J. London Math. Soc. (2) 0 (2021) 1–36 doi:10.1112/jlms.12429

Strengthened inequalities for the mean width and the ℓ-norm

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Abstract

Barthe proved that the regular simplex maximizes the mean width of convex bodies whose John ellipsoid (maximal volume ellipsoid contained in the body) is the Euclidean unit ball; or equivalently, the regular simplex maximizes the ℓ-norm of convex bodies whose Löwner ellipsoid (minimal volume ellipsoid containing the body) is the Euclidean unit ball. Schmuckenschläger verified the reverse statement; namely, the regular simplex minimizes the mean width of convex bodies whose Löwner ellipsoid is the Euclidean unit ball. Here we prove stronger stability versions of these results. We also consider related stability results for the mean width and the ℓ-norm of the convex hull of the support of centered isotropic measures on the unit sphere.

1. *Introduction*

In geometric inequalities and extremal problems, Euclidean balls and simplices often are the extremizers. A classical example is the isoperimetric inequality which states that Euclidean balls have smallest surface area among convex bodies (compact convex sets with non-empty interior) of given volume in Euclidean space $\mathbb{R}^n$, and Euclidean balls are the only minimizers. Another example is the Urysohn inequality which expresses the geometric fact that Euclidean balls minimize the mean width of convex bodies of given volume. To introduce the mean width, let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the scalar product and Euclidean norm in $\mathbb{R}^n$, and let B^n be the Euclidean unit ball centred at the origin with $\kappa_n = V(B^n) = \pi^{n/2}/\Gamma(1+n/2)$, where $V(\cdot)$ is the volume (Lebesgue measure) in $\mathbb{R}^n$. For a convex body K in $\mathbb{R}^n$, the support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of K is defined by $h_K(x) = \max_{y\in K}\langle x, y\rangle$ for $x \in \mathbb{R}^n$. Then the mean width of K is given by

$$W(K) = \frac{1}{n\kappa_n}\int_{S^{n-1}} (h_K(u) + h_K(-u))\, du,$$

where the integration over the unit sphere S^{n-1} is with respect to the $(n-1)$-dimensional Hausdorff measure (that coincides with the spherical Lebesgue measure in this case).

A prominent geometric extremal problem for which simplices are extremizers has been discovered and explored much more recently. First, recall that there exists a unique ellipsoid of maximal volume contained in K (which is called the John ellipsoid of K), and a unique ellipsoid of minimal volume containing K (which is called the Löwner ellipsoid of K). It has been shown by Ball [5] that simplices maximize the volume of K given the volume of the John ellipsoid of K, and thus simplices determine the extremal 'inner' volume ratio. For the dual

Received 5 February 2020; revised 22 November 2020.

2020 *Mathematics Subject Classification* 52A40 (primary), 52A38, 52B12, 26D15 (secondary).

K. Böröczky was supported by Hungarian National Research, Development and Innovation Office — NKFIH grants 129630 and 132002. This research of F. Fodor was supported by grant TUDFO/47138-1/2019-ITM of the Ministry for Innovation and Technology, Hungary, and by Hungarian National Research, Development and Innovation Office — NKFIH grants 129630.

problem, Barthe [11] proved that simplices minimize the volume of K given the volume of the Löwner ellipsoid of K, hence simplices determine the extremal 'outer' volume ratio (see also [52, 54]). In all these cases, equality was characterized by Barthe [11].

In this paper, we consider the mean width and the so called ℓ-norm. To define the latter, for a convex body $K \subset \mathbb{R}^n$ containing the origin in its interior, we set

$$\|x\|_K = \min\{t \geqslant 0 : x \in tK\}, \qquad x \in \mathbb{R}^n.$$

Furthermore, we write γ_n for the standard Gaussian measure in $\mathbb{R}^n$ which has the density function $x \mapsto \sqrt{2\pi}^{\,-n} e^{-\|x\|^2/2}$, $x \in \mathbb{R}^n$, with respect to Lebesgue measure. Then the ℓ-norm of K is given by

$$\ell(K) = \int_{\mathbb{R}^n} \|x\|_K \, \gamma_n(dx) = \mathbb{E}\|X\|_K,$$

where X is a Gaussian random vector with distribution γ_n. If the polar body of K is denoted by $K^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \ \forall y \in K\}$, then we obtain the relation

$$\ell(K) = \tfrac{\ell(B^n)}{2} W(K^\circ) \tag{1}$$

with

$$\lim_{n\to\infty} \frac{\ell(B^n)}{\sqrt{n}} = 1.$$

In addition, the ℓ-norm of K can be expressed in the form (see Barthe [10])

$$\ell(K) = \int_{\mathbb{R}^n} \mathbb{P}(\|X\|_K > t)\, dt = \int_0^\infty (1 - \gamma_n(tK))\, dt. \tag{2}$$

Let Δ_n be a regular simplex inscribed into B^n, and hence Δ_n° is a regular simplex circumscribed around B^n. Theorem 1.1 (i) is due to Barthe [10], and (ii) was proved by Schmuckenschläger [61].

Theorem 1.1 (Barthe '98, Schmuckenschläger '99). *Let K be a convex body in $\mathbb{R}^n$.*

(i) *If $B^n \supset K$ is the Löwner ellipsoid of K, then $\ell(K) \leqslant \ell(\Delta_n)$, and if $B^n \subset K$ is the John ellipsoid of K, then $W(K) \leqslant W(\Delta_n^\circ)$. Equality holds in either case if and only if K is a regular simplex.*

(ii) *If $B^n \subset K$ is the John ellipsoid of K, then $\ell(K) \geqslant \ell(\Delta_n^\circ)$, and if $B^n \supset K$ is the Löwner ellipsoid of K, then $W(K) \geqslant W(\Delta_n)$. Equality holds in either case if and only if K is a regular simplex.*

It follows from (1) and the duality of Löwner and John ellipsoids that the two statements in (i) are equivalent to each other, and the same is true for (ii).

The classical Urysohn inequality states that $(W(K)/2)^n \geqslant V(K)/\kappa_n$ with equality exactly when K is a ball. While a reverse form of the Urysohn inequality is still not known in general, we recall that Giannopoulos, Milman, Rudelson [32] proved a reverse Urysohn inequality, for zonoids, and Hug and Schneider [42] established reverse inequalities of other intrinsic and mixed volumes for zonoids and explored applications to stochastic geometry. A related classical open problem in convexity and probability theory is that among all simplices contained in the Euclidean unit ball, the inscribed regular simplex has the maximal mean width (see Litvak [50] for a comprehensive survey on this topic).

Let us discuss the range of $W(K)$ (and hence that of $\ell(K)$ by (1)) in Theorem 1.1. If K is a convex body in $\mathbb{R}^n$ whose Löwner ellipsoid is B^n, then the monotonicity of the mean width and Theorem 1.1 (i) yield

$$W(\Delta_n) \leqslant W(K) \leqslant W(B^n) = 2,$$

where, according to Böröczky [19], we have

$$W(\Delta_n) \sim 4\sqrt{\frac{2\ln n}{n}} \quad \text{as } n \to \infty.$$

In addition, if K is a convex body in $\mathbb{R}^n$ whose John ellipsoid is B^n, then

$$2 = W(B^n) \leqslant W(K) \leqslant W(\Delta_n^\circ)$$

with $W(\Delta_n^\circ) \sim 4\sqrt{2n\ln n}$.

An important concept in the proof of Theorem 1.1 is the notion of an isotropic measure on the unit sphere. Following Giannopoulos, Papadimitrakis [33] and Lutwak, Yang, Zhang [54], we call a Borel measure μ on the unit sphere S^{n-1} isotropic if

$$\int_{S^{n-1}} u \otimes u \, \mu(du) = \mathrm{I}_n, \tag{3}$$

where I_n is the identity map (or the identity matrix). Condition (3) is equivalent to

$$\langle x, x\rangle = \int_{S^{n-1}} \langle u, x\rangle^2 \, \mu(du) \quad \text{for } x \in \mathbb{R}^n.$$

In this case, equating traces of the two sides of (3), we obtain that $\mu(S^{n-1}) = n$. In addition, we say that the isotropic measure μ on S^{n-1} is centered if

$$\int_{S^{n-1}} u \, \mu(du) = o.$$

We observe that if μ is a centered isotropic measure on S^{n-1}, then for the cardinality $|\mathrm{supp}\,\mu|$ of the support of μ it holds that $|\mathrm{supp}\,\mu| \geqslant n+1$, with equality if and only if μ is concentrated on the vertices of some regular simplex and each vertex has measure $n/(n+1)$ (see [20, Lemma 10.2] for a quantitative version of this fact).

We recall that isotropic measures on $\mathbb{R}^n$ play a central role in the KLS conjecture by Kannan, Lovász and Simonovits [45] as well as in the analysis of Bourgain's hyperplane conjecture (slicing problem); see, for instance, Barthe and Cordero-Erausquin [13], Guedon and Milman [41], Klartag [46], Artstein-Avidan, Giannopoulos, Milman [2] and Alonso-Gutiérrez, Bastero [1].

The emergence of isotropic measures on S^{n-1} arises from Ball's crucial insight that John's characteristic condition [43, 44] for a convex body to have the unit ball as its John or Löwner ellipsoid (see [3, 5]) can be used to give the Brascamp–Lieb inequality a convenient form which is ideally suited for many geometric applications (see Section 2). John's characteristic condition (with the proof of the equivalence completed by Ball [6]) states that B^n is the John ellipsoid of a convex body K containing B^n if and only if there exist distinct unit vectors $u_1, \ldots, u_k \in \partial K \cap S^{n-1}$ and $c_1, \ldots, c_k > 0$ such that

$$\sum_{i=1}^k c_i u_i \otimes u_i = \mathrm{I}_n, \tag{4}$$

$$\sum_{i=1}^k c_i u_i = o. \tag{5}$$

In particular, the measure μ on S^{n-1} with support $\{u_1, \ldots, u_k\}$ and $\mu(\{u_i\}) = c_i$ for $i = 1, \ldots, k$ is isotropic and centered. In addition, B^n is the Löwner ellipsoid of a convex body $K \subset B^n$ if and only if there exist $u_1, \ldots, u_k \in \partial K \cap S^{n-1}$ and $c_1, \ldots, c_k > 0$ satisfying (4) and (5). According to John [44] (see also Gruber, Schuster [39]), we may assume that $k \leqslant n(n+3)/2$ in (4) and (5). It follows from John's characterization that B^n is the Löwner ellipsoid of a convex body $K \subset B^n$ if and only if B^n is the John ellipsoid of K°.

The finite Borel measures on S^{n-1} which have an isotropic linear image are characterized by Böröczky, Lutwak, Yang and Zhang [21], building on earlier work by Carlen and Cordero-Erausquin [23], Bennett, Carbery, Christ and Tao [17] and Klartag [47].

We write $\operatorname{conv} X$ to denote the convex hull of a set $X \subset \mathbb{R}^n$. We observe that if μ is a centered isotropic measure on S^{n-1}, then $o \in \operatorname{int} Z_\infty(\mu)$ for

$$Z_\infty(\mu) = \operatorname{conv} \operatorname{supp} \mu.$$

For the present purpose, the study of $Z_\infty(\mu)$ can be reduced to discrete measures, as Lemma 10.1 in Böröczky and Hug [20] states that for any centered isotropic measure μ, there exists a discrete centered isotropic measure μ_0 on S^{n-1} whose support is contained in the support of μ (see Lemma 2.1). It follows that Theorem 1.1 is equivalent to the following statements about isotropic measures proved by Li and Leng [48].

THEOREM 1.2 (Li and Leng '12). *If μ is a centered isotropic measure on S^{n-1}, then $\ell(Z_\infty(\mu)) \leqslant \ell(\Delta_n)$, $W(Z_\infty(\mu)^\circ) \leqslant W(\Delta_n^\circ)$, $\ell(Z_\infty(\mu)^\circ) \geqslant \ell(\Delta_n^\circ)$ and $W(Z_\infty(\mu)) \geqslant W(\Delta_n)$, with equality in either case if and only if $|\operatorname{supp} \mu| = n+1$.*

Results similar to Theorem 1.2 are proved by Ma [55] in the L_p setting.

The main goal of the present paper is to provide stronger stability versions of Theorems 1.1 and 1.2. Since our results use the notion of distance between convex bodies (and to fix the notation), we recall that the distance between compact subsets X and Y of $\mathbb{R}^n$ is measured in terms of the Hausdorff distance defined by

$$\delta_H(X,Y) = \max\{\max_{y\in Y} d(y,X), \max_{x\in X} d(x,Y)\},$$

where $d(x,Y) = \min\{\|x-y\| : y \in Y\}$. The Hausdorf distance defines a metric on the set of non-empty compact subsets of $\mathbb{R}^n$.

In addition, for convex bodies K and C, the symmetric difference distance of K and C is the volume of their symmetric difference; namely,

$$\delta_{\mathrm{vol}}(K,C) = V(K \setminus C) + V(C \setminus K).$$

Clearly, the symmetric difference distance also defines a metric on the set of convex bodies in $\mathbb{R}^n$. Both metrics induce the same topology on the space of convex bodies, but are not uniformly equivalent to each other (see [62, p. 71] and [63]).

Let $O(n)$ denote the orthogonal group (rotation group) of $\mathbb{R}^n$.

THEOREM 1.3. *Let B^n be the Löwner ellipsoid of a convex body $K \subset B^n$ in $\mathbb{R}^n$, let $c = n^{26n}$ and let $\varepsilon \in (0,1)$. If $\ell(K) \geqslant (1-\varepsilon)\ell(\Delta_n)$, then there exists a $T \in O(n)$ such that:*

(i) $\delta_{\mathrm{vol}}(K, T\Delta_n) \leqslant c\sqrt[4]{\varepsilon}$;
(ii) $\delta_H(K, T\Delta_n) \leqslant c\sqrt[4]{\varepsilon}$.

THEOREM 1.4. *Let B^n be the John ellipsoid of a convex body $K \supset B^n$ in $\mathbb{R}^n$ and let $\varepsilon > 0$. If $\ell(K) \leqslant (1+\varepsilon)\ell(\Delta_n^\circ)$, then there exists a $T \in O(n)$ such that:*

(i) $\delta_{\mathrm{vol}}(K, T\Delta_n^\circ) \leqslant c\sqrt[4]{\varepsilon}$ *for* $c = n^{27n}$;
(ii) $\delta_H(K, T\Delta_n^\circ) \leqslant c\sqrt[4n]{\varepsilon}$ *for* $c = n^{27}$.

Let us consider the optimality of the order of the estimates in Theorems 1.3 and 1.4. For Theorem 1.3 (i) and (ii), we use the following construction. We add an $(n+2)$nd vertex $v_{n+2} \in S^{n-1}$ to the $n+1$ vertices $v_1, \dots, v_{n+1}$ of Δ_n such that v_1 lies on the geodesic arc on S^{n-1} connecting v_2 and v_{n+2}, and such that $\angle(v_{n+2}, v_1) = c_1\varepsilon$ for a suitable $c_1 > 0$ depending on

n. The polytope $K = \mathrm{conv}\{v_1, \ldots, v_{n+2}\}$ satisfies $\ell(K) \geqslant (1-\varepsilon)\ell(\Delta_n)$ on the one hand, and $\delta_{\mathrm{vol}}(K, T\Delta_n) \geqslant c_2\varepsilon$ and $\delta_H(K, T\Delta_n) \geqslant c_2\varepsilon$ for a suitable $c_2 > 0$, depending on n, and for any $T \in O(n)$, on the other hand. Similarly, using the polar of this polytope K for Theorem 1.4 (i), possibly after decreasing c_1, we have $\ell(K^\circ) \leqslant (1+\varepsilon)\ell(\Delta_n^\circ)$ while $\delta_{\mathrm{vol}}(K^\circ, T\Delta_n^\circ) \geqslant c_3\varepsilon$ for a suitable $c_3 > 0$ depending on n and for any $T \in O(n)$. Finally, we consider the optimality of Theorem 1.4 (ii). Cutting off $n+1$ regular simplices of edge length $c_4 \sqrt[n]{\varepsilon}$ at the vertices of Δ_n°, for a suitable $c_4 > 0$ depending on n, results in a polytope $\widetilde{K}$ satisfying $\ell(\widetilde{K}) \leqslant (1+\varepsilon)\ell(\Delta_n^\circ)$ and $\delta_H(\widetilde{K}, T\Delta_n^\circ) \geqslant c_5 \sqrt[n]{\varepsilon}$ for any $T \in O(n)$ for some suitable $c_5 > 0$ depending on n.

We did not make an attempt to optimize the constants c that depend on n, but observe that the c is polynomial in n in Theorem 1.4 (ii).

In the case of the mean width, we have the following stability versions of Theorem 1.1.

Corollary 1.5. *Let K be convex body in $\mathbb{R}^n$.*

(i) *If B^n is the John ellipsoid of $K \supset B^n$ and $W(K) \geqslant (1-\varepsilon)W(\Delta_n^\circ)$ for some $\varepsilon \in (0,1)$, then there exists a $T \in O(n)$ such that $\delta_H(K, T\Delta_n^\circ) \leqslant c\sqrt[4]{\varepsilon}$ for $c = n^{27n}$.*

(ii) *If B^n is the Löwner ellipsoid of $K \subset B^n$ and $W(K) \leqslant (1+\varepsilon)W(\Delta_n)$ for some $\varepsilon > 0$, then there exists a $T \in O(n)$ such that $\delta_H(K, T\Delta_n) \leqslant c\sqrt[4n]{\varepsilon}$ for $c = n^{29}$.*

For the optimality of Corollary 1.5 (i), cutting off $n+1$ regular simplices of edge length $c_1\varepsilon$ at the vertices of Δ_n° for suitable $c_1 > 0$ depending on n results in a polytope K satisfying $W(K) \geqslant (1-\varepsilon)W(\Delta_n^\circ)$ and $\delta_H(K, T\Delta_n^\circ) \geqslant c_2\varepsilon$ for suitable $c_2 > 0$ depending on n and for any $T \in O(n)$. Concerning Corollary 1.5 (ii), let $v_1, \ldots, v_{n+1}$ be the vertices of Δ_n, and let $\widetilde{K}$ be the polytope whose vertices are $v_i, -(\frac{1}{n} + c_3\sqrt[n]{\varepsilon})v_i$ for $i = 1, \ldots, n+1$ for suitable $c_3 > 0$ depending on n in a way such that $W(\widetilde{K}) \leqslant (1+\varepsilon)W(\Delta)$. It follows that $\delta_H(K, T\Delta_n) \geqslant c_4\sqrt[n]{\varepsilon}$ for any $T \in O(n)$ and for a suitable $c_4 > 0$ depending on n.

We also have the following stronger form of Theorem 1.2 in the form of stability statements.

Theorem 1.6. *Let μ be a centered isotropic measure on the unit sphere S^{n-1}, let $c = n^{28n}$, and let $\varepsilon \in (0,1)$. If one of the conditions:*

(a) $\ell(Z_\infty(\mu)) \geqslant (1-\varepsilon)\ell(\Delta_n)$ *or*
(b) $W(Z_\infty(\mu)^\circ) \geqslant (1-\varepsilon)W(\Delta_n^\circ)$ *or*
(c) $\ell(Z_\infty(\mu)^\circ) \leqslant (1+\varepsilon)\ell(\Delta_n^\circ)$ *or*
(d) $W(Z_\infty(\mu)) \leqslant (1+\varepsilon)W(\Delta_n)$

is satisfied, then there exists a regular simplex with vertices $w_1, \ldots, w_{n+1} \in S^{n-1}$ such that

$$\delta_H(\mathrm{supp}\,\mu, \{w_1, \ldots, w_{n+1}\}) \leqslant c\,\varepsilon^{\frac{1}{4}}.$$

The proofs of Theorem 1.3 and Theorem 1.6 (a) and (b) are based on Proposition 7.1, which is the special case of Theorem 1.6 (a) for a discrete measure. In addition, a new stability version of Barthe's reverse of the Brascamp–Lieb inequality is required for a special parametric class of functions, which is derived in Section 6. In a similar vein, the proofs of Theorem 1.4 and Theorem 1.6 (c) and (d) are based on Proposition 9.1, which is the special case of Theorem 1.6 (c) for a discrete measure. In addition, we use and derive a stability version of the Brascamp–Lieb inequality for a special parametric class of functions (see also Section 6).

We note that our arguments are based on the rank one geometric Brascamp–Lieb and reverse Brascamp–Lieb inequalities (see Section 2), and their stability versions in a special case (see Section 6). Unfortunately, no quantitative stability version of the Brascamp–Lieb and reverse Brascamp–Lieb inequalities are known in general (see [**16**] for a certain weak stability version for higher ranks). On the other hand, in the case of the Borell–Brascamp–Lieb inequaliy (see [**8, 18, 22**]), stability versions were proved by Ghilli and Salani [**31**] and Rossi and Salani [**59**].

2. *Discrete isotropic measures and the* (*reverse*) *Brascamp-Lieb inequality*

For the purposes of this paper, the study of $Z_\infty(\mu)$ for centered isotropic measures on S^{n-1} can be reduced to the case when μ is discrete. Writing $|X|$ for the cardinality of a finite set X, we recall that Lemma 10.1 in Böröczky and Hug [**20**] states that for any centered isotropic measure μ, there exists a discrete centered isotropic measure μ_0 on S^{n-1} with $\operatorname{supp}\mu_0 \subset \operatorname{supp}\mu$ and $|\operatorname{supp}\mu_0| \leqslant \frac{n(n+3)}{2} + 1$. We use this statement in the following form.

Lemma 2.1. *For any centered isotropic measure μ on S^{n-1}, there exists a discrete centered isotropic measure μ_0 on S^{n-1} such that*

$$\operatorname{supp}\mu_0 \subset \operatorname{supp}\mu \quad \text{and} \quad |\operatorname{supp}\mu_0| \leqslant 2n^2.$$

The rank one geometric Brascamp–Lieb inequality (7) was identified by Ball [**3**] as an important case of the rank one Brascamp–Lieb inequality proved originally by Brascamp and Lieb [**22**]. In addition, the reverse Brascamp–Lieb inequality (8) is due to Barthe [**9, 11**]. To set up (7) and (8), let the distinct unit vectors $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$ satisfy

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n\,. \tag{6}$$

If $f_1, \ldots, f_k$ are non-negative measurable functions on $\mathbb{R}$, then the Brascamp–Lieb inequality states that

$$\int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(\langle x, u_i\rangle)^{c_i}\, dx \leqslant \prod_{i=1}^{k} \left(\int_{\mathbb{R}} f_i\right)^{c_i}, \tag{7}$$

and the reverse Brascamp–Lieb inequality is given by

$$\int_{\mathbb{R}^n}^{*} \sup_{x=\sum_{i=1}^{k} c_i\theta_i u_i} \prod_{i=1}^{k} f_i(\theta_i)^{c_i}\, dx \geqslant \prod_{i=1}^{k} \left(\int_{\mathbb{R}} f_i\right)^{c_i}, \tag{8}$$

where the star on the left-hand side denotes the upper integral. Here we always assume that $\theta_1, \ldots, \theta_k \in \mathbb{R}$ in (8). We note that $\theta_1, \ldots, \theta_k$ are unique if $k = n$ and hence $u_1, \ldots, u_n$ is an orthonormal basis.

It was proved by Barthe [**11**] that equality in (7) or in (8) implies that if none of the functions f_i is identically zero or a scaled version of a Gaussian, then there exists an origin symmetric regular crosspolytope in $\mathbb{R}^n$ such that $u_1, \ldots, u_k$ lie among its vertices. Conversely, we note that equality holds in (7) and (8) if either each f_i is a scaled version of the same centered Gaussian, or if $k = n$ and $u_1, \ldots, u_n$ form an orthonormal basis.

For a detailed discussion of the rank one Brascamp–Lieb inequality, we refer to Carlen and Cordero-Erausquin [**23**]. The higher rank case, due to Lieb [**49**], is reproved and further explored by Barthe [**11**]. Equality in the general version of the Brascamp-Lieb inequality is clarified by Bennett, Carbery, Christ, Tao [**17**]. In addition, Barthe, Cordero-Erausquin, Ledoux, Maurey (see [**14**]) develop an approach for the Brascamp–Lieb inequality via Markov semigroups in a quite general framework.

The fundamental papers by Barthe [**9, 11**] provided concise proofs of (7) and (8) based on mass transportation (see Ball [**7**] for a sketch in the case of (7)). Actually, the reverse Brascamp–Lieb inequality (8) seems to be the first inequality whose original proof is via mass transportation. During the argument in Barthe [**11**], the following four observations due to

Ball [3] (see also [11] for a simpler proof of (i)) play crucial roles: If $k \geqslant n$, $c_1, \dots, c_k > 0$ and $u_1, \dots, u_k \in S^{n-1}$ satisfy (6), then:

(i) for any $t_1, \dots, t_k > 0$, we have

$$\det\left(\sum_{i=1}^k t_i c_i u_i \otimes u_i\right) \geqslant \prod_{i=1}^k t_i^{c_i}; \tag{9}$$

(ii) for $z = \sum_{i=1}^k c_i \theta_i u_i$ with $\theta_1, \dots, \theta_k \in \mathbb{R}$, we have

$$\|z\|^2 \leqslant \sum_{i=1}^k c_i \theta_i^2; \tag{10}$$

(iii) for $i = 1, \dots, k$, we have

$$c_i \leqslant 1;$$

(iv) and it holds that

$$c_1 + \cdots + c_k = n. \tag{11}$$

Inequality (9) is called the Ball–Barthe inequality by Lutwak, Yang and Zhang [54], and Li and Leng [48].

3. *Review of the proof of the* (*reverse*) *Brascamp–Lieb inequality if all* $f_i = f$ *and* f *is log-concave*

Let $g(t) = \sqrt{2\pi}^{\,-1} e^{-t^2/2}$, $t \in \mathbb{R}$, be the standard Gaussian density (mean zero and variance one), and let f be a probability density function on $\mathbb{R}$ (here we restrict to log-concave functions to avoid differentiability issues). Let T and S be the transportation maps which are determined by

$$\int_{-\infty}^{x} f = \int_{-\infty}^{T(x)} g \qquad \text{and} \qquad \int_{-\infty}^{S(y)} f = \int_{-\infty}^{y} g.$$

Henceforth, we do not write the arguments and the Lebesgue measure in the integral if the meaning of the integral is unambiguous. As f is log-concave, there exists an open interval I such that f is positive on I and zero on the complement of the closure of I, and $T : I \to \mathbb{R}$ and $S : \mathbb{R} \to I$ are inverses of each other. In addition, for $x \in I$ and $y \in \mathbb{R}$, we have

$$f(x) = g(T(x))\, T'(x) \qquad \text{and} \qquad g(y) = f(S(y))\, S'(y). \tag{12}$$

For

$$\mathcal{C} = \{x \in \mathbb{R}^n : \langle u_i, x\rangle \in I \ \text{ for } i = 1, \dots, k\},$$

we consider the transformation $\Theta : \mathcal{C} \to \mathbb{R}^n$ with

$$\Theta(x) = \sum_{i=1}^k c_i\, T(\langle u_i, x\rangle)\, u_i, \qquad x \in \mathcal{C},$$

which satisfies

$$d\Theta(x) = \sum_{i=1}^k c_i\, T'(\langle u_i, x\rangle)\, u_i \otimes u_i.$$

It is known that $d\Theta$ is positive definite and $\Theta : \mathcal{C} \to \mathbb{R}^n$ is injective (see [9, 11]). Therefore, using first (12), then (i) with $t_i = T'(\langle u_i, x\rangle)$, and then the definition of Θ and (ii), the following argument leads to the Brascamp–Lieb inequality in this special case:

$$\begin{aligned}\int_{\mathbb{R}^n}\prod_{i=1}^k f(\langle u_i,x\rangle)^{c_i}\,dx &= \int_{\mathcal{C}}\prod_{i=1}^k f(\langle u_i,x\rangle)^{c_i}\,dx\\ &= \int_{\mathcal{C}}\left(\prod_{i=1}^k g(T(\langle u_i,x\rangle))^{c_i}\right)\left(\prod_{i=1}^k T'(\langle u_i,x\rangle)^{c_i}\right)dx\\ &\leqslant \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathcal{C}}\left(\prod_{i=1}^k e^{-c_iT(\langle u_i,x\rangle)^2/2}\right)\det\left(\sum_{i=1}^k c_iT'(\langle u_i,x\rangle)\,u_i\otimes u_i\right)dx\\ &\leqslant \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathcal{C}} e^{-\|\Theta(x)\|^2/2}\det\left(d\Theta(x)\right)dx\\ &\leqslant \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\|y\|^2/2}\,dy = 1.\end{aligned}$$

We note that the Brascamp–Lieb inequality (13) for an arbitrary non-negative log-concave function f follows by scaling; namely, (iv) implies

$$\int_{\mathbb{R}^n}\prod_{i=1}^k f(\langle x,u_i\rangle)^{c_i}\,dx \leqslant \left(\int_{\mathbb{R}} f\right)^n. \tag{13}$$

For the reverse Brascamp–Lieb inequality, we observe that

$$d\Psi(x) = \sum_{i=1}^k c_iS'(\langle u_i,x\rangle)\,u_i\otimes u_i$$

holds for the differentiable map $\Psi:\mathbb{R}^n\to\mathbb{R}^n$ with

$$\Psi(x) = \sum_{i=1}^k c_iS(\langle u_i,x\rangle)\,u_i, \qquad x\in\mathbb{R}^n.$$

In particular, $d\Psi$ is positive definite and $\Psi:\mathbb{R}^n\to\mathbb{R}^n$ is injective (see [**9, 11**]). Therefore, (i) and (12) lead to (for the first inequality, observe that Ψ is injective, but Ψ need not be surjective)

$$\begin{aligned}\int^*_{\mathbb{R}^n}\sup_{x=\sum_{i=1}^k c_i\theta_iu_i}\prod_{i=1}^k f(\theta_i)^{c_i}\,dx &\geqslant \int^*_{\mathbb{R}^n}\left(\sup_{\Psi(y)=\sum_{i=1}^k c_i\theta_iu_i}\prod_{i=1}^k f(\theta_i)^{c_i}\right)\det\left(d\Psi(y)\right)dy\\ &\geqslant \int_{\mathbb{R}^n}\left(\prod_{i=1}^k f(S(\langle u_i,y\rangle))^{c_i}\right)\det\left(\sum_{i=1}^k c_iS'(\langle u_i,y\rangle)\,u_i\otimes u_i\right)dy\\ &\geqslant \int_{\mathbb{R}^n}\left(\prod_{i=1}^k f(S(\langle u_i,y\rangle))^{c_i}\right)\left(\prod_{i=1}^k S'(\langle u_i,y\rangle)^{c_i}\right)dy\\ &= \int_{\mathbb{R}^n}\left(\prod_{i=1}^k g(\langle u_i,y\rangle)^{c_i}\right)dy = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\|y\|^2/2}\,dy = 1.\end{aligned}$$

Again, the reverse Brascamp–Lieb inequality (14) for an arbitrary non-negative log-concave function f follows by scaling and (iv); namely,

$$\int_{\mathbb{R}^n}^{*} \sup_{x=\sum_{i=1}^{k} c_i\theta_i u_i} \prod_{i=1}^{k} f(\theta_i)^{c_i}\, dx \geqslant \left(\int_{\mathbb{R}} f\right)^n. \tag{14}$$

We observe that (i) shows that the optimal factor in the geometric Brascamp–Lieb inequality and in its reverse form is 1.

4. *Observations on the stability of the Brascamp–Lieb inequality and its reverse*

This section summarizes certain stability forms of the Ball–Barthe inequality (9) based on work in Böröczky and Hug [**20**]. The first step is a stability version of the Ball–Barthe inequality (9) proved in [**20**].

LEMMA 4.1. *If $k \geqslant n+1$, $t_1,\ldots,t_k > 0$, $c_1,\ldots,c_k > 0$ and $u_1,\ldots,u_k \in S^{n-1}$ satisfy (6), then*

$$\det\left(\sum_{i=1}^{k} t_i c_i u_i \otimes u_i\right) \geqslant \theta^* \prod_{i=1}^{k} t_i^{c_i},$$

where

$$\theta^* = 1 + \frac{1}{2}\sum_{1\leqslant i_1<\ldots<i_n\leqslant k} c_{i_1}\cdots c_{i_n} \det[u_{i_1},\ldots,u_{i_n}]^2 \left(\frac{\sqrt{t_{i_1}\cdots t_{i_n}}}{t_0} - 1\right)^2,$$

$$t_0 = \sqrt{\sum_{1\leqslant i_1<\ldots<i_n\leqslant k} t_{i_1}\cdots t_{i_n} c_{i_1}\cdots c_{i_n} \det[u_{i_1},\ldots,u_{i_n}]^2}.$$

In order to estimate θ^* from below, we use the following observation from [**20**].

LEMMA 4.2. *If $a,b,x > 0$, then*

$$(xa-1)^2 + (xb-1)^2 \geqslant \frac{(a^2-b^2)^2}{2(a^2+b^2)^2}.$$

The combination of Lemmas 4.1 and 4.2 implies the following stability version of the Ball–Barthe inequality (9) that is easier to use.

COROLLARY 4.3. *If $k \geqslant n+1$, $t_1,\ldots,t_k > 0$, $c_1,\ldots,c_k > 0$ and $u_1,\ldots,u_k \in S^{n-1}$ satisfy (6), and there exist $\beta > 0$ and $n+1$ indices $\{i_1,\ldots,i_{n+1}\} \subset \{1,\ldots,k\}$ such that*

$$c_{i_1}\cdots c_{i_n} \det[u_{i_1},\ldots,u_{i_n}]^2 \geqslant \beta,$$

$$c_{i_2}\cdots c_{i_{n+1}} \det[u_{i_2},\ldots,u_{i_{n+1}}]^2 \geqslant \beta,$$

then

$$\det\left(\sum_{i=1}^{k} t_i c_i u_i \otimes u_i\right) \geqslant \left(1 + \frac{\beta(t_{i_1} - t_{i_{n+1}})^2}{4(t_{i_1} + t_{i_{n+1}})^2}\right) \prod_{i=1}^{k} t_i^{c_i}.$$

We may assume that $k \leqslant 2n^2$ (see Lemma 2.1), and thus the following observation from [**20**] can be used to estimate β in Corollary 4.3 from below.

Lemma 4.4. *If $k \geqslant n$, $c_1,\ldots,c_k>0$ and $u_1,\ldots,u_k \in S^{n-1}$ satisfy (6), then there exist $1 \leqslant i_1 < \ldots < i_n \leqslant k$ such that*

$$c_{i_1}\cdots c_{i_n}\det[u_{i_1},\ldots,u_{i_n}]^2 \geqslant \binom{k}{n}^{-1}.$$

5. *Discrete isotropic measures, orthonormal bases and approximation by a regular simplex*

According to Lemmas 3.2 and 5.1 in Böröczky and Hug [**20**], the following auxiliary results are available.

Lemma 5.1. *Let $v_1,\ldots,v_k \in \mathbb{R}^n \setminus \{0\}$ satisfy $\sum_{i=1}^k v_i \otimes v_i = \mathrm{I}_n$, and let $0<\eta<1/(3\sqrt{k})$. Assume for any $i \in \{1,\ldots,k\}$ that $\|v_i\| \leqslant \eta$ or there is some $j \in \{1,\ldots,n\}$ with $\angle(v_i,v_j) \leqslant \eta$. Then there exists an orthonormal basis $w_1,\ldots,w_n$ such that $\angle(v_i,w_i) < 3\sqrt{k}\,\eta$ for $i=1,\ldots,n$.*

Lemma 5.2. *Let $e \in S^{n-1}$ and let $\tau \in (0,1/(2n))$. If $w_1,\ldots,w_n$ is an orthonormal basis of $\mathbb{R}^n$ such that*

$$\frac{1}{\sqrt{n}}-\tau < \langle e,w_i\rangle < \frac{1}{\sqrt{n}}+\tau \quad \text{for } i=1,\ldots,n,$$

then there exists an orthonormal basis $\tilde{w}_1,\ldots,\tilde{w}_n$ such that $\langle e,\tilde{w}_i\rangle = \frac{1}{\sqrt{n}}$ and $\angle(w_i,\tilde{w}_i) < n\tau$ for $i=1,\ldots,n$.

Since $\sqrt{k}(n+1) < kn$ if $k>n\geqslant 2$, and $|\cos(\beta)-\frac{1}{\sqrt{n+1}}| \leqslant |\beta-\alpha|$ if $\alpha = \arccos\frac{1}{\sqrt{n+1}}$, we deduce from Lemmas 5.1 and 5.2 the following consequence.

Corollary 5.3. *Let $k>n\geqslant 2$, let $\tilde{u}_1,\ldots,\tilde{u}_k, e \in S^n$ in $\mathbb{R}^{n+1}$ and $\tilde{c}_1,\ldots,\tilde{c}_k>0$ satisfy $\sum_{i=1}^k \tilde{c}_i\tilde{u}_i\otimes\tilde{u}_i = \mathrm{I}_{n+1}$ and $\langle e,\tilde{u}_i\rangle = \frac{1}{\sqrt{n+1}}$ for $i=1,\ldots,k$, and let $0<\eta<1/(6kn)$. Assume for any $i\in\{1,\ldots,k\}$ that $\tilde{c}_i \leqslant \eta^2$ or there exists some $j\in\{1,\ldots,n+1\}$ with $\angle(\tilde{u}_i,\tilde{u}_j)\leqslant\eta$. Then there exists an orthonormal basis $\tilde{w}_1,\ldots,\tilde{w}_{n+1}$ of $\mathbb{R}^{n+1}$ such that $\langle e,\tilde{w}_i\rangle = \frac{1}{\sqrt{n+1}}$ and $\angle(\tilde{u}_i,\tilde{w}_i) < 3kn\eta$ for $i=1,\ldots,n+1$.*

For $\tilde{w}_1,\ldots,\tilde{w}_{n+1}\in S^n$ with $\langle e,\tilde{w}_i\rangle = \frac{1}{\sqrt{n+1}}$ for $i=1,\ldots,n+1$, the vectors $\tilde{w}_1,\ldots,\tilde{w}_{n+1}$ form an orthonormal basis of $\mathbb{R}^{n+1}$ if and only if their projection to $e^{\perp}$ form the vertices of a regular n-simplex. Therefore Corollary 5.3 provides information on how close $\mathrm{conv}\{\tilde{u}_1,\ldots,\tilde{u}_{n+1}\}$ is to some regular n-simplex. Lemma 5.2 in Böröczky and Hug [**20**] formulated this observation as follows.

Lemma 5.4. *Let Z be a polytope and let S be a regular simplex circumscribed to B^n. Assume that the facets of Z and S touch B^n at $u_1,\ldots,u_k$ and $w_1,\ldots,w_{n+1}$, respectively. Fix $\eta\in(0,1/(2n))$. If for any $i\in\{1,\ldots,k\}$, there exists some $j\in\{1,\ldots,n+1\}$ such that $\angle(u_i,w_j)\leqslant\eta$, then*

$$(1-n\eta)S \subset Z \subset (1+2n\eta)S.$$

Finally, we need to estimate the difference of Gaussian measures of certain polytopes $Z\subset S$. Since in our case $S\subset nB^n$, it is equivalent to estimate the volume difference up to a factor depending on n. Our first estimate of this kind is Lemma 5.3 in Böröczky and Hug [**20**].

Lemma 5.5. *Let Z be a polytope and let S be a regular simplex both circumscribed to B^n. Fix $\alpha = 9\cdot 2^{n+2}n^{2n+2}$ and $\eta\in(0,\alpha^{-1})$. Assume that the facets of Z and S touch B^n at*

$u_1,\dots,u_k$, $k\geqslant n+1$, and $w_1,\dots,w_{n+1}$, respectively. If $\angle(u_i,w_i)\leqslant\eta$ for $i=1,\dots,n+1$ and $\angle(u_k,w_i)\geqslant\alpha\eta$ for $i=1,\dots,n+1$, then

$$V(Z)\leqslant\left(1-\frac{\min_{i=1,\dots,n+1}\angle(u_k,w_i)}{2^{n+2}n^{2n}}\right)V(S).$$

Second, we prove another estimate concerning the volume difference of a convex body and a simplex.

LEMMA 5.6. *Let S be a regular simplex whose centroid is the origin, and let $M_1\subset S$ and $M_2\supset S$ be convex bodies. Suppose that there is some $\varepsilon\in(0,1)$ such that $M_1\not\supset(1-\varepsilon)S$ for (i) and $(1+\varepsilon)S\not\supset M_2$ for (ii), respectively. Then:*

(i) $V(S\setminus M_1)\geqslant\frac{n^n}{(n+1)^n}\,\varepsilon^n\,V(S)>\frac{1}{e}\,\varepsilon^n\,V(S)$;
(ii) $V(M_2\setminus S)\geqslant\frac{1}{n+1}\,\varepsilon\,V(S)$.

Proof. Let R be the circumradius of S, let $v_1,\dots,v_{n+1}$ be the vertices of S, and let $u_1,\dots,u_{n+1}$ be the corresponding exterior unit normals of the facets, and hence

$$v_i=-Ru_i\quad\text{for } i=1,\dots,n+1,\text{ and}\quad S=\left\{x\in\mathbb{R}^n:\langle x,u_i\rangle\leqslant\tfrac{R}{n}\quad\text{for } i=1,\dots,n+1\right\}.$$

For (i), there exists a v_i such that $(1-\varepsilon)v_i\notin M_1$, and hence there exists a closed halfspace H^+ with $(1-\varepsilon)v_i\in H^+$ and $H^+\cap M_1=\emptyset$. We observe that $(1-\varepsilon)v_i$ is the centroid of the simplex $S_\varepsilon=(1-\varepsilon)v_i+\varepsilon S\subset S$. Using Grünbaum's result [40] on minimal hyperplane sections of the simplex through its centroid, we obtain

$$V(S\setminus M_1)>V(S_\varepsilon\cap H^+)\geqslant\frac{n^n}{(n+1)^n}\,V(S_\varepsilon)=\frac{n^n}{(n+1)^n}\,\varepsilon^n\,V(S).$$

For (ii), there exists an $x_0\in M_2\setminus((1+\varepsilon)S)$, and hence there is a u_j such that $\langle x_0,u_j\rangle>\frac{(1+\varepsilon)R}{n}$. We write F_j to denote the facet of S with exterior unit normal u_j, and $|F_j|$ to denote the $(n-1)$-volume of F_j. It follows that

$$V(M_2\setminus S)\geqslant\frac{1}{n}\,\frac{\varepsilon R}{n}\,|F_j|=\frac{\varepsilon}{n+1}(n+1)\frac{1}{n}\frac{R}{n}|F_j|=\frac{\varepsilon}{n+1}\,V(S),$$

which completes the proof. □

REMARK. The estimates in (i) and in (ii) are optimal in the sense that there exist convex bodies M_1 and M_2 such that $V(S\setminus M_1)$ and $V(M_2\setminus S)$ are arbitrarily close to the first lower bound in (i) and the right-hand side in (ii), respectively. However, this will not be used in the following.

Finally, we provide some rough estimates that will be used repeatedly in the sequel.

LEMMA 5.7. *Let Δ_n be a regular simplex inscribed into B^n, and let $\Delta_n^\circ=-n\Delta_n$ be its polar. Then:*

(a) $\ell(\Delta_n)\leqslant\sqrt{n}^3$, $\ell(\Delta_n^\circ)\leqslant\sqrt{n}$;
(b) $V(\Delta_2)\leqslant 1.3$ *and* $V(\Delta_n)\leqslant 1$ *for* $n\geqslant 3$;
(c) $V(\Delta_n^\circ)=n^nV(\Delta_n)\geqslant(1+\frac{1}{n})^{\frac{n}{2}}>1$;
(d) $V(\Delta_n)\geqslant n^{-(n+2)}\ell(\Delta_n)$.

Proof. (a) Since $\frac{1}{n}B^n\subset\Delta_n$ and by an application of [65, (7)], we get

$$\ell(\Delta_n)\leqslant n\ell(B^n)=n\int_{\mathbb{R}^n}\|x\|\,\gamma_n(dx)=\frac{n^2}{\sqrt{2}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\leqslant\sqrt{n}^3.$$

For (b) and (c), we have

$$\frac{1}{n^n} \leqslant V(\Delta_n) = \left(1+\frac{1}{n}\right)^{\frac{n}{2}} \frac{\sqrt{n+1}}{n!} \leqslant \frac{\sqrt{e}\,\sqrt{n+1}}{n!} < 1,$$

where the upper bound on the right side only holds for $n \geqslant 3$.

(d) follows from (a) and (c). □

6. *On the derivatives of the transportation map*

Let f and h be probability density functions on $\mathbb{R}$ that are continuous and differentiable on the interiors of their supports, which are assumed to be intervals $I_f, I_h \subset \mathbb{R}$. Then there exists a transportation map $T : I_f \to I_h$ determined by

$$\int_{-\infty}^{x} f = \int_{-\infty}^{T(x)} h.$$

For $x \in I_f$, it follows that

$$T'(x) = \frac{f(x)}{h(T(x))}, \tag{15}$$

$$T''(x) = \frac{f(x)^2}{h(T(x))}\left(\frac{f'(x)}{f(x)^2} - \frac{h'(T(x))}{h(T(x))^2}\right). \tag{16}$$

Let g be the standard Gaussian density $g(t) = \sqrt{2\pi}^{-1} e^{-t^2/2}$, $t \in \mathbb{R}$, and for $s \in \mathbb{R}$, let g_s be the truncated Gaussian density

$$g_s(x) = \begin{cases} \left(\int_0^\infty g(t-s)\,dt\right)^{-1} g(x-s), & \text{if } x \geqslant 0,\\ 0, & \text{if } x < 0.\end{cases}$$

We frequently use that if $s \geqslant 0$, then

$$\frac{1}{2} \leqslant \int_0^\infty g(t-s)\,dt < 1.$$

We are going to apply (16) either in the case when $h = g$ and $f = g_s$, for some $s \in \mathbb{R}$, or when the roles of f, g are reversed. In particular, we consider the transport maps $\varphi_s : (0,\infty) \to \mathbb{R}$ and $\psi_s : \mathbb{R} \to (0,\infty)$ such that

$$\int_0^x g_s = \int_{-\infty}^{\varphi_s(x)} g \quad \text{and} \quad \int_{-\infty}^{y} g = \int_0^{\psi_s(y)} g_s.$$

Clearly, φ_s and ψ_s are inverses of each other for any given $s \in \mathbb{R}$.

Lemma 6.1. *Let $s \in [0, 0.15]$.*

(i) *If $x \in [0.74, 0.77]$, then $0 < \varphi_s(x) < 0.16$, $1.3 \leqslant \varphi_s'(x) \leqslant 2.05$ and $\varphi_s''(x) \leqslant -0.25$.*
(ii) *If $y \in [0, 0.15]$, then $0 < \psi_s(y) < 0.85$, $0.49 \leqslant \psi_s'(y) \leqslant 0.77$ and $\psi_s''(y) \geqslant 0.07$.*

Proof. We define $\alpha, \beta, \gamma, \delta, \xi > 0$ by the following integrals. The estimates for the values of $\alpha, \beta, \gamma, \delta, \xi > 0$ can be computed numerically.

$$\int_\delta^\infty g = \frac{7}{32}, \quad \text{thus } 0.77 < \delta < 0.78,$$
$$\int_\xi^\infty g = \frac{63}{256}, \quad \text{thus } 0.68 < \xi < 0.69,$$
$$\int_\alpha^\infty g = \frac{1}{4}, \quad \text{thus } 0.67 < \alpha < 0.68,$$
$$\int_\beta^\infty g = \frac{9}{32}, \quad \text{thus } 0.57 < \beta < 0.58,$$
$$\int_\gamma^\infty g = \frac{7}{16}, \quad \text{thus } 0.15 < \gamma < 0.16,$$

and therefore

$$\psi_0(0) = \alpha,$$
$$\psi_0(\gamma) = \delta > 0.77, \tag{17}$$
$$\psi_\gamma(0) = \gamma + \beta < 0.74, \tag{18}$$
$$\psi_\gamma(\gamma) = \gamma + \xi < 0.85. \tag{19}$$

First, we show that if $y \geqslant 0$, then the map $s \mapsto \psi_s(y) - s$, $s \geqslant 0$, is strictly decreasing and $\psi_s(y) - s > 0$.

In fact, by definition, we have

$$\int_{-\infty}^{y} g = \int_0^{\psi_s(y)} g_s = \left(\int_{-s}^{\infty} g \right)^{-1} \int_{-s}^{\psi_s(y)-s} g,$$

and hence

$$\int_{-\infty}^{y} g \int_{-s}^{\infty} g = \int_{-s}^{\infty} g - \int_{\psi_s(y)-s}^{\infty} g$$

or

$$\int_{-s}^{\infty} g \int_{y}^{\infty} g = \int_{\psi_s(y)-s}^{\infty} g. \tag{20}$$

The left-hand side of (20) is monotone increasing in s, hence the right-hand side is also increasing, which, in turn, implies that $\psi_s(y) - s$ is monotone decreasing as it is in the lower limit of the integral. Moreover, since the left side of (20) is less than $1/2$, it follows that $\psi_s(y) - s > 0$ for $y \geqslant 0$.

Now, we show that if $y \in [0, \gamma]$, then

$$\psi_s(y) \text{ is a monotone increasing function of } s \geqslant 0. \tag{21}$$

For the proof of (21), we show that if $0 \leqslant s < s'$, then the inequality

$$\int_0^{\psi_s(y)} g_{s'} \leqslant \int_0^{\psi_s(y)} g_s = \int_{-\infty}^{y} g \tag{22}$$

holds. The inequality (22) implies (21) because, by the positivity of $g_{s'}$, $\psi_{s'}(y) \geqslant \psi_s(y)$ must hold for

$$\int_0^{\psi_{s'}(y)} g_{s'} = \int_{-\infty}^{y} g$$

to be true. We set $x := \psi_s(y)$, $\Delta := s' - s \geqslant 0$ and define

$$A := \int_0^x g(\sigma - s)\, d\sigma, \qquad B := \int_0^\infty g(\sigma - s)\, d\sigma$$

and

$$a := \int_{-\Delta}^0 g(\sigma - s)\, d\sigma, \qquad b := \int_{x-\Delta}^x g(\sigma - s)\, d\sigma.$$

Note that

$$\int_0^x g_{s'} = \frac{\int_{-\Delta}^{x-\Delta} g(\tau - s)\, d\tau}{\int_{-\Delta}^{\infty} g(\tau - s)\, d\tau} = \frac{a + A - b}{a + B}$$

and the right-hand side of (22) equals A/B. Hence (22) is equivalent to

$$\frac{a + A - b}{a + B} \leqslant \frac{A}{B} \qquad \text{or} \qquad \frac{a}{b} \leqslant \frac{1}{1 - \frac{A}{B}}.$$

Since

$$\frac{A}{B} = \int_{-\infty}^y g \geqslant \frac{1}{2},$$

it is sufficient to show that $a/b \leqslant 2$.

By the symmetry of g, translation invariance of Lebesgue measure and inserting again $\Delta = s' - s$ and $x = \psi_s(y)$, we get

$$a = \int_s^{s'} g, \qquad b = \int_{s-\psi_s(y)}^{s'-\psi_s(y)} g.$$

Thus it remains to be shown that

$$\int_s^{s'} e^{-t^2/2}\, dt \leqslant \int_{s-\psi_s(y)}^{s'-\psi_s(y)} 2e^{-t^2/2}\, dt \tag{23}$$

for $0 \leqslant s < s'$ and $y \in [0, \gamma]$. To see this, we distinguish two cases.

If $s' - \psi_s(y) \leqslant 0$, then $2e^{-t^2/2} \geqslant 1$ for $t \in [s - \psi_s(y), s' - \psi_s(y)] \subset (-\infty, 0]$, since

$$\psi_s(y) - s \leqslant \psi_s(\gamma) - s \leqslant \psi_0(\gamma) - 0 = \psi_0(\gamma) < 0.78$$

and $2\exp(-0.5 \cdot 0.78^2) \geqslant 1.4 > 1$. Since $e^{-t^2/2} \leqslant 1$ for $t \in [s, s']$, the assertion follows in this case.

If $s' - \psi_s(y) > 0$, then by the previous reasoning and since $s - \psi_s(y) < 0$, we have

$$\int_s^{\psi_s(y)} e^{-t^2/2}\, dt \leqslant \int_{s-\psi_s(y)}^0 2e^{-t^2/2}\, dt, \tag{24}$$

and since $t \mapsto e^{-t^2/2}$, $t \geqslant 0$, is decreasing, we have

$$\int_{\psi_s(y)}^{s'} e^{-t^2/2}\, dt \leqslant \int_0^{s'-\psi_s(y)} e^{-t^2/2}\, dt, \tag{25}$$

so that (24) and (25) again imply (23). Thus, we have proved (22), and, in turn, (21).

We continue by proving the statements in (ii). We deduce from (17), (18) and (21) that $\psi_s(0) \leqslant \psi_s(\gamma) < 0.74$, $\psi_s(\gamma) \geqslant \psi_0(\gamma) > 0.77$, and hence

$$[0.74, 0.77] \subset \psi_s((0, \gamma)) \qquad \text{if } s \in [0, \gamma]. \tag{26}$$

We note that if $y \in [0,\gamma]$, then

$$\frac{g'(y)}{g(y)^2} = -\sqrt{2\pi}\, y e^{y^2/2} \geqslant -\sqrt{2\pi} \cdot 0.17. \tag{27}$$

On the other hand, if $0 \leqslant s \leqslant \gamma$ and $y \geqslant 0$, then

$$\frac{1}{2} \leqslant \int_{-s}^{\infty} g \quad \text{and} \quad \psi_s(y) - s \geqslant \psi_\gamma(y) - \gamma \geqslant \psi_\gamma(0) - \gamma = \beta > 0.57,$$

and therefore

$$\begin{aligned} \frac{g_s'(\psi_s(y))}{g_s(\psi_s(y))^2} &= -\sqrt{2\pi}\Big(\int_{-s}^{\infty} g\Big)(\psi_s(y) - s) e^{\frac{(\psi_s(y)-s)^2}{2}} \\ &\leqslant -\frac{\sqrt{2\pi}}{2}\beta\, e^{\frac{\beta^2}{2}} < -\sqrt{2\pi} \cdot 0.33. \end{aligned} \tag{28}$$

Combining (27) and (28), for $s, y \in [0,\gamma]$, we get

$$\frac{g'(y)}{g(y)^2} - \frac{g_s'(\psi_s(y))}{g_s(\psi_s(y))^2} \geqslant \sqrt{2\pi} \cdot 0.15. \tag{29}$$

If $s, y \in [0,\gamma]$, then

$$g_s(\psi_s(y)) \leqslant \frac{2}{\sqrt{2\pi}} \quad \text{and} \quad g(y) \geqslant \frac{e^{-\gamma^2/2}}{\sqrt{2\pi}} > \frac{0.98}{\sqrt{2\pi}}. \tag{30}$$

Hence, for $s, y \in [0,\gamma]$ we deduce from (16), (29) and (30) that

$$\psi_s''(y) \geqslant \frac{0.98^2}{2} \cdot 0.15 > 0.07. \tag{31}$$

In addition, (19) and (21) imply that if $s, y \in [0,\gamma]$, then

$$\psi_s(y) < 0.85. \tag{32}$$

To estimate the first derivative ψ_s', we use that (15) yields

$$\psi_s'(y) = \frac{g(y)}{g_s(\psi_s(y))}. \tag{33}$$

If $s, y \in [0,\gamma]$, then (30) and (33) yield

$$\psi_s'(y) \geqslant \frac{0.98/\sqrt{2\pi}}{2/\sqrt{2\pi}} = 0.49. \tag{34}$$

On the other hand, if $s, y \in [0,\gamma]$, then $0 < \psi_s(y) - s \leqslant \psi_0(y) - 0 \leqslant \psi_0(\gamma) = \delta < 0.78$, and hence

$$g_s(\psi_s(y)) = \frac{1}{\sqrt{2\pi}} \frac{e^{-(\psi_s(y)-s)^2/2}}{\int_{-s}^{\infty} g} \geqslant \frac{1}{\sqrt{2\pi}} \frac{e^{-0.78^2/2}}{\int_{-0.16}^{\infty} g} \geqslant \frac{1}{\sqrt{2\pi}} \cdot 1.3. \tag{35}$$

Hence we deduce from (33) that

$$\psi_s'(y) \leqslant \frac{1/\sqrt{2\pi}}{1.3/\sqrt{2\pi}} < 0.77. \tag{36}$$

We conclude (ii) from (31), (32), (34) and (36). This finishes the proof of Lemma 6.1(ii).

Finally, we prove part (i) of Lemma 6.1. Turning to φ_s, (26) yields

$$\varphi_s([0.74, 0.77]) \subset (0,\gamma) \quad \text{if } s \in [0,\gamma]. \tag{37}$$

It follows from (29) and (37) that if $s \in [0,\gamma]$ and $x \in [0.74, 0.77]$, then

$$\frac{g_s'(x)}{g_s(x)^2} - \frac{g'(\varphi_s(x)))}{g(\varphi_s(x))^2} \leqslant -\sqrt{2\pi} \cdot 0.15. \tag{38}$$

Now if $s \in [0,\gamma]$ and $x \in [0.74, 0.77]$, then we have

$$g_s(x) \geqslant \frac{\frac{1}{\sqrt{2\pi}} e^{-0.77^2/2}}{\int_{-0.16}^{\infty} g} > \frac{1.3}{\sqrt{2\pi}}, \qquad g(\varphi_s(x)) < \frac{1}{\sqrt{2\pi}}. \tag{39}$$

Hence, (39), (16) and (38) imply

$$\varphi_s''(x) \leqslant -1.3^2 \cdot 0.15 < -0.25. \tag{40}$$

To estimate the first derivative φ_s', we use that (15) yields

$$\varphi_s'(x) = \frac{g_s(x)}{g(\varphi_s(x))}. \tag{41}$$

If $s \in [0,\gamma]$ and $x \in [0.74, 0.77]$, then we conclude from (37) that

$$g(\varphi_s(x)) \geqslant \frac{e^{-\gamma^2/2}}{\sqrt{2\pi}} > \frac{0.98}{\sqrt{2\pi}} \quad \text{and} \quad g_s(x) \leqslant \frac{2}{\sqrt{2\pi}},$$

and hence (41) implies

$$\varphi_s'(x) \leqslant \frac{2/\sqrt{2\pi}}{0.98/\sqrt{2\pi}} < 2.05. \tag{42}$$

On the other hand, if $s \in [0,\gamma]$ and $x \in [0.74, 0.77]$, then we deduce from (39) and (41) that

$$\varphi_s'(x) > \frac{1.3/\sqrt{2\pi}}{1/\sqrt{2\pi}} = 1.3. \tag{43}$$

We conclude (i) from (40), (37), (42) and (43). □

In Proposition 6.2, we use the following notation. We fix an $e \in S^n \subset \mathbb{R}^{n+1}$, and identify $e^\perp \subset \mathbb{R}^{n+1}$ with $\mathbb{R}^n$. For $k \geqslant n+1$, let $u_1, \dots, u_k \in S^{n-1}$ and $c_1, \dots, c_k > 0$ be such that

$$\begin{aligned} \sum_{i=1}^k c_i u_i \otimes u_i &= \mathrm{I}_n, \\ \sum_{i=1}^k c_i u_i &= o. \end{aligned} \tag{44}$$

For each u_i, we consider

$$\begin{aligned} \tilde{u}_i &= \frac{\sqrt{n}}{\sqrt{n+1}} u_i + \tfrac{1}{\sqrt{n+1}} e \in S^n, \\ \tilde{c}_i &= \frac{n+1}{n} c_i, \end{aligned} \tag{45}$$

and hence (44) yields that

$$\sum_{i=1}^k \tilde{c}_i \, \tilde{u}_i \otimes \tilde{u}_i = \mathrm{I}_{n+1}.$$

Proposition 6.2. *With the above notation, let $k \leqslant 2n^2$, let $s \in [0, 0.15]$ and let $\varepsilon \in (0, n^{-56n})$. If*

$$\int_{\mathbb{R}^{n+1}} \prod_{i=1}^{k} g_s(\langle x, \tilde{u}_i \rangle)^{\tilde{c}_i}\, dx \geqslant 1 - \varepsilon, \text{ or} \tag{46}$$

$$\int^*_{\mathbb{R}^{n+1}} \sup_{x=\sum_{i=1}^k \tilde{c}_i \theta_i \tilde{u}_i} \prod_{i=1}^{k} g_s(\theta_i)^{\tilde{c}_i}\, dx \leqslant 1 + \varepsilon, \tag{47}$$

then there exists a regular simplex with vertices $w_1, \ldots, w_{n+1} \in S^{n-1}$ and $i_1 < \ldots < i_{n+1}$ such that $\angle(u_{i_j}, w_j) < n^{14n}\varepsilon^{1/4}$ for $j = 1, \ldots, n+1$.

Proof. According to Lemma 4.4, we may assume

$$\tilde{c}_1 \cdots \tilde{c}_{n+1} \det[\tilde{u}_1, \ldots, \tilde{u}_{n+1}]^2 \geqslant \binom{k}{n+1}^{-1}. \tag{48}$$

For $\eta = n^{10n}\varepsilon^{1/4} < 1$, we claim that if $i \in \{1, \ldots, k\}$, then

$$\tilde{c}_i \leqslant \eta^2, \text{ or there exists some } j \in \{1, \ldots, n+1\} \text{ with } \angle(\tilde{u}_i, \tilde{u}_j) \leqslant \eta. \tag{49}$$

We suppose that (49) does not hold, hence we may assume

$$\tilde{c}_{n+2} > \eta^2 \quad \text{and} \quad \angle(\tilde{u}_i, \tilde{u}_{n+2}) > \eta \quad \text{for } i = 1, \ldots, n+1.$$

We can write $\tilde{u}_{n+2} = \sum_{i=1}^{n+1} \lambda_i \tilde{u}_i$, where $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{R}$ are uniquely determined and satisfy $\lambda_1 + \cdots + \lambda_{n+1} = 1$. Hence we may assume that $\lambda_1 \geqslant \frac{1}{n+1}$. Therefore $\tilde{c}_{n+2} > \eta^2$, $\tilde{c}_1 \leqslant 1$ and (48) imply

$$\tilde{c}_2 \cdots \tilde{c}_{n+2} \det[\tilde{u}_2, \ldots, \tilde{u}_{n+2}]^2 \geqslant \binom{k}{n+1}^{-1} \frac{\eta^2}{(n+1)^2} \geqslant \frac{(n+1)!}{(2n^2)^{n+1}} \frac{\eta^2}{(n+1)^2}.$$

Here $\frac{(n+1)!}{(n+1)^2} \geqslant \frac{n^n}{(n+1)e^n} > \frac{n^{n-1}}{3^{n+1}}$, and thus

$$\tilde{c}_2 \cdots \tilde{c}_{n+2} \det[\tilde{u}_2, \ldots, \tilde{u}_{n+2}]^2 \geqslant \frac{\eta^2}{3^{n+1}2^{n+1}n^{n+3}} > \frac{\eta^2}{n^{4n+6}}. \tag{50}$$

In addition, $\angle(\tilde{u}_1, \tilde{u}_{n+2}) > \eta$ yields

$$\|\tilde{u}_1 - \tilde{u}_{n+2}\| > \eta/2. \tag{51}$$

We prove (49) separately for (46) and (47).

We start with the Brascamp–Lieb inequality; namely, we assume that (46) holds. We observe that if $i = 1, \ldots, k$, then

$$0.74 < \langle x, \tilde{u}_i \rangle < 0.77 \qquad \text{for } x \in 0.755\sqrt{n+1}\, e + 0.01B^n,$$

which can be checked by directly computing the inner product $\langle x, \tilde{u}_i \rangle$ using the definition of the vectors $\tilde{u}_i$ in (45).

Define

$$\Xi := 0.755\sqrt{n+1}\, e + 0.005\, \frac{\tilde{u}_1 - \tilde{u}_{n+2}}{\|\tilde{u}_1 - \tilde{u}_{n+2}\|} + 0.001B^n \subset 0.755\sqrt{n+1}\, e + 0.01B^n.$$

It follows using also (51), $\langle e, \tilde{u}_1 - \tilde{u}_{n+2}\rangle = 0$ and $V(B^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} > \frac{(2e\pi)^{\frac{n}{2}}}{n^{n/2}\sqrt{2\pi}e^{1/(6n)}} > \frac{1}{n^{n/2}}$ that

$$\langle x, \tilde{u}_1\rangle, \ldots, \langle x, \tilde{u}_k\rangle \in [0.74, 0.77] \quad \text{for } x \in \Xi, \tag{52}$$

$$\langle x, \tilde{u}_1\rangle - \langle x, \tilde{u}_{n+2}\rangle \geqslant 0.002\eta > 2^{-9}\eta \quad \text{for } x \in \Xi, \tag{53}$$

$$\Xi \subset \mathcal{C} := \{x \in \mathbb{R}^{n+1} : \langle \tilde{u}_i, x\rangle > 0 \; \forall i = 1, \ldots, k\}, \tag{54}$$

$$V(\Xi) = 0.001^n V(B^n) > \frac{1}{n^{11n}}, \tag{55}$$

where (54) is a consequence of (52). In addition, we consider the map $\Theta : \mathcal{C} \to \mathbb{R}^{n+1}$ with

$$\Theta(x) = \sum_{i=1}^{k} \tilde{c}_i \varphi_s(\langle \tilde{u}_i, x\rangle)\, \tilde{u}_i, \qquad x \in \mathcal{C},$$

which satisfies

$$d\Theta(x) = \sum_{i=1}^{k} \tilde{c}_i \varphi_s'(\langle \tilde{u}_i, x\rangle)\, \tilde{u}_i \otimes \tilde{u}_i.$$

As we have seen, $d\Theta$ is positive definite and $\Theta : \mathcal{C} \to \mathbb{R}^{n+1}$ is injective (see [**9, 11**]). Therefore, applying first (46), then (12), and after that the definition of Θ and (10), we obtain

$$\begin{aligned}
1 - \varepsilon &\leqslant \int_{\mathbb{R}^{n+1}} \prod_{i=1}^{k} g_s(\langle x, \tilde{u}_i\rangle)^{\tilde{c}_i}\, dx = \int_{\mathcal{C}} \prod_{i=1}^{k} g_s(\langle x, \tilde{u}_i\rangle)^{\tilde{c}_i}\, dx \\
&\leqslant \int_{\mathcal{C}} \left(\prod_{i=1}^{k} g(\varphi_s(\langle x, \tilde{u}_i\rangle))^{\tilde{c}_i}\right)\left(\prod_{i=1}^{k} \varphi_s'(\langle x, \tilde{u}_i\rangle)^{\tilde{c}_i}\right) dx \\
&= \left(\frac{1}{2\pi}\right)^{\frac{n+1}{2}} \int_{\mathcal{C}} \left(\prod_{i=1}^{k} e^{-\tilde{c}_i \varphi_s(\langle x, \tilde{u}_i\rangle)^2/2}\right)\left(\prod_{i=1}^{k} \varphi_s'(\langle x, \tilde{u}_i\rangle)^{\tilde{c}_i}\right) dx \\
&\leqslant \left(\frac{1}{2\pi}\right)^{\frac{n+1}{2}} \int_{\mathcal{C}} e^{-\|\Theta(x)\|^2/2}\left(\prod_{i=1}^{k} \varphi_s'(\langle x, \tilde{u}_i\rangle)^{\tilde{c}_i}\right) dx.
\end{aligned} \tag{56}$$

We deduce from (9) that

$$\prod_{i=1}^{k} \varphi_s'(\langle x, \tilde{u}_i\rangle)^{\tilde{c}_i} \leqslant \det\left(\sum_{i=1}^{k} \tilde{c}_i \varphi_s'(\langle x, \tilde{u}_i\rangle) \tilde{u}_i \otimes \tilde{u}_i\right) = \det(d\Theta(x)) \tag{57}$$

for any $x \in \mathcal{C}$.

If $s \in [0, 0.15]$ and $x \in \Xi$, then we can improve (57) using Corollary 4.3 based on (48) and (50) with

$$\beta_0 = \frac{\eta^2}{n^{4n+6}}.$$

Hence, applying first Corollary 4.3, then Lemma 6.1 (i), (52), (53) and finally $\eta < 1$, we get

$$\begin{aligned}\prod_{i=1}^{k}\varphi_s'(\langle x,\tilde u_i\rangle)^{\tilde c_i} &\leqslant \left(1+\frac{\beta_0(\varphi_s'(\langle x,\tilde u_1\rangle)-\varphi_s'(\langle x,\tilde u_{n+2}\rangle))^2}{4(\varphi_s'(\langle x,\tilde u_1\rangle)+\varphi_s'(\langle x,\tilde u_{n+2}\rangle))^2}\right)^{-1}\det(d\Theta(x))\\ &\leqslant \left(1+\frac{\beta_0(0.25(\langle x,\tilde u_1\rangle-\langle x,\tilde u_{n+2}\rangle))^2}{4(2\cdot 2.05)^2}\right)^{-1}\det(d\Theta(x))\\ &\leqslant \left(1+\frac{\eta^4 0.25^2 2^{-18}}{n^{4n+6}16\cdot 2.05^2}\right)^{-1}\det(d\Theta(x))\\ &\leqslant \left(1+\frac{\eta^4}{n^{4n+35}}\right)^{-1}\det(d\Theta(x)) \leqslant \left(1-\frac{\eta^4}{n^{4n+36}}\right)\det(d\Theta(x)).\end{aligned}$$

Moreover, if $s\in[0,0.15]$ and $x\in\Xi$, we deduce from (57) and Lemma 6.1(i) that

$$\det(d\Theta(x)) \geqslant \prod_{i=1}^{k}\varphi_s'(\langle x,\tilde u_i\rangle)^{\tilde c_i} \geqslant \prod_{i=1}^{k}1^{\tilde c_i}=1.$$

Thus if $s\in[0,0.15]$ and $x\in\Xi$, then

$$\prod_{i=1}^{k}\varphi_s'(\langle x,\tilde u_i\rangle)^{\tilde c_i} \leqslant \det(d\Theta(x)) - \frac{\eta^4}{n^{4n+36}}. \tag{58}$$

In addition, if $s\in[0,0.15]$ and $x\in\Xi$, then $\langle x,\tilde u_i\rangle\in[0.74,0.77]$ by (52) and hence $\varphi_s(\langle x,\tilde u_i\rangle)\subset(0,\gamma)$ by (37). Therefore, the definition of $\Theta(x)$, (10) and (11) imply

$$\|\Theta(x)\|^2 \leqslant \sum_{i=1}^{k}\tilde c_i\varphi_s(\langle\tilde u_i,x\rangle)^2 \leqslant \sum_{i=1}^{k}\tilde c_i 0.16^2 = 0.16^2(n+1),$$

and hence

$$\left(\frac{1}{2\pi}\right)^{\frac{n+1}{2}} e^{-\|\Theta(x)\|^2/2} \geqslant \left(\frac{1}{2\pi}\right)^{\frac{n+1}{2}} e^{-(n+1)0.16^2/2} > n^{-n-3}.$$

Applying first (54), using (57) and (58) in (56), then the substitution $z=\Theta(x)$, and finally also (55), we get

$$\begin{aligned}1-\varepsilon &\leqslant \left(\frac{1}{2\pi}\right)^{\frac{n+1}{2}}\int_{\mathcal C} e^{-\|\Theta(x)\|^2/2}\det(d\Theta(x))\,dx - \int_{\Xi}\frac{\eta^4}{n^{5n+39}}\,dx\\ &\leqslant \left(\frac{1}{2\pi}\right)^{\frac{n+1}{2}}\int_{\mathbb{R}^{n+1}} e^{-\|z\|^2/2}\,dz - \frac{1}{n^{11n}}\,\frac{\eta^4}{n^{5n+39}} \leqslant 1-\frac{\eta^4}{n^{39n}}.\end{aligned}$$

This contradicts $\eta = n^{10n}\varepsilon^{1/4}$, and hence we conclude (49) in the case of the Brascamp–Lieb inequality.

Now we consider the reverse Brascamp–Lieb inequality; namely, we assume that (47) holds. We observe that if $i\in\{1,\dots,k\}$, then

$$0\leqslant\langle x,\tilde u_i\rangle\leqslant 0.15 \quad \text{for } x\in 0.1\sqrt{n+1}\,e+0.05B^n,$$

and define

$$\widetilde\Xi := 0.1\sqrt{n+1}\,e + 0.03\,\frac{\tilde u_1-\tilde u_{n+2}}{\|\tilde u_1-\tilde u_{n+2}\|} + 0.01B^n \subset 0.1\sqrt{n+1}\,e+0.05B^n.$$

It follows using again (51), $\langle e, \tilde{u}_1 - \tilde{u}_{n+2}\rangle = 0$ and $V(B^n) > \frac{1}{n^{n/2}}$ that

$$\langle y, \tilde{u}_1\rangle, \ldots, \langle y, \tilde{u}_k\rangle \in [0, 0.15] \quad \text{for } y \in \widetilde{\Xi}, \tag{59}$$

$$\langle y, \tilde{u}_1\rangle - \langle y, \tilde{u}_{n+2}\rangle \geqslant 0.01\eta > 2^{-7}\eta \quad \text{for } y \in \widetilde{\Xi}, \tag{60}$$

$$V(\widetilde{\Xi}) = 0.01^n V(B^n) > \frac{1}{n^{8n}}. \tag{61}$$

In addition, we consider the map $\Psi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with

$$\Psi(y) = \sum_{i=1}^{k} \tilde{c}_i \psi_s(\langle \tilde{u}_i, y\rangle)\, \tilde{u}_i,$$

which satisfies

$$d\Psi(y) = \sum_{i=1}^{k} \tilde{c}_i \psi_s'(\langle \tilde{u}_i, y\rangle)\, \tilde{u}_i \otimes \tilde{u}_i.$$

As we have seen, $d\Psi$ is positive definite and $\Psi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is injective (see [**9, 11**]). Therefore, applying first (47), then the definition of Ψ, we obtain

$$\begin{aligned} 1 + \varepsilon &\geqslant \int^*_{\mathbb{R}^{n+1}} \sup_{x = \sum_{i=1}^k \tilde{c}_i \theta_i \tilde{u}_i} \prod_{i=1}^{k} g_s(\theta_i)^{\tilde{c}_i}\, dx \\ &\geqslant \int^*_{\mathbb{R}^{n+1}} \left(\sup_{\Psi(y) = \sum_{i=1}^k \tilde{c}_i \theta_i \tilde{u}_i} \prod_{i=1}^{k} g_s(\theta_i)^{\tilde{c}_i} \right) \det\left(d\Psi(y)\right) dy \\ &\geqslant \int_{\mathbb{R}^{n+1}} \left(\prod_{i=1}^{k} g_s(\psi_s(\langle \tilde{u}_i, y\rangle))^{\tilde{c}_i} \right) \det\left(\sum_{i=1}^{k} \tilde{c}_i \psi_s'(\langle \tilde{u}_i, y\rangle)\, \tilde{u}_i \otimes \tilde{u}_i \right) dy. \end{aligned} \tag{62}$$

Using (9), for $y \in \mathbb{R}^{n+1}$ we can bound the determinant in (62) from below by

$$\det\left(\sum_{i=1}^{k} \tilde{c}_i \psi_s'(\langle y, \tilde{u}_i\rangle) \tilde{u}_i \otimes \tilde{u}_i \right) \geqslant \prod_{i=1}^{k} \psi_s'(\langle y, \tilde{u}_i\rangle)^{\tilde{c}_i}. \tag{63}$$

If $s \in [0, 0.15]$ and $y \in \widetilde{\Xi}$, an application of Corollary 4.3 with $\beta_0 = \eta^2/n^{4n+6}$, Lemma 6.1 (ii), (59) and (60), allow us to improve (63) to get

$$\begin{aligned} \det\left(\sum_{i=1}^{k} \tilde{c}_i \psi_s'(\langle y, \tilde{u}_i\rangle) \tilde{u}_i \otimes \tilde{u}_i \right) &\geqslant \left(1 + \frac{\beta_0 (\psi_s'(\langle y, \tilde{u}_1\rangle) - \psi_s'(\langle y, \tilde{u}_{n+2}\rangle))^2}{4(\psi_s'(\langle y, \tilde{u}_1\rangle) + \psi_s'(\langle y, \tilde{u}_{n+2}\rangle))^2} \right) \prod_{i=1}^{k} \psi_s'(\langle y, \tilde{u}_i\rangle)^{\tilde{c}_i} \\ &\geqslant \left(1 + \frac{\beta_0 (0.07(\langle y, \tilde{u}_1\rangle - \langle y, \tilde{u}_{n+2}\rangle))^2}{4(2 \cdot 0.77)^2} \right) \prod_{i=1}^{k} \psi_s'(\langle y, \tilde{u}_i\rangle)^{\tilde{c}_i} \\ &\geqslant \left(1 + \frac{\eta^4 0.07^2 2^{-14}}{n^{4n+6} 16 \cdot 0.77^2} \right) \prod_{i=1}^{k} \psi_s'(\langle y, \tilde{u}_i\rangle)^{\tilde{c}_i} \\ &\geqslant \left(1 + \frac{\eta^4}{n^{4n+31}} \right) \prod_{i=1}^{k} \psi_s'(\langle y, \tilde{u}_i\rangle)^{\tilde{c}_i}. \end{aligned}$$

Moreover, if $s \in [0, 0.15]$ and $y \in \widetilde{\Xi}$, we deduce from Lemma 6.1(ii) and (11) that

$$\prod_{i=1}^{k} \psi_s'(\langle y, \tilde{u}_i\rangle)^{\tilde{c}_i} \geqslant \prod_{i=1}^{k} 0.49^{\tilde{c}_i} = 0.49^{n+1} > n^{-2n}.$$

Thus if $s \in [0, 0.15]$ and $y \in \widetilde{\Xi}$, then

$$\det\left(\sum_{i=1}^{k} \tilde{c}_i \psi_s'(\langle y, \tilde{u}_i\rangle)\tilde{u}_i \otimes \tilde{u}_i\right) \geqslant \prod_{i=1}^{k} \psi_s'(\langle y, \tilde{u}_i\rangle)^{\tilde{c}_i} + \frac{\eta^4}{n^{6n+31}}. \tag{64}$$

Further in (62), if $s \in [0, 0.15]$ and $y \in \widetilde{\Xi}$, then (59), Lemma 6.1 (ii), (11) and (35) imply

$$\prod_{i=1}^{k} g_s(\psi_s(\langle \tilde{u}_i, y\rangle))^{\tilde{c}_i} \geqslant \left(\frac{1.3}{\sqrt{2\pi}}\right)^{n+1} \geqslant 2^{-n-1} \geqslant n^{-n-1}. \tag{65}$$

Applying first (63), (64) and (65) in (62), and then (12) and (61), we deduce that if $s \in [0, 0.15]$, then

$$\begin{aligned} 1 + \varepsilon &\geqslant \int_{\mathbb{R}^{n+1}} \left(\prod_{i=1}^{k} g_s(\psi_s(\langle \tilde{u}_i, y\rangle))^{\tilde{c}_i}\right)\left(\prod_{i=1}^{k} \psi_s'(\langle \tilde{u}_i, y\rangle)^{\tilde{c}_i}\right) dy + \int_{\widetilde{\Xi}} \frac{\eta^4}{n^{7n+32}}\, dy \\ &\geqslant \int_{\mathbb{R}^{n+1}} \left(\prod_{i=1}^{k} g(\langle \tilde{u}_i, y\rangle)^{\tilde{c}_i}\right) dy + \frac{\eta^4}{n^{15n+31}} \\ &\geqslant \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{n+1}{2}} \int_{\mathbb{R}^{n+1}} e^{-\|y\|^2/2}\, dy + \frac{\eta^4}{n^{31n}} = 1 + \frac{\eta^4}{n^{31n}}. \end{aligned}$$

This contradicts $\eta = n^{10n}\varepsilon^{1/4}$, and hence we conclude (49) also in the case of the reverse Brascamp–Lieb inequality.

Now we return to the proof of Proposition 6.2. Since $k \leqslant 2n^2$ and $\varepsilon < n^{-56n}$, we have $\eta < 1/(6kn)$. Since (49) is available now, we can apply Corollary 5.3, which yields the existence of an orthonormal basis $\tilde{w}_1, \dots, \tilde{w}_{n+1}$ of $\mathbb{R}^{n+1}$ such that $\langle e, \tilde{w}_i\rangle = \frac{1}{\sqrt{n+1}}$ and $\|\tilde{u}_i - \tilde{w}_i\| \leqslant \angle(\tilde{u}_i, \tilde{w}_i) < 6n^3\eta$ for $\eta = n^{10n}\varepsilon^{1/4}$ and $i = 1, \dots, n+1$. Now we consider the vertices $w_1, \dots, w_{n+1} \in S^{n-1}$ of the regular simplex which are defined by the relations $\tilde{w}_i = \sqrt{\frac{n}{n+1}}\, w_i + \sqrt{\frac{1}{n+1}}\, e$ for $i = 1, \dots, n+1$. Therefore

$$\angle(u_i, w_i) < \frac{\pi}{2}\|u_i - w_i\| = \frac{\pi}{2}\sqrt{\frac{n+1}{n}}\,\|\tilde{u}_i - \tilde{w}_i\| < 18n^3\eta < n^{14n}\varepsilon^{1/4}$$

for $i = 1, \dots, n+1$. In turn, we conclude Proposition 6.2. □

We will actually use the Brascamp–Lieb inequality and its reverse for the function

$$\tilde{g}_s(t) = \mathbf{1}\{t \geqslant 0\} \exp\left(-\frac{(t-s)^2}{2}\right) \tag{66}$$

for $s \in \mathbb{R}$, that is,

$$\tilde{g}_s = \left(\int_{\mathbb{R}} \tilde{g}_s\right) g_s.$$

We note that if $s \geqslant 0$, then

$$\int_{\mathbb{R}} \tilde{g}_s \geqslant \frac{\sqrt{2\pi}}{2} > 1. \tag{67}$$

From Proposition 6.2 and (11), we deduce the following strengthened version of the Brascamp–Lieb inequality and its reverse for $\tilde{g}_s$.

COROLLARY 6.3. *Using the same notation as in Proposition 6.2, let $k \leqslant 2n^2$, let $s \in [0, 0.15]$ and let $\varrho \in (0,1)$.*

If for any regular simplex with vertices $w_1, \ldots, w_{n+1} \in S^{n-1}$ and any subset $\{i_1, \ldots, i_{n+1}\} \subset \{1, \ldots, k\}$, there exists $j \in \{1, \ldots, n+1\}$ such that $\angle(u_{i_j}, w_j) \geqslant \varrho$, then

$$\int_{\mathbb{R}^{n+1}} \prod_{i=1}^{k} \tilde{g}_s(\langle x, \tilde{u}_i \rangle)^{\tilde{c}_i} \, dx \leqslant (1 - n^{-56n} \varrho^4) \left(\int_{\mathbb{R}} \tilde{g}_s \right)^{n+1},$$

$$\int^*_{\mathbb{R}^{n+1}} \sup_{x = \sum_{i=1}^k \tilde{c}_i \theta_i \tilde{u}_i} \prod_{i=1}^{k} \tilde{g}_s(\theta_i)^{\tilde{c}_i} \, dx \geqslant (1 + n^{-56n} \varrho^4) \left(\int_{\mathbb{R}} \tilde{g}_s \right)^{n+1}.$$

7. *An almost regular simplex for Theorem 1.3 and Theorem 1.6 (a) and (b)*

The entire section is devoted to proving the following statement.

PROPOSITION 7.1. *Let $n+1 \leqslant k \leqslant 2n^2$, $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$ be such that*

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n,$$

$$\sum_{i=1}^{k} c_i u_i = o,$$

and $\ell(C) \geqslant (1-\varepsilon)\ell(\Delta_n)$ holds for $C = \mathrm{conv}\{u_1, \ldots, u_k\}$ and $\varepsilon \in (0, n^{-60n})$.

Then for $\eta = n^{15n} \varepsilon^{\frac{1}{4}} \in (0,1)$, there exists a regular simplex with vertices $w_1, \ldots, w_{n+1} \in S^{n-1}$ and $\{i_1, \ldots, i_{n+1}\} \subset \{1, \ldots, k\}$ such that

$$\angle(u_{i_j}, w_j) \leqslant \eta \quad \text{for } j = 1, \ldots, n+1.$$

We fix $e \in S^n \subset \mathbb{R}^{n+1}$, and identify $e^\perp \subset \mathbb{R}^{n+1}$ with $\mathbb{R}^n$. As before Proposition 6.2, for each u_i, we consider

$$\tilde{u}_i = \frac{\sqrt{n}}{\sqrt{n+1}} u_i + \frac{1}{\sqrt{n+1}} e \in S^n,$$

$$\tilde{c}_i = \frac{n+1}{n} c_i,$$

and hence

$$\sum_{i=1}^{k} \tilde{c}_i \, \tilde{u}_i \otimes \tilde{u}_i = \mathrm{I}_{n+1}, \tag{68}$$

$$\sum_{i=1}^{k} \tilde{c}_i = n+1. \tag{69}$$

Indirectly, we assume that Proposition 7.1 does not hold, and we aim at a contradiction. We deduce from Corollary 6.3 and (67) that if $s \in [0, 0.15]$, then

$$\int_{\mathbb{R}^{n+1}}^{*} \sup_{y=\sum_{i=1}^{k} \tilde{c}_i\theta_i\tilde{u}_i} \prod_{i=1}^{k} \tilde{g}_s(\theta_i)^{\tilde{c}_i}\,dy \geqslant (1+n^{-56n}\eta^4)\left(\int_{\mathbb{R}} \tilde{g}_s\right)^{n+1} > \left(\int_{\mathbb{R}} \tilde{g}_s\right)^{n+1} + n^{-56n}\eta^4. \quad (70)$$

Next we provide the following general auxiliary result (which holds independently of the indirect assumption).

Lemma 7.2. *If $s \in \mathbb{R}$ and C is defined as above, then:*

(i) $(2\pi)^{\frac{n}{2}} e^{-\frac{(n+1)s^2}{2}} \int_0^\infty e^{-\frac{r^2}{2}+sr\sqrt{n+1}}\gamma_n(r\sqrt{n}C)\,dr \geqslant \int_{\mathbb{R}^{n+1}}^{*} \sup_{y=\sum_{i=1}^{k} \tilde{c}_i\theta_i\tilde{u}_i} \prod_{i=1}^{k} \tilde{g}_s(\theta_i)^{\tilde{c}_i}\,dy;$

(ii) $(2\pi)^{\frac{n}{2}} e^{-\frac{(n+1)s^2}{2}} \int_0^\infty e^{-\frac{r^2}{2}+sr\sqrt{n+1}}\gamma_n(r\sqrt{n}\Delta_n)\,dr = \left(\int_{\mathbb{R}} \tilde{g}_s\right)^{n+1}.$

Proof. We consider the convex cone

$$\begin{aligned}\mathcal{C}_0 := &\left\{\sum_{i=1}^{k} \xi_i\tilde{u}_i : \xi_i \geqslant 0 \text{ for } i=1,\ldots,k\right\}\\ = &\left\{\sum_{i=1}^{k} r\sqrt{n}\lambda_i u_i + re : r \geqslant 0, \lambda_i \in [0,1], \sum_{i=1}^{k} \lambda_i = 1\right\}.\end{aligned}$$

Then we clearly have $x + re \in \mathcal{C}_0$ for $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$ if and only if $r \geqslant 0$ and $x \in r\sqrt{n}C$. If $y = \sum_{i=1}^{k} \tilde{c}_i\theta_i\tilde{u}_i$ for $\theta_1,\ldots,\theta_k \geqslant 0$, then $\langle y, e\rangle = (\sum_{i=1}^{k} \tilde{c}_i\theta_i)/\sqrt{n+1}$, and hence we deduce from (10), (68) and (69) that

$$\begin{aligned}\int_{\mathbb{R}^{n+1}}^{*} \sup_{y=\sum_{i=1}^{k} \tilde{c}_i\theta_i\tilde{u}_i} \prod_{i=1}^{k} \tilde{g}_s(\theta_i)^{\tilde{c}_i}\,dy &= \int_{\mathcal{C}_0}^{*} \sup_{y=\sum_{i=1}^{k} \tilde{c}_i\theta_i\tilde{u}_i,\theta_i\geqslant 0} e^{-\frac{1}{2}\sum_{i=1}^{k}\tilde{c}_i\theta_i^2+s\sum_{i=1}^{k}\tilde{c}_i\theta_i-\frac{s^2}{2}\sum_{i=1}^{k}\tilde{c}_i}\,dy\\ &\leqslant e^{-\frac{(n+1)s^2}{2}} \int_{\mathcal{C}_0} e^{-\frac{1}{2}\|y\|^2+s\langle y,e\rangle\sqrt{n+1}}\,dy\\ &= e^{-\frac{(n+1)s^2}{2}} \int_0^\infty \int_{r\sqrt{n}C} e^{-\frac{1}{2}(\|x\|^2+r^2)+sr\sqrt{n+1}}\,dx\,dr\\ &= (2\pi)^{\frac{n}{2}} e^{-\frac{(n+1)s^2}{2}} \int_0^\infty e^{-\frac{r^2}{2}+sr\sqrt{n+1}}\gamma_n(r\sqrt{n}C)\,dr,\end{aligned}$$

thus we have obtained (i).

For (ii), let $w_1,\ldots,w_{n+1}$ be the vertices of Δ_n, and let

$$\tilde{w}_i = \frac{\sqrt{n}}{\sqrt{n+1}}\,w_i + \frac{1}{\sqrt{n+1}}\,e$$

for $i = 1,\ldots,n+1$. Then $\tilde{w}_1,\ldots\tilde{w}_{n+1}$ form an orthonormal basis of $\mathbb{R}^{n+1}$, and hence

$$\sum_{i=1}^{n+1} \tilde{w}_i \otimes \tilde{w}_i = \mathrm{I}_{n+1}$$

(with $\tilde{c}_i = 1$ in this case). Moreover, for any $y \in \mathbb{R}^{n+1}$, there exist unique $\theta_1,\ldots,\theta_{n+1} \in \mathbb{R}$ satifying $y = \sum_{i=1}^{n+1} \theta_i\tilde{w}_i$, in fact, we have $\theta_i = \langle y, \tilde{w}_i\rangle$ and $\sum_{i=1}^{n+1} \theta_i^2 = \|y\|^2$ for $i = 1,\ldots,n+1$.

By the preceding argument, we deduce

$$(2\pi)^{\frac{n}{2}} e^{-\frac{(n+1)s^2}{2}} \int_0^\infty e^{-\frac{r^2}{2}+sr\sqrt{n+1}} \gamma_n(r\sqrt{n}\Delta_n)\,dr = \int^*_{\mathbb{R}^{n+1}} \sup_{y=\sum_{i=1}^k \theta_i \tilde{w}_i} \prod_{i=1}^{n+1} \tilde{g}_s(\theta_i)\,dy$$
$$= \left(\int_{\mathbb{R}} \tilde{g}_s\right)^{n+1},$$

where we used Fubini's theorem for the second equality. □

We apply the change of parameter $\tau = s\sqrt{n(n+1)}$ and the substitution $t = r\sqrt{n}$ in Lemma 7.2, and conclude with the help of the reverse Brascamp–Lieb inequality (14) that if $\tau \in \mathbb{R}$, then

$$\int_0^\infty e^{-\frac{1}{2n}(t-\tau)^2}\,\gamma_n(tC)\,dt = e^{\frac{-s^2(n+1)}{2}} \sqrt{n} \int_0^\infty e^{-\frac{r^2}{2}+sr\sqrt{n+1}} \gamma_n(r\sqrt{n}C)\,dr$$
$$\geqslant \frac{\sqrt{n}}{(2\pi)^{\frac{n}{2}}} \int^*_{\mathbb{R}^{n+1}} \sup_{y=\sum_{i=1}^k \tilde{c}_i\theta_i\tilde{u}_i} \prod_{i=1}^k \tilde{g}_s(\theta_i)^{\tilde{c}_i}\,dy$$
$$\geqslant \frac{\sqrt{n}}{(2\pi)^{\frac{n}{2}}} \left(\int_{\mathbb{R}} \tilde{g}_s\right)^{n+1}$$
$$= \int_0^\infty e^{-\frac{1}{2n}(t-\tau)^2}\,\gamma_n(t\Delta_n)\,dt.$$

Hence, we get

$$\int_0^\infty e^{-\frac{1}{2n}(t-\tau)^2}\,(1-\gamma_n(t\Delta_n))\,dt \geqslant \int_0^\infty e^{-\frac{1}{2n}(t-\tau)^2}\,(1-\gamma_n(tC))\,dt. \tag{71}$$

In addition, if $\tau \in [0, 0.15n] \subset [0, 0.15\sqrt{n(n+1)}]$, so that $s = \tau/\sqrt{n(n+1)} \in [0, 0.15)$, then using (70), instead of the reverse Brascamp–Lieb inequality (14), we obtain

$$\int_0^\infty e^{-\frac{1}{2n}(t-\tau)^2}\,\gamma_n(tC)\,dt \geqslant \int_0^\infty e^{-\frac{1}{2n}(t-\tau)^2}\,\gamma_n(t\Delta_n)\,dt + \frac{\sqrt{n}}{(2\pi)^{\frac{n}{2}}}\,n^{-56n}\eta^4,$$

and therefore

$$\int_0^\infty e^{-\frac{1}{2n}(t-\tau)^2}\,(1-\gamma_n(t\Delta_n))\,dt \geqslant \int_0^\infty e^{-\frac{1}{2n}(t-\tau)^2}\,(1-\gamma_n(tC))\,dt + \frac{\sqrt{n}}{(2\pi)^{\frac{n}{2}}}\,n^{-56n}\eta^4. \tag{72}$$

Integrating (71) for $\tau \in \mathbb{R}\setminus[0, 0.15n]$ and (72) for $\tau \in [0, 0.15n]$, we deduce that

$$\int_{-\infty}^\infty \int_0^\infty e^{-\frac{1}{2n}(t-\tau)^2}\,(1-\gamma_n(t\Delta_n))\,dt\,d\tau$$
$$\geqslant \int_{-\infty}^\infty \int_0^\infty e^{-\frac{1}{2n}(t-\tau)^2}\,(1-\gamma_n(tC))\,dt\,d\tau + 0.15n\,\frac{\sqrt{n}}{(2\pi)^{\frac{n}{2}}}\,n^{-56n}\eta^4. \tag{73}$$

Since for any $t \in \mathbb{R}$, we have

$$\int_{-\infty}^\infty e^{-\frac{1}{2n}(t-\tau)^2}\,d\tau = \sqrt{2\pi n},$$

we deduce from (2) and (73) that

$$\begin{aligned}
\ell(C) &= \int_0^\infty (1-\gamma_n(tC)\,dt \\
&= \frac{1}{\sqrt{2\pi n}}\int_{-\infty}^\infty\int_0^\infty e^{-\frac{1}{2n}(t-\tau)^2}\,(1-\gamma_n(tC))\,dt\,d\tau \\
&\leqslant \frac{1}{\sqrt{2\pi n}}\int_{-\infty}^\infty\int_0^\infty e^{-\frac{1}{2n}(t-\tau)^2}\,(1-\gamma_n(t\Delta_n))\,dt\,d\tau - \frac{0.15n}{(2\pi)^{\frac{n+1}{2}}}\,n^{-56n}\eta^4 \\
&= \ell(\Delta_n) - \frac{0.15n}{(2\pi)^{\frac{n+1}{2}}}\,n^{-56n}\eta^4. \qquad (74)
\end{aligned}$$

Hence, Lemma 5.7 (a), (74) and the hypothesis yield

$$(1-\varepsilon)\ell(\Delta_n) \leqslant \ell(C) < (1-n^{-60n}\eta^4)\ell(\Delta_n).$$

This contradicts $\eta = n^{15n}\varepsilon^{\frac{1}{4}}$, and in turn implies Proposition 7.1.

8. *Proof of Theorem* 1.3 *and of Theorem* 1.6 (a) *and* (b)

For Theorem 1.6, let μ be a centered isotropic measure on S^{n-1}, and let $K := Z_\infty(\mu)$, and hence $\operatorname{supp}\mu = \partial K \cap S^{n-1}$. In particular, under the assumptions of Theorem 1.3 and of Theorem 1.6 (a), we have $\ell(K) \geqslant (1-\varepsilon)\ell(\Delta_n)$. First, we assume that

$$0 < \varepsilon < n^{-100n}.$$

It follows from Lemma 2.1 and John's theorem that there exist $k \geqslant n+1$ with $k \leqslant 2n^2$, $u_1,\ldots,u_k \in \partial K \cap S^{n-1}$ and $c_1,\ldots,c_k > 0$ such that

$$\sum_{i=1}^k c_i u_i \otimes u_i = \mathrm{I}_n,$$

$$\sum_{i=1}^k c_i u_i = o.$$

We write μ_0 to denote the centered discrete isotropic measure with $\operatorname{supp}\mu_0 = \{u_1,\ldots,u_k\}$ and $\mu_0(\{u_i\}) = c_i$ for $i = 1,\ldots,k$, and define

$$C := Z_\infty(\mu_0) = \operatorname{conv}\{u_1,\ldots,u_k\}.$$

Since $\ell(C) \geqslant \ell(K) \geqslant (1-\varepsilon)\ell(\Delta_n)$ and $0 < \varepsilon < n^{-60n}$, it follows from Proposition 7.1 that we may assume that the vertices $w_1,\ldots,w_{n+1}$ of Δ_n satisfy

$$\angle(u_i,w_i) \leqslant \eta \ \text{ for } \eta = n^{15n}\varepsilon^{\frac{1}{4}} \quad \text{and} \quad i = 1,\ldots,n+1. \qquad (75)$$

For the simplex

$$S_0 = \operatorname{conv}\{u_1,\ldots,u_{n+1}\} \subset K,$$

we deduce from (75) and Lemma 5.4 (where we use $\eta < 1/(2n)$) that $S_0^\circ \subset (1+2n\eta)\Delta_n^\circ$, and hence

$$\widetilde{\Delta}_n := (1+2n\eta)^{-1}\Delta_n \subset S_0 \subset K. \qquad (76)$$

We note that

$$0 < \ell(\widetilde{\Delta}_n) - \ell(\Delta_n) = \int_{\mathbb{R}^n} \|x\|_{\widetilde{\Delta}_n}\, d\gamma_n(x) - \ell(\Delta_n)$$
$$\leqslant (1+2n\eta)\ell(\Delta_n) - \ell(\Delta_n) = 2n\eta\,\ell(\Delta_n). \tag{77}$$

Proof of Theorem 1.3. Let $\xi > 0$ be minimal such that

$$K \subset (1+\xi)\widetilde{\Delta}_n = (1+\xi)(1+2n\eta)^{-1}\Delta_n.$$

Then Lemmas 5.6 and 5.7 (d) imply

$$V(K \setminus \widetilde{\Delta}_n) \geqslant \frac{\xi}{n+1}\, V(\widetilde{\Delta}_n) = \frac{\xi}{(n+1)(1+2n\eta)^n}\, V(\Delta_n) > \frac{\xi}{n^{2n+4}}\, \ell(\Delta_n). \tag{78}$$

It follows from $K \subset B^n$, (76) and (78) that

$$\gamma_n(tK) \geqslant \gamma_n(t\widetilde{\Delta}_n) \quad \text{for } t > 0, \text{ and}$$

$$\gamma_n(tK) \geqslant \gamma_n(t\widetilde{\Delta}_n) + \frac{e^{-\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}}\, \frac{t^n \xi}{n^{2n+4}}\, \ell(\Delta_n) \quad \text{for } t \in (0,1],$$

where we used that $e^{-\frac{t^2}{2}} \geqslant e^{-\frac{1}{2}}$ for $t \in (0,1]$, and in turn we deduce from (2) that

$$\ell(\widetilde{\Delta}_n) - \ell(K) \geqslant \int_0^1 \gamma_n(tK) - \gamma_n(t\widetilde{\Delta}_n)\, dt \geqslant \int_0^1 \frac{t^n \xi}{n^{4n+4}}\, t\ell(\Delta_n)\, dt > n^{-(4n+5)}\ell(\Delta_n)\, \xi.$$

We conclude from (77) that

$$(1-\varepsilon)\ell(\Delta_n) \leqslant \ell(K) \leqslant (1 - n^{-(4n+5)}\xi + 2n\eta)\ell(\Delta_n),$$

and hence $\eta = n^{15n}\varepsilon^{\frac{1}{4}}$ implies

$$\xi \leqslant n^{4n+5}(2n\eta + \varepsilon) < n^{23n}\varepsilon^{\frac{1}{4}}.$$

It follows from (76) and the definition of ξ that

$$(1-2n\eta)\Delta_n \subset K \subset (1+\xi)\Delta_n. \tag{79}$$

Since $\Delta_n \subset B^n$, $\eta = n^{15n}\varepsilon^{\frac{1}{4}} < n^{23n}\varepsilon^{\frac{1}{4}} =: \tilde{\xi}$ and $\xi < \tilde{\xi}$, we conclude for the Hausdorff distance that $\delta_H(K, \Delta_n) < n^{23n}\varepsilon^{\frac{1}{4}}$.

To estimate the symmetric difference distance of K and Δ_n, Lemma 5.7 (b), (79) and $\xi < \tilde{\xi} \leqslant n^{-2n}$ yield

$$\delta_{\mathrm{vol}}(K, \Delta_n) \leqslant \Big((1+\tilde{\xi})^n - (1-\tilde{\xi})^n\Big) V(\Delta_n) \leqslant 2\tilde{\xi} n (1+\tilde{\xi})^{n-1} V(\Delta_n) < n^{25n}\varepsilon^{\frac{1}{4}},$$

which finishes the proof of Theorem 1.3 if $\varepsilon < n^{-100n}$. However, if $\varepsilon \geqslant n^{-100n}$, then Theorem 1.3 trivially holds as $\delta_{\mathrm{vol}}(M, \Delta_n) < \kappa_n$ and $\delta_H(M, \Delta_n) < 1$ for any convex body $M \subset B^n$ by the choice of the constant $c = n^{26n}$. □

Proof of Theorem 1.6 (a) and (b). We assume that $\ell(Z_\infty(\mu)) \geqslant (1-\varepsilon)\ell(\Delta_n)$ is available.

Let $\alpha_0 = 9 \cdot 2^{n+2} n^{2n+2}$ be the constant of Lemma 5.5. If for any $u \in \operatorname{supp}\mu$, there exists a w_i such that $\angle(u, w_i) \leqslant \alpha_0 \eta$, then

$$\delta_H(\operatorname{supp}\mu, \{w_1, \dots, w_{n+1}\}) \leqslant \alpha_0\eta < 9 \cdot 2^{n+2} n^{2n+2} n^{15n}\varepsilon^{\frac{1}{4}} < n^{22n}\,\varepsilon^{\frac{1}{4}}. \tag{80}$$

Therefore we indirectly assume that

$$\zeta := \max_{u\in \operatorname{supp}\mu} \min_{i=1,\dots,n+1} \angle(u, w_i) > \alpha_0\eta,$$

and hence there is some $u_0 \in \operatorname{supp}\mu$ such that $\min_{i=1}^{n+1} \angle(u_0, w_i) = \zeta$. Let

$$L := \operatorname{conv}\{u_0, u_1, \dots, u_{n+1}\}.$$

Lemma 5.5 and (75) imply

$$V(L^\circ) \leqslant \left(1 - \frac{\zeta}{2^{n+2}n^{2n}}\right) V(\Delta_n^\circ).$$

Since L is a polytope with $n+2$ vertices, it is shown in Meyer and Reisner [**56**] that

$$V(L)V(L^\circ) \geqslant V(\Delta_n)V(\Delta_n^\circ),$$

which proves a special case of the Mahler conjecture. Therefore we get

$$V(L) \geqslant \left(1 + \frac{\zeta}{2^{n+2}n^{2n}}\right) V(\Delta_n), \tag{81}$$

while readily

$$\widetilde{\Delta}_n \subset S_0 \subset L$$

holds for $\widetilde{\Delta}_n = (1 + 2n\eta)^{-1}\Delta_n$. It follows from this, $L \subset B^n$ and (81) that

$$\gamma_n(tL) > \gamma_n(t\widetilde{\Delta}_n) \quad \text{for } t > 0,$$

$$\gamma_n(tL) > \gamma_n(t\widetilde{\Delta}_n) + \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}^{\frac{n}{2}}} \frac{t^n\zeta}{2^{n+2}n^{2n}} V(\Delta_n) \quad \text{for } t \in (0, 1].$$

We deduce from (2) and Lemma 5.7 (d) that

$$\ell(\widetilde{\Delta}_n) - \ell(L) \geqslant \int_0^1 \gamma_n(tL) - \gamma_n(t\widetilde{\Delta}_n)\,dt > \int_0^1 \frac{t^n\zeta}{n^{5n+5}} \ell(\Delta_n)\,dt > n^{-8n}\ell(\Delta_n)\,\zeta.$$

We conclude from (77) that

$$(1 - \varepsilon)\ell(\Delta_n) \leqslant \ell(Z_\infty(\mu)) \leqslant \ell(L) \leqslant (1 - n^{-8n}\zeta + 2n\eta)\ell(\Delta_n),$$

and hence

$$\zeta \leqslant n^{8n}(2n\eta + \varepsilon) < n^{22n}\varepsilon^{\frac{1}{4}}.$$

Therefore in both cases (compare (80)), if $\ell(Z_\infty(\mu)) \geqslant (1 - \varepsilon)\ell(\Delta_n)$ for a centered isotropic measure μ on S^{n-1}, then

$$\delta_H(\operatorname{supp}\mu, \{w_1, \dots, w_{n+1}\}) < n^{28n}\,\varepsilon^{\frac{1}{4}}.$$

Since $\ell(Z_\infty(\mu)) \geqslant (1 - \varepsilon)\ell(\Delta_n)$ and $W(Z_\infty(\mu)^\circ) \geqslant (1 - \varepsilon)W(\Delta_n^\circ)$ are equivalent according to (1), we have verified the case $W(Z_\infty(\mu)^\circ) \geqslant (1 - \varepsilon)W(\Delta_n^\circ)$ of Theorem 1.6 as well, in the case $\varepsilon < n^{-100n}$. However, if $\varepsilon \geqslant n^{-100n}$, then Theorem 1.6 trivially holds since for any $x \in S^{n-1}$ there exists a vertex w of Δ_n with $\|x - w\| \leqslant \sqrt{2}$. □

9. *An almost regular simplex for Theorem 1.4 and Theorem 1.6 (c) and (d)*

The whole section is dedicated to proving the following statement.

PROPOSITION 9.1. *Let $n+1 \leqslant k \leqslant 2n^2$, $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$ be such that*

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n,$$

$$\sum_{i=1}^{k} c_i u_i = o,$$

and $\ell(C^\circ) \leqslant (1+\varepsilon)\ell(\Delta_n^\circ)$ holds for $C = \mathrm{conv}\{u_1, \ldots, u_k\}$ and $\varepsilon \in (0, n^{-60n})$.

Then for $\eta = n^{15n}\varepsilon^{\frac{1}{4}} \in (0,1)$, there exists a regular simplex with vertices $w_1, \ldots, w_{n+1} \in S^{n-1}$ and $\{i_1, \ldots, i_{n+1}\} \subset \{1, \ldots, k\}$ such that

$$\angle(u_{i_j}, w_j) \leqslant \eta \quad \text{for } j = 1, \ldots, n+1.$$

We recall from (66) that if $s \in \mathbb{R}$, then $\tilde{g}_s$ is defined by

$$\tilde{g}_s(t) = \mathbf{1}\{t \geqslant 0\} \exp\left(-\frac{(t-s)^2}{2}\right), \qquad t \in \mathbb{R}.$$

In this section, we slightly change the setup used in Proposition 6.2 and Corollary 6.3. As before Proposition 6.2, we fix an $e \in S^n \subset \mathbb{R}^{n+1}$, and identify $e^\perp \subset \mathbb{R}^{n+1}$ with $\mathbb{R}^n$. However, now, for each $u_i \in S^{n-1}$, we consider

$$\tilde{u}_i = -\frac{\sqrt{n}}{\sqrt{n+1}}\, u_i + \frac{1}{\sqrt{n+1}}\, e \in S^n,$$

$$\tilde{c}_i = \frac{n+1}{n}\, c_i,$$

and hence

$$\sum_{i=1}^{k} \tilde{c}_i\, \tilde{u}_i \otimes \tilde{u}_i = \mathrm{I}_{n+1}, \tag{82}$$

$$\sum_{i=1}^{k} \tilde{c}_i = n+1, \tag{83}$$

$$\sum_{i=1}^{k} \tilde{c}_i \tilde{u}_i = \frac{\sum_{i=1}^{k} \tilde{c}_i}{\sqrt{n+1}} \cdot e = \sqrt{n+1}\, e. \tag{84}$$

For the convex cone

$$\widetilde{\mathcal{C}} := \{z \in \mathbb{R}^{n+1} : \langle z, \tilde{u}_i \rangle \geqslant 0 \ \ i = 1, \ldots, k\},$$

the use of $-u_i$ instead of u_i in the definition of $\tilde{u}_i$ ensures that

$$x + r\, e \in \widetilde{\mathcal{C}} \text{ for } x \in \mathbb{R}^n \text{ and } r \in \mathbb{R} \ \text{ if and only if } \ r \geqslant 0 \text{ and } x \in \tfrac{r}{\sqrt{n}}\, C^\circ. \tag{85}$$

Moreover, we observe that if $C = \Delta_n$, then $k = n+1$, and $\tilde{u}_1, \ldots, \tilde{u}_{n+1}$ form an orthonormal basis of $\mathbb{R}^{n+1}$.

Since $-u_1, \ldots, -u_k$ satisfy the same conditions as $u_1, \ldots, u_k$, it follows that Corollary 6.3 remains true for the vectors $\tilde{u}_1, \ldots, \tilde{u}_k$ as defined in this section.

We suppose that Proposition 9.1 does not hold, and we seek a contradiction. From Corollary 6.3 and (67), we deduce that if $s \in [0, 0.15]$, then

$$\int_{\mathbb{R}^{n+1}} \prod_{i=1}^{k} \tilde{g}_s(\langle z, \tilde{u}_i\rangle)^{\tilde{c}_i}\, dz \leqslant (1 - n^{-56n}\eta^4)\left(\int_{\mathbb{R}} \tilde{g}_s\right)^{n+1} \leqslant \left(\int_{\mathbb{R}} \tilde{g}_s\right)^{n+1} - n^{-56n}\eta^4. \tag{86}$$

Next we state a counterpart to Lemma 7.2, which provides general relations independent of the indirect reasoning used to establish Proposition 9.1.

Lemma 9.2. *If $s \in \mathbb{R}$ and C is defined as above, then:*

(i) $(2\pi)^{\frac{n}{2}} e^{-\frac{(n+1)s^2}{2}} \int_0^\infty e^{-\frac{r^2}{2} + sr\sqrt{n+1}} \gamma_n(\frac{r}{\sqrt{n}}\, C^\circ)\, dr = \int_{\mathbb{R}^{n+1}} \prod_{i=1}^{k} \tilde{g}_s(\langle z, \tilde{u}_i\rangle)^{\tilde{c}_i}\, dz;$

(ii) $(2\pi)^{\frac{n}{2}} e^{-\frac{(n+1)s^2}{2}} \int_0^\infty e^{-\frac{r^2}{2} + sr\sqrt{n+1}} \gamma_n(\frac{r}{\sqrt{n}}\, \Delta_n^\circ)\, dr = \left(\int_{\mathbb{R}} \tilde{g}_s\right)^{n+1}.$

Proof. Applying first (82), (83), (84) and then (85), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}} \prod_{i=1}^{k} \tilde{g}_s(\langle z, \tilde{u}_i\rangle)^{\tilde{c}_i}\, dz &= \int_{\widetilde{C}} \exp\left(-\frac{1}{2}\sum_{i=1}^{k} \tilde{c}_i \langle z, \tilde{u}_i\rangle^2 + s\sum_{i=1}^{k} \tilde{c}_i\langle z, \tilde{u}_i\rangle - \frac{s^2}{2}\sum_{i=1}^{k}\tilde{c}_i\right) dz \\
&= e^{-\frac{(n+1)s^2}{2}} \int_{\widetilde{C}} e^{-\frac{\|z\|^2}{2} + s\sqrt{n+1}\langle z, e\rangle}\, dz \\
&= e^{-\frac{(n+1)s^2}{2}} \int_0^\infty \int_{\frac{r}{\sqrt{n}} C^\circ} e^{-\frac{\|x\|^2 + r^2}{2} + sr\sqrt{n+1}}\, dx\, dr \\
&= (2\pi)^{\frac{n}{2}} e^{-\frac{(n+1)s^2}{2}} \int_0^\infty e^{-\frac{r^2}{2} + sr\sqrt{n+1}} \gamma_n\left(\frac{r}{\sqrt{n}}\, C^\circ\right) dr.
\end{aligned}$$

For (ii), we observe that if we replace C by Δ_n in the argument above, then the analogues of $\tilde{u}_1, \ldots, \tilde{u}_{n+1}$ form an orthonormal basis of $\mathbb{R}^{n+1}$ and $\tilde{c}_i$ is replaced by 1. □

We apply the change of parameter $\tau = s\sqrt{\frac{n+1}{n}}$ and the substitution $t = r/\sqrt{n}$ in Lemma 9.2, and conclude with the help of the Brascamp–Lieb inequality (13) that if $\tau \in \mathbb{R}$, then

$$\begin{aligned}
\int_0^\infty e^{-\frac{n}{2}(t-\tau)^2}\, \gamma_n(tC^\circ)\, dt &= e^{-\frac{n\tau^2}{2}} \int_0^\infty e^{-\frac{nt^2}{2} + st\sqrt{n(n+1)}}\, \gamma_n(tC^\circ)\, dt \\
&= \frac{e^{\frac{-(n+1)s^2}{2}}}{\sqrt{n}} \int_0^\infty e^{-\frac{r^2}{2} + sr\sqrt{n+1}} \gamma_n\left(\frac{r}{\sqrt{n}} C^\circ\right) dr \\
&= \frac{1}{\sqrt{n}(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{n+1}} \prod_{i=1}^{k} \tilde{g}_s(\langle z, \tilde{u}_i\rangle)^{\tilde{c}_i}\, dz \\
&\leqslant \frac{1}{\sqrt{n}(2\pi)^{\frac{n}{2}}} \left(\int_{\mathbb{R}} \tilde{g}_s\right)^{n+1} \\
&= \int_0^\infty e^{-\frac{n}{2}(t-\tau)^2}\, \gamma_n(t\Delta_n^\circ)\, dt,
\end{aligned}$$

and hence

$$\int_0^\infty e^{-\frac{n}{2}(t-\tau)^2}\,(1-\gamma_n(t\Delta_n^\circ))\,dt \leqslant \int_0^\infty e^{-\frac{n}{2}(t-\tau)^2}\,(1-\gamma_n(tC^\circ))\,dt. \tag{87}$$

In addition, if $\tau\in[0,0.15]\subset[0,0.15\sqrt{\frac{n+1}{n}})$, which implies that $s\in[0,0.15)$, then using (86) instead of the Brascamp–Lieb inequality (13), we obtain

$$\begin{aligned}\int_0^\infty e^{-\frac{n}{2}(t-\tau)^2}\,\gamma_n(tC^\circ)\,dt &= \frac{1}{\sqrt{n}(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^{n+1}}\prod_{i=1}^k \tilde{g}_s(\langle z,\tilde{u}_i\rangle)^{\tilde{c}_i}\,dz\\ &\leqslant \frac{1}{\sqrt{n}(2\pi)^{\frac{n}{2}}}\left(\left(\int_{\mathbb{R}}\tilde{g}_s\right)^{n+1}-n^{-56n}\,\eta^4\right)\\ &= \int_0^\infty e^{-\frac{n}{2}(t-\tau)^2}\,\gamma_n(t\Delta_n^\circ)\,dt-\frac{n^{-56n}}{\sqrt{n}(2\pi)^{\frac{n}{2}}}\,\eta^4.\end{aligned}$$

Hence, we get

$$\int_0^\infty e^{-\frac{n}{2}(t-\tau)^2}\,(1-\gamma_n(t\Delta_n^\circ))\,dt \leqslant \int_0^\infty e^{-\frac{n}{2}(t-\tau)^2}\,(1-\gamma_n(tC^\circ))\,dt-\frac{0.15\,n^{-56n}}{\sqrt{n}(2\pi)^{\frac{n}{2}}}\,\eta^4. \tag{88}$$

Integrating (87) for $\tau\in\mathbb{R}\setminus[0,0.15]$ and (88) for $\tau\in[0,0.15]$, we deduce that

$$\begin{aligned}&\int_{-\infty}^\infty\int_0^\infty e^{-\frac{n}{2}(t-\tau)^2}\,(1-\gamma_n(t\Delta_n^\circ))\,dt\,d\tau\\ &\qquad\leqslant \int_{-\infty}^\infty\int_0^\infty e^{-\frac{n}{2}(t-\tau)^2}\,(1-\gamma_n(tC^\circ))\,dtd\tau-\frac{0.15\,n^{-56n}}{\sqrt{n}(2\pi)^{\frac{n}{2}}}\,\eta^4.\end{aligned} \tag{89}$$

Since for any $t\in\mathbb{R}$, we have

$$\int_{-\infty}^\infty e^{-\frac{n}{2}(t-\tau)^2}\,d\tau=\sqrt{\frac{2\pi}{n}},$$

we deduce from (2) and (89) that

$$\begin{aligned}\ell(C^\circ) &= \int_0^\infty(1-\gamma_n(tC^\circ))\,dt\\ &= \sqrt{\frac{n}{2\pi}}\int_{-\infty}^\infty\int_0^\infty e^{-\frac{n}{2}(t-\tau)^2}\,(1-\gamma_n(tC^\circ))\,dt\,d\tau\\ &\geqslant \sqrt{\frac{n}{2\pi}}\int_{-\infty}^\infty\int_0^\infty e^{-\frac{n}{2}(t-\tau)^2}\,(1-\gamma_n(t\Delta_n^\circ))\,dt\,d\tau+\frac{0.15\,n^{-56n}}{\sqrt{n}(2\pi)^{\frac{n}{2}}}\,\eta^4\\ &= \int_0^\infty(1-\gamma_n(t\Delta_n^\circ))\,dt+\frac{0.15\,n^{-56n}}{(2\pi)^{\frac{n+1}{2}}}\,\eta^4\\ &\geqslant \ell(\Delta_n^\circ)+n^{-60n}\eta^4\ell(\Delta_n^\circ),\end{aligned}$$

where Lemma 5.7 (a) was used in the last step. This shows that

$$(1+\varepsilon)\ell(\Delta_n^\circ)\geqslant\ell(C^\circ)>(1+n^{-60n}\eta^4)\ell(\Delta_n^\circ),$$

which contradicts $\eta=n^{15n}\varepsilon^{\frac14}$, and in turn implies Proposition 9.1.

10. *Proof of Theorem* 1.4 *and of Theorem* 1.6 (c) *and* (d)

For Theorem 1.6, let μ be a centered isotropic measure on S^{n-1}, and let $K = Z_\infty(\mu)^\circ$, and hence $\operatorname{supp}\mu = \partial K \cap S^{n-1}$. In particular, under the assumptions of Theorem 1.4 and of Theorem 1.6 (d), we have $\ell(K) \leqslant (1+\varepsilon)\ell(\Delta_n^\circ)$. First, we assume

$$\varepsilon < n^{-100n}.$$

It follows from Lemma 2.1 and John's theorem that there exist $k \geqslant n+1$ with $k \leqslant 2n^2$, $u_1, \dots, u_k \in \partial K \cap S^{n-1}$ and $c_1, \dots, c_k > 0$ such that

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n,$$

$$\sum_{i=1}^{k} c_i u_i = o.$$

We write μ_0 to denote the centered discrete isotropic measure with $\operatorname{supp}\mu_0 = \{u_1, \dots, u_k\}$ and $\mu_0(\{u_i\}) = c_i$ for $i = 1, \dots, k$, and define (again)

$$C := Z_\infty(\mu_0) = \operatorname{conv}\{u_1, \dots, u_k\} \subset K^\circ.$$

Since $\ell(C^\circ) \leqslant \ell(K) \leqslant (1+\varepsilon)\ell(\Delta_n^\circ)$, it follows from Proposition 9.1 that we may assume that the vertices $w_1, \dots, w_{n+1}$ of Δ_n satisfy

$$\angle(u_i, w_i) \leqslant \eta \quad \text{for } \eta = n^{15n}\varepsilon^{\frac{1}{4}} \quad \text{and} \quad i = 1, \dots, n+1. \tag{90}$$

We observe that $K \subset S_1 := S_0^\circ$, where S_1 is the polar of S_0 and the facets of

$$S_1 = \bigcap_{i=1}^{n+1} \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant 1\}$$

touch B^n at $u_1, \dots, u_{n+1}$. We deduce from (90) and Lemma 5.4 that

$$(1 - n\eta)\Delta_n^\circ \subset S_1 \subset (1 + 2n\eta)\Delta_n^\circ \subset 2\Delta_n^\circ. \tag{91}$$

We claim that

$$\delta_{\mathrm{vol}}(K, S_1) = V(S_1 \setminus K) \leqslant n^{23n}\varepsilon^{\frac{1}{4}}. \tag{92}$$

Using $\frac{1}{2n} S_1 \subset \frac{1}{n}\Delta_n^\circ \subset B^n$, (2) and Lemma 5.7 (a), we get

$$\begin{aligned}
\ell(K) - \ell(S_1) &= \int_0^\infty (\gamma_n(tS_1) - \gamma_n(tK))\, dt \geqslant \int_0^{\frac{1}{2n}} \frac{e^{-\frac{1}{2}}}{(2\pi)^{n/2}} t^n V(S_1 \setminus K)\, dt \\
&\geqslant \frac{e^{-\frac{1}{2}}}{(n+1)(2\pi)^{n/2}(2n)^{n+1}} V(S_1 \setminus K) \frac{\ell(\Delta_n^\circ)}{\sqrt{n}} > \frac{V(S_1 \setminus K)}{n^{6n}} \ell(\Delta_n^\circ).
\end{aligned} \tag{93}$$

In addition, (91) yields

$$\ell(S_1) - \ell(\Delta_n^\circ) \geqslant \ell((1+2n\eta)\Delta_n^\circ) - \ell(\Delta_n^\circ) = ((1+2n\eta)^{-1} - 1)\ell(\Delta_n^\circ) \geqslant -2n\eta\, \ell(\Delta_n^\circ). \tag{94}$$

We deduce from $\ell(K) \leqslant (1+\varepsilon)\ell(\Delta_n^\circ)$, (93) and (94) that

$$\varepsilon\, \ell(\Delta_n^\circ) \geqslant \ell(K) - \ell(\Delta_n^\circ) > \left(\frac{V(S_1 \setminus K)}{n^{6n}} - 2n\eta \right) \ell(\Delta_n^\circ).$$

Then $\eta = n^{15n}\varepsilon^{\frac{1}{4}}$ implies

$$V(S_1 \setminus K) < n^{6n}(\varepsilon + 2n\eta) < 4n \cdot n^{6n}\eta < n^{23n}\varepsilon^{\frac{1}{4}},$$

which proves (92).

Proof of Theorem 1.4. We start to deal with the symmetric volume distance of K and Δ°. Using (91), $(1+2n\eta)^n \leqslant 4/3$ and Lemma 5.7 (b), we get

$$\delta_{\text{vol}}(S_0, \Delta_n^\circ) \leqslant ((1+2n\eta)^n - (1-n\eta)^n)V(\Delta_n^\circ) < n\,4n\eta\,V(\Delta_n^\circ) < n^{19n}\varepsilon^{\frac{1}{4}}. \tag{95}$$

Combining (92) and (95), we get $\delta_{\text{vol}}(K, \Delta^\circ) \leqslant n^{24n}\varepsilon^{\frac{1}{4}}$, which proves Theorem 1.4(i) under the assumption $\varepsilon < n^{-100n}$.

In order to derive an upper bound for the Hausdorff distance of K and Δ°, we first show that the centroid σ_0 of S_0 satisfies

$$\sigma_0 \in 4n\eta\Delta_n^\circ. \tag{96}$$

To prove (96), we observe that

$$\Delta_n^\circ = \bigcap_{i=1}^{n+1}\{x \in \mathbb{R}^n : \langle x, w_i\rangle \leqslant 1\} = \text{conv}\{-nw_1, \dots, -nw_{n+1}\}.$$

For each $j \in \{1, \dots, n+1\}$, (91) yields that S_1 has a vertex v_j with

$$\langle -w_j, v_j\rangle \geqslant h_{(1-n\eta)\Delta_n^\circ}(-w_j) = (1-n\eta)n.$$

Since $S_1 \subset (1+2n\eta)\Delta_n^\circ$ and $\Delta_n^\circ + nw_j$ is homothetic to Δ_n° with o as the vertex with exterior normal $-w_j$, we have

$$\begin{aligned} v_j &\in \{x \in (1+2n\eta)\Delta_n^\circ : \langle -w_j, x\rangle \geqslant (1-n\eta)n\} \\ &= -(1+2n\eta)nw_j + 3n\eta((1+2n\eta)\Delta_n^\circ + (1+2n\eta)nw_j) \end{aligned} \tag{97}$$

for $j = 1, \dots, n+1$. Hence, the vertices $v_1, \dots, v_{n+1}$ are contained in mutually disjoint neighborhoods of $-nw_1, \dots, -nw_{n+1}$ and thus $S_1 = \text{conv}\{v_1, \dots, v_{n+1}\}$.

If $i = 1, \dots, n+1$, then $\langle w_i, w_j\rangle = \frac{-1}{n}$ for $j \neq i$ implies $\langle w_i, x\rangle \leqslant n+1$ for $x \in \Delta_n^\circ + nw_i$ and $\langle w_i, y\rangle \leqslant 0$ for $y \in \Delta_n^\circ + nw_j$ and $j \neq i$. Therefore (97) yields

$$\begin{aligned} (n+1)\langle w_i, \sigma_0\rangle = \langle w_i, v_i\rangle + \sum_{j\neq i}\langle w_i, v_j\rangle &\leqslant (1+2n\eta)[-n + 3n\eta(n+1)] + n(1+2n\eta) \\ &\leqslant 3n(n+1)(1+2n\eta)\eta \leqslant 4n(n+1)\eta, \end{aligned}$$

for $i = 1, \dots, n+1$, which proves the claim.

Note that $\sigma_0 \in 4n\eta\Delta_n^\circ \subset 4n^2\eta B^n \subset \text{int}(B^n) \subset K \subset S_1$, in particular $o \in \text{int}(K - \sigma_0)$ and $K - \sigma_0 \subset S_1 - \sigma_0$. Let $\xi \in [0, 1)$ be minimal such that

$$\sigma_0 + (1-\xi)(S_1 - \sigma_0) \subset K.$$

From (91), $\eta < 1/(4n^2)$ and Lemma 5.7 (c), we deduce that

$$V(S_1) \geqslant (1-n\eta)^n n^n V(\Delta_n) \geqslant \left[\left(1 - \frac{1}{4n}\right)^2\left(1 + \frac{1}{n}\right)\right]^{\frac{n}{2}} > 1.$$

Then it follows from Lemma 5.6(i) and (92) that

$$\frac{\xi^n}{e}V(S_1) \leqslant V(S_1 \setminus K) < n^{23n}\varepsilon^{\frac{1}{4}} < n^{23n}\varepsilon^{\frac{1}{4}}\,V(S_1),$$

and hence $\xi < n^{24}\varepsilon^{\frac{1}{4n}}$. We deduce from (96) that $-\sigma_0 \in 4n^2\eta\Delta_n^\circ$, and therefore

$$S_1 - \sigma_0 \subset (1+2n\eta)\Delta_n^\circ + 4n^2\eta\Delta_n^\circ \subset 2\Delta_n^\circ \subset 2nB^n.$$

Since $K \subset S_1 \subset K + \xi(S_1 - \sigma_0)$ by the definition of ξ, it follows that

$$\delta_H(S_1, K) \leqslant 2n\xi < n^{26}\varepsilon^{\frac{1}{4n}}.$$

On the other hand, $\eta = n^{15n}\varepsilon^{\frac{1}{4}}$, (91) and $\Delta_n^\circ \subset nB^n$ imply

$$\delta_H(S_1, \Delta_n^\circ) < 2n^2\eta < n^{17n}\varepsilon^{\frac{1}{4}}.$$

Since $n^{17n}\varepsilon^{\frac{1}{4}} < n^{26}\varepsilon^{\frac{1}{4n}}$ if $\varepsilon < n^{-100n}$, we have $\delta_H(K, \Delta_n^\circ) < n^{27}\varepsilon^{\frac{1}{4n}}$, which completes the proof of Theorem 1.4 if $\varepsilon < n^{-100n}$.

However, if $\varepsilon \geqslant n^{-100n}$, then Theorem 1.4 trivially holds as $\delta_{\rm vol}(M, \Delta_n) \leqslant n^n\kappa_n$ and $\delta_H(M, \Delta_n) \leqslant n$ for any convex body $M \subset nB^n$. Note that if B^n is the John ellipsoid of K, then $K \subset nB^n$, and $\kappa_n \leqslant 6$ for all $n \in \mathbb{N}$. □

Proof of Theorem 1.6 (c) and (d). Suppose $\ell(Z_\infty(\mu)^\circ) \leqslant (1+\varepsilon)\ell(\Delta_n^\circ)$.

Let $\alpha_0 = 9 \cdot 2^{n+2}n^{2n+2}$ be the constant of Lemma 5.5. If for any $u \in \operatorname{supp}\mu$ there exists a w_i such that $\angle(u, w_i) \leqslant \alpha_0\eta$, then

$$\delta_H(\operatorname{supp}\mu, \{w_1, \ldots, w_{n+1}\}) \leqslant \alpha_0\eta < 9 \cdot 2^{n+2}n^{2n+2}n^{15n}\varepsilon^{\frac{1}{4}} < n^{22n}\,\varepsilon^{\frac{1}{4}}.$$

Therefore we assume

$$\zeta := \max_{u \in \operatorname{supp}\mu}\ \min_{i=1,\ldots,n+1} \angle(u, w_i) > \alpha_0\eta,$$

and let $u_0 \in \operatorname{supp}\mu$ be such that $\min\{\angle(u_0, w_i) : i = 1, \ldots, n+1\} = \zeta$. Let

$$L = \operatorname{conv}\{u_0, u_1, \ldots, u_{n+1}\},$$

and hence $Z_\infty(\mu)^\circ = K \subset L^\circ \subset S_1$. Lemma 5.5 and (90) imply

$$V(L^\circ) \leqslant \left(1 - \frac{\zeta}{2^{n+2}n^{2n}}\right) V(\Delta_n^\circ),$$

thus

$$\delta_{\rm vol}(L^\circ, \Delta_n^\circ) \geqslant \frac{\zeta}{2^{n+2}n^{2n}}\, V(\Delta_n^\circ).$$

On the other hand, (91) and $(1+2n\eta)^n < 4/3$ yield

$$\delta_{\rm vol}(S_1, \Delta_n^\circ) \leqslant ((1+2n\eta)^n - (1-n\eta)^n)V(\Delta_n^\circ) < 4n^2\eta\, V(\Delta_n^\circ).$$

Therefore the triangle inequality implies

$$V(S_1 \setminus L^\circ) = \delta_{\rm vol}(S_1, L^\circ) \geqslant \left(\frac{\zeta}{2^{n+2}n^{2n}} - 4n^2\eta\right)V(\Delta_n^\circ).$$

Since $V(\Delta_n^\circ) > 1$ by Lemma 5.7 (c), we deduce from (92) that

$$n^{23n}\varepsilon^{\frac{1}{4}}\, V(\Delta_n^\circ) \geqslant V(S_1 \setminus Z_\infty(\mu)^\circ) \geqslant V(S_1 \setminus L^\circ) > \left(\frac{\zeta}{2^{n+2}n^{2n}} - 4n^2\eta\right)V(\Delta_n^\circ).$$

It follows from $\eta = n^{15n}\varepsilon^{\frac{1}{4}}$ that

$$\zeta < 2^{n+2}n^{2n}(n^{23n}\varepsilon^{\frac{1}{4}} + 4n^2\eta) < n^{28n}\varepsilon^{\frac{1}{4}},$$

which proves Theorem 1.6 in the case where $\ell(Z_\infty(\mu)^\circ) \leqslant (1+\varepsilon)\ell(\Delta_n^\circ)$ and $\varepsilon < n^{-100n}$.

Since $\ell(Z_\infty(\mu)^\circ) \leqslant (1+\varepsilon)\ell(\Delta_n^\circ)$ and $W(Z_\infty(\mu)) \leqslant (1+\varepsilon)W(\Delta_n)$ are equivalent according to (1), we have completed the proof of Theorem 1.6 if $\varepsilon < n^{-100n}$.

However, if $\varepsilon \geqslant n^{-100n}$, then Theorem 1.6 trivially holds as for any $x \in S^{n-1}$ there exists a vertex w of Δ_n with $\|x - w\| \leqslant \sqrt{2}$. □

11. *Proof of Corollary* 1.5

For the proof of Corollary 1.5, we need the following observation.

LEMMA 11.1. *If $\frac{1}{n} B^n \subset K, C \subset nB^n$ for convex bodies K and C in $\mathbb{R}^n$, then*

$$\tfrac{1}{n^2}\, \delta_H(K, C) \leqslant \delta_H(K^\circ, C^\circ) \leqslant n^2 \delta_H(K, C).$$

Proof. We also have $\frac{1}{n} B^n \subset K^\circ, C^\circ \subset nB^n$. First, we show

$$\delta_H(K^\circ, C^\circ) \leqslant n^2 \delta_H(K, C). \tag{98}$$

Since $K \subset C + \delta_H(K, C)B^n \subset C + n\delta_H(K, C)C = (1 + n\delta_H(K, C))C$, we have

$$C^\circ \subset (1 + n\delta_H(K, C))K^\circ \subset K^\circ + n^2 \delta_H(K, C)\, B^n.$$

By symmetry, we also have $K^\circ \subset C^\circ + n^2\delta_H(K, C)\, B^n$, and thus we have verified (98).

Changing the roles of K, C and their polars K°, C° in (98) (and using the bipolar theorem), we also deduce the inequality $\delta_H(K, C) \leqslant n^2 \delta_H(K^\circ, C^\circ)$. □

Since $W(K) = \frac{2}{\ell(B^n)}\, \ell(K^\circ)$ according to (1), we conclude Corollary 1.5 by combining Theorem 1.3(ii), Theorem 1.4(ii) and Lemma 11.1.

REMARK. The factor n^2 in Lemma 11.1 is optimal.

Acknowledgements. Open access funding enabled and organized by Projekt DEAL.

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