} and define the following sets.

$$\partial_c X_{x_{\text{base}}} := \{\alpha(\infty) \in \partial_c X \mid \alpha \text{ is a contracting geodesic ray and } \alpha(0) = x_{\text{base}}\} \quad (2.27.1)$$
$$\partial_c^n X_{x_{\text{base}}} = \{\gamma(\infty) \in \partial X \mid \gamma(0) = x_{\text{base}}, \gamma \text{ is an } n-\text{contracting geodesic ray}\}. \quad (2.27.2)$$

We equip the sets $\partial_c^n X_{x_{\text{base}}}$ with the subspace topology of the visual boundary and obtain topological spaces $\hat{\partial}_c^n X_{x_{\text{base}}}$. By definition of n-contracting geodesic rays, there is an inclusion map $\iota : \hat{\partial}_c^m X_{x_{\text{base}}} \to \hat{\partial}_c^n X_{x_{\text{base}}}$ for all $m < n$. Thus, we have $\partial_c X_{x_{\text{base}}} = \bigcup_{n\in\mathbb{N}} \partial_c^m X_{x_{\text{base}}}$. We interpret $\partial_c X_{x_{\text{base}}}$ as direct limit and equip $\partial_c X_{x_{\text{base}}}$ with the direct limit topology. In this topology, a set of boundary points is open if and only if it is open in $\hat{\partial}_c^n X_{x_{\text{base}}}$ for all $n \in \mathbb{N}$. This definition is independent of the base point. Thus, it induces a topology on $\partial_c X$. We call this topological space the *contracting boundary* of X and denote it by $\vec{\partial}_c X$. The set of contracting boundary points $\partial_c X$ is often called *contracting boundary* and equipped with different typologies. In this thesis, we speak of the contracting boundary of X if we mean $\partial_c X$ equipped with the direct limit topology. If X is hyperbolic, then the contracting boundary of X coincides with the Gromov boundary. In this sense, $\vec{\partial}_c X$ can be seen as a generalization of the Gromov boundary. The most important property of the contracting boundary is that it is a quasi-isometry invariant. The following is Theorem 3.11 in [CS15].

Theorem 2.28. *Let $f : X \to Y$ be a quasi-isometry of complete CAT(0) spaces. Then f induces a homeomorphism $\vec{\partial}_c f : \vec{\partial}_c X \to \vec{\partial}_c Y$.*

This implies that the contracting boundary can be defined for CAT(0) groups according to the Lemma of Švarc-Milnor.

Definition 2.29. The contracting boundary of a CAT(0) group G is the contracting boundary of a metric space on which G acts geometrically.

By the main theorem in [CS15], the contracting boundary $\vec{\partial}_c X$ has the following properties if X is a proper CAT(0) space.

Theorem 2.30. *If X is a proper CAT(0) space, then its contracting boundary is*

a) σ-compact, i.e., it is a countable union of compact subspaces;

b) a visibility space;

c) a quasi-isometry invariant.

The next theorem shows that contracting boundaries don't behave so nicely as the Gromov and visual boundary. It is Theorem 5.1 in [Mur19].

Theorem 2.31. *Let X be a complete proper CAT(0) space with a geometric group action such that $\vec{\partial}_c X$ contains at least two points. Then the following statements are equivalent:*

a) X is δ-hyperbolic.

b) $\vec{\partial}_c X_{x_{base}} \subseteq \partial_c^n X_{x_{base}}$ for some $n \in \mathbb{N}$.

c) The identity map $X \to X$ induces a homeomorphism of the visual boundary $\hat{\partial} X$ of X to the contracting boundary $\vec{\partial}_c X$ of X.

d) $\hat{\partial} X \subseteq \vec{\partial}_c X$, i.e. the visual boundary and the contracting boundary are the same

e) $\vec{\partial}_c X$ is compact.

f) $\vec{\partial}_c X$ is locally compact.

g) $\vec{\partial}_c X$ is first-countable, and in fact metrizable.

It is unpleasant that the topology of the contracting boundary is not second countable and thus not metrizable if the space is not hyperbolic. We remark that the set of equivalence classes of contracting geodesic rays can be equipped with another topology which is second countable and thus metrizable [CM19].

It is important for our considerations that the geometric action of a group on a complete CAT(0) space X induces an action of the group on its visual and contracting boundary by homeomorphisms. Thereby, the following theorem will be very useful. It is Proposition 4.5 and Corollary 4.7 of Murray in [Mur19, Cor. 4.7]; he transfers results of Hamenstädt [Ham09] about visual boundaries to contracting boundaries as far as possible.

Theorem 2.32. *Let γ be a contracting geodesic ray in a complete CAT(0) space X. Suppose that a group G acts cocompactly on X such that $\gamma(\infty)$ is not globally fixed by G, then its orbit is dense in the contracting boundary and the visual boundary of X.*

2.5 Subspaces of boundaries

In this section, we summarize our notation for different boundaries of CAT(0) spaces. Afterwards, we explain formally how we denote the boundaries of subspaces. If Z is a complete, convex subspace of a complete CAT(0) space X, we think of the boundaries of Z as embedded in corresponding boundaries of the ambient space X whenever possible. For instance, we think of the set $\partial Z = \{\gamma(\infty) \in \partial Z \mid \gamma \text{ is a geodesic ray in } Z\}$ as the embedded set $\{\gamma(\infty) \in \partial X \mid \gamma \subseteq Z\}$. On the other hand, we cannot think of the set $\partial_c Z = \{\gamma(\infty) \in \partial Z \mid \gamma \text{ is a contracting geodesic ray in } Z\}$ as embedded in $\partial_c X = \{\gamma(\infty) \in \partial Z \mid \gamma \text{ is a contracting geodesic ray in } X\}$. Indeed, a geodesic ray in Z that is contracting in Z might not be contracting in the ambient space X. Thus, we introduce notation and denote the set $\{\gamma(\infty) \in \partial_c X \mid \gamma \subseteq Z\}$ by $\partial_{c,X} Z$. If we study ∂Z or $\partial_{c,X} Z$, it is possible to forget the ambient space X of Z and to think of them as subsets of $\{\gamma(\infty) \in \partial Z \mid \gamma \text{ is a geodesic ray in } Z\}$. In this section, we write down formally what this means, and fix notation.

Let X be a complete CAT(0) space. Recall that $\partial X = \{\gamma(\infty) \mid \gamma \text{ is a geodesic ray in } X\}$ denotes the set of boundary points of X and that $\partial_c X = \{\gamma(\infty) \in \partial X \mid \gamma \text{ is contracting }\}$ denotes the set of contracting boundary points of X. If we equip ∂X with the cone-topology, we obtain the visual boundary of X, denoted by $\hat{\partial} X$. When we equip $\partial_c X$ with the subspace topology of $\hat{\partial} X$, we obtain the topological space $\hat{\partial}_c X$. If we equip $\partial_c X$ with the direct limit topology, we obtain the contracting boundary of X, denoted by $\vec{\partial}_c X$. Let Z be a subspace of X, i.e. Z is a subset of X equipped with the metric induced of the metric on X (i.e. Z is nonempty and the metric on Z is given by the restriction of the metric on X to the set $Z \times Z$). Recall that $\partial Z = \{\gamma(\infty) \in \partial Z \mid \gamma \text{ is a geodesic ray in } Z\}$ denotes the set of all equivalence classes of geodesic rays in Z. The inclusion map $\iota : Z \to X$ induces a bijection ι' between the set ∂Z and the set $\{\gamma(\infty) \in \partial X \mid \gamma \subseteq Z\}$. Usually, we work with the embedded set $\iota'(\partial Z)$. To simplify notation, we omit ι'. We emphasize when we mean by ∂Z the preimage of ι' and not the embedded set $\iota'(\partial Z)$. Recall that a *topological embedding* of a space Z into a space X is a function $f : Z \to X$ which maps X homeomorphically onto the subspace $f(Z)$ of X. The following fact follows from the definition of the cone topology (see [BH99, Example 8.11 (4) in Chapter II]).

Lemma 2.33. *Let X and Z be complete CAT(0) spaces. Every isometric embedding $\iota : Z \hookrightarrow X$ induces a topological embedding $\iota_* : \hat{\partial} Z \hookrightarrow \hat{\partial} X$ between their visual boundaries $\hat{\partial}_c Z$ and $\hat{\partial}_c X$.*

Let Z be a complete, convex subspace of a CAT(0) space X. Because Z is convex, Z is a CAT(0) space. Let $\hat{\partial} Z$ be the visual boundary of Z, i.e. we ignore the ambient space X and equip Z with the cone topology. By Lemma 2.33, the inclusion map $\iota : Z \to X$ induces a topological embedding $\iota_* : \hat{\partial} Z \hookrightarrow \hat{\partial} X$. Usually, we work with $\iota_*(\hat{\partial} Z)$. To simplify notation, we omit ι_* and denote the embedded visual boundary of Z by $\hat{\partial} Z$. We highlight the matter if we mean by $\hat{\partial} Z$ not the embedded visual boundary of Z in $\hat{\partial} X$. The question arises if the same can be done in the case of contracting boundaries. In general, this is not the case. For instance, the real line contains two contracting geodesic rays that start at 0. When we embed these rays isometrically in $\mathbb{R}^2$, these rays are convex subsets of $\mathbb{R}^2$ but they are not contracting in $\mathbb{R}^2$. Thus, we cannot think of the set $\partial_c Z$ as embedded in $\partial_c X$. Accordingly, we only write $\partial_c Z$, $\hat{\partial}_c Z$ or $\vec{\partial}_c Z$, if we ignore the ambient space X of Z. We always denote set $\partial Z = \{\gamma(\infty) \in \partial Z \mid \gamma$ is a contracting geodesic ray in $Z\}$ by $\partial_c Z$. If we equip this set with the subspace topology of the visual boundary of Z or with the direct limit topology, we obtain the topological space $\hat{\partial}_c Z$ and the contracting boundary $\vec{\partial}_c Z$ of Z respectively.

Definition 2.34. Let X be a complete CAT(0) space, Z a subspace of X and $\iota : Z \hookrightarrow X$ be the inclusion map. A geodesic $\gamma \subseteq Z$ is X*-contracting*, if $\iota(\gamma)$ is contracting in X.

Let X be a complete CAT(0) space and Z a subspace of X. Let

$$\partial_{c,X} Z \coloneqq \{\gamma(\infty) \in \partial_c Z \mid \gamma \text{ is } X\text{-contracting}\}$$

be the set of contracting boundary points in $\partial_c Z$ that are contracting in the ambient space X. If Z is a complete CAT(0) space, we can equip $\partial_{c,X} Z$ with the subspace topology of the visual boundary $\hat{\partial} Z$ of Z and with the contracting boundary $\vec{\partial}_c Z$ of Z. We denote these topological spaces by $\hat{\partial}_{c,X} Z$ and $\vec{\partial}_{c,X} Z$ respectively. The following lemma follows from the definition of the cone- and direct limit topology.

Lemma 2.35. *Let X be a complete CAT(0) space and Z a complete, convex subspace. The inclusion map $\iota : Z \hookrightarrow X$ induces topological embeddings $\iota_* : \hat{\partial}_{c,X} Z \hookrightarrow \hat{\partial}_c X$ and $\iota_{**} : \vec{\partial}_{c,X} Z \hookrightarrow \vec{\partial}_c X$.*

Proof. By Lemma 2.33, $\iota : Z \hookrightarrow X$ induces a topological embedding $\iota_* : \hat{\partial}_{c,X} Z \hookrightarrow \hat{\partial}_c X$. We choose a base point x_{base} in Z and show that the following map ι_{**} is a topological embedding.

$$\begin{aligned} \iota_{**} : \vec{\partial}_{c,X} Z_{x_{\text{base}}} &\hookrightarrow \vec{\partial}_c X_{x_{\text{base}}} \\ \gamma &\mapsto \gamma. \end{aligned}$$

Let $\gamma \in \vec{\partial}_{c,X} Z_{x_{\text{base}}}$. Then γ is an X-contracting geodesic ray in Z starting at x_{base}. Hence it is contained in the space $\vec{\partial}_c X_{x_{\text{base}}}$. Thus ι_{**} is a well-defined map. If γ and γ' are two X-contracting geodesic rays starting at x_{base} such that $\iota_{**}(\gamma) = \iota_{**}(\gamma')$, then $\gamma = \gamma'$. Hence f_* is injective. Let $C \coloneqq \iota_{**}(\vec{\partial}_{c,X} Z_{x_{\text{base}}})$. We have to show that ι_{**} and $\iota_{**}^{-1}|_C$ are continuous maps.

First, we show that the map ι_{**} is continuous. Let O be an open set in $\vec{\partial}_c X_{x_{\text{base}}}$, i.e. for each $k \in \mathbb{N}$ there exists a set $\hat{O}_k$ that is open in the visual boundary $\hat{\partial} X_{x_{\text{base}}}$ such that

$$O \cap \partial_c^k X = \hat{O}_k \cap \partial_c^k X \tag{2.35.1}$$

Hence,

$$\begin{aligned}
\iota_{**}^{-1}(O) &= \iota_{**}^{-1}(O \cap \iota_{**}(\partial_{c,X} Z_{x_{\text{base}}})) \\
&= \iota_{**}^{-1}(\bigcup_{k\in\mathbb{N}} (O \cap \partial_c^k X_{x_{\text{base}}}) \cap \iota_{**}(\partial_{c,X} Z_{x_{\text{base}}})) \\
&\overset{2.35.1}{=} \iota_{**}^{-1}(\bigcup_{k\in\mathbb{N}} (\hat{O}_k \cap \partial_c^k X_{x_{\text{base}}}) \cap \iota_{**}(\partial_{c,X} Z_{x_{\text{base}}})) \\
&= \bigcup_{k\in\mathbb{N}} \iota_{**}^{-1}(\hat{O}_k) \cap \iota_{**}^{-1}(\partial_c^k X_{x_{\text{base}}}) \cap \iota_{**}^{-1}(\iota(\partial_{c,X} Z_{x_{\text{base}}})) \\
&\overset{**}{=} \bigcup_{k\in\mathbb{N}} \iota_{**}^{-1}(\hat{O}_k) \cap \iota_{**}^{-1}(\partial_c^k X_{x_{\text{base}}}) \cap \partial_{c,X} Z_{x_{\text{base}}} \\
&= \bigcup_{k\in\mathbb{N}} \iota_{**}^{-1}(\hat{O}_k) \cap \partial_{c,X} Z_{x_{\text{base}}} \\
&= \iota_{**}^{-1}(\bigcup_{k\in\mathbb{N}} \hat{O}_k) \cap \partial_{c,X} Z_{x_{\text{base}}}
\end{aligned}$$

Thereby, $(**)$ holds as ι_{**} is injective and the second last inequality holds since

$$\iota_{**}^{-1}(\partial_c^k X_{x_{\text{base}}}) \subseteq \partial_{c,X} Z_{x_{\text{base}}}.$$

Indeed, let $\gamma \in \iota_{**}^{-1}(\partial_c^k X_{x_{\text{base}}})$. Then γ is a geodesic ray in Z that is k-contracting in the ambient space X. In particular, γ is X-contracting and hence, $\gamma \in \partial_{c,X} Z_{x_{\text{base}}}$.
Let $O' := \bigcup_{k\in\mathbb{N}} \hat{O}_k$. It remains to show that the preimage of any element in O' has an open neighborhood in $\vec{\partial}_{c,X} Z_{x_{\text{base}}}$ that is mapped by ι_{**} onto O'. Let γ be an X-contracting geodesic ray staring at x_{base} so that $\iota_{**}(\gamma) \subseteq O'$. The set O' is open in the visual boundary $\hat{\partial} X_{x_{\text{base}}}$ as union of open sets. Hence there exists an open set U in O' of the form $U(\iota_{**}(\gamma), r, \epsilon)$ as in Definition 2.18. As $x_{\text{base}} \in Z$ and as Z is a convex subset of X, the open set $\tilde{U}$ in $\hat{\partial} Z_{x_{\text{base}}}$ of the form $U(\gamma, r, \epsilon)$ satisfies $\iota_{**}(\tilde{U} \cap \partial_{c,X} Z_{x_{\text{base}}}) \subseteq U$. As the topology of the contracting boundary is finer than the cone-topology, the set $\tilde{U} \cap \partial_{c,X} Z_{x_{\text{base}}}$ is open in $\vec{\partial}_{c,X} Z_{x_{\text{base}}}$. Thus, $\tilde{U} \cap \partial_{c,X} Z_{x_{\text{base}}}$ is an open neighborhood of γ we were looking for $\iota_{**}^{-1}(O')$ is open in $\vec{\partial}_{c,X} Z_{x_{\text{base}}}$.

It remains to prove that $\iota_{**}^{-1}|_C$ is continuous where $C = \iota_{**}(\vec{\partial}_{c,X} Z_{x_{\text{base}}})$. Let O be an open set in $\vec{\partial}_{c,X} Z_{x_{\text{base}}}$. We have to prove that $\iota_{**}(O)$ is open in the space $C = \iota_{**}(\vec{\partial}_{c,X} Z_{x_{\text{base}}})$ equipped with the subspace topology of $\vec{\partial}_c X_{x_{\text{base}}}$. As O is an open set in $\vec{\partial}_{c,X} Z_{x_{\text{base}}}$, there exists an open set $\tilde{O}$ in $\vec{\partial}_c Z_{x_{\text{base}}}$ such that

$$O = \tilde{O} \cap \partial_{c,X} Z_{x_{\text{base}}}.$$

As $\tilde{O}$ is open in $\vec{\partial}_c Z_{x_{\text{base}}}$, for each $k \in \mathbb{N}$ there exists a set $\hat{O}_k \subseteq \hat{\partial} Z_{x_{\text{base}}}$ so that

$$\hat{O}_k \cap \partial_c^k Z_{x_{\text{base}}} = \tilde{O} \cap \partial_c^k Z_{x_{\text{base}}} \tag{2.35.2}$$

Hence,

$$\begin{aligned}
\iota_{**}(O) &= \iota_{**}(\tilde{O} \cap \partial_{c,X} Z_{x_{\text{base}}}) \\
&= \iota_{**}(\tilde{O} \cap \bigcup_{k \in \mathbb{N}} \partial_c^k Z_{x_{\text{base}}} \cap \partial_{c,X} Z_{x_{\text{base}}}) \\
&= \iota_{**}(\bigcup_{k \in \mathbb{N}} (\tilde{O} \cap \partial_c^k Z_{x_{\text{base}}}) \cap \partial_{c,X} Z_{x_{\text{base}}}) \\
&\overset{2.35.2}{=} \iota_{**}(\bigcup_{k \in \mathbb{N}} \hat{O}_k \cap \partial_c^k Z_{x_{\text{base}}} \cap \partial_{c,X} Z_{x_{\text{base}}}) \\
&= \bigcup_{k \in \mathbb{N}} \iota_{**}(\hat{O}_k \cap \partial_{c,X} Z_{x_{\text{base}}} \cap \partial_{c,X} Z_{x_{\text{base}}} \cap \partial_c^k Z_{x_{\text{base}}} \cap \hat{O}_k) \\
&\overset{***}{=} \bigcup_{k \in \mathbb{N}} \iota_{**}(\hat{O}_k \cap \partial_{c,X} Z_{x_{\text{base}}}) \cap \iota_{**}(\partial_{c,X} Z_{x_{\text{base}}} \cap \partial_c^k Z_{x_{\text{base}}} \cap \hat{O}_k) \\
&\overset{****}{=} \bigcup_{k \in \mathbb{N}} \iota_{**}(\hat{O}_k \cap \partial_{c,X} Z_{x_{\text{base}}}) \\
&= \iota_{**}(\bigcup_{k \in \mathbb{N}} \hat{O}_k \cap \partial_{c,X} Z_{x_{\text{base}}})
\end{aligned}$$

Thereby, $(***)$ holds as ι_{**} is injective and $(***)$ holds as

$$\iota_{**}(\partial_{c,X} Z_{x_{\text{base}}} \cap \partial_c^k Z_{x_{\text{base}}} \cap \hat{O}_k) \subseteq \iota_{**}(\hat{O}_k \cap \partial_{c,X} Z_{x_{\text{base}}})$$

.

Let $O' := \bigcup_{k \in \mathbb{N}} \hat{O}_k$. It remains to show that $\iota_{**}(O' \cap \partial_{c,X} Z_{x_{\text{base}}})$ is open in $C = \iota_{**}(\partial_{c,X} Z_{x_{\text{base}}})$ equipped with the subspace topology of $\vec{\partial}_c X_{x_{\text{base}}}$. Let γ be an X-contracting geodesic ray in $\partial_c X_{x_{\text{base}}}$ staring at x_{base} so that $\iota_{**}^{-1}(\gamma) \subseteq O' \cap \partial_{c,X} Z_{x_{\text{base}}}$. The set O' is open in the visual boundary $\hat{\partial} Z_{x_{\text{base}}}$ as union of open sets. Hence there is an open set $U \subseteq O' \subseteq \hat{\partial} Z_{x_{\text{base}}}$ of the form $U(\iota_{**}^{-1}(\gamma), r, \epsilon)$ as in Definition 2.18. Let $\tilde{U}$ be the open set in $\hat{\partial} X_{x_{\text{base}}}$ of the form $U(\gamma, r, \epsilon)$. As $x_{\text{base}} \in Z$ and since Z is convex, $\iota_{**}(U \cap \partial_{c,X} Z_{x_{\text{base}}}) \subseteq \tilde{U}$ and $\tilde{U} \cap C = \tilde{U} \cap \iota_{**}(\partial_{c,X} Z_{x_{\text{base}}}) \subseteq \iota_{**}(U \cap \partial_{c,X} Z_{x_{\text{base}}})$. Indeed, if α is a geodesic ray in $\tilde{U} \cap C$, then α is an X-contracting geodesic ray starting at x_{base} that is contained in Z such that $d(\gamma(t), \alpha(t)) < \epsilon$ for all $t \geq \epsilon$. Hence, $\alpha \in \iota_{**}(U \cap \partial_{c,X} Z_{x_{\text{base}}})$. As the topology of the contracting boundary is finer than the cone-topology, the set $\tilde{U} \cap C$ is open in C equipped with the subspace topology of $\vec{\partial}_c X$. Thus, $\tilde{U} \cap C$ is an open neighborhood of γ such that $\iota_{**}^{-1}(\tilde{U} \cap C) \subseteq O'$. Hence $\iota_{**}(O')$ is open in C equipped with the subspace topology of $\vec{\partial}_c X_{x_{\text{base}}}$. □

Usually, we work with the embedded spaces $\iota_*(\hat{\partial}_{c,X} Z)$ and $\iota_{**}(\vec{\partial}_{c,X} Z)$. To simplify notation, we omit ι_* and ι_{**} if we refer to the images of ι_* and ι_{**}, i.e. we denote the embedded spaces $\iota_*(\hat{\partial}_{c,X} Z)$ and $\iota_{**}(\vec{\partial}_{c,X} Z)$ by $\hat{\partial}_{c,X} Z$ and $\vec{\partial}_{c,X} Z$ respectively. It will be clear from the context whether we think of the spaces as embedded or not. We highlight the matter if we mean by $\hat{\partial}_{c,X} Z$ and $\vec{\partial}_{c,X} Z$ the preimages of ι_* and ι_{**} respectively.

In this thesis, we are interested in totally disconnected contracting boundaries. We say that a topological space X is *totally disconnected* if each of its connected components consists of a single point. The empty set does not contain any connected component and is totally disconnected.

In our considerations, we often work with totally disconnected subspaces. Suppose that x is contained in a totally disconnected subspace Y of X. We emphasize that the connected component of x in X might contain many points. For example, the rational numbers $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both totally disconnected subspaces of $\mathbb{R}$. But the ambient space $\mathbb{R}$ is connected. So, the connected component of each point in $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ in the ambient space $\mathbb{R}$ consists of $\mathbb{R}$.

A crucial property of the contracting boundary is that the direct limit topology on $\partial_c X$ is finer than the subspace topology of the visual boundary on $\partial_c X$, i.e. every set that is open in $\hat{\partial}_c X$ is open in $\vec{\partial}_c X$. Therefore, every connected component of $\vec{\partial}_c X$ is contained in a connected component of $\hat{\partial}_c X$. In particular, if $\hat{\partial}_c X$ is totally disconnected then $\vec{\partial}_c X$ is totally disconnected. We use this basic observation in the proofs of the main results of this thesis.

2.6 CAT(0) cube complexes

In this section, we introduce CAT(0) cube complexes. The notion of CAT(0) cube complexes has its origin in [Gro87]. Most of the following definitions and facts conform to [Sag14]. Other references are [HW08] and [Sag95]. We start with the following definition of the *standard Euclidean n-cube*. This definition is a variant of Example A.1.5. in [Dav08].

Definition 2.36. The standard Euclidean n-dimensional cube (n-cube) C^n is the cube $[0,1]^n \subseteq \mathbb{R}^n$. The sets of the form $C' = \{x \in C^n \mid x_i = \epsilon\}$, $\epsilon \in \{0,1\}$, $i = 1,\ldots,n$ are the *codimension 1 faces* of C^n. A *face* of C^n is a nonempty intersection of codimension 1 faces or C itself. Every face is isometric to a standard Euclidean cube of dimension $k \leq n$; we say that k is its dimension. Faces of dimension 0 are *vertices*, and faces of dimension 1 are *edges* of C^n.

An *n-cube* is a metric space isometric to the standard Euclidean n-cube. A space C is a *cube* if there exists n such that C is an n-cube. Let C be an n-cube and ϕ be an isometry from C to the standard n-cube C^n. Then every preimage of a face of C^n is a *face* of C. Let $\mathcal{M}$ be a set of disjoint cubes and $\mathcal{F}$ a set of collections of isometries between faces of cubes in $\mathcal{M}$. For every map $f : C' \to C''$ in $\mathcal{F}$, we identify $x \in C'$ with $f(x) \in C''$ and obtain the quotient space $X = \mathcal{M}/\mathcal{F}$. Let $q : \mathcal{M} \to X$ be the associated projecting map. The space X is a *cube complex* whose cubes are the images of the faces of the cubes in $\mathcal{M}$ under q. A *subcomplex* of X is a cube complex whose cubes are cubes of X. The k-skeleton of X is the k-dimensional subcomplex $X^{(k)}$ consisting of all cubes in X of dimension k or less than k. The 0-skeleton of a cube complex consists of vertices. The 1-skeleton of a cube complex is a (non-necessarily simplicial) graph. A *local edge* of $\mathcal{M}$ is a subinterval of length $\frac{1}{3}$ of an edge of a cube in $\mathcal{M}$ containing one of the endpoints of the edge. A local edge in X is the image of a local edge in $\mathcal{C}$ under q. The *link* of a vertex v in X , denoted by $\operatorname{lk}(v)$, is a simplicial complex obtained as follows. The vertices of $\operatorname{lk}(v)$ are the local edges of X that contain v. A set of local edges $\{e_0,\ldots,e_k\}$ spans a simplex if $\mathcal{M}$ contains a cube C such that

- $q^{-1}(e_0),\ldots,q^{-1}(e_k)$ are contained in C and
- $q^{-1}(e_0),\ldots,q^{-1}(e_k)$ share a vertex v of C.

The following is Definition 1.2 of [Sag14].

Definition 2.37. A *non-positively curved (NPC) cube complex* is a cube complex whose vertex links are simplicial flag complexes. A 1-connected NPC complex is called a *CAT(0)* cube complex.

If X is a non-positively curved (NPC) cube complex, the definition implies that no cube in $\mathcal{M}$ is glued to itself. Furthermore, for all $C \neq C' \in \mathcal{M}$ there is at most one gluing of C and C'. These two conditions must be satisfied if one defines a *cubical complex* as in Definition A.1.9 in [Dav08]. In particular, the map q restricted to one of its cubes is injective. If there exists $n \in \mathbb{N}$ such that every cube in X has dimension at most n,

X is finite-dimensional. If n is chosen minimal, n is the *dimension* of X. If no point in X is contained in infinitely many cubes, X is *locally finite.* This is the case if and only if the degree of every vertex in the 1-skeleton of X is finite. The cubes of X are isometric to Euclidean cubes and induce a metric on X. Indeed, we say that a curve is *rectifiable* if it can be subdivided into finitely many subcurves that are each contained in a cube of X. The *length* of a rectifiable curve is the sum of the length of all the subcurves in its subdivision. We define the distance between two points x and y in X as the infimum of the lengths of the rectifiable curves joining p and q. Suppose that X is a finite-dimensional cube complex. Then X equipped with this metric is a complete, geodesic metric space by [Bri91]. By Gromov's Link condition [Gro87], X equipped with this metric is locally CAT(0) if and only if X is NPC. The Cartan-Hadamard theorem [Car28; Had98] implies that if X is NPC and 1-connected, then X is a CAT(0) space. For more details, see [Sag95]. In this thesis, we are just interested in finite-dimensional cube complexes. We mention that the case where X is locally finite was studied by Moussong in [Mou88]. For the case that X is infinite-dimensional, Leary proved in [Lea13] that X is a geodesic metric space that is locally CAT(0). Furthermore, he examined when X is complete.

We obtain a metric on the 1-skeleton of X by restricting the metric on X to its 1-skeleton. It helps sometimes to examine this metric instead of the metric of X. This becomes clear if one considers so-called *hyperplanes* of X. Let $\sim$ be the equivalence relation on the set of midcubes of cubes generated by the condition that two midcubes are equivalent if they share a face. An equivalence class of this equivalence relation is a *hyperplane* H. The carrier of a hyperplane H, $\mathcal{N}(H)$, is the union of all cubes that contain a midcube of H. If a hyperplane H intersects an edge e of H, we say that H and e are dual. Each hyperplane corresponds to a set of edges that are dual to H and vice versa.

Lemma 2.38 (Theorem 4.10 of [Sag95])**.** *Suppose that X is a CAT(0) cube complex and H is a hyperplane in X. Then H does not self-intersect and $X \setminus H$ has exactly two components.*

If two hyperplanes are distinct and the carriers of two distinct hyperplanes intersect, we say that they are *adjacent.* Let H_0, H_1 and H three hyperplanes in X. By Lemma 2.38, X decomposes into two distinct half-spaces C_0 and C_1 if we delete H. We say that H lies *between* H_0 and H_1 if one of the two hyperplanes is contained in C_0 and the other one is contained in C_1. The following lemma implies that two disjoint hyperplanes have a hyperplane between them if they are not adjacent. Compare Theorem 4.6 and Theorem 4.13 of Sageev [Sag95].

Lemma 2.39 (Distance in 1-skeletons)**.** *Let u, v be two vertices in a CAT(0) cube complex X. The distance of u and v in the 1-skeleton of X coincides with the number of hyperplanes between u and v.*

2.7 Coxeter groups

In this section, we define Coxeter groups and list their properties as far as needed in this thesis. We are mainly interested in the right-angled case. Most of the following can be found in Davis's book about the geometry and topology of Coxeter groups [Dav08].

Definition 2.40. A *Coxeter matrix* $M = (m(s_i, s_j))_{1 \leq i,j \leq n}$ on a set S is an $S \times S$ symmetric matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that $m(s_i, s_j) = 1$ if $s_i = s_j$ and $m(s_i, s_j) \geq 2$ otherwise. Associated to M is a group W with presentation

$$\langle S \mid (s_i s_j)^{m(s_i,s_j)} = \text{id} \text{ for all } m(s_i, s_j) \neq \infty \rangle. \tag{2.40.1}$$

The pair (W, S) is a *Coxeter system* with *fundamental set of generators* S. The group W is a *Coxeter group*. If $m(s_i, s_j) \in \{2, \infty\}$ for all $1 \leq i, j \leq n$, $i \neq j$, (W, S) is *right-angled*. In this case, W is a *right-angled Coxeter group*.

Every Coxeter matrix leads to a presentation of a group as in Equation (2.40.1). On the other hand, every presentation as in Equation (2.40.1) leads to a Coxeter Matrix if $m(s_i, s_j) = m(s_i, s_j)$ for all $1 \leq i, j \leq n$. In this thesis, we are - as noted - mainly interested in right-angled Coxeter groups. Let (W, S) be a right-angled Coxeter system. Then the presentation in Equation (2.40.1) can be specified by means of the following simple graph Λ: The vertex set of Λ is S. Two vertices of Λ are adjacent if and only if they commute. Then the presentation of W can be written as

$$\langle V(\Lambda) \mid s_i^2 = \text{id} \text{ for all } s_i \in V(\Lambda), s_i s_j = s_j s_i \text{ for all } \{s_i, s_i\} \in E(\Lambda) \rangle. \tag{2.40.2}$$

On the other hand, if Λ is a finite, simple graph, it defines an associated right-angled Coxeter system $(W_\Lambda, V(\Lambda))$ where W_Λ is given by Equation (2.40.2). We say that Λ is the *defining graph* of the right-angled Coxeter system $(W_\Lambda, V(\Lambda))$. For short, we say that it is the defining graph of W_Λ.

Like Hosaka in [Hos03], we say that two Coxeter systems (W, S) and (W', S') are *isomorphic* if there exists a bijection $f : S \to S'$ such that $m(s,t) = m'(f(s), f(t))$ for each s, $t \in S$ where $m(s,t)$ and $m(s', t')$ denote the orders of st in W and $s't'$ in W' respectively. In general, a Coxeter group does not always determine its Coxeter system up to isomorphism. A Coxeter group may arise from two different presentations; i.e., there are examples of Coxeter systems (W, S) and (W', S') such that W and W' are isomorphic but the Coxeter systems (W, S) and (W', S') are not isomorphic. This is, not the case when the Coxeter group is right-angled. The following is Theorem 1 of Hosaka in [Hos03].

Theorem 2.41. *Every right-angled Coxeter group determines its Coxeter system up to isomorphism.*

It follows that every right-angled Coxeter group determines its defining graph up to isomorphism.

Definition 2.42. If W is a group that is generated by a set of elements of order two, then (W, S) is a *pre-Coxeter system.*

The following statement is a very useful tool for examining when a pre-Coxeter system is a Coxeter system. It is a mix of Theorem 3.2.16 and Theorem 3.3.4 in [Dav08]

Theorem 2.43. *The following conditions on a pre-Coxeter system (W, S) are equivalent.*

a) (W, S) is a Coxeter system.

b) (W, S) satisfies the Deletion Condition (D): If $\vec{g} = s_1, \ldots, s_k$ is a word in S with $k > l(\vec{g})$, then there are indices $i < j$ so that the subword $\vec{g'} = s_1, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_k$ is also an expression for g.

c) (W, S) satisfies the Exchange Condition (E): Given an S-reduced expression $\vec{g} = s_1, \ldots, s_k$ for $g \in W$ and an element $s \in S$, either $l(sg) = k + 1$ or there is an index i such that $g = ss_1 \cdots \hat{s}_i \cdots s_k$.

d) (W, S) satisfies the Folding Condition (F): Suppose $g \in W$ and s, $t \in S$ are such that $l(sg) = l(g) + 1$ and $l(gt) = l(g) + 1$. Then either $l(sgt) = l(g) + 2$ or $sgt = g$.

Let (W, S) be a Coxeter system. A subgroup of W is *special* if it is generated by a subset of S. If $T \subseteq S$, we denote the subgroup generated by T by W_T . Special subgroups have many important and interesting properties and play a crucial role in Chapter 5 of this thesis where we examine contracting boundaries of right-angled Coxeter groups. Let Λ be a simple, finite graph. The presentation of W_Λ given by Equation (2.40.2) shows that there is a one-to-one correspondence between the induced subgraphs of Λ and the special subgroups of W_Λ.

Lemma 2.44. *Let Λ be a finite simple graph. A subgroup of a right-angled Coxeter group W_Λ is special if and only if it has an induced subgraph of Λ as defining graph.*

The following statements are Proposition 4.1.1, corollary 4.1.2, and theorem 4.1.6 in [Dav08].

Proposition 2.45. *For each $g \in W$, there is a subset $S(g) \subseteq S$ so that for any reduced expression $s_1, \ldots, s_k$ for g, $S(g) = \{s_1, \ldots, s_k\}$. For each $T \subseteq S$, W_T consists of those elements $g \in W$ such that $S(g) \subseteq T$.*

It follows from this proposition that for each $T \subseteq S$, $W_T \cap S = T$ and that S is a minimal set of generators for W. The *rank* of the Coxeter system (W, S) is the number of elements in S.

Theorem 2.46.

a) For each $T \subseteq S$, (W_T, T) is a Coxeter system.

b) Let $(T_i)_{i\in I}$ be a family of subsets of S. If $T = \bigcap_{i\in I} T_i$, then

$$W_T = \bigcap_{i\in I} W_{T_i}.$$

c) Let T, T' be subsets of S and g, g' elements of W. Then $gW_T \subseteq g'W_{T'}$ (resp. $gW_T = g'W_{T'}$) if and only if $g^{-1}g' \in W_{T'}$ and $T \subseteq T'$ (resp. $T = T'$).

W acts on two associated complexes, namely on its *Coxeter complex* and on its *Davis complex.* In general, the action of W on its Coxeter complex does not need to be proper. Thus, we concentrate on the second associated complex, the *Davis complex.* We sketch the general idea how to obtain the Davis complex of a Coxeter system (W, S) and explain how this construction simplifies if W is a right-angled Coxeter group.

The Coxeter system (W, S) and W are called *finite* or *spherical*, if W is a finite group. As Davis, we denote the set of all spherical subsets of S by $\mathcal{S}$. This set is a poset, i.e. it is partially ordered by inclusion. The Davis complex is defined by means of the *nerve* of (W, S).

Definition 2.47. Let (W, S) be a Coxeter system. The poset $\mathcal{S}_{>\emptyset}$ of all nonempty spherical subsets is an abstract simplicial complex, i.e. if $T \in \mathcal{S}_{>\emptyset}$ and $T' \subseteq T$, $T' \neq \emptyset$, then $T' \in \mathcal{S}_{>\emptyset}$. This simplicial complex is the *nerve* of (W, S).

Let Λ be a defining graph for a right-angled Coxeter group W_Λ. Then the 1-skeleton of the nerve of $(W_\Lambda, V(\Lambda))$ is Λ. Furthermore, the nerve of $(W_\Lambda, V(\Lambda))$ is the flag complex determined by Λ: a finite, nonempty set T of vertices spans a simplex in the nerve of $(W_\Lambda, V(\Lambda))$ if and only if any two elements of T are adjacent.

A *spherical coset* is a coset of a spherical special subgroup in W. Like Davis [Dav08, p.125], we denote the set of all spherical cosets by $W\mathcal{S}$, i.e.

$$W\mathcal{S} = \bigcup_{T\in\mathcal{S}} W/W_T. \tag{2.47.1}$$

By Theorem 2.46, the union in Equation (2.47.1) is a disjoint union. Furthermore, $W\mathcal{S}$ is partially ordered by inclusion and W acts naturally on the poset $W\mathcal{S}$ such that the quotient poset is $\mathcal{S}$. A *geometric realization* is associated to any poset $\mathcal{P}$, i.e. there is a topological space associated to an abstract simplicial complex consisting of all finite chains in $\mathcal{P}$. For a Definition, see Appendix A.2 in [Dav08]. Let Σ be the geometric realization of the poset $W\mathcal{S}$. It is possible to equip Σ with a cell structure that is coarser than its simplicial structure. This leads to the results of Chapter 7 and Chapter 12 in [Dav08]. The following is Proposition 7.3.4 and Theorem 12.3.3 in [Dav08]. Theorem 12.3.3 in [Dav08] is Moussong's Theorem of [Mou88].

Theorem 2.48. *There is a natural cell structure on Σ so that*

- *its vertex set is W, its 1-skeleton is the Cayley graph, $Cay(W, S)$, and its 2-skeleton is a Cayley 2-complex,*

- *each cell is a so-called Coxeter polytope as defined in Definition 7.3.1 in [Dav08],*
- *the link of each vertex is isomorphic to the nerve of* (W, S)*,*
- *a subset of* W *is the vertex set of a cell if and only if it is a spherical coset,*
- *the poset of cells in* Σ *is* $W\mathcal{S}$*,*
- *there is a piecewise Euclidean metric on* Σ *that is CAT(0) and* W *acts properly and cocompactly by isometries on* Σ *equipped with this metric.*

Definition 2.49. The Davis complex $\Sigma(W, S)$ associated to a Coxeter system (W, S) is the geometric realization of $W\mathcal{S}$ equipped with the cell structure of Theorem 2.48

If W_Λ is a right-angled Coxeter group, Σ_Λ denotes the Davis complex of its associated right-angled Coxeter system. We say for the sake of simplicity, that Σ_Λ is the Davis complex of W_Λ or of Λ. In this situation, the definition of Σ_Λ simplifies. The Davis complex Σ_Λ is a cube complex constructed as follows. It has the Cayley graph $Cay(W_\Lambda, V(\Lambda))$ as one-skeleton. We attach cubes wherever possible, i.e. we fill in an n-cube wherever $Cay(W_\Lambda, S)$ contains the 1-skeleton of the n-cube. The resulting cube complex is the Davis complex of W_Λ. For a more detailed description how to construct Σ_Λ, see [Dav08, p.9-14]. The following is Theorem 12.2.1 and part of Corollary 12.6.3 in [Dav08]. It follows from Gromov's work [Gro87].

Theorem 2.50. *Suppose that* (W, S) *is a right-angled Coxeter system with defining graph* Λ*. Then*

a) The piecewise Euclidean cubical structure on Σ *is CAT(0).*

b) The piecewise hyperbolic structure on Σ *on which each cube is a regular hyperbolic cube of edge length* ϵ *is* $CAT(-1)$ *for some* $\epsilon > 0$ *if and only if* Λ *does not contain any induced* 4*-cycle.*

Furthermore, W *is word hyperbolic if and only if* Λ *does not contain any induced* 4*-cycle.*

As mentioned before, the behavior of special subgroups is very important in this thesis. Let Λ be a finite, simple graph and Λ' an induced subgraph. Recall that $W_{\Lambda'}$ is a special subgroup of W_Λ. By the construction of Σ and Item c) in Theorem 2.46, the Davis complex of Σ_Λ has a subcomplex that contains the identity of W_Λ as vertex and coincides with the Davis complex of $W_{\Lambda'}$. We say that we *embed* $\Sigma_{\Lambda'}$ *canonically* in Σ_Λ if we embed $\Sigma_{\Lambda'}$ isometrically in this subcomplex. Thereby, a vertex $g \in W_{\Lambda'}$ is identified with vertex $g \in W_\Lambda$. The trivial graph $(\emptyset, \emptyset)$ is an induced subgraph of Λ. The Davis complex of $(\emptyset, \emptyset)$ consists of a vertex – the identity of the trivial group. We embed this vertex canonically in Σ_Λ by identifying this vertex with the identity element of W_Λ. Let $\Sigma_{\Lambda'}$ be the canonically embedded Davis complex of $W_{\Lambda'}$ in Σ_Λ. As W_Λ acts on the Davis complex by isometries, $g\Sigma_{\Lambda'}$ is an isometrically embedded copy of $\Sigma_{\Lambda'}$ in Σ_Λ. Accordingly, for every left coset $gW_{\Lambda'}$ in $W_\Lambda/W_{\Lambda'}$ we obtain an embedded copy of $\Sigma_{\Lambda'}$ in Σ_Λ. We summarize these observations in the following lemma.

Lemma 2.51. *Let* Λ' *be an induced subgraph of a graph* Λ. *Every vertex of* Σ_Λ *is contained in an isometrically embedded Davis complex of* $W_{\Lambda'}$ *in* Σ_Λ. *The* 1*-skeleton of each such embedded Davis complex equipped with the word metric associated to* $(W_\Lambda, V(\Lambda))$ *is an isometrically embedded Cayley graph of* $W_{\Lambda'}$ *in* $\Sigma_\Lambda^{(1)}$ *equipped with the word metric associated to* $(W_{\Lambda'}, V(\Lambda'))$.

2.8 Amalgamated free products

In this section we define amalgamated free products of groups. We explain how group elements of an amalgamated free product can be represented uniquely. We define the Bass-Serre tree and extended Bass-Serre tree associated to an amalgamated free product. At the end of this section we define trees of groups and spaces. The following facts are based on [Ser03], [SW79], [BH99] and [SZ94]

Definition 2.52 ([SZ94], Def. 5.7.8)**.** Let G_0 and G_1 and H be groups and $\Phi_i : H \to G_i$, $i \in \{0,1\}$ monomorphisms. Let $\mathcal{N}\{\Phi_0(h)\Phi_1(h)^{-1} \mid h \in H\}$ be the normal subgroup of the free product of G_0 and G_1 that is generated by the conjugates of the elements $\{\Phi_0(h)\Phi_1(h)^{-1} \mid h \in H\}$. The group $G_0 * G_1/\mathcal{N}\{\Phi_0(h)\Phi_1(h)^{-1} \mid h \in H\}$ is called *amalgamated free product of* G_0 *and* G_1 *along* H and denoted by $G_0 *_H G_1$.

We say that a group G *splits over* H if it is isomorphic to an amalgamated free product along H. Let $G = G_0 *_H G_1$ be an amalgamated free product of two groups G_0 and G_1 along a group H. The group G_i, $i \in \{0,1\}$, is a subgroup of G by identifying it with its image under the natural map from G_i to $G_0 *_H G_1$. Let $U_0 = \Phi_0(H)$, and $U_1 = \Phi_1(H)$. By means of the isomorphism $\Phi = \Phi_1\Phi_0^{-1} : U_0 \to U_1$ we identify H with the subgroup $G_0 \cap G_1$ in $G_1 *_H G_2$. It can be found e.g. in [SZ94, Section 5.7.9] that this is possible.

Suppose that G_0 and G_1 are finitely presented. Presentations of G_0 and G_1 give rise to a generating set of G.

Lemma 2.53 (Section 5.7.8 (a) in [SZ94])**.** *If* $G_i = \langle S_i \mid R_i\rangle, i \in \{0,1\}$ *and* S_H *is a generating set for* H*, then* $G = \langle S_0 \cup S_1 | R_0 \cup R_1 \cup \{\Phi_0(h)\Phi_1(h)^{-1} \mid h \in S_H\}\rangle$*.*

It is possible to write group elements of G as words in $G_1 \cup G_2$. A word $\vec{g}$ in $G_1 \cup G_2$ representing $g \in G$ is a finite sequence $g_0, g_1 \ldots, g_k$ of elements $g_i \in G_1 \cup G_2$ such that $g = g_0 \cdot \ldots \cdot g_k$. An infinite word is an infinite sequence of elements in $G_1 \cup G_2$. The following Lemma merges Lemma 6.4 from Chapter 3 of [BH99] and the section before Theorem 1.6 in [SW79].

Lemma 2.54. *Let* $G = G_0 *_H G_1$*. Let* $a_1,\ldots,a_k \in G_0 \setminus H$ *and* $b_0,\ldots,b_{k-1} \in G_1 \setminus H$*,* $a_0 \in (G_0 \setminus H) \cup \{\mathrm{id}\}$*,* $b_k \in (G_1 \setminus H) \cup \{\mathrm{id}\}$ *and* $c \in H$*. Then* $a_0 b_0 \cdot \cdots \cdot a_k b_k c \neq 1$ *in* G*. When an element of* G *is expressed as such a product, it is said to be in reduced form. Every element of* $G \setminus \{id\}$ *can be written in reduced form.*

For simplicity we say that the reduced form of the identity is the identity itself.
We repeat how to yield a reduced form for an element $g \in G$ as in [SZ94][5.7.9 p. 135]. By definition of amalgamations, every element $g \in G$ can be written as a product of elements in $G_0 \cup G_1$. If there are two adjacent factors a and b that are both in the same group G_i for $i \in \{0,1\}$, we multiply them. This way we combine the two factors a and b to one factor $\tilde{a} = a \cdot b$. We proceed in this manner until g is written as a product $g_0 \cdot, \ldots, \cdot g_k$, $g_j \in G_0 \cup G_1$, $j \in \{0,\ldots,k\}$ such that for all $j \in \{0,\ldots,k-1\}$, g_j and g_{j+1} are contained in distinct groups G_i, $i \in \{0,1\}$. If there is a factor g_j which is contained in

U_0, we multiply $\Phi(g_j)$ with g_{j+1} and exchange g_j by $\Phi(g_j)g_{j+1}$. We proceed analogously if g_j is contained in U_2. Afterwards, no g_j is contained in U_1 or U_2, except for $j = k$. This way, g is written in reduced form as defined in Lemma 2.54.

Remark 2.55. The described procedure can be done the other way around. If we do so, we write every element $g \in G$ as a product of letters in which the letters alternate between G_0 and G_1 and just the first letter might be contained in U_0 or U_1. In [BH99], such a product is said to be in reduced form too. We have chosen a different definition for simplifying notation.

In general, there are many possibilities to write a group element in reduced form. The next lemma is about common features of such products in reduced forms.

Lemma 2.56. *Let $a_0b_0 \dots a_kb_kc$ and $a'_0b'_0 \dots a'_lb'_lc'$ be in reduced form and $g \in G$, $g \neq \mathrm{id}$ such that $g = a_0b_0 \dots a_kb_kc$ and $g = a'_0b'_0 \dots a'_lb'_lc'$. Then $k = l$ and for all $i \in \{0, \dots, k\}$ there are h, $h' \in H$ such that $a_iH = ha'_iH$, $h \in H$ and $b_iH = h'b'_iH$, $h' \in H$.*

Proof. We have by definition that $\mathrm{id} = a_0b_0 \dots a_kb_kcc'^{-1}b'^{-1}_l a'^{-1}_l \dots b'^{-1}_0 a'^{-1}_0$. If $b_kcc'^{-1}b'^{-1}_l$ would not be contained in H, $a_0b_0 \dots a_k(b_kcc'^{-1}b'^{-1}_l)a'^{-1}_l \dots b'^{-1}_0 a'^{-1}_0$ or $a_0b_0 \dots a_k(b_kc)(c'^{-1}b'^{-1}_l)a'^{-1}_l \dots b'^{-1}_0 a'^{-1}_0$ would be in reduced form – a contradiction to Lemma 2.54 as the product coincides with the identity. Hence, $b_kcc'^{-1}b'^{-1}_l$ is contained in H and there exists $h \in H$ such that $b_kH = hb'_lH$. The claim follows by repeating this argument. □

It is possible to represent every group element of G uniquely by a word in $G_1 \cup G_2$ whose letters (except for the last letter) are fixed representatives for G_0/H and G_1/H. This is the content of the next lemma. Recall that a word $\vec{g}$ in $G_1 \cup G_2$ representing $g \in G$ is a finite sequence $g_0, g_1 \dots, g_k$ of elements $g_i \in G_1 \cup G_2$ such that $g = g_0 \cdot \dots \cdot g_k$.

Lemma 2.57 (Thm 1.6 in[SW79], 5.7.9 p.235 in [SZ94])**.** *Let $G = G_0 *_H G_1$. Let R_0 and R_1 be sets of representatives for G_0/H and G_1/H such that the identity represents the coset H in G_0/H and G_1/H. Every element of an amalgamated free product $G = G_0 *_H G_1$ can be represented by a word $a_0, b_0 \dots a_k, b_k, c$ where $a_1, \dots, a_k \in R_0 \setminus \{\mathrm{id}\}$ and $b_0, \dots, b_{k-1} \in R_1 \setminus \{\mathrm{id}\}$, $a_0 \in R_0$, and $c \in H$. Every such word is uniquely determined and we call it (R_1, R_2)-reduced word for g.*

Next, we define the *Bass-Serre tree* corresponding to an amalgamated free product. That has its origin in Bass-Serre theory. See [Ser03]. We use the following combinatorial definition.

Definition 2.58 (Definition of the Bass-Serre tree associated to an amalgamated free product)**.** The Bass-Serre tree T associated to an amalgamated free product $G = G_0 *_H G_1$ is a tree T with vertex set

$$V(T) \coloneqq \{v_{gG_i} \mid gG_i \in G/G_i, i \in \{0,1\}\}$$

and edge set

$$E(T) \coloneqq \{e_{gH} \mid gH \in G/H\} \text{ where } e_{gH} \coloneqq \{v_{gG_0}, v_{gG_1}\}.$$

Let T be the Bass-Serre tree of $G = G_0 *_H G_1$. We define every edge to be isometric to the interval $[0, 1]$. This induces a metric on T in which every edge has length one. Let v_{gG_i}, $i \in \{0, 1\}$, $g \in G$ be a vertex of T and let $e_{gH} = \{v_{gG_0}, v_{gG_1}\}$ be an edge of T. We say that v_{gG_i} has *label* gG_i and that e_{gH} has *label* gH.

G acts isometrically on T by left multiplication. The edge whose vertices are labeled with G_0 and G_1 is a fundamental domain for this action. The group H is the stabilizer of this edge. The groups G_0 and G_1 are the stabilizer of the endpoints of this edge, i.e. of the vertices with label G_0 and G_1 respectively.

In the following, the cosets of H in G play an important role. Hence, we slightly vary the definition of the Bass-Serre tree.

Definition 2.59 (Extended Bass-Serre tree)**.** The *extended Bass-Serre tree* $\mathcal{T}_{ext}$ associated to an amalgamated free product $G = G_0 *_H G_1$ is obtained from the Bass-Serre tree T associated to $G = G_0 *_H G_1$ as defined in 2.58 by adding a vertex v_{gH} to the midpoint of every edge $\{v_{gG_0}, v_{gG_1}\}$ of T.

We metrisize $\mathcal{T}_{\text{ext}}$ by giving every edge length $\frac{1}{2}$, i.e. we define every edge to be isometric to $[0, \frac{1}{2}]$. The Bass-Serre tree T and the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$ are isometric via the isometry which maps for every $g \in G$ the edge $\{v_{gG_0}, v_{gG_1}\}$ in T onto the 2-path $v_{gG_0}, v_{gH}, v_{gG_1}$ in $\mathcal{T}_{\text{ext}}$. Accordingly, G acts isometrically on $\mathcal{T}_{\text{ext}}$ by left multiplication. The path v_{G_0}, v_H, v_{G_1} is a fundamental domain for this action. The group H is the stabilizer of this path. G_0 and G_1 are the stabilizers of v_{G_0} and v_{G_1} respectively. For every $gH \in G/H$ we denote the path $v_{gG_0}, v_{gH}, v_{gG_1}$ in $\mathcal{T}_{\text{ext}}$ by P_{gH}.

We finish this section with the definition of tree of groups and spaces.

Definition 2.60. (Tree of groups [Ser03, Chapter 4, Def.8]) Let G be a group. A *tree of groups* $\mathbb{T}_G$ over a group G consists of a tree $\mathbb{T}_G$, a *vertex group* G_v for each vertex v of $\mathbb{T}_G$, and an *edge group* G_e for each edge $e = \{v_0, v_1\}$ in $\mathcal{T}$, together with two monomorphisms $\Phi_{v_0} : G_e \hookrightarrow G_{v_0}$ and $\Phi_{v_1} : G_e \hookrightarrow G_{v_1}$.

Every tree of groups is associated to a *tree of spaces* that is obtained by exchanging every vertex group G_v with a *vertex space* X_v whose fundamental group is G_v and by exchanging every edge group G_e with an *edge space* X_e whose fundamental group is G_e.

Definition 2.61. (Tree of spaces [SW79, p155]) A *tree of spaces* $\mathbb{T}$ consists of a tree $\mathbb{T}$, a topological *vertex space* X_v for each vertex v of $\mathbb{T}$, and a topological *edge space* X_e for each edge $e = \{v_0, v_1\}$ in $\mathbb{T}$, together with two monomorphisms $\Phi_{v_0} : X_e \hookrightarrow X_{v_0}$ and $\Phi_{v_1} : X_e \hookrightarrow X_{v_1}$.

We remark that Scott and Wall don't demand that Φ_0 and Φ_1 are injective. For simplicity, we request Φ_0 and Φ_1 to be injective.

Definition 2.62. (Total space associated to a tree of spaces [SW79, p155]) Let $\mathbb{T}$ be a tree of spaces and

$$X := \left[\bigsqcup_{v \in V(\mathbb{T})} X_v \right] \sqcup \left[\bigsqcup_{\{v_0, v_1\} \in E(\mathbb{T})} [0,1] \times X_{\{v_0,v_1\}} \right]$$

The *total space* $X_{\mathbb{T}}$ associated to $\mathbb{T}$ is the quotient of X by the equivalence relation generated by

$$\bigcup_{\{v_0,v_1\} \in E(\mathbb{T})} [(0,x) \sim \Phi_0(x), (1,x) \sim \Phi_1(x) \ \forall x \in X_{\{v_0,v_1\}}].$$

Let $\mathbb{T}$ be a tree of spaces and $\pi(X_{\mathbb{T}})$ the fundamental group of the total space $X_{\mathbb{T}}$ associated to $\mathbb{T}$. Then $\mathbb{T}$ has an associated tree of groups $\mathbb{T}_{\pi(X_{\mathbb{T}})}$ by exchanging the vertex spaces and edge spaces with the fundamental groups of the vertex spaces and edge spaces. This way, the underlying trees of $\mathbb{T}$ and $\mathbb{T}_{\pi(X_{\mathbb{T}})}$ coincide. The *fundamental group* $G_{\mathbb{T}}$ of the tree of groups $\mathbb{T}_{\pi(X_{\mathbb{T}})}$ is defined to be the fundamental group of the total space $X_{\mathbb{T}}$. Scott and Wall show that the fundamental group $\pi(X_{\mathbb{T}})$ does not depend on the choice of the edge spaces and vertex spaces. Thus, every tree of group $\mathbb{T}_G$ has a fundamental group that is defined as the fundamental group of the total space of an associated tree of spaces. The fundamental group of a tree of groups is obtained by an iteration of amalgamated free products and coincides with the direct limit of its vertex and edge groups. See section 1.1 in [Ser03] for a Definition of direct limits and section 4.4 in [Ser03] for more details about trees of groups. We say that a group *splits as a tree of groups* if it is the fundamental group of a tree of groups.

3 Boundaries of CAT(0) spaces with block decomposition

In this chapter, we examine *CAT(0) spaces with block decompositions with thin and thick walls* and introduce a language for them. We investigate their properties, in particular itineraries of geodesic rays. Block decompositions with *thin walls* were defined by Mooney in [Moo10] as *CAT(0) spaces with block structure* and further studied by Ben-Zvi [BZ19] and Ben-Zvi and Kropholler [BZK19]. Roughly speaking, a CAT(0) space with block decomposition consists of convex subsets, called blocks, that are glued along walls in a certain way. Bridson and Haefliger also studied boundaries of spaces that are obtained by the gluing of blocks. See Section 11 of Chapter II in [BH99]. But unlike the mentioned authors above, Bridson and Haefliger thickened every wall A between two blocks to a thick wall $[0,1] \times A$. The two sides $\{0\} \times A$ and $\{1\} \times A$ of such a thick wall each are glued to one of the two blocks adjacent to the wall. Bridson and Haefliger call these thick walls *tubes*. Using tubes redounds to several advantages. The most important observation is that blocks don't intersect each other. Even better, the ϵ-neighborhoods of two blocks don't intersect each other if ϵ is less than $\frac{1}{2}$. Hence, two blocks have a minimum distance to each other, independently of the behavior of the blocks itself. Thus, we obtain a tree when we shrink blocks to points and shrink walls to intervals of length one. This is not the case when walls are not thickened. Indeed, in the non-thickened case, walls would be contained in more than one block, and so all blocks would be shrunk to one single point. This is the reason why it is possible to define itineraries of geodesic rays in spaces with thickened walls in a more natural way than in spaces where we don't thicken the walls. Motivated by this, we introduce CAT(0) spaces that have a block decomposition with *thick* walls. Recently, Ben-Zvi studied the spaces that arise from the construction of Bridson and Haefliger independently to this thesis. Ben-Zvi used the projection described above to define block decompositions with thin walls for such spaces in Example 6.8 in [BZ19]. The block decompositions in Example 6.8 in [BZ19] occur in our considerations as well.

This chapter is structured as follows. In the first section, we recall the original definition of a CAT(0) space with block structure. We say that such a space has a *block decomposition with thin walls*. We define the itinerary in such spaces X by means of a map of X to the tree mentioned above. We explain how this map and our definition of itineraries is related to the original definition of Croke and Kleiner in [CK00].

Afterwards, we define *block decompositions with thick walls*. We define itineraries of geodesic rays by means of a projection of the space to the tree mentioned above. Unlike in the case with thin walls, this projection map behaves very naturally. In the third section, we introduce a common language for CAT(0) spaces with thick or thin walls. We say that a space has a *block decomposition* if it has a block decomposition with thin or every wall is thick walls. We examine the properties of itineraries in such spaces. We consider itineraries of asymptotic geodesic rays in particular. Finally, we define itineraries of boundary points.

Section 3.4 is inspired by the study of cutpoints of Bowditch [Bow98a], Lemma 7 in Section 1.7 of [CK00] and by the example of Charney and Sultan that can be found in Section 4.2 of [CS15]. Recall that we refer to this example as the Cycle-Join-Example. We examine boundary points of walls (see Corollary 3.45). We show that the boundary of a wall behaves like a cutset of a topological space. A similar observation was recently made independently by Ben-Zvi and Kropholler in Lemma 3.1 of [BZK19]. Ben-Zvi and Kropholler were interested in path-connectedness; we study connectedness. Different to Lemma 3.1 in [BZK19], Corollary 3.45 can be applied to contracting boundaries.

Section 3.6 has its origin in section 11 of Chapter II in [BH99]. In this section, Bridson and Haefliger consider the question of when an amalgamated free product G of two CAT(0) groups G_0 and G_1 along a CAT(0) group H is itself a CAT(0) group. They prove that this is not always the case but that there are certain cases where such groups are CAT(0). They explain how to construct spaces on which certain amalgamated free products of CAT(0) groups along CAT(0) groups act geometrically. We recall this construction and observe that the obtained spaces have block decompositions with thick walls. An analog observation was mentioned in Example 6.8 in [BZ19]. We vary the construction and obtain block decompositions with thin walls if certain added conditions are satisfied. At the end of the first section, we list examples where the described construction can be done. In other words, we list examples of amalgamated free products of CAT(0) groups that act on a CAT(0) space with block decomposition geometrically.

This chapter is based on the research of the authors mentioned above and inspired by the Cycle-Join-Example of Charney and Sultan.

3.1 Block decompositions of CAT(0) spaces with thin walls

In this section, we introduce CAT(0) spaces that have a *decomposition with thin walls* and define *itineraries of geodesic rays* in such spaces. We explain how our definitions are related to the corresponding original concepts introduced of Croke and Kleiner in [CK00] and Mooney in [Moo10].

Mooney [Moo10] defines *block structures* on a CAT(0) space X. Motivated by the terminology of Definition 2.5 of Ben-Zvi and Kropholler in [BZK19], we call such a block structure a *block decomposition with thin walls* of X.

Definition 3.1 (Definition 3.1 in [Moo10])**.** Let X be a CAT(0) space and $\mathcal{B}$ a collection of closed, convex subspaces of X, called *blocks*. Let $\mathcal{A}$ be the collection of nontrivial intersections of blocks in $\mathcal{B}$, called *thin walls*. The ordered pair $(\mathcal{B}, \mathcal{A})$ is a *block decomposition with thin walls* of X if the following conditions are satisfied:

a) covering condition: $X = \bigcup_{B \in \mathcal{B}} B$,

b) parity condition: every block has a parity $(+)$ or $(-)$ such that two blocks intersect only if they have opposite parity,

c) ϵ-condition: there is an $\epsilon > 0$ such that two blocks intersect if and only if their ϵ-neighborhoods intersect.

We denote the set of blocks of parity $(-)$ by $\mathcal{B}^-$ and the blocks of parity $(+)$ by $\mathcal{B}^+$.

Ben-Zvi and Kropholler request in [BZK19] that each block intersects at least two other blocks. Because X is assumed to be a CAT(0) space, this additional condition implies that X does not have finite diameter. A block decomposition of X consists of blocks and thin walls. A thin wall is a nontrivial intersection of blocks. We observe that every thin wall is convex as an intersection of convex sets. The parity condition implies that only blocks of opposite parity intersect. In particular, every thin wall is contained in precisely two blocks. Because of that, every thin wall is closed as an intersection of finitely many closed sets. We say that a thin wall and a block are *adjacent* if they have a nonempty intersection. We say that two blocks B_0 and B_1 *share a thin wall* W if $W \cap B_0 \neq \emptyset$ and $W \cap B_1 \neq \emptyset$. Two blocks are *adjacent* if they share a wall. The intersection of the ϵ-condition implies that the ϵ-neighborhoods of two blocks don't overlap. In particular, every block has a lot of points that are contained in exactly one block.

We study the nerve of $\mathcal{B}$ as Mooney in [Moo10], The *nerve* of a collection $\mathcal{C}$ of sets is an (abstract) simplicial complex obtained as follows. We add for every set C in $\mathcal{C}$ a vertex v_C. A set of vertices M builds a simplex if $\bigcap_{v_C \in M} C \neq \emptyset$. For every block $B \in \mathcal{B}$, the nerve of $\mathcal{B}$ contains a vertex v_B. Two vertices are connected by an edge if and only if the corresponding blocks have nonempty intersection. The ϵ-condition together with the CAT(0) property of X implies that the nerve of $\mathcal{B}$ is a tree. This can be proven in the same way as Croke and Kleiner prove it for their example in [CK00]. We extend this tree and add a vertex v_A for every wall A that is adjacent to two blocks B_0 and B_1. We delete the edge connecting v_{B_0} and v_{B_1} and connect v_{B_0} and v_{B_1} with v_A respectively. In other words, we take the barycentric subdivision of the nerve of $\mathcal{B}$ and label every new vertex v with the wall that is adjacent to the two blocks occurring in the labels of the two vertices adjacent to v. We denote this tree, which is associated to $(\mathcal{B}, \mathcal{A})$, by $\mathcal{T}_{\mathcal{B},\mathcal{A}}$.

We recall how Croke and Kleiner define itineraries of geodesic rays or segments. They say that a geodesic ray or segment *enters* a block B if it passes through a point of B that is not contained in any wall. Croke and Kleiner define itineraries just for geodesic rays that start in a point that is not contained in any wall. They define the itinerary of a geodesic ray γ (that does not start in a wall) as the list $[B_1, B_2, \ldots]$ where B_i is the i^{th} block that γ enters. We remark that if γ enters a block B and $\gamma(t)$ is a point in B that is not contained in any other block, then there is $t' < t$ such that $\gamma(t')$ lies in B and is not contained in any other block. This is the case because of the ϵ-condition and because thin walls are closed sets. We vary the definition of itineraries. We define itineraries also for geodesic rays that start in a wall. If this happens, we add a vertex of the corresponding wall to the itinerary. Because walls play an important role, we include all the walls between two consecutive blocks to the itinerary. Furthermore, we define itineraries differently and use a projection for the definition. This will help us in transferring the concept of itineraries to the setting of block decompositions with thick walls in Section 3.2.

Every wall is contained in exactly two blocks by the parity condition. The ϵ-neighborhood of every wall A does not contain any other wall because of the ϵ-condition. Hence, the following natural projection from X to $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is well-defined.

Definition 3.2 (natural projection for block decompositions with thin walls)**.** Let X be a CAT(0) space that has a block decomposition with thin walls $(\mathcal{B}, \mathcal{A})$. Let $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ be the associated tree to $(\mathcal{B}, \mathcal{A})$. Let $p_{(\mathcal{B},\mathcal{A})} : X \to \mathcal{T}_{\mathcal{B},\mathcal{A}}$ be the projection that maps a point $x \in X \setminus \bigcup_{A \in \mathcal{A}} A$ to the vertex of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ corresponding to the block in $\mathcal{B}$ containing x and a point x in a wall $A \in \mathcal{A}$ to the vertex corresponding to the wall in $\mathcal{A}$ containing x.

Recall that a *generalized curve* is a continuous map c sending an interval $[0, b] \subset \mathbb{R}$ or a set $[0, \infty) \subset \mathbb{R}$ to X. We say that $c(0)$ is the point at which c *starts*. We denote the map c as well as its image under c by the letter c. We insert the domain of c for clarifying that we speak of the image of c and not of the map c only if the meaning of c is not clear. The image of a curve in X under the natural projection is a set of vertices in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. We defined it this way because every point in a wall is contained in two blocks. We use the natural projection for defining itineraries of generalized curves.

Definition 3.3 (itineraries in block decompositions with thin walls)**.** Let X be a CAT(0) space that has a block decomposition $(\mathcal{B}, \mathcal{A})$ with thin walls and c a generalized curve in X. The itinerary $I(c)$ is the following (possibly infinite) subgraph of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. A vertex v of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is contained in $I(c)$ if there exists $t \in \mathbb{R}$ such that $p_{(\mathcal{B},\mathcal{A})}(c(t)) = v$ and v

a) corresponds to a block or

b) corresponds to a wall A and the vertices adjacent to v are contained in the image of c under $p_{(\mathcal{B},\mathcal{A})}$. In other words, there are t_0 and $t_1 \in \mathbb{R}$ such that c intersects the blocks B_0 and B_1 adjacent to A at times t_0 and $t_1 \in \mathbb{R}$ respectively such that $p_{(\mathcal{B},\mathcal{A})}(c(t_0)) = v_{B_0}$ and $p_{(\mathcal{B},\mathcal{A})}(c(t_1)) = v_{B_1}$.

c) corresponds to a wall in which c starts.

Two vertices in $I(c)$ are connected by an edge if and only if they are adjacent in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$.

By definition, the itinerary $I(c)$ of a generalized curve is an induced subgraph of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. It is essential for the definition of itineraries that the ϵ-condition of the associated block decomposition is satisfied. Croke and Kleiner show by means of the ϵ-condition that $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is a tree. Furthermore, it might happen without the ϵ-condition that every point of X is contained in a wall. Then the image of the projection $p_{(\mathcal{B},\mathcal{A})}$ would not contain any vertex that corresponds to a block.

We say that a generalized curve *enters* a block B if its itinerary contains a vertex corresponding to B. We observe that this is equivalent to the definition of Croke and Kleiner. A generalized curve *enters* a block B if and only if c passes through a point of B that is not contained in any wall. We prove like Croke and Kleiner in Lemma 2 of [CK00] that the itinerary of a generalized curve is a subtree of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. Furthermore, the itinerary of a geodesic segment or ray is a path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. We recall their arguments for completeness.

Lemma 3.4. *The itinerary of a generalized curve in a CAT(0) space with a block decomposition with thin walls $(\mathcal{B}, \mathcal{A})$ is an induced subtree T of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. If c starts in a wall A, then v_A is the only vertex of degree one in $I(c)$ corresponding to a wall. If c does not start in a wall, every vertex of degree one in $I(c)$ corresponds to a block. If c is a geodesic segment or geodesic ray, then its itinerary is a (possibly) infinite graph-theoretical path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$.*

Proof. We follow the proof of Lemma 2 of Croke and Kleiner in [CK00]. Let c be a generalized curve in a CAT(0) space X that has a block decomposition with thin walls. We show that $I(c)$ is a connected subgraph of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. As $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is a tree, it follows that $I(c)$ is a subtree of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. If c is contained in the wall A, the itinerary consists of one vertex and the claim follows directly. Hence, we assume that c is not contained in a wall. It follows by the ϵ-condition that c enters at least one block B. We assume that B is the first block that c enters. Then there is a point $p = c(t)$ that is contained in B but not in any wall. If the whole generalized curve c is contained in B the claim follows. Let's assume that c is not contained in B. We observe that the topological frontier of any block is contained in the union of all walls that are contained in B. Hence, c passes through a point that is contained in a wall A of B and reaches a block B' afterwards that is adjacent to A. It follows from the ϵ-condition that c reaches a point of B' that is not contained in any wall, i.e., that c enters B'. It follows that $I(c)$ contains the vertices v_{B_0}, v_{B_1} and v_A. The two edges $\{v_B, v_A\}$ and $\{v_A, v_{B'}\}$ are contained in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. Hence, both edges are contained in $I(c)$. If c ends in B' we are done. Otherwise, we repeat the same argument for all two blocks that are consecutively entered by c and conclude that $I(c)$ contains a corresponding 2-path for each such two blocks. Because c is continuous, it follows that $I(c)$ is a subtree of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$.

Let A be a wall corresponding to a vertex in $I(c)$ in which c does not start. Then $I(c)$ contains two vertices v_{B_0} and v_{B_1} corresponding to the two blocks B_0 and B_1 that are adjacent to A. So, v_A has degree at least 2. Hence, a vertex corresponding to a wall A in $I(c)$ does not have degree one, if c does not start in A. If c starts in A, then the degree of v_A is one by definition.

If c is a geodesic segment or ray, then the intersection of c and a wall or a block is always a geodesic segment because every wall and every block is convex. Hence, c does not return to a block or a wall after it has left it at some point. It follows that $I(c)$ is a path. □

We summarize that the itinerary of a generalized curve can be obtained by the following procedure. First, we add all vertices according to blocks that c enters. If c enters a block B_0 and afterwards a block B_1, c passes through the wall A that is adjacent to B_0 and B_1. Then we add v_A and the edges $\{v_{B_0}, v_A\}$ and $\{v_A, v_{B_1}\}$. If c starts in a wall, we add to the itinerary of c the edge that connects the vertex corresponding to this wall to the first block that c enters.

We compare our definitions with the definitions of Croke and Kleiner. Recall that they define the itinerary of a geodesic ray γ (that does not start in a wall) as the list $L = [B_1, B_2, \dots]$ where B_i is the i^{th} block that γ enters. Let P be the path that corresponds to the itinerary according to our definition. We obtain the list L from P if we list all the blocks that occur in P as labels of vertices in the order given by the path P. On the other hand, the path P is obtained from the list L as follows: If γ starts in a wall, we add the wall as the first element to the list. For every two consecutive blocks in the list, we add the wall that is contained in the two blocks. Afterwards, we interpret this list as a path. We could define itineraries as a path in the nerve of $\mathcal{B}$ that does not contain vertices for walls. In the following sections, we will see that walls play a crucial role in our considerations. Hence, we include vertices for walls in our definition of itineraries.

We say that c *touches* a wall A if c interests A but v_A is not contained in the itinerary of c. These are the thin walls that are ignored in the itinerary of a geodesic ray.

Lemma 3.5. *Let C be an interval in $\mathbb{R}$ or $[0, \infty)$. Let X be CAT(0) space with a block decomposition with thin walls $(\mathcal{B}, \mathcal{A})$ and $c : C \to X$, a generalized curve that is not contained in a thin wall. Let A be a wall in which c does not start. The generalized curve c touches A if and only if for every $t \in C$ with $c(t) \in A$ holds one of the following*

a) $c(t')$ is contained in a side of A for all $t' \in C$ with $t' \geq t$,

b) there are a block $B \in \mathcal{B}$, an interval $(a, b) \subseteq C$ containing t and times t_0, $t_1 \in (a, b)$, $t_0 < t < t_1$ such that $c((a, b)) \subseteq B$ and $c(t_0) \notin A$, $c(t_1) \notin A$.

Proof. We assume that there exists $t \in \mathbb{R}$ such that $c(t) \in A$ and c does not touch A at time t. Then there is no $t' \in \mathbb{R}$ such that $c(t') \in A$ for all $t' \geq t$. Furthermore, there is no block $B \in \mathcal{B}$ and no interval $(a, b) \subseteq \mathbb{R}$ containing t and times t_0, $t_1 \in (a, b)$, $t_0 < t < t_1$ such that $c((a, b)) \subseteq B$ and $c(t_0) \notin A$, $c(t_1) \notin A$. Then t_0 and t_1 are contained in distinct blocks B_0 and B_1 for all choices of such intervals and times. As c is continuous and by definition of the natural projection, A is adjacent to B_0 and B_1 and both v_{B_0} and v_{B_1} are contained in the image of c under $p_{(\mathcal{B},\mathcal{A})}$. This is exactly then the case when v_A is contained in the itinerary of c.

On the other hand, let v_A be a vertex corresponding to a wall A that is contained in the itinerary of a generalized curve c. Then either c starts in A or A is contained in two distinct blocks B_0 and B_1 such that both v_{B_0} and v_{B_1} are contained in the image of c under $p_{(\mathcal{B},\mathcal{A})}$. By assumption, the first case does not occur. We consider the second case. As c is continuous and because of the ϵ-condition, there are $t_0, t, t_1 \in R$, $t_0 < t < t_1$ such that $c(t) \in A$, $c([t_0, t]) \subseteq B_0$ and $c([t, t_1]) \subseteq B_1$. It follows that c does not touch A. $\square$

3.2 Block decompositions of CAT(0) spaces with thick walls

In this section we define *block decompositions with thick walls* of CAT(0) spaces. We transfer the concepts of block decompositions with thin walls to this setting. In particular, we transfer the definition of itineraries of geodesic rays. Itineraries in CAT(0) spaces that have a block decomposition with thick walls behave more naturally than itineraries in CAT(0) spaces with thin walls. This is a reason why it might be easier to work in CAT(0) space having a block decomposition with thick walls instead of working in spaces that have only a block decomposition with thin walls. The concept of thick walls coincides with the concept of *tubes* as described in [BH99]. This section is inspired by section 11 of part II in [BH99].

Definition 3.6. Let X be a CAT(0) space and $\mathcal{B}$ a collection of closed, convex and pairwise disjoint subspaces of X called *blocks*. Let $\mathcal{A}$ be another collection of closed, convex subspaces of X, called *thick walls*, that each are a direct product of $[0,1]$ with a nonempty CAT(0) space. The ordered pair $(\mathcal{B},\mathcal{A})$ is a *block decomposition with thick walls* of X if the following conditions are satisfied:

a) covering condition: $X = \bigcup_{B\in\mathcal{B}} B \cup \bigcup_{A\in\mathcal{A}} A$,

b) parity condition: every block has a parity $(+)$ or $(-)$ such that two blocks intersect a thick wall simultaneously only if they have opposite parity,

c) wall condition: the *sides* $\{0\}\times Y$ and $\{1\}\times Y$ of every thick wall $[0,1]\times Y$ each are contained in a block. The intersection of two thick walls is empty or contained in one of their sides. The intersection of a thick wall and a block is empty or a side of a thick wall.

We denote the set of blocks of parity $(-)$ by $\mathcal{B}^-$ and the blocks of parity $(+)$ by $\mathcal{B}^+$.

Because every thick wall is convex, every side of a thick wall is convex too. Let $A=[0,1]\times Y$ be a thick wall in $\mathcal{A}$. The two sides of A each are contained in exactly one block B because of the wall condition and because every two blocks have an empty intersection. In particular, every side of $A=[0,1]\times Y$ is closed as the intersection of two closed sets. The *inside* of A is the set $(0,1)\times Y$. Each point inside A is an interior point of A. By the parity condition, the two blocks that intersect A have distinct parity. We assume without loss of generality that the block with parity $(-)$ intersects the side $\{0\}\times Y$ of A and that the other block of parity $(+)$ intersects the side $\{1\}\times Y$. Let S be a side of A. Arbitrarily many other thick walls may intersect S. Each of these thick walls has nonempty intersection with a block different to B. The parity condition implies that the parity of all these blocks is the parity opposite to B. We observe that the ϵ-neighborhoods of two blocks don't overlap for every $\epsilon < \frac{1}{2}$. Hence, the ϵ-condition of block decompositions with thin walls in Definition 3.1 is satisfied. We say that a thick wall and a block are *adjacent* if they have nonempty intersection. We say that two blocks B_0 and B_1 *share a thick wall* W if $W\cap B_0\neq\emptyset$ and $W\cap B_1\neq\emptyset$. Two blocks are *adjacent* if they share a wall. Unlike the situation of a block decomposition with thin walls, it is possible that walls overlap and that every point of a block is contained in a wall.

The following lemmas show how decompositions with thick and thin walls are related to each other. Let $(\mathcal{B}, \mathcal{A})$ be a block decomposition with thin walls of a CAT(0) space X. Associated to X is a (up to isometry uniquely determined) space $\bar{X}$ that is obtained by thickening every thin wall $A \in \mathcal{A}$ to the thick wall $\bar{A} := [0,1] \times A$. More precisely $\bar{X}$ is obtained from the set of blocks $\mathcal{B}$ in the following way. Let $\bar{\mathcal{B}}$ be a set of pairwise disjoint copies of all the blocks in $\mathcal{B}$. Let $\bar{B}_0$ and $\bar{B}_1$ be copies of two blocks B_0 and B_1 in $\mathcal{B}$ that intersect in a thin wall $A \in \mathcal{A}$. We observe that $\bar{B}_0$ and $\bar{B}_1$ each contain an isometrically embedded copy of A. For every such two blocks $\bar{B}_0$ and $\bar{B}_1$, we take a thickened wall $\bar{A} := [0,1] \times A$ and glue the two sides $\{0\} \times A$ and $\{1\} \times A$ of $\bar{A}$ to $\bar{B}_0$ and $\bar{B}_1$ along the two isometrically embedded copies of A in B_0 and B_1. The construction is unique up to the choice of copies. Hence, the space $\bar{X}$ is unique up to isometry.

Lemma 3.7. *Let $(\mathcal{B}, \mathcal{A})$ be a block decomposition with thin walls of a CAT(0) space X. Let $\bar{X}$ be a space obtained by thickening every thin wall $A \in \mathcal{A}$ to the thick wall $\bar{A} := [0,1] \times A$. Then $(\bar{\mathcal{B}}, \{\bar{A} \mid A \in \mathcal{A}\})$ is a block decomposition with thick walls of $\bar{X}$.*

Proof. Because $(\mathcal{B}, \mathcal{A})$ is a block decomposition of a CAT(0) space with thin walls, it has the properties given in Definition 3.1. Hence, $(\bar{\mathcal{B}}, \{\bar{A} \mid A \in \mathcal{A}\})$ is a block decomposition of $\bar{X}$ with thick walls, i.e., it satisfies all conditions of Definition 3.6. □

The converse procedure can have a very different behavior. If we have given a block decomposition with thick walls and shrink all thick walls to thin walls, the obtained space can have a completely different structure than the space where we started. We consider the following example: For every $i \in \mathbb{N}$, let B_i be a block that is isomorphic to the interval $[0,1] \subset \mathbb{R}$. Let the parity of B_i be $(+)$, if i is even and $(-)$ otherwise. For every i, let A_i be a thick wall isomorphic to the square $= [0,1] \times [0,1]$. We glue every interval B_i and B_{i+1} to two opposite sides of A_i such that both $A_i \cap B_i$ and $A_i \cap B_{i+1}$ are a side of the square A_i lying opposite to each other. The obtained space is isometric to $\mathbb{R} \times [0,1]$. It is a CAT(0) space with a block decomposition with thick walls. If we shrink all the thick walls to thin walls, the obtained space is the interval $[0,1]$. We interpret $[0,1]$ as a CAT(0) space with block decomposition that consists of a single block. So, we started with a space that had infinitely many blocks and ended with a space consisting of only one block. For sure, X and $[0,1]$ are not quasi-isometric.

Though a shrinking of thick walls might destroy the structure of a space, every CAT(0) space having a block decomposition with thick walls possesses an associated block decomposition with thin walls. Ben-Zvi mentioned the following block decomposition with thin walls for spaces that arise from the Equivalent Gluing Theorem of Bridson and Haefliger [BH99] in Example 6.8 in [BZ19]. Thereby, Ben-Zvi does not consider block decompositions with thick walls.

Lemma 3.8. *Every block decomposition with thick walls of a CAT(0) space X has an associated block decomposition with thin walls of X.*

Proof. Let $(\mathcal{B}, \mathcal{A})$ be a block decomposition with thick walls of a CAT(0) space X. Let B be a block of $\mathcal{B}$. Let M be the set of thick walls in $\mathcal{A}$ that are incident to B. Let $A = [0,1] \times Y$ be a thick wall in M. The intersection of B with A is a side $\{t\} \times Y$, $t \in \{0,1\}$ of A. If $t = 0$, let A' be the set $[0, \frac{1}{2}] \times Y$. Otherwise, let A' be the set $[\frac{1}{2}, 1] \times Y$. Let $\bar{B}$ be the union of B with the set $\bigcup_{A \in M} A'$. Let $\bar{\mathcal{B}} = \{\bar{B} \mid B \in \mathcal{B}\}$. Let the parity of every block $\bar{B} \in \bar{\mathcal{B}}$ be the parity of the corresponding block $B \in \mathcal{B}$. Let $\bar{\mathcal{A}}$ be the collection of nontrivial intersections of blocks in $\bar{\mathcal{B}}$. We prove that $(\bar{\mathcal{B}}, \bar{\mathcal{A}})$ is a block decomposition with thin walls. First, we observe that every $\bar{B} \in \bar{\mathcal{B}}$ is a closed, convex subspaces of X. By construction, $X = \bigcup_{\bar{B} \in \bar{\mathcal{B}}} \bar{B}$. By the wall- and parity condition, every block in $\bar{\mathcal{B}}$ has $(+)$ or $(-)$ parity such that two blocks intersect only if they have opposite parity. By the wall-condition, every two thin walls in $\bar{\mathcal{A}}$ have distance at least 1 to each other. Hence, two blocks intersect if and only if their $\frac{1}{2}$-neighborhoods intersect. As all conditions of Definition 3.1 are satisfied, $(\bar{\mathcal{B}}, \bar{\mathcal{A}})$ is a block decomposition with thin walls. □

Let $(\mathcal{B}, \mathcal{A})$ be a block decomposition with thick walls of a CAT(0) space X. We define an associated graph $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ as follows. The vertex set of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is the set $\{v_B \mid B \in \mathcal{B}\} \cup \{v_A \mid A \in \mathcal{A}\}$. Two vertices are connected by an edge if one vertex corresponds to a thick wall A and the other one to a block B such that $A \cap B \neq \emptyset$. By the parity condition, the graph is bipartite. The graph $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is not the nerve of $\mathcal{B} \cup \mathcal{A}$. Indeed, arbitrarily many thick walls may intersect each other but no two vertices corresponding to a wall are adjacent in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$.

Lemma 3.9. *The graph associated to a block decomposition with thick walls of a CAT(0) space X is isomorphic to the barycentric subdivision of the nerve of the block decomposition with thin walls associated to X. In particular, it is a tree.*

Proof. Let $(\mathcal{B}, \mathcal{A})$ be a block decomposition with thick walls of a CAT(0) space X. Let $(\bar{\mathcal{B}}, \bar{\mathcal{A}})$ be the block decomposition with thin walls associated to $(\mathcal{B}, \mathcal{A})$. Let $\mathcal{T}_{\bar{\mathcal{B}},\bar{\mathcal{A}}}$ be the associated tree. Recall that $\bar{\mathcal{B}}$ contains a block $\bar{B}$ for every block $B \in \mathcal{B}$ and that $\bar{\mathcal{A}}$ contains a thin wall $\bar{w}$ for every thick wall $w \in \mathcal{A}$. Thereby two blocks $\bar{B}_0$ and $\bar{B}_1$ intersect in a thin wall $\bar{A}$ if and only if the blocks B_0 and B_1 share the thick wall A. In $\mathcal{T}_{\bar{\mathcal{B}},\bar{\mathcal{A}}}$, a vertex corresponding to a block $\bar{B}$ is connected to a vertex corresponding to a thin wall $\bar{A}$ if and only if $\bar{A} \cap \bar{B} \neq \emptyset$. This is precisely the case when $B \cap A \neq \emptyset$. We exchange every label $\bar{B} \in \mathcal{B}$ of a vertex with label B and every label $\bar{A} \in \bar{\mathcal{A}}$ with label A and obtain the tree $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. □

We observe that $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ can be obtained from X by shrinking the blocks to vertices, shrinking the thick walls to edges, and taking the barycentric subdivision of the obtained tree. Accordingly, there is a nice projection map from X to $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. For defining this projection, we interpret $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ as a metric space by giving every edge the length $\frac{1}{2}$. Ben-Zvi [BZ19, Example 6.8] mentioned the following projection for spaces that arise from the Equivalent Gluing Theorem of Bridson and Haefliger [BH99].

Definition 3.10 (natural projection for block decompositions with thick walls)**.** Let X be a CAT(0) space with a block decomposition with thick walls $(\mathcal{B}, \mathcal{A})$ and $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ the associated tree to $(\mathcal{B}, \mathcal{A})$. We define a projection $p_{(\mathcal{B},\mathcal{A})} : X \to \mathcal{T}_{\mathcal{B},\mathcal{A}}$ as follows. If $x \in X$ is contained in a block B, $p_{(\mathcal{B},\mathcal{A})}(x)$ is the vertex v_B of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. Otherwise, x is contained on the inside of a thick wall A incident to two blocks B_0 and B_1. Then x is mapped to the point on the 2-path v_{B_0}, v_A, v_{B_1} in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ such that the distance of $p_{(\mathcal{B},\mathcal{A})}(x)$ to v_{B_i} in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ coincides with the distance of x to B_i in X, $i \in \{0, 1\}$.

The projection map is well-defined because of the properties that every block decomposition with thick walls has by definition. This projection map is more natural than the projection map in the case of a block decomposition with thin walls. Here, generalized curves in X are mapped to generalized curves in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. In a block decomposition with thin walls, the associated projection sends a generalized curve to a set of vertices.

As before, we mean by a generalized curve a continuous map c sending an interval $[0, b] \subset \mathbb{R}$ or a set $[0, \infty) \subset \mathbb{R}$ to X. We denote the map c as well as its image under c by the letter c. Only if the meaning of c is not clear, we insert the domain of c to clarify that we speak of the image of c and not of the map c.

Definition 3.11 (General itinerary of a generalized curve for block decompositions with thick walls)**.** Let c be a (possibly infinite) generalized curve in a CAT(0) space that has a block decomposition $(\mathcal{B}, \mathcal{A})$ with thick walls. The *general itinerary* $\tilde{I}(c)$ of c is the image of c under the natural projection $p_{(\mathcal{B},\mathcal{A})}$ from X into the tree $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ associated to $(\mathcal{B}, \mathcal{A})$.

Let c be a generalized curve in X. We observe like in the case of block decompositions with thin walls that the general itinerary $\tilde{I}(c)$ of c describes how c runs through the blocks and walls of X. The general itinerary $\tilde{I}(\gamma)$ is not always a graph-theoretical subgraph of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. For example, if c starts on the inside of a wall, $\tilde{I}(c)$ starts in an interior point of an edge of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. If c is a geodesic segment contained in a wall, $\tilde{I}(c)$ is a geodesic segment in an edge of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. We vary the definition in such a way that we obtain a graph-theoretical path.

Definition 3.12 (itineraries in block decompositions with thick walls)**.** Let c be a generalized curve in a CAT(0) space with a block decomposition with thick walls $(\mathcal{B}, \mathcal{A})$ and associated tree $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. If c is contained in a wall A, its itinerary is the trivial graph-theoretical path v_A in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. Otherwise, $I(c)$ is the subgraph of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ obtained from the general itinerary $\tilde{I}(c)$ as follows. An edge $\{v_A, v_B\}$ is contained in $I(c)$ if and only if there exists $d \in (0, \frac{1}{2}]$ such that $\tilde{I}(c)$ contains all points in the edge $\{v_A, v_B\}$ that have distance at most d to v_B.

We observe that $I(c)$ contains all vertices and edges of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ that are contained in the image of c under the natural projection $p_{(\mathcal{B},\mathcal{A})}$. Let c be a generalized curve that starts on the inside of a wall A. Let B_0 and B_1 be the two corresponding blocks that share A. Let us assume that c leaves A and reaches a point in B_0 afterwards. Then $I(c)$ contains the edge that connects vertex v_A with v_{B_0} but c does not contain the edge that connects v_A with B_1. The analogous matter happens when we consider a finite generalized curve that ends on the inside of a wall. Note that $\tilde{I}(c)$ might contain points in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ that are not contained in $I(c)$ and vice versa. For obtaining $I(c)$ from $\tilde{I}(c)$ we delete some subcurves from $\tilde{I}(c)$ or make some subcurves longer in such a way that we obtain a graph that represents how c passes through the corresponding blocks and walls. We say like before that a generalized curve *enters* a block B if v_B is contained in its itinerary. Unlike in the situation of a block decomposition with thin walls, this is not equivalent to the situation that c passes through a point of B that is contained in no thick wall. Every point in a block may be contained in a thick wall. In the situation here, a generalized curve enters a block B if and only if c intersects B at some point. We say that a generalized curve enters a block at time t if $c(t) \in B$ and there exists $\epsilon > 0$ such that $c(t) \notin B$ for all $t' \in (t-\epsilon, t)$. We say analogously that a generalized curve enters a thick wall A if v_A is contained in the itinerary of c. Recall that $p_{(\mathcal{B},\mathcal{A})}$ projects every side of a thick wall to the vertex corresponding to the block containing it. Furthermore, an edge $\{v_A, v_B\}$ is contained in $I(c)$ if and only if there exists $d \in (0, \frac{1}{2}]$ such that $\tilde{I}(c)$ contains all points in the edge $\{v_A, v_B\}$ that have distance at most d to v_B. Hence, v_A is contained in the itinerary of c if and only if c passes through an interior point of the thick wall, i.e., through a point of A that is not contained in a side of A.

We observe as Croke and Kleiner in Lemma 2 of [CK00] that the itinerary of a generalized curve is an induced subtree of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ and that the itinerary of a geodesic segment or ray is a path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$.

Lemma 3.13. *The itinerary of a generalized curve in a CAT(0) space with a block decomposition with thick walls $(\mathcal{B}, \mathcal{A})$ is an induced subtree of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. If c is a geodesic segment or a geodesic ray, then its itinerary is a graph-theoretical path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$.*

Proof. We follow the proof of Lemma 2 of Croke and Kleiner in [CK00]. Let c be a generalized curve in a CAT(0) space X that has a block decomposition with thick walls. We show that $I(c)$ is a connected subgraph of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. As $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is a tree, it follows that $I(c)$ is an induced subtree of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. If c is contained in a wall A, the itinerary consists of one vertex and the claim follows directly. Hence, we assume that c is not contained in a wall. Then c passes through a point that is contained in a block B, i.e., c enters at least one block B. Let B be the first block that c enters. If the whole generalized curve c is contained in B then the itinerary of B consists of v_B and the claim follows. Let's assume that c is not contained in B. We observe that the topological frontier of any block is contained in the union of all sides of walls that are contained in B. Hence, c passes through a point that is contained in a side of a wall A that is adjacent to B and reaches an interior point of A afterwards. As c is continuous, there exists $d \in [0, \frac{1}{2}]$ such that $\tilde{I}(c)$ contains all points in the edge $\{v_A, v_B\}$ that have distance at most d to v_B. Hence, $I(c)$ contains the edge $\{v_A, v_B\}$. If c ends in B', we are done. Otherwise, we repeat the same argument for every block and adjacent wall that are consecutively entered by c and conclude that $I(c)$ contains an edge for each such pair. Because c is continuous, it follows that $I(c)$ is a subtree of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. If c is a geodesic segment or a geodesic ray, then the intersection of c and a wall or a block is always a geodesic segment because every wall and every block is convex. Hence, c does not enter a block or a wall twice. It follows that $I(c)$ is a path. □

We say that c *touches* a wall A if c intersects A but v_A is not contained in the itinerary of c. These are the thin walls that are ignored by itineraries of geodesic rays.

Lemma 3.14. *Let $C = [0, b]$ be an interval in $\mathbb{R}$ or $[0, \infty)$. Let X be a CAT(0) space with a block decomposition with thick walls $(\mathcal{B}, \mathcal{A})$ and $c : C \to X$, a generalized curve that is not contained in a thick wall. The generalized curve c touches a wall A if and only if every $t \in C$ with $c(t) \in A$ satisfies one of the following statements.*

a) $c(t')$ is contained in a side of A for all $t' \in C$ with $t' \geq t$.

b) For all $t' \in C$ with $t' \leq t$ we have that $c(t')$ is contained in a side of A,

c) There are a block $B \in \mathcal{B}$, an interval $(a, b) \subseteq C$ containing t and times t_0, $t_1 \in (a, b)$, $t_0 < t < t_1$ such that $c((a, b)) \subseteq B$ and $c(t_0) \notin A$, $c(t_1) \notin A$.

Proof. We assume that there exists $t \in \mathbb{R}$ such that $c(t) \in A$ there is $t' \geq \mathbb{R}$ such that $c(t')$ is not contained in a side of A and there is $\tilde{t} \leq t$ such that $c(\tilde{t})$ is not contained in a side of A. Furthermore, we assume that there is no block $B \in \mathcal{B}$ and no interval $(a, b) \subseteq \mathbb{R}$ containing t and times t_0, $t_1 \in (a, b)$, $t_0 < t < t_1$ such that $c((a, b)) \subseteq B$ and $c(t_0) \notin A$, $c(t_1) \notin A$. Then t_0 and t_1 are contained in distinct blocks B_0 and B_1 for all choices of such intervals and times. As c is continuous, A has nonempty intersection with B_0 and B_1 such that both v_{B_0} and v_{B_1} are contained in the image of c under $p_{(\mathcal{B},\mathcal{A})}$. Then v_A is contained in the itinerary of c.

On the other hand, let v_A be a vertex corresponding to a wall A that is contained in the itinerary of a generalized curve c. By definition, A intersects two distinct blocks B_0 and B_1 such that both v_{B_0} and v_{B_1} are contained in the image of c under $p_{(\mathcal{B},\mathcal{A})}$. Because c is continuous, there are $t_0, t, t_1 \in R$, $t_0 < t < t_1$ such that $c(t) \in A$, $c([t_0, t]) \subseteq B_0$ and $c([t, t_1]) \subseteq B_1$. It follows that c does not touch A. ☐

Lemma 3.15. *If A is a thick wall of a CAT(0) space that has a block decomposition with thick walls and γ is a geodesic ray that is not contained in A and ends in A, then γ ends in a side of A.*

Proof. Since γ is not contained in A but ends in A, there exists t_0 such that $\gamma(t_0) \in A$ and $\gamma(t) \notin A$ for all $t < t_0$. By the definition of a block decomposition of thick walls, $\gamma(t_0)$ is contained in a side S of A. Because A is a direct product $[0, 1] \times Y$ of $[0, 1]$ with a CAT(0) space Y, $\gamma(t) = (c_1(t), c_2(t))$ for a generalized curve $c_1 : [t_0, \infty) \to [0, 1]$ and a generalized curve $c_2 : [t_0, \infty) \to Y$. By Proposition 5.3 in Chapter 1.5 of [BH99], c_1 and c_2 are linearly reparametrized geodesics on all compact intervals that are contained in $[t_0, \infty)$. This is in particular true for c_1. We observe that the only geodesic ray $c_1 : [t_0, \infty) \to [0, 1]$ having this property is the constant generalized curve $c_1 = t_0$. Because $\gamma(t_0)$ is contained in a side of A, the claim follows. ☐

The itinerary of a geodesic ray γ *starts* in vertex v if v is contained in $I(\gamma)$, has degree one in $I(\gamma)$ and $\gamma(0)$ is contained in the wall or block corresponding to v. We number the edges and vertices of $I(\gamma)$ according to their distance to v. The vertex v is the first vertex of $I(\gamma)$. Its corresponding number is 0. The first edge of $I(\gamma)$ is the edge incident to v. Its corresponding number is 0. If $I(\gamma)$ is finite, its *last vertex (edge)* is the vertex (edge) of $I(\gamma)$ that is most distant from v.

Corollary 3.16. *The itinerary of a geodesic ray γ in a CAT(0) space with block decomposition with thick walls starts in a thick wall if $\gamma(0)$ is contained in the interior of a thick wall. Otherwise, γ starts in a block. If the itinerary of γ is finite and γ is not contained in a wall, then it ends in a vertex corresponding to a block.*

Proof. Let γ be a geodesic ray in a CAT(0) space with block decomposition with thick walls. It follows directly from the definition of itineraries that $I(\gamma)$ starts with a vertex corresponding to a thick wall if $\gamma(0)$ is contained in the interior of a thick wall. Otherwise, $I(\gamma)$ starts in a block. Assume that $I(\gamma)$ is finite and not contained in A. Assume further that γ ends in a wall. By Corollary 3.22, there exists $t_0 \in R$ such that $\gamma(t)$ is contained in one of the two sides of A for all $t \geq t_0$. By definition, every side of a wall is contained in a block B. Recall that $p_{(\mathcal{B},\mathcal{A})}$ projects every block B to v_B. By the definition of the itinerary, the last vertex of $I(\gamma)$ corresponds to v_B. □

3.3 Itineraries of geodesic rays in CAT(0) spaces with block decomposition

In this section, we introduce a common language for CAT(0) spaces with a block decomposition with thin or thick walls. We study their properties and examine how itineraries of asymptotic geodesic rays behave. We define itineraries of boundary points of complete CAT(0) spaces at the end of this section.

Definition 3.17. We say that $(\mathcal{B}, \mathcal{A})$ is a *block decomposition* of a CAT(0) space X if it is a block decomposition with thin or thick walls of X.

A block decomposition is *trivial* if a block decomposition consists of only one block. Let $(\mathcal{B}, \mathcal{A})$ be a block decomposition of a CAT(0) space X. We call the elements of $\mathcal{B}$ *blocks* and the elements of $\mathcal{A}$ *walls*. Depending on whether $(\mathcal{B}, \mathcal{A})$ it is a decomposition with thin or thick walls, every *wall* $A \in \mathcal{A}$ is thin or thick. If A is a thick wall, then A is a direct product of $[0, 1]$ with a CAT(0) space Y. Then $\{0\} \times Y$ and $\{1\} \times Y$ are the two *sides* of A. If A is thin, then every *side of* A is the wall A itself. We say that a wall A and a block B are *adjacent* if they have nonempty intersection. In the case of a decomposition with thick walls, the intersection is contained in a side of A. Otherwise, the wall A is contained in B. We say that two blocks B_0 and B_1 *share a wall* W if $W \cap B_0 \neq \emptyset$ and $W \cap B_1 \neq \emptyset$. Two blocks are adjacent if they share a wall. Recall that every block decomposition with thick or thin walls has an associated tree $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. Thereby $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ contains a vertex for every wall and every block. A vertex corresponding to a block is connected to a vertex corresponding to a wall if the block and the wall are adjacent. Recall that every block decomposition with thin or thick walls has an associated projection $p_{(\mathcal{B},\mathcal{A})}$ from X to the tree $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. We defined the *itinerary of a generalized curve* in a block decomposition with thick or thin walls by means of this projection. Compare Definition 3.3 and Definition 3.12. Now we are interested in itineraries of geodesic rays.

Sometimes, it is useful for our considerations to study itineraries of bi-infinite geodesic rays as well.

Definition 3.18. Let $(\mathcal{B}, \mathcal{A})$ be a *block decomposition* of a CAT(0) space X and γ a geodesic ray in X. If $(\mathcal{B}, \mathcal{A})$ is a block decomposition with thin walls, the *itinerary* $I(\gamma)$ is the itinerary of γ as defined in Definition 3.3. Otherwise it is the itinerary of γ as defined in Definition 3.12. Suppose that $\gamma : \mathbb{R} \to X$ is a bi-infinite geodesic ray. Let γ^+ and γ^- be the two associated geodesic rays starting at $t = 0$, i.e., $\gamma^+ : [0, \infty) \to X$ and $\gamma^- : [0, \infty) \to X$ such that $\gamma^+(t) = \gamma(t)$ and $\gamma^-(t) = \gamma(-t)$. We define the itinerary of γ to be the union of the itinerary $I(\gamma^+)$ of γ^+ and the itinerary $I(\gamma^-)$ of γ^-.

The following lemma is a consequence of the last subsections.

Lemma 3.19. *The itinerary of a geodesic ray in a CAT(0) space with block decomposition is a possibly infinite graph-theoretical path. The itinerary of a bi-infinite geodesic ray is a possibly infinite graph-theoretical path or a bi-infinite graph-theoretical path.*

Proof. If γ is a geodesic ray, the claim follows from Lemma 3.4 and Lemma 3.13. Suppose that $\gamma : \mathbb{R} \to X$ is a bi-infinite geodesic ray and that γ^+ and γ^- are the two associated geodesic rays starting at $\gamma(0 =$. By definition of itineraries, the corresponding itineraries start at the same vertex of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. Thus, the itinerary of a bi-infinite geodesic ray coincides with the concatenation of two possibly infinite paths. So, it is a possibly infinite graph-theoretical path or a bi-infinite path. □

Let γ be a geodesic ray. The itinerary $I(\gamma)$ of γ describes how the geodesic ray runs through X. Unlike in the original definition of Croke and Kleiner, we defined itineraries for all geodesic rays, independently from whether they start in a wall or not. In the case of a block decomposition with thin walls, the itinerary of the geodesic ray γ starts with a vertex corresponding to a wall A if γ starts at a point in A. In the case of a block decomposition with thick walls an itinerary starts with a vertex corresponding to a wall $A = [0, 1] \times Y$ if the corresponding geodesic ray starts in an interior point $\{t\} \times Y$, $t \in (0, 1)$ of A. We say that $I(\gamma)$ *starts* in v if v is a vertex of $I(\gamma)$ of degree one and $\gamma(0)$ is contained in the wall or block corresponding to v. We number the edges and vertices of $I(\gamma)$ according to their distance to v. The vertex v is the first vertex of $I(\gamma)$. Its corresponding number is 0. The first edge of $I(\gamma)$ is the edge incident to v. Its corresponding number is 0. If $I(\gamma)$ is finite, its last vertex (edge) is the vertex (edge) of $I(\gamma)$ that is most distant from v. A geodesic ray γ *ends* in a wall A (in a side S of A, a block B) if there exists $t_0 \in \mathbb{R}$ such that $\gamma(t) \in A$ ($\gamma(t) \in S,\ \gamma(t) \in B$) for all $t \geq t_0$. A geodesic ray γ *starts* in a wall A (in a side S of A, B) if $\gamma(0) \in A$ ($\gamma(0) \in S,\ \gamma(0) \in B$). The itinerary of a geodesic ray that starts or ends in a wall might not start or end in the vertex corresponding to this wall. We say that γ *enters* a block B if $I(\gamma)$ contains the vertex v_B. If $(\mathcal{B}, \mathcal{A})$ is a block decomposition with thin walls, this is exactly then the case when γ passes through a point that is contained in B but not in any wall. This coincides with the original definition of itineraries of Croke and Kleiner. If $(\mathcal{B}, \mathcal{A})$ is a block decomposition with thick walls, the situation is different. Then γ enters a block B if and only if γ passes through a point in B. Thereby it is possible that every point of B

is contained in a wall. The geodesic ray γ *leaves* a wall A (a block B) if $I(\gamma)$ contains the vertex v_A (v_B) and v_A (v_B) is not the last vertex of $I(\gamma)$. We say that γ *passes through* a wall A (a block$B$), if v_A (v_B) is an inner vertex of $I(\gamma)$. This is exactly then the case when $I(\gamma)$ does not start with v_A (v_B) and γ enters and leaves A (B). The geodesic ray γ *touches* a wall A, if $\gamma \cap A$ is nonempty and v_A is not contained in $I(\gamma)$.

We study in the following how geodesic rays and vertices of their itineraries are related to each other. This is related to Lemma 3.2, 3.3 and 3.4 in [Moo10].

First, we analyze vertices corresponding to blocks in the itinerary of a geodesic ray.

Lemma 3.20 (Vertices corresponding to blocks in itineraries)**.** *Let γ be a geodesic ray intersecting a block B of a CAT(0) X space with block decomposition $(\mathcal{B}, \mathcal{A})$. Exactly one of the following statements is true.*

- *γ intersects B in a point that is not contained in any wall in $\mathcal{A}$,*
- *every point in $\gamma \cap B$ is contained in a wall in $\mathcal{A}$ and $(\mathcal{B}, \mathcal{A})$ is a block decomposition with thick walls,*
- *every point in $\gamma \cap B$ is contained in a wall in $\mathcal{A}$ and $(\mathcal{B}, \mathcal{A})$ is a block decomposition with thin walls.*

In the first two cases, v_B is contained in $I(\gamma)$. In the last case, v_B is not contained in $I(\gamma)$

Proof. One of the three situations occurs because of the definition of itineraries in spaces with block decomposition with thin and thick walls. See Definition 3.3 and Definition 3.12. No two of the three situations occur simultaneously. The remaining claim follows from the definition of itineraries. □

The next lemma concerns the case that a geodesic ray intersects a wall that is not contained in its itinerary. In particular, the itinerary of a geodesic ray that starts or ends in a wall might not start or end in the vertex corresponding to this wall.

Lemma 3.21. *Let X be a CAT(0) space with a block decomposition $(\mathcal{B}, \mathcal{A})$ and γ be a geodesic ray that is not contained in a wall. Let A be a wall. If X is a block decomposition with thin walls, we assume that γ does not start at A. The geodesic ray γ touches a wall A if and only if there exists $t \in \mathbb{R}$ with $c(t) \in A$ satisfying one of the following conditions.*

a) There exists $t' \geq t$ such that $\gamma(t')$ is contained in a side of A for all $t' \in \mathbb{R}$ with $t' \geq t$.

b) The wall A is thick and for all $t' \in \mathbb{R}$ with $t' \leq t$ we have that $\gamma(t')$ is contained in a side of A.

c) There are a block $B \in \mathcal{B}$, an interval $(a, b) \subseteq \mathbb{R}$ containing t and times t_0, t_1 $\in (a, b)$, $t_0 < t < t_1$ such that $c(a, b) \subseteq B$ and $c(t_0) \notin A$, $c(t_1) \notin A$.

Proof. Because blocks and walls are convex, the intersection of A and γ is a geodesic ray or segment. Hence, one of the three conditions is true if it is true for all $t \in \mathbb{R}$ with $c(t) \in A$. The claim follows from Lemma 3.5 and Lemma 3.14. □

The next corollary says when a vertex corresponding to a wall is contained in the itinerary of a geodesic ray.

Corollary 3.22 (Vertices corresponding to walls in itineraries)**.** *Let γ be a geodesic ray intersecting a wall A of a CAT(0) space X with block decomposition $(\mathcal{B}, \mathcal{A})$. Exactly one of the following statements is true:*

a) A is thick and γ starts in A but not in one of its sides.

b) A is thin and γ starts in A.

c) γ is not contained in A, γ intersects both sides of A and does not start in A.

d) γ does not start in A and ends in A.

e) A is thick and γ starts in a side of A.

f) There are a block $B \in \mathcal{B}$, an interval $(a, b) \subseteq \mathbb{R}$ containing t and times t_0, $t_1 \in (a, b)$, $t_0 < t < t_1$ such that $c((a, b)) \subseteq B$ and $c(t_0) \notin A$, $c(t_1) \notin A$.

In the first two cases, $I(\gamma)$ starts with v_A. In the third case, v_A is an inner vertex of $I(\gamma)$. In the last three cases, v_A is not contained in $I(\gamma)$.

Proof. If γ does not start in A and γ does not end in A, then the itinerary of γ contains v_A by the definition of the itinerary. If γ does not pass through A, then v_A is not an inner vertex of $I(\gamma)$. Then v_A is either the first vertex or the last vertex of $I(\gamma)$ or it is not contained in $I(\gamma)$. In the first case, exactly one of the items a) and b) is satisfied by the definition of the itinerary in CAT(0) spaces with block decomposition with thick and thin walls. The second case in which $I(\gamma)$ ends with v_A does not occur. Indeed, the last vertex of the itinerary of a geodesic ray corresponds to a block if the ray is not completely contained in a wall. See Lemma 3.4 and Corollary 3.16. In the remaining case, γ touches A and the claim follows from Lemma 3.21. □

We study how the starting point of a geodesic ray is related to the first vertex of its itinerary in the next two lemmas

Lemma 3.23 (First vertices of itineraries)**.** *Let γ be a geodesic ray in a CAT(0) space with block decomposition. If γ starts in the interior point of a thick wall A, then $I(\gamma)$ starts with v_A. If γ starts in a thin wall A, then $I(\gamma)$ starts with v_A. Otherwise, there exists a unique block B in which γ starts, and the first vertex of $I(\gamma)$ is v_B.*

Proof. If γ starts in the interior of a thick wall or in a thin wall, then $I(\gamma)$ starts with the vertex corresponding to this wall because of Corollary 3.22. Otherwise, the definition of block decompositions of thick and thin walls implies that there exists a unique block B in which γ starts. By definition of the itinerary, v_B is the first vertex of $I(\gamma)$. □

Lemma 3.24 (itineraries of geodesic rays having a common starting point)**.** *Let x be a point in a CAT(0) space with block decomposition $(\mathcal{B}, \mathcal{A})$. Every (contracting) geodesic (ray) γ issuing from x starts with the same vertex of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$.*

Proof. Assume first, that X is a block decomposition with thick walls. If x is contained in a block B, then every geodesic ray issuing from x starts in B. By definition of the itinerary, any itinerary of a geodesic ray starting at x has v_B as the first vertex. Otherwise, x is contained in the interior of a thick wall A. By definition of the itinerary, any itinerary of a geodesic ray γ starting at x has v_A as the first vertex. Now we consider the case that X is a block decomposition with thin walls. If x is contained in a wall A, then the first vertex of every itinerary of a geodesic ray starting at x is v_A. Otherwise, x is contained in exactly one block B, and the first vertex of the corresponding itinerary is v_B. See Lemma 3.4. □

The next lemma analyzes properties of last vertices of finite itineraries.

Lemma 3.25 (Last vertices of finite itineraries)**.** *Let γ be a geodesic ray in a CAT(0) space X with a block decomposition whose itinerary is finite. If γ is contained in a wall A, then $I(\gamma)$ consists of the vertex v_A. Otherwise, the last vertex of γ corresponds to a block in which γ ends. If γ ends in a wall A, then the last vertex of $I(\gamma)$ corresponds to a block adjacent to A.*

Proof. The claim follows from the definition of itineraries in CAT(0) spaces with thick and thin walls and from Lemma 3.4 and Lemma 3.13 and Corollary 3.16. □

We study inner vertices of itineraries of geodesic rays that correspond to walls.

Lemma 3.26 (Inner vertices of itineraries corresponding to walls)**.** *Let γ be a geodesic ray in a CAT(0) space with block decomposition and A be a wall. If v_A is an inner vertex of the itinerary of γ, then γ intersects the sides of A and does not end in one of its sides. There exists $t_0 \in \mathbb{R}$ such that $\gamma(t_0) \in A$ and $\gamma(t) \notin A$ for all $t \geq t_0$.*

Proof. If the wall is thin, the claim follows from the definition of itineraries. Otherwise, let γ be a geodesic ray having v_A as inner vertex. Let B_0 and B_1 be the two blocks adjacent to A. Because v_A is an inner vertex of the itinerary of γ, the two edges $\{v_{B_0}, v_A\}$ and $\{v_{B_1}, v_A\}$ are contained in $I(\gamma)$. This means that the natural projection maps a subgeodesic of γ to the path v_{B_0}, A, v_{B_1}. Since this subgeodesic of γ reaches and leaves the interior of A through a side of A, it intersects a side of A at least twice. Because A and its sides are convex, these two sides are distinct. By the convexity of the sides of A and Lemma 3.15, γ does not end in a side of A. Furthermore, there exists $t_0 \in \mathbb{R}$ such that $\gamma(t_0) \in A$ and $\gamma(t) \notin A$ for all $t \geq t_0$. □

We say that a geodesic ray in X *switches between blocks forever*, if for all $t \in \mathbb{R}$ there exists $t_0 > t$ and $t_1 > t$ such that $\gamma(t_0)$ lies in a block with parity $(-)$ and $\gamma(t_1)$ lies in block with parity $(+)$. Recall that a geodesic ray *ends* in A (in a side S of A, a block B) if there exits $t_0 \in \mathbb{R}$ such that $\gamma(t) \in A$ $(\gamma(t) \in S,\ \gamma(t) \in B)$ for all $t \geq t_0$.

Lemma 3.27. *Every geodesic ray γ in a CAT(0) space with block decomposition*

a) is contained in a wall or

b) ends in a block or

c) switches between blocks forever.

Proof. The claim follows from Lemma 3.4, Lemma 3.13 and Corollary 3.16. □

Let X be a complete CAT(0) space with a block decomposition. Then the equivalence classes of asymptotic geodesic rays in X are independent of the choice of the base point and the visual boundary as well as the contracting boundary are well-defined for X. We observe that every block, every side of a wall and every wall are complete as closed subsets of the complete metric space X. We examine how the itineraries of asymptotic geodesic rays behave in the following lemmas,

The following Lemma can be proven in the same way as Lemma 3.4 of Mooney in [Moo10].

Lemma 3.28 (Lemma 3.4 in [Moo10])**.** *Let γ be a geodesic ray in a CAT(0) space with block decomposition whose itinerary is infinite. Then $\lim_{t\to\infty} d(\gamma(t), B) = \infty$ for every block B.*

This lemma implies that a geodesic ray ending in a block cannot be asymptotic to a geodesic ray with infinite itinerary. If ξ is a point in ∂X, then all its representatives have finite itineraries or all its representatives have infinite itineraries. Furthermore, if two geodesic rays have infinite itineraries and are asymptotic to each other, then they coincide from a vertex v on. These consequences are listed by Mooney in Corollary 3.5 of [Moo10]. In the following, we proof Corollary 3.5 of Mooney differently and generalize his results.

Lemma 3.29. *Let X be a complete CAT(0) space with block decomposition. If two geodesic rays γ and γ' are asymptotic and their itineraries start with the same vertex, then $I(\gamma) = I(\gamma')$.*

Proof. Let γ and γ' be two asymptotic geodesic rays such that $I(\gamma)$ and $I(\gamma')$ start at the same vertex of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. For achieving a contradiction, we assume that $I(\gamma)$ and $I(\gamma')$ are distinct. Let v be the last vertex shared by $I(\gamma)$ and $I(\gamma')$. The vertex v corresponds to a block B. Indeed, recall that every vertex in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ corresponding to a wall has degree two. If v corresponds to a wall, v has degree two. As $I(\gamma)$ and $I(\gamma')$ are distinct, one of the two itineraries, say $I(\gamma)$ would be contained in the other and v would be the last vertex of $I(\gamma)$ – a contradiction to Lemma 3.25. Hence, $v = v_B$ where B is a block. Then there exists a wall A such that one of the two itineraries, say $I(\gamma)$, contains the edge $\{v_A, v_B\}$, but $I(\gamma)$ doesn't do so. By the last two lemmas, v_A is an inner vertex of $I(\gamma)$ and γ passes through the wall A, but γ' doesn't do so. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the two trees we obtain by removing the two edges incident to v_A form $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. By construction,

X decomposes into two disjoint subspaces X_0' and X_1' if we delete A from X such that X_i' is covered by the union of all walls and blocks that correspond to vertices in $\mathcal{T}_i$ for $i \in \{0,1\}$. Let X_0 be the union of X_0' and A and X_1 be the union of X_1' and A. As both spaces are isometrically embedded in X, they are CAT(0). Furthermore, they are complete, as they are closed subspaces of the complete space X. By our considerations, γ' is included in one of them, say X_0. Let $t_0 \in \mathbb{R}$ be the time where γ leaves A. Then $\gamma|[t_0,\infty]$ is contained in the other space X_1 and not contained in X_0. Let x be a point on the geodesic ray γ which is contained in X_0. By Proposition 8.2 in Chapter II of [BH99], there exists a geodesic ray α in X_0 starting at x which is asymptotic to γ'. By assumption, this geodesic ray is asymptotic to γ. As X_0 is isometrically embedded in X, α is a geodesic ray in X. By Proposition 8.2 in Chapter II of [BH99], there exists $\tilde{t}$ such that $\alpha = \gamma|[\tilde{t},\infty]$ in X – a contradiction to the fact that α is contained in X_0 and $\gamma(t)$ is not contained in X_0 if t is large. □

Let I_0 and I_1 be two (possibly infinite) paths in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ and v a vertex contained in I_0 and I_1. Let $i \in \{0,1\}$. If I_i is finite, let I_i' be the path connecting v with the last vertex of I_i. If I_i is infinite, let I_i' be the unique infinite subpath of I_i starting with v. We say that I_0 and I_1 *coincide from the vertex v on*, if $I_0' = I_1'$.

Lemma 3.30. *Let X be a complete CAT(0) space with block decomposition. If two geodesic rays γ and γ' are asymptotic and their itineraries have a vertex v in common, then $I(\gamma)$ and $I(\gamma')$ coincide from the vertex v on.*

Proof. The claim follows from Lemma 3.29. □

Lemma 3.31. *Let X be a complete CAT(0) space with block decomposition. If two geodesic rays γ and γ' are asymptotic and their itineraries don't share a vertex, then $I(\gamma)$ and $I(\gamma')$ are finite and γ and γ' are asymptotic to a geodesic ray ending in a wall. The unique shortest path P in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ connecting $I(\gamma)$ with $I(\gamma')$ connects the last vertex of $I(\gamma)$ with the last vertex of $I(\gamma')$. Every wall corresponding to a vertex of P contains a geodesic ray that is asymptotic to γ and γ'.*

Proof. Let γ and γ' two asymptotic geodesic rays whose itineraries don't share a vertex. Let P be the shortest path connecting $I(\gamma)$ and $I(\gamma')$. As $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is a tree there exists exactly one such path. Let v be the endvertex of P that is contained in $I(\gamma)$ and v' be the endvertex of P that is contained in $I(\gamma)$. Let A be a wall corresponding to a vertex v_A on P. Such a vertex exists because P contains at least one edge and every edge of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ has an endvertex corresponding to a wall. The vertex v_A is not an inner vertex of $I(\gamma)$ and not an inner vertex of $I(\gamma')$ as the degree of v_A in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is two. Hence, neither γ nor γ' passes through A. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the two trees we obtain by removing the two edges incident to v_A form $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. By definition of a block decomposition, X decomposes into two spaces X_0' and X_1' if we delete A from X such that X_i' is covered by the union of all walls and blocks that correspond to vertices in $\mathcal{T}_i$ for $i \in \{0,1\}$. Let X_0 be the union of X_0' with A and X_1' be the union of X_1' and A. As both spaces are isometrically embedded in X, they are CAT(0). Furthermore, they are complete, since they are closed subspaces of the complete space X. As γ and γ' don't pass through A, γ is contained in one of both spaces, say in X_0 and γ' is contained in the other space X_1. Let p be a point in A. By Proposition 8.2 in Chapter II of [BH99], there exists a geodesic ray γ_p in X_0 starting at p which is asymptotic to γ. As X_0 is isometrically embedded in X, γ_p is a geodesic ray in X. Analogously, there exists a geodesic ray γ'_p in X_1 starting at p that is asymptotic to γ'. As X_1 is isometrically embedded in X, γ'_p is a geodesic ray in X. Thus, $\gamma_p = \gamma'_p$. Accordingly, every wall corresponding to a vertex of P contains a geodesic ray which is asymptotic to γ and γ'. It follows that v and v' are no inner vertices of $I(\gamma)$ and $I(\gamma')$. If γ (γ') is contained in a wall, then its itinerary is the trivial path v (v'). Otherwise, v (v') correspond to a block containing A by Lemma 3.25. In every case v and v' are the last vertices of the itineraries of γ and γ'. □

The last lemmas result in a generalization of Corollary 3.5 in [Moo10] given by the next two corollaries.

Corollary 3.32. *Let X be a complete CAT(0) space with block decomposition. Let γ and γ' be two asymptotic geodesic rays. Then their itineraries both are finite or both are infinite. If γ and γ' are not asymptotic to a geodesic ray ending in a wall, then there is a vertex v such that $I(\gamma)$ and $I(\gamma')$ coincide from vertex v on.*

Proof. If γ and γ' are asymptotic and their itineraries share a vertex v, then their itineraries coincide from vertex v on by Lemma 3.30. In particular, $I(\gamma)$ is finite if and only if $I(\gamma')$ is finite. If γ and γ' are asymptotic and their itineraries don't share a vertex, then they have both a finite itinerary and both are asymptotic to a geodesic ray in a wall by Lemma 3.31. □

Corollary 3.33. *Let X be a complete CAT(0) space with block decomposition. Let γ and γ' be two asymptotic geodesic rays with finite itineraries. Let v and v' be the last vertices of $I(\gamma)$ and $I(\gamma')$ respectively. Let P be the unique shortest path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ connecting v and v'. If $v \neq v'$, then every wall corresponding to a vertex of P contains a geodesic ray asymptotic to γ and γ'.*

Proof. The claim follows directly from Lemma 3.31. □

The last statements show that itineraries of geodesic rays can be used for understanding boundary points of a complete CAT(0) space with block decomposition. We choose a base point x_{base} of X. By Lemma 3.24, every itinerary of a (contracting) geodesic (ray) γ issuing from x_{base} starts with the same vertex of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. We denote this vertex by v_{base}. Recall that ∂X denotes the set of equivalence classes of geodesic rays in X.

Definition 3.34. Let X be a complete CAT(0) space with block decomposition. Let $\xi \in \partial X$. The *Itinerary* $I(\xi)$ of ξ is the itinerary of the geodesic ray based at x_{base} that represents ξ.

By Corollary 3.33, the itinerary of a point ξ in the boundary of X depends on the choice of x_{base}. If γ and γ' are two geodesic rays starting at different points that both are asymptotic to a geodesic ray ending in a wall, then it may happen that the itineraries of $\gamma(\infty)$ and $\gamma'(\infty)$ don't end in the same vertex. However, if γ and γ' are not asymptotic to a geodesic ray ending in a wall, then their itineraries coincide from a vertex v on. See Corollary 3.32. In this case, they end in the same vertex if $I(\gamma)$ and $I(\gamma')$ are finite. If $I(\gamma)$ and $I(\gamma')$ are infinite, they have always an infinite path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ in common.

The following can be proven as Lemma 3.7 of Mooney in [Moo10].

Lemma 3.35 (Lemma 3.7 of Mooney in [Moo10])**.** *Let X be a complete CAT(0) space with block decomposition. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of geodesic rays all having a common base point and infinite itinerary. Suppose that $(\gamma_n)_{n\in\mathbb{N}}$ converges in the visual boundary of X to a geodesic ray γ with infinite itinerary. Then for every block B corresponding to a vertex v_B in $I(\xi)$, we have $v_B \in I(\xi_n)$ for large enough n.*

Recall that $\hat{\partial}\mathbb{X}$ and $\hat{\partial}\mathcal{T}_{\mathcal{B},\mathcal{A}}$ denote the visual boundary of X and $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ respectively. By Mooney's Corollary 3.8 in [Moo10], the last lemma results in the following.

Corollary 3.36. *([Moo10, Cor. 3.8]) Let X be a complete CAT(0) space with block decomposition $(\mathcal{B},\mathcal{A})$. Let $\Phi : \hat{\partial}X \to \hat{\partial}\mathcal{T}_{\mathcal{B},\mathcal{A}}$ be the map that sends a point $\xi \in \hat{\partial}X$ to the point of the visual boundary of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ that is determined by the itinerary of ξ. The map Φ is continuous. For all ξ_1, $\xi_2 \in \hat{\partial}\mathbb{X}$ and all geodesic rays γ_1 and γ_2 representing ξ_1 and ξ_2, $I(\gamma_1)$ and $I(\gamma_1)$ coincide from a vertex v on if and only if $\Phi(\xi_1) = \Phi(\xi_2)$.*

We finish this section by studying isometries acting on spaces with block decomposition. Suppose that g is an isometry of X that sends blocks to blocks and walls to walls. Then g acts naturally on $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ as a graph-automorphism. Furthermore, g acts on $\hat{\partial}X$ $(\hat{\partial}_c X, \vec{\partial}_c X)$ as a homeomorphism. Let $\xi \in \partial X$ $(\partial_c X)$ and γ be a representative of ξ. The isometry g maps ξ to the equivalence class of $g \cdot \gamma$.

Lemma 3.37. *Let X be a complete CAT(0) space with block decomposition and g be an isometry of X that sends blocks to blocks and walls to walls. Let $\xi \in \partial X$ $(\partial_c X)$ and γ be its representative that starts at x_{base}. If γ has finite itinerary and does not end in a wall, then the itinerary of $g \cdot \xi$ is the path connecting v_{base} with the last vertex of $I(g \cdot \gamma)$. If γ ends in a wall, the itinerary of $g \cdot \xi$ is an initial subpath of the path connecting v_{base} with the last vertex of $I(g \cdot \gamma)$. If the itinerary of γ is infinite, the itinerary of $g \cdot \xi$ coincides with $I(g \cdot \gamma)$ from a vertex v in $I(g \cdot \gamma)$ on.*

Proof. Let $\tilde{\gamma}$ be the geodesic ray asymptotic to $g \cdot \gamma$ that starts at x_{base}. If γ does not end in a wall, $I(\tilde{\gamma})$ and $I(g \cdot \gamma)$ share a vertex v by Lemma 3.31. It follows from Lemma 3.30 that $I(\tilde{\gamma})$ and $I(g \cdot \gamma)$ coincide from vertex v on. Because of that, the itinerary of $\tilde{\gamma}$ connects v_{base} with the last vertex of $I(g \cdot \gamma)$. There exists exactly one such path and this path is the itinerary of $\tilde{\gamma}$. If γ ends in a wall, it follows from Lemma 3.31 that the itinerary of $g \cdot \xi$ is an initial subpath of the path connecting v_{base} with the last vertex of $I(g \cdot \gamma)$. If the itinerary of γ is infinite, $I(\tilde{\gamma})$ and $I(g \cdot \gamma)$ share a vertex v by Lemma 3.31. It follows from Lemma 3.30 that $I(\tilde{\gamma})$ and $I(g \cdot \gamma)$ coincide from vertex v on. □

3.4 The boundary points of every wall behave like a cutset

In this section we prove that the boundary points of every wall in a CAT(0) space X with a nontrivial block decomposition behave like a cutset. By this, we mean the following. Let Ξ be the contracting boundary $\vec{\partial}_c X$ of X, the visual boundary $\hat{\partial} X$ of X or the subspace $\hat{\partial}_c X$ of $\hat{\partial} X$ that consists of all equivalence classes of contracting geodesic rays. Suppose that A is a wall of X. If X is nontrivial, X decomposes into two spaces X_0 and X_1 if we delete A. Suppose that Ξ has a connected component κ that contains a boundary point of X_0 and a boundary point of X_1. We prove that then κ contains a boundary point of A. In this situation, the deletion of the boundary points of A decompose Ξ into more connected components than before. Thus, A "behaves like a cutset". This leads to Lemma 3.50, a key-lemma of this thesis. This key-lemma implies that every connected component with two boundary points with distinct itinerary contains a boundary point of a wall. This section is inspired by Croke's and Kleiner's example in [CK00] and the Cycle-Join-Example of Charney and Sultan in Section 4.2 in [CS15] that was one of the main motivations for this thesis. See Section 5.1 for more details concerning this example. Furthermore, this section is inspired by the study of cutpoints of Bowditch [Bow98a], the research based on this as summarized in Section 1.1.2 and Lemma 7 in Section 1.7 of [CK00]. Recently, Ben-Zvi and Kropholler proved independently a similar statement concerning path-components of visual boundaries in Lemma 3.1 of [BZK19]. We cite this lemma as Lemma 3.44 and compare it with our lemma.

We begin with some lemmas concerning complete CAT(0) spaces. We apply them to CAT(0) spaces with block decompositions afterwards. Suppose that Z is a complete, convex subspace of a complete CAT(0) space X. We use the notation established in Chapter 2. We summarized our notation concerning boundaries in Notation 1.1. In the following, we consider ∂Z as embedded in ∂X, i.e., we mean by ∂Z the embedded set $\{\gamma(\infty) \in \partial X \mid \gamma \subseteq Z\}$. Recall that a geodesic ray $\gamma \subseteq Z$ is X-*contracting*, if it is contracting in the ambient space X. We denote by $\partial_{c,X} Z$ the set $\{\gamma(\infty) \in \partial_c X \mid \gamma \subseteq Z\}$ of equivalence classes of X-contracting geodesic rays in Z. If we equip $\partial_{c,X} Z$ with the subspace topology of the visual boundary and contracting boundary of X, we obtain the topological spaces $\hat{\partial}_{c,X} Z$ and $\vec{\partial}_{c,X} Z$ respectively.

Remark 3.38. If we ignore the ambient space X, ∂Z denotes the set

$$\{\gamma(\infty) \mid \gamma \text{ is a geodesic ray in } Z\}.$$

By Lemma 2.35, $\hat{\partial}_{c,X} Z$ and $\vec{\partial}_{c,X} Z$ are homeomorphic to $\{\gamma(\infty) \in \partial Z \mid \gamma \text{ is X-contracting}\}$ equipped with the subspace topology of $\hat{\partial} Z$ and $\vec{\partial}_c Z$ respectively. For more details, see Section 2.5.

The following two lemmas are basics of topology. We prove them for completeness.

Lemma 3.39. *Let X be a topological space and Y be a topological subspace of X such that the set Y is open in X. If a connected component κ of X contains two points of distinct connected components of Y, then κ contains at least one point of $X \setminus Y$.*

Proof. Let x and y be two points of distinct connected components of Y and M' be a set in X containing x and y. Suppose that $M' \subseteq Y$ contains x and y. Then, M' is not connected in Y. Thus, there exist two nonempty open sets O_0 and O_1 in X such that $O_0 \cap Y$ and $O_1 \cap Y$ are nonempty and disjoint and $M' = (Y \cap O_0) \sqcup (Y \cap O_1)$. Because Y is open in X, M' is the union of two open disjoint sets in X and M' is not connected in X. □

Lemma 3.40. *Let X be a topological space and M a subset of X which is open and closed. Then every connected component of X is contained in M or M^c.*

Proof. Let K be a connected component of X. By assumption, $K = (K \cap M) \sqcup (K \cap M^c)$. Because K is a connected component, K is closed. Hence, both $(K \cap M)$ and $(K \cap M^c)$ are closed sets as intersection of closed sets. It follows that either $(K \cap M)$ is empty or $(K \cap M^c)$ is empty. This implies that K is either contained in M or in M^c. □

The following lemmas are basics in CAT(0) geometry. We prove them for completeness. Let α be a geodesic ray in a complete CAT(0) space X. Recall that $U(\alpha(\infty), r, \epsilon)$ denotes the set of points in ∂X whose representatives starting at $\alpha(0)$ are (ϵ, r)-close to α, i.e., $d(\alpha(t), \gamma(t)) < \epsilon \ \forall\ t \leq r$.

Lemma 3.41. *Let X be a complete CAT(0) space, Z a complete, convex subspace, and γ a geodesic ray starting at a point $z \in Z$. Either γ is completely contained in Z or there exists an $\epsilon > 0, r > 0$ such that every geodesic ray $\tilde{\gamma}$ that starts at z and represents an element in $U(\gamma(\infty), r, \epsilon)$ leaves Z at some point, i.e., there exists $t_0 \in \mathbb{R}$ such that $\tilde{\gamma}(t) \notin Z$ for all $t \geq t_0$.*

Proof. Let γ be a geodesic ray in X starting at a point $z \in Z$. Suppose that for all $\epsilon > 0$, $r > 0$ there exists a representative $\tilde{\gamma}$ of a point in $U(\gamma, r, \epsilon)$ that starts at z and does not leave Z. Because Z is convex, the whole geodesic ray $\tilde{\gamma}$ is contained in Z. Let $\tilde{\gamma}_k$ be a representative of such a boundary point in $U(\gamma, k, \frac{1}{k})$, i.e., $\tilde{\gamma}_k$ is contained in Z and $\tilde{\gamma}_k(\infty) \in U(\gamma, k, \frac{1}{k})$. Then the sequence $(\tilde{\gamma}_k(\infty))_{k \in \mathbb{N}}$ converges to $\gamma(\infty)$ in the visual boundary of X. Because $(\gamma_k)_{k \in \mathbb{N}} \subseteq Z$, and Z is complete, the point-wise limit of this sequence is a geodesic ray that is contained in Z. Thus, γ is contained in Z. □

The following corollary can be found as a remark in Example 8.11 (4) in Chapter II in [BH99]. We prove it for completeness

Corollary 3.42. *Let X be a complete CAT(0) space and Z a complete, convex subspace of X. Then ∂Z ($\hat{\partial}_{c,X} Z$) is closed in $\hat{\partial} X$ ($\hat{\partial}_c X$ and $\vec{\partial}_c X$).*

Proof. First, we study the visual boundary $\hat{\partial} X$ of X. If $\partial Z = \partial X$, the claim is obvious. Thus, we assume that $\partial X \setminus \partial Z \neq \emptyset$. Let $\xi \in \partial X \setminus \partial Z$. Let γ' be a representative of ξ that starts in Z. Because Z is convex and γ' is not asymptotic to a geodesic ray in Z, γ' leaves Z at some point, i.e., there exists $t_0 \in \mathbb{R}$ such that $\gamma'(t) \notin Z$ for all $t \geq t_0$. By Lemma 3.41, there exists an $\epsilon > 0, r > 0$ such that every geodesic ray $\tilde{\gamma}$ that starts at z and represents an element in $U(\gamma(\infty), r, \epsilon)$ leaves Z at some point. Thus, no boundary point in $U(\gamma(\infty), r, \epsilon)$ is contained in ∂Z and ξ has an open neighborhood in $\hat{\partial} X$, that is contained in $\partial X \setminus \partial Z$. It follows that ∂Z is closed in $\hat{\partial} X$. Now suppose that $\xi \in \partial_c X$. Then, the set of contracting geodesic rays in $U(\gamma(\infty), r, \epsilon)$ is an open set of $\hat{\partial}_c X$. Because the direct limit topology is finer than the cone topology, the set of all contracting geodesic rays in $U(\gamma(\infty), r, \epsilon)$ is also open in the contracting boundary $\vec{\partial}_c X$ of X. Thus, ξ has an open neighborhood in $\hat{\partial}_c X$ and $\vec{\partial}_c X$ that is contained in $\partial X \setminus \partial Z$. Hence, $\hat{\partial}_{c,X} Z$ is closed in $\hat{\partial}_c X$ and $\vec{\partial}_c X$. □

The following observation is inspired by Croke's and Kleiner's example in [CK00] and the Cycle-Join-Example of Charney and Sultan.

Corollary 3.43. *Let X be a complete CAT(0) space and X_0, X_1 closed subsets such that the intersection $Z = X_0 \cap X_1$ is convex and $X = X_0 \cup X_1$. If a connected component κ in $\hat{\partial} X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) contains a boundary point in ∂X_0 and in ∂X_1, then κ contains a boundary point in ∂Z.*

Proof. We prove the claim for $\hat{\partial} X$. The remaining cases can be proven analogously. Let ξ_0 and ξ_1 be two boundary points in ∂X_0 and ∂X_0 respectively. If ξ_0 or ξ_1 are contained in ∂Z , the claim is obvious. Thus, we assume that neither ξ_0 nor ξ_1 are contained in ∂Z. Let $\mathcal{Y} := \partial X \setminus \partial Z$ and M' be a subset of $\mathcal{Y}$ containing ξ_0 and ξ_1. Because $X = X_0 \cup X_1$ and $Z = X_0 \cap X_1$ is convex, X_0 and X_1 are convex. By Corollary 3.42, ∂X_0 and ∂X_0 are closed in $\hat{\partial} X$. It follows that $\mathcal{Y}$ is open in $\hat{\partial} X$. Furthermore, $\partial X_i \setminus \partial Z = \partial X \setminus \partial X_j$, i, $j \in \{0, 1\}$, $i \neq j$. Hence, $\partial X_0 \setminus \partial Z$ and $\partial X_1 \setminus \partial Z$ are open in $\hat{\partial} X$. Thus, $\partial X_0 \cap \mathcal{Y}$ and $\partial X_1 \cap \mathcal{Y}$ are closed and open in $\mathcal{Y}$ equipped with the subspace topology of X. By Lemma 3.40, ξ_0 and ξ_1 are contained in distinct connected components of $\mathcal{Y}$. By Lemma 3.39, M' is not connected in X. □

Recently, Ben-Zvi and Kropholler proved a statement similar to Corollary 3.43 independently. They proved the following [BZK19, Lemma 3.1].

Lemma 3.44. *([BZK19, Lemma 3.1]) Let X be a proper, complete CAT(0) space and let A and B be closed subsets of X. Suppose that there exists a closed subset C such that any geodesic from A to B passes through C. Then any path in the boundary between $\hat{\partial} A$ and $\hat{\partial} B$ passes through $\hat{\partial} C$.*

Note that the set C in the lemma of Ben-Zvi and Kropholler contains $A \cap B$ because otherwise, $A \cap B$ contains a curve from A to B that does not pass through C. Assume that $A \cap B$ is convex. In this case, the lemma of Ben-Zvi follows from Corollary 3.43 because every path connected set is connected.

Next, we apply Corollary 3.43 to block decompositions of CAT(0) spaces. Let $(\mathcal{B}, \mathcal{A})$ be a block decomposition of a complete CAT(0) space X. Recall that every block $B \in \mathcal{B}$ is a closed, convex, complete subset of X. The set of walls consists either of thin walls or of thick walls. In the first case, every wall is the intersection of two blocks of distinct parity. In the second case, every wall is a direct product $[0, 1] \times Y$ of $[0, 1]$ with a CAT(0) space Y and the two sides $\{0\} \times Y$ and $\{1\} \times Y$ are glued at blocks of distinct parity. Recall that a side of a thin wall is the wall itself. Every wall is a convex, complete subset of X.

Corollary 3.45. *Let X be a complete CAT(0) space with block decomposition. Let A be a wall and X_0 and X_1 be two connected components of the space obtained by deleting A from X. Then any connected component in $\hat{\partial}X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) containing a boundary point in ∂X_0 and a boundary point in ∂X_1 contains a boundary point of ∂A.*

Proof. Because blocks are closed sets, $X_0 \cup A$ and $X_1 \cup A$ are closed. Furthermore, every wall is convex and closed and $X = (X_0 \cup A) \cup (X_1 \cup A)$ and $A = (X_0 \cup A) \cap (X_1 \cup A)$. Thus, the claim follows from Corollary 3.43. □

In the following, we examine the consequences of Corollary 3.45 for connected components containing boundary points with distinct itineraries. Let X be a complete CAT(0) space with block decomposition. We choose a base point x_{base} of X and study geodesic rays starting at x_{base}. Recall that there is a tree $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ associated to the block decomposition $(\mathcal{B}, \mathcal{A})$ of X. By Definition 3.34, the *itinerary* $I(\xi)$ of an element $\xi \in \partial X$ ($\partial_c X$) is the itinerary of the geodesic ray γ representing ξ that starts in x_{base}. It is a (possibly infinite) path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ that describes how γ runs through the walls and blocks of X. Every itinerary of a (contracting) geodesic (ray) issuing from x_{base} starts in the same vertex v_{base} of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. Like Charney and Sultan in the Cycle-Join-Example in Section 4.2 of [CS15], we write $I_0 \leq I_1$ if I_0 is an initial subpath of I_1. Like them, we define the following sets.

Definition 3.46. Let I be a (possibly infinite) path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ starting at v_{base}.

$$U(I) := \{\gamma(\infty) \in \partial X \mid \gamma(0) = x_{\text{base}} \text{ and } I \leq I(\gamma)\} \tag{3.46.1}$$

$$\hat{U}(I) := \{\gamma(\infty) \in \partial X \mid \gamma(0) = x_{\text{base}} \text{ and } I = I(\gamma)\} \tag{3.46.2}$$

$$U_c(I) := \{\gamma(\infty) \in \partial_c X \mid \gamma(0) = x_{\text{base}} \text{ and } I \leq I(\gamma)\} \tag{3.46.3}$$

$$\hat{U}_c(I) := \{\gamma(\infty) \in \partial_c X \mid \gamma(0) = x_{\text{base}} \text{ and } I = I(\gamma)\}. \tag{3.46.4}$$

We remark that the definition is independent of a topology on ∂X or $\partial_c X$. Neither the definition of itineraries of geodesic rays nor the sets $U(I)$ and $U_c(I)$ depend on the topology on ∂X or $\partial_c X$.

We observe similar as in the Cycle-Join-Example of Charney and Sultan, that the sets $U(I)$ can be characterized as follows.

Lemma 3.47. *Let X be a complete CAT(0) space with block decomposition $(\mathcal{B}, \mathcal{A})$. Let I be a path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ of length at least two starting with v_{base} and ending in a vertex corresponding to a block B. Let A be the wall corresponding to the second last vertex of I. Let S be the side of A that has a nonempty intersection with B. The set $U(I)$ ($U_c(I)$) consists of all equivalence classes of (contracting) geodesic rays in X based at x_{base} that intersect the side S at least once and don't end in S.*

Proof. If a geodesic ray starts in x_{base} and intersects the side S of A at least once and does not end in S, its itinerary starts with I. Indeed, because γ does not end in S and intersects S, the natural projection maps a point in B to the vertex v_B in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. As $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is a tree, there exists exactly one path connecting v_{base} with v_B and this path is an initial subpath of $I(\gamma)$. As it contains I, $I \leq I(\gamma)$. On the other hand, let ξ be an element in $U(I)$ and γ its representative starting at x_{base}. As the itinerary of γ contains I as initial subpath, the vertex v_A is an inner vertex of the itinerary of γ, i.e., γ passes through A. By Lemma 3.26, γ intersects S and does not end in S. □

Corollary 3.48. *Let X be a complete CAT(0) space with block decomposition $(\mathcal{B}, \mathcal{A})$. Let I be a path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ of length at least two starting with v_{base} and ending in a vertex corresponding to a block B. Let A be the wall corresponding to the second last vertex of I. Let X_0 and X_1 be the two connected components of the space obtained by deleting A from X. The set $U(I)$ ($U_c(I)$) consists of all equivalence classes of (X-contracting) geodesic rays in $X_0 \setminus A$ or $X_1 \setminus A$.*

Proof. Let I be a path of length at least two ending in a vertex corresponding to a block and A the wall corresponding to the second last vertex of I. Let X_0 and X_1 be the both spaces we obtain when we delete A. Recall that every wall is convex. Thus, if a geodesic ray γ intersects A and does not end in A then it is not asymptotic to a geodesic ray in A. Thus, it follows from Lemma 3.47 that $U(I)$ coincides with $\partial X_0 \setminus \partial A$ or $\partial X_i \setminus \partial A$. Analogously one proves that $U_c(I)$ coincides with the set of all equivalence classes of X-contracting geodesic rays in $X_0 \setminus A$ or $X_0 \setminus A$. □

Definition 3.49. Let X be a CAT(0) space with block decomposition $(\mathcal{B}, \mathcal{A})$. Let I_0 and I_1 be two paths in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ starting with v_{base}. Let I' be the subgraph of $I_0 \cup I_1$ consisting of all edges that lie in I_0 or I_1 but not in I_0 and I_1 simultaneously. We say that a vertex v *is between* I_0 and I_1 if it is contained in I' and say that I' is the *path between* I_0 *and* I_1.

If I_0 and I_1 are finite, then I' is the unique path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ connecting the last vertex of I_0 with the last vertex of I_1. Otherwise, I' is an infinite path. This path contains all vertices of I_0 and I_1 that are not contained in I_0 and I_1 simultaneously. Also, if $I_0 \cup I_1$ contains a vertex of degree 3, this vertex is contained in I' as well.

The following key-lemma is a helpful tool for our considerations in this thesis. It is related to Lemma 7 in Section 1.7 of [CK00]

Lemma 3.50 (Key-lemma)**.** *Let X be a complete CAT(0) space with block decomposition $(\mathcal{B}, \mathcal{A})$. Let κ be a connected component of a subspace of $\hat{\partial} X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) containing two points with different itineraries. For every vertex between their itineraries corresponding to a wall A there exists a point $\xi \in \partial A$ such that $\xi \in \kappa$.*

Proof. Because connected components of subspaces of topological spaces are contained in connected components of the origin space, it is enough to show the statement for a connected component of $\hat{\partial} X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$). We prove the claim for $\hat{\partial} X$. The claim in brackets follows analogously. Let γ_0 and γ_1 be two geodesic rays starting at x_{base} with different itineraries and κ be a connected component of $\hat{\partial} X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) containing $\gamma_0(\infty)$ and $\gamma_1(\infty)$. For achieving a contradiction, we assume that there is a vertex between $I(\gamma_0)$ and $I(\gamma_1)$ corresponding to a wall A such that $\kappa \cap \partial A = \emptyset$. Let X_0 and X_1 be the two connected components of the space obtained by deleting A from X. Because κ does not contain any point in ∂A, neither $\gamma_0(\infty)$ nor $\gamma_1(\infty)$ is contained in ∂A. Thus, $\gamma_0(\infty)$ and

$\gamma_1(\infty)$ are contained in $\partial X_0 \cup \partial X_1 \setminus \partial A$. Because neither $\gamma_0(\infty)$ nor $\gamma_1(\infty)$ is contained in ∂A and as v_A has degree two, the path between $I(\gamma_0)$ and $I(\gamma_1)$ contains both edges incident to v_A. Thus, v_A is not incident to a vertex that is contained in $I(\gamma_0)$ and $I(\gamma_1)$ simultaneously. Hence, v_A is an inner vertex of $I(\gamma_0)$ or $I(\gamma_1)$ but not an inner vertex of both itineraries simultaneously. By Lemma 3.26, exactly one of both geodesic rays, say γ_0, runs through A (i.e., there exists $t_0 \in \mathbb{R}$ such that $\gamma_0(t_0) \in X_0$ for all $t \geq t_0$). Thus, $\gamma_0(\infty)$ is contained in $\partial X_0 \setminus \partial A$ and the other point $\gamma_1(\infty)$ is contained in $\partial X_1 \setminus \partial A$. By Corollary 3.45, the connected component κ contains a geodesic ray that is contained in ∂A – a contradiction.

□

Lemma 3.51. *Let X be a complete CAT(0) space with block decomposition $(\mathcal{B}, \mathcal{A})$. Let v and w be two vertices in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ and P the unique path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ connecting v and w. Let K_v be the block or wall corresponding to v and K_w be the block or wall corresponding to w. Suppose that K_v contains a geodesic ray γ_v and K_w contains a geodesic ray γ_w such that $\gamma_v(\infty)$ and $\gamma_w(\infty)$ are contained in a common connected component κ of $\hat{\partial}X$ ($\hat{\partial}_cX$, $\vec{\partial}_cX$). Then all walls and blocks corresponding to vertices of P contain a geodesic ray whose equivalence class is contained in κ.*

Proof. Let X, v, w, P, γ_w, γ_v K_v , K_w and κ be as in the claim. First, we observe that it is sufficient to show that all walls corresponding to vertices of P contain a geodesic ray whose equivalence class is contained in κ. Indeed, assume that P is not trivial. By Lemma 3.50, all walls corresponding to vertices of P contain a geodesic ray whose equivalence class is contained in κ. Every vertex of P corresponding to a block B is adjacent to a vertex corresponding to a wall A, i.e., A and B intersect in a side S. If a geodesic ray γ is contained in this side S of A, it is contained in B. Otherwise, A is a thick wall isometric to $S \times [0, 1]$. Then A contains a geodesic ray that is asymptotic to γ. Hence, if each wall corresponding to a vertex of P contains a geodesic ray whose equivalence class is contained in κ, then all walls and blocks corresponding to a vertex of P each contain a geodesic ray whose equivalence class is contained in κ.
Recall that every geodesic ray in a side of a thick wall A is asymptotic to a geodesic ray in the interior of A. Thus, if K_v (K_w) is a thick wall, then the itinerary of γ_v (γ_w) is the trivial path consisting of the vertex v (w). Because of this, if K_v is a thick wall, we assume without loss of generality that γ_v is contained in the interior of the wall K_v. We proceed analogously with γ_w.

By definition of the itinerary, the itinerary of any geodesic ray in a thin wall and the itinerary of any geodesic ray in the interior of a thick wall is trivial, i.e., consists of a single vertex. Let $\tilde{\gamma}_v$ and $\tilde{\gamma}_w$ be the geodesic rays starting at x_{base} that are asymptotic to γ_v and γ_w respectively. By definition, $I(\gamma_v(\infty)) = I(\tilde{\gamma}_v)$ and $I(\gamma_w(\infty)) = I(\tilde{\gamma}_w)$ respectively. Because of Corollary 3.32, both $I(\tilde{\gamma}_v)$ and $I(\tilde{\gamma}_w)$ are finite paths. Let $\tilde{v}$ be the last vertex of $I(\gamma_v(\infty))$. Let P_v be the unique path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ connecting $\tilde{v}$ and v. If $v \neq \tilde{v}$, every vertex of P_v corresponding to a wall contains a geodesic ray asymptotic to γ_v because of Lemma 3.31. Analogously let $\tilde{w}$ be the last vertex of $I(\gamma_w(\infty))$. Let P_w be the unique shortest path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ connecting $\tilde{w}$ and w. If $w \neq \tilde{w}$, every vertex of P_w that corresponds to a wall contains a geodesic ray asymptotic to γ_w according to Lemma 3.31. Let v' be the vertex on P closest to $\tilde{v}$ and w' be the vertex on P closest to $\tilde{w}$. Let P'_v be the subpath of P connecting v with v'. Analogously, let P'_w be the subpath of P connecting w with w'. As $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is a tree, P'_v is contained in P_v and P'_w is contained in P_w. Then all vertices in P'_v and P'_w satisfy the desired property. If P'_v and P'_w cover P, we are done. Otherwise, P contains a path linking v' and w' that has an edge in common neither with P'_v nor with P'_w. Then the unique path between $\tilde{v}$ and $\tilde{w}$ contains this path and the claim follows from Lemma 3.50.

□

3.5 Types of connected components

Let X be a complete CAT(0) space with a block decomposition $(\mathcal{B}, \mathcal{A})$ and $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ its associated tree. Let Ξ be the contracting boundary $\vec{\partial}_c X$ of X, the visual boundary $\hat{\partial} X$ of X or the subspace $\hat{\partial}_c X$ of $\hat{\partial} X$ that consists of all equivalence classes of contracting geodesic rays. In this section, we study connected components of Ξ. We will see that Ξ has two different types of connected components and analyze these two types. The classification used is motivated by the Cycle-Join-Example of Charney and Sultan in [CS15]. In this example, Charney and Sultan calculate the contracting boundary of a right-angled Coxeter group. They observe that the contracting boundary of this right-angled Coxeter group has two types of connected components. They characterize these two types and conclude that the contracting boundary of the considered group is totally disconnected. Motivated by this, we classify connected components into two types and study their properties. This classification is based on itineraries of geodesic rays. The study of itineraries of geodesic rays has its origin in Croke's und Kleiner's example [CK00]. Thereby, the behavior of boundaries of blocks and their relation to geodesic rays not lying in the boundary of a block plays an important role. This is also the case for the for papers generalizing the example of Croke and Kleiner. See for example [Moo10], [Wil05] and [BZK19].

We use the notation established in Chapter 2. We summarized our notation concerning boundaries in Notation 1.1. Recall that X is a complete CAT(0) space with a block decomposition $(\mathcal{B}, \mathcal{A})$ and $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ its associated tree. Recall that ∂X ($\partial_c X$) denotes the set of (contracting) boundary points without a topology. Our goal is to classify the connected components of the contracting boundary $\vec{\partial}_c X$ of X, of the visual boundary $\hat{\partial} X$ of X and of the subspace $\hat{\partial}_c X$ of $\hat{\partial} X$ consisting of all equivalence classes of contracting geodesic rays. We use itineraries for that purpose. The itinerary of a geodesic ray is a (possibly infinite) path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ that describes how the ray runs through the blocks and walls of X. Like before, we choose a base point x_{base} of X. The itinerary of every (contracting) geodesic (ray) γ issuing from x_{base} starts with the same vertex v_{base} of $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. By Definition 3.34, the *itinerary* $I(\xi)$ of an element $\xi \in \partial X$ ($\partial_c X$) is the itinerary of the geodesic ray representing ξ that starts at x_{base}. Let I be a (possibly infinite) path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ starting with v_{base}. Recall from Definition 3.46 that

$$\hat{U}(I) \coloneqq \{\gamma(\infty) \in \partial X \mid \gamma(0) = x_{\text{base}} \text{ and } I = I(\gamma)\}$$
$$\hat{U}_c(I) \coloneqq \{\gamma(\infty) \in \partial_c X \mid \gamma(0) = x_{\text{base}} \text{ and } I = I(\gamma)\}.$$

We define two different types of connected components motivated by the Cycle-Join-Example in Section 4.2 of [CS15].

Definition 3.52. A connected component κ of $\hat{\partial} X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) is of type 1, if $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ contains a path I starting with v_{base} such that κ is contained in $\hat{U}(I)$. If I is finite, κ is of type 1_f. If I is infinite, κ is of type 1_∞. Otherwise, if κ is not of 1, then κ is of *type* 2.

Remark 3.53. Let κ be a connected component in $\hat{\partial}_c X$ or $\vec{\partial}_c X$. Then, every geodesic ray in κ is contracting. Thus, κ is contained in $\hat{U}(I)$ if and only if κ is contained in $\hat{U}_c(I)$.

Lemma 3.54. *Every connected component of $\hat{\partial}X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) is of type* 1 *or type* 2 *and not of type* 1 *and type* 2 *simultaneously. Every connected component of type* 1 *is of type* 1_f *or of type* 1_∞ *and not of both types simultaneously. If a connected component is of type* 1 *(type* 2*), the connected components of its elements are of type* 1 *(type* 2*).*

Proof. The Lemma follows directly from the definition of the itinerary and from Corollary 3.32. □

Because the cone topology is finer than the direct limit topology, information about types in the cone topology gives us information about the corresponding types in the direct limit topology and vice versa.

Lemma 3.55. *If a connected component κ of an element $\xi \in \partial_c X$ is of type* 1 *in $\hat{\partial}_c X$, then it is of type* 1 *in $\vec{\partial}_c X$. If a connected component κ of an element $\xi \in \partial_c X$ is of type* 2 *in $\vec{\partial}_c X$, then it is of type* 2 *in $\hat{\partial}_c X$.*

Proof. Because the direct limit topology is finer than the cone topology, every connected component of $\vec{\partial}_c X$ is contained in a connected component of $\hat{\partial}_c X$. This implies the claim. □

First, we study connected components of type 1. We say that a connected component κ of $\hat{\partial}X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) *comes from* the boundary ∂B of a block B, if the representative of every point in κ that starts at x_{base} ends in B.

Lemma 3.56. *If a connected component κ of $\hat{\partial}X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) is of type 1_f, then κ comes from the boundary ∂B of a block B.*

Proof. If all elements of a connected component have one finite itinerary, then either the itinerary is a trivial path or the itinerary ends with a vertex corresponding to a block. See Lemma 3.25. In both cases, there exists a block B such that every geodesic ray representing an element of κ ends in B. □

We say that a point $\xi \in \partial X$ *comes from* the boundary ∂B of a block B if ξ ends in B. If ξ is contracting in X, then $\xi \cap B$ is contracting in B. The converse is not true in general. It might happen that a geodesic ray is contracting in a block B but not in the ambient space X. Recall that a geodesic $\gamma \subseteq B$ is *X-contracting*, if it is contracting in the ambient space X. Recall that we denote by $\partial_{c,X} B$ the set $\{\gamma(\infty) \in \partial_c X \mid \gamma \subseteq B\}$ of equivalence classes of X-contracting geodesic rays in B. If we equip $\partial_{c,X} B$ with the subspace topology of the visual boundary and contracting boundary of X, we obtain the topological spaces $\hat{\partial}_{c,X} B$ and $\vec{\partial}_{c,X} B$ respectively. By Lemma 2.35, $\hat{\partial}_{c,X} B$ and $\vec{\partial}_{c,X} B$ are homeomorphic to the set of equivalence classes of X-contracting geodesic rays in B equipped with the subspace topology of the visual and contracting boundary of B respectively.

Lemma 3.57. *If a connected component κ of $\hat{\partial}X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) comes from a block B of X, then κ is homeomorphic to a connected component of $\hat{\partial}B$ ($\hat{\partial}_{c,X} B$, $\vec{\partial}_{c,X} B$).*

Proof. Let κ be a connected component of $\hat{\partial} X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) that comes from a block B of X. Because the visual and contracting boundary are independent of the base point we can assume without loss of generality that x_{base} is contained in B. This way, we can consider κ as subset of ∂B. Thus, κ is homeomorphic to a connected component of $\hat{\partial} B$ ($\hat{\partial}_{c,X} B$, $\vec{\partial}_{c,X} B$). $\square$

Lemma 3.58. *A connected component κ of $\hat{\partial} X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) is of type 1_∞ if and only if each point in κ has infinite itinerary if and only if no boundary point in κ comes from the boundary of a block.*

Proof. Let κ be a connected component containing two points with infinite itineraries I_0 and I_1. For every vertex between I_0 and I_1 corresponding to a wall A, κ contains an element of ∂A ($\partial_{c,X} A$) By Lemma 3.50. Every such element has finite itinerary. Hence, κ contains at least one element with infinite and one element of finite itinerary if I_0 and I_1 are distinct. It follows that κ is of type 1_∞ if and only if every point in κ has infinite itinerary.

If the itinerary of each point in κ is infinite, no point in κ comes from a block as the itinerary of each point in κ enters infinitely many blocks by Lemma 3.20. On the other hand, if a point in κ comes from a block, then its itinerary is finite by Lemma 3.57. $\square$

The last lemma implies the following.

Lemma 3.59. *If the itinerary of a point ξ in $\hat{\partial} X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) is infinite, then all points in the connected component of ξ have the same itinerary as ξ, or the connected component of ξ contains a point of finite itinerary.*

Proof. Let κ be the connected component of ξ. If κ does not contain any point with finite itinerary, then the itinerary of each point in κ is infinite. By Lemma 3.58 all points in κ have the same itinerary. As ξ is contained in κ, their itineraries coincide with the itinerary of ξ. $\square$

The next corollary is a direct consequence of the last lemma. It shows that Lemma 7 in Section 1.7 of [CK00] is true in the general setting we consider here.

Corollary 3.60. *Let $c : [0,1] \to \hat{\partial} X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) be a path, and $c(0)$ a point with infinite itinerary in $\hat{\partial} X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$). Then $c(t)$ has the same itinerary as $c(0)$ for all $t \in [0.1]$, or there is $t \in [0,1]$ such that $c(t)$ has finite itinerary.*

Let I be an infinite path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ starting with v_{base}. It is difficult to understand the set $\hat{U}(I)$. It can contain uncountable many elements. But in special cases, the set $\hat{U}(I)$ consists of a single point. Then this point is the only point with itinerary I. Ben-Zvi and Kropholler call a geodesic ray *lonely* if it is the only geodesic ray starting at x_{base} with its itinerary. They prove that certain geodesic rays in special Salvetti complexes of right-angled Artin groups are lonely. See Lemma 3.6 in [BZK19]. In later sections, we will examine lonely points as well. We will study special cases in which every connected component of type 1_∞ consists of at most one single point, i.e., we will study the case where every nonempty connected component of type 1_∞ consists of at most one point that is lonely. This property has important consequences. If every connected component of type 1_∞ consists of at most one single point, then every connected component of size at least two contains at least one geodesic ray of finite itinerary. Indeed, otherwise, it would consist of points whose itineraries are infinite. By Lemma 3.58, all these points would have the same itinerary – a contradiction.

Now, we consider connected components of type 2. Recall that a connected component κ is if type 2 if it contains at least two boundary points with distinct itineraries. The key-lemma of the last section Lemma 3.50 leads to the following observation.

Lemma 3.61. *If a connected component κ of $\hat{\partial} X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) is of type* 2*, then there is a wall A and a point $\xi \in \partial A$ ($\xi \in \partial_{c,X} A$) such that $\kappa = \kappa(\xi)$.*

Proof. Let κ be a connected component of $\hat{\partial} X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) that is of type 2. By assumption, κ contains two points with different itineraries I_0 and I_1. For every vertex between I_0 and I_1 corresponding to a wall A, κ contains an element of ∂A ($\partial_{c,X} A$) according to Lemma 3.50. By Lemma 3.25, I_0 and I_1 consist of a vertex or end in a vertex corresponding to a block. Besides, $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is bipartite. Hence, there exists at least one vertex between I_0 and I_1 that corresponds to a wall. □

Figure 3.1 summarizes the classification of connected components in $\hat{\partial}_c X$ and $\vec{\partial}_c X$. Because $\vec{\partial}_c X$ is finer than $\hat{\partial}_c X$, every connected component of an element ξ that is of type 1 in $\hat{\partial}_c X$ is also of type 1 in $\vec{\partial}_c X$. Hence, the classification pictured in Figure 3.1 leads to the classification pictured in Figure 4.3. Thereby, if γ is a geodesic ray in X, we denote the connected component of $\gamma(\infty)$ by $\kappa(\gamma(\infty))$. If it is not clear from the context which topology we consider, we write $\hat{\kappa}(\gamma(\infty))$ and $\vec{\kappa}(\gamma(\infty))$ if we mean the cone topology and the direct limit topology respectively.

We have seen that every connected component of type 2 contains a boundary point with finite itinerary. The question remains: When does every connected component of type 2 contain a boundary point with infinite itinerary?

Question 8. When does a connected component of type 2 contain a boundary point with infinite itinerary?

The last lemmas show the following equivalence

Lemma 3.62. *Each connected component of type* 2 *contains a boundary point with infinite itinerary if and only if exactly one of the following is satisfied.*

a) All connected components are of type 1*.*

b) The connected component of every point with a representative ending in a wall contains a boundary point with infinite itinerary.

The following Lemma characterizes connected components in the situation of Lemma 3.62.

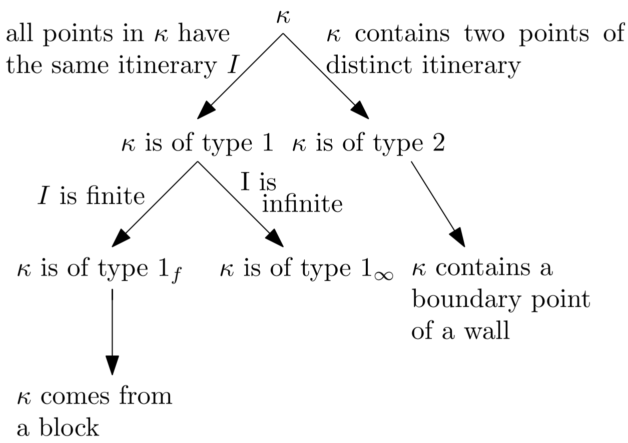

Figure 3.1 Possible types of a connected component. The letter X denotes a complete CAT(0) space with block decomposition and $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is its associated tree. The letter κ denotes a connected component in $\hat{\partial}X$, $\vec{\partial}_c X$ or $\hat{\partial}_c X$. The arrows denote implications valid under the conditions of the labels at the arrows.

Lemma 3.63. *Let κ be a connected component κ of $\hat{\partial}X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$). We assume that every connected component of type 2 contains a geodesic ray with infinite itinerary. Then the following is true*

a) κ is of type 1_f if and only if each element in κ has finite itinerary.

b) κ is of type 1_∞ if and only if the itinerary of each element in κ is infinite.

c) κ is of type 2 if and only if κ contains an element with finite itinerary and an element with infinite itinerary.

We finish this section by considering denseness properties of connected components of the visual and contracting boundary of X in the case that there is a group acting cocompactly on X. Let γ be a contracting geodesic ray in a complete CAT(0) space X. Suppose that a group G acts cocompactly on X such that $\gamma(\infty)$ is not globally fixed by G. By Proposition 4.5 and Corollary 4.7 in [Mur19] of Murray (Theorem 2.32), the orbit of $\gamma(\infty)$ is dense in the contracting boundary and the visual boundary of X.

Definition 3.64. Let X be a CAT(0) space with block decomposition. Let $\mathcal{I}_\infty$ be the set of all points of the visual (contracting) boundary of X whose connected components are of type 1_∞. Let $\mathcal{I}_f$ be the set of all points of the visual (contracting) boundary of X whose connected components are of type 1_f. Let $\mathcal{I}_2$ be the set of all points of the visual (contracting) boundary of X whose connected components are of type 2.

Corollary 3.65. *Let X be a complete CAT(0) space with a block decomposition and G a group acting cocompactly on X. If the set $\mathcal{I}_\infty$ ($\mathcal{I}_f$, $\mathcal{I}_2$) contains a point ξ that is not globally fixed by G and contracting, then $\mathcal{I}_\infty$ ($\mathcal{I}_f$, $\mathcal{I}_2$) is dense in the visual and contracting boundary of X.*

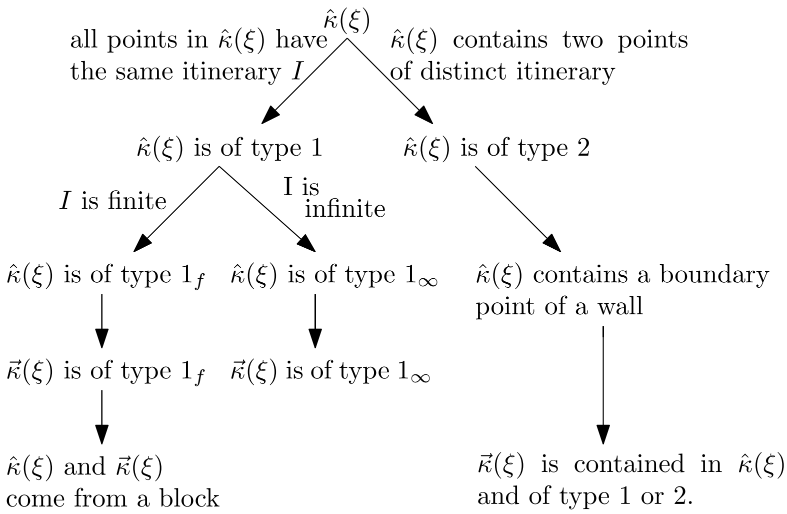

Figure 3.2 Possible types of a connected component of an element ξ in $\partial_c X$ where X is a CAT(0) space with block decomposition and $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is its associated tree. The connected component of ξ in $\hat{\partial}_c X$ is denoted by $\hat{\kappa}(\xi)$ and the connected component of ξ in $\vec{\partial}_c X$ is denoted by $\vec{\kappa}(\xi)$. The arrows denote implications valid under the conditions of the attached labels.

3.6 Actions on CAT(0) spaces with block decomposition

This section has its origin in Section 11 of Chapter II in [BH99]. In this section, Bridson and Haefliger consider the question of when an amalgamated free product $G = G_0 *_H G_1$ of two CAT(0) groups G_0 and G_1 along a CAT(0) group H is itself a CAT(0) group. They prove that this is not always the case but that there are certain cases where such groups are CAT(0). They explain how to construct spaces on which certain amalgamated free products of CAT(0) groups act geometrically. We recall this construction and observe that the obtained spaces have block decompositions with thick walls. An analog observation was mentioned in Example 6.8 in [BZ19]. We vary the construction and obtain block decompositions with thin walls if certain added conditions are satisfied. At the end of this section we list examples when the described construction can be done, i.e., we list examples of amalgamated free products of CAT(0) groups that act on CAT(0) spaces with block decomposition.

The following lemma shows properties that have to be satisfied if an amalgamated free product acts on a CAT(0) space.

Lemma 3.66. *Let $G = G_0 *_H G_1$ be an amalgamated free product of two groups G_0, G_1 and H. Let Φ_i be monomorphisms of $H \hookrightarrow G_i$, $i \in \{0, 1\}$. Suppose that G acts geometrically on a proper CAT(0) space X. Let X_i be a subspace of X that is invariant under G_i and Y be a space that is invariant under H. Then the actions of G_0, G_1 and H on X_0, X_1 and Y each are proper. Furthermore, the map $f_i : Y \to X_i$, $hx \mapsto \Phi(h)x$ is an Φ_i-equivariant isometry, $i \in \{0, 1\}$.*

Proof. The claim follows from Proposition 8.5, part I in [BH99] and the definition of equivariant maps. □

Let $G = G_0 *_H G_1$ be an amalgamated free product of two CAT(0) groups along a CAT(0) group. The lemma shows that G does not act geometrically on a CAT(0) space X if the following happens. Suppose that the space X contains two spaces X_0 and X_1 respectively that both contain an isometrically embedded copy of a space Y. Assume that X_0 is invariant under G_0, that X_1 is invariant under X_1 and that both copies of Y in X_0 and X_1 are invariant under H. Suppose that these two copies of Y don't fit together in the sense that there is no Φ_0-equivariant or Φ_1-equivariant isometry of Y to X_0 or X_1. Then X is not CAT(0) or G does not act on X geometrically.

Bridson and Haefliger use this observation for proving in Proposition 6.10 of Γ.6 in part III of [BH99] that there is an amalgamated free product of two CAT(0) groups that is not CAT(0). By the lemma above, invariant subspaces under G_i, $i \in \{0, 1\}$ might not be CAT(0) spaces. Furthermore, the action of G_i, $i \in \{0, 1\}$ might not be cocompact. So, in general, it is difficult to say when a free amalgamated product acts geometrically on a CAT(0) space. However, Bridson and Haefliger formulate conditions under which an amalgamated free product of CAT(0) groups is itself CAT(0): suppose that G is an amalgamated free product of two CAT(0) groups G_0 and G_1 acting on proper CAT(0) spaces X_0 and X_1 respectively. Assume that H acts geometrically on a proper CAT(0) space Y. If there are Φ_i-equivariant maps $Y \hookrightarrow X_i$ like in the lemma above, then G is CAT(0).

For proving this, Bridson and Haefliger construct a CAT(0) space X on which G acts geometrically. This construction is similar to the construction of total spaces associated to amalgamated free products in Bass-Serre theory. Compare [SW79]. The constructed spaces turn out to have block decompositions with thick walls. We recall the construction of X in the language of block decompositions. Afterwards we explain when we can shrink the thick walls of X to thin walls such that the group acts still geometrically on the resulting space.

The following convention and definitions can be found in Theorem 11.18, chapter II of [BH99].

Convention 3.67. Let G_0, G_1 and H be groups that act properly and cocompactly by isometries on proper CAT(0) spaces X_0, X_1 and Y respectively. Suppose that for $j \in \{0,1\}$ there exist a monomorphism $\Phi_j : H \hookrightarrow G_j$ and a Φ_j-equivariant isometric embedding $f_j : Y \hookrightarrow X_j$. Let $G = G_0 *_H G_1$. We assume that the amalgamated free product $G = G_0 *_H G_1$ is not trivial, i.e., that H is not isomorphic to G_0 or G_1.

Remark 3.68. In the assumptions of Theorem 11.18 of Bridson and Haefliger, the groups X_0, X_1 and Y are not assumed to be proper but to be complete. By the Hopf-Rinow Theorem, a length-spaces is proper if and only if it is complete and locally compact. So, properness implies that X_0, X_1 and Y are complete. We use a stronger condition than Bridson and Haefliger because then all spaces in this chapter are proper. Bridson and Haefliger work with a slightly stronger version of proper actions. With this stronger version of proper actions, every space that admits a geometric action is itself proper. This follows from Exercise 8.4 (1) in [BH99]. In case that a group acts geometrically on a proper length space, the stronger version of proper actions Bridson and Haefliger use coincides with the usual definition. For more details, see Remark 2.3.

The groups G_0 and G_1 are subgroups of $G = G_0 *_H G_1$. Let $U_0 = \Phi_0(H)$, and $U_1 = \Phi_1(H)$. By means of the isomorphism $\Phi = \Phi_1\Phi_0^{-1} : U_0 \to U_1$ we identify H with the subgroup $G_0 \cap G_1$ in $G = G_0 *_H G_1$. For simplifying the notation, we omit the embeddings of H in the groups G_0, G_1 and G and write H if we mean the embedded group H in G_0, G_1 or G. We denote by $\mathcal{T}_{\text{ext}}$ the extended Bass-Serre tree of $G = G_0 *_H G_1$ as defined in Definition 2.59.

Let X be the space $(G \times X_0) \bigsqcup (G \times [0,1] \times Y) \bigsqcup (G \times X_1)$.
We study the equivalence relation on X generated by

$$(gg_0, x_0) \sim (g, g_0.x_0),\ (gg_1, x_1) \sim (g, g_1.x_1),\ (gh, t, y) \sim (g, t, h.y),$$
$$(g, f_0(y)) \sim (g, 0, y),\ (g, f_1(y)) \sim (g, 1, y)$$

for all $g \in G$, $\gamma_0 \in G_0$, $g_1 \in G_1$, $h \in H$, $x_0 \in X_0$, $x_1 \in X_1$, $t \in [0,1]$, $y \in Y$.

Definition 3.69. Let G_0, X_0, G_1, X_1, H, Y and $G = G_0 *_H G_1$ be as in Convention 3.67. Let $\mathcal{X} = \mathcal{X}(G_0, X_0, G_1, X_1, H, Y)$ be the quotient of X by the equivalence relation $\sim$.

With help of the proof of Theorem 11.18 in chapter II in [BH99], we observe that $\mathcal{X}$ is a CAT(0) space that has a block decomposition with thick walls. This statement is related to Example 6.8 in [BZ19].

Lemma 3.70. *Let G_0, X_0, G_1, X_1, H, Y and $G = G_0 *_H G_1$ be as in Convention 3.67 and $\mathcal{X} = \mathcal{X}(G_0, X_0, G_1, X_1, H, Y)$ as in Definition 3.69. The space $\mathcal{X}$ is a CAT(0) space that has a block decomposition with thick walls $(\mathcal{B}, \mathcal{A})$ such that*

a) for every coset gG_0 of G_0 in G, $\mathcal{B}$ contains a block $B^{(gG_0)}$ that is isometric to X_0 and has parity $(-)$,

b) for every coset gG_1 of G_1 in G, $\mathcal{B}$ contains a block $B^{(gG_1)}$ that is isometric to X_1 and has parity $(+)$,

c) for every coset gH of H in G, $\mathcal{A}$ contains a thick wall $A^{(gH)}$ isometric to $[0,1] \times Y$,

d) the two sides $A_0^{(gH)} = (\{0\} \times Y)^{(gH)}$ and $A_1^{(gH)} = (\{1\} \times Y)^{(gH)}$ of any thick wall $A^{(gH)} = ([0,1] \times Y)^{(gH)}$ in $\mathcal{A}$ are contained in the blocks $B^{(gG_0)}$ and $B^{(gG_1)}$ respectively.

e) the tree $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ associated to $(\mathcal{B}, \mathcal{A})$ is isometric to the extended Bass-Serre tree $\mathcal{T}_{ext}$ associated to G.

Proof. Let $i \in \{0,1\}$ and $\mathcal{B}_i$ be the quotient of $G \times X_i$ by the restriction of the equivalence relation above. For every coset $gG_i \in G/G_i$, $\mathcal{B}_i$ contains a copy isometric to X_i. Indeed, let $(g, x_i) \in G \times X_i$ and $[(g, x_i)]$ be its equivalence class in $\mathcal{B}_i$. Then $[(g, x_i)] = gG_i \times \{x_i\}$ and $\bigcup_{x_i \in X_i} [(g, x_i)] = \bigcup_{x_i \in X_i} gG_i \times \{x_i\} = gG_i \times X_i$. Every such copy corresponds to a block as defined in the claim. The set $\mathcal{B} \coloneqq \mathcal{B}_0 \cup \mathcal{B}_1$ is the set of all blocks. By the definition of the equivalence relation above, the blocks in $\mathcal{B}_i$ are pairwise disjoint. Like above we see that for every coset gH of H in G/H, the quotient $\mathcal{A}$ of $G \times [0,1] \times Y$ by the equivalence relation above contains a copy isometric to $[0,1] \times Y$. These are the walls as defined in the claim. Furthermore, the equivalence relation implies that two sides of every thick wall are contained in a block. The intersection of two thick walls is empty or contained in one of their sides. The intersection of a thick wall and a block is empty or a side of a thick wall. For every $g \in G$, the block $B^{(gG_i)}$ is glued to $A^{(gH)}$ along its sides $A_i^{(gH)}$, $i \in \{0,1\}$. If we give all blocks in the set $\mathcal{B}_0$ parity $(-)$ and all blocks in the set $\mathcal{B}_1$ parity $(+)$, the parity condition is satisfied. From the definition of the extended Bass-Serre tree of G follows that $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ and $\mathcal{T}_{\text{ext}}$ are isometric.

It remains to show that $\mathcal{X}$ is CAT(0). Like in the proof of Theorem 11.18 in part II in [BH99], we show that we can construct $\mathcal{X}$ inductively by means of $\mathcal{T}_{\text{ext}}$. Then it follows from Theorem 11.3 in part II of [BH99], that $\mathcal{X}$ is CAT(0). Recall that there is a natural projection from the space $\mathcal{X}$ to $\mathcal{T}_{\text{ext}}$. See Definition 3.10. We can construct the space $\mathcal{X}$ by means of the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$ as follows. We choose a vertex v of $\mathcal{T}_{\text{ext}}$. Its preimage under $p_{(\mathcal{B},\mathcal{A})}$ is a block $B^{(gG_i)}$ isometric to X_i with label gG_i, $g \in G$. The preimage of every outgoing 2-path v, v', v'' of v is a wall $A^{(g'H)}$ with label $g'H$ such that $g'G_i = gG_i$. We glue the wall $A^{(g'H)}$ to the block $B^{(gG_i)} = B^{(g'G_i)}$ along the side

$A_i^{(gH)}$ in $A^{(g'H)}$. The preimage of v'' under p is the block $B^{(g'G_j)}$, $i, j \in \{0,1\}$, $j \neq i$. We glue this block to $A^{(g'H)}$ along the side $A_j^{(g'H)}$ in the wall $A^{(g'H)}$. We repeat the same procedure for any other outgoing 2-path of v. We continue in this manner for vertices corresponding to blocks of larger and larger distance to v. It follows from Theorem 11.3 in part II of [BH99], that $\mathcal{X}$ is CAT(0). □

We summarize some properties of $\mathcal{X}$ from the proof of Theorem 11.18 in part II of [BH99]. The group G acts by left multiplication on the set of blocks $\mathcal{B}$ and the set of walls $\mathcal{A}$. For every $g, g' \in G$, $i \in \{0,1\}$, G maps the blocks $B^{(g'G_0)}$, $B^{(g'G_1)}$ and the wall $A^{(g'H)}$ to the blocks $B^{(gg'G_0)}$, $B^{(gg'G_1)}$ and $A^{(gg'H)}$ respectively. The stabilizer of $B^{(gG_i)}$ in G is the group gG_ig^{-1}. The stabilizer of $A^{(gH)}$ in G is the group gHg^{-1}. The group gG_ig^{-1} acts on the block $B^{(gG_i)}$ geometrically. Analogously, the group gHg^{-1} acts on the wall $A^{(gH)}$ geometrically. Accordingly, on every block of parity $(-)$ acts a conjugate of G_0 in G, on every bock of parity $(+)$ acts a conjugate of G_1 in G and on every wall acts a conjugate of H in G. Hence, the actions of G_0, G_1 and H on X_0, X_1 and Y induce an action $\cdot_G$ of G on $\mathcal{X}$. If $x_i \in B^{(\mathrm{id}\, G_i)}$ and $g \in G$, then $g \cdot_G x_i$ is contained in $B^{(gG_i)}$ and $g \cdot_G g_i \cdot_G x_i = g \cdot_G (g_i \cdot_{G_i} x_i)$ for all $g_i \in G_i$. Analogously, if $y \in A^{(\mathrm{id}\, H)}$ and $g \in G$, then $g \cdot_G y$ is contained in $A^{(gH)}$ and $g \cdot_G h \cdot_G y = g \cdot_G (h \cdot_H x_0)$ for all $h \in H$.

We describe by means of the group action of G on $\mathcal{X}$ how walls and blocks are glued to each other. For that purpose we identify $B^{(\mathrm{id}\, G_i)}$ with the space X_i, $i \in \{0,1\}$ and $A^{(\mathrm{id}\, H)}$ with the space $[0,1] \times Y$. Let $g \in G$. Every wall $A^{(gH)}$ is glued to $B^{(gG_0)}$ and $B^{(gG_1)}$ along the sides $A_0^{(gH)}$ and $A_1^{(gH)}$ of $A^{(gH)}$ according to the gluing maps $\tilde{f}_i$, $i \in \{0,1\}$ defined as follows:

$$\begin{aligned} \tilde{f}_i : A_i^{(gH)} &\to B^{(gG_i)} \\ y_i &\mapsto g \cdot f_i(g^{-1} \cdot y_i). \end{aligned}$$

Remark 3.71. (stabilizers of sides of walls) Let S be a side of a wall A in $\mathcal{X}$. Then S is contained in a block. Let's assume that it is the block $B^{(\mathrm{id}\, G_0)}$. Then the stabilizer of $B^{(\mathrm{id}\, G_0)}$ in G is G_0 and G_0 acts geometrically on $B^{(\mathrm{id}\, G_0)}$. Clearly, the side S is invariant under H. Hence, the stabilizer of S in G_0 contains H. The stabilizer of S in G_0 may be larger than H. However, if H' is the stabilizer of S in G_0, then $[H' : H] < \infty$. Indeed, by assumption, S is invariant under H'. Thus, every conjugate of H in H' contributes an element to the stabilizer of a point x in S. If $[H' : H] = \infty$, there are infinitely many such conjugates and the stabilizers of points in S is infinite. Then the action of G_0 on X_0 is not proper. Accordingly, there is only a limited number of walls that share a common side.

Theorem 3.72. *([BH99, Thm 11.18 in part II]) If all conditions of Convention 3.67 are satisfied, then the space $\mathcal{X} = \mathcal{X}(G_0, X_0, G_1, X_1, H, Y)$ is proper and G acts properly and cocompactly by isometries on $\mathcal{X}$.*

The action in Theorem 3.72 is induced by the actions of G_0 on X_0, G_1 on X_1 and H on Y as described above.

Remark 3.73. In Theorem 11.18 in part II in [BH99], Bridson and Haefliger state that the space $\mathcal{X}$ is complete and not proper. Properness follows because we assume X_0, X_1 and Y to be proper. See Remark 3.68.

The natural question arises what happens if we shrink all thick walls to thin walls. Does G still act on the obtained space geometrically? We observe that three problems might occur. First, the stabilizer of $f_i(Y)$ in G_i does not have to be equal to H, $i \in \{0, 1\}$. Let's for example assume that the stabilizer H' of $f_0(Y)$ is a subgroup of G_0 that contains H. Let B be a block of $\mathcal{X}$ and A be a thick wall adjacent to B. The intersection of B and A is a side S of A. This side S is stabilized by a conjugate of H' in G_0. Hence, for every coset of H in H' a thick wall A is glued to B along S. There are $[H' : H]$ thick walls adjacent to B that each contain the side S of A. By assumption, $[H' : H] \geq 2$. If we shrink all thick walls to thin walls, B intersects afterwards at least two other blocks in the resulting space. Then at least 3 blocks intersect each other in a wall. This contradicts the parity condition of block decompositions with thin walls.

Secondly, it might happen that the action is not proper anymore when we shrink thick walls of $\mathcal{X}$ to thin walls. If x is a point in a block B of $\mathcal{X}$ and g is a group element of G, then there are two possibilities. Either $g \cdot x$ is contained in B or $g \cdot x$ is moved to another block. In the first case, g is contained in the stabilizer of B, i.e., in a conjugate of G_0 or G_1 in G. By assumption, this conjugate acts properly on B. If $g \cdot x$ lies in another block, then the distance of $g \cdot x$ and x is at least 2. Indeed, by the parity condition, x is mapped to a block of the same parity. Hence, we have to pass through at least two walls to move from x to $g \cdot x$. This is the reason why the action of G on $\mathcal{X}$ is proper. If we shrink all thick walls to thin walls, it might happen that x and $g \cdot x$ are shrunk to the same point in $\mathcal{X}$. If this happens with infinitely many group elements, the stabilizer of x is not finite anymore. Then the action of G on the space with shrunk walls is not proper anymore.

A third problem is that the isometric embedded space $f_i(Y)$ in X_i could overlap an image of $f_i(Y)$ under a group element $g_i \in G_i$, $i \in \{0, 1\}$. In this case, a block B intersects at least two other blocks if we shrink all thick walls to thin walls. Then the parity condition of CAT(0) spaces with thin walls is not satisfied.
To avoid that these problems occur, we formulate the following convention.

Convention 3.74. Like in Convention 3.67 let G_0, G_1 and H be groups that act properly and cocompactly by isometries on proper CAT(0) spaces X_0, X_1 and Y respectively. Like in Convention 3.67 suppose that for $j \in \{0, 1\}$ there exist a monomorphism $\Phi_j : H \hookrightarrow G_j$ and a Φ_j-equivariant isometric embedding $f_j : Y \hookrightarrow X_j$. In addition, we assume that the following conditions are satisfied:

a) stabilizer condition: For $j \in \{0, 1\}$, H is the stabilizer of $f_j(Y)$ in G_j.

b) ϵ-condition: For $j \in \{0, 1\}$ there is an $\epsilon_j > 0$ so that for all two group elements g_j and g'_j in G_j we have: The subsets $g_j f_j(Y)$ and $g'_j f_j(Y)$ of X_j have nonempty intersection if and only if their ϵ_j-neighborhoods have nonempty intersection.

Let X' be the space $(G \times X_0) \bigsqcup (G \times Y) \bigsqcup (G \times X_1)$.

We study the equivalence relation on X' generated by

$$(gg_0, x_0) \sim (g, g_0.x_0),\ (gg_1, x_1) \sim (g, g_1.x_1),\ (gh, y) \sim (g, h.y),$$
$$(g, f_0(y)) \sim (g, y),\ (g, f_1(y)) \sim (g, y)$$

for all $g \in G$, $\gamma_0 \in G_0$, $g_1 \in G_1$, $h \in H$, $x_0 \in X_0$, $x_1 \in X_1$, $y \in Y$.

Definition 3.75. Let G_0, X_0, G_1, X_1, H, Y be as in Convention 3.74.
Let $\mathcal{X}' = \mathcal{X}'(G_0, X_0, G_1, X_1, H, Y)$ be the quotient of X' by the equivalence relation $\sim$.

We observe that $\mathcal{X}'$ is the space we obtain from $\mathcal{X}$ as defined in Definition 3.69 by shrinking all thick walls to thin walls.

Lemma 3.76. *Let G_0, X_0, G_1, X_1, H, Y and $G = G_0 *_H G_1$ be as in Convention 3.74 and $\mathcal{X}' = \mathcal{X}'(G_0, X_0, G_1, X_1, H, Y)$ as in Definition 3.75. The space $\mathcal{X}'$ is a CAT(0) space with a block decomposition with thin walls $(\mathcal{B}, \mathcal{A})$ such that*

a) for every coset gG_0 of G_0 in G, $\mathcal{B}$ contains a block $B^{(gG_0)}$ that is isometric to X_0 and has parity $(-)$,

b) for every coset gG_1 of G_1 in G, $\mathcal{B}$ contains a block $B^{(gG_1)}$ that is isometric to X_1 and has parity $(+)$,

c) for every coset gH of H in G, $\mathcal{A}$ contains a thin wall $A^{(gH)}$ isometric to Y,

d) any thin wall $A^{(gH)}$ in $\mathcal{A}$ is contained in the blocks $B^{(gG_0)}$ and $B^{(gG_1)}$,

e) the tree associated to $(\mathcal{B}, \mathcal{A})$ is isometric to the extended Bass-Serre tree $\mathcal{T}_{ext}$ associated to G.

Proof. Let $i \in \{0, 1\}$ and $\mathcal{B}_i$ be the quotient of $G \times X_i$ by the restriction of the equivalence relation above. Like in the proof of Lemma 3.70 we observe that for every coset $gG_i \in G/G_i$ $\mathcal{B}_i$ contains a copy isometric to X_i. Every such copy corresponds to a block as defined in the claim. The set $\mathcal{B} := \mathcal{B}_0 \cup \mathcal{B}_1$ is the set of all blocks. The union of all blocks covers $\mathcal{X}'$. For every coset gH of H in G/H, the quotient $\mathcal{A}$ of $G \times Y$ by the equivalence relation above contains a copy isometric to Y. These are the thin walls as defined in the claim. For every $g \in G$, the block $B^{(gG_0)}$ is glued to $B^{(gG_1)}$ along $A^{(gH)}$. Since H is the stabilizer of Y in G_0 and G_1 respectively, every wall A is contained in at most two blocks. Because of the ϵ-condition of Convention 3.74, A does not intersect any wall different to A. If we give all blocks in the set $\mathcal{B}_0$ parity $(-)$ and all blocks in the set $\mathcal{B}_1$ parity $(+)$, the parity condition is satisfied. Furthermore, two blocks intersect if and only if their ϵ-neighborhoods intersect. From the definition of the extended Bass-Serre tree of G follows that $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ and $\mathcal{T}_{\text{ext}}$ are isometric.

It remains to show that $\mathcal{X}'$ is CAT(0). Like in the proof of Theorem 11.18 in part II in [BH99], we show that we can construct $\mathcal{X}'$ inductively by means of the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$. Then it follows from Theorem 11.3 in part II of [BH99], that $\mathcal{X}'$ is CAT(0).

Recall that there is a natural projection from the space $\mathcal{X}'$ to $\mathcal{T}_{\text{ext}}$. See Definition 3.2. We can construct the space $\mathcal{X}'$ by means of $\mathcal{T}_{\text{ext}}$ as follows. We choose a vertex v of $\mathcal{T}_{\text{ext}}$ whose label is a coset of G_0 or G_1. Because of the ϵ-condition of Convention 3.74, its preimage under $p_{(\mathcal{B},\mathcal{A})}$ contains exactly one block $B^{(gG_i)}$ isometric to X_i with label gG_i, $g \in G$. Because of the ϵ-condition of Convention 3.74 and the parity condition, the preimage of every outgoing 2-path v, v', v'' of v contains only one wall $A^{(g'H)}$ with label $g'H$ such that $g'G_i = gG_i$. This wall $A^{(g'H)}$ is isometrically embedded in $B^{(gG_i)} = B^{(g'G_i)}$. Because of the ϵ-condition, the preimage of v'' under p contains the block $B^{(g'G_j)}$, $i, j \in \{0,1\}$, $j \neq i$ in which $A^{(g'H)}$ is isometrically embedded as well. We glue $B^{(g'G_j)}$ and $B^{(g'G_i)}$ along the isometrically embedded walls. We repeat the same procedure for any other outgoing 2-path of v. We continue in this manner for vertices corresponding to blocks of larger and larger distance to v. □

Corollary 3.77. *If all conditions of Convention 3.74 are satisfied, then the space $\mathcal{X}'$ as defined in Definition 3.75 is proper and G acts properly and cocompactly by isometries on $\mathcal{X}'$.*

Proof. By Lemma 3.76, $\mathcal{X}'$ is a CAT(0) space with block decomposition with thin walls. Like before, the action of G is induced by the actions of G_0, G_1 and H on X_0, X_1 and Y respectively. We follow the proof of Theorem 11.18 in part II in [BH99]. We show first that the action is proper. If x is a point in a block B of $\mathcal{X}'$ and g is a group element of G, then there are two possibilities. Either $g \cdot x$ is contained in B or $g \cdot x$ is moved to another block. In the first case, g is contained in the stabilizer of B, i.e., in a conjugate of G_0 or G_1 in G. By assumption, this conjugate acts properly on B. If $g \cdot x$ lies in another block, then the distance of $g \cdot x$ and x is at least $\min\{\epsilon_1, \epsilon_2\}$ where ϵ_1 and ϵ_2 are as defined in Convention 3.74. Indeed, x is mapped to a block that has the same parity as B. By the parity condition, no blocks of the same parity are adjacent. Hence, the geodesic ray connecting x and gx intersects at least one wall in B and another wall that is not contained in B. The distance between these two walls is by assumption $\min\{\epsilon_1, \epsilon_2\}$. Thus, the action of G on $\mathcal{X}'$ is proper. The action of G on $\mathcal{X}'$ is cocompact since the actions of G_0, G_1 and H on X_0, X_1 and Y are cocompact and $G \cdot (B^{(\operatorname{id} G_0)} \cup B^{(\operatorname{id} G_1)})$ covers $\mathcal{X}'$. The space $\mathcal{X}'$ is proper because G acts geometrically on $\mathcal{X}'$. See Remark 3.68 and Exercise 8.4 (1) in [BH99]. □

Motivated by the group actions in this section, we study in this thesis group actions on CAT(0) spaces with block decomposition of the following form.

Convention 3.78. Let G_0, G_1 and H be groups acting geometrically on proper CAT(0) spaces X_0, X_1 and Y respectively. Suppose that $G = G_0 *_H G_1$ acts geometrically on a proper CAT(0) space $\mathbb{X} = \mathbb{X}(G_0, X_0, G_1, X_1, H, Y)$ with block decomposition $(\mathcal{B}, \mathcal{A})$ satisfying the following conditions.

a) For every coset gG_0 of G_0 in G, $\mathcal{B}$ contains a block $B^{(gG_0)}$ that is isometric to X_0 and has parity $(-)$.

b) For every coset gG_1 of G_1 in G, $\mathcal{B}$ contains a block $B^{(gG_1)}$ that is isometric to X_1 and has parity $(+)$.

c) For every coset gH of H in G, $\mathcal{A}$ contains a wall $A^{(gH)}$. If $(\mathcal{B}, \mathcal{A})$ is a block decomposition with thin walls, $A^{(gH)}$ is isometric to Y. Otherwise, $A^{(gH)}$ is isometric to $[0, 1] \times Y$.

d) Any wall $A^{(gH)}$ in $\mathcal{A}$ is adjacent to the blocks $B^{(gG_0)}$ and $B^{(gG_1)}$.

e) The tree $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ associated to $(\mathcal{B}, \mathcal{A})$ is isometric to the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$ associated to $G = G_0 *_H G_1$. We identify $\mathcal{T}_{\text{ext}}$ with $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ and say that a vertex with label gH in $\mathcal{T}_{\text{ext}}$ corresponds to the wall $A^{(gH)}$. Analogously a vertex with label gG_i corresponds to the block $B^{(gG_i)}$.

f) The stabilizer of $B^{(gG_i)}$ in G is gG_ig^{-1} for all $g \in G$, $i \in \{0, 1\}$. The stabilizer of every side of $A^{(gH)}$ in G is gHg^{-1} for all $g \in G$. The action of the stabilizer G_i on $B^{(\text{id}\, G_i)}$ is given by the action of G_i on X_i, $i \in \{0, 1\}$. The action of the stabilizer H on every side of $A^{(\text{id}\, H)}$ is given by the action of H on Y.

We denote the set of all blocks of parity $(-)$ by $\mathcal{B}^-$ and the set of all blocks of parity $(+)$ by $\mathcal{B}^+$.

Item e) and Item f) imply that the action of G on $\mathbb{X}$ is compatible with the action of G on the extended Bass-Serre tree. Hence, the stabilizer of any block in G is a conjugate of G_0 or G_1 and the stabilizer of any wall is a conjugate of H. Thus, on every block acts a conjugate of G_0 or G_1 and on every wall acts a conjugate of H. Item f) ensures that the corresponding actions come from the actions of G_0 on X_0, G_1 on X_1 and H on Y. Let G, G_0, G_1, H, X_0, X_1 and Y as in Convention 3.67. We have seen above that there exists a CAT(0) space $\mathbb{X}$ with block decomposition such that all conditions of Convention 3.78 are satisfied. By Theorem 3.72, $\mathbb{X}$ can always be chosen as a space having a block decomposition $\mathcal{X}$ with thick walls. If not only Convention 3.67 but also Convention 3.74 is satisfied, then $\mathbb{X}$ can be chosen as a space having a block decomposition with thin walls $\mathcal{X}'$. See Corollary 3.77. We finish this chapter by listing a few examples where Convention 3.67 is satisfied. In such cases, a space as in Convention 3.78 exists.

Example 3.79 (Situations in which a space as in Convention 3.78 exists)**.** In the following situations, Convention 3.67 is satisfied and a space as in Convention 3.78 exists.

a) Amalgamated free products of CAT(0) groups along groups that contain a cyclic subgroup of finite index. See Corollary 11.19 in Section 11 of part II in [BH99].

b) Davis complexes of amalgamated free products of right-angled Coxeter groups along special subgroups. See Section 5.2 in Chapter 5.

c) Universal covers of Salvetti complexes of certain right-angled Artin groups. See for example [CK00], [Wil05], [Moo10], and [BZK19].

Item a) includes amalgamated free products along groups that contain an infinite cyclic subgroup of finite index. This is equivalent to being quasi-isometric to $\mathbb{Z}$ (see Lemma 8.40 of part I in [BH99]). In Section 4.4, we study contracting boundaries of such amalgamated free products. In Chapter 5 of this thesis, we will see that Davis complexes of infinite right-angled Coxeter groups have nontrivial block decompositions with thin walls. We will study contracting boundaries of right-angled Coxeter groups by examining such block decompositions.

4 Contracting boundaries of amalgamated free products of CAT(0) groups

In this chapter, we calculate contracting boundaries of amalgamated free products that act geometrically on a CAT(0) space with block decomposition.

In Section 4.1, we analyze a theorem that was recently proven independently by Ben-Zvi and Kropholler and compare this theorem with the focus of this thesis. The theorem of Ben-Zvi and Kropholler provides examples for visual boundaries that are not path connected but contain a big path-component and belong to CAT(0) spaces admitting a geometric action of a free amalgamated product of CAT(0) groups. For showing their theorem, Ben-Zvi and Kropholler use a similar cutset property as we examine in Chapter 3. Our variant of the cutset property enables us to formulate an analog to the theorem of Ben-Zvi and Kropholler for contracting boundaries of CAT(0) spaces with block decomposition. We investigate the meaning of this variant and explain why the focus of this variant is different form the viewpoint of this thesis.

In Section 4.2, we generalize the Cycle-Join-Example of Charney and Sultan in Section 4.2 of [CS15]. We study amalgamated free products acting on a CAT(0) space with block decomposition whose walls don't contain contracting geodesic rays. We prove in Theorem 4.10 that such spaces have disconnected contracting boundaries and calculate the connected components of such spaces. This leads to examples of amalgamated free products that have totally disconnected contracting boundaries. See Corollary 4.11.

Section 4.3 concerns boundaries of proper CAT(0) spaces. We study connected components of points in subspaces of visual boundaries whose representatives are oriented axes for rank-one isometries. In Theorem 4.24 we prove that the boundary points associated to an axis for a rank-one isometry are either both contained in a common connected component or that they consist of single points. We conclude several consequences for the contracting boundary of X. As a preparation for Section 4.4, we finish this section with the study of axes for rank-one isometries that are contained in CAT(0) spaces with block decomposition on which a group acts geometrically.

In Section 4.4, we study contracting boundaries of amalgamated free products of CAT(0) groups along groups that are quasi-isometric to $\mathbb{Z}$. By Corollary 11.19 in Section 11 of Chapter II in [BH99] such groups act geometrically on a CAT(0) space with block decomposition. Recall that we study in Section 4.2 the case where walls don't contain geodesic rays that are contracting in the ambient space. Hence, we concentrate on the remaining case where every wall contains a geodesic ray which is contracting in the ambient space. In this situation, two types of connected components can occur. We characterize these two types of connected components. In Corollary 4.39, we conclude that connected components of type 1 are either single points or come from the boundaries of blocks. Afterwards we study connected components of type 2. Using the results of Section 4.3, we show the following: If there exists a connected component of type 2, then the set of connected components of type 2 is bijective to a set of pairwise isometric edge-disjoint subtrees of $\mathcal{T}_{\text{ext}}$ that cover $\mathcal{T}_{\text{ext}}$. The amalgamated free product acts on this set of trees transitively. See Theorem 4.50 and Lemma 4.49. At last, we investigate consequences for the case where G_0 and G_1 have totally disconnected boundaries and finish the section with Question 11.

4.1 A variant of a theorem of Ben-Zvi and Kropholler

Recently, Ben-Zvi and Kropholler [BZK19, Thm 3.2] proved a theorem that provides examples for visual boundaries that are not path connected but contain a big path-component and belong to CAT(0) spaces admitting a geometric action of a free amalgamated product. To show their theorem, they prove that the boundary points of walls behave like cutsets of path-components. See Lemma 3.1 in [BZK19]. In Corollary 3.43, we proved independently a variant of this lemma concerning connected components of visual and contracting boundaries. This enables us to prove an analog to the theorem of Ben-Zvi and Kropholler for contracting boundaries of CAT(0) spaces with block decompositions. See Theorem 4.2. We analyze this variant and compare it with the considerations in this thesis.

Like Ben-Zvi and Kropholler we say that a subspace X_H of a topological space X *separates* two sets X_0 and X_1 if every path from X_0 to X_1 passes through X_H. The *limit set* $\Lambda(H)$ of a subgroup H of $\mathrm{Iso}(X)$ is the set of accumulation points in $\hat{\partial}X$ $(\hat{\partial}_c X, \vec{\partial}_c X)$ of an orbit of the action of G on X. Suppose that X is a CAT(0) space with block decomposition $(\mathcal{B}, \mathcal{A})$. Like Ben-Zvi and Kropholler, we say that X has a *connected block decomposition* if $\bigcup_{B\in\mathcal{B}} \hat{\partial}B$ is path connected. In this case, we call the path component of $\bigcup_{B\in\mathcal{B}} \hat{\partial}B$ *nexus* of X and denote it by $Nex(X)$. Recall that a CAT(0) space is proper if and only if it is complete and locally compact because of the Hopf-Rinow theorem.

Ben-Zvi and Kropholler proved the following theorem.

Theorem 4.1. *([BZK19, Thm 3.2]) Let $G = G_0 *_H G_1$ be a CAT(0) group acting geometrically on a proper CAT(0) space X. Suppose that G_0 and H act geometrically on subspaces X_0 and X_H, respectively. Furthermore, suppose that X_H and its translates separate X_0 from the rest of X. Lastly, suppose that X_0 satisfies the following:*

a) X_0 has a block decomposition $(\mathcal{B}, \mathcal{A})$ with thin walls such that $\bigcup_{B\in\mathcal{B}} \hat{\partial}B$ is nonempty and path connected

b) $\hat{\partial}X_0$ is not path connected and

c) $\Lambda(H)$ is contained in $Nex(X_0)$

Then $\hat{\partial}X$ is not path connected.

This theorem is a consequence of Lemma 3.1 of Ben-Zvi and Kropholler in [BZK19].

Lemma. *([BZK19, Lemma 3.1]) Let X be a proper CAT(0) space and let A and B be closed subsets of X. Suppose there exists a closed subset C such that any geodesic from A to B passes through C. Then any path in the boundary between $\hat{\partial}A$ and $\hat{\partial}B$ passes through $\hat{\partial}C$.*

In Corollary 3.43, we proved independently the following variant of the lemma of Ben-Zvi and Kropholler.

Lemma (Corollary 3.43)**.** *Let X be a complete CAT(0) space and X_0, X_1 closed subsets such that the intersection $Z = X_0 \cap X_1$ is convex and $X = X_0 \cup X_1$. If a connected component κ in $\hat{\partial} X$ ($\hat{\partial}_c X$, $\vec{\partial}_c X$) contains a boundary point in ∂X_0 and in ∂X_1, then κ contains a boundary point in ∂Z.*

Unlike the Lemma of Ben-Zvi and Kropholler, the last corollary holds not only for visual boundaries but also for contracting boundaries. In particular, it can be applied to path-components of contracting boundaries of CAT(0) spaces with block decomposition. Indeed, suppose that X is a complete CAT(0) space with block decomposition $(\mathcal{B}, \mathcal{A})$. Let A be a wall. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the two trees we obtain by removing the two edges incident to v_A form $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. X decomposes into two disjoint subspaces X_0' and X_1' if we delete A from X such that X_i' is covered by the union of all walls and blocks that correspond to vertices in $\mathcal{T}_i$ for $i \in \{0, 1\}$. Let X_0 be the union of X_0' and A and X_1 be the union of X_1' and A. Then, $X = (X_0 \cup A) \cup (X_1 \cup A)$ and $(X_0 \cup A) \cap (X_1 \cup A) = A$ and we can apply the lemma above: Any path in the contracting boundary of X that links a boundary point of X_0 with a boundary point of X_1 passes through the boundary of the wall A. It follows, that Ben-Zvi's and Kropholler's Theorem 3.2 of [BZK19] can be formulated for contracting boundaries of CAT(0) spaces with block decomposition. For completeness, we formulate this variant not only for contracting boundaries but for visual boundaries as well. In the case of visual boundaries, the next theorem is a direct consequence of the theorem of Ben-Zvi and Kropholler.

Suppose that B is a block of a proper CAT(0) X with block decomposition. We use notation as in Notation 1.1. Recall that $\partial_{c,X} B$ denotes the set $\{\gamma(\infty) \in \partial_c X \mid \gamma \subseteq B\}$. The corresponding topological subspaces of $\hat{\partial}_c X$ and $\vec{\partial}_c X$ are denoted by $\hat{\partial}_{c,X} B$ and $\vec{\partial}_{c,X} B$ respectively.

Theorem 4.2. *(Variant of Theorem 3.2 in [BZK19]) Let $G = G_0 *_H G_1$ be a CAT(0) group acting geometrically on a proper CAT(0) space X with block decomposition. Suppose that G_0 and H act geometrically on a block $\mathbf{B}$ and a wall A of X respectively. Furthermore, suppose A and its translates to separate $\mathbf{B}$ from the rest of X. Lastly, suppose that $\mathbf{B}$ satisfies the following*

a) $\mathbf{B}$ *has a block decomposition $(\mathcal{B}, \mathcal{A})$ such that $\bigcup_{B\in\mathcal{B}} \hat{\partial} B$ ($\bigcup_{B\in\mathcal{B}} \hat{\partial}_{c,X} B$, $\bigcup_{B\in\mathcal{B}} \vec{\partial}_{c,X} B$) is nonempty and path connected,*

b) $\hat{\partial}\mathbf{B}$ ($\hat{\partial}_{c,X}\mathbf{B}, \vec{\partial}_{c,X}\mathbf{B}$) *is not path connected and*

c) $\Lambda(H)$ *is contained in the path component of* $\hat{\partial}\mathbf{B}$ ($\hat{\partial}_{c,X}\mathbf{B}, \vec{\partial}_{c,X}\mathbf{B}$) *that contains* ($\bigcup_{B\in\mathcal{B}} \hat{\partial}_{c,X} B$, $\bigcup_{B\in\mathcal{B}} \vec{\partial}_{c,X} B$)

Then $\hat{\partial} X$ ($\hat{\partial}_c X$ $\vec{\partial}_c X$) is not path connected.

Proof. All statements concerning the visual boundary of X follow from Theorem 3.2 in [BZK19]. The remaining claim concerning the contracting boundary follows word by word from Theorem 3.2 in [BZK19]. The only difference is that one applies Corollary 3.43

instead of Lemma 3.1 in [BZK19]. For completeness, we repeat the argumentation of Ben-Zvi and Kropholler. By assumption, $\vec{\partial}_{c,X}\mathbf{B}$ is not path connected and $\mathbf{B}$ has a block decomposition $(\mathcal{B}, \mathcal{A})$ such that $\bigcup_{B\in\mathcal{B}} \vec{\partial}_{c,X}\mathbf{B}$ is path connected. We denote the path component of $\bigcup_{B\in\mathcal{B}} \vec{\partial}_{c,X}\mathbf{B}$ by $Nex(\mathbf{B})$. By assumption, there is a point p in $Nex(\mathbf{B})$ and a point q in $\vec{\partial}_{c,X}\mathbf{B}$ that cannot be connected by a curve in $\vec{\partial}_{c,X}\mathbf{B}$. For achieving a contradiction, we suppose that there is a curve γ connecting p and q in $\vec{\partial}_{c,X}\mathbf{B}$. By the choice of p and q, γ is not contained in $\vec{\partial}_{c,X}\mathbf{B}$. Thus, γ contains two subcurves in $\vec{\partial}_{c,X}\mathbf{B}$ that connect p and q to boundary points that are not contained in $\vec{\partial}_{c,X}\mathbf{B}$ respectively. To do so, the two subcurves have to pass through a boundary point of a wall because of Corollary 3.43. So, there are two curves in $\vec{\partial}_{c,X}\mathbf{B}$ connecting p and q to a boundary point of a wall respectively. By assumption, the boundary points of any wall in $\mathbf{B}$ are contained in $Nex(\mathbf{B})$. Since $Nex(\mathbf{B})$ is path connected, we obtain a path from p to q in $\vec{\partial}_{c,X}\mathbf{B}$ – a contradiction. □

Suppose that an amalgamated free product G acts geometrically on a CAT(0) space X that has a block decomposition $(\mathcal{B}, \mathcal{A})$ and satisfies all conditions of Theorem 4.2. In the case of visual boundaries, the example of Croke and Kleiner in [CK00] satisfies the conditions of the block $\mathbf{B}$ in the theorem above. The Nexus of the Croke-Kleiner example is path connected but the visual boundary of the whole space is not path connected. If we study the contracting boundary of the Croke-Kleiner example, this is not true anymore. Indeed, any geodesic ray in the Nexus of the Croke-Kleiner example is not contracting. Also, the generalizations of the example of Croke and Kleiner studied by Mooney [Moo10], Wilson [Wil05], and Ben-Zvi and Kropholler [BZK19] have no contracting geodesic ray in their Nexus. It is an interesting question if there are contracting boundaries that satisfy the assumptions of Theorem 4.2 above. It might be possible that such examples don't exist. If there exist examples satisfying the conditions of the theorem above, then the contracting boundary contains a big connected component. Indeed, by assumption, every block $\tilde{\mathbf{B}}$ that is isometric to $\mathbf{B}$ has a block decomposition $(\tilde{\mathcal{B}}, \tilde{\mathcal{A}})$ such that $\bigcup_{\tilde{B}\in\tilde{\mathcal{B}}} \vec{\partial}_c\tilde{B}$ is contained in a path-component κ. Recall that we are mainly interested what for contracting boundaries of amalgamated free products have totally disconnected contracting boundaries. So, it is interesting for us if we can reformulate the theorem of Ben-Zvi and Kropholler such that we don't assume the existence of such a large path-component κ. We observe that this is not possible. Indeed, there are CAT(0) spaces with path connected visual and contracting boundaries that have a block decomposition such that every block has totally disconnected visual and contracting boundary.

Example 4.3. Let Σ_C be the Davis complex corresponding to a cycle C of lengths at least 5. Then Σ_C is quasi-isometric to the hyperbolic plane. Accordingly, the visual and contracting boundary of Σ_C is a one-sphere, i.e., a path connected topological space. The Davis complex Σ_C is a CAT(0) space with block decomposition. To see this, we decompose the cycle C into two subpaths of length at least two such that both paths share only their end vertices. The blocks of Σ_C are Davis complexes of these two subpaths. Compare Proposition 5.28. The visual and contracting boundary of a Davis complex corresponding to a path of length at least two is totally disconnected. Hence, all blocks have totally disconnected contracting and visual boundaries.

As the focus of this thesis does not coincide with the focus of the theorem of Ben-Zvi and Kropholler, the result of the next section differ to the considerations here.

4.2 Generalization of an example of Charney and Sultan

In this section, we generalize the Cycle-Join-Example of Charney and Sultan in Section 4.2 of [CS15] to amalgamated free products of CAT(0) groups. In their example, Charney and Sultan calculate the contracting boundary of a right-angled Coxeter group W. For that aim, they examine the contracting boundary of its Davis complex X. This Davis complex X is a CAT(0) space with block decomposition. All blocks of one parity (+) of X have an empty contracting boundary and all blocks of the other parity (-) have a 1-sphere S^1 as contracting boundary. Thereby, a dense set of points in each such sphere corresponds to geodesic rays that are not contracting in the ambient space X. Thus, every block of parity $(-)$ contributes a totally disconnected subset of a 1-sphere to the contracting boundary of X. With help of this observation, Charney and Sultan prove that the contracting boundary of X is totally disconnected. For a more precise explanation of the example, see Section 5.1. The crucial point in their proof is that no wall contains any geodesic ray that is contracting in the ambient space. Inspired by this, we transfer their considerations to the setting of amalgamated free products of CAT(0) groups that act on a CAT(0) space with block decomposition and study the case that no wall contains any geodesic ray that is contracting in the ambient space. In contrast to the Cycle-Join-Example of Charney and Sultan, we allow that all blocks have nonempty contracting boundary. In Theorem 4.10, we prove that the contracting boundary of such a space is not connected and calculate the connected components of its contracting boundary in terms of contracting boundaries of their blocks. In case that the blocks have totally disconnected contracting boundaries, we prove that the whole space has totally disconnected contracting boundary. See Corollary 4.11. Afterwards, we prove that the conditions of Theorem 4.10 and Corollary 4.11 simplify if the amalgamated free product is a Coxeter group. This is a preparation for Chapter 5. In Chapter 5, we apply Theorem 4.10 to show that the contracting boundaries of a certain class of right-angled Coxeter groups are totally disconnected. See Corollary 5.38. This way, Theorem 4.10 leads to examples of amalgamated free products of CAT(0) groups that have totally disconnected contracting boundaries.

A related statement to Theorem 4.10 is Theorem 3.2 of Ben-Zvi and Kropholler in [BZK19]. In the last section, we cited this theorem and proved a variant of it that concerns not only path-components of visual boundaries but also path-components of contracting boundaries. Ben-Zvi's and Kropholler's theorem concerns contracting boundaries that are not path connected but contain a big path-component. Theorem 4.10 leads to examples of amalgamated free products that have totally disconnected contracting boundaries. We observe that the theorem of Ben-Zvi and and Kropholler and Theorem 4.10 complement each other.

We use the notation established in Chapter 2 and Chapter 3 and maintain the notation of the last sections. We summarized our notation concerning boundaries in Notation 1.1. Let G_0, G_1 and H be groups acting geometrically on proper CAT(0) spaces X_0, X_1 and Y respectively. Suppose that $G = G_0 *_H G_1$ acts geometrically on a proper CAT(0) space $\mathbb{X} = \mathbb{X}(G_0, X_0, G_1, X_1, H, Y)$ with block decomposition as in Convention 3.78. Recall that a geodesic ray in a wall is $\mathbb{X}$-contracting if it is contracting in the ambient space $\mathbb{X}$.

We suppose that the walls of $\mathbb{X}$ don't contain $\mathbb{X}$-contracting geodesic rays and transfer the argumentation of Charney and Sultan to our setting. It is sufficient to suppose that a side of one wall in $\mathbb{X}$ does not contain any $\mathbb{X}$-contracting geodesic ray. This assumption implies that no wall of $\mathbb{X}$ does contain an $\mathbb{X}$-contracting geodesic ray. Indeed, let A be a wall. Let's assume that one side of A does not contain any $\mathbb{X}$-contracting geodesic ray. Then, the whole wall A does not contain any $\mathbb{X}$-contracting geodesic ray according to the stability Lemma 3.8 from Bestvina and Fujiwara in [BF09] (See Lemma 2.25). Isometries preserve the contracting-property of geodesic rays: If γ is an $\mathbb{X}$-contracting geodesic ray, then its image under an isometry is an $\mathbb{X}$-contracting geodesic ray as well. Hence, no geodesic ray in any wall $g \cdot A$, is $\mathbb{X}$-contracting in $\mathbb{X}$ for any $g \in G$. As G acts transitively on the set of wall, no wall contains an $\mathbb{X}$-contracting geodesic ray. Hence, it is sufficient to assume that a side of one wall does not contain any contracting geodesic ray.

In Chapter 3, we introduced itineraries of geodesic rays. See Definition 3.18. The itinerary of a geodesic ray in $\mathbb{X}$ is a (possibly infinite) path in the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$ of $G = G_0 *_H G_1$ that describes how the ray runs through the blocks and walls of $\mathbb{X}$. We choose a base point x_{base} of $\mathbb{X}$. The itinerary of every (contracting) geodesic (ray) γ issuing from x_{base} starts in the same vertex v_{base} of $\mathcal{T}_{\text{ext}}$. By Definition 3.34, the *itinerary* $I(\xi)$ of an element $\xi \in \partial\mathbb{X}$ ($\partial_c\mathbb{X}$) is the itinerary of the geodesic ray representing ξ that starts at x_{base}. Let I be a (possibly infinite) path in $\mathcal{T}_{\text{ext}}$ starting with v_{base}. Recall from Definition 3.46 that

$$\hat{U}(I) := \{\gamma(\infty) \in \partial\mathbb{X} \mid \gamma(0) = x_{\text{base}} \text{ and } I = I(\gamma)\}$$
$$\hat{U}_c(I) := \{\gamma(\infty) \in \partial_c\mathbb{X} \mid \gamma(0) = x_{\text{base}} \text{ and } I = I(\gamma)\}$$

We saw in Section 3.5 that there are two different types of connected components of $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$). A connected component κ of $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$) is of type 1 if there exists a (possibly infinite) path in $\mathcal{T}_{\text{ext}}$ such that κ is contained in $\hat{U}(I)$. We remark that a connected component in $\hat{\partial}_c\mathbb{X}$ or $\vec{\partial}_c\mathbb{X}$ is contained in $\hat{U}(I)$ if and only if it is contained in $\hat{U}_c(I)$. If κ is not of type 1, it is of type 2. Like Charney and Sultan in the Cycle-Join-Example, we show that all connected components of $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$) are of type 1.

Lemma 4.4. *Let G, G_0, G_1 H, X_0, X_1, Y and $\mathbb{X}$ be as in Convention 3.78. If a side of one wall in $\mathbb{X}$ does not contain any $\mathbb{X}$-contracting geodesic ray, then every connected component of $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$) is of type* 1.

Proof. By Lemma 3.61, a connected component κ of $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$) of type 2 contains a point of $\{\xi \mid \xi \in \partial A^{(gH)}, g \in G\}$ ($\{\xi \mid \xi \in \partial_{c,\mathbb{X}} A^{(gH)}, g \in G\}$). By assumption, there exists such a wall that one of its sides does not contain any contracting geodesic ray. By the stability Lemma 3.8 from Bestvina and Fujiwara in [BF09] (Lemma 2.25), the whole wall does not contain any contracting geodesic ray. As G acts transitively on the set of walls and because being contracting is preserved under isometries, no wall of $\mathbb{X}$ contains a contracting geodesic ray. Hence, no geodesic ray in $\mathbb{X}$ ends in a wall. It follows that the set $\partial A^{(gH)}$ ($\partial_{c,\mathbb{X}} A^{(gH)}$) is empty for all $g \in G$. Hence, the contracting (visual) boundary of $\mathbb{X}$ does not contain any connected component of type 2. □

Let κ be a connected component in $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$) of type 1. Then, there exists a (possibly infinite) path in $\mathcal{T}_{\text{ext}}$ such that κ is contained in $U(I)$ ($U_c(I)$). Recall that κ is of type 1_f if I is finite and κ is of type 1_∞ if I is infinite. Connected components of type 1_f are well-understood. Indeed, every connected component κ of 1_f comes from a block, i.e., there exists a block B such that κ is homeomorphic a connected component of $\hat{\partial}B$ ($\hat{\partial}_{c,B}\mathbb{X}$, $\vec{\partial}_{c,B}\mathbb{X}$). See Lemma 3.57. Hence, we concentrate on the case that κ is of type 1_∞. We show that $|\kappa| \leq 1$ if certain conditions are satisfied.

Lemma 4.5. *Let γ_1 and γ_2 be two geodesic rays parametrized proportionally to arc length in a CAT(0) space (X, d) and $t_1, t_2 \in \mathbb{R}_+$, $t_1 < t_2$ such that*

$$d(\gamma_1(t_2), \gamma_2(t_2)) > d(\gamma_1(t_1), \gamma_2(t_1)),$$

then $d(\gamma_1(t), \gamma_2(t))$ is a monotone increasing function in t.

Proof. The statement follows directly from the convexity of the metric of CAT(0) spaces, see [BH99, Chapter II.2, Prop 2.2]. □

We prove the following lemma as a warm-up.

Lemma 4.6. *Let G, G_0, G_1 H, X_0, X_1, Y and $\mathbb{X}$ be as in Convention 3.78. If Y has bounded diameter and I is an infinite path of $\mathcal{T}_{ext}$ starting with v_{base}, then $|\hat{U}(I)| \leq 1$ and $|\hat{U}_c(I))| \leq 1$.*

Proof. We follow the argumentation of Charney and Sultan in the Cycle-Join-Example. Let D be the diameter of Y. Because Y has bounded diameter, every wall in $\mathbb{X}$ has bounded diameter. Let α and β be two (contracting) geodesic rays in $U(I)$. As I is infinite, α and β pass through the same set of walls in $\mathbb{X}$. Let A be such a wall with label gH of arbitrarily large distance to x_{base}. Then both geodesic rays contain points x_α and x_β in A. As Y has the bounded diameter D, the distance of x_α and x_β is bounded by D. Thereby, D does not depend on g. It follows from Lemma 4.5 that α and β are asymptotic to each other. □

Recall that $\partial_{c,\mathbb{X}}B$ denotes the set $\{\gamma(\infty) \in \partial_c\mathbb{X} \mid \gamma \subseteq B\}$. The corresponding topological subspaces of $\hat{\partial}_c\mathbb{X}$ and $\vec{\partial}_c\mathbb{X}$ are denoted by $\hat{\partial}_{c,\mathbb{X}}B$ and $\vec{\partial}_{c,\mathbb{X}}B$ respectively. The last lemma has the following well-known consequence. Compare for instance [MS15].

Corollary 4.7. *Let G_0, G_1 and H be groups acting geometrically on proper CAT(0) spaces X_0, X_1 and Y respectively such that Y has bounded diameter. Suppose that $G_0 *_H G_1$ acts geometrically on a proper CAT(0) space $\mathbb{X} = \mathbb{X}(G_0, X_0, G_1, X_1, H, Y)$ with block decomposition as in Convention 3.78. Suppose that κ is a connected component of $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}, \vec{\partial}_c\mathbb{X}$). Then one of the following is satisfied.*

a) The connected component κ consists of a single point.

b) For all $B \in \mathcal{B}^-$, κ is homeomorphic to a connected component of $\hat{\partial}B$ ($\hat{\partial}_{c,\mathbb{X}}B$, $\vec{\partial}_{c,\mathbb{X}}B$)

c) For all $B \in \mathcal{B}^+$, κ is homeomorphic to a connected component of $\hat{\partial}B$ ($\hat{\partial}_{c,\mathbb{X}}B$, $\vec{\partial}_{c,\mathbb{X}}B$)

Proof. By Lemma 4.4, every connected component κ of $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}, \vec{\partial}_c\mathbb{X}$) is of type 1. If κ is of type 1_f, then there is a block B so that κ is homeomorphic to a connected component of $\hat{\partial}B$ ($\hat{\partial}_c B, \vec{\partial}_c B$) by Lemma 3.57. The Block B is either isometric to every block in $\mathcal{B}^-$ or to every block in $\mathcal{B}^+$. In the first case we have that for all $B \in \mathcal{B}^-$, κ is homeomorphic to a connected component of $\hat{\partial}B$ ($\hat{\partial}_{c,\mathbb{X}}B$, $\vec{\partial}_{c,\mathbb{X}}B$). In the second case we have that for all $B \in \mathcal{B}^+$, κ is homeomorphic to a connected component of $\hat{\partial}B$ ($\hat{\partial}_{c,\mathbb{X}}B$, $\vec{\partial}_{c,\mathbb{X}}B$). If κ is of type 1_∞, there exists an infinite path I in $\mathcal{T}_{\text{ext}}$ starting with v_{base} such that $\kappa \in \hat{U}(I)$. By Lemma 4.6, $\hat{U}(I)$ consists of at most one single point. Hence, $|\kappa| \leq 1$. □

In the following, we want to prove that we have a similar situation as in the last corollary if there is a wall without an $\mathbb{X}$-contracting geodesic ray. To do so, it is important to prove that $|\hat{U}_c(I))| \leq 1$ for all infinite paths in $\mathcal{T}_{\text{ext}}$ starting with v_{base}. In the Cycle-Join-Example, Charney and Sultan argue in a similar way like in the proof of Lemma 4.6 that $|\hat{U}_c(I))| \leq 1$ for all infinite paths in $\mathcal{T}_{\text{ext}}$ starting with v_{base}. Thereby, they use that the space in their example is a CAT(0) cube complex where all blocks of one parity have empty contracting boundary. In this section, we don't work in CAT(0) cube complexes. Furthermore, we allow that all blocks have nonempty contracting boundary. We merely assume that the walls of $\mathbb{X}$ don't contain geodesic rays that are contracting in $\mathbb{X}$. We will see in the following what a price we have to pay for that. We will add an extra condition which makes it possible to argue in a similar way as Charney and Sultan that for every infinite path in $\mathcal{T}_{\text{ext}}$ starting with v_{base}, $|\hat{U}_c(I))| \leq 1$. At the end of this section, we will see that this extra condition is satisfied if G is a Coxeter group.

Let R_1 and R_2 be sets of representatives for the left cosets of H in G_0 and G_1 respectively such that the identity represents the coset H in G_0/H and G_1/H respectively. By Lemma 2.57, very element of G can be represented by a unique (R_1, R_2)-reduced word. Let $g \in G$ and $\vec{g} := a_0, b_0, \ldots, a_k, b_k, c$ be the (R_1, R_2)-reduced word representing $g \in G$. Let $\vec{\tilde{g}}$ be the word $a_0, b_0, \ldots a_k, b_k$ representing $\tilde{g} = a_0 \cdot b_0 \cdot \ldots \cdot a_k \cdot b_k$. Let $[x_{\text{base}}, \tilde{g}x_{\text{base}}]$ be the geodesic segment in $\mathbb{X}$ connecting x_{base} with $\tilde{g}x_{\text{base}}$ and let $[\tilde{g}x_{\text{base}}, gx_{\text{base}}]$ be the geodesic segment in $\mathbb{X}$ connecting $\tilde{g}x_{\text{base}}$ and gx_{base}. Let the curve $c(g, R_1, R_2)$ be the concatenation of these two geodesic segments connecting x_{base} with gx_{base}.

Definition 4.8. Let G, G_0, G_1 H, X_0, X_1, Y and $\mathbb{X}$ be as in Convention 3.78. We say that $\mathbb{X}$ satisfies the (quasi-geodesic)-property (QG) if there exist

- $K \geq 1$, $L \geq 0$ and
- R_1 and R_2 of representatives for the left cosets of H in G_0 and G_1 respectively such that the identity represents the coset H in G_0/H and G_1/H respectively

so that for all $g \in G$ the curve $c(g, R_1, R_2)$ is a (K, L)-quasi-geodesic.

Similar to Charney and Sultan in the Cycle-Join-Example, we prove the following lemma.

Lemma 4.9. *Let G, G_0, G_1 H, X_0, X_1, Y and $\mathbb{X}$ be as in Convention 3.78. Let I be an infinite path in $\mathcal{T}_{ext}$ starting with v_{base}. Suppose that $\mathbb{X}$ has such a wall that one of its sides does not contain any ($\mathbb{X}$-contracting) geodesic ray. If $\mathbb{X}$ satisfies the property (QG) as defined in Definition 4.8, then $|\hat{U}(I)| \leq 1$ ($|\hat{U}_c(I)| \leq 1$).*

Proof. We argue in a similar way like Charney and Sultan in the Cycle-Join-Example. If no wall contains a geodesic ray, every wall is bounded (This follows from the Theorem of Arzelà-Ascoli, see Lemma 2.7). Then the claim follows from Lemma 4.6. We concentrate on the remaining claim in brackets, i.e., we show that $|\hat{U}_c(I)| \leq 1$ if one side of a wall does not contain any $\mathbb{X}$-contracting geodesic ray. We choose the base point x_{base} in $A^{(\text{id}\, H)}$. Let α and β be two contracting geodesic rays in $\hat{U}_c(I)$. We have to show that α and β are asymptotic to each other. As I is infinite, α and β pass through the same set of walls in $\mathbb{X}$. For any such A with label gH, let $x^g_\alpha \in A \cap \alpha$ and $x^g_\beta \in A \cap \beta$. Let γ be a geodesic segment connecting x^g_α and x^g_β. We show that the length of γ is uniformly bounded by a constant which does not depend on g. It follows that α and β are asymptotic to each other because of Lemma 4.5.

We choose sets R_1 and R_2 of representatives for the left cosets of H in G_1 and for the left cosets of H in G_2 respectively such that they satisfy the condition stated in the definition of the property (QG) in Definition 4.8. Let $a_0, b_0, \ldots, a_k, b_k$ be the (R_1, R_2)-reduced word for g. (Without loss of generality it does not end with $c \in H \setminus \{\text{id}\}$. Indeed, recall that g is a representative for the coset corresponding to the wall A. If the word ends with $c \in H \setminus \{\text{id}\}$, we delete $c \in H$ and choose the group element obtained by deleting c as representative for the coset corresponding to A). As G acts cocompactly on $\mathbb{X}$, there exists a constant D and group elements $\tilde{g}$ and $\hat{g}$ such that $d(x^g_\alpha, \tilde{g}x_{\text{base}}) < D$ and $d(x^g_\beta, \hat{g}x_{\text{base}}) < D$. By the structure of $\mathbb{X}$ and the group action of G described in Definition 3.69, there exist $\hat{h} \in H$ and $\tilde{h} \in H$ such that $\tilde{g}x_{\text{base}} = g\tilde{h}x_{\text{base}}$ and $\hat{g}x_{\text{base}} = g\hat{h}x_{\text{base}}$. Then the (R_1, R_2)-reduced word of $\hat{g}$ is $a_0, b_0, \ldots, a_k, b_k, \hat{h}$ and the (R_1, R_2)-reduced word of $\tilde{g}$ is $a_0, b_0, \ldots, a_k, b_k, \tilde{h}$.

Let $[x_{\text{base}}, gx_{\text{base}}]$ be the geodesic segment connecting x_{base} with gx_{base}, let $s^g_\alpha := [gx_{\text{base}}, \tilde{g}x_{\text{base}}]$ be the geodesic segment connecting gx_{base} and $\tilde{g}x_{\text{base}}$ and let $[\tilde{g}x_{\text{base}}, x^g_\alpha]$ be the geodesic segment connecting $\tilde{g}x_{\text{base}}$ and x^g_α. Let c^g_α be the concatenation of these three geodesic segments connecting x_{base} with x^g_α. Define analogously the segment s^g_β and the curve c^g_β for x^g_β. As $\mathbb{X}$ satisfies the property (QG), c^g_α and c^g_β are (K, L)-quasi-geodesics with endpoints on α and β respectively. As α and β are Morse, there exists a constant $D_1 = D_1(K, L)$ such that c^g_α is contained in the D_1-neighborhood $U_{D_1}(\alpha)$ and c^g_β is contained in the D_1-neighborhood $U_{D_1}(\beta)$. By the stability Lemma 3.8 from Bestvina and Fujiwara in [BF09] (Lemma 2.25), every geodesic segment in $U_{D_1}(\alpha)$ and $U_{D_1}(\beta)$ is C-contracting where C is a constant which depends just on D_1 and the contracting constants of α and β. In particular, the segment s^g_α is C-contracting, where C does not depend on g. For achieving a contradiction, we assume that the length of γ is not uniformly bounded by a constant. Then either the length of the segment s^g_α or the length of s^g_β would not be uniformly bounded by a constant. We assume without loss of generality that this is the case for s^g_α. Then there exists a sequence $(s^{g_i}\alpha)_{i\in\mathbb{N}}$ of segments whose lengths are monotone increasing. The associated sequence $(g_i^{-1} \cdot s^{g_i}\alpha)_{i\in\mathbb{N}}$ is a sequence of segments starting at the base point x_{base} of $\mathbb{X}$. Recall that $x_{\text{base}} \in A^{(\text{id}\, H)}$. Because all walls are convex and by the definition of the segments s^g_α, every segment in the sequence $(g_i^{-1} \cdot s^{g_i}\alpha)_{i\in\mathbb{N}}$ is contained in A. By the Theorem of Arzelà-Ascoli (see Lemma 2.7), a subsequence of $(g_i^{-1} \cdot s^{g_i}\alpha)_{i\in\mathbb{N}}$ converges uniformly on compact sets to a geodesic ray γ starting at x_{base}. Because the wall A is convex and complete, γ is contained in A. By the stability Lemma 3.8 from Bestvina and Fujiwara in [BF09] (Lemma 2.25), γ is contracting. It follows that the wall A contains a contracting geodesic ray – a contradiction. □

Let G_0, G_1 and H be groups acting geometrically on proper CAT(0) spaces X_0, X_1 and Y respectively. Furthermore we suppose that $G = G_0 *_H G_1$ acts geometrically on a proper CAT(0) space $\mathbb{X} = \mathbb{X}(G_0, X_0, G_1, X_1, H, Y)$ as in Convention 3.78. Let $(\mathcal{B}, \mathcal{A})$ be the corresponding block decomposition of $\mathbb{X}$. Recall that $\mathcal{B}^-$ denotes the set of blocks of parity $(-)$ and that $\mathcal{B}^+$ denotes the set of blocks of parity $(+)$. Every block of parity $(-)$ is isometric to X_0 and every block of parity $(+)$ is isometric to X_1. Let B be a block. We use the notation established in Chapter 2 and Chapter 3 and maintain the notation of the last sections. We summarized our notation concerning boundaries in Notation 1.1. Accordingly, $\partial_{c,\mathbb{X}}B$ denotes the set $\{\gamma(\infty) \in \partial_c\mathbb{X} \mid \gamma \subseteq B\}$. The corresponding topological subspaces of $\hat{\partial}_c\mathbb{X}$ and $\vec{\partial}_c\mathbb{X}$ are denoted by $\hat{\partial}_{c,\mathbb{X}}B$ and $\vec{\partial}_{c,\mathbb{X}}B$ respectively. By Lemma 2.35, $\hat{\partial}_{c,\mathbb{X}}B$ and $\vec{\partial}_{c,\mathbb{X}}B$ are homeomorphic to the set of equivalence classes of $\mathbb{X}$-contracting geodesic rays in B equipped with the subspace topology of the visual and contracting boundary of B respectively.

The last lemmas result in the following generalization of the example of Charney and Sultan in section 4.2 of [CS15].

Theorem 4.10 (Generalization of the example of Charney and Sultan)**.** *Let G_0, G_1 and H be groups acting geometrically on proper CAT(0) spaces X_0, X_1 and Y respectively. Suppose that $G_0 *_H G_1$ acts geometrically on a proper CAT(0) space $\mathbb{X} = \mathbb{X}(G_0, X_0, G_1, X_1, H, Y)$ with block decomposition as in Convention 3.78. Assume that*

1. *one side of a wall in $\mathbb{X}$ does not contain any geodesic ray that is contracting in $\mathbb{X}$ and that*
2. *$\mathbb{X}$ satisfies property (QG) as defined in Definition 4.8.*

Suppose that κ is a connected component of $\vec{\partial}_c\mathbb{X}$ $(\hat{\partial}_c\mathbb{X})$. Then

1. *κ consists of a single point or*
2. *for all $B \in \mathcal{B}^-$, κ is homeomorphic to a connected component of $\vec{\partial}_{c,\mathbb{X}}B$ $(\hat{\partial}_{c,\mathbb{X}}B)$ or*
3. *for all $B \in \mathcal{B}^+$, κ is homeomorphic to a connected component of $\vec{\partial}_{c,\mathbb{X}}B$ $(\hat{\partial}_{c,\mathbb{X}}B)$.*

Proof. By Lemma 4.4, every connected component κ of $\vec{\partial}_c\mathbb{X}$ $(\hat{\partial}_c\mathbb{X})$ is of type 1. If κ is of type 1_f, then there is a block B so that κ is homeomorphic to a connected component of $\vec{\partial}_{c,\mathbb{X}}B$ $(\hat{\partial}_{c,\mathbb{X}}B)$ according to Lemma 3.57. The Block B is either isometric to every block in $\mathcal{B}^-$ or to every block in $\mathcal{B}^+$. In the first case we have that for all $B \in \mathcal{B}^-$, κ is homeomorphic to a connected component of $\hat{\partial}B$ ($\hat{\partial}_{c,\mathbb{X}}B$, $\vec{\partial}_{c,\mathbb{X}}B$). In the second case we have that for all $B \in \mathcal{B}^+$, κ is homeomorphic to a connected component of $\hat{\partial}B$ ($\hat{\partial}_{c,\mathbb{X}}B$, $\vec{\partial}_{c,\mathbb{X}}B$). If κ is of type 1_∞, there exists an infinite path I in $\mathcal{T}_{\text{ext}}$ starting with v_{base} such that $\kappa \in \hat{U}_c(I)$. By Lemma 4.9, $\hat{U}_c(I)$ consists of at most one single point. Hence, $|\kappa| \leq 1$. □

A related statement to Theorem 4.10 was recently proved independently by Ben-Zvi and Kropholler in Theorem 3.2 of [BZK19]. Ben-Zvi's and Kropholler's theorem and Theorem 4.10 complement each other in a nice way. We cited the theorem of Ben-Zvi and Kropholler in as Theorem 4.1. It leads to examples of not path connected visual boundaries that belong to CAT(0) spaces admitting geometric actions of amalgamated free products of CAT(0) groups. In Theorem 4.2, we proved a variant of the theorem of Ben-Zvi and Kropholler for contracting boundaries. If a space satisfies the conditions of this variant, then its contracting boundary has a large path-component. By assumption, such a space would have many blocks with large path connected subsets in its contracting boundary. Theorem 4.10 formulated above has another focus. In contrast to the variant of Ben-Zvi and Kropholler, we don't assume that the contracting boundary of a block has a large path connected component. We observed in Example 4.3 that the theorem of Ben-Zvi and Kropholler cannot be formulated for the case that blocks have totally disconnected contracting boundaries. The variant of the theorem of Ben-Zvi and Kropholler cannot be used for understanding how contracting boundaries of CAT(0) spaces with block decompositions look like when their blocks have totally disconnected contracting boundaries. In the next Corollary, we will see that Theorem 4.10 can be used to examine such a situation.

Corollary 4.11. *Let G, G_0, G_1 H, X_0, X_1, Y and $\mathbb{X}$ be as in Convention 3.78. If all assumptions of Theorem 4.10 are satisfied and $\hat{\partial}_{c,\mathbb{X}}X_0$ and $\hat{\partial}_{c,\mathbb{X}}X_1$ ($\vec{\partial}_{c,\mathbb{X}}X_0$ and $\vec{\partial}_{c,\mathbb{X}}X_1$) each are totally disconnected, then $\hat{\partial}_c\mathbb{X}$ ($\vec{\partial}_c\mathbb{X}$) is totally disconnected.*

Proof. The claim follows directly from Theorem 4.10. □

We finish this section by studying the question of when property (QG) as defined in Definition 4.8 is satisfied. We will see that the following property (SN) of the group G implies that the space $\mathbb{X}$ has property (QG). At the end of this section, we will prove that Coxeter groups have property (SN).

Definition 4.12. We say that $G = G_0 *_H G_1$ satisfies the (shortest-normal-form)-property (SN) if there exist

a) a generating set S of G,

b) sets R_1 and R_2 of representatives for the left cosets of H in G_0 and G_1 respectively so that the identity represents the coset H in G_0/H and G_1/H

that satisfy the following: for every (R_1, R_2)-reduced word $a_0, b_0, \ldots, a_k, b_k, c$ exist S-reduced words $\vec{a_i}$, $\vec{b_i}$, $\vec{c}$ in S representing a_i, b_i and c respectively, $i \in \{0, \ldots, k\}$, so that $\vec{a_1}, \vec{b_1}, \vec{a_2}, \vec{b_2}, \ldots, \vec{b_k}, \vec{c}$ is an S-reduced word in S.

Lemma 4.13. *Let G, G_0, G_1 H, X_0, X_1, Y and $\mathbb{X}$ be as in Convention 3.78. If G satisfies the property (SN) as defined in Definition 4.12, then the space $\mathbb{X}$ satisfies the property (QG) as defined in Definition 4.8.*

Proof. Suppose that property (SN) is satisfied. Let S, R_1 and R_2 be as in Definition 4.12. Let

$g \in G$,

$a_0, b_0, \ldots, a_k, b_k, c$ *its* (R_1, R_2)*-reduced word and*

$\vec{g} := \vec{a_1}, \vec{b_1}, \vec{a_2}, \vec{b_2}, \ldots, \vec{b_k}, \vec{c}$ be the corresponding S-reduced word in S.

Let γ_g^1 be the corresponding edge-path in Cay(G,S) connecting the identity with the group element g. Let $F : Cay(G, S) \to \mathbb{X}$ be the orbit map which sends every group element g to $g \cdot x_{\text{base}}$. By the Lemma of Swarc-Milnor (Theorem 2.6), there exist $K \geq 1$, $L \geq 0$ such that F is a (K, L)-quasi-isometry. Hence, the geodesic γ_g^1 is mapped to a (K, L)-quasi-geodesic in $\mathbb{X}$. The curve $c(g, R_1, R_2)$ as defined in Definition 4.8 is obtained by straightening a part of this quasi-geodesic. Hence, it is a (K, L)-quasi-geodesic. □

Lemma 4.14. *Let G, G_0, G_1 H, X_0, X_1, Y and $\mathbb{X}$ be as in Convention 3.78. If G is a Coxeter group, then G satisfies property (SN) as defined in Definition 4.12.*

Proof. Let S be a fundamental set of generators for G. Our goal is to apply Lemma 4.13. For doing so, we choose a set R_i of representatives for the left cosets of H in G_i, $i \in \{0, 1\}$, so that the distance of every representative to the identity is minimal in the word-metric d_S: if $a \in R_0$ ($b \in R_1$) represents aH (bH), then $d_S(\text{id}, a) \leq d_S(\text{id}, g)$ for all $g \in aH$ ($d_S(\text{id}, b) \leq d_S(\text{id}, g)$ for all $g \in bH$). Let $g \in G$ and $\vec{g} := a_0, b_0, \ldots, a_k, b_k, c$ its (R_1, R_2)-reduced word. We choose S-reduced words $\vec{a_i}$, $\vec{b_i}$, $\vec{c}$ in S representing a_i, b_i and c respectively, $i \in \{0, \ldots, k\}$. We show that $\vec{w} := \vec{a_0}, \vec{b_0}, \vec{a_1}, \vec{b_1}, \ldots, \vec{b_k}, \vec{c}$ is an S-reduced word in S representing g. For achieving a contradiction we assume that $\vec{w}$ is not reduced. Then there exists another S-reduced word $\vec{w}'$ representing g. As G is a Coxeter group, it satisfies the Deletion property. Hence, $\vec{w}'$ is obtained from w by deleting some letters. Then $\vec{w}' = \vec{a_0}', \vec{b_0}', \vec{a_1}', \vec{b_1}', \ldots, \vec{b_k}', \vec{c}'$ where $\vec{a_i}'$, $\vec{b_i}'$ and $\vec{c}'$ are obtained from $\vec{a_i}$, $\vec{b_i}$ and $\vec{c}$ by deleting some letters, $i \in \{0 \ldots, k\}$. Let a_i', b_i' and c' be the group elements represented by $\vec{a_i}'$, $\vec{b_i}'$ and $\vec{c}'$ respectively, $i \in \{0, \ldots, k\}$. If $i \in \{1, \ldots, k\}$ exists such that a_i' or b_i' are contained in H, then a reduced form for g exists that has fewer factors than $\vec{g}$ has – a contradiction to Lemma 2.56. Hence, $a_i' \in G_0 \setminus H$ and $b_i' \in G_1 \setminus H$ for all $i \in \{1, \ldots, k\}$. By Lemma 2.56 and because the (R_1, R_2)-reduced word for g is unique, there exists $h \in H$ such that $a_0' = a_0 h$. If $\vec{a_0}'$ is shorter than $\vec{a_0}$, then $\vec{a_0}'$ is a shorter representative for the coset aH – a contradiction. Hence, $\vec{a_0} = \vec{a_0}'$. We consider $\vec{a_0}\vec{b_0}'$. Because the (R_1, R_2)-reduced word for g is unique, there exists $h \in H$ such that $b_0' = b_0 h$. If $\vec{b_0}'$ is shorter than $\vec{b_0}$, then $\vec{b_0}'$ is a shorter representative for the coset $b_0 H$ – a contradiction. We continue in this manner and see that $\vec{a_i}' = \vec{a_i}'$ and $\vec{b_i}' = \vec{b_i}$ for all $i \in \{1, \ldots, k\}$. Then $\vec{c}$ and $\vec{c}'$ represent the same word and coincide as well. It follows that w and w' are equal – a contradiction. □

Corollary 4.15. *Let G, G_0, G_1 H, X_0, X_1, Y and $\mathbb{X}$ be as in Convention 3.78. If G is a Coxeter group, the space $\mathbb{X}$ satisfies the property (QG) as defined in Definition 4.8.*

Proof. We assume that G is a Coxeter group. By Lemma 4.14, G satisfies property (SN) as defined in Definition 4.12. By Lemma 4.13, $\mathbb{X}$ satisfies property (QG) as defined in Definition 4.8. □

We conclude that the conditions of Theorem 4.10 simplify if G is a Coxeter group. In Chapter 5 of this thesis, we apply Theorem 4.10 to a class of right-angled Coxeter groups. See Theorem 5.32. This leads to a class consisting of right-angled Coxeter groups with totally disconnected contracting boundaries. See Corollary 5.38.

4.3 Boundary points of axes for rank-one isometries

Let X be a proper CAT(0) space. Suppose that X contains an axis γ for an axial rank-one isometry g. Let $\gamma^+(\infty)$ and $\gamma^-(\infty)$ be the associated boundary points. Assume further that g acts as a homeomorphism on a subspace Z of $\hat{\partial}X$. Based on the results of Hamenstädt in [Ham09], we examine connected components of Z that contain $\gamma^+(\infty)$ or $\gamma^-(\infty)$. We will see in Theorem 4.24 that there occur only two extreme cases. Either both $\gamma^+(\infty)$ and $\gamma^-(\infty)$ are contained in a common connected component or the connected component of $\gamma^+(\infty)$ and $\gamma^-(\infty)$ each consist of a single point. Inspired by this observation, we study the behavior of rank-one-isometries in CAT(0) spaces with block decomposition and define essential axes for rank-one isometries in Definition 4.30. For us, the described result is interesting because it can be applied to the subspace $\hat{\partial}_c X$ of the visual boundary of X consisting of all equivalence classes of contracting geodesic rays in X. Recall that the contracting boundary of X has a finer topology than the visual boundary of X. Thus, every connected component of the contracting boundary of X is contained in a connected component of $\hat{\partial}_c X$. So, the study of $\hat{\partial}_c X$ allows s to draw consequences for the contracting boundary of X. In Section 4.4, we will use this for examining contracting boundaries of amalgamated free products along groups that are quasi-isometric to $\mathbb{Z}$. As a preparation for Section 4.4, we finish this section with the study of axes for rank-one isometries that are contained in CAT(0) spaces with block decomposition on which a group acts geometrically.

We use the notation established in Chapter 2 and Chapter 3 and maintain the notation of the last sections. We summarized our notation concerning boundaries in Notation 1.1. In the following notation, we recap what axial isometries are and fix notation.

Notation 4.16. An isometry g of a CAT(0) space X is called *axial* if the displacement function of g assumes a minimum d on X and if g does not have a fix point. Let $\mathrm{Min}(g) \subseteq X$ be the set in X on which the displacement function of g is minimal. By Proposition 6.2 and Theorem 6.8 in Part II of [BH99], the set $\mathrm{Min}(g)$ is isometric to a direct product $C \times \mathbb{R}$ where C is a closed and convex subset of $\mathrm{Min}(g)$. Furthermore, the isometry g acts by translations on all sets of the form $\{x\} \times \mathbb{R}$ in $\mathrm{Min}(g)$. Every such set is called *axis* for g. Every axis $\gamma \subset X$ is the image of a biinfinite geodesic ray $\gamma^+ : \mathbb{R} \to X$ such that $g\gamma^+(t) = \gamma^+(t+d)$ for all $t \in \mathbb{R}$. Such a biinfinite geodesic ray is called *oriented axis* for g. Let g^{-1} be the inverse of g. It is $\mathrm{Min}(g) = \mathrm{Min}(g^{-1})$ and g^{-1} acts by the reverse translations on all sets of the form $\{x\} \times \mathbb{R}$ in $\mathrm{Min}(g)$. The reverse biinfinite geodesic ray $\gamma^- : \mathbb{R} \to X$ sending t to $\gamma^+(-t)$ is an oriented axis for g^{-1}. We denote the corresponding geodesic rays that start at $\gamma^+(0)$ and $\gamma^-(0)$ respectively by $\gamma^+_{\geq 0}$ and $\gamma^-_{\geq 0}$. If x is a point on γ we denote the geodesic rays $\gamma^+|_{[t,\infty)}$ and $\gamma^-|_{[-t,\infty)}$ with $\gamma^+(t) = x$ and $\gamma^-(-t) = x$ by γ^+_x and γ^-_x. Let $\gamma^+(\infty)$ be the equivalence class of $\gamma^+_{\geq 0}$ and $\gamma^-(\infty)$ be equivalence class of $\gamma^-_{\geq 0}$. If γ and γ' are two distinct axes for g, then γ and γ' are asymptotic to each other because every axis is of the form $\{x\} \times \mathbb{R} \subseteq C \times \mathbb{R}$ where C is convex and closed in $\mathrm{Min}(g)$. Hence, every axial isometry is associated to two points in the visual boundary of X. Furthermore, an axis for g is contracting if and only if all axes for g are contracting.

We define rank-one isometries like Hamenstädt in [Ham09]. This Definition is based on the definition of B-rank-one isometries of Bestvina und Fujiwara in Definition 5.1 of [BF09].

Definition 4.17. An axial isometry g of a proper CAT(0) space X is called *rank-one* if there is a contracting axis for g.

A *flat half-plane* F in X is a subspace of X that is isometric to a Euclidean half-plane and totally geodesic embedded in X, i.e., every geodesic in F is also a geodesic in X. The following statement is proven by Bestvina and Fujiwara.

Theorem 4.18. *([BF09, Thm 5.4]) An axis for an axial isometry in a proper CAT(0) space is contracting if and only if it fails to bound a flat half-plane.*

This leads to the following characterization of rank-one isometries.

Corollary 4.19. *An axial isometry g of a proper CAT(0) space is rank-one if and only if there is an axis for γ which does not bound a flat half-plane.*

Suppose that a hyperbolic group acts geometrically on a proper geodesic metric space. Then every of its elements of infinite order acts with *North-South Dynamics* on X. A proof can be found for instance in [Bal95]. We use the Definition as in [Ham09].

Definition 4.20 (North-South Dynamics)**.** A homeomorphism g of a compact space K is said to act with *North-South Dynamics* if there are two fixed points $a \neq b \in K$ for the action of g such that for every neighborhood U of a, V of b there is some $k > 0$ such that $g^k(K - V) \subseteq U$ and $g^{-k}(K - U) \subseteq V$. The point a is the *attracting fixed point* for g, and b is the *repelling fixed point*.

Ballmann proves in Lemma 3.3 of chapter III in [Bal95] that every rank-one isometry of a proper CAT(0) space X acts with North-South Dynamics on $\hat{\partial} X$. For compact orbispaces, this is theorem A in [BB95] of Ballmann and Brin. Hamenstädt shows that the reversed implication is also true.

Lemma 4.21. *([Ham09, Lem. 4.4]) An axial isometry g of a proper CAT(0) space is rank-one if and only if g acts with North-South Dynamics on the visual boundary of X.*

Murray found in [Mur19] an example of a space X in which a rank-one isometry does not act with North-South Dynamics on the contracting boundary of X. In Theorem 4.2, he formulates a weaker variant of North-South Dynamics that holds for contracting boundaries. Recently, Liu [Liu19, Cor. 6.8] proved that so-called *Morse isometries* of proper metric spaces satisfy this weaker North-South Dynamics as well.

Definition 4.22 (weak North-South Dynamics)**.** A homeomorphism g of a compact space X is said to act with *weak North-South Dynamics* if there are two fixed points $a \neq b \in K$ for the action of g such that for every neighborhood U of a and every compact set K in $X \setminus \{a\}$ there is some $k > 0$ such that $g^k(K) \subseteq U$.

Theorem 4.23 (Corollary 4.3 in [Mur19])**.** *Let X be a proper CAT(0) space and let G be a group acting geometrically on it. If g is a rank-one isometry in G, γ is an axis for h_α, U is an open neighborhood of $\gamma^+(\infty)$ and K is a compact set in $\vec{\partial}_c X \setminus \{\gamma^-(\infty)\}$ then $\gamma^k(K) \subseteq U$ for sufficiently large k.*

Recall that $\hat{\partial}X$ denotes the visual boundary of X and that $\hat{\partial}_c X$ denotes the subspace of $\hat{\partial}X$ that consists of all equivalence classes of contracting geodesic rays in X. If g is an isometry of a proper CAT(0) space X, then g induces a homeomorphism $\phi^g : \hat{\partial}X \to \hat{\partial}X$ on the visual boundary of X. Thereby ϕ^g sends the equivalence class of a geodesic ray γ to the equivalence class of $g\gamma$. Isometries map contracting geodesic rays to contracting geodesic rays. Hence, the image of $\hat{\partial}_c X$ under ϕ^g is contained in $\hat{\partial}_c X$ and $\phi^g|_{\hat{\partial}_c X} : \hat{\partial}_c X \to \hat{\partial}_c X$ is a homeomorphism. So, g acts on $\hat{\partial}_c X$ as a homeomorphism. Hence, the following theorem is interesting for our consideration. It is a direct consequence of Lemma 4.21.

Theorem 4.24. *Let g be an axial rank-one isometry of a proper CAT(0) space X and γ an axis for g. Suppose that Z is a subspace of the visual boundary of X containing $\gamma^+(\infty)$ and $\gamma^-(\infty)$ such that g acts on Z as a homeomorphism. Let $\kappa(\gamma^+(\infty))$ and $\kappa(\gamma^-(\infty))$ be the connected components of $\gamma^+(\infty)$ and $\gamma^-(\infty)$ in Z respectively. Then either*

a) $|\kappa(\gamma^+(\infty))| = |\kappa(\gamma^-(\infty))| = 1$ *or*

b) $\kappa(\gamma^+(\infty)) = \kappa(\gamma^-(\infty))$.

If Z is not connected, then every open neighborhood of $\gamma^+(\infty)$ ($\gamma^-(\infty)$) contains a connected component. If Z is not connected and contains more than two points, then every open neighborhood of $\gamma^+(\infty)$ ($\gamma^-(\infty)$) contains a connected component that does not contain $\gamma^+(\infty)$ ($\gamma^-(\infty)$).

Proof. Suppose that $\kappa(\gamma^+(\infty)) \cap \kappa(\gamma^-(\infty)) = \emptyset$. We have to show that $|\kappa(\gamma^+(\infty))| = |\kappa(\gamma^-(\infty))| = 1$. If $\kappa(\gamma^+(\infty)) \cap \kappa(\gamma^-(\infty)) = \emptyset$, then there exists an open neighborhood V of $\gamma^-(\infty)$ in $\hat{\partial}X$ such that $\kappa(\gamma^+(\infty)) \subseteq \partial X \setminus V$. By Lemma 4.21, g acts with North-South Dynamics on $\hat{\partial}X$. Hence, for all open neighborhoods U of γ^+ in $\hat{\partial}X$, there exists $k > 1$ such that $g^k\kappa(\gamma^+(\infty)) \subseteq U$. As g acts on γ^+ by translations, $g^k\gamma^+(\infty) = \gamma^+(\infty)$. Hence, $\gamma^+(\infty) \in g^k\kappa(\gamma^+(\infty))$. As g acts as a homeomorphisms on Z, the image of $\kappa(\gamma^+(\infty))$ under g^k is a connected component in Z. As it contains $\gamma^+(\infty)$, $\kappa(\gamma^+(\infty)) = g^k \cdot \kappa(\gamma^+(\infty)) \subseteq U$. We conclude that $\kappa(\gamma^+(\infty))$ is contained in any open neighborhood of $\gamma^+(\infty)$. Hence, $|\kappa(\gamma^+(\infty))| = 1$. It follows analogously that $|\kappa(\gamma^-(\infty))| = 1$.
Suppose that Z is not connected. It remains to show that every open neighborhood of $\gamma^+(\infty)$ ($\gamma^-(\infty)$) contains a whole connected component and that this connected component does not contain $\gamma^+(\infty)$ ($\gamma^-(\infty)$) if Z has more than two points. For symmetry reasons, it is sufficient to prove the claim for $\gamma^+(\infty)$. As before, we study the connected component $\kappa(\gamma^-(\infty))$ in Z. If Z is not connected, there exists a connected component κ such that $\kappa(\gamma^-(\infty)) \cap \kappa = \emptyset$. Then there exists an open neighborhood V of $\gamma^-(\infty)$ such that $\kappa \subseteq \partial X \setminus V$. By Lemma 4.21, g acts with North-South Dynamics on $\hat{\partial}X$. Hence, for all open neighborhoods U of $\gamma^+(\infty)$ in $\hat{\partial}X$, there exists $k > 1$ such

that $g^k\kappa \subseteq U$. As g acts as a homeomorphism on Z, $g^k\kappa$ is a connected component of Z. It remains to show that $\gamma^+(\infty)$ is not contained in $g^k\kappa$ if Z has more than two points. To prove this, we assume that $g^k\kappa$ contains $\gamma^+(\infty)$. Then κ contains $g^{-k}\gamma^+(\infty)$ and because g^k acts by translations on γ, $g^{-k}\gamma^+(\infty) = \gamma^+(\infty)$. By the choice of κ, $\gamma^-(\infty)$ is not contained in κ. Thus, $\gamma^-(\infty)$ and $\gamma^+(\infty)$ are not contained in a common connected component. By our considerations above, the connected component of $\gamma^+(\infty)$ and $\gamma^-(\infty)$ each consist of a single point. Then κ consists of the point $\gamma^+(\infty)$. Recall that κ was chosen as an arbitrary connected component that does not contain $\gamma^-(\infty)$. As each such connected component coincides with the connected component of $\gamma^+(\infty)$ and since the connected component of $\gamma^-(\infty)$ consists of a single point, we conclude that Z consists of two single points equipped with the discrete topology.

□

Theorem 4.24 holds for contracting boundaries if we add the condition that $\kappa(\gamma^+(\infty))$ and $\kappa(\gamma^-(\infty))$ are contained in a compact subset of the contracting boundary of X. Indeed, suppose that $\kappa(\gamma^+(\infty))$ and $\kappa(\gamma^-(\infty))$ are contained in a compact subset of the contracting boundary of X. As closed subsets of compact sets are compact, both the connected component of $\gamma^-(\infty)$ and $\gamma^+(\infty)$ are compact. We apply the weak North-South Dynamics of rank-one isometries according to Theorem 4.23 instead of the North-South Dynamics of rank-one isometries according to Lemma 4.21. The claim follows by repeating the proof above word by word.

Theorem 4.24 implies the following corollary.

Corollary 4.25. *Let g be an axial rank-one isometry of a proper CAT(0) space X and γ an axis for g. Either $\hat{\partial}_c X$ has a connected component containing $\gamma^+(\infty)$ and $\gamma^-(\infty)$ simultaneously or the connected components of $\gamma^+(\infty)$ and $\gamma^-(\infty)$ in $\hat{\partial}_c X$ and $\vec{\partial}_c X$ each consist of a single point.*

Proof. As g is rank-one, g admits a contracting axis. As all axes for an axial rank-one isometry are asymptotic to each other, every axis for g is contracting. Hence, $\gamma^+(\infty)$ and $\gamma^-(\infty)$ are contained in $\hat{\partial}_c X$. Furthermore, g acts on $\hat{\partial}_c X$ as a homeomorphism. The claim follows from Theorem 4.24 and the fact that the direct limit topology is finer than the subspace topology of the visual boundary. □

Definition 4.26. A topological space X is locally connected at a point $x \in X$ if for every open set U containing x there exists a connected, open set V with $x \in V \subseteq U$. The space X is called *locally connected* if it is locally connected at x for all $x \in X$.

The following corollary is another consequence of Theorem 4.24.

Corollary 4.27. *Suppose that a group G acts by isometries on a proper CAT(0) space X. Suppose that Z is a disconnected subspace of the visual boundary of X containing both boundary points associated to an axis for an axial rank-one isometry g such that g acts on Z as a homeomorphism. If Z contains more than two points then Z is not locally connected.*

Proof. If $\hat{\partial}X$ is not connected, it contains at least two connected components. Let γ be an axis for a rank-one isometry in the group acting on X. By Theorem 4.24, every open neighborhood of $\gamma(\infty)^+$ ($\gamma(\infty)^-$) contains a whole connected component not containing $\gamma(\infty)^+$ (analogously $\gamma(\infty)^-$). Hence, $\hat{\partial}X$ is not locally connected at $\gamma(\infty)^+$ ($\gamma(\infty)^-$). □

The last corollary has the following consequence.

Corollary 4.28. *Suppose that a group G acts by isometries on a proper CAT(0) space X such that G contains an axial rank-one isometry. Suppose that the visual boundary of X is locally connected and contains at least three points. Then X is one-ended.*

Proof. Suppose that $\hat{\partial}_c X$ is locally connected. By Corollary 4.27, $\hat{\partial}_c X$ is connected. The visual boundary $\hat{\partial}_c X$ is connected if and only if X is one-ended. So, if $\hat{\partial}_c X$ is locally connected then X is one-ended. □

The question arises what happens if we study the other direction of this implication. Suppose that X is one-ended. When is the visual boundary $\hat{\partial}_c X$ locally connected? This is related to the following open question that can be found in Dani's survey [Dan18]. Dani gives an overview of the known results concerning this topic in her survey.

Question 9 (Question 4.7 in [Dan18])**.** Is there a CAT(0) group that acts on two different CAT(0) spaces, such that the one has locally connected boundary and the other has non-locally connected boundary?

In the following, we study consequences of Theorem 4.24 for actions of amalgamated free products of CAT(0) groups on CAT(0) spaces with block decomposition. Let G_0, G_1 and H be groups acting geometrically on proper CAT(0) spaces X_0, X_1 and Y respectively. Let $\mathbb{X}$ be a CAT(0) space with block decomposition on which $G = G_0 *_H G_1$ acts geometrically such that all conditions of Convention 3.78 are satisfied. Recall that the tree associated to this block decomposition is the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$ of $G = G_0 *_H G_1$. In the following, we study axes for rank-one isometries in $\mathbb{X}$. Let α be a geodesic rays in $\mathbb{X}$. Recall that $I(\alpha)$ denotes the itinerary of α as introduced in Chapter 3. See Definition 3.18. It is a path in the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$ of $G = G_0 *_H G_1$ that describes how α passes through the walls and blocks of $\mathbb{X}$.

Lemma 4.29. *Let $G = G_0 *_H G_1$, X_0, X_1 Y and $\mathbb{X}$ be as in Convention 3.78. Let γ be an axis for an axial isometry in $G = G_0 *_H G_1$. Then one of the following statements holds.*

a) $I(\gamma^+_{\geq 0})$ and $I(\gamma^-_{\geq 0})$ are infinite and they have just their first vertices in common.

b) γ is contained in a block or a wall of $\mathbb{X}$ and there exists a vertex in $\mathcal{T}_{ext}$ such that the itinerary of $\gamma^+_{\geq 0}$ and $\gamma^-_{\geq 0}$ consists of v.

Proof. Let γ be an axis for an axial isometry ϕ in G and d the translation length of ϕ. By Lemma 3.24, the itinerary of $\gamma^+_{\geq 0}$ and $\gamma^-_{\geq 0}$ start with the same vertex of $\mathcal{T}_{\text{ext}}$. If γ is contained in a wall or block of $\mathbb{X}$, the itinerary of $\gamma^+_{\geq 0}$ and $\gamma^-_{\geq 0}$ each consist of one vertex.

We assume that γ is not contained in any wall or block of $\mathbb{X}$. Then γ leaves at least one wall or block K at a time t_0 in $\mathbb{X}$. As ϕ acts by translations on γ, γ leaves at every time $z \cdot d + t_0$, $z \in \mathbb{Z}$ a block or wall isometric to K. As walls and blocks are convex and γ is a bi-infinite geodesic ray, γ passes through every block and wall at most once. Thus, γ passes through infinitely many distinct blocks of $\mathbb{X}$. Hence, both $\gamma^+_{\geq 0}$ and $\gamma^-_{\geq 0}$ are infinite and have at most their first vertices in common. □

Let γ be a bi-infinite geodesic ray. By Definition 3.18, the itinerary of γ is the union of $I(\gamma^+_{\geq 0})$ and $I(\gamma^-_{\geq 0})$. This union is a bi-infinite path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. Suppose that γ is an axis for an axial rank-one isometry and that γ is not contained in a wall or in a block. By Lemma 4.29, the itinerary of γ is infinite. The following lemmas show that such axes influence the structure of the visual and contracting boundary of $\mathbb{X}$ a lot. Thus, we call such geodesic rays *essential*.

Definition 4.30 (essential axes for rank-one isometries)**.** Let γ be an a bi-infinite geodesic ray in a CAT(0) space with block decomposition $(\mathcal{B}, \mathcal{A})$ with associated tree $\mathcal{T}_{\mathcal{B},\mathcal{A}}$. We call γ *essential* in $(\mathcal{B}, \mathcal{A})$, if γ is an axis for a rank-one isometry and has an infinite itinerary.

Let x_{base} be a base point of $\mathbb{X}$. Recall that the itinerary of $\gamma^+_{\geq 0}(\infty)$ ($\gamma^-_{\geq 0}(\infty)$) is defined as the itinerary of the representative of $\gamma^+_{\geq 0}(\infty)$ ($\gamma^-_{\geq 0}(\infty)$) that starts at x_{base}. By Corollary 3.32, $I(\gamma^+_{\geq 0}(\infty))$ and $I(\gamma^-_{\geq 0}(\infty))$ coincide with $I(\gamma^+_{\geq 0})$ and $I(\gamma^-_{\geq 0})$ from some vertex on. So, if γ is essential, $I(\gamma^+_{\geq 0}(\infty))$ and $I(\gamma^-_{\geq 0}(\infty))$ are infinite paths. In Section 3.5, we characterized connected components of different type. We say that a connected component is of type 1 if all geodesic rays in κ have the same itinerary. Otherwise, it is of 2. A connected component of type 1 is of type 1_f if all geodesic rays in κ have finite itinerary. Otherwise, it is of type 1_∞. See Definition 3.52. Let γ be an essential axis for a rank-one isometry in $\mathbb{X}$. The following lemmas characterize the connected components of the oriented axes $\gamma^+_{\geq 0}(\infty)$ and $\gamma^-_{\geq 0}(\infty)$ of γ. As the itineraries of $\gamma^+_{\geq 0}(\infty)$ and $\gamma^-_{\geq 0}(\infty)$ are infinite, the connected components of $\gamma^+_{\geq 0}(\infty)$ and $\gamma^-_{\geq 0}(\infty)$ are of type 1_∞ or of type 2. Let $\gamma^+_{\geq 0}$ one of its oriented axes. We study first the case where the connected component of $\gamma^+_{\geq 0}(\infty)$ is of type 1_∞. Then we study the case where it is of type 2.

Recall that $\hat{\partial}\mathbb{X}$ denotes the visual boundary of $\mathbb{X}$, that $\hat{\partial}_c\mathbb{X}$ denotes the subspace of $\hat{\partial}\mathbb{X}$ consisting of all equivalence classes of contracting geodesic rays and that $\vec{\partial}_c\mathbb{X}$ denotes the contracting boundary of $\mathbb{X}$. If x is a point in a topological space, then $\kappa(x)$ denotes its connected component.

Lemma 4.31. *Let $G = G_0 *_H G_1$, X_0, X_1 Y and $\mathbb{X}$ be as in Convention 3.78. Let γ be an essential axis for a rank-one isometry in G and γ^+ one of its oriented axes. If the type of the connected component of $\gamma^+(\infty)$ in $\hat{\partial}_c\mathbb{X}$ is 1_∞, then the connected component of $\gamma^+(\infty)$ in $\vec{\partial}_c\mathbb{X}$ is of type 1_∞. Furthermore, the connected component of $\gamma^+(\infty)$ in $\hat{\partial}_c\mathbb{X}$ and the connected component of $\gamma^+(\infty)$ in $\vec{\partial}_c\mathbb{X}$ each consist of a single point.*

Proof. Let γ be an axis for a rank-one isometry that is not contained in a block or a wall. Then both $\gamma^+(\infty)$ and $\gamma^-(\infty)$ have infinite itinerary by Lemma 4.29. Hence, both the connected components of $\gamma^+(\infty)$ and $\gamma^-(\infty)$ in $\hat{\partial}_c\mathbb{X}$ are of type 1_∞ or of type 2. If the type of the connected component of $\gamma^+(\infty)$ in $\hat{\partial}_c\mathbb{X}$ is 1_∞, then it does not contain a geodesic ray whose itinerary is different to $I(\gamma^+(\infty))$. By Lemma 4.29, the itinerary of $\gamma^+(\infty)$ and $\gamma^-(\infty)$ are distinct. It follows that the connected components of $\gamma^+(\infty)$ and $\gamma^-(\infty)$ have empty intersection. By Corollary 4.25, the connected component of $\gamma^+(\infty)$ and $\gamma^-(\infty)$ in $\vec{\partial}_c\mathbb{X}$ and $\hat{\partial}_c\mathbb{X}$ each consist of a single point. □

Now we study the case where the connected component of $\gamma^+_{\geq 0}(\infty)$ is of type 2.

Lemma 4.32. *Let $G = G_0 *_H G_1$, X_0, X_1 Y and $\mathbb{X}$ be as in Convention 3.78. Let γ be an essential axis for a rank-one isometry in G and γ^+ one of its oriented axes. Suppose that the connected component $\kappa(\gamma^+(\infty))$ of $\gamma^+(\infty)$ in $\hat{\partial}_c\mathbb{X}$ is of type 2. Then $\kappa(\gamma^+(\infty))$ contains $\gamma^-(\infty)$. Furthermore, for every vertex of $I(\gamma)$ that corresponds to a wall A, $\kappa(\gamma^+(\infty))$ contains an equivalence class of a geodesic ray contained in A.*

Proof. If the connected component $\kappa(\gamma^+)$ of γ^+ is of type 2, then it contains at least two points. By Corollary 4.25 follows, that $\kappa(\gamma^+(\infty))$ contains $\gamma^-(\infty)$. Let I be the itinerary of γ. If I is trivial, we are done. Hence, we assume that I is not trivial. By Lemma 4.29, I is a bi-infinite path in the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$. By Corollary 3.32, the itineraries of $\gamma^+_{\geq 0}$ and $\gamma^-_{\geq 0}$ coincide with $I(\gamma^+_{\geq 0}(\infty))$ and $I(\gamma^-_{\geq 0}(\infty))$ from some vertex on. Because paths between two points in trees are unique, I coincides with the path between $I(\gamma^+_{\geq 0}(\infty))$ and $I(\gamma^-_{\geq 0}(\infty))$ as defined in Definition 3.49. By Lemma 3.50, $\kappa(\gamma^+(\infty))$ contains for every vertex of I that correspond to a wall A an equivalence class of a geodesic ray contained in A. □

Figure 4.1 summarizes the properties of a connected component of an equivalence class of an oriented axes for a rank-one isometry in $\hat{\partial}_c\mathbb{X}$.

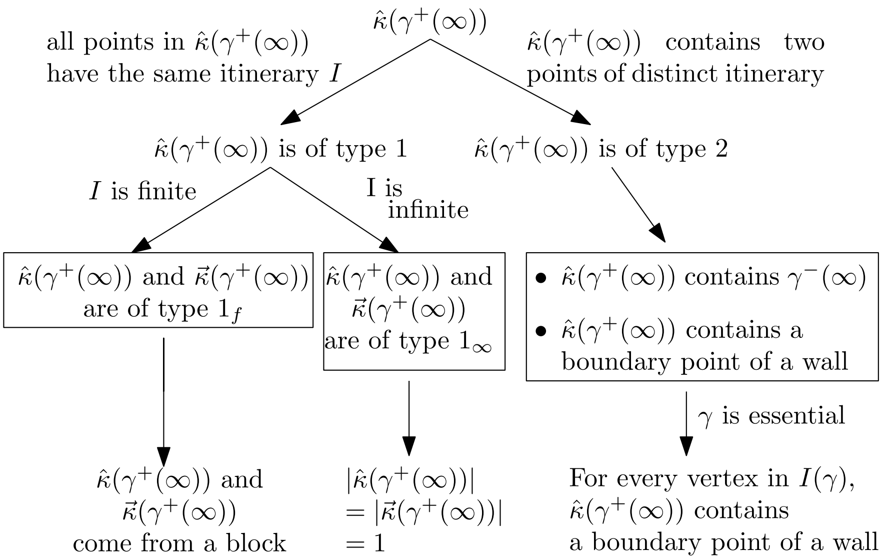

Figure 4.1 The letter γ denotes an axis for a rank-one isometry in a CAT(0) space $\mathbb{X}$ as in Convention 3.78. The terms $\hat{\kappa}(\gamma^+(\infty))$ and $\vec{\kappa}(\gamma^+(\infty))$ denote the connected component of $\gamma^+(\infty)$ in $\hat{\partial}_c\mathbb{X}$ and $\vec{\partial}_c\mathbb{X}$ respectively. The arrows denote implications. The property at a peak follows if the conditions at the arrows are satisfied.

4.4 Amalgamated free products along groups quasi-isometric to $\mathbb{Z}$

In this section, we study contracting boundaries of amalgamated free products $G = G_0 *_H G_1$ where G_0 and G_1 are CAT(0) groups and H is quasi-isometric to $\mathbb{Z}$. Bridson and Haefliger proved in Corollary 11.19 of part II in [BH99] that G is a CAT(0) group. For showing this, they construct a CAT(0) space on which G acts geometrically by means of Theorem 11.18 in part II of [BH99]. We have seen in Section 3.6, that the so constructed spaces are CAT(0) spaces with block decomposition. Thus, Corollary 11.19 of part II in [BH99] implies the following theorem.

Theorem 4.33 ([BH99, Cor 11.19, part II])**.** *Let G_0, G_1 and H be groups acting each geometrically on a proper CAT(0) space. Suppose that H is quasi-isometric to $\mathbb{Z}$. Then $G = G_0 *_H G_1$ acts geometrically on a proper CAT(0) space with block decomposition such that all properties of Convention 3.78 are satisfied.*

We use the notation established in Chapter 2 and Chapter 3 and maintain the notation of the last sections. We summarized our notation concerning boundaries in Notation 1.1. Let G_0, G_1, H, X_0, X_1, Y and $\mathbb{X}$ be as in Convention 3.78, i.e., X_0, X_1 and Y are proper CAT(0) spaces on which G_0, G_1 and H act geometrically and $\mathbb{X}$ is a CAT(0) space with block decomposition on which $G = G_0 *_H G_1$ acts geometrically. Because of the Hopf-Rinow Theorem, X_0, X_1 and Y are complete and locally compact. Suppose that H is quasi-isometric to $\mathbb{Z}$. In the following, we examine the contracting boundary of $G_0 *_H G_1$ by studying the contracting boundary of $\mathbb{X}$. Suppose that a wall of $\mathbb{X}$ does not contain a geodesic ray that is contracting in $\mathbb{X}$. Then we know already what is going on. Indeed, we can apply our results of Section 4.2. Thus, we concentrate on the remaining case that every wall contains a contracting geodesic ray. We examine properties of connected components that occur in the contracting boundary of G and study consequences for the case in which we assume that the contracting boundaries of G_0 and G_1 are totally disconnected.

Since H is quasi-isometric to $\mathbb{Z}$, H contains an infinite cyclic subgroup of finite index by Lemma 8.40 of part I in [BH99]. It follows from Chapter 6 of Part II in [BH99], that H contains an axial isometry. At the beginning of Section 4.3, we defined axial isometries; we use notation as in Notation 4.16 to denote them. Recall that an axis for g is contracting if and only if all axes for g are contracting. As g acts by translation on every associated axis, every axis for g is contracting if and only if one of its oriented axes is contracting. By Definition 4.17, an axis for an axial isometry admitting a contracting axis is called rank-one.

We want to get a feeling for the situation of this section. Therefore, we sketch the proof of Theorem 4.33 that says that every amalgamated free product of CAT(0) groups is a CAT(0) group.

Sketch of the proof for Theorem 4.33. Let X_0, X_1 and Y be CAT(0) spaces on which G_0, G_1 and H act geometrically. We follow the proof of Corollary 11.19 of part II in [BH99].

By Lemma 8.40 of part I in [BH99], H contains an infinite cyclic subgroup of finite index. Thus, both X_0 and X_1 contain an axis $c_0 : \mathbb{R} \to X_0$ and $c_1 : \mathbb{R} \to X_1$ of an axial isometry τ. The image of H in the isometry group of X_i acts either simultaneously on c_0 and c_1 as an infinite Dihedral group, or it acts simultaneously by translations. It is possible to rescale the metric on X_0 in both cases so that there is a H-equivariant isometry sending $c_1(t)$ to $c_2(t)$. If we set $Y = \mathbb{R}$ and $f_j(t) = c_j(t)$, then f_j is an H-invariant isometry, $j \in \{0, 1\}$. This implies that blocks isometric to X_0 or X_1 can be glued along the images of c_1 and c_2. To obtain a space we are looking for, we glue these blocks not directly to each other but with help of tubes. In our language such tubes are thick walls. The maps above satisfy Convention 3.67. Theorem 3.72 implies that it is possible to construct a CAT(0) space on which G acts geometrically by gluings of the described form. We have seen in Lemma 3.70 that the obtained space is a CAT(0) space with a block decomposition with thick walls. It satisfies all conditions of Convention 3.78. □

Recall that we study the case where $G = G_0 *_H G_1$ acts geometrically on a CAT(0) space $\mathbb{X}$ with block decomposition satisfying the Convention 3.78. Recall further that H is quasi-isometric to $\mathbb{Z}$. Hence, H contains an axial isometry h_α for an axis α in Y. The proof-sketch shows that the walls in $\mathbb{X}$ can be chosen as axes isometric to α. We allow that walls can have a slightly different form. We assume that $Min(h_\alpha) = Y$. Then $Y = C \times \mathbb{R}$ where C is a closed and convex subset of $\mathrm{Min}(h_\alpha)$ and where every $\{x\} \times \mathbb{R}$ in $C \times \mathbb{R}$ is an axis for an axial isometry. Recall that every wall A in $\mathbb{X}$ is isometric to Y or $[0, 1] \times Y$. We denote the blocks and walls in $\mathbb{X}$ as in Convention 3.78. For every coset of G_i in G, the space $\mathbb{X}$ has a block $B^{(gG_i)}$ isometric to X_i, $i \in \{0, 1\}$ and for every coset of H in G, the space $\mathbb{X}$ has a wall $A^{(gH)}$ isometric to $[0, 1] \times Y$ or Y. The group H acts geometrically on the wall $A^{(\mathrm{id}\, H)}$. Accordingly, $A^{(\mathrm{id}\, H)}$ contains an axis for the rank-one isometry $h_\alpha \in H$. We mean this axis when we write α in the following. We choose a base point of $\mathbb{X}$ that is contained in α. This way, both the itineraries of $\alpha^+(\infty)$ and $\alpha^-(\infty)$ consist of the vertex v_{base} in the extended Bass-Serre tree $\mathcal{T}_{\mathrm{ext}}$ associated to $G = G_0 *_H G_1$. We suppose without loss of generality that $\alpha(0) = x_{\mathrm{base}}$. Recall that we assume that every wall contains a geodesic ray that is contracting in $\mathbb{X}$. This implies that h_α and all its conjugates in G are rank-one isometries.

Lemma 4.34. *Suppose that $\mathbb{X}$ has a wall containing a geodesic ray that is contracting in $\mathbb{X}$. Then every conjugate of h_α in G is an axial rank-one isometry of G and every wall A contains an axis for a conjugate of h_α.*

Proof. Let A be a wall of $\mathbb{X}$. By definition of $\mathbb{X}$, a conjugate of H acts geometrically on A. According to Chapter 6 of Part II in [BH99], A contains an axial isometry g acting by translations on an axis γ for g. As every wall is quasi-isometric to $\mathbb{Z}$, every geodesic ray in A is asymptotic to one of the oriented axes γ^+ and γ^- of γ. Thereby, γ is contracting if and only if one of its oriented axes is contracting. Indeed, being contracting is preserved under isometries, and g acts by translations on γ. □

We summarize our assumptions.

Convention 4.35. Let G_0 and G_1 and H be groups acting geometrically on CAT(0) spaces X_0, X_1 and Y respectively. Let Y be quasi-isometric to $\mathbb{Z}$. Then H contains an axial isometry h_α. We assume that h_α is rank-one and that $Min(h_\alpha) = Y$. Let $\mathbb{X}$ be a CAT(0) space with block decomposition associated to G, X_0, X_1 and Y as in Convention 3.78. Let α be an axis for h_α which is contained in $A^{(\mathrm{id}\, H)}$. We choose a base point x_{base} that is contained in α. Then both the itineraries of $\alpha^+(\infty)$ and $\alpha^-(\infty)$ consist of the vertex v_{base} in the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$ associated to $G = G_0 *_H G_1$. We suppose without loss of generality that $\alpha(0) = x_{\text{base}}$.

Our task is to calculate the contracting boundary of $\mathbb{X}$. We use the notation established in Chapter 2. We summarized our notation concerning boundaries in Notation 1.1. We introduced itineraries of geodesic rays in Chapter 3. The itinerary of a geodesic ray in $\mathbb{X}$ is a (possibly infinite) path in the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$ of $G = G_0 *_H G_1$ that describes how the ray runs through the blocks and walls of $\mathbb{X}$. See Definition 3.18. We choose a base point x_{base} of $\mathbb{X}$. The itinerary of every (contracting) geodesic (ray) γ issuing from x_{base} starts in the same vertex v_{base} of $\mathcal{T}_{\text{ext}}$. By Definition 3.34, the *itinerary* $I(\xi)$ of an element $\xi \in \partial\mathbb{X}$ ($\partial_c\mathbb{X}$) is the itinerary of the geodesic ray representing ξ that starts in x_{base}. Let I be a (possibly infinite) path in $\mathcal{T}_{\text{ext}}$ starting in v_{base}. Recall from Definition 3.46 that

$$\hat{U}(I) := \{\gamma(\infty) \in \partial\mathbb{X} \mid \gamma(0) = x_{\text{base}} \text{ and } I = I(\gamma)\}$$
$$\hat{U}_c(I) := \{\gamma(\infty) \in \partial_c\mathbb{X} \mid \gamma(0) = x_{\text{base}} \text{ and } I = I(\gamma)\}$$

We saw in Section 3.5 that there are two different types of connected components in $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$). A connected component κ of $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$) is of type 1 if there exists a (possibly infinite) path in $\mathcal{T}_{\text{ext}}$ such that κ is contained in $\hat{U}(I)$. We remark that a connected component in $\hat{\partial}_c\mathbb{X}$ or $\vec{\partial}_c\mathbb{X}$ is contained in $\hat{U}(I)$ if and only if it is contained in $\hat{U}_c(I)$. Otherwise, κ is of type 2.

First, we study connected components of type 1. Then, we examine connected components of type 2. If ξ is a point in $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$), we denote its connected component by $\kappa(\xi)$.

Lemma 4.36. *Let ξ be a point in $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$). For all $g \in G$ we have $\kappa(g\xi) = g\kappa(\xi)$.*

Proof. Clearly, $g\kappa(\xi)$ contains $g\xi$. As G acts on $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}\vec{\partial}_c\mathbb{X}$) by homeomorphisms, connected components are mapped to connected components. Hence, $g\kappa(\xi) = \kappa(g\xi)$. □

Connected components of type 1

We study connected components of $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$) of type 1. Every connected component of type 1 is of type 1_f or 1_∞. Connected components of type 1_f are well-understood. Every such connected component comes from a block B by Lemma 3.56, so it is homeomorphic to a connected component of $\hat{\partial}B$ ($\hat{\partial}_{c,\mathbb{X}}B$, $\vec{\partial}_{c,\mathbb{X}}B$) by Lemma 3.57. Hence, we concentrate on the examination of connected components of type 1_∞. A connected component κ is of type type 1_∞ if there exists an infinite path I in the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$

staring at v_{base} such that κ is contained in $\hat{U}(I)$. We show that $|\hat{U}(I)| \leq 1$, i.e., that connected components of type 1_f consist of single points. For that purpose, we use the methods of Murray's proof of Proposition 4.5 in [Mur19]. Recall that h_α is an axial isometry contained in H and that α is an axis for h_α passing through x_{base} at time $t = 0$. Recall that we have chosen x_{base} in the wall $A^{(\text{id}\, H)}$.

Lemma 4.37. *Let G, G_0, X_0, G_1, X_1, H, Y and $\mathbb{X}$ be as in Convention 4.35. Let I be an infinite path in $\mathcal{T}_{ext}$ starting with v_{base} and $\xi \in \partial\mathbb{X}$ a point having itinerary I. Let g_iH be the coset of H in G that is the label of the i^{th} vertex of I corresponding to a wall. Then $\lim_{i\to\infty} g_i\alpha^+(\infty) = \lim_{i\to\infty} g_i\alpha^-(\infty) = \xi(\infty)$ in the visual boundary of $\mathbb{X}$.*

Proof. The proof uses the methods of Murray's proof of Proposition 4.5 in [Mur19]. Recall that h_α is an axial isometry contained in H and that α is an axis for h_α passing through x_{base}. Recall that we have chosen x_{base} in the wall $A^{(\text{id}\, H)}$ such that $\alpha(0) = x_{\text{base}}$ and that α^+ and α^- are oriented axes for α. (The axis α in our proof plays the role of α in Murray's proof). Let $\xi \in \partial\mathbb{X}$ and β be a representative of ξ that starts at $x_{\text{base}} = \alpha(0)$. Then $\xi = \beta(\infty)$ and β have the itinerary I. (The role of β in Murray's proof is the same as in our proof). Hence, β enters $A^{(g_iH)}$ for all $i \geq 1$. Let $t_1 = 0$ and $(t_i)_{i\geq 2}$ be the sequence of times where $\beta(t_i) \cap (A)^{(g_iH)} \neq \emptyset$ and $\beta(t_i - \epsilon) \notin A$ for all $\epsilon > 0$. (The time t_i plays the role of the time i in Murray's proof). Let $(\eta^+_{t_i})_{i\in\mathbb{N}}$ and $(\eta^-_{t_i})_{i\in\mathbb{N}}$ be the two representatives of $g_i\alpha^+(\infty)$ and $g_i\alpha^-(\infty)$ that start at $x_{\text{base}} = \alpha(0)$ ($(\eta^+_{t_i})_{i\in\mathbb{N}}$ and $(\eta^+_{t_i})_{i\in\mathbb{N}}$ play the roles of $(k_i)_{i\in N}$ in Murray's proof). We have to prove that $(\eta^+_{t_i}(\infty))_{i\in\mathbb{N}}$ and $(\eta^-_{t_i}(\infty))_{i\in\mathbb{N}}$ converge to $\beta(\infty)$ in the visual boundary of $\mathbb{X}$.

Let t_α be the translation length of h_α. We assume without loss of generality that $d(g_ix_{\text{base}}, \beta(t_i)) = d(g_i\alpha(0), \beta(t_i)) < t_\alpha$. Indeed, if $d(g_ix_{\text{base}}, \beta(t_i)) = d(g_i\alpha(0), \beta(t_i)) \geq t_\alpha$, there exists $k \in \mathbb{Z}$ such that $d(h^k_\alpha g_ix_{\text{base}}, \beta(t_i)) = d(h^k_\alpha g_i\alpha(0), \beta(t_i)) < t_\alpha$. Then we exchange g_i with $h^k_\alpha g_i$ and choose $(h^k_\alpha g_i)_i \in \mathbb{N}$ as sequence of representatives of the cosets $(g_iH)_i \in \mathbb{N}$. (In Murray's proof, the constant corresponding to t_α is C). Recall that $g_i\alpha$ is F-contracting for some $F > 0$. By Theorem 2.24, $g_i\alpha$ is δ_F-slim for some $\delta_F > 0$. Assume that we know that the following analog of Lemma 4.6 in [Mur19] is true:

For all $i \in \mathbb{N}$,

$$d(\eta^+_{t_i}(t_i), \beta(t_i)) \leq 2(3\delta_F + t_\alpha) \text{ and } d(\eta^-_{t_i}(t_i), \beta(t_i)) \leq 2(3\delta_F + t_\alpha). \tag{4.37.1}$$

Recall that the sets $U(\beta(\infty), \epsilon, r)$ form a neighborhood basis for $\beta(\infty)$ in $\hat{\partial}\mathbb{X}$ (see Definition 2.18). Let

$$N(\epsilon, r) := \max\{r, \frac{2r(3\delta_F + t_\alpha)}{\epsilon}\}. \tag{4.37.2}$$

Using Equation (4.37.1) and the convexity of the metric, we conclude as Murray that $d(\eta_{t_i}(r), \beta(r)) \leq \frac{r}{t_i} d(\eta_{t_i}(t_i), \beta(t_i)) \leq \frac{r}{t_i} 2(3\delta_F + t_\alpha) \leq \epsilon$. Thus, every $\eta_{t_i}(\infty)$ is contained in $U(\beta(\infty), \epsilon, r)$ for all i such that $t_i \geq N$. It follows that $(\eta^+_{t_i}(\infty))_{i\in\mathbb{N}}$ and $(\eta^-_{t_i}(\infty))_{i\in\mathbb{N}}$ converge to $\beta(\infty)$ in the visual boundary of $\mathbb{X}$.

It remains to prove Equation (4.37.1). We follow Murray's argumentation in his proof of Lemma 4.6 in [Mur19]. We conclude as Murray by means of Lemma 2.23 that it is sufficient to prove that $d(\eta_{t_i}^+, \beta(t_i)) \leq 3\delta_F + t_\alpha$ and $d(\eta_{t_i}^-, \beta(t_i)) \leq 3\delta_F + t_\alpha$. Recall that α is δ_F-slim. By Lemma 2.22, the projection $\pi_{g_i\alpha}(x_{\text{base}})$ of x_{base} on $g_i\alpha$ satisfies

$$d(\eta_i^+, \pi_{g_i\alpha}(x_{\text{base}})) \leq \delta_F \text{ and } d(\eta_i^-, \pi_{g_i\alpha}(x_{\text{base}})) \leq \delta_F. \tag{4.37.3}$$

Let $\tilde{t}_i$ be such that $\pi_{g_i\alpha}(x_{\text{base}}) = g_i\alpha^+(\tilde{t}_i) = g_i\alpha^-(-\tilde{t}_i)$. Let us assume that $\tilde{t}_i$ is negative. It follows from Equation (4.37.3) and the convexity of the distance function that $d(g_i\alpha(0), \eta_i^+) = d(g_i x_{\text{base}}, \eta_i^+) \leq \delta_F$. Then $d(g_i\alpha(0), \eta_i^-) = d(g_i x_{\text{base}}, \eta_i^-) \leq d(g_i x_{\text{base}}, \eta_i^+) + d(\eta_i^+, \pi_{g_i\alpha}(x_{\text{base}})) + d(\pi_{g_i\alpha}(x_{\text{base}}), \eta_i^-) \leq 3\delta_F$. Thereby the third last and second last inequalities follow from Equation (4.37.3). If $\tilde{t}_i$ is positive, it follows analogously that $d(g_i x_{\text{base}}, \eta_i^-) \leq \delta_F$ and $d(g_i x_{\text{base}}, \eta_i^+) \leq 3\delta_F$. By our choice of g_i it is $d(g_i x_{\text{base}}, \beta(t_i)) \leq t_\alpha$. We conclude that $d(\eta_i^+, \beta(t_i)) \leq d(\eta_i^+, g_i x_{\text{base}}) + d(g_i x_{\text{base}}, \beta(t_i)) \leq 3\delta_F + t_\alpha$. Analogously $d(\eta_i^-, \beta(t_i)) \leq 3\delta_F + t_\alpha$.

□

Corollary 4.38. *Let G, G_0, X_0, G_1, X_1, H, Y and $\mathbb{X}$ be as in Convention 4.35. Let I be an infinite path in $\mathcal{T}_{ext}$ starting with v_{base}. Then $|\hat{U}(I)| \leq 1$ ($|\hat{U}_c(I))| \leq 1$).*

Proof. Let I be an infinite path in $\mathcal{T}_{\text{ext}}$ starting with v_{base} and $g_i H$ be the label of the i^{th} vertex of I corresponding to a wall. We assume that $\hat{U}(I)$ ($\hat{U}_c(I)$) contains an element ξ. By Lemma 4.37, $\lim_{i\to\infty}(g_i\alpha^+(\infty)) = \lim_{i\to\infty}(g_i\alpha^-(\infty) = \xi$ in the visual boundary of $\mathbb{X}$. As the limit of a sequence is unique, it follows that $|\hat{U}(I)| \leq 1$ ($|\hat{U}_c(I)| \leq 1$). □

Let B be a block of $\mathbb{X}$. Recall that every block B is isometric to X_0 or X_1. The set $\mathcal{B}^-$ denotes the set of blocks isometric to X_0 and the set $\mathcal{B}^+$ denotes the set of blocks isometric to X_1. We use notation as in Notation 1.1. Accordingly, $\partial_{c,\mathbb{X}}B$ denotes the set $\{\gamma(\infty) \in \partial_c\mathbb{X} \mid \gamma \subseteq B\}$. The corresponding topological subspaces of $\hat{\partial}_c\mathbb{X}$ and $\vec{\partial}_c\mathbb{X}$ are denoted by $\hat{\partial}_{c,\mathbb{X}}B$ and $\vec{\partial}_{c,\mathbb{X}}B$ respectively. By Lemma 2.35, $\hat{\partial}_{c,\mathbb{X}}B$ and $\vec{\partial}_{c,\mathbb{X}}B$ are homeomorphic to the set of equivalence classes of $\mathbb{X}$-contracting geodesic rays in B equipped with the subspace topology of the visual and contracting boundary of B respectively.

Corollary 4.39. *Let G, G_0, X_0, G_1, X_1, H, Y and $\mathbb{X}$ be as in Convention 4.35. If κ is a connected component of $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$) of type 1, then*

a) κ consists of a single point or

b) for all $B \in \mathcal{B}^-$, κ is homeomorphic to a connected component of $\hat{\partial}B$ ($\hat{\partial}_{c,\mathbb{X}}B$, $\vec{\partial}_{c,\mathbb{X}}B$) or

c) for all $B \in \mathcal{B}^+$, κ is homeomorphic to a connected component of $\hat{\partial}B$ ($\hat{\partial}_{c,\mathbb{X}}B$, $\vec{\partial}_{c,\mathbb{X}}B$).

Proof. As each block is isometric to X_0 or X_1 and G acts transitively on the set $\mathcal{B}^+$ and $\mathcal{B}^-$ respectively, the claim follows from Lemma 3.57 and Corollary 4.38. □

Connected components of type 2

In this subsection, we study connected components of $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$) of type 2. A connected component is of type 2 if it contains at least two points whose itineraries are distinct. Recall that h_α is an axial isometry contained in H and that γ is an axis for h_α passing through x_{base}. Recall that we have chosen x_{base} in the wall $A^{(\text{id}\, H)}$. As before, we denote the connected component of a point x in a topological space by $\kappa(x)$.

Lemma 4.40. *Let G, G_0, X_0, G_1, X_1, H, Y and $\mathbb{X}$ be as in Convention 4.35. If κ is a connected component of $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$) of type 2, there exists $g \in G$ such that $g\alpha^-(\infty)$ or $g\alpha^+(\infty)$ is contained in κ. In particular, κ is homeomorphic to $\kappa(\alpha^+(\infty))$ or $\kappa(\alpha^-(\infty))$.*

Proof. If κ is of type 2, then it contains two points of a distinct itinerary. By Lemma 3.50, there exists $g \in G$ such that $g\alpha^-(\infty)$ or $g\alpha^+(\infty)$ is contained in κ. As G acts by homeomorphism on $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$), $g \cdot \kappa(\alpha^-(\infty)) = \kappa(g \cdot \alpha^-(\infty))$ and $g \cdot \kappa(\alpha^+(\infty)) = \kappa(g \cdot \alpha^+(\infty))$ and κ is homeomorphic to $\kappa(\alpha^+(\infty))$ or $\kappa(\alpha^-(\infty))$. □

Corollary 4.41. *Let G, G_0, X_0, G_1, X_1, H, Y and $\mathbb{X}$ be as in Convention 4.35. Let ξ be a point in $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$). If the itinerary of ξ is infinite, then $|\kappa(\xi)| = 1$ or there exists $g \in G$ such that $\kappa(\xi) = \kappa(g\alpha^+(\infty))$ or $\kappa = \kappa(g\alpha^-(\infty))$.*

Proof. If $\kappa(\xi)$ is of type 1, $|\kappa(\xi)| = 1$ by Corollary 4.38. Otherwise, κ is of type 2. The claim follows from Lemma 4.40. □

Recall that $\partial_c\mathbb{X}$ denotes the set of equivalence classes of contracting geodesic rays in $\mathbb{X}$ and that $\hat{\partial}_c\mathbb{X}$ denotes the space we obtain when we equip $\partial_c\mathbb{X}$ with the subspace topology of the visual boundary $\hat{\partial}\mathbb{X}$ of $\mathbb{X}$. The last lemmas show that the oriented axes α^+ and α^- play an important role. Every connected component of type 2 in $\hat{\partial}_c\mathbb{X}$ and $\vec{\partial}_c\mathbb{X}$ is homeomorphic to $\kappa(\alpha^+(\infty))$ or $\kappa(\alpha^-(\infty))$. As the direct limit topology is finer than the cone topology we know that every connected component of the contracting boundary $\vec{\partial}_c\mathbb{X}$ is contained in a connected component of the topological space $\hat{\partial}_c\mathbb{X}$. So, the study of $\hat{\partial}_c\mathbb{X}$ allows us to deduce properties of connected components of the contracting boundary of $\mathbb{X}$. As α is an axis for a rank-one isometry, we can apply the results of Section 4.3. By Corollary 4.25, either $|\kappa(\alpha^+(\infty))| = |\kappa(\alpha^-(\infty))| = 1$ or $\kappa(\alpha^+(\infty)) = \kappa(\alpha^-(\infty))$. In this case, the statements above are very powerful. It is a hint that it might be true that there occur always two extreme cases. Either all connected components are of type 1 or every connected component of type 2 is large. Let G, G_0, X_0, G_1, X_1, H, Y and $\mathbb{X}$ be as in Convention 4.35. In the following, we study connected components of type 2 in $\hat{\partial}_c\mathbb{X}$. We are mainly interested in the topological space $\hat{\partial}_c\mathbb{X}$. Since all the following results hold for the visual boundary of $\mathbb{X}$ as well, we formulate them for $\hat{\partial}\mathbb{X}$ as well.

Lemma 4.42. *Suppose that $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$) contains a connected component κ of type 2. Then $\kappa(\alpha^+(\infty)) = \kappa(\alpha^-(\infty))$ and there exists $g \in G$ such that $g\alpha^-(\infty)$ and $g\alpha^+(\infty)$ are contained in κ. In particular, κ is homeomorphic to $\kappa(\alpha^+(\infty)) = \kappa(\alpha^-(\infty))$.*

Proof. Let κ be a connected component of type 2. By Lemma 3.61, there exists $g \in G$ such that $g\alpha^-(\infty)$ or $g\alpha^+(\infty)$ is contained in κ. As κ is of type 2, it contains at least two elements. Hence, it follows from Corollary 4.25, that κ contains $g\alpha^-(\infty)$ and $g\alpha^+(\infty)$. Because of Lemma 4.36, $\kappa(\alpha^+(\infty)) = \kappa(\alpha^-(\infty))$ and κ is homeomorphic to $\kappa(\alpha^+(\infty)) = \kappa(\alpha^-(\infty))$. □

Lemma 4.43. *Suppose that $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$) contains a connected component κ of type 2. A connected component of $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c\mathbb{X}$) is of type 2, if and only if there exists $g \in G$ such that $g\alpha^-(\infty)$ or $g\alpha^+(\infty)$ is contained in κ.*

Proof. If κ is a connected component of type 2, then there exists $g \in G$ such that $g\alpha^-(\infty)$ or $g\alpha^+(\infty)$ is contained in κ by Lemma 4.43. On the other hand suppose that a connected component κ contains $g\alpha^-(\infty)$ or $g\alpha^+(\infty)$, $g \in G$. By assumption, there exists a connected component κ' of type 2. Because of Lemma 4.42, there exists $\hat{g} \in G$ such that $\hat{g}\alpha^-(\infty)$ and $\hat{g}\alpha^+(\infty)$ are contained in κ'. Since G acts by homeomorphisms on $\hat{\partial}_c\mathbb{X}$, $\kappa = g\hat{g}^{-1}\kappa'$. As κ' is of type 2, κ is of type 2 as well. □

Using the last lemma, we prove the existence of a bijection between the connected components of type 2 and an edge-disjoint set of trees that cover the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$ and are pairwise isometric and edge-disjoint.

Definition 4.44. Let $\Xi \in \{\hat{\partial}\mathbb{X}, \hat{\partial}_c\mathbb{X}\}$. Let $g \in G$. Let $T_{g\cdot\alpha} = T_{g\cdot\alpha}(\Xi)$ be the subgraph of $\mathcal{T}_{\text{ext}}$ induced by all vertices whose corresponding wall or block in $\mathbb{X}$ contains a geodesic ray γ such that $\gamma(\infty) \in \kappa(g \cdot \alpha^+(\infty))$ in Ξ.

In the following, let $\Xi \in \{\hat{\partial}\mathbb{X}, \hat{\partial}_c\mathbb{X}\}$. If we write $T_{g\cdot\alpha}$, we always mean the associated tree $T_{g\cdot\alpha}(\Xi)$ as defined above.

Remark 4.45. If there exists a connected component of type 2, $\kappa(g\cdot\alpha^+(\infty)) = \kappa(g\cdot\alpha^-(\infty))$ for all $g \in G$ by Lemma 4.43. Thus, we denote the tree in the last definition by $T_{g\cdot\alpha}$ and not by $T_{g\cdot\alpha^+}$.

Lemma 4.46. *For all $g \in G$, $T_{g\cdot\alpha} = g \cdot T_\alpha$*

Proof. Let $g \in G$. Recall that G acts on $\mathcal{T}_{\text{ext}}$ by isometries and that the action of G on $\mathbb{X}$ is compatible with the action of G on $\mathcal{T}_{\text{ext}}$. Furthermore, G acts by homeomorphisms on Ξ. Hence, $g\kappa(\gamma^+(\infty)) = \kappa(g\gamma^+(\infty))$ for all $g \in G$. Thus, $g \cdot T_\alpha = T_{g\cdot\alpha}$. □

Lemma 4.47. *For all $g \in G$, $T_{g\cdot\alpha}$ is a tree.*

Proof. Because of Lemma 4.46, it is sufficient proving the statement for $g = \text{id}$. We show that T_α is a connected subgraph of $\mathcal{T}_{\text{ext}}$. Then it is a tree as a subgraph of a tree. Let v and w be two vertices in $\mathcal{T}_{\text{ext}}$ that don't coincide and are not adjacent. Let P be the unique path in $\mathcal{T}_{\text{ext}}$ connecting v and w. We have to show that T_α contains P. Let K_v be the block or wall corresponding to v and K_w be the block or wall corresponding to w. As v and w lie in T_α, K_v and K_w contain a geodesic ray γ_v and γ_w respectively so that both $\gamma_v(\infty)$ and $\gamma_w(\infty)$ are contained in $\kappa(\alpha^+(\infty)) = \kappa(\alpha^-(\infty))$. By Lemma 3.51, all walls and blocks corresponding to vertices of P contain a geodesic ray whose equivalence class is contained in $\kappa(\alpha^+(\infty)) = \kappa(\alpha^-(\infty))$. Hence, P is contained in T_α. □

Lemma 4.48. *Let g, $g' \in G$. Suppose that Ξ contains a connected component of type* 2. *If $\kappa(g\alpha^+(\infty)) \neq \kappa(g'\alpha^+(\infty))$, then $T_{g\cdot\alpha}$ and $T_{g'\cdot\alpha}$ don't share an edge.*

Proof. Assume that $T_{g\cdot\alpha}$ and $T_{g'\cdot\alpha}$ share an edge $e = \{v, w\}$. One vertex of e, say v, corresponds to a wall A by definition of $\mathcal{T}_{\text{ext}}$. As v is contained in $T_{g\cdot\alpha}$ and $T_{g'\cdot\alpha}$, the wall A contains a geodesic ray γ such that $\gamma(\infty) \in \kappa(g\alpha^+) \cap \kappa(g'\alpha^+)$. Let $\hat{g}H$ be the label of A. Then γ is asymptotic to $\hat{g}\alpha^+_{\geq 0}$ or $\hat{g}\alpha^-_{\geq 0}$. Suppose that $\kappa(\hat{g}\alpha^+(\infty)) = \kappa(\hat{g}\alpha^-(\infty))$. Then both $\hat{g}\alpha^+(\infty)$ and $\hat{g}\alpha^+(\infty)$ are contained in $\kappa(g'\alpha^+)$ and $\kappa(g\alpha^+)$. It follows that $\kappa(g\alpha^+(\infty)) = \kappa(g'\alpha^+(\infty))$.

It remains to study the case that $\kappa(\hat{g}\alpha^+(\infty)) \neq \kappa(\hat{g}\alpha^-(\infty))$. Then, $|\kappa(\hat{g}\alpha^+(\infty))| = 1$ and $|\kappa(\hat{g}\alpha^-(\infty))| = 1$ because of Theorem 4.24 and Corollary 4.25. As G acts by homeomorphisms on Ξ, $|\kappa(\alpha^+(\infty))| = |\kappa(\alpha^-(\infty))| = 1$. Then Ξ does not contain any connected component of type 2 according to Lemma 4.42. □

We define an equivalence relation on G. For g_0, $g_1 \in G$, We say that $g_0 \sim g_1$ if and only if $T_{g_0\cdot\alpha} = T_{g_1\cdot\alpha}$. Let M be a set of representatives for the equivalence classes corresponding to this equivalence relation. By Lemma 4.48, all trees in $\{T_{g\cdot\alpha} \mid g \in M\}$ are pairwise edge-disjoint if Ξ contains a connected component of type 2.

Lemma 4.49. *Suppose that Ξ contains a connected component of type* 2. *Then the trees in the set $\{T_{g\cdot\alpha} \mid g \in M\}$ are pairwise edge-disjoint, isometric, and cover $\mathcal{T}_{ext}$, i.e., every edge of $\mathcal{T}_{ext}$ is contained in an edge of a tree in $\{T_{g\cdot\alpha} \mid g \in M\}$. Furthermore, G acts on the set $\{T_{g\cdot\alpha} \mid g \in M\}$ transitively.*

Proof. Recall that α is an axis contained in $A^{(\mathrm{id}\,H)}$. Accordingly, both adjacent blocks $B^{(\mathrm{id}\,G_0)}$ and $B^{(\mathrm{id}\,G_1)}$ contain a geodesic ray asymptotic to α. Thus, T_α contains the path $v_{\mathrm{id}\,G_0}, v_{\mathrm{id}\,H}, v_{\mathrm{id}\,G_1}$. This is a fundamental domain for the action of G on $\mathcal{T}_{\text{ext}}$. By Lemma 4.46, $T_{g\cdot\alpha} = gT_\alpha$ for all $g \in G$. Hence, $\bigcup_{g\in G} T_{g\alpha}$ covers $\mathcal{T}_{\text{ext}}$. Let g_1, $g_2 \in M$. By Lemma 4.48, $T_{g_1\cdot\alpha} = T_{g_2\cdot\alpha}$ or $T_{g_1\cdot\alpha}$ and $T_{g_2\cdot\alpha}$ are edge-disjoint. The trees in the set $\{T_{g\cdot\alpha} \mid g \in M\}$ are isometric because the action of G on $\mathbb{X}$ is compatible with the action of G on $\mathcal{T}_{\text{ext}}$ and $T_{g\cdot\alpha} = gT_\alpha$ according to Lemma 4.46. By Lemma 4.46, G acts on the set of trees $\{T_{g\cdot\alpha} \mid g \in M\}$ transitively. □

Theorem 4.50. *Let G, G_0, X_0, G_1, X_1, H, Y and $\mathbb{X}$ be as in Convention 4.35. Let $\Xi \in \{\hat{\partial}\mathbb{X}, \hat{\partial}_c\mathbb{X}\}$. Suppose that Ξ contains a connected component of type* 2. *Then the set of connected components of type* 2 *is bijective to the set of edge-disjoint subtrees $\{T_{g\cdot\alpha}(\Xi) \mid g \in M\}$ of $\mathcal{T}_{ext}$ covering $\mathcal{T}_{ext}$.*

Proof. By Lemma 4.42, the existence of a connected component of type 2 implies that $\alpha^+(\infty)$ and $\alpha^-(\infty)$ are contained in a common connected component. Furthermore, for every connected component κ of type 2 exists $g \in G$ such that $g\alpha^+(\infty)$ and $g\alpha^-(\infty)$ are contained in κ. Hence, the set of connected components of type 2 coincides with the set $\{\kappa(g\alpha^+(\infty)) \mid g \in G\}$. Let g' and $\hat{g}$ be two group elements of G. If $g'\alpha^+(\infty)$ and $\hat{g}\alpha^+(\infty)$ are contained in the same connected component, then $T_{g'\cdot\alpha} = T_{\hat{g}\cdot\alpha}$. Indeed, if $\kappa(g'\alpha^+(\infty)) = \kappa(\hat{g}\alpha^+(\infty)$, then both $T_{g'\cdot\alpha}$ and $T_{\hat{g}\cdot\alpha^+}$ contain the vertex $v_{g'H}$ and the two edges incident to $v_{g'H}$. Hence, the trees $T_{g'\cdot\alpha}$ and $T_{\hat{g}\cdot\alpha}$ overlap. By Lemma 4.49, they coincide. On the other hand, let $T_{g'\cdot\alpha}$ and $T_{\hat{g}\cdot\alpha}$ be two distinct trees in the set $\{T_{g\cdot\alpha} \mid g \in M\}$. Then $\kappa(g'\alpha^+(\infty))$ and $\kappa(\hat{g}\alpha^+(\infty))$ are distinct. Indeed, otherwise their trees would not be edge-disjoint by the same argumentation as above. Thus, the set of connected components of type 2 coincides with the set $\{\kappa(g\alpha^+(\infty)) \mid g \in M\}$ and the map ϕ sending a connected component $\kappa(g\alpha(\infty))$, $g \in M$, to the tree $T_{g\cdot\alpha}$ is well-defined and a bijection. □

Figure 4.2 summarizes the classification of connected components of $\hat{\partial}_c\mathbb{X}$ ($\hat{\partial}\mathbb{X}$) resulting from this section. As $\vec{\partial}_c\mathbb{X}$ is finer than $\hat{\partial}_c\mathbb{X}$, every connected component of an element ξ that is of type 1 in $\hat{\partial}_c\mathbb{X}$ is also of type 1 in $\vec{\partial}_c\mathbb{X}$. Hence, the classification pictured in Figure 4.2 leads to Figure 4.3.

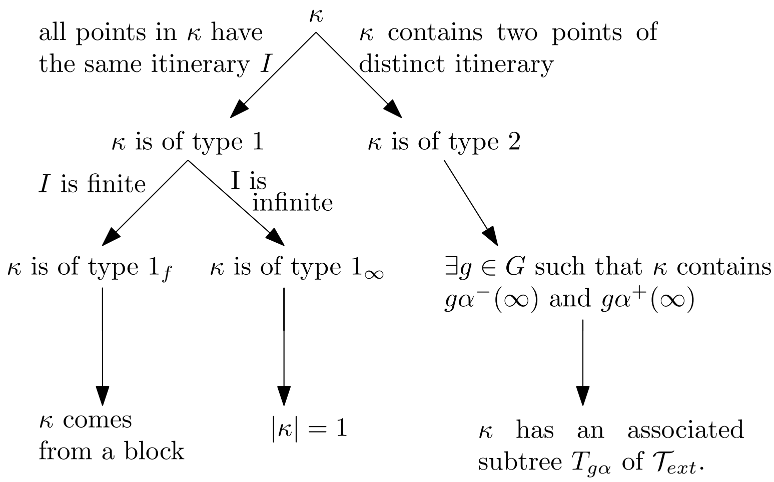

Figure 4.2 Possible types of a connected component κ in $\hat{\partial}_c\mathbb{X}$ ($\hat{\partial}\mathbb{X}$) where $\mathbb{X}$ is as in Convention 4.35. The arrows denote implications which are valid under the conditions of the labels at the arrows.

Recently, Ben-Zvi and Kropholler achieved a result that can be applied to some of the situations we study in this section. They studied CAT(0) spaces X on which an amalgamated free product acts geometrically. Their Theorem 3.2 of [BZK19] gives examples in which the visual boundary of X is not path connected. We cited this theorem in Section 4.1 and saw that a variant of this theorem is true for contracting boundaries of CAT(0) spaces with block decomposition. See Theorem 4.1 and Theorem 4.2. We apply these theorems to our situation where the walls of $\mathbb{X}$ are quasi-isometric to $\mathbb{Z}$. Recall that the *limit set* $\Lambda(H)$ of the subgroup H of G is the set of accumulation points in $\hat{\partial}X$ ($\hat{\partial}_cX, \vec{\partial}_cX$) of an orbit of the action of H on X. If B is a block, $\partial_{c,\mathbb{X}}B$ denotes the set $\{\gamma(\infty) \in \partial_c\mathbb{X} \mid \gamma \subseteq B\}$. The corresponding topological subspaces of $\hat{\partial}_c\mathbb{X}$ and $\vec{\partial}_c\mathbb{X}$ are denoted by $\hat{\partial}_{c,\mathbb{X}}B$ and $\vec{\partial}_{c,\mathbb{X}}B$ respectively. The theorem of Ben-Zvi and Kropholler and its variant for contracting boundaries implies that $\hat{\partial}\mathbb{X}$ ($\vec{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$) is not path connected if $\mathbb{X}$ has a block $\mathbf{B}$ such that the following conditions are satisfied:

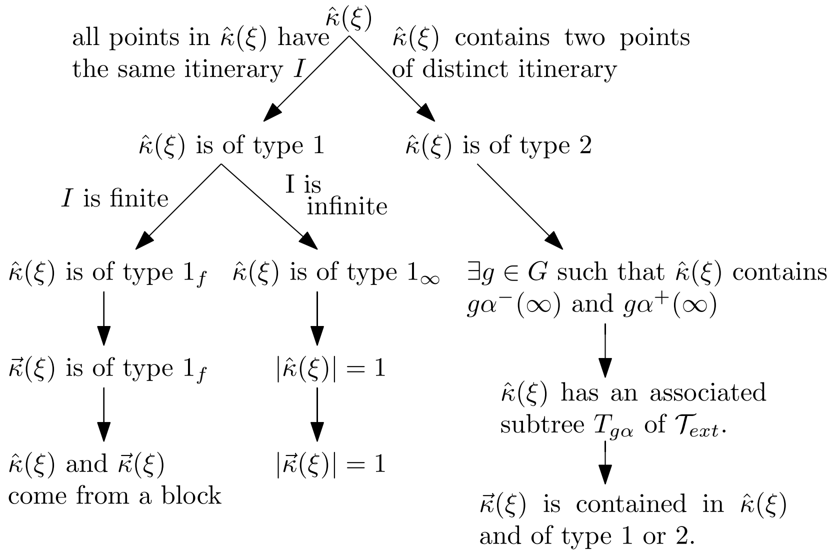

Figure 4.3 Possible types of a connected component of an element ξ in $\partial_c\mathbb{X}$ where $\mathbb{X}$ is as in Convention 4.35. The connected component of ξ in $\hat{\partial}_c\mathbb{X}$ is denoted by $\hat{\kappa}(\xi)$ and the connected component of ξ in $\vec{\partial}_c\mathbb{X}$ is denoted by $\vec{\kappa}(\xi)$. The arrows denote implications under the conditions of the labels of the arrows.

a) $\mathbf{B}$ has a block decomposition $(\mathcal{B}, \mathcal{A})$ such that $\bigcup_{B\in\mathcal{B}} \hat{\partial} B$ ($\bigcup_{B\in\mathcal{B}} \hat{\partial}_{c,\mathbb{X}} B$, $\bigcup_{B\in\mathcal{B}} \vec{\partial}_{c,\mathbb{X}} B$) is nonempty and path connected,

b) $\hat{\partial}\mathbf{B}$ ($\hat{\partial}_{c,\mathbb{X}}\mathbf{B}, \vec{\partial}_{c,\mathbb{X}}\mathbf{B}$) is not path connected and

c) $\Lambda(H)$ is contained in the path component of $\hat{\partial}\mathbf{B}$ ($\hat{\partial}_{c,\mathbb{X}}\mathbf{B}, \vec{\partial}_{c,\mathbb{X}}\mathbf{B}$) that contains $\bigcup_{B\in\mathcal{B}} \hat{\partial} B$ ($\bigcup_{B\in\mathcal{B}} \hat{\partial}_{c,\mathbb{X}} B$, $\bigcup_{B\in\mathcal{B}} \vec{\partial}_{c,\mathbb{X}} B$)

Let us assume that there exists such a block $\mathbf{B}$. Then $\hat{\partial}\mathbf{B}$ ($\vec{\partial}_{c,\mathbb{X}}\mathbf{B}$), $\vec{\partial}_c\mathbf{B}$) is not path connected, but it has a path component κ containing all the points coming from blocks in the decomposition $(\mathcal{B}, \mathcal{A})$ of $\mathbf{B}$. The last assumption says the following. All boundary points of the blocks in the block decomposition $(\mathcal{B}, \mathcal{A})$ of $\mathbf{B}$ and all boundary points associated to axes of the form $g \cdot \alpha$ that are contained in $\mathbf{B}$ are contained in a common path component κ. In the case that we study $\hat{\partial}_c\mathbb{X}$ or $\hat{\partial}\mathbb{X}$, the considerations of this section imply that this path component κ is very large if there exists a connected component of type 2. Indeed, κ contains in such a situation for every vertex in T_α corresponding to a block $\mathbf{B}'$ isometric to $\mathbf{B}$ all boundary points associated to axes for the form $g \cdot \alpha$ that are contained in $\mathbf{B}'$. Furthermore, κ contains all boundary points coming from the associated block decomposition of $\mathbf{B}'$. Simultaneously, Theorem 4.1 implies that $\hat{\partial}\mathbb{X}$ ($\vec{\partial}_c\mathbb{X}$, $\vec{\partial}_c\mathbb{X}$) is not path connected. Recall that in the case of contracting boundaries, we don't know an example satisfying the conditions listed above.

We observed in Example 4.3 that the theorem of Ben-Zvi and Kropholler cannot be used for understanding the case that the visual or contracting boundaries of the blocks are totally disconnected. We finish this section by examining this situation. We consider the following question.

Question 10. Let G, G_0, X_0, G_1, X_1, H, Y and $\mathbb{X}$ be as in Convention 4.35. Suppose that $\vec{\partial}_c X_0$ and $\vec{\partial}_c X_1$ are totally disconnected. When is $\vec{\partial}_c \mathbb{X}$ totally disconnected?

The following lemma is a consequence of our considerations.

Lemma 4.51. *Let G, G_0, X_0, G_1, X_1, H, Y and $\mathbb{X}$ be as in Convention 4.35. Suppose that $\hat{\partial} X_0$ and $\hat{\partial} X_1$ ($\hat{\partial}_c X_0$ and $\hat{\partial}_c X_1$, $\vec{\partial}_c X_0$ and $\vec{\partial}_c X_1$) are totally disconnected. Then the following statements are equivalent.*

a) $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c \mathbb{X}$, $\vec{\partial}_c \mathbb{X}$) is totally disconnected.

b) $|\kappa(\alpha^+(\infty))| = |\kappa(\alpha^-(\infty))| = 1$ in $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c \mathbb{X}$, $\vec{\partial}_c \mathbb{X}$).

c) Every connected component of $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c \mathbb{X}$, $\vec{\partial}_c \mathbb{X}$) is of type 1.

Proof. We show that Item a) $\Rightarrow$ Item b) $\Rightarrow$ Item c) $\Rightarrow$ Item a). Clearly, $|\kappa(\alpha^+(\infty))| = |\kappa(\alpha^-(\infty))| = 1$ if $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c \mathbb{X}$, $\vec{\partial}_c \mathbb{X}$) is totally disconnected. Suppose that $|\kappa(\alpha^+(\infty))| = |\kappa(\alpha^-(\infty))| = 1$ in $\hat{\partial}\mathbb{X}$ ($\hat{\partial}_c \mathbb{X}$, $\vec{\partial}_c \mathbb{X}$). Let κ be a connected component. Then there exists a path I in $\mathcal{T}_{\text{ext}}$ starting with v_{base} such that $\kappa \in \hat{U}_c(I)$ ($\kappa \in \hat{U}_c(I)$). Indeed, if this would not be the case, κ would contain an orbit point of $\alpha^+(\infty)$ or $\alpha^-(\infty)$ by Lemma 3.50 and either $\kappa(\alpha^+(\infty))$ or $\kappa(\alpha^-(\infty))$ would contain more than one single point – a contradiction. Hence, κ is a connected component of type 1. By Corollary 4.39, a connected component κ of type 1 consists of a single point or there exists a block such that κ is homeomorphic to a connected component of $\hat{\partial} B$ ($\hat{\partial}_{c,\mathbb{X}} B$,$\vec{\partial}_{c,\mathbb{X}} B$). By assumption, $\hat{\partial} X_0$ ($\hat{\partial}_c X_0$, $\vec{\partial}_c X_0$) and $\hat{\partial} X_1$ ($\hat{\partial}_c X_1$, $\vec{\partial}_c X_1$) are totally disconnected. Hence, κ consists of a single point. □

The following corollary is a direct consequence of our considerations.

Corollary 4.52. *Suppose that $\vec{\partial}_c X_0$ and $\vec{\partial}_c X_1$ are totally disconnected. If $\vec{\partial}_c \mathbb{X}$ is not totally disconnected, then $\hat{\partial}_c \mathbb{X}$ has a connected component that contains $\alpha^+(\infty)$ and $\alpha^-(\infty)$.*

Proof. Suppose that $\vec{\partial}_c X_0$ and $\vec{\partial}_c X_1$ are totally disconnected. Assume that $\vec{\partial}_c \mathbb{X}$ is not totally disconnected. By Lemma 4.51, $\vec{\partial}_c \mathbb{X}$ contains a connected component of type 2. Since the direct limit topology is finer than the subspace topology of the visual boundary, $\hat{\partial}\mathbb{X}$ contains a connected component of type 2. (See Lemma 3.55). By Lemma 4.51, $\kappa(\alpha^+(\infty))$ or $\kappa(\alpha^-(\infty))$ does not consist of a single point in $\hat{\partial}\mathbb{X}$. By Corollary 4.25, $\hat{\partial}_c \mathbb{X}$ has a connected component that contains $\alpha^+(\infty)$ and $\alpha^-(\infty)$. □

Let G, G_0, X_0, G_1, X_1, H, Y and $\mathbb{X}$ be as in Convention 4.35. Suppose that $\vec{\partial}_c X_0$ and $\vec{\partial}_c X_1$ are totally disconnected. The last corollary says that there are only two cases. Either $\vec{\partial}_c \mathbb{X}$ is totally disconnected or $\hat{\partial}_c \mathbb{X}$ has a connected component that contains $\alpha^+(\infty)$ and $\alpha^-(\infty)$. In the visual boundary, these boundary points are far apart from each other. It is reasonable that a connected component has to be very large if it contains the two boundary points $\alpha^+(\infty)$ and $\alpha^-(\infty)$. Thus, the last corollary can be seen as a hint that either $\hat{\partial}_c\mathbb{X}$ is totally disconnected or $\hat{\partial}_c\mathbb{X}$ has a large connected component. By Lemma 4.51, this large connected component is of type 2. For understanding how large such a connected component is, we study Theorem 4.50. By Theorem 4.50, the set of connected components of type 2 in $\hat{\partial}_c\mathbb{X}$ is bijective to a set of trees $\{T_{g\cdot\alpha} \mid g \in G\}$ as defined in Definition 4.44. The trees in the set $\{T_{g\cdot\alpha} \mid g \in G\}$ are pairwise isometric to each other, they cover $\mathcal{T}_{\text{ext}}$ and are pairwise edge-disjoint. The question arises of how large these trees are. For developing an answer, we study essential axes for rank-one isometries. Let γ be an axis for a rank-one isometry. Recall that the itinerary $I(\gamma)$ of γ in a CAT(0) space $\mathbb{X}$ with block decomposition is the union of $I(\gamma^+_{\geq 0})$ and $I(\gamma^-_{\geq 0})$. By Definition 4.30, we call γ *essential* if its itinerary is a bi-infinite path in $\mathcal{T}_{\text{ext}}$. Lemma 4.29 implies that $I(\gamma)$ is either a bi-infinite path in the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$ or trivial. So, γ is essential if and only if it is not contained in a block or a wall of $\mathbb{X}$. Suppose that the equivalence class of one of its oriented axes does not consist of a single point. The following lemma says that the essential axis γ leads to the existence of a bi-infinite path P in $\mathcal{T}_{\text{ext}}$ that is contained in T_α.

Lemma 4.53. *Let G, G_0, X_0, G_1, X_1, H, Y and $\mathbb{X}$ be as in Convention 4.35. Let γ be an essential axis for a rank-one isometry in G and γ^+ one of its oriented axes. Suppose that the connected component of $\gamma^+(\infty)$ in $\hat{\partial}_c\mathbb{X}$ does not consist of a single point. Then $\hat{\partial}_c\mathbb{X}$ has a connected component κ of type 2 containing $\gamma^+(\infty)$ and $\gamma^-(\infty)$. Furthermore, for every vertex of $I(\gamma)$ with label $\bar{g}H$, κ contains the points $\bar{g}\alpha^+(\infty)$ and $\bar{g}\alpha^-(\infty)$. For all $g \in G$, there exists $\hat{g} \in G$ such that $g\hat{g}I(\gamma)$ is contained in $T_{g\cdot\alpha}$.*

Proof. If $\gamma^+(\infty)$ and $\gamma^-(\infty)$ have distinct connected components in $\hat{\partial}_c\mathbb{X}$, then both of their connected components consist of a single point by Corollary 4.25. Suppose that the connected components of $\gamma^+(\infty)$ and $\gamma^-(\infty)$ in $\hat{\partial}_c\mathbb{X}$ are not distinct. Let κ be the connected component in $\hat{\partial}_c\mathbb{X}$ that contains $\gamma^+(\infty)$ and $\gamma^-(\infty)$. Because γ is essential, both $I(\gamma^+_{\geq 0})$ and $I(\gamma^-_{\geq 0})$ are infinite and distinct. Thus, κ is of type 2. For every vertex of $I(\gamma)$ corresponding to a wall A, κ contains an equivalence class of a geodesic ray contained in A according to Lemma 4.32. For every wall A in $\mathbb{X}$ exists $g \in G$ such that $g\alpha \in A$ and every geodesic ray in A is asymptotic to $g\alpha^+(\infty)$ or $g\alpha^-(\infty)$. If $g\alpha^+(\infty)$ or $g\alpha^-(\infty)$ is contained in κ, the connected component of $g\alpha^+(\infty)$ or $g\alpha^-(\infty)$ does not consist of a single point. Then both are contained in the same connected component because of Corollary 4.25. Thus, for every vertex of $I(\gamma)$ with label gH, κ contains the points $g\gamma^+(\infty)$ and $g\gamma^-(\infty)$. Then $I(\gamma)$ is contained in $T_{\hat{g}\cdot\alpha}$ where $\hat{g}H$ is the label of a vertex of $I(\gamma)$. Indeed, not just every vertex corresponding to a wall but also every vertex corresponding to a block in $I(\gamma)$ contains a geodesic ray whose equivalence class is contained in $\kappa(\hat{g}\alpha^+(\infty))$. Every vertex of $I(\gamma)$ corresponding to a block B is adjacent to a vertex of $I(\gamma)$ which corresponds to a wall A and A and B intersect in a side S of A. Let β be a geodesic ray in A. If β is contained in S, it is contained in B. Otherwise, A is a thick wall isometric to $S \times [0, 1]$. Then B contains a geodesic ray that is asymptotic to β. Thus, $I(\gamma)$ is contained in $T_{\hat{g}\cdot\alpha}$ where $\hat{g}H$ is the label of a vertex of $I(\gamma)$. By Lemma 4.46, $T_{g\cdot\alpha} = gT_\alpha$ for all $g \in G$. Hence, every tree $T_{g\cdot\alpha}$, $g \in G$, contains the path $g\hat{g}^{-1}P$. □

Suppose that the tree T_α contains a bi-infinite path P. Let $g \in G$. If $g \cdot P$ and T_α have an edge in common, then $g \cdot P$ is contained in T_α because G acts on the set of edge-disjoint trees $\{T_{g\cdot\alpha} \mid g \in G\}$ transitively. See Lemma 4.49. So, if T_α contains a lot of bi-infinite paths, then a lot of translates of these bi-infinite paths are contained in T_α as well. Then T_α is a large subtree of $\mathcal{T}_{\text{ext}}$. If T_α is large enough, there is no cover of $\mathcal{T}_{\text{ext}}$ with at least two edge-disjoint trees that are each isometric to T_α. Then $T_\alpha = \mathcal{T}_{\text{ext}}$. In this situation, $\hat{\partial}_c\mathbb{X}$ is connected.

Lemma 4.54. *Suppose that $G = G_0 *_H G_1$ is not trivial i.e., that H is not isomorphic to G_0 or G_1. If $T_\alpha = \mathcal{T}_{ext}$, then $\hat{\partial}_c\mathbb{X}$ ($\hat{\partial}\mathbb{X}$) is connected.*

Proof. If $T_\alpha = \mathcal{T}_{\text{ext}}$, there exists a connected component κ so that every block and every wall in $\mathcal{T}_{\text{ext}}$ contains a geodesic ray whose equivalence class is contained in κ. In particular, κ contains the orbit of $\alpha^+(\infty)$. As $G_0 *_H G_1$ is not trivial, $\alpha^+(\infty)$ is not globally fixed by G. By Theorem 2.32 [Ham09], [Mur19] the orbit of $\alpha^+(\infty)$ is dense in $\hat{\partial}_c\mathbb{X}$. As connected components are closed, they contain all their limit points. Thus, κ contains $\hat{\partial}_c\mathbb{X}$. □

We summarize our considerations. Let G, G_0, X_0, G_1, X_1, H, Y and $\mathbb{X}$ be as in Convention 4.35. Suppose that $\vec{\partial}_c X_0$ and $\vec{\partial}_c X_1$ are totally disconnected. Assume that $\vec{\partial}_c\mathbb{X}$ is not totally disconnected. As of Corollary 4.52, $\hat{\partial}_c\mathbb{X}$ has a connected component that contains $\alpha^+(\infty)$ and $\alpha^-(\infty)$. It is reasonable that the connected component containing $\alpha^+(\infty)$ and $\alpha^-(\infty)$ is large. Lemma 4.53 says that large connected components containing $\alpha^+(\infty)$ and $\alpha^-(\infty)$ arise if $\mathbb{X}$ contains an essential axis such that the connected components of the two associated boundary points don't consist of two single points. Might it be true that the existence of an essential axis is necessary to obtain such a connected component? Then, the answer to the following question is positive.

Question 11. Let G_0 and G_1 CAT(0) groups and H a group quasi-isometric to $\mathbb{Z}$. Suppose that $\vec{\partial}_c G_0$ and $\vec{\partial}_c G_1$ are totally disconnected. Are the following statements equivalent?

a) The contracting boundary of $G = G_0 *_H G_1$ is totally disconnected or empty.

b) G acts geometrically on a CAT(0) space X such that the connected component of every equivalence class of an oriented essential axis in $\hat{\partial}_c X$ consists of a single point.

Theorem 5.58 in Section 5.4 and Corollary 5.59 are applications of our considerations in this section. We study contracting boundaries of right-angled Coxeter groups W_Λ that can be written as amalgamated free products along a group quasi-isometric to $\mathbb{Z}$. Theorem 5.58 says that the contracting boundary of W_Λ is either totally disconnected or $\hat{\partial}_c\Sigma_\Lambda$ has a large connected component. Thereby, Σ_Λ denotes the Davis complex of W_Λ.

5 Contracting boundaries of right-angled Coxeter groups

In this chapter, we study the question of which right-angled Coxeter groups have totally disconnected contracting boundaries. Let Λ be a simplicial graph with vertex set S of size n and edge set E. Associated to Λ is the right-angled Coxeter group

$$W_\Lambda = \langle S \mid s^2 = \text{id} \text{ for all } s \in S, ss' = s's \text{ for all } \{s, s'\} \in E\rangle. \tag{5.0.1}$$

The set S is a *fundamental generating set* and Λ is the *defining* graph of the Coxeter system (W_Λ, S). For short, we say that Λ is the defining graph of W_Λ. We are interested in the question: When is the contracting boundary $\vec{\partial}_c W_\Lambda$ of W_Λ totally disconnected? For that purpose, we have to study the contracting boundary of a space on which W_Λ acts geometrically. Such a space is the Davis complex Σ_Λ of W_Λ. We use the notation established in Chapter 2. We summarized our notation concerning boundaries in Notation 1.1. Throughout this chapter, we assume that every graph is simplicial. We define the Davis complex of a simplicial graph to be the Davis complex of its corresponding right-angled Coxeter group.

In the first section, we summarize what is known about right-angled Coxeter groups with totally disconnected contracting boundaries and complement a known result with a proof by Lazarovich (see Proof 5.23) presented to me in a discussion we had.

Section 5.2 and Section 5.3 concern the first main result of this chapter: the discovery of a new graph class $\mathcal{J}$ of so-called join-decomposable graphs defined in Definition 5.37 that correspond to right-angled Coxeter groups with totally disconnected contracting boundaries (see Corollary 5.38). As a preparation for this result, we prove in Section 5.2 that every Davis complex has a block decomposition with thin walls (Proposition 5.28) using our considerations in Chapter 4.

The second main result of this chapter concerns the question of how the contracting boundary of a right-angled Coxeter group changes if we glue a path of length at least two on its defining graph. We study this question in Section 5.4 and obtain our second main result in Theorem 5.58.

The penultimate section, Section 5.5, is joint work with Graeber, Lazarovich and Stark. We sketch some examples proving that the Burst-Cycle-Conjecture is wrong in general.

In the last section, we summarize the results of this chapter, explain how the results are related to each other, and state a new conjecture.

5.1 A conjecture about contracting boundaries of right-angled Coxeter groups

In this section, we present what is known about a conjecture about totally disconnected contracting boundaries of right-angled Coxeter groups of Tran in [Tra19, Conj. 1.14]. As in the introduction, we refer to this conjecture as the Burst-Cycle-Conjecture. We start this section with an example of Charney and Sultan [CS15, Sec.4.2] for motivating the Burst-Cycle-Conjecture. As in the introduction, we refer to this example as the Cycle-Join-Example. Afterwards study known results concerning the Burst-Cycle-Conjecture from different perspectives. At the end of this section, we complement a known result with a proof presented to me by Lazarovich (see Proof 5.23).

We use the notation established in Chapter 2. We summarized our notation concerning boundaries in Notation 1.1. Recall, we assume throughout this chapter that all graphs are simplicial. Furthermore, the Davis complex of a graph Λ is the Davis complex Σ_Λ of the corresponding right-angled Coxeter group W_Λ. A subgraph Λ' of a graph Λ is *induced* if every edge of Λ with endvertices in $V(\Lambda')$ is an edge of Λ'. If Λ' is an induced subgraph of Λ, then $W_{\Lambda'}$ is a special subgroup of W_Λ and the Davis complex $\Sigma_{\Lambda'}$ can be isometrically embedded in Σ_Λ so that its 1-skeleton contains the identity vertex of Σ_Λ. Compare Lemma 2.51. In such a situation, we say that $\Sigma_{\Lambda'}$ is canonically embedded in Σ_Λ. The contracting boundary of W_Λ is denoted by $\vec{\partial}_c W_\Lambda$. We calculate $\vec{\partial}_c W_\Lambda$ by examining the contracting boundary of Σ_Λ, denoted by $\vec{\partial}_c \Sigma_\Lambda$.

5.1.1 The Cycle-Join-Example of Charney and Sultan

For motivating The Burst-Cycle-Conjecture [Tra19, Conj. 1.14], we recap Section 4.2 in [CS15]. In this subsection, Charney and Sultan calculate the contracting boundaries of two certain right-angled Coxeter groups and prove that one of them contains a 1-sphere and the other one has totally disconnected contracting boundary. Because the contracting boundary is a quasi-isometry invariant, they conclude that the two groups are not quasi-isometric. We are interested in the example of having totally disconnected contracting boundary. This is the example we refer to as the Cycle-Join-Example. The corresponding defining graph Λ is pictured in Figure 5.1.

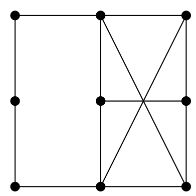

Figure 5.1 The defining graph of a right-angled Coxeter group studied of Charney and Sultan in Section 4.2 of [CS15].

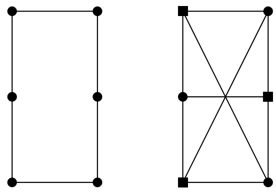

Figure 5.2 Decomposition of the graph in Figure 5.1 into two induced subgraphs Λ_0 (left) and Λ_1 (right).

We sketch Charney's and Sultan's proof that the contracting boundary of W_Λ is totally disconnected. First, they decompose Λ into two induced subgraphs Λ_0 and Λ_1 as in Figure 5.2. The graph Λ_0 is a 6-cycle. Its Davis complex is quasi-isometric to the hyperbolic plane. Thus, the contracting boundary of W_{Λ_0} is a 1-sphere. We consider the other graph Λ_1. Let Λ_0' be the subgraph of Λ_1 induced by the three vertices having the form of a square in Figure 5.2. Let Λ_1' be the graph of Λ_1 induced by the three remaining vertices. The graphs Λ_0' and Λ_1' are both empty graphs on three vertices and Λ is the *nontrivial join* of Λ_0' and Λ_1'.

Definition 5.1. The *join* of two vertex disjoint graphs Λ_0 and Λ_1 is the graph obtained by connecting every vertex of Λ_0 with every vertex of Λ_1 by an edge. A *nontrivial join* is a join of two graphs that each contain at least two non-adjacent vertices.

Let $i \in \{0, 1\}$. Each pair of non-adjacent vertices in Λ_i' generates an infinite Dihedral group D_∞ that is a special subgroup of W'_{Λ_i}. Thus, $\Sigma_{\Lambda_i'}$ is an infinite CAT(0) space. The Davis complex of a join of two graphs is the direct product of their Davis complexes. Accordingly, Σ_{Λ_1} is a direct product of two infinite CAT(0) spaces. By the definition of the product metric, every geodesic ray in a direct product of two infinite CAT(0) spaces bounds a Euclidean half plane. No geodesic ray bounding a Euclidean half plane is contracting. Thus, the contracting boundary of W_{Λ_1} is empty.

We summarize that the graph Λ in Figure 5.1 decomposes into two induced subgraphs Λ_0 and Λ_1 so that $\vec{\partial}_c W_{\Lambda_0}$ is a 1-sphere and $\vec{\partial}_c W_{\Lambda_1}$ is empty.

Let $\Lambda_* := \Lambda_0 \cap \Lambda_1$. The graph Λ_* is a path of length 2. The Davis complex of a 2-path is pictured in Figure 5.3.

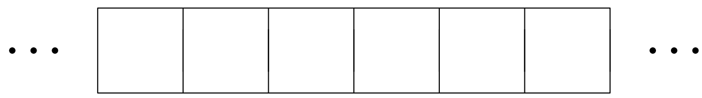

Figure 5.3 The Davis complex of a 2-path.

The group W_Λ can be written as amalgamated free product $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$. This splitting is associated to a block decomposition of the Davis complex of W_Λ into two types of blocks that are isometric to the Davis complexes of Λ_0 and Λ_1 respectively. Indeed, recall that Λ_0, Λ_1, and Λ_* are induced subgraphs of Λ and that the Davis complex of an induced subgraph of a graph Λ is isometrically embedded in the ambient Davis complex of Λ. Accordingly, for every coset gW_{Λ_i}, $g \in W_\Lambda$, Σ_Λ contains a block isometric to Σ_{Λ_i} and for every coset gW_{Λ_*} $g \in W_\Lambda$, Σ_Λ contains a copy isometric to Σ_{Λ_*}. Such a copy of Σ_{Λ_*} is a *wall* of the block decomposition. Two blocks of different types have either an empty intersection or they share a wall. We observe that every wall is contained in a block isometric to Σ_{Λ_1}. Recall that Σ_{Λ_1} is a direct product of two infinite CAT(0) spaces. Hence, none of both boundary points associated to a wall is contracting. Let $\vec{\partial}_c B$ be the contracting boundary of a block that is isometric to Σ_{Λ_0}. Recall that Σ_{Λ_0} is isometric to the hyperbolic plane. If we ignore the ambient Davis complex of Λ, $\vec{\partial}_c B$ is a 1-sphere. If we don't ignore the ambient complex, the situation is different. The block B contains infinitely many walls and the corresponding boundary points are not contracting in Σ_Λ. Charney and Sultan argue that the equivalence classes of geodesic rays in B that are contained in walls, build a dense subset of $\vec{\partial}_c B$. If we remove a dense set from a 1-sphere, we obtain a totally disconnected set. Charney and Sultan conclude that every block isometric to Σ_{Λ_0} contributes a totally disconnected subspace to the contracting boundary of W_Λ. Because the contracting boundary of every block isometric to Σ_{Λ_1} is empty, such a block contributes nothing to the contracting boundary of W_Λ. All other boundary points correspond to geodesic rays that switch between blocks forever. Charney and Sultan prove that every connected component of each such point consists of a single point. This completes the proof that the contracting boundary of W_Λ is totally disconnected.

5.1.2 From the Cycle-Join-Example to the Burst-Cycle-Conjecture

Let Λ be the defining graph of the Cycle-Join-Example and Λ_0 and Λ_1 as in Figure 5.2. The argumentation in the Cycle-Join-Example of Charney and Sultan does not depend on the length of the cycle Λ_0 with the exception that its length has to be larger than 4. Furthermore, the exact form of the graph Λ_1 is not important. The only crucial property of Λ_1 is that it is a join of two graphs each containing two non-adjacent vertices. Motivated by this, we look at the following situation.

Definition 5.2. Let C be a cycle of length at least 5 and C_4 a 4-cycle sharing with C either a path of length two or two vertices that are neither adjacent in C nor adjacent in C_4. Then C_4 is a *glued tetragon* on C.

Two examples of a 5-cycle a, b, c, d, e with a glued tetragon are pictured in Figure 5.4.

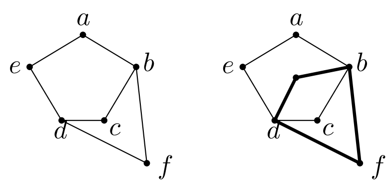

Figure 5.4 Two 5-cycles with a glued tetragon. Left: The 5-cycle and 4-cycle share the 2-path b, c, d. Right: The 5-cycle and 4-cycle share the two vertices b and d.

Every 4-cycle is a join of two empty graphs on two vertices. Moreover, the Davis complex of a 4-cycle is isometric to the Euclidean plane. See Figure 5.5.

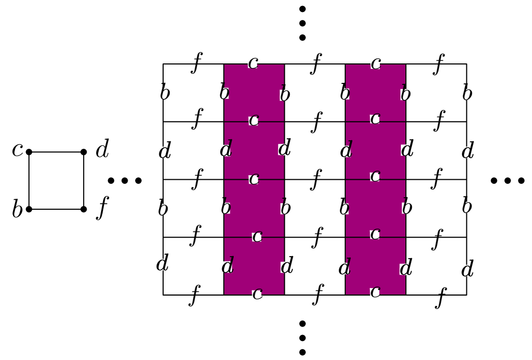

Figure 5.5 Left: a 4-cycle. Right: its Davis complex. The pink stripes are subcomplexes isometric to the Davis complex of the 2-path b, c, d.

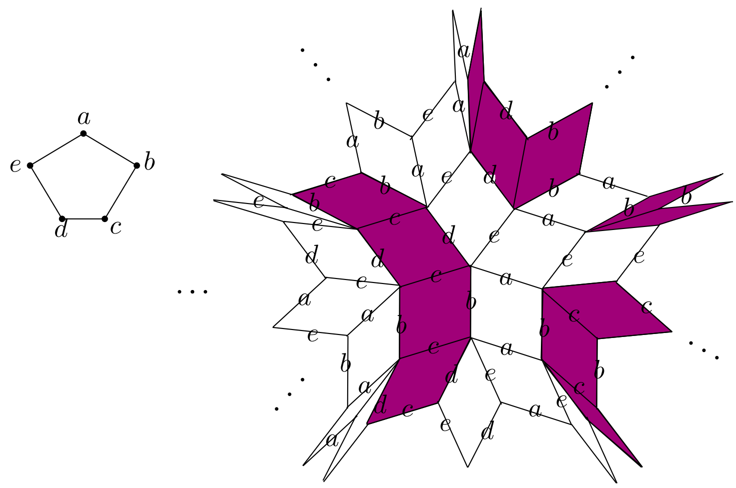

Figure 5.6 Left: a 5-cycle. Right: Its Davis complex. The pink stripes are subcomplexes isometric to the Davis complex of the 2-path b, c, d.

The Davis complex of a cycle of length at least 5 with a glued tetragon consists of two types of blocks. One type is isometric to the Davis complex of a 4-cycle; such a Davis complex is isometric to the Euclidean plane. The other one is quasi-isometric to the Davis complex of a cycle of length at least 5; such a Davis complex is quasi-isometric to the hyperbolic plane. See Figure 5.6. Charney's and Sultan's argumentation in the Cycle-Join-Example implies that the corresponding contracting boundary is totally disconnected.

Lemma 5.3 (Charney and Sultan)**.** *If* Λ *is a cycle of length at least* 5 *with a glued tetragon, then* W_Λ *has totally disconnected contracting boundary.*

The lemma above motivates the following definition of *burst* cycles in graphs. For that purpose, recall that an edge of Λ is a *diagonal* of a cycle C if it connects two non-consecutive vertices of C. A cycle is *induced* if it does not have diagonals.

Definition 5.4 (burst cycles)**.** We say that a cycle in a graph Λ is *burst* in Λ if one of the following three conditions is satisfied:

- C has length 3 or 4,
- C has a diagonal, i.e., two non-consecutive vertices of C are connected by an edge,
- the vertex set of C contains a pair of non-adjacent vertices of an induced 4-cycle.

A cycle is *intact*, if it is not burst.

Let C be a burst cycle in a graph Λ and W_C its corresponding right-angled Coxeter group. Let Σ_Λ be the Davis complex of Λ and Σ_C be the canonically embedded Davis complex of C in Σ_Λ. Let $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_C$ be the subspace of $\vec{\partial}_c\Sigma_\Lambda$ consisting of all equivalence classes of geodesic rays in Σ_C that are contracting in the ambient space Σ_Λ. The subspace $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_C$ is either empty or a non-empty totally disconnected subspace of the contracting boundary of Σ_Λ. Indeed, if C has length 3, Σ_C has a finite diameter. If C has length 4, Σ_C is isometric to $\mathbb{R}^2$. In both cases, the contracting boundary of Σ_C is empty. By Lemma 2.35, $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_C$ is homeomorphic to a subspace of $\vec{\partial}_c\Sigma_C$. Thus, $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_C$ is empty. Otherwise, C is a cycle of length at least 5 with a glued tetragon C_4. Let $\Sigma_{C\cup C_4}$ be the canonically embedded Davis complex of $C\cup C_4$ and $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{C\cup C_4}$ be the subspace of $\vec{\partial}_c\Sigma_\Lambda$ that contains all equivalence classes of contracting geodesic rays that are contained in $\Sigma_{C\cup C_4}$. By Lemma 2.35, $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{C\cup C_4}$ is homeomorphic to a subspace of $\vec{\partial}_c\Sigma_{C\cup C_4}$. As $C\cup C_4$ is a Charney-Sultan-graph, $\vec{\partial}_c\Sigma_{C\cup C_4}$ is totally disconnected. Thus, $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{C\cup C_4}$ is totally disconnected or empty. Because $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_C$ is contained in $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{C\cup C_4}$, $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_C$ is totally disconnected or empty. Suppose that $\xi \in \vec{\partial}_{c,\Sigma_\Lambda}\Sigma_C$. The question arises if the connected component of ξ in the contracting boundary of Σ_Λ consists of a single point. Note that the connected component of a point in a totally disconnected subspace Y of a topological space X might consist of many points. For example, both the rational numbers $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are totally disconnected subspaces of $\mathbb{R}$. But the connected component of each point in $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ in $\mathbb{R}$ consists of the whole space $\mathbb{R}$. However, it seems to be reasonable that such a situation does not occur if Σ_Λ is the Davis complex of a graph missing intact cycles. We refer to the following conjecture formulated by Tran in [Tra19, Conjecture 1.14] as the Burst-Cycle-Conjecture (see Conjecture 1.16).

Conjecture 5.5 (The Burst-Cycle-Conjecture in [Tra19] (Conjecture 1.14))**.** *Every cycle in the defining graph Λ of a right-angled Coxeter group W_Λ is burst if and only if the contracting boundary of W_Λ is totally disconnected.*

5.1.3 Right-angled Coxeter groups with empty contracting boundaries

If the contracting boundary of a right-angled Coxeter group is empty, the contracting boundary is totally disconnected as it does not contain any connected component. In this subsection, we examine which right-angled Coxeter groups have empty contracting boundaries and observe that they satisfy the Burst-Cycle-Conjecture.

At first, we assume that a graph Λ is a clique. Then the associated group W_Λ is finite and its contracting boundary is empty. Suppose that Λ is a nontrivial join. The associated group W_Λ is the direct product of two infinite right-angled Coxeter groups. Hence, the contracting boundary of W_Λ is empty. This can also be proven by means of divergence. Behrstock, Hagen and Sisto proved in Proposition 2.11 in [BHS17] that W_Λ has linear divergence if and only if Λ is a nontrivial join. The triangle-free case was proven by Dani and Thomas in [DT15a]. The divergence of W_Λ is an upper bound for the lower divergence of any geodesic ray in Σ_Λ. By the characterization of contracting geodesic rays of Charney and Sultan in [CS15] (Theorem 2.24), a geodesic ray is contracting if and only if it has superlinear lower divergence. This is the case if and only if it has at least quadratic lower divergence. So, if a graph Λ is a nontrivial join, W_Λ has linear divergence and then every geodesic ray in Σ_Λ has at most linear lower divergence and is not contracting.

On the other hand, suppose that Λ is neither a clique nor a nontrivial join. Then the contracting boundary of W_Λ is not empty by Caprace's and Sageev's considerations in [CS11, Cor. B]. If Λ is triangle-free, Nguyen and Tran observed this fact in [NT19] and the general case was for instance studied by Levcovitz [Lev18]. He proved in Theorem 7.3 of [Lev18] that the divergence of W_Λ is at least quadratic. For that purpose, he proved that the divergence of a certain bi-infinite geodesic ray in Σ_Λ is at least quadratic. Because this bi-infinite geodesic ray is periodic, it follows that its lower divergence is at least quadratic. Then it is contracting according to the characterization of contracting geodesic rays by Charney and Sultan (Theorem 2.24). We explain what this bi-infinite geodesic ray looks like and sketch a proof that it is contracting: The bi-infinite geodesic ray is obtained as follows. Let S be the vertex set of Λ. We choose a word $s_0, \dots, s_k$ in S such that for every generator $s \in S$, there exists an i such that $s_i = s$. Furthermore, s_0 and s_k don't commute and s_i and s_{i+1} don't commute for all $1 \leq i < k$. Then the bi-infinite word $\dots s_0, \dots, s_k, s_0, \dots s_k \dots$ defines a bi-infinite path in the 1-skeleton in Σ_Λ. This path is a bi-infinite geodesic ray γ in Σ_Λ. Levcovitz proved that the two hyperplanes $\bar{H}$ and $\hat{H}$ dual to s_0 and s_k are strongly separated, i.e., there exists no hyperplane intersecting $\bar{H}$ and $\hat{H}$ simultaneously. Because $s_0 \cdots s_k$ acts by translation on γ, the translates of $\bar{H}$ and $\hat{H}$ under $(s_0 \cdots s_k)^i$, $i \in \mathbb{Z}$, build a hyperplane sequence $(H_i)_{i \in \mathbb{Z}}$ intersecting γ at points x_i, $i \in \mathbb{Z}$. By construction, $d(x_i, x_{i+1}) \leq k$ for all $i \in \mathbb{Z}$. Furthermore, H_i and H_{i+1} are strongly separated. By Theorem 4.2 of [CS15] (cited below as Theorem 5.19) γ is contracting. The following theorem summarizes these observations.

Theorem 5.6. *Let* Λ *be a graph and* W_Λ *its associated right-angled Coxeter group. The following statements are equivalent.*

a) Λ *is a nontrivial join or a clique.*

b) W_Λ *has at most linear divergence.*

c) The contracting boundary of W_Λ *is empty.*

d) The lower divergence of every geodesic ray in Σ_Λ *is at most linear.*

Finally, we observe that non-trivial joins and cliques don't contain any intact cycle. Thus, they satisfy the Burst-Cycle-Conjecture.

5.1.4 Spheres in contracting boundaries coming from intact cycles

In this subsection, we examine one direction of the Burst-Cycle-Conjecture: If W_Λ is totally disconnected, then Λ does not contain any intact cycle. It turns out that this statement is true. Indeed, every intact cycle of a graph Λ contributes a 1-sphere to the contracting boundary of W_Λ. This follows from the next theorem about *stability* introduced by Durham and Taylor in [DT15b]. See Definition 2.14 for a formal definition of stability.

Theorem 5.7 (Tran [Tra19], Genevois [Gen20], Russell, Spriano and Tran [RST18])**.** *Let* Λ' *be an induced subgraph of a graph* Λ *and* $\Sigma_{\Lambda'}$ *be the canonically embedded Davis complex of* Λ' *in* Σ_Λ*. No pair of non-adjacent vertices in* $V(\Lambda')$ *are contained in an induced* 4*-cycle in* Λ *if and only if* $W_{\Lambda'}$ *is a stable subgroup of* W_Λ*.*

The following corollary is a direct consequence of Theorem 5.7 and implies that one direction of Conjecture 5.5 is true.

Corollary 5.8. *Suppose that a graph* Λ *contains a intact cycle* C*. Then every geodesic ray in the canonically embedded Davis complex* Σ_C *of* C *in* Σ_Λ *is contracting in* $\vec{\partial}_c\Sigma_\Lambda$*. The subspace of* $\vec{\partial}_c\Sigma_\Lambda$ *consisting of all contracting geodesic rays in* Σ_C *is homeomorphic to a* 1*-sphere.*

Theorem 5.7 was proven several times. In the case of triangle-free graphs, it is Corollary 7.12 in Tran's paper [Tra19]. In Corollary 7.14, Tran concluded that one direction of Conjecture 5.5 is true for triangle-free graphs. For general graphs, Theorem 5.7 follows from Proposition 4.9 in [Gen20]. Russell, Spriano and Tran formulated another proof of Genevois' proposition in Theorem 7.5 of [RST18]. Theorem 5.7 is related to an example of Behrstock in [Beh19]. Behrstock studied a certain right-angled Coxeter group whose defining graph has an intact cycle of length 5. He proved independently to the mentioned theorems that the corresponding special subgroup is stable. At the end of this section, in Proof 5.23, we add to the proofs of Theorem 5.7 another one of Lazarovich, presented to me in a discussion we had. This proof is similar to the proof in [Gen20]. Both proofs use the behavior of hyperplanes in CAT(0) cube complexes. Whereas Genevois used so-called grids of hyperplanes, the proof of Lazarovich applies the characterization of contracting geodesic rays in CAT(0) cube complexes that was given by Charney and Sultan in the paper introducing contracting boundaries [CS15, Thm 4.2].

Remark 5.9. Proposition 4.9 of Genevois in [Gen20] and Theorem 7.5 of Russell, Spriano and Tran in [RST18] are handling not stability but *strong quasiconvexity*. So-called *strongly quasiconvex subgroups* of finitely generated groups were introduced by Tran in [Tra19] and independently by Genevois in [Gen20] under the name *Morse subgroups*. Similar objects are *N-stable* subsets of geodesic metric spaces studied by Cordes and Hume in [CH17]. For a definition of strong quasiconvexity see Definition 2.15. Genevois and Russell, Spriano and Tran showed that a special subgroup $W_{\Lambda'}$ of a right-angled Coxeter group W_Λ is strongly quasiconvex if and only if its defining graph does not contain two non-adjacent vertices of an induced 4-cycle that is not completely contained in Λ'. By Theorem 4.8 in [Tra19] (cited as Theorem 2.16), an infinite subgroup of a finitely generated group is stable if and only if it is hyperbolic and strongly quasi-convex in the ambient group. By Theorem 2.50, this implies that a special subgroup $W_{\Lambda'}$ of W_Λ is *stable* if and only if its defining subgraph Λ' does not contain a pair of non-adjacent vertices that are contained in an induced 4-cycle in Λ.

Suppose that C is an good cycle in a graph Λ. By Corollary 5.8, $\vec{\partial}_c W_\Lambda$ contains a sphere. This sphere comes from the canonically embedded Davis complex Σ_C in Σ_Λ. Indeed, Σ_C is quasi-isometric to the hyperbolic plane. Its contracting boundary is a sphere and this sphere is topologically embedded in the contracting boundary of Σ_Λ. More general, let Λ' be an induced subgraph of a graph Λ and $\Sigma_{\Lambda'}$ be the canonically embedded Davis complex of Λ' in Σ_Λ. Suppose no pair of non-adjacent vertices in Λ' is contained in an induced 4-cycle. Then the contracting boundary of $W_{\Lambda'}$ is topologically embedded in the contracting boundary of W_Λ. Indeed, there exists $D > 0$ such that every geodesic ray in $\Sigma_{\Lambda'}$ is D-contracting in the ambient complex Σ_Λ. By the characterization of contracting geodesic rays of Charney and Sultan (See Theorem 2.24) in [CS15], this is equivalent to the property that there exists M such that all geodesic rays in $\Sigma_{\Lambda'}$ are M-Morse. On the other hand, if there is a geodesic ray in $\Sigma_{\Lambda'}$ that is not contracting, or equivalently not Morse, two non-adjacent vertices in Λ' are contained in an induced 4-cycle of Λ'. Accordingly, Theorem 5.7 has the following simple and useful consequence. It also follows from Theorem 7.9 in [Lev18].

Lemma 5.10. *Let Λ be a graph and u and v two non-adjacent vertices of Λ that are not contained in an induced 4-cycle. Let $\Lambda_{u,v}$ be the subgraph of Λ induced by u and v. let $\Sigma_{\Lambda_{u,v}} \subseteq \Sigma_\Lambda$ be the canonically embedded Davis complex of $\Lambda_{u,v}$ in the Davis complex Σ_Λ of Λ. Then every geodesic ray that is contained in $g\Sigma_{\Lambda_{u,v}}$, $g \in W_\Lambda$ is contracting in Σ_Λ for all $g \in W_\Lambda$.*

Proof. The claim follows from Theorem 5.7. ☐

A statement related to Theorem 5.7 concerning mapping class groups were studied by Kim in [Kim19]. Russell, Spriano and Tran [RST18, Corollary 7.4] united and expanded the work of Tran [Tra19], Genevois [Gen20], Nguyen-Tran [NT19] and Kim [Kim19]. They proved that every strongly quasiconvex subset of any group listed below (Item a) - Item f)) is either hyperbolic or coarsely covers the entire space. In particular, if H is a strongly quasiconvex subgroup in any of the following groups, then H is either stable or finite index. Tran provided in [Tra19] a counterexample proving that not all right-angled Coxeter groups have this property. Accordingly, in the following list occur not all right-angled Coxeter groups but those whose defining graphs are *strongly* $\mathcal{CFS}$. We will take a closer look at this graph class further below.

a) The Teichmüller space of finite type, oriented surface with the Teichmüller metric

b) The Teichmüller space of finite type, oriented surface of complexity at least 6 with the Weil-Petersson metric

c) The mapping class group of a finite type, oriented surface

d) A right-angled Artin group with connected defining graph

e) A right-angled Coxeter group with strongly $\mathcal{CFS}$ defining graph

f) The fundamental group of a non-geometric graph manifold

5.1.5 Groups with quadratic divergence satisfying the Burst-Cycle-Conjecture

We examine the Burst-Cycle-Conjecture for right-angled Coxeter groups that have quadratic divergence. Recall that the contracting boundary of a right-angled Coxeter group W_Λ is empty if and only if W_Λ has at most linear divergence and that every other right-angled Coxeter group has at least quadratic divergence. Recall further that the divergence of W_Λ is an upper bound of the lower divergence of a geodesic ray in Σ_Λ and that a geodesic ray is contracting if and only if its lower divergence is at least quadratic.

In [DT15a], Dani and Thomas introduced a class of so-called $\mathcal{CFS}$ graphs. They proved that the divergence of a right-angled Coxeter group whose defining graph Λ is triangle-free is quadratic if and only if Λ is $\mathcal{CFS}$. Levcovitz proved this statement for general graphs in Theorem 7.4 of [Lev18].

For defining when a graph Λ is $\mathcal{CFS}$, we consider its *four-cycle graph*. The *four-cycle graph* Λ^4 of Λ is obtained as follows. The vertices of Λ^4 are the induced cycles of length four. Two vertices are connected by an edge if the corresponding 4-cycles have a pair of vertices in common that are not adjacent in Λ. The *support* of a subgraph K of Λ^4 is the set of vertices of Λ that are contained in a 4-cycle corresponding to a vertex of K. The following is a generalization of the original Definition of Dani and Thomas that was introduced in [Beh+18].

Definition 5.11 ($\mathcal{CFS}$)**.** A graph Λ is $\mathcal{CFS}$ if it is a join of two graphs Δ and K where Δ is a nontrivial subgraph of Λ and K is a clique (it is allowed that this clique is trivial, i.e., $(\emptyset, \emptyset)$) so that Δ^4 has a connected component whose support coincides with the vertex set $V(\Delta)$ of Δ.

Russell, Spriano and Tran studied graphs that are *strongly* $\mathcal{CFS}$.

Definition 5.12 (strongly $\mathcal{CFS}$)**.** If a graph Λ is $\mathcal{CFS}$ and Λ^4 is connected, then it is *strongly* $\mathcal{CFS}$.

Remark 5.13. The original definition can be obtained by assuming that the clique K is the trivial graph $(\emptyset, \emptyset)$. If K is not empty and Λ is the join of K and a subgraph Δ, W_Δ is of finite index in W_Λ. Hence W_Λ and W_Δ are quasi-isometric and their contracting boundaries coincide.

Intuitively, a $\mathcal{CFS}$ graph contains a lot of induced 4-cycles and one could expect that its contracting boundary is totally disconnected. However, the example of Behrstock in [Beh19] mentioned above proved that this is wrong. This example is a right-angled Coxeter group whose defining graph contains an intact cycle and is $\mathcal{CFS}$. Thus, its contracting boundary contains a 1-sphere and the subgroup associated to the intact cycle is stable. Russell, Spriano and Tran expanded this example in [RST18]. They showed that any right-angled Coxeter group (respectively hyperbolic right-angled Coxeter group) is an infinite index strongly quasiconvex subgroup (respectively stable subgroup) of a $\mathcal{CFS}$ right-angled Coxeter group (see Proposition 7.6 in [RST18]). They concluded that the quasi-isometry classification of a right-angled Coxeter group whose defining graph is $\mathcal{CFS}$ might be no simpler than the quasi-isometry classification of all right-angled Coxeter groups.

We come back to the question of how the contracting boundary of a right-angled Coxeter group looks like whose defining graph is $\mathcal{CFS}$. Nguyen and Tran studied the following graph class in [NT19].

Definition 5.14. Let $\mathcal{G}$ be the graph class consisting of all graphs that are $\mathcal{CFS}$, connected, triangle-free, planar, and having at least 5 vertices and no separating vertices or edges.

They show that W_Λ has totally disconnected contracting boundary if $\Lambda \in \mathcal{G}$ using Corollary 3.12 in [NT19]. For formulating this corollary, we define *suspensions*. A graph Λ is a *suspension* of three points if it is the join of an empty graph on two vertices u and v and an empty graph on three vertices a_1, a_2 and a_3. Recall that the contracting boundary of a join of two graphs is empty if both contain two non-adjacent vertices. In particular, the contracting boundary of a right-angled Coxeter group with a suspension of three points as defining graph is empty. The following is Corollary 3.12 in [NT19].

Theorem 5.15 (Nguyen and Tran)**.** *Let Λ be a connected, triangle-free, planar graph that has at least* 5 *vertices and no separating vertices or edges. The right-angled Coxeter group W_Λ splits as a tree of groups satisfying the following:*

- *each vertex group T_v is W_C where C is the suspension of three distinct points or T_v is a relatively hyperbolic group with respect to a collection of $D_\infty \times D_\infty$ subgroups of T_v.*
- *each edge group is $D_\infty \times D_\infty$.*

Moreover, all vertex groups are isomorphic to a right-angled Coxeter group of the suspension of three distinct points if and only if Λ is $\mathcal{CFS}$.

Let $\Lambda \in \mathcal{G}$. Theorem 5.15 says that W_Λ splits as a tree of groups whose vertex groups have empty contracting boundaries. By the argumentation in the Cycle-Join-Example explained above, the contracting boundary of a right-angled Coxeter group with totally disconnected contracting boundary stays totally disconnected if we glue a non-trivial join on its defining graph. Thus, the next corollary follows, as mentioned by Nguyen and Tran on page 3 of [NT19].

Corollary 5.16 (Nguyen and Tran)**.** *Let $\Lambda \in \mathcal{G}$. The contracting boundary of W_Λ is empty if and only if Λ is a suspension of at least three vertices. Otherwise, the contracting boundary of W_Λ is nonempty and totally disconnected. In particular, $\mathcal{G}$ satisfies the Burst-Cycle-Conjecture 5.5.*

Remark 5.17. We remark that Corollary 5.16 is Corollary 1.11 in the first version of Theorem 5.15 on arXiv.

5.1.6 A proof of Theorem 5.7 by Lazarovich

As announced, we will finish this section with a proof of Theorem 5.7 that was presented to me by Lazarovich. For formulating this proof, we use the notation of Section 2.6. As a preparation for the announced proof, we recall some background of CAT(0) cube complexes.

Let X be a CAT(0) cube complex. Let $\sim$ be the equivalence relation on the set of midplanes generated by the condition that two midplanes are equivalent if they share a face. Recall that a *hyperplane* in X is the union of all the midplanes in an equivalence class of this equivalence relation. The *carrier* $\mathcal{N}(H)$ of a hyperplane is the union of all cubes which contain H. Every hyperplane is convex, does not intersect itself, and is itself a CAT(0) cube complex. If two hyperplanes are distinct and the carriers of two distinct hyperplanes intersect, we say that they are *adjacent*. Let H_0, H_1 and H be three hyperplanes in X. By Lemma 2.38, X decomposes into two distinct half-spaces C_0 and C_1 if we delete H. We say that H *separates* H_0 and H_1 if one of the two hyperplanes is contained in C_0 and the other one is contained in C_1. In such a situation, H *lies between* H_0 and H_1. Lemma 2.39 implies that two disjoint hyperplanes that are not adjacent have a hyperplane between them. Like Charney and Sultan in Definition 4.1 in [CS15], we define when two hyperplanes are *k-separated*. This notion was introduced by Behrstock and Charney [BC12] and is related to the rank rigidity theorem of Caprace and Sageev[CS11].

Definition 5.18 ([CS15])**.** Two hyperplanes H_0, H_1 are *k-separated* if they are disjoint and at most k hyperplanes intersect both H_0 and H_1. If H_0 and H_1 are 0-separated, they are *strongly separated*.

We cite Theorem 4.2 of Charney and Sultan in [CS15].

Theorem 5.19 (Theorem 4.2 in [CS15]). *Let X be a uniformly locally finite CAT(0) cube complex, i.e., the 1-skeleton of X has bounded valence ν. There exist $r > 0$, $k \geq 0$ (depending only on D and ν) such that a geodesic ray γ in X is D-contracting if and only if γ crosses an infinite sequence of hyperplanes H_1, H_2, $H_3, \ldots$ at points $x_i = \gamma \cap H_i$ satisfying the following conditions:*

a) H_i, H_{i+1} are k-separated and

b) $d(x_i, x_{i+1}) < r$.

Definition 5.20. We say that a hyperplane sequence $(H_i)_{i\in\mathbb{N}}$ is a (r, k)-*good* hyperplane sequence for γ if it is a hyperplane sequence as in Theorem 5.19. A hyperplane sequence $(H_i)_{i\in\mathbb{N}}$ is *good for* γ if there are $r > 0$, $k \geq 0$ such that $(H_i)_{i\in\mathbb{N}}$ is (r, k)-good.

Recall that the link of a vertex v in a graph Λ is the subgraph induced by all vertices adjacent to v.

Lemma 5.21. *Two vertices v and w in a graph are not contained in an induced 4-cycle if and only if the graph induced by $\mathrm{lk}(v) \cap \mathrm{lk}(w)$ is complete.*

Proof. Let v and w be two vertices in a graph Λ that are not contained in an induced 4-cycle. For arriving at a contradiction we assume that $\mathrm{lk}(v) \cap \mathrm{lk}(w)$ does not induce a complete graph. Then the graph induced by $\mathrm{lk}(v) \cap \mathrm{lk}(w)$ contains two non-adjacent vertices v' and w'. By definition, both v' and w' are connected to v and w by an edge. But then v, w, v' and w' induce a 4-cycle – a contradiction. On the other hand, we assume that the graph induced by $\mathrm{lk}(v)\cap\mathrm{lk}(w)$ is complete. Let C be a 4-cycle containing v an w. If v and w would be adjacent, $\mathrm{lk}(v) \cap \mathrm{lk}(w)$ would be empty. Hence, v and w have distance two in C. Thus, the other two vertices of C are contained in $\mathrm{lk}(v) \cap \mathrm{lk}(w)$. As $\mathrm{lk}(v) \cap \mathrm{lk}(w)$ induces a complete graph, there is an edge between these two vertices. So, C has a diagonal and is not induced. □

Recall that the Davis complex of a right-angled Coxeter group with defining graph Λ is the CAT(0) cube complex obtained by gluing in Euclidean cubes in the Cayley graph corresponding to Λ whenever possible. Recall that every edge connecting two vertices g and h in the Cayley graph is labeled by a generator s of W_Λ if $gs = h$. This label corresponds to a vertex of Λ. A hyperplane H intersects opposite edges of cubes. Thus, all edges intersected by a hyperplane have the same label. We say that this is the label of H.

Lemma 5.22. *Let H and H' be two adjacent hyperplanes in a Davis complex of a graph Λ with label s and t respectively. The number of hyperplanes intersecting H and H' simultaneously coincides with the number of hyperplanes in the Davis complex of $lk(s) \cap \mathrm{lk}(t)$.*

Proof. Let H_0 and H_1 be two adjacent hyperplanes with label s and label t. Let H be a hyperplane intersecting H_0 and H_1. Then H intersects an edge in the carrier of H_0 and an edge in the carrier of H_1. The label l of H coincides with the label of these two edges. The hyperplane H consists of a maximal connected set of midplanes of cubes that intersect edges with label l. This set is isometric to the Davis complex of the star S_s of s, ie. the subgraph of Λ induced by s and all its adjacent vertices. Thus, s is the label of a vertex in the intersection of S_s and S_t. This is the subgraph of Λ induced by all the vertices in $\mathrm{lk}(s) \cap \mathrm{lk}(t)$. On the other hand, let H be a hyperplane in $\Sigma_{\mathrm{lk}(s)\cap\mathrm{lk}(t)}$. Then the label of H commutes with s and t simultaneously. Because H_0 and H_1 have label s and t respectively, they are isometric to the Davis complex of the stars S_s and S_t respectively. The intersection of these two complexes coincides with $\Sigma_{\mathrm{lk}(s)\cap\mathrm{lk}(t)}$. Hence, for every hyperplane in $\Sigma_{\mathrm{lk}(s)\cap\mathrm{lk}(t)}$ we obtain a hyperplane intersecting H_0 and H_1 simultaneously. □

We attain the further proof for Theorem 5.7 presented to me by Lazarovich, now.

Proof 5.23 (of Theorem 5.7 by Lazarovich). First, we assume that no pair of two non-adjacent vertices in Λ' is contained in an induced 4-cycle in Λ. By Theorem 2.50, $W_{\Lambda'}$ is hyperbolic and acts geometrically on $\Sigma_{\Lambda'}$. By Theorem 2.12, there exists $M : [1,\infty) \times [0,\infty) \to [0,\infty)$ such that all geodesic rays in $\Sigma_{\Lambda'}$ are M-Morse in $\Sigma_{\Lambda'}$. By Theorem 2.14 of Charney and Sultan in [CS15], there exists $C > 0$ such that all geodesic rays in $\Sigma_{\Lambda'}$ are C-contracting in $\Sigma_{\Lambda'}$. We have to show that there exists $D > 0$ such that all geodesic rays in $\Sigma_{\Lambda'}$ are D-contracting in the ambient complex Σ_Λ. Let α be a geodesic ray in $\Sigma_{\Lambda'}$ starting at the vertex corresponding to the identity of W_Λ. By assumption, α is C-contracting in $\Sigma_{\Lambda'}$. Then α has an (r,k)-good hyperplane sequence $(H_i)_{i\in\mathbb{N}}$. Let $i \in \mathbb{N}$. Then H_i and H_{i+1} intersect γ at point x_i and x_{i+1} respectively. If H_i and H_{i+1} are not adjacent, there is a hyperplane H between them because of Lemma 2.39. This hyperplane intersects γ in its subsegment between x_i and x_{i+1}. Indeed otherwise, the geodesic segment of γ connecting x_i with x_{i+1} would lie in $X \setminus H$ – a contradiction to the fact that H separates H_i and H_{i+1}. We add H to the hyperplane sequence $(H_i)_{i\in\mathbb{N}}$ and obtain a new hyperplane sequence $(\hat{H}_j)_{j\in\mathbb{N}}$. More precisely, let $\hat{H}_j := H_i$ for all $j \in \mathbb{N}$, $j \leq i$. Let $\hat{H}_{i+1} := H$ and $\hat{H}_j := H_{i+1}$ for all $j \geq i+2$. We continue in this manner until every hyperplane between H_i and H_{i+1} is contained in the new constructed hyperplane sequence. We repeat this procedure for all $i \in \mathbb{N}$. This way we obtain a hyperplane sequence $(\bar{H}_i)_{i\in\mathbb{N}}$ intersecting γ at points $\bar{x}_i = \gamma \cap \bar{H}_i$ in $\Sigma_{\Lambda'}$ such that for all $i \in \mathbb{N}$

- $d(x_i, x_{i+1}) < r$
- $\bar{H}_i$ and $\bar{H}_{i+1}$ are adjacent.

The label of every hyperplane in this sequence corresponds to a vertex in Λ' and every such vertex corresponds to an edge in the 1-skeleton of the Davis complex $\Sigma_{\Lambda'}$. We embed $\Sigma_{\Lambda'}$ canonically in Σ_Λ. As hyperplanes of CAT(0) cube complexes are convex and because $\Sigma_{\Lambda'}$ is isometrically embedded in Σ_Λ, every hyperplane $\bar{H}_i$ in $\Sigma_{\Lambda'}$ defines a unique hyperplane $\tilde{H}_i$ in Σ_Λ such that $\bar{H}_i$ is isometrically embedded in $\tilde{H}_i$ for all $i \in \mathbb{N}$. So, the hyperplane sequence $\tilde{H}_1, \tilde{H}_2, \ldots,$ is a hyperplane sequence in Σ_Λ. Let k' be the number of vertices in Λ. We show that $\tilde{H}_i$ and $\tilde{H}_{i+1}$ are k'-separated in Σ_Λ for all $i \in \mathbb{N}$. Then the claim follows from Theorem 5.19. Let $i \in \mathbb{N}$. First, we show that $\tilde{H}_i$ and $\tilde{H}_{i+1}$ are disjoint. Let s be the label of $\tilde{H}_i$ and t be the label of $\tilde{H}_{i+1}$ respectively. Because γ

is contained in $\Sigma_{\Lambda'}$, s and t are contained in Λ'. If s and t would coincide, $\tilde{H}_i$ and $\tilde{H}_{i+1}$ would not be adjacent as no incident edges in a Cayley graph are labeled with the same generator. Thus, s and t are distinct. If $\tilde{H}_i$ and $\tilde{H}_{i+1}$ intersect, s and t commute. Then s and t are connected by an edge in Λ. Because Λ' is an induced subgraph of Λ, this edge would also be contained in Λ'. Then $\bar{H}_i$ and $\bar{H}_{i+1}$ intersect in $\Sigma_{\Lambda'}$ – a contradiction. It follows that $\tilde{H}_i$ and $\tilde{H}_{i+1}$ are disjoint. By construction, they are adjacent. It remains to show that at most k' hyperplanes of Σ_Λ intersect $\tilde{H}_i$ and $\tilde{H}_{i+1}$ simultaneously. Because $\tilde{H}_i$ and $\tilde{H}_i$ are adjacent, we can apply Lemma 5.22 and conclude that the number of hyperplanes intersecting H_i and H_i' simultaneously coincides with the number of hyperplanes in $\Sigma_{\mathrm{lk}(s)\cap\mathrm{lk}(t)}$. By assumption, s and t are not contained in an induced 4-cycle. By Lemma 5.21, $\mathrm{lk}(s) \cap \mathrm{lk}(t)$ is a complete graph. The Davis complex of $\mathrm{lk}(s) \cap \mathrm{lk}(t)$ is isometric to a k'-dimensional cube and contains k' hyperplanes. Thus, $(\tilde{H}_i)_{i\in\mathbb{N}}$ is a (r, k')-good hyperplane sequence for γ where k' coincides with the number of vertices in Λ.

Suppose on the other hand that Λ' contains two non-adjacent vertices s and t that are contained in an induced 4-cycle C_4. Then the subgraph Λ_* of Λ induced by s and t is empty. Let Σ_{Λ_*} be the canonically embedded Davis complex of Σ_{Λ_*} in Σ_Λ. This Davis complex Σ_{Λ_*} is isometric to $\mathbb{R}$, and the group generated by s and t is the infinite Dihedral group acting on Σ_{Λ_*}. Let α be a geodesic ray that is contained in Σ_{Λ_*} and starts at the vertex corresponding to the identity. Then α is isometrically embedded in Σ_{Λ_*} and Σ_{Λ_*} is isometrically embedded in the Davis complex of C_4. The Davis complex of a 4-cycle is isometric to a Euclidean plane. Thus, α is not contracting. □

5.2 Block decompositions of Davis complexes

This section is a preparation for the first main results of this chapter. We show in Proposition 5.28 that every Davis complex of a non-spherical right-angled Coxeter group has a nontrivial block decomposition with thin walls that can be obtained by a so-called *proper separation* of its defining graph. Block decompositions of CAT(0) spaces were introduced of Mooney in [Moo10] as *CAT(0) spaces with block structure*. We studied block decompositions of CAT(0) spaces in Chapter 3 and introduced Mooney's concept as *block decomposition with thin walls* in Definition 3.1. In Chapter 4, we examined contracting boundaries of amalgamated free products acting geometrically on spaces that have a block decomposition with thin or thick walls. Thus, Proposition 5.28 enables us to apply our results of Chapter 4 to right-angled Coxeter groups.

We use chapter 8 of David's Book [Dav08] about the geometry and topology of Coxeter groups as background for this section. We recall his statements in the setting of right-angled Coxeter groups. We remark that Davis considered Coxeter groups in general and proved analogous statements for their nerves. We use the notation established in Chapter 2. We summarized our notation concerning boundaries in Notation 1.1. Recall that we assume that every graph is simplicial. We introduce the further notation we need now.

Notation 5.24. The following terminology is motivated by the concept of separating sets in Graph theory. See [Wes01] and [Die17]. We say that a subgraph Λ_* of a graph Λ separates two subgraphs Λ_0 and Λ_1 of Λ if every path linking a vertex of Λ_0 with a vertex of Λ_1 contains a vertex of Λ_*. In this case, Λ_* is a *separating subgraph* of Λ. Suppose that Λ contains two induced subgraphs Λ_0 and Λ_1 such that $\Lambda = \Lambda_0 \cup \Lambda_1$. Then every path that connects a vertex of Λ_0 with a vertex of Λ_1 contains a vertex of $\Lambda_* = \Lambda_0 \cap \Lambda_1$. So, $\Lambda_* = \Lambda_0 \cap \Lambda_1$ is a separating subgraph of Λ. We observe that Λ_* is an induced subgraph of Λ. We say that an unordered pair $\{\Lambda_0, \Lambda_1\}$ of two induced subgraphs of a graph Λ is a *separation* of Λ if $\Lambda = \Lambda_0 \cup \Lambda_1$. The intersection $\Lambda_* = \Lambda_0 \cap \Lambda_1$ is the *separating subgraph* associated to the separation of Λ into the two subgraphs Λ_0 and Λ_1. A separation $\{\Lambda_0, \Lambda_1\}$ of Λ is called *proper* if both $V(\Lambda_0) \setminus V(\Lambda_1)$ and $V(\Lambda_1) \setminus V(\Lambda_0)$ are nonempty. It is possible that $\Lambda_0 \cap \Lambda_1$ is the trivial graph $(\emptyset, \emptyset)$. In this situation, Λ has two connected components Λ_0 and Λ_1, and $\{\Lambda_0, \Lambda_1\}$ is a proper separation with the separating subgraph $(\emptyset, \emptyset)$. We say that the deletion of a subgraph Λ' of Λ decomposes Λ into more than one connected component if $\Lambda \setminus \Lambda'$ is not connected. The trivial graph $(\emptyset, \emptyset)$ decomposes Λ into more than one connected component if Λ is not connected.

Lemma 5.25. *A graph Λ contains an induced subgraph whose deletion decomposes Λ into more than one connected component if and only if Λ has a proper separation into two induced subgraphs.*

Proof. Suppose that a graph Λ contains an induced subgraph Λ_*, whose deletion decomposes Λ into more than one connected component. Let $\mathcal{C}$ be the collection of these connected components. We decompose $\mathcal{C}$ into two collections of connected components $\mathcal{C}_0$ and $\mathcal{C}_1$ such that $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$ and $\mathcal{C}_0 \cap \mathcal{C}_1 = \emptyset$. Let Λ_0 be the subgraph of Λ induced by

$V(\Lambda_*)$ and all vertices that are contained in a connected component in $\mathcal{C}_0$. Analogously, let Λ_1 be the subgraph of Λ induced by $V(\Lambda_*)$ and all vertices that are contained in a connected component in $\mathcal{C}_1$. Then, Λ_0 and Λ_1 are two induced subgraphs such that $\Lambda_* = \Lambda_0 \cap \Lambda_1$ and $V(\Lambda) = V(\Lambda_1) \cup V(\Lambda_0)$. By the choice of Λ_0 and Λ_1, the deletion of Λ_* decomposes Λ into the graphs $\Lambda_0 \setminus \Lambda_*$ and $\Lambda_1 \setminus \Lambda_*$. Thus, Λ does not contain any edge connecting a vertex of Λ_0 with a vertex of Λ_1. So, $\Lambda = \Lambda_0 \cup \Lambda_1$ and $\{\Lambda_0, \Lambda_1\}$ is a proper separation of Λ with separating subgraph Λ_*.

Suppose that Λ contains two induced subgraphs Λ_0 and Λ_1 such that $\Lambda = \Lambda_0 \cup \Lambda_1$ and $\Lambda_* = \Lambda_0 \cap \Lambda_1$. First, we show that Λ_* is an induced subgraph of Λ. Let e be an edge in Λ. As $\Lambda = \Lambda_1 \cup \Lambda_0$, this edge is either contained in Λ_1 or in Λ_0. If e is not contained in Λ_0 and Λ_1 simultaneously, then e is not contained in Λ_*. Hence, an edge is contained in Λ_* if and only if it is an edge of Λ. Thus, Λ_* is an induced subgraph. It remains to show that Λ_* decomposes Λ into more than one connected component. By assumption, Λ_0 and Λ_1 don't coincide with Λ_*. As Λ_* is induced, both Λ_0 and Λ_1 contain a vertex that is not contained in Λ_*. Thus, both the graphs $\Lambda \setminus \Lambda_1$ and $\Lambda \setminus \Lambda_0$ contain at least one vertex. As $\Lambda = \Lambda_1 \cup \Lambda_0$, there is no edge that connects a vertex of $\Lambda \setminus \Lambda_1$ with a vertex of $\Lambda \setminus \Lambda_0$. Thus, every path connecting a vertex in $\Lambda \setminus \Lambda_1$ with a vertex in $\Lambda \setminus \Lambda_0$ contains a vertex of Λ_*. Hence, we obtain at least two connected components when we delete Λ_* from Λ. □

Lemma 5.26. *Every graph that is not complete has a proper separation.*

Proof. If Λ is not complete, it has at least one vertex v that is not connected to all other vertices of Λ. Let Λ_0 be the star of v, i.e., the graph induced by v and all the vertices that are adjacent to v. Let Λ_1 be the graph that is induced by all vertices except for v. Let $\Lambda_* = \Lambda_0 \cap \Lambda_1$. By definition, Λ_* is induced by all vertices that are adjacent to v. Furthermore, there is no edge that connects a vertex of $V(\Lambda_i) \setminus V(G_*)$ with a vertex of $V(\Lambda_j) \setminus V(G_*)$, i, $j \in \{0, 1\}$, $i \neq j$. The graph $\Lambda_0 \setminus \Lambda_*$ consists of v. By assumption, Λ contains a vertex w that is not adjacent to w. This vertex is contained in $\Lambda_1 \setminus \Lambda_*$. Thus, $\{\Lambda_0, \Lambda_1\}$ is a proper separation of Λ. □

Suppose that Λ is a graph that is not complete. By Lemma 5.26, Λ has a proper separation into two induced subgraphs Λ_0 and Λ_1 with separating subgraph $\Lambda_* = \Lambda_0 \cap \Lambda_1$. We observe like Davis in Proposition 8.8.1 of chapter 8 in [Dav08] that W_Λ can be written as an amalgamated free product $W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$. We remark that in the following statement, $\Lambda_* = \Lambda_0 \cap \Lambda_1$ is allowed to be the trivial graph $(\emptyset, \emptyset)$. Then, $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$ is the free product of W_{Λ_0} and W_{Λ_1}.

Lemma 5.27. *Suppose that a graph Λ has a proper separation $\{\Lambda_0, \Lambda_1\}$ with the separating subgraph $\Lambda_* = \Lambda_0 \cap \Lambda_1$. Then, $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$.*

Proof. Let S_0, S_1 and S_* be the vertex sets of Λ_0, Λ_1 and Λ_* respectively. By definition, each of W_{Λ_0}, W_{Λ_1} and W_{Λ_*} has a group presentation as in Equation (5.0.1). Let R_0 be the relations of the group presentation of W_{Λ_0} and R_1 the relations of the group presentation of W_{Λ_1}. Because Λ_* is an induced subgraph of Λ_0 and Λ_1, W_{Λ_*} is a special subgroup of W_{Λ_0} and W_{Λ_1} by Lemma 2.51. Let $\iota_j : W_{\Lambda_*} \hookrightarrow W_{\Lambda_j}$, $j \in \{0,1\}$ be the corresponding canonical embeddings of W_{Λ_*} in W_{Λ_j}, $j \in \{0,1\}$. By Lemma 2.53, $W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1} = \langle S_0 \cup S_1 | R_0 \cup R_1 \cup \{\iota_0(s)\iota_1(s)^{-1} \mid s \in S_*\}\rangle$. As W_{Λ_*} is a special subgroup of W_{Λ_0} and W_{Λ_1}, each relation $\iota_0(s)\iota_1(s)^{-1}$ is trivial and $W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1} = \langle S_0 \cup S_1 | R_0 \cup R_1\rangle$. By definition, this right-angled Coxeter group has $\Lambda = \Lambda_0 \cup \Lambda_1$ as defining graph. □

Suppose that $\{\Lambda_0, \Lambda_1\}$ is a proper separation of a graph Λ. We will use this separation to decompose the Davis complex of Λ into blocks that are isometric to the Davis complex of Λ_0 and Λ_1. For definitions of blocks and block decompositions of CAT(0) spaces, see Chapter 3.

In the following statement, $\Lambda_* = \Lambda_0 \cap \Lambda_1$ is allowed to be the trivial graph $(\emptyset, \emptyset)$. Recall that the Davis complex of the trivial graph is a vertex and that we identify this vertex with the identity vertex of Σ_Λ when we embed $\Sigma_{(\emptyset,\emptyset)}$ in Σ_Λ conically. Furthermore, if $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$, then $\mathcal{T}_{\text{ext}}$ denotes its extended Bass-Serre tree as defined in Definition 2.59.

Proposition 5.28. *Let $\{\Lambda_0, \Lambda_1\}$ be a proper separation of a graph Λ into two induced subgraphs Λ_0 and Λ_1 with separating subgraph Λ_*. Let Σ_{Λ_0}, Σ_{Λ_1} and Σ_{Λ_*} be the canonically embedded Davis complexes of Λ_0, Λ_1 and Λ_* in the Davis complex Σ_Λ of Λ. Then*

$$(\{g\Sigma_{\Lambda_0} \mid g \in W_\Lambda\} \cup \{g\Sigma_{\Lambda_1} \mid g \in W_\Lambda\},\ \{g\Sigma_{\Lambda_*} \mid g \in W_\Lambda\})$$

is a block decomposition with thin walls of Σ_Λ. All blocks of parity $(-)$ and $(+)$ are of the form $g\Sigma_{\Lambda_0}$ and $g\Sigma_{\Lambda_1}$, $g \in W_\Lambda$, respectively. The action of $W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$ on Σ_Λ with this block decomposition satisfies all properties of Convention 3.78.

Proof. By Lemma 5.27, $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$. We prove in step 1 that the conditions of Convention 3.74 are satisfied. Then Corollary 3.77 implies that W_Λ acts geometrically on a CAT(0) space $\mathcal{X}'$ with block decomposition with thin walls satisfying all properties of Convention 3.78. In step 2 we show that this space $\mathcal{X}'$ is isometric to the Davis complex of W_Λ.

Step 1: As a first step we prove that the conditions of Convention 3.74 are satisfied. The groups W_{Λ_0}, W_{Λ_1} and W_{Λ_*} act properly and cocompactly by isometries on the Davis complexes Σ_{Λ_0}, Σ_{Λ_1} and Σ_{Λ_*} respectively. Because Λ_* is an induced subgraph of Λ_0 and Λ_1, W_{Λ_*} is a special subgroup of W_{Λ_0} and W_{Λ_1} by Lemma 2.51. Let $j \in \{0,1\}$. Let $f_j : \Sigma_{\Lambda_*} \to \Sigma_{\Lambda_j}$ be the canonical isometric embeddings of Σ_{Λ_*} in Σ_{Λ_j}. Let $\iota_j : W_{\Lambda_*} \hookrightarrow W_{\Lambda_j}$ be the inclusions of W_{Λ_*} to W_{Λ_j}. Then f_j is ι_j-equivariant. We show that the stabilizer-condition is satisfied. Suppose that H is a special subgroup of W_Λ and $g \in W_\Lambda$ is not contained in H. Then gH is not contained in H by Theorem 2.46. The action of W_Λ on its Davis complex is induced by the action of W_Λ on itself by left multiplication. Thus, W_{Λ_*} is the stabilizer of Σ_{Λ_*} in W_{Λ_0} and W_{Λ_1} respectively. It remains to prove the ϵ-condition. Let $\epsilon \in (0, \frac{1}{4})$ and g_i and g_i' be two group elements in W_{Λ_i}. If the subsets $g_i\Sigma_{\Lambda_*}$ and $g_i'\Sigma_{\Lambda_*}$ intersect in Σ_{Λ_i}, then their ϵ-neighborhoods intersect as well. Assume on the other hand that the ϵ-neighborhoods of $g_i\Sigma_{\Lambda_*}$ and $g_i'\Sigma_{\Lambda_*}$ intersect in Σ_{Λ_i}. Suppose that there does not exist a hyperplane H that separates $g_i\Sigma_{\Lambda_*}$ and $g_i'\Sigma_{\Lambda_*}$, i.e., if we delete H, $g_i\Sigma_{\Lambda_*}$ and $g_i'\Sigma_{\Lambda_*}$ lie in a common connected component of the resulting space. It follows from Lemma 2.39, that the distance of the 1-skeletons of $g_i\Sigma_{\Lambda_*}$ and $g_i'\Sigma_{\Lambda_*}$ is zero. In particular, $g_i\Sigma_{\Lambda_*}$ and $g_i'\Sigma_{\Lambda_*}$ intersect. For achieving a contradiction, we assume that there is a hyperplane H separating $g_i\Sigma_{\Lambda_*}$ and $g_i'\Sigma_{\Lambda_*}$. By Lemma 2.39, the distance of $g_i\Sigma_{\Lambda_*}$ and $g_i'\Sigma_{\Lambda_*}$ in the 1-skeleton of Σ_Λ is at least 1. Then, the interior of the carrier of H lies between $g_i\Sigma_{\Lambda_*}$ and $g_i'\Sigma_{\Lambda_*}$ and the distance of $g_i\Sigma_{\Lambda_*}$ and $g_i'\Sigma_{\Lambda_*}$ is at least 1. Because $\epsilon < \frac{1}{4}$ the ϵ-neighborhoods of $g_i\Sigma_{\Lambda_*}$ and $g_i'\Sigma_{\Lambda_*}$ don't intersect–a contradiction. We conclude that all conditions of Convention 3.74 are satisfied. By Corollary 3.77, $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$ acts on the space $\mathcal{X}'(W_{\Lambda_0}, \Sigma_{\Lambda_0}, W_{\Lambda_1}, \Sigma_{\Lambda_1}, W_{\Lambda_*}, \Sigma_{\Lambda_*})$ as defined in Definition 3.75. By Lemma 3.76, this space has a block decomposition $(\mathcal{B}, \mathcal{A})$ with thin walls satisfying Convention 3.78.

Step 2: In this second step we prove that the space $\mathcal{X}'$ and Σ_Λ are isometric. Recall that $\mathcal{X}'$ can be constructed by means of the extended Bass-Serre tree $\mathcal{T}_{\text{ext}}$ of $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$ as follows. We choose a vertex v of $\mathcal{T}_{\text{ext}}$ whose label is a coset of W_{Λ_0} or W_{Λ_1}. Its preimage under the natural projection is a block $\Sigma_{\Lambda_i}^{(gW_i)}$ isometric to Σ_{Λ_i} with label gW_{Λ_i}, $g \in W_\Lambda$. Let v, v', v'' be an outgoing two-path of v. Then the preimage of v' is a wall $(\Sigma_{\Lambda_*})^{(g'W_*)}$ isometric to Σ_{Λ_*} with label $g'W_{\Lambda_*}$ such that $g'W_{\Lambda_i} = gW_{\Lambda_i}$. This wall is isometrically embedded in $\Sigma_{\Lambda_i}^{(gW_i)} = \Sigma_{\Lambda_i}^{(g'W_i)}$. The preimage of v'' under p is the block $\Sigma_{\Lambda_i}^{(g'W_j)}$, $i, j \in \{0, 1\}$, $j \neq i$. The wall $(\Sigma_{\Lambda_*})^{(g'W_*)}$ is isometrically embedded in $\Sigma_{\Lambda_i}^{(g'W_j)}$, $i, j \in \{0, 1\}$, $j \neq i$. We glue the block $\Sigma_{\Lambda_i}^{(g'W_j)}$ to the block $\Sigma_{\Lambda_i}^{(g'W_i)}$ along $(\Sigma_{\Lambda_*})^{(g'W_*)}$. We repeat the same procedure for all other outgoing 2-paths of v. We continue in this manner for vertices corresponding to blocks of increasing distance to v. We obtain the Davis complex of Λ through this construction. Indeed, the Davis complex of W_Λ has the Cayley graph of W_Λ as 1-skeleton. Hence, for every coset gW_{Λ_*}, Σ_Λ contains an isometrically embedded copy $g\Sigma_{\Lambda_0}$ of Σ_{Λ_0} and $g\Sigma_{\Lambda_0}$ of Σ_{Λ_1}. As $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$, the intersection of $g\Sigma_{\Lambda_0}$ and $g\Sigma_{\Lambda_1}$ in Σ_Λ is a copy of Σ_{Λ_*}. Thus, all blocks and thin walls of $(\mathcal{B}, \mathcal{A})$ are isometrically embedded in Σ_Λ. The action of W_Λ on Σ_Λ is induced by the action of W_Λ on itself and the action of W_Λ on the set of embedded blocks of the form Σ_{Λ_i} for $i \in \{0, 1\}$ is induced by the action of W_Λ on the left cosets of W_{Λ_i}. These actions are compatible with the action of $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$ on its Bass-Serre tree. Hence, two embedded blocks in Σ_Λ have non-empty intersection if and only if their corresponding vertices in the Bass-Serre tree share an edge. Thus, $\mathcal{X}'$ is isometrically embedded in Σ_Λ. On the other hand, $\mathcal{X}'$ consists of all blocks and walls in $(\mathcal{B}, \mathcal{A})$. Thus, Σ_Λ is isometrically embedded in $\mathcal{X}'$. Hence, Σ_Λ and $\mathcal{X}'$ are isometric. $\square$

Recall that a block decomposition of a CAT(0) space is trivial, if it has only one block.

Corollary 5.29. *Every Davis complex is finite or has a nontrivial block decomposition with thin walls.*

Proof. If Λ is not complete, Σ_Λ is not finite. By Lemma 5.26, Λ has a proper separation. By Lemma 5.27, W_Λ can be written as an amalgamated free product. By Proposition 5.28 the Davis complex of W_Λ has a nontrivial block decomposition with thin walls. □

The main idea of the proofs in this chapter is to decompose Davis complexes using Proposition 5.28 in different ways, to apply results of Chapter 4 and to use inductive arguments. For speaking of different decompositions of a Davis complex, we use the following terminology.

Definition 5.30. Suppose that a graph Λ has an induced subgraph Λ_* whose deletion decomposes the graph in at least two connected components. Every block decomposition of Σ_Λ as in Proposition 5.28 whose walls are of the form $g\Sigma_{\Lambda_*}$, $g \in W_\Lambda$, is a *block decomposition of* Σ_Λ *along* Λ_*. If $\{\Lambda_0, \Lambda_1\}$ is a separation of Λ, there is exactly one block decomposition of Σ_Λ along $\Lambda_* = \Lambda_0 \cap \Lambda_1$. This is *the* block decomposition associated to $\{\Lambda_0, \Lambda_1\}$.

Let Λ be a graph that is not complete. By Lemma 5.26, Λ has a proper separation into two induced subgraphs Λ_0 and Λ_1 with separating subgraph $\Lambda_* = \Lambda_0 \cap \Lambda_1$. By Lemma 5.27, $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$. Let $i \in \{0,1\}$. If Λ_i has a proper separation into two graphs such that one of them contains Λ_*, we can repeat this procedure for Λ_i and can write W_{Λ_i} as an amalgamated free product. We can continue in this matter as long as possible. This way W_Λ decomposes as a *tree of groups*. Indeed, let Λ be a graph with at least one vertex that is not complete. By Lemma 5.26, Λ has a proper separation into two induced subgraphs Λ_0 and Λ_1 with separating subgraph $\Lambda_* = \Lambda_0 \cap \Lambda_1$. By Lemma 5.27, $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$. Let $i \in \{0,1\}$. If Λ_i has a proper separation into two graphs H_0 and H_1 such that one of them, say H_0, contains Λ_*, we add to our tree an edge and label it with the groups whose defining graphs are H_0, H_1 and $H_0 \cap H_1$. We identify the vertex of H_0 with the vertex of Λ_i and remove the old label W_{Λ_i} from this vertex. We repeat this procedure as long as possible. This way we obtain a tree of groups $\mathbb{T}_{W_\Lambda}$ associated to W_Λ. Trees of groups are a powerful tool to prove statements for amalgamated free products with help of an induction. With help of such trees, one can decompose the associated Davis complex. This way, one obtains a *tree of spaces* and each space of this tree is a Davis complex.

We finish this section with a statement proven by Davis in Proposition 8.8.2 of [Dav08]. We say that a graph Λ is *splittable* if it contains a complete subgraph whose deletion decomposes Λ into more than one connected component. A graph is non-splittable if it is not splittable. Davis proves in Theorem 8.7.2 and Proposition 8.8.2 of [Dav08] that every graph Λ can be decomposed as a tree of subgraphs where the graphs associated to each edge is a complete graph (or empty) and for any vertex v the associated subgraph is a non-splittable graph. The corresponding tree of right-angled Coxeter groups can be used to calculate the ends of W_Λ. A *ray* in a topological space X is a map $r : [0,\infty) \to X$. A ray $r : [0,\infty) \to X$ is *proper* if and only if for any compact $C \subseteq X$, there is a positive integer N so that $r([N,\infty) \subseteq X - C$. Two proper rays $r_0 : [0,\infty) \to X$ and $r_1 : [0,\infty) \to X$ determine the same *end* if for any compact subset $C \subseteq X$, there is a positive inter N

so that $r_0([N, \infty))$ and $r_1([N, \infty))$ are contained in the same path component of $X \setminus C$. This is an equivalence relation on the set of proper rays. An end of W_Λ is an end of its Cayley graph. It is well-defined up to a canonical homeomorphism. Davis proves in Theorem 8.7.2 of [Dav08], that a right-angled Coxeter group is one-ended if and only if every induced subgraph is non-splittable. Thus, the decomposition of a graph into non-splittable subgraphs implies that every right-angled Coxeter group has an associated tree of groups where each vertex group is a 0- or 1-ended special subgroup and each edge group is a finite special subgroup. Davis considers not only right-angled Coxeter groups. We cite Proposition 8.8.2 in its general version concerning general Coxeter systems.

Lemma 5.31. *([Dav08, Prop. 8.8.2]) Any Coxeter system decomposes as a tree of groups, where each vertex group is a* 0*- or* 1*-ended special subgroup and each edge group is a finite special subgroup.*

This lemma shows that it is sufficient to study the boundaries of Davis complexes of graphs that are non-splittable. Indeed, boundaries corresponding to splittings over finite subgroups are well understood. We remark that we consider in Corollary 4.7 such situations.

5.3 Right-angled Coxeter groups satisfying the conjecture

We saw in the last section that a Davis complex of a non-complete graphs has a nontrivial block decomposition with thin walls. We use this observation and apply Theorem 4.10, one of our main results of Chapter 4. This leads to a variant of Theorem 4.10 for right-angled Coxeter groups, stated as Theorem 5.32. By means of Theorem 5.32, we define a graph class $\mathcal{J}$ of so-called *join-decomposable graphs* and prove in Corollary 5.38 that each graph in this class correspond to a right-angled Coxeter group whose contracting boundary is totally disconnected. In addition, $\mathcal{J}$ satisfies the Burst-Cycle-Conjecture [Tra19, Conjecture 1.14] (see Corollary 5.39).

We use the notation established in Chapter 2. We summarized our notation concerning boundaries in Notation 1.1. For concepts concerning proper separations of graphs, we use Notation 5.24. Recall that we assume that all graphs are simplicial. We define the Davis complex Σ_Λ of a graph Λ to be the Davis complex of the right-angled Coxeter group W_Λ that has Λ as defining graph. If Λ' is an induced subgraph of Λ, then $W_{\Lambda'}$ is a special subgroup of W_Λ and the Davis complex $\Sigma_{\Lambda'}$ can be isometrically embedded in Σ_Λ such that its 1-skeleton contains the identity vertex of Σ_Λ. Compare Lemma 2.51. In such a situation, we say that $\Sigma_{\Lambda'}$ is canonically embedded in Σ_Λ.

The following theorem is a generalization of the example of Charney and Sultan we discussed in Section 5.1. Recall that we refer to this example as the Cycle-Join-Example. Its defining graph Λ is pictured in Figure 5.7. The group W_Λ has totally disconnected contracting boundary.

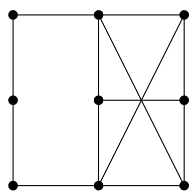

Figure 5.7 The defining graph of a right-angled Coxeter group studied by Charney and Sultan in Section 4.2 of [CS15]. We refer to this example as the Cycle-Join-Example.

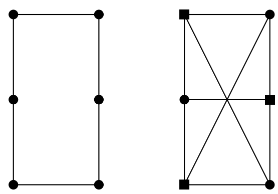

Figure 5.8 Decomposition of the graph in Figure 5.1 into two induced subgraphs Λ_0 (left) and Λ_1 (right).

Charney and Sultan use three observations for proving that the contracting boundary is totally disconnected. First, they decompose Λ into two graphs Λ_0 and Λ_1 as pictured in Figure 5.8 such that the contracting boundaries of W_{Λ_0} and W_{Λ_1} are known. Secondly, they use that the contracting boundary of Λ_1 is empty. Thirdly, they observe that the canonically embedded Davis complex of $\Lambda_* = \Lambda_0 \cap \Lambda_1$ in Σ_Λ does not contain any geodesic ray that is contracting in the ambient Davis complex of Λ. The next theorem is a generalization of this example. It says that we can calculate the contracting boundary of a right-angled Coxeter group if its defining graph Λ has a decomposition into two induced subgraphs Λ_0 and Λ_1 similar to the example of Charney and Sultan. Such a decomposition has the following form. It is a proper separation of Λ, i.e., it contains two induced subgraphs Λ_0 and Λ_1 such that Λ is the union of these two graphs and $\Lambda \setminus \Lambda_i$ is not empty, $i \in \{0,1\}$. Differently to the example of Charney and Sultan, we allow that both graphs Λ_0 and Λ_1 have nonempty contracting boundary. We only force that the canonically embedded Davis complex of $\Lambda_* = \Lambda_0 \cap \Lambda_1$ does not contain any contracting geodesic ray in the ambient Davis complex of Λ. We suppose in addition that the contracting boundaries of W_{Λ_0} and W_{Λ_1} are known. Under these assumptions, we calculate the contracting boundary of W_Λ. Recall that Nguyen and Tran considered a certain graph class $\mathcal{G}$ consisting of graphs whose corresponding right-angled Coxeter groups have totally disconnected contracting boundary. By means of the following theorem, we will define a larger graph class $\mathcal{J}$ with this property. The reason why $\mathcal{J}$ is larger than $\mathcal{G}$ is that we allow both groups W_{Λ_0} and W_{Λ_1} to have nonempty contracting boundary.

Let $\{\Lambda_0, \Lambda_1\}$ be a proper separation of a graph Λ and $i \in \{0,1\}$. Let Σ_{Λ_i} be the canonically embedded Davis complex of Λ_i in Σ_Λ. We use notation as in Notation 1.1. We think of boundaries of Σ_{Λ_i} as embedded in corresponding boundaries of Σ_Λ whenever possible. Note that this is not possible if we study contracting boundaries. Indeed, a geodesic ray γ in Σ_{Λ_i} might be contracting in Σ_{Λ_i} but not in the ambient Davis complex Σ_Λ. We say that $\gamma \subseteq \Sigma_{\Lambda_i}$ is Σ_Λ-contracting if it is contracting in the ambient Davis complex Σ_Λ and denote the set $\{\gamma(\infty) \in \partial_c \Sigma_\Lambda \mid \gamma \subseteq \Sigma_{\Lambda_i}\}$ by $\partial_{c,\Sigma_\Lambda} \Sigma_{\Lambda_i}$. If we equip $\partial_{c,\Sigma_\Lambda} \Sigma_{\Lambda_i}$ with the subspace topology of the visual- and contracting boundary of Σ_Λ, we obtain the topological spaces $\hat{\partial}_{c,\Sigma_\Lambda} \Sigma_{\Lambda_i}$ and $\vec{\partial}_{c,\Sigma_\Lambda} \Sigma_{\Lambda_i}$ respectively. By Lemma 2.35, $\hat{\partial}_{c,\Sigma_\Lambda} \Sigma_{\Lambda_i}$ and $\vec{\partial}_{c,\Sigma_\Lambda} \Sigma_{\Lambda_i}$ are homeomorphic to the set $\{\gamma(\infty) \in \partial_c \Sigma_{\Lambda_i} \mid \gamma \text{ is } \Sigma_\Lambda\text{-contracting}\}$ equipped with the subspace topology of the visual and contracting boundary of Σ_{Λ_i} respectively.

For proving the following theorem, we apply Theorem 4.10, one of our main results of Chapter 4. Recall that Theorem 4.10 is a generalization of the example of Chaney and Sultan to amalgamated free products of CAT(0) groups. Hence, the following theorem can be understood as a variant of Theorem 4.10 for right-angled Coxeter groups.

Let $\Lambda_* = \Lambda_0 \cap \Lambda_1$ be the separating subgraph of the separation $\{\Lambda_0, \Lambda_1\}$. By Theorem 5.6, W_{Λ_*} has empty contracting boundary if and only if Λ is a clique or a nontrivial join. We remark that the separating subgraph Λ_* of the proper separation $\{\Lambda_0, \Lambda_1\}$ might be the trivial graph $(\emptyset, \emptyset)$. Such a graph is a clique on 0 vertices. The Davis complex of $(\emptyset, \emptyset)$ consists of a vertex. If we embed this vertex in Σ_Λ canonically, we identify this vertex with the vertex corresponding to the identity vertex in Σ_Λ.

Theorem 5.32 (Variant of Theorem 4.10 for right-angled Coxeter groups)**.** *Let Λ be a graph with a proper separation $\{\Lambda_0, \Lambda_1\}$ with separating subgraph Λ_*. Suppose that Λ_* satisfies one of the following two conditions.*

a) Λ_* is contained in a clique.

b) Λ_* is contained in a nontrivial join of two induced subgraphs of Λ.

Let Σ_{Λ_0} and Σ_{Λ_1} be the canonically embedded Davis complexes of Λ_0 and Λ_1 in Σ_Λ. Then every connected component of $\vec{\partial}_c\Sigma_\Lambda$ $(\hat{\partial}_c\Sigma_\Lambda)$

a) consists of a single point or

b) is homeomorphic to a connected component of $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda_0}$ $(\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda_0})$ or

c) is homeomorphic to a connected component of $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda_1}$ $(\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda_1})$.

Proof. We write W_Λ as amalgamated free product $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$ by means of Lemma 5.27. By Proposition 5.28,

$$(\{g\Sigma_{\Lambda_0} \mid g \in W_\Lambda\} \cup \{g\Sigma_{\Lambda_1} \mid g \in W_\Lambda\},\ \{g\Sigma_{\Lambda_*} \mid g \in W_\Lambda\})$$

is a block decomposition with thin walls of Σ_Λ. All blocks of parity $(-)$ and $(+)$ are of the form $g\Sigma_{\Lambda_0}$ and $g\Sigma_{\Lambda_1}$, $g \in W_\Lambda$, respectively. Furthermore, the action of $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$ on Σ_Λ with this block decomposition satisfies all properties of Convention 3.78. By Corollary 4.15, Σ_Λ satisfies the property (QG) as defined in Definition 4.8. By assumption, every wall is contained in an isometrically embedded copy of a Davis complex of a clique or a nontrivial join, and such a complex has empty contracting boundary by Theorem 5.6. Thus, no wall of the block decomposition contains a geodesic ray that is contracting in the ambient Davis complex Σ_Λ. Hence, the claim follows from Theorem 4.10. □

Before we define a certain class of right-angled Coxeter groups with totally disconnected contracting boundaries by means of this theorem, we apply Theorem 5.32 to some simple examples.

Definition 5.33. We say that two vertices u and v build a *separating vertex pair* of a graph Λ if the deletion of the two vertices u and v decompose a connected component of Λ into more than one connected component.

We apply Theorem 5.32 to the situation where a separating vertex pair is contained in an induced 4-cycle or in a clique.

Example 5.34. Let Λ be a graph with a *separating vertex pair* $\{u, v\}$. Then Λ has a proper separation $\{\Lambda_0, \Lambda_1\}$ into two induced subgraphs Λ_0 and Λ_1 such that the separating subgraph Λ_* is induced by u and v. Let Σ_{Λ_*} be the canonically embedded Davis complex of Σ_{Λ_*} in Σ_Λ. If u and v are adjacent, Σ_{Λ_*} is isometric to a square. It does not contain any geodesic ray then. If u and v are not adjacent, Σ_{Λ_*} is isometric to $\mathbb{R}$. If u and v are contained in a join of two graphs that each contain two non-adjacent vertices, u and v are contained in an induced 4-cycle. The Davis complex of a 4-cycle is isometric to the Euclidean plane. So, Σ_{Λ_*} is contained in a Euclidean plane $E \subseteq \Sigma_\Lambda$ and no geodesic ray in Σ_{Λ_*} is contracting in the ambient Davis complex. We can apply Theorem 5.32 in both situations. Every connected component of $\vec{\partial}_c W_\Lambda$ consists of a single point or can be topologically embedded in a connected component of $\vec{\partial}_c W_{\Lambda_0}$ or $\vec{\partial}_c W_{\Lambda_1}$.

Definition 5.35. Let Λ be a connected graph that is the union of two induced subgraphs Λ_0 and Λ_1. Suppose that Λ_0 is a cycle of length at least 5 and that Λ_1 is a nontrivial join of two induced subgraphs of Λ. Then Λ is a *Charney-Sultan-graph*.

Suppose that Λ is a Charney-Sultan-graph. The argumentation of section 4.2 in [CS15] can be used to show that the contracting boundary of W_Λ is totally disconnected. We prove this fact again by applying Theorem 5.32. In addition, we apply Theorem 5.32 to right-angled Coxeter groups that have trees and empty graphs as defining graphs.

Example 5.36 (Graphs with totally disconnected contracting boundaries)**.**
Let Λ be a graph satisfying one of the following conditions.

a) Λ is an empty graph on at least two vertices.

b) Λ is a tree with at least two edges.

c) Λ is a Charney-Sultan-graph.

Then $\vec{\partial}_c \Sigma_\Lambda$ and $\hat{\partial}_c \Sigma_\Lambda$ are totally disconnected.

Proof. We prove the statements for the contracting boundary of $\vec{\partial}_c \Sigma_\Lambda$. The statement for $\hat{\partial}_c \Sigma_\Lambda$ can be proven analogously. First, let Λ be an empty graph. The Davis complex of a vertex is isometric to an interval of length one and its contracting boundary is empty. The Davis complex of an empty graph with two vertices is isometric to $\mathbb{R}$. Its contracting boundary consists of two points and is totally disconnected. We assume that the contracting boundary of every Davis complex of an empty graph with n vertices is totally disconnected. Let Λ be an empty graph on $n + 1$ vertices. The graph Λ is the union of two empty subgraphs Λ_0 and Λ_1 with less than n vertices such that $\Lambda_* = \Lambda_0 \cap \Lambda_1$ is a vertex. By induction hypothesis, the contracting boundary of $\vec{\partial}_c \Sigma_{\Lambda_i}$, $i \in \{0, 1\}$ is totally disconnected. By Theorem 5.32, $\vec{\partial}_c \Sigma_\Lambda$ is totally disconnected. Because of Lemma 5.10, the contracting boundary of Σ_Λ is not empty. Indeed, Λ contains at least two non-adjacent vertices that are not contained in an induced 4-cycle.
Next, let Λ be a tree. The Davis complex of an edge is isometric to a square and its contracting boundary is empty. The Davis complex of a path of length 2 is isometric to $[0, 1] \times \mathbb{R}$. Its contracting boundary consists of two points and is totally disconnected.

We assume that the contracting boundary of every Davis complex of a tree with $n \geq 2$ edges is totally disconnected. Let Λ be a tree with $n+1$ edges. The tree Λ is the union of two induced subtrees Λ_0 and Λ_1 with less than n edges such that $\Lambda_* = \Lambda_0 \cap \Lambda_1$ is a vertex. By induction hypothesis, the contracting boundary of $\vec{\partial}_c \Sigma_{\Lambda_i}$, $i \in \{0,1\}$ is totally disconnected. By Theorem 5.32, $\vec{\partial}_c \Sigma_\Lambda$ is totally disconnected. Because of Lemma 5.10, the contracting boundary of Σ_Λ is not empty. Indeed, Λ has at least two edges. Thus, it contains at least two non-adjacent vertices that are not contained in an induced 4-cycle. At last, let Λ be a Charney-Sultan-graph. We follow the proof of Charney and Sultan in [CS15, sec.4.2] to show that we are able to apply Theorem 5.32 to Λ. By definition, Λ is the union of two induced subgraphs Λ_0 and Λ_1 where Λ_0 is a cycle of length at least 5 and Λ_1 is the join of two graphs that each contain two vertices that are not adjacent in Λ. The intersection $\Lambda_* = \Lambda_0 \cap \Lambda_1$ of Λ_0 and Λ_1 is an induced subgraph of Λ_0 and Λ_1 and contains two non-adjacent vertices in Λ as well. Let Σ_{Λ_*}, Σ_{Λ_0} and Σ_{Λ_1} be the canonically embedded Davis complexes of Λ_*, Λ_0 and Λ_1 in Σ_Λ respectively. The Davis complex of Λ_1 is a direct product of two CAT(0) spaces of infinite diameter and hence empty. The Davis complex of a cycle of length at least 5 is quasi-isometric to the hyperbolic plane. Thus, the contracting boundary of Σ_{Λ_0} coincides with the visual boundary of Σ_{Λ_0} and is a 1-sphere S. We think of S as embedded in the visual boundary of Σ_Λ. The sphere S is contained in the contracting boundary of Σ_Λ not as a whole because many geodesic rays in Σ_{Λ_0} are not contracting in Σ_Λ. Indeed, every geodesic ray in Σ_{Λ_*} is contained in Σ_{Λ_1} and thus is not contracting in Σ_Λ. Let M be the intersection of $\{\gamma(\infty) \in \partial\Sigma_\Lambda \mid \gamma \subseteq \Sigma_{\Lambda_0}\}$ and $\bigcup_{g \in W_{\Lambda_0}} \{\gamma(\infty) \in \partial\Sigma_\Lambda \mid \gamma \subseteq g \cdot \Sigma_{\Lambda_*}\}$. Like Charney and Sultan, we observe that M is dense in the embedded sphere S. If we delete a dense set from S, we obtain a totally disconnected set. Thus, $\vec{\partial}_{c,\Sigma_\Lambda} \Sigma_{\Lambda_0}$ is totally disconnected. The claim follows from Theorem 5.32. □

We use our considerations to define a class of graphs that correspond to right-angled Coxeter groups with totally disconnected contracting boundaries. The idea is to define this graph class inductively. At first, we add cliques and nontrivial joins. The corresponding right-angled Coxeter groups have empty contracting boundaries by Theorem 5.6. Then we add the graphs we have considered in Example 5.36. The example above shows that the corresponding right-angled Coxeter groups have totally disconnected contracting boundaries. We apply Theorem 5.32 to such graphs and obtain new graphs whose right-angled Coxeter groups have totally disconnected contracting boundaries. We add these graphs to our graph class.

Definition 5.37. Let $\mathcal{J}$ be the smallest class of finite graphs such that

a) each finite graph without edges is contained in $\mathcal{J}$;

b) each finite tree is contained in $\mathcal{J}$;

c) each Charney-Sultan graph is contained in $\mathcal{J}$;

d) each clique is contained in $\mathcal{J}$;

e) each nontrivial join of two graphs is contained in $\mathcal{J}$;

f) the union of two graphs Λ_0, $\Lambda_1 \in \mathcal{J}$ is contained in $\mathcal{J}$ if $\Lambda_0 \cap \Lambda_1$ is an induced subgraph of $\Lambda_0 \cup \Lambda_1$ so that one of the following three conditions is satisfied:
   - $\Lambda_0 \cap \Lambda_1$ is empty.
   - $\Lambda_0 \cap \Lambda_1$ is contained in a clique of $\Lambda_0 \cup \Lambda_1$.
   - $\Lambda_0 \cap \Lambda_1$ is contained in a nontrivial join of two induced subgraphs of $\Lambda_0 \cup \Lambda_1$.

A graph in $\mathcal{J}$ is called *Join-decomposable.*

Corollary 5.38. *Let* Λ *be a join-decomposable graph. If* Λ *is a clique or a nontrivial join, the contracting boundary of* W_Λ *is empty. In the remaining case, the contracting boundary of* W_Λ *is nonempty and totally disconnected.*

Proof. Let Λ be join-decomposable. If Λ is as in Item a), Item b) or Item c) of Definition 5.37 and not a clique, then $\vec{\partial}_c W_\Lambda$ is totally disconnected by Example 5.36. If Λ is as in Item d) or Item e), then $\vec{\partial}_c W_\Lambda$ is empty by Theorem 5.6. Suppose that Λ is as in Item f). Then Λ has a proper separation into two induced subgraphs Λ_0 and Λ_1 with separating subgraph $\Lambda_* = \Lambda_0 \cap \Lambda_1$ and Λ_* is contained in a clique or in a nontrivial join of two induced subgraphs of Λ. By Theorem 5.32, every connected component of $\vec{\partial}_c W_\Lambda$ consists of a single point or can be topologically embedded in a connected component of $\vec{\partial}_c W_{\Lambda_0}$ or $\vec{\partial}_c W_{\Lambda_1}$. Let $i \in \{0, 1\}$. If Λ_i is not a clique, not a nontrivial join and not a graph as in Item a), Item b) or Item c) of Definition 5.37, Λ_i is as in in Item f). Then we repeat the same procedure for Λ_i. By definition of $\mathcal{J}$, Λ is finite. Furthermore, every graph-pair in Item f) is a proper separation of a graph. Each graph of such a graph pair contains less vertices than the union of both graphs. So, this procedure ends at some point. Hence, every connected component of $\vec{\partial}_c W_\Lambda$ consists of a single point or can be topologically embedded in a connected component of the contracting boundary of a right-angled Coxeter group whose defining graph is as in Item a), Item b) or Item c) or Item d) or Item e) of Definition 5.37. Every such connected component consists of a single point. It follows that the contracting boundary of W_Λ is totally disconnected. By Theorem 5.6, $\vec{\partial}_c W_\Lambda$ is empty if and only if Λ is a clique or a nontrivial join. Thus, in all remaining cases, W_Λ is nonempty and totally disconnected. $\square$

Recall that a cycle C in a graph is *burst*, if one of the following three conditions is satisfied:

- C has length 3 or 4,
- C has a diagonal, i.e., two non-consecutive vertices of C are connected by an edge,
- the vertex set of C contains a pair of non-adjacent vertices of an induced 4-cycle.

A cycle is intact if it is not burst. Recall the Burst-Cycle-Conjecture 5.5: Every cycle in the defining graph Λ of a right-angled Coxeter group W_Λ is burst if and only if the contracting boundary of W_Λ is totally disconnected. By construction, Λ does not contain any intact cycle. This can be proven inductively: If Λ is as in Item a), Item b), Item c), Item d), or Item e), it follows directly by definition. Otherwise we can write Λ as $\Lambda = \Lambda' \cup \Lambda''$ where $\{\Lambda', \Lambda''\}$ is a proper separation of Λ and $\Lambda' \cap \Lambda''$ is empty or contained in a clique or in a nontrivial join. Suppose that Λ' and Λ'' don't contain intact cycles. In order to obtain a contradiction we assume that Λ has an intact cycle C. Then C has at least one vertex in Λ'' and at least one vertex in Λ'. As $\{\Lambda', \Lambda''\}$ is a proper separation, two non-adjacent vertices u and v of C are contained in $\Lambda' \cap \Lambda''$. If $\Lambda' \cap \Lambda''$ is a clique, u and v are adjacent and C is not induced. Hence, u and v are contained in a nontrivial join. But then u and v are contained in an induced 4-cycle – a contradiction. Thus, Λ does not contain any intact cycle and Corollary 5.38 implies that the Burst-Cycle-Conjecture 5.5 is true for all right-angled Coxeter groups whose defining graphs are join-decomposable.

Corollary 5.39. *The Burst-Cycle-Conjecture 5.5 is true for every right-angled Coxeter group whose defining graph is join-decomposable.*

We list a few graphs that produce join-decomposable graphs if we glue them on a join-decomposable graph. For instance, one obtains a join-decomposable graph if we glue glued tetragons to paths.

Definition 5.40. Let P be a path of length at least 2 and C_4 a 4-cycle sharing with P either a path of length two or two vertices that are neither adjacent in P nor adjacent in C_4. Then $P \cup C_4$ is a *path with a glued tetragon.*

Example 5.41 (Allowed gluings)**.** Suppose that Λ is a join-decomposable graph. The graph $\Lambda \cup \Delta$ is join-decomposable if Δ is an induced subgraph of $\Lambda \cup \Delta$ and one of the following conditions is satisfied.

a) Δ is a tree and $\Lambda \cap \Delta$ is a vertex.

b) Δ is a clique or a join of two graphs that each contain two non-adjacent vertices.

c) Δ is a Charney-Sultan graph and $\Lambda \cap \Delta$ is contained in a clique or in a nontrivial join of two induced subgraphs of $\Lambda \cup \Delta$.

d) Δ is a 4-cycle and $\Lambda \cap \Delta$ is an edge, a 2-path or consists of two vertices that are not adjacent in $\Lambda \cup \Delta$.

e) Δ is a path P of length at least 2, $\Lambda \cap \Delta$ consists of the endvertices u and v of P and u and v are either adjacent or contained in an induced 4-cycle.

f) Δ is a path of length at least 2 with a glued tetragon such that $\Lambda \cap \Delta$ consists of the endvertices u and v of P.

Proof. The first four examples follow directly from the definition of the graph class $\mathcal{J}$. Let Δ be a path P of length at least 2 with a glued tetragon C_4 such that $\Lambda \cap \Delta$ consists of the endvertices u and v of P. Then we can decompose $\Lambda \cap \Delta$ in the following way. By definition, $P \cap C_4$ consists either of two vertices or of a 2-path P'. In the first case, let u' and v' be the two vertices that are contained in $P \cap C_4$. In the second case, let u' and v' be the endvertices of P'. Without loss of generality, u' is closer to u than v'. Let $P_{u,u'}$ be the subpaths of P connecting u and u'. Analogously, let $P_{v,v'}$ be the path connecting v and v'. Then $\Lambda \cup \Delta$ can be obtained in the following way. Let Λ' be the union of Λ and $P_{u,u'}$. Then $\Lambda \cap P_{u,u'}$ consists of a vertex. A vertex is a clique. Thus, $\Lambda \cup P_{u,u'}$ is contained in $\mathcal{J}$. Let Λ'' be the union of Λ' and $P_{v,v'}$. With the same argumentation as before, Λ'' is contained in $\mathcal{J}$. Let $\bar{\Lambda}$ be the union of C_4 and the subpath of P connecting u' and v'. By construction, Λ is the union of Λ'' and $\bar{\Lambda}$. The intersection of Λ'' and $\bar{\Lambda}$ is contained in C_4 and C_4 is a join of two non-adjacent vertices. Thus, Λ is contained in $\mathcal{J}$. □

Recall that Nguyen and Tran proved, that W_Λ has totally disconnected or empty contracting boundary if Λ is a connected, triangle-free, planar graph that has at least 5 vertices, no separating vertices or edges, and is $\mathcal{CFS}$. Recall further that we denoted the class of such graphs by $\mathcal{G}$. See Definition 5.14 and Corollary 5.16. By Proposition 3.11 in [NT19] and the Definition of $\mathcal{J}$, every such graph is join-decomposable.

Lemma 5.42. *If $\Lambda \in \mathcal{G}$, then Λ is join-decomposable.*

Proof. By Proposition 3.11, Λ is a tree of graphs whose vertices correspond to suspensions of exactly three points and whose edges correspond to a 4-cycle. Thus, Λ can be obtained by gluing suspensions along 4-cycles successively. By definition of $\mathcal{J}$, Λ is join-decomposable. □

Next, we study a right-angled Coxeter group that is not planar and hence not contained in the graph class $\mathcal{G}$. This example was examined by Russell, Spriano and Tran in Example 7.7 [RST18]. It's defining graph Λ is pictured in Figure 5.9. Russell, Spriano and Tran observe that W_Λ is $\mathcal{CFS}$ but not strongly $\mathcal{CFS}$, i.e., there does not exist a nontrivial subgraph Δ whose four-cycle graph Δ^4 is connected and whose support coincides with the vertex set $V(\Delta)$ of Δ such that Λ is the join of Δ and a (possibly trivial) clique. Indeed, the red-colored 4-cycle does not have two-non-adjacent vertices with another induced 4-cycle in common. Furthermore, the special subgroup induced by the red 4-cycle is a strongly quasi-convex, virtually $\mathbb{Z}^2$ subgroup. In addition, Russell,

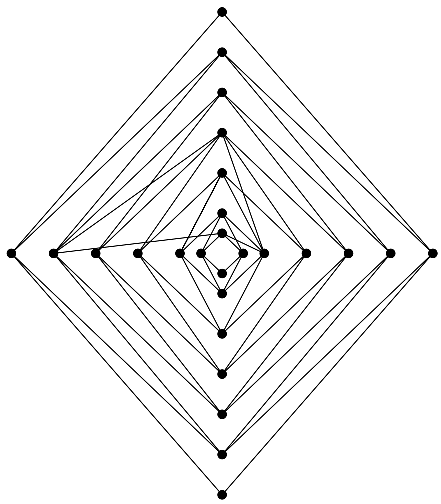

Figure 5.9 Defining graph of a right-angled Coxeter group studied in [RST18, Example 7.7]

Spriano and Tran observe that W_Λ is not quasi-isometric to a right-angled Artin group. The contracting boundary of W_Λ was unknown (See the tabular in Example 7.7 of [RST18]). The decomposition pictured in Figure 5.10 shows that Λ can be obtained by means of allowed gluings. The graph Λ is pictured in the left upper corner. We decompose Λ from left to right and above to bottom. In the first step (second graph in the first row), we decompose Λ into a green and a black subgraph. The intersection graph consists of two red-colored vertices. The red vertices are contained in a 4-cycle, namely the green colored one. We delete the green 4-cycle and obtain the third graph in the first row. We continue in this manner. In every second step, we decompose the graph into a green and a black graph. The intersection of these two graphs is always a graph consisting of single vertices, marked by the thick red vertices. These red vertices are either contained in an induced 4-cycle or in another nontrivial join or in a clique. In every second step, we delete the green subgraph and continue to decompose the obtained graph in the next step. Finally, we end up with a 4-cycle. By definition, a 4-cycle is join-decomposable. We conclude that Λ is join-decomposable. As Λ is neither a clique nor a nontrivial join, W_Λ has totally disconnected contracting boundary.

We consider an important gluing of a path on a join-decomposable graph that might not produce a join-decomposable graph.

Example 5.43 (Forbidden gluing)**.**
Suppose that a Graph Λ is obtained from a join-decomposable graph Λ' by gluing a path P of length at least two along its endvertices to two non-adjacent vertices u and v i.e., $\Lambda = \Lambda' \cup P$ and $\Lambda' \cap P$ is the empty graph on u and v. If u and v are not contained in an induced 4-cycle in Λ, Λ might not be join-decomposable. We consider such examples.

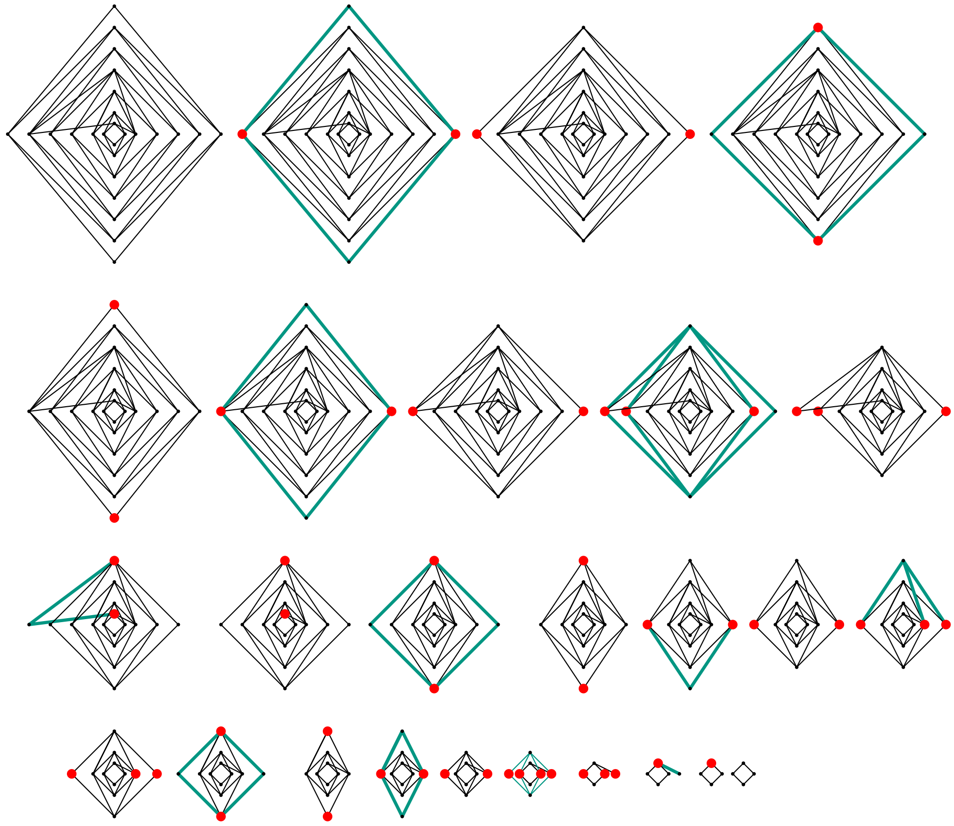

Figure 5.10 Decomposition of the graph in the left upper corner. This Decomposition shows that the graph in the left upper corner is join-decomposable.

- Consider the graph in Figure 5.11. A simple case-by-case analysis shows that $\Lambda = \Lambda' \cup P$ is not join-decomposable. The red edges build an intact cycle Λ. Accordingly, $\vec{\partial}_c W_\Lambda$ contains a sphere.
- Let Λ, Λ' and P as on the right or on the left side in Figure 5.12. We observe that Λ' is join-decomposable. Accordingly, Λ' does not contain any intact cycle and $\vec{\partial}_c W_{\Lambda'}$ is totally disconnected. By a simple case-by-case analysis, we observe that Λ does not contain any intact cycle and that Λ is not join-decomposable. In Section 5.5, we will study the contracting boundaries of the groups having the graphs in Figure 5.12 as defining graphs. See Section 5.5.1 and Section 5.5.2.

Let Λ' be join-decomposable. Suppose that u and v are two vertices that are not adjacent and not contained in an induced 4-cycle. Assume further that every path in Λ' connecting u and v has a glued tetragon. Then there is no path of length two that connects u and v. Let Λ be a graph obtained from Λ' by gluing the endvertices of a path P of length at least two to Λ. As u and v are not connected by a 2-path in Λ',

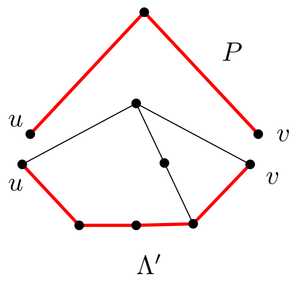

Figure 5.11 A graph that is not join-decomposable and contains an intact cycle (red).

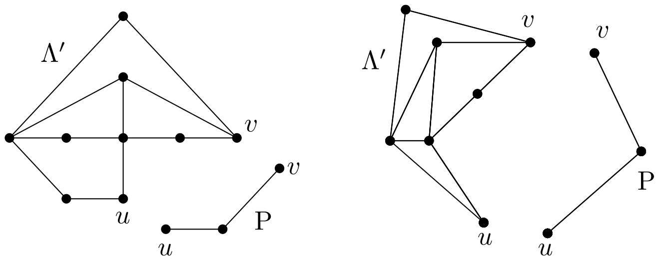

Figure 5.12 Left and Right: The union of the path and the pictured graph is a graph that is not join-decomposable and does not contain any intact cycle.

u and v are not contained in an induced 4-cycle in Λ. Since every path connecting u and v in Λ' has a glued tetragon, every cycle in Λ containing P is burst. Furthermore, every cycle in Λ' is burst because no join-decomposable graph contains an intact cycle. Thus, every cycle in Λ is burst. See Figure 5.12 for two such examples. One may get the idea to extend the graph class $\mathcal{J}$ by allowing gluings of the described form. One may hope that all right-angled Coxeter groups with defining graphs in this extended graph class satisfy the Burst-Cycle-Conjecture 5.5. However, the right-angled Coxeter groups whose defining graphs are as pictured in Figure 5.12 do not have totally disconnected contracting boundaries. Accordingly, Conjecture 5.5 is wrong in general. In Section 5.5 we will sketch proofs that both contracting boundaries contain a sphere. This is joint work with Graeber, Lazarovich and Stark. See Section 5.5.1 and Section 5.5.2. That the contracting boundaries in these examples contain a sphere, implies the following: It might happen that the contracting boundary of W_Λ has a sphere in its contracting boundary though both $\vec{\partial}_c W_P$ and $\vec{\partial}_c W_{\Lambda'}$ are totally disconnected. In this situation, gluing a path on a graph produces an example of totally disconnected topological spaces whose union contains a sphere. In the next chapter, we study this situation more precisely. We examine how contracting boundaries of right-angled Coxeter groups are influenced by gluing paths on their defining graphs.

5.4 Gluing paths on graphs

In this section, we study the question of how the contracting boundary of a right-angled Coxeter group changes if we glue a path of a length of at least two to two distinct vertices of the defining graph. If the endvertices of the path are not adjacent, such a group can be written as an amalgamated free product along a group that is quasi-isometric to $\mathbb{Z}$. In Section 4.4 we studied contracting boundaries of amalgamated free products of CAT(0) groups along groups quasi-isometric to $\mathbb{Z}$. Thus, this section can be seen as an application of Section 4.4. In addition, we use our considerations concerning boundary points of rank-one isometries in Theorem 4.24. The main result of this section is Theorem 5.58. It states a Dichotomy.

We use the notation established in Chapter 2. We summarized our notation concerning boundaries in Notation 1.1. For concepts concerning proper separations of graphs, we use Notation 5.24. Recall that we assume that all graphs in this chapter are simplicial. We define the Davis complex Σ_Λ of a graph Λ to be the Davis complex of the right-angled Coxeter group W_Λ that has Λ as defining graph. Recall that $\vec{\partial}_c\Sigma_\Lambda$ denotes the contracting boundary of Σ_Λ and that $\hat{\partial}_c\Sigma_\Lambda$ denotes the set of equivalence classes of contracting geodesic rays equipped with the subspace topology of the visual boundary $\hat{\partial}\Sigma_\Lambda$ of Σ_Λ. If Λ' is an induced subgraph of Λ, then $W_{\Lambda'}$ is a special subgroup of W_Λ and the Davis complex $\Sigma_{\Lambda'}$ can be isometrically embedded in Σ_Λ such that its 1-skeleton contains the identity vertex of Σ_Λ. Compare Lemma 2.51. In such a situation, we say that $\Sigma_{\Lambda'}$ is canonically embedded in Σ_Λ. We think of boundaries of $\Sigma_{\Lambda'}$ as embedded in corresponding boundaries of Σ_Λ whenever possible. Note that this is not possible if we study contracting boundaries. Indeed, a geodesic ray γ in $\Sigma_{\Lambda'}$ might be contracting in $\Sigma_{\Lambda'}$ but not in the ambient Davis complex Σ_Λ. We say that $\gamma \subseteq \Sigma_{\Lambda'}$ is Σ_Λ-contracting if it is contracting in the ambient Davis complex Σ_Λ and denote by $\partial_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ the set $\{\gamma(\infty) \in \partial_c\Sigma_\Lambda \mid \gamma \subseteq \Sigma_{\Lambda'}\}$. If we equip $\partial_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ with the subspace topology of the visual- and contracting boundary of Σ_Λ, we obtain the topological spaces $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ and $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ respectively. By Lemma 2.35, $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ and $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ are homeomorphic to the set $\{\gamma(\infty) \in \partial_c\Sigma_{\Lambda'} \mid \gamma \text{ is } \Sigma_\Lambda\text{-contracting}\}$ equipped with the subspace topology of the visual and contracting boundary of $\Sigma_{\Lambda'}$ respectively.

Our goal is to study the contracting boundary of a right-angled Coxeter group whose defining graph is obtained by gluing on a path of a length of at least two to a given graph. Recall that two or more paths in a graph are *independent* if none of them contains an inner vertex of another. Motivated by this, we define when a path is *independent* in a graph.

Definition 5.44. A path P in a graph Λ is *independent* in Λ if Λ has a vertex that is not contained in P, the path P has a length of at least 2 and every inner vertex of P has degree two.

Remark 5.45. The definition of independent paths implies that the endvertices of an independent path are distinct. As a reminder, we will sometimes repeat this assumption.

Recall that we say that we delete a vertex v of a graph Λ if we delete v from the vertex set $V(\Lambda)$ and all edges incident to v from the edge set $E(\Lambda)$. Suppose that a graph Λ has an independent path P. If we delete all inner vertices of P, we obtain an induced subgraph $\bar{P}$. We observe that Λ is obtained from $\bar{P}$ by gluing the endvertices of P to $\bar{P}$.

We repeat the definition of proper separations of graphs. Suppose that a graph Λ has two induced subgraphs Λ_0 and Λ_1 such that $\Lambda = \Lambda_0 \cup \Lambda_1$. Then the unordered pair $\{\Lambda_0, \Lambda_1\}$ is a *separation* of Λ into the two subgraphs Λ_0 and Λ_1. If both $V(\Lambda_0) \setminus V(\Lambda_1)$ and $V(\Lambda_1) \setminus V(\Lambda_0)$ are not empty, the separation is called *proper*. The graph $\Lambda_* = \Lambda_0 \cap \Lambda_1$ is called *separating subgraph*. Recall that two vertices s and t build a *separating vertex pair* of a graph Λ if the deletion of s and t decomposes a connected component of Λ into more than one connected components. In such a situation, there are two induced subgraphs Λ_0 and Λ_1 of Λ such that $\Lambda = \Lambda_0 \cup \Lambda_1$ and $\Lambda_* = \Lambda_0 \cap \Lambda_1$ is the graph induced by s and t. In other words, $\{\Lambda_0, \Lambda_1\}$ is a proper separation of Λ into the two induced subgraphs Λ_0 and Λ_1.

Lemma 5.46. *Let P be a path in a graph Λ. The following statements are equivalent.*

a) Λ has a vertex that is not contained in P, P has a length of at least two and every inner vertex of P has degree two.

b) Λ has a vertex that is not contained in P and P has a length of at least two. In addition, if P' is a path in Λ that is not contained in P, then P' does not contain an inner vertex of P.

c) Λ has a proper separation into the subgraph induced by P and the subgraph obtained from Λ by deleting the inner vertices of P.

Proof. If P has a length of at least two and every inner vertex of P has degree two, none of its vertices are an inner vertex of another path in Λ that is not contained in P. On the other hand, if Λ would contain a path P' with an inner vertex lying in P then an inner vertex of P would have degree three. Let Λ_0 be the graph induced by the path P and Λ_1 be the graph induced by all vertices of Λ that are no inner vertices of P. Then $\Lambda = \Lambda_0 \cup \Lambda_1$ and $\Lambda_0 \cap \Lambda_1$ is the graph induced by the endvertices of Λ. Then $\{\Lambda_0, \Lambda_1\}$ is a proper separation of Λ and the endvertices of P build a separating vertex pair. On the other hand, suppose that Λ has a proper separation into P and the graph obtained from Λ by deleting all inner vertices of P. Then P has a length of at least two because otherwise, the separation is not proper. Furthermore, P is independent of all paths in Λ that are not contained in P. □

Let P be an independent path in a graph Λ with endvertices s and t. Let $\bar{P}$ be the graph obtained from Λ by deleting all inner vertices of P. In other words, Λ is obtained from $\bar{P}$ by gluing P on $\bar{P}$. If s and t are adjacent or contained in an induced 4-cycle, we know already how the contracting boundary of $\bar{P}$ changes when we glue P on $\bar{P}$. Indeed, the following lemma follows from Theorem 5.32 and our considerations in Section 5.3. Suppose that Λ' is an induced subgraph of Λ and that the Davis complex of Λ' is canonically embedded in the Davis complex of Λ. Recall that $\partial_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ denotes the set $\{\gamma(\infty) \in \partial_c\Sigma_\Lambda \mid \gamma \subseteq \Sigma_{\Lambda'}\}$. The corresponding topological subspaces of $\hat{\partial}_c\Sigma_\Lambda$ and $\vec{\partial}_c\Sigma_\Lambda$ are denoted by $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ and $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ respectively.

Lemma 5.47. *Let Λ be a graph with an independent path P with endvertices s and t. Suppose that*

- *s and t are adjacent or*
- *s and t are contained in an induced 4-cycle.*

Let Λ_P be the subgraph of Λ induced by P and $\bar{P}$ be the graph obtained from Λ by deleting all inner vertices of P. Then every connected component of $\hat{\partial}_c\Sigma_\Lambda$ ($\vec{\partial}_c\Sigma_\Lambda$)

a) consists of a single point or

b) is homeomorphic to a connected component of $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda_P}$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda_P}$) or

c) is homeomorphic to a connected component of $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$).

If s and t coincide or are adjacent, the contracting boundary of Λ_P is empty or a 1-sphere. Otherwise, the contracting boundary of Λ_P is totally disconnected.

Proof. By assumption, the two vertices s and t are either contained in an edge or they are contained in an induced 4-cycle. Hence, we can apply Theorem 5.32. By Theorem 5.32, every connected component of the contracting boundary of Σ_Λ consists either of a single point or is homeomorphic to a connected component of $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda_P}$ or to a connected component of $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$. If s and t are not adjacent, then the graph Λ_P is a path. By Example 5.36, the contracting boundary of Σ_{Λ_P} is totally disconnected. Otherwise, Λ_P is a cycle. If this cycle hat length three or four, the contracting boundary of Σ_{Λ_P} is empty. Otherwise, Σ_{Λ_P} is quasi-isometric to the hyperbolic plane and the contracting boundary of Σ_{Λ_P} is homeomorphic to a 1-sphere. The claim concerning $\hat{\partial}_c\Sigma_\Lambda$ follows analogously. ☐

Remark 5.48. Recall that the endvertices of an independent path are distinct by definition. We can transfer the argumentation of Lemma 5.47 to the situation where the endvertices agree. In this case, one glues a cycle on a graph such that the intersection of the graph and the cycle is a single vertex.

Our goal is to understand the case where the endvertices of P are not adjacent and not contained in an induced 4-cycle. For that purpose, we study the structure of an independent path P in a graph Λ. Let $P = v_0, \ldots, v_k$ be a path. Let i, j, $l \in \{0, \ldots, k\}$. We say that vertex v_j *lies between* v_i and v_l if $i \leq j \leq l$. Let v_i and v_j be two non-adjacent vertices in P. Then the vertices between v_i and v_j induce an independent path P' of Λ. In particular, v_i and v_j build a separating vertex pair of Λ.

Lemma 5.49. *Suppose that P is an independent path in a graph Λ. Every two non-adjacent vertices of P are the endvertices of an independent path of Λ and build a separating vertex pair of Λ.*

Proof. Let $\{s, t\}$ be a pair of vertices of P that are not adjacent in Λ. If s and t are the endvertices of P, they build a separating pair by Lemma 5.46. Otherwise, s, t and the vertices between s and t induce a subpath of P. This subpath is independent in Λ. By Lemma 5.46, the endvertices s and t of this path build a separating vertex pair. □

It is an important observation, that every pair of two non-adjacent vertices of P are endvertices of an independent path. Therefore, every pair of vertices of P is associated to some data that helps to understand the structure of the contracting boundary of the associated right-angled Coxeter group. In the following, we introduce these data and corresponding notation. Let Λ be a graph with an independent path P. Let Σ_Λ be its Davis complex. Let $\{s, t\}$ be a pair of non-adjacent vertices of P. Let $W_{s,t}$ be the special subgroup of W_Λ generated by s and t. The group $W_{s,t}$ is isomorphic to the infinite Dihedral group, and its canonically embedded Davis complex in Σ_Λ is a bi-infinite geodesic ray $\alpha_{s,t}$ in Σ_Λ intersecting the identity vertex id of W_Λ. It is contained in the 1-skeleton of Σ_Λ. It is a geodesic ray in the metric of the 1-skeleton as well as in the metric of the ambient complex Σ_Λ. In other words, $\alpha^1_{s,t} = \alpha_{s,t}$. The edges of $\alpha^1_{s,t}$ build a path in the 1-skeleton of Σ_Λ, i.e., in the Cayley graph of W_Λ. These edges are labeled with the letters s and t and they alternate. More precisely, the word associated to $\alpha^1_{s,t}$ is the bi-infinite word $\dots ststst \dots$ i.e., the bi-infinite word whose letters alternate between s and t. The group elements st and ts act by translations on $\alpha_{s,t}$. The corresponding translation length is two. So, the group elements st and $(st)^{-1} = ts$ are axial isometries and $\alpha_{s,t}$ is an axis for them. We denote the corresponding oriented axes by $\alpha^+_{s,t}$ and $\alpha^-_{s,t}$, i.e., $\alpha^+_{s,t}$ and $\alpha^-_{s,t}$ are two bi-infinite geodesic rays $\alpha^+_{s,t} : \mathbb{R} \to \Sigma_\Lambda$ and $\alpha^-_{s,t} : \mathbb{R} \to \Sigma_\Lambda$ such that $\alpha^+_{s,t}(x + 2) = st\alpha^+_{s,t}(x)$ and $\alpha^-_{s,t}(x + 2) = ts\alpha^-_{s,t}(x)$ for all $x \in \mathbb{R}$. We parameterize $\alpha^+_{s,t}$ and $\alpha^-_{s,t}$ so that $\alpha^+_{s,t}(0)$ and $\alpha^-_{s,t}(0)$ is the identity vertex id in the one-skeleton of Σ_Λ. Let $\alpha^+_{s,t}(\infty)$ and $\alpha^-_{s,t}(\infty)$ be the equivalence classes of $\alpha^+_{s,t}|_{[0,\infty)}$ and $\alpha^-_{s,t}|_{[0,\infty)}$ respectively. The axis $\alpha_{s,t}$ is invariant under $W_{s,t}$. It is $s\alpha^+_{s,t}(\infty) = \alpha^-_{s,t}(\infty)$ and $t\alpha^+_{s,t}(\infty) = \alpha^-_{s,t}(\infty)$. We denote the connected component of $\alpha^+_{s,t}(\infty)$ in $\hat{\partial}_c\Sigma_\Lambda$ and $\vec{\partial}_c\Sigma_\Lambda$ by $\hat{\kappa}(\alpha^+_{s,t}(\infty))$ and $\vec{\kappa}(\alpha^+_{s,t}(\infty))$ respectively. Analogously, we denote the connected component of $\alpha^-_{s,t}(\infty)$ in $\hat{\partial}_c\Sigma_\Lambda$ and $\vec{\partial}_c\Sigma_\Lambda$ by $\hat{\kappa}(\alpha^-_{s,t}(\infty))$ and $\vec{\kappa}(\alpha^-_{s,t}(\infty))$ respectively. Let $P_{s,t}$ be the independent path of Λ induced by all vertices of P that lie between s and t. Let $\bar{P}_{s,t}$ be the graph we obtain by deleting all inner vertices of $P_{s,t}$ from Λ. Let $\Lambda_{s,t}$ be the subgraph of Λ induced by s and t. As s and t are not adjacent, the graph $\Lambda_{s,t}$ is an empty graph consisting of s and t. It is $\Lambda_{s,t} = \bar{P}_{s,t} \cap P_{s,t}$ and its Davis complex is $\alpha_{s,t}$. Since s and t are not adjacent, $P_{s,t}$ is an induced subgraph of Λ. By Lemma 5.46, $\{\bar{P}_{s,t}, P_{s,t}\}$ is a proper separation of Λ. We write W_Λ as amalgamated free product $W_\Lambda = W_{\bar{P}_{s,t}} *_{W_{\Lambda_{s,t}}} W_{P_{s,t}}$ by means of Lemma 5.27. By Proposition 5.28,

$$(\mathcal{B}_{s,t}, \mathcal{A}_{s,t}) \coloneqq (\{g\Sigma_{\bar{P}_{s,t}} \mid g \in W_\Lambda\} \cup \{g\Sigma_{P_{s,t}} \mid g \in W_\Lambda\},\ \{g\alpha_{s,t} \mid g \in W_\Lambda\}) \tag{5.49.1}$$

is a block decomposition with thin walls of Σ_Λ. All blocks of parity $(-)$ and $(+)$ are of the form $g\Sigma_{\bar{P}_{st}}$ and $g\Sigma_{P_{st}}$, $g \in W_\Lambda$, respectively. Every wall is of the form $g\alpha_{s,t}$, $g \in W_\Lambda$. The block decomposition satisfies all properties of Convention 3.78. In particular, the tree associated to $(\mathcal{B}_{s,t}, \mathcal{A}_{s,t})$ is the extended Bass-Serre tree $\mathcal{T}_{\{s,t\}}$ of $W_\Lambda = W_{\bar{P}_{s,t}} *_{W_{\Lambda_{s,t}}} W_{P_{s,t}}$.

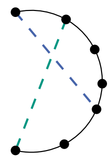

Figure 5.13 Each of the dashed edges marks a non-adjacent vertex pair. As the dashed edges cross, the corresponding non-adjacent vertex pairs cross.

Recall that the itinerary of a geodesic ray is a path in the extended Bass-Serre tree that describes how the ray runs through the blocks and the walls of the space. The axis $\alpha_{s,t}$ is contained in a wall of $(\mathcal{B}_{s,t}, \mathcal{A}_{s,t})$. Accordingly, the itinerary of $\alpha^{+}_{s,t}|_{[0,\infty)}(\infty)$ and $\alpha^{-}_{s,t}|_{[0,\infty)}(\infty)$ in the extended Bass-Serre tree $\mathcal{T}_{\{s,t\}}$ of $W_\Lambda = W_{\bar{P}_{s,t}} *_{W_{\Lambda_{s,t}}} W_{P_{s,t}}$ is trivial. We observe that $W_{s,t}$ is quasi-isometric to $\mathbb{Z}$. Thus, we can apply our results of Section 4.4, in which we studied contracting boundaries of amalgamated free products of CAT(0) groups along groups which are quasi-isometric to $\mathbb{Z}$. Let us assume that the endvertices of P are not contained in an induced 4-cycle. As P is independent, no pair of non-adjacent vertices in P is contained in an induced 4-cycle. Then $\alpha_{s,t}$ is contracting because of Lemma 5.10. Accordingly, st and ts are rank-one isometries and we can apply our considerations of Section 4.3. For that purpose, it helps to study *crossing vertex pairs* of P.

Definition 5.50. Let $\{u, v\}$ and $\{u', v'\}$ be two vertex pairs of non-adjacent vertices in a path P. We say that $\{u, v\}$ *crosses* $\{u', v'\}$ if u, v, u' and v' are pairwise distinct and exactly one of the two vertices u and v lies between u' and v'.

By definition, a pair of vertices $\{u, v\}$ crosses $\{u', v'\}$ if and only if $\{u', v'\}$ crosses $\{u, v\}$. In Figure 5.13, a crossing vertex pair is pictured. The two dashed edges connect two vertices that build a non-adjacent vertex pair. As the dashed edges cross, the corresponding vertex pairs cross.

As before, let Λ be a graph with an independent pat P and s and t two non-adjacent vertices on P. Recall that the itinerary of a geodesic ray in a CAT(0) space with block decomposition $(\mathcal{B}, \mathcal{A})$ with associated tree $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ is a (possibly infinite) path in $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ that describes how the geodesic ray runs through the walls and the blocks of the space. See Definition 3.18 in Chapter 3. Let $(\mathcal{B}_{s,t}, \mathcal{A}_{s,t})$ be a block decomposition of Σ_Λ as in Equation (5.49.1). Recall that then $\mathcal{T}_{\mathcal{B},\mathcal{A}}$ coincides with the Bass-Serre tree $\mathcal{T}_{\{s,t\}}$ associated to $W_\Lambda = W_{\bar{P}} *_{W_{\Lambda_{s,t}}} W_P$. Let γ be an axis for an axial isometry in Σ_Λ. Recall that the itinerary of γ is the union of the itineraries $I(\gamma^{+}_{\geq 0})$ and $I(\gamma^{-}_{\geq 0})$ of $\gamma^{+}_{\geq 0}$ and $\gamma^{-}_{\geq 0}$ respectively. By Lemma 4.29, $I(\gamma)$ is either a bi-infinite path in $\mathcal{T}_{\text{ext}}$ or trivial. By Definition 4.30, we call γ *essential* in $(\mathcal{B}_{s,t}, \mathcal{A}_{s,t})$, if γ is an axis for a rank-one isometry such that $I(\gamma)$ is a bi-infinite path in $\mathcal{T}_{\{s,t\}}$. In Section 4.3 and Section 4.4, we have seen that essential rays have special properties. These special properties are crucial for the proof of the main theorem of this section. The following lemmas implies the existence of essential geodesic rays if the endvertices of P are not contained in an induced 4-cycle.

Recall that a *geodesic* is an isometric embedding of a possibly infinite interval into a metric space and that a subgeodesic ray γ' of a geodesic γ is a geodesic whose image is isometric to $[0, \infty)$.

Lemma 5.51. *Let Λ be a graph with an independent path P and $\{s, t\}$ and $\{s', t'\}$ two vertex pairs of P that cross each other. If we delete $\alpha_{s,t}$ from Σ_Λ, Σ_Λ decomposes into two subcomplexes that each contain a subgeodesic ray of $\alpha_{s',t'}$.*

Proof. By Lemma 5.49, $\{s, t\}$ and $\{s', t'\}$ are separating vertex pairs. Hence, s, t, s' and t' are pairwise distinct and $\alpha_{s,t}$ and $\alpha_{s',t'}$ intersect only in the identity vertex in Σ^1_Λ. The Davis complex Σ_Λ has two block decompositions $(\mathcal{B}_{s,t}, \mathcal{A}_{s,t})$ and $(\mathcal{B}_{s',t'}, \mathcal{A}_{s',t'})$ as defined in Equation (5.49.1). As $\{s, t\}$ and $\{s', t'\}$ cross, exactly one of the two vertices s and t lies between s' and t'. Thus, the canonically embedded Davis complex e'_s of s' and the canonically embedded Davis complex e'_t of t' in Σ_Λ are in blocks of different type in $(\mathcal{B}_{s,t}, \mathcal{A}_{s,t})$. The complex $e_{s'}$ is the edge in the one-skeleton of Σ_Λ that is labeled with s'. The complex $e_{t'}$ is the edge in the one-skeleton of Σ_Λ that is labeled with t'. The one-skeleton of $\alpha_{s',t'}$ is associated to the bi-infinite word $\ldots s't's't' \ldots$. Accordingly, $\alpha^1_{s',t'}$ contains $e_{s'}$ and $e_{t'}$. The two blocks of different parity that contain $e_{s'}$ and $e_{t'}$ share the wall $\alpha_{s,t}$. So, if we delete $\alpha_{s,t}$, Σ_Λ decomposes into two subcomplexes such that one of them contains $e_{s'}$ and the other contains $e_{t'}$. As $\alpha_{s,t}$ and $\alpha_{s',t'}$ intersect only in the identity vertex of Σ^1_Λ, each component contains a subgeodesic ray of $\alpha_{s',t'}$. $\square$

Lemma 5.52. *Let Λ be a graph with an independent path P. Let $\{s, t\}$ and $\{s', t'\}$ be two vertex pairs of P that cross each other. Let $(\mathcal{B}_{s,t}, \mathcal{A}_{s,t})$ and $(\mathcal{B}_{s',t'}, \mathcal{A}_{s',t'})$ be the associated block decompositions as in Equation* (5.49.1)*. The itinerary of the axis $\alpha_{s,t}$ in the Bass-Serre tree $\mathcal{T}_{\{s',t'\}}$ associated to $(\mathcal{B}_{s',t'}, \mathcal{A}_{s',t'})$ is a bi-infinite path. Analogously, the itinerary of the axis $\alpha_{s',t'}$ in the Bass-Serre tree $\mathcal{T}_{\{s,t\}}$ associated to $(\mathcal{B}_{s,t}, \mathcal{A}_{s,t})$ is a bi-infinite path.*

Proof. By Lemma 5.51, the deletion of the wall $\alpha_{s',t'}$ in $(\mathcal{B}_{s',t'}, \mathcal{A}_{s',t'})$ decomposes Σ_Λ into two subcomplexes that each contain a subgeodesic ray of $\alpha_{s,t}$. Thus, $\alpha_{s,t}$ is not contained in a wall or a block of $(\mathcal{B}_{s',t'}, \mathcal{A}_{s',t'})$. Then its itinerary in $\mathcal{T}_{\{s',t'\}}$ is infinite by Lemma 4.29. Analogously, the itinerary of $\alpha_{s',t'}$ in $\mathcal{T}_{\{s,t\}}$ is infinite. $\square$

Corollary 5.53. *Let Λ be a graph with an independent path P. Suppose that the endvertices of P are not contained in an induced* 4*-cycle. Let $\{s, t\}$ and $\{s', t'\}$ be two vertex pairs of P that cross each other. Then $\alpha_{s,t}$ is essential in $(\mathcal{B}_{s',t'}, \mathcal{A}_{s',t'})$ and $\alpha_{s',t'}$ is essential in $(\mathcal{B}_{s,t}, \mathcal{A}_{s,t})$.*

Proof. As the endvertices of P are not contained in an induced 4-cycle, $\alpha_{s,t}$ and $\alpha_{s',t'}$ are axes for rank-one isometries by Lemma 5.10 and the claim follows directly from Lemma 5.52. $\square$

In the situation of Corollary 5.53, it is useful to know how the itineraries of $\alpha_{s,t}$ and $\alpha_{s',t'}$ look like.

Lemma 5.54. *Let Λ be a graph with an independent path P. Let $\{s,t\}$ and $\{s',t'\}$ be two vertex pairs of P that cross each other. Let $(\mathcal{B}_{s,t}, \mathcal{A}_{s,t})$ and $(\mathcal{B}_{s',t'}, \mathcal{A}_{s',t'})$ be the associated block decompositions as in Equation* (5.49.1)*. Let $\mathcal{T}_{\{s,t\}}$ and $\mathcal{T}_{\{s',t'\}}$ be the corresponding extended Bass-Serre trees. The itinerary of $\alpha_{s,t}$ in $(\mathcal{B}_{s',t'}, \mathcal{A}_{s',t'})$ is the bi-infinite path in $\mathcal{T}_{\{s',t'\}}$ induced by all vertices corresponding to cosets $g \cdot W_{s',t'}$ such that $g \in W_{s,t}$. Analogously, the itinerary of $\alpha_{s',t'}$ in $(\mathcal{B}_{s,t}, \mathcal{A}_{s,t})$ is the bi-infinite path in $\mathcal{T}_{\{s',t'\}}$ induced by all vertices corresponding to cosets $g \cdot W_{s,t}$ such that $g \in W_{s',t'}$.*

Proof. The bi-infinite geodesic ray $\alpha_{s',t'}$ is contained in the one-skeleton of Σ_Λ and $\dots s't's't' \dots$ is its associated bi-infinite word. Thus, the infinite path $\alpha^1_{s',t'}$ intersects every translate $g\alpha_{s,t}$, $g \in W_{s',t'}$. With the same argumentation as in the proof of Lemma 5.52, we see for all $g \in W_{s',t'}$ that the deletion of $g\alpha_{s,t}$ decomposes Σ_Λ into two subcomplexes that each contain a subgeodesic ray of $\alpha_{s',t'}$. So, $\alpha_{s',t'}$ passes through every such bi-infinite geodesic ray $g\alpha_{s,t}$. Accordingly, the itinerary of $\alpha_{s',t'}$ in $(\mathcal{B}_{s,t}, \mathcal{A}_{s,t})$ is the bi-infinite path in the extended Bass-Serre tree $\mathcal{T}_{\{s,t\}}$ that is induced by all vertices corresponding to cosets $g \cdot W_{s,t}$ such that $g \in W_{s',t'}$. Analogously, the itinerary of $\alpha_{s,t}$ in $(\mathcal{B}_{s',t'}, \mathcal{A}_{s',t'})$ is the bi-infinite path in the Bass-Serre tree $\mathcal{T}_{\{s',t'\}}$ that is induced by all vertices corresponding to cosets $g \cdot W_{s',t'}$ such that $g \in W_{s,t}$. ☐

Let Λ be a graph with an independent path P whose endvertices are distinct, not adjacent, and not contained in an induced 4-cycle. Let $\alpha_{s,t}$ be an axis corresponding to a non-adjacent vertex pair $\{s,t\}$ of P. Then $\alpha_{s,t}$ is essential in every block decomposition associated to a non-adjacent vertex pair of P that crosses $\{s,t\}$. The following lemma examines how such essential rays corresponding to crossing vertex pairs of vertices in P influence each other. Recall that $\vec{\partial}_c\Sigma_\Lambda$ denotes the contracting boundary of Σ_Λ and that $\hat{\partial}_c\Sigma_\Lambda$ denotes the set of equivalence classes of contracting geodesic rays equipped with the subspace topology of the visual boundary $\hat{\partial}\Sigma_\Lambda$ of Σ_Λ. Let γ be a geodesic ray in the Davis complex Σ_Λ of Λ. We denote the connected component of γ in $\hat{\partial}_c\Sigma_\Lambda$ by $\hat{\kappa}(\gamma(\infty))$. Analogously, $\vec{\kappa}(\gamma(\infty))$ denotes the connected component of γ in $\vec{\partial}_c\Sigma_\Lambda$.

Lemma 5.55 (Crossing Lemma)**.** *Let Λ be a graph with an independent path P of a length of at least three whose endvertices are distinct and neither adjacent in Λ nor contained in an induced 4-cycle in Λ. Let $\{s,t\}$ and $\{s',t'\}$ be two vertex pairs in P that cross each other. Let Λ' be the subgraph of Λ induced by s, t, s' and t'. Exactly one of the following two possibilities holds.*

a) $|\hat{\kappa}(g\cdot\alpha^+_{s',t'}(\infty))| = |\hat{\kappa}(g\cdot\alpha^+_{s,t}(\infty))| = |\vec{\kappa}(g\cdot\alpha^+_{s',t'}(\infty))| = |\vec{\kappa}(g\cdot\alpha^+_{s,t}(\infty))| = 1 \ \forall\ g \in W_\Lambda$

b) $\hat{\partial}_c\Sigma_\Lambda$ *has a connected component that contains* $W_{\Lambda'} \cdot \alpha^+_{s',t'}(\infty)$ *and* $W_{\Lambda'} \cdot \alpha^+_{s,t}(\infty)$

Proof. By Lemma 5.49, both $\{s,t\}$ and $\{s',t'\}$ are separating vertex pairs. The Davis complexes of the two vertex pairs are two axes $\alpha_{s,t}$ and $\alpha_{s',t'}$ of the axial isometries st and $s't'$ respectively. As the endvertices of P are not contained in an induced 4-cycle, no induced 4-cycle in Λ contains one of the two vertex pairs $\{s,t\}$ and $\{s',t'\}$. Thus, st and $s't'$ are rank-one isometries by Lemma 5.10. Recall that $\alpha_{s,t}$ and $\alpha_{s',t'}$ denote the corresponding axes in Σ_Λ which contain the identity vertex id in the 1-skeleton of Σ_Λ.

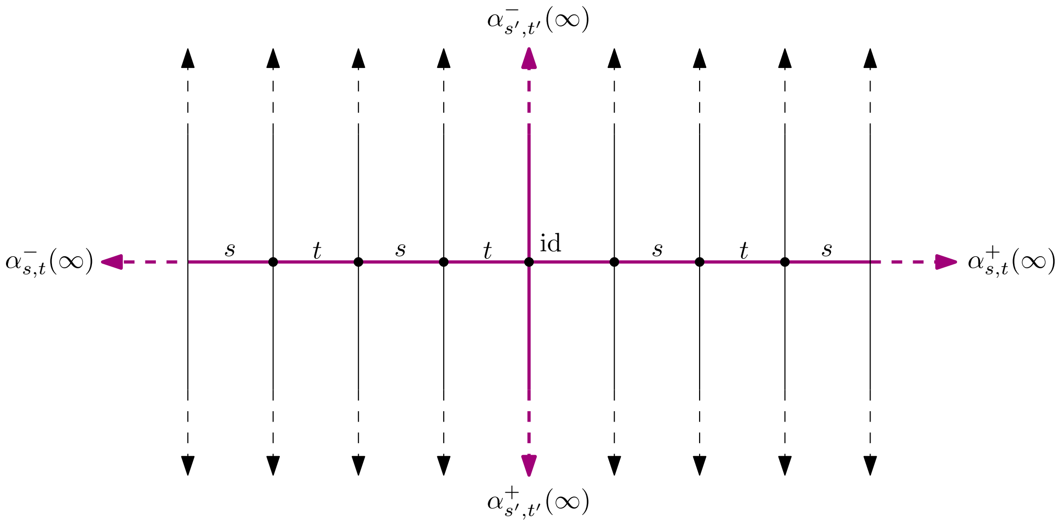

Figure 5.14 The violet rays are the axes $\alpha_{s,t}$ and $\alpha_{s',t'}$. The black rays are translates of $\alpha_{s',t'}$.

In case 1, we study the situation where the connected component $\kappa(\alpha^+_{s,t}(\infty))$ of $\alpha^+_{s,t}(\infty)$ in $\hat{\partial}_c\Sigma_\Lambda$ contains more than one single point. We show that then $\kappa(\alpha^+_{s,t}(\infty))$ contains $W_{\Lambda'} \cdot \alpha^+_{s',t'}(\infty)$ and $W_{\Lambda'} \cdot \alpha^+_{s,t}(\infty)$. In case 2, we study what happens when the connected component $\kappa(\alpha^+_{s,t}(\infty))$ of $\alpha^+_{s,t}(\infty)$ in $\hat{\partial}_c\Sigma_\Lambda$ consists of a single point. We prove that then $|\hat{\kappa}(g \cdot \alpha^+_{s',t'}(\infty))| = |\hat{\kappa}(g \cdot \alpha^+_{s,t}(\infty))| = |\vec{\kappa}(g \cdot \alpha^+_{s',t'}(\infty))| = |\vec{\kappa}(g \cdot \alpha^+_{s,t}(\infty))| = 1 \; \forall \; g \in W_\Lambda$. In Figure 5.14, the axes $\alpha_{s,t}$ and $\alpha_{s',t'}$ and translates of $\alpha_{s',t'}$ are pictured.

Case 1: Suppose that the connected component $\kappa(\alpha^+_{s,t}(\infty))$ of $\alpha^+_{s,t}(\infty)$ in $\hat{\partial}_c\Sigma_\Lambda$ contains more than one single point. We show that $\kappa(\alpha^+_{s,t}(\infty))$ contains $W_{\Lambda'} \cdot \alpha^+_{s',t'}(\infty)$ and $W_{\Lambda'} \cdot \alpha^+_{s,t}(\infty)$. We show this by induction on the word length of g in $W_{\Lambda'}$.

Induction base: Since the set of vertices $\{s, t, s', t'\}$ is a fundamental generating set of $W_{\Lambda'}$, s, t, s' and t' are the only words in $W_{\Lambda'}$ of length one. By Theorem 4.24, the connected component $\kappa(\alpha^+_{s,t}(\infty))$ contains $\kappa(\alpha^-_{s,t}(\infty))$. It is $s\alpha^+_{s,t}(\infty) = \alpha^-_{s,t}(\infty)$ and $t\alpha^+_{s,t}(\infty) = \alpha^-_{s,t}(\infty)$. Thus, $\kappa(\alpha^+_{s,t}(\infty))$ contains $s\alpha^+_{s,t}(\infty)$ and $t\alpha^+_{s,t}(\infty)$. Next, we show that $\kappa(\alpha^+_{s,t}(\infty))$ contains $\alpha^+_{s',t'}(\infty)$, $\alpha^-_{s',t'}(\infty)$, $s\alpha^+_{s',t'}(\infty)$ and $t\alpha^+_{s',t'}(\infty)$. Indeed, by Lemma 5.52, $\alpha_{s,t}$ is essential in $(\mathcal{B}_{s',t'}, \mathcal{A}_{s',t'})$. Let $I(\alpha_{s,t})$ be the bi-infinite itinerary of $\alpha_{s,t}$ in the extended Bass-Serre tree $\mathcal{T}_{\{s',t'\}}$ associated to $(\mathcal{B}_{s',t'}, \mathcal{A}_{s',t'})$. By Lemma 4.53, $g\alpha^+_{s',t'}(\infty) \in \kappa(\alpha^+_{s,t}(\infty))$ and $g\alpha^-_{s',t'}(\infty) \in \kappa(\alpha^+_{s,t}(\infty))$ for all $g \in W_\Lambda$ such that $v_{gW_{s',t'}}$ is contained in $I(\alpha_{s,t})$. By Lemma 5.54, $I(\alpha_{s,t})$ is induced by all vertices corresponding to cosets $gW_{s',t'}$ such that $g \in W_{s,t}$. In particular, $\alpha^+_{s',t'}(\infty)$, $\alpha^-_{s',t'}(\infty)$, $s\alpha^+_{s',t'}(\infty)$, $t\alpha^+_{s',t'}(\infty) \in \kappa(\alpha^+_{s,t}(\infty))$. For proving the induction base it remains to show that $\kappa(\alpha^+_{s,t}(\infty))$ contains $s'\alpha^+_{s',t'}(\infty)$ and $t'\alpha^+_{s',t'}(\infty)$, $s'\alpha^+_{s,t}(\infty)$ and $t'\alpha^+_{s,t}(\infty)$. We have seen already that

$\alpha^+_{s',t'}(\infty)$ is contained in $\kappa(\alpha^+_{s,t}(\infty))$. Thus, $\kappa(\alpha^+_{s,t}(\infty)) = \kappa(\alpha^+_{s',t'}(\infty))$. In particular, $\kappa(\alpha^+_{s',t'}(\infty)$ contains more than one point. We change the rules of s, t and s', t' in the argumentation above and conclude that $\kappa(\alpha^+_{s,t}(\infty)) = \kappa(\alpha^+_{s',t'}(\infty))$ contains $s'\alpha^+_{s',t'}(\infty)$ and $t'\alpha^+_{s',t'}(\infty)$, $s'\alpha^+_{s,t}(\infty)$ and $t'\alpha^+_{s,t}(\infty)$.

Induction step: Assume that $\kappa(\alpha^+_{s,t}(\infty))$ contains $\hat{g}\alpha^+_{s,t}(\infty)$ for all $\hat{g} \in W_{\Lambda'}$ with a word length of at most k, $k \in \mathbb{N}$. Let $g \in W_{\Lambda'}$ be an element with word length $k+1$. Let $\vec{g} = s_0, \dots, s_k$ be a word in S associated to g. Let $h = s_1 \cdot s_2 \cdots s_k$. Then $g = s_0 \cdot h$. By induction hypothesis, $\kappa(\alpha^+_{s,t}(\infty))$ contains $h\alpha^+_{s,t}(\infty)$ and $h\alpha^-_{s,t}(\infty)$. As s_0 acts as a homeomorphism on $\hat{\partial}_c\Sigma_\Lambda$, $\kappa(s_0\alpha^+_{s,t}(\infty))$ contains $s_0h\alpha^+_{s,t}(\infty) = g\alpha^+_{s,t}(\infty)$ and contains $s_0h\alpha^-_{s,t}(\infty) = g\alpha^-_{s,t}(\infty)$. By induction hypothesis, $s_0\alpha^+_{s,t}(\infty)$ is contained in $\kappa(\alpha^+_{s,t}(\infty))$. Thus, $\kappa(s_0\alpha^+_{s,t}(\infty))$=$\kappa(\alpha^+_{s,t}(\infty))$. Hence, $\kappa(\alpha^+_{s,t}(\infty))$ contains $g\alpha^+_{s,t}(\infty)$ and $g\alpha^-_{s,t}(\infty)$.

Case 2: Suppose that the connected component $\kappa(\alpha^+_{s,t}(\infty))$ of $\alpha^+_{s,t}(\infty)$ in $\hat{\partial}_c\Sigma_\Lambda$ consists of a single point. Let $g \in W_\Lambda$. As W_Λ acts by homeomorphisms on $\hat{\partial}_c\Sigma_\Lambda$, the connected component of $g\alpha^+_{s,t}(\infty)$ in $\hat{\partial}_c\Sigma_\Lambda$ consists of a single point. As the direct limit topology is finer than the subspace topology of the visual boundary, the connected component of $g\alpha^+_{s,t}(\infty)$ in $\vec{\partial}_c\Sigma_\Lambda$ consist of a single point too. Suppose we know that the connected component of $\alpha^+_{s',t'}(\infty)$ in $\hat{\partial}_c\Sigma_\Lambda$ consists of a single point. Then the connected component of $g\alpha^+_{s',t'}(\infty)$ in $\hat{\partial}_c\Sigma_\Lambda$ consists of a single point for all $g \in W_\Lambda$ as W_Λ acts by homeomorphisms on $\hat{\partial}_c\Sigma_\Lambda$. As the direct limit topology is finer than the subspace topology of the visual boundary, this implies that the connected component of $g\alpha^+_{s',t'}(\infty)$ in $\vec{\partial}_c\Sigma_\Lambda$ consists of a single point for all $g \in W_\Lambda$. Thus, it remains to prove that the connected component of $\alpha^+_{s',t'}(\infty)$ in $\hat{\partial}_c\Sigma_\Lambda$ consists of a single point. To achieve a contradiction, suppose that the connected component $\kappa(\alpha^+_{s',t'}(\infty))$ of $\alpha^+_{s',t'}(\infty)$ in $\hat{\partial}_c\Sigma_\Lambda$ contains more than one single point. By Lemma 5.52, the axis $\alpha_{s',t'}$ is essential in $(\mathcal{B}_{s,t}, \mathcal{A}_{s,t})$. Let $I(\alpha_{s',t'})$ be the bi-infinite itinerary of $\alpha_{s',t'}$ in the extended Bass-Serre tree $\mathcal{T}_{\{s,t\}}$ associated to $(\mathcal{B}_{s,t}, \mathcal{A}_{s,t})$. By Lemma 4.53, $g\alpha^+_{s,t}(\infty) \in \kappa(\alpha^+_{s',t'}(\infty))$ and $g\alpha^-_{s,t}(\infty) \in \kappa(\alpha^+_{s',t'}(\infty))$ for all $g \in W_\Lambda$ such that $v_{gW_{s,t}}$ is contained in $I(\alpha_{s',t'})$. By Lemma 5.54, $I(\alpha_{s',t'})$ is induced by all vertices corresponding to cosets $gW_{s,t}$ such that $g \in W_{s',t'}$. In particular, $\kappa(\alpha^+_{s',t'}(\infty))$ contains $\alpha^+_{s,t}(\infty)$ – a contradiction to the assumption that the connected component of $\alpha^+_{s,t}(\infty)$ in $\hat{\partial}_c\Sigma_\Lambda$ consists of a single point. □

Let Λ be a graph with an independent path P. In the following proposition, we will apply the Crossing Lemma several times. A domino effect leads to an interesting behavior of connected components of $\vec{\partial}_c\Sigma_\Lambda$. We need the following lemma for the proof of the proposition.

Recall that the Davis complex of an induced subgraph Λ' in a graph Λ is canonically embedded in the Davis complex of Λ. Recall further that $\partial_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ denotes the set $\{\gamma(\infty) \in \partial_c\Sigma_\Lambda \mid \gamma \subseteq \Sigma_{\Lambda'}\}$. The corresponding topological subspaces of $\hat{\partial}_c\Sigma_\Lambda$ and $\vec{\partial}_c\Sigma_\Lambda$ are denoted by $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ and $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ respectively.

Lemma 5.56. *Let Λ be a graph with an independent path P with distinct endvertices s and t that are not adjacent and not contained in an induced 4-cycle. Let $\bar{P}$ be the graph obtained from Λ by deleting all inner vertices of P. Suppose that the connected component of $\alpha_{s,t}^+$ in $\hat{\partial}_c\Sigma_\Lambda$ consists of a single point.*

Then every connected component of $\hat{\partial}_c\Sigma_\Lambda$ and $\vec{\partial}_c\Sigma_\Lambda$ of an element in $W_\Lambda \cdot \partial_{c,\Sigma_\Lambda}\Sigma_P$ consists of a single point. Furthermore, every connected component of $\hat{\partial}_c\Sigma_\Lambda$ ($\vec{\partial}_c\Sigma_\Lambda$) consists of

- *a single point or*
- *is homeomorphic to a connected component of $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$).*

Proof. Let $(\mathcal{B}_{s,t}, \mathcal{A}_{s,t})$ be the block decomposition associated to $\{P, \bar{P}\}$ as defined in Equation (5.49.1). Recall that every connected component in $\hat{\partial}_c\Sigma_\Lambda$ is of type 1 or 2 as defined in Definition 3.52. If $|\hat{\kappa}(\alpha_{s,t}^+(\infty))| = 1$, $\hat{\partial}_c\Sigma_\Lambda$ does not contain any connected component of type 2 because of Lemma 4.42. Since the direct limit topology is finer than the cone topology, $\vec{\partial}_c\Sigma_\Lambda$ does not contain any connected component of type 2. Compare Lemma 3.55. Hence, every connected component of $\hat{\partial}_c\Sigma_\Lambda$ and $\vec{\partial}_c\Sigma_\Lambda$ is of type 1. Every itinerary of a geodesic ray ending in Σ_P is finite. Thus, every connected component κ of $\hat{\partial}_c\Sigma_\Lambda$ ($\vec{\partial}_c\Sigma_\Lambda$) that contains an equivalence class of a contracting geodesic ray in Σ_P is of type 1_f. By Lemma 3.57, κ is homeomorphic to a connected component of $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_P$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_P$). By Lemma 2.35, $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_P$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_P$) is homeomorphic to a subspace of $\hat{\partial}_c\Sigma_P$ ($\vec{\partial}_c\Sigma_P$). By Example 5.36, $\hat{\partial}_c\Sigma_P$ ($\vec{\partial}_c\Sigma_P$) is totally disconnected. Thus, κ consists of a single point. As W_Λ acts by homeomorphisms on $\hat{\partial}_c\Sigma_\Lambda$, every connected component of an element in $W_\Lambda \cdot \partial_{c,\Sigma_\Lambda}P$ consists of a single point. By Corollary 4.39, every connected component of $\hat{\partial}_c\Sigma_\Lambda$ ($\vec{\partial}_c\Sigma_\Lambda$) consists of a single point or is homeomorphic to a connected component of $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$). □

Let Λ be a graph with an independent path P and $\{s, t\}$ be a pair of non-adjacent vertices of P. Let γ be a geodesic ray in the Davis complex Σ_Λ of Λ. Recall that $\hat{\kappa}(\gamma(\infty))$ denotes the connected component of γ in $\hat{\partial}_c\Sigma_\Lambda$. Analogously, $\vec{\kappa}(\gamma(\infty))$ denotes the connected component of γ in $\vec{\partial}_c\Sigma_\Lambda$. The following proposition arises from a Domino effect by applying Lemma 5.55 several times. Combinatorial, the Domino effect can be explained as follows. Say that we have a path as in Figure 5.15. The dashed edges that connect non-consecutive vertices of the path mark non-adjacent vertex pairs. If two such edges cross, the corresponding vertex pairs cross. We start with a vertex pair defined by the green dashed edge in the left picture of Figure 5.15. This edge is crossed by the dashed edges in the next picture. All the corresponding vertex pairs cross the first vertex pair. In the next picture, all the dashed blue edges cross at least one dashed black edge. So, every vertex pair corresponding to a blue dashed edge crosses a vertex pair corresponding to a black dashed edge. In the last picture, all non-adjacent vertex pairs are connected by a dashed edge to each other except for the two endvertices of the path.

Suppose that Λ' is an induced subgraph of Λ and that the Davis complex of Λ' is canonically embedded in the Davis complex of Λ. Recall that $\partial_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ denotes the set

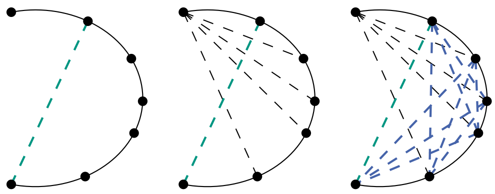

Figure 5.15 The dashed edges mark non-adjacent vertex pairs. If two dashed edges cross, the corresponding vertex pairs cross. By a Domino effect, all non-adjacent vertex pairs are contained in the right picture except for the pair of the endvertices of the path.

$\{\gamma(\infty) \in \partial_c\Sigma_\Lambda \mid \gamma \subseteq \Sigma_{\Lambda'}\}$. The corresponding topological subspaces of $\hat{\partial}_c\Sigma_\Lambda$ and $\vec{\partial}_c\Sigma_\Lambda$ are denoted by $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ and $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ respectively. Let γ be a geodesic ray in the Davis complex Σ_Λ of Λ. Recall that $\hat{\kappa}(\gamma(\infty))$ denotes the connected component of γ in $\hat{\partial}_c\Sigma_\Lambda$. Analogously, $\vec{\kappa}(\gamma(\infty))$ denotes the connected component of γ in $\vec{\partial}_c\Sigma_\Lambda$.

Proposition 5.57 (Domino effect)**.** *Let Λ be a graph that contains an independent path P whose endvertices are distinct and not adjacent. Let Σ_P be the canonically embedded Davis complex of P in Σ_Λ. One of the following statements holds.*

a) If κ is a connected component in $\hat{\partial}_c\Sigma_\Lambda$ or $\vec{\partial}_c\Sigma_\Lambda$ such that $\kappa \cap W_\Lambda \cdot \partial_{c,\Sigma_\Lambda}\Sigma_P \neq \emptyset$, then $|\kappa| = 1$.

b) Every geodesic ray in Σ_P is contracting in Σ_Λ. If s and t are two non-adjacent vertices in P and at least one of them is an inner vertex of P, then

$$|\vec{\kappa}(g\alpha^+_{s,t}(\infty))| = |\hat{\kappa}(g\alpha^+_{s,t}(\infty))| = 1 \;\forall\; g \in W_\Lambda.$$

Otherwise, if s and t are the endvertices of P, then $\hat{\kappa}(\alpha^+_{s,t}(\infty)) = \hat{\kappa}(\alpha^-_{s,t}(\infty))$.

c) The following two statements are satisfied.

a) Every geodesic ray in Σ_P is contracting in Σ_Λ. If s and t are two non-adjacent vertices in P, then

$$\hat{\kappa}(\alpha^+_{s,t}(\infty)) = \hat{\kappa}(\alpha^-_{s,t}(\infty)).$$

b) Suppose that $\{s,t\}$ and $\{s',t'\}$ are two non-adjacent vertex pairs that contain each an inner vertex of P. Let Λ' be the subgraph of Λ induced by s, t, s' and t'. Then $\hat{\partial}_c\Sigma_\Lambda$ has a connected component that contains $W_{\Lambda'} \cdot \alpha^+_{s',t'}(\infty)$ and $W_{\Lambda'} \cdot \alpha^+_{s,t}(\infty)$.

Proof. Suppose that P has length 2. Then Σ_P is quasi-isometric to $\mathbb{Z}$. If the endvertices of P are contained in an induced 4-cycle, no geodesic ray in Σ_P is contracting in the ambient Davis complex Σ_Λ and Item a) of the claim is satisfied. Otherwise, the two geodesic rays in Σ_P starting at the identity vertex are contracting because of Lemma 5.10.

Then the two connected components of the two boundary points associated to Σ_P are single point or they are contained in a common connected component of $\hat{\partial}_c\Sigma_\Lambda$ because of Corollary 4.25. So, if Item a) is not satisfied, then *Item b*) and *Item c*) are satisfied.

From now on we suppose that P has a length of at least three. Let s and t be the endvertices of P. If s and t are contained in an induced 4-cycle, Item a) is satisfied because of Lemma 5.47.

From now on we assume that s and t are not contained in an induced 4-cycle. Then no pair of non-adjacent vertices of P is contained in an induced 4-cycle. By Theorem 5.7, every geodesic ray in Σ_P is contracting in Σ_Λ. If $|\hat{\kappa}(\alpha^+_{s,t}(\infty))| = 1$, Lemma 5.56 implies that Item a) is satisfied.

From now on we assume that $|\hat{\kappa}(\alpha^+_{s,t}(\infty))| > 1$. By Corollary 4.25, $\hat{\kappa}(\alpha^+_{s,t}(\infty)) = \hat{\kappa}(\alpha^-_{s,t}(\infty))$. Let $\{s', t'\}$ be a vertex pair of non-adjacent vertices in P containing at least one inner vertex of P. In case 1, we assume that $|\hat{\kappa}(\alpha^+_{s',t'}(\infty))| = 1$ and proof that then Item b) is satisfied. Afterwards we study in case 2 the situation where $\hat{\kappa}(\alpha^+_{s',t'}(\infty))$ consists more than one point and prove that then Item c) is satisfied. For proving both cases, we use a domino effect and apply Lemma 5.55 several times.

Case1: Suppose that $|\hat{\kappa}(\alpha^+_{s',t'}(\infty))| = 1$. As the direct limit topology is finer than the subspace topology of the visual boundary, $|\vec{\kappa}(\alpha^+_{s',t'}(\infty))| = 1$. Since W_Λ acts by homeomorphisms on $\hat{\partial}_c\Sigma_\Lambda$ and $\vec{\partial}_c\Sigma_\Lambda$, $|\vec{\kappa}(g\alpha^+_{s',t'}(\infty))| = |\hat{\kappa}(g\alpha^+_{s',t'}(\infty))| = 1 \;\forall\; g \in W_\Lambda$. Suppose that $\tilde{s}$ and $\tilde{t}$ are two non-adjacent vertices of P such that $\{s', t'\}$ and $\{\tilde{s}, \tilde{t}\}$ cross. Then Lemma 5.55 implies that $|\vec{\kappa}(g\alpha^+_{\tilde{s},\tilde{t}}(\infty))| = |\hat{\kappa}(g\alpha^+_{\tilde{s},\tilde{t}}(\infty))| = 1 \;\forall\; g \in W_\Lambda$. Assume that $\tilde{s}$ and $\tilde{t}$ are two non-adjacent vertices in P such that $\{s', t'\}$ and $\{\tilde{s}, \tilde{t}\}$ don't cross. If $\tilde{s}$ and $\tilde{t}$ are the endvertices of P, then $\kappa(\alpha^+_{\tilde{s},\tilde{t}}(\infty)) = \kappa(\alpha^-_{\tilde{s},\tilde{t}}(\infty))$ by assumption. Otherwise, either $\tilde{s}$ or $\tilde{t}$ is an inner vertex of P. Then a vertex u of P lies between $\tilde{s}$ and $\tilde{t}$ on P. Recall that either s' or t' is an inner vertex of P and that $\{s', t'\}$ and $\{\tilde{s}, \tilde{t}\}$ don't cross. Thus, either s' or t' lies between u and an endvertex of P. Let w be this endvertex of P. Then $\{u, w\}$ crosses $\{\tilde{s}, \tilde{t}\}$ and $\{s', t'\}$. We apply Lemma 5.55 to $\{u, w\}$ and $\{s', t'\}$ and afterwards to $\{u, w\}$ and $\{\tilde{s}, \tilde{t}\}$ and conclude that $\vec{\kappa}(g\alpha^+_{\tilde{s},\tilde{t}}(\infty))| = |\hat{\kappa}(g\alpha^+_{\tilde{s},\tilde{t}}(\infty))| = 1 \;\forall\; g \in W_\Lambda$. It follows that Item b) is satisfied.

Case2: Suppose that $\hat{\kappa}(\alpha^+_{s',t'}(\infty))$ contains more than one point. By Corollary 4.25, $\hat{\kappa}(\alpha^+_{s',t'}(\infty)) = \hat{\kappa}(\alpha^-_{s',t'}(\infty))$. Suppose that $\tilde{s}$ and $\tilde{t}$ are two non-adjacent vertices of P such that $\{s', t'\}$ and $\{\tilde{s}, \tilde{t}\}$ cross. Let Λ' be the subgraph of Λ induced by s', t', $\tilde{s}$ and $\tilde{t}$. By Lemma 5.55 $\hat{\partial}_c\Sigma_\Lambda$, has a connected component that contains $W_{\Lambda'} \cdot \alpha^+_{s',t'}(\infty)$ and $W_{\Lambda'} \cdot \alpha^+_{\tilde{s},\tilde{t}}(\infty)$. Assume that $\tilde{s}$ and $\tilde{t}$ are two non-adjacent vertices in P such that $\{s', t'\}$ and $\{\tilde{s}, \tilde{t}\}$ don't cross. If $\tilde{s}$ and $\tilde{t}$ are the endvertices of P, $\kappa(\alpha^+_{\tilde{s},\tilde{t}}(\infty)) = \kappa(\alpha^-_{\tilde{s},\tilde{t}}(\infty))$ by assumption. Otherwise, either $\tilde{s}$ or $\tilde{t}$ is an inner vertex of P. Then a vertex u of P lies between $\tilde{s}$ and $\tilde{t}$ in P. Recall that either s' or t' is an inner vertex of P and that $\{s', t'\}$ and $\{\tilde{s}, \tilde{t}\}$ don't cross. Thus, either s' or t' lies between u and an endvertex of P. Let w be this endvertex of P. Then $\{u, w\}$ crosses $\{\tilde{s}, \tilde{t}\}$ and $\{s', t'\}$. We apply Lemma 5.55 to $\{s', t'\}$ and $\{u, w\}$. Afterwards we apply Lemma 5.55 to $\{u, w\}$ and $\{\tilde{s}, \tilde{t}\}$ and conclude that $\hat{\partial}_c\Sigma_\Lambda$ has a connected component that contains $W_{\Lambda'} \cdot \alpha^+_{\tilde{s},\tilde{t}}(\infty)$ and $W_{\Lambda'} \cdot \alpha^+_{s,t}(\infty)$ for all $g \in W_{\Lambda'}$. It follows that Item c) is satisfied. □

The goal of this section is to understand how the contracting boundary of a right-angled Coxeter group changes if we glue a path of a length of at least two on its defining graph. Let P be an independent path in a graph Λ. Let $\bar{P}$ be the graph obtained from Λ by deleting all inner vertices of P. In other words, we obtain Λ by gluing P on $\bar{P}$. If the endvertices of P are adjacent or contained in an induced 4-cycle, Lemma 5.47 says how the contracting boundary of $W_{\bar{P}}$ changes when we glue P on $\bar{P}$.

The case remains that the endvertices of P are not adjacent. The following theorem says that there occur only two extreme cases. Either all new arising connected components in the contracting boundary $\vec{\partial}_c\Sigma_\Lambda$ of Σ_Λ are single points or there arises a large connected component in $\hat{\partial}_c\Sigma_\Lambda$ that contains a set bijective to the visual boundary of Σ_P. Thereby, $\hat{\partial}_c\Sigma_\Lambda$ denotes the subspace of the visual boundary $\hat{\partial}\Sigma_\Lambda$ of Σ_Λ that consists of all equivalence classes of contracting geodesic rays in Σ_Λ.

Suppose that Λ' is an induced subgraph of Λ and that the Davis complex of Λ' is canonically embedded in the Davis complex of Λ. Recall that $\partial_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ denotes the set $\{\gamma(\infty) \in \partial_c\Sigma_\Lambda \mid \gamma \subseteq \Sigma_{\Lambda'}\}$. The corresponding topological subspaces of $\hat{\partial}_c\Sigma_\Lambda$ and $\vec{\partial}_c\Sigma_\Lambda$ are denoted by $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ and $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ respectively. By Lemma 2.35, $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ and $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\Lambda'}$ are homeomorphic to the set $\{\gamma(\infty) \in \partial_c\Sigma_{\Lambda'} \mid \gamma \text{ is } \Sigma_\Lambda\text{-contracting}\}$ equipped with the subspace topology of the visual and contracting boundary of $\Sigma_{\Lambda'}$ respectively.

Theorem 5.58 (Gluing paths on graphs)**.** *Let Λ be a graph that contains an independent path P with distinct endvertices s and t that are not adjacent. Let $\bar{P}$ be the graph obtained from Λ by deleting all inner vertices of P. Let Σ_P and $\Sigma_{\bar{P}}$ be the canonically embedded Davis complexes of P and $\bar{P}$ in Σ_Λ respectively. Let $\alpha_{s,t}$ be the axis for the axial isometry st that intersects the identity-vertex of Σ_Λ. One of the following statements holds.*

a) Every geodesic ray in Σ_P is contracting in Σ_Λ and for each $g \in W_\Lambda$ there exists a connected component in $\hat{\partial}_c\Sigma_\Lambda$ containing $g \cdot \partial\Sigma_P$.

b) For all $\xi \in W_\Lambda \cdot \partial_{c,\Sigma_\Lambda}\Sigma_P \setminus W_\Lambda \cdot \alpha^+_{s,t}(\infty)$, the connected component of ξ in $\hat{\partial}_c\Sigma_\Lambda$ and $\vec{\partial}_c\Sigma_\Lambda$ consists of a single point.

Suppose that Item b) is satisfied. Then every connected component of $\vec{\partial}_c\Sigma_\Lambda$ consists of a single point or is homeomorphic to a connected component of $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$. Analogously, every connected component of $\hat{\partial}_c\Sigma_\Lambda$ consists of a single point or is homeomorphic to a connected component of $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$.

Proof. Suppose that P has length 2. Then Σ_P is quasi-isometric to $\mathbb{Z}$. If the endvertices of P are contained in an induced 4-cycle, no geodesic ray in Σ_P is contracting in the ambient Davis complex Σ_Λ and Item b) of the claim is satisfied. Otherwise, every geodesic ray in Σ_P is contracting because of Lemma 5.10. Then the two connected components of the two boundary points associated to Σ_P are single point or they are contained in a common connected component of $\hat{\partial}_c\Sigma_\Lambda$ because of Corollary 4.25. In particular, if Item a) is not satisfied, then Item b) of the claim is satisfied.

From now on we suppose that P has a length of at least three. We apply Proposition 5.57. Proposition 5.57 says that Item a), Item b) or Item c) of Proposition 5.57 is satisfied. Item a) of Proposition 5.57 implies directly Item b) of the claim. It remains to study what happens if Item b) or Item c) of Proposition 5.57 is satisfied. In case 1, we study the situation where Item b) of Proposition 5.57 is satisfied. We prove that then Item b) of the claim is satisfied and conclude that then every connected component of $\vec{\partial}_c\Sigma_\Lambda$ ($\hat{\partial}_c\Sigma_\Lambda$) consists of a single point or is homeomorphic to a connected component of $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$ ($\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$). In case 2, we examine what happens when Item c) of Proposition 5.57 is satisfied. We prove that this implies Item a) of the claim.

<u>Case 1:</u> Suppose that P has a length of at least three and that Item b) of Proposition 5.57 is satisfied. As P has a length of at least three, P has at least one inner vertex $\tilde{s}$. Let $\tilde{t}$ be a vertex of P that is not adjacent to $\tilde{s}$. Without loss of generality, the distance of $\tilde{s}$ to the endvertex s of P is less than the distance of $\tilde{t}$ to s. By assumption, $|\kappa(\alpha^+_{\tilde{s},\tilde{t}}(\infty))| = |\kappa(\alpha^-_{\tilde{s},\tilde{t}}(\infty))| = 1$. By Lemma 5.49, $\{\tilde{s},\tilde{t}\}$ is a separating vertex pair of Λ. Let $\Lambda_{\tilde{s},\tilde{t}}$ be the induced subgraph of Λ that consists of $\tilde{s}$ and $\tilde{t}$. Let $P_{\tilde{s},\tilde{t}}$ be the independent path induced by $\tilde{s}$ and $\tilde{t}$ and all vertices of P that lie between $\tilde{s}$ and $\tilde{t}$. Let $\bar{P}_{\tilde{s},\tilde{t}}$ be the graph obtained from Λ by deleting all inner vertices of $P_{\tilde{s},\tilde{t}}$. Let $\alpha_{\tilde{s},\tilde{t}}$ be the associated bi-infinite axis for the isometry $\tilde{s}\tilde{t}$. By assumption, $|\kappa(\alpha^+_{\tilde{s},\tilde{t}}(\infty))| = |\kappa(\alpha^-_{\tilde{s},\tilde{t}}(\infty))| = 1$. By Lemma 5.56, every connected component of an element in $W_\Lambda \cdot \partial_{c,\Sigma_\Lambda} P_{\tilde{s},\tilde{t}}$ consists of a single point in $\hat{\partial}_c\Sigma_\Lambda$ and $\vec{\partial}_c\Sigma_\Lambda$. Furthermore, every connected component of $\hat{\partial}_c\Sigma_\Lambda$ ($\vec{\partial}_c\Sigma_\Lambda$) consists of a single point or is homeomorphic to a connected component of $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}_{\tilde{s},\tilde{t}}}$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}_{\tilde{s},\tilde{t}}}$). By Lemma 2.35, $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}_{\tilde{s},\tilde{t}}}$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}_{\tilde{s},\tilde{t}}}$) is homeomorphic to a subspace of $\hat{\partial}_c\Sigma_{\bar{P}_{\tilde{s},\tilde{t}}}$ ($\hat{\partial}_c\Sigma_{\bar{P}_{\tilde{s},\tilde{t}}}$). Let us consider $\hat{\partial}_c\Sigma_{\bar{P}_{\tilde{s},\tilde{t}}}$ ($\vec{\partial}_c\Sigma_{\bar{P}_{\tilde{s},\tilde{t}}}$). Let $P_{s,\tilde{s}}$ be the subpath of P connecting s and $\tilde{s}$. Analogously let $P_{\tilde{t},t}$ be the subpath of P connecting $\tilde{t}$ and t. Let $\bar{P}$ be the graph obtained from Λ by deleting all inner vertices of P. Then $\bar{P}_{\tilde{s},\tilde{t}} = \bar{P} \cup P_{\tilde{s},s} \cup P_{\tilde{t},t}$. The deletion of s separates $\bar{P}_{\tilde{s},\tilde{t}} = \bar{P} \cup P_{\tilde{s},s} \cup P_{\tilde{t},t}$ into two connected components. Accordingly, $\{\bar{P} \cup P_{\tilde{t},t}, P_{\tilde{s},s}\}$ is a proper separation of $\bar{P}_{\tilde{s},\tilde{t}}$. The separating subgraph of this separation consists of the vertex s. By Proposition 5.28,

$$\{ \bigcup_{g \in W_{\bar{P}_{\tilde{s},\tilde{t}}}} g\Sigma_{\bar{P}\cup P_{\tilde{t},t}} \cup \bigcup_{g \in W_{\bar{P}_{\tilde{s},\tilde{t}}}} g\Sigma_{P_{\tilde{s},s}}, \bigcup_{g \in W_{\bar{P}_{\tilde{s},\tilde{t}}}} g\Sigma_s\}$$

is a block decomposition with thin walls of $\Sigma_{\bar{P}_{\tilde{s},\tilde{t}}}$. By Theorem 5.32, every connected component of $\hat{\partial}_c\Sigma_{\bar{P}_{\tilde{s},\tilde{t}}}$ ($\vec{\partial}_c\Sigma_{\bar{P}_{\tilde{s},\tilde{t}}}$) consists either of a single point or is homeomorphic to a connected component of $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{P_{\tilde{s},s}}$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{P_{\tilde{s},s}}$) or of $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}\cup P_{\tilde{t},t}}$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}\cup P_{\tilde{t},t}}$). By Lemma 2.35, $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{P_{\tilde{s},s}}$ is homeomorphic to a subspace of $\hat{\partial}_c\Sigma_{P_{\tilde{s},s}}$. The visual and contracting boundary of the Davis complex of a path is empty or totally disconnected. Hence, $|\vec{\kappa}(\gamma(\infty))| = |\hat{\kappa}(\gamma(\infty))| = 1$ for every contracting geodesic ray γ in $\Sigma_{P_{\tilde{s},s}}$. It remains to consider $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}\cup P_{\tilde{t},t}}$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}\cup P_{\tilde{t},t}}$). By Lemma 2.35, $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}\cup P_{\tilde{t},t}}$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}\cup P_{\tilde{t},t}}$) is homeomorphic to a subspace of $\hat{\partial}_c\Sigma_{\bar{P}\cup P_{\tilde{t},t}}$ ($\vec{\partial}_c\Sigma_{\bar{P}\cup P_{\tilde{t},t}}$). We observe that $\{\bar{P}, P_{\tilde{t},t}\}$ is a proper separation of $\bar{P} \cup P_{\tilde{t},t}$ whose separating subgraph consists of the vertex t. We repeat the argumentation above and conclude that every connected component of $\hat{\partial}_c\Sigma_{\bar{P}\cup P_{\tilde{t},t}}$ ($\vec{\partial}_c\Sigma_{\bar{P}\cup P_{\tilde{t},t}}$) consists either of a single point or is homeomorphic to a connected component of $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{P_{\tilde{t},t}}$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{P_{\tilde{t},t}}$) or of $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$). As above, we conclude that $|\hat{\kappa}(\gamma(\infty))| = 1$ and $|\vec{\kappa}(\gamma(\infty))| = 1$ for every contracting geodesic ray γ in $\Sigma_{P_{\tilde{t},t}}$. In summary, we have proven that for all $\xi \in W_\Lambda \cdot \partial_{c,\Sigma_\Lambda}\Sigma_P \setminus W_\Lambda \cdot \alpha^+_{s,t}(\infty)$, $|\hat{\kappa}(\xi)| = |\vec{\kappa}(\xi)| = 1$. Recall that every connected component of $\hat{\partial}_c\Sigma_\Lambda$ ($\vec{\partial}_c\Sigma_\Lambda$) consists of a single point or is homeomorphic to a connected component of $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}_{\tilde{s},\tilde{t}}}$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}_{\tilde{s},\tilde{t}}}$). By our considerations, every connected component of $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}_{\tilde{s},\tilde{t}}}$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}_{\tilde{s},\tilde{t}}}$) consists either of a single point or is homeomorphic to a connected component of $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$ ($\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$). Hence, every connected component of $\vec{\partial}_c\Sigma_\Lambda$ consists of a single point or is homeomorphic to a connected component of $\vec{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$. Analogously, every connected component of $\hat{\partial}_c\Sigma_\Lambda$ consists of a single point or is homeomorphic to a connected component of $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_{\bar{P}}$.

Case 2: Suppose that P has a length of at least three and that Item c) of Proposition 5.57 is satisfied. As P has a length of at least 3, P has at least one inner vertex $\tilde{s}$. Let $\tilde{t}$ be a vertex of P that is not adjacent to $\tilde{s}$. By assumption, every geodesic ray in Σ_P is contracting in Σ_Λ. Thus, $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_P = \hat{\partial}_c\Sigma_P$ and $\tilde{s}\tilde{t}$ is a rank-one isometry of W_Λ. Recall that $\alpha_{\tilde{s},\tilde{t}}$ denotes the axis for $\tilde{s}\tilde{t}$ intersecting the identity vertex in the one-skeleton of Σ_Λ. Suppose we would have proven that a connected component $\hat{\kappa}$ of $\hat{\partial}_c\Sigma_\Lambda$ contains $W_P \cdot \alpha^+_{\tilde{s},\tilde{t}}(\infty)$. As $\alpha^+_{\tilde{s},\tilde{t}}(\infty)$ is not globally fixed by W_P, the orbit $W_P \cdot \alpha^+_{\tilde{s},\tilde{t}}(\infty)$ is dense in $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_P$ by Theorem 2.32 of Murray. This also follows from Theorem 1.1 in [Ham09] of Hamenstädt. As connected components are closed, $\hat{\kappa}$ contains $\hat{\partial}_{c,\Sigma_\Lambda}\Sigma_P$. Since $\hat{\partial}\Sigma_P = \hat{\partial}_{c,\Sigma_\Lambda}\Sigma_P$, $\hat{\partial}_c\Sigma_\Lambda$ has a connected component containing $\partial\Sigma_P$. As W_Λ acts by homeomorphisms on $\hat{\partial}_c\Sigma_\Lambda$, there exists a connected component in $\hat{\partial}_c\Sigma_\Lambda$ containing $g \cdot \partial\Sigma_P$ for each $g \in W_\Lambda$ and Item a) is satisfied. It remains to prove that a connected component $\hat{\kappa}$ of $\hat{\partial}_c\Sigma_\Lambda$ contains $W_P \cdot \alpha^+_{\tilde{s},\tilde{t}}(\infty)$.

We observe that it is sufficient to prove that $\{s \cdot \alpha^+_{\tilde{s},\tilde{t}}(\infty) \mid s \in V(P) \cup \{\mathrm{id}\}\}$ is contained in a connected component of $\hat{\partial}_c\Sigma_\Lambda$. Indeed, suppose that $\{s \cdot \alpha^+_{\tilde{s},\tilde{t}}(\infty) \mid s \in V(P) \cup \{\mathrm{id}\}\}$ is contained in a connected component of $\hat{\partial}_c\Sigma_\Lambda$. As W_P acts by homeomorphisms on $\hat{\partial}_c\Sigma_P$, $M_g \coloneqq \{gs \cdot \alpha^+_{\tilde{s},\tilde{t}}(\infty) \mid s \in V(P) \cup \{\mathrm{id}\}\}$ is contained in a connected component of $\hat{\partial}_c\Sigma_P$ for every $g \in W_P$. Suppose that $g, h \in W_P$ are two group element that are adjacent in the 1-skeleton of Σ_P. As $\{s \mid s \in V(P)\}$ generates W_P, $M_g \cap M_h \neq \emptyset$. Indeed, the 1-skeleton of Σ_P is the Cayley graph of $(W_P, V(P))$ and g and h are adjacent if and only if there exists $s \in V(P)$ such that $g = hs$ (or the other way around). Then M_g and M_h share $h \cdot \alpha^+_{\tilde{s},\tilde{t}}(\infty)$. It follows that there is a connected component that contains all sets

M_g, $g \in W_P$. Then $W_P \cdot \alpha^+_{\tilde{s},\tilde{t}}(\infty)$ is contained in a connected component of $\hat{\partial}_c\Sigma_\Lambda$.

It remains to prove that $\{s \cdot \alpha^+_{\tilde{s},\tilde{t}}(\infty) \mid s \in V(P) \cup \{\mathrm{id}\}\}$ is contained in a connected component of $\hat{\partial}_c\Sigma_\Lambda$. To achieve a contradiction, suppose that there exists a vertex s' in the 1-skeleton of Σ_P adjacent to the identity vertex id such that $\alpha^+_{\tilde{s},\tilde{t}}(\infty)$ and $s'\alpha^+_{\tilde{s},\tilde{t}}(\infty)$ lie in different connected components. Then s' is the label of the edge between id and s' in the one-skeleton of Σ_P. Recall that $\tilde{s}$ is an inner vertex of P. Thus, the pair of endvertices of P does not coincide with $\{s', \tilde{s}\}$. Furthermore, s' is not simultaneously adjacent to $\tilde{s}$ and s'. Indeed otherwise, the subgraph of P induced by s', $\tilde{s}$ and $\tilde{t}$ would be a path of length 2. The Davis complex of a 2-path is isometric to $[0,1] \times \mathbb{R}$ and $\alpha^+_{\tilde{s},\tilde{t}}(\infty)$ and $s'\alpha^+_{\tilde{s},\tilde{t}}(\infty)$ would be asymptotic – a contradiction. Without loss of generality let s' be not adjacent to $\tilde{s}$. We apply Item c) of Proposition 5.57 to the vertex pairs $\{\tilde{s}, \tilde{t}\}$ and $\{\tilde{s}, s'\}$. Let $\bar{\Lambda}$ be the graph induced by $\tilde{s}, \tilde{t}$ and s'. Then, $\hat{\partial}_c\Sigma_\Lambda$ has a connected component that contains $W_{\bar{\Lambda}} \cdot \alpha^+_{\tilde{s},\tilde{t}}(\infty)$ and $W_{\bar{\Lambda}} \cdot \alpha^+_{\tilde{s},s'}(\infty)$ for all $g \in W_{\bar{\Lambda}}$. In particular, $\alpha^+_{\tilde{s},\tilde{t}}(\infty)$ and $s'\alpha^+_{\tilde{s},\tilde{t}}(\infty)$ are contained in a common connected component – a contradiction. □

We study the consequences of the last theorem for the where W_P and $W_{\bar{P}}$ have totally disconnected contracting boundaries.

Corollary 5.59. *Let Λ be a graph that contains an independent path P whose endvertices are not adjacent. Let $\bar{P}$ be the graph obtained from Λ by deleting all inner vertices of P. Suppose that the contracting boundaries of $W_{\bar{P}}$ and W_P are totally disconnected. Then exactly one of the following is true*

a) The contracting boundary of W_Λ is totally disconnected or empty and the topological space $\hat{\partial}_c\Sigma_\Lambda$ is totally disconnected or empty.

b) The topological space $\hat{\partial}_c\Sigma_\Lambda$ has a connected component that contains a set bijective to the visual boundary of Σ_P.

Proof. By Theorem 5.58, there occur just two extreme cases. Suppose that Item b) of Theorem 5.58 is true. Then, $\hat{\partial}_c\Sigma_\Lambda$ and $\vec{\partial}_c\Sigma_\Lambda$ are empty or totally disconnected. Otherwise, Item a) of Theorem 5.58 is true. Then the topological space $\hat{\partial}_c\Sigma_\Lambda$ has a connected component that contains a set bijective to the visual boundary of Σ_P. □

The last statements show how the contracting boundary of a right-angled Coxeter group changes if we glue a path on its defining graph. We see that there are just two extreme cases. The reason for this dichotomy lies in the behavior of essential axes for rank-one isometries. They influence the structure of the contracting boundary of a right-angled Coxeter group a lot. In Section 4.4, we studied the role of essential axes for rank-one isometries for splittings along groups quasi-isometric to $\mathbb{Z}$ in general. The last result can be seen as an application of our considerations in Section 4.4. If we are in the situation of Corollary 5.59, either all connected components in $\vec{\partial}_c W_\Lambda$ and $\hat{\partial}_c\Sigma_\Lambda$ are single points or $\hat{\partial}_c\Sigma_\Lambda$ contains a large connected component κ. This connected component contains all equivalence classes of geodesic rays that are contained in the canonically embedded Davis complex Σ_P in Σ_Λ. Suppose that we are in the second case. It is an interesting question how large this connected component κ is. By our considerations in Section 4.4, it is a connected component of type 2. Associated to this connected component is a subtree $T_{\alpha_{s,t}}$ of the Bass-Serre tree $\mathcal{T}_{\text{ext}}$ associated to $W_\Lambda = W_{\bar{P}} *_{W_{\Lambda_{s,t}}} W_P$. See Definition 4.44. The larger this tree $T_{\alpha_{s,t}}$ is, the larger is the connected component κ. It would be interesting to understand how large this tree is and to answer Question 11 for the situation we consider in this section.

Recall that we are interested in the Burst-Cycle-Conjecture 5.5. In Corollary 5.39, we have seen that the Burst-Cycle-Conjecture 5.5 is true for join-decomposable graphs. We denoted the class of join-decomposable graphs by $\mathcal{J}$. See Definition 5.37. We are interested in the question whether there is a larger graph class satisfying the Burst-Cycle-Conjecture 5.5. We considered, therefore, the following situation at the end of Section 5.3. Let Λ' be a join-decomposable. Suppose that u and v are two vertices that are not adjacent and not contained in an induced 4-cycle. Assume further that every path in Λ' connecting u and v has a glued tetragon. We observed that every cycle in Λ is burst, i.e., not intact. By Theorem 5.58, there are only two extreme situations. In the situation of Item b) in Theorem 5.58, the contracting boundary of W_Λ is totally disconnected. We can add Λ to the graph class $\mathcal{J}$. If Λ is not join-decomposable, we obtain a larger graph class satisfying Conjecture 5.5. Otherwise, we are in the situation of Item a) in Theorem 5.58. Then $\hat{\partial}_c\Sigma_\Lambda$ has a connected component that contains the whole visual boundary of Σ_P. If this connected component is also a connected component of the contracting boundary of Σ_Λ, Λ is a counterexample to the Burst-Cycle-Conjecture 5.5. The question occurs whether such examples exist. Indeed, there are some. We will sketch such examples in Section 5.5.1 and Section 5.5.2 of Section 5.5. These examples are joint work with Graeber, Lazarovich and Stark.

5.5 Counterexamples to the conjecture

This section is an outlook. It is joint work with Graeber, Lazarovich and Stark. Our research leads to counterexamples to Conjecture 5.5 that was formulated by Tran in [Tra19, Conj. 1.14]. Recall that we refer to this conjecture as the Burst-Cycle-Conjecture. We sketch three counterexamples and explain resulting consequences. The first example in Section 5.5.1 was found by Graeber. The two other examples described in Section 5.5.3 and Section 5.5.3 were inspired by this first example.

First, we recall the Burst-Cycle-Conjecture 5.5. We call a cycle C in a graph *burst*, if one of the following three conditions is satisfied:

- C has length 3 or 4,
- C has a diagonal, i.e., two non-consecutive vertices of C are connected by an edge,
- the vertex set of C contains a pair of non-adjacent vertices of an induced 4-cycle.

A cycle is *intact*, if it is not burst.

The Burst-Cycle-Conjecture 5.5 says that every right-angled Coxeter group with defining graph Λ has totally disconnected contracting boundary if and only if every cycle in Λ is burst. We saw in Section 5.1 that one direction of this conjecture is true. If Λ contains an intact cycle, then the contracting boundary of W_Λ contains a 1-sphere. This was proven several times. See Theorem 5.7 for more details. Furthermore, it is known that the contracting boundary of a right-angled Coxeter group is empty if and only if its defining graph is a nontrivial join or a clique. See Theorem 5.6 for more details. Nguyen and Tran [NT19] showed, that each graph in the graph class $\mathcal{G}$, which is defined in Definition 5.14, correspond to a right-angled Coxeter group with totally disconnected contracting boundary. In Section 5.3, we extended this graph class to the graph class $\mathcal{J}$ defined in Definition 5.37. It remains to study the Burst-Cycle-Conjecture for graphs that are not contained in $\mathcal{J}$.

The following examples show that the Burst-Cycle-Conjecture is wrong in general and lead to a new conjecture about the contracting boundary of right-angled Coxeter groups. First, we sketch a 3-dimensional example, i.e., a right-angled Coxeter group whose defining graph Λ contains a cycle of length three and no clique with more than three vertices. We say that a cycle of length 3 is a *triangle*. Afterwards, we discuss two 2-dimensional examples, i.e., right-angled Coxeter groups whose defining graphs don't contain triangles. The contracting boundaries of these Coxeter groups contain 1-spheres. So, they are not totally disconnected. Their defining graphs satisfy different conditions. All of them don't contain an intact cycle.

We use the notation established in Chapter 2. We summarized our notation concerning boundaries in Notation 1.1. For concepts concerning proper separations of graphs, we use Notation 5.24. Recall that the Davis complex Σ_Δ of a graph Δ is the Davis complex of the Right-angled Coxeter group W_Δ. The set of boundary points of Σ_Δ is denoted by $\partial\Sigma_\Delta$ and its subset of contracting boundary points is denoted by $\partial_c\Sigma_\Delta$. The contracting boundary of W_Δ is denoted by $\vec{\partial}_c W_\Delta$. We calculate it by examining the contracting boundary of Σ_Δ, denoted by $\vec{\partial}_c\Sigma_\Delta$. Because the topology of the contracting boundary is finer than the topology of the visual boundary, it is interesting to understand the visual boundary of Σ_Δ, denoted by $\hat{\partial}\Sigma_\Delta$. In particular, it is useful studying the subspace of the visual boundary of Σ_Δ that consists of contracting boundary points, denoted by $\hat{\partial}_c\Sigma_\Delta$. If Λ is an induced subgraph of Δ, Σ_Δ contains an isometrically embedded copy of Σ_Λ that contains the identity vertex of Σ_Δ. In this case, we say that Σ_Λ is canonically embedded in Σ_Δ. Suppose that Σ_Λ is canonically embedded in the Davis complex of Δ. We think of boundaries of Σ_Λ as embedded in corresponding boundaries of Σ_Δ whenever possible. Note that this is not possible if we study contracting boundaries. Indeed, a geodesic ray γ in Σ_Λ might be contracting in Σ_Λ but not in the ambient Davis complex Σ_Δ. We say that $\gamma \subseteq \Sigma_\Lambda$ is Σ_Δ-contracting if it is contracting in the ambient Davis complex Σ_Δ and denote the set $\{\gamma(\infty) \in \partial_c\Sigma_\Delta \mid \gamma \subseteq \Sigma_\Lambda\}$ by $\partial_{c,\Sigma_\Delta}\Sigma_\Lambda$. If we equip $\partial_{c,\Sigma_\Delta}\Sigma_\Lambda$ with the subspace topology of the visual- and contracting boundary of Σ_Δ, we obtain the topological spaces $\hat{\partial}_{c,\Sigma_\Delta}\Sigma_\Lambda$ and $\vec{\partial}_{c,\Sigma_\Delta}\Sigma_\Lambda$ respectively. By Lemma 2.35, $\hat{\partial}_{c,\Sigma_\Delta}\Sigma_\Lambda$ and $\vec{\partial}_{c,\Sigma_\Delta}\Sigma_\Lambda$ are homeomorphic to the set $\{\gamma(\infty) \in \partial_c\Sigma_\Lambda \mid \gamma \text{ is } \Sigma_\Delta\text{-contracting}\}$ equipped with the subspace topology of the visual and contracting boundary of Σ_Λ respectively.

5.5.1 A three-dimensional counterexample

In this subsection, we consider a 3-dimensional counterexample to the Burst-Cycle-Conjecture 5.5 that was found by Graeber. It is the right-angled Coxeter group with the defining graph Δ as pictured in Figure 5.16. All cycles in Δ are burst. We sketch the

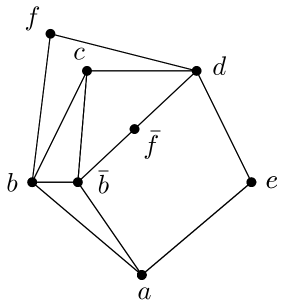

Figure 5.16 The graph Δ.

proof that the contracting boundary of the corresponding right-angled Coxeter group contains a 1-sphere. This proves that the Burst-Cycle-Conjecture 5.5 is wrong in general.

Before we sketch the proof, we study the graph Δ for getting a better understanding of the significance of the example. Recall that we found a graph class $\mathcal{J}$ in Corollary 5.38 whose corresponding right-angled Coxeter groups have totally disconnected or empty contracting boundaries. We called the graphs in this graph class join-decomposable. We observe that Δ is not join-decomposable. Recall that the class of join-decomposable graphs is defined recursively. A case-by-case analysis shows that every decomposition of Δ in induced subgraphs leads to a forbidden gluing, i.e., the decomposition contains a graph that is not contained in $\mathcal{J}$. For example, we study a decomposition of Δ in paths.

Definition 5.60. A graph Λ is *path-decomposable* if it contains induced subgraphs $\Lambda_0, \dots, \Lambda_k$ such that

- Λ_0 is a clique
- For all $i \in \{1, \dots, k\}$, Λ_i is obtained from Λ_{i-1} by identifying the endvertices of P with one vertex of Λ_{i-1} or Λ_i is obtained from Λ_{i-1} by identifying the endvertices of P with two distinct vertices of Λ_i,
- $\Lambda_k = \Lambda$.

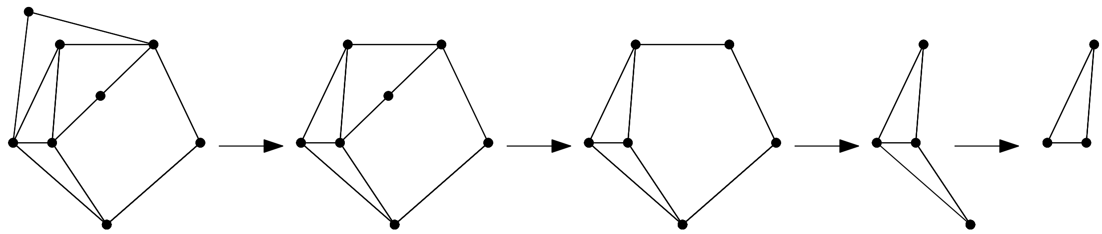

Figure 5.17 Delta can be obtained from a clique by successive gluings of paths.

The graph Δ is path-decomposable. Figure 5.17 shows a decomposition of Δ in paths. In this decomposition, the graph in the middle of Figure 5.17 is not contained in $\mathcal{J}$. We denote this graph by Λ. See Figure 5.18.

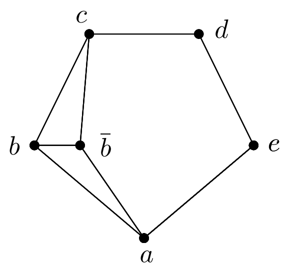

Figure 5.18 The graph Λ.

We observe that Λ is obtained from Δ by deleting two 2-paths. These two paths are contained in the only two 4-cycles of Δ. So, in the first two steps of the decomposition of Δ pictured in Figure 5.17, the two induced 4-cycles of Δ vanish. In particular, Λ does not contain any induced 4-cycle, and W_Λ is hyperbolic by Theorem 2.50. In the first two steps of the decomposition pictured in Figure 5.17, we can apply Theorem 5.32.

This way, we observe that all connected components of the contracting boundary of W_Δ are either single points or homeomorphic to a connected component that can be topologically embedded in the contracting boundary of W_Λ. We will see that such a connected component contains a whole 1-sphere.

Let us consider a decomposition of Δ into an induced subgraph and a path P as pictured in Figure 5.19. We observe that P is an independent path in Δ. The induced subgraph

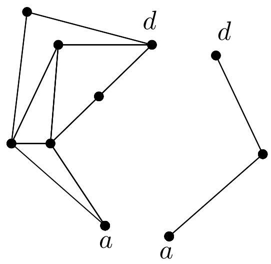

Figure 5.19 Decomposition of Δ into an induced subgraph and a path that connects vertex a and vertex d. Every path in the induced subgraph that links a and d contains two non-adjacent vertices that are contained in an induced 4-cycle.

obtained from Δ by deleting the inner vertices of P does not contain intact cycles and is join-decomposable. Thus, the contracting boundary of the corresponding right-angled Coxeter group is totally disconnected (it is not empty because there are two non-adjacent vertices that are not contained in an induced 4-cycle). The vertices a and d generate an infinite Dihedral group. The elements ad and da are axial isometries with an axis $\alpha_{a,d}$ intersecting the identity vertex of Σ_Δ. By Lemma 5.10, $\alpha_{a,d}$ is contracting in Σ_Δ. Let $\alpha^-_{a,d}(\infty)$ and $\alpha^+_{a,d}(\infty)$ be the corresponding boundary points. Recall that $\hat{\partial}_c\Sigma_\Delta$ denotes the subspace of the visual boundary of Σ_Δ that consists of all contracting boundary points. By Theorem 5.58, the contracting boundary of W_Δ is either totally disconnected, or $\alpha^-_{a,d}(\infty)$ and $\alpha^+_{a,d}(\infty)$ are contained in a common connected component of $\hat{\partial}_c\Sigma_\Delta$. We will see that we are in the second situation: $\hat{\partial}_c\Sigma_\Delta$ contains a connected component containing $\alpha^-_{a,d}(\infty)$ and $\alpha^+_{a,d}(\infty)$. Moreover, the contracting boundary of Σ_Δ contains a connected component containing $\alpha^-_{a,d}(\infty)$ and $\alpha^+_{a,d}(\infty)$ too. Indeed, both $\alpha^-_{a,d}(\infty)$ and $\alpha^+_{a,d}(\infty)$ are contained in a 1-sphere that is contained in $\vec{\partial}_c\Sigma_\Delta$. We remark that we can exchange the path P with a longer path, i.e., we can add some vertices on P without changing the situation. If P is longer, then the visual boundary of the Davis complex of this path is contained in a 1-sphere that is contained in the contracting boundary of Σ_Δ.

Having these considerations in mind, we sketch the proof that the contracting boundary of Σ_Δ contains a 1-sphere. This can be proven in the following steps. We define at first a space $\mathcal{H}$ that is quasi-isometric to the hyperbolic plane. Its contracting boundary is a 1-sphere. We embed $\mathcal{H}$ isometrically in the Davis complex Σ_Δ of Λ. Because Λ is an induced subgraph of Δ, Σ_Δ is isometrically embedded in the Davis complex Σ_Δ of Δ.

Thus, $\mathcal{H}$ is not only isometrically embedded in Σ_Δ but also in Σ_Δ. We prove that there exists $D > 0$ such that every geodesic ray in $\mathcal{H}$ is D-contracting in the Davis complex of Δ. This implies that the contracting boundary of $\mathcal{H}$ is topologically embedded in the contracting boundary of W_Δ. As $\mathcal{H}$ is quasi-isometric to the hyperbolic plane, the contracting boundary of $\mathcal{H}$ is a 1-sphere. Thus, the contracting boundary $\vec{\partial}_c W_\Delta$ contains a 1-sphere.

We define at first the space $\mathcal{H}$. For that purpose, we consider a certain subgroup of W_Λ. By means of Theorem 2.43 the following lemma can be proven.

Lemma 5.61. *The group elements a, $b\bar{b}$, c, d and e of W_Λ generate a right-angled Coxeter group with a 5-cycle K as defining graph.*

Let K be the cycle pictured in Figure 5.20.

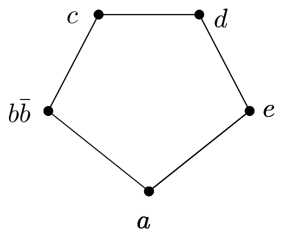

Figure 5.20 The cycle K corresponding to the subgroup of W_Λ generated by a, $b\bar{b}$, c, d and e .

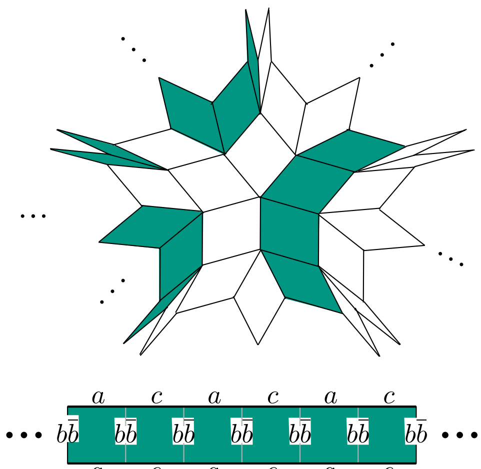

Figure 5.21 Above: the Davis complex of the 5-cycle K. The green strips are blocks isometric to the Davis complex of P_0. The white subcomplexes are blocks isometric to the Davis complex of P_1. Below: The Davis complex of P_0.

The Davis complex of K is quasi-isometric to the hyperbolic plane. See Figure 5.21. We obtain the space $\mathcal{H}$ by a transformation of Σ_K. The obtained space is quasi-isometric to Σ_K and therefore quasi-isometric to $\mathbb{H}$. For describing this transformation, we write W_K as amalgamated free product. Let A be the graph consisting of the vertices a and c. The vertices a and c are not adjacent and build a separating vertex pair of K, i.e., K decomposes into more than one connected component if we delete a and c. Let P_0 and P_1 be the corresponding paths as in Figure 5.22.

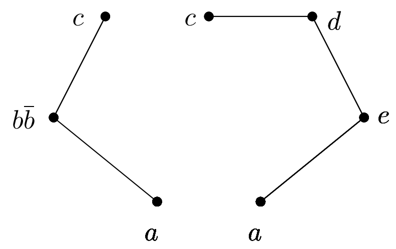

Figure 5.22 Left: path P_0. Right: path P_1

We write W_K as amalgamated product: $W_K = W_{P_0} *_{W_A} W_{P_1}$. By Proposition 5.28, Σ_K has a block decomposition with thin walls

$$(\mathcal{B}_{a,c}, \mathcal{A}_{a,c}) \coloneqq (\{g\Sigma_{P_0} \mid g \in W_K\} \cup \{g\Sigma_{P_1} \mid g \in W_K\},\ \{g\Sigma_A \mid g \in W_K\})$$

All blocks of parity $(-)$ and $(+)$ are of the form $g\Sigma_{P_0}$ and $g\Sigma_{P_1}$ respectively. The action of $W_K = W_{P_0} *_{W_A} W_{P_1}$ on Σ_K with this block decomposition satisfies all properties of Convention 3.78. So, for every $g \in W_K$, the blocks $g\Sigma_{\Lambda_0}$ and $g\Sigma_{\Lambda_1}$ are glued together along $g\Sigma_A$. The Davis complex of P_0 is a stripe consisting of 2-dimensional cubes. See Figure 5.23.

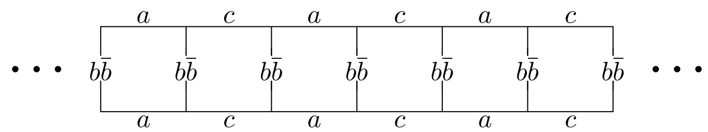

Figure 5.23 The Davis complex of P_0.

The Davis complex of P_1 is a 2-dimensional cube complex that has a form of a thickened tree of valence 4. See Figure 5.24. The Davis complex of Λ_* is a bi-infinite geodesic ray in the 1-skeleton of the Davis complex whose associated bi-infinite word is $\dots acacac\dots$. See Figure 5.25. For every coset gW_{P_i} in W_K/W_{P_i}, the Davis complex of W_K contains a block isometric to Σ_{P_i}, $i \in \{0,1\}$ and for every coset gW_A in W_K/W_A the Davis complex of W_K contains a wall isometric to Σ_A. The Davis complex of K consists of all these blocks and walls. Let $g \in W_K$. Every wall Σ_A with label gW_A is glued to the block $g\Sigma_{P_0}$ and to the block $g\Sigma_{P_1}$. This way we obtain the Davis complex of K as pictured in Figure 5.21.

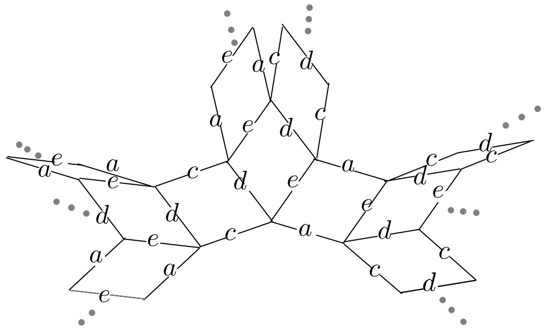

Figure 5.24 The Davis complex of P_1.

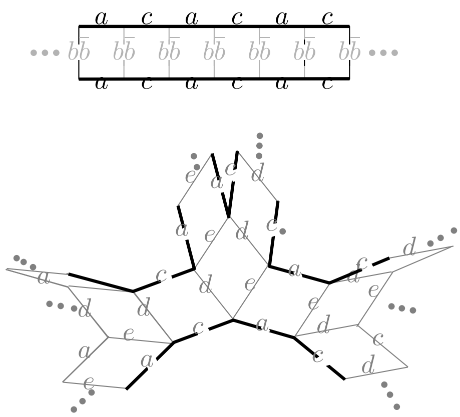

Figure 5.25 The black edges are contained in isometrically embedded copies of the Davis complex of A in the Davis complex of P_0 and P_1.

Let Σ_0' be the space obtained from Σ_{P_0} by scaling every cube of Σ_{P_0} to a filled tetrahedron such that the length of any edge labeled with a or c does not change and such that the edges labeled with $b\bar{b}$ are of length $\sqrt{2}$ afterwards. See Figure 5.26.

This way, every cube of the scaled Davis complex of P_0 can be isometrically embedded into a 3-dimensional cube of side lengths one as pictured in Figure 5.27.

Definition 5.62. Let $\mathcal{H}$ be the space obtained from Σ_K by transforming every block B isometric to Σ_{P_0} to a copy of Σ_0' by the procedure described above: We scale every cube C in B so that the angles stay the same as before and

- each edge labeled with a or c has length 1 afterwards and
- each edge labeled with $b\bar{b}$ has length $\sqrt{2}$ afterwards.

We observe that $\mathcal{H}$ has a block decomposition with thin walls where every block is isometric to the scaled Davis complex Σ_0' of P_0 or to the Davis complex of P_1.

Our goal is to explain how $\mathcal{H}$ can be embedded isometrically in Σ_Λ. For that purpose, we study the Davis complex of Λ as pictured in Figure 5.18. Let Λ_0, Λ_* and Λ_1 be the

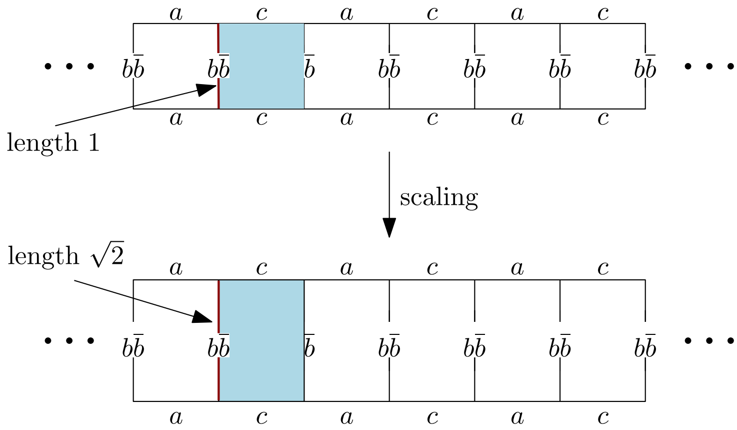

Figure 5.26 Above: Davis complex of P_0. Below: scaled Davis complex Σ_0' of P_0.

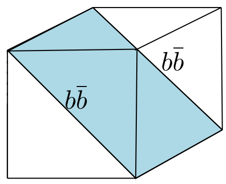

Figure 5.27 A filled tetrahedron in the scaled Davis complex of P_0 embedded in a 3-dimensional cube of side lengths one.

graphs in Figure 5.28. We observe that Λ_0, Λ_* and Λ_1 are induced subgraphs of Δ. We

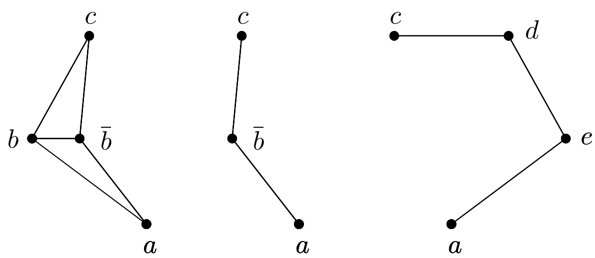

Figure 5.28 Left: The graph Λ_0. Middle: The path Λ_*. Right: The graph Λ_1.

write W_Λ as the amalgamated free product $W_\Lambda = W_{\Lambda_0} *_{\Lambda_*} W_{\Lambda_1}$. By Proposition 5.28, Σ_Λ has a block decomposition with thin walls

$$(\mathcal{B}, \mathcal{A}) := (\{g\Sigma_{\Lambda_0} \mid g \in W_\Lambda\} \cup \{g\Sigma_{\Lambda_1} \mid g \in W_\Lambda\},\ \{g\Sigma_{\Lambda_*} \mid g \in W_\Lambda\})$$

All blocks of parity $(-)$ and $(+)$ are of the form $g\Sigma_{\Lambda_0}$ and $g\Sigma_{\Lambda_1}$ respectively. The action of $W_\Lambda = W_{\Lambda_0} *_{W_{\Lambda_*}} W_{\Lambda_1}$ on Σ_Λ satisfies all properties of Convention 3.78. So, for every $g \in W_\Lambda$, the blocks $g\Sigma_{\Lambda_0}$ and $g\Sigma_{\Lambda_1}$ are glued together along $g\Sigma_{\Lambda_*}$. The Davis complex of

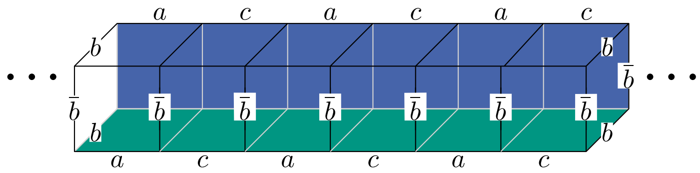

Figure 5.29 The Davis complex of Σ_{Λ_0}.

Λ_0 is a stripe consisting of 3-dimensional cubes. See Figure 5.29. The Davis complex of Λ_* is a stripe consisting of 2-dimensional cubes. It is isometric to the Davis complex of P_0 as pictured in Figure 5.23. The Davis complex of Λ_1 is a 2-dimensional cube complex that has a form of a thickened tree of valence 4. It is isometric to the Davis complex of P_1 as pictured in Figure 5.24. For every coset gW_{Λ_i} in W_K/W_{Λ_i}, the Davis complex of W_Λ contains a block isometric to Σ_{Λ_i}, $i \in \{0, 1\}$ and for every coset gW_{Λ_*} in W_Λ/W_{Λ_*} the Davis complex of W_Λ contains a wall isometric to Σ_{Λ_*}. The Davis complex of Λ consists of all these blocks and walls. Let $g \in W_\Lambda$. Every wall Σ_{Λ_*} with label gW_{Λ_*} is glued to the block $g\Sigma_{\Lambda_0}$ and to the block $g\Sigma_{\Lambda_1}$. This way, we obtain the Davis complex of Λ as pictured in Figure 5.30.

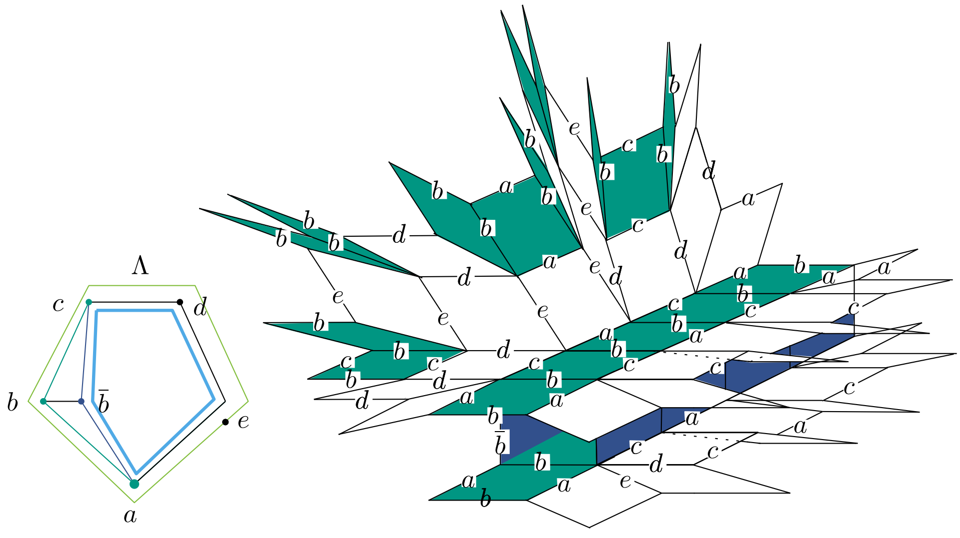

Figure 5.30 Left: the graph Λ. Right: block decomposition of the Davis complex of Λ. The white subcomplexes are blocks isometric to the Davis complex of Σ_{Λ_1}. The green strips are walls. The blue and green strips are isometrically embedded in blocks isometric to Σ_{Λ_0}.

We embed now the space $\mathcal{H}$ in Σ_Δ isometrically. Recall that $\mathcal{H}$ consists of blocks that each are isometric to Σ_0' or Σ_{P_1} as pictured in Figure 5.26 and Figure 5.24 respectively. The Davis complex of Σ_Δ consists of blocks isometric to Σ_{Λ_0} as pictured in Figure 5.29 and Σ_{Λ_1} isometric to Σ_{P_1} as pictured in Figure 5.24. Every Σ_{P_1}-block in $\mathcal{H}$ is isometric to the Σ_{Λ_1}-blocks in Σ_Δ. For all $g \in W_K \subseteq W_\Lambda$, we identify the block $g\Sigma_{P_1}$ in $\mathcal{H}$ with the block $g\Sigma_{\Lambda_1}$ in Σ_Δ. In addition, we embed the block $g\Sigma_0'$ of $\mathcal{H}$ in the block $g\Sigma_{\Lambda_0} \subseteq \Sigma_\Delta$ by identifying $g\Sigma_0'$ with the diagonal of the cube complex $g\Sigma_{\Lambda_0}$ as pictured in Figure 5.31. This way, $\mathcal{H}$ is isometrically embedded in Σ_Λ.

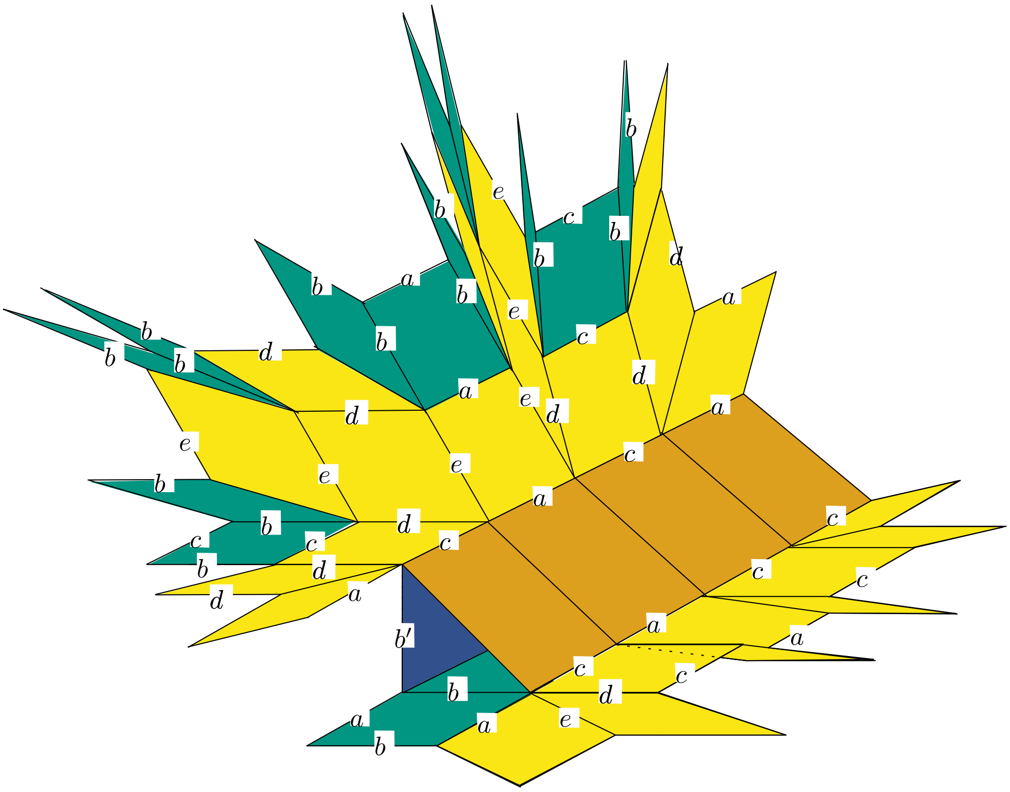

Figure 5.31 The orange space is the complex Σ_0' embedded in Σ_Δ. The space $\mathcal{H}$ contains the orange complex and the two yellow subcomplexes. These yellow subcomplexes are blocks isometric to Σ_{Λ_1}.

Lemma 5.63. *The subcomplex $\mathcal{H}$ is isometrically embedded in Σ_Λ.*

Because Λ is an induced subgraph of Δ, it follows that $\mathcal{H}$ is isometrically embedded in Σ_Δ.

Corollary 5.64. *The subcomplex $\mathcal{H}$ is isometrically embedded in Σ_Δ.*

It remains to show that there exists $D > 0$ such that all geodesic rays in the embedded complex $\mathcal{H}$ are D-contracting. Recall that $\mathcal{H}$ has a block decomposition. The blocks of $\mathcal{H}$ are isometrically embedded in the blocks of Σ_Λ that each are isometric to Σ_{Λ_0} or Σ_{Λ_1}. We observe that neither Λ_0 nor Λ_1 have two non-adjacent vertices that are contained in an induced 4-cycle of Δ. By Theorem 2.50, Σ_{Λ_0} and Σ_{Λ_1} are hyperbolic. So, $\mathcal{H}$ is contained in a subspace of Σ_Δ that consists of hyperbolic blocks. We observe further that Δ contains only two induced 4-cycles, namely b, c, d, f and $\bar{b}, c, d, \bar{f}$. By symmetrical reasons, it does not matter which of these two cycles we examine. Let Σ_{Δ_4}

be the canonically embedded Davis complex of the cycle b, c, d, f. We want to understand how Σ_{Δ_4} and $\mathcal{H}$ intersect. For that purpose, we consider the canonically embedded Davis complex of the path b, c, d. It is isometric to a stripe F as pictured in Figure 5.23. We observe that the stripe F is contained in the canonically embedded Davis complex Σ_{Λ_5} of the 5-cycle a, b, c, d, e. The space $\mathcal{H}$ is isometrically embedded in Σ_Δ such that $\mathcal{H}$ only shares with Σ_{Λ_5} the canonically embedded Davis complex Σ_{Λ_1} of Λ_1. The Strip F only shares with Σ_{Λ_1} one two-dimensional cube. Hence, $\mathcal{H}$ only has one two-dimensional cube C with F in common. See Figure 5.32. The violet lines mark copies of the stripe F. The two adjacent cubes of C in F are contained in blocks isometric to Σ_{Λ_0}. The space $\mathcal{H}$ shares a tetrahedron with these two blocks and this tetrahedron is embedded in the diagonal of a cube. This leads to the observation that $\mathcal{H}$ only shares three hyperplanes with Σ_{Δ_4}.

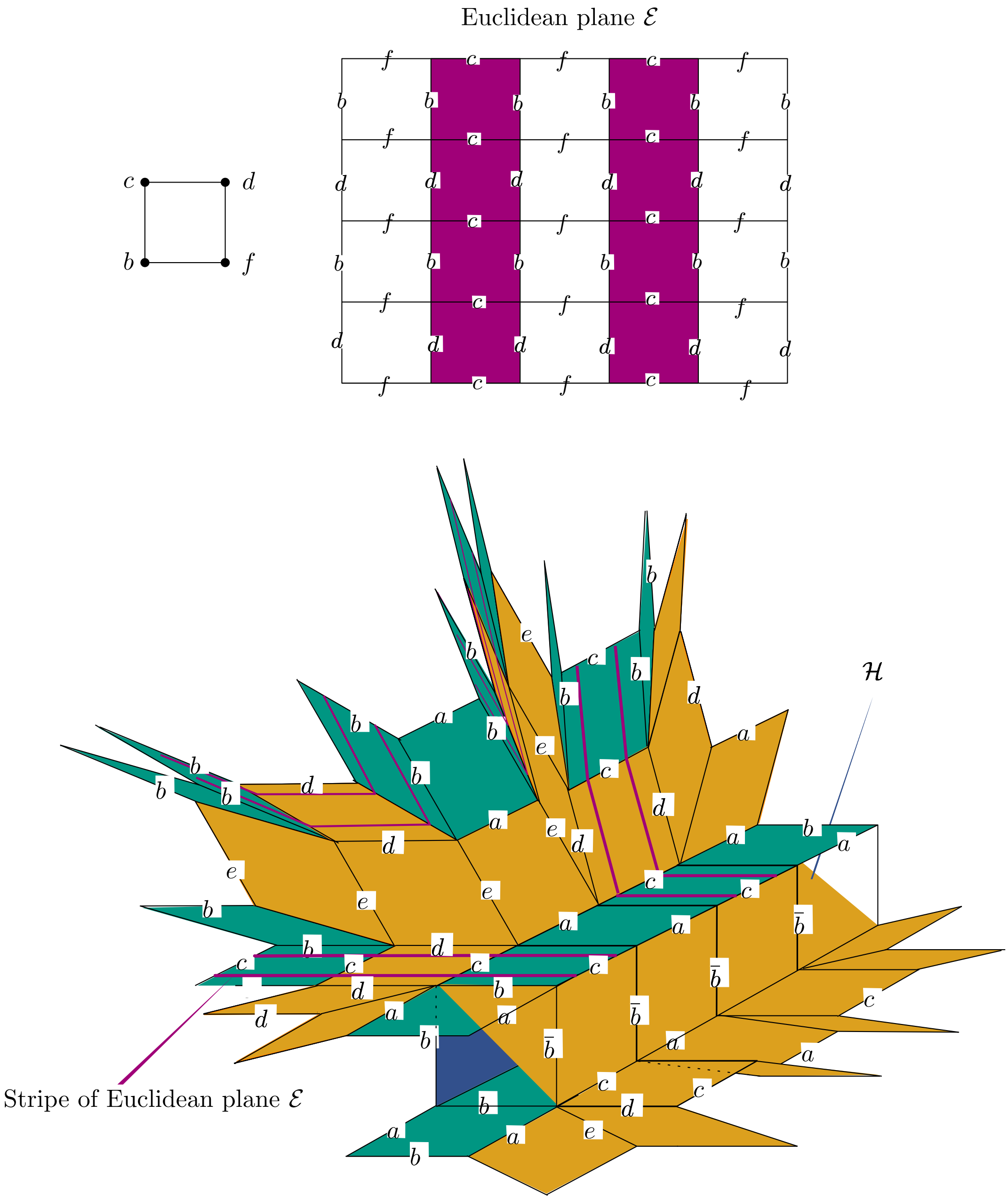

Figure 5.32 Above: a 4-cycle with its Davis complex. Bottom: embedded space $\mathcal{H}$ (orange) in Σ_Δ. The Euclidean plane $\mathcal{E}$ denotes the canonically embedded Davis complex of the pictured 4-cycle. The violet edges mark the intersection of $\mathcal{E}$ and translates of $\mathcal{E}$ with pictured cubes.

This motivates the following lemma.

Lemma 5.65. *Let M be an isometrically embedded Davis complex in W_Δ whose defining graph is an induced* 4*-cycle of* Δ*. There are at most* 3 *hyperplanes in* Σ_Δ *intersecting* M *and* $\mathcal{H}$*.*

This means that no Euclidean plane in Σ_Δ coming from a 4-cycle in Δ affects the contracting boundary of $\mathcal{H}$. Using the characterization of contracting geodesic rays by Charney and Sultan as stated in Theorem 5.19, it follows that there exists $D > 0$ such that every geodesic ray in $\mathcal{H}$ is D-contracting.

Corollary 5.66. *There exists $D > 0$ such that every geodesic ray in $\mathcal{H}$ is D-contracting*

This has the consequence, that the contracting boundary of W_Δ contains a 1-sphere.

Theorem 5.67. *The contracting boundary of W_Δ contains a* 1*-sphere.*

We remark that Δ is path-decomposable, i.e. not all path-decomposable graphs satisfy the Burst-Cycle-Conjecture 5.5.

Corollary 5.68. *There is a path-decomposable graph that does not satisfy the Burst-Cycle-Conjecture 5.5.*

5.5.2 A two-dimensional counterexample

In this subsection, we consider a counterexample to the Burst-Cycle-Conjecture 5.5 that is two-dimensional, i.e., the defining graph does not contain a triangle. The defining graph of this example is pictured in Figure 5.33 and denoted by Δ'. All cycles in Δ' are burst. We sketch the proof that the contracting boundary of the corresponding right-angled Coxeter group contains a 1-sphere.

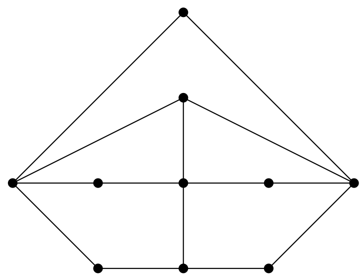

Figure 5.33 The graph Δ'.

The graph Δ' is obtained from the 3-dimensional example studied in the last subsection by *doubling along a vertex*. Let Δ be the defining graph of the three-dimensional counterexample pictured in Figure 5.16. We double Δ along b: at first, we take two copies of Δ. Let S be the star of b in both copies of Δ, i.e., the graph consisting of b and all edges incident to b. We identify these two stars. Afterwards, we delete vertex b and all of its incident edges. The resulting graph is Δ'. See Figure 5.34.

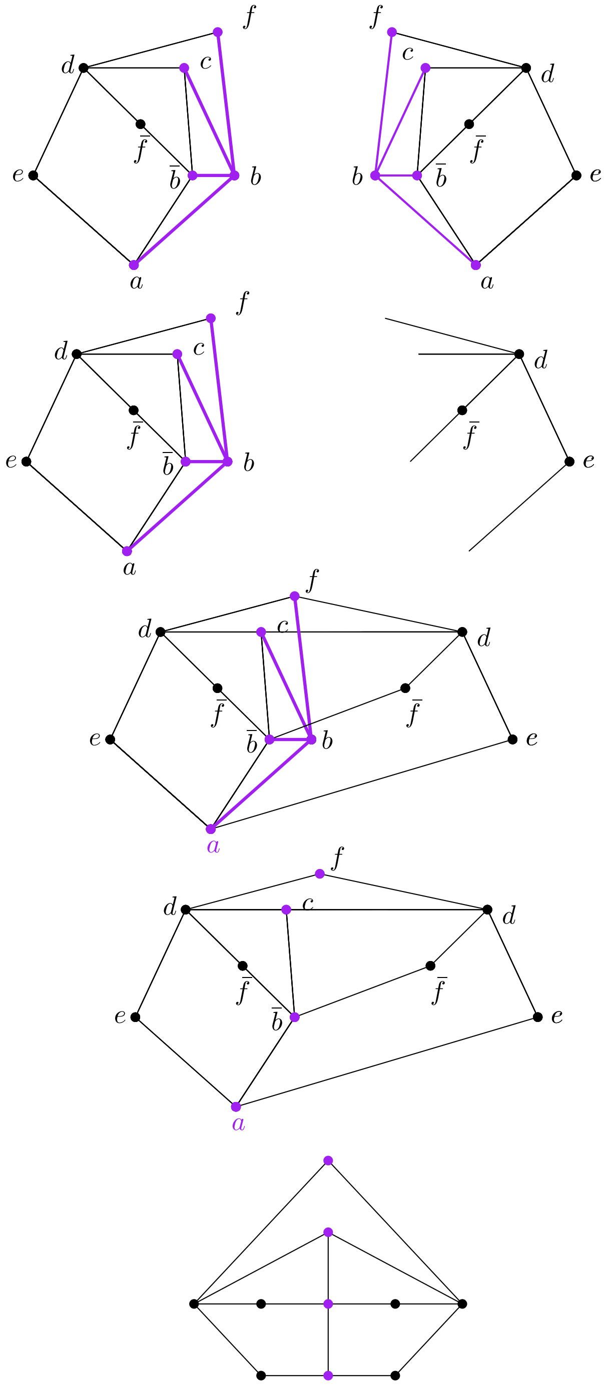

Figure 5.34 Double of Δ along vertex b. First row: two copies of Δ. Second and third row: identification of the star of b in both copies of Δ. Forth row: deletion of vertex b and all edges incident to b. Last row: the resulting graph Δ'.

Because Δ' is obtained from Δ by doubling along a vertex, $W_{\Delta'}$ is a subgroup of W_Δ of index two. That was proven by Dani and Thomas in [DT17, Sec.2, Version 1 on arXiv]. We repeat the proof. Let $\Phi : W_\Delta \to \mathbb{Z}/2\mathbb{Z}$ be the homeomorphism sending b to the generator of $\mathbb{Z}/2\mathbb{Z}$ and a, $\bar{b}$, c, d and e to the identity of $\mathbb{Z}/2\mathbb{Z}$. Then the kernel $\ker(\Phi)$ of Φ is generated by the elements $\{s, bsb \mid s \in \{a, \bar{b}, c, d, e\}\}$. We observe that, $\ker(\Phi)$ is isomorphic to $W_{\Delta'}$. Because $\ker(\Phi)$ is an index 2 subgroup, the Davis complex of $W_{\Delta'}$ and W_Δ are quasi-isometric. Hence, their contracting boundaries coincide. Because the contracting boundary of W_Δ contains a 1-sphere, $W_{\Delta'}$ contains a 1-sphere as well.

Theorem 5.69. *The contracting boundary of $W_{\Delta'}$ contains a* 1*-sphere.*

For understanding the significance of this counterexample, we study the graph Δ'. We observe first that Δ' does not contain triangles. Each cycle of Δ' is burst. Furthermore, Δ' is path-decomposable. In Figure 5.35, a decomposition of Δ' is pictured. Thus,

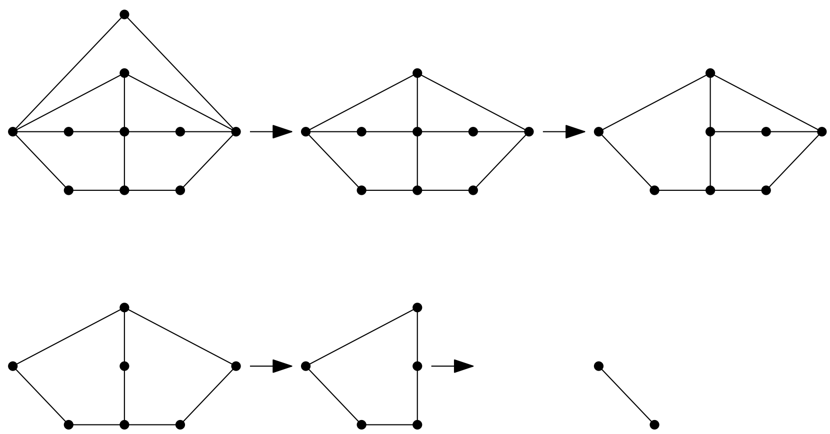

Figure 5.35 The graph Δ' is path-decomposable, because it can be obtained by starting with a clique (left above) and adding paths of length at least 2 successively.

not all triangle-free path-decomposable graphs satisfy the Burst-Cycle-Conjecture 5.5. Furthermore, Δ' is planar. We remark that this implies that $W_{\Delta'}$ acts properly on a contractible 3-manifold [DO01].

Corollary 5.70. *There is a planar, triangle-free, path-decomposable graph that does not satisfy the Burst-Cycle-Conjecture 5.5.*

We continue our study of Δ' using our considerations in Section 5.3 and Section 5.4. Recall that we have proven in Corollary 5.38 that the contracting boundary of every join-decomposable graph is totally disconnected. The graph class $\mathcal{J}$ of join-decomposable graphs is defined recursively. Similar to the 3-dimensional example, a case-by-case analysis shows that Δ' is not join-decomposable. For example, in the decomposition pictured in Figure 5.35, the left bottom graph is not join-decomposable. We denote this graph by Λ'. See Figure 5.36. By studying the decomposition in Figure 5.35 we observe

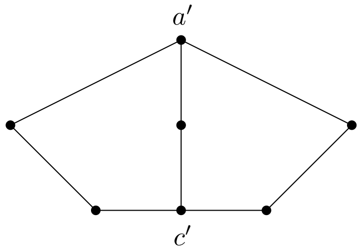

Figure 5.36 The graph Λ'.

that Λ' can be obtained from Δ' by deleting three 2-paths. The deletion of every such 2-path implies that an induced 4-cycle of Δ' vanishes. Each time deleting one of these three two-paths, we can apply Theorem 5.32. We conclude that every connected component of the contracting boundary of $W_{\Delta'}$ consists of a point or can be topologically embedded in the contracting boundary of $W_{\Lambda'}$. Corollary 5.70 shows that one such connected component contains a 1-sphere. The graph Λ' does not contain any induced 4-cycle. By Theorem 2.50, $W_{\Lambda'}$ is hyperbolic. We remark that Λ' is a θ-graph, see [DST18].

We come back to the graph Δ' and consider the decomposition of Δ' into an induced subgraph and a path P pictured in Figure 5.37. We observe that P is an independent path

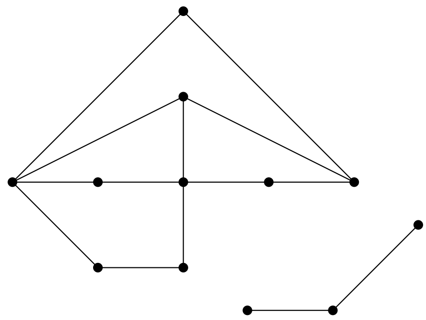

Figure 5.37 Decomposition of Δ' into an induced subgraph and a path P. Every path in the induced subgraph that links the endvertices of P contains two non-adjacent vertices that are contained in an induced 4-cycle.

in Δ. The induced subgraph obtained from Δ by deleting the inner vertices of P does not contain intact cycles. Furthermore, it is join-decomposable, i.e., the contracting boundary of the corresponding right-angled Coxeter group is totally disconnected. (it is not empty because two non-adjacent vertices are not contained in an induced 4-cycle). Let Σ_P be the canonically embedded Davis complex of P. Recall that $\hat{\partial}_c\Sigma_{\Delta'}$ denotes the subspace of the visual boundary of Σ_Δ consisting of all contracting boundary points. As in the 3-dimensional example, we observe that Σ_P has two boundary points, apply Theorem 5.58 and conclude that either the contracting boundary of $W_{\Delta'}$ is totally disconnected or there exists a connected component in $\hat{\partial}_c\Sigma_{\Delta'}$ that contains both boundary points of Σ_P simultaneously. We will see that we are in the second situation. Moreover, not only

$\hat{\partial}_c\Sigma_{\Delta'}$, but also the contracting boundary $\vec{\partial}_c\Sigma_{\Delta'}$ has a connected component containing both boundary points. Both boundary points are contained in a 1-sphere.

We remark that we can exchange the path P with a longer path and are still in the same situation. If P is longer, then the visual boundary of the Davis complex of this path is contained in a 1-sphere that is contained in the contracting boundary of $\Sigma_{\Delta'}$.

In the following, we study the question of where we find a 1-sphere in the contracting boundary of $\Sigma_{\Delta'}$. we will see similarly as in the first counterexample that $\Sigma_{\Lambda'}$ contains a space $\mathcal{H}'$ quasi-isometric to the hyperbolic plane $\mathbb{H}$, and the contracting boundary of $\Sigma_{\Delta'}$ contains the boundary of this space. This time, the space $\mathcal{H}'$ is a subcomplex of $\Sigma_{\Delta'}$. It is isometric to the Davis complex of a 5-cycle. We proceed similar as in the first counterexample in Section 5.5.1. We concentrate on the induced subgraph Λ' of Δ'. We define an isometrically embedded subcomplex $\mathcal{H}'$ in the Davis complex $\Sigma_{\Lambda'}$ of Λ'. Because Λ' is an induced subgraph of Δ', $\Sigma_{\Lambda'}$ is isometrically embedded in the Davis complex $\Sigma_{\Delta'}$ of Δ'. Thus, $\mathcal{H}'$ is not only isometrically embedded in $\Sigma_{\Lambda'}$ but also in $\Sigma_{\Delta'}$. We prove that there exists $D > 0$ such that every geodesic ray in $\mathcal{H}'$ is D-contracting in the Davis complex of Δ'. This implies that the contracting boundary of $\mathcal{H}'$ is topologically embedded in the contracting boundary of $W_{\Delta'}$. As $\mathcal{H}'$ is quasi-isometric to the hyperbolic plane, the contracting boundary of $\mathcal{H}'$ is a 1-sphere. Thus, the contracting boundary $\vec{\partial}_c W_{\Delta'}$ contains a 1-sphere.

Our goal is to define the subcomplex $\mathcal{H}'$ of $\Sigma_{\Lambda'}$. For that purpose, we study the Davis complex of Λ'. Let Λ'_0, Λ'_1, P'_0, P'_1 and P'_* be the graphs pictured in Figure 5.38. We

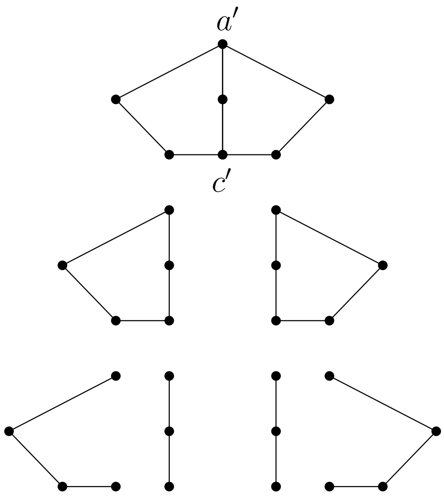

Figure 5.38 First row: The graph Λ'. Second row from left and right: The graph Λ'_1 and Λ'_2. Third row from left to right: The graph P'_0, P'_*, P'_* and P'_1.

write $W_{\Lambda'}$ as the amalgamated free product $W_{\Lambda'} = W_{\Lambda'_0} *_{\Lambda'_*} W_{\Lambda'_1}$. By Proposition 5.28, $\Sigma_{\Lambda'}$ has a block decomposition with thin walls

$$(\mathcal{B}', \mathcal{A}') := (\{g\Sigma_{\Lambda'_0} \mid g \in W_{\Lambda'}\} \cup \{g\Sigma_{\Lambda'_1} \mid g \in W_{\Lambda'}\},\ \{g\Sigma_{P'_*} \mid g \in W_{\Lambda'}\}).$$

All blocks of parity $(-)$ and $(+)$ are of the form $g\Sigma_{\Lambda'_0}$ and $g\Sigma_{\Lambda'_1}$ respectively. The action of $W_{\Lambda'} = W_{\Lambda'_0} *_{W_{P'_*}} W_{\Lambda'_1}$ on $\Sigma_{\Lambda'}$ satisfies all properties of Convention 3.78. So, for every $g \in W_{\Lambda'}$, the blocks $g\Sigma_{\Lambda'_0}$ and $g\Sigma_{\Lambda'_1}$ are glued together along $g\Sigma_{P'_*}$. The graphs Λ'_0 and Λ'_1 both are cycles of length 5 and their Davis complexes are 2-dimensional cube complexes as pictured in Figure 5.21. They are quasi-isometric to the hyperbolic plane. The Davis complex of P'_* is a stripe consisting of 2-dimensional cubes. It is isometric to the Davis complex of P_0 as pictured in Figure 5.23. For every coset $gW_{\Lambda'_i}$ in $W_{\Lambda'}/W_{\Lambda'_i}$, the Davis complex of $W_{\Lambda'}$ contains a block isometric to $\Sigma_{\Lambda'_i}$, $i \in \{0, 1\}$ and for every coset $gW_{P'_*}$ in $W_{\Lambda'}/W_{P'_*}$ the Davis complex of $W_{\Lambda'}$ contains a wall isometric to $\Sigma_{P'_*}$. The Davis complex of Λ' consists of all these blocks and walls. Let $g \in W_{\Lambda'}$. Every wall $\Sigma_{P'_*}$ with label gW_{Λ_*} is glued to the block $g\Sigma_{\Lambda'_0}$ and to the block $g\Sigma_{\Lambda'_1}$. This way we obtain the Davis complex of Λ'. In Figure 5.39, two blocks of $\Sigma_{\Lambda'}$ are pictured. In Figure 5.40, a section of the subcomplex $\mathcal{H}'$ contained in $\Sigma_{\Lambda'}$ is pictured.

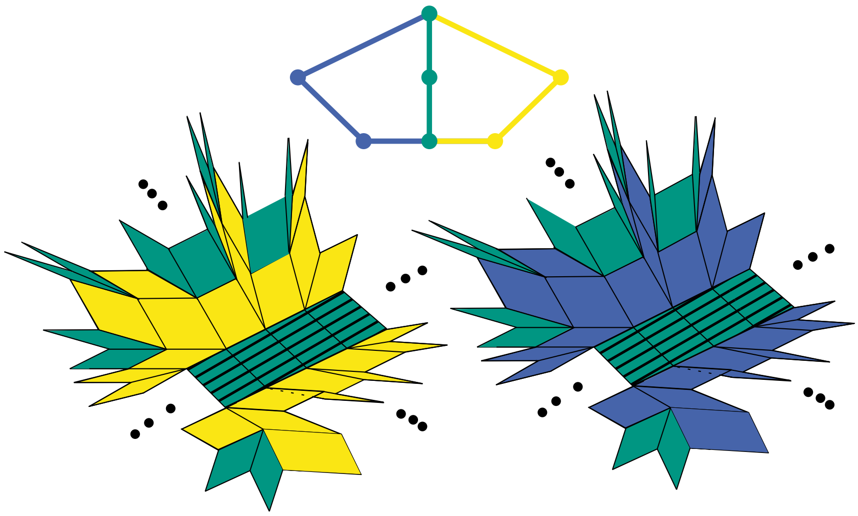

Figure 5.39 Above: the graph Λ'. Second row: Two blocks of distinct parity in the Davis complex of Λ'. Both blocks are isometric to a Davis complex of a 5-cycle in Λ'. Left: The block corresponds to the 5-cycle in Λ' that consists of the yellow and the green path in Λ'. Right: The block corresponds to the 5-cycle in Λ' that consists of the blue and the green path. Both blocks contain the same dashed green strip.

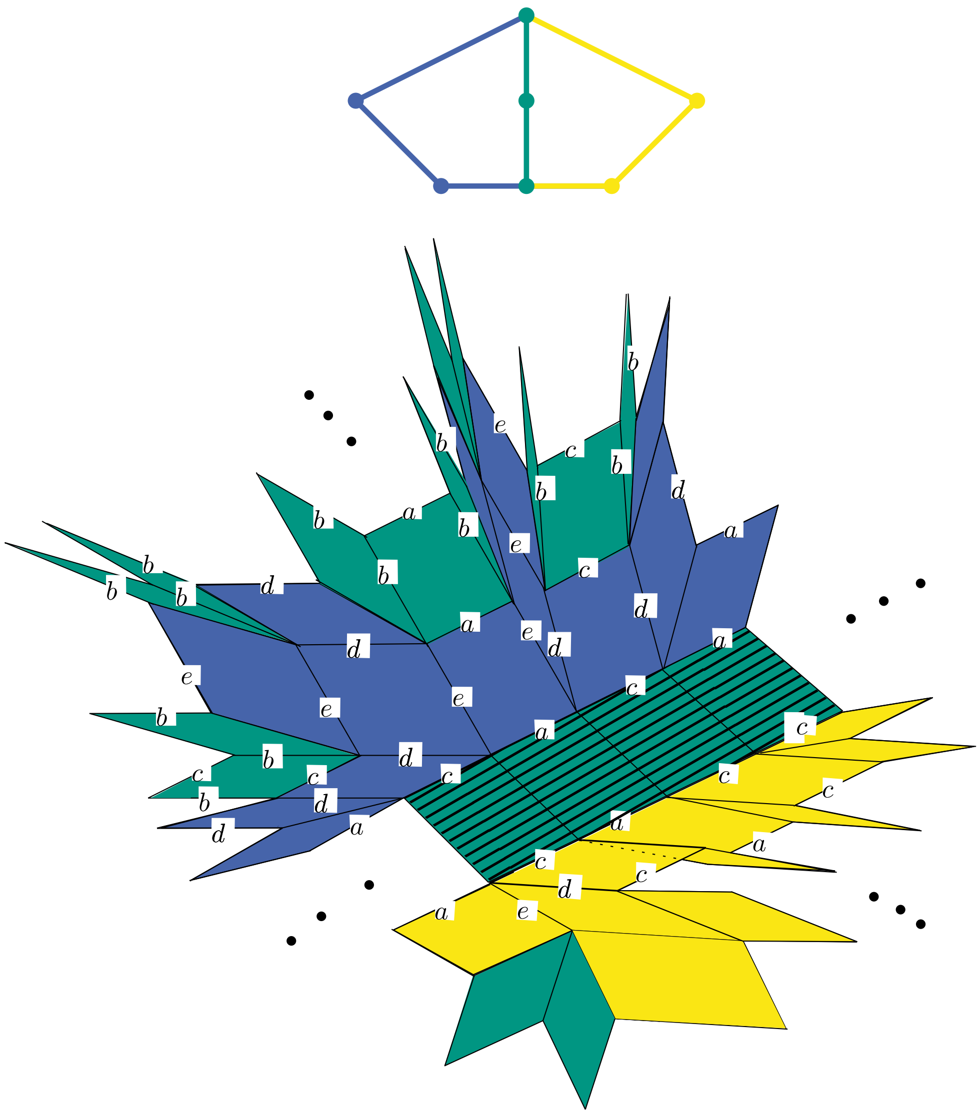

Figure 5.40 Above: the graph Λ'. Second row: a section of the subcomplex $\mathcal{H}'$ contained in $\Sigma_{\Lambda'}$.

We study the blocks of the described block decomposition $(\mathcal{B}', \mathcal{A}')$ and observe that each such block has itself a block decomposition. Indeed, the blocks of $(\mathcal{B}', \mathcal{A}')$ are isometric to a Davis complex of a 5-cycle. In the first counterexample in Section 5.5.1 we have seen a block decomposition of such a Davis complex. We repeat this block decomposition. Let $i \in \{0, 1\}$. We study the Davis complex of Λ'_i. Let a' and c' be the vertices pictured in Figure 5.36. Let A' be the graph consisting of the vertices a' and c'. The vertices a' and c' are not adjacent and build a separating vertex pair of the graph Λ' . Let P'_i and P'_* be the corresponding paths as in Figure 5.38. We write $W_{\Lambda'_i}$ as the amalgamated product $W_{\Lambda'_i} = W_{P'_i} *_{W_{A'}} W_{P'_*}$. By Proposition 5.28, $\Sigma_{\Lambda'_i}$ has a corresponding block decomposition with thin walls. This block decomposition is the same as the block decomposition of Σ_K described in the first counterexample in Section 5.5.1. The Davis complex of P'_* is a stripe consisting of 2-dimensional cubes. It is isometric to the Davis complex of P_0 pictured in See Figure 5.23. The Davis complex of P_i is a 2-dimensional cube complex that has a form of a thickened tree of valence 4. It is isometric to the Davis complex of P_1 as pictured in Figure 5.24. The Davis complex of A' is a bi-infinite geodesic ray in the 1-skeleton of the Davis complex whose associated bi-infinite word is $\ldots a'c'a'c'a'c' \ldots$. See Figure 5.25. For every $g \in W_{\Lambda'_i}$, the blocks $g\Sigma_{\Lambda'_0}$ and $g\Sigma_{\Lambda'_1}$ are glued together along $g\Sigma_{A'}$.

Note that P'_0, P'_1 and P'_* are induced subgraphs of Λ'. Hence, if B is a block in $\{\mathcal{B}', \mathcal{A}'\}$, then every block of B is isometrically embedded in the Davis complex of Λ'. These blocks correspond to the green, yellow and blue subcomplexes as pictured in in Figure 5.39.

Now we define the subcomplex $\mathcal{H}'$ of $\Sigma_{\Lambda'}$. Recall that it is isometric to a Davis complex of a 5-cycle. We explain the idea by means of Figure 5.39 and Figure 5.40. First, we choose a block B isometric to $\Sigma_{\Lambda'_0}$ in $(\mathcal{B}', \mathcal{A}')$. Suppose that this block is the subcomplex quasi-isometric to the hyperbolic plane pictured in the second row on the right-hand side in Figure 5.39. So, B consists of green strips and blue thickened trees. We choose one blue thickened tree T contained in B. This subcomplex T is adjacent to many green subcomplexes. We choose all these adjacent green strips and add them to our complex. Each such green stripe F is adjacent to another block B' of $\Sigma_{\Lambda'}$. The block B' contains a thickened yellow tree that is adjacent to F. We add this yellow three to our complex. In B', a lot of green strips are adjacent to this yellow tree. We add all these stripes to our complex. Each such green stripe F is adjacent to another block B'' of $\Sigma_{\Lambda'}$. The block B'' contains a thickened blue tree that is adjacent to F. We add this blue tree to our complex. We continue in this manner and obtain a subcomplex isometric to a Davis complex of a 5-cycle. This is the subcomplex $\mathcal{H}'$. In the second row of Figure 5.40, a section of $\mathcal{H}'$ is pictured. We explain this procedure formally. We choose a block isometric to $\Sigma_{\Lambda'_0}$ in $\{\mathcal{B}', \mathcal{A}'\}$. In this block we choose a block C_0 isometric to $\Sigma_{P'_0}$ and we choose all blocks isometric to $\Sigma_{P'_*}$ that are adjacent to C_0. Let $\mathcal{H}'_0$ be the subcomplex consisting of these blocks. Let $j = (i \mod 2)$. In the i^{th} step, we choose for every block B_* isometric to $\Sigma_{P'_*}$ in $\mathcal{H}'_{i-1} \setminus \mathcal{H}'_{i-2}$ a block B^j_* in $\{\mathcal{B}', \mathcal{A}'\}$ isometric to $\Sigma_{\Lambda'_j}$ which is adjacent to $\mathcal{H}'_{i-1}$. Then B^j_* contains a block C^j_* isometric to $\Sigma_{P'_j}$ which is adjacent to $\mathcal{H}'_{i-1}$. We add this block to $\mathcal{H}'_{i-1}$. Afterwards we add all blocks of B^j_* to $\mathcal{H}'_{i-1}$ that are isometric to $\Sigma_{P'_*}$ and adjacent to C^j_* . We add this block to the complex. The obtained complex is $\mathcal{H}'_i$. The complex $\mathcal{H}'$ is the union of all complexes $\mathcal{H}'_i$, $i \in \mathbb{N}$. By construction, $\mathcal{H}'$ is isometric to a Davis complex of a 5-cycle.

Remark 5.71. It is possible to vary the definition of $\mathcal{H}'$. We can choose an arbitrary number n_g for every coset $gW_{\Lambda'_i}$ in $W_{\Lambda'}$ and can define $\mathcal{H}'$ so that it shares either 0 or n_g blocks isometric to $\Sigma_{P'_i}$ with the block $g\Sigma_{\Lambda_i}$.

We observe that $\mathcal{H}'$ consists of blocks isometric to the Davis complexes of P_0', P_1' and P_*'. We observe that none of these three paths contains two non-adjacent vertices that are contained in an induced 4-cycle. By Theorem 2.50, each of these Davis complexes is hyperbolic. By Theorem 5.7, all geodesic rays in a block isometric to the Davis complex of such a path is contracting in $\Sigma_{\Delta'}$. Accordingly, $\mathcal{H}'$ consists of hyperbolic blocks whose geodesic rays are contracting in the ambient complex. One observes similar as in the three-dimensional example that the induced 4-cycles in Δ' glued on Λ' share at most two hyperplanes with $\mathcal{H}'$.

Lemma 5.72. *Let M be an isometrically embedded Davis complex in $\Sigma_{\Delta'}$ whose defining graph is an induced 4-cycle of Δ'. There are at most 2 hyperplanes in $\Sigma_{\Delta'}$ intersecting M and $\mathcal{H}'$.*

Thus, no Euclidean plane coming from an induced 4-cycle in Δ' affects $\mathcal{H}'$. Using the characterization of contracting geodesic rays by Charney and Sultan as stated in Theorem 5.19, it follows that there exists $D > 0$ such that every geodesic ray in $\mathcal{H}'$ is D-contracting.

Corollary 5.73. *There exists $D > 0$ such that every geodesic ray in $\mathcal{H}'$ is D-contracting.*

Corollary 5.74. *The contracting boundary of $\mathcal{H}'$ is topologically embedded in the contracting boundary of $W_{\Delta'}$.*

The example shows that not all 2-dimensional path-decomposable graphs satisfy the Burst-Cycle-Conjecture 5.5. We want to understand, why. For that purpose, we consider the following example.

Example 5.75. Recall that the subcomplex $\mathcal{H}'$ of $\Sigma_{\Delta'}$ consists of blocks isometric to $\Sigma_{P_i'}$, $i \in \{0, 1\}$ and $\Sigma_{P_*'}$ in $\Sigma_{\Delta'}$. Recall that no two non-adjacent vertices of the paths P_0', P_1' and P_*' are contained in an induced 4-cycle. By Theorem 5.7, all geodesic rays in a block isometric to the Davis complex of such a path are contracting in $\Sigma_{\Delta'}$. Let P be a path of length two. If we glue the endvertices of P to two non-adjacent vertices of P_0', the path P_0' has a glued tetragon. We glue P on the path P_0' as in Figure 5.41.

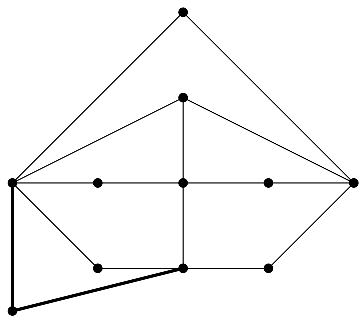

Figure 5.41 The graph Δ' with a glued two-path. The glued two-path is thickened.

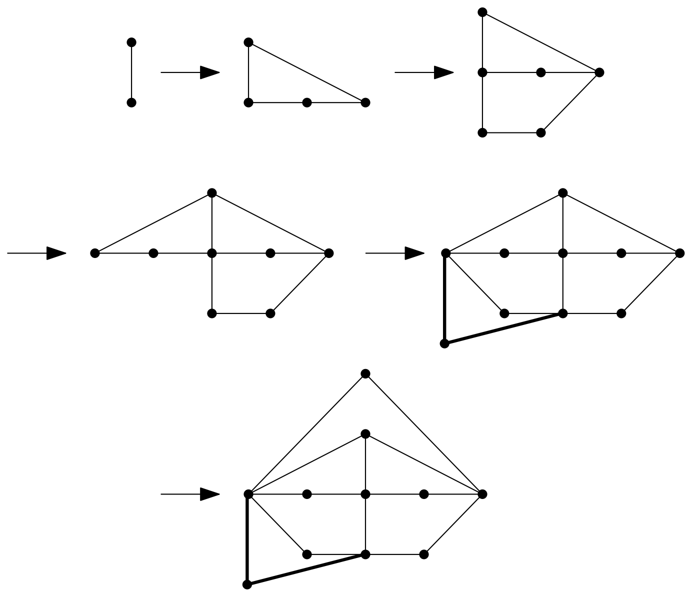

Figure 5.42 Decomposition of $\Delta' \cup P$. The path P is thickened. This decomposition shows that $\Delta' \cup P$ is join-decomposable.

The decomposition of $P \cup \Delta'$ pictured in Figure 5.42 shows that $P \cup \Delta'$ is join-decomposable. Thus, the right-angled Coxeter group with $P \cup \Delta'$ as defining graph has totally disconnected contracting boundary by Corollary 5.38. So, the gluing of the path P 'destroys' the sphere that was in the contracting boundary of $W_{\Delta'}$. The question arises: Does the Burst-Cycle-Conjecture 5.5 become true if we forbid the existence of such paths P'_0, P'_1 and P'_*? We consider this question in the next subsection.

Remark 5.76*.* Let us consider the graph $\bar{\Delta}'$ obtained from Δ' by adding vertices to the blue edge as pictured in Figure 5.43. The same argumentation as for the graph Δ' shows

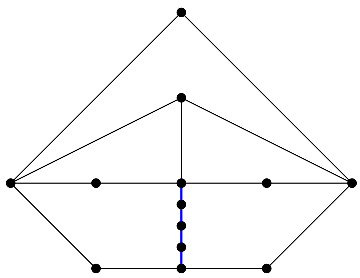

Figure 5.43 The pictured graph is obtained from Δ' by adding vertices to the blue edge. The contracting boundary of the corresponding right-angled Coxeter group contains a 1-sphere.

that the contracting boundary of the right-angled Coxeter group with defining graph $\bar{\Delta}'$ contains a 1-sphere.

5.5.3 A second two-dimensional counterexample

In this subsection, we consider a second 2-dimensional counterexample to a variant of the Burst-Cycle-Conjecture. This variant is more restrictive than the original conjecture. The defining graph Δ'' of the counterexample is *totally burst*, i.e., Δ'' is neither a clique nor a nontrivial join, does not contain any intact cycle and satisfies an additional condition. We sketch the proof that the contracting boundary of $W_{\Delta''}$ group contains a 1-sphere.

Recall that two or more paths in a graph are *independent* if none of them contains an inner vertex of another. If we consider the last two counterexamples, we see that their defining graphs Δ and Δ' share a common property. The graph Δ (Δ') contains three paths that are independent to each other and link a and c (a' and c'). Thereby, none of these paths contains two non-adjacent vertices that are contained in an induced 4-cycle. Furthermore, two of the three pairs of the three paths build an induced cycle of length at least 5. See Figure 5.44

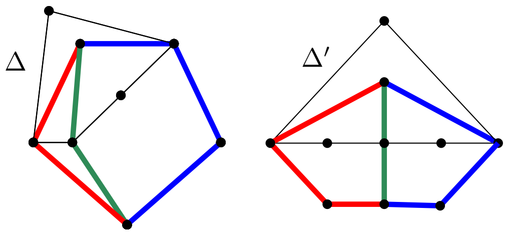

Figure 5.44 The two first counterexamples contain three independent paths that have no glued tetragon and link two vertices of the graph. These are thickened and colored red, green and blue. Left: The union of the red and the blue path and the union of the green and the blue path are induced 5-cycles. Right: The union of any two paths is an induced cycle of length at least 5.

We saw in Example 5.75, that the contracting boundary of $W_{\Delta'}$ becomes totally disconnected if we add a glued tetragon to one of these paths. The question arises of whether the Burst-Cycle-Conjecture becomes true if we use a stricter condition for the defining graph than before.

Definition 5.77. A graph Λ is *totally burst*, if all its cycles are burst and Λ does not contain a pair of non-adjacent vertices u and v that are linked by three paths P_0, P_1 such that

a) P_0, P_1 and P_2 are independent to each other,

b) for each $i \in \{1, 2, 3\}$ no pair of non-adjacent vertices in P_i is contained in an induced 4-cycle of Λ,

c) two of the three pairs of the three paths build an induced cycle of length at least 5, i.e., there exists $i \in \{1, 2, 3\}$ such that for $j \in \{1, 2, 3\} \setminus \{i\}$, $P_i \cup P_j$ is an induced cycle of length at least 5.

A graph that is not totally burst is *pretty intact*.

The spheres in the first two examples came from three paths as in Definition 5.77 We consider the question of whether the Burst-Cycle-Conjecture 5.5 becomes true if we demand that a graph is totally burst. We prove that this is not the case. There exists a triangle-free totally burst graph whose contracting boundary contains a 1-sphere.

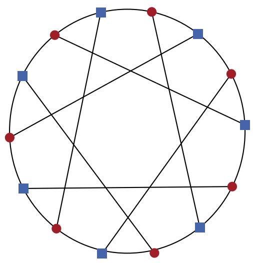

Figure 5.45 The Heawood graph Λ'.

Definition 5.78. Let Λ'' be the Heawood graph as pictured in Figure 5.45. The Heawood graph is bipartite. We color its vertices in two colors such that adjacent vertices have different colors (In Figure 5.45, we mark the two colors by different forms of the vertices). For every non-adjacent pair of vertices u and v of distinct color in Λ'' we add two independent paths of length 2 to the graph and identify their endvertices with u and v.

Every non-adjacent pair of vertices u and v of distinct color in Λ'' is contained in an induced 6-cycle in Δ''. The graph Λ'' has girth 6, i.e., the length of the shortest cycle in Λ'' is 6. Hence, we obtain Δ'' by adding three induced 4-cycles to every 6-cycle of the Heawood graph as pictured in Figure 5.46. The graph Δ'' is totally burst.

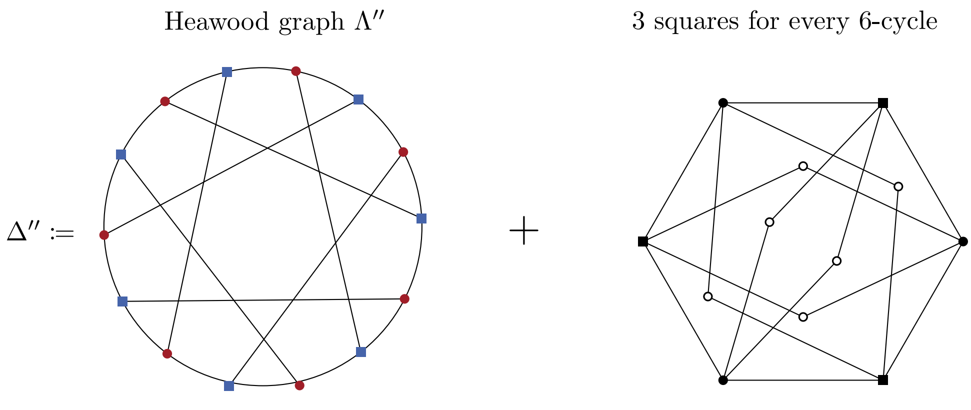

Figure 5.46 The graph Δ''.

It can be proven like in the examples before that the contracting boundary of the Davis complex of Δ'' contains a 1-sphere. That can be shown in the following steps. We embed a Davis complex of a 6-cycle in the Davis complex of Λ''. This embedded subcomplex $\mathcal{H}''$ is quasi-isometric to the hyperbolic plane. Its contracting boundary is a 1-sphere. Because Λ'' is an induced subgraph, this implies that $\mathcal{H}''$ is isometrically embedded in the Davis complex of Δ''. As in the examples before it can be proven that there exists $D > 0$ such that every geodesic ray in $\mathcal{H}''$ is D-contracting in the Davis complex of Δ''. This implies that the contracting boundary of $\mathcal{H}''$ is contained in the Davis complex of Δ''. As $\mathcal{H}''$ is quasi-isometric to the hyperbolic plane, the contracting boundary of $\mathcal{H}''$ is a 1-sphere. For the proofs of these steps, we need two properties of the Heawood graph.

Lemma 5.79. *Every 3-path of Λ'' is contained of a 6-cycle of Λ''.*

Lemma 5.80. *For every 2-path P in a 6-cycle C of Λ'' there is another 6-cycle C' of Λ'' such that $C \cap C' = P$.*

We denote the set of 6-cycles in the Heawood graph by $\mathcal{C}$. Let K'' be a 6-cycle. We embed the Davis complex of K'' in $\Sigma_{\Lambda''}$ such that the embedded subcomplex $\mathcal{H}''$ with its vertex set $\mathcal{V}(\mathcal{H}'')$ satisfies the following properties concerning the *stars of vertices* in $\mathcal{H}''$. If v is a vertex in the one-skeleton of $\mathcal{H}''$, we denote the subcomplex of $\mathcal{H}''$ consisting of all cubes that contain vertex v by *star(v)*.

- $\forall\, v \in \mathcal{V}(\mathcal{H}'')\ \exists\, C_v \in \mathcal{C}$: star(v) is contained in Σ_{C_v}
- $\forall$ adjacent $u, v \in \mathcal{V}(\mathcal{H}'') : C_u \neq C_v$
- Let Σ_4 be an isometrically embedded Davis complex in $\Sigma_{\Delta''}$ whose defining graph is an induced 4-cycle of Δ''. Then $\mathcal{H}''$ and Σ_4 share at most 2 edges.
- $\mathcal{H}''$ is isometrically embedded in $\Sigma_{\Lambda''}$.

We embed the Davis complex of a 6-cycle in $\Sigma_{\Lambda''}$ by an inductive construction. We start the construction by choosing a 6-cycle C of Λ''. We choose a vertex of the canonically embedded Davis complex of C in $\Sigma_{\Lambda''}$. Let $\mathcal{H}_1$ be the star of this vertex. In the i^{th} step, we complete all stars of vertices of $\mathcal{H}''_{i-1}$ in such a way that the degree of every vertex of $\mathcal{H}''_{i-1}$ is 6 in $\mathcal{H}''_i$. For that purpose, we choose suitable squares from the Davis complex of Λ'', such that the properties stated above are satisfied. This is possible because of Lemma 5.79 and Lemma 5.80. The complex $\mathcal{H}''$ is the union of all complexes $\mathcal{H}_i$, $i \in \mathbb{N}$. Because Λ'' is an induced subgraph of Δ'', its Davis complex is isometrically embedded in $\Sigma_{\Delta''}$.

Corollary 5.81. *The subcomplex $\mathcal{H}''$ is isometrically embedded in $\Sigma_{\Delta''}$.*

One observes similar as in the examples before that Euclidean planes in $\Sigma_{\Delta''}$ coming from induced 4-cycles don't affect $\mathcal{H}''$.

Lemma 5.82. *Let M be an isometrically embedded Davis complex in $\Sigma_{\Delta''}$ whose defining graph is an induced 4-cycle of Δ''. There are at most 2 hyperplanes in $\Sigma_{\Delta''}$ intersecting M and $\mathcal{H}''$.*

Using the characterization of contracting geodesic rays by Charney and Sultan as stated in Theorem 5.19, it follows that there exists $D > 0$ such that every geodesic ray in $\mathcal{H}$ is D-contracting in $\Sigma_{\Delta''}$.

Corollary 5.83. *There exists $D > 0$ such that every geodesic ray in $\mathcal{H}''$ is D-contracting in $\Sigma_{\Delta''}$.*

Theorem 5.84. *The contracting boundary of $W_{\Delta''}$ contains a 1-sphere.*

This example shows that the Burst-Cycle-Conjecture 5.5 does not become true if we reformulate it for totally burst graphs. Because Δ'' is triangle-free, not all triangle-free graphs satisfy the reformulated conjecture. We observe, however, that the Heawood graph is not path-decomposable. Thus, Δ'' is not path-decomposable. It is an interesting question of whether one can find an example of a totally burst graph whose corresponding right-angled Coxeter group is not totally disconnected.

5.6 Summary of the results of this chapter and a new conjecture

In this section, we summarize the results of the chapter and formulate a new conjecture about contracting boundaries of right-angled Coxeter groups. This chapter was motivated by an example studied by Charney and Sultan in Section 4.2 of [CS15] (see Figure 5.7) and a conjecture that was formulated by Tran in [Tra19] (See Conjecture 5.5). As before, we refer to the example as the Cycle-Join-Example and we refer to the conjecture as the Burst-Cycle-Conjecture. Throughout this section, all graphs are supposed to be simplicial.

Recall that a cycle C in a graph Λ is burst if one of the following three conditions is satisfied:

- C has length 3 or 4,
- C has a diagonal, i.e., two non-consecutive vertices of C are connected by an edge,
- the vertex set of C contains a pair of non-adjacent vertices of an induced 4-cycle.

A cycle is *intact*, if it is not burst.

One part of Conjecture 1.14 in [Tra19] says that the contracting boundary of a right-angled Coxeter group is not totally disconnected if its defining graph contains an intact cycle. That was proven in the last years. Indeed, every intact cycle leads to the existence of a 1-sphere in the contracting boundary of the corresponding right-angled Coxeter group. This was proven several times. It follows from Corollary 7.12 of [Tra19] in the case of triangle-free graphs. It follows from Proposition 4.9 of Genevois in [Gen20] for arbitrary graphs. Russell, Spriano and Tran formulated another proof of Genevois statement in Theorem 7.5 of [RST18]. At the end of Section 5.1, in Proof 5.23, we added another proof of Lazarovich, presented to me in a discussion we had.

It remained to consider the case that no cycle in a graph is intact. The easiest examples of such graphs are cliques and nontrivial joins. The corresponding right-angled Coxeter groups have empty contracting boundaries. If a graph is neither a clique nor a nontrivial join, its contracting boundary contains at least one element. This follows from [CS11]. See Theorem 5.6 for more details. In [NT19], Nguyen and Tran discovered a graph class $\mathcal{G}$ defined in Definition 5.14 that satisfies the Burst-Cycle-Conjecture. See Section 5.1 for details. Our goal was to find a larger graph class satisfying the conjecture.

For that purpose, we examined contracting boundaries of right-angled Coxeter groups whose defining graphs satisfy certain properties. First, we observed in Section 5.2 that we can use proper separations of defining graphs for decomposing Davis complexes in blocks that are itself Davis complexes of induced subgraphs. Block decompositions of CAT(0) spaces were introduced by Mooney in [Moo10] as *CAT(0) spaces with block structure* and we studied them in Chapter 3. Using such block decompositions, we proved Theorem 5.32, a generalization of the Cycle-Join-Example, in Section 5.3. We proved that we can calculate the contracting boundary of a right-angled Coxeter group if its defining graph has a proper separation satisfying certain conditions. In the proof, we applied Theorem 4.10, our generalization of the Cycle-Join-Example in the setting of amalgamated free products of CAT(0) groups. Thus, Theorem 5.32 can be understood as a variant of Theorem 4.10 in the setting of right-angled Coxeter groups.

We used Theorem 5.32 to define a graph class $\mathcal{J}$ of *join-decomposable* graphs. It contains the graph class $\mathcal{G}$ mentioned above. We showed in Corollary 5.38 that the contracting boundary of every join-decomposable graph is totally disconnected. This means that the graph class $\mathcal{J}$ satisfies the Burst-Cycle-Conjecture 5.5. Theorem 5.32 and Corollary 5.38 are the first main result of this chapter.

We considered the question of whether there is a larger graph class satisfying the Burst-Cycle-Conjecture 5.5. Motivated by this question, we studied how the contracting boundary of a right-angled Coxeter group $W_{\Lambda'}$ with defining graph Λ' changes when we glue the endvertices of a path P of length at least two to two non-adjacent vertices of Λ'. Then the corresponding right-angled Coxeter group can be written as an amalgamated free product along a group that is quasi-isometric to $\mathbb{Z}$. Thus, we could apply the results of Section 4.4. With help of our result about boundary points of rank-one-isometries in Theorem 4.24, we proved our second main result of this chapter, Theorem 5.58. It says that there occur just two extreme cases. In the first case, all "new connected components" are single points, i.e., if a connected component of the contracting boundary of $W_{\Lambda' \cup P}$ consists of more than one point, then it can be topologically embedded in the contracting boundary of $W_{\Lambda'}$. If the contracting boundary of $W_{\Lambda'}$ is totally disconnected, this means that the contracting boundary of $W_{\Lambda' \cup P}$ is totally disconnected. In the second case, the visual boundary of the Davis complex of P can be topologically embedded in a connected component κ of $\hat{\partial}_c \Sigma_{\Lambda' \cup P}$, the subspace of the visual boundary of $\Sigma_{\Lambda' \cup P}$ that consists of contracting boundary points. Suppose that we are in the second situation. If such a large connected component κ is connected in the contracting boundary of $\Sigma_{\Lambda' \cup P}$ and $\Lambda' \cup P$ does not contain any intact cycle, $\Lambda' \cup P$ is a counterexample to the Burst-Cycle-Conjecture 5.5.

In the next section, we sketched such examples. This outlook-section was joint work with Graeber, Lazarovich and Stark. We explained three counterexamples to the Burst-Cycle-Conjecture 5.5. The first example in Section 5.5.1 was found by Graeber. The two other examples described in Section 5.5.3 and Section 5.5.3 were inspired by this first example. The contracting boundary of each of the three examples contains a 1-sphere though each of the defining graphs does not contain any intact cycle. The first two examples in Section 5.5.1 and Section 5.5.2 can be obtained by gluing a path on a graph like described above. The third example has other interesting properties.

We recap a few interesting properties of the three counterexamples. The defining graphs Δ and Δ' of the first two examples in Section 5.5.1 and Section 5.5.2 both are path-decomposable. The graph Δ contains triangles. The graph Δ' is triangle-free and planar. We sketched proofs that the contracting boundaries of both examples contain a 1-sphere.We wanted to understand why the spheres occur in the contracting boundaries of the right-angled Coxeter groups W_Δ and $W_{\Delta'}$ with defining graphs Δ and Δ'. For that aim we studied Δ and Δ'. We observed that both graphs contain three paths P_0, P_1 and P_2 connecting two non-adjacent vertices u and v such that

a) P_0, P_1 and P_2 are independent to each other,

b) for all $i \in \{1,2,3\}$, no pair of non-adjacent vertices in P_i are contained in an induced 4-cycle of Λ,

c) two of the three pairs of the three paths build an induced cycle of length at least 5, i.e., there exists $i \in \{1,2,3\}$ such that for $j \in \{1,2,3\} \setminus \{i\}$, $P_i \cup P_j$ is an induced cycle of length at least 5.

The spheres in the first two examples came from three such paths. This leaded to the question whether the Burst-Cycle-Conjecture 5.5 might be true for right-angled Coxeter groups whose defining graphs don't contain three paths with the listed properties. We called a graph without intact cycles that does not contain such paths *totally burst.* We studied the question whether the contracting boundary of a right-angled Coxeter group is totally disconnected if its defining graph is totally burst. In Section 5.5.3 we saw an example where this is not the case. The defining graph Δ'' of this example is totally burst and triangle-free. We sketched the proof that the contracting boundary of $W_{\Delta''}$ contains a 1-sphere. This means that the Burst-Cycle-Conjecture 5.5 does not become true if we forbid the existence of paths as described above. The following table visualizes the properties of the three counterexamples studied in Section 5.5.

properties	Δ	Δ'	Δ''
contracting boundary contains 1-sphere	yes	yes	yes
triangle-free	no	yes	yes
cycles are burst	yes	yes	yes
totally burst	no	no	yes
path-decomposable	yes	yes	no

We see in the tabular, that the graph Δ'' is not path-decomposable. It is an interesting question of whether there is an example of a totally burst, path-decomposable graph whose corresponding right-angled Coxeter group is not totally disconnected. If there does not exist such an example, the contracting boundary of every right-angled Coxeter group whose defining graph is path-decomposable and totally burst, is totally disconnected.

Question 12. Suppose that Λ is a path-decomposable, totally burst graph. Is the contracting boundary of W_Λ totally disconnected?

Let Λ be a path-decomposable, totally burst graph. If we can prove that Λ is join-decomposable, then the contracting boundary of W_Λ is totally disconnected by Corollary 5.38 and the answer to Question 12 is positive. Thus, Question 12 leads to the following question.

Question 13. Suppose that Λ is path-decomposable. Is Λ totally burst if and only if Λ is join-decomposable?

Suppose that Λ is path-decomposable. Suppose further that Λ is totally burst. It is reasonable that there is a decomposition of Λ in paths such that every induced subgraph obtained by deleting such paths is join-decomposable. If this is true, Λ is join-decomposable. Then one direction of Question 13 is true. Let us consider the other direction. Suppose that Λ is path-decomposable but not join-decomposable. Is it true that the contracting boundary of W_Λ is not totally disconnected? A hint that this might be the case is provided by Theorem 5.58. Recall that Theorem 5.58 concerns gluings of paths on graphs. It shows that there occur only two extreme cases. One case implies the existence of a large connected component in $\hat{\partial}_c\Sigma_\Lambda$. In addition, the examples studied in Section 5.5 have spheres in their contracting boundaries. The spheres came from three paths having the properties listed above. This leads to the following question.

Question 14. Suppose that Λ is path-decomposable and not join-decomposable. Does Λ contain three paths P_0, P_1 and P_2 connecting two non-adjacent vertices u and v such that

a) P_0, P_1 and P_2 are independent to each other,

b) for all $i \in \{1, 2, 3\}$, no pair of non-adjacent vertices in P_i are contained in an induced 4-cycle of Λ,

c) two of the three pairs of the three paths build an induced cycle of length at least 5, i.e., there exists $i \in \{1, 2, 3\}$ such that for $j \in \{1, 2, 3\} \setminus \{i\}$, $P_i \cup P_j$ is an induced cycle of length at least 5.

Let Λ be a path-decomposable graph that is not join-decomposable. If the answer to Question 14 is positive, considerations similar to the arguments in Section 5.5.1 and Section 5.5.2 might imply that the contracting boundary of W_Λ contains a 1-sphere.

Question 15. Suppose that Λ is path-decomposable and not join-decomposable. Does the existence of three paths in Λ as in Question 14 imply that the contracting boundary of W_Λ contains a sphere?

It seems to be reasonable that the answers to the last three questions are positive. Hence, we finish this thesis with the following new conjecture. If the answers to the last three questions are positive, then the following conjecture holds.

Conjecture 5.85. *Let Λ be a path-decomposable graph. The following statements are equivalent.*

- *The contracting boundary of W_Λ is totally disconnected.*
- *Λ is totally burst.*
- *Λ is join-decomposable.*
- *The contracting boundary of W_Λ does not contain a 1-sphere.*

This conjecture concerns only path-decomposable graphs. For general graphs, we ask the following question.

Question 16. Suppose that Λ is a graph that is not join-decomposable. When does it contain a 1-sphere?

For answering this question, variants of the graph Δ'' as defined in Section 5.5.3 are relevant. The graph Δ'' was obtained from the Heawood graph by adding certain induced 4-cycles. We ask now what happens if we add more induced 4-cycles to the Heawood graph. In Section 5.5.3 we constructed a certain subcomplex $\mathcal{H}''$ of the Davis complex of $W_{\Delta''}$ whose contracting boundary is a sphere and topologically embedded in the contracting boundary of $W_{\Delta''}$. Suppose that $\mathcal{H}''$ does not share an unbounded set with any embedded Davis-complex of a 6-cycle in $\mathcal{C}$. In this case, we can add as many induced 4-cycles to Λ'' as we want, and no such 4-cycle affects $\mathcal{H}''$. Then the corresponding contracting boundary always contains a 1-sphere.

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