Random mosaics in hyperbolic space

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der KIT-Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

DISSERTATION

von

Felix Herold, M.Sc.
aus Darmstadt

Tag der mündlichen Prüfung: 17.06.2020

Referent: Prof. Dr. Daniel Hug

Referent: Prof. Dr. Günter Last

Publications

The results of Section 3 and the corresponding parts of Chapter 1 and 2 of this thesis are the subject of the following paper:

> Felix Herold, Daniel Hug, and Christoph Thäle, *Does a central limit theorem hold for the k-skeleton of Poisson hyperplanes in hyperbolic space?*, arXiv:1911.02120 (2019).

The paper is uploaded on arXiv on 5. November 2019 and is submitted to a journal.

Danksagungen

Diese Doktorarbeit ist während meiner Zeit am Institut für Stochastik des Karlsruher Instituts für Technologie entstanden.

Zuerst möchte ich meinen beiden Betreuern Daniel Hug und Günter Last für ihre Ratschläge und ihr umfangreiches Feedback danken. Mein besonderer Dank geht an meinen Doktorvater Daniel Hug, der jedes Komma dieser Arbeit kennt und mich bereits während meiner Bachelor- und Masterarbeit begleitet hat.

Ganz herzlich möchte ich mich auch bei allen Institutsmitarbeiterinnen und Mitarbeitern dafür bedanken, dass ihr mich so herzlich empfangen und mir immer den Rücken freigehalten habt. Vielen Dank auch für die unkomplizierte Zusammenarbeite an all diejenigen, mit denen ich die Freude hatte gemeinsam eine Lehrveranstaltung zu betreuen. Ich möchte auch Danke sagen an alle Mitstreiter auf dem Fußballplatz, die gemeinsam den 3-fachen Turniersieg erkämpft haben. Insbesondere danke ich allen Doktoranden der letzten Jahre für eine unvergessliche Zeit auf Konferenzen, bei Promotionsfeiern oder einfach im Alltag in der Kaffeepause und auf dem Gang.

Ein spezieller Dank geht an Christoph Thäle, der mir Dank seiner Einladung und vielen Gesprächen einen ergebnisreichen und rundum gelungenen Aufenthalt tief im Westen ermöglicht hat. Außerdem danke ich dem gesamten Institut für Stochastik in Bochum für die schöne Zeit, die hilfreichen Anmerkungen, ihre Mithilfe und Nächte voller Kaffee.

Ich möchte auch allen Freunden danken, die ich auf meinem Weg hierher getroffen habe. Vielen Dank für eine wunderbare Abizeit, lange Lerntage im Z10, viele Kilometer im Wald, große Emotionen auf der Bühne, Bubs Trips und dafür, dass ihr meine Liebe zur Mathematik teilt oder eben auch zu anderen Dingen.

Vielen Dank an meine gesamte Familie, dass ihr die beste Familie der Welt seid, mir negative Zahlen beigebracht habt, mir immer zuhört, mich immer unterstützt und mir meine Macken verzeiht. Auch vielen Dank an meine neue Familie, dafür, dass ihr mich von Anfang an so herzlich aufgenommen habt.

Mein abschließender Dank gilt meiner geliebten Ehefrau Eva dafür, dass sie immer an meiner Seite steht und stehen wird.

Contents

CHAPTER 1

Introduction

1.1 Random mosaics

Random mosaics in $\mathbb{R}^d$ form a class of mathematical objects that have been under intensive investigation in stochastic geometry during the last decades. By a mosaic or tessellation of $\mathbb{R}^d$ one understands a system of convex polytopes which cover the whole space and have non intersecting interior. In addition to intrinsic mathematical curiosity, a major reason for continuing interest in random tessellations is that they provide highly relevant models for practical applications, for example, in telecommunication or materials science [34, 84, 85, 94]. One of the principal random tessellation models in Euclidean space is induced by a Poisson process of hyperplanes. Other famous models are the Voronoi tessellation, Delaunay tessellation and STIT tessellation.

We emphasize that the present work contributes to a recent and active line of current mathematical research in stochastic geometry on models in non-Euclidean spaces. The dissertation [111] deals with aspects of convex geometry in spherical spaces. Further concrete examples we mention the studies about spherical convex hulls and convex hulls on half-spheres in [5, 50, 68] and convex cones in [23]. Central limit theorems for the volume of random convex hulls in spherical space, hyperbolic spaces and Minkowski geometries were obtained in [8], asymptotic normality of very general so-called stabilizing functionals of Poisson point processes on manifolds was considered in [90]. Again more specifically, the papers [9, 31, 79, 87] study various aspects of random geometric graphs in hyperbolic spaces, including central limit theorems for a number of parameters. The paper [114] deals with visibility in the hyperbolic plane and [116] treats the Busemann-Petty problem in spaces of constant curvature. Random tessellations of the unit sphere by great hyperspheres are the content of [2, 42, 46, 75], while so-called random splitting tessellations in spherical spaces were introduced and investigated in [26, 47]. The paper [17] is concerned with properties of the Poisson-Voronoi tessellation on general Riemannian manifolds. Finally, the geometry of random fields on the sphere is studied in the monograph [70] and invariant random fields on spaces with a group action are described in [69]. In a similar vein, it is pointed out in [66] that a systematic study of the invariance properties of probability distributions under a general group action is missing. The book [66] therefore explores Markov processes whose distributions are invariant under the action of a Lie group. It should be pointed out that studying models in non-Euclidean spaces leads to a deeper understanding of the interplay of probability theory and geometry. So one will see that in some cases the results of analogue problems in hyperbolic space turn out to be quite similar, whereas in other cases the outcome is strikingly different.

1.2 Overview

This dissertation is structured as follows. In Chapter 1 we introduce the main topics which are dealt with in Chapter 3, 4 and 5. We further sketch the construction of the different mosaics in hyperbolic or Euclidean space. In a next step we do a first comparison of the outcomes in different geometries. Chapter 1 also contains some of the results developed in the main chapters as well as a short mentioning of the background theory needed to develop these results.

Chapter 2 will provide concepts and definitions from different fields of mathematics needed in this work. We will start with a section containing some basic notations from probability theory. In Section 2.2 we introduce hyperbolic space as the d-dimensional complete, simply connected Riemannian space of constant curvature -1. This introduction involves a short part containing the history of hyperbolic space as well as a presentation of several of the most important models of hyperbolic space. Section 2.3 introduces some concepts from (Euclidean) stochastic geometry and also the analogues in hyperbolic space needed in this work. Since we are aiming to subdivide $\mathbb{H}^d$, we introduce in a next section the space of k-planes in $\mathbb{H}^d$, namely the space of totally geodesic k-dimensional subspaces. We also introduce an invariant measure μ_k on this space. Here invariance refers to the action of isometries $I(\mathbb{H}^d)$ of $\mathbb{H}^d$. We investigate this measure and state some properties known from the literature. The following sections are dedicated to introduce concepts which are exclusively needed in one of the main Chapters 3, 4 and 5. We start in Section 2.5 with the concepts needed for Chapter 3. This part of the work is based on the theory of U-statistics as well as literature about normal approximation bounds for U-statistics. As mentioned in the beginning of this work, the content of Section 2.5 is already uploaded as a preprint on arXiv [41] and submitted to a journal. The subsequent section contains an introduction and an intuitive definition of splitting tessellations of hyperbolic space, whereas the last section of Chapter 2 deals with some basic definitions and results needed in Chapter 5, which deals with Kendall's problem in hyperbolic space.

Chapter 3 contains the results and proofs concerning the investigation of the k-skeleton of a Poisson hyperplane tessellation in hyperbolic space. As mentioned, this chapter exists as a preprint on arXiv [41] and is submitted to a journal. Its joint work with Daniel Hug and

Christoph Thäle which was initiated during my one month stay in Bochum. The main question in Chapter 3 is whether a central limit theorem holds for the k-dimensional Hausdorff measure of the k-skeleton of the Poisson hyperplane tessellation. Here we investigate two scenarios. In the first we increase the intensity of the underlying Poisson hyperplane tessellation. In the second we keep the intensity fixed and increase the observation window. Besides the concern, whether or not a central limit theorem holds, other objects are investigated. More or less on the way to the central limit theorem we get explicit results of the expected Hausdorff measure of the k-skeleton $k = 0, \dots, d-1$ and an integral representation of the variance of this functional. Also the limit behaviour of the variance is of interest. Besides this we state the covariance structure of the vector containing all d values of the k-skeleton, $k = 0, \dots, d-1$. Further insights concerning the interplay of these functionals are given by investigating the K- and the pair correlation-function.

Chapter 4 deals with the introduction and investigation of hyperbolic splitting tessellations. This model is a hyperbolic version of the so-called STIT-tessellation in the Euclidean space. We start with the definition of this process. Moreover, results concerning the capacity function are derived. This allows us to prove the existence of a splitting tessellation on the whole space $\mathbb{H}^d$ as well as first moments of the total k-dimensional Hausdorff measure of the k-skeleton. Also second moments and their limit behaviour are studied in this chapter. In the last part we turn to show that the introduced splitting tessellations are mixing. A possible application of these results is showing central limit theorems of functionals based on this process.

In Chapter 5 the so-called Kendall problem is explored in hyperbolic space. It asks for the shape of a cell, given that it is in some sense big. We start with some basic results showing the challenges in hyperbolic space concerning this problem. Later the problem is considered for the zero cell, i.e. the almost surely uniquely determined cell containing the origin of a Poisson hyperplane tessellation. Moreover, we consider the typical cell of the same process. In a last step the typical cell of a Poisson-Voronoi tessellation in hyperbolic space is studied.

1.3 Hyperplane tessellations

In $\mathbb{R}^d$ with $d \geq 2$ and in the stationary and isotropic case, the construction of a Poisson hyperplane tessellation can be described as follows. Fix a parameter $t > 0$ and consider a stationary Poisson point process on the real line with intensity t. To each point p_i of the Poisson process we attach independently of each other and independently of the underlying Poisson process a random vector u_i which is uniformly distributed on the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$. Then to each pair $(p_i, u_i) \in \mathbb{R} \times \mathbb{S}^{d-1}$ we associate the hyperplane $H_i := \{x \in \mathbb{R}^d : \langle x, u_i \rangle = p_i\}$ and call the random collection of all such hyperplanes a (stationary and isotropic) Poisson hyperplane process in $\mathbb{R}^d$ with intensity t. The random hyperplanes H_i almost surely divide the space $\mathbb{R}^d$ into countably many random convex polytopes. The collection of all these polytopes is a (stationary and isotropic) Poisson hyperplane tessellation in $\mathbb{R}^d$ with intensity t. We remark that the intensity parameter t, roughly speaking, controls the expected surface content of the Poisson hyperplane tessellation per unit volume. More precisely, $t = \mathbb{E}\mathcal{H}^{d-1}(Z \cap [0,1]^d)$, where $Z = \bigcup_{i=1}^{\infty} H_i$ is the random union set induced by the Poisson hyperplane process and $\mathcal{H}^{d-1}$ stands for the $(d-1)$-dimensional Hausdorff measure.

For Poisson hyperplane tessellations many first- and second-order quantities are explicitly available for a broad class of functionals and also a comprehensive central limit theory has been developed over the last 15 years, cf. [37, 39, 65, 96] and [103, Chapter 10] as well as the many references cited therein. In the literature, central limit theorems for functionals of Poisson hyperplanes have been considered in two different set-ups. In a first setting the tessellation is restricted to a fixed (usually convex) observation window and the asymptotic behaviour is explored when the intensity t of the underlying Poisson process is increased. Alternatively, the intensity is kept fixed, while the size of the observation window is increased. By a simple scaling relation both set-ups are equivalent when homogeneous functionals (such as intrinsic volumes, positive powers of intrinsic volumes or integrals with respect to support measures) of the tessellation are considered, see [65, Corollary 6.2]. While the spherical analogues of Poisson hyperplane tessellations, namely Poisson great hypersphere tessellations, were investigated, for example, in [2, 47, 42, 46, 44, 75], only few results seem to be available for such tessellations in standard spaces of constant negative curvature, see [49, 91, 100, 114]. The spherical space of constant positive curvature especially features by its compactness, which in turn implies that Poisson great hypersphere tessellations almost surely consist of only finitely many spherical random polytopes. In contrast, Poisson hyperplane tessellations in a standard space of constant negative curvature display a number of striking new phenomena that cannot be observed in their Euclidean or spherical counterparts. It is the purpose of the present work to uncover some of the anticipated and remarkable new phenomena. We confine ourselves to the study of the total volume (in the appropriate dimension) of the intersection processes induced by Poisson hyperplanes in a (hyperbolic convex) test set. We explicitly identify the expectation and the covariance structure of these functionals by making recourse to general formulas for and structural properties of Poisson U-statistics and to Crofton-type formulas from hyperbolic integral geometry. In addition and more importantly, we study probabilistic limit theorems for these functionals in the two asymptotic regimes described above for the Euclidean set-up. While the central limit theorems for growing intensity and fixed observation window are a direct consequence of general central limit theorems for Poisson U-statistics [65, 96, 108, 109], it will turn out that the limit theory in the other regime, that is, when the intensity is kept fixed and the size of the observation window is increased, is fundamentally different. We will prove that here a central limit theorem in fact holds in space dimensions $d = 2$ and $d = 3$. On the other hand, we will show that a central limit theorem fails for all space dimensions $d \geq 4$ if the total $(d-1)$-volume of the union of all hyperplanes is considered. For the total volume of intersection processes of arbitrary order this will be proved for technical reasons only for dimensions $d \geq 7$. We emphasize that this remarkable and surprising new feature is a consequence of the negative curvature of the underlying space and has no counterpart in the Euclidean or spherical set-up. Another interesting and unexpected feature is observed in this regime for the asymptotic covariance matrix of the vector of k-volumes of the k-skeletons, $k = 0, \dots, d-1$. This matrix turns out to have full rank for $d = 2$, but it has rank one in dimension $d \geq 3$. In addition, we will study the situation in which the intensity and the size of the observation window are increased *simultaneously*. In this case it will turn out that in all situations where the central limit theorem fails for fixed intensity, the Gaussian fluctuations are in fact preserved as soon as the intensity tends to infinity, independently of the behaviour of the size of the observation window (as long as it is bounded from below). As anticipated above, the proofs of our results concerning first- and second-order properties of the total volume of intersection processes rely on general formulas for U-statistics of Poisson point processes as presented in [64] and on tools from hyperbolic integral geometry as developed in [15, 32, 101, 110]. The central limit theorems we consider will be of quantitative nature, that is, we will provide explicit bounds on the quality (speed) of normal approximation measured in terms of both the Wasserstein and the Kolmogorov distance. Their proofs are based on general normal approximation bounds that have been derived in [27, 96, 109] using the Malliavin-Stein technique on Poisson spaces (see collection [88] for a representative overview concerning this method). This directly implies the central limit theorem for fixed windows and growing intensities. On the other hand, for fixed intensity and when the window is a hyperbolic ball B_r of radius r around a fixed point in $\mathbb{H}^d$, crucial building blocks of these bounds are Crofton-type integrals of the form

$$\int_{A_h(d,k)} \mathcal{H}^k(H \cap B_r)^l \, \mu_k(dH),$$

where $A_h(d,k)$ denotes the space of k-dimensional totally geodesic subspaces of $\mathbb{H}^d$ and μ_k is the suitably normalized invariant measure on $A_h(d,k)$ (all terms will be explained in detail in Section 2). While in the Euclidean case the asymptotic behaviour of such integrals, as $r \to \infty$, is quite straightforward, this is not the case in the hyperbolic set-up. In contrast to the Euclidean case, it will turn out that their behaviour crucially depends on whether $l(k-1)$ is less than, greater than or equal to $d-1$ (see Lemma 3.2.5). In essence, the latter is an effect of the negative curvature, which in turn causes an exponential growth of volume of linearly expanding balls in $\mathbb{H}^d$. To show that a central limit theorem fails in higher space dimensions is arguably the most technical part of this work. We do this by showing that the fourth cumulant of the centred and normalized total volume of the intersection processes does not converge to 0, which in turn is the fourth cumulant of a standard Gaussian distribution. However, to bring this in contradiction with a central limit theorem we need to argue that the fourth power of the total volume is uniformly integrable, which in turn will be established by consideration of their fifths moments. This requires a fine analysis of combinatorial moment formulas for U-statistics of Poisson processes. In essence and in contrast to the lower dimensional cases $d = 2$ and $d = 3$, the failure of the central limit theorem for space dimensions $d \geq 4$ is due to the fact that in these dimensions the contribution of single hyperplanes is asymptotically not negligible anymore.

1.4 Splitting tessellations

Splitting tessellations are introduced in [81] and are more commonly known as STIT-tessellations in the Euclidean space. In contrast to most other famous models, their cells are not face to face. This fact makes them a useful model in many applications such as investigating earthquake distributions [51] or the indoor propagation of 5G waves [83]. Other possible applications are presented in [80]. Besides their use in applied science, splitting tessellation are challenging and fruitful models in a mathematical point of view. Their connection with other famous models such as hyperplane tessellations makes them a highly observed topic in the last decade. It turned out that introducing a splitting tessellation as a continuous time pure jump Markov process is an elegant way to deduce a great number of results. We will follow this approach in our work.

As in the Euclidean and spherical set up, the splitting tessellation $Y_t = Y_t(W)$ for $t \geq 0$ inside a fixed compact and convex window W at time t can also be constructed recursively. This introductions helps to develop an intuition for the process. Most of the time we choose the observation window W to be a geodesic ball of radius $r > 0$ and centre $p \in \mathbb{H}^d$. At a starting time zero the tessellation consists of a single cell, namely the convex set W. After an exponential waiting time with parameter $\mu_{d-1}(\mathbb{H}_{d-1}\langle W \rangle)$ a uniform random $(d-1)$-dimensional totally geodesic subspace (hyperbolic hyperplane) divides W into two cells, where $\mu_{d-1}(\mathbb{H}_{d-1}\langle W \rangle)$ is the measure of the set of hyperplanes hitting W (for more details on this measure see Section 2.4). The splitting process continuous independently in both cells, whereas the waiting times depend on the measure of all hyperplanes having nonempty intersection with the respective cell. The construction of Y_t is more formally described by:

1 Initiation:

At time zero we set $Y_0 := \{W\}$, $\tau_0 = 0$ and take a counter $n = n(t)$, describing the number of cells, to be equal to one.

2 Recursion:

Assume that the counter is $n \geq 1$ and the corresponding random waiting time τ_{n-1} as well as the random tessellation $Y_{\tau_{n-1}}$ have been realized. Generate the random time τ_n such that $\tau_n - \tau_{n-1}$ has the same distribution as an exponential random variable with parameter $\sum_{c \in Y_{\tau_{n-1}}} \mu_{d-1}(\mathbb{H}_{d-1}\langle c \rangle)$.

If $\tau_n \leq t$, we

- pick a cell $c_n \in Y_{\tau_{n-1}}$ at random, were each cell $c \in Y_{\tau_{n-1}}$ is picked with probability proportional to $\mu_{d-1}(\mathbb{H}_{d-1}\langle c \rangle)$,
- choose a uniform randomly generated c_n-hyperplane H_n,
- set $Y_{\tau_n} := \oslash(c_n, H_n, Y_{\tau_{n-1}})$, which means keeping all cells except of c_n and replace it with the two cells $c_n \cap H_n^+$ and $c_n \cap H_n^-$ (see Definition 2.6.2),
- increase the counter n by one and repeat the recursion step.

If $\tau_n > t$ give out the random tessellation $Y_{\tau_{n-1}}$.

As mentioned above, a similar model is known in the Euclidean space. It is studied intensively in various works such as [82, 74, 106, 107]. The Euclidean counterpart is known as a so-called STIT-tessellation, since it is stable under iterations. In [26] the model is introduced in the spherical setting for dimension 2. First effects resulting from the different curvature are described in this work. More recently Hug and Thäle [47] extended this research on spherical spaces of arbitrary dimensions. Besides isotropic splitting tessellations also the more general case of arbitrary direction distribution is treated therein. To the best of my knowledge the present work is the first that investigates a similar model in hyperbolic space. On one hand, lacking the vanishing curvature of the Euclidean space, it somehow behaves as the spherical model. On the other hand, the fact that hyperbolic space is, in contrast to spherical space, unbounded leads to more closeness to the results in Euclidean space.

We start in Chapter 4 with giving an alternative, less algorithmic definition of the model. To do so we define the process as a continuous time pure jump Markov process. This gives us the chance to derive martingale properties for a broad class of functionals depending on the process. In a next step the capacity functional is defined and considered in detail. It is used in order to show the existence of a splitting tessellation on the whole space $\mathbb{H}^d$ and not just inside a fixed observation window. Restricting ourselves again onto a fixed window, we derive several expected values of functionals depending on the splitting process. Such functionals are for example the expected k-dimensional Hausdorff measure of the k-skeleton and the expected volume of the typical- (in dimension 2) and Crofton cell (in arbitrary dimensions). It turns out that the expected k-dimensional Hausdorff measure coincides with the ones in the Euclidean and spherical space, whereas the expected volume of the typical and Crofton cell show fundamentally different behaviour. Further restricting the observation window to take the shape of a ball gives us the opportunity to treat second moments. We calculate the variance of the total surface area of the $(d-1)$-skeleton. The limit behaviour of the variance is considered for growing time t and also for growing radius of the spherical intersection window W. The behaviour is compared with the results for Poisson hyperplane tessellations in Chapter 3. As in the previous chapter one can observe major differences to the results in Euclidean and spherical spaces. As shown in Chapter 3, the behaviour of the limit variance heavily depends on the dimension of the surrounding space. In the last part of Chapter 4 a mixing property of the process is shown. Such properties can be used in order to show central limit theorems in further research.

Chapter 4 makes use of the theory for continuous time pure jump Markov processes. Several works deal with this theory such as [13, Chapter 15], [25] [54, Chapter 12], [63, p. 19, Chapter 2.5] and [67, Chapter 1]. Further we apply theory of random closed sets and its connection to the capacity functional which can be found in [103]. In order to derive the results concerning expected values, we argue with the Crofton-type formula 2.4 and the representation of the invariant measure of hyperplanes in hyperbolic space 2.3.

1.5 Kendall's problem

In Chapter 5 we start by again considering Poisson hyperplane tessellations. Later also the Poisson Voronoi tessellation is treated. The just mentioned Voronoi tessellation can be constructed by realizing a (invariant) Poisson point process X on $\mathbb{H}^d$ and associating with each point $x \in X$ all points of $\mathbb{H}^d$ that have less distance to x than to every other point in X. This time the so-called Kendall problem is treated. In [112] the original form of this problem is recalled. The original problem asks for the conditional law for the shape of the zero cell C_0 of an isotropic and stationary Poisson line process, where the given area tends to infinity. The conjecture stated that the shape concentrates at the circular shape in the limit. Heuristic arguments by Miles supported this conjecture. A proof was first given by Kovalenko (see [57] for the simplified version) for the Euclidean plane. Later many different generalizations and modifications were studied. In the literature various works deal with related problems such as [16], [33], [11]. All variants have in common that one is interested in the shape of a cell given that it is somehow big. Recently the problem was considered on d-dimensional spherical spaces [42], [95]. To the best of my knowledge the present work is the first that approaches the problem in hyperbolic space. Due to the curvature many results that hold in the Euclidean set up cannot be transferred to hyperbolic space.

We start by proving two basic results showing that there are major differences as well as similarities to the Euclidean set up. A first example shows that the convergence of the shape of a cell does not hold in general for increasing size of the cell. In a second step we show that the shape of the Crofton cell of a Poisson hyperplane tessellation, given that it contains a ball with centre in the origin, converges to the shape of a ball as the intensity of the process tends to infinity. We also develop several useful results concerning the continuity of several functionals needed later on. Further inequalities of isoperimetric type, which are typical tools in this context, are shown. These results include more specific inequalities in lower dimensions and a more general inequality for arbitrary dimensions and size-/ hitting functionals. Also approximation results are transferred to the hyperbolic setting. More precisely we approximate convex sets with polytopes having in some sense few vertices. The end of Section 5.2 is dedicated to show that the Crofton cell is with high probability contained in a ball with some radius $r > 0$ and centre p. This is needed in order to restrict ourselves to a fixed observation window. In Section 5.3 we show that the shape of the Crofton cell tends to the shape of a ball, given that its volume exceeds a certain value $a > 0$, as the intensity of the Poisson hyperplane process tends to infinity. Later also the limit distribution of the volume of the Crofton cell is shown. In the next section we recall theory for stationary random measures on homogeneous spaces, specialized for our context. We use this theory to transfer the results for the Crofton cell to the typical cell of a Poisson hyperplane tessellation. In the following part, dedicated to the Poisson-Voronoi tessellation, we start by formally introducing the model. We also show that the shape of the typical cell tends to the shape of a ball, given that it contains a ball of fixed radius $a > 0$ around its generating point, as the intensity of the underlying Poisson point process tends to infinity. In this scenario also explicit rates of convergence can be shown.

The proofs of the theorems concerning the Crofton cell of a Poisson hyperplane process use techniques which already proved to be helpful in the Euclidean and spherical setting. These techniques include approximating results containing arguments from [14]. Since hyperbolic space is in contrast to spherical space not compact, we have to restrict ourselves to bounded sets. The results allowing us this restriction contain several geometric arguments. Specialized results of isoperimetric type rely on a recently proved Bonnesen-style inequality in spaces of constant curvature [19]. The more general isoperimetric inequality is based on research developed in [29]. The part concerning the typical cell heavily depends on theory of measures on homogeneous spaces which are treated for example in [62, 99]. The proofs for typical cells in Poisson-Voronoi tessellations again use some geometric ideas.

Chapter 2

Basics

In this chapter we introduce the most important definitions and fix the notations needed in this work. We will also state and partially prove some general results. We start in Section 2.3 with concepts from probability theory. In the subsequent section we introduce hyperbolic space and focus on the history of hyperbolic space, its models and definitions related to it. The following Section 2.3 deals with some general concepts from stochastic geometry and their adaptation in hyperbolic setting. In Section 2.4 we introduce a measure on the space of k-planes, state some of its properties and use it to define a Poisson hyperplanes process. Also the concept of hyperbolic quermassintegrals is discussed in this section. The following sections are introducing concepts for the main chapters, namely Chapter 3, 4 and 5.

2.1 Probability theory

We let $(\Omega, \mathcal{A}, \mathbb{P})$ be the underlying probability space with σ-Algebra $\mathcal{A}$ and probability measure $\mathbb{P}$. We will always assume $(\Omega, \mathcal{A}, \mathbb{P})$ to be rich enough to carry all random objects in this work. By $\mathbb{E}$, $\mathbb{V}\mathrm{ar}$, $\mathbb{C}\mathrm{ov}$ we denote the expectation, variance and covariance, respectively. For a sub σ-Algebra $\tilde{\mathcal{A}} \subseteq \mathcal{A}$, we denote the conditional probability by $\mathbb{P}(\cdot\,|\,\tilde{\mathcal{A}})$. Convergence in distribution is indicated by $\overset{d}{\to}$ and equality in distribution by $\overset{d}{=}$. We say that an event A happens almost surely, with respect to some probability measure $\mathbb{P}$, whenever $\mathbb{P}(A) = 1$ holds. A d-dimensional, real valued random vector $\mathbf{N}$ is said to be normally distributed with some (positive definite) covariance matrix $\Sigma = (\sigma_{i,j})_{i,j=1}^d$ and expectation $\mu \in \mathbb{R}^d$ if its density is given by

$$f_{\mathbf{N}}(x) = \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} \exp(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)), \qquad x \in \mathbb{R}^d.$$

2.2 Hyperbolic space and its models

In Subsection 2.2.1 we start with a small overview of the history of hyperbolic space and introduce some models of hyperbolic space in the following subsections. We will also point out their advantages and disadvantages for our different proposes. In Subsection 2.2.5 we deal with the most basic concepts from hyperbolic geometry. Also hyperbolic trigonometry is treated in Subsection 2.2.6.

2.2.1 History of hyperbolic space

The history of hyperbolic geometry begins more than 2000 years ago, when Euclid formulated his five postulates for plane geometry. While the first four postulates seemed pretty convincing and were easy to understand, the fifth stood out. In its original form it was formulated like:

> *If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on the side on which the angles are less than two right angles.*

A more modern formulation that is proven to be equivalent reads like

> *Through a point outside a given infinite straight line there is one and only one infinite straight line parallel to the given line.*

Also many geometric results did not need for the fifth postulate to hold. Therefore many mathematicians tried and failed to derive the fifth postulate from the first four. Carl Friedrich Gauß was the first to formulate a so-called non-Euclidean geometry which arises by denying the fifth postulate. Since he never published his results, the ideas had to be rediscovered by Nikolai Lobachevsky and János Bolyai in the beginning of the 19-th century. Now the fifth or parallel postulate in hyperbolic space reads like:

> *Through a point outside a given line there are infinitely many lines parallel to the given line.*

The postulates quoted here are taken from [92, Chapter 1].

2.2.2 Hyperboloid model

Hyperbolic d-space $\mathbb{H}^d$ is the unique d-dimensional complete simply-connected Riemannian space of constant curvature -1. Many books define it via the hyperboloid model. To do so the so-called Lorentzian inner product is used. For two points $x, y \in \mathbb{R}^{d+1}$ it is defined to be the real number

$$x \circ y = -x_1 y_1 + x_2 y_2 + \ldots . + x_{d+1} y_{d+1}.$$

Using the inner product one can define the set $\mathcal{F}^d := \{x \in \mathbb{R}^{d+1} : \ x \circ x = -1\}$. Since this set is the union of two disconnected hyperboloids, one defines hyperbolic d-space $\mathbb{H}^d$ as the positive half of $\mathcal{F}^d$, namely $\mathcal{F}^d_+$. The construction comes from the duality to the sphere of radius $r > 0$, which has constant curvature r^{-2}. Interpreting $\|x\|_L := \sqrt{x \circ x}$ as a distance (possibly imaginary), hyperbolic space is a sphere of unit imaginary radius and therefore has negative constant curvature. Obviously $\| \cdot \|_L$ is not a norm even though it is called Lorentzian norm. Technically it is a quasinorm. The space $\mathbb{H}^d$ carries a metric, the so-called hyperbolic distance function $d_h : \mathbb{H}^d \times \mathbb{H}^d \to [0, \infty)$. It is defined via the Lorentzian time like angle $\eta(x, y)$ between two points $x, y \in \mathbb{H}^d$ (see [92, Chapter 3.2]). An alternative approach to define a metric on $\mathbb{H}^d$ is via the Infimum of all C^1-paths (see [1, Chapter 3.4] for the 2-dimensional case). Unlike to the Euclidean case there exists no distinguishable origin. We can therefore pick an arbitrary point as the origin. In the hyperboloid model this will be $(1, 0, \ldots, 0) \in \mathcal{F}^d_+$.

Since the hyperboloid representation of hyperbolic space is in many situations quite unhandy, there exist several models of hyperbolic space. Each of them has their specific advantages and disadvantages. The most famous models are the conformal ball model (interior of the disk model), projective disk model (Beltrami, Klein model), the Half-space model and the already mentioned hyperboloid model. In this work we will focus on the first two models. To see a description of all model plus the hemisphere model, we refer to [18, Chapter 7].

2.2.3 Conformal ball model

The conformal ball model, also called Poincaré disk model or interior of a disk model, represents $\mathbb{H}^d$ in the interior of the d-dimensional unit ball $\mathcal{D}^d = B_{euc}(0,1)^o \subseteq \mathbb{R}^d$. The bijection between $\mathcal{F}^d_+$ and $\mathcal{D}^d$ is given by

$$\pi \colon F^d_+ \to \mathcal{D}^d, \ (x_1, x_2, \ldots, x_{d+1}) \mapsto \left(\frac{x_2}{1 + x_1}, \cdots, \frac{x_{d+1}}{1 + x_1} \right).$$

The map is visualized for the 1-dimensional case in Figure 2.2.2. By the choice of the origin, the Euclidean and the hyperbolic origin coincide. The metric in $\mathcal{F}^d_+$ transfers to $\mathcal{D}^d$ via the inverse of π, namely

$$d_{\mathcal{D}}(x, y) := d_h(\pi^{-1}(x), \pi^{-1}(y)), \qquad x, y \in \mathcal{D}^d.$$

It can also be calculated by

$$d_{\mathcal{D}}(x, y) = \operatorname{arcosh}(2\varphi(x, y) + 1), \qquad x, y \in \mathcal{D}^d,$$

with

$$\varphi(x, y) = \frac{\|x - y\|^2_{euc}}{(1 - \|x\|^2_{euc})(1 - \|y\|^2_{euc})}$$

(see [92, Chapter 4.5] for a proof). The greatest advantage of the conformal ball model is already implemented in its name. Since the projection preserves angles, the angle between two hyperbolic lines is the Euclidean angle between their conformal representation ([92, Chapter

1.2]). Furthermore, hyperbolic balls are represented by Euclidean balls (but with possibly shifted centre) and hyperbolic m-planes are represented by the intersection of $\mathcal{D}^d$ and m-dimensional linear subspaces of $\mathbb{R}^d$ or an m-sphere of $\mathbb{R}^d$ orthogonal to $\partial\mathcal{D}^d$ (see [92, Theorem 4.5.3, 4.5.4]). Figure 2.2.1 shows some hyperbolic objects frequently needed in this work, represented in $\mathcal{D}^2$.

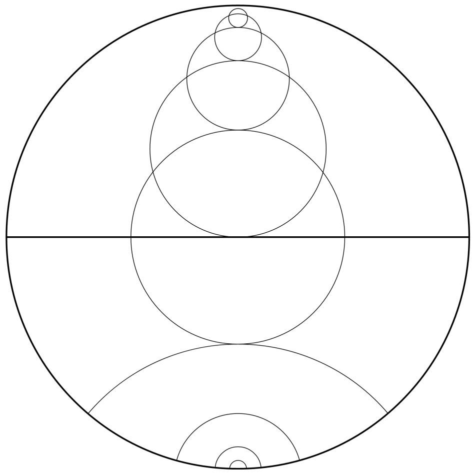

Figure 2.2.1: Balls of radius 1 and hyperplanes in the conformal ball model.

2.2.4 Projective disk model

The projective disk model, Beltrami or Klein model also represents $\mathbb{H}^d$ in the interior of the open unit ball $B_{euc}(0,1)^o \subseteq \mathbb{R}^d$. In contrast to the conformal ball model it does not preserve angles. The bijection from the hyperboloid is given by

$$\tilde{\pi} : \mathcal{F}_+^d \to B_{euc}(0,1)^o,\ (x_1,\dots,x_{d+1}) \mapsto \left(\frac{x_2}{x_1},\dots,\frac{x_{d+1}}{x_1}\right).$$

Again the Euclidean and the hyperbolic origin coincide. Its greatest advantage is that m-planes are the (non-empty) intersection of $B_{euc}^o(0,1)$ with Euclidean affine m-planes [92, Theorem 6.1.4]. This fact makes the model useful in context of convexity. Since convexity of a hyperbolic set is equal to convexity (in Euclidean sense) of its Euclidean representation in the projective disk model many results can be easily transferred. Also other results are easily transferable from the Euclidean setting via the projective disk model as long as they are of principle nature such as existence results.

2.2.5 Geometric concepts in hyperbolic space

Recall that by $\mathbb{H}^d$ we denote the hyperbolic space of dimension d. Let $p \in \mathbb{H}^d$ be an arbitrary (fixed) point. We will also refer to p as the origin. For $x \in \mathbb{H}^d$ we denote by $T_x\mathbb{H}^d$ the tangent space to $\mathbb{H}^d$ at x. We use the notation $\exp_x : T_x\mathbb{H}^d \to \mathbb{H}^d$ for the exponential map. Recall that $d_h(\cdot,\cdot)$ is the hyperbolic metric function. We write $B_h(z,r) = \{x \in \mathbb{H}^d : d_h(x,z) \le r\}$ for

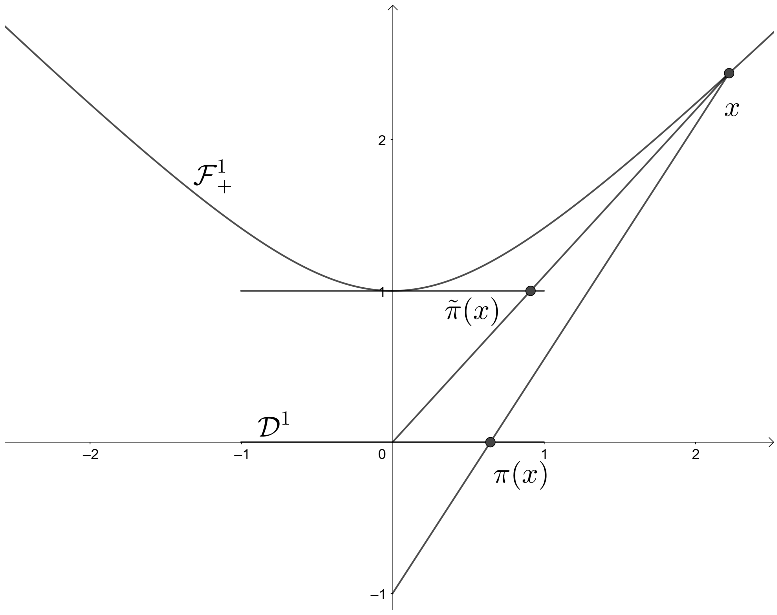

Figure 2.2.2: Projections between the models.

the hyperbolic ball with centre $z \in \mathbb{H}^d$ and radius $r \geq 0$ and put $B_r = B(p,r)$, where p is the fixed reference point. If it is clear from the context, we will omit the index indicating the surrounding space. For $a > 0$ we denote by B^a the unique hyperbolic ball with centre in p such that $\mathcal{H}^d(B^a) = a$ holds. In this work the s-dimensional Hausdorff measure $\mathcal{H}^s$, $s \geq 0$, is understood with respect to the metric space $(\mathbb{H}^d, d_h)$. The hyperbolic Hausdorff distance between two sets C, C' is defined by

$$\delta_h(C,C') \coloneqq \max\left\{\min_{x\in C}\max_{y\in C'} d_h(x,y), \min_{x\in C'}\max_{y\in C} d_h(x,y)\right\}.$$

For $k \in \{0,1,\dots,d-1\}$ a k-dimensional totally geodesic subspace of $\mathbb{H}^d$ is called a k-plane and especially $(d-1)$-planes are called hyperplanes. Here a space H is called totally geodesic if for any two points $x, y \in H$ the (unique) geodesic connecting x and y is contained in H. The space of k-planes in $\mathbb{H}^d$ is denoted by $A_h(d,k)$. For more information on this space, we refer to Section 2.4. For a fixed hyperplane $H \in A_h(d,d-1)$ we denote by H^+, H^- the two closed half-spaces into which $\mathbb{H}^d$ is divided by H.

For later reference we need a formula for the surface area of a hyperbolic ball $B(z,r)$. It is given by

$$\mathcal{H}^{d-1}(\partial B(z,r)) = \omega_d \sinh^{d-1}(r),$$

where ω_d is the surface area of the $(d-1)$-dimensional unit ball and cosh and sinh are the hyperbolic cosine and sine, which are given by

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}, \qquad x \in \mathbb{R},$$

respectively. Moreover, the volume of a hyperbolic ball of radius r is given by

$$\mathcal{H}^d(B(z,r)) = \omega_d \int_0^r \sinh^{d-1}(s)\ ds. \tag{2.1}$$

We refer to Sections 3.3 and 3.4 and especially to formulas (3.25) and (3.26) in the monograph [20]. For the special case $d = 2$, we thus get $\mathcal{H}^2(B(z,r)) = 2\pi(\cosh(r) - 1)$. We will frequently make use of the fact that $\cosh(x), \sinh(x) \in \Theta(e^x)$, as $x \to \infty$, where $\Theta(\cdot)$ stands for the usual Landau symbol. We also use the notations $f \in \mathcal{O}(g)$ to indicate that f grows at most with the same speed as g and $f = o(g)$ to indicate that g grows faster than f. Additionally we will use the following inequalities.

Lemma 2.2.1. *The function* sinh *satisfies the inequalities*

(a) $\sinh(x) \geq e^{x-3}$ *for* $x \geq 0.1$, *(b)* $\sinh(x) \geq x$ *for* $x \geq 0$.

Proof. (a) By the definition of the hyperbolic sine function, we get

$$\frac{2\sinh(x)}{e^{x-3}} = e^3 - e^{-2x+3} = e^3(1 - e^{-2x}) \geq 2 \quad \text{for } x \geq 0.1,$$

since $\exp(2x) \geq (1 - 2\exp(-3))^{-1}$ for $x \geq 0.1$.

(b) This follows from the definition of sinh by basic calculus.

□

2.2.6 Hyperbolic Trigonometry

In this section we present some trigonometric formulas in hyperbolic space. Their proofs can be found in [92, Chapter 3.5] and will therefore be omitted in this work. We start with the general results and then formulate the special cases of right-angled hyperbolic triangles.

Theorem 2.2.2 (The first law of cosines)**.** *Let* α, β, γ *be the angles of a hyperbolic triangle and* a, b, c *the lengths of the opposite sides, then the first law of cosines*

$$\cos(\gamma) = \frac{\cosh(a)\cosh(b) - \cosh(c)}{\sinh(a)\sinh(b)}$$

holds.

Theorem 2.2.3 (Right-angled triangles)**.** *Let* α, β, γ *be the angles of a hyperbolic triangle and* a, b, c *the lengths of the opposite sides with* $\gamma = \pi/2$. *Then the following equations hold:*

$$a) \qquad \cos(\alpha) = \frac{\tanh(b)}{\tanh(c)}$$

$$b) \qquad \sin(\alpha) = \frac{\sinh(a)}{\sinh(c)}$$

2.3 Stochastic and convex geometry

By $B_{euc}(x,r)$ we denote the Euclidean ball with centre $x \in \mathbb{R}^d$ and radius $r \geq 0$. If it is clear from the context we omit the indicator of the space. The Hausdorff distance between two nonempty compact Euclidean subsets $C, C' \subset \mathbb{R}^d$ is defined to be

$$\delta_{euc}(C,C') := \max\left\{\min_{x\in C}\max_{y\in C'} d_{euc}(x,y), \min_{x\in C'}\max_{y\in C} d_{euc}(x,y)\right\}.$$

Let $\mathbb{S}^d$ be the d-dimensional unit sphere embedded in $\mathbb{R}^{d+1}$ and let $B_s(u,\alpha),\ u \in \mathbb{S}^d,\ \alpha > 0$ be the spherical cap with centre u and radius α. The natural distance function on $\mathbb{S}^d$ is denoted by $d_s(\cdot,\cdot)$. The Lebesgue measure on $\mathbb{S}^d$ is denoted by σ_d. For the surface area of the unit sphere $\mathbb{S}^d$ we write $\omega_{d+1} := \sigma_d(\mathbb{S}^d)$. Further let κ_d be the volume of the unit ball. These values are given by $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ and $\kappa_d = \pi^{d/2}/\Gamma(1+d/2)$ and are therefore connected by $\omega_d = d\,\kappa_d$. Here Γ stands for the Gamma-function. Further we denote by $\mathbb{S}_p^d := \{u \in T_p\mathbb{H}^d :\ d_h(\exp_p(u),p) = 1\}$ the set of directions in the tangent space of $\mathbb{H}^d$. With a slight abuse of notation, we use the notations $\sigma_{d-1}(\cdot),\ B_s(\cdot,\cdot)$ and $d_s(\cdot)$ on $\mathbb{S}_p^{d-1}$ as well.

We write ∂A for the boundary of $A \subseteq \mathbb{H}^d$ and use the notation $\operatorname{int}(A)$ or A° for its interior. The operator $\operatorname{relint}(A)$ refers to the relative interior of A and $\operatorname{cl}(A)$ to its closure.

Let E be a topological space. We denote by $\mathcal{F}(E)$ the space of closed subsets of E and by $\mathcal{F}'(E)$ the space of nonempty closed subsets of E. We write $\mathcal{F}_{lf}(E)$ for the set containing all elements of $\mathcal{F}(E)$ which are additionally locally finite subsets of E. Further let $\mathcal{C}(E),\ \mathcal{O}(E)$ be the set of compact and open subsets of E, respectively. Whenever it is clear from the context, we will omit the notation of the underlying space E. For a subset $A \subseteq E$, we denote by

$$\mathcal{F}^A := \{F \in \mathcal{F}(E) :\ F \cap A = \emptyset\}.$$

This definition is used in order to define a topology on $\mathcal{F}(E)$. It is known as the Fell topology and is generated by the system

$$\{\mathcal{F}^C :\ C \in \mathcal{C}(E)\}.$$

For a more extensive introduction, we refer to [103, Chapter 2]. A set $A \subseteq \mathbb{H}^d$ is called convex iff for all $x, y \in A$ the unique geodesic connecting x and y is contained in A. Let $\mathcal{F}_{conv}(\mathbb{H}^d)$ be the set of all closed, nonempty, convex sets. Besides this definition of convexity in hyperbolic space there exists the more restrictive h-convexity which will not play an important role in this work. Further let $\mathcal{K}_h^d$ be the set of all convex bodies (compact, nonempty, convex sets) in d-dimensional hyperbolic space and let $\mathcal{K}_{h,0}^d$ be the set of convex bodies containing the origin p. Denote by $\operatorname{conv}(A),\ A \subseteq \mathbb{H}^d$ the convex hull, namely the smallest convex set that contains A.

Lemma 2.3.1. *The space $\mathcal{K}_h^d$ is a locally compact topological space with countable base.*

Proof. Since $\mathbb{H}^d$ is a locally compact space with a countable base, [103, Theorem 12.2.1] shows that $\mathcal{F}(\mathbb{H}^d)$, the system of closed subsets of $\mathbb{H}^d$, is a compact space with a countable base with respect to the Fell topology. Since [103, Theorem 12.3.4] transfers to the hyperbolic setting, the topology induced by the hyperbolic Hausdorff metric and the subspace topology induced by the Fell topology on $\mathcal{K}_h^d$ coincide. For a given $K \in \mathcal{K}_h^d$, let $U \subseteq \mathbb{H}^d$ be an open and relatively compact set with $K \subset U$. Then $\overline{U} := \mathrm{cl}(U)$ is compact and $K \in \mathcal{F}_U \subset \mathcal{F}_{\overline{U}} = \mathcal{F} \setminus \mathcal{F}^{\overline{U}}$, where $\mathcal{F}_U$ is open and $\mathcal{F} \setminus \mathcal{F}^{\overline{U}}$ is closed and therefore compact. □

The following lemmas allow us to transfer results from the Euclidean to the hyperbolic setting. We write $\tilde{\pi} : \mathbb{H}^d \to B_{euc}^{d,\circ}$ for an isometric diffeomorphism which allows us to identify hyperbolic space and the projective disk model. Lemma 2.3.2 shows that the model preserves convexity. Although this is well known, we indicate the proof to show how we can proceed in other situations. For this, let $\mathcal{K}_{euc}^d$ be the set of Euclidean convex bodies.

Lemma 2.3.2. *Let $K \subseteq \mathbb{H}^d$ be a hyperbolic set, then the following equivalence holds*

$$K \in \mathcal{K}_h^d \iff \pi(K) \in \mathcal{K}_{euc}^d.$$

Proof. Fix $K \in \mathcal{K}_h^d$ and pick any $x, y \in \pi(K)$. Since K is convex, the hyperbolic segment $[\tilde{\pi}^{-1}(x), \tilde{\pi}^{-1}(y)]$ lies completely in K. Since geodesics are mapped onto lines in the model, we get

$$\tilde{\pi}([\tilde{\pi}^{-1}(x), \tilde{\pi}^{-1}(y)]) = [x, y] \subset \pi(K).$$

Therefore we get that $\tilde{\pi}(K)$ is convex. The other direction can be shown similarly. □

The projective disk model preserves basic topological features. This is expressed in the next lemma; see [93, Theorem 6.2.2].

Lemma 2.3.3. *Let $K \in \mathcal{K}_h^d$. Then $x \in \mathrm{int}(K)$ if and only if $\tilde{\pi}(x) \in \mathrm{int}(\tilde{\pi}(K))$. The same holds for* $\mathrm{relint}\,(\cdot)$, $\mathrm{cl}(\cdot)$ *and* $\partial(\cdot)$.

Moreover, if $x \in \mathrm{int}(K)$ ($x \in \mathrm{relint}\,(K)$) and $y \in \mathrm{cl}(K)$, then $[x, y) \subseteq \mathrm{int}(K)$ ($[x, y) \subseteq \mathrm{relint}\,(K)$).

Now we state Blaschke's selection theorem in the hyperbolic setting. A direct proof can be based on Blaschke's selection theorem in the Euclidean space and the representation of hyperbolic space in the projective disk model. Lemma 2.3.4 is also a very special case of [4, Satz 4.6], which holds in a complete Riemannian space with non-positive sectional curvature.

Lemma 2.3.4. *Let $(K_i)_{i\in\mathbb{N}}$ be a sequence of convex bodies, all lying in a geodesic ball of radius $R > 0$. Then there exists a subsequence $(K_{i_k})_{k\in\mathbb{N}}$ and a convex body $K_0 \in \mathcal{K}_h^d$, $K_0 \subseteq B_R$, such that*

$$K_{i_k} \xrightarrow{k\to\infty} K_0$$

holds in the hyperbolic Hausdorff distance.

In the following we introduce some definitions from [92]. We define a *side* of a convex set $A \in \mathcal{K}_{conv}^d$ as a nonempty, maximal and convex subset of ∂A. Here the maximality means that for a side S of A there exists no further nonempty subset $\tilde{S} \subseteq \partial A \setminus S$ such that $\tilde{S} \cup S$ is still convex. We use this definition in order to define a convex polyhedron.

Definition 2.3.5 (Polyhedron)**.** *A convex polyhedron P in $\mathbb{H}^d$ is a nonempty, closed, convex subset of $\mathbb{H}^d$ such that the collection $\mathcal{S}$ of its sides is locally finite in $\mathbb{H}^d$. We denote the set of hyperbolic polyhedrons of dimension at most d by $\mathbb{PD}^d$.*

The following results concerning polyhedrons are available from the literature:

- A d-dimensional convex polyhedron is the intersection of the (at most countably many) half-spaces bounded by the hyperplanes which are determined by the sides of P and contain P [92, Theorem 6.3.2].
- Let $\mathcal{G}$ be a family of closed half-spaces of $\mathbb{H}^d$ such that $\{\partial G : G \in \mathcal{G}\}$ is locally finite (and hence at most countable) and $\cap\{G : G \in \mathcal{G}\} \neq \varnothing$. Then $\cap\{G : G \in \mathcal{G}\}$ is a convex polyhedron [92, Ex. 6.3 (2)].
- A d-dimensional convex polyhedron P is compact if and only if P has at least $d+1$ sides, P has only finitely many sides and each side of P is compact [92, Theorem 6.3.6].
- A convex polyhedron P in $\mathbb{H}^d$ is compact if and only if P has only finitely many vertices (zero-dimensional faces) and P is the convex hull of its vertices [92, Theorem 6.3.17].

In a next step we want to define the i-faces of a polyhedron $P \in \mathbb{PD}^d$. In order to give a proper definition of i-faces, we start with the following definition which is based on the definition in Euclidean space.

Definition 2.3.6 (Support set)**.** *Let $A \subseteq \mathbb{H}^d$ be a closed and convex set and let H be a supporting hyperplane. Here a hyperplane $H \in A_h(d, d-1)$ is said to support A if A is contained in at least one of the two half-spaces H^+, H^- and if the intersection $H \cap A \neq \varnothing$ is nonempty. Then the set $A \cap H$ is called a support set (of A).*

One can use the definition of support sets in order to define the i-faces of a polyhedron.

Definition 2.3.7 (i-face)**.** *The support sets of a polyhedron $P \in \mathbb{PD}^d$ are called faces. A face F of P is called an i-face for $i \in \{0, \dots, d-1\}$ if its dimension $\dim(F)$ is equal to i. We will denote the set of i-faces of a polyhedron $P \in \mathbb{PD}^d$ by $\mathcal{F}_i(P)$. The set $\mathcal{F}_0(P)$ will also be referred to as the vertices of P.*

We are now in the position to define polytopes in hyperbolic space the following way.

Definition 2.3.8. *A polytope in $\mathbb{H}^d$ is a convex polyhedron P in $\mathbb{H}^d$ such that P has only finitely many vertices $x_1, \dots, x_n$ and fulfills*

$$P = \operatorname{conv}\{x_1, \dots, x_n\}.$$

We denote the set of hyperbolic polytopes of dimension at most d by $\mathbb{P}^d$.

One can show that the definition above coincides with the one in Euclidean space, where polytopes are defined as the convex hull of finitely many points (see [102, p. 3]).

Lemma 2.3.9. *A set P is a hyperbolic polytope if and only if P is the convex hull of finitely many points.*

Proof. First let P be a polytope. By definition it has only finitely many vertices and P is the convex hull of its vertices. To show the other implication let $\tilde{P} = \operatorname{conv}\{x_1, \dots, x_n\}$ be the convex hull of finitely many points. In a first step we show that $\tilde{P}$ is a convex polyhedron. By definition it is nonempty and convex. To show that $\tilde{P}$ is closed and that the collection of its sides is locally finite, we project $\tilde{P}$ into the projective disk model. Now $\tilde{\pi}(\tilde{P}) = \operatorname{conv}\{\tilde{\pi}(x_1), \dots, \tilde{\pi}(x_n)\}$ is a polytope in the Euclidean sense. Since $\tilde{\pi}(\tilde{P})$ is closed, $\tilde{P}$ is closed, too. Again using that $\tilde{\pi}(\tilde{P})$ is a polytope we know that it has a finite number of sides. Therefore the number of sides of $\tilde{P}$ is finite and consequently the set of its sides is locally finite. Since the vertices of $\tilde{\pi}(\tilde{P})$ are among $\tilde{\pi}(x_1), \dots, \tilde{\pi}(x_n)$, their number is finite and so is the number of vertices of $\tilde{P}$ which are among $x_1, \dots, x_n$. Finally, since $\tilde{\pi}(\tilde{P})$ is the convex hull of its vertices, the same holds true for $\tilde{P}$. □

Now we know that the Euclidean and hyperbolic definition of a polytope coincide. Hence we know that a set $P \subseteq \mathbb{H}^d$ is a polytope if and only if $\tilde{\pi}(P)$ is a Euclidean polytope. This allows us to transfer many results about polytopes into the hyperbolic setting. We start with transferring [102, Theorem 2.4.3].

Lemma 2.3.10. *Every hyperbolic polytope is the intersection of finitely many closed half-spaces.*

Proof. Let P be a hyperbolic polytope. Since $\tilde{\pi}(P)$ is a Euclidean polytope, it is by [102, Theorem 2.4.3] the intersection of finitely many closed half-spaces $H_1^+, \dots, H_n^+$. Therefore P is the intersection of $\tilde{\pi}^{-1}(H_1^+), \dots, \tilde{\pi}^{-1}(H_n^+)$. □

Lemma 2.3.11. *For every $r > 0$ one can find a hyperbolic polytope $P = P(r)$ such that $B_r \subseteq P(r)$ and $\partial P(r) \cap B_r = \varnothing$ holds.*

Proof. We consider the projection of B_r in the projective disk model. It is a Euclidean ball of radius $\tilde{r} \in (0,1)$ and centre 0. Now there exists a Euclidean polytope $\tilde{P}$ such that $B_{euc}(0,\tilde{r}) \subseteq \tilde{P} \subseteq B_{euc}(0,1)$ with $\partial\tilde{P} \cap B_{euc}(0,\tilde{r}) = \partial\tilde{P} \cap \partial B_{euc}(0,1) = \varnothing$. Therefore the hyperbolic polytope $P := \tilde{\pi}^{-1}(\tilde{P})$ fulfills the desired properties. □

Lemma 2.3.12. *Every bounded, nonempty intersection of finitely many closed half-spaces is a polytope in $\mathbb{H}^d$.*

Proof. Let P be the intersection of finitely many hyperbolic half-spaces $H_1^+, \dots, H_n^+$ such that P is bounded. Hence $\tilde{\pi}(P)$ is the intersection of finitely many Euclidean half-spaces $\tilde{\pi}(H_1^+), \dots, \tilde{\pi}(H_n^+)$. Additionally, we know that $\tilde{\pi}(P)$ is bounded. Therefore $\tilde{\pi}(P)$ is by [102, Theorem 2.4.6] a polytope. Hence P is a hyperbolic polytope. □

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space. We denote by $\mathbf{N}$ the space of all counting measures on $\mathbb{X}$. Furthermore, we denote by $\mathcal{N}$ the σ-field which is generated by the collection of all subsets of $\mathbf{N}$ of the form

$$\{\mu \in \mathbf{N} : \mu(B) = k\}, \qquad B \in \mathcal{X},\ k \in \mathbb{N}_0.$$

We call a random element η of $(\mathbf{N},\mathcal{N})$ a point process. The space of all simple counting measures is denoted by $\mathbf{N}_s$.

Definition 2.3.13. *Let λ be a s-finite measure on $\mathbb{X}$. A Poisson process with intensity measure λ is a point process on $\mathbb{X}$ that fulfills the following properties:*

i) For every $B \in \mathcal{X}$, the random variable $\eta(B)$ is Poisson distributed with parameter $\lambda(B)$.

ii) For every natural number $m \in \mathbb{N}$ and pairwise disjoint sets $B_1, \dots, B_m \in \mathcal{X}$ the random variables $\eta(B_1), \dots, \eta(B_m)$ are stochastically independent.

Fore background on Poisson processes, we refer to [64].

Hyperbolic tessellations partition the space $\mathbb{H}^d$ into countably many, non-overlapping d-dimensional hyperbolic convex sets. By $\mathbb{T}^d$ we denote the set of all tessellations. We start with the definition of a mosaic respectively of a tessellation. Here and in the following, we use mosaic and tessellation synonymously. In contrast to the Euclidean counterpart, we allow the sets or cells of a mosaic to be unbounded. One underlying reasons for this is to include some cases of Poisson hyperplane mosaics in hyperbolic space with low intensity. For a definition in the Euclidean case see [103, Definition 10.1.1].

Definition 2.3.14 (Mosaic, Tessellation)**.** *A mosaic (tessellation) in $\mathbb{H}^d$ is a countable system m of subsets of $\mathbb{H}^d$ which satisfies the following conditions:*

1. *The system m is a locally finite system of nonempty closed sets, i.e. $m \in \mathcal{F}_{lf}(\mathcal{F}'(\mathbb{H}^d))$.*
2. *The sets $c \in m$ are convex and have interior points.*
3. *The sets $c \in m$ cover the whole space*

$$\bigcup_{c \in m} c = \mathbb{H}^d.$$

4. *Two different sets $c_1, c_2 \in m$ have no common interior point, i.e.* $\operatorname{int}(c_1) \cap \operatorname{int}(c_2) = \emptyset$.

The set that contains all skeletons belonging to a mosaic is denoted by $\mathcal{F}^d_{h,skel}$. We are aiming to show that the set $\mathbb{T}^d$ of all tessellations is a Borel set in $\mathcal{F}(\mathcal{F}'(\mathbb{H}^d))$. This is done in the forthcoming lemma. The proof is based on the one in Euclidean space (see [103, Lemma 10.1.2]).

Lemma 2.3.15. *The set $\mathbb{T}^d$ of all tessellations is a Borel set in $\mathcal{F}(\mathcal{F}'(\mathbb{H}^d))$.*

Proof. The idea of the proof is to rewrite the set $\mathbb{T}^d$ and to show the measurability of the new representation. In order to do so, we need to define the following set. For $r > 0$ let $\mathcal{K}^{(d-1)}_{r,h}$ be the set

$$\mathcal{K}^{(d-1)}_{r,h} := \{K \in \mathcal{F}_{conv}(\mathbb{H}^d) : \ \dim(K) \le d-1, \ K \cap B_r \neq \emptyset\}$$

of lower dimensional closed convex subsets of $\mathbb{H}^d$ which intersect B_r. Now we can rewrite $\mathbb{T}^d$ as

$$\mathbb{T}^d = \left\{ m \in \mathcal{F}_{lf}(\mathcal{F}'(\mathbb{H}^d)) : \bigcup_{c \in m} c = \mathbb{H}^d \right\}$$
$$\bigcap_{r=1}^{\infty} \left\{ m \in \mathcal{F}_{lf}(\mathcal{F}'(\mathbb{H}^d)) : m \cap \mathcal{K}_{r,h}^{(d-1)} = \varnothing, \ \sum_{c \in m} \mathcal{H}^d(c \cap B_r) = \mathcal{H}^d(B_r) \right\}. \tag{2.2}$$

To show that the right hand side is a Borel set in $\mathcal{F}(\mathcal{F}'(\mathbb{H}^d))$, we start with showing that $\mathcal{F}_{lf}(\mathcal{F}'(\mathbb{H}^d))$ is a Borel set in $\mathcal{F}(\mathcal{F}'(\mathbb{H}^d))$. In order to do so, we rewrite $\mathcal{F}_{lf}(\mathcal{F}'(\mathbb{H}^d))$ as

$$\mathcal{F}_{lf}(\mathcal{F}'(\mathbb{H}^d)) = \bigcap_{r=1}^{\infty} \bigcup_{n=1}^{\infty} \{ F \in \mathcal{F}'(\mathbb{H}^d) : \ |F \cap B_r^\circ| \le n \}.$$

For $r, n \in \mathbb{N}$ the sets $\{F \in \mathcal{F}(\mathbb{H}^d) : \ |F \cap B_r^\circ| \le n\}$ are closed. This can be shown by using [103, Theorem 12.2.2] as one picks a sequence $F_1, F_2, \ldots$ of elements in $\{F \in \mathcal{F}(\mathbb{H}^d) : \ |F \cap B_r^\circ| \le n\}$ for fixed $r, n \in \mathbb{N}$ which converges to F. By making use of (c_1) in Theorem 12.2.2 one can show that the limit of the sequence has to lie in $\{F \in \mathcal{F}(\mathbb{H}^d) : \ |F \cap B_r^\circ| \le n\}$ as well. This fact immediately gives the measurability of $\mathcal{F}_{lf}(\mathcal{F}'(\mathbb{H}^d))$. We now turn to show that the mappings in (2.2) are measurable. We start with the mapping

$$f : \mathcal{F}_{lf}(\mathcal{F}'(\mathbb{H}^d)) \to \mathcal{F}'(\mathbb{H}^d), \ m \mapsto \bigcup_{c \in m} c.$$

Since the σ-Algebra of Borel sets of $\mathcal{F}'(\mathbb{H}^d)$ is generated by the system $\{\mathcal{F}^C : \ C \in \mathcal{C}\}$, it is enough to consider their preimage. So fix $C \in \mathcal{C}$. We get $\bigcup_{c \in m} c \in \mathcal{F}^C$ iff no set in m hits C. This in turn is equal to $m(\mathcal{F}^C) = 0$. Since the set of systems fulfilling this is measurable, we get the measurability of f. Further the mapping $m \mapsto m \cap \mathcal{K}_{r,h}^{(d-1)}$ is by [103, Theorem 12.2.6 (a)] upper semi continuous and hence measurable. The measurability of the remaining map $m \mapsto \sum_{c \in m} \mathcal{H}^d(c \cap B_r)$ follows directly from Campbell‘s theorem (see for example [103, Theorem 3.1.2]). Together this shows the statement. □

2.4 The space of k-planes

Let $I(\mathbb{H}^d)$ denote the isometry group of $\mathbb{H}^d$ and let $I(\mathbb{H}^d, p)$ denote the subgroup of isometries which fix p. We remark that in the conformal ball model, $I(\mathbb{H}^d)$ can be identified with the group of Möbius transformations of B_{euc}^d, see [92, Corollary 4.5.1]. We denote by $G_h(d,k)$ the compact space of k-dimensional totally geodesic subspaces containing the origin p. We recall that in the conformal ball model, all elements of $G_h(d,k)$ arise as follows. If p coincides with the centre o of B_{euc}^d, then an element of $G_h(d,k)$ is the intersection of B_{euc}^d with a k-dimensional Euclidean linear subspace of $\mathbb{R}^d$. If otherwise $p \neq o$, then an element of $G_h(d,k)$ is the intersection of B_{euc}^d with a k-dimensional Euclidean sphere in $\mathbb{R}^d$ through p which is orthogonal to the boundary of B_{euc}^d, cf. [92, Theorem 4.5.3]. Up to a scaling factor, $G_h(d,k)$ carries a unique regular Borel measure ν_k which is invariant under $I(\mathbb{H}^d, p)$. Since $G_h(d,k)$ is compact we can normalize ν_k such that $\nu_k(G_h(d,k)) = 1$. We denote by $A_h(d,k)$ the space of k-dimensional planes in

$\mathbb{H}^d$. In the conformal ball model all elements of $A_h(d,k)$ can be represented as intersections with B^d_{euc} of either k-dimensional Euclidean linear subspace of $\mathbb{R}^d$ or k-dimensional Euclidean spheres in $\mathbb{R}^d$ that are orthogonal to the boundary of B^d_{euc}. On $A_h(d,k)$ there exists a unique (up to scaling) $I(\mathbb{H}^d)$-invariant measure. In contrast to $G_h(d,k)$, the larger space $A_h(d,k)$ is not compact. Each k-plane $H \in A_h(d,k)$ is uniquely determined by its orthogonal subspace L_{d-k} passing through the origin p and the intersection point $\{x\} = H \cap L_{d-k}$. Using these facts, Santaló [101, Equation (17.41)] (see also [110, Proposition 2.1.6], [32, Equation (9)]) provides a useful representation of an isometry invariant measure on $A_h(d,k)$, which we use here with a different normalization. For a Borel set $B \subset A_h(d,k)$, it is given by

$$\mu_k(B) = \int_{G_h(d,d-k)} \int_L \cosh^k(d_h(x,p)) \mathbb{1}\{H(L,x) \in B\} \mathcal{H}^{d-k}(dx)\ \nu_{d-k}(dL), \tag{2.3}$$

where $H(L,x)$ is the k-plane orthogonal to L passing through x.

Remark 2.4.1. The current normalization of the measure μ_k differs from the normalization of the measure dL_k used in [101] by the constant $\omega_d \cdots \omega_{d-k+1}/(\omega_k \cdots \omega_1)$. This also affects the constants in the formulas from hyperbolic integral geometry taken from [101]. The reason for the present normalization is to simplify a comparison of our results to corresponding results in Euclidean and spherical space.

We use this measure in order to define a Poisson hyperplane process in hyperbolic space. For $t > 0$ we let η_t be a Poisson process on the space of hyperplanes in $\mathbb{H}^d$. The intensity measure is t times the invariant measure μ_{d-1}. The Poisson process η_t will be referred to as a (hyperbolic) Poisson hyperplane process with intensity t. It induces a Poisson hyperplane tessellation in $\mathbb{H}^d$ with (possibly unbounded) hyperbolic cells. According to [101, Equation (14.69)] the measure μ_k satisfies the following Crofton-type formula. In fact, the discussion in [15, Section 7] allows us to state the result not only for sets bounded by smooth submanifolds (as in [101]), but for much more general sets, which include arbitrary convex sets as a very special case. The following lemma holds for $\mathcal{H}^{d+i-k}$ measurable sets $W \subset \mathbb{H}^d$ which are Hausdorff $(d+i-k)$-rectifiable. Following [15, Definition 5.13], we say that a set $W \subset \mathbb{H}^d$ is ℓ-rectifiable if ℓ is an integer with $0 < \ell \le d$ and W is the image of some bounded subset of $\mathbb{R}^\ell$ under a Lipschitz map from $\mathbb{R}^\ell$ to $\mathbb{H}^d$. A set $W \subset \mathbb{H}^d$ is Hausdorff ℓ-rectifiable provided that $\mathcal{H}^\ell(W) < \infty$ and if there exist ℓ-rectifiable subsets $B_1, B_2, \dots$ of $\mathbb{H}^d$ such that $\mathcal{H}^\ell(W \setminus \bigcup_{i\ge1} B_i) = 0$. Clearly, any Borel set W which is contained in an ℓ-dimensional plane is Hausdorff ℓ-rectifiable if it satisfies $\mathcal{H}^\ell(W) < \infty$.

Lemma 2.4.1. *Let $0 \le i \le k \le d-1$, and let $W \subset \mathbb{H}^d$ be a Borel set which is Hausdorff $(d+i-k)$-rectifiable. Then*

$$\int_{A_h(d,k)} \mathcal{H}^i(W \cap E)\ \mu_k(dE) = \frac{\omega_{d+1}\,\omega_{i+1}}{\omega_{k+1}\,\omega_{d-k+i+1}} \mathcal{H}^{d+i-k}(W). \tag{2.4}$$

Remark 2.4.2. Strictly speaking the case $k = i$ is not covered by [15]. Although the framework in [15] should extend to this marginal case, we prefer to provide an elementary direct argument for the case $k = i$. In this case, the left side of (2.4) defines an isometry invariant Borel measure on $\mathbb{H}^d$. Therefore in order to confirm (2.4) in this case, it is sufficient to show that the equality holds for $W = B_r$, $r \geq 0$. Since equality holds for $r = 0$ and in view of (2.1), it is sufficient to show that $\omega_d \sinh^{d-1}(r)$ is the derivative with respect to r of the function defined by

$$\begin{aligned}
h(r) :=& \int_{A_h(d,k)} \mathcal{H}^i(B_r \cap E)\ \mu_k(dE) \\
=& \int_{G_h(d,d-k)} \int_L \cosh^k(d_h(x,p))\, \mathcal{H}^i(B_r \cap H(L,x))\ \mathcal{H}^{d-k}(dx)\ \nu_{d-k}(dL) \\
=& \int_{\tilde{L}} \cosh^k(d_h(x,p))\, \mathcal{H}^i(B_r \cap H(\tilde{L},x))\ \mathcal{H}^{d-k}(dx) \\
=& \int_0^r \int_{\mathbb{S}_{\tilde{L}}^{d-k-1}} \sinh^{d-k-1}(t) \cosh^k(t)\, \mathcal{H}^i(B_r \cap H(\tilde{L}, \exp_p(tu)))\ \sigma_{d-k-1}(du)\ dt \\
=&\omega_k \int_0^r \int_{\mathbb{S}_{\tilde{L}}^{d-k-1}} \sinh^{d-k-1}(t) \cosh^k(t) \int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(t)}\right)} \sinh^{k-1}(s)\, \sigma_{d-k-1}(du)\ ds\ dt \\
=&\omega_k \omega_{d-k} \int_0^r \sinh^{d-k-1}(t) \cosh^k(t) \int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(t)}\right)} \sinh^{k-1}(s)\, ds\, dt,
\end{aligned}$$

where we used (2.3) and (3.7) and denoted by $\tilde{L}$ an arbitrary $(d-k)$-dimensional linear subspace and by $\mathbb{S}_{\tilde{L}}^{d-k-1}$ the set of direction spanning the linear space $\tilde{L}$. The differential of h can be determined by basic rules of calculus. Using that $\operatorname{arcosh}(\cosh(r)/\cosh(r)) = 0$, we thus obtain

$$h'(r) = \omega_k \omega_{d-k} \int_0^r \sinh^{d-k-1}(t) \sinh(r) \big(\cosh^2(r) - \cosh^2(t)\big)^{\frac{k-1}{2}} \cosh(t)\, dt.$$

The substitution $\sinh(t) = \sinh(r) \cdot x$ leads to

$$h'(r) = \omega_k \omega_{d-k} \int_0^1 x^{d-k-1} \big(1 - x^2\big)^{\frac{k-2}{2}}\ dx\ \sinh^{d-1}(r) = \omega_d \sinh^{d-1}(r),$$

as was to be shown.

Remark 2.4.3. Although both sides of (2.4.1) define measures with respect to their dependence on a Borel set $W \subset \mathbb{H}^d$, for $k \neq i$ the equality in (2.4.1) in general does not extend from $(d+i-k)$-rectifiable sets to general Borel sets. This is due to deep classical results in the structure theory of geometric measure theory, see [30, p. 2] or [78, Chapter 3] for an introduction and [30, Theorem 3.3.13] for the general treatment. In fact, in the Euclidean setting, for $i = 0$, $k \in \{1, \ldots, d-1\}$ and for a general Borel set $W \subset \mathbb{R}^d$, the right side of (2.4.1) is always as large as the left side with equality if and only if W is $(d-k)$-rectifiable.

We will frequently make use of the following transformation formula.

Lemma 2.4.2. *Let $k \in \{0, \ldots, d-1\}$, and let $f : A_h(d,k) \to \mathbb{R}$ be a non-negative measurable function satisfying $f(H_1 \cap \ldots \cap H_{d-k}) = 0$ if dim$(H_1 \cap \ldots \cap H_{d-k}) \neq k$. Then*

$$\int_{A_h(d,d-1)^{d-k}} f(H_1 \cap \ldots \cap H_{d-k})\, \mu_{d-1}^{d-k}(d(H_1, \ldots, H_{d-k})) = c(d,k) \int_{A_h(d,k)} f(E)\ \mu_k(dE)$$

with

$$c(d,k) = \frac{\omega_{k+1}}{\omega_{d+1}} \left(\frac{\omega_{d+1}}{\omega_d} \right)^{d-k}.$$

Proof. Let $\varphi \in I(\mathbb{H}^d)$ be an arbitrary isometry, and let B a measurable subset of $A_h(d,k)$. Then we have

$$\begin{aligned}
&\mu_{d-1}^{d-k}(\{(H_1,\dots,H_{d-k}) \in A_h(d,d-1)^{d-k} : \ H_1 \cap \dots \cap H_{d-k} \in \varphi B\}) \\
&\quad = \mu_{d-1}^{d-k}(\{(H_1,\dots,H_{d-k}) \in A_h(d,d-1)^{d-k} : \ H_1 \cap \dots \cap H_{d-k} \in B\})
\end{aligned}$$

by the isometry invariance of μ_{d-1}. Since up to a multiplicative constant, μ_k is the only isometry invariant measure on $A_h(d,k)$, the formula follows up to the determination of the constant, which is independent of the function f. We do this by choosing

$$f(H_1,\dots,H_{d-k}) = \begin{cases} \mathcal{H}^k(H_1 \cap \dots \cap H_{d-k} \cap W) & : \dim(H_1 \cap \dots \cap H_{d-k}) = k, \\ 0 & : \text{otherwise}, \end{cases}$$

where $W \in \mathcal{K}_h^d$ is a fixed convex body with $\mathcal{H}^d(W) = 1$. We compute

$$\begin{aligned}
&\int_{A_h(d,d-1)^{d-k}} f(H_1 \cap \dots \cap H_{d-k} \cap W) \ \mu_{d-1}^{d-k}(d(H_1,\dots,H_{d-k})) \\
&\quad = \left(\prod_{i=k}^{d-1} \frac{\omega_{d+1}}{\omega_d} \frac{\omega_{i+1}}{\omega_{i+2}} \right) \mathcal{H}^d(W) = \frac{\omega_{k+1}}{\omega_{d+1}} \left(\frac{\omega_{d+1}}{\omega_d} \right)^{d-k}
\end{aligned}$$

by a $(d-k)$-fold application of the Crofton formula (2.4) with the choice $k = d-1$ and (successively) $i = k, k+1, \dots, d-1$ there. On the other hand, applying directly the Crofton formula with $i = k$, we get

$$\int_{A_h(d,k)} \mathcal{H}^k(W \cap E) \ \mu_k(dE) = \mathcal{H}^d(W) = 1.$$

A comparison yields the constant and proves the assertion of the lemma. □

In what follows we use the convention that $\dim(\varnothing) = -1$.

Lemma 2.4.3. *Fix* $d \ge 2$ *and let* $n \in \{1,\dots,d\}$*. Then* $\dim(H_1 \cap \dots \cap H_n) \in \{-1, d-n\}$ *holds for* μ_{d-1}^n*-almost all* $(H_1,\dots,H_n) \in A_h(d,d-1)^n$.

Proof. We apply induction over n and start by observing that for $n = 1$ there is nothing to show. For $n \ge 2$ we have

$$\mu_{d-1}^{n-1}(\{(H_1,\dots,H_{n-1}) \in A_h(d,d-1)^{n-1} : \ \dim(H_1 \cap \dots \cap H_{n-1}) \notin \{-1, d-(n-1)\}\}) = 0$$

by the induction hypothesis. Let us introduce the abbreviation $L_{d-k} := H_1 \cap \dots \cap H_k$ for

$H_1, \ldots, H_k \in A_h(d, d-1)$ and $k \in \{1, \ldots, d\}$. We obtain

$$\begin{aligned}
&\mu_{d-1}^n(\{(H_1, \ldots, H_n) \in A_h(d, d-1)^n : \dim(H_1 \cap \ldots \cap H_n) \notin \{-1, d-n\}\}) \\
&\quad = \int_{A_h(d,d-1)^n} \mathbf{1}\{\dim(L_{d-n}) \notin \{-1, d-n\}\} \, \mu_{d-1}^n(d(H_1, \ldots, H_n)).
\end{aligned}$$

We decompose the indicator function as follows:

$$\begin{aligned}
&\mathbf{1}\{\dim(L_{d-n}) \notin \{-1, d-n\}\} \\
&= \mathbf{1}\{\dim(L_{d-n}) \notin \{-1, d-n\},\ \dim(L_{d-(n-1)}) = d-(n-1)\} \\
&\quad + \mathbf{1}\{\dim(L_{d-n}) \notin \{-1, d-n\},\ \dim(L_{d-(n-1)}) = -1\} \\
&\quad + \mathbf{1}\{\dim(L_{d-n}) \notin \{-1, d-n\},\ \dim(L_{d-(n-1)}) \notin \{-1, d-(n-1)\}\}.
\end{aligned} \tag{2.5}$$

Since the second indicator function on the right-hand side is identically equal to zero, we arrive at

$$\begin{aligned}
&\mu_{d-1}^n(\{(H_1, \ldots, H_n) \in A_h(d, d-1)^n : \dim(H_1 \cap \ldots \cap H_n) \notin \{-1, d-n\}\}) \\
&\le \int_{A_h(d,d-1)^n} \mathbf{1}\{\dim(L_{d-n}) \notin \{-1, d-n\},\ \dim(L_{d-(n-1)}) = d-(n-1)\} \\
&\qquad\qquad + \mathbf{1}\{\dim(L_{d-(n-1)}) \notin \{-1, d-(n-1)\}\} \, \mu_{d-1}^n(d(H_1, \ldots, H_n)).
\end{aligned}$$

By the induction hypothesis and Fubini's theorem we get

$$\int_{A_h(d,d-1)^n} \mathbf{1}\{\dim(L_{d-(n-1)}) \notin \{-1, d-(n-1)\}\} \, \mu_{d-1}^n(d(H_1, \ldots, H_n)) = 0,$$

which covers the case of the third indicator function on the right-hand side of (2.5). Finally, we write $c(H_1, \ldots, H_{n-1})$ for an arbitrary point chosen on $H_1 \cap \ldots \cap H_{n-1}$ (in a measurable way). Then, again by Fubini's theorem, we conclude for the first indicator function on the right-hand side of (2.5) that

$$\begin{aligned}
&\mu_{d-1}^n(\{(H_1, \ldots, H_n) \in A_h(d, d-1)^n : \dim(L_{d-n}) \notin \{-1, d-n\}, \\
&\qquad\qquad\qquad\qquad \dim(L_{d-(n-1)}) = d-(n-1)\}) \\
&\le \int_{A_h(d,d-1)^{n-1}} \int_{A_h(d,d-1)} \mathbf{1}\{H_1 \cap \ldots \cap H_{n-1} \subseteq H_n,\ H_1 \cap \ldots \cap H_{n-1} \neq \varnothing\} \\
&\qquad\qquad\qquad\qquad \mu_{d-1}(dH_n) \, \mu_{d-1}^{n-1}(d(H_1, \ldots, H_{n-1})) \\
&\le \int_{A_h(d,d-1)^{n-1}} \int_{A_h(d,d-1)} \mathbf{1}\{c(H_1, \ldots, H_{n-1}) \in H_n\} \, \mu_{d-1}(dH_n) \, \mu_{d-1}^{n-1}(d(H_1, \ldots, H_{n-1})) \\
&= \int_{A_h(d,d-1)^{n-1}} 0 \, \mu_{d-1}^{n-1}(d(H_1, \ldots, H_{n-1})) = 0.
\end{aligned}$$

This completes the proof. □

2.4.1 Quermassintegrals

For a convex domain $K \in \mathcal{K}_h^d$, the quermassintegrals are in [110, Definition 2.2.1] defined by

$$W_k(K) := \frac{(d-k)\omega_k \cdots \omega_1}{d\,\omega_{d-1} \cdots \omega_{d-k}} \int_{\mathcal{L}_k} \chi(L_k \cap K)\; dL_k, \qquad k = 1, \dots, d-1.$$

Here $\mathcal{L}_k$ is the notation for the space of k-dimensional totally geodesic subspaces of $\mathbb{H}^d$ in [110]. Further dL_k stands for the invariant measure on $\mathcal{L}_k$. The map $\chi(\cdot)$ is the Euler characteristic. Choosing $k = 0$, one ends up with the hyperbolic volume functional $\mathcal{H}^d$. The measure dL_k is unique up to a constant factor. For normalization the same interpretation given in [32] is used. In order to give a definition with respect to our measure μ_k, we have to change the normalization factor to

$$W_k(K) := \frac{(d-k)\omega_d}{d\,\omega_{d-k}} \int_{A_h(d,k)} \chi(H \cap K)\; \mu_k(dH), \qquad k = 1, \dots, d-1.$$

This definition coincides with the definition of quermassintegrals in the Euclidean case.

2.5 Hyperplane tessellations

2.5.1 Poisson U-statistics

In this section we define and introduce the concepts needed in Chapter 3. Besides Poisson U-statistics this includes also the theory of normal approximation bounds.

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space, which is supplied with a σ-finite measure μ. Let η be a proper Poisson process on $\mathbb{X}$ with intensity measure μ (we refer to [64] for a formal construction). Further, fix $m \in \mathbb{N}$ and let $h : \mathbb{X}^m \to \mathbb{R}$ be a non-negative, measurable and symmetric function, which is integrable with respect to μ^m, the m-fold product measure of μ. By a Poisson U-statistic (of order m and with kernel h) we understand a random variable of the form

$$\mathscr{U} = \sum_{(x_1,\dots,x_m) \in \eta_{\neq}^m} h(x_1, \dots, x_m),$$

where $\eta_{\neq}^m$ is the collection of all m-tuples of distinct points of η, see [64]. Functionals of this type have received considerable attention in the literature, especially in connection with applications in stochastic geometry, see, for example, [27, 48, 59, 60, 65, 88, 96, 108, 109]. In the following, we will frequently use the following consequence of the multivariate Mecke equation for Poisson functionals [64, Theorem 4.4]. Namely, the expectation $\mathbb{E}\mathscr{U}$ of the Poisson U-statistic $\mathscr{U}$ is given by

$$\mathbb{E}\mathscr{U} = \mathbb{E} \sum_{(x_1,\dots,x_m) \in \eta_{\neq}^m} h(x_1, \dots, x_m) = \int_{\mathbb{X}^m} h(x_1, \dots, x_m)\, \mu^m(d(x_1, \dots, x_m)). \tag{2.6}$$

In the present work we need a formula for the centred moments of the Poisson U-statistics $\mathscr{U}$ as well as a bound for the Wasserstein and the Kolmogorov distance of a normalized version of $\mathscr{U}$ and a standard Gaussian random variable. To state such results, we need some more notation. Following [64, Chapter 12], for an integer $n \in \mathbb{N}$ we let Π_n and Π_n^* be the set of partitions and sub-partitions of $[n] := \{1, \dots, n\}$, respectively. We recall that by a sub-partition of $\{1, \dots, n\}$ we understand a family of non-empty disjoint subsets (called blocks) of $\{1, \dots, n\}$ and that a sub-partition σ is called a partition if $\bigcup_{J \in \sigma} J = \{1, \dots, n\}$. For $\sigma \in \Pi_n^*$ we let $|\sigma|$ be the number of blocks of σ and $\|\sigma\| = \left|\bigcup_{J \in \sigma} J\right|$ be the number of elements of $\bigcup_{J \in \sigma} J$. In particular, a partition $\sigma \in \Pi_n$ satisfies $\|\sigma\| = n$. For $\ell \in \mathbb{N}$ and $n_1, \dots, n_\ell \in \mathbb{N}$, let $n := n_1 + \dots + n_\ell$ and define

$$J_i := \{j \in \mathbb{N} : n_1 + \dots + n_{i-1} < j \le n_1 + \dots + n_i\}, \qquad i \in \{1, \dots, \ell\},$$

and $\pi := \{J_i : i \in \{1, \dots, \ell\}\}$. Next, we introduce two classes of sub-partitions of $[n]$ by

$$\Pi^*(n_1, \dots, n_\ell) := \{\sigma \in \Pi_n^* : |J \cap J'| \le 1 \text{ for all } J \in \sigma \text{ and } J' \in \pi\},$$
$$\Pi_{\ge 2}^*(n_1, \dots, n_\ell) := \{\sigma \in \Pi^*(n_1, \dots, n_\ell) : |J| \ge 2 \text{ for all } J \in \sigma\}.$$

In the same way the two classes of partitions $\Pi(n_1, \dots, n_\ell)$ and $\Pi_{\ge 2}(n_1, \dots, n_\ell)$ of $[n]$ are defined (just by omitting the upper index * in the above definition). From now on we assume that $n_1 = \dots = n_\ell = m \in \mathbb{N}$ and define, for $\sigma \in \Pi^*(m, \dots, m)$ (where here and below m appears ℓ times),

$$[\sigma] := \{i \in [\ell] : \text{there exists a block } J \in \sigma \text{ such that } J \cap \{m(i-1)+1, \dots, mi\} \ne \varnothing\}$$

as well as

$$\Pi_{\ge 2}^{**}(m, \dots, m) := \{\sigma \in \Pi_{\ge 2}^*(m, \dots, m) : [\sigma] = [\ell]\}.$$

The sub-partitions $\sigma \in \Pi_{\ge 2}^{**}(m, \dots, m)$ of $[m\ell]$ are easy to visualize as diagrams (cf. [113, Chapter 4]). The $m\ell$ elements of $[m\ell]$ are arranged in an array of ℓ rows and m columns, where $1, \dots, m$ form the first row, $m+1, \dots, 2m$ the second etc. The blocks of σ are indicated by closed curves, where the elements enclosed by a curve are meant to belong to the same block. Then the condition that $\sigma \in \Pi_{\ge 2}^{**}(m, \dots, m)$ can be expressed by the following three requirements:

(i) all blocks of σ have at least two elements,

(ii) each block of σ contains at most one element from each row,

(iii) in each row there is at least one element that belongs to some block of σ.

For an example and a counterexample we refer to Figure 2.5.1.

For two functions $g_1 : \mathbb{X}^{\ell_1} \to \mathbb{R}$ and $g_2 : \mathbb{X}^{\ell_2} \to \mathbb{R}$ with $\ell_1, \ell_2 \in \mathbb{N}$), we denote by $g_1 \otimes g_2 : \mathbb{X}^{\ell_1 + \ell_2} \to \mathbb{R}$ their usual tensor product. We are now in the position to rephrase the following formula for the centred moments of the Poisson U-statistic $\mathscr{U}$ (see [64, Proposition 12.13]):

$$\mathbb{E}[(\mathscr{U} - \mathbb{E}\mathscr{U})^\ell] = \sum_{\sigma \in \Pi_{\ge 2}^{**}(m, \dots, m)} \int_{\mathbb{X}^{m\ell + |\sigma| - \|\sigma\|}} (h^{\otimes \ell})_\sigma \, d\mu^{m\ell + |\sigma| - \|\sigma\|}, \tag{2.7}$$

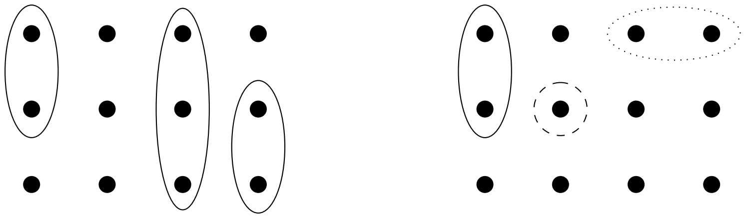

Figure 2.5.1: Left panel: Sub-partition from $\Pi^{**}_{\geq 2}(4,4,4)$. Right panel: Example of a sub-partition not belonging to $\Pi^{**}_{\geq 2}(4,4,4)$. In fact, the block indicated by the dashed curve contradicts condition (i), the block indicated by the dotted curve contradicts condition (ii) and since no element from the last row is contained in any block also condition (iii) is violated

where $h^{\otimes \ell}$ is the ℓ-fold tensor product of h with itself and $(h^{\otimes \ell})_\sigma : \mathbb{X}^{m\ell+|\sigma|-\|\sigma\|} \to \mathbb{R}$ stands for the function that arises from $h^{\otimes \ell}$ by replacing all variables that are in the same block of σ by a new, common variable. Here, we implicitly assume that the function h is such that all integrals that appear on the right-hand side are well-defined. This formula will turn out to be a crucial tool in the proof of Theorem 3.1.5 (c).

2.5.2 Normal approximation bounds

In this section, we continue to use the notation and the set-up of the preceding section. But since we turn to normal approximation bounds for Poisson U-statistics, some further notation is required. For $u, v \in \{1, \dots, m\}$ we let $\Pi^{con}_{\geq 2}(u,u,v,v)$ be the class of partitions in $\Pi_{\geq 2}(u,u,v,v)$ whose diagram is connected, which means that the rows of the diagram cannot be divided into two subsets, each defining a separate diagram (cf. [113, page 47]). More formally, there are no sets $A, B \subset [4]$ with $A \cup B = [4]$, $A \cap B = \emptyset$ and such that each block either consists of elements from rows in A or of elements from rows in B, see Figure 2.5.2 for an example and a counterexample. We can now introduce the quantities

$$M_{u,v}(h) := \sum_{\sigma \in \Pi^{con}_{\geq 2}(u,u,v,v)} \int_{\mathbb{X}^{|\sigma|}} (h_u \otimes h_u \otimes h_v \otimes h_v)_\sigma \; d\mu^{|\sigma|}, \tag{2.8}$$

where

$$h_u(x_1, \dots, x_u) = \binom{m}{u} \int_{\mathbb{X}^{m-u}} h(x_1, \dots, x_u, \tilde{x}_1, \dots, \tilde{x}_{m-u}) \, \mu^{m-u}(d(\tilde{x}_1, \dots, \tilde{x}_{m-u})) \tag{2.9}$$

for $u \in \{1, \dots, m\}$ (again, we implicitly assume that h is such that the integrals appearing in (2.8) are well-defined). To measure the distance between two real-valued random variables X, Y (or, more precisely, their laws), the Kolmogorov distance

$$d_K(X,Y) := \sup_{s \in \mathbb{R}} |\mathbb{P}(X \leq s) - \mathbb{P}(Y \leq s)|$$

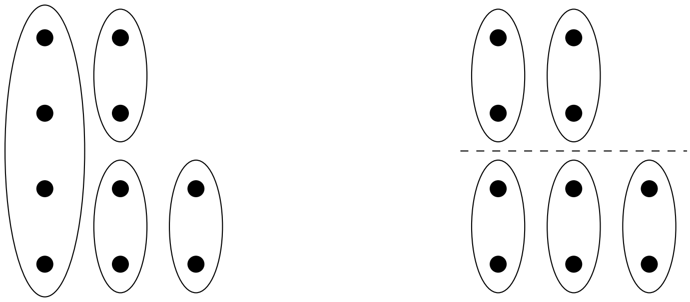

Figure 2.5.2: Left panel: Partition from $\Pi^{con}_{\geq 2}(2,2,3,3)$. Right panel: Example of a partition not belonging to $\Pi^{con}_{\geq 2}(2,2,3,3)$. In fact, the diagram is not connected as indicated by the dashed line

and the Wasserstein distance

$$d_W(X,Y) := \sup_{\varphi \in \mathrm{Lip}(1)} |\mathbb{E}\varphi(X) - \mathbb{E}\varphi(Y)|$$

are used, where Lip(1) denotes the space of Lipschitz functions $\varphi : \mathbb{R} \to \mathbb{R}$ with a Lipschitz constant less than or equal to one. It is well known that convergence with respect to the Kolmogorov or the Wasserstein distance implies convergence in distribution. We are now in the position to rephrase a quantitative central limit theorem for Poisson U-statistics. Namely, [96, Theorem 4.7] and [109, Therorem 4.2] state that there exists a constant $c_m \in (0,\infty)$, depending only on m (the order of the Poisson U-statistic), such that

$$d\left(\frac{\mathscr{U} - \mathbb{E}\mathscr{U}}{\sqrt{\mathbb{V}\mathrm{ar}(\mathscr{U})}}, N\right) \leq c_m \sum_{u,v=1}^{m} \frac{\sqrt{M_{u,v}(h)}}{\mathbb{V}\mathrm{ar}(\mathscr{U})}, \tag{2.10}$$

where $d(\,\cdot\,,\,\cdot\,)$ stands for either the Wasserstein or the Kolmogorov distance. Here, one can choose $c_m = 2m^{7/2}$ for the Wasserstein distance and $c_m = 19m^5$ for the Kolmogorov distance.

Finally, we turn to a multivariate normal approximation for Poisson U-statistics. For integers $p \in \mathbb{N}$ and $m_1,\dots,m_p \in \mathbb{N}$, and for each $i \in \{1,\dots,p\}$, let

$$\mathscr{U}_i = \sum_{(x_1,\dots,x_{m_i}) \in \eta_{\neq}^{m_i}} h^{(i)}(x_1,\dots,x_{m_i})$$

be a Poisson U-statistic of order m_i based on a kernel function $h^{(i)} : \mathbb{X}^{m_i} \to \mathbb{R}$ satisfying the same assumptions as above. We form the p-dimensional random vector $\mathbf{U} := (\mathscr{U}_1,\dots,\mathscr{U}_p)$ and our goal is to compare $\mathbf{U}$ with a p-dimensional Gaussian random vector $\mathbf{N}$. To do this, we use the so-called d_2- and d_3-distance, which are defined as

$$d_2(\mathbf{U},\mathbf{N}) := \sup_{h \in C^2} \big|\mathbb{E}\varphi(\mathbf{U}) - \mathbb{E}\varphi(\mathbf{N})\big|$$
$$d_3(\mathbf{U},\mathbf{N}) := \sup_{h \in C^3} \big|\mathbb{E}\varphi(\mathbf{U}) - \mathbb{E}\varphi(\mathbf{N})\big|,$$

respectively. Here, C^2 is the space of function $\varphi : \mathbb{R}^p \to \mathbb{R}$ which are twice partially continuously differentiable and satisfy

$$\sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\|x - y\|} \le 1 \qquad \text{and} \qquad \sup_{x \neq y} \frac{\|\nabla\varphi(x) - \nabla\varphi(y)\|_{\mathrm{op}}}{\|x - y\|} \le 1,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^p$ and $\|\cdot\|_{op}$ stands for the operator norm. Moreover, C^3 is the space of functions $\varphi : \mathbb{R}^p \to \mathbb{R}$ which are thrice partially continuously differentiable and satisfy

$$\max_{1 \le i \le j \le p} \sup_{x \in \mathbb{R}^p} \left| \frac{\partial^2 \varphi(x)}{\partial x_i \partial x_j} \right| \le 1 \qquad \text{and} \qquad \max_{1 \le i \le j \le k \le p} \sup_{x \in \mathbb{R}^p} \left| \frac{\partial^3 \varphi(x)}{\partial x_i \partial x_j \partial x_k} \right| \le 1.$$

Moreover, similarly to the quantities $M_{u,v}(h)$ introduced in (2.8), for $i, j \in \{1, \dots, p\}$, $u \in \{1, \dots, m_i\}$ and $v \in \{1, \dots, m_j\}$ we define

$$M_{u,v}(h^{(i)}, h^{(j)}) := \sum_{\pi \in \Pi_{\ge 2}^{con}(u,u,v,v)} \int_{\mathbb{X}^{|\pi|}} (h_u^{(i)} \otimes h_u^{(i)} \otimes h_v^{(j)} \otimes h_v^{(j)})_\pi \, d\mu^{|\pi|},$$

where $h_u^{(i)}$ and $h_v^{(j)}$ are given by (2.9). This allows us to state the following multivariate normal approximation bound from [108, Theorem 6.3] (see also [97, Equation (5.1)]). Namely, if $\mathbf{N}$ is a centred Gaussian random vector with covariance matrix $\Sigma = (\sigma_{i,j})_{i,j=1}^p$, then

$$\begin{aligned} d_3(\mathbf{U} - \mathbb{E}\mathbf{U}, \mathbf{N}) \le& \frac{1}{2} \sum_{i,j=1}^p |\sigma_{i,j} - \mathbb{C}\mathrm{ov}(\mathscr{U}_i, \mathscr{U}_j)| \\ &+ \frac{p}{2} \left(\sum_{n=1}^p \sqrt{\mathbb{V}\mathrm{ar}(\mathscr{U}_n)} + 1 \right) \sum_{i,j=1}^p \sum_{u=1}^{m_i} \sum_{v=1}^{m_j} m_i^{7/2} \sqrt{M_{u,v}(h^{(i)}, h^{(j)})}. \end{aligned} \tag{2.11}$$

If the covariance matrix Σ is positive definite then also

$$\begin{aligned} d_2(\mathbf{U} - \mathbb{E}\mathbf{U}, \mathbf{N}) \le& \|\Sigma^{-1}\|_{op} \|\Sigma\|_{op}^{1/2} \sum_{i,j=1}^p |\sigma_{i,j} - \mathbb{C}\mathrm{ov}(\mathscr{U}_i, \mathscr{U}_j)| \\ &+ \frac{p\sqrt{2\pi}}{4} \|\Sigma^{-1}\|_{op}^{3/2} \|\Sigma\|_{op} \left(\sum_{i=1}^p \sqrt{\mathbb{V}\mathrm{ar}(\mathscr{U}_i)} + 1 \right) \sum_{i,j=1}^p \sum_{u=1}^{m_i} \sum_{v=1}^{m_j} m_i^{7/2} \sqrt{M_{u,v}(h^{(i)}, h^{(j)})}, \end{aligned} \tag{2.12}$$

where again $\|\cdot\|_{op}$ stands for the operator norm. We remark that although the bound for $d_2(\mathbf{U} - \mathbb{E}\mathbf{U}, \mathbf{N})$ is not explicitly stated in the literature, it directly follows from [89, Theorem 3.3] together with the computations in [108, Chapters 5 and 6] for the d_3-distance.

2.6 Splitting tessellations

In Chapter 4 two different set ups are considered. In a first step we study splitting tessellations inside a fixed convex window $W \in \mathcal{K}_h^d$. In a second step we will show that the process can be extended to the whole space $\mathbb{H}^d$. In order to handle technical problems arising due to boundary effects we introduce some definitions concerning tessellations inside a fixed window W. We will often omit reference to the convex set W if it does not play an important role or is clear from the context. We define the set $\mathbb{P}_W^d$ by

$$\mathbb{P}_W^d := \{P \cap W : \ P \in \mathbb{PD}^d\},$$

and call the elements of $\mathbb{P}_W^d$ semipolytopes (in W). Here $\mathbb{PD}^d$ is the set of d-dimensional polyhedrons in hyperbolic space. For a definition of polyhedrons, we refer to Section 2.3. For the special case $W = B_r, \ r \geq 0$ we will shorten the notation by defining $\mathbb{P}_r^d := \mathbb{P}_{B_r}^d$.

Definition 2.6.1. *By a tessellation T of W we understand a finite collection of d-dimensional semi-polytopes such that*

i) $\bigcup_{c \in T} c = W$

ii) For $c_1, c_2 \in T$ with $c_1 \neq c_2$ the intersection $\operatorname{int}(c_1) \cap \operatorname{int}(c_2)$ *is empty.*

The set of all tessellations of W is denoted by $\mathbb{T}^d(W)$. The space $\mathbb{T}^d(W)$ can be equipped with a σ-field. Analogue to Lemma 2.3.15 one can show that $\mathbb{T}^d(W)$ is a Borel subset of $\mathcal{F}(\mathcal{F}'(\mathbb{H}^d))$.

Next we will define the so-called splitting operator $\oslash$. It defines the tessellation which results if one splits a single cell of the tessellation by a hyperplane and replaces it with the two arising parts.

Definition 2.6.2. *For a convex set $W \in \mathcal{K}_h^d$, $T \in \mathbb{T}^d(W)$, $c \in \mathbb{P}_W^d$ and $H \in A_h(d, d-1)$, we define the operator $\oslash : \mathbb{P}_W^d \times A_h(d, d-1) \times \mathbb{T}^d \to \mathbb{T}^d$ by*

$$\oslash(c, H, T) := \begin{cases} (T \setminus \{c\}) \cup \{c \cap H^+, c \cap H^-\} & : c \in T, \\ T & : \textit{otherwise}, \end{cases}$$

where H^+ and H^- are the two closed half-spaces determined by H.

If $c \in T$ and $H \cap c = \varnothing$ holds, then the splitting operator keeps the tessellations invariant, since in this case one of the two closed half-spaces contains c. Similar to [47, Lemma 2.6], the (Borel-) measurability of the operator can be shown the following way. One first shows that the set $\{(c, T) \in \mathbb{P}_W^d \times \mathbb{T}^d(W) : \ c \in T\}$ is measurable. The argumentation to prove this transfers from [47] to our setting as well. Furthermore, the map $(c, T) \mapsto T \setminus \{c\}$ is measurable, since removing one cell from T corresponds to subtracting the dirac measure on c from the measure η. Here $\eta \in \mathbf{N}(\mathbb{H}^d)$ is the simple counting measure which takes the value 1 if $c \in T$ holds and 0 otherwise. In a last step we start by arguing that the set $\{(c, H) \in \mathbb{P}_W^d \times \mathbb{H}_{d-1}(\langle W \rangle)\}$ is open and therefore measurable. On this set the map $\mathbb{P}_W^d \times \mathbb{H}_{d-1}(\langle W \rangle) \to \mathcal{N}_h(\mathbb{H}^d), \ (c, H) \mapsto \partial_{c \cap H^+} + \partial_{c \cap H^-}$ is measurable. Now $\oslash$ is measurable as the combination of measurable maps.

We denote by $\mathbb{T}_{split}^d(W)$, the set of all tessellations of W resulting from as splitting process. We state the following lemma without a proof.

Lemma 2.6.3. *The set $\mathbb{T}_{split}^d(W)$ of all splitting tessellations is a Borel set in $\mathbb{T}^d(W)$.*

2.7 Kendall's problem

In this section we consider some concepts which are needed in order to formulate and solve Kendall's problem in hyperbolic space. We start with some terms that allow us to speak more precisely about the size and the shape of a cell. We also state and prove the reverse Hölder inequality since it is frequently used in Chapter 5.

The *Crofton cell* or zero cell C_0 of an isometry invariant random tessellation is defined as the cell containing the origin p (the fixed reference point). In our model, this cell is almost surely uniquely determined. For the introduction of the concept of the typical cell, which can be interpreted as a statistical, spatial average of the cells in the tessellation, we refer to Section 5.4 and the literature cited there.

Since our aim is to provide (quantitative) information about the shape of Crofton and typical cells of Poisson hyperplane tessellations in hyperbolic space, we also require various geometric concepts. In particular, we will use the notions of a deviation function, a size functional, and a hitting functional for comparing the shapes of cells to the spherical shape, to quantify the size of cells and to describe the hyperplane process in more geometric terms. The corresponding concepts have already proved to be particularly useful in Euclidean and spherical spaces.

An upper semicontinuous function $\vartheta : \mathcal{K}_h^d \to [0,\infty)$ is said to be a *deviation function* for a class $\mathcal{G} \subseteq \mathcal{K}_h^d$, if for all $K \in \mathcal{K}_h^d$ with $\mathcal{H}^d(K) > 0$, the equality $\vartheta(K) = 0$ holds if and only if $K \in \mathcal{G}$. In the following, we will call a deviation function for the class of hyperbolic balls simply a deviation function. A canonical example for an isometry invariant deviation functional is

$$\bar{\vartheta}(K) := \inf\{\delta_h(K,B) :\ B \text{ is a hyperbolic ball}\},$$

where δ_h is the hyperbolic Hausdorff distance (for a definition see Section 2.2.5). The required properties can easily be verified. For a convex body $K \in \mathcal{K}_h^d$, we define the centred inball radius r_c by

$$r_c(K) := \begin{cases} \sup\{r \geq 0 :\ B_h(c,r) \subseteq K\}, & c \in K, \\ 0, & \text{otherwise.} \end{cases}$$

and the centred circumradius R_c by

$$R_c(K) := \inf\{r \geq 0 :\ K \subseteq B_h(c,r)\}.$$

Here one could replace the supremum (infimum) by the maximum (minimum) respectively. For the special case $c = p$, we use the notation r_0 and R_0. Using these definitions, a specific deviation function for the class of balls with centre in the origin is

$$\vartheta_0\colon\ \mathcal{K}_h^d \to [0,\infty),\ K \mapsto R_0(K) - r_0(K)$$

Clearly, $\vartheta_0(K) = 0$ holds if and only if K is a hyperbolic ball with centre in p. The continuity of ϑ_0 follows from Lemmas 5.2.4 and 5.2.5 below. Furthermore, ϑ_0 is invariant under isometries fixing p.

For the isoperimetric inequalities, which are based on a Bonnesen-style inequality, we introduce for $K \in \mathcal{K}_h^d$ the notation

$$r_{in}(K) := \sup\{r \geq 0 : B_h(c,r) \subseteq K \text{ for some } c \in K\}.$$

Further, we denote by $R_{out}(K)$ the circumradius of K which is defined by

$$R_{out}(K) := \inf\{r \geq 0 : K \subseteq B_h(x,r) \text{ for some } x \in \mathbb{H}^d\}.$$

Note that $\tilde{\vartheta}(\cdot) := R_{out}(\cdot) - r_{in}(\cdot)$ is a deviation function; more precisely, it is an isometry invariant deviation function for balls with arbitrary centre. As in the case of the centred inball- and circumradius, it is possible to replace inf(sup) by min(max) in the definitions of r_{in} and R_{out}. The continuity of $\tilde{\vartheta}$ is shown in Lemmas 5.2.6, 5.2.7. We denote by $C_r(K) := \{c \in K : r_c(K) = r_{in}(K)\}$ the set of points realizing the inball radius. Using this notation, we can define the geometric functional

$$\vartheta_r(K) := \inf\{R_c(K) - r_c(K) : c \in C_r(K)\}.$$

This functional is no deviation functional since it is lower semicontinuous but not upper semicontinuous (for a proof see Lemma 5.2.11 and the following remark). Nevertheless it will be used for the statement of a result of isoperimetric type (see Theorem 5.2.17).

A function $\Sigma : \mathcal{K}_h^d \to [0,\infty)$ that is continuous, not identically zero and increasing under inclusion is called a *size functional*. Some examples are the hyperbolic volume $\mathcal{H}^d$ or the hyperbolic inball radius $\Sigma_r := r_{in}$ (we use the different notation in order to indicate that it takes the role of a size functional). Lemma 5.2.2 shows the continuity of $\mathcal{H}^d$, the continuity of the inball radius follows from Lemma 5.2.7. Another important functional which arises naturally in this context is the so-called *hitting-functional* $\Phi : \mathcal{K}_h^d \to [0,\infty)$. The hitting functional depends on the underlying point process and roughly describes how likely it is that a body is hit by the (hyperbolic) hyperplanes of the point process. It is continuous with respect to the Hausdorff metric and fulfills $\Phi(K) > 0$ for all K containing more than one point. In this work, the hitting functional will be a constant multiple of the $(d-1)$-dimensional quermaßintegral W_{d-1} whenever we investigate the behaviour of cells in Poisson hyperplane tessellations. The continuity of W_{d-1} will be shown in Lemma 5.2.1. For the case of Poisson Voronoi tessellations, we use a different hitting functional which will be introduced in Section 5.5.

The concept of a deviation function is used for defining the convergence of the shape of cells in tessellations. A random set $Z = Z(t)$ is said to converge in probability to the shape of a ball as $t \to \infty$ if for an appropriate (geometrically meaningful) deviation function ϑ and for all $\varepsilon > 0$ we have

$$\lim_{t\to\infty} \mathbb{P}_t^*(\vartheta(Z) > \varepsilon) = 0.$$

In the present work, the parameter t indicates the dependence on the intensity of the underlying Poisson hyperplane process η_t inducing the tessellation. The probability measure $\mathbb{P}_t^*$ is given by a conditional probability, where the condition implies a restriction on the size of the random cells under consideration.

For a fixed hitting functional Φ, size functional Σ and a number $a > 0$ we define the *isoperimetric constant*

$$\tau(\Phi, \Sigma, a) := \min\{\Phi(K) : K \in \mathcal{K}_{h,0}^d,\ \Sigma(K) \geq a\},$$

where $\mathcal{K}_{h,0}^d \subseteq \mathcal{K}_h^d$ is the set of convex bodies containing the origin. We will see that this minimum is indeed attained. Further, we omit the dependence on Φ, Σ if they are clear from the context.

We will use the so-called reverse Hölder inequality in the proofs of Chapter 5. It follows easily in a few steps by the classical Hölder inequality, which can be found for example in [56, Theorem 7.16].

Lemma 2.7.1. *Let $p \in (1, \infty)$, $(X, \mathcal{F}, \mu)$ a measurable space and $f, g : X \to \mathbb{R}$ such that $\int_X f(x)g(x)\ \mu(dx) < \infty$ and $\int_X g(x)^{-\frac{1}{p-1}}\ \mu(dx) < \infty$ hold with $g(x) \neq 0$ for μ-almost all $x \in X$. Then the reverse Hölder inequality*

$$\int_X f(x)g(x)\ \mu(dx) \geq \left(\int_X f(x)^{1/p}\ \mu(dx)\right)^p \left(\int_X g(x)^{-1/(p-1)}\ \mu(dx)\right)^{-(p-1)}$$

holds.

Proof. We define $q := \frac{p}{p-1}$ which fulfills $\frac{1}{p} + \frac{1}{q} = \frac{1}{p} + \frac{p-1}{p} = 1$. Applying Hölder's inequality for the functions $(f\,g)^{1/p}$ and $g^{-1/p}$ in the penultimate step we get

$$\begin{aligned}
&\left(\int_X f(x)g(x)\ \mu(dx)\right)^{1/p} \left(\int_X g(x)^{-1/(p-1)}\ \mu(dx)\right)^{(p-1)/p} \\
&= \left(\int_X f(x)g(x)\ \mu(dx)\right)^{1/p} \left(\int_X g(x)^{-q/p}\ \mu(dx)\right)^{1/q} \\
&= \left(\int_X \left((f(x)g(x))^{1/p}\right)^p\ \mu(dx)\right)^{1/p} \left(\int_X (g(x)^{-1/p})^q\ \mu(dx)\right)^{1/q} \\
&\geq \int_X (f(x)g(x))^{1/p} g(x)^{-1/p}\ \mu(dx) \\
&\geq \int_X f(x)^{1/p}\ \mu(dx).
\end{aligned}$$

Taking the p-th power and dividing by $\left(\int_X g(x)^{-1/(p-1)}\ \mu(dx)\right)^{(p-1)}$ yields the result. □

CHAPTER 3

Hyperbolic Poisson hyperplane tessellations

This chapter is structured as follows. In the next section we recall several important notations in $\mathbb{H}^d$ and present our main results. We start in Section 3.1.1 with expectations and continue in Section 3.1.2 with second-order characteristics associated with the total volume of intersections processes. Our limit theorems will be discussed in Section 3.1.3. All remaining sections are devoted to the proofs of our results. In Section 3.2 we present the proofs for first- and second-order parameters and also carry out a detailed covariance analysis, which is needed for our multivariate central limit theory. Our results on generalizations of the K-function and the pair-correlation function are established in Section 3.3. All univariate limit theorems are proved in Section 3.4, while the arguments for the multivariate central limit theorems are provided in the final Section 3.5. For the necessary background material on hyperbolic geometry and hyperbolic integral geometry we refer to Chapter 2 and especially to Subsections 2.5.1 and 2.5.2 for the background material on Poisson U-statistics.

3.1 Main results

3.1.1 First-order quantities

Recall that we denote by $\mathbb{H}^d$, for $d \geq 2$, the d-dimensional hyperbolic space of constant curvature -1, which is supplied with the hyperbolic metric $d_h(\,\cdot\,,\,\cdot\,)$. We refer to Section 2.2.5 above for further background material on hyperbolic geometry and for a description of the conformal ball model for $\mathbb{H}^d$. Let $p \in \mathbb{H}^d$ be an arbitrary (fixed) point, also referred to as the origin. For $r \geq 0$ we denote by $B_r = \{x \in \mathbb{H}^d : d_h(x,p) \leq r\}$ the hyperbolic ball around p with radius r. A set $K \subset \mathbb{H}^d$ is called a hyperbolic convex body, provided that K is non-empty, compact and if with each pair of points $x, y \in K$ the (unique) geodesic connecting x and y is contained in K.

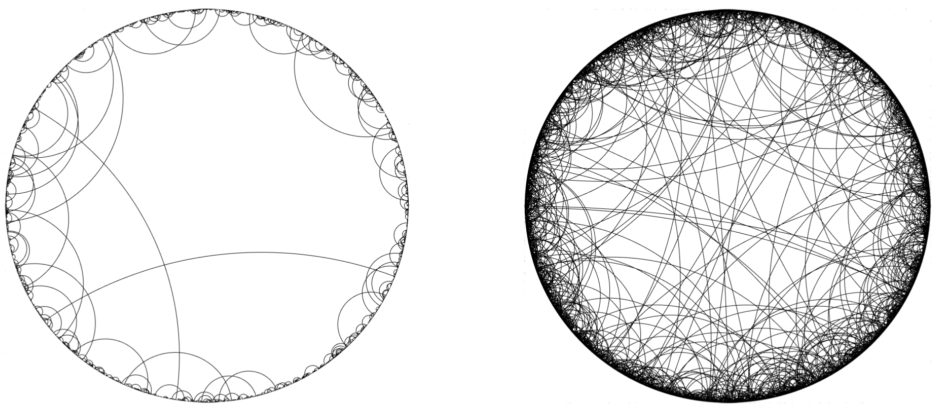

Figure 3.1.1: Two realizations of a Poisson hyperplane tessellation in $\mathbb{H}^2$ of different intensities represented in the conformal ball model

The space of hyperbolic convex bodies is denoted by $\mathcal{K}_h^d$. Recall that for $k \in \{0, 1, \dots, d-1\}$ a k-dimensional totally geodesic subspace of $\mathbb{H}^d$ is called a k-plane and especially $(d-1)$-planes are called hyperplanes. The space of k-planes in $\mathbb{H}^d$ is denoted by $A_h(d,k)$. The space $A_h(d,k)$ carries a measure μ_k, which is invariant under isometries of $\mathbb{H}^d$ (see Section 2.2.5 for the present normalization of this measure). For $s \geq 0$ we denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure with respect to the intrinsic metric of $\mathbb{H}^d$ as a Riemannian manifold. Finally, we write $\omega_k = 2\pi^{k/2}/\Gamma(k/2)$, $k \in \mathbb{N}$, for the surface area of the k-dimensional unit ball in the Euclidean space $\mathbb{R}^k$.

For $t > 0$, let η_t be a Poisson process on the space $A_h(d, d-1)$ of hyperplanes in $\mathbb{H}^d$ with intensity measure $t\mu_{d-1}$. We refer to η_t as a (hyperbolic) Poisson hyperplane process with intensity t. It induces a Poisson hyperplane tessellation in $\mathbb{H}^d$, i.e., a subdivision of $\mathbb{H}^d$ into (possibly unbounded) hyperbolic cells (generalized polyhedra), see Figure 3.1.1. For $i \in \{0, \dots, d-1\}$ we consider the intersection process $\xi_t^{(i)}$ of order $d-i$ of the Poisson hyperplane process η_t given by

$$\xi_t^{(i)} := \frac{1}{(d-i)!} \sum_{(H_1,\dots,H_{d-i}) \in \eta_{t,\neq}^{d-i}} \delta_{H_1 \cap \dots \cap H_{d-i}} \mathbf{1}\{\dim(H_1 \cap \dots \cap H_{d-i}) = i\},$$

where $\eta_{t,\neq}^{d-i}$ is the set of $(d-i)$-tuples of different hyperplanes supported by η_t, $\delta_{(\cdot)}$ denotes the Dirac measure and $\dim(\cdot)$ stands for the dimension of the set in the argument. In this work we are interested in random variables of the form

$$\begin{aligned} F_{W,t}^{(i)} &:= \int \mathcal{H}^i(E \cap W)\, \xi_t^{(i)}(dE) \\ &= \frac{1}{(d-i)!} \sum_{(H_1,\dots,H_{d-i}) \in \eta_{t,\neq}^{d-i}} \mathcal{H}^i(H_1 \cap \dots \cap H_{d-i} \cap W)\, \mathbf{1}\{\dim(H_1 \cap \dots \cap H_{d-i}) = i\}, \end{aligned} \tag{3.1}$$

where $W \subset \mathbb{H}^d$ is a (fixed) Borel set in $\mathbb{H}^d$. In other words, $F_{W,t}^{(i)}$ measures the total i-volume (i.e., the i-dimensional Hausdorff measure) of the intersection process $\xi_t^{(i)}$ within W. For example,

$$F_{W,t}^{(d-1)} = \sum_{H\in\eta_t} \mathcal{H}^{d-1}(H\cap W) = \mathcal{H}^{d-1}\Big(\bigcup_{H\in\eta_t} H\cap W\Big)$$

is the total surface content of the union of all hyperplanes of η within W. On the other hand,

$$F_{W,t}^{(0)} = \frac{1}{d!}\sum_{(H_1,\dots,H_d)\in\eta_{t,\neq}^d} \mathbf{1}\{H_1\cap\ldots\cap H_d\in W, \mathrm{d}im(H_1\cap\ldots\cap H_d)=0\}$$

is the total number of vertices in W of the Poisson hyperplane tessellation, i.e., the total number of intersection points induced by the hyperplanes of η_t. In the Euclidean case these random variables have received particular attention in the literature, see e.g. [35, 37, 48, 52, 53, 65, 73, 96, 103] and the references cited therein. As in the Euclidean case, we will start by investigating the expectation of $F_{W,t}^{(i)}$.

Theorem 3.1.1 (Expectation)**.** *If $W\subset\mathbb{H}^d$ is a Borel set, $t>0$ and $i\in\{0,1,\dots,d-1\}$, then*

$$\mathbb{E}F_{W,t}^{(i)} = \frac{\omega_{i+1}}{\omega_{d+1}}\left(\frac{\omega_{d+1}}{\omega_d}\right)^{d-i}\frac{t^{d-i}}{(d-i)!}\mathcal{H}^d(W).$$

Remark 3.1.1. In comparison with the Euclidean and spherical case we observe that precisely the same formula holds in these spaces. This is not surprising, since the proof of Theorem 3.1.1 is based only on the multivariate Mecke formula for Poisson processes and a recursive application of Crofton's formula from integral geometry, see Section 3.2. Since the latter holds for any standard space of constant curvature $\kappa\in\{-1,0,1\}$ with the same constant (cf. [15, 101]), independently of the curvature κ, the result of Theorem 3.1.1 holds simultaneously for all standard spaces of constant curvature $\kappa\in\{-1,0,1\}$. In other words, this means that the expectation $\mathbb{E}F_{W,t}^{(i)}$ is not an appropriate quantity to 'feel' or to 'detect' the curvature of the underlying space. For this we will use second-order characteristics.

3.1.2 Second-order quantities

In a next step, we describe the covariance structure of the functionals $F_{W,t}^{(i)}$, $i\in\{0,1,\dots,d-1\}$, introduced in (3.1). The following explicit representation for the covariances will be derived from the Fock space representation of Poisson U-statistics.

Theorem 3.1.2 (Covariances)**.** *Let $W\subset\mathbb{H}^d$ be a Borel set, let $t>0$, and let $i,j\in\{0,1,\dots,d-1\}$. Then*

$$\mathbb{C}ov(F_{W,t}^{(i)},F_{W,t}^{(j)}) = \sum_{n=1}^{\min\{d-i,d-j\}} c_{i,j,n,d}\, t^{2d-i-j-n}\int_{A_h(d,d-n)}\mathcal{H}^{d-n}(E\cap W)^2\,\mu_{d-n}(dE)$$

with

$$c_{i,j,n,d} = \frac{1}{n!}\frac{1}{\omega_{d+1}\,\omega_{d-n+1}}\frac{\omega_{i+1}}{(d-i-n)!}\frac{\omega_{j+1}}{(d-j-n)!}\left(\frac{\omega_{d+1}}{\omega_d}\right)^{2d-i-j-n}.$$

Remark 3.1.2. Since Theorem 3.1.2 follows from the general Fock space representation of Poisson U-statistics, the formula for $\mathbb{C}\mathrm{ov}(F_{W,t}^{(i)}, F_{W,t}^{(j)})$ is formally the same for all spaces of constant curvature $\kappa \in \{-1, 0, 1\}$. However, the curvature properties of the underlying space are hidden in the integral-geometric expression

$$J_k(W) := \int_{A_h(d,k)} \mathcal{H}^k(E \cap W)^2 \, \mu_k(dE),$$

for $k \in \{0, \ldots, d-1\}$. In fact, if $\kappa \in \{-1, 0\}$ and if we replace W by a ball B_r of radius r around an arbitrary fixed point, we can consider the asymptotic behaviour of $J_k(B_r)$, as $r \to \infty$, which is quite different in these two cases (note that in spherical spaces with constant curvature $\kappa = 1$ the range of r is bounded). While in the Euclidean case $\kappa = 0$, $J_k(B_r)$ behaves like a constant multiple of r^{d+k} for all choices of k, in the hyperbolic case $\kappa = -1$ we will show that $J_k(B_r)$ behaves like a constant multiple of $e^{(d-1)r}$ if $2k - 1 < d$, like a constant multiple of $re^{(d-1)r}$ if $2k - 1 = d$ and like a constant multiple of $e^{2(k-1)r}$ if $2k - 1 > d$, see Lemma 3.2.5 below. In this sense we can say that second-order properties of the functionals $F_{W,t}^{(i)}$ are sensitive to the curvature of the underlying space.

Continuing the discussion of second-order properties of Poisson hyperplane tessellations in $\mathbb{H}^d$, we now introduce and describe the K-function and the pair-correlation function of the i-dimensional Hausdorff measure restricted to the i-skeleton of the tessellation. In the Euclidean case these two functions have turned out to be essential tools in the second-order analysis of stationary random measures (see the original paper [98] and the recent monograph [3] as well as the references cited therein). To be precise, for $i \in \{0, 1, \ldots, d-1\}$ and fixed $t > 0$, we first consider the i-skeleton of the Poisson hyperplane tessellation in $\mathbb{H}^d$ with intensity t, which is defined as the random closed set

$$\mathrm{skel}_i := \bigcup_{\substack{(H_1,\ldots,H_{d-i}) \in \eta_{t,\neq}^{d-i} \\ dim(H_1 \cap \ldots \cap H_{d-i}) = i}} H_1 \cap \ldots \cap H_{d-i}.$$

The i-dimensional Hausdorff measure on skel_i is denoted by $\mathbf{M}_i$. It is a stationary random measure on $\mathbb{H}^d$, that is, its distribution is invariant under isometries of $\mathbb{H}^d$. Its intensity is defined by $\lambda_i = \mathbb{E} F_{B,t}^{(i)}$, where $B \subset \mathbb{H}^d$ is an arbitrary Borel set with $\mathcal{H}^d(B) = 1$. It follows from Theorem 3.1.1 that

$$\lambda_i = \frac{\omega_{i+1}}{\omega_{d+1}} \left(\frac{\omega_{d+1}}{\omega_d} \right)^{d-i} \frac{t^{d-i}}{(d-i)!}. \tag{3.2}$$

The K-function of the random measure $\mathbf{M}_i$ is defined by

$$K_i(r) := \frac{1}{\lambda_i^2} \mathbb{E} \int_{\mathbb{H}^d} \int_B \mathbb{1}\{0 < d_h(x, y) \le r\} \, \mathbf{M}_i(dy) \, \mathbf{M}_i(dx), \qquad r > 0. \tag{3.3}$$

The condition $d_h(x, y) > 0$ is usually omitted in the definition of the K-function of a diffuse stationary random measure. For $i \in \{1, \ldots, d-1\}$, the proof of the following more general Theorem 3.1.3 will show that $K_i(r)$ remains indeed unchanged if we drop the condition $d_h(x, y) > 0$. For $i = 0$, however, the random measure $\mathbf{M}_i$ is a stationary point process in $\mathbb{H}^d$

and then the restriction $d_h(x,y)>0$ is common. The proof of Theorem 3.1.3 will also show that the summands corresponding to indices $n\in\{0,\dots,d-1\}$ in (3.4) are not affected by the restriction, but the summand with $n=d$ will be zero.

If we define $K_i(B,r)$ as in (3.3), but for a general measurable set $B\subset\mathbb{H}^d$, it follows from the stationarity of η_t that the measure $K_i(\,\cdot\,,r)$ is isometry invariant and hence a constant multiple of $\mathcal{H}^d(\,\cdot\,)$, provided it is locally finite. In Theorem 3.1.3, this will be shown and the constant will be determined. We will also see that $K_i(r)$ is differentiable, which allows us to consider the pair-correlation function

$$g_i(r) := \frac{1}{\omega_d \sinh^{d-1}(r)}\,\frac{dK_i}{dr}(r), \qquad r>0.$$

Roughly speaking it describes the probability of finding a point on the i-skeleton at geodesic distance r from another point belonging to skel_i.

More generally and in analogy to the covariances considered in Theorem 3.1.2, we will consider the mixed K-function K_{ij} for $i,j\in\{0,\dots,d-1\}$. For $r>0$ and a measurable set $B\subset\mathbb{H}^d$ with $\mathcal{H}^d(B)=1$ it is defined by

$$\begin{aligned}K_{ij}(r) &= \frac{1}{\lambda_i\lambda_j}\,\mathbb{E}\int_{\mathbb{H}^d}\int_B \mathbb{1}\{0<d_h(x,y)\le r\}\,\mathbf{M}_j(dy)\,\mathbf{M}_i(dx)\\ &= \frac{1}{\lambda_i\lambda_j}\,\mathbb{E}\int_{\mathrm{skel}_i}\int_{\mathrm{skel}_j\cap B} \mathbb{1}\{0<d_h(x,y)\le r\}\,\mathcal{H}^j(dy)\,\mathcal{H}^i(dx)\end{aligned}$$

and describes the random measure $\mathbf{M}_i$ as seen from a typical point of $\mathbf{M}_j$, in the sense of Palm distribution. In particular, we retrieve the ordinary K-function by the special choice $j=i$. The mixed pair-correlation function g_{ij} is then defined in the obvious way by differentiation of K_{ij}, namely,

$$g_{ij}(r) := \frac{1}{\omega_d \sinh^{d-1}(r)}\,\frac{dK_{ij}}{dr}(r), \qquad r>0.$$

As in the case of the K-function, the condition that $0<d_h(x,y)$ can be omitted if $i\ge 1$ or $j\ge 1$.

Theorem 3.1.3 (Mixed K-function and mixed pair-correlation function)**.** *If $i,j\in\{0,1,\dots,d-1\}$, $t>0$ and $r>0$, then*

$$\begin{aligned}K_{ij}(r) &= \sum_{n=0}^{m(d,i,j)} n!\binom{d-i}{n}\binom{d-j}{n}\frac{\omega_{d+1}\omega_{d-n}}{\omega_{d-n+1}}\left(\frac{\omega_d}{\omega_{d+1}}\frac{1}{t}\right)^n\int_0^r \sinh^{d-n-1}(s)\,ds,\\ g_{ij}(r) &= 1+\sum_{n=1}^{m(d,i,j)} n!\binom{d-i}{n}\binom{d-j}{n}\frac{\omega_{d-n}}{\omega_{d-n+1}}\left(\frac{\omega_d}{\omega_{d+1}}\right)^{n-1}\frac{1}{(t\sinh(r))^n},\end{aligned} \tag{3.4}$$

where $m(d,i,j):=\min\{d-i,d-j,d-1\}$.

In (3.4) we restrict the summation to $n\le d-1$ in order to avoid an undefined expression which arises for $i=j=0$ and $n=d$. Alternatively, for $n=d$ the factor $\omega_{d-n}=\omega_0$ is $\omega_0=2/\Gamma(0)=0$ and the product with the infinite integral can be defined to be zero.

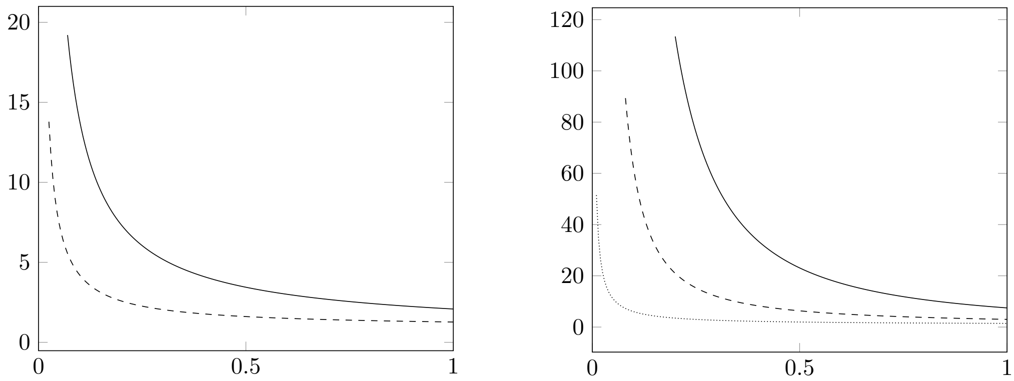

Figure 3.1.2: Left panel: The pair-correlation functions g_0 (solid curve) and g_1 (dashed curve) for $d = 2$ and $t = 1$. Right panel: The pair-correlation functions g_0 (solid curve), g_1 (dashed curve) and g_2 (dotted curve) for $d = 3$ and $t = 1$

In the special case $d = 2$ and for $i = j$ we thus obtain

$$g_0(r) = 1 + \frac{4}{\pi t}\frac{1}{\sinh(r)} \qquad \text{and} \qquad g_1(r) = 1 + \frac{1}{\pi t}\frac{1}{\sinh(r)},$$

and for $d = 3$ and again $i = j$ we get

$$\begin{aligned} g_0(r) &= 1 + \frac{9}{2t}\frac{1}{\sinh(r)} + \frac{36}{\pi^2 t^2}\frac{1}{\sinh^2(r)}, \\ g_1(r) &= 1 + \frac{2}{t}\frac{1}{\sinh(r)} + \frac{4}{\pi^2 t^2}\frac{1}{\sinh^2(r)}, \\ g_2(r) &= 1 + \frac{1}{2t}\frac{1}{\sinh(r)}, \end{aligned}$$

see Figure 3.1.2.

Remark 3.1.3. An inspection of the proof shows that Theorem 3.1.3 is based only on Crofton's formula and Lemma 2.4.2, which in turn is also based on Crofton's formula. However, since the latter holds for any space of constant curvature $\kappa \in \{-1, 0, 1\}$ with the same constant (cf. [15, 101]), independently of the curvature κ, Theorem 3.1.3 remains valid also in spherical and Euclidean spaces of curvature $\kappa = 1$ and $\kappa = 0$, respectively. Namely, defining the modified sine function

$$\mathrm{sn}_\kappa(r) := \begin{cases} \sin(r) & : \kappa = 1, \\ r & : \kappa = 0, \\ \sinh(r) & : \kappa = -1, \end{cases}$$

we obtain

$$K_{ij}(r) = \sum_{n=0}^{m(d,i,j)} n! \binom{d-i}{n}\binom{d-j}{n} \frac{\omega_{d+1}\omega_{d-n}}{\omega_{d-n+1}} \left(\frac{\omega_d}{\omega_{d+1}}\frac{1}{t}\right)^n \int_0^r \mathrm{sn}_\kappa^{d-n-1}(s)\, ds$$

and

$$g_{ij}(r) = 1 + \sum_{n=1}^{m(d,i,j)} n! \binom{d-i}{n} \binom{d-j}{n} \frac{\omega_{d-n}}{\omega_{d-n+1}} \left(\frac{\omega_d}{\omega_{d+1}} \right)^{n-1} \frac{1}{(t\,\mathrm{sn}_\kappa(r))^n}$$

for $r > 0$ if $\kappa \in \{-1, 0\}$ and $0 < r < \pi$ if $\kappa = 1$. For $i = j = d-1$ and $\kappa = 1$ these formulas have been proved in [47, Section 6.2] based on a different normalization. Moreover, for $\kappa = 0$ the formula for $g_0(r)$ appears as the identity (3.15) in [38], while $g_{d-1}(r)$ can be found in [104, Section 7]. As already explained in [39], for general $i \in \{0, 1, \ldots, d-1\}$ it can in principle be deduced from an explicit formula for the second-order moments of the total volume of intersection processes, see [72, p. 164].

3.1.3 Limit theorems

Our next result is a central limit theorem for $F_{W,t}^{(i)}$, for a fixed hyperbolic convex body W, when the intensity parameter t tends to infinity. We will measure the distance between (the laws of) two random variables by the Wasserstein and the Kolmogorov distance. For their definitions we refer to Section 2.5.1.

Theorem 3.1.4 (CLT, growing intensity)**.** *Let $d \geq 2$, $i \in \{0, 1, \ldots, d-1\}$ and let $W \in \mathcal{K}_h^d$ be a fixed hyperbolic convex body with non-empty interior. Let N be a standard Gaussian random variable, and let $d(\cdot, \cdot)$ denote either the Wasserstein or the Kolmogorov distance. Then there exists a constant $c \in (0, \infty)$ such that*

$$d\left(\frac{F_{W,t}^{(i)} - \mathbb{E} F_{W,t}^{(i)}}{\sqrt{\mathbb{V}ar F_{W,t}^{(i)}}}, N \right) \leq c\, t^{-1/2}$$

for all $t \geq 1$.

As already explained in the introduction, the central limit problem for $F_{W,t}^{(i)}$ can also be approached in another set-up, which in the Euclidean case is equivalent to the one just discussed, but turns out to be fundamentally different in hyperbolic space. More precisely, we turn now to the case, where the intensity t is fixed, while the size of the observation window is increased. We do this only in the case of spherical windows in $\mathbb{H}^d$. In other words, we choose for W the hyperbolic ball B_r (around the origin p) and write $F_{r,t}^{(i)}$ instead of $F_{B_r,t}^{(i)}$ in this case. Our next result is a central limit theorem for $F_{r,t}^{(i)}$ for dimension $d = 2$ in part (a) and for $d = 3$ in part (b). Moreover, it turns out that a central limit theorem for $F_{r,t}^{(i)}$ is no longer valid in any space dimension $d \geq 4$, see part (c). We emphasize that this surprising phenomenon is in sharp contrast to the Euclidean case [37, 65, 96] and is an effect of the negative curvature.

Theorem 3.1.5 (CLT, growing spherical window)**.** *Let $t \geq 1$, let N be a standard Gaussian random variable, and let $d(\cdot, \cdot)$ denote either the Wasserstein or the Kolmogorov distance.*

(a) If $d = 2$, then there is a constant $c_2 \in (0,\infty)$ only depending on t such that

$$d\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{\mathbb{V}ar F_{r,t}^{(i)}}}, N\right) \le c_2\, r^{1-i}\, e^{-r/2}$$

for $i \in \{0,1\}$ and $r \ge 1$.

(b) If $d = 3$, then there is a constant $c_3 \in (0,\infty)$ only depending on t such that

$$d\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{\mathbb{V}ar F_{r,t}^{(i)}}}, N\right) \le \begin{cases} c_3\, r^{-1} & : i = 2, \\ c_3\, r^{-1/2} & : i \in \{0,1\}, \end{cases}$$

for $r \ge 1$.

(c) If $d \ge 4$ and $i = d-1$ or if $d \ge 7$ and $i \in \{0,1,\dots,d-1\}$, then a central limit theorem for $(F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)})/\sqrt{\mathbb{V}ar F_{r,t}^{(i)}}$ does not hold for $r \to \infty$.

Remark 3.1.4. (i) The restriction imposed on the parameters d, i in Theorem 3.1.5 (c) is the result of a number of technical obstacles one needs to overcome in its proof. We strongly believe that a central limit theorem in fact fails for all $d \ge 4$ and all choices of $i \in \{0,1,\dots,d-1\}$. However, we have to leave this as an open problem for future work. For some remarks about the potential limiting distribution in Theorem 3.1.5 (c) we refer to Remark 3.4.1.

(ii) It is instructive to rewrite the normal approximation bounds in Theorem 3.1.5 (a) and (b) as follows. For $d = 2$ and $i \in \{0,1\}$ we have that

$$d\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{\operatorname{Var} F_{r,t}^{(i)}}}, N\right) \le \hat{c}_2\, \frac{\log^{1-i} \mathcal{H}^2(B_r)}{\sqrt{\mathcal{H}^2(B_r)}}, \qquad r \ge 1,$$

and for $d = 3$ we have, again for $r \ge 1$,

$$d\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{\operatorname{Var} F_{r,t}^{(i)}}}, N\right) \le \hat{c}_3 \begin{cases} \frac{1}{\log \mathcal{H}^3(B_r)} & : i = 2, \\ \frac{1}{\sqrt{\log \mathcal{H}^3(B_r)}} & : i \in \{0,1\}. \end{cases}$$

Here $\hat{c}_2, \hat{c}_3 \in (0,\infty)$ are again constants only depending on t. This means that in dimension $d = 2$ and for $i = 0$ the speed of convergence is the same as in the Euclidean case, up to the logarithmic factor. Moreover, it shows that $d = 3$ is the critical dimension for the central limit theorem, which only holds in this case with a rate of convergence which is very much slowed down.

Theorem 3.1.4 shows that for fixed radius r and increasing intensity t a central limit theorem for $F_{r,t}^{(i)}$ with $i \in \{0,1,\dots,d-1\}$ holds. On the other hand, according to Theorem 3.1.5 (c) the central limit theorem breaks down for dimensions $d \ge 4$ (if the total surface area is considered) or $d \geq 7$ (for general $i \in \{0,1,\dots,d-1\}$) if the intensity t stays fixed and $r \to \infty$. Against this background the question arises whether in these cases the central limit behaviour can be preserved if the intensity t and the radius r tend to infinity *simultaneously*. In fact, the following result states that this is indeed the case. More precisely, it says that, independently of the behaviour of r, the central limit theorem holds as soon as $t \to \infty$ (and r is bounded from below by 1).

Theorem 3.1.6 (CLT for simultaneous growth of intensity and window)**.** *Let $d \geq 4$ and $i = d-1$ or $d \geq 7$ and $i \in \{0,1,\dots,d-1\}$. Also, let N be a standard Gaussian random variable. Then there is a constant $c \in (0,\infty)$ such that*

$$d\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{\mathbb{V}ar F_{r,t}^{(i)}}}, N\right) \leq \frac{c}{\sqrt{t}}$$

for all $r \geq 1$ and $t \geq 1$, where $d(\,\cdot\,,\,\cdot\,)$ denotes either the Wasserstein or the Kolmogorov distance.

Remark 3.1.5. In dimensions $d = 2$ and $d = 3$ we also have normal approximation bounds that simultaneously involve the two parameters t and r. In fact, for $d = 2$ the bounds (3.24) and (3.28) below show that

$$d\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{\mathbb{V}\mathrm{ar}\, F_{r,t}^{(i)}}}, N\right) \leq c\, t^{-1/2}\, r^{1-i} e^{-r/2}$$

holds for all $t \geq 1$, $r \geq 1$ and $i \in \{0,1\}$. Similarly, for $d = 3$ the estimates (3.30), (3.34) and (3.35) prove that

$$d\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{\mathbb{V}\mathrm{ar}\, F_{r,t}^{(i)}}}, N\right) \leq c \cdot \begin{cases} t^{-1/2} r^{-1} & : i = 2, \\ t^{-1/2} r^{-1/2} & : i \in \{0,1\}, \end{cases}$$

for all $t \geq 1$ and $r \geq 1$. In both cases, $d(\,\cdot\,,\,\cdot\,)$ stands for either the Wasserstein or the Kolmogorov distance. This way we recover Theorem 3.1.4 for $d = 2$ and $d = 3$ in the special case where $W = B_r$ with r fixed and we recover Theorem 3.1.5 (a) and (b) by fixing t.

Finally, let us turn to the multivariate set-up. To compare the distance between the distributions of (the laws of) two random vectors we use what is known as the d_2- and the d_3-distance; for their definition we refer to Section 2.5.2 below. We approach the multivariate central limit theorem by considering, as above, two different settings. To handle the central limit problem for a fixed window $W \in \mathcal{K}_h^d$ and growing intensities we define for $t > 0$ the d-dimensional random vector

$$\mathbf{F}_{W,t} := \left(\frac{F_{W,t}^{(0)} - \mathbb{E}F_{W,t}^{(0)}}{t^{d-1/2}}, \dots, \frac{F_{W,t}^{(i)} - \mathbb{E}F_{W,t}^{(i)}}{t^{d-i-1/2}}, \dots, \frac{F_{W,t}^{(d-1)} - \mathbb{E}F_{W,t}^{(d-1)}}{t^{1/2}}\right).$$

Moreover, for $i, j \in \{0,1,\dots,d-1\}$ we introduce the asymptotic covariances and the asymptotic covariance matrix of the random vector $\mathbf{F}_{W,t}$, as $t \to \infty$, by

$$\tau_W^{i,j} := \lim_{t\to\infty} \mathbb{C}\mathrm{ov}\left(\frac{F_{W,t}^{(i)} - \mathbb{E}F_{W,t}^{(i)}}{t^{d-i-1/2}}, \frac{F_{W,t}^{(j)} - \mathbb{E}F_{W,t}^{(j)}}{t^{d-j-1/2}}\right), \qquad T_W := \left(\tau_W^{i,j}\right)_{i,j=0}^{d-1}.$$

The existence of the limit and the precise value of $\tau_W^{i,j}$ follows from (3.6) below. It is easy to see that T_W has rank one, as in Euclidean space.

In view of Theorem 3.1.5, for fixed intensity $t > 0$ and a sequence of growing spherical windows, taking $W = B_r$ for $r > 0$ we put

$$\mathbf{F}_{r,t} := \begin{cases} \left(\frac{F_{r,t}^{(0)} - \mathbb{E}F_{r,t}^{(0)}}{e^{r/2}}, \frac{F_{r,t}^{(1)} - \mathbb{E}F_{r,t}^{(1)}}{e^{r/2}}\right) & : d = 2, \\ \left(\frac{F_{r,t}^{(0)} - \mathbb{E}F_{r,t}^{(0)}}{\sqrt{r}\, e^{r}}, \frac{F_{r,t}^{(1)} - \mathbb{E}F_{r,t}^{(1)}}{\sqrt{r}\, e^{r}}, \frac{F_{r,t}^{(2)} - \mathbb{E}F_{r,t}^{(2)}}{\sqrt{r}\, e^{r}}\right) & : d = 3, \\ \left(\frac{F_{r,t}^{(0)} - \mathbb{E}F_{r,t}^{(0)}}{e^{r(d-2)}}, \dots, \frac{F_{r,t}^{(d-1)} - \mathbb{E}F_{r,t}^{(d-1)}}{e^{r(d-2)}}\right) & : d \geq 4, \end{cases}$$

and define the asymptotic covariance matrix $\Sigma_d = \left(\sigma_d^{i,j}\right)_{i,j=0}^{d-1}$ of the random vector $\mathbf{F}_{r,t}$, as $r \to \infty$, for $d \geq 2$ by

$$\sigma_d^{i,j} := \begin{cases} \lim_{r\to\infty} \mathbb{C}\mathrm{ov}\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{e^{r/2}}, \frac{F_{r,t}^{(j)} - \mathbb{E}F_{r,t}^{(j)}}{e^{r/2}}\right) & : d = 2, \\ \lim_{r\to\infty} \mathbb{C}\mathrm{ov}\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{r}\, e^{r}}, \frac{F_{r,t}^{(j)} - \mathbb{E}F_{r,t}^{(j)}}{\sqrt{r}\, e^{r}}\right) & : d = 3, \\ \lim_{r\to\infty} \mathbb{C}\mathrm{ov}\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{e^{r(d-2)}}, \frac{F_{r,t}^{(j)} - \mathbb{E}F_{r,t}^{(j)}}{e^{r(d-2)}}\right) & : d \geq 4. \end{cases}$$

The covariance matrices Σ_d are explicitly given by (3.8) for $d = 2$, (3.16) for $d = 3$ and (3.17) for $d \geq 4$ below. Moreover, in Section 3.2.5 we determine convergence rates. In particular, we will show that Σ_2 has full rank (is positive definite) and Σ_d has rank one for $d \geq 3$. We remark that this is in sharp contrast to the corresponding result in Euclidean spaces, where the asymptotic covariance matrix has rank one for all $d \geq 2$, see [37, Theorem 5.1 (ii)]. Note that the dependence of these limits on the fixed intensity $t > 0$ is not made explicit by our notation, but this dependence is shown in Lemmas 20, 21 and 23.

In order to state the multivariate central limit theorem, we use the d_2 and the d_3 distance for random vectors (see Section 2.5.2 for explicit definitions).

Theorem 3.1.7 (Multivariate CLT). *(a) Let $d \geq 2$ and $W \in \mathcal{K}_h^d$. Let N_{T_W} be a d-dimensional centred Gaussian random vector with covariance matrix T_W. Then there exists a constant $c \in (0, \infty)$ such that*

$$d_3(\boldsymbol{F}_{W,t}, N_{T_W}) \leq c\, t^{-1/2}$$

for all $t \geq 1$.

(b) *Fix $t \geq 1$ and let $d = 2$. Let N_{Σ_2} be a 2-dimensional centred Gaussian random vector with covariance matrix Σ_2. Then there exists a constant $c_2 \in (0,\infty)$ such that*

$$d_j(\boldsymbol{F}_{r,t}, N_{\Sigma_2}) \leq c_2\, r\, e^{-r/2}$$

for all $r \geq 1$ and $j \in \{2,3\}$.

(c) *Fix $t \geq 1$ and let $d = 3$. Let N_{Σ_3} be a 3-dimensional centred Gaussian random vector with covariance matrix Σ_3. Then there exists a constant $c_3 \in (0,\infty)$ such that*

$$d_3(\boldsymbol{F}_{r,t}, N_{\Sigma_3}) \leq c_3\, r^{-1/2}$$

for all $r \geq 1$.

Remark 3.1.6. After having seen that in the univariate case the central limit theorem for $d \geq 4$ can be preserved by a simultaneous growth of the intensity t and the radius r, the question arises whether such a phenomenon also holds in the multivariate set-up. This is in fact the case, but we decided not to present the details for brevity.

3.2 Proofs I – Expectations and variances

3.2.1 Representation as a Poisson U-statistic

We recall that η_t, for $t > 0$, is a Poisson hyperplane process in $\mathbb{H}^d$ with intensity measure $t\mu_{d-1}$. Moreover, for a Borel set $W \subset \mathbb{H}^d$ and $i \in \{0,1,\dots,d-1\}$ we recall from (3.1) the definition of the functional $F_{W,t}^{(i)}$. To shorten our notation we put

$$f^{(i)}(H_1,\dots,H_{d-i}) := \begin{cases} \frac{1}{(d-i)!}\mathcal{H}^i(H_1 \cap \ldots \cap H_{d-i} \cap W) & : \dim(H_1 \cap \ldots \cap H_{d-i}) = i, \\ 0 & : \text{otherwise}, \end{cases}$$

which allows us to rewrite $F_{W,t}^{(i)}$ as

$$F_{W,t}^{(i)} = \sum_{(H_1,\dots,H_{d-i}) \in \eta_{t,\neq}^{d-i}} f^{(i)}(H_1,\dots,H_{d-i}).$$

In other words, $F_{W,t}^{(i)}$ is a Poisson U-statistic of order $d-i$ and with kernel $f^{(i)}$. It is well known (see [64, 59, 60, 65, 96]) that Poisson U-statistics admit a Fock space representation having only a finite number of terms. This leads to the variance and covariance representations

$$\mathbb{V}\mathrm{ar}(F_{W,t}^{(i)}) = \sum_{n=1}^{d-i} t^{2(d-i)-n} n! \| f_n^{(i)} \|_n^2, \tag{3.5}$$

where the functions $f_n^{(i)} : A_h(d, d-1)^n \to [0, \infty)$ are given by

$$f_n^{(i)}(H_1, \dots, H_n) := \binom{d-i}{n} \int_{A_h(d,d-1)^{d-i-n}} f^{(i)}(H_1, \dots, H_n, \tilde{H}_1, \dots, \tilde{H}_{d-i-n}) \times \mu_{d-1}^{d-i-n}(d(\tilde{H}_1, \dots, \tilde{H}_{d-i-n})),$$

recall (2.9), and we write $\| \cdot \|_n$ for the norm in the L^2-space $L^2(\mu_{d-1}^n)$ with respect to the n-fold product measure of μ_{d-1}. Similarly, for $i, j \in \{0, 1, \dots, d-1\}$ the covariance $\mathbb{C}\mathrm{ov}(F_{W,t}^{(i)}, F_{W,t}^{(j)})$ can be represented as

$$\mathbb{C}\mathrm{ov}(F_{W,t}^{(i)}, F_{W,t}^{(j)}) = \sum_{n=1}^{\min\{d-i,d-j\}} t^{2d-i-j-n} n! \langle f_n^{(i)}, f_n^{(j)} \rangle_n, \tag{3.6}$$

where $\langle \cdot, \cdot \rangle_n$ denotes the standard scalar product in $L^2(\mu_{d-1}^n)$.

3.2.2 Expectations: Proof of Theorem 3.1.1

Theorem 3.1.1 is a consequence of the transformation formula in Lemma 2.4.2 and the Crofton formula in Lemma 2.4.1 with $k = i$ there. In fact, using (2.6) we obtain

$$\begin{aligned}
\mathbb{E} F_{W,t}^{(i)} &= t^{d-i} \int_{A_h(d,d-1)^{d-i}} f^{(i)}(H_1, \dots, H_{d-i}) \, \mu_{d-1}^{d-i}(d(H_1, \dots, H_{d-i})) \\
&= \frac{t^{d-i}}{(d-i)!} \int_{A_h(d,d-1)^{d-i}} \mathcal{H}^i(H_1 \cap \ldots \cap H_{d-i} \cap W) \, \mu_{d-1}^{d-i}(d(H_1, \dots, H_{d-i})) \\
&= c(d, i) \frac{t^{d-i}}{(d-i)!} \int_{A_h(d,i)} \mathcal{H}^i(E \cap W) \, \mu_i(dE) \\
&= c(d, i) \frac{t^{d-i}}{(d-i)!} \mathcal{H}^d(W) \\
&= \frac{\omega_{i+1}}{\omega_{d+1}} \left(\frac{\omega_{d+1}}{\omega_d} \right)^{d-i} \frac{t^{d-i}}{(d-i)!} \mathcal{H}^d(W),
\end{aligned}$$

and the proof is complete. □

Remark 3.2.1. The measure $W \mapsto \mathbb{E} F_{W,t}^{(i)}$ is isometry invariant. One could argue that it must be a constant multiple of $\mathcal{H}^d$, if one knows that it is also locally finite. Theorem 3.1.1 shows that this is indeed the case and also yields the constant.

3.2.3 Variances: Proof of Theorem 3.1.2

To investigate the variance of $F_{W,t}^{(i)}$ we use the representation as a Poisson U-statistic, especially (3.5). We start by simplifying the kernel functions $f_n^{(i)}$.

Lemma 3.2.1. *Let $n \in \{1, \dots, d-i\}$. Let $W \subset \mathbb{H}^d$ be a bounded Borel set. If $H_1, \dots, H_n \in A_h(d, d-1)$ are n hyperplanes satisfying* $\dim(H_1 \cap \ldots \cap H_n) = d - n$*, then*

$$f_n^{(i)}(H_1, \dots, H_n) = c(i, n, d) \, \mathcal{H}^{d-n}(H_1 \cap \ldots \cap H_n \cap W)$$

with

$$c(i,n,d) := \frac{\binom{d-i}{n}}{(d-i)!}\frac{\omega_{i+1}}{\omega_{d-n+1}}\left(\frac{\omega_{d+1}}{\omega_d}\right)^{d-n-i}.$$

Proof. We use the definition of $f_n^{(i)}$, the transformation formula in Lemma 2.4.2 and the Crofton formula (2.4) (in the general form indicated before the statement of Lemma 2.4.1). Putting $L_{d-n} := H_1 \cap \ldots \cap H_n$, this gives

$$\begin{aligned}
&\binom{d-i}{n}^{-1} f_n^{(i)}(H_1,\ldots,H_n)\\
&= \frac{1}{(d-i)!}\int_{A_h(d,d-1)^{d-i-n}} \mathcal{H}^i(L_{d-n}\cap \tilde{H}_1 \cap \ldots \cap \tilde{H}_{d-i-n}\cap W)\,\mu_{d-1}^{d-i-n}(d(\tilde{H}_1,\ldots,\tilde{H}_{d-i-n}))\\
&= \frac{c(d,i+n)}{(d-i)!}\int_{A_h(d,i+n)} \mathcal{H}^i(L_{d-n}\cap W\cap E)\,\mu_{i+n}(dE)\\
&= \frac{c(d,i+n)}{(d-i)!}\frac{\omega_{d+1}\,\omega_{i+1}}{\omega_{i+n+1}\,\omega_{d-n+1}}\mathcal{H}^{d-n}(L_{d-n}\cap W)\\
&= \frac{1}{(d-i)!}\frac{\omega_{i+1}}{\omega_{d-n+1}}\left(\frac{\omega_{d+1}}{\omega_d}\right)^{d-n-i}\mathcal{H}^{d-n}(H_1\cap\ldots\cap H_n\cap W).
\end{aligned}$$

Here we used that since L_{d-n} is $(d-n)$-dimensional, the intersection $L_{d-n}\cap W$ is Hausdorff $(d-n)$-rectifiable. ☐

For the variances and covariances, we need the L^2-norms and the scalar products of these functions.

Corollary 3.2.2. *Let $W \subset \mathbb{H}^d$ be a bounded Borel set. If $n \in \{1,\ldots,\min\{d-i,d-j\}\}$, then*

$$\langle f_n^{(i)}, f_n^{(j)}\rangle_n = c(d,n,i,j)\int_{A_h(d,d-n)} \mathcal{H}^{d-n}(E\cap W)^2\,\mu_{d-n}(dE).$$

Especially, the choice $W = B_r$ for some $r>0$ yields

$$\langle f_n^{(i)}, f_n^{(j)}\rangle_n = c(d,n,i,j)\,\omega_n\int_0^r \cosh^{d-n}(s)\sinh^{n-1}(s)\,\mathcal{H}^{d-n}(L_{d-n}(s)\cap B_r)^2\,ds,$$

where $c(d,n,i,j) := c(d,d-n)\,c(i,n,d)\,c(j,n,d)$ and $L_{d-n}(s)$ for $s\in[0,r]$ is an arbitrary $(d-n)$-dimensional totally geodesic subspace which satisfies $d_h(L_{d-n}(s),p)=s$.

Proof. The first claim is a direct consequence of the previous lemma and the transformation formula from Lemma 2.4.2.

The second claim follows by combining the previous result with (2.3) and using geodesic spherical coordinates in the $(d-n)$-dimensional planes L_{d-n} through p (see [20, Proposition 3.1 and Equation (3.22)]). ☐

Proof of Theorem 3.1.2. This is now a direct consequence of (3.5) and Corollary 3.2.2. ☐

3.2.4 Variance: Asymptotic behaviour

In this section we look at the variance of $F_{r,t}^{(i)} = F_{B_r,t}^{(i)}$ in the asymptotic regime, as $r\to\infty$. We divide our analysis into the three different cases $d=2$, $d=3$ and $d\geq 4$. Before, we start with a number of preprations.

PRELIMINARIES

The following lemma will be repeatedly applied below.

Lemma 3.2.3. *If* $r > 0$ *and* $0 \le s \le r$*, then*

$$0 \le \operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right) - (r-s) \le \log(2).$$

Proof. We start by proving the lower bound which is equivalent to $\cosh(r) - \cosh(s)\cosh(r-s) \ge 0$. By definition of cosh, sinh and since $0 \le s \le r$ we have

$$\cosh(r) - \cosh(s)\cosh(r-s) = \sinh(s)\sinh(r-s) \ge 0.$$

This yields the lower bound. Next, we turn to the upper bound. We use the logarithmic representation $\operatorname{arcosh}(x) = \log(x + \sqrt{x^2-1})$ of the arcosh-function and the fact that $\cosh(r)/\cosh(s) \ge 1$ for $r \ge s \ge 0$. Then we get

$$\begin{aligned}
\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right) - (r-s) &= \log\left(\frac{\cosh(r)}{\cosh(s)} + \sqrt{\frac{\cosh^2(r)}{\cosh^2(s)} - 1}\right) - (r-s) \\
&= \log\left(\frac{e^s\cosh(r)}{e^r\cosh(s)} + \sqrt{\frac{e^{2s}\cosh^2(r)}{e^{2r}\cosh^2(s)} - \frac{e^{2s}}{e^{2r}}}\right) \\
&= \log\left(\frac{e^s(e^r+e^{-r})}{e^r(e^s+e^{-s})} + \sqrt{\frac{e^{2s}(e^r+e^{-r})^2}{e^{2r}(e^s+e^{-s})^2} - \frac{e^{2s}}{e^{2r}}}\right) \\
&= \log\left(\frac{1+e^{-2r}}{1+e^{-2s}} + \sqrt{\frac{e^{2s}(e^{2r}+2+e^{-2r}-e^{2s}-2-e^{-2s})}{e^{2r}(e^{2s}+2+e^{-2s})}}\right) \\
&= \log\left(\frac{1+e^{-2r}}{1+e^{-2s}} + \sqrt{\frac{1+e^{-4r}-e^{2s-2r}-e^{-4s}}{1+2e^{-2s}+e^{-2s-2r}}}\right) \\
&\le \log(2),
\end{aligned}$$

where the last inequality holds because both terms in the argument of the log function are bounded from above by 1 for $s \in [0, r]$. □

Moreover, we frequently apply the following upper and lower bounds for $\mathcal{H}^i(L_i(s) \cap B_r)$. As before, let $L_i(s)$ denote an arbitrary measurable choice of an i-dimensional totally geodesic subspace satisfying $d_h(L_i(s), p) = s$, $i \in \{1, \dots, d-1\}$. The following lemma concerns the case $i \in \{2, \dots, d-1\}$.

Lemma 3.2.4. *If* $i \in \{2, \dots, d-1\}$ *and* $0 \le s \le r$*, then*

$$\mathcal{H}^i(L_i(s) \cap B_r) \le \frac{\omega_i}{i-1} e^{(r-s)(i-1)}.$$

If, in addition, $0 \le s \le r - 1/2$ *then*

$$\frac{\omega_i}{e^{3(i-1)}(i-1)} e^{(r-s)(i-1)} \le \mathcal{H}^i(L_i(s) \cap B_r).$$

Proof. We start by noting that $L_i(s) \cap B_r$ is an i-dimensional hyperbolic ball of radius $\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right)$ for $i \in \{1, \ldots, d-1\}$, see [92, Theorem 3.5.3]. Thus we get

$$\mathcal{H}^i(L_i(s) \cap B_r) = \omega_i \int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right)} \sinh^{i-1}(u)\, du \tag{3.7}$$

for $i \in \{1, \ldots, d-1\}$. Hence, using Lemma 3.2.3 and for $i \in \{2, \ldots, d-1\}$ we get

$$\begin{aligned} \mathcal{H}^i(L_i(s) \cap B_r) &\le \omega_i \int_0^{r-s+\log(2)} \sinh^{i-1}(u)\, du \\ &\le \frac{\omega_i}{2^{i-1}} \int_0^{r-s+\log(2)} e^{u(i-1)}\, du \\ &\le \frac{\omega_i}{2^{i-1}(i-1)} 2^{i-1} e^{(r-s)(i-1)} \\ &= \frac{\omega_i}{i-1} e^{(r-s)(i-1)}. \end{aligned}$$

On the other hand, Lemma 3.2.3 and Lemma 2.2.1 imply that

$$\begin{aligned} \mathcal{H}^i(L_i(s) \cap B_r) &\ge \omega_i \int_0^{r-s} \sinh^{i-1}(u)\, du \\ &= \omega_i \left(\int_{1/2}^{r-s} \sinh^{i-1}(u)\, du + \int_0^{1/2} \sinh^{i-1}(u)\, du \right) \\ &\ge \omega_i \left(\int_{1/2}^{r-s} \left(\frac{e^u}{e^3}\right)^{i-1} du + \int_0^{1/2} u^{i-1}\, du \right) \\ &= \frac{\omega_i}{e^{3(i-1)}(i-1)} \left(e^{(r-s)(i-1)} - e^{(i-1)/2} \right) + \frac{1}{2^i} \frac{\omega_i}{i} \\ &\ge \frac{\omega_i}{e^{3(i-1)}(i-1)} e^{(r-s)(i-1)}, \end{aligned}$$

where we used that $s \le r - 1/2$ to obtain the equality in the third line. The last inequality is due to

$$\frac{1}{2^i} \frac{\omega_i}{i} - \frac{\omega_i}{e^{(5/2)(i-1)}(i-1)} = \omega_i \left(\frac{1}{i\, 2^i} - \frac{1}{e^{(5/2)(i-1)}(i-1)} \right) \ge 0.$$

The positivity of the last term holds for $i \ge 2$, since $2^{i+1} \le e^{(5/2)(i-1)}$ implies that

$$2^i \le \frac{i-1}{i} e^{(5/2)(i-1)},$$

which is equivalent to the desired inequality. □

We will need later the following lemma.

Lemma 3.2.5. *Let* $r \ge 1$. *For* $k \in \{0, 1, \ldots, d-1\}$ *and* $0 \le s \le r$, *let* $L_k(s) \in A_h(d,k)$ *be a*

k-dimensional totally geodesic subspace such that $d_h(L_k(s),p) = s$. Then for any $l \in \mathbb{N}$ there exist constants $c, C > 0$, depending only on k, l and d, such that

$$c\, g(k,l,d,r) \le \omega_{d-k} \int_0^r \cosh^k(s) \sinh^{d-1-k}(s)\, \mathcal{H}^k(L_k(s) \cap B_r)^l\, ds$$
$$= \int_{A_h(d,k)} \mathcal{H}^k(H \cap B_r)^l\, \mu_k(dH) \le C\, g(k,l,d,r)$$

with

$$g(k,l,d,r) = \begin{cases} \exp(r(d-1)) & : l(k-1) < d-1, \\ r \exp(r(d-1)) & : l(k-1) = d-1, \\ \exp(r\, l(k-1)) & : l(k-1) > d-1. \end{cases}$$

Proof. The asserted equality of the two integral expressions is clear from the argument for the second claim in Corollary 3.2.2.

For $k = 0$ the integral is just the volume of a geodesic ball of radius r which can be bounded from above and below by a positive constant times $\exp(r(d-1))$.

In the following, we repeatedly use that the intersection $L_k(s) \cap B_r$ is a k-dimensional hyperbolic ball of radius $\operatorname{arcosh}(\cosh(r)/\cosh(s))$. The constants c and C used in the calculations below only depend on k, l, d and may vary from line to line. Suppose that $k \ge 2$. The substitution $u = r - s$ and an application of Lemma 3.2.3 yield

$$\begin{aligned}
&\int_0^r \cosh^k(s) \sinh^{d-1-k}(s)\, \mathcal{H}^k(L_k(s) \cap B_r)^l\, ds \\
&= \int_0^r \cosh^k(r-u)\, \sinh^{d-1-k}(r-u)\, \mathcal{H}^k(L_k(r-u) \cap B_r)^l\, du \\
&= \int_0^r \cosh^k(r-u)\, \sinh^{d-1-k}(r-u) \left(\omega_k \int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(r-u)}\right)} \sinh^{k-1}(s)\ ds \right)^l du \\
&\le C \int_0^r e^{k(r-u)}\, e^{(d-1-k)(r-u)} \left(2^{-(k-1)} \int_0^{u+\log(2)} e^{(k-1)s}\ ds \right)^l du \\
&\le C \int_0^r e^{(d-1)(r-u)} \left(\frac{1}{k-1} e^{(u+\log(2))(k-1)} \right)^l du \\
&\le C e^{r(d-1)} \int_0^r e^{u(l(k-1)-(d-1))}\, du \\
&\le C g(k,l,d,r).
\end{aligned}$$

To obtain the lower bound, we first show for $u \ge 0.2$ that

$$\begin{aligned}
\int_0^u \sinh^{k-1}(s) \ge \int_{0.1}^u \sinh^{k-1}(s)\ ds &\ge \int_{0.1}^u e^{(k-1)(s-3)}\ ds \\
&\ge \frac{e^{-3(k-1)}}{k-1} \left(e^{(k-1)u} - e^{0.1(k-1)} \right) \\
&\ge \frac{e^{0.1(k-1)}}{k-1}\, e^{-3(k-1)} \left(e^{(k-1)(u-0.1)} - 1 \right) \\
&\ge \frac{e^{0.1(k-1)}}{k-1}\, e^{-3(k-1)} \frac{1}{20} e^{(k-1)(u-0.1)}.
\end{aligned}$$

Now we substitute again $u = r - s$. An application of Lemma 2.2.1 and the lower bound from Lemma 3.2.3 then yield

$$\begin{aligned}
&\int_0^r \cosh^k(s)\sinh^{d-1-k}(s)\,\mathcal{H}^k(L_k(s)\cap B_r)^l\,ds\\
&= \int_0^r \cosh^k(r-u)\,\sinh^{d-1-k}(r-u)\,\mathcal{H}^k(L_k(r-u)\cap B_r)^l\,du\\
&= \int_0^r \cosh^k(r-u)\,\sinh^{d-1-k}(r-u)\left(\omega_k\int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(r-u)}\right)}\sinh^{k-1}(s)\;ds\right)^l du\\
&\geq c\int_0^{r-0.1} e^{k(r-u)}\,e^{(d-1-k)(r-u-3)}\left(\int_0^u \sinh^{k-1}(s)\;ds\right)^l du\\
&\geq ce^{r(d-1)}\int_{0.2}^{r-0.1} e^{-u(d-1)}\,e^{l(k-1)(u-0.1)}\,du\\
&= ce^{r(d-1)}\int_{0.2}^{r-0.1} e^{u(l(k-1)-(d-1))}\,du\\
&\geq cg(k,l,d,r).
\end{aligned}$$

For $k = 1$, the proof is almost the same except that we simply use that $\int_0^a \sinh^{k-1}(s)\,ds = a$ for $a \geq 0$. □

The planar case $d = 2$

Although we present a very detailed covariance analysis in Section 3.2.5 we will separately investigate the asymptotic behaviour of the variances in Lemmas 3.2.6 – 3.2.8. In fact while the results of Section 3.2.5 are necessary for the multivariate central limit theorems, the variance analysis we carry out here is already sufficient for the unvariate cases. In this and the following two sections, c_i will denote a positive constant only depending on the dimension and a counting parameter $i \in \mathbb{N}_0$. If it additionally depends on another parameter $n \in \mathbb{N}_0$, we indicate this by writing, for instance, $c_{i,n}$ or $c_i(n)$. The value of this constant may change from occasion to occasion.

Lemma 3.2.6. *Let $d = 2$, $i \in \{0,1\}$ and $t \geq t_0 > 0$. Then there are constants $c^{(i)}(2,t_0), C^{(i)}(2,t_0) \in (0,\infty)$ such that for all $r \geq 1$,*

$$c^{(i)}(2,t_0)\,t^{3-2i}\,e^r \leq \mathbb{V}ar(F_{r,t}^{(i)}) \leq C^{(i)}(2,t_0)\,t^{3-2i}\,e^r.$$

Proof. For $i \in \{0,1\}$ and $n = 1$, Corollary 3.2.2 and Lemma 3.2.5 yield

$$c_i\,e^r \leq \|f_1^{(i)}\|_1^2 = c_i\int_0^r \cosh(s)\,\mathcal{H}^1(L_1(s)\cap B_r)^2\;ds \leq C_i\,e^r.$$

Similarly, for $i = 0$ and $n = 2$ we obtain

$$\|f_2^{(0)}\|_2^2 = c_0\int_0^r \sinh(s)\mathcal{H}^0(L_1(s)\cap B_r)^2\;ds = c_0\int_0^r \sinh(s)\;ds = c_0\big(\cosh(r)-1\big).$$

From (3.5) we now deduce that

$$c(t^2+t^3)e^r \le c_1 t^3 e^r + c_2 t^2 e^r \le \mathbb{V}\mathrm{ar}(F_{r,t}^{(0)}) \le c_1 t^3 e^r + c_2 t^2 e^r \le C(t^2+t^3)e^r.$$

Using that $t \ge t_0 > 0$, the assertion follows for $i = 0$. The case $i = 1$ is obtained in the same way, but requires bounds for only one summand in (3.5). □

THE SPATIAL CASE $d = 3$

Lemma 3.2.7. *Let $d = 3$, $i \in \{0,1,2\}$ and $t \ge t_0 > 0$. Then there are constants $c^{(i)}(3,t_0), C^{(i)}(3,t_0) \in (0,\infty)$ such that for all $r \ge 1$,*

$$c^{(i)}(3,t_0)\, t^{5-2i}\, re^{2r} \le \mathbb{V}ar(F_{r,t}^{(i)}) \le C^{(i)}(3,t_0)\, t^{5-2i}\, re^{2r}.$$

Proof. Corollary 3.2.2 and Lemma 3.2.5 imply the upper bound

$$\mathbb{V}\mathrm{ar}(F_{r,t}^{(i)}) - \sum_{n=2}^{3-i} t^{6-2i-n} n! \|f_n^{(i)}\|_n^2 = t^{5-2i} \|f_1^{(i)}\|_1^2 \le c_i\, t^{5-2i}\, re^{2r}.$$

It remains to determine the asymptotic behaviour in r of the terms $\|f_2^{(i)}\|_2^2$ and $\|f_3^{(i)}\|_3^2$. Corollary 3.2.2 and Lemma 3.2.5 yield

$$c_i\, e^{2r} \le \|f_2^{(i)}\|_2^2 \le C_i\, e^{2r} \qquad \text{and} \qquad c_i\, e^{2r} \le \|f_3^{(i)}\|_3^2 \le C_i\, e^{2r}.$$

To deduce the lower bound, it is sufficient to derive a lower bound for the term $\|f_1^{(i)}\|_1^2$. But

$$\mathbb{V}\mathrm{ar}(F_{r,t}^{(i)}) \ge t^{5-2i} \|f_1^{(i)}\|_1^2 \ge c_i\, t^{5-2i}\, re^{2r}.$$

This completes the proof. □

THE HIGHER DIMENSIONAL CASE $d \ge 4$

Lemma 3.2.8. *Let $d \ge 4$, $i \in \{0,1,\dots,d-1\}$, and $t \ge t_0 > 0$. Then there are positive constants $c^{(i)}(d,t_0)$, $C^{(i)}(d,t_0) \in (0,\infty)$ such that for all $r \ge 1$,*

$$c^{(i)}(d,t_0)\, t^{2(d-i)-1}\, e^{2r(d-2)} \le \mathbb{V}ar(F_{r,t}^{(i)}) \le C^{(i)}(d,t_0)\, t^{2(d-i)-1}\, e^{2r(d-2)}.$$

Proof. Combining Corollary 3.2.2 with Lemma 3.2.5, we obtain

$$\mathbb{V}\mathrm{ar}(F_{r,t}^{(i)}) - \sum_{n=d-1}^{d-i} t^{2(d-i)-n} n! \|f_n^{(i)}\|_n^2 \le \sum_{n=1}^{\min\{d-2,\ d-i\}} c_{i,n}\, t^{2(d-i)-n}\, g(d-n,2,d,r).$$

For $n = 1 \le \min\{d-2,\ d-i\}$, we have $g(d-1,2,d,r) \le C_i \exp(r2(d-2))$, since $2(d-2)-(d-1) = d-3 > 0$. If $2(d-n-1)-(d-1) = d-1-2n > 0$, then $g(d-n,2,d,r) \le g(d-1,2,d,r)$. For the remaining cases, we use that $\exp(r(d-1))$ is of lower order than $\exp(2r(d-2))$ for $d \ge 4$. Moreover, as in the case $d = 3$ it follows that $\|f_{d-1}^{(i)}\|_{d-1}^2$ and $\|f_d^{(i)}\|_d^2$ are of order at most $e^{r(d-1)}$.

This yields the upper bound.

The lower bound is again derived by just taking into account $\|f_1^{(i)}\|_1^2$ and by applying the lower bound $g(d-1,2,d,r) \geq c_i \exp(r2(d-2))$ from Lemma 3.2.5. □

3.2.5 Covariance analysis

In this section we prepare the proof of Theorem 3.1.7 by an asymptotic analysis of the covariance structure of the random vector $\mathbf{F}_{r,t}$ in dimensions $d = 2$ and $d = 3$.

The planar case $d = 2$

The following lemma describes the rate of convergence, as $r \to \infty$, of the suitably scaled covariances to the asymptotic covariance matrix $\Sigma_d = (\sigma_d^{i,j})_{i,j=0}^{d-1}$ for $d = 2$.

Lemma 3.2.9. *Let $d = 2$ and $t \geq t_0 > 0$. There is a positive constant $c_{cov}(2,t_0) \in (0,\infty)$ such that if $r \geq 1$, then*

$$\left|\sigma_2^{i,j} - \mathbb{C}ov\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{e^{r/2}}, \frac{F_{r,t}^{(j)} - \mathbb{E}F_{r,t}^{(j)}}{e^{r/2}}\right)\right| \leq c_{cov}(2,t_0)\, t^{3-i-j} r^2 e^{-r}, \qquad i,j \in \{0,1\}.$$

Moreover,

$$\Sigma_2 = \begin{pmatrix} t^2\left(\left(\frac{4}{\pi}\right)^2 ta + \frac{1}{\pi}\right) & \frac{8}{\pi}t^2 a \\ \frac{8}{\pi}t^2 a & 4ta \end{pmatrix} \tag{3.8}$$

and $a = 4 \cdot G$ with Catalan's constant $G \approx 0.915965594$. In particular, Σ_2 is positive definite with $\det(\Sigma_2) = \frac{4}{\pi}t^3 a$.

Proof. Since $\mathbf{F}_{r,t}$ is a vector of Poisson U-statistics the covariance representation (3.6) shows that, for $i,j \in \{0,1\}$,

$$\mathbb{C}\mathrm{ov}\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{e^{r/2}}, \frac{F_{r,t}^{(j)} - \mathbb{E}F_{r,t}^{(j)}}{e^{r/2}}\right) = e^{-r} \sum_{n=1}^{\min\{2-i,2-j\}} t^{4-i-j-n} n! \langle f_n^{(i)}, f_n^{(j)} \rangle_n$$

and it remains to compute the scalar products. Using (3.7) and Corollary 3.2.2 we get

$$\begin{aligned}
\langle f_1^{(i)}, f_1^{(j)} \rangle_1 &= c(2,1,i,j) \cdot 2 \cdot 4 \int_0^r \cosh(s)\, \operatorname{arcosh}^2\left(\frac{\cosh(r)}{\cosh(s)}\right) ds \\
&= c(2,1,i,j) \cdot 2 \cdot 4 \int_0^r \cosh(r-s)\, \operatorname{arcosh}^2\left(\frac{\cosh(r)}{\cosh(r-s)}\right) ds \\
&= c_1^{(i,j)} \int_0^r (e^{r-s} + e^{s-r})\, \operatorname{arcosh}^2\left(e^s\left(\frac{1+e^{-2r}}{1+e^{2(s-r)}}\right)\right) ds
\end{aligned}$$

with $c_1^{(i,j)} = 4 \cdot c(i,1,2)c(j,1,2)$. We have $c(0,1,2) = 2/\pi$ and $c(1,1,2) = 1$, and hence

$$c_1^{(0,0)} = 4\left(\frac{2}{\pi}\right)^2 = \left(\frac{4}{\pi}\right)^2, \quad c_1^{(1,1)} = 4, \quad c_1^{(1,0)} = c_1^{(0,1)} = 4 \cdot \frac{2}{\pi} = \frac{8}{\pi}.$$

Furthermore, again by Corollary 3.2.2

$$\langle f_2^{(i)}, f_2^{(j)} \rangle_2 = c(2,2,i,j) \cdot 2 \int_0^r \sinh(s)\, ds = c_2^{(i,j)}(e^r + e^{-r} - 2)$$

with $c_2^{(i,j)} = (2/\pi)c(i,2,2)c(j,2,2)$. In particular, $c_2^{(0,0)} = 1/(2\pi)$. In the following, we use that

$$\operatorname{arcosh}\left(e^s\left(\frac{1+e^{-2r}}{1+e^{2(s-r)}}\right)\right) \le \operatorname{arcosh}(e^s) \le s + \log(2). \tag{3.9}$$

For $(i,j) \in \{(0,1),(1,0),(1,1)\}$ we then deduce from the dominated convergence theorem that

$$\begin{aligned}\sigma_2^{i,j} &= \lim_{r\to\infty} c_1^{(i,j)} t^{3-i-j} \int_0^r (e^{-s} + e^{-2r+s}) \operatorname{arcosh}^2\left(e^s\left(\frac{1+e^{-2r}}{1+e^{2(s-r)}}\right)\right) ds \\ &= c_1^{(i,j)} t^{3-i-j} \int_0^\infty e^{-s} \operatorname{arcosh}^2(e^s)\; ds =: c_1^{(i,j)} t^{3-i-j} \cdot a\end{aligned}$$

and, in addition we have

$$\sigma_2^{0,0} = c_1^{(0,0)} t^3 \cdot a + 2t^2\, c_2^{(0,0)}.$$

Since $a = 4 \cdot \mathrm{G}$ by the following Remark 3.2.2, we obtain the specific values of $\sigma_2^{i,j}$ for $i,j \in \{0,1\}$, and hence the determinant of the asymptotic covariance matrix Σ_2 given in (3.8).

Next we prove the asserted rates of convergence. For $(i,j) \in \{(0,1),(1,0),(1,1)\}$, we get

$$\begin{aligned}&\left|\sigma_2^{i,j} - \mathbb{C}\mathrm{ov}\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{e^{r/2}}, \frac{F_{r,t}^{(j)} - \mathbb{E}F_{r,t}^{(j)}}{e^{r/2}}\right)\right| \\ &= \left|c_1^{(i,j)} t^{3-i-j} \cdot a - c_1^{(i,j)} t^{3-i-j} \int_0^r (e^{-s} + e^{-2r+s}) \operatorname{arcosh}^2\left(e^s\left(\frac{1+e^{-2r}}{1+e^{2(s-r)}}\right)\right) ds\right| \\ &\le c_1^{(i,j)} t^{3-i-j} \int_0^r e^{-s}\left(\operatorname{arcosh}^2(e^s) - \operatorname{arcosh}^2\left(e^s\left(\frac{1+e^{-2r}}{1+e^{2(s-r)}}\right)\right)\right) ds \qquad (3.10) \\ &\quad + c_1^{(i,j)} t^{3-i-j} \int_0^r e^{-2r+s} \operatorname{arcosh}^2\left(e^s\left(\frac{1+e^{-2r}}{1+e^{2(s-r)}}\right)\right) ds \qquad (3.11) \\ &\quad + c_1^{(i,j)} t^{3-i-j} \int_r^\infty e^{-s} \operatorname{arcosh}^2(e^s)\; ds. \qquad (3.12)\end{aligned}$$

Applying the second inequality in (3.9) to the expression in (3.12) we get

$$\int_r^\infty e^{-s} \operatorname{arcosh}^2(e^s)\; ds \le \int_r^\infty e^{-s}(\log(2)+s)^2\; ds \le c\, r^2 e^{-r}. \tag{3.13}$$

Using (3.9) for the expression in (3.11) we obtain

$$\begin{aligned}\int_0^r e^{-2r+s} \operatorname{arcosh}^2\left(e^s\left(\frac{1+e^{-2r}}{1+e^{2(s-r)}}\right)\right) ds &\le \int_0^r e^{-2r+s} \operatorname{arcosh}^2(e^s)\; ds \\ &\le \int_0^r e^{-2r+s}(s+\log(2))^2\, ds \\ &\le c\, r^2 e^{-r}. \qquad (3.14)\end{aligned}$$

Finally, we treat the expression in (3.10). An application of the mean value theorem in the first and (3.9) in the second to last step yields

$$\begin{aligned}
&\int_0^r e^{-s}\left(\operatorname{arcosh}^2(e^s) - \operatorname{arcosh}^2\left(e^s\left(\frac{1+e^{-2r}}{1+e^{2(s-r)}}\right)\right)\right) ds \\
&\le \int_0^r e^{-s}\left(\left(e^s - e^s\left(\frac{1+e^{-2r}}{1+e^{2(s-r)}}\right)\right)\max_{z\in\left[e^s\left(\frac{1+e^{-2r}}{1+e^{2(s-r)}}\right),e^s\right]}\frac{d}{dz}(\operatorname{arcosh}^2(z))\right) ds \\
&\le \int_0^r \left(\frac{e^{2(s-r)}-e^{-2r}}{1+e^{2(s-r)}}\right)\frac{2\operatorname{arcosh}(e^s)}{\sqrt{\left(\frac{e^s+e^{-2r+s}}{1+e^{2(s-r)}}\right)^2-1}}\, ds \\
&= \int_0^r e^{-2r}\left(e^{2s}-1\right)\frac{2\operatorname{arcosh}(e^s)}{\sqrt{e^{2s}-1+e^{2(s-2r)}-e^{-4(r-s)}}}\, ds \\
&\le \frac{1}{\sqrt{1-e^{-2r}}}\int_0^r e^{-2r}\left(e^{2s}-1\right)\frac{2\operatorname{arcosh}(e^s)}{\sqrt{e^{2s}-1}}\, ds \\
&\le ce^{-2r}\int_0^r \sqrt{e^{2s}-1}\,\operatorname{arcosh}(e^s)\, ds \\
&\le ce^{-2r}\int_0^r e^s(s+\log(2))\, ds \\
&\le cre^{-r}. \qquad (3.15)
\end{aligned}$$

Thus, a combination of (3.13), (3.14) and (3.15) yields the result for $(i,j) \in \{(0,1),(1,0),(1,1)\}$. Finally, if $(i,j) = (0,0)$ we obtain the result by additionally taking into account that

$$|c_2^{(0,0)}(1+e^{-2r}-2e^{-r}) - c_2^{(0,0)}| \le c\,e^{-r}.$$

This completes the proof. □

Remark 3.2.2. The relation $a = 4\mathrm{G}$ can be confirmed by Maple. It is not clear to us how Maple verifies this relation. Since we could not find the current integral representation of the Catalan constant in one of the lists available to us, we provide a short derivation. We first use the substitution $t = \exp(-\operatorname{arcosh}(e^s))$ or $e^s = \frac{1}{2}(t^{-1}+t)$ and then expand $(1+t^2)^{-2}$ into a Taylor series under the integral sign. This leads to

$$a = \int_0^\infty e^{-s}\operatorname{arcosh}^2(e^s) = \int_0^1 \frac{1-t^2}{(1+t^2)^2}(\ln t)^2\, dt = 2\int_0^1 \sum_{i=0}^\infty (-1)^i(i+1)t^{2i}(1-t^2)(\ln t)^2\, dt.$$

By the substitution $t = e^y$ we obtain

$$\int_0^1 t^{2i}(\ln t)^2\, dt = \frac{2}{(2i+1)^3}.$$

Hence we can interchange summation and integration to get

$$\begin{aligned} a &= 4\left(\sum_{i=0}^{\infty}(-1)^i(i+1)\frac{1}{(2i+1)^3} - \sum_{i=0}^{\infty}(-1)^i(i+1)\frac{1}{(2i+3)^3}\right) \\ &= 4\left(\frac{1}{2}\mathrm{G} + \frac{1}{2}\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)^3} - \frac{1}{2}(-\mathrm{G}+1) + \frac{1}{2}\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+3)^3}\right) \\ &= 4\left(\frac{1}{2}\mathrm{G} + \frac{1}{2}\mathrm{G} + \frac{1}{2} - \frac{1}{2}\right) = 4\mathrm{G}. \end{aligned}$$

THE SPATIAL CASE $d = 3$

Now we turn to the case $d = 3$ and again describes the rate of convergence, as $r \to \infty$, of the suitably scaled covariances to the asymptotic covariance matrix $\Sigma_d = (\sigma_d^{i,j})_{i,j=0}^{d-1}$.

Lemma 3.2.10. *Let $d = 3$ and $t \geq t_0 > 0$. There exists a positive constant $c_{cov}(3, t_0) \in (0, \infty)$ such that*

$$\left|\sigma_3^{i,j} - \mathbb{C}ov\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{r}\, e^r}, \frac{F_{r,t}^{(j)} - \mathbb{E}F_{r,t}^{(j)}}{\sqrt{r}\, e^r}\right)\right| \leq c_{cov}(3, t_0)\, t^{5-i-j}\, r^{-1}, \qquad i, j \in \{0, 1, 2\},$$

for $r \geq 1$. The matrix Σ_3 has rank one and is explicitly given by

$$\Sigma_3 = 2\pi^2 \begin{pmatrix} \frac{\pi^2}{2^8}t^5 & \frac{\pi^2}{2^6}t^4 & \frac{\pi}{2^4}t^3 \\ \frac{\pi^2}{2^6}t^4 & \frac{\pi^2}{2^4}t^3 & \frac{\pi}{2^2}t^2 \\ \frac{\pi}{2^4}t^3 & \frac{\pi}{2^2}t^2 & t \end{pmatrix}. \tag{3.16}$$

Proof. For $i, j \in \{0, 1, 2\}$, the covariance formula for Poisson U-statistics yields that

$$\mathbb{C}\mathrm{ov}\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{r}\, e^r}, \frac{F_{r,t}^{(j)} - \mathbb{E}F_{r,t}^{(j)}}{\sqrt{r}\, e^r}\right) = r^{-1}e^{-2r}\sum_{n=1}^{\min\{3-i,3-j\}} t^{6-i-j-n} n! \langle f_n^{(i)}, f_n^{(j)}\rangle_n.$$

As in the planar case $d = 2$ we compute the scalar products. We let $L_2(s)$ be a 2-dimensional subspace in $\mathbb{H}^3$ having distance $s \geq 0$ from the origin p. For $n = 1$ Corollary 3.2.2 and Equation (3.7) yield

$$\begin{aligned} \langle f_1^{(i)}, f_1^{(j)}\rangle_1 &= \omega_1 c(3,1,i,j)\int_0^r \cosh^2(s)\,\mathcal{H}^2(L_2(s)\cap B_r)^2\, ds \\ &= \omega_2^2\omega_1 c(3,1,i,j)\int_0^r \cosh^2(s)\left(\int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right)} \sinh(u)\, du\right)^2 ds \\ &= \omega_2^2\omega_1 c(3,1,i,j)\int_0^r \cosh^2(s)\left(\frac{\cosh(r)}{\cosh(s)} - 1\right)^2 ds \\ &= \omega_2^2\omega_1 c(3,1,i,j)\int_0^r (\cosh(r) - \cosh(s))^2\, ds \\ &= \omega_2^2\omega_1 c(3,1,i,j)\frac{1}{2}\left(r + 2r\cosh^2(r) - 3\sinh(r)\cosh(r)\right). \end{aligned}$$

In addition, using Lemma 2.4.2 and Lemma 3.2.5, we obtain

$$\langle f_2^{(i)}, f_2^{(j)} \rangle_2 \le c\, e^{2r} \qquad \text{and} \qquad \langle f_3^{(i)}, f_3^{(j)} \rangle_3 \le c\, e^{2r}.$$

Since $c(3,2) = 1$, $c(0,1,3) = \pi/16$, $c(1,1,3) = \pi/4$ and $c(2,1,3) = 1$, we obtain $c(3,1,0,0) = \pi^2/2^8$, $c(3,1,0,1) = \pi^2/2^6$, $c(3,1,0,2) = \pi/2^4$, $c(3,1,1,1) = \pi^2/2^4$, $c(3,1,1,2) = \pi/2^2$ and $c(3,1,2,2) = 1$. Moreover, we have

$$\begin{aligned}\sigma_3^{i,j} &= \lim_{r\to\infty} t^{5-i-j}\, \omega_2^2 \omega_1 c(3,1,i,j) \frac{1}{2} r^{-1} e^{-2r} \left(r + 2r\cosh^2(r) - 3\sinh(r)\cosh(r)\right)\\ &= t^{5-i-j}\, \omega_2^2 \omega_1 c(3,1,i,j) \frac{1}{4}\\ &= t^{5-i-j}\, 2\pi^2 c(3,1,i,j).\end{aligned}$$

Therefore we conclude that the asymptotic covariance matrix Σ_3 is given by (3.16). Clearly, Σ_3 has rank one. Moreover, we obtain

$$\begin{aligned}&\left|\sigma_3^{i,j} - \mathbb{C}\mathrm{ov}\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{r}\, e^r}, \frac{F_{r,t}^{(j)} - \mathbb{E}F_{r,t}^{(j)}}{\sqrt{r}\, e^r}\right)\right|\\ &\le t^{5-i-j}\, 4\pi^2 c(3,1,i,j) \left|1/2 - r^{-1} e^{-2r}\left(r + 2r\cosh^2(r) - 3\sinh(r)\cosh(r)\right)\right|\\ &\qquad\qquad + r^{-1} e^{-2r} \sum_{n=2}^{\min\{3-i,3-j\}} t^{6-i-j-n}\, n! \langle f_n^{(i)}, f_n^{(j)} \rangle_n\\ &\le c_{cov}(3, t_0)\, t^{5-i-j}\, r^{-1},\end{aligned}$$

where we used that $|1/2 - r^{-1} e^{-2r}\left(r + 2r\cosh^2(r) - 3\sinh(r)\cosh(r)\right)|$ is bounded from above by a constant multiple of r^{-1} as $r \to \infty$. This completes the proof. □

The case $d \ge 4$

In order to describe explicitly the limit covariance matrix $\Sigma(d)$ for $d \ge 4$ we need the following lemma.

Lemma 3.2.11. *For $\alpha > 0$,*

$$\int_0^\infty \cosh^{-\alpha}(x)\; dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha+1}{2})}.$$

Proof. Substituting first $u = e^x$ and then $\tan(z) = u$, and using $(\tan^2(x) + 1)^{-1} = \cos^2(x)$, we get

$$\int_0^\infty \cosh^{-\alpha}(x)\; dx = 2^\alpha \int_1^\infty \frac{u^{\alpha-1}}{(u^2+1)^\alpha}\; du = 2^\alpha \int_{\pi/4}^{\pi/2} \sin^{\alpha-1}(z) \cos^{\alpha-1}(z)\; dz =: I_\alpha.$$

The trigonometric identity $2\sin\alpha\cos\alpha = \sin(2\alpha)$ and the substitution $y = 2z$ yield

$$I_\alpha = 2\int_{\pi/4}^{\pi/2} \sin^{\alpha-1}(2z)\ dz = \int_0^{\pi/2} \sin^{\alpha-1}(y)\ dy = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha+1}{2})}.$$

This completes the argument. □

Depending on the dimension, we will bound the speed of convergence by means of the function

$$h(d,r) = \begin{cases} e^{-r} & : d = 4,\\ re^{-2r} & : d = 5,\\ e^{-2r} & : d \geq 6. \end{cases}$$

Lemma 3.2.12. *Let $d \geq 4$ and $t \geq t_0 > 0$. There exists a positive constant $c_{cov}(d,t_0) \in (0,\infty)$ such that*

$$\left|\sigma_d^{i,j} - \mathbb{C}ov\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{e^{r(d-2)}}, \frac{F_{r,t}^{(j)} - \mathbb{E}F_{r,t}^{(j)}}{e^{r(d-2)}}\right)\right| \leq c_{cov}(d,t_0)\, t^{2d-1-i-j}\, h(d,r),$$

for $r \geq 1$ and any $i,j \in \{0,\dots,d-1\}$. The matrix Σ_d has rank one and its entries are explicitly given by

$$\sigma_d^{i,j} = t^{2d-1-i-j}\, c(i,1,d)\, c(j,1,d)\, \frac{\omega_{d-1}\omega_d}{4^{d-2}(d-3)(d-2)}, \qquad i,j \in \{0,\dots,d-1\}, \tag{3.17}$$

where the constants $c(i,1,d), c(j,1,d)$ are introduced in Lemma 3.2.1.

Proof. Recall that

$$\mathbb{C}\mathrm{ov}\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{e^{(d-2)r}}, \frac{F_{r,t}^{(j)} - \mathbb{E}F_{r,t}^{(j)}}{e^{(d-2)r}}\right) = e^{-2(d-2)r} \sum_{n=1}^{\min\{d-i,d-j\}} t^{2d-i-j-n} n! \langle f_n^{(i)}, f_n^{(j)}\rangle_n \tag{3.18}$$

for $r \geq 1$. In a first step, we bound from above the summands with $n \geq 2$. For this, let $n \in \{2,\dots,\min\{d-i,d-j\}\}$. Lemma 3.2.5 implies that

$$e^{-2(d-2)r}\,\langle f_n^{(i)}, f_n^{(j)}\rangle_n \leq c\, e^{-2(d-2)r}\, g(d-n,2,d,r)$$

with some constant c, not depending on r. For $2(d-n-1) < d-1$ we obtain from Lemma 3.2.5 that

$$c\, e^{-2(d-2)r} g(d-n,2,d,r) \leq c\, e^{r(-2d+4)}\, e^{r(d-1)} \leq c\, e^{r(-d+3)} \leq c\, h(d,r).$$

Note that $2(d-n-1) = d-1$ implies that d is odd, hence $d \geq 5$, and therefore

$$c\, e^{-2(d-2)r} g(d-n,2,d,r) \leq c\, e^{r(-2d+4)}\, r\, e^{r(d-1)} \leq c\, r\, e^{r(-d+3)} \leq c\, h(d,r).$$

For $2(d-n-1) > d-1$ we get

$$c\, e^{-2(d-2)r} g(d-n,2,d,r) \leq c\, e^{r(-2d+4)}\, e^{2r(d-n-1)} \leq c\, e^{r(-2n+2)} \leq c\, h(d,r),$$

since $n \geq 2$.

Now we examine the remaining term corresponding to $n = 1$ in (3.18). By Corollary 3.2.2 and (3.7) we get

$$\begin{aligned}
&e^{-2(d-2)r}\,\langle f_1^{(i)}, f_1^{(j)}\rangle_1 \\
&= \frac{c(d,1,i,j)\,\omega_1}{e^{2(d-2)r}} \int_0^r \cosh^{d-1}(s)\,\mathcal{H}^{d-1}(L_{d-1}(s)\cap B_r)^2\;ds \\
&= \frac{c(d,1,i,j)\,\omega_1}{e^{2(d-2)r}} \int_0^r \cosh^{d-1}(s)\left(\omega_{d-1}\int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right)} \sinh^{d-2}(u)\;du\right)^2\;ds \\
&= \frac{c(d,1,i,j)\,\omega_1\,\omega_{d-1}^2}{e^{2(d-2)r}} \int_0^r \cosh^{d-1}(s) \\
&\qquad \times\left(\int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right)} \sum_{k=0}^{d-2}\frac{(-1)^k}{2^{d-2}}\binom{d-2}{k} e^{u(d-2-2k)}\;du\right)^2\;ds \qquad (3.19)\\
&= \frac{c(d,1,i,j)\,\omega_1\,\omega_{d-1}^2}{4^{d-2}e^{2(d-2)r}} \int_0^r \cosh^{d-1}(s) \\
&\qquad \times\left(\sum_{k=0}^{d-2}(-1)^k\binom{d-2}{k}\int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right)} e^{u(d-2-2k)}\;du\right)^2\;ds.
\end{aligned}$$

The quadratic term in brackets in (3.19) is given by

$$\begin{aligned}
\sum_{(k_1,k_2)\in\{0,\dots,d-2\}^2} (-1)^{k_1+k_2}\binom{d-2}{k_1}\binom{d-2}{k_2}\int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right)} e^{u_1(d-2-2k_1)}\;du_1 \\
\times\int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right)} e^{u_2(d-2-2k_2)}\;du_2.
\end{aligned}$$

Next, we provide and upper bound for the summands obtained for $(k_1,k_2)\in\{0,\dots,d-2\}^2\setminus\{(0,0)\}$. Without loss of generality we assume $k_2 \geq 1$. Then we get

$$\begin{aligned}
&e^{-2(d-2)r}\int_0^r \cosh^{d-1}(s)\int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right)} e^{u_1(d-2-2k_1)}\;du_1\int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right)} e^{u_2(d-2-2k_2)}\;du_2\;ds \qquad (3.20)\\
&\leq c\,e^{-2r(d-2)}\int_0^r e^{s(d-1)}\int_0^{r-s+\log(2)} e^{u_1(d-2-2k_1)}\;du_1\int_0^{r-s+\log(2)} e^{u_2(d-2-2k_2)}\;du_2\;ds \\
&\leq c\,e^{-2r(d-2)}\int_0^r e^{s(d-1)}\,e^{(r-s)(d-2)}\,e^{(r-s)(d-4)}\;ds \\
&\leq c\,e^{-2r}\int_0^r e^{s(-d+5)}\;ds \leq c\,h(d,r).
\end{aligned}$$

for $d \geq 5$. For $d = 4$ the third line is

$$c\,e^{-4r}\int_0^r e^{3s}\,e^{2(r-s)}(r-s+\log(2))\;ds = c\,e^{-2r}\int_0^r (r-s+\log(2))\,e^s\;ds \leq c\,h(4,r).$$

Therefore we can concentrate on the summand corresponding to $k = 0$ in (3.19). In the following we will make use of the logarithmic representation $\operatorname{arcosh}(x) = \log(x+\sqrt{x^2-1})$ of the arcosh-function in order to evaluate the inner integral. Then we get

$$\begin{aligned}
&\cosh^{-2(d-2)}(r)\int_0^r \cosh^{d-1}(s)\left(\int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right)} e^{u(d-2)}\,du\right)^2 ds\\
&=\frac{\cosh^{-2(d-2)}(r)}{(d-2)^2}\int_0^r \cosh^{d-1}(s)\left(\left(\frac{\cosh(r)}{\cosh(s)}+\sqrt{\frac{\cosh^2(r)}{\cosh^2(s)}-1}\right)^{d-2}-1\right)^2 ds \qquad (3.21)\\
&=(d-2)^{-2}\int_0^r \cosh^{-(d-3)}(s)\left(\left(1+\sqrt{1-\frac{\cosh^2(s)}{\cosh^2(r)}}\right)^{d-2}-\left(\frac{\cosh(s)}{\cosh(r)}\right)^{d-2}\right)^2 ds.
\end{aligned}$$

For $r\to\infty$ this expression converges to a constant. To get the correct rate stated in the lemma we observe that

$$\begin{aligned}
&\left|\sigma_d^{i,j}-\mathbb{C}\mathrm{ov}\left(\frac{F_{r,t}^{(i)}-\mathbb{E}F_{r,t}^{(i)}}{e^{r(d-2)}},\frac{F_{r,t}^{(j)}-\mathbb{E}F_{r,t}^{(j)}}{e^{r(d-2)}}\right)\right|\\
&\le\left|\sigma_d^{i,j}-e^{-2(d-2)r}\,t^{2d-1-i-j}\langle f_1^{(i)},f_1^{(j)}\rangle_1\right|+e^{-2(d-2)r}\sum_{n=2}^{\min\{d-i,d-j\}}t^{2d-i-j-n}n!\langle f_n^{(i)},f_n^{(j)}\rangle_n.
\end{aligned}$$

We have already shown that the second summand satisfies the asserted upper bound. It follows from (3.20) that it remains to consider

$$\begin{aligned}
&\left|\sigma_d^{i,j}-\frac{\beta}{e^{2r(d-2)}}\int_0^r\cosh^{d-1}(s)\left(\int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right)}e^{u(d-2)}\,du\right)^2 ds\right|\\
&\le\left|\sigma_d^{i,j}-\frac{\beta}{4^{d-2}\cosh^{2(d-2)}(r)}\int_0^r\cosh^{d-1}(s)\left(\int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right)}e^{u(d-2)}\,du\right)^2 ds\right| \qquad (3.22)\\
&\quad+\beta\left|\frac{1}{4^{d-2}\cosh^{2(d-2)}(r)}-\frac{1}{e^{2r(d-2)}}\right|\int_0^r\cosh^{d-1}(s)\left(\int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right)}e^{u(d-2)}\,du\right)^2 ds,
\end{aligned}$$

where we set

$$\beta := t^{2d-1-i-j}\,\frac{c(d,1,i,j)\,\omega_1\,\omega_{d-1}^2}{4^{d-2}}.$$

For the second summand, observe that

$$\left|\frac{1}{4^{d-2}\cosh^{2(d-2)}(r)}-\frac{1}{e^{2r(d-2)}}\right|\le e^{-2r(d-2)}\left(1-(1+e^{-2r})^{-2(d-2)}\right)\le c\,e^{-2r(d-1)}.$$

Since by (3.21) the integral in the second summand of (3.22) is of the order $e^{2r(d-2)}$, the second summand is at most of the order $\beta\,e^{-2r}$.

It remains to show the decay of the first summand in (3.22). This is done by using the same steps as in (3.21) and by splitting up the limit covariance $\sigma_d^{i,j}$. Lemma 3.2.11 and basic calculus show that the asserted entries of the asymptotic covariance matrix can be written in the form

$$\sigma_d^{i,j} = \frac{\beta}{(d-2)^2} \int_0^\infty \cosh^{-(d-3)}(s)\ ds.$$

Then we get

$$\left| \sigma_d^{i,j} - \frac{\beta}{4^{d-2}\cosh^{2(d-2)}(r)} \int_0^r \cosh^{d-1}(s) \left(\int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(s)}\right)} e^{u(d-2)}\ du \right)^2 ds \right| \leq I_1 + I_2,$$

where

$$I_1 := \frac{\beta}{(d-2)^2} \int_r^\infty \cosh^{-(d-3)}(s)\ ds \leq c\,\beta\, e^{-(d-3)r}.$$

and

$$\begin{aligned} I_2 :&= \frac{\beta}{(d-2)^2\, 4^{d-2}} \int_0^r \cosh^{-(d-3)}(s) \\ &\quad \times \left| 2^{2(d-2)} - \left(\left(1 + \sqrt{1 - \frac{\cosh^2(s)}{\cosh^2(r)}} \right)^{d-2} - \left(\frac{\cosh(s)}{\cosh(r)} \right)^{d-2} \right)^2 \right| ds. \end{aligned}$$

It remains to provide an upper bound for I_2. For this we expand the square and use the triangle inequality to get $I_2 \leq I_3 + I_4$, where

$$\begin{aligned} I_3 &\leq \beta \int_0^r \cosh^{-(d-3)}(s) \left(2 \left(1 + \sqrt{1 - \frac{\cosh^2(s)}{\cosh^2(r)}} \right)^{d-2} \left(\frac{\cosh(s)}{\cosh(r)} \right)^{d-2} + \left(\frac{\cosh(s)}{\cosh(r)} \right)^{2(d-2)} \right) ds \\ &\leq c\,\beta \int_0^r e^{-s(d-3)} \left(e^{(s-r)(d-2)} + e^{(s-r)(2d-4)} \right) ds \\ &\leq c\,\beta\, e^{-r(d-2)} \int_0^r e^s\ ds + c\,\beta\, e^{-r(2d-4)} \int_0^r e^{s(d-1)}\ ds \leq c\,\beta\, h(d,r), \end{aligned}$$

with some constant c. Here we also used that

$$\frac{\cosh(s)}{\cosh(r)} = \frac{e^s + e^{-s}}{e^r + e^{-r}} \leq \frac{2\,e^s}{e^r} = 2\,e^{s-r}, \qquad 0 \leq s \leq r. \tag{3.23}$$

In order to provide an upper bound for I_4, we use the mean value theorem and the inequality $1 - \sqrt{1-x} \leq x$, for $x \in [0,1]$, to get

$$\left| 2^{2(d-2)} - \left(1 + \sqrt{1-x} \right)^{2(d-2)} \right| \leq 2(d-2)2^{2d-5} x.$$

Hence we obtain

$$\begin{aligned} I_4 &\leq \beta \int_0^r \cosh^{-(d-3)}(s) \left| 2^{2(d-2)} - \left(1 + \sqrt{1 - \frac{\cosh^2(s)}{\cosh^2(r)}} \right)^{2(d-2)} \right| ds \\ &\leq c\,\beta \int_0^r \cosh^{-(d-3)}(s) \frac{\cosh^2(s)}{\cosh^2(r)}\, ds \leq c\,\beta e^{-2r} \int_0^r e^{s(-d+5)}\ ds \leq c\,\beta\, h(d,r), \end{aligned}$$

where also (3.23) was used. This concludes the proof. □

3.3 PROOFS II – MIXED K-FUNCTION AND MIXED PAIR-CORRELATION FUNCTION

Let $r > 0$, $i, j \in \{0, \ldots, d-1\}$ and let $B \subset \mathbb{H}^d$ be measurable with $\mathcal{H}^d(B) = 1$. Then

$$K_{ij}(r) = \frac{1}{\lambda_i \lambda_j} \mathbb{E} \int_{\mathrm{skel}_i} \int_{\mathrm{skel}_j \cap B} \mathbb{1}\{0 < d_h(x,y) \le r\}\, \mathcal{H}^j(dy)\, \mathcal{H}^i(dx).$$

Already at this point we see that the condition $0 < d_h(x,y)$ can be omitted if $i \ge 1$ or $j \ge 1$. Requiring that $x \in \mathrm{skel}_i$ and $y \in \mathrm{skel}_j$ means that there exist

$$(H_1, \ldots, H_{d-i}) \in \eta_{t,\ne}^{d-i} \qquad \text{and} \qquad (G_1, \ldots, G_{d-j}) \in \eta_{t,\ne}^{d-j}$$

such that $x \in H_1 \cap \ldots \cap H_{d-i}$ and $y \in G_1 \cap \ldots \cap G_{d-j}$. However, some of the hyperplanes of the first $(d-i)$-tuple may coincide with some of the hyperplanes of the second $(d-j)$-tuple. We denote by $n \in \{0, 1, \ldots, d-i\}$ the number of common hyperplanes. Then we obtain the representation

$$K_{ij}(r) = \frac{1}{\lambda_i \lambda_j} \sum_{n=0}^{\min\{d-i,d-j\}} \alpha(d,i,j,n)\, \mathbb{E} \sum_{(H_1,\ldots,H_{d-i},G_1,\ldots,G_{d-j-n}) \in \eta_{t,\ne}^{2d-i-j-n}} \int_{H_1 \cap \ldots \cap H_{d-i}} \times \int_{H_1 \cap \ldots \cap H_n \cap G_1 \cap \ldots G_{d-j-n} \cap B} \mathbb{1}\{0 < d_h(x,y) \le r\}\, \mathcal{H}^j(dy)\, \mathcal{H}^i(dx)$$

with the combinatorial coefficient given by

$$\alpha(d,i,j,n) = \frac{1}{n!(d-i-n)!(d-j-n)!}.$$

Note that if $n = 0$ we interpret the second integral as an integral over the set $G_1 \cap \ldots \cap G_{d-j} \cap B$ and if $n = d-j$ we understand that the integral ranges over $H_1 \cap \ldots \cap H_{d-j} \cap B$. Moreover, if $i = j = 0$, then the summand obtained for $n = d$ is zero, since almost surely $x, y \in H_1 \cap \ldots \cap H_d$ and $d_h(x,y) > 0$ cannot be satisfied simultaneously. Hence the summation can be restricted to $n \le m(d,i,j)$ in the following. An application of (2.6) leads to

$$\begin{aligned} &K_{ij}(r) \\ &= \frac{1}{\lambda_i \lambda_j} \sum_{n=0}^{m(d,i,j)} \alpha(d,i,j,n) t^{2d-i-j-n} \int_{A_h(d,d-1)^{2d-i-j-n}} \int_{H_1 \cap \ldots \cap H_{d-i}} \int_{H_1 \cap \ldots \cap H_n \cap G_1 \cap \ldots G_{d-j-n} \cap B} \\ &\quad \times \mathbb{1}\{0 < d_h(x,y) \le r\}\, \mathcal{H}^j(dy)\, \mathcal{H}^i(dx)\, \mu_{d-1}^{2d-i-j-n}(d(H_1, \ldots, H_{d-i}, G_1, \ldots, G_{d-j-n})) \\ &= \frac{1}{\lambda_i \lambda_j} \sum_{n=0}^{m(d,i,j)} \alpha(d,i,j,n) t^{2d-i-j-n} \int_{A_h(d,d-1)^n} \int_{A_h(d,d-1)^{d-i-n}} \int_{A_h(d,d-1)^{d-j-n}} \\ &\quad \times \int_{H_1 \cap \ldots \cap H_{d-i}} \int_{H_1 \cap \ldots \cap H_n \cap G_1 \cap \ldots G_{d-j-n} \cap B} \mathbb{1}\{0 < d_h(x,y) \le r\}\, \mathcal{H}^j(dy)\, \mathcal{H}^i(dx) \\ &\quad \times \mu_{d-1}^{d-j-n}(d(G_1, \ldots, G_{d-j-n}))\, \mu_{d-1}^{d-i-n}(d(H_{n+1}, \ldots, H_{d-i}))\, \mu_{d-1}^{n}(d(H_1, \ldots, H_n)), \end{aligned}$$

where we have used Fubini's theorem to split the integration over $A_h(d,d-1)^{2d-i-j-n}$ in the form $A_h(d,d-1)^n \times A_h(d,d-1)^{d-i-n} \times A_h(d,d-1)^{d-j-n}$. The first group of hyperplanes comprises the n common hyperplanes $H_1,\dots,H_n$, while the second and the third group is associated with the $(d-i-n)$-tuple $H_{n+1},\dots,H_{d-i}$ and the $(d-j-n)$-tuple $G_1,\dots,G_{d-j-n}$, respectively. We now apply Lemma 2.4.2 successively to each of the three outer integrals. Together with Fubini's theorem this gives

$$
\begin{aligned}
&K_{ij}(r)\\
&= \frac{1}{\lambda_i\lambda_j}\sum_{n=0}^{m(d,i,j)} \alpha(d,i,j,n)\beta(d,i,j,n)t^{2d-i-j-n}\int_{A_h(d,d-n)}\int_{A_h(d,i+n)}\int_{A_h(d,j+n)}\\
&\qquad\times \int_{E\cap F}\int_{B\cap E\cap G} \mathbb{1}\{0<d_h(x,y)\le r\}\,\mathcal{H}^j(dy)\,\mathcal{H}^i(dx)\,\mu_{j+n}(dG)\,\mu_{i+n}(dF)\,\mu_{d-n}(dE)\\
&= \frac{1}{\lambda_i\lambda_j}\sum_{n=0}^{m(d,i,j)} \alpha(d,i,j,n)\beta(d,i,j,n)t^{2d-i-j-n}\int_{A_h(d,d-n)}\int_{A_h(d,j+n)}\int_{B\cap E\cap G}\\
&\qquad\times \int_{A_h(d,i+n)}\int_{E\cap F} \mathbb{1}\{0<d_h(x,y)\le r\}\,\mathcal{H}^i(dx)\,\mu_{i+n}(dF)\,\mathcal{H}^j(dy)\,\mu_{j+n}(dG)\,\mu_{d-n}(dE),
\end{aligned}
$$

where $\beta(d,i,j,n) := c(d,d-n)c(d,i+n)c(d,j+n)$.

For the two innermost integrals we get

$$
\begin{aligned}
&\int_{A_h(d,i+n)}\int_{E\cap F} \mathbb{1}\{0<d_h(x,y)\le r\}\,\mathcal{H}^i(dx)\,\mu_{i+n}(dF)\\
&\qquad= \int_{A_h(d,i+n)} \mathcal{H}^i(\{x\in E\cap F : 0<d_h(x,y)\le r\})\,\mu_{i+n}(dF)\\
&\qquad= \int_{A_h(d,i+n)} \mathcal{H}^i(E\cap (B(y,r)\setminus\{y\})\cap F)\,\mu_{i+n}(dF).
\end{aligned}
$$

Since $y\in E$, the intersection $E\cap(B(y,r)\setminus\{y\})$ has dimension $d-n$ and we can apply Crofton's formula to conclude that

$$
\int_{A_h(d,i+n)}\int_{E\cap F} \mathbb{1}\{0<d_h(x,y)\le r\}\,\mathcal{H}^i(dx)\,\mu_{i+n}(dF) = \frac{\omega_{d+1}\omega_{i+1}}{\omega_{i+n+1}\omega_{d-n+1}}\mathcal{H}^{d-n}(E\cap B(y,r)).
$$

Here we also used that $\mathcal{H}^{d-n}(E\cap(B(y,r)\setminus\{y\})) = \mathcal{H}^{d-n}(E\cap B(y,r))$, since $d-n\ge 1$. Moreover, since $y\in E$ the value of $\mathcal{H}^{d-n}(E\cap B(y,r))$ is independent of the choice of E and y, and is given by the $(d-n)$-dimensional Hausdorff measure

$$
\mathcal{H}^{d-n}(B_r^{d-n}) = \omega_{d-n}\int_0^r \sinh^{d-n-1}(s)\,ds
$$

of a $(d-n)$-dimensional geodesic ball B_r^{d-n} of radius r. We thus arrive at

$$\begin{aligned}K_{ij}(r) &= \frac{1}{\lambda_i\lambda_j}\sum_{n=0}^{m(d,i,j)}\alpha(d,i,j,n)\beta(d,i,j,n)\frac{\omega_{d+1}\omega_{i+1}}{\omega_{i+n+1}\omega_{d-n+1}}\mathcal{H}^{d-n}(B_r^{d-n})t^{2d-i-j-n}\\&\quad\times\int_{A_h(d,d-n)}\int_{A_h(d,j+n)}\int_{B\cap E\cap G}1\,\mathcal{H}^j(dy)\,\mu_{j+n}(dG)\,\mu_{d-n}(dE)\\&= \frac{1}{\lambda_i\lambda_j}\sum_{n=0}^{m(d,i,j)}\alpha(d,i,j,n)\beta(d,i,j,n)\frac{\omega_{d+1}\omega_{i+1}}{\omega_{i+n+1}\omega_{d-n+1}}\mathcal{H}^{d-n}(B_r^{d-n})t^{2d-i-j-n}\\&\quad\times\int_{A_h(d,d-n)}\int_{A_h(d,j+n)}\mathcal{H}^j(B\cap E\cap G)\,\mu_{j+n}(dG)\,\mu_{d-n}(dE).\end{aligned}$$

The two remaining integrals can be evaluated by using twice the Crofton formula. Indeed, noting that for μ_{d-n}-almost all $E\in A_h(d,d-n)$ the set $B\cap E$ is either empty or has dimension $d-n$ we find that

$$\begin{aligned}&\int_{A_h(d,d-n)}\int_{A_h(d,j+n)}\mathcal{H}^j(B\cap E\cap G)\,\mu_{j+n}(dG)\,\mu_{d-n}(dE)\\&\quad=\frac{\omega_{d+1}\omega_{j+1}}{\omega_{j+n+1}\omega_{d-n+1}}\int_{A_h(d,d-n)}\mathcal{H}^{d-n}(B\cap E)\,\mu_{d-n}(dE)\\&\quad=\frac{\omega_{d+1}\omega_{j+1}}{\omega_{j+n+1}\omega_{d-n+1}}\mathcal{H}^d(B).\end{aligned}$$

Since $\mathcal{H}^d(B)=1$ we finally conclude that

$$\begin{aligned}K_{ij}(r) &= \frac{1}{\lambda_i\lambda_j}\sum_{n=0}^{m(d,i,j)}\alpha(d,i,j,n)\beta(d,i,j,n)\frac{\omega_{d+1}^2\omega_{i+1}\omega_{j+1}}{\omega_{d-n+1}^2\omega_{i+n+1}\omega_{j+n+1}}t^{2d-i-j-n}\mathcal{H}^{d-n}(B_r^{d-n})\\&= \frac{1}{\lambda_i\lambda_j}\sum_{n=0}^{m(d,i,j)}\alpha(d,i,j,n)\beta(d,i,j,n)\frac{\omega_{d+1}^2\omega_{i+1}\omega_{j+1}}{\omega_{d-n+1}^2\omega_{i+n+1}\omega_{j+n+1}}\\&\qquad\qquad\times\omega_{d-n}t^{2d-i-j-n}\int_0^r\sinh^{d-n-1}(s)\,ds.\end{aligned}$$

Simplification of the constant by means of the constants given in (3.2) and Lemma 2.4.2 completes the proof for the mixed K-function K_{ij}. The formula for the mixed pair-correlation function follows by differentiation. This completes the proof of Theorem 3.1.3. □

3.4 Proofs III – Univariate limit theorems

3.4.1 The case of growing intensity: Proof of Theorem 3.1.4

The central limit theorem is in this case a direct consequence of the central limit theorem for general Poisson U-statistics stated as Corollary 4.3 in [109] (see also [27]). □

3.4.2 The case of growing windows: Proof of Theorem 3.1.5

Our strategy in the proof of Theorem 3.1.5 (a) and (b) can be summarized as follows. The normal approximation bound (2.10) for general U-statistics of Poisson processes is given by a sum involving terms of the type $M_{u,v}$, which are defined in (2.8) and (2.9) and which in turn are given as sums of integrals over partitions $\sigma \in \Pi_{\geq 2}^{con}(u,u,v,v)$. In applying these normal approximation bounds to the Euclidean counterparts of the functionals $F_{r,t}^{(i)}$ it was possible to extract a common scaling factor from each of the integrals in $M_{u,v}$ and to treat the number of terms, that is, the number of elements of $\Pi_{\geq 2}^{con}(u,u,v,v)$ as a constant, see [65, 96]. In the hyperbolic set-up this is no longer possible and each integral in the definition of $M_{u,v}$ needs a separate treatment. In fact, it will turn out that these integrals exhibit different asymptotic behaviours as functions of r, as $r \to \infty$. For the analysis, we have to determine explicitly the partitions in $\Pi_{\geq 2}^{con}(u,u,v,v)$ and for each such partition we have to provide a bound for the resulting integral. Since $\mu = t\mu_{d-1}$, we can bound the dependence with respect to the intensity $t \geq 1$ by $4(d-i) - 2(u+v) + |\sigma|$ for each $\sigma \in \Pi_{\geq 2}^{con}(u,u,v,v)$.

To show that a central limit theorem fails in higher space dimensions $d \geq 4$ is the most technical part in the proof of Theorem 3.1.5. We do this by showing that the fourth cumulant of the centred and normalized total volume $F_{r,t}^{(i)}$ is bounded away from 0 by an absolute and strictly positive constant and hence does not converge to 0. The latter in turn is the fourth cumulant of a standard Gaussian random variable. However, in view of the well known expression of the fourth cumulant in terms of the first four centred moments this approach can only work if we can ensure that the sequence of random variables

$$\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{\mathbb{V}\mathrm{ar}(F_{r,t}^{(i)})}}\right)^4$$

is uniformly integrable. We will prove that this is indeed the case by showing that their fifths moments are uniformly bounded. This requires a very careful analysis of the combinatorial formula (2.7) for the centred moments of U-statistics of Poisson processes.

The representation of a U-statistic will be as in Section 3.2.1. In the following computations, c will be a positive constant only depending on the dimension and whose value may change from occasion to occasion.

The planar case $d = 2$: Proof of Theorem 3.1.5 (a)

As indicated above, we will use the bound (2.10). We distinguish the cases $i = 0$ and $i = 1$. In the following, we can assume that $r, t \geq 1$.

For $i = 1$, which corresponds to the total edge length in B_r, it is enough to bound $M_{1,1}(f^{(1)})$. For this we note that $\Pi_{\geq 2}^{con}(1,1,1,1)$ only consists of the trivial partition $\sigma_1 = \{1,2,3,4\}$, see Figure 3.4.1 (left panel). Thus, using Lemma 3.2.5, we have that

$$M_{1,1}(f^{(1)}) = c\,t \int_{A_h(2,1)} \mathcal{H}^1(H \cap B_r)^4 \, \mu_1(dH) \leq c\,t\,e^r.$$

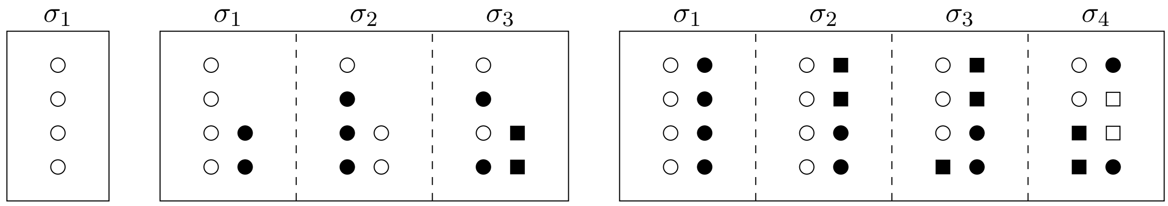

Figure 3.4.1: Left panel: Illustration of the partition in $\Pi^{con}_{\geq 2}(1,1,1,1)$. Middle panel: Illustration of the partitions in $\Pi^{con}_{\geq 2}(1,1,2,2)$. Right panel: Illustration of the partitions in $\Pi^{con}_{\geq 2}(2,2,2,2)$

Together with the lower variance bound from Lemma 3.2.6 this yields

$$d\left(\frac{F^{(1)}_{r,t} - \mathbb{E}F^{(1)}_{r,t}}{\sqrt{\mathbb{V}\mathrm{ar}(F^{(1)}_{r,t})}}, N\right) \leq c\,\frac{\sqrt{te^r}}{tc^{(1)}(2)\,e^r} \leq c\,t^{-1/2}\,e^{-r/2}. \tag{3.24}$$

Here we used that the exponent of t is given by $4(2-1) - 2(1+1) + 1 = 1$.

Next, we deal with the case $i = 0$, which corresponds to the total vertex count in B_r. In this situation, we need to bound the terms $M_{1,1}(f^{(0)}), M_{1,2}(f^{(0)}), M_{2,2}(f^{(0)})$. For $M_{1,1}(f^{(0)})$ we can argue as in the case $i = 1$, since $\Pi^{con}_{\geq 2}(1,1,1,1)$ only consists of the single partition σ_1, see Figure 3.4.1 (left panel). This allows us to conclude that

$$M_{1,1}(f^{(0)}) = c\,t^5 \int_{A_h(2,1)} \mathcal{H}^1(H \cap B_r)^4\, \mu_1(dH) \leq c\,t^5\, e^r,$$

where we used that the exponent of t is given by $4(2-0) - 2(1+1) + 1 = 5$.

To deal with $M_{1,2}(f^{(0)})$ we observe that, up to renumbering of the elements, $\Pi^{con}_{\geq 2}(1,1,2,2)$ consists of precisely three partitions σ_1, σ_2 and σ_3, which are illustrated in Figure 3.4.1 (middle panel). For σ_1 we obtain, using Crofton's formula and Lemma 3.2.5,

$$\begin{aligned}
&\int_{A_h(2,1)^2} \mathcal{H}^1(H_1 \cap B_r)^2\, \mathcal{H}^0(H_1 \cap H_2 \cap B_r)^2\, \mu_1^2(d(H_1,H_2)) \\
&\quad = \int_{A_h(2,1)^2} \mathcal{H}^1(H_1 \cap B_r)^2\, \mathcal{H}^0(H_1 \cap H_2 \cap B_r)\, \mu_1^2(d(H_1,H_2)) \\
&\quad = c\int_{A_h(2,1)} \mathcal{H}^1(H_1 \cap B_r)^3\, \mu_1(dH_1) \leq c\,e^r.
\end{aligned} \tag{3.25}$$

Moreover, for the partition σ_2 we compute, using twice that $\mathcal{H}^1(H \cap B_r) \leq 2r$ for each $H \in A_h(2,1)$ and again Crofton's formula,

$$\begin{aligned}
&\int_{A_h(2,1)^2} \mathcal{H}^1(H_1 \cap B_r)\, \mathcal{H}^1(H_2 \cap B_r)\, \mathcal{H}^0(H_1 \cap H_2 \cap B_r)^2\, \mu_1^2(d(H_1,H_2)) \\
&\quad \leq 4r^2 \int_{A_h(2,1)^2} \mathcal{H}^0(H_1 \cap H_2 \cap B_r)\, \mu_1^2(d(H_1,H_2)) \leq c\,r^2\, e^r,
\end{aligned} \tag{3.26}$$

and for partition σ_3 we get

$$\begin{aligned}
&\int_{A_h(2,1)^3} \mathcal{H}^1(H_1 \cap B_r)\,\mathcal{H}^1(H_2 \cap B_r)\,\mathcal{H}^0(H_1 \cap H_3 \cap B_r)\,\mathcal{H}^0(H_2 \cap H_3 \cap B_r)\,\mu_1^3(d(H_1, H_2, H_3))\\
&\quad \le 2r \int_{A_h(2,1)^2} \mathcal{H}^1(H_2 \cap B_r)\,\mathcal{H}^1(H_3 \cap B_r)\,\mathcal{H}^0(H_2 \cap H_3 \cap B_r)\,\mu_1^2(d(H_2, H_3))\\
&\quad \le 4r^2 \int_{A_h(2,1)} \mathcal{H}^1(H_3 \cap B_r)^2\,\mu_1(dH_3) \le c\,r^2\,e^r. \qquad (3.27)
\end{aligned}$$

This yields that $M_{1,2}(f^{(0)}) \le c\,t^5\,(e^r + 2r^2 e^r) \le c\,t^5\,r^2 e^r$ (recall that $r, t \ge 1$). Here we used that the exponent of t is given by $4(2-0) - 2(2+1) + \max\{2,3\} = 5$.

Now we deal with the term $M_{2,2}(f^{(0)})$, which involves a summation over partitions in $\Pi^{con}_{\ge 2}(2,2,2,2)$. Up to renumbering of the elements, there are precisely four such partitions σ_1, σ_2, σ_3 and σ_4, which are illustrated in Figure 3.4.1 (right panel). For σ_1 we compute

$$\begin{aligned}
\int_{A_h(2,1)^2} \mathcal{H}^0(H_1 \cap H_2 \cap B_r)^4\,\mu_1^2(d(H_1, H_2)) &= \int_{A_h(2,1)^2} \mathcal{H}^0(H_1 \cap H_2 \cap B_r)\,\mu_1^2(d(H_1, H_2))\\
&= c \int_{A_h(2,1)} \mathcal{H}^1(H_1 \cap B_r)\,\mu_1(dH_1) \le c\,e^r,
\end{aligned}$$

where we used Crofton's formula and Lemma 3.2.5. Similarly, for σ_2 and σ_3 we get

$$\begin{aligned}
&\int_{A_h(2,1)^3} \mathcal{H}^0(H_1 \cap H_2 \cap B_r)^2\,\mathcal{H}^0(H_1 \cap H_3 \cap B_r)^2\,\mu_1^3(d(H_1, H_2, H_3))\\
&\quad = \int_{A_h(2,1)^3} \mathcal{H}^0(H_1 \cap H_2 \cap B_r)\,\mathcal{H}^0(H_1 \cap H_3 \cap B_r)\,\mu_1^3(d(H_1, H_2, H_3))\\
&\quad = c \int_{A_h(2,1)} \mathcal{H}^1(H_1 \cap B_r)^2\,\mu_1(dH_1) \le c\,e^r,
\end{aligned}$$

and, additionally using that $\mathcal{H}^0(H_1 \cap H_2 \cap B_r) \le 1$ for μ_1^2-almost all $(H_1, H_2) \in A_h(2,1)^2$,

$$\begin{aligned}
&\int_{A_h(2,1)^3} \mathcal{H}^0(H_1 \cap H_2 \cap B_r)^2\,\mathcal{H}^0(H_1 \cap H_3 \cap B_r)\,\mathcal{H}^0(H_2 \cap H_3 \cap B_r)\,\mu_1^3(d(H_1, H_2, H_3))\\
&\quad \le \int_{A_h(2,1)^3} \mathcal{H}^0(H_1 \cap H_3 \cap B_r)\,\mathcal{H}^0(H_2 \cap H_3 \cap B_r)\,\mu_1^3(d(H_1, H_2, H_3))\\
&\quad = c \int_{A_h(2,1)} \mathcal{H}^1(H_3 \cap B_r)^2\,\mu_1(dH_3) \le c\,e^r.
\end{aligned}$$

Finally, we deal with σ_4. Using once more that $\mathcal{H}^0(H_1 \cap H_2 \cap B_r) \le 1$ for μ_1^2-almost all $(H_1, H_2) \in A_h(2,1)^2$ and also that $\mathcal{H}^1(H \cap B_r) \le 2r$ for each $H \in A_h(2,1)$, and again Crofton's formula together with Lemma 3.2.5, we obtain

$$\begin{aligned}
&\int_{A_h(2,1)^4} \mathcal{H}^0(H_1 \cap H_2 \cap B_r)\,\mathcal{H}^0(H_1 \cap H_3 \cap B_r)\,\mathcal{H}^0(H_3 \cap H_4 \cap B_r)\\
&\qquad\qquad \times \mathcal{H}^0(H_2 \cap H_4 \cap B_r)\,\mu_1^4(d(H_1, H_2, H_3, H_4))\\
&\quad \le c \int_{A_h(2,1)^2} \mathcal{H}^1(H_3 \cap B_r)\,\mathcal{H}^0(H_3 \cap H_4 \cap B_r)\,\mathcal{H}^1(H_4 \cap B_r)\,\mu_1^2(d(H_3, H_4))\\
&\quad \le c\,r \int_{A_h(2,1)} \mathcal{H}^1(H_4 \cap B_r)^2\,\mu_1(dH_4) \le c\,r\,e^r.
\end{aligned}$$

Altogether, this yields that $M_{2,2}(f^{(0)}) \le ct^4\,(e^r + e^r + e^r + re^r) \le ct^4\,re^r$, where the exponent of t follows from $4\cdot 2 - 2\cdot 4 + \max\{2,3,4\} = 4$.

Combining the bounds for $M_{1,1}(f^{(0)})$, $M_{1,2}(f^{(0)})$ and $M_{2,2}(f^{(0)})$ with the lower variance bound provided by Lemma 3.2.6 we deduce from (2.10) that

$$d\left(\frac{F_{r,t}^{(0)} - \mathbb{E}F_{r,t}^{(0)}}{\sqrt{\mathbb{V}\mathrm{ar}(F_{r,t}^{(0)})}}, N\right) \le c\,\frac{\sqrt{t^5e^r} + \sqrt{t^5r^2e^r} + \sqrt{t^4re^r}}{t^3c^{(0)}(2)e^r} \le c\,t^{-1/2}\,re^{-r/2}. \tag{3.28}$$

This completes the proof of Theorem 3.1.5 (a). □

The spatial case $d = 3$: Proof of Theorem 3.1.5 (b)

The following lemma will be used repeatedly in deriving upper bounds for integrals. For $H \in A_h(3,2)$ we write $L_1(H)$ for an arbitrary 1-dimensional subspace in H which satisfies $d_h(H,p) = d_h(L_1(H),p)$.

Lemma 3.4.1. *Let $d = 3$ and $a, b \ge 0$. If $r \ge 1$, then*

$$I(a,b) := \int_{A_h(3,2)} \mathcal{H}^2(H\cap B_r)^a\,\mathcal{H}^1(L_1(H)\cap B_r)^b\,\mu_2(dH) \le c\begin{cases}\exp(2r) & : 0 \le a < 2,\\ r^{b+1}\exp(2r) & : a = 2,\\ r^b\exp(ar) & : a > 2,\end{cases}$$

where $c = c(a,b)$ is a constant depending only on a and b.

Proof. We use the definition (2.3) of the measure μ_2, Lemma 3.2.4 and the argument in the proof of Lemma 3.2.5 to get

$$I(a,b) \le c\int_0^r e^{2s}e^{(r-s)a}(r-s+\log 2)^b\,ds.$$

If $0 \le a < 2$, then

$$I(a,b) \le c\,e^{2r}\int_0^r e^{s(a-2)}(s+\log 2)^b\,ds \le c\,e^{2r}.$$

This also shows that $I(2,b) \le ce^{2r}r^{b+1}$. For $a > 2$, we get

$$I(a,b) \le c\,e^{ar}\int_0^r e^{s(2-a)}(r-s+\log 2)^b\,ds \le c\,r^be^{ar},$$

which completes the argument. □

For $d = 3$ we need to distinguish the cases $i = 2$, $i = 1$ and $i = 0$. If $i = 2$ there is only one partition σ_1 (compare with the left panel of Figure 3.4.1) and we obtain

$$\int_{A_h(3,2)} \mathcal{H}^2(H\cap B_r)^4\,\mu_2(dH) \le c\,g(2,4,3,r) \le c\,e^{4r}. \tag{3.29}$$

This proves that $M_{1,1}(f^{(2)}) \le c\,t\,e^{4r}$ and together with the lower variance bound from Lemma

3.2.7 and (2.10) this yields

$$d\left(\frac{F_{r,t}^{(2)} - \mathbb{E}F_{r,t}^{(2)}}{\sqrt{\mathbb{V}\mathrm{ar}(F_{r,t}^{(2)})}}, N\right) \le c\,\frac{\sqrt{te^{4r}}}{tc^{(2)}(3)e^{2r}r} \le c\,t^{-1/2}\,r^{-1}. \tag{3.30}$$

To deal with the case $i = 1$, we need to bound $M_{1,1}(f^{(1)})$, $M_{1,2}(f^{(1)})$ and $M_{2,2}(f^{(1)})$. As in the case $d = 2$, to bound $M_{1,1}(f^{(1)})$ we can argue as for $i = 2$ to obtain $M_{1,1}(f^{(1)}) \le c t^5 e^{4r}$. Next, we consider $M_{1,2}(f^{(1)})$, which requires an analysis of the integrals resulting from the three partitions σ_1, σ_2 and σ_3 shown in the middle panel of Figure 3.4.1. For σ_1 we compute

$$\begin{aligned}
&\int_{A_h(3,2)^2} \mathcal{H}^2(H_1\cap B_r)^2\,\mathcal{H}^1(H_1\cap H_2\cap B_r)^2\,\mu_2^2(d(H_1,H_2))\\
&\quad\le \int_{A_h(3,2)^2} \mathcal{H}^2(H_1\cap B_r)^2\,\mathcal{H}^1(L_1(H_1)\cap B_r)\mathcal{H}^1(H_1\cap H_2\cap B_r)\,\mu_2^2(d(H_1,H_2))\\
&\quad\le c\,I(3,1)\le c\,r e^{3r},
\end{aligned} \tag{3.31}$$

where we used the Crofton formula and Lemma 3.4.1. Arguing similarly for the partition σ_2 from the middle panel of Figure 3.4.1 we obtain

$$\begin{aligned}
&\int_{A_h(3,2)^2} \mathcal{H}^2(H_1\cap B_r)\,\mathcal{H}^2(H_2\cap B_r)\,\mathcal{H}^1(H_1\cap H_2\cap B_r)^2\,\mu_2^2(d(H_1,H_2))\\
&\le c\int_{A_h(3,2)^2} \mathcal{H}^2(H_1\cap B_r)\,\mathcal{H}^2(H_2\cap B_r)\,\mathcal{H}^1(L_1(H_1)\cap B_r)\,\mathcal{H}^1(L_1(H_2)\cap B_r)\,\mu_2^2(d(H_1,H_2))\\
&\le c\,I(1,1)^2\le c\,e^{4r},
\end{aligned} \tag{3.32}$$

and for σ_3 we get

$$\begin{aligned}
&\int_{A_h(3,2)^3} \mathcal{H}^2(H_1\cap B_r)\,\mathcal{H}^2(H_2\cap B_r)\,\mathcal{H}^1(H_1\cap H_3\cap B_r)\,\mathcal{H}^1(H_2\cap H_3\cap B_r)\,\mu_2^3(d(H_1,H_2,H_3))\\
&\le \int_{A_h(3,2)^3} \mathcal{H}^2(H_1\cap B_r)\,\mathcal{H}^2(H_2\cap B_r)\,\mathcal{H}^1(L_1(H_1)\cap B_r)\,\mathcal{H}^1(H_2\cap H_3\cap B_r)\,\mu_2^3(d(H_1,H_2,H_3))\\
&\le c\int_{A_h(3,2)^2} \mathcal{H}^2(H_1\cap B_r)\,\mathcal{H}^2(H_2\cap B_r)^2\,\mathcal{H}^1(L_1(H_1)\cap B_r)\,\mu_2^2(d(H_1,H_2))\\
&\le c\,I(1,2)\,g(2,2,3,r)\le c\,r\,e^{4r}.
\end{aligned} \tag{3.33}$$

We thus conclude that $M_{1,2}(f^{(1)}) \le c t^5\,(re^{3r} + e^{4r} + re^{4r}) \le c t^5\,re^{4r}$.

Finally, we deal with $M_{2,2}(f^{(1)})$ for which an analysis of the four partitions σ_1, σ_2, σ_3 and σ_4 shown in the right panel of Figure 3.4.1 is necessary. For σ_1 we have

$$\begin{aligned}
&\int_{A_h(3,2)^2} \mathcal{H}^1(H_1\cap H_2\cap B_r)^4\,\mu_2^2(d(H_1,H_2))\\
&\quad\le \int_{A_h(3,2)^2} \mathcal{H}^1(H_1\cap H_2\cap B_r)\,\mathcal{H}^1(L_1(H_1)\cap B_r)^3\,\mu_2^2(d(H_1,H_2))\le c\,I(1,3)\le c\,e^{2r},
\end{aligned}$$

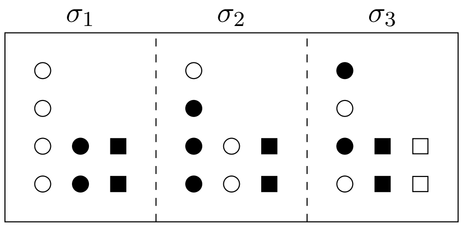

Figure 3.4.2: Illustration of the partition in $\Pi_{\geq 2}^{con}(1,1,3,3)$

where we also used Crofton's formula. We continue with σ_2 and get, by similar arguments,

$$\begin{aligned}
&\int_{A_h(3,2)^3} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)^2 \, \mathcal{H}^1(H_1 \cap H_3 \cap B_r)^2 \, \mu_2^3(d(H_1,H_2,H_3)) \\
&\quad \leq \int_{A_h(3,2)^3} \mathcal{H}^1(L_1(H_1) \cap B_r)^2 \, \mathcal{H}^1(H_1 \cap H_2 \cap B_r) \, \mathcal{H}^1(H_1 \cap H_3 \cap B_r) \, \mu_2^3(d(H_1,H_2,H_3)) \\
&\quad = c \int_{A_h(3,2)} \mathcal{H}^1(L_1(H_1) \cap B_r)^2 \, \mathcal{H}^2(H_1 \cap B_r)^2 \, \mu_2(dH_1) \\
&\quad \leq c\, I(2,2) \leq c\, r^3 \, e^{2r}.
\end{aligned}$$

Moreover, for σ_3 and σ_4 we have the bounds

$$\begin{aligned}
&\int_{A_h(3,2)^3} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)^2 \, \mathcal{H}^1(H_1 \cap H_3 \cap B_r) \, \mathcal{H}^1(H_2 \cap H_3 \cap B_r) \, \mu_2^3(d(H_1,H_2,H_3)) \\
&\quad \leq c \int_{A_h(3,2)^3} \mathcal{H}^1(L_1(H_1) \cap B_r)^3 \, \mathcal{H}^1(H_2 \cap H_3 \cap B_r) \, \mu_2^3(d(H_1,H_2,H_3)) \\
&\quad \leq c\, \mathcal{H}^3(B_r) \, I(0,3) \leq c\, e^{4r}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{A_h(3,2)^4} \mathcal{H}^1(H_1 \cap H_2 \cap B_r) \, \mathcal{H}^1(H_1 \cap H_3 \cap B_r) \, \mathcal{H}^1(H_3 \cap H_4 \cap B_r) \\
&\qquad\qquad \times \mathcal{H}^1(H_2 \cap H_4 \cap B_r) \, \mu_2^4(d(H_1,H_2,H_3,H_4)) \\
&\leq \int_{A_h(3,2)^4} \mathcal{H}^1(L_1(H_1) \cap B_r)^2 \, \mathcal{H}^1(H_2 \cap H_4 \cap B_r) \, \mathcal{H}^1(H_3 \cap H_4 \cap B_r) \, \mu_2^4(d(H_1,H_2,H_3,H_4)) \\
&= c \int_{A_h(2,3)^2} \mathcal{H}^1(L_1(H_1) \cap B_r)^2 \, \mathcal{H}^2(H_4 \cap B_r)^2 \, \mu_2^2(d(H_1,H_4)) \\
&\leq c\, I(0,2) \, g(2,2,3,r) \leq c\, r\, e^{4r}.
\end{aligned}$$

Altogether this gives $M_{2,2}(f^{(1)}) \leq c t^4 \left(e^{2r} + r^3 e^{2r} + e^{4r} + r e^{4r}\right) \leq c t^4 \, r e^{4r}$. The estimates for $M_{1,1}(f^{(1)})$, $M_{1,2}(f^{(1)})$ and $M_{2,2}(f^{(1)})$ together with Lemma 3.2.7 and (2.10) show that

$$d\left(\frac{F_{r,t}^{(1)} - \mathbb{E}F_{r,t}^{(1)}}{\sqrt{\mathbb{V}\mathrm{ar}(F_{r,t}^{(1)})}}, N\right) \leq c \, \frac{\sqrt{t^5 e^{4r}} + \sqrt{t^5 r e^{4r}} + \sqrt{t^4 r e^{4r}}}{t^3 c^{(1)}(3) e^{2r} r} \leq c\, t^{-1/2} \, r^{-1/2}. \tag{3.34}$$

Finally, we need to treat the case of $F_{r,t}^{(0)}$, which requires to find upper bounds for the terms $M_{u,v}(f^{(0)})$ with $(u,v) \in \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$. We have $M_{1,1}(f^{(0)}) \leq c t^9 \, e^{4r}$

from (3.29). To treat $M_{1,2}(f^{(0)})$ we need to consider the partitions σ_1, σ_2 and σ_3 shown in the middle panel of Figure 3.4.1 and to obtain upper bounds for the three integrals which are already treated in (3.31), (3.32) and (3.33). This implies that $M_{1,2}(f^{(0)}) \le c\,t^9\, re^{4r}$. Next, we deal with $M_{1,3}(f^{(0)})$, which can be expressed as a sum over the three partitions σ_1, σ_2 and σ_3 shown in Figure 3.4.2. For σ_1, using that $\mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r) \le 1$ for μ_2^3-almost all $(H_1, H_2, H_3) \in A_h(3,2)^3$, we have that

$$\begin{aligned}
&\int_{A_h(3,2)^3} \mathcal{H}^2(H_1 \cap B_r)^2\, \mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)^2\, \mu_2^3(d(H_1,H_2,H_3)) \\
&\quad = \int_{A_h(3,2)^3} \mathcal{H}^2(H_1 \cap B_r)^2\, \mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)\, \mu_2^3(d(H_1,H_2,H_3)) \\
&\quad = c \int_{A_h(3,2)} \mathcal{H}^2(H_1 \cap B_r)^3\, \mu_2(dH_1) \le c\, g(2,3,3,r) \le c\, e^{3r},
\end{aligned}$$

where we also used Crofton's formula and Lemma 3.2.5. Similarly, for σ_2 we obtain

$$\begin{aligned}
&\int_{A_h(3,2)^3} \mathcal{H}^2(H_1 \cap B_r)\, \mathcal{H}^2(H_2 \cap B_r)\, \mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)^2\, \mu_2^3(d(H_1,H_2,H_3)) \\
&\quad = \int_{A_h(3,2)^3} \mathcal{H}^2(H_1 \cap B_r)\, \mathcal{H}^2(H_2 \cap B_r)\, \mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)\, \mu_2^3(d(H_1,H_2,H_3)) \\
&\quad \le c\, \mathcal{H}^3(B_r)\, I(1,1) \le c\, e^{4r},
\end{aligned}$$

and for σ_3 we have that

$$\begin{aligned}
&\int_{A_h(3,2)^4} \mathcal{H}^2(H_1 \cap B_r)\, \mathcal{H}^2(H_2 \cap B_r)\, \mathcal{H}^0(H_1 \cap H_3 \cap H_4 \cap B_r) \\
&\qquad\qquad \times \mathcal{H}^0(H_2 \cap H_3 \cap H_4 \cap B_r)\, \mu_2^4(d(H_1,H_2,H_3,H_4)) \\
&\quad \le \int_{A_h(3,2)^4} \mathcal{H}^2(H_1 \cap B_r)\, \mathcal{H}^2(H_2 \cap B_r)\, \mathcal{H}^0(H_1 \cap H_3 \cap H_4 \cap B_r)\, \mu_2^4(d(H_1,H_2,H_3,H_4)) \\
&\quad = c\, \mathcal{H}^3(B_r) \int_{A_h(3,2)} \mathcal{H}^2(H_1 \cap B_r)^2\, \mu_2(dH_1) \le c\, e^{2r}\, g(2,2,3,r) \le c\, re^{4r}.
\end{aligned}$$

This proves that $M_{1,3}(f^{(0)}) \le c\,t^8\,(e^{3r} + e^{4r} + re^{4r}) \le c\,t^8\, re^{4r}$.

The next term is $M_{2,2}(f^{(0)})$. However, up to a constant, this term is the same as $M_{2,2}(f^{(1)})$, which was already bounded above. This yields that $M_{2,2}(f^{(0)}) \le c\,t^8\, re^{4r}$ and it remains to consider $M_{2,3}(f^{(0)})$ and $M_{3,3}(f^{(0)})$.

In order to deal with $M_{2,3}(f^{(0)})$, up to renumbering of the elements precisely the 12 partitions $\sigma_1, \dots, \sigma_{12}$ in $\Pi_{\ge 2}^{con}(2,2,3,3)$ have to be considered, see Figure 3.4.3. Using that $\mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r) \le 1$ for μ_2^3-almost all $(H_1, H_2, H_3) \in A_h(3,2)^3$ we find for σ_1 that

$$\begin{aligned}
&\int_{A_h(3,2)^3} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)^2\, \mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)^2\, \mu_2^3(d(H_1,H_2,H_3)) \\
&\quad = \int_{A_h(3,2)^3} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)^2\, \mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)\, \mu_2^3(d(H_1,H_2,H_3)).
\end{aligned}$$

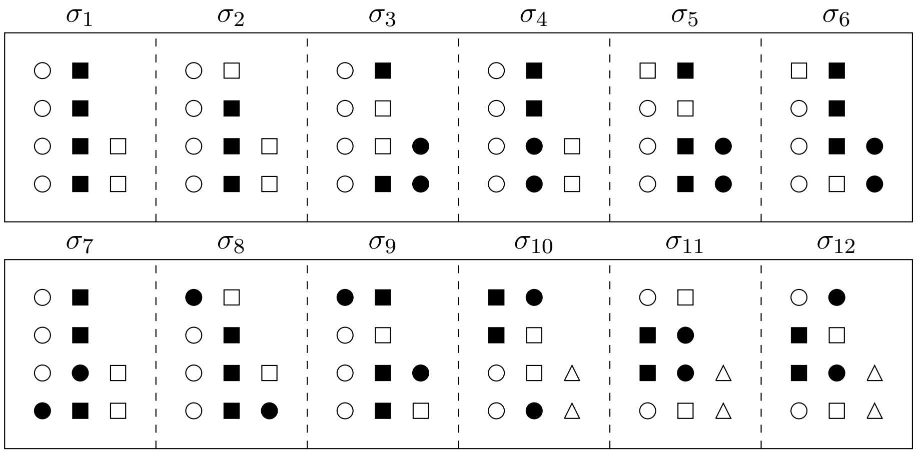

Figure 3.4.3: Illustration of the partition in $\Pi^{con}_{\geq 2}(2,2,3,3)$

Applying now Crofton's formula, we obtain the upper bound

$$c\int_{A_h(3,2)^2}\mathcal{H}^1(H_1\cap H_2\cap B_r)^3\,\mu_2^3(d(H_1,H_2))\leq c\,I(1,2)\leq c\,e^{2r}.$$

The same arguments also lead to bounds for the remaining partitions $\sigma_2,\dots,\sigma_{12}$. As for σ_1, the first step is always to bound the 0-dimensional Hausdorff measure $\mathcal{H}^0(\,\cdot\,)$ of the intersection of the three planes corresponding to the last row of the partition by 1, which is a valid estimate for μ_2^3-almost all triples of planes. For this reason we systematically skip this first step in our following computations and only show how to deal with the integral of the three remaining terms

$$\begin{aligned}&\mathcal{H}^1(\text{intersection of the 2 planes corresponding to the first row})\\ &\times\mathcal{H}^1(\text{intersection of the 2 planes corresponding to the second row})\\ &\quad\times\mathcal{H}^0(\text{intersection of the 3 planes corresponding to the third row}).\end{aligned}$$

For σ_2 we get

$$\begin{aligned}&\int_{A_h(3,2)^3}\mathcal{H}^1(H_1\cap H_2\cap B_r)\,\mathcal{H}^1(H_1\cap H_3\cap B_r)\mathcal{H}^0(H_1\cap H_2\cap H_3\cap B_r)\,\mu_2^3(d(H_1,H_2,H_3))\\ &\leq c\int_{A_h(3,2)^3}\mathcal{H}^1(L_1(H_1)\cap B_r)^2\,\mathcal{H}^0(H_1\cap H_2\cap H_3\cap B_r)\,\mu_2^3(d(H_1,H_2,H_3))\\ &\leq c\,I(1,2)\leq c\,e^{2r},\end{aligned}$$

for σ_3 we get

$$\begin{aligned}
&\int_{A_h(3,2)^4} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\, \mathcal{H}^1(H_1 \cap H_3 \cap B_r)\, \mathcal{H}^0(H_1 \cap H_3 \cap H_4 \cap B_r)\, \mu_2^4(d(H_1, H_2, H_3, H_4)) \\
&\quad \le c \int_{A_h(3,2)^3} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\, \mathcal{H}^1(L_1(H_1) \cap B_r)\, \mathcal{H}^1(L_1(H_3) \cap B_r)\, \mu_2^3(d(H_1, H_2, H_3)) \\
&\quad \le c\, I(1,1)\, I(0,1) \le c\, e^{4r},
\end{aligned}$$

for σ_4 we get

$$\begin{aligned}
&\int_{A_h(3,2)^4} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)^2\, \mathcal{H}^0(H_1 \cap H_3 \cap H_4 \cap B_r)\, \mu_2^4(d(H_1, H_2, H_3, H_4)) \\
&\quad \le c \int_{A_h(3,2)^2} \mathcal{H}^1(L_1(H_1) \cap B_r)\, \mathcal{H}^1(L_1(H_2) \cap B_r)\, \mathcal{H}^2(H_1 \cap B_r)\, \mu_2^2(d(H_1, H_2)) \\
&\quad \le c\, I(1,1)\, I(0,1) \le c\, e^{4r},
\end{aligned}$$

for σ_5 we get

$$\begin{aligned}
&\int_{A_h(3,2)^4} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\, \mathcal{H}^1(H_1 \cap H_3 \cap B_r)\, \mathcal{H}^0(H_2 \cap H_3 \cap H_4 \cap B_r)\, \mu_2^4(d(H_1, H_2, H_3, H_4)) \\
&\quad \le c \int_{A_h(3,2)^3} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\, \mathcal{H}^1(L_1(H_3) \cap B_r)^2\, \mu_2^3(d(H_1, H_2, H_3)) \\
&\quad \le c\, \mathcal{H}^3(B_r)\, I(0,2) \le c\, e^{4r},
\end{aligned}$$

for σ_6 we get

$$\begin{aligned}
&\int_{A_h(3,2)^4} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\, \mathcal{H}^1(H_2 \cap H_3 \cap B_r) \mathcal{H}^0(H_2 \cap H_3 \cap H_4 \cap B_r)\, \mu_2^4(d(H_1, H_2, H_3, H_4)) \\
&\quad \le c \int_{A_h(3,2)^3} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\, \mathcal{H}^1(L_1(H_3) \cap B_r)^2\, \mu_2^3(d(H_1, H_2, H_3)),
\end{aligned}$$

which is the same as for σ_5 and thus bounded by e^{4r}. For σ_7 we have

$$\begin{aligned}
&\int_{A_h(3,2)^4} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)^2\, \mathcal{H}^0(H_1 \cap H_3 \cap H_4 \cap B_r)\, \mu_2^4(d(H_1, H_2, H_3, H_4)) \\
&\quad \le c \int_{A_h(3,2)^2} \mathcal{H}^1(L_1(H_1) \cap B_r)\, \mathcal{H}^1(L_1(H_2) \cap B_r)\, \mathcal{H}^2(H_1 \cap B_r)\, \mu_2^2(d(H_1, H_2)) \\
&\quad \le c\, I(1,1)\, I(0,1) \le c\, e^{4r},
\end{aligned}$$

for σ_8 we have

$$\begin{aligned}
&\int_{A_h(3,2)^4} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\, \mathcal{H}^1(H_3 \cap H_4 \cap B_r)\, \mathcal{H}^0(H_2 \cap H_3 \cap H_4 \cap B_r)\, \mu_2^4(d(H_1, H_2, H_3, H_4)) \\
&\quad \le \int_{A_h(3,2)^4} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\, \mathcal{H}^1(H_3 \cap H_4 \cap B_r)\, \mu_2^4(d(H_1, H_2, H_3, H_4)) \\
&\quad = c\, \mathcal{H}^3(B_r)^2 \le c\, e^{4r},
\end{aligned}$$

for σ_9 we have

$$\begin{aligned}
&\int_{A_h(3,2)^4} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\,\mathcal{H}^1(H_3 \cap H_4 \cap B_r)\,\mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)\,\mu_2^4(d(H_1,H_2,H_3,H_4))\\
&\quad \le \int_{A_h(3,2)^4} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\,\mathcal{H}^1(H_3 \cap H_4 \cap B_r)\,\mu_2^4(d(H_1,H_2,H_3,H_4))\\
&\quad = c\,\mathcal{H}^3(B_r)^2 \le c\,e^{4r}.
\end{aligned}$$

Next, for σ_{10} we get

$$\begin{aligned}
&\int_{A_h(3,2)^5} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\,\mathcal{H}^1(H_1 \cap H_3 \cap B_r)\\
&\qquad\qquad \times \mathcal{H}^0(H_3 \cap H_4 \cap H_5 \cap B_r)\,\mu_2^5(d(H_1,H_2,H_3,H_4,H_5))\\
&\quad \le c \int_{A_h(3,2)^3} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\,\mathcal{H}^1(L_1(H_3) \cap B_r)\,\mathcal{H}^2(H_3 \cap B_r)\,\mu_2^3(d(H_1,H_2,H_3))\\
&\quad \le c\,\mathcal{H}^3(B_r)\,I(1,1) \le c\,e^{4r},
\end{aligned}$$

for σ_{11} we get

$$\begin{aligned}
&\int_{A_h(3,2)^5} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\,\mathcal{H}^1(H_3 \cap H_4 \cap B_r)\\
&\qquad\qquad \times \mathcal{H}^0(H_3 \cap H_4 \cap H_5 \cap B_r)\,\mu_2^5(d(H_1,H_2,H_3,H_4,H_5))\\
&\quad = c \int_{A_h(3,2)^4} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\,\mathcal{H}^1(H_3 \cap H_4 \cap B_r)^2\,\mu_2^4(d(H_1,H_2,H_3,H_4))\\
&\quad \le c\,\mathcal{H}^3(B_r) \int_{A_h(3,2)^2} \mathcal{H}^1(L_1(H_3) \cap B_r)\,\mathcal{H}^1(H_3 \cap H_4 \cap B_r)\,\mu_2^2(d(H_3,H_4))\\
&\quad = c\,\mathcal{H}^3(B_r) \int_{A_h(3,2)} \mathcal{H}^1(L_1(H_3) \cap B_r)\,\mathcal{H}^2(H_3 \cap B_r)\,\mu_2(dH_3) \le c\,\mathcal{H}^3(B_r)\,I(1,1) \le c\,e^{4r}
\end{aligned}$$

and for σ_{12} we get

$$\begin{aligned}
&\int_{A_h(3,2)^5} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\,\mathcal{H}^1(H_3 \cap H_4 \cap B_r)\\
&\qquad\qquad \times \mathcal{H}^0(H_2 \cap H_3 \cap H_5 \cap B_r)\,\mu_2^5(d(H_1,H_2,H_3,H_4,H_5))\\
&\quad \le c\,\mathcal{H}^3(B_r) \int_{A_h(3,2)} \mathcal{H}^2(H_3 \cap B_r)\,\mathcal{H}^1(L_1(H_3) \cap B_r)\,\mu_2(dH_3) \le c\,e^{2r}\,I(1,1) \le c\,e^{4r}.
\end{aligned}$$

Altogether this yields that $M_{2,3}(f^{(0)}) \le c\,t^7\,(2\,e^{2r} + 10\,e^{4r}) \le c\,t^7\,e^{4r}$.

Finally, we deal with the term $M_{3,3}(f^{(0)})$. This requires to consider the partitions in $\Pi_{\ge 2}^{con}(3,3,3,3)$. Up to renumbering of the elements there are precisely 11 partitions $\sigma_1,\dots,\sigma_{11}$ of this type and they are shown in Figure 3.4.4. The analysis of the resulting integrals works the same way as above. Especially, we use once again systematically that $\mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r) \le 1$ for μ_2^3-almost all $(H_1,H_2,H_3) \in A_h(3,2)^3$ and apply this to the term corresponding to the last

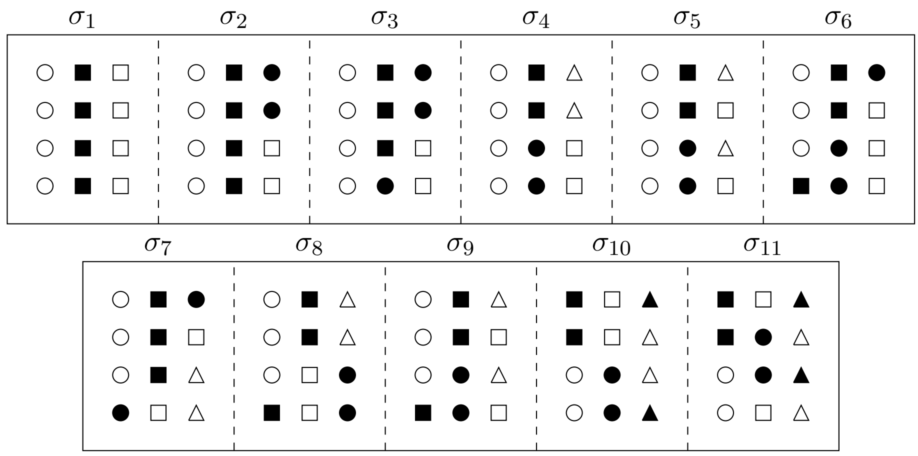

Figure 3.4.4: Illustration of the partition in $\Pi^{con}_{\geq 2}(3,3,3,3)$

row of each of the partitions. This leaves us with integrals over

$$\mathcal{H}^0(\text{intersection of the 3 planes corresponding to the first row})$$
$$\times\, \mathcal{H}^0(\text{intersection of the 3 planes corresponding to the second row})$$
$$\times\, \mathcal{H}^0(\text{intersection of the 3 planes corresponding to the third row}),$$

which in turn can be bounded using Crofton's formula, Lemma 3.2.5 and Lemma 3.4.1. For σ_1 this yields

$$\begin{aligned}&\int_{A_h(3,2)^3} \mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)^3\, \mu_2^3(d(H_1,H_2,H_3))\\ &\quad= \int_{A_h(3,2)^3} \mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)\, \mu_2^3(d(H_1,H_2,H_3)) = c\,\mathcal{H}^3(B_r) \leq c\,e^{2r},\end{aligned}$$

for σ_2 and σ_3 we obtain

$$\begin{aligned}&\int_{A_h(3,2)^4} \mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)^2\, \mathcal{H}^0(H_1 \cap H_2 \cap H_4 \cap B_r)\, \mu_2^4(d(H_1,H_2,H_3,H_4))\\ &\quad= \int_{A_h(3,2)^4} \mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)\, \mathcal{H}^0(H_1 \cap H_2 \cap H_4 \cap B_r)\, \mu_2^4(d(H_1,H_2,H_3,H_4))\\ &\quad= c\int_{A_h(3,2)^2} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)^2\, \mu_2^2(d(H_1,H_2)) \leq c\,I(1,1) \leq c\,e^{2r},\end{aligned}$$

for σ_4 we obtain

$$\begin{aligned}&\int_{A_h(3,2)^5} \mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)^2\, \mathcal{H}^0(H_1 \cap H_4 \cap H_5 \cap B_r)\, \mu_2^5(d(H_1,H_2,H_3,H_4,H_5))\\ &\quad= \int_{A_h(3,2)^5} \mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)\, \mathcal{H}^0(H_1 \cap H_4 \cap H_5 \cap B_r)\, \mu_2^5(d(H_1,H_2,H_3,H_4,H_5))\\ &\quad= c\int_{A_h(3,2)} \mathcal{H}^2(H_1 \cap B_r)^2\, \mu_2(dH_1) \leq c\,g(2,2,3,r) \leq c\,r e^{2r},\end{aligned}$$

for σ_5 we have

$$\begin{aligned}\int_{A_h(3,2)^5} &\mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)\,\mathcal{H}^0(H_1 \cap H_2 \cap H_4 \cap B_r)\\ &\qquad\times \mathcal{H}^0(H_1 \cap H_3 \cap H_5 \cap B_r)\,\mu_2^5(d(H_1,H_2,H_3,H_4,H_5))\\ &\le c\int_{A_h(3,2)} \mathcal{H}^2(H_1 \cap B_r)\,\mathcal{H}^1(L_1(H_1)\cap B_r)^2\,\mu_2(dH_1) \le c\,I(1,2) \le c\,e^{2r},\end{aligned}$$

for σ_6 we have

$$\begin{aligned}\int_{A_h(3,2)^4} &\mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)\,\mathcal{H}^0(H_1 \cap H_2 \cap H_4 \cap B_r)\\ &\qquad\times \mathcal{H}^0(H_1 \cap H_3 \cap H_4 \cap B_r)\,\mu_2^4(d(H_1,H_2,H_3,H_4))\\ &\le \int_{A_h(3,2)^4} \mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)\,\mathcal{H}^0(H_1 \cap H_2 \cap H_4 \cap B_r)\,\mu_2^4(d(H_1,H_2,H_3,H_4))\\ &= c\int_{A_h(3,2)^2} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)^2\,\mu_2^2(d(H_1,H_2)) \le c\,e^{2r}\end{aligned}$$

by the same argument as for σ_2 and σ_3. For σ_7 we have

$$\begin{aligned}\int_{A_h(3,2)^5} &\mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)\,\mathcal{H}^0(H_1 \cap H_2 \cap H_4 \cap B_r)\\ &\qquad\times \mathcal{H}^0(H_1 \cap H_2 \cap H_5 \cap B_r)\,\mu_2^5(d(H_1,H_2,H_3,H_4,H_5))\\ &= c\int_{A_h(3,2)^2} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)^3\,\mu_2(d(H_1,H_2))\\ &\le c\int_{A_h(3,2)^2} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\,\mathcal{H}^1(L_1(H_1)\cap B_r)^2\,\mu_2^2(d(H_1,H_2)) \le c\,I(1,2) \le c\,e^{2r},\end{aligned}$$

for σ_8 we obtain

$$\begin{aligned}\int_{A_h(3,2)^5} &\mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)^2\,\mathcal{H}^0(H_1 \cap H_4 \cap H_5 \cap B_r)\,\mu_2^5(d(H_1,H_2,H_3,H_4,H_5))\\ &= \int_{A_h(3,2)^5} \mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)\,\mathcal{H}^0(H_1 \cap H_4 \cap H_5 \cap B_r)\,\mu_2^5(d(H_1,H_2,H_3,H_4,H_5))\\ &= c\int_{A_h(3,2)} \mathcal{H}^2(H_1 \cap B_r)^2\,\mu_2(dH_1) \le c\,g(2,2,3,r) \le c\,re^{2r},\end{aligned}$$

for σ_9 we get

$$\begin{aligned}\int_{A_h(3,2)^5} &\mathcal{H}^0(H_1 \cap H_2 \cap H_3 \cap B_r)\,\mathcal{H}^0(H_1 \cap H_2 \cap H_4 \cap B_r)\\ &\qquad\times \mathcal{H}^0(H_1 \cap H_3 \cap H_5 \cap B_r)\,\mu_2^5(d(H_1,H_2,H_3,H_4,H_5))\\ &\le c\int_{A_h(3,2)^3} \mathcal{H}^1(H_1 \cap H_2 \cap B_r)\,\mathcal{H}^1(H_1 \cap H_3 \cap B_r)\,\mu_2^3(d(H_1,H_2,H_3))\\ &= c\int_{A_h(3,2)} \mathcal{H}^2(H_1 \cap B_r)^2\,\mu_2(dH_1) \le c\,g(2,2,3,r) \le c\,re^{2r},\end{aligned}$$

for σ_{10} we obtain

$$\begin{aligned}\int_{A_h(3,2)^6} &\mathcal{H}^0(H_1\cap H_2\cap H_3\cap B_r)\,\mathcal{H}^0(H_1\cap H_2\cap H_4\cap B_r)\\ &\qquad\times\mathcal{H}^0(H_4\cap H_5\cap H_6\cap B_r)\,\mu_2^6(d(H_1,\dots,H_6))\\ &\le c\int_{A_h(3,2)^4}\mathcal{H}^0(H_1\cap H_2\cap H_3\cap B_r)\,\mathcal{H}^2(H_4\cap B_r)\,\mu_2^4(d(H_1,H_2,H_3,H_4))\\ &= c\,\mathcal{H}^3(B_r)^2\le c\,e^{4r}\end{aligned}$$

and, finally, for σ_{11} we have

$$\begin{aligned}\int_{A_h(3,2)^6} &\mathcal{H}^0(H_1\cap H_2\cap H_3\cap B_r)\,\mathcal{H}^0(H_1\cap H_4\cap H_5\cap B_r)\\ &\qquad\times\mathcal{H}^0(H_3\cap H_4\cap H_6\cap B_r)\,\mu_2^6(d(H_1,\dots,H_6))\\ &= c\int_{A_h(3,2)^3}\mathcal{H}^1(H_1\cap H_3\cap B_r)\,\mathcal{H}^1(H_1\cap H_4\cap B_r)\,\mathcal{H}^1(H_3\cap H_4\cap B_r)\,\mu_2^3(d(H_1,H_3,H_4))\\ &\le c\int_{A_h(3,2)^3}\mathcal{H}^1(L_1(H_1)\cap B_r)^2\,\mathcal{H}^1(H_3\cap H_4\cap B_r)\,\mu_2^3(d(H_1,H_3,H_4))\\ &= c\,\mathcal{H}^3(B_r)\,I(0,2)\le c\,e^{4r}.\end{aligned}$$

We thus conclude that $M_{3,3}(f^{(0)})\le c\,t^6\,(6e^{2r}+3\,re^{2r}+2e^{4r})\le c\,t^6\,e^{4r}$. An application of the upper bounds for $M_{u,v}(f^{(0)})$ with $(u,v)\in\{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$ and the lower bound for the variance from Lemma 3.2.7 in (2.10) shows that

$$d\left(\frac{F_{r,t}^{(0)}-\mathbb{E}F_{r,t}^{(0)}}{\sqrt{\mathbb{V}\mathrm{ar}(F_{r,t}^{(0)})}},N\right)\le c\,\frac{3\sqrt{t^9e^{4r}}+3\sqrt{t^9re^{4r}}}{t^5c^{(1)}(3)e^{2r}r}\le c\,t^{-1/2}\,r^{-1/2}\tag{3.35}$$

and the proof of Theorem 3.1.5 (b) is complete. □

The higher dimensional cases $d\ge 4$: Proof of Theorem 3.1.5 (c)

In order to show that for $d\ge 4$ and $i=d-1$ and for $d\ge 7$ and $i\in\{0,\dots,d-1\}$ non of the centred and normalized functionals $F_{r,t}^{(i)}$ converges in distribution to a Gaussian random variable, as $r\to\infty$, we will argue that the fourth cumulant

$$\mathrm{cum}_4:=\mathbb{E}\left(\widetilde{F_{r,t}^{(i)}}\right)^4-3,\qquad\qquad \widetilde{F_{r,t}^{(i)}}:=\frac{F_{r,t}^{(i)}-\mathbb{E}F_{r,t}^{(i)}}{\sqrt{\mathbb{V}\mathrm{ar}(F_{r,t}^{(i)})}}$$

does not converge to zero, which is the value of the fourth cumulant of a standard Gaussian random variable. We start with the following crucial, but rather technical result, which is based on the formula (2.7) for the centred moments of a Poisson U-statistic.

Lemma 3.4.2. *Let $d\ge 4$, $i\in\{0,1,\dots,d-1\}$ and $t\ge t_0>0$. If $d\in\{4,5,6\}$ and $i=d-1$ or if $d\ge 7$, then*

$$\sup_{r\ge 1}\mathbb{E}\left(\widetilde{F_{r,t}^{(i)}}\right)^5<\infty.$$

$b(\cdot)$	$c(\cdot)$
1	2
1	2
2	1
1	3
0	4

Figure 3.4.5: Left panel: The two types of (sub-)partitions in $\Pi^{**}_{\geq 2}(1,1,1,1,1)$. Right panel: Example of a sub-partition σ from $\Pi^{**}_{\geq 2}(4,4,4,4,4)$ with $m(\sigma) = 3$

Proof. We start by explaining our method by considering the case $i = d-1$. In this situation

$$\mathbb{E}\Big(\widetilde{F^{(d-1)}_{r,t}}\Big)^5 = \frac{\mathbb{E}\big(F^{(d-1)}_{r,t} - \mathbb{E}F^{(d-1)}_{r,t}\big)^5}{(\mathbb{V}\mathrm{ar}(F^{(d-1)}_{r,t}))^{5/2}} \leq c\,\frac{\mathbb{E}\big(F^{(d-1)}_{r,t} - \mathbb{E}F^{(d-1)}_{r,t}\big)^5}{e^{5r(d-2)}},$$

where we used the variance bound from Lemma 3.2.8, which is available since $t \geq t_0$ and $r \geq 1$. For the centred fifth moment, (2.7) implies that

$$\mathbb{E}\big(F^{(d-1)}_{r,t} - \mathbb{E}F^{(d-1)}_{r,t}\big)^5 = \sum_{\sigma\in\Pi^{**}_{\geq 2}(1,1,1,1,1)} t^{5-|\sigma|+\|\sigma\|} \int_{A_h(d,d-1)^{5-|\sigma|-\|\sigma\|}} \big((f^{(d-1)})^{\otimes 5}\big)_\sigma \, d\mu^{5-|\sigma|+\|\sigma\|}_{d-1}.$$

The set $\Pi^{**}_{\geq 2}(1,1,1,1,1)$ consists only of two types of sub-partitions of $\{1,2,3,4,5\}$, which are actually partitions, see Figure 3.4.5. The first type only consists of one partition, namely the trivial partition, only containing the single block $\{1,2,3,4,5\}$. The second type contains $\binom{5}{2} = 10$ partitions having precisely two blocks, one of size 2 and the other of type 3. Since the integrals corresponding to these partitions all yield the same contribution, we can restrict our computations to $\{\{1,2,3\},\{4,5\}\}$, for example. Thus,

$$\begin{aligned}\mathbb{E}\big(F^{(d-1)}_{r,t} - \mathbb{E}F^{(d-1)}_{r,t}\big)^5 &= t^9 \int_{A_h(d,d-1)} \mathcal{H}^{d-1}(H\cap B_r)^5\,\mu_{d-1}(dH)\\ &\quad + 10t^8 \int_{A_h(d,d-1)^2} \mathcal{H}^{d-1}(H_1\cap B_r)^3\,\mathcal{H}^{d-1}(H_2\cap B_r)^2\,\mu^2_{d-1}(d(H_1,H_2)).\end{aligned}$$

By Lemma 3.2.5 we have

$$\int_{A_h(d,d-1)} \mathcal{H}^{d-1}(H\cap B_r)^5\,\mu_{d-1}(dH) \leq c\,g(d-1,5,d,r) \leq c\,e^{5r(d-2)},$$

since $5(d-2) - (d-1) = 4d - 9 > 0$. Again by Lemma 3.2.5 we obtain

$$\begin{aligned}&\int_{A_h(d,d-1)^2} \mathcal{H}^{d-1}(H_1\cap B_r)^3\,\mathcal{H}^{d-1}(H_2\cap B_r)^2\,\mu^2_{d-1}(d(H_1,H_2))\\ &\qquad \leq c\,g(d-1,3,d,r)\,g(d-1,2,d,r) \leq c\,e^{3r(d-2)}e^{2r(d-2)} \leq c\,e^{5r(d-2)},\end{aligned}$$

since $d > 3$. Thus we get

$$\sup_{r\geq 1} \mathbb{E}\left(\widetilde{F_{r,t}^{(i)}}\right)^5 \leq c \sup_{r\geq 1} \frac{e^{5r(d-2)} + e^{5r(d-2)}}{e^{5r(d-2)}} = c < \infty.$$

This proves the claim for $i = d-1$.

Now we fix $i \in \{0, 1, \dots, d-2\}$ arbitrarily and assume that $d \geq 7$. Furthermore, we fix an arbitrary partition $\sigma \in \Pi_{\geq 2}^{**}(d-i, d-i, d-i, d-i, d-i)$. We denote by $m(\sigma) \in \{2, 3, 4, 5\}$ the size of the maximal block of σ and represent σ as a diagram. The elements of this diagram are labelled $a_{p,q}$. Here, $p \in \{1, \dots, 5\}$ represents the row number and $q \in \{1, \dots, d-i\}$ stands for the column number. Without loss of generality we can and will assume that the maximal block of σ sits in the left upper corner of the diagram of σ, that is, the maximal block is of the form $\{a_{1,1}, \dots, a_{m(\sigma),1}\}$. To each row $p \in \{1, \dots, 5\}$ we associate two numbers $b(p)$ and $c(p)$ in the following way. By $b(p)$ we denote the number of elements of row p in position

$$(p, q) \in (\{1, \dots, m(\sigma)\} \times \{2, \dots, d-i\}) \cup (\{m(\sigma)+1, \dots, 5\} \times \{1, \dots, d-i\})$$

which are contained in a block of σ that has at least one element in a row below p, and we let $c(p)$ be the number of elements in position (p, q) (with the same restrictions as above) in row p not contained in any block of σ that has at least one element in a row below p, see Figure 3.4.5 for an example. Note that $b(5) = 0$, $c(5) = d-i$ if $m(\sigma) < 5$, and $c(p) = d-i-b(p)-1$ if $p \in \{1, \dots, m(\sigma)\}$. Our task is to show that the integral (in symbolic notation)

$$\begin{aligned}\mathscr{I} &:= \int \cdots \int \left(\left(f^{(i)}\right)^{\otimes 5}\right)_\sigma \\ &= \int \cdots \int f^{(i)}(H_1, G_1, \dots, G_{b(1)}, K_1, \dots, K_{c(1)}) \\ &\qquad\qquad \times f^{(i)}(\dots)\, f^{(i)}(\dots)\, f^{(i)}(\dots)\, f^{(i)}(\dots)\, \mu_{d-1}(dH_1) \dots\end{aligned}$$

is bounded by a constant multiple of $e^{5(d-2)r}$, which is the order of $(\mathbb{V}\mathrm{ar}(F_{r,t}^{(i)}))^{5/2}$. We first integrate with respect to the hyperplanes $K_1, \dots, K_{c(1)}$, which do not appear in any of the arguments of the other four functions $f^{(i)}(\dots)$. By Crofton's formula this gives $c\,\mathcal{H}^{d-1-b(1)}(B_r \cap H_1 \cap G_1 \cap \dots \cap G_{b(1)})$. Now we replace $H_1 \cap G_1 \cap \dots \cap G_{b(1)}$ by a $(d-1-b(1))$-dimensional subspace $L_{d-1-b(1)}(s_1)$ having distance $s_1 = d_h(H_1, p)$ from p. This leads to

$$\mathcal{H}^{d-1-b(1)}(B_r \cap H_1 \cap G_1 \cap \dots \cap G_{b(1)}) \leq \mathcal{H}^{d-1-b(1)}(B_r \cap L_{d-1-b(1)}(s_1)). \tag{3.36}$$

Then $G_1, \dots, G_{b(1)}$ are active integration variables for rows below the first row. Repeating the same argument for $p = 2, \dots, m(\sigma)$, we arrive at (again in symbolic notation)

$$\begin{aligned}\mathscr{I} \leq c \int \cdots \int \mathcal{H}^{d-1-b(1)}(B_r \cap L_{d-1-b(1)}(s_1)) \cdots \mathcal{H}^{d-1-b(m(\sigma))}(B_r \cap L_{d-1-b(m(\sigma))}(s_1)) \\ \times f^{(i)}(\dots) \cdots f^{(i)}(\dots)\, \mu_{d-1}(dH_1) \dots,\end{aligned}$$

where $f^{(i)}(\dots)$ appears $5 - m(\sigma)$ times. From now on we distinguish the following two cases:

(a) there is no block that contains precisely two elements from the rows below $m(\sigma)$,

(b) there exists a block that contains precisely two elements from the rows below $m(\sigma)$.

We start by treating case (a). If $m(\sigma) = 2$, then all blocks of σ have two elements. In particular, no element of row $p \geq 3$ can be in a (2-element) block with another element in a block below. Hence, we have $c(p) = d - i$ for $p \geq 3$. If $m(\sigma) = 3$, then an element of row $p = 4$ cannot be in a common block with an element of row 5 due to assumption (a). Hence $c(4) = c(5) = d - i$. This shows that $c(p) = d - i$ for $p \in \{m(\sigma)+1, \dots, 5\}$. We can thus carry out the $5 - m(\sigma)$ integrals involving the functions $f^{(i)}(\dots)$, which by Crofton's formula and Lemma 3.2.5 leads to the upper bound

$$\mathcal{H}^d(B_r)^{5-m(\sigma)} \leq c\, e^{(5-m(\sigma))(d-1)r}. \tag{3.37}$$

The only remaining integral in $\mathscr{I}$ is

$$\mathscr{J} := \int_0^r \cosh^{d-1}(s)\, \mathcal{H}^{d-1-b(1)}(B_r \cap L_{d-1-b(1)}(s)) \cdots \mathcal{H}^{d-1-b(m(\sigma))}(B_r \cap L_{d-1-b(m(\sigma))}(s))\, ds.$$

To proceed, we define for $p \in \{1, \dots, m(\sigma)\}$ the function

$$g_p(s) := e^{-r(d-2)} \cdot \begin{cases} e^{(r-s)(d-2-b(p))} & : d-1-b(p) \geq 2, \\ r - s + \log(2) & : d-1-b(p) = 1, \\ 1 & : d-1-b(p) = 0. \end{cases}$$

Then, Lemma 3.2.3, (3.7) and Lemma 3.2.4 imply that

$$\mathscr{J} \leq c\, e^{m(\sigma)(d-2)r}\, \mathscr{K} \qquad \text{with} \qquad \mathscr{K} := \int_0^r \cosh^{d-1}(s)\, g_1(s) \cdots g_{m(\sigma)}(s)\, ds. \tag{3.38}$$

We let

$$\begin{aligned} Z_{01} &:= \{p \in \{1, \dots, m(\sigma)\} : d-1-b(p) \in \{0,1\}\}, \\ Z_1 &:= \{p \in \{1, \dots, m(\sigma)\} : d-1-b(p) = 1\}. \end{aligned}$$

Then

$$\mathscr{K} \leq c\, e^{-r(d-2)|Z_{01}| - r\sum_{p=1, p\notin Z_{01}}^{m(\sigma)} b(p)} \int_0^r (r - s + \log(2))^{|Z_1|}\, e^{sE}\, ds, \tag{3.39}$$

where the exponent E is given by

$$E := (d-1) - (d-2)(m(\sigma) - |Z_{01}|) + \sum_{p=1, p\notin Z_{01}}^{m(\sigma)} b(p).$$

If $E < 0$ the integral in (3.39) is bounded by a constant times $r^{|Z_1|}$. In view of (3.37) and (3.38) we conclude that

$$\mathscr{I} \le c\, e^{(5-m(\sigma))(d-1)r}\, e^{m(\sigma)(d-2)r}\, e^{-(d-2)|Z_{01}|r - r\sum_{p=1,p\notin Z_{01}}^{m(\sigma)} b(p)}\, r^{|Z_1|}. \tag{3.40}$$

In order to bound $\mathscr{I}$ from above by a constant times $e^{5(d-2)r}$, we use the decomposition

$$e^{5(d-2)r} = e^{(5-m(\sigma))(d-1)r}\, e^{m(\sigma)(d-2)r}\, e^{-(5-m(\sigma))r}. \tag{3.41}$$

A comparison of the exponents in (3.40) and (3.41) shows that if $E < 0$, then it is sufficient to prove that

$$(d-2)|Z_{01}| + \sum_{p=1,p\notin Z_{01}}^{m(\sigma)} b(p) \begin{cases} \ge 5 - m(\sigma) & : \text{if } |Z_1| = 0, \\ > 5 - m(\sigma) & : \text{if } |Z_1| > 0. \end{cases}$$

If $|Z_{01}| > 0$, then $(d-2)|Z_{01}| \ge 4 > 5 - m(\sigma)$ for $d \ge 6$. If $|Z_{01}| = 0$, then also $|Z_1| = 0$, and in this case it is sufficient to show that $\sum_{p=1}^{m(\sigma)} b(p) \ge 5 - m(\sigma)$. To see this, note that, for any $m(\sigma) \in \{2,\dots,5\}$, under condition (a) we know that for $5 - m(\sigma)$ of the positions $(p,q) \in \{1,\dots,m(\sigma)\} \times \{2,\dots,d-i\}$ there has to be a block containing the element at (p,q) and exactly one element at $(p',q') \in \{m(\sigma)+1,\dots,5\} \times \{1,\dots,d-i\}$, since each row has to be visited by some block. But this implies the required inequality.

Next, suppose that $E = 0$. Then the integral in (3.39) is bounded by a polynomial in r of degree at most $|Z_1| + 1$ and another comparison of exponents in (3.40) and (3.41) implies that in this case we need to prove that

$$(d-2)|Z_{01}| + \sum_{p=1,p\notin Z_{01}}^{m(\sigma)} b(p) > 5 - m(\sigma). \tag{3.42}$$

Using the assumption that $E = 0$, we see that in this case

$$(d-2)|Z_{01}| + \sum_{p=1,p\notin Z_{01}}^{m(\sigma)} b(p) = m(\sigma)(d-2) - (d-1).$$

This shows that the inequality in (3.42) is equivalent to $(d-1)(m(\sigma)-1) > 5$, which is always satisfied for $d \ge 7$.

Finally, we suppose that $E > 0$ in which case a comparison of the exponents in (3.40) and (3.41) shows that we have to verify that

$$(d-2)|Z_{01}| + \sum_{p=1,p\notin Z_{01}}^{m(\sigma)} b(p) - (d-1) + (d-2)(m(\sigma) - |Z_{01}|) - \sum_{p=1,p\notin Z_{01}}^{m(\sigma)} b(p) \ge 5 - m(\sigma).$$

After simplification, this is equivalent to $(d-1)(m(\sigma)-1) \ge 5$, which holds for $d \ge 6$. This completes the argument in case (a) for $d \ge 7$.

We turn now to case (b), where we have to distinguish the sub-cases $m(\sigma) = 2$ and $m(\sigma) = 3$. We start with the case $m(\sigma) = 2$. Then, arguing as at the beginning of the proof for case (a), we have

$$\mathscr{I} \le c\, \mathscr{I}_1 \mathscr{I}_2 \mathcal{H}^d(B_r)$$

with

$$\mathscr{I}_j := \int_0^r \cosh^{d-1}(s)\, \mathcal{H}^{d-1-\bar{b}(2j-1)}(B_r \cap L_{d-1-\bar{b}(2j-1)}(s))\, \mathcal{H}^{d-1-\bar{b}(2j)}(B_r \cap L_{d-1-\bar{b}(2j)}(s))\, ds$$

for $j \in \{1,2\}$, where $\bar{b}(i) = b(i)$ for $i \in \{1,2,4\}$ and $\bar{b}(3) = b(3) - 1 \ge 0$. Moreover, without loss of generality, we can assume that $b(1) \ge 1$. Similarly to (3.38), for $j \in \{1,2\}$ we get

$$\mathscr{I}_j \le e^{2(d-2)r}\, \mathscr{K}_j \qquad \text{with} \qquad \mathscr{K}_j := \int_0^r \cosh^{d-1}(s)\, g_{2j-1}(s)\, g_{2j}(s)\, ds.$$

For $j \in \{1,2\}$ we let

$$\begin{aligned} Z_{01}^j &:= \{p \in \{2j-1, 2j\} : d-1-\bar{b}(p) \in \{0,1\}\}, \\ Z_1^j &:= \{p \in \{2j-1, 2j\} : d-1-\bar{b}(p) = 1\}. \end{aligned}$$

Then

$$\mathscr{K}_j \le c\, e^{-r(d-2)|Z_{01}^j| - r\sum_{p=2j-1, p\notin Z_{01}^j}^{2j} \bar{b}(p)} \int_0^r (r - s + \log(2))^{|Z_1^j|}\, e^{sE_j}\, ds, \tag{3.43}$$

where the exponents E_j, $j \in \{1,2\}$, are given by

$$E_j := (d-1) - (d-2)(2 - |Z_{01}^j|) + \sum_{p=2j-1, p\notin Z_{01}^j}^{2j} \bar{b}(p).$$

We will show that $\mathscr{K}_1$ is bounded by a constant multiple of e^{-r} and $\mathscr{K}_2$ by a constant. Then we can conclude that

$$\mathscr{I} \le c\, e^{(d-1)r} \mathscr{I}_1 \mathscr{I}_2 \le c\, e^{(d-1)r} e^{4(d-2)r} e^{-r} \le e^{5(d-2)r}.$$

We first consider $\mathscr{K}_1$. For $E_1 < 0$ the integral in (3.43) is bounded by a constant multiple of $r^{|Z_1^1|}$. Therefore it is sufficient to compare the exponents and to show that

$$(d-2)|Z_{01}^1| + \sum_{p=1, p\notin Z_{01}^1}^{2} b(p) \begin{cases} \ge 1 & : |Z_1^1| = 0, \\ > 1 & : |Z_1^1| > 0. \end{cases}$$

Since $b(1) \ge 1$ and $d \ge 4$, this is satisfied.

Next, suppose that $E_1 = 0$. In this case, the integral in (3.43) is bounded by a polynomial in r and we have to show the inequality

$$(d-2)|Z_{01}^1| + \sum_{p=1, p\notin Z_{01}^1}^{2} b(p) > 1. \tag{3.44}$$

Using the assumption that $E_1 = 0$, we get

$$(d-2)|Z_{01}^1| + \sum_{p=1, p\notin Z_{01}^1}^{2} b(p) = -(d-1) + 2(d-2) = d-3.$$

Hence (3.44) is true for $d \geq 5$.

Finally, we suppose that $E_1 > 0$. Then we have to show that

$$(d-2)|Z_{01}^1| + \sum_{p=1, p\notin Z_{01}^1}^{2} b(p) - (d-1) + (d-2)(2 - |Z_{01}|^j) - \sum_{p=1, p\notin Z_{01}^1}^{2} b(p) \geq 1.$$

After simplifications this is equivalent to $d \geq 4$.

Now we prove that $\mathscr{K}_2$ is bounded by a constant. For $E_2 < 0$, a comparison of the exponents in (3.43) shows that we need that

$$(d-2)|Z_{01}^2| + \sum_{p=3, p\notin Z_{01}^2}^{4} \bar{b}(p) \begin{cases} \geq 0 & : |Z_1^2| = 0, \\ > 0 & : |Z_1^2| > 0, \end{cases}$$

which is trivially satisfied.

For $E_2 = 0$ the required inequality is

$$(d-2)|Z_{01}^2| + \sum_{p=3, p\notin Z_{01}^2}^{4} \bar{b}(p) > 0,$$

which is equivalent to $-(d-1) + 2(d-2) > 0$, that is, to $d \geq 4$.

Finally, if $E_2 > 0$ then we have to verify that

$$(d-2)|Z_{01}^2| + \sum_{p=3, p\notin Z_{01}^2}^{4} \bar{b}(p) - (d-1) + (d-2)(2 - |Z_{01}|^2) - \sum_{p=3, p\notin Z_{01}^2}^{4} \bar{b}(p) \geq 0.$$

Again simplification yields that this is equivalent to $d \geq 3$.

Now we turn to the case $m(\sigma) = 3$. Then we have

$$\mathscr{I} \leq c\, \mathscr{I}_3 \mathscr{I}_4$$

with

$$\mathscr{I}_3 := \int_0^r \cosh^{d-1}(s) \prod_{i=1}^{3} \mathcal{H}^{d-1-b(i)}(B_r \cap L_{d-1-b(i)}(s))\, ds,$$
$$\mathscr{I}_4 := \int_0^r \cosh^{d-1}(s) \prod_{i=4}^{5} \mathcal{H}^{d-1-\bar{b}(i)}(B_r \cap L_{d-1-\bar{b}(i)}(s))\, ds,$$

where $0 \leq \bar{b}(4) := b(4) - 1 \leq d - i - 1 \leq d - 1$ and $\bar{b}(5) = 0$. We will prove that $\mathscr{I}_3 \leq c\, e^{3(d-2)r}$ and $\mathscr{I}_4 \leq c\, e^{2(d-2)r}$, which in turn proves that $\mathscr{I} \leq c\, e^{5(d-2)r}$.

As in the proof of case (a) (and for $m(\sigma) = 3$ there), we obtain

$$\mathscr{I}_3 \le c\,e^{3(d-2)r}\mathscr{K}_3 \qquad \text{with} \qquad \mathscr{K}_3 := \int_0^r \cosh^{d-1}(s) g_1(s) g_2(s) g_3(s)\, ds.$$

We show that $\mathscr{K}_3 \le c$. For this, we proceed as before and obtain

$$\mathscr{K}_3 \le c\,e^{-r(d-2)|Z_{01}^3| - r\sum_{p=1, p\notin Z_{01}^3}^3 b(p)} \int_0^r (r - s + \log(2))^{|Z_1^3|}\, e^{sE_3}\, ds,$$

where

$$Z_{01}^3 := \{p \in \{1, \dots, 3\} : d - 1 - b(p) \in \{0, 1\}\}, \quad Z_1^3 := \{p \in \{1, \dots, 3 : d - 1 - b(p) = 1\}$$

and

$$E_3 := (d-1) - (d-2)(3 - |Z_{01}^3|) + \sum_{p=1, p\notin Z_{01}^3}^3 b(p).$$

If $E_3 \le 0$, then

$$r^{|Z_{01}^3|} e^{-r(d-2)|Z_{01}^3|} e^{-r\sum_{p=1, p\notin Z_{01}^3}^3 b(p)} \le c$$

provided that

$$(d-2)|Z_{01}^3| + \sum_{p=1, p\notin Z_{01}^3}^3 b(p) \begin{cases} \ge 0 & : |Z_1^3| = 0, \\ > 0 & : |Z_1^3| > 0. \end{cases}$$

This is obviously true, since $|Z_{01}^3| \ge |Z_1^3|$ and $d \ge 4$. Hence, if $E_3 \le 0$, then $\mathscr{K}_3 \le c$.

If $E_3 > 0$, then $\mathscr{K}_3 \le c$ follows provided that

$$(d-2)|Z_{01}^3| + \sum_{p=1, p\notin Z_{01}^3}^3 b(p) - E_3 \ge 0.$$

The latter is equivalent to $(d-2)3 - (d-1) \ge 0$, that is, to $2d \ge 5$. Thus we have shown that $\mathscr{I}_3 \le c\,e^{3(d-2)}$. In order to show that $\mathscr{I}_4 \le c\,e^{2(d-2)}$, we distinguish several cases.

If $\bar{b}(4) < d - 3$, then

$$\begin{aligned} \mathscr{I}_4 &\le c \int_0^r e^{s(d-1)} e^{(r-s)(d-2-\bar{b}(4))} e^{(r-s)(d-2)}\, ds \\ &\le c\,e^{(2(d-2)-\bar{b}(4))r} \int_0^r e^{s(-d+3+\bar{b}(4))}\, ds \le c\,e^{2(d-2)r}. \end{aligned}$$

If $\bar{b}(4) = d - 3$, then

$$\mathscr{I}_4 \le c\,e^{(2(d-2)-d+3)r} r = c\,r e^{r(d-1)} \le c\,e^{2(d-2)r},$$

since $d - 1 < 2(d-2)$ for $d \ge 4$.

If $\bar{b}(4) = d - 2$, then

$$\mathscr{I}_4 \le c \int_0^r e^{s(d-1)} (r - s + \log(2)) e^{(r-s)(d-2)}\, ds \le c\,e^{r(d-1)}.$$

If $\bar{b}(4) = d-1$, then

$$\mathscr{I}_4 \le c \int_0^r e^{s(d-1)} e^{(r-s)(d-2)}\, ds \le c\, e^{r(d-1)}.$$

Thus in all cases we have $\mathscr{I}_4 \le c\, e^{2(d-2)r}$, which completes the proof. □

Proof of Theorem 3.1.5 (c). Let d and i be as in the statement of Theorem 3.1.5 (c), and suppose to the contrary that $\widetilde{F_{r,t}^{(i)}}$ converges in distribution, as $r \to \infty$, to a standard Gaussian random variable N. As a consequence of Lemma 3.4.2, the family of random variables $\big((\widetilde{F_{r,t}^{(i)}})^4\big)_{r\ge 1}$ is uniformly integrable, which implies that $\mathbb{E}(\widetilde{F_{r,t}^{(i)}})^4 \to \mathbb{E}N^4 = 3$, as $r \to \infty$. Thus, we would also have that

$$\mathrm{cum}_4 = \mathbb{E}\left(\widetilde{F_{r,t}^{(i)}}\right)^4 - 3 \to \mathbb{E}N^4 - 3 = 0, \tag{3.45}$$

as $r \to \infty$. On the other hand, from [109, page 112] we know that

$$\frac{M_{1,1}(f^{(i)})}{(\mathbb{V}\mathrm{ar}(F_{r,t}^{(i)}))^2} \le \mathrm{cum}_4\,.$$

In addition, we have the following lower bound for $M_{1,1}(f^{(i)})$:

$$M_{1,1}(f^{(i)}) = ct^{4(d-1-i)+1} \int_{A_h(d,d-1)} \mathcal{H}^{d-1}(\tilde{H}_1 \cap B_r)^4\ \mu_{d-1}(d\tilde{H}_1) \ge c\, g(d-1,4,d,r) \ge c\, e^{4r(d-2)},$$

since $4(d-2)-(d-1) > 0$, which follows from our assumption that $d \ge 4$, and since $i \le d-1$ and $t \ge 1$. In combination with Lemma 3.2.8 we thus find that

$$\mathrm{cum}_4 \ge \frac{M_{1,1}(f^{(i)})}{(\mathbb{V}\mathrm{ar}(F_{r,t}^{(i)}))^2} \ge \frac{c}{c^{(i)}(d)}\frac{e^{4r(d-2)}}{e^{4r(d-2)}} = c > 0,$$

which is a contradiction to (3.45). Consequently, the family of random variables $\big(\widetilde{F_{r,t}^{(i)}}\big)_{r\ge 1}$ cannot satisfy a central limit theorem as $r \to \infty$. □

Remark 3.4.1. Let $d \ge 4$ and $i = d-1$ or $d \ge 7$ and $i \in \{0,1,\dots,d-1\}$. For such d and i the proof of Theorem 3.1.5 (c) in combination with [10, Corollary 4.7.19], a corollary of the Eberlein-Šmulian theorem, shows that there exists a subsequence $\widetilde{F_{r_k,t}^{(i)}}$ such that $\widetilde{F_{r_k,t}^{(i)}}$ converges in distribution and in L^4 to some limiting random variable X, say. Especially this implies that $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$ and $\mathbb{E}X^m < \infty$ for $m \in \{3,4\}$. In particular, this rules out for X the classical α-stable distributions for any $0 < \alpha < 2$ and, since we have shown that $\mathrm{cum}_4(X) > 0$, also a Gaussian distribution. We leave the determination of the distribution of the limiting random variable X as a challenging open problem for future research.

3.4.3 The case of simultaneous growth of intensity and window: Proof of Theorem 3.1.6

According to Lemma 3.4.2 we have that, for any fixed $t \geq 1$,

$$\sup_{r\geq 1} \mathbb{E}\left(\widetilde{F_{r,t}^{(i)}}\right)^5 < \infty, \qquad \text{where} \qquad \widetilde{F_{r,t}^{(i)}} = \frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{\mathbb{V}\mathrm{ar}(F_{r,t}^{(i)})}}$$

and where d and i are as in the statement of Theorem 3.1.6. Then, taking $t = 1$, by Hölder's inequality it follows that

$$\sup_{r\geq 1} \mathbb{E}\left(\widetilde{F_{r,1}^{(i)}}\right)^4 \leq \sup_{r\geq 1}\left(\mathbb{E}\left(\widetilde{F_{r,1}^{(i)}}\right)^5\right)^{4/5} < \infty. \tag{3.46}$$

Next, we recall the definition of the integrals $M_{u,v}(h)$, $u, v \in \{1, \dots, m\}$, from (2.8) that are associated with a general Poisson U-statistic of order $m \in \mathbb{N}$ with kernel function h. In order to emphasize the role of the measure these integrals are taken with, we will write $M_{u,v}(h\,;\,\mu)$ in what follows. By definition of the integrated kernels in (2.9) we have that

$$M_{u,v}(f^{(i)}\,;\,t\mu_{d-1}) \leq t^{4(d-i-1)+1} M_{u,v}(f^{(i)}\,;\,\mu_{d-1}) \tag{3.47}$$

for any $t \geq 1$ and any fixed $r \geq 1$. In fact, $f_u^{(i)}$ and $f_v^{(i)}$ contribute twice the factor t^{d-i-u} and twice the factor t^{d-i-v} by (2.9), respectively, and the integral in (2.8) leads to an additional factor $t^{|\sigma|}$. By the choice $u = v = 1$ we maximize the resulting exponent and see that their product is bounded by $t^{4(d-i-1)+1}$. Indeed, if $u = v = 1$ we necessarily have that $|\sigma| = 1$ since σ has to be connected. On the other hand, if $u + v \geq 3$ then $|\sigma| \leq u + v$ and hence

$$\begin{aligned} 2(d-i-u) + 2(d-i-v) + |\sigma| &\leq 2(d-i-u) + 2(d-i-v) + u + v \\ &= 4(d-i-1) - (u+v) + 4 \\ &\leq 4(d-i-1) + 1. \end{aligned}$$

Now, we apply the normal approximation bound (2.10) to the Poisson U-statistic $F_{r,t}^{(i)}$. Together with (3.47) and the lower and the upper variance bound from Lemma 3.2.8 this yields

$$\begin{aligned} d\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{\mathbb{V}\mathrm{ar}\,F_{r,t}^{(i)}}}, N\right) &\leq c \sum_{u,v=1}^{d-i} \frac{\sqrt{M_{u,v}(f^{(i)};\,t\mu_{d-1})}}{\mathbb{V}\mathrm{ar}(F_{r,t}^{(i)})} \\ &\leq c \sum_{u,v=1}^{d-i} \frac{t^{2(d-i-1)+1/2}}{t^{2(d-i)-1}} \frac{\sqrt{M_{u,v}(f^{(i)}\,;\,\mu_{d-1})}}{\mathbb{V}\mathrm{ar}(F_{r,1}^{(i)})} \\ &= \frac{c}{\sqrt{t}} \sum_{u,v=1}^{d-i} \frac{\sqrt{M_{u,v}(f^{(i)}\,;\,\mu_{d-1})}}{\mathbb{V}\mathrm{ar}(F_{r,1}^{(i)})} \end{aligned}$$

for any $t \geq 1$ and $r \geq 1$. Note that the expression in the sum has now become a function of the parameter r only. We can now apply for any $u, v \in \{1, \dots, d-i\}$ the estimate

$$\frac{\sqrt{M_{u,v}(f^{(i)}\,;\,\mu_{d-1})}}{\mathbb{V}\mathrm{ar}(F_{r,1}^{(i)})} \le \sqrt{\mathbb{E}\left(\widetilde{F_{r,1}^{(i)}}\right)^4 - 3}$$

from the discussion after [109, Corollary 4.3] (see also [60, Proposition 3.8]). This leads to the bound

$$d\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{\mathbb{V}\mathrm{ar}\, F_{r,t}^{(i)}}}, N\right) \le \frac{c}{\sqrt{t}}\sqrt{\mathbb{E}\left(\widetilde{F_{r,1}^{(i)}}\right)^4 - 3} \le \frac{c}{\sqrt{t}}\sqrt{\mathbb{E}\left(\widetilde{F_{r,1}^{(i)}}\right)^4}.$$

However, in view of (3.46) the last expression is bounded by $c/\sqrt{t}$ for all $t \ge 1$ and $r \ge 1$. This completes the proof of Theorem 3.1.6. □

3.5 Proofs IV – Multivariate limit theorems

3.5.1 The case of growing intensity: Proof of Theorem 3.1.7 (a)

This is a direct consequence of [65, Theorem 5.2]. □

3.5.2 The case of growing windows: Proof of Theorem 3.1.7 (b) and (c)

The planar case $d = 2$: Proof of Theorem 3.1.7 (b)

Our goal is to use (2.11). The first term in (2.11) is bounded by a constant multiple of $r^2 e^{-r}$ by Lemma 3.2.9. To evaluate the second term we have to combine the lower variance bound from Lemma 3.2.6 with upper bounds for the terms $M_{1,1}$, $M_{1,2}$ and $M_{2,2}$. In the proof of Theorem 3.1.5 (a) we have already shown that $M_{1,1}(f^{(i)}, f^{(i)}) \le ce^r$ for $i \in \{0,1\}$ and $M_{2,2}(f^{(0)}, f^{(0)}) \le cre^r$, which implies that

$$M_{1,1}(e^{-r/2} f^{(i)}, e^{-r/2} f^{(i)}) \le c\, e^{-2r}\, e^r = c\, e^{-r},$$
$$M_{2,2}(e^{-r/2} f^{(0)}, e^{-r/2} f^{(0)}) \le c\, r\, e^{-2r}\, e^r = c\, r\, e^{-r}.$$

Finally, up to a constant factor an upper bound for $M_{1,2}(e^{-r/2} f^{(i)}, e^{-r/2} f^{(0)})$, for $i \in \{0,1\}$, is given by
$M_{1,2}(e^{-r/2} f^{(0)}, e^{-r/2} f^{(0)})$, which is equal to

$$e^{-2r} M_{1,2}(f^{(0)}) \le c\, e^{-2r}\, (e^r + 2r^2\, e^r) \le c\, r^2\, e^{-r}.$$

Thus we conclude from (2.11) that

$$d_3(\mathbf{F}_{r,t}, N_{\Sigma_2}) \le c\,(r^2\, e^{-r} + e^{-r/2} + r^{1/2}\, e^{-r/2} + r\, e^{-r/2}) \le c\, r\, e^{-r/2}.$$

Since the covariance matrix Σ_2 is invertible, $\|\Sigma_2^{-1}\|_{op}\|\Sigma_2\|_{op}^{1/2}$ and $\|\Sigma_2^{-1}\|_{op}\|^{3/2}\Sigma_2\|_{op}$ are positive and finite constants only depending on t. Together with (2.12) this also implies that

$$d_2(\mathbf{F}_{r,t}, N_{\Sigma_2}) \leq c\, r\, e^{-r/2}.$$

and completes the proof of Theorem 3.1.7 (b). □

The spatial case $d = 3$: Proof of Theorem 3.1.7 (c)

Our goal is again to use the normal approximation bound (2.11). By Lemma 3.2.10 the first term in (2.11) is bounded from above by a constant multiple of r^{-1}. Next, it remains to provide upper bounds for the terms

$$M_{u,v} \qquad \text{for} \qquad (u,v) \in \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}.$$

As in the planar case $d = 2$ all integrals which are involved have already been treated in the proof of the univariate limit theorem. Thus, using the bounds derived in the proof of Theorem 3.1.5 (b) we can complete the proof in dimension $d = 3$. □

CHAPTER 4

Splitting tessellations in hyperbolic spaces

In this chapter we investigate a splitting model in hyperbolic space. For further background we refer to Chapter 2. We will start by formally defining the model in Section 4.1. Section 4.2 and Section 4.3 contain certain general properties of this process that will be needed later on. In Section 4.4 we introduce a famous geometric concept, namely the capacity functional which is helpful for making statements about the distribution of the process. Later in Section 4.5 and 4.6 first and second order properties of functionals depending on the process are regarded. Here we are mainly interested in the k-dimensional Hausdorff measure of the k-skeleton. Finally, Section 4.7 shows that the process is mixing.

4.1 Definition of the model

In Section 1.4 we already gave an intuitive definition. In this section we define the model in a more formal way as a continuous time pure jump Markov process. This definition is similar to the one used for random fragmentation processes and branching Markov chains (see [7]). For the context of splitting tessellations, it already appears in [105]. The theory of pure jump Markov processes is introduced for example in [13, Chapter 15], [25] [54, Chapter 12], [63, p. 19, Chapter 2.5]. Recall that we denote by $\mathbb{H}_{d-1}[W]$ the set of hyperplanes, having nonempty intersection with a window W.

Definition 4.1.1. *By an isotropic splitting process* $(Y_t)_{t\geq 0}$ *in hyperbolic space inside a fixed window* W *we understand the continuous time pure jump Markov process on the space of*

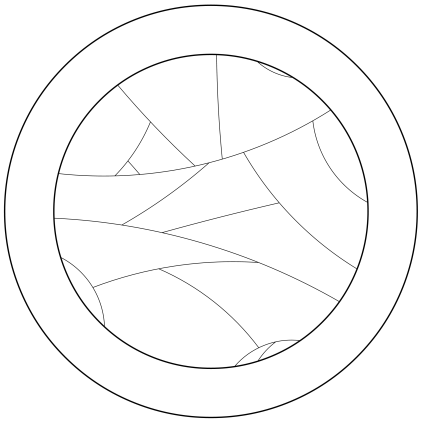

Figure 4.1.1: A realization of a splitting tessellation inside the window $W = B_2$, represented in the Conformal ball model.

tessellations with generator

$$(\mathcal{A}f)(T) := \sum_{c \in T} \int_{\mathbb{H}_{d-1}[W]} [f(\oslash(c, H, T)) - f(T)]\ \mu_{d-1}(dH)$$

for $T \in \mathbb{T}^d$ *and* $f : \mathbb{T}^d \to \mathbb{R}$ *being a bounded and measurable function. The random tessellation* Y_t *will be called splitting tessellation at time* t*.*

Remark 4.1.1. One can show the existence of such a process, as done in [13, Chapter 15, Section 6], and its uniqueness [13, Proposition 15.38]. Also variations are possible, such as starting with a fixed tessellation $\bar{Y} \in \mathbb{T}^d$ at time $t = 0$ instead of the whole window W. Also the splitting hyperplanes can be chosen with a different directional distribution which would renounce the isotropy property.

4.2 Auxiliary results

The following lemmas will be used in Section 4.3 in order to show a martingale property for certain stochastic processes depending on the splitting process. The first Lemma 4.2.1 gives bounds for the distribution of the number of cells after a certain time $t \geq 0$. The second Lemma 4.2.2 is more technical and deals with the underlying σ-fields.

Lemma 4.2.1. *The number of cells in a splitting tessellation inside a window* W *at time* $t \geq 0$ *is stochastically dominated by a random variable having a geometric distribution with parameter* $\exp(-t\,\mu_{d-1}(\mathbb{H}_{d-1}\langle W \rangle))$*.*

Proof. We can interpret the number of cells $|Y_t|$ as a continuous time pure jump Markov process in $\mathbb{N}$ with intensity rate

$$\sum_{c \in Y_t} \mu_{d-1}(\mathbb{H}_{d-1}\langle c \rangle)$$

and jump height 1. We further introduce M_t as a continuous time pure jump Markov process in $\mathbb{N}$ with intensity rate

$$\sum_{c\in Y_t}(\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)-\mu_{d-1}(\mathbb{H}_{d-1}\langle c\rangle))+\sum_{i=1}^{M_t}\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)\geq 0,$$

and jump height 1. We further let the jump times of M_t be independent from the ones in $|Y_t|$ and set $M_0=0$. Further both processes are constructed on the same probability space. The sum of both processes $|Y_t|+M_t$ is a continuous time pure jump Markov process in $\mathbb{N}$ with intensity rate

$$\begin{aligned}\sum_{c\in Y_t}\mu_{d-1}(\mathbb{H}_{d-1}\langle c\rangle)+\sum_{c\in Y_t}(\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)-\mu_{d-1}(\mathbb{H}_{d-1}\langle c\rangle))+\sum_{i=1}^{M_t}\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)\\ =\sum_{i=1}^{|Y_t|+M_t}\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle),\end{aligned}$$

i.e. a Yule-Furry process with birth rate $\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)$. This shows that $|Y_t|$ is stochastically dominated by a fitting Yule-Furry process. Such a process is geometrical distributed with parameter $\exp(-t\,\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle))$. To see this, let $N(t):=|Y_t|+M_t$ be the random number of individuals at time t in this process. Further define $p_n(t):=\mathbb{P}(N(t)=n)$ as the probability to have n individuals at time t. By the definition of the process, a given individual has probability

$$\begin{aligned}\int_0^h e^{-\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)(h-s)}\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)e^{-\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)s}\,ds\\ =h\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)e^{-\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)h}\end{aligned}$$

of splitting into exactly two within the time interval $(t,t+h)$ and probability $e^{-\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)h}$ of not splitting within this time interval. Adding that the splitting happens independently for all individuals, yields

$$\begin{aligned}p_n(t+h)&=\sum_{k=0}^{n}p_k(t)\,\mathbb{P}(N(t+h)=n\mid N(t)=k)\\ &=p_n(t)\,e^{-n\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)h}+p_{n-1}(t)\,(n-1)h\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)e^{-n\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)h}\\ &\quad+\sum_{k=0}^{n-2}p_k(t)\,\mathbb{P}(N(t+h)=n\mid N(t)=k).\end{aligned}$$

One can easily show that $\mathbb{P}(N(t+h)=n\mid N(t)=k)=o(h)$ holds for every $n\in\mathbb{N}$ and $k\leq n-2$.

Thus we get

$$\begin{aligned}
p_n'(t) &= \lim_{h\searrow 0} \frac{p_n(t+h)-p_n(t)}{h} \\
&= \lim_{h\searrow 0} h^{-1}\, p_n(t) \left(-1 + \sum_{j=0}^{\infty} \frac{(-n\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)h)^j}{j!} \right) \\
&\qquad + p_{n-1}(t)(n-1)\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle) e^{-n\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)h} \\
&\qquad + \sum_{k=0}^{n-2} p_k(t)\, h^{-1}\mathbb{P}(N(t+h)=n \mid N(t)=k) \\
&= \lim_{h\searrow 0} h^{-1}\, p_n(t) \sum_{j=1}^{\infty} \frac{(-n\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)h)^j}{j!} \\
&\qquad + p_{n-1}(t)(n-1)\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle) e^{-n\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)h} \\
&= -p_n(t)\, n\, \mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle) + p_{n-1}(n-1)\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle).
\end{aligned}$$

Adding the initial condition $p_1(0) = 1$ gives the differential equation

$$\begin{aligned}
&p_n'(t) = -p_n(t)\, n\, \mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle) + p_{n-1}(n-1)\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle) \\
&p_1(0) = 1.
\end{aligned}$$

The solution of this is given by

$$p_n(t) = e^{-t\,\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}(1 - e^{-t\,\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)(n-1)})$$

which in turn is the geometric distribution with parameter $\exp(-t\,\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle))$. □

The space $\mathbb{T}^d$ can be equipped with a σ-field. Lemma 4.2.2 will show that two different approaches lead to the same σ-field. In order to formulate this lemma, let $i_s : \mathbf{N}_s(E) \to \mathcal{F}_{lf}(E),\ \eta \mapsto \text{supp}(\eta)$ be the map that assigns to each simple counting measure on E its support, where E is an arbitrary locally compact space with a countable base. The space $\mathcal{F}_{lf}(E)$ can be equipped with the subspace topology of the Fell topology on the whole space of closed subsets of E, denoted by $\mathcal{T}_{lf}$. On the other hand, $\mathbf{N}_s(E)$ will be equipped with the vague topology, i.e. the coarsest topology such that the mapping $\eta \mapsto \int_E g(x)\ \eta(dx)$ is continuous for every continuous, non-negative function $g : E \to [0,\infty)$. This topology will be denoted by $\mathcal{T}_{vg}$. The following lemma is an analogue to [47, Lemma 2.4]. It is used in the proof of Lemma 4.3.2.

Lemma 4.2.2. *Let $\mathcal{B}_{lf}$ and $\mathcal{B}_{vg}$ be the σ-fields generated by $\mathcal{T}_{lf}$ and $\mathcal{T}_{vg}$, respectively. Then $\mathcal{B}_{vg} = i_s^{-1}(\mathcal{B}_{lf})$ and $i_s(\mathcal{B}_{vg}) = \mathcal{B}_{lf}$. In particular, the σ-fields induced on $\mathbb{T}^d$ by the vague topology and by the Fell topology coincide, and $\mathbb{T}^d$ is a Borel space.*

Proof. Let $E := \mathcal{F}_{conv}(\mathbb{H}^d)$ be the space of nonempty closed convex subsets of $\mathbb{H}^d$. In a first step we show that $i_s :\ \mathbf{N}_s(E) \to \mathcal{F}_{lf}(E)$ is continuous. To do so let $\eta,\ \eta_n,\ n \in \mathbb{N}$ be elements of $\mathbf{N}_s(E)$ such that $\eta_n \to \eta$ as $n \to \infty$. By [103, Theorem 12.2.2] we know that $i_s(\eta_h) \to i_s(\eta)$

is equivalent to (i) and (ii) to hold with

(i) If $A \in \mathcal{O}(\mathbf{N}_s(E))$ and $A \cap i_s(\eta) \neq \varnothing,$ then $A \cap i_s(\eta_j) \neq \varnothing$ for almost all $j \in \mathbb{N}$.

(ii) If $B \in \mathcal{C}(\mathbf{N}_s(E))$ and $B \cap i_s(\eta) = \varnothing,$ then $B \cap i_s(\eta_j) = \varnothing$ for almost all $j \in \mathbb{N}$.

To show (i) we let A be an open subset of $\mathbf{N}_s(E)$ such that $A \cap i_s(\eta) \neq \varnothing$ and assume that $A \cap i_s(\eta_j) = \varnothing$ for infinitely many $j \in \mathbb{N}$. By the Portmanteau-Theorem (see [28, p. 385]) the convergence of counting measures $\eta_n,\ n \in \mathbb{N}$ implies

$$\liminf_{n\to\infty} \eta_n(A) \geq \eta(A) \geq 1.$$

By our assumption we get on the other hand $\liminf_{n\to\infty} \eta_n(A) = 0$ which is a contradiction. The same way (ii) can be shown.

Using the continuity of i_s we get $i_s^{-1}(\mathcal{T}_{lf}) \subset \mathcal{T}_{vg}$. This directly yields the inclusion property for the induced σ-fields $i_s^{-1}(\mathcal{B}_{lf}) \subset \mathcal{B}_{vg}$ and hence also for the subspace σ-fields on $\mathbb{T}^d$. It remains to show that the other inclusion $\mathcal{B}_{vg} \subset i_s^{-1}(\mathcal{B}_{lf})$ also holds (since this again transfers the result to the intersection with $\mathbb{T}^d$). The desired property can be shown using [103, Lemma 3.1.4].

Applying a hyperbolic version of [103, Lemma 10.1.2] gives the second claim. □

4.3 Martingales

Similar to the Euclidean and spherical case we will rely on the martingale property of certain random processes depending on the splitting process. For completeness reasons and since analogues such as [106, Proposition 2] contain some inaccuracies, we will present some detailed proofs in this work. The inaccuracies mentioned above are fixed in the spherical work [47]. The proofs presented below base on this work. The following underlying lemma is taken from [25, Proposition 14.13]. For the definition of a Markov process and its domain, we refer to this work as well.

Lemma 4.3.1. *Let E be a Borel space and let $(X_t)_{t\geq 0}$ be a Markov process with values in E and with generator $\mathcal{L}$ whose domain is $D(\mathcal{L})$. Further, let $f \in D(\mathcal{L})$. Then the random process*

$$f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)\, ds, \qquad t \geq 0,$$

is a martingale with respect to the filtration induced by $(X_t)_{t\geq 0}$. If $(X_t)_{t\geq 0}$ is a jump process with bounded intensity function, then $\mathcal{F}_b(E) = D(\mathcal{L})$.

By Lemma 4.2.2 we know that the space of tessellations $\mathbb{T}^d$ is indeed a Borel space, as a Borel subset of the Polish space $\mathcal{F}(\mathcal{F}'(\mathbb{H}^d))$. Therefore we will choose E to be the space of tessellations of $\mathbb{H}^d$ in a first application of Lemma 4.3.1. Further the generator defined in 4.1.1 will play the role of the generator $\mathcal{L}$ and the splitting process $(Y_t)_{t\geq 0}$ will be the Markov process $(X_t)_{t\geq 0}$. Since the intensity function of our jump process, denoted by $\lambda(Y_t)$, is not bounded (the sum over all hitting values of all cells increases over time) and since the desired functionals are not necessarily bounded, we will need to work with some sort of localization argument. The idea is to introduce a second Markov process that realizes the same values as the original process, but stops at a certain time. This process will fulfill the requirements in Lemma 4.3.1. Now letting this stopping time run to infinity will first show a local martingale property and then the proper martingale property for the original process.

Proposition 4.3.2. *Let $\phi : \mathbb{P}_W^d \to \mathbb{R}$ be bounded and measurable, and define*

$$\Sigma_\phi(T) := \sum_{c \in T} \phi(c) = \int_{\mathbb{P}_W^d} \phi(P)\ \mu_T(dP), \qquad T \in \mathbb{T}^d.$$

Then the stochastic process

$$M_t(\phi) := \Sigma_\phi(Y_t) - \Sigma_\phi(Y_0) - \int_0^t (\mathcal{A}\Sigma_\phi)(Y_s)\ ds, \qquad t \geq 0,$$

is a martingale with respect to $\mathcal{Y}$, where $\mathcal{Y} := (\mathcal{Y}_t)_{t\geq 0}$ is the filtration corresponding to the family of σ-fields $\mathcal{Y}_t = \sigma(Y_s : 0 \leq s \leq r)$.

Remark 4.3.1. Let $k \in \{0, \dots, d\}$. A function that assigns the total k-dimensional Hausdorff measures of all k-faces of a cell is not necessarily bounded. But if we restrict ourself to a bounded observation window they are.

Proof. In a first step we show that $\Sigma_\phi : \mathbb{T}^d \to \mathbb{R}$ is measurable. To show this we let $B \in \mathcal{B}(\mathbb{R})$, then

$$\Sigma_\phi^{-1}(B) = \{T \in \mathbb{T}^d :\ \Sigma_\phi(T) \in B\} = \mathbb{T}^d \cap i_h\left(\{\eta \in \mathcal{N}_s(\mathcal{K}_h^d) : \int_{\mathcal{K}_h^d} \phi(K)\ \eta(dK) \in B\}\right).$$

This is contained in $\mathbb{T}^d \cap \mathcal{B}_{lf}$ since the map $\eta \mapsto \int_{\mathcal{K}_h^d} \phi(K)\ \eta(dK)$ is by definition of ϕ (non-negative, continuous) and the definition of the vague topology a continuous function. Therefore the set $\{\eta \in \mathcal{N}_s(\mathcal{K}_h^d) : \int_{\mathcal{K}_h^d} \phi(K)\ \eta(dK) \in B\}$ is contained in $\mathcal{B}_{vg}$. By Lemma 4.2.2 the image $i_h(A)$ lies in $\mathcal{B}_{lf}$, whenever $A \in \mathcal{B}_{vg}$. Since the function ϕ is assumed to be bounded, one can find a real value $\alpha := \sup\{|\phi(c)| : c \in \mathbb{P}_W^d\} < \infty$ depending on the functional ϕ and the observation window W. Using this value, we introduce a truncated version of the functional Σ_ϕ. Namely for every $N \in \mathbb{N}$ we define for $\wedge$ denoting the minimum

$$\Sigma_\phi^N(T) := (\Sigma_\phi(T) \wedge (N\alpha)) \vee (-(N\alpha)), \qquad T \in \mathbb{T}^d, \tag{4.1}$$

which is measurable since Σ_ϕ is measurable and bounded via $|\Sigma_\phi^N(T)| \leq N\alpha$. Besides the truncated functional we introduce for each $N \in \mathbb{N}$ a truncated Markov jump process $(Y_t)_{t\geq 0}$. Here the truncation is with respect to the transition kernel $q^N(T, \cdot) := (\lambda(T) \wedge N)\pi(T, \cdot)$ and the generator, which takes the form $\mathcal{A}^N := (\lambda \wedge N)(\pi - I) = (\lambda \wedge N)\lambda^{-1}\mathcal{A}$. This way we bound the intensity function by N, since it takes the value $\lambda \wedge N$ by the above definitions. This makes sure that $D(\mathcal{A}^N) = \mathcal{F}_b(\mathbb{T}^d)$, which is not necessarily the case for processes with unbounded intensity function. Using the results in [54, Chapter 12], we can construct $(Y_t)_{t\geq 0}$ and its truncated version $(Y_t^N)_{t\geq 0}$ on the same probability space. Denoting $J_k,\ k\in\mathbb{N}$ as the time of the k-th jump of the process, we know that for all times $s<J_N$ the transition kernel and the generator of the original and the truncated process coincide. Since they are constructed on the same probability space this means that $Y_s^N = Y_s$ holds almost surely for all $s<J_N$. Applying Lemma 4.3.1 on the truncated process and the truncated functional Σ_ϕ^N yields that

$$M_t^N(\phi) := \Sigma_\phi^N(Y_t^N) - \Sigma_\phi^N(Y_0^N) - \int_0^\infty (\mathcal{A}^N\Sigma_\phi^N)(Y_s^N)\,ds, \qquad t\geq 0,$$

is a $\mathcal{Y}^N$-martingale, where $\mathcal{Y}^N$ is the filtration generated by $(Y_t^N)_{t\geq 0}$. Here Lemma 4.3.1 is applicable since $(Y_t^N)_{t\geq 0}$ has a bounded rate function and since Σ_ϕ^N is bounded.

In a next step we are aiming to transfer the martingale property to our original stochastic process $M_t(\phi)$. The idea is to take the limit of N to infinity and since this way $M_t(\phi)$ and $M_t^N(\phi)$ will coincide more and more the martingale property will transfer locally to the original process. In a next step we will show that the local martingale property can indeed be extended to a proper one.

In order to derive the local martingale property, we define for every $N\in\mathbb{N}$ the (almost surely finite) random variable $\tau_N := \inf\{t\geq 0:\ \lambda(Y_t)\geq N\}$ as the time of the $(N-1)$-th jump. We will show that τ_N is a stopping time with respect to both filtrations $\mathcal{Y}$ and $\mathcal{Y}^N$. To do so we consider the event

$$\{\tau_N > t\} = \{|Y_t|\leq N-1\} = \{|Y_t^N|\leq N-1\}\in\mathcal{Y}_t\cap\mathcal{Y}_t^N.$$

Since the optional stopping theorem (see for example [56]) states that stopping at a certain stopping-time does not change the martingale property, we know that $\widetilde{M}_t^N(\phi) := M_{\tau_N\wedge t}^N(\phi),\ t\geq 0$ defines a martingale as well. Next we will show that this stopped and truncated process is equal to the stopped original process $(M_t^{\tau_N}(\phi))_{t\geq 0}$, which is defined via $M_t^{\tau_N}(\phi) := M_{\tau_N\wedge t}(\phi)$ for $t\geq 0$. To do so we start by considering $\Sigma_\phi(Y_{\tau_N\wedge t})$ for $t\geq 0$. Since $\tau_N\wedge t = J_{N-1}\wedge t\leq J_{N-1}<J_N$ holds almost surely we derive

$$|\Sigma_\phi(Y_{\tau_N\wedge t})| = \sum_{c\in Y_{\tau_N\wedge t}}\phi(c)\leq |Y_{\tau_N\wedge t}|\alpha\leq N\alpha$$

and therefore by the definition of Σ_ϕ^N in (4.1) the original and the truncated functional are identical in the sense that

$$\Sigma_\phi^N(Y_{\tau_N\wedge t}^N) = \Sigma_\phi^N(Y_{\tau_N\wedge t}) = \Sigma_\phi(Y_{\tau_N\wedge t}). \tag{4.2}$$

Since for $s<\tau_N$ there have been at most $N-2$ jumps in the process, even $|Y_s|\leq N-1$ holds. This gives a similar result to (4.2) for Σ_ϕ and Σ_ϕ^N evaluated at a split process, namely since

$$|\Sigma_\phi(\oslash(c,H,Y_s^N))| = |\Sigma_\phi(\oslash(c,H,Y_s))|\leq |Y_s+1|\alpha\leq N\alpha$$

holds for all $H \in A_h(d, d-1)$ the equation

$$\Sigma_\phi^N(\oslash(c, H, Y_s^N)) = \Sigma_\phi^N(\oslash(c, H, Y_s)) = \Sigma_\phi(\oslash(c, H, Y_s))$$

follows with $H \in A_h(d, d-1)$, $s < \tau_N$. Since $\Sigma_\phi^N(Y_0^N) = \Sigma_\phi(Y_0)$ obviously holds true, we remain to show the equivalence of the integral. In order to do so we consider the integrand. For $s < \tau_N$ our definition of $\mathcal{A}^N$ yields

$$\begin{aligned}
(\mathcal{A}^N \Sigma_\phi^N)(Y_s^N) &= \frac{\lambda(Y_s^N)}{\lambda(Y_s)} \int_{\mathcal{N}_h(\mathbb{T}^d)} \int_{A_h(d,d-1)} [\Sigma_\phi^N(\oslash(c, H, Y_s^N)) - \Sigma_\phi^N(Y_s^N)] \; \mu_{d-1}(dH) \; \mu_{Y_s^N}(dc) \\
&= \int_{\mathcal{N}_h(\mathbb{T}^d)} \int_{A_h(d,d-1)} [\Sigma_\phi(\oslash(c, H, Y_s)) - \Sigma_\phi(Y_s)] \; \mu_{d-1}(dH) \; \mu_{Y_s}(dc) \\
&= (\mathcal{A}\Sigma_\phi)(Y_s).
\end{aligned}$$

Thus all summands of $\widetilde{M}_t^N(\phi)$ and $M_t^{\tau_N}(\phi)$ are equal and we can conclude

$$\widetilde{M}_t^N(\phi) = M_t^{\tau_N}(\phi), \qquad t \geq 0.$$

Therefore it remains to show that $\tau_N \to \infty$ diverges almost surely for $N \to \infty$. This follows from Lemma 4.2.1. Therefore $(M_t^{\tau_N}(\phi))_{t \geq 0}$ is a local $\mathcal{Y}$-martingale.

In a last step we show that the local $\mathcal{Y}$ martingale property of $(M_t(\phi))_{t \geq 0}$ extends to a proper martingale property. By [55, Definition 4.8, Problem 5.19(i)] we have to show that $M_t^{\tau_N}$ converges in $\mathcal{L}^1$ to M_t as N tends to infinity for every $t \geq 0$. By the dominated convergence theorem it is sufficient to show that

$$\mathbb{E} \sup_{N \in \mathbb{N}} |M_t^{\tau_N}| < \infty$$

for all $t \geq 0$. First we show

$$\begin{aligned}
\left| \int_0^t (\mathcal{A}\Sigma_\phi)(Y_{\tau_N \wedge s}) \; ds \right| &\leq \int_0^t \sum_{c \in Y_{\tau_N \wedge s}} \int_{\mathbb{H}_{d-1}[c]} \left| \Sigma_\phi(\oslash(c, H, Y_{\tau_N \wedge s})) - \Sigma_\phi(Y_{\tau_N \wedge s}) \right| \; \mu_{d-1}(dH) \; ds \\
&\leq \int_0^t \sum_{c \in Y_s} \mu_{d-1}(\mathbb{H}_{d-1}\langle W \rangle) 3\alpha \; ds \\
&\leq 3\alpha \, t \, \mu_{d-1}(\mathbb{H}_{d-1}\langle W \rangle) \, |Y_t|.
\end{aligned}$$

Using this and the results from Lemma 4.2.1 we conclude

$$\begin{aligned}
\mathbb{E} \sup_{N \in \mathbb{N}} |M_t^{\tau_N}| &\leq \mathbb{E} \sup_{N \in \mathbb{N}} \left| \Sigma_\phi(Y_{\tau_N \wedge t}) - \Sigma_\phi(Y_0) - \int_0^t (\mathcal{A}\Sigma_\phi)(Y_{\tau_N \wedge s}) \; ds \right| \\
&\leq \mathbb{E} \left[\sup_{N \in \mathbb{N}} \left| \Sigma_\phi(Y_{\tau_N \wedge t}) \right| + \left| \Sigma_\phi(Y_0) \right| + \left| \int_0^t (\mathcal{A}\Sigma_\phi)(Y_{\tau_N \wedge s}) \; ds \right| \right] \\
&\leq \mathbb{E} \left[\sup_{N \in \mathbb{N}} |Y_{\tau_N \wedge t}| \, \alpha + \alpha + t|Y_t| \mu_{d-1}(\mathbb{H}_{d-1}\langle W \rangle) 3\alpha \right] \\
&\leq \mathbb{E} \left[|Y_t| \, \alpha + \alpha + t|Y_t| \mu_{d-1}(\mathbb{H}_{d-1}\langle W \rangle) 3\alpha \right] < \infty.
\end{aligned}$$

The finiteness holds since, as mentioned above, the number of cells is stochastically bounded by a geometrically distributed random variable. Therefore the expected number of cells at time t is finite. □

Again similar to the spherical setting one can show the following result, which will be used to investigate higher moments.

Proposition 4.3.3. *Let $\phi : \mathbb{P}^d_W \to \mathbb{R}$ be bounded and measurable, let $b \in C^1([0,\infty))$, and define*

$$\Psi_\phi(T,t) := (\Sigma_\phi(T) - b(t))^2, \qquad T \in \mathbb{T}^d,\ t \geq 0.$$

Then the stochastic process

$$N_t(\Psi_\phi) := \Psi_\phi(Y_t,t) - \Psi_\phi(Y_0,0) - \int_0^t (\mathcal{A}\,\Psi_\phi(\cdot,s))(Y_s) + \frac{\partial \Psi_\phi}{\partial s}(\cdot,s)(Y_s)\ ds,$$

is a martingale with respect to $\mathcal{Y}$.

Proof. The proof is similar to the one stated in [47] □

4.4 The capacity functional

In this section we will investigate some properties of the capacity functional of a splitting tessellation and calculate concrete values in some special situations. In a next step we will use the theory for capacity functions in order to expand the process to the whole space $\mathbb{H}^d$.

4.4.1 Capacity functional for splitting tessellations

It is often useful to the union of all cell boundaries instead of the tessellation itself. Therefore we define

$$Z_t := Z_{Y_t} := \bigcup_{c \in Y_t} \partial c, \qquad t \geq 0$$

as this union. This set will be called the skeleton of Y_t. Recall that $\mathcal{F}^d_{h,skel}$ is the set of skeletons of a tessellation. Our first aim is to show that Z_t is a random closed set as defined in [103, chapter 2], namely to show that Z_t is a measurable map from the underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into the measurable space $(\mathcal{F}(\mathbb{H}^d), \mathcal{B}(\mathcal{F}(\mathbb{H}^d)))$ of closed sets. Additionally, we will show that for $W = B_r$ the distribution of Z_t is isotropic for all $t \geq 0$, i.e. invariant under all isometries of hyperbolic space fixing the origin. Clearly, it is not invariant under all isometries of hyperbolic space, since Z_t is restricted to B_r. In Theorem 4.4.4 we will show that invariance with respect to isometries holds locally, that means that $Z_t \cap C \overset{d}{=} Z_t \cap \varphi(C)$, as long as $C, \varphi(C) \subseteq W$ are compact sets and $\varphi \in I(\mathbb{H}^d)$ is an isometry.

In order to apply several results from [103] one has to show that $\mathbb{H}^d$ is locally compact and has a countable base. Both properties are easy to show. The set

$$\mathcal{B} := \{B_h(\exp_p(x), q_0)^\circ :\ x = q_1 b_1 + \dots + q_d b_d,\ q_1, \dots, q_d \in \mathbb{Q},\ q_o \in \mathbb{Q}^+\}$$

gives a countable base. Here $\{b_1,\dots,b_d\}$ is an arbitrary basis of $T_p\mathbb{H}^d$. To show that $\mathbb{H}^d$ is locally compact we show that it is Hausdorff and that every point has a compact neighborhood. The Hausdorff property comes with the metric and closed balls of radius 1 fulfill the second requirement.

Lemma 4.4.1. *For every $t\geq 0$, $W=B_r$, $r>0$ the set Z_t is an isotropic random closed set in $\mathbb{H}^d$.*

Proof. We show the measurability of Z_t by decomposing Z_t into three maps, which we will show to be measurable. The map $Y_t:(\Omega,\mathcal{A},\mathbb{P})\to(\mathcal{N}(\mathcal{K}_h^d),\mathcal{B}_{vg})$ is by definition measurable. To show that the map $(\mathcal{N}(\mathcal{K}_h^d),\mathcal{B}_{vg})\to(\mathcal{N}(\mathcal{F}(\mathbb{H}^d)),\mathcal{B}_{vg}^*)$, $\sum_c\delta_c\mapsto\sum_c\delta_{\partial(c)}$, which assigns to each simple counting measure on $\mathcal{K}_h^d$ the corresponding counting measure of its boundaries, is measurable, we apply [103, Theorem 12.2.6]. It states that the map $(\mathcal{K}_h^d,\mathcal{B}(\mathcal{K}_h^d))\to(\mathcal{F}(\mathbb{H}),\mathcal{B}(\mathcal{F}(\mathbb{H}))), c\mapsto\partial(c)$ is lower semicontinuous and therefore measurable. Finally, the union map $(\mathcal{N}(\mathcal{F}(\mathbb{H}^d)),\mathcal{B}_{vg}^*)\to(\mathcal{F}(\mathbb{H}),\mathcal{B}(\mathcal{F}(\mathbb{H}))),\sum_F\delta_F\mapsto\cup_F F$ is measurable. The proof presented in [103, Theorem 3.6.2] transfers to the hyperbolic case. Composing these measurable maps yields the measurability of Z_t.

The isotropy of Z_t is based on the isotropy of Y_t, whereas Y_t is isotropic due to the isotropy of the measure μ_{d-1}. To show that Y_t is isotropic we aim to show that $Y_t\overset{d}{=}\varphi Y_t$ for all $\varphi\in I(\mathbb{H}^d,p)$. Here $I(\mathbb{H}^d,p)$ is the group of isometries fixing p, namely $I(\mathbb{H}^d,p):=\{\varphi\in I(\mathbb{H}^d):\ \varphi\circ p=p\}$. By [13, Proposition 15.38] it is sufficient to show that the generators of Y_t and φY_t coincide. By $\varphi T:=\{\varphi c: c\in T\}$ we refer to the rotated tessellation $T\in\mathbb{T}^d$. Now let $f:\mathbb{T}^d\to\mathbb{R}$ be a bounded and measurable map. The rotational invariance of μ_{d-1} gives

$$\begin{aligned}(\mathcal{A}f)(\varphi T)&=\sum_{c\in\varphi T}\int_{\mathbb{H}_{d-1}[c]}[f(\oslash(c,H,\varphi T))-f(\varphi T)]\ \mu_{d-1}(dH)\\&=\sum_{\varphi^{-1}c\in T}\int_{\mathbb{H}_{d-1}[c]}[f(\oslash(c,H,\varphi T))-f(\varphi T)]\ \mu_{d-1}(dH)\\&=\sum_{\tilde{c}\in\varphi T}\int_{\mathbb{H}_{d-1}[\tilde{c}]}[f(\oslash(\varphi\tilde{c},\varphi H,\varphi T))-f(\varphi T)]\ \mu_{d-1}(dH)=(\mathcal{A}(f\circ\varphi))(T).\end{aligned}$$

Using the limit representation of the infinitesimal operator $\mathcal{A}$ gives

$$\begin{aligned}(\mathcal{A}^{\varphi}f)(T)&=\lim_{t\searrow 0}t^{-1}(\mathbb{E}[f(\varphi Y_t)|\varphi Y_0=T]-f(T))\\&=\lim_{t\searrow 0}t^{-1}(\mathbb{E}[(f\circ\varphi)(Y_t)|Y_0=\varphi^{-1}(T)]-(f\circ\varphi)(\varphi^{-1}T))\\&=(\mathcal{A}(f\circ\varphi))(\varphi^{-1}(T)).\end{aligned}$$

Combining both equalities gives $(\mathcal{A}f)(T)=(\mathcal{A}^{\varphi}f)(T)$ and by [13, Proposition 15.38] the equality in distribution of the rotated tessellation. □

Next we will consider the capacity functional $T_t(C):\mathcal{C}(W)\to[0,1]$ of the random closed set Z_t. It is defined by

$$T_t(C):=T_{Y_t(W)}(C):=\mathbb{P}(Z_t\cap C\neq\varnothing)$$

for $C \in \mathcal{C}(W)$, where $\mathcal{C}(W)$ is the set of all compact subsets of W. With a slight abuse of notation we will also write $T_{Z_{Y_t(W)}}$ for $T_{Y_t(W)}$. For simplicity reasons we will compute the value of $U_t(C)$ which is defined as

$$U_t(C) := U_{Y_t(W)}(C) := 1 - T_t(C) = \mathbb{P}(Z_t \cap C = \varnothing).$$

We want to point out that $U_t(C)$ depends W, even though we omit the dependence on W in its notation. Using [103, Theorem 2.1.3] we will be able to make some invariance statements about the process. The next theorem gives an explicit form of the U-functional evaluated for connected sets which lie in the interior of $W = B_r$. An Euclidean analogue is proven in [81].

Theorem 4.4.2. *Let $W = B_r$, $r > 0$ be a fixed window. Let $C \in \mathcal{C}(W)$ be connected and such that $C \cap \partial W = \varnothing$ holds. Then*

$$U_t(C) = \exp(-t\,\mu_{d-1}(\mathbb{H}_{d-1}\langle C\rangle)), \qquad t \geq 0.$$

Proof. Consider the map $\phi : \mathbb{P}_r^d \to \mathbb{R},\ c \mapsto \mathbb{1}\{C \subseteq c\}$. Since ϕ is measurable and bounded, the process

$$\sum_{c \in Y_t} \phi(c) - \phi(B_r) - \int_0^t \sum_{c \in Y_s} \int_{\mathbb{H}_{d-1}[c]} [\phi(c \cap H^+) + \phi(c \cap H^-) - \phi(c)]\ \mu_{d-1}(dH)\ ds \tag{4.3}$$

is by Proposition 4.3.2 a martingale with respect to the filtration $\mathcal{Y}$ induced by $(Y_t)_{t \geq 0}$. First define

$$\xi_t := \sum_{c \in Y_t} \phi(c) = \sum_{c \in Y_t} \mathbb{1}\{C \subseteq c\}.$$

We aim to show that $\xi_t = \mathbb{1}\{C \cap Z_t = \varnothing\}$ holds $\mathbb{P}$-almost surely. We show this by distinguishing three different cases:

Case 1: Let $\xi_t \geq 2$. This implies that there exist two cells $c_1, c_2 \in Y_t$ such that $C \subseteq c_1$, $C \subseteq c_2$ and therefore $C \subseteq c_1 \cap c_2$ holds. Since Y_t is a tessellation this implies $C \subseteq \partial(c_1) \subseteq Z_t$. Therefore using Lemma 4.4.1 one gets for $z_0 \in C$ and $\varphi \in I(\mathbb{H}^d, p)$

$$\mathbb{P}(z_0 \in \varphi^{-1} Z_t) = \mathbb{P}(z_0 \in Z_t) \geq \mathbb{P}(C \subseteq Z_t) \geq \mathbb{P}(\xi_t \geq 2).$$

If $z_0 = p$ we immediately get $\mathbb{P}(z_0 \in Z_t) = 0$. For $z_0 \neq p$ and $d_h(z_0, p) = s > 0$ we integrate over all hyperbolic rotations to get

$$\begin{aligned}\mathbb{P}(\xi_t \geq 2) &\leq \int_{I(\mathbb{H}^d,p)} \mathbb{E}[\mathbb{1}\{\varphi z_0 \in Z_t\}]\ \nu(d\varphi)\\ &= \mathbb{E}\left[\int_{I(\mathbb{H}^d,p)} \mathbb{1}\{\varphi z_0 \in Z_t\}\ \nu(d\varphi)\right]\\ &= \frac{\mathbb{E}[\mathcal{H}^{d-1}(\partial B_{\|z_0\|_h} \cap Z_t)]}{\mathcal{H}^{d-1}(\partial B_{\|z_0\|_h})} = 0.\end{aligned}$$

Here the last equality is due to Theorem 4.5.2.

Case 2: Let $\xi_t = 0$. We assume $Z_t \cap C = \varnothing$. Therefore we get $C \subseteq \cup_{c \in Y_t} \operatorname{int}(c)$. Since $\xi_t = 0$

there are two cells $c_1, c_2 \in Y_t$ such that $C \cap c_1 \neq \varnothing$ and $C \cap c_2 \neq \varnothing$. Thus C is not connected. This is a contradiction and we conclude that $\mathbb{1}\{Z_t \cap C = \varnothing\} = 0$ holds in this case.

Case 3: Let $\xi_t = 1$. Therefore there exists exactly one cell $c \in Y_t$ such that $C \subseteq c$. If $C \nsubseteq \operatorname{int} c$, then there exists a time $s \le t$ such that at time s a hyperplane H appears such that $C \cap H \neq \varnothing$ and $C \subseteq H^+$ or $C \subseteq H^-$ holds for the first time. A hyperplane having these properties will be called a supporting hyperplane of C. Using the representation of the measure μ_{d-1} stated in (2.3) we derive

$$\begin{aligned}\mu_{d-1}&\{H \in A_h(d, d-1) : C \cap H \neq \varnothing,\ C \subseteq H^+ \text{ or } C \subseteq H^-\}\\ &= \int_{G_h(d,1)} \int_L \cosh^{d-1}(d_h(x,p))\, \mathbb{1}\{H(L,x) \text{ supports } C\}\ \mathcal{H}^1(dx)\ \nu_1(dL)\\ &= \int_{G_h(d,1)} 0\ \nu_1(dL)\\ &= 0.\end{aligned}$$

Therefore the probability of such an event is 0 as well. Thus having $\xi_t = 1$ implies that C is $\mathbb{P}$-almost surely contained in the interior of a single cell and therefore $Z_t((T)) \cap C = \varnothing$ holds $\mathbb{P}$-almost surely as well.

Together these three steps prove that $\xi_t = \mathbb{1}\{C \cap Z_t = \varnothing\}$ holds $\mathbb{P}$-almost surely. We now progress with the proof of the desired equality. Taking the expectation in equation (4.3) leads to

$$U_t(C) = 1 + \int_0^t \mathbb{E} \sum_{c \in Y_s} \int_{\mathbb{H}_{d-1}[c]} [\mathbb{1}(C \subseteq c \cap H^+) + \mathbb{1}(C \subseteq c \cap H^-) - \mathbb{1}(C \subseteq c)]\, \mu_{d-1}(dH)\ ds$$

since $\phi(B_r) = 1$ and since the expression in (4.3) is a martingale. For $s \in [0,t]$, multiplication of the sum in the expression above with ξ_s does not change the result. To show this let $\xi_s = 0$. In this case there is no $c \in Y_s$ such that any of the indicator functions take value 1, namely

$$\mathbb{1}(C \subseteq c \cap H^+) + \mathbb{1}(C \subseteq c \cap H^-) - \mathbb{1}(C \subseteq c) = 0, \qquad c \in Y_s.$$

Since we showed $\xi_s \in \{0,1\}$ a.s. there is nothing more to show. Now the sum

$$\sum_{c \in Y_s} \int_{\mathbb{H}_{d-1}[c]} [\mathbb{1}(C \subseteq c \cap H^+) + \mathbb{1}(C \subseteq c \cap H^-) - \mathbb{1}(C \subseteq c)]\, \mu_{d-1}(dH)$$

has almost surely the form

$$\begin{aligned}\xi_s &\sum_{c \in Y_s} \int_{\mathbb{H}_{d-1}[c]} [\mathbb{1}(C \subseteq c \cap H^+) + \mathbb{1}(C \subseteq c \cap H^-) - \mathbb{1}(C \subseteq c)]\, \mu_{d-1}(dH)\\ &= \xi_s \int_{\mathbb{H}_{d-1}[c_0]} [\mathbb{1}(C \subseteq c_0 \cap H^+) + \mathbb{1}(C \subseteq c_0 \cap H^-) - \mathbb{1}(C \subseteq c_0)]\, \mu_{d-1}(dH)\\ &= -\xi_s \int_{\mathbb{H}_{d-1}[c_0]} \mathbb{1}\{C \cap H \neq \varnothing\}\ \mu_{d-1}(dH)\\ &= -\mu_{d-1}(\mathbb{H}_{d-1}\langle C\rangle)\mathbb{1}\{Z_s \cap C = \varnothing\}.\end{aligned}$$

The first equality is due to the result above that there is almost surely at most one cell containing C. For the second equality consider the case $H \cap C = \varnothing$. This gives $C \subseteq \operatorname{int}(H^+)$ or $C \subseteq \operatorname{int}(H^-)$ and therefore the integrand takes the value 0 in this case. Now let $H \cap C \neq \varnothing$. We can assume for $C \subseteq c_0$ to hold, since otherwise the whole expression takes the value zero anyway. Therefore it remains to show that both indicator functions $\mathbb{1}(C \subseteq c_0 \cap H^+)$, $\mathbb{1}(C \subseteq c_0 \cap H^-)$ are μ_{d-1}-almost surely 0. This is the case since in order for C to be contained in one of the half-spaces H^+ or H^- and C having nonempty intersection with H, would mean for H to touch C. This is a negligible event under μ_{d-1}. Taking this into account we get

$$\begin{aligned}
&\int_0^t \mathbb{E} \sum_{c \in Y_s} \int_{\mathbb{H}_{d-1}[c]} \left[\mathbb{1}(C \subseteq c \cap H^+) + \mathbb{1}(C \subseteq c \cap H^-) - \mathbb{1}(C \subseteq c)\right] \mu_{d-1}(dH) ds \\
&\quad = -\mu_{d-1}(\mathbb{H}_{d-1}\langle C \rangle) \int_0^t \mathbb{P}(Z_s \cap C = \varnothing)\ ds \\
&\quad = -\mu_{d-1}(\mathbb{H}_{d-1}\langle C \rangle) \int_0^t U_s(C)\ ds
\end{aligned}$$

and therefore

$$U_t(C) = 1 - \mu_{d-1}(\mathbb{H}_{d-1}\langle C \rangle) \int_0^t U_s(C)\ ds.$$

Adding the condition $U_0(C) = 1$ gives the unique solution

$$U_t(C) = \exp(-t\, \mu_{d-1}(\mathbb{H}_{d-1}\langle C \rangle)).$$

□

Remark 4.4.1. The expression on the right hand side in the theorem above does not depend on W anymore. This will be needed in order to expand the process to $\mathbb{H}^d$. It should also be pointed out, that $U_t(C) = 0$ holds trivially for every $C \in \mathcal{C}(W)$ with $C \cap \partial W \neq \varnothing$. This is due to the fact that ∂W is almost surely contained in Z_t for every $t \geq 0$.

Corollary 4.4.3. *For the special case of $C = \overline{x,y}$, $x, y \in B(0,r)^\circ$ being a hyperbolic segment, the capacity functional takes the form*

$$U_t(C) = \exp\left(-t\, \frac{\omega_{d+1}\, \omega_1}{\omega_d\, \omega_2}\, d_h(x,y)\right).$$

Proof. The proof follows easily by Theorem 4.4.2 and the Crofton-type formula (2.4) which shows that

$$\mu_{d-1}(\mathbb{H}_{d-1}\langle \overline{x,y} \rangle) = \frac{\omega_{d+1}\, \omega_1}{\omega_d\, \omega_2}\, d_h(x,y).$$

□

Now we will give a result for the capacity function of sets, containing more than one connected component. Since its proof is similar to the proof given on the sphere, we will omit it in this work and refer to [47, Theorem 3.5]. It is worthy to mention that in order to show the formula for the capacity functional for more than one connected component, a little more general version of Theorem 4.4.2 is needed, as done in [47]. There the result (for spherical splitting tessellation) is shown for an arbitrary initial tessellation. It turns out that the formula does not depend on the initial tessellation. In Euclidean space an analogue result is stated in [81, Lemma 4]. It is worthy to mention that the arguments used there are of a different form than in [47].

In order to state the result, we need some more notations. First let $\bar{C}$ be the hyperbolic closed convex hull of C, namely

$$\bar{C} := \mathrm{cl}\Big(\bigcap_{A \in \mathcal{K}_h^d:\ C \subset A} A \Big).$$

Also let $\mathbb{H}_{d-1}\langle B_1 | B_2 \rangle$ be the set of hyperplanes that (properly) separate two sets B_1, B_2. More formally it is given by

$$\mathbb{H}_{d-1}\langle B_1 | B_2 \rangle := \{ H \in A_h(d, d-1) :\ \bar{B}_1 \cap H = \bar{B}_2 \cap H = \varnothing,\ \overline{B_1 \cup B_2} \cap H \neq \varnothing \}.$$

Last we consider closed subset $C = C_1 \cup \ldots \cup C_n \in \mathcal{C}(\mathbb{H}^d),\ n \in \mathbb{N}$ with $C_1, \ldots, C_n \in \mathcal{C}(\mathbb{H}^d)$ being closed and connected subsets of $\mathbb{H}^d$. Now $\Pi(C)$ is the set of proper partitions of C. This means that $\{P, \hat{P}\} \in \Pi(C)$ if and only if there exists a set $J \subseteq \{1, \ldots, n\},\ 0 < |J| < n$ such that

$$P = \bigcup_{j \in J} C_j, \qquad \hat{P} = \bigcup_{j \in [n] \setminus J} C_j.$$

Theorem 4.4.4. *Let $W = B_r,\ r > 0$ be a fixed window. Let $C \in \mathcal{C}(W)$ be such that there exists an integer $n \in \mathbb{N}$ and pairwise disjoint, nonempty connected subsets $C_1, \ldots, C_n \in \mathcal{C}(W)$ with $C = C_1 \cup \ldots \cup C_n$ and $C \cap \partial W = \varnothing$. Then the value $U_t(C)$ is given by*

$$U_t(C) = e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle \bar{C} \rangle)} + \sum_{\{P, \hat{P} \in \Pi(C)\}} \mu_{d-1}(\mathbb{H}_{d-1}\langle P | \hat{P} \rangle) \times \int_0^t e^{-s\mu_{d-1}(\mathbb{H}_{d-1}\langle \bar{C} \rangle)} U_{t-s}(P) U_{t-s}(\hat{P})\ ds$$

for any $t \geq 0$.

Proof. See [47, Theorem 3.5]. Recall that the cited proof relies on a little more general version of Theorem 4.4.2. □

Remark 4.4.2. It should be pointed out that Theorem 4.4.4 gives the value of the capacity functional on the class of sets that can be written as a finite union of pairwise disjoint, nonempty connected closed subsets of B_r. One can show that this class is indeed rich enough to determine the value of $T(C)$ for all $C \in \mathcal{C}(W)$. Indeed the class of sets which can be written as a finite union of pairwise disjoint, nonempty, connected closed subsets is a separating class (for a definition see [77, Definition 1.1.48]). To see this, we see that the class of sets which can be written as a finite union of pairwise disjoint, nonempty, connected *open* subsets forms a basis of the standard topology on $\mathbb{H}^d$. Now applying [77, Proposition 1.1.53] yields the separating property and the unique determination of $T_t(C)$ for all $C \in \mathcal{C}(B_r)$. Having this we can conclude that the capacity functional is invariant under isometries that keep the evaluated set in the interior of W.

4.4.2 Expansion of the process

Using the results developed in Section 4.4 enables us to expand the process to the whole space $\mathbb{H}^d$. We do this by using the fact that the distribution of a random closed set Z is determined by its capacity functional $T_Z : \mathcal{C}(\mathbb{H}^d) \to [0,1]$ (see [103, Theorem 2.1.3]). In a second step we show that the map that assigns to every skeleton the corresponding tessellation is measurable. This shows that the distribution of a random tessellation is determined by the capacity functional of its skeleton.

Theorem 4.4.5. *For every $t \geq 0$ there exists a random tessellation $\mathbf{Y}_t$ of $\mathbb{H}^d$ with*

$$Z_{\mathbf{Y}_t} \cap K \overset{d}{=} Z_{Y_t(B_n)} \cap K$$

for every $K \in \mathcal{C}(\mathbb{H}^d)$ and $n \in \mathbb{H}^d$ such that $K \subseteq B_{n-1/2}$ holds.

Proof. Let $t \geq 0$ and $B_j,\ j = 1, 2, \ldots$ be a sequence of hyperbolic balls with centre in p and radius $j \in \mathbb{N}$. We define the random closed set $\tilde{Z}_i,\ i \in \mathbb{N}$ by

$$\tilde{Z}_i := Z_{Y_t(B_i)} \cap B_{i-1/2}.$$

For every $i \in \mathbb{N}$ and $j > i$ we have

$$\tilde{Z}_j \cap B_{i-1/2} \overset{d}{=} \tilde{Z}_i.$$

To show this let $C \in \mathcal{C}(B_{i-1/2})$, then

$$\begin{aligned} T_{(\tilde{Z}_j \cap B_{i-1/2})}(C) = T_{(Z_{Y_t(B_j)} \cap B_{j-1/2} \cap B_{i-1/2})}(C) = T_{Z_{Y_t(B_j)}}(C) &= T_{Y_t(B_j)}(C) \\ &\overset{(*)}{=} T_{Y_t(B_i)}(C) \\ &= T_{Z_{Y_t(B_i)}}(C) \\ &= T_{Z_{Y_t(B_i)} \cap B_{i-1/2}}(C) \\ &= T_{\tilde{Z}_i}(C) \end{aligned}$$

holds. Here we used simple consequence of the definition of the capacity functional. The equality labeled by $(*)$ follows from Theorem 4.4.4 since there it was shown that the capacity functional is independent of the choice of the underlying window W. Applying [103, Theorem 2.3.1] gives the existence of a random closed set $\mathbf{Y}_t$ in $\mathbb{H}^d$ such that

$$Z_{\mathbf{Y}_t} \cap B_{j-1/2} \overset{d}{=} Z_{Y_t(B_j)} \cap B_{j-1/2}, \qquad j \in \mathbb{N}.$$

Now take an arbitrary compact set $K \subseteq \mathbb{H}^d$. For every $j \in \mathbb{N}$ with $K \subseteq B_{j-1/2}$ we know

$$\begin{aligned} Z_{\mathbf{Y}_t} \cap K = (Z_{\mathbf{Y}_t} \cap K) \cap B_{j-1/2} = (Z_{\mathbf{Y}_t} \cap B_{j-1/2}) \cap K \overset{d}{=} (Z_{Y_t(B_j)} \cap B_{j-1/2}) \cap K \\ = Z_{Y_t(B_j)} \cap K. \end{aligned}$$

□

Having the existence of such a tessellation $\mathbf{Y}_t$ one can show that it is invariant.

Theorem 4.4.6. *The tessellation $\mathbf{Y}_t$ is invariant with respect to $I(\mathbb{H}^d)$ for all $t \geq 0$.*

Proof. For every $C \in \mathcal{C}(\mathbb{H}^d)$ and $\varphi \in I(\mathbb{H}^d)$ there exists an integer $j = j(C,\varphi) \in \mathbb{N}$ such that $C \cup \varphi \circ C \subseteq B_{j-1/2}$. Now using the results from Theorem 4.4.4 and Theorem 4.4.5 we get

$$\begin{aligned} T_{\mathbf{Y}_t}(\varphi \circ C) = T_{Z_{\mathbf{Y}_t}}(\varphi \circ C) = T_{Z_{\mathbf{Y}_t} \cap B_{j-1/2}}(\varphi \circ C) &= T_{Z_{Y(B_j)} \cap B_{j-1/2}}(\varphi \circ C) \\ &= T_{Z_{Y(B_j)} \cap B_{j-1/2}}(C) \\ &= T_{Z_{\mathbf{Y}_t} \cap B_{j-1/2}}(C) \\ &= T_{Z_{\mathbf{Y}_t}}(C) \\ &= T_{\mathbf{Y}_t}(C). \end{aligned}$$

□

Remark: The famous and name giving scaling property, which is derived for Euclidean space in [81, Lemma 5], does not hold in the hyperbolic case.

Lemma 4.4.7. *The map $h : \mathcal{F}^d_{h,skel} \to \mathbb{T}^d$ that assigns to every $(d-1)$-skeleton the corresponding tessellation is measurable.*

Proof. We decompose the map h. Recall that $\mathcal{F}^d_{h,skel}$ is the set of skeletons of tessellations in $\mathbb{H}^d$. Now let $y \in \mathbb{H}^d$ be an arbitrary point. We show that the map

$$\varphi_y : \mathcal{F}^d_{h,skel} \to \mathcal{F}'_{conv}(\mathbb{H}^d), \ Z \mapsto \begin{cases} \mathrm{cl}\,(\mathrm{ccomp}\,(y, Z^c)) & : \text{if } y \notin Z, \\ \mathbb{H}^d & : \text{otherwise.} \end{cases}$$

is upper semicontinuous. By [103, Theorem 12.2.5] it is sufficient to show that $\limsup(\varphi_y(Z_i)) \subseteq \varphi_y(Z)$ for all $Z, Z_1, \ldots \in \mathcal{F}^d_{h,skel}$ such that $Z_i \to Z$. We let $x \in \limsup(\varphi_y(Z_i))$, then there exists a subsequence $(i_k)_{k\in\mathbb{N}}$ and $x_{i_k} \in \varphi_y(Z_{i_k})$ such that $x_{i_k} \to x$ holds for $k \to \infty$. We assume that $x \notin \varphi_y(Z)$ holds. If y lies in Z, then our assumption would be contradicted since $y \in Z$ implies $\varphi_y(Z) = \mathbb{H}^d$. Since Z is the skeleton of a tessellation, we know that the cells are convex. Therefore by our assumption $x \notin \varphi_y(Z)$ we know that there exists a $z \in (x,y)$ with $z \in Z$ since otherwise x would lie in the same connected component as y. Since $z \in Z$ and $Z_i \to Z$ hold, there exists a converging sequence $z_i \to z$ with $z_i \in Z_i$, $i \in \mathbb{N}$. Further since y is contained in the interior of $\varphi_y(Z)$, we get the existence of an open neighbourhood $U = U(y)$ of y such that $U \subseteq \mathrm{int}\,(\varphi_y(Z))$ holds. By the convergence of Z_i we know that for almost all

$i \in \mathbb{N}$ this neighbourhood U is contained in $\operatorname{int}(\varphi_y(Z_i))$ as well. Further by $x_i \to x$ and $z_i \to z$, we know that almost all lines $(l(x_i, z_i))_{i\in\mathbb{N}}$ through x_i and z_i have nonempty intersection with U. Therefore almost all x_i are not contained in $\varphi_y(Z_i)$. Therefore our assumption is wrong and we know $x \in \varphi(Z)$. This in turn shows the upper semicontinuity of φ_y.

In a next step we show that the map

$$\psi : \mathcal{F}(\mathbb{H}^d) \times \mathbb{H}^d \to \mathbb{R},\ (F, x) \mapsto \mathbb{1}\{x \notin F\}$$

is lower semicontinuous. Again we apply [103, Theorem 12.2.5]. For $(F_i, x_i) \to (F, x)$ we need to show $\liminf_{i\to\infty} \mathbb{1}\{x_i \notin F_i\} \geq \mathbb{1}\{x \notin F\}$. If $\liminf_{i\to\infty} \mathbb{1}\{x_i \notin F_i\} = 1$ holds, there is nothing to show. Therefore assume $\liminf_{i\to\infty} \mathbb{1}\{x_i \notin F_i\} = 0$. This implies the existence of a subsequence $(i_k)_{k\in\mathbb{N}}$ with $\mathbb{1}\{x_{i_k} \notin F_{i_k}\} = 0$. Therefore $x_{i_k} \in F_{i_k}$ holds and [103, Theorem 12.2.2] tells us that therefore $x \in F$ also holds true which in turn gives $\mathbb{1}\{x \notin F\} = 0$.

Let $\mathbb{Q}_p^d := \{q_1 u_1 + \ldots + q_d u_d :\ u_1, \ldots, u_d \in \mathbb{Q}\}$, where $u_1, \ldots, u_d$ is an arbitrary basis of $T_p\mathbb{H}^d$. We now get that

$$g : \mathcal{F}^d_{h,skel} \to N(\mathcal{F}'_{conv}(\mathbb{H}^d)),\ Z \mapsto \sum_{y=\exp_p(q):\ q\in\mathbb{Q}_p^d} \mathbb{1}\{y \notin Z\}\, \delta_{\varphi_y(Z)}.$$

is measurable as the composition of measurable maps. In a last step we argue that the map $i : N(\mathcal{F}'_{conv}(\mathbb{H}^d)) \mapsto \mathcal{F}(\mathcal{F}'_{conv}(\mathbb{H}^d)),\ \eta \mapsto \operatorname{supp}(\eta)$ is measurable by [103, Lemma 3.1.4] and so is therefore the map

$$h : \mathcal{F}^d_{h,skel} \to \mathbb{T}^d,\ Z \mapsto \operatorname{supp}(g(Z)).$$

In order to show that h is actually the map, assigning to each $(d-1)$-skeleton its corresponding tessellation we let $c \in T$ be a cell of a tessellation T. Further ∂c is contained in Z_T and since c has interior points, there exists a $q \in \mathbb{Q}_p^d$ with $\exp(q) \in \operatorname{int}(c)$. Therefore we know that c is contained in the support of $g(Z_T)$. Now assume that $g(Z_T)$ contains a set $\tilde{c}$, which is not contained in T. Therefore there exists a point $x = \exp_p(q)$ with $q \in \mathbb{Q}_p^d$ which is in the interior of at least two cells of T. This is a contradiction to the definition of a tessellation. □

4.5 Expectations

In this section we consider several expected values. In Section 4.5.1 the expected i-dimensional Hausdorff measure of the i-skeleton in the interior of the observation window and including its boundary is considered for $i \in \{0, \ldots, d-1\}$. In Section 4.5.2 we investigate the expected volume of the Crofton cell. For dimension $d = 2$ also the expected volume of the typical cell can be calculated. For this special case and a spherical intersection window we will also compare the limit behaviour of the typical and the Crofton cell for growing radius r.

4.5.1 Expected i-dimensional Hausdorff measure of the i-skeleton

This section contains results for the expected i-dimensional Hausdorff measure of the i-skeleton of all cells of a splitting tessellation. For this purpose, we first recall the notation

$$\Sigma_f(Y_t) = \sum_{c \in Y_t} f(c)$$

for a measurable function $f : \mathbb{P}_W^d \to [0,\infty)$. With a slight abuse of notation, we also write $\Sigma_f(t) := \Sigma_f(Y_t)$. Now we make a special choice for f, namely 1/2-times the i-dimensional Hausdorff measure of all i-faces. Recall that for a polyhedron $P \in \mathbb{PD}^d$ we denote by $\mathcal{F}_i(P)$ the set of all i-faces of P. We enlarge this definition for semi-polytopes. First we know by the construction of a cell c of a splitting tessellation in W that it is the intersection of finitely many half-spaces with W, i.e. the intersection of a polyhedral set with W and hence a semi-polytope. We define the set $\mathcal{F}_i(c)$ of all i-faces of c to be

$$\mathcal{F}_i(c) := \{F \cap W : \ F \in \mathcal{F}_i(P(c))\},$$

where $P(c)$ is the polyhedral set inducing c. Now define

$$f_i(c) := \sum_{F \in \mathcal{F}_i(c)} \frac{1}{2} \mathcal{H}^i(F \setminus \partial W), \qquad i = 0, \dots, d$$

as the sum over all i-faces $F \in \mathcal{F}_i(c)$ of c, where the i-dimensional Hausdorff measure of $F \setminus \partial W$ is summed up. As special cases one has $f_d(c) = \mathcal{H}^d(c \setminus \partial W)/2 = \mathcal{H}^d(c)/2$, $f_{d-1}(c) = \mathcal{H}^{d-1}(\partial c \setminus \partial W)/2$ and $f_0(c)$ is the number of vertices of c in the interior of W divided by two. The additional factor is used in order to actually get the total measure of the skeleton. Since each i-dimensional face is contained in exactly two cells, the value $\mathbb{E}[\Sigma_{f_i}(t)]$ is the expected Hausdorff measure of the i-skeleton for $i = 0, \dots, d-1$. Also one has to keep in mind that $\mathbb{E}[\Sigma_{f_d}(t)] = \frac{1}{2}\mathcal{H}^d(W)$ holds.

Instead of asking for the expected i-dimensional Hausdorff measure of the process restricted to the interior of the window W, one can include the boundary of W as well. To do so, we define $\tilde{f}_i(c) := f_i(c) + \hat{f}_i(c)$ for $i = 0, \dots, d-1$ with

$$\hat{f}_i(c) = \sum_{F \in \mathcal{F}_{i+1}(c)} \frac{1}{2} \mathcal{H}^i(F \cap \partial W).$$

This functional allows us to calculate the total Hausdorff measure of the i-skeleton on the boundary which is induced by all $(i+1)$-faces of the process. As above, one has to keep in mind that $\mathbb{E}[\Sigma_{\hat{f}_{d-1}}(t)] = \frac{1}{2}\mathcal{H}^{d-1}(\partial W)$ holds. The measurability of f_i and $\hat{f}_i$ can be shown with the help of [117, Corollary 2.1.4]. For the two functionals f_i and $\hat{f}_i$ one gets the following result.

Lemma 4.5.1. *Let $W \in \mathcal{K}_h^d$ and $t \geq 0$. Then*

$$\mathbb{E}[\Sigma_{f_i}(t)] = \mathbb{E}[\Sigma_{f_i}(Y_t)] = \frac{2\,\omega_{d+1}\,\omega_{i+1}}{\omega_d\,\omega_{i+2}} \int_0^t \mathbb{E}[\Sigma_{f_{i+1}}(Y_s)]\, ds, \qquad i = 0, \dots, d-1$$

and

$$\mathbb{E}[\Sigma_{\hat{f}_i}(t)] = \mathbb{E}[\Sigma_{\hat{f}_i}(Y_t)] = \frac{2\,\omega_{d+1}\,\omega_{i+1}}{\omega_d\,\omega_{i+2}} \int_0^t \mathbb{E}[\Sigma_{\hat{f}_{i+1}}(Y_s)]\;ds, \qquad i = 0,\dots,d-2$$

hold.

Proof. We let g_i be either f_i or $\hat{f}_i$ for $i = 0,\dots,d-2$ and f_i for $i = d-1$. Now we take the expectation of the martingale in Proposition 4.3.2 with $\phi(c) = g_i(c)$ to derive

$$\begin{aligned}\mathbb{E}[\Sigma_{g_i}(Y_t)] &= \mathbb{E}\Sigma_{g_i}(Y_0) + \mathbb{E}\int_0^t (\mathcal{A}\Sigma_{g_i})(Y_s)\;ds\\ &= \int_0^t \mathbb{E}\sum_{c\in Y_s}\int_{\mathbb{H}_{d-1}\langle c\rangle} [\Sigma_{g_i}(\oslash(c,H,Y_s)) - \Sigma_{g_i}(Y_s)]\;\mu_{d-1}(dH)\;ds\\ &= \int_0^t \mathbb{E}\sum_{c\in Y_s}\int_{\mathbb{H}_{d-1}\langle c\rangle} 2g_i(c\cap H)\;\mu_{d-1}(dH)\;ds. \end{aligned} \tag{4.4}$$

To show the last equality, we fix a cell c and a hyperplane $H \in \mathbb{H}_{d-1}\langle c\rangle$. Then we get μ_{d-1} almost surely

$$\begin{aligned}&2(f_i(c\cap H^-) + f_i(c\cap H^+) - f_i(c))\\ &\quad= \sum_{F\in\mathcal{F}_i(c\cap H^-)} \mathcal{H}^i(F\setminus\partial W) + \sum_{F\in\mathcal{F}_i(c\cap H^+)} \mathcal{H}^i(F\setminus\partial W) - \sum_{F\in\mathcal{F}_i(c)} \mathcal{H}^i(F\setminus\partial W)\\ &\quad= \sum_{F\in\mathcal{F}_i(c\cap H^-),\;F\subseteq H} \mathcal{H}^i(F\setminus\partial W) + \sum_{F\in\mathcal{F}_i(c\cap H^-),\;F\notin H} \mathcal{H}^i(F\setminus\partial W)\\ &\quad+ \sum_{F\in\mathcal{F}_i(c\cap H^+),\;F\subseteq H} \mathcal{H}^i(F\setminus\partial W) + \sum_{F\in\mathcal{F}_i(c\cap H^+),\;F\notin H} \mathcal{H}^i(F\setminus\partial W)\\ &\quad- \sum_{F\in\mathcal{F}_i(c)} \mathcal{H}^i((F\cap H^-)\setminus\partial W) - \sum_{F\in\mathcal{F}_i(c)} \mathcal{H}^i((F\cap H^+)\setminus\partial W)\\ &\quad= \sum_{F\in\mathcal{F}_i(c\cap H^-),\;F\subseteq H} \mathcal{H}^i(F\setminus\partial W) + \sum_{F\in\mathcal{F}_i(c\cap H^+),\;F\subseteq H} \mathcal{H}^i(F\setminus\partial W)\\ &\quad= 2\sum_{F\in\mathcal{F}_i(c\cap H),\;F\subseteq H} \mathcal{H}^i(F\setminus\partial W)\\ &\quad= 4f_i(c\cap H).\end{aligned}$$

The equality $\hat{f}_i(c\cap H^-) + \hat{f}_i(c\cap H^+) - \hat{f}_i(c) = 2\hat{f}_i(c\cap H)$ can be shown for μ_{d-1} almost all hyperplanes the same way by

$$
\begin{aligned}
&2(\hat{f}_i(c\cap H^-)+\hat{f}_i(c\cap H^+)-\hat{f}_i(c))\\
&\quad=\sum_{F\in\mathcal{F}_{i+1}(c\cap H^-)}\mathcal{H}^i(F\cap\partial W)+\sum_{F\in\mathcal{F}_{i+1}(c\cap H^+)}\mathcal{H}^i(F\cap\partial W)-\sum_{F\in\mathcal{F}_{i+1}(c)}\mathcal{H}^i(F\cap\partial W)\\
&\quad=\sum_{F\in\mathcal{F}_{i+1}(c\cap H^-),\ F\subseteq H}\mathcal{H}^i(F\cap\partial W)+\sum_{F\in\mathcal{F}_{i+1}(c\cap H^-),\ F\nsubseteq H}\mathcal{H}^i(F\cap\partial W)\\
&\quad+\sum_{F\in\mathcal{F}_{i+1}(c\cap H^+),\ F\subseteq H}\mathcal{H}^i(F\cap\partial W)+\sum_{F\in\mathcal{F}_{i+1}(c\cap H^+),\ F\nsubseteq H}\mathcal{H}^i(F\cap\partial W)\\
&\quad-\sum_{F\in\mathcal{F}_{i+1}(c)}\mathcal{H}^i((F\cap H^-)\cap\partial W)-\sum_{F\in\mathcal{F}_{i+1}(c)}\mathcal{H}^i((F\cap H^+)\cap\partial W)\\
&\quad=\sum_{F\in\mathcal{F}_{i+1}(c\cap H^-),\ F\subseteq H}\mathcal{H}^i(F\cap\partial W)+\sum_{F\in\mathcal{F}_{i+1}(c\cap H^+),\ F\subseteq H}\mathcal{H}^i(F\cap\partial W)\\
&\quad=2\sum_{F\in\mathcal{F}_{i+1}(c\cap H),\ F\subseteq H}\mathcal{H}^i(F\cap\partial W)\\
&\quad=4\hat{f}_i(c\cap H).
\end{aligned}
$$

Using this gives

$$\Sigma_{g_i}(\oslash(c,H,Y_s))-\Sigma_{g_i}(Y_s)=g_i(c\cap H^-)+g_i(c\cap H^+)-g_i(c)=2g_i(c\cap H).$$

Continuing the calculations in (4.4) with $g_i=f_i$ yields

$$
\begin{aligned}
&\int_0^t\mathbb{E}\sum_{c\in Y_s}\int_{\mathbb{H}_{d-1}\langle c\rangle}2f_i(c\cap H)\ \mu_{d-1}(dH)\ ds\\
&\quad=\int_0^t\mathbb{E}\sum_{c\in Y_s}\int_{\mathbb{H}_{d-1}\langle c\rangle}\sum_{F\in\mathcal{F}_i(c\cap H)}\mathcal{H}^i(F\setminus\partial W)\ \mu_{d-1}(dH)\ ds\\
&\quad\overset{(*)}{=}\int_0^t\mathbb{E}\sum_{c\in Y_s}\sum_{G\in\mathcal{F}_{i+1}(c)}\int_{\mathbb{H}_{d-1}\langle c\rangle}\mathcal{H}^i((G\cap H)\setminus\partial W)\ \mu_{d-1}(dH)\ ds\\
&\quad\overset{(**)}{=}\int_0^t\mathbb{E}\sum_{c\in Y_s}\sum_{G\in\mathcal{F}_{i+1}(c)}\frac{\omega_{d+1}\,\omega_{i+1}}{\omega_d\,\omega_{i+2}}\mathcal{H}^{i+1}(G\setminus\partial W)\ ds\\
&\quad=\frac{2\,\omega_{d+1}\,\omega_{i+1}}{\omega_d\,\omega_{i+2}}\int_0^t\mathbb{E}\sum_{c\in Y_s}f_{i+1}(c)\ ds\\
&\quad=\frac{2\,\omega_{d+1}\,\omega_{i+1}}{\omega_d\,\omega_{i+2}}\int_0^t\mathbb{E}\Sigma_{f_{i+1}}(Y_s)\ ds.
\end{aligned}
$$

Equality $(*)$ holds since $F\in\mathcal{F}_i(c\cap H)$ is either the intersection of an $(i+1)$-dimensional or an i-dimensional face of c with H. For the latter we get for $\tilde{G}\in\mathcal{F}_i(c)$

$$\int_{\mathbb{H}_{d-1}\langle c\rangle}\mathcal{H}^i(\tilde{G}\cap H\setminus\partial W)\ \mu_{d-1}(dH)=\frac{\omega_{d+1}\omega_{i+1}}{\omega_d\omega_{i+2}}\mathcal{H}^{i+1}(\tilde{G})=0,\qquad i=0,\dots,d-1.$$

The second marked identity $(**)$ is due to the Crofton-type formula (2.3). With minor changes, the same calculations and arguments can be applied for $g = \hat{f}_i$

$$\begin{aligned}
&\int_0^t \mathbb{E} \sum_{c\in Y_s} \int_{\mathbb{H}_{d-1}\langle c\rangle} 2\hat{f}_i(c\cap H)\ \mu_{d-1}(dH)\ ds \\
&\quad = \int_0^t \mathbb{E} \sum_{c\in Y_s} \int_{\mathbb{H}_{d-1}\langle c\rangle} \sum_{F\in\mathcal{F}_{i+1}(c\cap H)} \mathcal{H}^i(F\cap \partial W)\ \mu_{d-1}(dH)\ ds \\
&\quad = \int_0^t \mathbb{E} \sum_{c\in Y_s} \sum_{G\in\mathcal{F}_{i+2}(c)} \int_{\mathbb{H}_{d-1}\langle c\rangle} \mathcal{H}^i((G\cap H)\cap \partial W)\ \mu_{d-1}(dH)\ ds \\
&\quad = \int_0^t \mathbb{E} \sum_{c\in Y_s} \sum_{G\in\mathcal{F}_{i+2}(c)} \frac{\omega_{d+1}\,\omega_{i+1}}{\omega_d\,\omega_{i+2}} \mathcal{H}^{i+1}(G\cap \partial W)\ ds \\
&\quad = \frac{2\,\omega_{d+1}\,\omega_{i+1}}{\omega_d\,\omega_{i+2}} \int_0^t \mathbb{E} \sum_{c\in Y_s} \hat{f}_{i+1}(c)\ ds \\
&\quad = \frac{2\,\omega_{d+1}\,\omega_{i+1}}{\omega_d\,\omega_{i+2}} \int_0^t \mathbb{E}\Sigma_{\hat{f}_{i+1}}(Y_s)\ ds.
\end{aligned}$$

□

Since Y_t is for all $t \geq 0$ almost surely a tessellation one clearly gets

$$\mathbb{E}[\Sigma_{f_d}(Y_t)] = \frac{1}{2}\mathcal{H}^d(W\setminus \partial W) = \frac{1}{2}\mathcal{H}^d(W), \qquad t\geq 0 \tag{4.5}$$

and

$$\mathbb{E}[\Sigma_{\hat{f}_{d-1}}(Y_t)] = \frac{1}{2}\mathcal{H}^{d-1}(W\cap \partial W) = \frac{1}{2}\mathcal{H}^{d-1}(\partial W), \qquad t\geq 0. \tag{4.6}$$

Using (4.5) and Lemma 4.5.1, we derive the following theorem.

Theorem 4.5.2. *For $k = 1,\dots,d$ the expected $(d-k)$-dimensional Hausdorff measure at time $t \geq 0$ of a splitting tessellation in W is given by*

$$\mathbb{E}[\Sigma_{f_{d-k}}(t)] = \mathbb{E}[\Sigma_{f_{d-k}}(Y_t)] = \frac{\omega_{d-k+1}}{\omega_d}\left(\frac{2\,\omega_{d+1}}{\omega_d}\right)^{k-1}\frac{t^k}{k!}\,\mathcal{H}^d(W).$$

Remark 4.5.1. The expected total $(d-1)$-dimensional Hausdorff measure of the $(d-1)$-skeleton is the same as in the case of a Poisson hyperplane process investigated in Theorem 3.1.1. For $k > 1$ the results differ by the factor 2^{k-1}. The same behaviour can be seen for Euclidean splitting tessellations. To see this, one combines [106, Theorem 2] with the expected Hausdorff measure of the skeleton in Euclidean space, also stated in Theorem 3.1.1.

Proof. We prove this by induction on k. For $k = 0$ the result is obviously true (compare equation (4.5)). Now let $k \geq 1$ and assume that it holds for $k-1$. By Lemma 4.5.1 with $g_i = f_i$

and the hypothesis for $k-1$ we derive

$$\begin{aligned}\mathbb{E}[\Sigma_{f_{d-k}} Y_t] &= \frac{2\,\omega_{d+1}\,\omega_{d-k+1}}{\omega_d\,\omega_{d-k+2}} \int_0^t \mathbb{E}[\Sigma_{f_{d-(k-1)}}(Y_s)]\; ds \\ &= \frac{2\,\omega_{d+1}\,\omega_{d-k+1}}{\omega_d\,\omega_{d-k+2}} \int_0^t \frac{\omega_{d-k+2}}{\omega_d} \left(\frac{2\,\omega_{d+1}}{\omega_d}\right)^{k-2} \frac{s^{k-1}}{(k-1)!}\, \mathcal{H}^d(W)\; ds \\ &= \frac{\omega_{d-k+1}}{\omega_d} \left(\frac{2\,\omega_{d+1}}{\omega_d}\right)^{k-1} \frac{t^k}{k!}\, \mathcal{H}^d(W).\end{aligned}$$

□

Theorem 4.5.3. *For $k = 2, \dots, d$ the expected $(d-k)$-dimensional Hausdorff measure at time $t \geq 0$ including its boundary is given by*

$$\mathbb{E}[\Sigma_{\tilde{f}_{d-k}}(Y_t)] = \frac{\omega_{d-k+1}}{\omega_d} \left(\frac{2\,\omega_{d+1}}{\omega_d}\right)^{k-1} \left(\frac{t^k}{k!}\, \mathcal{H}^d(W) + \frac{t^{k-1}}{2\,(k-1)!} \mathcal{H}^{d-1}(\partial W)\right).$$

Proof. First one gets

$$\mathbb{E}[\Sigma_{\tilde{f}_{d-k}}(Y_t)] = \mathbb{E}[\Sigma_{f_{d-k}}(Y_t)] + \mathbb{E}[\Sigma_{\hat{f}_{d-k}}(Y_t)]$$

by the definition of $\tilde{f}$. Since the first summand was already treated in Theorem 4.5.2, we concentrate on the second. Let $k \in \{2, \dots, d\}$ then by Lemma 4.5.1 with $g_{d-k} = \hat{f}_{d-k}$ we derive

$$\mathbb{E}[\Sigma_{\hat{f}_{d-k}}(Y_t)] = \frac{2\,\omega_{d+1}\,\omega_{d-k+1}}{\omega_d\,\omega_{d-k+2}} \int_0^t \mathbb{E}[\Sigma_{\hat{f}_{d-(k-1)}}(Y_s)]\; ds. \tag{4.7}$$

As in the proof of Theorem 4.5.2, we apply induction on k. The base case for the forthcoming induction argument is the surface area. In this case we get by (4.6)

$$\mathbb{E}[\Sigma_{\hat{f}_{d-1}}(Y_t)] = \frac{1}{2}\mathcal{H}^{d-1}(\partial W).$$

Now let $k \geq 2$ and assume that

$$\mathbb{E}[\Sigma_{\hat{f}_{d-\ell}}(Y_t)] = \frac{\omega_{d-\ell+1}}{\omega_d} \left(\frac{2\,\omega_{d+1}}{\omega_d}\right)^{\ell-1} \frac{t^{\ell-1}}{2\,(\ell-1)!}\, \mathcal{H}^{d-1}(\partial W)$$

holds true for $\ell = k-1$. Now we derive

$$\begin{aligned}\mathbb{E}[\Sigma_{\hat{f}_{d-k}} Y_t] &= \frac{2\,\omega_{d+1}\,\omega_{d-k+1}}{\omega_d\,\omega_{d-k+2}} \int_0^t \mathbb{E}[\Sigma_{\hat{f}_{d-(k-1)}}(Y_s)]\; ds \\ &= \frac{2\,\omega_{d+1}\,\omega_{d-k+1}}{\omega_d\,\omega_{d-k+2}} \int_0^t \frac{\omega_{d-k+2}}{\omega_d} \left(\frac{2\,\omega_{d+1}}{\omega_d}\right)^{k-2} \frac{s^{k-2}}{2\,(k-2)!}\, \mathcal{H}^{d-1}(\partial W)\; ds \\ &= \frac{\omega_{d-k+1}}{\omega_d} \left(\frac{2\,\omega_{d+1}}{\omega_d}\right)^{k-1} \frac{t^{k-1}}{2\,(k-1)!}\, \mathcal{H}^{d-1}(\partial W).\end{aligned}$$

Combining this with the results from Theorem 4.5.2 finishes the proof. □

4.5.2 Expected volume of the typical and Crofton cell

In this section we make the special choice $W = B_r$ for some $r > 0$. We will call the almost surely uniquely determined cell at time $t \geq 0$ containing p the Crofton cell $C_0 = C_0(t)$. We are aiming to give two results on the expected $(d-1)$-dimensional Hausdorff measure of the Crofton and the typical cell of the hyperbolic cell splitting tessellation. We will investigate their limit behaviour in t. To describe the limit behaviour, we write $f \sim g$ if $f(t)/g(t) \to 1$ as $t \to \infty$. For a definition of the typical cell, we refer to Section 5.4.

Lemma 4.5.4. *The expected d-dimensional Hausdorff measure of the Crofton cell of a splitting tessellation within B_r is*

$$\mathbb{E}[\mathcal{H}^d(C_0)] = \omega_d \int_0^r \sinh^{d-1}(s) \exp(-t \, \frac{\omega_{d+1}\,\omega_1}{\omega_d\,\omega_2} s) \; ds.$$

Proof. We calculate directly by using spherical coordinates, Fubini and the isotropy of the process

$$\begin{aligned}
\mathbb{E}[\mathcal{H}^d(C_0)] &= \mathbb{E} \int_{B_r} \mathbb{1}\{x \in C_0\} \; \mathcal{H}^d(dx) \\
&= \mathbb{E} \int_{\mathbb{S}_p^{d-1}} \int_0^r \sinh^{d-1}(s) \mathbb{1}\{\exp_p([p, su]) \subseteq C_0\} \; ds \; du \\
&= \int_{\mathbb{S}_p^{d-1}} \mathbb{E} \int_0^r \sinh^{d-1}(s) \mathbb{1}\{\exp_p([p, su]) \subseteq C_0\} \; ds \; du \\
&= \int_{\mathbb{S}_p^{d-1}} \mathbb{E} \int_0^r \sinh^{d-1}(s) \mathbb{1}\{\exp_p([p, sv]) \subseteq C_0\} \; ds \; du \\
&= \omega_d \, \mathbb{E} \int_0^r \sinh^{d-1}(s) \mathbb{1}\{\exp_p([p, sv]) \subseteq C_0\} \; ds \\
&= \omega_d \, \mathbb{E} \int_0^r \sinh^{d-1}(s) \mathbb{1}\{\exp_p([p, sv]) \cap Z_t = \varnothing\} \; ds \\
&= \omega_d \int_0^r \sinh^{d-1}(s) \exp(-t \, \frac{\omega_{d+1}\,\omega_1}{\omega_d\,\omega_2} s) \; ds,
\end{aligned}$$

where $v \in \mathbb{S}_p^{d-1}$ is an arbitrary direction. Here the last equality is due to Corollary 4.4.3. □

We now investigate this value for two different set ups. In the first one we fix the radius of the intersection window and increase the time t. In the second scenario we fix the time and increase the radius r of the intersection window.

Lemma 4.5.5. *Let $r > 0$ be fixed and let C_0 be the Crofton cell of a splitting tessellation within B_r. Then*

$$\mathbb{E}[\mathcal{H}^d(C_0)] \sim \omega_d \, (d-1)! \left(\frac{\omega_d\,\omega_2}{\omega_{d+1}\,\omega_1} \right)^d t^{-d}$$

as $t \to \infty$. In particular the limit is independent of r.

Proof. To show the limit behaviour, we pick an arbitrary value $a \in (d/(d+2), 1)$ and set

$c := \frac{\omega_{d+1}\omega_1}{\omega_d\omega_2}$. Consider the following integral for t big enough, such that $t^{-a} \le r$ holds

$$\int_0^r t^d \sinh^{d-1}(s)\exp(-tcs)\ ds = \int_0^{t^{-a}} t^d \sinh^{d-1}(s)\exp(-tcs)\ ds + \underbrace{\int_{t^{-a}}^r t^d \sinh^{d-1}(s)\exp(-tcs)\ ds}_{\le r\sinh^{d-1}(r)\,t^d\exp(-t^{1-a}c)}. \tag{4.8}$$

Since the second term is additionally non-negative, it vanishes for growing t. In a next step we investigate the expression $t^d(\sinh^{d-1}(s) - s^{d-1})$ for $s \in [0, t^{-a}]$. First one can easily see that $\sinh^{d-1}(s) - s^{d-1} \ge 0$ holds. Additionally we get for $t \ge 1$

$$\begin{aligned}
t^d(\sinh^{d-1}(s) - s^{d-1}) &= t^d\left(\left(\sum_{k=0}^{\infty}\frac{s^{1+2k}}{(1+2k)!}\right)^{d-1} - s^{d-1}\right)\\
&= t^d \sum_{(k_1,\dots,k_{d-1})\in\mathbb{N}_0^{d-1}\setminus\{0\}}\prod_{i=1}^{d-1}\frac{s^{1+2k_i}}{(1+2k_i)!}\\
&\le t^d t^{-(d+1)a}\sum_{(k_1,\dots,k_{d-1})\in\mathbb{N}_0^{d-1}\setminus\{0\}}\prod_{i=1}^{d-1}\frac{1}{(1+2k_i)!}\\
&\le t^{-(d+1)a+d}\sinh^{d-1}(1).
\end{aligned}$$

Here we used the series expansion of the sinh-function. In the penultimate step we made use of the relations $s \le t^{-a} \le 1$ and the fact that every summand in the second line contains the factor s^{d+1}. Now we use the calculations above in order to replace the sinh-function with s^{d-1} via

$$\int_0^{t^{-a}} t^d \sinh^{d-1}(s)\exp(-tcs)\ ds = \int_0^{t^{-a}} t^d s^{d-1}\exp(-tcs)\ ds + \underbrace{\int_0^{t^{-a}} t^d(\sinh^{d-1}(s) - s^{d-1})\exp(-tcs)\ ds}_{\le t^{-(d+2)a+d}\sinh^{d-1}(1)}. \tag{4.9}$$

Here the exponential function was bounded by 1. The remaining term can be evaluated by substituting $u = tcs$

$$\begin{aligned}
\int_0^{t^{-a}} t^d s^{d-1}\exp(-tcs)\ ds &= \int_0^{ct^{-a+1}} t^d\frac{u^{d-1}}{(tc)^{d-1}}\exp(-u)\frac{1}{tc}\ du\\
&= \int_0^{ct^{-a+1}}\frac{u^{d-1}}{c^d}\exp(-u)\ du\\
&= c^{-d}\left[-\Gamma(d,u)\right]_0^{ct^{-a+1}}\\
&= c^{-d}(-\Gamma(d,ct^{-a+1}) + (d-1)!) \xrightarrow{t\to\infty}\frac{(d-1)!}{c^d}.
\end{aligned} \tag{4.10}$$

Combining (4.8)-(4.10) with the results from Lemma 4.5.5 yields the desired result. □

Remark 4.5.2. The results derived in Lemma 4.5.4 and 4.5.5 can easily be transferred to the setting of Crofton cells of a Poisson hyperplane tessellation. This is based on the fact that the capacity function of segments is the same in both models. Having this one can compare the limit behaviour with the expected volume of Euclidean Crofton cells of Poisson hyperplane tessellations. This value is calculated in [103, Theorem 10.4.9]. It coincides with the limit case in hyperbolic space which can be shown by

$$\begin{aligned}\omega_d\,(d-1)!\left(\frac{\omega_d\,\omega_2}{\omega_{d+1}\,\omega_1}\right)^d t^{-d} &= d!\kappa_d\left(\frac{\omega_{d+1}\omega_1}{\omega_d\omega_2}t\right)^{-d}\\ &= d!\kappa_d\left(\frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)d\kappa_d 2\pi}t\right)^{-d}\\ &= d!\kappa_d\left(\frac{2\pi^{(d-1)/2}}{\Gamma((d+1)/2)d\kappa_d}t\right)^{-d}\\ &= d!\kappa_d\left(\frac{2\kappa_{d-1}}{d\kappa_d}t\right)^{-d}.\end{aligned}$$

In a next step we consider the expected volume of the Crofton cell for fixed time $t > 0$ and growing radius of the observation window.

Lemma 4.5.6. *For $t \le (d-1)\frac{\omega_d\omega_2}{\omega_{d+1}\omega_1}$ the expected volume of the Crofton cell diverges to infinity for $r \to \infty$ and for $t > (d-1)\frac{\omega_d\omega_2}{\omega_{d+1}\omega_1}$ it converges for growing radius r.*

Proof. Let the time $t > 0$ be such that $t \le (d-1)\frac{\omega_d\omega_2}{\omega_{d+1}\omega_1}$ holds. Using Lemma 4.5.4, Lemma 2.2.1 and choosing $r \ge 1$ we see

$$\begin{aligned}\mathbb{E}[\mathcal{H}^d(C_0)] &= \omega_d\int_0^r \sinh^{d-1}(s)\exp(-t\,\frac{\omega_{d+1}\,\omega_1}{\omega_d\,\omega_2}s)\;ds\\ &\ge \omega_d\int_1^r \sinh^{d-1}(s)\exp(-t\,\frac{\omega_{d+1}\,\omega_1}{\omega_d\,\omega_2}s)\;ds\\ &\ge \omega_d\int_1^r \exp((s-3)(d-1) - st\frac{\omega_{d+1}\omega_1}{\omega_d\omega_2})\;ds\\ &\ge \omega_d\int_1^r \exp((s-3)(d-1) - s(d-1))\;ds \xrightarrow{r\to\infty} \infty.\end{aligned}$$

Now let $t > (d-1)\frac{\omega_d\omega_2}{\omega_{d+1}\omega_1}$, then there exists an $\varepsilon > 0$ such that $t = \varepsilon + (d-1)\frac{\omega_d\omega_2}{\omega_{d+1}\omega_1}$. This yields

$$\begin{aligned}\mathbb{E}[\mathcal{H}^d(C_0)] &= \omega_d\int_0^r \sinh^{d-1}(s)\exp(-t\,\frac{\omega_{d+1}\,\omega_1}{\omega_d\,\omega_2}s)\;ds\\ &\le \frac{\omega_d}{2^{d-1}}\int_0^r \exp((d-1)s)\exp(-\varepsilon\,\frac{\omega_{d+1}\,\omega_1}{\omega_d\,\omega_2}s-(d-1)s)\;ds\\ &= \frac{\omega_d}{2^{d-1}}\int_0^r \exp(-\varepsilon\,\frac{\omega_{d+1}\,\omega_1}{\omega_d\,\omega_2}s)\;ds.\end{aligned}$$

This expression is bounded in r. We get the desired result by monotone convergence. □

Remark 4.5.3. As mentioned in the remark above, the results concerning the expected volume of the Crofton cell are transferable into the setting of the expected volume of the Crofton cell of a Poisson hyperplane tessellation. Considering the different normalization of the measure μ_{d-1}, the threshold found in Lemma 4.5.6 in dimension 2 coincides with the one established in [114] (for Poisson hyperplane tessellations in dimension 2). In this work the authors asked for the probability of having visibility into infinity in an arbitrary direction from p. Below this threshold, the probability was positive and above the threshold it is zero.

Using the calculations done in Section 4.5.1 we get the expected number of cells and therefore the expected area of the typical cell. For a formal definition of the typical cell, we refer to Section 5.4.

Lemma 4.5.7. *Let C be the typical cell of a splitting tessellation inside a 2-dimensional ball with radius r at time $t > 0$. Then*

$$\mathbb{E}[\mathcal{H}^2(C)] = t^{-2}\pi \left(1 + t^{-1}\frac{\sinh(r)}{\cosh(r)-1} + \pi\mathcal{H}^2(B_r)^{-1}t^{-2}\right)^{-1}.$$

Proof. First we get by the Crofton-type formula in Lemma 2.4.1 that for a convex body $K \in \mathcal{K}_h^d$

$$\begin{aligned}\mu_1(\mathbb{H}_1\langle K\rangle) = \int_{A_h(2,1)} \mathbb{1}\{H \cap K \neq \varnothing\}\,\mu_1(dH) &= \frac{1}{2}\int_{A_h(2,1)} \mathcal{H}^0(H\cap\partial K)\,\mu_1(dH)\\ &= \frac{\omega_3\,\omega_1}{2\,\omega_2^2}\mathcal{H}^1(\partial K)\\ &= \pi^{-1}\mathcal{H}^1(\partial K)\end{aligned}$$

holds. Using Theorem 4.5.2 in the penultimate line, the expected number of cells is given by

$$\begin{aligned}\mathbb{E}|Y_t| = \mathbb{E}\sum_{c\in Y_t} 1 = \mathbb{E}|Y_0| + \int_0^t \mathbb{E}\sum_{c\in Y_s}\int_{\mathbb{H}_1\langle c\rangle} 1\,\mu_1(dH)\,ds\\ = 1 + \int_0^t \mathbb{E}\sum_{c\in Y_s}\mu_1(\mathbb{H}_1\langle c\rangle)\,ds\\ = 1 + \int_0^t \mathbb{E}\sum_{c\in Y_s}\pi^{-1}\mathcal{H}^1(\partial c)\,ds\\ = 1 + \pi^{-1}\int_0^t \partial B_r + 2\,\mathbb{E}\Sigma_{f_1}(s)\,ds\\ = 1 + \pi^{-1}\int_0^t 2\pi\sinh(r) + 2\,\mathbb{E}\Sigma_{f_1}(s)\,ds\\ = 1 + 2\sinh(r)\,t + \pi^{-1}\int_0^t 2\,s\,\mathcal{H}^2(B_r)\,ds\\ = 1 + 2\sinh(r)\,t + \pi^{-1}\mathcal{H}^2(B_r)\,t^2.\end{aligned}$$

Therefore the expected 2-dimensional Hausdorff measure of the typical cell in dimension 2 is

$$\begin{aligned}\mathbb{E}[\mathcal{H}^2(C)] &= \frac{\mathcal{H}^2(B_r)}{1 + 2\sinh(r)\,t + \pi^{-1}\mathcal{H}^2(B_r)\,t^2}\\ &= t^{-2}\pi\left(1 + t^{-1}\frac{\sinh(r)}{\cosh(r)-1} + \frac{1}{2(\cosh(r)-1)}t^{-2}\right)^{-1} \leq t^{-2}\pi.\end{aligned}$$

□

Remark 4.5.4. We would like to point out that the expected 2-dimensional Hausdorff measure converges for growing radius r. The behaviour for large r and small intensities $t \leq \pi/2$ is

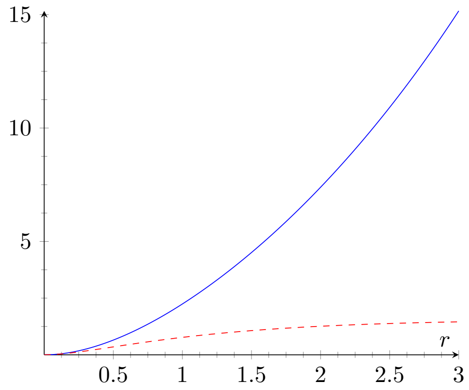

Figure 4.5.1: Comparison of the expected area of the Crofton cell (straight blue line) and the expected area of the typical cell (dashed red line) of a splitting tessellation inside a ball of radius r at time $t = 1$.

fundamentally different in contrast to the expected 2-dimensional Hausdorff measure of the Crofton cell (see figure 4.5.1).

The same behaviour occurs for higher dimensions (with different threshold depending on the dimension d). We show this by bounding the expected number of cells in arbitrary dimension from below.

Lemma 4.5.8. *The expected volume of the typical cell of a d-dimensional splitting tessellation is bounded in r for any fixed time $t > 0$.*

Proof. We give a lower bound for the expected number of cells at time t by

$$\begin{aligned}\mathbb{E}|Y_t| = \mathbb{E}|Y_0| + \int_0^t \mathbb{E} \sum_{c \in Y_s} \int_{\mathbb{H}_{d-1}\langle c \rangle} 1 \; \mu_{d-1}(dH)\; ds &\geq 1 + \int_0^t \mu_{d-1}(\mathbb{H}_{d-1}\langle B_r \rangle)\; ds \\ &= 1 + 2t \int_0^r \cosh^{d-1}(\ell)\; d\ell \\ &\geq 1 + \frac{2t}{2^{d-1}} \int_0^r e^{\ell(d-1)}\; d\ell \\ &= 1 + \frac{t}{2^{d-2}(d-1)}(e^{r(d-1)} - 1).\end{aligned}$$

Therefore the expected d-dimensional Hausdorff measure of the typical cell C in arbitrary dimension is bounded from above by some constant c for any radius $r > 0$, since

$$\mathbb{E}[\mathcal{H}^d(C)] = \frac{\mathcal{H}^d(B_r)}{\mathbb{E}|Y_t|} \leq \frac{\mathcal{H}^d(B_r)}{1 + \frac{t}{2^{d-2}(d-1)}(e^{r(d-1)} - 1)} \leq c$$

holds. □

4.6 Higher moments

After investigating various first moments of functionals depending on splitting tessellations, we aim to give results for the variance of the surface area. We start in Section 4.6.1 with an integral representation for the variance of the hyperbolic surface area of a splitting tessellation. Later in Section 4.6.2 and 4.6.3 we will use this representation to investigate the variance of the total surface area for growing time t and growing radius r respectively. In this section we will fix the intersection window W to be a ball of radius $r > 0$.

4.6.1 Variance of the surface area

Theorem 4.6.1. *For $t \geq 0$ the variance of the surface area of the splitting tessellation $\Sigma_{f_{d-1}}(t)$ within a ball of radius $r > 0$ is given by*

$$\begin{aligned}&\mathbb{V}ar(\Sigma_{f_{d-1}}(t))\\ &= c_d \int_{\mathbb{H}_{d-1}\langle B_r\rangle} \int_{H\cap B_r} \int_{H\cap B_r} \frac{1-\exp(-c_d^{-1}\, d_h(x,y)\, t)}{d_h(x,y)}\, \mathcal{H}^{d-1}(dx)\, \mathcal{H}^{d-1}(dy)\, \mu_{d-1}(dH)\end{aligned}$$

with

$$c_d := \frac{\omega_d\, \omega_2}{\omega_{d+1}\, \omega_1}.$$

Remark 4.6.1. For other spaces of constant curvature comparable results are established. For the Euclidean case see [104, Theorem 4.1] and for the spherical results see [47, Theorem 5.4].

Proof. To show this we define

$$\bar{\Sigma}_{f_{d-1}}(t) := \Sigma_{f_{d-1}}(t) - \mathbb{E}\Sigma_{f_{d-1}}(t) = \sum_{c\in Y_t} f_{d-1}(c) - \mathcal{H}^d(B_r)\, t,$$

where we used Theorem 4.5.2 to evaluate the expectation. To shorten the notation, we define for $T \in \mathbb{T}^d$ and $t \geq 0$

$$g(T,t) := \left(\sum_{c\in T} f_{d-1}(c) - \mathcal{H}^d(B_r)\, t\right)^2$$

and using this notation yields $\bar{\Sigma}_{f_{d-1}}(t)^2 = g(Y_t, t)$. For the special case $t = 0$ we get $g(Y_0, 0) = 0$. The derivative of g with respect to the time component and a fixed tessellation T gives

$$\frac{\partial g}{\partial s}(T,s) = -2\left(\sum_{c\in T} f_{d-1}(c) - \mathcal{H}^d(B_r)\, s\right)\mathcal{H}^d(B_r).$$

Inserting $T = Y_s$ yields

$$\frac{\partial g}{\partial s}(Y_s, s) = -2\bar{\Sigma}_{f_{d-1}}(s)\, \mathcal{H}^d(B_r).$$

Now Proposition 4.3.3 shows that

$$\begin{aligned}\bar{\Sigma}_{f_{d-1}}(t)^2 - 0 - \int_0^t \sum_{c\in Y_s} \int_{\mathbb{H}_{d-1}[c]} \Bigg[\Bigg(&\sum_{\bar{c}\in Y_s\setminus\{c\}\cup\{c\cap H^+\}\cup\{c\cap H^-\}} f_{d-1}(\bar{c}) - \mathcal{H}^d(B_r)\, s\Bigg)^2\\ &- \left(\sum_{\bar{c}\in Y_s} f_{d-1}(\bar{c}) - \mathcal{H}^d(B_r)\, s\right)^2\Bigg]\mu_{d-1}(dH) - 2\bar{\Sigma}_{f_{d-1}}(s)\mathcal{H}^d(B_r)\; ds\end{aligned}$$

is a $\mathcal{Y}$-martingale. Here we chose $b(t) = t\,\mathcal{H}^d(B_r)$ and $\phi = f_{d-1}$. Taking a closer look at the expression in brackets gives

$$\begin{aligned}
&\Bigg(\sum_{\bar{c}\in Y_s\setminus\{c\}\cup\{c\cap H^+\}\cup\{c\cap H^-\}} f_{d-1}(\bar{c}) - \mathcal{H}^d(B_r)\,s\Bigg)^2 - \Bigg(\sum_{\bar{c}\in Y_s} f_{d-1}(\bar{c}) - \mathcal{H}^d(B_r)\,s\Bigg)^2\\
&=\big(\bar{\Sigma}_{f_{d-1}}(s) - f_{d-1}(c) + f_{d-1}(c\cap H^+) + f_{d-1}(c\cap H^-)\big)^2 - \big(\bar{\Sigma}_{f_{d-1}}(s)\big)^2\\
&=\big(\bar{\Sigma}_{f_{d-1}}(s) + 2f_{d-1}(c\cap H)\big)^2 - \big(\bar{\Sigma}_{f_{d-1}}(s)\big)^2\\
&=4f_{d-1}(c\cap H)^2 + 4f_{d-1}(c\cap H)\,\bar{\Sigma}_{f_{d-1}}(s).
\end{aligned}$$

To simplify the martingale, we use the Crofton-type formula stated in (2.4)

$$\begin{aligned}
\sum_{c\in Y_s}\int_{\mathbb{H}_{d-1}[c]} 4f_{d-1}(c\cap H)\ \mu_{d-1}(dH) &= 4\sum_{c\in Y_s}\int_{\mathbb{H}_{d-1}[c]} \frac{1}{2}\mathcal{H}^{d-1}((c\cap H)\setminus\partial B_r)\ \mu_{d-1}(dH)\\
&= 4\sum_{c\in Y_s}\frac{1}{2}\mathcal{H}^d(c\setminus\partial B_r)\\
&= 2\,\mathcal{H}^d(B_r)
\end{aligned}$$

and therefore

$$\bar{\Sigma}_{f_{d-1}}(t)^2 - \int_0^t \sum_{c\in Y_s}\int_{\mathbb{H}_{d-1}[c]} 4f_{d-1}(c\cap H)^2\ \mu_{d-1}(dH)\ ds$$

is a $\mathcal{Y}$-martingale. Taking the expectation and using the definition of f_{d-1} leads to

$$\begin{aligned}
\mathbb{V}\mathrm{ar}(\Sigma_{f_{d-1}}(t)) = \mathbb{E}(\bar{\Sigma}_{f_{d-1}}(t)^2) &= \mathbb{E}\int_0^t \sum_{c\in Y_s}\int_{\mathbb{H}_{d-1}[c]} 4f_{d-1}(c\cap H)^2\ \mu_{d-1}(dH)\ ds\\
&= \int_0^t \mathbb{E}\sum_{c\in Y_s}\int_{\mathbb{H}_{d-1}[c]} \mathcal{H}^{d-1}(c\cap H)^2\ \mu_{d-1}(dH)\ ds.
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}(\Sigma_{f_{d-1}}(t))\\
&\quad= \int_0^t \mathbb{E}\sum_{c\in Y_s}\int_{\mathbb{H}_{d-1}[c]}\Big(\int_H \mathbb{1}\{x\in c\}\ \mathcal{H}^{d-1}(dx)\Big)\Big(\int_H \mathbb{1}\{y\in c\}\ \mathcal{H}^{d-1}(dy)\Big)\ \mu_{d-1}(dH)\ ds\\
&\quad= \int_0^t \mathbb{E}\sum_{c\in Y_s}\int_{\mathbb{H}_{d-1}[c]}\int_H\int_H \mathbb{1}\{x,y\in c\}\ \mathcal{H}^{d-1}(dx)\ \mathcal{H}^{d-1}(dy)\ \mu_{d-1}(dH)\ ds\\
&\quad= \int_0^t \int_{\mathbb{H}_{d-1}[B_r]}\int_{H\cap B_r}\int_{H\cap B_r} \mathbb{E}\sum_{c\in Y_s}\mathbb{1}\{x,y\in c\}\ \mathcal{H}^{d-1}(dx)\ \mathcal{H}^{d-1}(dy)\ \mu_{d-1}(dH)\ ds\\
&\quad= \int_0^t \int_{\mathbb{H}_{d-1}[B_r]}\int_{H\cap B_r}\int_{H\cap B_r} \mathbb{P}(\overline{x,y}\cap Z_s = \varnothing)\ \mathcal{H}^{d-1}(dx)\ \mathcal{H}^{d-1}(dy)\ \mu_{d-1}(dH)\ ds.
\end{aligned}$$

Since by Corollary 4.4.3 the probability is given by

$$\mathbb{P}(\overline{x,y}\cap Z_s = \varnothing) = \exp\Big(-\frac{\omega_{d+1}\,\omega_1}{\omega_d\,\omega_2}\,d_h(x,y)\,s\Big).$$

Defining $c_d := \frac{\omega_d \, \omega_2}{\omega_{d+1} \, \omega_1}$ we can continue as

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}(\Sigma_{f_{d-1}}(t)) \\
&= \int_0^t \int_{\mathbb{H}_{d-1}[B_r]} \int_{H \cap B_r} \int_{H \cap B_r} \exp(-c_d^{-1} \, d_h(x,y) \, s) \; \mathcal{H}^{d-1}(dx) \; \mathcal{H}^{d-1}(dy) \; \mu_{d-1}(dH) \; ds \\
&= c_d \int_{\mathbb{H}_{d-1}[B_r]} \int_{H \cap B_r} \int_{H \cap B_r} \frac{1 - \exp(-c_d^{-1} \, d_h(x,y) \, t)}{d_h(x,y)} \; \mathcal{H}^{d-1}(dx) \; \mathcal{H}^{d-1}(dy) \; \mu_{d-1}(dH).
\end{aligned}$$

□

4.6.2 Limit behaviour of the variance of the total surface area for growing time t

Having the representation developed in Section 4.6.1 gives us the opportunity to make some statements about the limit behaviour of the variance for growing time t. We distinguish between $d = 2$ and $d \geq 3$. The first one is covered by Corollary 4.6.4 and the second by 4.6.5.

Before we can start with the main results, we first have to establish some auxiliary results needed in the proofs.

Lemma 4.6.2. *Let $a > 0$. The function $f_a : (0, \infty) \to \mathbb{R}$ with*

$$f_a(x) = \frac{1 - e^{-ax}}{x}$$

is decreasing.

Proof. First of all, we see that f_a is continuously differentiable. Therefore it remains to show

$$f'(x) = \frac{axe^{-ax} + e^{-ax} - 1}{x^2} \leq 0.$$

Using the series expansion of the exponential function we get $ax + 1 - e^{ax} \leq 0$ and trough the following equivalence relation

$$ax + 1 - e^{ax} \leq 0 \iff axe^{-ax} + e^{-ax} - 1 \leq 0 \iff \frac{axe^{-ax} + e^{-ax} - 1}{x^2} \leq 0$$

the desired monotonicity. □

The next result deals with two inequalities for integrals showing up in the proofs of Corollary 4.6.4 and 4.6.5.

Lemma 4.6.3. *Let $a \in [-b, b]$ with $b > 0$ and let $f : [0, \infty) \to [0, \infty)$ be a decreasing function. Then the following relations hold*

$$\begin{aligned}
(i): \quad & \int_{-b}^{b} f(|b - y|) \; dy \leq \int_{-b}^{b} f(|a - y|) \; dy, \\
(ii): \quad & \int_{-b}^{b} f(|0 - y|) \; dy \geq \int_{-b}^{b} f(|a - y|) \; dy.
\end{aligned}$$

Proof. Due to symmetry we can assume a to be non-negative. We start with showing (i). This is done by the following equivalence relations, where the first one holds true due to the monotonicity of f

$$\begin{aligned} & \int_0^{b-a} f(u+b+a)\, du \leq \int_0^{b-a} f(u)\, du \\ \iff & \int_{b+a}^{2b} f(u)\, du \leq \int_0^{b-a} f(u)\, du \\ \iff & \int_0^{2b} f(u)\, du \leq \int_0^{b+a} f(u)\, du + \int_0^{b-a} f(u)\, du \\ \iff & \int_{-b}^{b} f(b-y)\, dy \leq \int_{-b}^{a} f(a-y)\, dy + \int_a^b f(y-a)\, dy \\ \iff & \int_{-b}^{b} f(|b-y|)\, dy \leq \int_{-b}^{b} f(|a-y|)\, dy. \end{aligned}$$

Inequality (ii) is due to

$$\begin{aligned} & \int_b^{b+a} f(u-a)\, du \geq \int_b^{b+a} f(u)\, du \\ \iff & \int_{b-a}^{b} f(u)\, du \geq \int_b^{b+a} f(u)\, du \\ \iff & \int_0^{b} f(u)\, du \geq \int_0^{b-a} f(u)\, du + \int_b^{b+a} f(u)\, du \\ \iff & 2\int_0^{b} f(u)\, du \geq \int_0^{b-a} f(u)\, du + \int_0^{b+a} f(u)\, du \\ \iff & \int_{-b}^{b} f(|y|)\, dy \geq \int_a^{b} f(y-a)\, dy + \int_{-b}^{a} f(a-y)\, dy \\ \iff & \int_{-b}^{b} f(|0-y|)\, dy \geq \int_{-b}^{b} f(|a-y|)\, dy. \end{aligned}$$

□

Corollary 4.6.4. *Let $d = 2$ and $r > 0$ be fixed. Then the variance $\mathbb{V}ar(\Sigma_{f_1}(t))$ grows with logarithmic speed in t and is therefore unbounded in t.*

Proof. Let $0 < \varepsilon < r$ and define the set A of hyperplanes having at least intersection length 2ε with B_r

$$A := A(\varepsilon, r) := \{H \in A_h(2,1) : \mathcal{H}^1(H \cap B_r) \geq 2\varepsilon\}.$$

By our choice of ε this set is nonempty. To shorten the notation, we write

$$c := c(r,H) := \frac{\mathcal{H}^1(B_r \cap H)}{2} = \operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(d_h(H,p))}\right).$$

Now we are able to give a lower bound for the variance of the total edge length

$$\begin{aligned}
\mathbb{V}\mathrm{ar}(\Sigma_{f_1}(t)) &= c_1 \int_{\mathbb{H}_1[B_r]} \int_{H\cap B_r} \int_{H\cap B_r} \frac{1-\exp(-c_1^{-1}\, d_h(x,y)t)}{d_h(x,y)}\; \mathcal{H}^1(dx)\; \mathcal{H}^1(dy)\; \mu_1(dH) \\
&= c_1 \int_{\mathbb{H}_1[B_r]} \int_{-c}^{c} \int_{-c}^{c} \frac{1-\exp(-c_1^{-1}\, d_e(x,y)t)}{d_e(x,y)}\; dx\; dy\; \mu_1(dH) \\
&\geq c_1 \int_{A} \int_{-c}^{c} \int_{-c}^{c} \frac{1-\exp(-c_1^{-1}\, d_e(x,y)t)}{d_e(x,y)}\; dx\; dy\; \mu_1(dH) \\
&\geq c_1 \int_{A} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \frac{1-\exp(-c_1^{-1}\, d_e(x,y)t)}{d_e(x,y)}\; dx\; dy\; \mu_1(dH) \\
&= c_1\, \mu_1(A) \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \frac{1-\exp(-c_1^{-1}\,|x-y|t)}{|x-y|}\; dx\; dy. \qquad (4.11)
\end{aligned}$$

We now apply Lemma 4.6.3 (i) with $b=\varepsilon$ and $f(u) = \frac{1-\exp(-c_1^{-1}ut)}{u}$. Due to Lemma 4.6.2 the requirements of this lemma are fulfilled. This allows us to bound the two integrals in (4.11) further by

$$\begin{aligned}
\int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \frac{1-\exp(-c_1^{-1}\,|x-y|t)}{|x-y|}\; dx\; dy &\geq \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \frac{1-\exp(-c_1^{-1}\,|\varepsilon-y|\,t)}{|\varepsilon-y|}\; dy\; dx \\
&= 2\varepsilon \int_{-\varepsilon}^{\varepsilon} \frac{1-\exp(-c_1^{-1}\,|\varepsilon-y|\,t)}{|\varepsilon-y|}\; dy \\
&= 2\varepsilon \int_{0}^{2\varepsilon} \frac{1-\exp(-c_1^{-1}\,u\,t)}{u}\; du \\
&= 2\,\varepsilon \int_{0}^{1} \frac{1-\exp(-2c_1^{-1}\varepsilon\, v\, t)}{v}\; dv \\
&= 2\varepsilon(\gamma + \ln(c_1^{-1}2\varepsilon t) + E_1(c_1^{-1}2\varepsilon t)). \qquad (4.12)
\end{aligned}$$

Here γ is the Euler-Mascheroni constant and $E_1(t) := \int_t^\infty s^{-1}e^{-s}\, ds$ (see [86, Chapter 6]) is a finite number. Combining (4.11) and (4.12) yields the lower bound for the variance

$$\mathbb{V}\mathrm{ar}(\Sigma_{f_1}(t)) \geq 2\, c_1 \varepsilon\, \mu_1(A)(\gamma + \ln(c_1^{-1}2\varepsilon t) + E_1(c_1^{-1}2\varepsilon t)).$$

An upper bound is given by

$$\begin{aligned}
\mathbb{V}\mathrm{ar}(\Sigma_{f_1}(t)) &= c_1 \int_{\mathbb{H}_1[B_r]} \int_{H\cap B_r} \int_{H\cap B_r} \frac{1-\exp(-c_1^{-1}\, d_h(x,y)t)}{d_h(x,y)}\; \mathcal{H}^1(dx)\; \mathcal{H}^1(dy)\; \mu_1(dH) \\
&= c_1 \int_{\mathbb{H}_1[B_r]} \int_{-c}^{c} \int_{-c}^{c} \frac{1-\exp(-c_1^{-1}\, d_e(x,y)t)}{d_e(x,y)}\; dx\; dy\; \mu_1(dH) \\
&\leq c_1 \int_{\mathbb{H}_1[B_r]} \int_{-r}^{r} \int_{-r}^{r} \frac{1-\exp(-c_1^{-1}\, d_e(x,y)t)}{d_e(x,y)}\; dx\; dy\; \mu_1(dH) \\
&= c_1\, \mu_1(\mathbb{H}_1[B_r]) \int_{-r}^{r} \int_{-r}^{r} \frac{1-\exp(-c_1^{-1}\,|x-y|t)}{|x-y|}\; dx\; dy. \qquad (4.13)
\end{aligned}$$

Now applying Lemma 4.6.3 (ii) with $b=r$ and $f(u) = \frac{1-\exp(-c_1^{-1}ut)}{u}$ which fulfills again due to

Lemma 4.6.2 the requirements, yields

$$\begin{aligned}\int_{-r}^{r}\int_{-r}^{r}\frac{1-\exp(-c_1^{-1}|x-y|t)}{|x-y|}\,dx\,dy &\le \int_{-r}^{r}\int_{-r}^{r}\frac{1-\exp(-c_1^{-1}|0-y|t)}{|0-y|}\,dx\,dy\\ &= 2r\int_{-r}^{r}\frac{1-\exp(-c_1^{-1}|y|t)}{|y|}\,dx\,dy\\ &= 4r\int_{0}^{r}\frac{1-\exp(-c_1^{-1}u\,t)}{u}\,du\\ &= 4r\int_{0}^{1}\frac{1-\exp(-c_1^{-1}r\,v\,t)}{v}\,dv\\ &= 4r(\gamma+\ln(c_1^{-1}r\,t)+E_1(c_1^{-1}r\,t)). \end{aligned} \tag{4.14}$$

Combining (4.13) and (4.14) finishes the proof. □

For $d \ge 3$ we get a different limit behaviour.

Corollary 4.6.5. *Let $d \ge 3$ and $r > 0$ be fixed. Then the variance $\mathbb{V}ar(\Sigma_{f_{d-1}}(t))$ is bounded in $t \ge 0$.*

Proof. To shorten the notation, we define $\tilde{c} := c_d\,\mu_{d-1}(\mathbb{H}_{d-1}\langle B_r\rangle)$. Further let L_p be an arbitrary $(d-1)$-dimensional subspace containing the origin. Similar as in the proof of Corollary 4.6.4 one gets an upper bound by

$$\begin{aligned}&\mathbb{V}\mathrm{ar}(\Sigma_{f_{d-1}}(t))\\ &\quad= c_d\int_{\mathbb{H}_{d-1}[B_r]}\int_{H\cap B_r}\int_{H\cap B_r}\frac{1-\exp(-c_d^{-1}\,d_h(x,y)t)}{d_h(x,y)}\,\mathcal{H}^{d-1}(dx)\,\mathcal{H}^{d-1}(dy)\,\mu_{d-1}(dH)\\ &\quad\overset{(*)}{\le} c_d\int_{\mathbb{H}_{d-1}[B_r]}\int_{L_p\cap B_r}\int_{L_p\cap B_r}\frac{1-\exp(-c_d^{-1}\,d_h(x,y)t)}{d_h(x,y)}\,\mathcal{H}^{d-1}(dx)\,\mathcal{H}^{d-1}(dy)\,\mu_{d-1}(dH)\\ &\quad= \tilde{c}\int_{L_p\cap B_r}\int_{L_p\cap B_r}\frac{1-\exp(-c_d^{-1}\,d_h(x,y)t)}{d_h(x,y)}\,\mathcal{H}^{d-1}(dx)\,\mathcal{H}^{d-1}(dy)\end{aligned} \tag{4.15}$$

In order to justify $(*)$ let $R := R(H,r)$ be the radius of the $(d-1)$-dimensional ball $H\cap B_r$ and let $\tilde{B}(R) = \tilde{B}(R, L_p)$ be the $(d-1)$-dimensional ball in L_p having midpoint p and radius R. Since $\tilde{B}(R)$ and $H \cap B_r$ have the same radius, there exists an isometry $\varphi = \varphi(\tilde{B}(R), H\cap B_r) \in I(\mathbb{H}^d)$

such that $\varphi(H \cap B_r) = \tilde{B}(R)$. Using these definitions, we get

$$\begin{aligned}
&\int_{H\cap B_r}\int_{H\cap B_r} \frac{1-\exp(-c_d^{-1}\, d_h(x,y)t)}{d_h(x,y)}\, \mathcal{H}^{d-1}(dx)\, \mathcal{H}^{d-1}(dy)\\
&\quad= \int_{\varphi(H\cap B_r)}\int_{\varphi(H\cap B_r)} \frac{1-\exp(-c_d^{-1}\, d_h(\varphi^{-1}(x),\varphi^{-1}(y))t)}{d_h(\varphi^{-1}(x),\varphi^{-1}(y))}\, \mathcal{H}^{d-1}(dx)\, \mathcal{H}^{d-1}(dy)\\
&\quad= \int_{\tilde{B}(R)}\int_{\tilde{B}(R)} \frac{1-\exp(-c_d^{-1}\, d_h(\varphi^{-1}(x),\varphi^{-1}(y))t)}{d_h(\varphi^{-1}(x),\varphi^{-1}(y))}\, \mathcal{H}^{d-1}(dx)\, \mathcal{H}^{d-1}(dy)\\
&\quad= \int_{\tilde{B}(R)}\int_{\tilde{B}(R)} \frac{1-\exp(-c_d^{-1}\, d_h(x,y)t)}{d_h(x,y)}\, \mathcal{H}^{d-1}(dx)\, \mathcal{H}^{d-1}(dy)\\
&\quad\le \int_{L_p\cap B_r}\int_{L_p\cap B_r} \frac{1-\exp(-c_d^{-1}\, d_h(x,y)t)}{d_h(x,y)}\, \mathcal{H}^{d-1}(dx)\, \mathcal{H}^{d-1}(dy)
\end{aligned}$$

for every $r > 0$ and $H \in \mathbb{H}_{d-1}[B_r]$. Here the last inequality used that for every $r > 0$ and our choice of $L_p \in G_h(d, d-1)$, the $(d-1)$-dimensional ball $\tilde{B}(R)$ is contained in $L_p \cap B_r$. In a next step and since the set $L_p \cap B_r$ is contained in the $(d-1)$-dimensional subspace L_p, we can find a $(d-1)$-dimensional subspace $T_p^{d-1}M$ of the tangent space T_pM at p such that $\exp_p(T_p^{d-1}M) = L_p$ holds. Now we use spherical coordinates on $T_p^{d-1}M$ in order to rewrite the outer integration over $L_p \cap B_r$ in (4.15), i.e. every point in $T_p^{d-1}M$ is represented by its spherical direction $u \in \mathbb{S}_p^{d-2}$ and its distance from the origin $s \ge 0$. Further applying Fubini's theorem allows us to obtain

$$\begin{aligned}
&\int_{L_p\cap B_r}\int_{L_p\cap B_r} \frac{1-\exp(-c_d^{-1}\, d_h(x,y)t)}{d_h(x,y)}\, \mathcal{H}^{d-1}(dx)\, \mathcal{H}^{d-1}(dy)\\
&\quad= \int_0^r \int_{\mathbb{S}_p^{d-2}} \sinh^{d-2}(s) \int_{L_p\cap B_r} \frac{1-\exp(-c_d^{-1}\, d_h(\exp_p(su),y)t)}{d_h(\exp_p(su),y)}\, \mathcal{H}^{d-1}(dy)\, \sigma_{d-2}(du)\, ds\\
&\quad= \omega_{d-1} \int_0^r \sinh^{d-2}(s) \int_{L_p\cap B_r} \frac{1-\exp(-c_d^{-1}\, d_h(\exp_p(sv),y)t)}{d_h(\exp_p(sv),y)}\, \mathcal{H}^{d-1}(dy)\, ds. \qquad (4.16)
\end{aligned}$$

Here the last equality is obtained by first fixing an arbitrary direction $v \in \mathbb{S}_p^{d-2}$. Now for every $u \in \mathbb{S}_p^{d-2}$ there exists an isometry $\varphi_{u,v} \in I(\mathbb{H}^d)^p$ fixing the origin such that $\varphi_{u,v}(\exp_p(su)) = \exp_p(sv)$ holds for any $s \in [0, r]$. We will use this isometry to get rid of the dependence of u in the inner integral in the second line of (4.16) via

$$\begin{aligned}
&\int_{L_p\cap B_r} \frac{1-\exp(-c_d^{-1}\, d_h(\exp_p(su),y)t)}{d_h(\exp_p(su),y)}\, \mathcal{H}^{d-1}(dy)\\
&\quad= \int_{L_p\cap B_r} \frac{1-\exp(-c_d^{-1}\, d_h(\varphi_{u,v}(\exp_p(su)),\varphi_{u,v}(y))t)}{d_h(\varphi_{u,v}(\exp_p(su)),\varphi_{u,v}(y))}\, \mathcal{H}^{d-1}(dy)\\
&\quad= \int_{L_p\cap B_r} \frac{1-\exp(-c_d^{-1}\, d_h(\exp_p(sv),\varphi_{u,v}(y))t)}{d_h(\exp_p(sv),\varphi_{u,v}(y))}\, \mathcal{H}^{d-1}(dy)\\
&\quad= \int_{L_p\cap B_r} \frac{1-\exp(-c_d^{-1}\, d_h(\exp_p(sv),y)t)}{d_h(\exp_p(sv),y)}\, \mathcal{H}^{d-1}(dy).
\end{aligned}$$

In order to give further bounds we have to obtain results similar to the ones in Lemma 4.6.3 but for higher dimensions. More precisely we are aiming to show that for any $s \in [0, r]$

$$\int_{L_p \cap B_r} \frac{1 - \exp(-c_d^{-1} \, d_h(\exp_p(sv), y)t)}{d_h(\exp_p(sv), y)} \, \mathcal{H}^{d-1}(dy) \\ \leq \int_{L_p \cap B_r} \frac{1 - \exp(-c_d^{-1} \, d_h(p, y)t)}{d_h(p, y)} \, \mathcal{H}^{d-1}(dy) \tag{4.17}$$

holds. To show this let $A \coloneqq A(s, r) \coloneqq B_h(\exp_p(sv), r) \cap (B_r \cap L_p)$ be the set of points in $(B_r \cap L_p)$ having distance at most r from $\exp_p(sv)$. Further there exists an isometry $\varphi \in \mathbb{I}(L_p)$ with $\varphi(\exp_p(sv)) = p$. Therefore this isometry fulfills $\varphi(A) \subseteq B_r \cap L_p$. The integrands in (4.17) only depend on the distance between y and $\exp_p(sv)$ and on the distance between y and p. Thus we get

$$\int_A \frac{1 - \exp(c_d^{-1} d_h(\exp_p(sv), y)t)}{d_h(\exp_p(sv), y)} \mathcal{H}^{d-1}(dy) = \int_{\varphi(A)} \frac{1 - \exp(c_d^{-1} d_h(p, y)t)}{d_h(p, y)} \mathcal{H}^{d-1}(dy). \tag{4.18}$$

Now for any $x \in (L_p \cap B_r) \setminus A$ we get by definition of A that $d_h(\exp_p(sv), x) > r$ holds, whereas for all $x \in (L_p \cap B_r) \setminus \varphi(A)$ the distance $d_h(p, x)$ is bounded by r. Since the integrand is by Lemma 4.6.2 decreasing in the distance and since the sets $(L_p \cap B_r) \setminus A$ and $(L_p \cap B_r) \setminus \varphi(A)$ have the same $(d-1)$-dimensional Hausdorff measure we get

$$\int_{(L_p \cap B_r) \setminus A} \frac{1 - \exp(-c_d^{-1} \, d_h(\exp_p(sv), y)t)}{d_h(\exp_p(sv), y)} \, \mathcal{H}^{d-1}(dy) \\ \leq \int_{(L_p \cap B_r) \setminus \varphi(A)} \frac{1 - \exp(-c_d^{-1} \, d_h(p, y)t)}{d_h(p, y)} \, \mathcal{H}^{d-1}(dy)$$

and together with equation (4.18) the desired relation (4.17) follows. Now combining (4.15), (4.16) and (4.17) yields

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}(\Sigma_{f_{d-1}}(t)) \\
&\quad \leq \tilde{c}\,\omega_{d-1} \int_0^r \sinh^{d-2}(s) \int_{L_p \cap B_r} \frac{1 - \exp(-c_d^{-1} \, d_h(p, y)t)}{d_h(p, y)} \, \mathcal{H}^{d-1}(dy) \, ds \\
&\quad = \tilde{c}\,\mathcal{H}^{d-1}(L_p \cap B_r) \int_{L_p \cap B_r} \frac{1 - \exp(-c_d^{-1} \, d_h(p, y)t)}{d_h(p, y)} \, \mathcal{H}^{d-1}(dy) \\
&\quad = \tilde{c}\,\mathcal{H}^{d-1}(L_p \cap B_r) \int_{\mathbb{S}_p^{d-2}} \int_0^r \sinh^{d-2}(s) \frac{1 - \exp(-c_d^{-1} \, d_h(p, \exp_p(su))t)}{d_h(p, \exp_p(su))} \, ds \, \sigma_{d-2}(du) \\
&\quad = \tilde{c}\,\mathcal{H}^{d-1}(L_p \cap B_r)\,\omega_{d-1} \int_0^r \sinh^{d-2}(s) \frac{1 - \exp(-c_d^{-1} \, d_h(p, \exp_p(sv))t)}{d_h(p, \exp_p(sv))} \, ds \\
&\quad = \tilde{c}\,\mathcal{H}^{d-1}(L_p \cap B_r)\,\omega_{d-1} \int_0^r \sinh^{d-2}(s) \frac{1 - \exp(-c_d^{-1} \, st)}{s} \, ds.
\end{aligned}$$

Here we used spherical coordinates to replace the integration over $L_p \cap B_r$. This can be further simplified by using

$$\frac{\sinh(s)}{s} = \sum_{n=0}^{\infty} \frac{s^{2n}}{(2n+1)!} \le \sum_{n=0}^{\infty} \frac{s^{2n}}{(2n)!} = \cosh(s).$$

Finally, we conclude

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}(\Sigma_{f_{d-1}}(t)) \\
&\le \tilde{c}\,\mathcal{H}^{d-1}(L_p \cap B_r)\,\omega_{d-1} \int_0^r \sinh^{d-3}(s)\cosh(s)(1-\exp(-c_d^{-1}\,st)\;ds \\
&\le \tilde{c}\,\mathcal{H}^{d-1}(L_p \cap B_r)\,\omega_{d-1} \int_0^r \sinh^{d-3}(s)\cosh(s)\;ds \\
&< \infty.
\end{aligned}$$

□

Remark: The limit behaviour of the variance of the surface area for growing time t is the same as in the spherical setting, as shown in [47]. Heuristically one might argue that the underlying reason for this is that due to the bounded observation window the curvature of the underlying space does not play an important role. If one likes to compare the behaviour of the variance for growing time t with the one in Euclidean space, one has to consider [104, Theorem 3] and rescale the observation window. Doing so it follows that in Euclidean space and stationary set up the variance grows with logarithmic speed in t for $d = 2$ and converges to a constant in the limit case for $d \ge 3$.

To show the behaviour in Euclidean space let $Y_{t,euc}(W)$ be the stationary Euclidean splitting tessellation (i.e. STIT-tessellation) at time $t > 0$ inside a fixed compact window $W \subset \mathbb{R}^d$. For proper definition of this process see for example [81]. Further let $Y_{t,euc}(\mathbb{R}^d)$ be the corresponding splitting tessellation of the whole space at time $t > 0$. Using [81, Theorem 1, Lemma 5] we show in a first step

$$\begin{aligned}
Y_{t,euc}(W) \overset{d}{=} Y_{t,euc}(\mathbb{R}^d) \cap W \overset{d}{=} t^{-1}Y_{1,euc}(\mathbb{R}^d) \cap W = t^{-1}\left(Y_{1,euc}(\mathbb{R}^d) \cap tW\right) \\
\overset{d}{=} t^{-1}Y_{1,euc}(tW).
\end{aligned}$$

Now applying for $d = 2$ the results from [104, Theorem 3] gives

$$\begin{aligned}
\mathbb{V}\mathrm{ar}(\mathcal{H}^1(Y_{t,euc}(W))) &= \mathbb{V}\mathrm{ar}(\mathcal{H}^1(t^{-1}Y_{1,euc}(tW))) \\
&= \mathbb{V}\mathrm{ar}(t^{-1}\mathcal{H}^1(Y_{1,euc}(tW))) \\
&= t^{-2}\;\mathbb{V}\mathrm{ar}(\mathcal{H}^1(Y_{1,euc}(tW))) \\
&\sim \pi\,\mathcal{H}^2(W)\;\log(t),
\end{aligned}$$

where $\mathcal{H}^1(Y_{t,euc}(W))$ is the total surface area of the process at time t. The same way we get for $d \geq 3$ and the total surface area $\mathcal{H}^{d-1}(Y_{t,euc}(W))$

$$\begin{aligned}\mathbb{V}\mathrm{ar}(\mathcal{H}^{d-1}(Y_{t,euc}(W))) &= \mathbb{V}\mathrm{ar}(\mathcal{H}^{d-1}(t^{-1}Y_{1,euc}(tW))) \\ &= \mathbb{V}\mathrm{ar}(t^{-(d-1)}\mathcal{H}^{d-1}(Y_{1,euc}(tW))) \\ &= t^{-2(d-1)}\,\mathbb{V}\mathrm{ar}(\mathcal{H}^{d-1}(Y_{1,euc}(tW))) \\ &\sim \frac{d-1}{2}E_2(W),\end{aligned}$$

where $E_2(W)$ is the 2-energy of W (i.e. a constant), given by

$$E_2(W) = \int_W \int_W \|x-y\|^{-2}\,dx\,dy.$$

4.6.3 Limit behaviour of the variance of the total surface area for growing radius r

In this section we will investigate how the variance of the surface area behaves for growing spherical windows. We will start with the 2-dimensional case and treat higher dimensions afterwards in Lemma 4.6.7.

Lemma 4.6.6. *Let $d = 2$ and $t > 0$ be fixed. Then there are constants $c^{(1)}(2,t)$, $C^{(1)}(2,t) \in (0,\infty)$ such that for $r \geq 1$*

$$c^{(1)}(2,t)\,e^r \leq \mathbb{V}ar(\Sigma_{f_1}(t)) \leq C^{(1)}(2,t)\,e^r.$$

Proof. Using Theorem 4.6.1 we get a representation for the variance. Applying the representation for the measure μ_1 stated in (2.3) yields

$$\mathbb{V}\mathrm{ar}(\Sigma_{f_1}(t)) = c_2 \int_0^r \cosh(s) \int_{H_1(s)\cap B_r} \int_{H_1(s)\cap B_r} \frac{1-\exp(-a\,d_h(x,y))}{d_h(x,y)}\,\mathcal{H}^1(dx)\,\mathcal{H}^1(dy)\,ds,$$

where we denote by $a = a(t) = \omega_1\,\omega_3\,\omega_2^{-2} t$ and let $H_1(s)$ be an arbitrary totally geodesic line at distance s from the origin. We will use Lemma 3.2.5 which gives bounds on the intersection length of $H_1(s)$ with B_r and Lemma 4.6.2 showing that the integrand is decreasing. Further we apply Lemma 4.6.3 (ii). This yields

$$\begin{aligned}&\mathbb{V}\mathrm{ar}(\Sigma_{f_1}(t)) \\ &\leq c_2 \int_0^r \cosh(s) \int_{-(r-s+\log(2))}^{r-s+\log(2)} \int_{-(r-s+\log(2))}^{r-s+\log(2)} \frac{1-\exp(-a\,|x-y|)}{|x-y|}\,dx\,dy\,ds \\ &\leq c_2 \int_0^r \cosh(s) \int_{-(r-s+\log(2))}^{r-s+\log(2)} \int_{-(r-s+\log(2))}^{r-s+\log(2)} \frac{1-\exp(-a\,|0-y|)}{|0-y|}\,dx\,dy\,ds \\ &= 4\,c_2 \int_0^r \cosh(s)(r-s+\log(2)) \int_0^{r-s+\log(2)} \frac{1-\exp(-a\,y)}{y}\,dy\,ds. \end{aligned} \tag{4.19}$$

Here the integrand is bounded by

$$\frac{1-\exp(-a\,y)}{y} \leq a, \qquad y \in (0,\infty).$$

Combining this inequality with (4.19) gives

$$\begin{aligned}\mathbb{V}\mathrm{ar}(\Sigma_{f_1}(t)) &\le 4\,c_2 \int_0^r \cosh(s)(r-s+\log(2)) \int_0^{r-s+\log(2)} a\ dy\ ds\\ &= 4a\,c_2 \int_0^r \cosh(s)(r-s+\log(2))^2\ ds\\ &\le 4a\,c_2 \int_0^r e^s(r-s+\log(2))^2\ ds\\ &\le C^{(1)}(2,t)\,e^r.\end{aligned}$$

In order to derive the lower bound, we use $r \ge 1$ and do the following calculation using the bounds from Lemma 3.2.5 on the intersection length and Lemma 4.6.3 (i)

$$\begin{aligned}\mathbb{V}\mathrm{ar}(\Sigma_{f_1}(t)) &\ge c_2 \int_0^r \cosh(s) \int_{-(r-s)}^{r-s} \int_{-(r-s)}^{r-s} \frac{1-\exp(-a\,|x-y|)}{|x-y|}\ dx\ dy\ ds\\ &\ge c_2 \int_0^r \cosh(s) \int_{-(r-s)}^{r-s} \int_{-(r-s)}^{r-s} \frac{1-\exp(-a\,|r-s-y|)}{|r-s-y|}\ dx\ dy\ ds\\ &= 2\,c_2 \int_0^r \cosh(s)(r-s) \int_{-(r-s)}^{r-s} \frac{1-\exp(-a\,|r-s-y|)}{|r-s-y|}\ dy\ ds\end{aligned}$$

In a next step we decrease the integration area of the first integral and rewrite the second integration. This gives

$$\begin{aligned}\mathbb{V}\mathrm{ar}(\Sigma_{f_1}(t)) &\ge 2\,c_2 \int_0^{r-0,5} \cosh(s)(r-s) \int_0^{2(r-s)} \frac{1-\exp(-a\,u)}{u}\ du\ ds\\ &\ge c_2 \int_0^{r-0,5} e^s(r-s) \int_0^{2(r-s)} \frac{1-\exp(-a\,u)}{u}\ du\ ds\\ &\ge c_2 \int_0^{r-0,5} e^s(r-s) \int_0^{\min\{2(r-s),1\}} \frac{1-\exp(-a\,u)}{u}\ du\ ds\\ &= c_2 \int_0^{r-0,5} e^s(r-s) \int_0^1 \frac{1-\exp(-a\,u)}{u}\ du\ ds\\ &\ge c^{(1)}(2,t)\,e^r.\end{aligned}$$

The last inequality holds, since the second integral does not depend on s anymore and is therefore a constant only depending on t and the dimension $d = 2$. □

Now we turn to the case of higher dimensions.

Lemma 4.6.7. *Let $d \ge 3$, $t > 0$ be fixed and $r \ge 1$. Then there are constants $c^{(d-1)}(d,t)$, $C^{(d-1)}(d,t) \in (0,\infty)$ such that*

$$c^{(2)}(3,t)\,e^{2r} \le \mathbb{V}ar(\Sigma_{f_2}(t)) \le C^{(2)}(3,t)\log(r)\,e^{2r}, \qquad d = 3$$

and

$$c^{(d-1)}(d,t)\,r^{-1}e^{2r(d-2)} \le \mathbb{V}ar(\Sigma_{f_{d-1}}(t)) \le C^{(d-1)}(d,t)\,r^{-1}e^{2r(d-2)}, \qquad d \ge 4.$$

Proof. In this proof c will indicate a constant depending only on d and t which may vary from line to line. We will start with showing the **lower bound**. Using Theorem 4.6.1 and the representation for the measure μ_{d-1} in (2.3) we get a lower bound for the variance

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}(\Sigma_{f_{d-1}})\\
=&c_d \int_0^r \cosh^{d-1}(s) \int_{L_{d-1}(s)\cap B_r} \int_{L_{d-1}(s)\cap B_r} \frac{1-\exp(-a\, d_h(x,y))}{d_h(x,y)}\, \mathcal{H}^{d-1}(dx)\, \mathcal{H}^{d-1}(dy)\, ds\\
\geq&\frac{c_d}{2^{d-1}} \int_0^r e^{s(d-1)} \int_{L_{d-1}(s)\cap B_r} \int_{L_{d-1}(s)\cap B_r} \frac{1-\exp(-a\, d_h(x,y))}{d_h(x,y)}\, \mathcal{H}^{d-1}(dx)\, \mathcal{H}^{d-1}(dy)\, ds,
\end{aligned}$$

where we set $a := a(t,d) := \omega_{d+1}\,\omega_1\,(\omega_d\,\omega_2)^{-1}t$. By using that the integrand is by Lemma 4.6.2 decreasing we derive for any $s\in[0,r],\ x,y\in L_{d-1}(s)\cap B_r$

$$\frac{1-\exp(-a\, d_h(x,y))}{d_h(x,y)} \geq \frac{1-\exp(-2ra)}{2r} \geq \frac{1-\exp(-2a)}{2r}. \tag{4.20}$$

Now let $d\geq 4$. We make use of (4.20) and apply the inequalities derived in Lemma 3.2.4 to get

$$\begin{aligned}
\mathbb{V}\mathrm{ar}(\Sigma_{f_{d-1}}) &\geq c \int_0^r e^{s(d-1)} \int_{L_{d-1}(s)\cap B_r} \int_{L_{d-1}(s)\cap B_r} \frac{1-\exp(-2a)}{2r}\, \mathcal{H}^{d-1}(dx)\, \mathcal{H}^{d-1}(dy)\, ds\\
&= c \int_0^r e^{s(d-1)} (\mathcal{H}^{d-1}(L_{d-1}(s)\cap B_r))^2 r^{-1}\, ds\\
&\geq c \int_0^{r-1/2} e^{s(d-1)} (\mathcal{H}^{d-1}(L_{d-1}(s)\cap B_r))^2 r^{-1}\, ds\\
&\geq c \int_0^{r-1/2} e^{s(d-1)} e^{2(r-s)(d-2)} r^{-1}\, ds\\
&= c\, r^{-1} e^{2r(d-2)} \int_0^{r-1/2} e^{s(-d+3)}\, ds\\
&\geq c^{(d-1)}(d,t) r^{-1} e^{2r(d-2)}.
\end{aligned} \tag{4.21}$$

For $d=3$ we can use the same arguments to get

$$\mathbb{V}\mathrm{ar}(\Sigma_{f_2}) \geq c^{(2)}(3,t)\, e^{2r}.$$

The only difference for $d=3$ is that the integral in the penultimate line of (4.21) is not bounded in r and therefore cancels out the term r^{-1}.

Now we show the **upper bound**. We once again start with the case $d\geq 4$. As above we derive the following representation of the variance

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}(\Sigma_{f_{d-1}})\\
=&c_{d-1} \int_0^r \cosh^{d-1}(s) \int_{L_{d-1}(s)\cap B_r} \int_{L_{d-1}(s)\cap B_r} \frac{1-\exp(-a\, d_h(x,y))}{d_h(x,y)}\, \mathcal{H}^{d-1}(dx)\, \mathcal{H}^{d-1}(dy)\, ds.
\end{aligned}$$

Using the same arguments as in the proof of Corollary 4.6.5 (see equation (4.17)) this can be further bounded by

$$
\begin{aligned}
&\mathbb{V}\mathrm{ar}(\Sigma_{f_{d-1}})\\
&\quad\le c\int_0^r \cosh^{d-1}(s)\int_{L_{d-1}(s)\cap B_r}\int_{L_{d-1}(s)\cap B_r}\frac{1-\exp(-a\,d_h(p,y))}{d_h(p,y)}\,\mathcal{H}^{d-1}(dx)\,\mathcal{H}^{d-1}(dy)\,ds\\
&\quad= c\int_0^r \cosh^{d-1}(s)\mathcal{H}^{d-1}(L_{d-1}(s)\cap B_r)\int_{L_{d-1}(s)\cap B_r}\frac{1-\exp(-a\,d_h(p,y))}{d_h(p,y)}\,\mathcal{H}^{d-1}(dy)\,ds\\
&\quad\le c\int_0^r e^{s(d-1)}e^{(r-s)(d-2)}\int_{L_{d-1}(s)\cap B_r}\frac{1-\exp(-a\,d_h(p,y))}{d_h(p,y)}\,\mathcal{H}^{d-1}(dy)\,ds,\\
&\quad= c\,e^{r(d-2)}\int_0^r e^{s}\int_{L_{d-1}(s)\cap B_r}\frac{1-\exp(-a\,d_h(p,y))}{d_h(p,y)}\,\mathcal{H}^{d-1}(dy)\,ds,
\end{aligned}
$$

where the last inequality used Lemma 3.2.4 and an upper bound for the cosh-function. Now we use spherical coordinates on $L_{d-1}(s)\cap B_r$ and Lemma 4.6.2 to derive

$$
\begin{aligned}
&\mathbb{V}\mathrm{ar}(\Sigma_{f_{d-1}})\\
&\quad\le c\,e^{r(d-2)}\int_0^r e^{s}\int_0^{r-s+\log(2)}\sinh^{d-2}(\ell)\frac{1-\exp(-a\,\ell)}{\ell}\,d\ell\,ds\\
&\quad= c\,e^{r(d-2)}\int_0^r e^{s}\bigg(\int_0^{r/4}\sinh^{d-2}(\ell)\frac{1-\exp(-a\,\ell)}{\ell}\,d\ell\\
&\qquad\quad+\int_{r/4}^{r-s+\log(2)}\sinh^{d-2}(\ell)\frac{1-\exp(-a\,\ell)}{\ell}\,d\ell\bigg)ds\\
&\quad\le c\,e^{r(d-2)}\int_0^r e^{s}\bigg(\int_0^{r/4}\sinh^{d-2}(\ell)\,a\,d\ell+\int_{r/4}^{r-s+\log(2)}\sinh^{d-2}(\ell)r^{-1}\,d\ell\bigg)ds\\
&\quad\le c\,e^{r(d-2)}\int_0^r e^{s}\bigg(\int_0^{r/4}\sinh^{d-2}(\ell)\,a\,d\ell+\int_{0}^{r-s+\log(2)}\sinh^{d-2}(\ell)r^{-1}\,d\ell\bigg)ds\\
&\quad\le c\,e^{r(d-2)}\int_0^r e^{s}\left(a\,e^{\frac{r}{4}(d-2)}+r^{-1}e^{(r-s)(d-2)}\right)ds\\
&\quad= c\,e^{r(d-2)}\int_0^r (a\,e^{\frac{r}{4}(d-2)}e^{s}+r^{-1}e^{r(d-2)}e^{s(-d+3)})ds\\
&\quad\le c\,a\,e^{r\left(\frac{5d}{4}-\frac{3}{2}\right)}+c\,r^{-1}e^{2r(d-2)}\\
&\quad\le C^{(d-1)}(d,t)\,r^{-1}e^{2r(d-2)}.
\end{aligned}
$$

Here we depended on $d\ge 4$ in the third ultimate line in order to bound the integral $\int_0^r e^{s(-d+3)}\,ds$ by a constant. Further we used $d\ge 4$ in order to show that the second summand in the penultimate line is indeed the leading term for $r\to\infty$.

Now we turn to the case $d=3$. Here we have to be a little bit more careful. In order to shorten the notation, we consider the two inner integrals first. Using the same arguments as in the proof of Corollary 4.6.5 (see equation (4.17)) and spherical coordinates on $L_2(s)\cap B_r$ we get

$$
\begin{aligned}
&\int_{L_2(s)\cap B_r}\int_{L_2(s)\cap B_r}\frac{1-\exp(-a\,d_h(x,y))}{d_h(x,y)}\,\mathcal{H}^2(dx)\,\mathcal{H}^2(dy)\\
&\quad\le\int_{L_2(s)\cap B_r}\int_{L_2(s)\cap B_r}\frac{1-\exp(-a\,d_h(p,y))}{d_h(p,y)}\,\mathcal{H}^2(dx)\,\mathcal{H}^2(dy)\\
&\quad=\mathcal{H}^2(L_2(s)\cap B_r)\int_{L_2(s)\cap B_r}\frac{1-\exp(-a\,d_h(p,y))}{d_h(p,y)}\,\mathcal{H}^2(dy)\\
&\quad\le c\mathcal{H}^2(L_2(s)\cap B_r)\int_0^{r-s+\log(2)}\sinh(\ell)\frac{1-\exp(-a\,\ell)}{\ell}\,d\ell.
\end{aligned}
\tag{4.22}
$$

Now we use Lemma 3.2.4 and the monotonicity of the integrand (shown in Lemma 4.6.2) to continue the inequalities in (4.22)

$$
\begin{aligned}
&\mathcal{H}^2(L_2(s)\cap B_r)\int_0^{r-s+\log(2)}\sinh(\ell)\frac{1-\exp(-a\,\ell)}{\ell}\,d\ell\\
&\le c\,e^{r-s}\int_0^{r-s+\log(2)}\sinh(\ell)\frac{1-\exp(-a\,\ell)}{\ell}\,d\ell\\
&\le c\,e^{r-s}\int_0^{r-s+\log(2)}e^{\ell}\,\frac{1-\exp(-a\,\ell)}{\ell}\,d\ell\\
&= c\,e^{r-s}\left(\int_0^{\log(2)}e^{\ell}\,\frac{1-\exp(-a\,\ell)}{\ell}\,d\ell+\int_{\log(2)}^{r-s+\log(2)}e^{\ell}\,\frac{1-\exp(-a\,\ell)}{\ell}\,d\ell\right)\\
&\le c\,e^{r-s}\Bigg(\underbrace{\int_0^{\log(2)}e^{\ell}a\;d\ell}_{\le c}+\underbrace{\int_{\log(2)}^{r-s+\log(2)}e^{\ell}\,\ell^{-1}\;d\ell}_{(*)}\Bigg).
\end{aligned}
\tag{4.23}
$$

The first integral can be easily bounded. For the second integral $(*)$ we get

$$
\int_{\log(2)}^{r-s+\log(2)}e^{\ell}\ell^{-1}\;d\ell=\mathrm{Ei}(r-s+\log(2))-\mathrm{Ei}(\log(2))
\tag{4.24}
$$

where $\mathrm{Ei}(\cdot)$ is the exponential integral (see [86, Chapter 6]). Combining (4.21)-(4.23) and the representation of the variance gives

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}(\Sigma_{f_2})&\le c\int_0^r\cosh^2(s)\int_{L_2(s)\cap B_r}\int_{L_2(s)\cap B_r}\frac{1-\exp(-a\,d_h(x,y))}{d_h(x,y)}\,\mathcal{H}^2(dx)\,\mathcal{H}^2(dy)\,ds\\
&\le c\int_0^r\cosh^2(s)e^{r-s}(1+\mathrm{Ei}(r-s+\log(2))-\mathrm{Ei}(\log(2))\,ds\\
&\le c\,e^r\int_0^r e^s(1+\mathrm{Ei}(r-s+\log(2))-\mathrm{Ei}(\log(2)))\,ds\\
&= c\,e^r\int_0^r e^{r-s}(1+\mathrm{Ei}(s+\log(2))-\mathrm{Ei}(\log(2)))\,ds\\
&= c\,e^{2r}\int_0^r e^{-s}(1+\mathrm{Ei}(s+\log(2))-\mathrm{Ei}(\log(2)))\,ds.
\end{aligned}
\tag{4.25}
$$

Since the constant terms in the inner brackets are asymptotically negligible, it remains to solve the integral

$$\int_0^r e^{-s}\mathrm{Ei}(s+\log(2)) = [2\log(s+\log(2)) - e^{-s}\mathrm{Ei}(s+\log(2))]_0^r$$
$$= 2\log(r+\log(2)) - e^{-r}\mathrm{Ei}(r+\log(2)) - 2\log(\log(2)) + \mathrm{Ei}(\log(2)).$$

All summands are in $\mathcal{O}(\log(r))$ except the first one. Combining this with (4.25) yields

$$\mathbb{V}\mathrm{ar}(\Sigma_{f_2}) \leq C^{(2)}(3,t)\log(r)\,e^{2r}.$$

□

Remark 4.6.2. The results presented in this section are almost similar to the ones for hyperbolic Poisson hyperplane tessellations (see Section 3.2). For $d = 2$ the limit behaviour in r is the very same, whereas for $d = 3$ an additional factor $\log(r)$ appears in the upper bound. For $d \geq 4$ the lower and the upper bound contain the additional factor r^{-1}.

4.7 Mixing property

Similar to the Euclidean case, considered in [58], we will investigate the mixing property of hyperbolic splitting tessellations. The first Subsection 4.7.1 gives an introduction to the concept and states some definitions and a lemma needed to prove the main result. The result and its proof are treated in Section 4.7.2.

4.7.1 Mixing introduction

The concept of mixing is investigated for many different random mosaics (some classical results can be found in [103, Chapter 10.5]). Heuristically speaking, a mosaic is called mixing, if the dependency between two events vanishes for growing distance. Formally a Euclidean random mosaic X is called (α-) mixing, if for all compact sets $C_1, C_2 \in \mathcal{C}(\mathcal{F}'(\mathbb{R}))$

$$\lim_{\|x\|\to\infty} \mathbb{P}(X \cap C_1 = \varnothing,\ X \cap (C_2 + x) = \varnothing) = \mathbb{P}(X \cap C_1 = \varnothing)\,\mathbb{P}(X \cap C_2 = \varnothing)$$

holds. For different models the speed of this convergence differs. While for Euclidean stationary Voronoi mosaics the β-mixing rate (for definitions in this case see [71, Chapter 2]) decays exponentially fast in $\|x\|$ (see [36]), the β-mixing rate for Euclidean STIT tessellations decays at most linearly in $\|x\|$ (see [71, Theorem 5.3]). This is due to the fact that early cracks have a long range and therefore imply long range dependencies in the mosaic. Mixing properties play an important role in fields such as Poisson approximation (see [21]). Also central limit theorems ([36]) and extreme value properties ([22]) rely on mixing properties.

In order to state an auxiliary lemma, recall the definition of the capacity function and the U-function

$$T_t(C) = \mathbb{P}(Z_t \cap C \neq \varnothing), \qquad U_t(C) = \mathbb{P}(Z_t \cap C = \varnothing), \qquad C \in \mathcal{C}(\mathbb{H}^d)$$

For a compact set $A \in \mathcal{C}(\mathbb{H}^d)$ and $a > 0$ we let $r = r(A) \in \mathbb{N}$ be the smallest value such that $A \subseteq \operatorname{int}(B_r)$ holds and define the constant

$$\lambda_{A,a} := \exp(a\,\mu_{d-1}(\mathbb{H}_{d-1}\langle B_{r(A)}\rangle)), \tag{4.26}$$

only depending on a, A and the space dimension d. Having this definition, we can prove the following lemma, which will be used in the proof of Theorem 4.7.2.

Lemma 4.7.1. *For $A, B \in \mathcal{C}(\mathbb{H}^d)$, $0 \le t < a$ the following inequality holds*

$$|U_{a-t}(A)U_{a-t}(B) - U_a(A)U_a(B)| \le t(\lambda_{A,a} + \lambda_{B,a})$$

where $\lambda_{A,a}$, $\lambda_{B,a}$ are defined in (4.26).

Proof. First the expression is rewritten by the capacity functional

$$\begin{aligned}
&|U_{a-t}(A)U_{a-t}(B) - U_a(A)U_a(B)|\\
=&(1 - T_{a-t}(A))(1 - T_{a-t}(B)) - (1 - T_a(A))(1 - T_a(B))\\
=&(T_a(A) - T_{a-t}(A)) + (T_a(B) - T_{a-t}(B)) - (T_a(A)T_a(B) - T_{a-t}(A)T_{a-t}(B))\\
\le&(T_a(A) - T_{a-t}(A)) + (T_a(B) - T_{a-t}(B)).
\end{aligned}$$

Let $r = r(A) \in \mathbb{N}$ be the smallest value such that $A \subseteq \operatorname{int}(B_r)$ holds. We consider both summands separately and restrict ourself to the process $Y_a = Y_a(B_r)$ on B_r. Now denote by N_{a-t} the number of hyperplanes that appear up to time $a - t$ for $t \in [0, a]$. For $n \in \mathbb{N}_0$ we set $p_n := \mathbb{P}(N_{a-t} = n) > 0$. By construction of the process the tessellation almost surely contains $N_{a-t} + 1$ cells at time $a - t$. We will call them $c_1, \dots, c_{N_{a-t}+1}$ and define $A_i := A \cap c_i$, $i = 1, \dots, N_{a-t} + 1$. This gives

$$\begin{aligned}
T_a(A) - T_{a-t}(A) &= U_{a-t}(A) - U_a(A)\\
&= \mathbb{P}(Z_{a-t} \cap A = \varnothing) - \mathbb{P}(Z_a \cap A = \varnothing)\\
&= \mathbb{P}(Z_{a-t} \cap A = \varnothing) - \mathbb{P}(Z_{a-t} \cap A = \varnothing)\mathbb{P}(Z_a \cap A = \varnothing \mid Z_{a-t} \cap A = \varnothing)\\
&= U_{a-t}(A)\mathbb{P}(A \cap Z_a \ne \varnothing \mid A \cap Z_{a-t} = \varnothing)\\
&\le \mathbb{P}(A \cap Z_a \ne \varnothing \mid A \cap Z_{a-t} = \varnothing),
\end{aligned} \tag{4.27}$$

where we used the probability of the complementary event in the penultimate step. Now we are aiming to further bound the expression in the equation above. To do so we condition on the number of hyperplanes that appear up to time $a-t$

$$
\begin{aligned}
&\mathbb{P}(A \cap Z_a \neq \varnothing \mid A \cap Z_{a-t} = \varnothing) \\
&\quad = \sum_{n\geq 0} p_n \, \mathbb{P}(A \cap Z_a \neq \varnothing \mid N_{a-t} = n,\ A \cap Z_{a-t} = \varnothing) \\
&\quad = \sum_{n\geq 0} p_n \, \mathbb{P}(\exists i \in \{1,\dots,n+1\}:\ A_i \cap Z_a \neq \varnothing \mid N_{a-t} = n,\ A \cap Z_{a-t} = \varnothing) \\
&\quad \leq \sum_{n\geq 0} p_n \sum_{i=1}^{n+1} \mathbb{P}(A_i \cap Z_a \neq \varnothing \mid N_{a-t} = n,\ A \cap Z_{a-t} = \varnothing) \\
&\quad \overset{(*)}{=} \sum_{n\geq 0} p_n \sum_{i=1}^{n+1} \mathbb{P}(A_i \cap Z_a \neq \varnothing \mid N_{a-t} = n,\ A_i \cap Z_{a-t} = \varnothing) \\
&\quad \leq \sum_{n\geq 0} p_n \sum_{i=1}^{n+1} \mathbb{P}(\mathrm{conv}(A_i) \cap Z_a \neq \varnothing \mid N_{a-t} = n,\ A_i \cap Z_{a-t} = \varnothing) \\
&\quad = \sum_{n\geq 0} p_n \sum_{i=1}^{n+1} \mathbb{P}(\mathrm{conv}(A_i) \cap Z_a \neq \varnothing \mid N_{a-t} = n,\ \mathrm{conv}(A_i) \cap Z_{a-t} = \varnothing).
\end{aligned}
\tag{4.28}
$$

Here the last equality holds, since $\{A_i \cap Z_{a-t} = \varnothing\} = \{\mathrm{conv}(A_i) \cap Z_{a-t} = \varnothing\}$ holds. The relation $\{A_i \cap Z_{a-t} = \varnothing\} \supseteq \{\mathrm{conv}(A_i) \cap Z_{a-t} = \varnothing\}$ holds trivially. In order to show

$$
\{A_i \cap Z_{a-t} = \varnothing\} \subseteq \{\mathrm{conv}(A_i) \cap Z_{a-t} = \varnothing\},
$$

we assume that there exists a realisation such that $A_i \cap Z_{a-t} = \varnothing$ and $\mathrm{conv}(A_i) \cap Z_{a-t} \neq \varnothing$ hold. This implies that there exist two points $x_1, x_2 \in A_i \subseteq c_i$ and an $y \in (x_1, x_2) \in \mathrm{conv}(A_i)$ such that $y \in Z_{a-t},\ x_1, x_2 \notin Z_{a-t}$ holds. This in turn implies that x_1 and x_2 are not contained in the same cell, since the cells are convex. This is a contradiction to our choice of x_1 and x_2.

In order to justify the equality marked by $(*)$ in (4.28), we do the following calculations. To shorten the notation, we define the event B_i for $i \in \{1,\dots,n+1\}$ by

$$
B_i \coloneqq B_i(a,t) \coloneqq \bigcap_{k \in \{1,\dots,n+1\} \setminus \{i\}} \{A_k \cap Z_{a-t} = \varnothing\}.
$$

Using this, one can see

$$
\begin{aligned}
&\mathbb{P}(A_i \cap Z_a \neq \varnothing \mid N_{a-t} = n,\ A \cap Z_{a-t} = \varnothing) \\
&\quad = \frac{\mathbb{P}(A_i \cap Z_a \neq \varnothing,\ N_{a-t} = n,\ A_1 \cap Z_{a-t} = \varnothing, \dots,\ A_{n+1} \cap Z_{a-t} = \varnothing)}{\mathbb{P}(\ N_{a-t} = n,\ A_1 \cap Z_{a-t} = \varnothing, \dots,\ A_{n+1} \cap Z_{a-t} = \varnothing)} \\
&\quad = \frac{\mathbb{P}(A_i \cap Z_a \neq \varnothing,\ N_{a-t} = n,\ A_i \cap Z_{a-t} = \varnothing, B_i)}{\mathbb{P}(\ N_{a-t} = n,\ A_i \cap Z_{a-t} = \varnothing, B_i)} \\
&\quad = \frac{\mathbb{P}(A_i \cap Z_a \neq \varnothing,\ N_{a-t} = n,\ A_i \cap Z_{a-t} = \varnothing)}{\mathbb{P}(N_{a-t} = n,\ A_i \cap Z_{a-t} = \varnothing)} \\
&\qquad \times \frac{\mathbb{P}(B_i \mid A_i \cap Z_a \neq \varnothing,\ N_{a-t} = n,\ A_i \cap Z_{a-t} = \varnothing)}{\mathbb{P}(B_i \mid N_{a-t} = n,\ A_i \cap Z_{a-t} = \varnothing)}.
\end{aligned}
\tag{4.29}
$$

Here the first factor is equal to $\mathbb{P}(A_i \cap Z_a \neq \varnothing \mid N_{a-t} = n,\ A_i \cap Z_{a-t} = \varnothing)$. Therefore it remains to show that the second factor is equal to 1. To do so, we let $C_i \coloneqq C_i(a,t)$ be the event that the process hits A_i within the time interval $(a-t,a]$. Using this, we get

$$\begin{aligned}\mathbb{P}(B_i \mid A_i \cap Z_a \neq \varnothing,\ N_{a-t} = n,\ A_i \cap Z_{a-t} = \varnothing)\\ = \mathbb{P}(B_i \mid C_i,\ N_{a-t} = n,\ A_i \cap Z_{a-t} = \varnothing)\\ = \mathbb{P}(B_i \mid N_{a-t} = n,\ A_i \cap Z_{a-t} = \varnothing),\end{aligned}$$

where the last equality holds, since B_i only depends on the behaviour of the process within the time interval $[0,a-t]$ and C_i on the other hand only on its behaviour within the time interval $(a-t,a]$. Therefore the second factor in the last line of (4.29) is equal to zero and therefore $(*)$ holds.

Now we take a closer look at the probability of the event

$$\{\mathrm{conv}(A_i) \cap Z_a \neq \varnothing \mid N_{a-t} = n,\ \mathrm{conv}(A_i) \cap Z_{a-t} = \varnothing\}$$

and hence the probability of hitting the set $\mathrm{conv}(A_i)$ within th time interval $(a-t,a]$, given that $\mathrm{conv}(A_i)$ is contained in the interior of a cell c_i at time $a-t$. By the definition of the process, this probability is the same as the probability of $\tilde{Y}_t(c_i)$ hitting $\mathrm{conv}(A_i)$, where $\tilde{Y}_t(c_i)$ is a splitting process within the cell c_i. This probability does not depend on c_i and is given by

$$\begin{aligned}\mathbb{P}(\mathrm{conv}(A_i) \cap Z_a \neq \varnothing \mid N_{a-t} = n,\ \mathrm{conv}(A_i) \cap Z_{a-t} = \varnothing) &= \mathbb{P}(\mathrm{conv}(A_i) \cap Z_{\tilde{Y}_t(c_i)} \neq \varnothing)\\ &= 1 - U_t(\mathrm{conv}(A_i))\\ &\leq 1 - U_t(\mathrm{conv}(A))\\ &= 1 - e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle \mathrm{conv}(A)\rangle)}.\end{aligned}$$

We combine this inequality with (4.27) and (4.28) to derive

$$\begin{aligned}T_a(A) - T_{a-t}(A) &\leq \sum_{n\geq 0} p_n \sum_{i=1}^{n+1} \left(1 - e^{-t\,\mu_{d-1}(\mathbb{H}_{d-1}\langle \mathrm{conv}(A)\rangle)}\right)\\ &\leq \sum_{n\geq 0} p_n \sum_{i=1}^{n+1} t\,\mu_{d-1}(\mathbb{H}_{d-1}\langle \mathrm{conv}(A)\rangle)\\ &= t\,\mu_{d-1}(\mathbb{H}_{d-1}\langle \mathrm{conv}(A)\rangle) \sum_{n\geq 0} p_n(n+1)\\ &= t\,\mu_{d-1}(\mathbb{H}_{d-1}\langle \mathrm{conv}(A)\rangle)\mathbb{E}[N_{a-t}+1].\end{aligned}$$

Here we used $1 - \exp(-x) \leq x,\ x \geq 0$ for the second inequality. By Lemma 4.2.1, we know that the number of cells at a certain time $t > 0$ can be bounded by a fitting random variable, having geometric distribution with parameter $\exp(-a\,\mu_{d-1}(\mathbb{H}_{d-1}\langle()\rangle B_r))$. Therefore we get the following inequality

$$\mathbb{E}[N_{a-t}] \leq \mathbb{E}[N_a] \leq \exp(a\,\mu_{d-1}(\mathbb{H}_{d-1}\langle()\rangle B_r)).$$

Finally, using the definition of $\lambda_{A,a}$, given in (4.26), shows the claim

$$T_a(A) - T_{a-t}(A) \leq t\,\lambda_{A,a}.$$

□

4.7.2 Mixing main result

For a fixed compact set $A \in \mathcal{C}$, we define $\mathcal{F}^A$ to be the event that A gets hit by the process $(\mathbf{Y}_t)_{t\geq 0}$. Now we are in the position to state and prove the main theorem of Section 4.7.

Theorem 4.7.2. *Let $x \in \mathbb{H}^d$, $a > 0$ and $\tau_x \in I(\mathbb{H}^d)$ be an isometry with $\tau_x(p) = x$. Further let $A, B \in \mathcal{C}(\mathbb{H}^d)$ such that A and B do not just contain a single point. Then*

$$|\mathbb{P}_a(\mathcal{F}^A \cap \mathcal{F}^{\tau_x B}) - \mathbb{P}_a(\mathcal{F}^A)\cdot\mathbb{P}_a(\mathcal{F}^B)| \in \mathcal{O}(d_h(p,x)^{-1}). \tag{4.30}$$

Proof. The first results that are used to prove the theorem will be stated for two general sets $A, B \in \mathcal{C}(\mathbb{H}^d)$ instead of considering the two sets $A, \tau_x B$ directly. Let us consider the splitting process $(Y_t)_{t\geq 0}$ inside the window $W = \mathrm{conv}(A \cup B)$. We set

$$t_1 := \inf\{t \geq 0 \mid Y_0 \neq Y_t\}$$

as the time of the first appearance of a hyperplane dividing W. This hyperplane will be called $H_1 \in \mathbb{H}_{d-1}[W]$. Now consider the event

$$\Gamma_{A,B} := \Gamma_{A,B}(a) := \{H_1 \in \langle A|B\rangle,\ t_1 \leq a\},$$

where $\langle A|B\rangle$ is the set of hyperplanes that separate A and B. Thus $\Gamma_{A,B}$ is the event that A and B are separated until time a by the very first hyperplane of the process. Heuristically speaking the event $\Gamma_{A,B}$ is very likely to happen for A, B being far away from each other. We use this event in order to give an upper bound. In a first step we bound the difference in the theorem by

$$\begin{aligned} |\mathbb{P}_a(\mathcal{F}^A \cap \mathcal{F}^{\tau_x B}) - \mathbb{P}_a(\mathcal{F}^A)\cdot\mathbb{P}_a(\mathcal{F}^B)| \leq{}& |\mathbb{P}_a(\mathcal{F}^A \cap \mathcal{F}^B \cap \Gamma_{A,B}) - \mathbb{P}_a(\mathcal{F}^A,\ \mathcal{F}^B)| \\ &+ |\mathbb{P}_a(\mathcal{F}^A \cap \mathcal{F}^B \cap \Gamma_{A,B}) - \mathbb{P}_a(\mathcal{F}^A)\cdot\mathbb{P}_a(\mathcal{F}^B)|. \end{aligned} \tag{4.31}$$

We consider the first summand in the expression above. Knowing that $\Gamma_{A,B}$ takes place, the behaviour of the process is independent on both sides of H_1 and therefore also the probability of Y_a hitting A, B respectively. We first show

$$\begin{aligned}
|\mathbb{P}_a(\mathcal{F}^A \cap \mathcal{F}^B \cap \Gamma_{A,B}) - \mathbb{P}_a(\mathcal{F}^A \cap \mathcal{F}^B)| &= |\mathbb{P}_a(\mathcal{F}^A \cap \mathcal{F}^B \cap \Gamma^c_{A,B})| \\
&\leq \mathbb{P}(\Gamma^c_{A,B}) \\
&= \mathbb{P}(H_1 \notin \langle A|B\rangle \ \cup \ t_1 > a) \\
&\leq \mathbb{P}(t_1 > a) + \mathbb{P}(H_1 \notin \langle A|B\rangle) \\
&= \mathbb{P}(t_1 > a) + \mathbb{P}(H_1 \cap (\operatorname{conv}(A) \cup \operatorname{conv}(B)) \neq \emptyset).
\end{aligned}$$

The reason for this is that a hyperplane $H \in \mathbb{H}_{d-1}\langle \operatorname{conv}(A \cup B)\rangle$ either separates A and B or intersects one of the two sets $\operatorname{conv}(A)$, $\operatorname{conv}(B)$. The first term in the expression above is given by

$$\mathbb{P}(t_1 > a) = e^{-a\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} \tag{4.32}$$

and the second one by

$$\mathbb{P}(H_1 \cap (\operatorname{conv}(A) \cup \operatorname{conv}(B)) \neq \emptyset) = \frac{\mu_{d-1}(\mathbb{H}_{d-1}\langle \operatorname{conv}(A) \cup \operatorname{conv}(B)\rangle)}{\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}. \tag{4.33}$$

Now we turn to the second summand in (4.31), namely the difference of $\mathbb{P}_a(\mathcal{F}^A,\ \mathcal{F}^B, \Gamma_{A,B})$ and $\mathbb{P}_a(\mathcal{F}^A) \cdot \mathbb{P}_a(\mathcal{F}^B)$. We have

$$\begin{aligned}
&\mathbb{P}_a(\mathcal{F}^A \cap \mathcal{F}^B \cap \Gamma_{A,B}) \\
&\quad = \int_0^a \mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle) e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}\, \mathbb{P}(\mathcal{F}^A \cap \mathcal{F}^B \cap H_1 \in \langle A|B\rangle \mid t_1 = t)\ dt \\
&\quad = \int_0^a \mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle) e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}\, \mathbb{P}(H_1 \in \langle A|B\rangle \mid t_1 = t) \\
&\qquad\qquad \times \mathbb{P}(\mathcal{F}^A \cap \mathcal{F}^B \mid H_1 \in \langle A|B\rangle \cap t_1 = t)\ dt \\
&\quad = \int_0^a \mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle) e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}\, \frac{\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle)}{\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}\, \mathbb{P}_{a-t}(\mathcal{F}^A)\, \mathbb{P}_{a-t}(\mathcal{F}^B) \\
&\quad = \mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle) \int_0^a e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}\, \mathbb{P}_{a-t}(\mathcal{F}^A)\, \mathbb{P}_{a-t}(\mathcal{F}^B)\ dt \\
&\quad = \mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle) \int_0^a e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} U_{a-t}(A) U_{a-t}(B)\ dt.
\end{aligned}$$

Using the value of the integral

$$\int_0^\infty e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} \mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)\ dt = 1$$

gives the following

$$
\begin{aligned}
&|\mathbb{P}_a(\mathcal{F}^A \cap \mathcal{F}^B \cap \Gamma_{A,B}) - \mathbb{P}_a(\mathcal{F}^A) \cdot \mathbb{P}_a(\mathcal{F}^B)| \\
=&\Big|\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle) \int_0^a e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} U_{a-t}(A)U_{a-t}(B)\ dt \\
&- U_a(A)U_a(B) \int_0^\infty e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} \mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)\ dt\Big| \\
=&\Big|\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle) \int_0^a e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} U_{a-t}(A)U_{a-t}(B)\ dt \\
&\qquad - \mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle) \int_0^\infty e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} U_a(A)U_a(B)\ dt\Big| \\
&+ \Big|\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle) \int_0^\infty e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} U_a(A)U_a(B)\ dt \\
&\qquad - U_a(A)U_a(B) \int_0^\infty e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} \mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)\ dt\Big| \\
\le&\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle) \int_0^a e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} |U_{a-t}(A)U_{a-t}(B) - U_a(A)U_a(B)|\ dt \qquad \text{(I)} \\
&+ \left|-\frac{\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle)}{\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} + 1\right| \mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle) \int_0^\infty e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} U_a(A)U_a(B)\ dt \qquad \text{(II)} \\
&+ \mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle) \int_a^\infty e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}\ dt. \qquad \text{(III)}
\end{aligned}
$$

We are aiming to control expression (I)-(III). We define $\delta(A,B,a) := \lambda_{A,a} + \lambda_{B,a}$, where $\lambda_{A,a}$ and $\lambda_{B,a}$ are taken from Lemma 4.7.1. This lemma also provides

$$|U_{a-t}(A)U_{a-t}(B) - U_a(A)U_a(B)| \le (\lambda_{A,a} + \lambda_{B,a})t = t \cdot \delta(A,B,a)$$

and therefore an upper bound for (I)

$$
\begin{aligned}
&\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle) \int_0^a e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} |U_{a-t}(A)U_{a-t}(B) - U_a(A)U_a(B)|\ dt \\
&\quad \le \mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle)\delta(A,B,a) \int_0^a e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} t\ dt \\
&\quad = \mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle)\delta(A,B,a) \left[-\frac{e^{-t\,\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}(t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle) + 1)}{\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)^2}\right]_0^a \\
&\quad = \mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle) \frac{\delta(A,B,a)}{\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)^2} \left(-e^{-a\,\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}(a\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle) + 1) + 1\right) \\
&\quad \le \mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle) \frac{\delta(A,B,a)}{\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)^2}.
\end{aligned}
$$

Term (II) can be bounded by

$$\begin{aligned}
&\left|-\frac{\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle)}{\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}+1\right|\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)\int_0^\infty e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}U_a(A)U_a(B)\,dt\\
&\quad=\left|-\frac{\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle)}{\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}+1\right|U_a(A)U_a(B)\\
&\quad\le\left|-\frac{\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle)}{\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}+1\right|\\
&\quad=1-\frac{\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle)}{\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)},
\end{aligned}$$

where the first step used that the U-functional is bounded by 1. The last term (III) simplifies via

$$\begin{aligned}
\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle)\int_a^\infty e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}\,dt&=\frac{\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle)}{\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}e^{-a\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}\\
&\le e^{-a\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}.
\end{aligned}$$

Combining the inequalities for $(I)-(III)$ gives

$$\begin{aligned}
&|\mathbb{P}_a(\mathcal{F}^A\cap\mathcal{F}^B\cap\Gamma_{A,B})-\mathbb{P}_a(\mathcal{F}^A)\cdot\mathbb{P}_a(\mathcal{F}^B)|\\
\le&\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle)\frac{\delta(A,B,a)}{\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)^2}+\left(1-\frac{\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle)}{\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}\right)+e^{-a\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}.
\end{aligned}\tag{4.34}$$

To show the result one has to control the growth of $\mu_{d-1}(\mathbb{H}_{d-1}\langle A|\tau_x B\rangle)$ and $\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)$ in terms of $d_h(x,p)$. Since A,B are nonempty there exist $x_A\in A,\ x_B\in B$. Without loss of generality one has $x_B=p$ and therefore $x=\tau_h(x_B)\in\tau_x B$. This implies $[x_A,x]\subseteq W$. Since

$$d_h(x_A,x)\ge d_h(x,p)-d_h(x_A,p)$$

the hyperbolic length of this interval is at least $d_h(x,p)-d_h(x_A,p)$. Therefore one has by the Crofton-type formula in Lemma 2.4.1

$$\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)\ge\mu_{d-1}(\mathbb{H}_{d-1}\langle x_A,x\rangle)\ge\frac{\omega_{d+1}\omega_1}{\omega_d\omega_2}(d_h(x,p)-d_h(x_A,p)).$$

Now we focus one the difference of $\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)$ and $\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle)$. First one knows that

$$0\le\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)-\mu_{d-1}(\mathbb{H}_{d-1}\langle A|B\rangle)\le\mu_{d-1}(\mathbb{H}_{d-1}\langle\mathrm{conv}(A)\rangle)+\mu_{d-1}(\mathbb{H}_{d-1}\langle\mathrm{conv}(B)\rangle).$$

Now defining $\chi(A,B,a)$ as

$$\chi(A,B,a):=\delta(A,B,a)+2\mu_{d-1}(\mathbb{H}_{d-1}\langle\mathrm{conv}(A)\rangle)+2\mu_{d-1}(\mathbb{H}_{d-1}\langle\mathrm{conv}(B)\rangle)$$

and using (4.32), (4.33), (4.34) we conclude for the second summand in (4.31)

$$\begin{aligned}|\mathbb{P}_a(\mathcal{F}^A \cap \mathcal{F}^B) - \mathbb{P}_a(\mathcal{F}^A) \cdot \mathbb{P}_a(\mathcal{F}^B)| \leq & |\mathbb{P}_a(\mathcal{F}^A \cap \mathcal{F}^B \cap \Gamma_{A,B}) - \mathbb{P}_a(\mathcal{F}^A \cap \mathcal{F}^B)| \\ & + |\mathbb{P}_a(\mathcal{F}^A \cap \mathcal{F}^B \cap \Gamma_{A,B}) - \mathbb{P}_a(\mathcal{F}^A) \cdot \mathbb{P}_a(\mathcal{F}^B)| \\ \leq & \frac{\chi(A,B,a) + 2\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle) \cdot e^{-a\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)}}{\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} \\ = & \frac{\chi(A,B,a) + o(1)}{\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)} \in \mathcal{O}(d_h(x,p)^{-1}),\end{aligned}$$

where we used that $\mu_{d-1}(\mathbb{H}_{d-1}\langle W\rangle)$ is bounded from below by the measure of a line segment of length $d_h(p,x)$ minus a constant. □

CHAPTER 5

Kendall's problem in hyperbolic spaces

This chapter is dedicated to a hyperbolic version of the so-called Kendall problem. In Section 5.1 we get some first insights showing that we cannot expect the same results as in the Euclidean case to hold. On the other hand, it is shown in this section that in some sense Euclidean and hyperbolic results are alike. The second Section 5.2 contains various lemmas and auxiliary results needed in the proofs of the main theorems. Such auxiliary results include continuity properties for various functionals, inequalities of isoperimetric type and approximation of convex bodies by polytopes. A crucial result is developed in Subsection 5.3.1, allowing us to focus on a region around the origin p, whenever we investigate the behaviour of the Crofton cell. The latter is considered in Section 5.3. We investigate the limit behaviour (for increasing intensity of the hyperplane process) of its shape, given that it exceeds a certain volume. Also the asymptotic distribution of the volume of the Crofton cell is considered. The following Section 5.4 transfers the results from the Crofton cell to the typical cell of a Poisson hyperplane mosaic. The last Section 5.5 is dedicated to investigating the behaviour of the typical cell in a Poisson-Voronoi mosaic.

5.1 First insights

Theorems 5.1.1 and 5.1.3 give results for the behaviour of the Crofton cell of a Poisson hyperplane mosaic in hyperbolic space conditioned on the event that it includes the ball B_a. In the Euclidean case, for constant intensity $t > 0$, the shape of the zero cell of a Poisson hyperplane tessellation converges to the shape of a ball as a tends to infinity. This follows from [44, Theorem 1] by choosing Σ to be the Euclidean centred inball radius and $b = \infty$. Surprisingly, in hyperbolic space a corresponding fact is no longer true, as we can see from

Theorem 5.1.3. The first theorem is restricted to the hyperbolic plane.

Before we can state and prove the results, we need to define the restriction of a hyperplane process η_t to a closed subset $A \in \mathcal{F}(\mathbb{H}^d)$ to be the set of hyperplanes that have nonempty intersection with A, namely

$$\eta_{t|A} := \{H \in \eta_t : \ H \cap A \neq \emptyset\}.$$

The rest of the process is denoted by $\overline{\eta_{t|A}}$, it contains all hyperplanes of η_t not intersecting A, i.e.

$$\overline{\eta_{t|A}} := \eta_t \setminus \eta_{t|A} = \{H \in \eta_t : \ H \cap A = \emptyset\}.$$

Likewise, we define the random sets

$$Z_{t|A} := \bigcup_{H \in \eta_{t|A}} H, \qquad \overline{Z_{t|A}} := \bigcup_{H \in \overline{\eta_{t|A}}} H.$$

Theorem 5.1.1. *Let η_t be an isometry invariant Poisson line process in $\mathbb{H}^2$ with intensity $t \in (0, \pi/2)$. Then the shape of the zero cell C_0, given it contains B_a, does not converge to the shape of a ball as a tends to infinity.*

Proof. Let $\mathcal{R}$ be the set of hyperbolic rays starting a p. The set $\mathcal{R}$ can be constructed by mapping all rays of $T_p\mathbb{H}^d$ that start at the origin into $\mathbb{H}^d$ via the exponential function $\exp_p$. Since the intensity is chosen low enough we know by [114, Section 4.3] that the probability that there exists a ray not intersecting η_t is strictly positive

$$\mathbb{P}(\{R \in \mathcal{R} : \ R \cap Z_t = \emptyset\} \neq \emptyset) > 0.$$

Note that the normalization of the measure on the set of lines used in [114] and μ_1 differ by the constant $\omega_2/\omega_1 = \pi$ (see Section 2.4 for more details). An alternative notation for this event above is

$$\{\{R \in \mathcal{R} : \ R \cap Z_t = \emptyset\} \neq \emptyset\} = \bigcap_{i=1}^{\infty} \{R_0(C_0 \cap B_i) = i\}.$$

Now we consider the probability

$$\begin{aligned}
&\mathbb{P}(\{R \in \mathcal{R} : \ R \cap Z_t = \emptyset\} \neq \emptyset, \ B_a \subseteq C_0) \\
&\quad = \mathbb{P}(\{R \in \mathcal{R} : \ R \cap Z_{t|B_a} = \emptyset, \ R \cap \overline{Z_{t|B_a}} = \emptyset\} \neq \emptyset, \ Z_{t|B_a} = \emptyset) \\
&\quad = \mathbb{P}(\{R \in \mathcal{R} : \ R \cap \overline{Z_{t|B_a}} = \emptyset\} \neq \emptyset, \ Z_{t|B_a} = \emptyset) \\
&\quad = \mathbb{P}(\{R \in \mathcal{R} : \ R \cap \overline{Z_{t|B_a}} = \emptyset\} \neq \emptyset)\, \mathbb{P}(Z_{t|B_a} = \emptyset) \\
&\quad = \mathbb{P}(\{R \in \mathcal{R} : \ R \cap \overline{Z_{t|B_a}} = \emptyset\} \neq \emptyset)\, \mathbb{P}(B_a \subseteq C_0).
\end{aligned}$$

Here we used the Poisson property of η_t for the third equality. Since $\overline{Z_{t|B_a}}$ is almost surely contained in Z_t for every $a > 0$ we get by using the equality above

$$\begin{aligned}\mathbb{P}(\{R \in \mathcal{R}:\ R \cap Z_t = \varnothing\} \neq \varnothing \,|\, B_a \subseteq C_0) &= \frac{\mathbb{P}(\{R \in \mathcal{R}:\ R \cap Z_t = \varnothing\} \neq \varnothing,\ B_a \subseteq C_0)}{\mathbb{P}(B_a \subseteq C_0)}\\ &= \mathbb{P}(\{R \in \mathcal{R}:\ R \cap \overline{Z_{t|B_a}} = \varnothing\} \neq \varnothing)\\ &\geq \mathbb{P}(\{R \in \mathcal{R}:\ R \cap Z_t = \varnothing\} \neq \varnothing) > 0.\end{aligned}$$

Since the last value does not depend on a, the shape of the zero cell does not converge to the shape of a ball as a tends to infinity. □

In order to show an analogue result for higher dimensions, we first state and prove a useful lemma which deals with the intersection of a Poisson hyperplane process in dimension d with a fixed 2-dimensional linear subspace.

Lemma 5.1.2. *Let η_t be a Poisson hyperplane process with intensity $t > 0$ in dimension $d > 2$ and let $L \in G_h(d, 2)$ be an arbitrary 2-dimensional totally geodesic subspace containing p. Then the intersection process*

$$\eta_t \cap L := \{H \cap L:\ H \in \eta_{t|L}\}$$

is almost surely a Poisson line process in L with intensity $\tilde{t} = \frac{\omega_{d+1}\,\omega_2}{\omega_d\,\omega_3} t$.

Proof. In a first step we show that the elements of $\eta_t \cap L$ are almost surely 1-dimensional totally geodesic subspaces. First we see that the elements of $\eta_t \cap L$ are by definition not the empty set. Therefore we know that their dimension is either 1 or 2 since they are the intersection of a 2-dimensional and a $(d-1)$-dimensional subspace. Using the Crofton type formula in Lemma 2.4.1 we get

$$\begin{aligned}\int_{A_h(d,d-1)} \mathbb{1}\{\dim(L \cap H) = 2\}\ (t\,\mu_{d-1})(dH) &\leq t \int_{A_h(d,d-1)} \mathcal{H}^2(L \cap H \cap B_1)\ \mu_{d-1}(dH)\\ &= t\,\frac{\omega_{d+1}\,\omega_3}{\omega_d\,\omega_4}\,\mathcal{H}^3(L \cap B_1) = 0.\end{aligned}$$

Here the first inequality used that $\dim(L \cap H) = 2$ implies that the intersection $L \cap H$ is equal to L. Then the integrand $\mathbb{1}\{\dim(L \cap H) = 2\}$ can be bounded by $\mathcal{H}^2(L \cap H \cap B_1) = 2\,\pi(\cosh(1) - 1) \geq 1$. Now let $A \in \mathcal{B}(A_h(2,1))$ be a Borel set of lines in L. We define the Borel set $B := B(A, L) := \{H \in A_h(d, d-1):\ H \cap L \in A\}$. Now one can show that $\eta_t \cap L$ is a Poisson process since

$$\begin{aligned}\mathbb{P}((\eta_t \cap L)(A) = k) = \mathbb{P}(|\{\tilde{H} \in \eta_t \cap L:\ \tilde{H} \in A\}| = k) &= \mathbb{P}(|\{H \in \eta_{t|L}:\ H \cap L \in A\}| = k)\\ &= \mathbb{P}(|\{H \in \eta_t:\ H \in B\}| = k)\\ &= \mathbb{P}(\eta_t(B) = k)\end{aligned}$$

holds for all $k \in \mathbb{N}_0$ and since η_t is a Poisson process. In a next step we show that $\eta_t \cap L$ is an isometry invariant Poisson line process. To do so let $\varphi \in I(\mathbb{H}^d, L)$ be an isometry fixing L. Let

$k \in \mathbb{N}_0$, then the invariance of $\eta_t \cap L$ under isometries is shown by

$$\begin{aligned}\mathbb{P}((\eta_t \cap L)(\varphi A) = k) &= \mathbb{P}(|\{\tilde{H} \in \eta_t \cap L : \ \tilde{H} \in \varphi A\}| = k)\\ &= \mathbb{P}(|\{H \in \eta_{t|L} : \ H \cap L \in \varphi A\}| = k)\\ &= \mathbb{P}(|\{H \in \eta_{t|L} : \ \varphi^{-1}(H \cap L) \in A\}| = k)\\ &= \mathbb{P}(|\{H \in \eta_{t|L} : \ \varphi^{-1}(H) \cap L \in A\}| = k)\\ &= \mathbb{P}(|\{H \in \eta_{t|L} : \ H \cap L \in A\}| = k)\\ &= \mathbb{P}(|\{\tilde{H} \in \eta_t \cap L : \ \tilde{H} \in A\}| = k)\\ &= \mathbb{P}((\eta_t \cap L)(A) = k).\end{aligned}$$

Here we used the invariance of η_t for the third last equality. We derive the intensity $\tilde{t}$ by measuring the set of hyperplanes hitting a segment $[p,x]$ of length r in L, $x \in L$. This measure is given by Lemma 2.4.1 as

$$t\, \mu_{d-1}(\mathbb{H}_{d-1}\langle[p,x]\rangle) = \frac{\omega_{d+1}\, \omega_1}{\omega_d\, \omega_2} rt.$$

Therefore we get the intensity by

$$\tilde{t}\, \mu_1(\mathbb{H}_1\langle[0,x]\rangle) = \tilde{t}\, \frac{\omega_3\, \omega_1}{\omega_2^2} \mathcal{H}^1([p,x]) = \tilde{t}\, \frac{\omega_3\, \omega_1}{\omega_2^2}\, r = \frac{\omega_{d+1}\, \omega_1}{\omega_d\, \omega_2} rt.$$

This yields $\tilde{t} = \frac{\omega_{d+1}\omega_2}{\omega_d \omega_3} t$ which finishes the proof. □

The same effect shown for dimension 2 in Theorem 5.1.1 occurs for higher dimensional hyperplane mosaics in hyperbolic d-space. The proof transfers the result from dimension 2 into higher dimensions.

Theorem 5.1.3. *Let η_t be an invariant Poisson hyperplane process in $\mathbb{H}^d$ with $t < \omega_d\, \omega_2/(\omega_{d+1}\omega_1)$. Then the shape of the zero cell C_0, given it contains B_a, does not converge to the shape of a ball as a tends to infinity.*

Proof. Take a fixed 2-dimensional totally geodesic submanifold $L \in G_h(d,2)$ containing p. Almost surely L is not a subset of Z_t. Therefore $\eta_t \cap L = \{H \cap L : \ H \in \eta_{t|L}\}$ is by Lemma 5.1.2 a Poisson line process in the hyperbolic plane L. By Theorem 5.1.1 we know that the probability that the zero cell $\hat{C}_0$ in $\eta_t \cap L$ is unbounded is positive. An unbounded zero cell in the mosaic generated by $\eta_t \cap L$ leads to an unbounded zero cell of the mosaic generated by η_t. This implies that there is no convergence of the shape of the zero cell to the shape of a ball. □

Remark 5.1.1. In the coming associated paper, a related result to Theorem 5.1.3 is shown. Its advantage is that it does not require an upper bound on the intensity t of the underlying Poisson hyperplane process.

The latter theorems show major differences in the behaviour of large Crofton cells in Euclidean compared to the ones in hyperbolic space. One could show that there are also

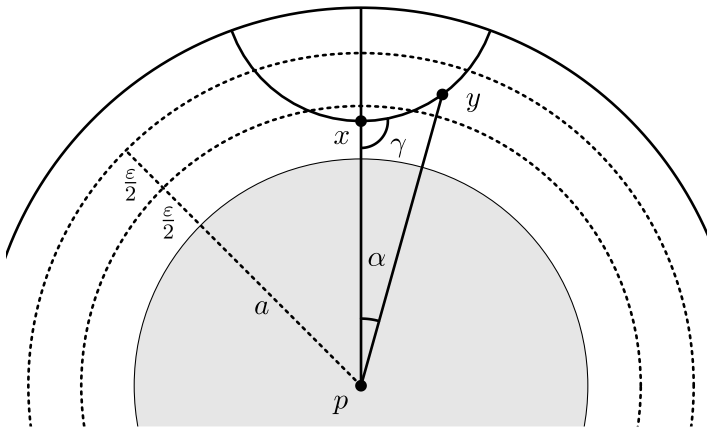

Figure 5.1.1: Illustration for the proof of Lemma 5.1.4 with $x = \exp_p(ru)$ and $y = \exp_p(\tilde{r}\tilde{u})$

similarities to the Euclidean case, when it comes to the asymptotic shape of the Crofton cell. Before we can state the theorem showing the behaviour of the zero cell, conditioned on the event that it includes B_a, for increasing intensity t, we need to show a useful geometric lemma.

Lemma 5.1.4. *Let $a, \varepsilon > 0$ and $u \in \mathbb{S}_p^{d-1}$. Further define $\tilde{\beta} := \sqrt{2 - \frac{2\tanh(a+\varepsilon/2)}{\tanh(a+\varepsilon)}}$. Then*

$$[p, \exp_p((a+\varepsilon)\tilde{u})] \cap H(\exp_p(ru)) \neq \varnothing$$

holds for every $\tilde{u} \in B_s(u, \tilde{\beta})$ and every $r \in (a, a+\varepsilon/2]$.

Proof. Let $r \in (a, a+\varepsilon/2]$ and $\alpha \in (0, \tilde{\alpha}]$. Since $\cos(t) \geq 1 - t^2/2$ for $t \in \mathbb{R}$, we get

$$\cos(\tilde{\beta}) \geq 1 - \frac{1}{2}\left(2 - \frac{2\tanh(a+\varepsilon/2)}{\tanh(a+\varepsilon)}\right) = \frac{\tanh(a+\varepsilon/2)}{\tanh(a+\varepsilon)}.$$

Hence, by the monotonicity of cos and tanh,

$$\frac{\tanh(r)}{\cos(\alpha)} \leq \frac{\tanh(a+\varepsilon/2)}{\cos(\tilde{\alpha})} \leq \tanh(a+\varepsilon) < 1.$$

Therefore there is a unique number $\tilde{r} \in (0, a+\varepsilon]$ such that

$$\tanh(\tilde{r}) = \frac{\tanh(r)}{\cos(\alpha)}. \tag{5.1}$$

Let $x := \exp_p(ru) \in \mathbb{H}^d$, and for $\tilde{u} \in \mathbb{S}_p^{d-1}$ with $d_s(u, \tilde{u}) = \alpha \in (0, \tilde{\alpha}]$ (for $\tilde{u} = u$ the assertion of the lemma is clearly true) we define $y := \exp_p(\tilde{r}\,\tilde{u})$. We will show that $y \in H(\exp_p(ru))$. Let γ denote the angle at x of the hyperbolic triangle determined by p, x, y. For an illustration see Figure 5.1.1. Then the assertion follows once we have proved that $\cos(\gamma) = 0$. For this, let $\tilde{a} := d_h(x, y)$. Then [92, Thm. 3.5.3] yields

$$\cosh(\tilde{a}) = \cosh(r)\cosh(\tilde{r}) - \sinh(r)\sinh(\tilde{r})\cos(\alpha). \tag{5.2}$$

Moreover, it follows from [92, (3.5.3)], (5.1) and (5.2) that

$$\begin{aligned}\sinh(\tilde{a})\sinh(r)\cos(\gamma) &= \cosh(\tilde{r}) - \cosh(\tilde{a})\cosh(r)\\ &= \cosh(\tilde{r}) - \cosh^2(r)\cosh(\tilde{r}) + \sinh(r)\cosh(r)\sinh(\tilde{r})\cos(\alpha)\\ &= \cosh(\tilde{r})(1-\cosh^2(r)) + \sinh(r)\cosh(r)\sinh(\tilde{r})\frac{\tanh(r)}{\tanh(\tilde{r})}\\ &= -\cosh(\tilde{r})\sinh^2(r) + \sinh^2(r)\cosh(\tilde{r}) = 0,\end{aligned}$$

which shows that $\cos(\gamma) = 0$. □

Theorem 5.1.5. *Let η_t be an isometry invariant Poisson hyperplane process in $\mathbb{H}^d$ and $a > 0$. Then the shape of the Crofton cell C_0 of the induced hyperplane tessellation X_t, given C_0 contains B_a, converges to the shape of a ball, as t tends to infinity.*

Proof. Let $\varepsilon > 0$. In order to prove the convergence of the shape of the Crofton cell to the shape of a geodesic ball, we show that the probability $\mathbb{P}_t(\vartheta_0(C_0) \geq \varepsilon \mid B_a \subseteq C_0)$ converges to zero for every $\varepsilon > 0$ as t tends to infinity.

We start by observing that

$$\begin{aligned}\mathbb{P}_t(\vartheta_0(C_0) \geq \varepsilon,\ B_a \subseteq C_0) &= \mathbb{P}_t(R_0(C_0) - r_0(C_0) \geq \varepsilon,\ B_a \subseteq C_0)\\ &\leq \mathbb{P}_t(R_0(C_0) \geq a + \varepsilon,\ B_a \subseteq C_0). \qquad (5.3)\end{aligned}$$

Let $\tilde{\beta}$ be as defined in Lemma 5.1.4. By applying the results in [12, Chapter 6] for $\mathbb{S}_p^{d-1}$, we know that there exists a natural number $n = n(\tilde{\beta})$ and n directions $u_1, \dots, u_n \in \mathbb{S}_p^{d-1}$ such that the spherical caps (geodesic balls)

$$B_s(u_i, \tilde{\beta}/2) = \{v \in \mathbb{S}_p^{d-1} : \ d_s(v, u_i) \leq \tilde{\beta}/2\}, \qquad i = 1, \dots, n,$$

cover $\mathbb{S}_p^{d-1}$. Using this and (5.3) gives

$$\begin{aligned}&\mathbb{P}_t(\vartheta_0(C_0) \geq \varepsilon,\ B_a \subseteq C_0)\\ &\quad \leq \mathbb{P}_t(\exists u \in \mathbb{S}_p^{d-1} : \ \exp_p((a+\varepsilon)u) \in C_0,\ B_a \subseteq C_0)\\ &\quad = \mathbb{P}_t(\exists i \in \{1, \dots, n\} : \exists u \in B_s(u_i, \tilde{\beta}/2) : \ \exp_p((a+\varepsilon)u) \in C_0,\ B_a \subseteq C_0)\\ &\quad \leq \sum_{i=1}^{n} \mathbb{P}_t(\exists u \in B_s(u_i, \tilde{\beta}/2) : \ \exp_p((a+\varepsilon)u) \in C_0,\ B_a \subseteq C_0)\\ &\quad = n\,\mathbb{P}_t(\exists u \in B_s(u_1, \tilde{\beta}/2) : \ \exp_p((a+\varepsilon)u) \in C_0,\ B_a \subseteq C_0).\end{aligned}$$

We define

$$D(u_1, \tilde{\beta}) := \{H(\exp_p(ru)) : \ r \in (a, a+\varepsilon/2],\ u \in B_s(u_1, \tilde{\beta}/2)\}.$$

Now, by using the results from Lemma 5.1.4, we get the inequality

$$\begin{aligned}&\mathbb{P}_t(\exists u\in B_s(u_1,\tilde\beta/2):\ \exp_p((a+\varepsilon)u)\in C_0,\ B_a\subseteq C_0)\\&\le \mathbb{P}_t(\eta_t\cap D(u_1,\tilde\beta)=\varnothing,\ B_a\subseteq C_0),\end{aligned}$$

Clearly, we have $c:=c(a,\varepsilon,\tilde\beta):=\mu_{d-1}(D(u_1,\tilde\beta))>0$. Since

$$D(u_1,\tilde\beta)\cap\mathbb{H}_{d-1}\langle B_a\rangle=\varnothing,$$

the independence property of the Poisson process yields

$$\mathbb{P}_t(\eta_t\cap D(u_1,\tilde\beta)=\varnothing,\ B_a\subseteq C_0)=\mathbb{P}_t(\eta_t\cap D(u_1,\tilde\beta)=\varnothing)\,\mathbb{P}_t(B_a\subseteq C_0).$$

Thus we conclude that

$$\begin{aligned}\mathbb{P}_t(\vartheta_0(C_0)\ge\varepsilon\mid B_a\subseteq C_0)&=\frac{\mathbb{P}_t(\vartheta_0(C_0)\ge\varepsilon,\ B_a\subseteq C_0)}{\mathbb{P}_t(B_a\subseteq C_0)}\\&\le n\,\frac{\mathbb{P}_t(\eta_t\cap D(u_1,\tilde\beta)=\varnothing)\,\mathbb{P}_t(B_a\subseteq C_0)}{\mathbb{P}_t(B_a\subseteq C_0)}\\&=n\,\mathbb{P}_t(\eta_t\cap D(u_1,\tilde\beta)=\varnothing)\\&=n\,e^{-tc}\overset{t\to\infty}{\longrightarrow}0,\end{aligned}$$

where we used that η_t is a Poisson process for the last equality. □

5.2 Auxiliary results

In this section we show various useful results. More precisely, in Section 5.2.1 we show the continuity of various functions which are used in the main theorems. Section 5.2.2 is dedicated to results of isoperimetric type. Section 5.2.3 contains the approximation of convex sets by polytopes.

5.2.1 Continuity results

Before we can turn to stability results in hyperbolic space, we have to show that the functionals W_{d-1} and $\mathcal{H}^d$ are continuous on the set of all convex bodies $\mathcal{K}_h^d$ with respect to the Hausdorff distance. We will also prove the continuity of the (centred) inball radius r_0 and the (centred) circumradius R_0 as functions of convex bodies in $\mathbb{H}^d$. Furthermore the continuity of the circum- and inball radius are shown as well as the lower semicontinuity of ϑ_r. We then turn to show the continuity of the circumcentre c_h.

Lemma 5.2.1. *The functional W_{d-1} is continuous on $\mathcal{K}_h^d$ with respect to the Hausdorff metric.*

Proof. Let $K,K_i\in\mathcal{K}_h^d$ for $i\in\mathbb{N}$ with $\delta_h(K,K_i)\to0$ as $i\to\infty$. Since $\chi(H\cap K_i)=\mathbb{1}\{H\cap K_i\}$ for $H\in A_h(d,d-1)$, we show that $\mathbb{1}\{H\cap K_i\neq\varnothing\}\to\mathbb{1}\{H\cap K\neq\varnothing\}$ for μ_{d-1}-almost all $H\in A_h(d,d-1)$. We distinguish two cases (and can use e.g. the projective disc model). If

$H \in A_h(d,d-1)$ is such that $H \cap \operatorname{relint}(K) \neq \varnothing$ and $K \notin H$, then $H \cap K_i \neq \varnothing$ for sufficiently large i, and therefore $\mathbb{1}\{H \cap K_i \neq \varnothing\} \to \mathbb{1}\{H \cap K \neq \varnothing\}$ as $i \to \infty$. If $H \cap K = \varnothing$, then $H \cap K_i = \varnothing$ for all sufficiently large i. Therefore one gets convergence of the indicator function in this case as well.

It remains to be shown that all other hyperplanes have zero measure. For the set of hyperplanes containing K this is clear. Hence we consider the set

$$A := \{H \in A_h(d,d-1) : \ H \cap \operatorname{relint}(K) = \varnothing, \ H \cap K \neq \varnothing\}.$$

By the isometry invariance of μ_{d-1}, we can assume that $p \in \operatorname{relint}(K)$. Let $u \in \mathbb{S}_p^{d-1}$ be a fixed direction and define

$$l_K(u) := \sup\{r \geq 0 : \ H(\exp_p(ru)) \cap K \neq \varnothing\}, \tag{5.4}$$

where $H(\exp_p(ru))$ is the hyperplane orthogonal to $[p, \exp_p(ru)]$ that contains $\exp_p(ru)$. Let $r \in [0, l_K(u))$. By Lemma 2.3.3 and [92, Ex. 6.1 (5)] we have $H(\exp_p(ru)) \cap \operatorname{relint}(K) \neq \varnothing$, and therefore $H(\exp_p(ru)) \notin A$. Hence the representation of μ_{d-1} in (2.3) yields

$$\begin{aligned}
&\int_{A_h(d,d-1)} \mathbb{1}\{H \in A\}\, \mu_{d-1}(dH) \\
&\qquad \leq \omega_d^{-1} \int_{\mathbb{S}_p^{d-1}} \int_{l_K(u)}^{l_K(u)} \cosh^{d-1}(r) \mathbb{1}\{H(\exp_p(ru)) \in A\}\, dr\, \sigma_{d-1}(du) = 0.
\end{aligned}$$

The result follows from the dominated convergence theorem, since $\mu_{d-1}(\mathbb{H}_{d-1}\langle B_s \rangle) < \infty$ for $s \geq 0$. □

The following lemma states the continuity of the d-dimensional Hausdorff measure (volume) on $\mathcal{K}_h^d$ in hyperbolic space.

Lemma 5.2.2. *The functional $\mathcal{H}^d$ is continuous on $\mathcal{K}_h^d$ with respect to the Hausdorff metric.*

Proof. This is a special case of [4, Satz 4.7], in view of Lemma 4.1 and Korollar 4.2 in [4]. □

For the proof of the continuity of r_0 and R_0, the following lemma is useful. For $K \in \mathcal{K}_h^d$ and $x \in \mathbb{H}^d$, let $d_h(K,x) := \min\{d_h(z,x) : z \in K\}$ denote the distance from x to K. By the unique footpoint property of compact, convex sets in $\mathbb{H}^d$, there exists exactly one point $z \in K$ for which $d_h(K,x) = d_h(z,x)$, the metric projection of x onto K. Let $\varepsilon \geq 0$. Then $K_\varepsilon := \{y \in \mathbb{H}^d : d_h(K,y) \leq \varepsilon\}$ is the parallel set of K at distance ε.

Lemma 5.2.3. *Let $q \in \mathbb{H}^d$, $K \in \mathcal{K}_h^d$ and $\varepsilon_1, \varepsilon_2 \geq 0$ with $\varepsilon_1 \geq \varepsilon_2$. If $B_h(q,\varepsilon_1) \subseteq K_{\varepsilon_2}$, then $B_h(q, \varepsilon_1 - \varepsilon_2) \subseteq K$.*

Proof. Suppose there is some $x \in B_h(q, \varepsilon_1 - \varepsilon_2) \setminus K$. Let z be the nearest point from x in K with $\delta := d_h(K,x) > 0$, and let $u \in \mathbb{S}_z^{d-1}$ be the unique unit vector in $T_z\mathbb{H}^d$ for which $\exp_z(\delta u) = x$. By hyperbolic trigonometry and the convexity of K (a separation/support property), it follows that $d_h(K, \exp_z(tu)) = t = d_h(z, \exp_z(tu))$ for $t \geq 0$. Then $y := \exp_z((\delta + \varepsilon_2)u)$ satisfies

$d_h(K, y) = \delta + \varepsilon_2 > \varepsilon_2$, hence $y \notin K_{\varepsilon_2}$. On the other hand, $d_h(x, y) = \varepsilon_2$ and $d_h(q, x) \leq \varepsilon_1 - \varepsilon_2$, hence $d_h(q, y) \leq \varepsilon_1$. But then $y \in B_h(q, \varepsilon_1) \subseteq K_{\varepsilon_2}$, a contradiction. □

Lemma 5.2.4. *The centred inball radius functional $r_0 : \mathcal{K}_h^d \to [0, \infty)$ is continuous with respect to the Hausdorff metric.*

Proof. Let $K, K_i \in \mathcal{K}_h^d$ for $i \in \mathbb{N}$ with $\delta_h(K_i, K) \to 0$ as $i \to \infty$. If $r_0(K) = 0$, then clearly K is contained in a hyperplane and therefore $r_0(K_i) \to 0$ as $i \to \infty$. Now let $r_0(K) > 0$. For $\varepsilon \in (0, r_0(K))$ we have $B_h(p, r_0(K)) \subseteq K \subseteq (K_i)_\varepsilon$ if $i \geq i_0(\varepsilon)$. Hence, by Lemma 5.2.3 we get $B_h(p, r_0(K) - \varepsilon) \subseteq K_i$ for $i \geq i_0(\varepsilon)$. This shows that $\liminf_{i\to\infty} r_0(K_i) \geq r_0(K)$.

Now suppose that $\limsup_{i\to\infty} r_0(K_i) > r_0(K)$. Then there is some $\varepsilon > 0$ such that $r_0(K_i) \geq r_0(K) + 2\varepsilon$ for infinitely many $i \in \mathbb{N}$, hence $B_h(p, r_0(K) + 2\varepsilon) \subseteq K_i \subseteq K_\varepsilon$ for infinitely many $i \in \mathbb{N}$. Using again Lemma 5.2.3, we see that $B_h(p, r_0(K) + \varepsilon) \subseteq K$, a contradiction. This proves that $\limsup_{i\to\infty} r_0(K_i) \leq r_0(K)$. □

Lemma 5.2.5. *The centred circumradius functional $R_0 : \mathcal{K}_h^d \to [0, \infty)$ is continuous with respect to the Hausdorff metric.*

Proof. Let $K, K_i \in \mathcal{K}_h^d$ for $i \in \mathbb{N}$ with $\delta_h(K_i, K) \to 0$ as $i \to \infty$. Clearly, we have $K \subseteq B_h(p, R_0(K))$. If $\varepsilon > 0$, then $K_i \subseteq K_\varepsilon \subseteq B_h(p, R_0(K) + \varepsilon)$ for $i \geq i_0(\varepsilon)$. This shows that $\limsup_{i\to\infty} R_0(K_i) \leq R_0(K) + \varepsilon$, and thus $\limsup_{i\to\infty} R_0(K_i) \leq R_0(K)$. In particular, if $R_0(K) = 0$, then $\lim_{i\to\infty} R_0(K_i) = 0 = R_0(K)$.

Now let $R_0(K) > 0$ and suppose that $\liminf_{i\to\infty} R_0(K_i) = R_0(K) - 3\varepsilon \leq R_0(K)$ for some $\varepsilon > 0$. Then there are infinitely many $i \in \mathbb{N}$ such that $K_i \subseteq B_h(p, R_0(K) - 2\varepsilon)$, and thus $K \subseteq (K_i)_\varepsilon \subseteq B_h(p, R_0(K) - \varepsilon)$ for infinitely many $i \in \mathbb{N}$, a contradiction. □

We can now turn to the circum- and inball radius.

Lemma 5.2.6. *The functional R_{out} is continuous with respect to the Hausdorff metric.*

Proof. Let $(K_i)_{i\in\mathbb{N}}$ be a sequence of compact convex sets with $K_i \to K$ as $i \to \infty$. There exists some $c_K \in \mathbb{H}^d$ with $K \subseteq B_h(c_K, R_{out}(K))$. Let $\varepsilon > 0$. By our choice of $(K_i)_{i\in\mathbb{N}}$ there exists an integer $i(\varepsilon)$ such that $K_i \subseteq K_\varepsilon \subseteq B_h(c_K, R_{out}(K) + \varepsilon)$ holds for every $i \geq i_0(\varepsilon)$. This implies $\limsup_{i\to\infty} R_{out}(K_i) \leq R_{out}(K) + \varepsilon$ and therefore $\limsup_{i\to\infty} R_{out}(K_i) \leq R_{out}(K)$.

We now assume that $\liminf_{i\to\infty} R_{out}(K_i) = R_{out}(K) - 3\varepsilon$ holds for some $\varepsilon > 0$. This implies $R_{out}(K_i) \leq R_{out}(K) - 2\varepsilon$ for infinitely many $i \in \mathbb{N}$. Further there exists a sequence $(c_i)_{i\in\mathbb{N}}$ in $\mathbb{H}^d$ such that $K_i \subseteq B_h(c_i, R_{out}(K) - 2\varepsilon)$ holds for infinitely many $i \in \mathbb{N}$. By our choice of $(K_i)_{i\in\mathbb{N}}$ there exist infinitely many $i \in \mathbb{N}$ with $K \subseteq (K_i)_\varepsilon \subseteq B_h(c_i, R_{out}(K) - \varepsilon)$ and therefore $R_{out}(K) \leq R_{out}(K) - \varepsilon$, a contradiction. This shows $\liminf_{i\to\infty} R_{out}(K_i) \geq R_{out}(K)$ and together with the first part the continuity of R_{out}. □

Lemma 5.2.7. *The functional r_{in} is continuous with respect to the Hausdorff metric.*

Proof. Let $(K_i)_{i\in\mathbb{N}}$ be a sequence of compact convex sets with $K_i \to K$ as $i \to \infty$. Now assume $\limsup_{i\to\infty} r_{in}(K_i) \geq r_{in}(K) + 4\varepsilon$ for some $\varepsilon > 0$. Therefore there exists an (ordered) index set $I \subseteq \mathbb{N}$

with $r_{in}(K_i) \geq r_{in}(K) + 3\varepsilon$ for all $i \in I$ and $|I| = \infty$. This yields the existence of a sequence $(c_i)_{i\in I}$ such that $B_h(c_i, r_{in} + 3\varepsilon) \subseteq K_i$ holds for all $i \in I$. Since the sequence $(c_i)_{i\in I}$ is bounded, there exists a convergent subsequence, i.e. an index set $J \subseteq I$ of infinite cardinality with $c_j \to c$ as $j \to \infty$ for $j \in J$. This and our choice of $(K_i)_{i\in\mathbb{N}}$ implies

$$B_h(c, r_{in}(K) + 2\varepsilon) \subseteq B_h(c_j, r_{in}(K) + 3\varepsilon) \subseteq K_j \subseteq K_\varepsilon$$

for all $j \in J$ exceeding a certain threshold $j_0(\varepsilon)$. This in turn implies $r_{in}(K) \geq r_{in}(K) + \varepsilon$ by Lemma 5.2.3, a contradiction and thus $\limsup_{i\to\infty} r_{in}(K_i) \leq r_{in}(K)$.
For the case $r_{in}(K) = 0$ there is nothing more to show. We can therefore assume $r_{in}(K) > 0$. Let $\varepsilon \in (0, r_{in}(K))$. There exists some $c \in \mathbb{H}^d$ with $B_h(c, r_{in}(K)) \subseteq K$. By our choice of $(K_i)_{i\in\mathbb{N}}$ there further exists some $i_0(\varepsilon) \in \mathbb{N}$ with $K \subseteq (K_i)_\varepsilon$ for all $i \geq i_0(\varepsilon)$ and hence $B_h(c, r_{in}(K)) \subseteq (K_i)_\varepsilon$ for all $i \geq i_0(\varepsilon)$. We can therefore conclude with Lemma 5.2.3 $B_h(c, r_{in}(K) - \varepsilon) \subseteq K_i$ for all $i \geq i(\varepsilon)$ and thus

$$\liminf_{i\to\infty} r_{in}(K_i) \geq r_{in}(K) - \varepsilon.$$

Since we can choose ε arbitrarily small, the assertion follows. □

Before we can prove the lower semicontinuity of ϑ_r, we have to show a line of helping results first.

Lemma 5.2.8. *The set*

$$C_{in}(K) = \{c \in K : \ r_c(K) = r_{in}(K)\}$$

is a compact subset of K.

Proof. Let $(c_i)_{i\in\mathbb{N}}$ be a converging subsequence in $C_{in}(K)$ with limit in $c \in \mathbb{H}^d$. Therefore $B_h(c_i, r_{in}(K)) \subseteq K$ holds. This again implies $B_h(c, r_{in}(K))^\circ \subseteq K$ and since K is closed also $B_h(c, r_{in}(K)) \subseteq K$. In conclusion $c \in C_{in}(K)$ holds. □

Lemma 5.2.9. *The map* $(c, K) \mapsto R_c(K)$ *is continuous on* $\mathbb{H}^d \times \mathcal{K}_h^d$.

Proof. Let $c_i \to c$ and $K_i \to K$ as $i \to \infty$. Then the relation $K_i \subseteq B_h(c_i, R_{c_i}(K_i))$ holds for all $i \in \mathbb{N}$. Let $\varepsilon > 0$ and $r := \liminf_{i\to\infty} R_{c_i}(K_i)$. This implies $R_{c_i}(K_i) \leq r + \varepsilon$ and therefore also

$$K_i \subseteq B_h(c_i, r + \varepsilon)$$

for infinitely many $i \in \mathbb{N}$. By our choice of $(c_i)_{i\in\mathbb{N}}$ we further get $B_h(c_i, r + \varepsilon) \subseteq B_h(c, r + 2\varepsilon)$ for all $i \in \mathbb{N}$ exceeding some threshold $i_1(\varepsilon)$ and by the convergence of $(K_i)_{i\in\mathbb{N}}$ we get the relation $K \subseteq (K_i)_\varepsilon$ for all $i \in \mathbb{N}$ exceeding a certain threshold $i_2(\varepsilon)$. Together with the considerations above also

$$K \subseteq (K_i)_\varepsilon \subseteq B_h(c, r + 3\varepsilon)$$

holds for infinitely many $i \in \mathbb{N}$. This again implies $R_c(K) \leq r + 3\varepsilon$ and thus $\liminf_{i\to\infty} R_{c_i}(K_i) \geq R_c(K) - 3\varepsilon$. Since ε can be chosen arbitrarily small, we get $\liminf_{i\to\infty} R_{c_i}(K_i) \geq R_c(K)$.

Now assume that $\limsup_{i\to\infty} R_{c_i}(K_i) \geq R_c(K) + 3\varepsilon$ holds for some $\varepsilon > 0$. By our choice of $(K_i)_{i\in\mathbb{N}}$ and of $(c_i)_{i\in\mathbb{N}}$ there exists some $i_0(\varepsilon) \in \mathbb{N}$ such that

$$K_i \subseteq K_\varepsilon \subseteq B_h(c, R_c(K) + \varepsilon) \subseteq B_h(c_i, R_c(K) + 2\varepsilon)$$

holds for all $i \geq i_0(\varepsilon)$. This implies $\limsup_{i\to\infty} R_{c_i}(K_i) \leq R_c(K) + 2\varepsilon$ and thus a contradiction. Therefore $\limsup_{i\to\infty} R_{c_i}(K_i) \leq R_c(K)$ follows. □

Lemma 5.2.10. *The functional $(c, K) \mapsto r_c(K)$ is continuous on $\mathbb{H}^d \times \mathcal{K}_h^d$.*

Proof. Let $c_i \to c$ and $K_i \to K$ as $i \to \infty$. We assume $\limsup_{i\to\infty} r_{c_i}(K_i) \geq r_c(K) + 4\varepsilon$ for some $\varepsilon > 0$. This implies $r_{c_i}(K_i) \geq r_c(K) + 3\varepsilon$ and thus $B_h(c_i, r_c(K) + 3\varepsilon) \subseteq K_i$ for infinitely many $i \in \mathbb{N}$. This in turn implies $B_h(c, r_c(K) + 2\varepsilon) \subseteq K_i$ for infinitely many $i \in \mathbb{N}$ and therefore by Lemma 5.2.3 and our choice of $(K_i)_{i\in\mathbb{N}}$ the relation $B_h(c, r_c(K) + \varepsilon) \subseteq K$. This is a contradiction and shows $\limsup_{i\to\infty} r_{c_i}(K_i) \leq r_c(K)$.
For the case $r_c(K) = 0$, there is nothing left to show. For $r_c(K) > 0$, we chose $\varepsilon \in (0, r_c(K)/2)$. This implies $B_h(c_i, r_c(K)) \subseteq B_h(c, r_c(K) + \varepsilon) \subseteq K_\varepsilon \subseteq (K_i)_{2\varepsilon}$ and therefore by Lemma 5.2.3 also

$$B_h(c_i, r_c(K) - 2\varepsilon) \subseteq K_i$$

for infinitely many $i \in \mathbb{N}$. This further implies

$$\liminf_{i\to\infty} r_{c_i}(K_i) \geq r_c(K) - 2\varepsilon$$

and thus $\liminf_{i\to\infty} r_{c_i}(K_i) \geq r_c(K)$. □

Lemma 5.2.11. *The functional ϑ_r is lower semicontinuous and therefore measurable.*

Proof. Let $K_i \to K$ as $i \to \infty$. Then there are $c_i \in C_{in}(K_i)$ such that

$$\vartheta_r(K_i) = R_{c_i}(K_i) - r_{c_i}(K_i).$$

Now chose $I \subseteq \mathbb{N}$ with $|I| = \infty$ such that $\vartheta_r(K_i) \to \liminf_{j\to\infty} \vartheta_r(K_j)$ as $i \to \infty$ and $i \in I$. Since $(c_i)_{i\in I}$ is bounded, there exists a subset $J \subseteq I$ with $|J| = \infty$ such that $c_j \to c$ as $j \to \infty$ and $j \in J$ for some $c \in \mathbb{H}^d$. The limit point c is contained in $C_{in}(K)$ since by Lemmas 5.2.10 and 5.2.7

$$r_c(K) = \lim_{j\to\infty, j\in J} r_{c_j}(K_j) = \lim_{j\to\infty, j\in J} r_{in}(K_j) = r_{in}(K)$$

holds. This implies by using Lemmas 5.2.9, 5.2.10 and $c \in C_{in}(K)$

$$\liminf_{j\to\infty} \vartheta_r(K_j) = \lim_{j\to\infty,\ j\in J} \vartheta_r(K_j) = \lim_{j\to\infty,\ j\in J} \left(R_{c_j}(K_j) - r_{c_j}(K_j)\right) = R_c(K) - r_c(K) \geq \vartheta_r(K).$$

□

Remark 5.2.1. There are examples showing that the functional ϑ_r is not upper semicontinuous and thus not continuous.

In order to show the continuity of the circumcentre, we first show the following lemma.

Lemma 5.2.12. *The circumcentre $c_h(K)$ of a convex body $K \in \mathcal{K}_h^d$ is uniquely determined and contained in K.*

Proof. The first statement is well known. It follows by observing that the intersection of two different geodesic balls of equal radius is contained in a geodesic ball of smaller radius (by basic hyperbolic trigonometry).

Without loss of generality we can assume that $c_h(K) = p$ and $R_{out}(K) = r > 0$. Suppose that $p \notin K$. Consideration of the situation in the Beltrami-Klein model yields the existence of a Euclidean hyperplane $\tilde{H}$ such that (the representation of) K is contained in $\tilde{H}^+$ and the origin is in the interior of $\tilde{H}^-$. We therefore know that there exists a hyperbolic hyperplane H with $K \subseteq H^+$ and $p \in \operatorname{int}(H^-)$. We denote by $x \in \mathbb{H}^d$ the unique point fulfilling $H = H(x)$. We have $d_h(p,x) \in (0,r]$. Now let $z \in K = K \cap H^+$ be an arbitrary point in K. We consider the hyperbolic triangle determined by p, x, z and denote the angle at x by γ. Since $z \in H^+$ we know that $\gamma \in [\frac{\pi}{2}, \pi]$. Therefore [92, Theorem 3.5.3] yields

$$\cosh(d_h(p,z)) \geq \cosh(d_h(p,x)) \cosh(d_h(x,z))$$

and thus

$$d_h(x,z) \leq \cosh^{-1}\left(\frac{\cosh(d_h(p,z))}{\cosh(d_h(p,x))}\right) \leq \cosh^{-1}\left(\frac{\cosh(r)}{\cosh(d_h(p,x))}\right) < r.$$

This implies that $K \subseteq B_h(x,\tilde{r})$ for some $\tilde{r} < r$, a contradiction. □

Lemma 5.2.13. *The map c_h, assigning to each convex body its corresponding circumcentre, is continuous on $\mathcal{K}_h^d$.*

Proof. Let $K_i \to K$ as $i \to \infty$. We assume $c_h(K_i) \not\to c_h(K)$ as $i \to \infty$. By Lemma 5.2.12 the circumcentre $c_h(K_i)$ is contained in K_i for every $i \in \mathbb{N}$. This implies that the sequence $(c_h(K_i))_{i\in\mathbb{N}}$ is bounded and therefore contains a convergent subsequence $(c_h(K_{i_j}))_{j\in\mathbb{N}}$ with limit in $c \neq c_h(K)$. We obviously have $R_{out}(K_i) = R_{c_h(K_i)}(K_i)$ for all $i \in \mathbb{N}$. We therefore infer by Lemmas 5.2.6 and 5.2.9

$$R_{c_h(K)}(K) = R_{out}(K) = \lim_{j\to\infty} R_{out}(K_{i_j}) = \lim_{j\to\infty} R_{c_h(K_{i_j})}(K_{i_j}) = R_c(K).$$

Since the circumcentre is uniquely defined, we get $c = c_h(K)$, a contradiction. □

5.2.2 Isoperimetric results

In this section, we establish several inequalities of isoperimetric type. Recall that we denote by B^a, for $a > 0$, the unique hyperbolic ball with centre in p and $\mathcal{H}^d(B^a) = a$. The following theorem will be applied in the situation where $\mathcal{H}^d$ is the size functional and W_{d-1} is proportional to the hitting functional.

Theorem 5.2.14. *Let ϑ be a deviation function and $a > 0$. Then there exists a function $f_{a,\vartheta}\colon [0,\infty) \to [0,\infty)$ with $f_{a,\vartheta}(0) = 0$, $f_{a,\vartheta}(t) > 0$ for $t > 0$ and*

$$W_{d-1}(K) \geq (1 + f_{a,\vartheta}(\varepsilon))W_{d-1}(B^a)$$

for $\varepsilon > 0$ and $K \in \mathcal{K}^d_{h,0}$ with $\mathcal{H}^d(K) \geq a$ and $\vartheta(K) \geq \varepsilon$.

Proof. For $n \in \mathbb{N}$ we define the set

$$\tilde{\mathcal{K}}(a,n) \coloneqq \{K \in \mathcal{K}^d_{h,0} :\ W_{d-1}(K) \leq nW_{d-1}(B^a)\}, \qquad a > 0,$$

of all convex bodies $K \in \mathcal{K}^d_h$ with $p \in K$ and for which $W_{d-1}(K)$ is bounded from above. By the definition of W_{d-1} and an application of Lemma 2.4.1 with $k = d-1$ and $i = 0$, it follows that $W_{d-1}(I_r) = \omega_{d+1}(d\omega_2)^{-1}r$ for any interval I_r of length r. Hence every $K \in \tilde{\mathcal{K}}(a,n)$ is contained in $B_{\tilde{r}}$ for some $\tilde{r} = \tilde{r}(d,a,n) > 0$. Then Lemmas 2.3.4 and 5.2.1 imply that $\tilde{\mathcal{K}}(a,n)$ is compact. Now we consider the set

$$\mathcal{K}(a,n) \coloneqq \{K \in \tilde{\mathcal{K}}(a,n) : \mathcal{H}^d(K) \geq a\}.$$

Since by Lemma 5.2.2 the functional $\mathcal{H}^d$ is continuous, $\mathcal{K}(a,n)$ is compact as well. Clearly, $B^a \in \mathcal{K}(a,n)$ and hence $\mathcal{K}(a,n) \neq \varnothing$. The functional W_{d-1} is continuous and attains its minimum on $\mathcal{K}(a,n)$. The results in [29] show that this minimum is attained precisely by geodesic balls of volume a. Now consider the set

$$\mathcal{K}_{a,n,\varepsilon} \coloneqq \{K \in \mathcal{K}(a,n) : \vartheta(K) \geq \varepsilon\}.$$

Since ϑ is upper semicontinuous, $\mathcal{K}_{a,n,\varepsilon}$ is compact. We can now chose $n \in \mathbb{N}$ such that $K_{a,n,\varepsilon}$ is none empty. If there exists no such n then there exists no $K \in \mathcal{K}^d_{h,0}$ with $\mathcal{H}^d(K) \geq a$ and $\vartheta(K) \geq \varepsilon$, hence there is nothing to show. Now W_{d-1} attains its minimum on $\mathcal{K}_{a,n,\varepsilon}$. Clearly, the inequality

$$\tau_{a,\varepsilon} \coloneqq \min_{K \in \mathcal{K}_{a,n,\varepsilon}} W_{d-1}(K) \geq \min_{K \in \mathcal{K}(a,n)} W_{d-1}(K) \eqqcolon \tau_a \tag{5.5}$$

holds for all $\varepsilon > 0$ since $\mathcal{K}_{a,n,\varepsilon} \neq \varnothing$. Assume that $\tau_{a,\varepsilon} = \tau_a$ holds. This implies the existence of a body $K \in \mathcal{K}_{a,n,\varepsilon}$ such that $\mathcal{H}^d(K) \geq a$ and $W_{d-1}(K) = \tau_a$. Therefore K is a ball and hence $\vartheta(K) = 0$. Since this is a contradiction, the inequality in (5.5) is strict. Finally, we define $f_{a,\vartheta}(t) \coloneqq \tau_{a,t}/\tau_a - 1$ for $t \in (0,\infty)$ and $f_{a,\vartheta}(0) \coloneqq 0$. The function $f_{a,\vartheta}$ then has the desired properties. □

Remark 5.2.2. The proof in [29] uses two-point symmetrization. Roughly speaking the argument shows that if a convex body K is not a ball, then the two point symmetrization T does not increase the $(d-1)$-th quermassintegral, while the volume is preserved. This means that for a convex body $K \in \mathcal{K}^d_h$

$$W_{d-1}(K) \geq W_{d-1}(T(K)) \quad \text{and} \quad \mathcal{H}^d(K) = \mathcal{H}^d(T(K))$$

holds true. In [29] F. Gao, D. Hug and R. Schneider focus on the spherical case but point out that the argument works in the hyperbolic case as well. For the characterization of the equality case it is used that two-point symmetrization preserves the surface area as well and that geodesic balls are the only extremal bodies (in the class of compact convex sets) for the isoperimetric problem in hyperbolic space.

In [115], G. Wang and C. Xia prove that the minimum of W_k on $\mathcal{K}$ given W_l is achieved precisely by geodesic balls for $0 \le l < k \le n-1$. Here $\mathcal{K}$ is the set of all h-convex bodies with smooth boundary.

Theorem 5.2.14 can be extended to arbitrary size functionals as done in [42]. We often write f_a instead of $f_{a,\vartheta}$ if the underlying deviation functional is clear from the context.

The following result is based on a Bonnesen-style inequality established in [19]. It yields a specific stability function $f_a : [0,1] \to [0,1]$ in the special 2-dimensional case of Theorem 5.2.14. We will frequently use that $W_1(K) = \mathcal{H}^1(\partial K)/2$ holds for $K \in \mathcal{K}_h^2$ with interior points (positive volume).

Theorem 5.2.15. *Let $a > 0$ and $\varepsilon \in [0,1]$. Let $K \in \mathcal{K}_h^2$ with $\mathcal{H}^2(K) \ge a$ and $R_{out}(K) - r_{in}(K) = \varepsilon$. Then for*

$$f_a : [0,1] \to [0,1], \ s \mapsto \min\{(12\cosh^2(r_a + 2)\sinh^2(r_a))^{-1} s^2, 1\} \tag{5.6}$$

with $r_a := \operatorname{arcosh}(1 + a/2\pi)$ the inequality

$$W_1(K) \ge (1 + f_a(\varepsilon))W_1(B^a)$$

holds.

Proof. Let $r_a := \operatorname{arcosh}(1 + a/2\pi)$ be the radius of a hyperbolic circle with area a. In order to prove the theorem, we distinguish two cases. First, assume that $r_{in}(K) \ge r_a + 1$. In this case we get

$$W_1(K) \ge W_1(B_{r_{in}(K)}) = \pi \sinh(r_{in}(K)) \ge \pi \sinh(r_a + 1) \ge 2\pi \sinh(r_a) = 2W_1(B^a),$$

where we used that $\sinh(x+1) \ge 2\sinh(x)$ for $x \ge 0$. Therefore the claim holds in this case.

Now suppose that $r_{in}(K) < r_a + 1$. As in [19], we define

$$\Delta_{-1}(K) := 4W_1(K)^2 - (4\pi + \mathcal{H}^2(K))\mathcal{H}^2(K) \le 4W_1(K)^2 - (4\pi + a)a. \tag{5.7}$$

For the right hand side we obtain

$$\begin{aligned}(4\pi + a)a &= (4\pi + 2\pi(\cosh(r_a) - 1))(2\pi(\cosh(r_a) - 1)) \qquad (5.8)\\ &= 4\pi^2(\cosh(r_a) + 1)(\cosh(r_a) - 1)\\ &= 4\pi^2(\cosh^2(r_a) - 1)\\ &= (2\pi \sinh(r_a))^2\\ &= 4\, W_1(B^a)^2. \qquad (5.9)\end{aligned}$$

We define $g : \mathcal{K}_h^2 \to [0, \infty)$ such that it fulfills

$$W_1(K) = (1 + g(K))W_1(B^a). \qquad (5.10)$$

By [29], we know that g is non-negative. If $g(K) \geq 1$ holds, then there is nothing to show. Thus we consider $K \in \mathcal{K}_h^2$ with $g(K) < 1$. Then we get

$$\begin{aligned}4\left(W_1(K)^2 - W_1(B^a)^2\right) &= 4\left((1 + g(K))^2 W_1(B^a)^2 - W_1(B^a)^2\right)\\ &= 4\left(2g(K) + g(K)^2)W_1(B^a)\right)^2\\ &\leq 12 g(K) W_1(B^a)^2. \qquad (5.11)\end{aligned}$$

By [19, Corollary 2.1] we have

$$\Delta_{-1}(K) \geq \frac{\pi^2}{\cosh^2(R_{out}(K))\cosh^2(r_{in}(K))}\left(\sinh(R_{out}(K)) - \sinh(r_{in}(K))\right)^2.$$

Combining this with (5.7)-(5.11), we deduce that

$$12 g(K) W_1(B^a)^2 \geq \frac{\pi^2}{\cosh^2(R_{out}(K))\cosh^2(r_{in}(K))}\left(\sinh(R_{out}(K)) - \sinh(r_{in}(K))\right)^2. \qquad (5.12)$$

The last factor on the right-hand side can be bounded from below by

$$\begin{aligned}&\sinh(R_{out}(K)) - \sinh(r_{in}(K))\\ &\quad= \sinh(r_{in}(K) + \varepsilon) - \sinh(r_{in}(K))\\ &\quad= \sinh(r_{in}(K))\cosh(\varepsilon) + \sinh(\varepsilon)\cosh(r_{in}(K)) - \sinh(r_{in}(K))\\ &\quad\geq \sinh(\varepsilon)\cosh(r_{in}(K))\\ &\quad\geq \varepsilon \cosh(r_{in}(K)), \qquad (5.13)\end{aligned}$$

where an addition theorem for sinh was used. Finally, by using (5.12), (5.13) and $W_1(B^a) =$

$\pi \sinh(r_a)$, we get

$$\begin{aligned}
g(K) &\geq \frac{\pi^2}{12\, W_1(B^a)^2 \cosh^2(R_{out}(K))}\,\varepsilon^2 \\
&= \frac{1}{12 \cosh^2(R_{out}(K)) \sinh^2(r_a)}\,\varepsilon^2 \\
&= \frac{1}{12 \cosh^2(r_{in}(K)+\varepsilon) \sinh^2(r_a)}\,\varepsilon^2 \\
&\geq \frac{1}{12 \cosh^2(r_a+2) \sinh^2(r_a)}\,\varepsilon^2,
\end{aligned}$$

where we used $r_{in}(K) < r_a + 1$ in the last line. Combining the last inequality with (5.10) finishes the proof. □

Before we can state and prove a result for the special case where $\Sigma = \Sigma_r$ and $\vartheta = \vartheta_r$, we need to show the following lemma, which is used in the proof.

Lemma 5.2.16. *Let* $a > 0$, $\varepsilon \in [0,1]$ *and recall*

$$\tilde{\beta} = \sqrt{2 - \frac{2\tanh(a+\varepsilon/2)}{\tanh(a+\varepsilon)}}.$$

Then

$$1 > \tilde{\beta} \geq \varepsilon^{\frac{1}{2}} \left(\cosh(a+1/2)\sinh(a+1)\right)^{-\frac{1}{2}}.$$

Proof. By using an identity of the sinh-function in the third row, we get

$$\begin{aligned}
h(a,\varepsilon) := 1 - \frac{\tanh(a+\varepsilon/2)}{\tanh(a+\varepsilon)} &= 1 - \frac{\sinh(a+\varepsilon/2)\cosh(a+\varepsilon)}{\cosh(a+\varepsilon/2)\sinh(a+\varepsilon)} \\
&= \frac{\cosh(a+\varepsilon/2)\sinh(a+\varepsilon) - \sinh(a+\varepsilon/2)\cosh(a+\varepsilon)}{\cosh(a+\varepsilon/2)\sinh(a+\varepsilon)} \\
&= \frac{\sinh(\varepsilon/2)}{\cosh(a+\varepsilon/2)\sinh(a+\varepsilon)} \\
&\geq \frac{\varepsilon}{2\,\cosh(a+\varepsilon/2)\sinh(a+\varepsilon)} \\
&\geq \frac{\varepsilon}{2\,\cosh(a+1/2)\sinh(a+1)},
\end{aligned} \tag{5.14}$$

from which the lower bound follows. For the upper bound, it is sufficient to consider $\varepsilon \in (0,1]$. Then (5.14) and the monotonicity of sinh and cosh yield

$$h(a,\varepsilon) \leq \frac{\sinh(\varepsilon/2)}{\cosh(\varepsilon/2)\sinh(\varepsilon)} = 2\frac{e^{\varepsilon}}{(e^{\varepsilon}+1)^2} < \frac{1}{2},$$

which implies the upper bound. □

Now we are in the position to prove the following result.

Theorem 5.2.17. *Let $K \in \mathcal{K}_h^d$, $\varepsilon \in [0,1]$ and $a > 0$. If $\Sigma_r(K) \geq a$ and $\vartheta_r(K) \geq \varepsilon$, then there is a constant $c_1 = c_1(a,d)$ such that*

$$W_{d-1}(K) \geq \left(1 + c_1\, \varepsilon^{\frac{d+1}{2}}\right) W_{d-1}(B_a),$$

where c_1 is given by

$$c_1 = c_1(a,d) = \frac{\omega_{d-1}}{2\,a\,3^{d-2}\omega_d(\cosh(a+1/2)\sinh(a+1))^{(d-1)/2}}.$$

Proof. There is a point $\tilde{c} \in C_r(K)$ with $R_{\tilde{c}}(K) - r_{\tilde{c}}(K) = \vartheta_r(K)$. Further, there is an isometry $\varphi_{\tilde{c}} \in I(\mathbb{H}^d)$ such that $\varphi_{\tilde{c}}(\tilde{c}) = p$. Let $\tilde{K} := \varphi_{\tilde{c}}(K)$ be the isometric image body. It contains B_a and since $\vartheta_r(K) \geq \varepsilon$, it also contains a point $z_0 = \exp_p((a+\varepsilon)u)$ for some $u \in \mathbb{S}_p^{d-1}$. Furthermore, since $\tilde{K}$ is convex, it follows that $[\exp_p(au), \exp_p((a+\varepsilon)u)] \subseteq \tilde{K}$. Therefore

$$W_{d-1}(K) = W_{d-1}(\tilde{K}) \geq W_{d-1}(\operatorname{conv}_h(B_a \cup [\exp_p(au), \exp_p((a+\varepsilon)u)])),$$

where conv_h is the hyperbolic convex hull operator. The definition of W_{d-1} implies that the value $W_{d-1}(K)$ can be bounded from below by

$$\begin{aligned} W_{d-1}(K) &\geq \frac{\omega_d}{d\,\omega_1} \int_{A_h(d,d-1)} \chi(H \cap (\operatorname{conv}_h(B_a \cup [\exp_p(au), \exp_p((a+\varepsilon)u)]))) \; \mu_{d-1}(dH) \\ &\geq W_{d-1}(B_a) + \frac{\omega_d}{d\,\omega_1} \int_{A_h(d,d-1) \setminus \mathbb{H}_{d-1}\langle B_a \rangle} \chi(H \cap [\exp_p(au), \exp_p((a+\varepsilon)u)]) \; \mu_{d-1}(dH). \end{aligned}$$

Now recall $\tilde{\beta} = \sqrt{2 - \frac{2\tanh(a+\varepsilon/2)}{\tanh(a+\varepsilon)}}$ and

$$c(\tilde{\beta}) := \frac{1}{\omega_d}\, \sigma_{d-1}(\{v \in \mathbb{S}_p^{d-1} :\; d_s(u,v) \leq \tilde{\beta}\}) \in [0,1].$$

By Lemma 5.1.4 we know that every hyperplane in

$$D(u,\tilde{\beta}) := \{H(\exp_p(rv)) : r \in (a, a+\varepsilon/2],\; v \in B_s(u,\tilde{\beta})\}$$

has nonempty intersection with $[p, \exp_p((a+\varepsilon)u)]$ and therefore also with $[\exp_p(au), \exp_p((a+\varepsilon)u)]$. Therefore, by using $D(u,\tilde{\beta}) \subseteq A_h(d,d-1) \setminus \mathbb{H}_{d-1}\langle B_a \rangle$, the representation of μ_{d-1} in (2.3) and

$$\begin{aligned} W_{d-1}(B_a) &= \frac{\omega_d}{d\,\omega_1} \int_{A_h(d,d-1)} \chi(H \cap B_a) \; \mu_{d-1}(dH) \\ &= \frac{\omega_d}{2\,d} 2 \int_0^a \cosh^{d-1}(t) \; dt \\ &\leq \frac{\omega_d\, a}{d} \cosh^{d-1}(a). \end{aligned}$$

we get

$$
\begin{aligned}
&W_{d-1}(K) - W_{d-1}(B_a)\\
&\quad \ge \frac{\omega_d}{d\,\omega_1}\int_{D(u,\tilde\beta)} 1\ \mu_{d-1}(dH)\\
&\quad = \frac{\omega_d}{2\,d}\int_{G_h(d,1)}\int_L \cosh^{d-1}(d_h(x,p))\,\mathbb{1}\{H(L,x)\in D(u,\tilde\beta)\}\ \mathcal{H}^1(dx)\ \nu_1(dL)\\
&\quad = \frac{\omega_d}{2\,d}\,c(\tilde\beta)\,2\int_a^{a+\varepsilon/2}\cosh^{d-1}(s)\ ds\\
&\quad \ge \frac{\omega_d\,c(\tilde\beta)}{d}\,\frac{\varepsilon}{2}\cosh^{d-1}(a)\\
&\quad \ge \frac{c(\tilde\beta)}{2\,a}\,\varepsilon\,W_{d-1}(B_a).
\end{aligned}
\tag{5.15}
$$

We take a closer look at $c(\tilde\beta)$. Here we use spherical coordinates on the sphere, Hölder's inequality and the elementary inequality $1-\cos(t)\ge t^2/3$ for $t\in[0,2]$ to get

$$
\begin{aligned}
c(\tilde\beta) = \frac{\omega_{d-1}}{\omega_d}\int_0^{\tilde\beta}\sin^{d-2}(t)\ dt &\ge \frac{\tilde\beta\,\omega_{d-1}}{\omega_d}\left(\frac{1}{\tilde\beta}\int_0^{\tilde\beta}\sin(t)\ dt\right)^{d-2} = \frac{\tilde\beta\,\omega_{d-1}}{\omega_d}\left(\frac{1-\cos(\tilde\beta)}{\tilde\beta}\right)^{d-2}\\
&\ge \frac{\omega_{d-1}}{3^{d-2}\omega_d}\,\tilde\beta^{d-1}.
\end{aligned}
$$

The inequality in Lemma 5.2.16 then yields

$$
c(\tilde\beta) \ge \frac{\omega_{d-1}}{3^{d-2}\omega_d}\,\tilde\beta^{d-1} \ge \frac{\omega_{d-1}}{3^{d-2}\omega_d}(\cosh(a+1/2)\sinh(a+1))^{-\frac{d-1}{2}}\,\varepsilon^{\frac{d-1}{2}}, \tag{5.16}
$$

Combination of the results in (5.16) with inequality (5.15) gives

$$
W_{d-1}(K) - W_{d-1}(B_a) \ge \frac{c(\tilde\beta)}{2\,a}\varepsilon W_{d-1}(B_a) \ge c_1(a,d)\,\varepsilon^{\frac{d+1}{2}}\,W_{d-1}(B_a)
$$

where $c_1(a,d)$ is given by

$$
c_1(a,d) = \frac{\omega_{d-1}}{2\,a\,3^{d-2}\omega_d(\cosh(a+1/2)\sinh(a+1))^{(d-1)/2}}.
$$

□

5.2.3 Approximation results

In this section, we are aiming to approximate convex bodies with polytopes having a controlled number of vertices. The following lemma is an analogue to [42, Theorem 5.1] and yields bounds for the Hausdorff distance between a hyperbolically convex set and an approximating polytope. In contrast to the Euclidean or spherical case, it involves the circumradius of the convex body. In fact, in hyperbolic space a polytope exceeding a certain volume has to have a certain number of vertices.

Lemma 5.2.18. *Let $K\in\mathcal{K}_h^d$ with $R_0(K)\le r$. Then there are constants k_0 and b_0, depending only on the dimension d, such that for all $k\ge k_0$, there is a hyperbolic polytope Q with at most*

k vertices, which can be chosen on the boundary of K, such that

$$\delta_h(K,Q) \le \cosh^2(r) b_0 k^{-\frac{2}{d-1}}.$$

The mapping $K \mapsto Q(K)$ can be chosen to be measurable.

Proof. We consider the projective disk model. We know that hyperbolic m-planes are precisely the intersections of Euclidean m-planes with $B_{euc}^{d,\circ}$, in the model space. Therefore polytopes are represented as intersections of Euclidean polytopes which are contained in $B_{euc}^{d,\circ}$. By [14], there is an integer $k_0 \in \mathbb{N}$ and a non-negative number $b_0 \ge 0$ such that for $k \ge k_0$ there is a polytope $Q_0 \subseteq B_{euc}^{d,\circ}$ with at most k vertices on the boundary of $\tilde{\pi}(K)$ and

$$\delta_{euc}(\tilde{\pi}(K), Q_0) \le b_0 k^{-\frac{2}{d-1}}.$$

Here δ_{euc} is the Euclidean Hausdorff distance. Since the furthest point of K has hyperbolic distance at most r from the origin, we can bound the maximal Euclidean distance $\tilde{r}$ of $\tilde{\pi}(K)$ from the origin from above. These two distances are related by

$$\cosh(r) = \frac{1}{\sqrt{1-\tilde{r}^2}}, \tag{5.17}$$

where we used the metric d_D in the projective disk model (see [92, Theorem 6.1.1]). Let $x, y \in B_{euc}^d(0,\tilde{r})$, and set $z(t) := (1-t)x + ty$ for $t \in [0,1]$. We write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ for the Euclidean scalar product and norm in $\mathbb{R}^d$. Then [92, Theorem 6.1.5] and (5.17) yield

$$\begin{aligned} d_D(x,y) &\le \int_0^1 \frac{\sqrt{(1-\|z(t)\|^2)\|y-x\|^2 + \langle z(t), y-x\rangle^2}}{1-\|z(t)\|^2}\, dt \\ &\le \int_0^1 \frac{\sqrt{(1-\|z(t)\|^2)\|y-x\|^2 + \|z(t)\|^2\|y-x\|^2}}{1-\|z(t)\|^2}\, dt \\ &= \int_0^1 \frac{\|y-x\|}{1-\|z(t)\|^2}\, dt \le \frac{1}{1-\tilde{r}^2}\, d_{euc}(x,y) \\ &= \cosh^2(r) d_{euc}(x,y). \end{aligned} \tag{5.18}$$

We now define $Q := \tilde{\pi}^{-1}(Q_0)$. Having this one can give an estimate for the hyperbolic Hausdorff distance between K and Q:

$$\begin{aligned} \delta_h(K,Q) &= \max\left\{\max_{x\in K}\min_{y\in Q} d_h(x,y), \max_{x\in Q}\min_{y\in K} d_h(x,y)\right\} \\ &= \max\left\{\max_{x\in\tilde{\pi}(K)}\min_{y\in Q_0} d_D(x,y), \max_{x\in Q_0}\min_{y\in\tilde{\pi}(K)} d_D(x,y)\right\} \\ &\le \max\left\{\max_{x\in\tilde{\pi}(K)}\min_{y\in Q_0} \cosh^2(r) d_{euc}(x,y), \max_{x\in Q_0}\min_{y\in\tilde{\pi}(K)} \cosh^2(r) d_{euc}(x,y)\right\} \\ &= \cosh^2(r)\delta_{euc}(\tilde{\pi}(K), Q_0) \\ &\le \cosh^2(r) b_0 k^{-\frac{2}{(d-1)}}. \end{aligned}$$

Since the polytope Q_0 in the projective disc model can be chosen in a measurable way (see [42]) this is true for Q as well. □

Theorem 5.2.19. *Let Σ be a size functional, Φ a hitting functional and let $a, r > 0$ and $\alpha \in (0,1)$ be real numbers. Then there is an integer $\nu \in \mathbb{N}$, depending only on $a, d, \alpha, r, \Sigma, \Phi$, such that for every hyperbolic polytope P with $\Sigma(P) \geq a$ and $R_0(P) \leq r$ there is a hyperbolic polytope $Q = Q(P)$ satisfying $\operatorname{ext}(Q) \subseteq \operatorname{ext}(P)$, $f_0(Q) \leq \nu$ and*

$$\Phi(Q) \geq (1-\alpha)\Phi(P).$$

Furthermore, the map $P \mapsto Q(P)$ can be chosen to be measurable.

Proof. The proof is similar to the one of [42, Theorem 5.2]. The functional Φ is continuous on the compact set $\mathcal{K}^d_{h,r}$ with respect to the hyperbolic Hausdorff metric and therefore uniformly continuous. Define the constant $\varepsilon := \alpha\,\tau(\Phi, \Sigma, a)$. Using the uniform continuity, it follows that there is a $\delta_r(\varepsilon) > 0$ such that $|\Phi(K) - \Phi(K')| \leq \varepsilon$ for all $K, K' \in \mathcal{K}^d_{h,r}$ such that $\delta_h(K, K') \leq \delta_r(\varepsilon)$.

Now let P be a hyperbolic polytope of size $\Sigma(P) \geq a$ and $R_0(P) \leq r$. By Lemma 5.2.18 there exists a hyperbolic polytope $Q = Q(P)$ and an integer $\nu = \nu(\alpha, a, d, r, \Sigma, \tau)$, such that $\operatorname{ext}(Q) \subseteq \operatorname{ext}(P)$, $f_0(Q) \leq \nu$ and $\delta_h(Q, P) \leq \delta_r(\varepsilon)$. Since $\Sigma(P) \geq a$ we get $\Phi(P) \geq \tau(\Phi, \Sigma, a)$ and therefore

$$\Phi(P) - \Phi(Q) \leq |\Phi(P) - \Phi(Q)| \leq \varepsilon = \alpha\,\tau(\Phi, \Sigma, a) \leq \alpha\,\Phi(P).$$

This gives the desired inequality. The measurability follows from Lemma 5.2.18. □

5.3 Zero cell

In this section, we will investigate Kendall's problem in hyperbolic space under the assumption that $t > \hat{c}_d$, where $\hat{c}_d = \frac{(d-1)\omega_d\,\omega_2}{\omega_{d+1}\omega_1}$. In particular, we will show that under this assumption all cells of the tessellation X_t are almost surely bounded.

5.3.1 Bounding of the zero cell

For the proofs of the main results of Section 5.3, we distinguish for a suitable radius $r > 0$ whether $C_0 \subseteq B_r$ or not. In a first step, we then provide an upper bound for the probability that $C_0 \nsubseteq B_r$. This is achieved in two steps. First, we show that the boundary of B_r can be covered by a certain number of hyperbolic caps depending on the radius $r > 0$. By a hyperbolic cap C of radius $b > 0$ we mean the intersection of the boundary of a hyperbolic ball B of radius $r > b$ and a second hyperbolic ball of radius b with centre on the boundary of B. In the second step, we show that the probability that the process contains a hyperplane that separates such a cap from the origin decays much faster than the number of required caps grows.

Lemma 5.3.1. *The boundary of B_r, $r > 0$, can be covered by $c_2(d)e^{r(d-1)}$ hyperbolic caps of radius 1.*

Proof. The map $h : \partial B_r \to \mathbb{S}_p^{d-1},\ x \mapsto \frac{\exp_p^{-1}(x)}{\|\exp_p^{-1}(x)\|}$, provides a bijection between ∂B_r and $\mathbb{S}_p^{d-1}$. We are aiming to give an upper bound for the spherical radius r_s of

$$h(\partial B_r \cap B_h(\exp_p(ru), 1)),$$

where $u \in \mathbb{S}_p^{d-1}$. We denote by $x \in \mathbb{H}^d$ an intersection point of ∂B_r and $\partial B_h(\exp_p(ru), 1)$. The value r_s is given as the angle α (measured at p of the hyperbolic triangle determined by the points $(p, \exp_p(ru), x)$. By the hyperbolic law of cosines ([92, Theorem 3.5.3]) this radius is given via

$$\cosh(1) = \cosh^2(r) - \cos(r_s)\sinh^2(r),$$

which yields

$$\begin{aligned} r_s = \arccos\left(\frac{\cosh^2(r) - \cosh(1)}{\sinh^2(r)}\right) &= \arccos\left(1 - \frac{\sinh^2(r) - \cosh^2(r) + \cosh(1)}{\sinh^2(r)}\right) \\ &= \arccos\left(1 - \frac{\cosh(1) - 1}{\sinh^2(r)}\right) \\ &\geq \sqrt{2}\sqrt{\frac{\cosh(1) - 1}{\sinh^2(r)}}. \end{aligned} \tag{5.19}$$

Here the last inequality is equivalent to $\cos(z) \geq 1 - \frac{1}{2}z^2$ for $z \in \mathbb{R}$. Using the definition of sinh and (5.19), we get

$$r_s \geq \frac{2\sqrt{2}}{e^r - e^{-r}}\sqrt{\cosh(1) - 1} \geq 2\sqrt{2}e^{-r}\sqrt{\cosh(1) - 1} \geq 2\sqrt{2}e^{-r}\sqrt{1/2} = 2e^{-r}.$$

By [12, Chapter 6] we know that $\mathbb{S}^{d-1}$ and therefore also $\mathbb{S}_p^{d-1}$ can be covered by $c_3(d)/r_s^{d-1}$ caps of radius $r_s > 0$. Therefore ∂B_r can be covered by

$$\frac{c_3(d)}{r_s^{d-1}} \leq \frac{c_3(d)}{(2\,e^{-r})^{d-1}} = c_3(d)2^{-(d-1)}e^{r(d-1)}$$

hyperbolic caps of radius 1. Hence the assertion follows by choosing $c_2(d) = c_3(d)2^{-(d-1)}$. □

The following lemma uses the result of Lemma 5.3.1 to give an arbitrarily fast exponential decay of a probability used in the proof of Theorem 5.3.5. Recall that $\hat{c}_d = (d-1)\omega_d\omega_2/(2\omega_{d+1})$.

Lemma 5.3.2. *Let $d \geq 2$, $\kappa > 1$, $t \geq \kappa\hat{c}_d$ and $a > 0$. Then for every $c > 0$, there exists a real number $r = r(d, a, c, \kappa) > 0$ such that*

$$\mathbb{P}_t(R_0(C_0) \geq r) \leq e^{-t(c+\tau(a))} \qquad \textit{and} \qquad \mathbb{P}_t\left(R_0(C_0) \geq r \mid \mathcal{H}^d(C_0) \geq a\right) \leq e^{-ct}.$$

Proof. Let $\kappa > 1$ be fixed. Let $r > 1$ and $u \in \mathbb{S}_p^{d-1}$. We denote by C the hyperbolic cap

$$C := C(u, r) := \partial B_r \cap B_h(\exp_p(ru), 1).$$

Now consider the set

$$\mathbb{H}_{d-1}\langle C\,|\,p\rangle := \{H \in A_h(d,d-1):\ p \subseteq H^-,\ C \subseteq H^+,\ (\{p\}\cup C)\cap H = \varnothing\}$$

of hyperplanes that separate the origin p from C. Since $\mathbb{H}_{d-1}\langle C\,|\,p\rangle = \mathbb{H}_{d-1}\langle \mathrm{conv}_h(\{p\}\cup C)\rangle \setminus (\mathbb{H}_{d-1}\langle\{p\}\rangle \cup \mathbb{H}_{d-1}\langle C\rangle)$, the measure of this set is bounded from below by

$$\begin{aligned}\mu_{d-1}(\mathbb{H}_{d-1}\langle C\,|\,p\rangle) &= \mu_{d-1}(\mathbb{H}_{d-1}\langle \mathrm{conv}_h(\{p\}\cup C)\rangle) - \mu_{d-1}(\mathbb{H}_{d-1}\langle\{p\}\rangle) - \mu_{d-1}(\mathbb{H}_{d-1}\langle C\rangle)\\ &\geq \mu_{d-1}(\mathbb{H}_{d-1}\langle[p,\exp_p(ru)]\rangle) - \mu_{d-1}(\mathbb{H}_{d-1}\langle B_h(\exp_p(ru),1)\rangle)\\ &= \frac{\omega_{d+1}\,\omega_1}{\omega_d\,\omega_2}\,r - \mu_{d-1}(\mathbb{H}_{d-1}\langle B_1\rangle),\end{aligned}$$

where the last we used the fact that $\mu_{d-1}(\mathbb{H}_{d-1}\langle\{p\}\rangle) = 0$ and the Crofton type formula (2.4). Therefore there exists a real number $\tilde{r} = \tilde{r}(d,\kappa) > 1$ such that for all $r \geq \tilde{r}$ the inequality

$$\mu_{d-1}(\mathbb{H}_{d-1}\langle C\,|\,p\rangle) \geq \frac{\omega_{d+1}\omega_1}{\omega_d\omega_2}\frac{1+\kappa}{2\kappa}r$$

holds. Lemma 5.3.1 shows that one can cover ∂B_r by $n = c_2(d)e^{r(d-1)}$ hyperbolic caps $C_1,\ldots,C_n$ of radius 1. Hence we get for $r \geq \tilde{r}$

$$\begin{aligned}\mathbb{P}_t(R_0(C_0)\geq r) = \mathbb{P}_t(\partial B_r \cap C_0 \neq \varnothing) &= \mathbb{P}_t\left(\left(\bigcup_{i=1}^n C_i\right)\cap C_0 \neq \varnothing\right)\\ &\leq \sum_{i=1}^n \mathbb{P}_t(C_i\cap C_0\neq\varnothing)\\ &= n\,\mathbb{P}_t(C_1\cap C_0\neq\varnothing).\end{aligned}$$

If $C_1 \cap C_0 \neq \varnothing$, then no hyperplane $H \in \eta_t$ separates C_1 from p. By inserting the value of n and using the Poisson property of the hyperplane process, we conclude that

$$\begin{aligned}\mathbb{P}_t(R_0(C_0)\geq r) &\leq c_2(d)e^{r(d-1)}e^{-t\mu_{d-1}(\mathbb{H}_{d-1}\langle C_1\,|\,p\rangle)}\\ &\leq c_2(d)e^{r(d-1)}e^{-t\frac{\omega_{d+1}\omega_1}{\omega_d\omega_2}\frac{1+\kappa}{2\kappa}r}\\ &= c_2(d)\exp\left(r(d-1)\left(1-\frac{t}{\hat{c}_d}\frac{1+\kappa}{2\kappa}\right)\right).\end{aligned} \tag{5.20}$$

Now let $\tilde{c} = \tilde{c}(d) > 0$ be such that $c_2(d) \leq e^{\tilde{c}t}$ for $t \geq \hat{c}_d$. Note that $1 - \frac{t}{\hat{c}_d}\frac{1+\kappa}{2\kappa} \leq 0$ since $t \geq \kappa\hat{c}_d$. Then the choice $r = \max\left\{\tilde{r}, (c+\tilde{c}+\tau(a))2\kappa(\kappa-1)^{-1}\hat{c}_d(d-1)^{-1}\right\}$ implies that

$$\begin{aligned}\mathbb{P}_t(R_0(C_0)\geq r) &\leq c_2(d)\,\exp\left((c+\tilde{c}+\tau(a))\frac{2\kappa}{\kappa-1}\hat{c}_d - (c+\tilde{c}+\tau(a))\frac{2\kappa}{\kappa-1}\frac{1+\kappa}{2\kappa}t\right)\\ &\leq e^{\tilde{c}t}\exp\left((c+\tilde{c}+\tau(a))\left(\frac{2}{\kappa-1}t - \frac{1+\kappa}{\kappa-1}t\right)\right)\\ &= \exp\left(-t\left(c+\tau(a)\right)\right).\end{aligned} \tag{5.21}$$

In the present setting, we have $\Phi(K) = \mu_{d-1}(\mathbb{H}_{d-1}\langle K\rangle)$ for $K \in \mathcal{K}_h^d$. Then the results of Section

5.2.2 show that

$$\tau(a) = \min\{\Phi(K) : K \in \mathcal{K}_h^d, \mathcal{H}^d(K) \geq a\} = \Phi(B^a) = \mu_{d-1}(\mathbb{H}_{d-1}\langle B^a\rangle)$$

for $a > 0$. Hence

$$\mathbb{P}_t\left(\mathcal{H}^d(C_0) \geq a\right) \geq \mathbb{P}_t\left(\eta_t(\mathbb{H}_{d-1}\langle B^a\rangle) = 0\right) = \exp\left(-t\mu_{d-1}(\mathbb{H}_{d-1}\langle B^a\rangle)\right) = \exp(-t\tau(a)). \tag{5.22}$$

By the definition of the conditional probability and (5.21) we thus get

$$\begin{aligned}\mathbb{P}_t\left(R_0(C_0) \geq r \mid \mathcal{H}^d(C_0) \geq a\right) &= \frac{\mathbb{P}_t\left(R_0(C_0) > r,\ \mathcal{H}^d(C_0) \geq a\right)}{\mathbb{P}_t\left(\mathcal{H}^d(C_0) \geq a\right)} \\ &\leq \frac{\mathbb{P}_t\left(R_0(C_0) > r\right)}{\mathbb{P}_t\left(\mathcal{H}^d(C_0) \geq a\right)} \\ &\leq \frac{e^{-t(c+\tau(a))}}{e^{-t\tau(a)}} \\ &= e^{-ct}.\end{aligned}$$

As one can see, the choice of $r = r(d, a, c, \kappa)$ only depends on d, a, c and κ. □

Lemma 5.3.3. *If $t > \hat{c}_d$, then almost surely the cells of X_t are bounded. In particular, in this case also C_0 is almost surely bounded.*

Proof. For any fixed $t > \hat{c}_d$ there is some $\kappa > 1$ such that $t \geq \kappa\hat{c}_d$. For $n \in \mathbb{N}$ we consider the events $A_n := \{R_0(C_0) \geq n\}$. Since $t \geq \kappa\hat{c}_d$ we have $(1 - \frac{t}{\hat{c}_d}\frac{1+\kappa}{2\kappa}) \leq \frac{1-\kappa}{2} < 0$, and hence (5.20) implies that

$$\mathbb{P}_t(A_n) \leq c_2(d)\exp\left(-n(d-1)\frac{\kappa-1}{2}\right).$$

Summation over all $n \in \mathbb{N}$ yields

$$\sum_{n=1}^{\infty}\mathbb{P}_t(A_n) \leq \sum_{n=1}^{\infty} c_2(d)\exp\left(-n(d-1)\frac{\kappa-1}{2}\right) < \infty.$$

Now the Borel–Cantelli Lemma implies that $\mathbb{P}_t(\limsup_{n\to\infty} A_n) = 0$. Since $\{C_0 \text{ unbounded}\} \subseteq \limsup A_n$, this proves that C_0 is almost surely bounded if $t > \hat{c}_d$.

Let $A \subset \mathbb{H}^d$ be a countable, dense subset. Then $\mathbb{P}$-almost surely $\eta_t(\mathcal{F}_A) = 0$, hence $A \cap \partial C = \varnothing$ for each $C \in X_t$ holds almost surely. By isometry invariance and the preceding part of the proof, we can conclude that the unique cell of X_t containing a given point $a \in A$ (in its interior) is almost surely bounded. Since cells have nonempty interiors, each cell contains at least one of the points. This proves that all cells are almost surely bounded if $t > \hat{c}_d$. □

5.3.2 Large Crofton cells

The proof of the main theorem of this section (Theorem 5.3.5) is based on a combination of the auxiliary results provided in Section 5.3.1 and the following lemma. Since the deviation function ϑ is fixed in the following, we define

$$\mathcal{K}_{a,\varepsilon} := \mathcal{K}_{a,\varepsilon}(\vartheta) := \{K \in \mathcal{K}_h^d : \mathcal{H}^d(K) \geq a, \ \vartheta(K) \geq \varepsilon\}$$

as the set of convex bodies having size at least $a > 0$ and for which the deviation functional is bounded from below by $\varepsilon > 0$. Lemma 5.3.2 implies that the probability that $C_0 \in \mathcal{K}_{a,\varepsilon}(\vartheta)$ and $R_0(C_0) \leq r$, for a fixed $r > 0$, decays exponentially in t at a specific speed. Note that if $C_0 \in \mathcal{K}_{a,\varepsilon}$, then the Crofton cell is bounded and therefore has at least $d+1$ bounding hyperbolic hyperplanes. In the preceding section, it was proved that the Crofton cell (and in fact all cells of X_t) is almost surely bounded if $t > \hat{c}_d$, but in the lemma no such restriction is imposed. However, we already showed in Lemma 5.1.3 that, for sufficiently small intensity t, with positive probability the Crofton cell is indeed unbounded.

Lemma 5.3.4. *Let C_0 be the Crofton cell of a hyperplane tessellation X_t induced by an isometry invariant hyperbolic Poisson hyperplane process η_t with intensity $t > 0$. Further, let $a, \varepsilon, r > 0$ and $\aleph \in (0,1)$. Then there is a constant $c_4 > 0$, depending only on $a, d, r, \aleph$ and ε, such that*

$$\mathbb{P}_t(C_0 \in \mathcal{K}_{a,\varepsilon}(\vartheta), \ R_0(C_0) \leq r) \leq c_4 \exp\left(-\left(1 + \aleph f_{a,\vartheta}(\varepsilon)\right)\tau(a)t\right).$$

Proof. Let $N \in \mathbb{N}$ and $H_1, \ldots, H_N \in \mathbb{H}_{d-1}\langle B_r\rangle$ be such that $p \notin H_i$ for $i = 1, \ldots, N$, which will be the case almost surely in the following. Define $H_{(N)} := (H_1, \ldots, H_N)$ and let $P(H_{(N)})$ denote the (hyperbolic) Crofton cell of the tessellation induced by $H_1, \ldots, H_N$. In the following, we consider $H_1, \ldots, H_N \in \mathbb{H}_{d-1}\langle B_r\rangle$ such that $P(H_{(N)}) \in \mathcal{K}_{a,\varepsilon}$ and $P(H_{(N)}) \subseteq B_r$. This implies that $N \geq d+1$ (as pointed out before).

Since ϑ is fixed, we simply write $\mathcal{K}_{a,\varepsilon}$ instead of $\mathcal{K}_{a,\varepsilon}(\vartheta)$. Let $f_a = f_{a,\vartheta}$ be the stability function from Theorem 5.2.14. Define $\overline{\aleph} := (1-\aleph)/2 \in (0, 1/2)$ and $\alpha := \overline{\aleph} f_a(\varepsilon)/(1 + f_a(\varepsilon))$. Hence we have $(1-\alpha)(1 + f_a(\varepsilon)) = 1 + \overline{\alpha}$ with $\overline{\alpha} = (1 - \overline{\aleph}) f_a(\varepsilon)$. For $r > 0$ we set

$$\hat{\tau}(r) := \mu_{d-1}(\mathbb{H}_{d-1}\langle B_r\rangle) = \frac{2dW_{d-1}(B_r)}{\omega_d}.$$

Moreover, we note that

$$\tau(a) = \mu_{d-1}(\mathbb{H}_{d-1}\langle B^a\rangle) = \frac{2dW_{d-1}(B^a)}{\omega_d}.$$

Then, by Lemma 5.2.18 and Theorem 5.2.19, there are at most $\nu = \nu(d, a, \varepsilon, r, \aleph)$ vertices of $P(H_{(N)}) \in \mathcal{K}_{a,\varepsilon}$ with $P(H_{(N)}) \subseteq B_r$ such that the hyperbolic convex hull $Q(H_{(N)})$ of these vertices satisfies

$$
\begin{aligned}
\hat{\tau}(r) &= \frac{2dW_{d-1}(B_r)}{\omega_d} \\
&\geq \frac{2d\,W_{d-1}(Q(H_{(N)}))}{\omega_d} \\
&\geq (1-\alpha)\frac{2d\,W_{d-1}(P(H_{(N)}))}{\omega_d} \\
&\geq (1-\alpha)(1+f_a(\varepsilon))\frac{2d\,W_{d-1}(B^a)}{\omega_d} \\
&= (1+\overline{\alpha})\tau(a).
\end{aligned}
\tag{5.23}
$$

By Lemma 5.2.19 we can assume that the map $(H_1, \dots, H_N) \mapsto Q(H_{(N)})$ is measurable. For fixed $r > 0$, we consider the restriction of the isometry invariant measure μ_{d-1} to $\mathbb{H}_{d-1}\langle B_r\rangle$ and normalize it to get

$$
\tilde{\mu}_{d-1} := \frac{\mu_{d-1} \llcorner \mathbb{H}_{d-1}\langle B_r\rangle}{\mu_{d-1}(\mathbb{H}_{d-1}\langle B_r\rangle)}.
$$

Since μ_{d-1} is isometry invariant, every vertex of $Q(H_{(N)})$ lies $\mathbb{P}_t$-almost surely in exactly d of these hyperplanes. The remaining hyperplanes which do not hit any vertex of $Q(H_{(N)})$ do not hit $Q(H_{(N)})$. Hence the number of hyperplanes hitting $Q(H_{(N)})$ is $j \in \{d+1, \dots, d\nu\}$. Without loss of generality we can assume $H_l \cap Q(H_{(N)}) \neq \varnothing$ for $l = 1, \dots, j$. Hence there are subsets $J_1, \dots, J_\nu$ of $\{1, \dots, j\}$, each of cardinality d, such that

$$
\bigcap_{l \in J_i} H_l, \qquad i = 1, \dots, \nu,
$$

are the vertices of $Q(H_{(N)})$. In the following, let $\sum_{(J_1,\dots,J_\nu)}$ denote the sum over all ν-tuples of subsets of $\{1, \dots, j\}$ with d elements. For $K \in \mathcal{K}_h^d$ with $K \subseteq B_r$ we get

$$
\int_{\mathbb{H}_{d-1}\langle B_r\rangle} \mathbb{1}\{H \cap K = \varnothing\}\ \tilde{\mu}_{d-1}(dH) = 1 - \frac{W_{d-1}(K)}{W_{d-1}(B_r)}.
$$

If $\mathbb{P}_t(C_0 \in \mathcal{K}_{a,\varepsilon}, R_0(C_0) \leq r) = 0$, then there is nothing to show. In the complementary case, we have $\eta_t(\mathbb{H}_{d-1}\langle B_r\rangle) \geq d+1$ $\mathbb{P}_t$-almost surely. Basic properties of the Poisson process then yield that for $N \geq d+1$ we have

$$
\begin{aligned}
&\mathbb{P}_t(C_0 \in \mathcal{K}_{a,\varepsilon}, R_0(C_0) \leq r \mid \eta_t(\mathbb{H}_{d-1}\langle B_r\rangle) = N) \\
&= \int_{\mathbb{H}_{d-1}\langle B_r\rangle^N} \mathbb{1}\{P(H_{(N)}) \in \mathcal{K}_{a,\varepsilon}, P(H_{(N)}) \subseteq B_r\} \; \tilde{\mu}_{d-1}^N(d(H_1,\dots,H_N)) \\
&\leq \sum_{j=d+1}^{d\nu} \binom{N}{j} \int_{\mathbb{H}_{d-1}\langle B_r\rangle^N} \mathbb{1}\{P(H_{(N)}) \in \mathcal{K}_{a,\varepsilon}, P(H_{(N)}) \subseteq B_r\} \; \mathbb{1}\{H_l \cap Q(H_{(N)}) \neq \varnothing,\ l = 1,\dots,j\} \\
&\qquad \times \mathbb{1}\{H_l \cap Q(H_{(N)}) = \varnothing,\ l = j+1,\dots,N\} \; \tilde{\mu}_{d-1}^N(d(H_1,\dots,H_N)) \\
&\leq \sum_{j=d+1}^{d\nu} \binom{N}{j} \sum_{(J_1,\dots,J_\nu)} \int_{\mathbb{H}_{d-1}\langle B_r\rangle^j} \int_{\mathbb{H}_{d-1}\langle B_r\rangle^{N-j}} \mathbb{1}\left\{\frac{2dW_{d-1}(\mathrm{conv}_h \bigcup_{k=1}^{\nu} \bigcap_{i\in J_k} H_i)}{\omega_d} \geq (1+\overline{\alpha})\tau(a)\right\} \\
&\qquad \times \mathbb{1}\{H_l \cap \mathrm{conv}_h \bigcup_{k=1}^{\nu} \bigcap_{i\in J_k} H_i = \varnothing,\ l = j+1,\dots,N\} \; \tilde{\mu}_{d-1}^{N-j}(d(H_{j+1},\dots,H_N)) \\
&\qquad \times \mathbb{1}\{\mathrm{conv}_h \bigcup_{k=1}^{\nu} \bigcap_{i\in J_k} H_i \subseteq B_r\} \; \tilde{\mu}_{d-1}^{j}(d(H_1,\dots,H_j)) \\
&\leq \sum_{j=d+1}^{d\nu} \binom{N}{j} \sum_{(J_1,\dots,J_\nu)} \int_{\mathbb{H}_{d-1}\langle B_r\rangle^j} \mathbb{1}\left\{\frac{2d\,W_{d-1}(\mathrm{conv}_h \bigcup_{k=1}^{\nu} \bigcap_{i\in J_k} H_i)}{\omega_d} \geq (1+\overline{\alpha})\tau(a)\right\} \\
&\qquad \times \mathbb{1}\{\mathrm{conv}_h \bigcup_{k=1}^{\nu} \bigcap_{i\in J_k} H_i \subseteq B_r\} \left[1 - \frac{W_{d-1}(\mathrm{conv}_h \bigcup_{k=1}^{\nu} \bigcap_{i\in J_k} H_i)}{W_{d-1}(B_r)}\right]^{N-j} \tilde{\mu}_{d-1}^{j}(d(H_1,\dots,H_j)) \\
&\leq \sum_{j=d+1}^{d\nu} \binom{N}{j} \binom{j}{d}^{\nu} \left[1 - \frac{(1+\overline{\alpha})\tau(a)}{\hat{\tau}(r)}\right]^{N-j},
\end{aligned} \tag{5.24}
$$

where the last inequality used that for at least one tuple $(J_1,\dots,J_\nu)$ the integral in the line preceding (5.24) is positive and therefore a tuple $(H_1,\dots,H_j)$ of hyperplanes exists such that

$$
1 \geq \frac{W_{d-1}(\mathrm{conv}_h \bigcup_{k=1}^{\nu} \bigcap_{i\in J_k} H_i)}{W_{d-1}(B_r)} \geq \frac{(1+\overline{\alpha})\tau(a)\omega_d}{2d\,W_{d-1}(B_r)} = \frac{(1+\overline{\alpha})\tau(a)}{\hat{\tau}(r)}.
$$

Hence, if $\mathbb{P}_t(C_0 \in \mathcal{K}_{a,\varepsilon},\ R_0(C_0) \leq r) > 0$, then summation over N gives

$$
\begin{aligned}
&\mathbb{P}_t(C_0 \in \mathcal{K}_{a,\varepsilon},\ R_0(C_0) \leq r) \\
&= \sum_{N=d+1}^{\infty} \mathbb{P}_t(C_0 \in \mathcal{K}_{a,\varepsilon},\ R_0(C_0) \leq r \mid \eta_t(\mathbb{H}_{d-1}\langle B_r\rangle) = N)\, \mathbb{P}_t(\eta_t(\mathbb{H}_{d-1}\langle B_r\rangle) = N) \\
&\leq \sum_{N=d+1}^{\infty} \sum_{j=d+1}^{d\nu} \binom{N}{j} \binom{j}{d}^{\nu} \left[1 - \frac{(1+\overline{\alpha})\tau(a)}{\hat{\tau}(r)}\right]^{N-j} \frac{(t\hat{\tau}(r))^N}{N!} \exp(-t\hat{\tau}(r)) \\
&= \sum_{j=d+1}^{d\nu} \binom{j}{d}^{\nu} \exp(-t\hat{\tau}(r)) \frac{(t\hat{\tau}(r))^j}{j!} \sum_{N=j}^{\infty} \left[1 - \frac{(1+\overline{\alpha})\tau(a)}{\hat{\tau}(r)}\right]^{N-j} \frac{(t\hat{\tau}(r))^{N-j}}{(N-j)!} \\
&= \exp[-t\hat{\tau}(r) + t\hat{\tau}(r) - (1+\overline{\alpha})\tau(a)t] \sum_{j=d+1}^{d\nu} \binom{j}{d}^{\nu} \frac{(t\hat{\tau}(r))^j}{j!} \\
&\leq \sum_{j=d+1}^{d\nu} \binom{j}{d}^{\nu} \frac{(t\hat{\tau}(r))^j}{j!} \exp(-(1+(1-\overline{\aleph})f_a(\varepsilon))\tau(a)t).
\end{aligned}
$$

But then we get in any case

$$\mathbb{P}_t(C_0 \in \mathcal{K}_{a,\varepsilon}, R_0(C_0) \le r) \le \overline{c_4} \max\{1, t\hat{r}(r)\}^{d\nu} \exp(-(1 + (1 - \overline{\aleph}) f_a(\varepsilon))\tau(a)t),$$

where

$$\overline{c_4} = \overline{c_4}(\varepsilon, a, d, r, \aleph) = \sum_{j=d+1}^{d\nu} \binom{j}{d}^{\nu} \frac{1}{j!},$$

and therefore also

$$\mathbb{P}_t(C_0 \in \mathcal{K}_{a,\varepsilon}, R_0(C_0) \le r) \le c_4 \exp(-(1 + (1 - 2\overline{\aleph}) f_a(\varepsilon))\tau(a)t),$$

which proves the lemma, since $(1 - 2\overline{\aleph}) = \aleph$. □

We are now in position to state and proof the main theorem of this section.

Theorem 5.3.5. *Let C_0 be the Crofton cell of the hyperbolic Poisson hyperplane tessellation X_t induced by the isometry invariant Poisson hyperplane process η_t with intensity $t \ge \kappa \hat{c}_d$, where $\kappa > 1$ (is fixed). Let ϑ be a deviation functional for hyperbolic balls. Further let $a > 0$, $\aleph \in (0,1)$ and $\varepsilon \in (0,1]$. Then there is a constant $c_5 > 0$ such that*

$$\mathbb{P}_t(\vartheta(C_0) \ge \varepsilon \mid \mathcal{H}^d(C_0) \ge a) \le c_5 \exp\left(-\aleph f_{a,\vartheta}(\varepsilon)\tau(a)t\right),$$

where c_5 depends only on $a, d, \varepsilon, \kappa, \aleph$.

Proof. The proof of Theorem 5.3.5 can now be done by combining the results from Lemmas 5.3.2 and 5.3.4. Let $a > 0$, $\varepsilon \in (0,1]$, $\aleph \in (0,1)$ and $\kappa > 1$ be fixed. For a given $r > 0$ (to be specified below), we split the probability into two parts, that is,

$$\begin{aligned}\mathbb{P}_t\left(\vartheta(C_0) \ge \varepsilon \mid \mathcal{H}^d(C_0) \ge a\right) = \mathbb{P}_t&\left(\vartheta(C_0) \ge \varepsilon,\ R_0(C_0) > r \mid \mathcal{H}^d(C_0) \ge a\right) \\ &+ \mathbb{P}_t\left(\vartheta(C_0) \ge \varepsilon,\ R_0(C_0) \le r \mid \mathcal{H}^d(C_0) \ge a\right).\end{aligned}$$

Lemma 5.3.2 implies that the first summand can be bounded from above by $\exp(-ct)$, for an arbitrary $c > 0$, as long as $r = r(d, a, c, \kappa)$ is chosen big enough. For the second summand we get

$$\mathbb{P}_t\left(\vartheta(C_0) \ge \varepsilon,\ R_0(C_0) \le r \mid \mathcal{H}^d(C_0) \ge a\right) = \frac{\mathbb{P}_t\left(\vartheta(C_0) \ge \varepsilon,\ R_0(C_0) \le r,\ \mathcal{H}^d(C_0) \ge a\right)}{\mathbb{P}_t\left(\mathcal{H}^d(C_0) \ge a\right)}.$$

The numerator was considered in Lemma 5.3.4, for the denominator we already know by (5.22) that

$$\mathbb{P}_t\left(\mathcal{H}^d(C_0) \ge a\right) \ge \exp(-\tau(a)t).$$

Combining these observations, we obtain that

$$\begin{aligned}\mathbb{P}_t(\vartheta(C_0) \geq \varepsilon \mid \mathcal{H}^d(C_0) \geq a) &\leq \exp(-ct) + c_4(\varepsilon, a, d, r, \aleph)\frac{\exp\left(-(1+\aleph f_a(\varepsilon))\tau(a)t\right)}{\exp(-\tau(a)t)} \\ &= \exp(-ct) + c_4(\varepsilon, a, d, r, \aleph)\exp\left(-\aleph f_a(\varepsilon)\tau(a)t\right).\end{aligned}$$

Choosing $c = \aleph f_a(\varepsilon)\tau(a)$, we get

$$\mathbb{P}_t(\vartheta(C_0) \geq \varepsilon \mid \mathcal{H}^d(C_0) \geq a) \leq (1 + c_4(\varepsilon, a, d, r, \aleph))\exp\left(-\aleph f_a(\varepsilon)\tau(a)t\right). \tag{5.25}$$

By Lemma 5.3.2 and our choice of c the required choice of r only depends on $d, a, \varepsilon, \kappa, \aleph$, and hence c_4 also depends only on $d, a, \varepsilon, \kappa, \aleph$. Thus the choice $c_5 = 1 + c_4$ yields

$$\mathbb{P}_t(\vartheta(C_0) \geq \varepsilon \mid \mathcal{H}^d(C_0) \geq a) \leq c_5 \exp\left(-\aleph f_a(\varepsilon)\tau(a)t\right)$$

which completes the proof of the theorem. □

By the same argument as in the proof of Lemma 5.3.4, but using Theorem 5.2.17 instead of Theorem 5.2.14, we obtain the following lemma.

Lemma 5.3.6. *Let C_0 be the Crofton cell of a hyperplane tessellation X_t induced by an isometry invariant hyperbolic Poisson hyperplane process η_t with intensity $t > 0$. Further, let $a, \varepsilon, r > 0$ and $\aleph \in (0,1)$. Then there is a constant $c_6 > 0$, depending only on $a, d, r, \aleph$ and ε, such that*

$$\mathbb{P}_t(\Sigma_r(C_0) \geq a,\ \vartheta_r(C_0) \geq \varepsilon,\ R_0(C_0) \leq r) \leq c_6 \exp\left(-\left(1 + \aleph c_1 \varepsilon^{\frac{d+1}{2}}\right)\hat{\tau}(a)t\right).$$

As a consequence, we obtain the following more specific version of Theorem 5.3.5.

Theorem 5.3.7. *Let C_0 be the Crofton cell of the hyperbolic Poisson hyperplane tessellation X_t induced by the isometry invariant Poisson hyperplane process η_t with intensity $t \geq \kappa\hat{c}_d$, where $\kappa > 1$ (is fixed). Let ϑ be a deviation functional for hyperbolic balls. Further let $a > 0$, $\aleph \in (0,1)$ and $\varepsilon \in (0,1]$. Then there is a constant $c_7 > 0$ such that*

$$\mathbb{P}_t(\vartheta_r(C_0) \geq \varepsilon \mid \Sigma_r(C_0) \geq a) \leq c_7 \exp\left(-\aleph c_1 \varepsilon^{\frac{d+1}{2}}\hat{\tau}(a)t\right),$$

where c_7 depends only on $a, d, \varepsilon, \kappa, \aleph$.

In the hyperbolic plane, a specific deviation result (with explicit rate function) can be obtained from Theorem 5.2.15 with $\mathcal{H}^2$ as the size functional and ϑ_r as the deviation functional.

Theorem 5.3.8 describes the limit behaviour of the probability that $\mathcal{H}^d(C_0)$ exceeds the threshold a for growing intensity t. The proof is based on the same techniques as used in the proof of Lemma 5.3.4. Therefore we will be rather brief in our presentation.

Theorem 5.3.8. *Let $a > 0$ (be fixed) and let η_t, X_t, C_0 be as in Theorem 5.3.5. Then*

$$\lim_{t\to\infty} t^{-1} \ln \mathbb{P}_t(\mathcal{H}^d(C_0) \geq a) = -\tau(a).$$

Proof. First, we provide a lower bound for the limes inferior of the probability. This is done by using (5.22) to get

$$t^{-1} \ln(\mathbb{P}_t(\mathcal{H}^d(C_0) \geq a)) \geq t^{-1} \ln(\exp(-\tau(a)\, t)) = -\tau(a).$$

Next we derive an upper bound for $\mathbb{P}_t(\mathcal{H}^d(C_0) \geq a)$. Let $t \geq 2\hat{c}_d$ (say). Then Lemma 5.3.2 shows that for $c = 1$ there is some $r = r(d, a, 1, 2)$ such that

$$\mathbb{P}_t(R_0(C_0) > r) \leq \exp(-t(1 + \tau(a))),$$

and hence

$$\begin{aligned} \mathbb{P}_t(\mathcal{H}^d(C_0) \geq a) &= \mathbb{P}_t(\mathcal{H}^d(C_0) \geq a,\ R_0(C_0) \leq r) + \mathbb{P}_t(\mathcal{H}^d(C_0) \geq a,\ R_0(C_0) > r) \\ &\leq \mathbb{P}_t(\mathcal{H}^d(C_0) \geq a,\ R_0(C_0) \leq r) + \exp(-t(1 + \tau(a))). \end{aligned} \tag{5.26}$$

Next we deal with the first summand on the right-hand side of (5.26). Let $H_1, \dots, H_N \in \mathbb{H}_{d-1}\langle B_r \rangle$ and $H_{(N)} = (H_1, \dots, H_N)$. If $P(H_{(N)}) \in \mathcal{K}_{a,0}$ and $P(H_{(N)}) \subseteq B_r$, then Theorem 5.2.19 implies that for each $k \in (0, 1)$ there exists a polytope $Q = Q(k)$ such that $W_{d-1}(Q(k)) \geq (1 - k/2) W_{d-1}(B^a)$, $\mathrm{ext}(Q(k)) \subseteq \mathrm{ext}(P(H_{(N)}))$ and $f_0(Q(k)) \leq \nu = \nu(d, a, k)$ (compare the proof of Lemma 5.3.4). By the same calculations as the ones leading to (5.24) in the proof of Lemma 5.3.4, we see that if $\mathbb{P}_t(C_0 \in \mathcal{K}_{a,0},\ R_0(C_0) \leq r) > 0$, then for $N \geq d + 1$ we obtain

$$\mathbb{P}_t(C_0 \in \mathcal{K}_{a,0},\ R_0(C_0) \leq r \mid \eta_t(\mathbb{H}_{d-1}\langle B_r \rangle) = N) \leq \sum_{j=d+1}^{d\nu} \binom{N}{j} \binom{j}{d}^{\nu} \left[1 - \frac{(1 - k/2)\tau(a)}{\hat{\tau}(r)}\right]^{N-j}.$$

Following again the argument in the proof of Lemma 5.3.4, we sum over all $N \geq d + 1$ and thus in any case we get that

$$\mathbb{P}_t(C_0 \in \mathcal{K}_{a,0},\ R_0(C_0) \leq r) \leq c_8 \exp(-(1 - k)\tau(a)\, t), \tag{5.27}$$

where the constant c_8 only depends on a, d and k. Thus we deduce

$$\begin{aligned} \mathbb{P}_t(\mathcal{H}^d(C_0) \geq a) &\leq c_8 \exp(-(1 - k)\tau(a)\, t) + \exp(-t(1 + \tau(a))) \\ &= \exp(-\tau(a)t)\, (c_8 \exp(k\tau(a)t) + \exp(-t)) \\ &\leq \exp(-\tau(a)t)(c_8 + 1) \exp(k\tau(a)t). \end{aligned} \tag{5.28}$$

From this we conclude that

$$\limsup_{t\to\infty} t^{-1} \ln(\mathbb{P}_t(\mathcal{H}^d(C_0) \geq a)) \leq -(1 - k)\tau(a).$$

Since k can be chosen arbitrarily in $(0,1)$, it follows that indeed we also have

$$\limsup_{t\to\infty} t^{-1}\ln(\mathbb{P}_t(\mathcal{H}^d(C_0)\geq a))\leq -\tau(a).$$

Together with the matching lower bound for the limes inferior the assertion of the theorem follows. □

In the same way as Theorem 5.3.8 was deduced from the arguments in Theorem 5.3.5, we obtain the following result via Theorem 5.3.7.

Theorem 5.3.9. *Let $a>0$ (be fixed) and let η_t, X_t, C_0 be as in Theorem 5.3.5. Then*

$$\lim_{t\to\infty} t^{-1}\ln\mathbb{P}_t(\Sigma_r(C_0)\geq a) = -\hat{\tau}(a).$$

5.4 Typical cell

After having studied the Crofton cell it is natural to ask for the behaviour of the typical cell. A quite general setting for describing typical objects in homogeneous spaces is developed in [61, 62]. In these contributions, stationary random measures in homogeneous spaces are studied. We will define the typical cell of an isometry invariant particle process, and hence in particular of an invariant hyperplane tessellation, in hyperbolic space by specializing the more general concepts developed in [61, 62] to the present setting. A generic relation between the typical and the Crofton cell of isometry invariant tessellations will then allow us to transfer the results from the last chapter to the typical cell of an isometry invariant Poisson hyperplane tessellation.

5.4.1 Typical particles of invariant processes in hyperbolic space

Let $I(\mathbb{H}^d)$ denote the group of isometries of $\mathbb{H}^d$. It is well known that $I(\mathbb{H}^d)$ is a locally compact, second countable Hausdorff space and a Lie group which acts continuously and transitively on $\mathbb{H}^d$. Hence, up to a multiplicative constant there exists a uniquely determined Haar measure λ on $I(\mathbb{H}^d)$. Since $I(\mathbb{H}^d)$ is unimodular (see [40, Chap. X, Prop. 1.4] or [6, Prop. C 4.11]) λ is left invariant, right invariant and inversion invariant. We will choose the normalization of λ such that

$$\mathcal{H}^d = \lambda\circ\pi_x^{-1} = \int_{I(\mathbb{H}^d)} \mathbb{1}\{\varphi(x)\in\cdot\}\,\lambda(d\varphi),$$

where $\pi_x: I(\mathbb{H}^d)\to\mathbb{H}^d$, $\varphi\mapsto\varphi(x)$, and $x\in\mathbb{H}^d$ (the right-hand side is indeed independent of x). In the following, we also write φx instead of $\varphi(x)$ for $\varphi\in I(\mathbb{H}^d)$ and $x\in\mathbb{H}^d$. Proceeding as in [42], we consider the isotropy group $I(\mathbb{H}^d,p) := \{\varphi\in I(\mathbb{H}^d): \varphi(p)=p\}$ of isometries fixing p and denote by $\kappa(p,\cdot)$ the $I(\mathbb{H}^d)$ invariant probability measure on this compact subgroup. Defining $\kappa(p, I(\mathbb{H}^d)\setminus I(\mathbb{H}^d,p)) := 0$ this measure is extended to $I(\mathbb{H}^d)$. More generally, for

$x \in \mathbb{H}^d$ we define

$$I(\mathbb{H}^d)_{p,x} := \{\varphi \in I(\mathbb{H}^d) : \ \varphi(p) = x\}$$

as the set of isometries that map p to x. Choosing an arbitrary $\varphi_x \in I(\mathbb{H}^d)_{p,x}$, we define

$$\kappa(x, B) := \int \mathbb{1}\{\varphi_x \circ \varphi \in B\} \ \kappa(p, d\varphi), \qquad B \in \mathcal{B}(I(\mathbb{H}^d)).$$

This definition is independent of the choice of φ_x (see [62]). Since $x \mapsto \varphi_x$ can be chosen as a measurable map, κ is a stochastic transition kernel from $\mathbb{H}^d$ to itself. Moreover, $\kappa(x, \cdot)$ is concentrated on $I(\mathbb{H}^d)_{p,x}$.

We assume that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ comes with a measurable flow $\{\theta_\varphi : \varphi \in I(\mathbb{H}^d)\}$, that is, a measurable map $\Omega \times I(\mathbb{H}^d) \to \Omega$, $(\omega, \varphi) \mapsto \theta_\varphi(\omega)$, which leaves $\mathbb{P}$ invariant, that is to say, $\mathbb{P} \circ \theta_\varphi = \mathbb{P}$ for $\varphi \in I(\mathbb{H}^d)$. (In our application, a canonical state space can be chosen so that this assumption is fulfilled, see also [62]). In this case, a random measure ξ on $\mathbb{H}^d$ is called invariant under the flow if

$$\xi(\theta_\varphi \omega, \varphi B) = \xi(\omega, B), \qquad \omega \in \Omega, \ \varphi \in I(\mathbb{H}^d), \ B \in \mathcal{B}(\mathbb{H}^d). \tag{5.29}$$

Here φB is defined pointwise $\varphi B := \{\varphi x : \ x \in B\}$. Let $w : \mathbb{H}^d \to [0, \infty)$ be a measurable (weight) function with $\int_{\mathbb{H}^d} w(x) \, \mathcal{H}^d(dx) = 1$, and hence also

$$\int_{I(\mathbb{H}^d)} w(\varphi(x)) \, \lambda(d\varphi) = 1, \qquad x \in \mathbb{H}^d. \tag{5.30}$$

Assume that

$$\mathbb{E} \int_{\mathbb{H}^d} w(x) \, \xi(dx) \in (0, \infty).$$

Then the Palm measure of an invariant random measure ξ is the finite measure on Ω defined by

$$\mathbb{P}_\xi(A) := \int_\Omega \int_{\mathbb{H}^d} \int_{I(\mathbb{H}^d)} \mathbb{1}\left\{\theta_\varphi^{-1}\omega \in A\right\} w(x) \ \kappa(x, d\varphi) \ \xi(\omega, dx) \ \mathbb{P}(d\omega), \qquad A \in \mathcal{F}. \tag{5.31}$$

This definition is independent of the choice of w, which follows from the refined Campbell theorem (see [61, Theorem 3.6]). Note that in general $\mathbb{P}_\xi$ is not a probability measure.

We use these concepts and results to motivate the definition of the distribution of the typical cell of a particle process. Let X' be an isometry invariant particle process on $\mathbb{H}^d$ having nonempty compact convex particles. By this we mean a point process in $\mathcal{K}_h^d$ satisfying

$$X'(\theta_\varphi \omega) = \varphi X'(\omega), \qquad \omega \in \Omega, \ \varphi \in I(\mathbb{H}^d).$$

In addition, we require a measurable centre function $c_h : \mathcal{K}_h^d \to \mathbb{H}^d$ satisfying

$$c_h(\varphi K) = \varphi c_h(K), \qquad K \in \mathcal{K}_h^d, \ \varphi \in I(\mathbb{H}^d). \tag{5.32}$$

An example is the circumcentre (see Lemma 5.2.13). Using a fixed centre function, we associate with X' the marked point process (random measure) ξ', living on the product space $\mathbb{H}^d \times \mathcal{K}_h^d$, which is defined by

$$\xi'(\omega)(\cdot) := \int_{\mathcal{K}_h^d} \int_{I(\mathbb{H}^d)} \mathbb{1}\{(c_h(K), \varphi^{-1}K) \in \cdot\}\, \kappa(c_h(K), d\varphi) X'(\omega, dK).$$

For notational convenience, we write $\xi'(\omega, \cdot) := \xi'(\omega)(\cdot)$. Let $A \in \mathcal{B}(\mathcal{K}_h^d)$ and $B \in \mathcal{B}(\mathbb{H}^d)$. Let

$$\mathcal{K}_{h,cp}^d := \{K \in \mathcal{K}_h^d :\ c_h(K) = p\}$$

be the set of all convex bodies having p as their center. Then ξ' is concentrated on $\mathbb{H}^d \times \mathcal{K}_{h,cp}^d$. Intuitively, the marked point process ξ' is the collection of all pairs of centers and "shapes" of bodies $K \in X'$, where the shape of K is obtained by moving K so that its center is at p. Since a unique selection of a motion is not available in the present setting, we use the probability kernel $\kappa(c(K), \cdot)$ for a uniform random choice of an isometry mapping $c(K)$ to p. Finally, note that the measure ξ' is invariant under motions in $I(\mathbb{H}^d)$ in the sense that

$$\xi'(\theta_\psi \omega, (\psi B) \times A) = \xi'(\omega, B \times A), \qquad \psi \in I(\mathbb{H}^d). \tag{5.33}$$

The intensity of the particle process X' is defined by

$$\gamma_{X'} := \mathbb{E}\left[\int_{\mathcal{K}_h^d} w(c_h(K))\ X'(dK)\right].$$

The definition is independent of w. We assume that $\gamma_{X'} \in (0, \infty)$. Then the Palm measure of ξ' is the measure on $\Omega \times \mathcal{K}_h^d$ which is given by

$$\mathbb{P}_{\xi'}(\cdot) := \int_\Omega \int_{\mathbb{H}^d \times \mathcal{K}_{h,cp}^d} \int_{I(\mathbb{H}^d)} \mathbb{1}\{(\theta_\varphi^{-1}\omega, K) \in \cdot\} w(x)\, \kappa(x, d\varphi)\, \xi'(\omega, d(x, K))\, \mathbb{P}(d\omega). \tag{5.34}$$

The definition is independent of the choice of the weight function w. Moreover, ξ' is concentrated on the product space $\Omega \times \mathcal{K}_{h,cp}^d$.

After these preparations, we define the distribution of the typical particle of the isometry invariant particle process X' with intensity $\gamma_{X'} \in (0, \infty)$ as the mark distribution of the Palm measure of ξ'. Explicitly, it is given by

$$\mathbb{P}_C(\cdot) := \frac{1}{\gamma_{X'}} \mathbb{E} \int_{\mathcal{K}_h^d} \int_{I(\mathbb{H}^d)} \mathbb{1}\{\varphi^{-1}K \in \cdot\} w(c_h(K))\, \kappa(c_h(K), d\varphi)\, X'(dK).$$

This is a probability measure which is concentrated on $\mathcal{K}_{h,cp}^d$. A random convex body C which has distribution $\mathbb{P}_C(\cdot)$ is called typical particle of X'.

Clearly, the typical particle C is not stationary, but its distribution still has some symmetry property. We state this as a lemma, the proof is straightforward.

Lemma 5.4.1. *Let X' be an isometry invariant particle process on $\mathbb{H}^d$ with intensity $\gamma_{X'} \in (0, \infty)$, and let C denote the typical particle of X'. Then the distribution of C is invariant* *under isometries fixing p, namely*

$$\mathbb{P}(\psi C \in \cdot) = \mathbb{P}(C \in \cdot), \qquad \psi \in I(\mathbb{H}^d, p).$$

In addition, the following disintegration result holds, which relates the intensity measure of X' to the distribution of the typical particle.

Theorem 5.4.2. *Let X' be an invariant particle process on $\mathbb{H}^d$ with intensity $\gamma_{X'} \in (0, \infty)$, and let C denote the typical particle of X'. If $f : \mathcal{K}^d_h \to [0, \infty)$ is measurable then*

$$\int_\Omega \int_{\mathcal{K}^d_h} f(K)\, X'(\omega, dK)\, \mathbb{P}(d\omega) = \gamma_{X'} \int_{\mathcal{K}^d_{h,cp}} \int_{I(\mathbb{H}^d)} f(\varphi K)\, \lambda(d\varphi)\, \mathbb{P}_C(dK). \tag{5.35}$$

Moreover, $\mathbb{P}_C$ is the uniquely determined probability measure on $\mathcal{K}^d_h$ which satisfies (5.35)*, is concentrated on $\mathcal{K}^d_{h,cp}$ and invariant under isometries fixing p.*

Proof. To verify the asserted relation, we start from the expression on the right-hand side and use (in this order) the definition of $\mathbb{P}_C$, the right invariance of λ, the fact that $\kappa(c_h(K), \cdot)$ is a probability measure, Fubini's theorem, the isometry invariance of X' and again Fubini's theorem to get

$$\begin{aligned}
&\gamma_{X'} \int_{\mathcal{K}^d_{h,cp}} \int_{I(\mathbb{H}^d)} f(\varphi K)\, \lambda(d\varphi)\, \mathbb{P}_C(dK) \\
&= \mathbb{E} \int_{\mathcal{K}^d_h} \int_{I(\mathbb{H}^d)} \int_{I(\mathbb{H}^d)} f(\varphi(\psi^{-1} K))\, \lambda(d\varphi)\, w(c_h(K))\, \kappa(c_h(K), d\psi)\, X'(dK) \\
&= \mathbb{E} \int_{\mathcal{K}^d_h} \int_{I(\mathbb{H}^d)} \int_{I(\mathbb{H}^d)} f(\bar\varphi K)\, \lambda(d\bar\varphi)\, w(c_h(K))\, \kappa(c_h(K), d\psi)\, X'(dK) \\
&= \mathbb{E} \int_{\mathcal{K}^d_h} \int_{I(\mathbb{H}^d)} f(\bar\varphi K) w(c_h(K))\, \lambda(d\bar\varphi)\, X'(dK) \\
&= \int_{I(\mathbb{H}^d)} \mathbb{E} \int_{\mathcal{K}^d_h} f(\bar\varphi K) w(c_h(K))\, X'(dK)\, \lambda(d\bar\varphi) \\
&= \int_{I(\mathbb{H}^d)} \mathbb{E} \int_{\mathcal{K}^d_h} f(\overline{K}) w(c_h(\bar\varphi^{-1} \overline{K}))\, X'(d\overline{K})\, \lambda(d\bar\varphi) \\
&= \mathbb{E} \int_{\mathcal{K}^d_h} f(\overline{K}) \int_{I(\mathbb{H}^d)} w(c_h(\bar\varphi^{-1} \overline{K}))\, \lambda(d\bar\varphi)\, X'(d\overline{K}) \\
&= \mathbb{E} \int_{\mathcal{K}^d_h} f(\overline{K})\, X'(d\overline{K}),
\end{aligned}$$

where we used (5.30), $c_h(\bar\varphi^{-1} \overline{K}) = \bar\varphi^{-1} c_h(\overline{K})$ and the inversion invariance of λ in the last step.

For the uniqueness, we consider another probability measure $\mathbb{P}^*$ on $\mathcal{K}^d_h$ which satisfies (5.35), is concentrated on $\mathcal{K}^d_{h,cp}$ and invariant under isometries fixing p. The map $\mathbb{H}^d \to I(\mathbb{H}^d)$, $x \mapsto \varphi_x \in I(\mathbb{H}^d)_{p,x}$, can be chosen to be measurable and by [62, (2.9)] we have

$$\lambda(\cdot) = \int_{\mathbb{H}^d} \kappa(x, \cdot)\, \mathcal{H}^d(dx). \tag{5.36}$$

Since $\mathbb{P}_C$ and $\mathbb{P}^*$ both satisfy (5.35), it follows that

$$\int_{\mathcal{K}^d_{h,cp}} \int_{I(\mathbb{H}^d)} f(\rho K)\, \lambda(d\rho)\, \mathbb{P}_C(dK) = \int_{\mathcal{K}^d_{h,cp}} \int_{I(\mathbb{H}^d)} f(\rho K)\, \lambda(d\rho)\, \mathbb{P}^*(dK)$$

for all measurable functions $f : \mathcal{K}_h^d \to [0, \infty)$. Using Fubini's theorem, (5.36) and the definition of the kernel κ, we obtain

$$\begin{aligned}
&\int_{\mathcal{K}_{h,cp}^d} \int_{I(\mathbb{H}^d)} f(\rho K)\, \lambda(d\rho)\, \mathbb{P}_C(dK) \\
&= \int_{I(\mathbb{H}^d)} \int_{\mathcal{K}_{h,cp}^d} f(\rho K)\, \mathbb{P}_C(dK)\, \lambda(d\rho) \\
&= \int_{\mathbb{H}^d} \int_{I(\mathbb{H}^d)} \int_{\mathcal{K}_{h,cp}^d} f(\varphi_x \circ \psi K)\, \mathbb{P}_C(dK)\, \kappa(p, d\psi)\, \mathcal{H}^d(dx) \\
&= \int_{\mathbb{H}^d} \int_{\mathcal{K}_{h,cp}^d} f(\varphi_x K)\, \mathbb{P}_C(dK)\, \mathcal{H}^d(dx),
\end{aligned}$$

where we used in the last step that $\mathbb{P}_C$ is invariant under isometries fixing p and $\kappa(p, \cdot)$ is a probability measure. Since $\mathbb{P}^*$ has the same properties as $\mathbb{P}_C$, we thus get

$$\int_{\mathbb{H}^d} \int_{\mathcal{K}_{h,cp}^d} f(\varphi_x K)\, \mathbb{P}_C(dK)\, \mathcal{H}^d(dx) = \int_{\mathbb{H}^d} \int_{\mathcal{K}_{h,cp}^d} f(\varphi_x K)\, \mathbb{P}^*(dK)\, \mathcal{H}^d(dx). \tag{5.37}$$

Let $h : \mathcal{K}_h^d \to [0, \infty)$ be an arbitrary measurable function. Then we define

$$f(K) := w(c_h(K)) \cdot h\left(\varphi_{c_h(K)}^{-1} K\right), \qquad K \in \mathcal{K}_h^d.$$

If $K \in \mathcal{K}_{h,cp}^d$ and $x \in \mathbb{H}^d$, then $c_h(\varphi_x K) = \varphi_x c_h(K) = \varphi_x(p) = x$, and hence

$$f(\varphi_x K) = w(c_h(\varphi_x K)) h\left(\varphi_{c_h(\varphi_x K)}^{-1} \varphi_x K\right) = w(x) h\left(\varphi_x^{-1} \varphi_x K\right) = w(x) h(K).$$

Thus with this particular choice of f we obtain from (5.37) that

$$\int_{\mathbb{H}^d} \int_{\mathcal{K}_{h,cp}^d} w(x) h(K)\, \mathbb{P}_C(dK)\, \mathcal{H}^d(dx) = \int_{\mathbb{H}^d} \int_{\mathcal{K}_{h,cp}^d} w(x) h(K)\, \mathbb{P}^*(dK)\, \mathcal{H}^d(dx),$$

and therefore

$$\int_{\mathcal{K}_{h,cp}^d} h(K)\, \mathbb{P}_C(dK) = \int_{\mathcal{K}_{h,cp}^d} h(K)\, \mathbb{P}^*(dK),$$

by the normalization (5.30). Since h was arbitrary, this proves the asserted uniqueness. □

Now we specify these results to the situation where X' is the particle process determined by an isometry invariant random tessellation of hyperbolic space (into hyperbolic convex polytopes having nonempty interiors). It follows from (5.35) that $\mathbb{P}$-almost surely p is contained in (the interior of) precisely one cell of X', since the $\mathcal{H}^d$ measure of the boundary of each $K \in X'$ is zero. We denote by C_0 the almost surely unique cell of X' containing the origin in its interior.

Lemma 5.4.3. *Let $f : \mathcal{K}_h^d \mapsto [0, \infty)$ be measurable and isometry invariant. Let X' be an isometry invariant tessellation of $\mathbb{H}^d$ (with positive and finite intensity $\gamma_{X'}$). Further let C_0, C be the Crofton and the typical cell of the tessellation, respectively. Then*

$$\mathbb{E}\left[f(C_0)\right] = \gamma_{X'} \mathbb{E}\left[f(C) \mathcal{H}^d(C)\right].$$

Proof. A proof can be found in [62, Corollary 8.4]. For the convenience of the reader, we provide a short derivation as an application of Theorem 5.4.2. We start from the left-hand side and use the invariance of f for the second inequality, hence

$$\begin{aligned}\mathbb{E}\left[f(C_0)\right] &= \mathbb{E}\int_{\mathcal{K}_h^d} f(K)\mathbb{1}\{p \in K\}\, X'(dK)\\ &= \gamma_{X'}\int_{\mathcal{K}_{h,cp}^d}\int_{I(\mathbb{H}^d)} f(\varphi K)\mathbb{1}\{p\in\varphi K\}\,\lambda(d\varphi)\,\mathbb{P}_C(dK)\\ &= \gamma_{X'}\int_{\mathcal{K}_{h,cp}^d} f(K)\int_{I(\mathbb{H}^d)}\mathbb{1}\{p\in\varphi K\}\,\lambda(d\varphi)\,\mathbb{P}_C(dK)\\ &= \gamma_{X'}\int_{\mathcal{K}_{h,cp}^d} f(K)\mathcal{H}^d(K)\,\mathbb{P}_C(dK),\end{aligned}$$

since

$$\int_{I(\mathbb{H}^d)}\mathbb{1}\{p\in\varphi K\}\,\lambda(d\varphi) = \int_{I(\mathbb{H}^d)}\mathbb{1}\{\varphi^{-1}p\in K\}\,\lambda(d\varphi) = \mathcal{H}^d(K),$$

since λ is inversion invariant. □

The lemma is well known in Euclidean space (see [103, Theorem 10.4.1]), in spherical space a proof is given in [42, Theorem 9.2].

5.4.2 Large typical cells

The statement of Theorem 5.4.5 for the typical cell corresponds to the statement of Theorem 5.3.5 for the Crofton cell. In this section we will deduce Theorem 5.4.5 from Theorem 5.3.5 via the connection between the Crofton and the typical cell provided in Lemma 5.4.3.

In the following, we always assume that the intensity t of the underlying isometry invariant Poisson hyperplane process η_t satisfies $t > \hat{c}_d$. This is required to ensure that the cells of the resulting tessellation X_t in $\mathbb{H}^d$ are almost surely bounded and thus X_t fits into the framework of Section 5.4.1. The next lemma shows in particular that the intensity of the particle process X_t is well defined, that is, we have $\gamma_{X_t} \in (0,\infty)$. In fact, we will need a better upper bound for the dependence of γ_{X_t} on t in the following.

Recall that $B^1 = B_{r_0}$ is a geodesic ball of volume 1 and radius r_0. We set

$$\mathbb{E}\left(\eta_t(\mathcal{F}_{B^1})\right) = t\cdot\int_{A_h(d,d-1)}\mathbb{1}\left\{H\cap B^1 \neq \varnothing\right\}\mu_{d-1}(dH) =: t\cdot b_1,$$

with $b_1 := \mu_{d-1}(\mathbb{H}_{d-1}\langle B^1\rangle)$.

Lemma 5.4.4. *For $t > \hat{c}_d$,*

$$0 < \gamma_{X_t} \le \sum_{j=0}^{d}\frac{(b_1 t)^j}{j!} < \infty.$$

In particular, $\ln(\gamma_{X_t}) = o(t)$ *as* $t\to\infty$.

Proof. We already know from Lemma 5.3.3 that the cells of X_t are almost surely bounded, hence the center function is well defined. Choosing the weight function as the indicator function of B_{r_0}, we have

$$\gamma_{X_t} = \mathbb{E} \int_{\mathcal{K}_h^d} \mathbb{1}\{c_h(K) \in B_{r_0}\} \, X_t(dK),$$

where $c_h(K)$ is the circumcentre of K. By a result of Miles [76], the number $\nu(\ell)$ of cells of the cell decomposition of $\mathbb{R}^d$ induced by $\ell \in \mathbb{N}_0$ hyperplanes in general position which meet B_{r_0} (considered as a subset of B_1) is bounded from above by

$$\nu(\ell) \le \sum_{j=0}^{d} \binom{\ell}{j}.$$

Interpreting the corresponding situation for a tessellation of hyperbolic space by hyperbolic hyperplanes in the Beltrami–Klein model, we see that the same upper bound holds. Now we use that by Lemma 5.2.12 $c_h(K) \in K$ for $K \in \mathcal{K}_h^d$ and

$$\begin{aligned} \sum_{K \in X_t} \mathbb{1}\{c_h(K) \in B_{r_0}\} &\le \sum_{K \in X_t} \mathbb{1}\{K \cap B_{r_0} \neq \varnothing\} \le \sum_{\ell=0}^{\infty} \mathbb{1}\left\{\eta_t(\mathcal{F}_{B_{r_0}}) = \ell\right\} \cdot \nu(\ell) \\ &\le \sum_{\ell=0}^{\infty} \mathbb{1}\left\{\eta_t(\mathcal{F}_{B_{r_0}}) = \ell\right\} \sum_{j=0}^{d} \binom{\ell}{j} \end{aligned}$$

to obtain

$$\begin{aligned} \gamma_{X_t} &\le \sum_{\ell=0}^{\infty} \mathbb{E}\left[\mathbb{1}\left\{\eta_t(\mathcal{F}_{B_{r_0}}) = \ell\right\}\right] \sum_{j=0}^{d} \binom{\ell}{j} = \sum_{\ell=0}^{\infty} \sum_{j=0}^{d} \frac{(b_1 t)^\ell}{\ell!} e^{-b_1 t} \binom{\ell}{j} \\ &= \sum_{j=0}^{d} \sum_{\ell=j}^{\infty} \frac{1}{j!(\ell-j)!} (b_1 t)^j (b_1 t)^{\ell-j} e^{-b_1 t} = \sum_{j=0}^{d} \frac{(b_1 t)^j}{j!}, \end{aligned}$$

which proves the asserted upper bound.

To show that $\gamma_{X_t} > 0$, suppose the contrary. Then

$$\mathbb{E} \int_{\mathcal{K}_h^d} \mathbb{1}\{c_h(K) \in B_r\} \, X_t(dK) = 0$$

for each $r > 0$, and by monotone convergence also

$$\mathbb{E} \int_{\mathcal{K}_h^d} \mathbb{1}\{c_h(K) \in \mathbb{H}^d\} \, X_t(dK) = 0.$$

This yields a contradiction, since X_t has infinitely many unbounded cells (for $t > \hat{c}_d$). □

Theorem 5.4.5. *Let C be the typical cell of the hyperbolic Poisson hyperplane tessellation X_t induced by the isometry invariant Poisson hyperplane process η_t with intensity $t \ge \kappa \hat{c}_d$, where $\kappa > 1$ (is fixed). Let ϑ be an isometry invariant deviation functional for hyperbolic balls. Further let $a > 0$, $\aleph \in (0,1)$ and $\varepsilon \in (0,1]$. Then there is a constant $c_9 > 0$ such that*

$$\mathbb{P}_t(\vartheta(C) \ge \varepsilon \mid \mathcal{H}^d(C) \ge a) \le c_9 \exp\left(-\aleph f_{a,\vartheta}(\varepsilon) \tau(a) t\right),$$

where c_9 depends only on $a, d, \varepsilon, \kappa, \aleph$.

Proof. The minimal circumradius was defined (in an isometry invariant way) by

$$R_{out}(K) := \inf\{r \geq 0 : K \subseteq B_h(x,r) \text{ for some } x \in \mathbb{H}^d\}, \qquad K \in \mathcal{K}_h^d.$$

Let $\aleph \in (0,1)$ be fixed. Assuming $t \geq \kappa\hat{c}_d$ for some fixed $\kappa > 1$, we want to provide an upper bound for the probability

$$\mathbb{P}_t(\vartheta(C) \geq \varepsilon \mid \mathcal{H}^d(C) \geq a) = \frac{\mathbb{P}_t(\vartheta(C) \geq \varepsilon,\ \mathcal{H}^d(C) \geq a)}{\mathbb{P}_t(\mathcal{H}^d(C) \geq a)}. \tag{5.38}$$

For this, we first provide a lower bound for the denominator. Let $r > 0$. Since $\mathcal{H}^d$ and R_{out} are isometry invariant, Lemma 5.4.3 yields

$$\begin{aligned}
\mathbb{P}_t(\mathcal{H}^d(C) \geq a) &\geq \mathbb{P}_t(\mathcal{H}^d(C) \geq a,\ R_{out}(C) \leq r)\\
&= \mathbb{E}_t\left[\mathbb{1}\{\mathcal{H}^d(C) \geq a,\ R_{out}(C) \leq r\}\frac{\mathcal{H}^d(C)}{\mathcal{H}^d(C)}\right]\\
&= \frac{1}{\gamma_{X_t}}\mathbb{E}_t\left[\mathbb{1}\{\mathcal{H}^d(C_0) \geq a,\ R_{out}(C_0) \leq r\}\mathcal{H}^d(C_0)^{-1}\right]\\
&\geq \frac{1}{\gamma_{X_t}\mathcal{H}^d(B_r)}\mathbb{E}_t\left[\mathbb{1}\{\mathcal{H}^d(C_0) \geq a,\ R_{out}(C_0) \leq r\}\right]\\
&= \frac{1}{\gamma_{X_t}\mathcal{H}^d(B_r)}\mathbb{P}_t(\mathcal{H}^d(C_0) \geq a,\ R_{out}(C_0) \leq r)\\
&\geq \frac{1}{\gamma_{X_t}\mathcal{H}^d(B_r)}\left(\mathbb{P}_t(\mathcal{H}^d(C_0) \geq a) - \mathbb{P}_t(R_{out}(C_0) > r)\right).
\end{aligned} \tag{5.39}$$

An application of Lemma 5.3.2 with $c = \ln(2)/(\kappa\hat{c}_d)$ shows that there is an $r_1(d,a,\kappa) > 0$ such that for $r \geq r_1(d,a,\kappa)$ we have

$$\mathbb{P}_t(R_{out}(C_0) > r) \leq \mathbb{P}_t(R_0(C_0) > r) \leq e^{-t(c+\tau(a))}$$

if $t \geq \kappa\hat{c}_d$ (which is part of the assumption). Hence, combining this with equation (5.39) and (5.22), we get

$$\begin{aligned}
\mathbb{P}_t(\mathcal{H}^d(C) \geq a) &\geq \frac{1}{\gamma_{X_t}\mathcal{H}^d(B_r)}\left(\mathbb{P}_t(\mathcal{H}^d(C_0) \geq a) - e^{-\frac{t}{\kappa\hat{c}_d}\ln(2) - \tau(a)t}\right)\\
&\geq \frac{1}{\gamma_{X_t}\mathcal{H}^d(B_r)}\left(e^{-\tau(a)t} - \frac{1}{2}e^{-\tau(a)t}\right)\\
&= \frac{1}{2\gamma_{X_t}\mathcal{H}^d(B_r)}e^{-\tau(a)t},
\end{aligned} \tag{5.40}$$

since $t/(\kappa\hat{c}_d) \geq 1$.

Now we turn to the numerator in (5.38). We apply Lemma 5.4.3, use that $\mathcal{H}^d$ and ϑ are isometry invariant, and thus we get

$$\begin{aligned}\mathbb{P}_t(\mathcal{H}^d(C) \geq a,\ \vartheta(C) \geq \varepsilon) &= \mathbb{E}_t\left[\mathbb{1}\{\mathcal{H}^d(C) \geq a,\ \vartheta(C) \geq \varepsilon\}\frac{\mathcal{H}^d(C)}{\mathcal{H}^d(C)}\right] \\ &= \frac{1}{\gamma_{X_t}}\mathbb{E}_t\left[\mathbb{1}\{\mathcal{H}^d(C_0) \geq a,\ \vartheta(C_0) \geq \varepsilon\}\mathcal{H}^d(C_0)^{-1}\right] \\ &\leq \frac{1}{a\gamma_{X_t}}\mathbb{P}_t\left(\mathcal{H}^d(C_0) \geq a,\ \vartheta(C_0) \geq \varepsilon\right). \qquad (5.41)\end{aligned}$$

Finally, by Lemma 5.3.2 with $c = \aleph f_{a,\vartheta}(\varepsilon)\tau(a)$ and by an application of Lemma 5.3.4 it follows that there is an $r_2(d,a,\kappa,\aleph,\varepsilon) > 0$ such that for $r \geq r_2(d,a,\kappa,\aleph,\varepsilon)$ we have

$$\begin{aligned}\mathbb{P}_t\left(\mathcal{H}^d(C_0) \geq a,\ \vartheta(C_0) \geq \varepsilon\right) &= \mathbb{P}_t\left(\mathcal{H}^d(C_0) \geq a,\ \vartheta(C_0) \geq \varepsilon,\ R_0(C_0) > r\right) \\ &\quad + \mathbb{P}_t\left(\mathcal{H}^d(C_0) \geq a,\ \vartheta(C_0) \geq \varepsilon,\ R_0(C_0) \leq r\right) \\ &\leq \mathbb{P}_t\left(R_0(C_0) > r\right) + \mathbb{P}_t\left(C_0 \in \mathcal{K}_{a,\varepsilon}, R_0(C_0) \leq r\right) \qquad (5.42) \\ &\leq e^{-\left(\aleph f_{a,\vartheta}(\varepsilon)\tau(a)+\tau(a)\right)t} + c_4 e^{-\left(1+\aleph f_{a,\vartheta}(\varepsilon)\right)\tau(a)t} \\ &\leq (1+c_4)e^{-\tau(a)t}e^{-\aleph f_{a,\vartheta}(\varepsilon)\tau(a)t}. \qquad (5.43)\end{aligned}$$

Now we choose $r := \max\{r_1(d,a,\kappa), r_2(d,a,\kappa,\aleph,\varepsilon)\}$. Hence, c_4 depends only on $d,a,\varepsilon,\kappa,\aleph$.

Combination of (5.40), (5.41) and (5.43) then yields

$$\mathbb{P}_t(\vartheta(C) \geq \varepsilon \mid \mathcal{H}^d(C) \geq a) \leq c_9 \exp\left(-\aleph f_{a,\vartheta}(\varepsilon)\tau(a)t\right),$$

where $c_9 = 2\,\mathcal{H}^d(B_r)(1+c_4)a^{-1}$ depends only on $d,a,\varepsilon,\kappa,\aleph$. □

5.4.3 Asymptotic volume distribution of typical cells

In order to establish the asserted asymptotic behaviour of the distribution of the typical cell, as the intensity goes to infinity, we proceed similarly as for the Crofton cell. However, at one point an additional argument is required, since the intensity γ_{X_t} of the associated particle process does not cancel out and depends on the intensity t of the underlying hyperbolic hyperplane process.

Theorem 5.4.6. *Let $a > 0$ (be fixed) and let η_t, X_t, C be as in Theorem 5.4.5. Then*

$$\lim_{t\to\infty} t^{-1} \ln \mathbb{P}_t(\mathcal{H}^d(C) \geq a) = -\tau(a).$$

Proof. Throughout the argument, we assume that $t \geq \kappa\hat{c}_d$ for some fixed $\kappa > 1$. By (5.40) we have

$$\mathbb{P}_t(\mathcal{H}^d(C) \geq a) \geq \frac{1}{2\gamma_{X_t}\mathcal{H}^d(B_r)}e^{-\tau(a)t},$$

where $r = r(d,a,\kappa)$ is independent of t. Hence,

$$\liminf_{t\to\infty} t^{-1} \ln \mathbb{P}_t(\mathcal{H}^d(C) \geq a) \geq -\tau(a),$$

where Lemma 5.4.4 was used.

On the other hand, as in the derivation of (5.41) and by (5.28) we get

$$\begin{aligned}\mathbb{P}_t(\mathcal{H}^d(C) \geq a) &\leq \frac{1}{a\gamma_{X_t}} \mathbb{P}_t\left(\mathcal{H}^d(C_0) \geq a\right) \\ &\leq \frac{c_8+1}{a\gamma_{X_t}} \exp(-\tau(a)t)\exp(k\tau(a)t),\end{aligned}$$

where $k \in (0,1)$ can be chosen arbitrarily and c_8 depends only on d, a, k. Then again by Lemma 5.4.4 we get

$$\limsup_{t\to\infty} t^{-1} \ln \mathbb{P}_t(\mathcal{H}^d(C) \geq a) \leq -\tau(a)(1-k).$$

□

5.5 Voronoi tessellation

After considering the behaviour of large cells in hyperbolic Poisson-hyperplane tessellations, it is natural to take a look at another famous model for generating mosaics, namely the decomposition of $\mathbb{H}^d$ by Voronoi cells. Their behaviour, in the context of Kendall's problem, is already considered in Euclidean space [43, 45] and in the spherical case [42]. We start with formally defining a Poisson-Voronoi tessellation in hyperbolic space. We then state a characterisation for Poisson point processes in $\mathbb{H}^d$. This gives us the opportunity to show that the distribution of the typical cell is the same as the distribution of the zero cell of a Voronoi tessellation depending on another point process. Next we state and prove an inequality of isoperimetric type for the context of the Poisson-Voronoi tessellation. After proving several auxiliary lemmas, the main result is stated in Theorem 5.5.5. It considers the probability that the deviation of the typical cell of a Poisson-Voronoi tessellation, given that it contains a ball of radius $a > 0$, exceeds a certain value.

For a given locally finite point process $X \subseteq \mathbb{H}^d$, the Voronoi cell generated by its nucleus $x \in X$ is defined by

$$C(x, X) := \{z \in \mathbb{H}^d : \ d_h(z,x) \leq d_h(z,y), \text{ for all } y \in X\}.$$

The cells are isometry invariant in the given way

$$C(\varphi x, \varphi X) = \varphi C(x, X), \qquad \varphi \in I(\mathbb{H}^d).$$

The aim of this section is to get results for the behaviour of large typical cells of the Voronoi tessellation. The deviation function in this context is the difference between the centred circumradius R_0 and the centred inradius r_0 of the cell, denoted by ϑ_0. Since the typical cell C will take values in $\mathcal{K}^d_{h,0}$, we can measure the size of the cells by their centred inradius r_0.

The functional $\tilde{U}$ is defined by

$$\tilde{U}(A) := \mathcal{H}^d(\{x \in \mathbb{H}^d : H(\exp_p(\exp_p^{-1}(x)/2)) \cap A \neq \varnothing\}), \qquad A \in \mathcal{B}(\mathbb{H}^d).$$

For a given radius $r > 0$ one also defines

$$\tilde{\Phi}(A) := \tilde{\Phi}(A, r) := \frac{\tilde{U}(A)}{\mathcal{H}^d(B_{2r})} \in [0,1], \qquad A \subseteq B_r.$$

For simplicity reasons write $\hat{\tau}(a) := \tilde{U}(B_a) = \mathcal{H}^d(B_{2a})$ for $a \geq 0$. To show results about the typical cell one first has to derive its distribution. In order to do so, one uses the results from Section 5.4 and a famous result by Slivnyak (for a proof see [103, Theorem 3.3.5]) generalized by Gentner in [24, Theorem 4.21].

Theorem 5.5.1. *Let X be an isometry invariant point process on $\mathbb{H}^d$ with positive and finite intensity $\gamma_h > 0$. Let $\mathbb{P}_X$ denote its Palm distribution. Then X is a Poisson point process, if and only if*

$$\mathbb{P}(X \in A) = \mathbb{P}(X + \delta_p \in A), \qquad A \in \mathcal{N}(\mathbb{H}^d).$$

Here the Palm distribution is defined as the normalized Palm measure, given in (5.31)

$$\mathbb{P}_X(A) := \frac{1}{\gamma_h} \int_\Omega \int_{\mathbb{H}^d} \int_{I(\mathbb{H}^d)} \mathbb{1}\{\theta_\varphi^{-1}\omega \in A\} w(x)\ \kappa(x, d\varphi)\ X(\omega, dx)\ \mathbb{P}(d\omega).$$

In this case the refined Campbell Theorem (see [61, Theorem 3.6]) takes the form

$$\mathbb{E} \int_{\mathbb{H}^d} \int_{I(\mathbb{H}^d)} f(\theta_\varphi^{-1}, \varphi) \kappa(x, d\varphi) X(dx) = \gamma_h \mathbb{E}_X \int_{I(\mathbb{H}^d)} f(\theta_{id}, \varphi) \lambda(d\varphi) \tag{5.44}$$

for any measurable map $f : \Omega \times I(\mathbb{H}^d) \to [0, \infty)$. In order to define the distribution of the typical cell C, we take a look at the random measure ξ on $\mathbb{H}^d \times \mathcal{K}_h^d$ which is given by

$$\xi(\omega) := \int_{\mathbb{H}^d} \int_{I(\mathbb{H}^d)} \delta_{(x, \varphi^{-1}C(x, X(\omega)))}\ \kappa(x, d\varphi)\ X(\omega, dx).$$

This definition is related to the measure defined in (5.34). Therefore by the invariance property of ξ (see (5.33))

$$\xi(\theta_\psi \omega, (\psi B) \times A) = \xi(\omega, B \times A), \qquad A \subseteq \mathcal{K}_h^d,\ B \subseteq \mathbb{H}^d,\ \psi \in I(\mathbb{H}^d)$$

holds, where A and B have to be measurable. Since X is a Poisson point process with intensity measure $\mathbb{E}[X(\cdot)] = \gamma_h \mathcal{H}^d(\cdot)$, the distribution of C is given as the mark distribution of ξ, namely

$$\mathbb{P}(C \in \cdot) = \frac{1}{\gamma_h} \int_\Omega \int_{\mathbb{H}^d} \int_{I(\mathbb{H}^d)} \mathbb{1}\{\varphi^{-1}C(x, X(\omega)) \in \cdot\} w(x)\ \kappa(x, d\varphi)\ X(\omega, dx)\ \mathbb{P}(d\omega).$$

Using the results in Section 5.4.1 and the refined Campbell Theorem (5.44) in the second line, this can be transformed to

$$\begin{aligned}\mathbb{P}(C \in \cdot) &= \frac{1}{\gamma_h} \int_\Omega \int_{\mathbb{H}^d} \int_{I(\mathbb{H}^d)} \mathbb{1}\{\varphi^{-1} C(\varphi(p), X(\omega)) \in \cdot\} w(\varphi(p))\ \kappa(x, d\varphi)\ X(\omega, dx)\ \mathbb{P}(d\omega) \\ &= \frac{1}{\gamma_h} \gamma_h \int_\Omega \int_{I(\mathbb{H}^d)} \mathbb{1}\{\varphi^{-1} C(\varphi(p), X(\theta_\varphi \omega)) \in \cdot\} w(\varphi(p))\ \lambda(d\varphi)\ \mathbb{P}_X(d\omega) \\ &= \int_\Omega \int_{I(\mathbb{H}^d)} \mathbb{1}\{C(p, X(\omega)) \in \cdot\} w(\varphi(p))\ \lambda(d\varphi)\ \mathbb{P}_X(d\omega) \\ &= \mathbb{P}_X(C(p, X) \in \cdot) \int_{I(\mathbb{H}^d)} w(\varphi(p))\ \lambda(d\varphi) \\ &= \mathbb{P}(C(p, X + \delta_p) \in \cdot).\end{aligned}$$

Therefore the distribution of the typical cell is given by the distribution of the Crofton cell $\tilde{C}_0$ of a special process, namely the hyperplane process

$$\tilde{\eta}_\gamma := \{H(\exp_p(\exp_p^{-1}(x)/2)) :\ x \in X\}. \tag{5.45}$$

We will use this connection later in this chapter. Remark that this hyperplane process is not isometry invariant anymore.

Theorem 5.5.2. *Let $K \in \mathcal{K}^d_{h,0}$, $\varepsilon \in [0,1]$ and $a > 0$. If $r_0(K) \geq a$ and $\vartheta_0(K) \geq \varepsilon$ holds, then*

$$\tilde{U}(K) \geq (1 + c_{10}(a,d)\varepsilon^{(d+1)/2})\, \tilde{U}(B_a),$$

holds, where $c_{10}(a,d)$ is given by

$$c_{10}(a,d) = \min\{1, \frac{\omega_{d-1}}{2a\,\omega_d 3^{d-2}(\cosh(a+1/2)\sinh(a+1))^{(d-1)/2}}\}.$$

Proof. If $r_0(K) \geq a$ and $\vartheta_0(K) \geq \varepsilon$, then there exists a direction $u \in \mathbb{S}_p^{d-1}$ with $z_0 := \exp_p((a + \varepsilon)u) \in K$. Since K is convex, also $I := [\exp_p(au), \exp_p((a+\varepsilon)u)] \subseteq K$ holds. Thus we get

$$\tilde{U}(K) \geq \tilde{U}(B_a \cup I) = \tilde{U}(B_a) + \mathcal{H}^d(\{x \in \mathbb{H}^d \setminus B_{2a} :\ H(\exp_p(\exp_p^{-1}(x)/2)) \cap I \neq \varnothing\}). \tag{5.46}$$

Further recall $\tilde{\beta} = \sqrt{2 - \frac{2\tanh(a+\varepsilon/2)}{\tanh(a+\varepsilon)}}$ and

$$c(\tilde{\beta}) = \frac{1}{\omega_d} \sigma_{d-1}(\{x \in \mathbb{S}_p^{d-1} :\ d_s(u,x) \leq \tilde{\beta}\}) \in [0,1].$$

We argue as in the proof of Theorem 5.2.17. We know, by using Lemma 5.1.4, that every hyperplane in

$$D(u,\tilde{\beta}) := \{H(\exp_p(ru_1)) :\ r \in (a, a+\varepsilon/2],\ u_1 \in B_s(u,\tilde{\beta})\}$$

has nonempty intersection with $[p, \exp_p((a+\varepsilon)u)]$ and therefore also with $[\exp_p(au), \exp_p((a+\varepsilon)u)]$. We define $A_a(\varepsilon)$ as the set of points such that the hyperplane having equal distance from p and this point is contained in $D(u,\tilde{\beta})$. More precisely define

$$A_a(\varepsilon) := \{z \in \mathbb{H}^d : \{y \in \mathbb{H}^d : d_h(y,z) = d_h(y,p)\} \in D(u,\tilde{\beta})\}.$$

By the construction rule of the Poisson Voronoi mosaic, we get

$$A_a(\varepsilon) = \{\exp_p(ru_0) : 2a < r \le 2a+\varepsilon,\ d_s(u,u_0) \le \tilde{\beta}\}.$$

Therefore we get by (5.46)

$$\tilde{U}(K) \ge \tilde{U}(B_a) + \mathcal{H}^d(A_a(\varepsilon)). \tag{5.47}$$

Before we can give a lower bound for $\mathcal{H}^d(A_a(\varepsilon))$, we derive, by using spherical coordinates on $\mathbb{H}^d$

$$\mathcal{H}^d(\{x \in \mathbb{H}^d : d_h(p,x) \in (2a, 2a+\varepsilon]\}) = \omega_d \int_{2a}^{2a+\varepsilon} \sinh^{d-1}(t)\ dt \ge \omega_d\, \varepsilon \sinh^{d-1}(2a). \tag{5.48}$$

By the definition of $\tilde{U}$

$$\tilde{U}(B_a) = \mathcal{H}^d(B_{2a}) = \omega_d \int_0^{2a} \sinh^{d-1}(t)\ dt \le 2a\,\omega_d \sinh^{d-1}(2a)$$

holds. Therefore, using (5.48) and the definitions above, the hyperbolic volume of $A_a(\varepsilon)$ is bounded from below by

$$\mathcal{H}^d(A_a(\varepsilon)) = \omega_d\, c(\tilde{\beta}) \int_{2a}^{2a+\varepsilon} \sinh^{d-1}(t)\ dt \ge c(\tilde{\beta})\,\omega_d\, \varepsilon \sinh^{d-1}(2a) \ge \frac{c(\tilde{\beta})}{2a}\,\varepsilon\, \tilde{U}(B_a). \tag{5.49}$$

By equation (5.16) we get

$$c(\tilde{\beta}) \ge \frac{\omega_{d-1}}{\omega_d} \left(\frac{1}{3}\right)^{d-2} c_{11}(a,d)\varepsilon^{\frac{d-1}{2}} \tag{5.50}$$

with $c_{11}(a,d) = (\cosh(a+1/2)\sinh(a+1))^{-(d-1)/2}$. Combining the results for $c(\tilde{\beta})$ in (5.50) with the inequalities in (5.49) and (5.47) gives

$$\begin{aligned}\tilde{U}(K) - \tilde{U}(B_a) \ge \frac{c(\tilde{\beta})}{2a}\varepsilon\, \tilde{U}(B_a) &\ge \left(\frac{1}{3}\right)^{d-2} \frac{\omega_{d-1} c_{11}(a,d)}{2a\omega_d} \varepsilon^{\frac{d+1}{2}} \tilde{U}(B_a)\\ &\ge \min\{1, \left(\frac{1}{3}\right)^{d-2} \frac{\omega_{d-1} c_{11}(a,d)}{2a\omega_d}\} \varepsilon^{\frac{d+1}{2}} \tilde{U}(B_a)\\ &= c_{10}(a,d)\varepsilon^{\frac{d+1}{2}} \tilde{U}(B_a),\end{aligned}$$

where $c_{10}(a,d)$ is given by

$$c_{10}(a,d) = \min\{1, \frac{\omega_{d-1}}{2a\,\omega_d 3^{d-2}(\cosh(a+1/2)\sinh(a+1))^{(d-1)/2}}\}.$$

□

We want to use Theorem 5.5.2 to get results for the probability of a typical cell to have a certain deviation from the ball, given that it exceeds a certain size. Before that one needs to somehow bound the size of a typical cell in Poisson Voronoi mosaics. Recall that $\tilde{C}_0$ is the zero cell of the hyperplane process $\tilde{\eta}_t$, defined in (5.45).

Lemma 5.5.3. *Let X be an isometry invariant Poisson point process of intensity $\gamma \geq 1$ generating the Voronoi mosaic. In this case there exists a constant $\tilde{r} = \tilde{r}(d) \geq 2$ such that*

$$\mathbb{P}(R_0(\tilde{C}_0) > r) \leq \exp\left(-2\gamma\kappa_d \sinh^{d-1}(r - d/(d-1))\right)$$

for all $r \geq \tilde{r}$.

Proof. Consider a fixed direction $u_0 \in \mathbb{S}_p^{d-1}$. Let $z_0 = \exp_p(ru_0) \in \mathbb{H}^d$ be the point in that direction having distance r from the origin. Since $\tilde{C}_0$ is convex, we get that the event $\{z_0 \in \tilde{C}_0\}$ is the same as $\{[p, z_0] \subseteq \tilde{C}_0\}$. The probability of this event is given by

$$\mathbb{P}(z_0 \in \tilde{C}_0) = \mathbb{P}(d_h(z_0, p) \leq d_h(z_0, X)) = \mathbb{P}(B(z_0, r)^\circ \cap X = \varnothing) = \exp(-\gamma\,\mathcal{H}^d(B(z_0, r))).$$

Let $r \geq 2/(d-1)$ then

$$\begin{aligned}
\mathcal{H}^d(B_r) &= d\kappa_d \int_0^r \sinh^{d-1}(t)\; dt \\
&\geq d\kappa_d \int_{r-2/(d-1)}^r \sinh^{d-1}(t)\; dt \\
&= d\kappa_d \int_0^{1/(d-1)} \sinh^{d-1}(r - 1/(d-1) + t) + \sinh^{d-1}(r - 1/(d-1) - t)\; dt \\
&\geq d\kappa_d \int_0^{1/(d-1)} 2\sinh^{d-1}(r - 1/(d-1))\; dt \\
&= \frac{2d\kappa_d}{d-1} \sinh^{d-1}(r - 1/(d-1))
\end{aligned} \tag{5.51}$$

gives

$$\mathbb{P}(z_0 \in \tilde{C}_0) \leq \exp\left(-\gamma \cdot \frac{2d\kappa_d}{d-1} \sinh^{d-1}(r - 1/(d-1))\right).$$

In the next step one considers the neighbourhood of z_0, namely $B(z_0, 1) \cap \delta B_r$. The probability of $\tilde{C}_0$ having nonempty intersection with this set can be calculated by

$$\begin{aligned}
\mathbb{P}\left(\exists x \in B(z_0, 1) \cap \delta B_r :\; x \in \tilde{C}_0\right) &= \mathbb{P}\left(\exists x \in B(z_0, 1) \cap \delta B_r : B(x, r)^\circ \cap X = \varnothing\right) \\
&\leq \mathbb{P}(B(z_0, r-1)^\circ \cap X = \varnothing) \\
&= \exp\left(-\gamma \cdot \mathcal{H}^d(B_{r-1})\right) \\
&\leq \exp\left(-\gamma \cdot \frac{2d\kappa_d}{d-1} \sinh^{d-1}(r - d/(d-1))\right).
\end{aligned}$$

By Lemma 5.3.1 we know that there exists a constant $c_2(d)$ such that δB_r can be covered by $c_2(d)e^{r(d-1)}$ balls of radius 1 with centre on ∂B_r, namely

$$\delta B_r \subseteq \bigcup_{x \in I} B(x, 1),$$

where I denotes the set of centres of these balls. This gives, for r being big enough,

$$\begin{aligned}\mathbb{P}\left(\exists x \in \delta B_r:\ x \in \tilde{C}_0\right) &= \mathbb{P}\left(\exists z_0 \in I : \exists x \in B(z_0,1) \cap \delta B_r : B(x,r)^\circ \cap X = \varnothing\right)\\ &\le \sum_{z_0 \in I} \mathbb{P}\left(\exists x \in B(z_0,1) \cap \delta B_r : B(x,r)^\circ \cap X = \varnothing\right)\\ &= |I|\,\mathbb{P}\left(\exists x \in B(z,1) \cap \delta B_r : B(x,r)^\circ \cap X = \varnothing\right)\\ &\le |I|\,\mathbb{P}(B(z,r-1)^\circ \cap X = \varnothing)\\ &= |I| \exp\left(-\gamma \cdot \mathcal{H}^d(B_{r-1})\right)\\ &\le \exp\left(-\gamma \cdot \frac{2d\kappa_d}{d-1} \sinh^{d-1}(r-d/(d-1)) + \log(c_2(d))r(d-1)\right)\\ &\le \exp\left(-\gamma \cdot 2\kappa_d \sinh^{d-1}(r-d/(d-1))\right),\end{aligned}$$

where z is an arbitrary centre from I. In the last step we used that for r being big enough the inequality

$$\frac{2\gamma\kappa_d}{d-1} \sinh^{d-1}(r-d/(d-1)) \ge \log(c_2(d))r(d-1)$$

holds. □

We define the set $\tilde{\mathcal{K}}_{a,\varepsilon}$ to be

$$\tilde{\mathcal{K}}_{a,\varepsilon} := \{K \in \mathcal{K}_h^d :\ r_0(K) \ge a,\ \vartheta_0(K) \ge \varepsilon\}.$$

Theorem 5.5.4. *Let $a > 0$ and $\varepsilon \in [0,1]$. Let further X be a homogeneous Poisson point process with intensity $\gamma \ge 1$. Then there exist constants $c_{10}, c_{12} > 0$ and $\nu \in \mathbb{N}$ such that*

$$\mathbb{P}(C \in \tilde{\mathcal{K}}_{a,\varepsilon},\ R_0(C) \le r) \le c_{12} \max\{1, \gamma\hat{\tau}(r)\}^{d\nu} \exp(-(1 + c_{10}(a,d)\varepsilon^{(d+1)/2}/3)\hat{\tau}(a)\gamma),$$

where ν only depends on a, d, ε and r, $c_{10} = c_{10}(a,d)$ is taken from Theorem 5.5.2 and c_{12} is given by

$$c_{12} = c_{12}(a,d,\varepsilon) = \sum_{j=d+1}^{d\nu} \binom{j}{d}^{\nu} \frac{1}{j!} + \sum_{N=0}^{d} \frac{1}{N!}.$$

Proof. Consider the hyperplane process $\tilde{\eta}_\gamma$, described in (5.45). Since the distribution of C and $\tilde{C}_0$ are equivalent one focuses on $\mathbb{P}(\tilde{C}_0 \in \tilde{\mathcal{K}}_{a,\varepsilon},\ R_0(\tilde{C}_0 \le r))$. For $N \in \mathbb{N}$ and $H_1, \dots, H_N \in \mathbb{H}_{d-1}\langle B_r\rangle$ we define $H_{(N)} := (H_1, \dots, H_N)$ and let $P(H_{(N)})$ denote the hyperbolic Crofton cell of the tessellation induced by $H_1, \dots, H_N$. Assume that $P(H_{(N)}) \in \tilde{\mathcal{K}}_{a,\varepsilon}$. Further define $\tilde{\alpha} := c_{10}(a,d)\varepsilon^{(d+1)/2}/(2 + c_{10}(a,d)\varepsilon^{(d+1)/2})$, where the constant $c_{10}(a,d)$ is taken from Theorem 5.5.2. This leads to

$$(1-\tilde{\alpha})(1 + c_{10}(a,d)\varepsilon^{(d+1)/2}) = 1 + \tilde{\alpha}$$

and since $c_{10}(a,d)\varepsilon^{(d+1)/2} \le 1$ also

$$\tilde{\alpha} \ge 3^{-1} \cdot c_{10}(a,d)\varepsilon^{(d+1)/2}$$

holds. By Theorem 5.2.19, there are at most $\nu = \nu(a,d,\varepsilon,r)$ vertices of $P(H_{(N)})$ such that the hyperbolic convex hull $Q(H_{(N)})$ of these vertices satisfies

$$\begin{aligned}\tilde{\Phi}(Q(H_{(N)})) &\geq (1-\tilde{\alpha})\tilde{\Phi}(P(H_{(N)}))\\ &\geq (1-\tilde{\alpha})(1+c_{10}(a,d)\varepsilon^{(d+1)/2})\tilde{\Phi}(B_a)\\ &= (1+\tilde{\alpha})\tilde{\Phi}(B_a).\end{aligned}$$

Here the second inequality used the result of Theorem 5.5.2. Continuing in the same way as in Lemma 5.3.4 gives the result. It should be pointed out that the arguments in the proof of Lemma 5.3.4 transfer to the current setting, even thought a different measure on the set of hyperplanes is used. □

Combining the results in this section, one gets the main result concerning the limit shape of large typical cells.

Theorem 5.5.5. *Let $a > 0$ and $\varepsilon \in [0,1]$. Let further X be a homogeneous Poisson point process with intensity $\gamma \geq 1$. Then there exist constants c_{10}, c_{12}, ν such that*

$$\mathbb{P}(\vartheta_0(C) \geq \varepsilon \mid r_0(C) \geq a) \leq (1 + c_{12}\max\{1,\gamma\hat{\tau}(r)\}^{d\nu})\exp(-c_{10}(a,d)\varepsilon^{(d+1)/2}\hat{\tau}(a)\gamma/3),$$

where c_{12}, ν only depend on a,d,ε and $c_{10} = c_{10}(a,d)$ is explicitly given in Theorem 5.5.2.

Proof. We use the alternative representation of the distribution of the typical cell, split the probability and apply Theorem 5.5.3 and 5.5.4

$$\begin{aligned}\mathbb{P}(\vartheta_0(C) \geq \varepsilon \mid r_0(C) \geq a) =&\mathbb{P}(\vartheta_0(\tilde{C}_0) \geq \varepsilon \mid r_0(\tilde{C}_0) \geq a)\\ =&\mathbb{P}(\vartheta_0(\tilde{C}_0) \geq \varepsilon,\ R_0(\tilde{C}_0) \leq r \mid r_0(\tilde{C}_0) \geq a)\\ &+ \mathbb{P}(\vartheta_0(\tilde{C}_0) > \varepsilon,\ R_0(\tilde{C}_0) > r \mid r_0(\tilde{C}_0) \geq a)\\ \leq&\frac{\mathbb{P}(\vartheta_0(\tilde{C}_0) \geq \varepsilon,\ R_0(\tilde{C}_0) \leq r,\ r_0(\tilde{C}_0) \geq a)}{\mathbb{P}(r_0(\tilde{C}_0) \geq a)} + \frac{\mathbb{P}(R_0(\tilde{C}_0) > r)}{\mathbb{P}(r_0(\tilde{C}_0) \geq a)}\\ \leq&\frac{c_{12}\max\{1,\gamma\hat{\tau}(r)\}^{d\nu}\exp(-(1+c_{10}(a,d)\varepsilon^{(d+1)/2}/3)\hat{\tau}(a)\gamma)}{\exp(-\hat{\tau}(a)\gamma)}\\ &+\frac{\exp(-2\gamma\kappa_d\sinh^{d-1}(r-d/(d-1)))}{\exp(-\hat{\tau}(a)\gamma)}\\ \leq&(1+c_{12}\max\{1,\gamma\hat{\tau}(r)\}^{d\nu})\exp(-c_{10}(a,d)\varepsilon^{(d+1)/2}\hat{\tau}(a)\gamma/3).\end{aligned}$$

Here the last inequality is fulfilled for picking r large enough, depending on d,a,ε. □

Remark 5.5.1. With a few more steps one can get rid of the dependence on γ of the first factor in Theorem 5.5.5. Since

$$\gamma \mapsto (1 + c_{12}\max\{1,\gamma\hat{\tau}(r)\}^{d\nu})\exp(-c_{10}(a,d)\varepsilon^{(d+1)/2}\hat{\tau}(a)\gamma/6)$$

is bounded one can find a constant c_{13} depending on d, a, ε that fulfills

$$\mathbb{P}(\vartheta_0(C) \geq \varepsilon \mid r_0(C) \geq a) \leq c_{13} \exp(-c_{10}(a,d)\varepsilon^{(d+1)/2}\hat{\tau}(a)\gamma/6).$$

Bibliography

[1] James W. Anderson, *Hyperbolic geometry*, 2. ed., corr. print ed., Springer undergraduate mathematics series, Springer, London, 2007.

[2] E. Arbeiter and M. Zähle, *Geometric measures for random mosaics in spherical spaces*, Stochastics and Stochastic Reports **46** (2007), no. 1-2, 63–77.

[3] Adrian Baddeley, Ege Rubak, and Rolf Turner, *Spatial point patterns: Methodology and applications with R*, A Chapman & Hall book, CRC Press Taylor & Francis Group, Boca Raton and London and New York, 2016.

[4] Victor Bangert, *Konvexe Mengen in Riemannschen Mannigfaltigkeiten*, Mathematische Zeitschrift **162** (1978), no. 3, 263–286.

[5] Imre Bárány, Daniel Hug, Matthias Reitzner, and Rolf Schneider, *Random points in halfspheres*, Random Structures & Algorithms **50** (2017), no. 1, 3–22.

[6] Riccardo Benedetti and Carlo Petronio, *Lectures on Hyperbolic Geometry*, Universitext, Springer, Berlin and Heidelberg, 1992.

[7] Jean Bertoin, *Different aspects of a random fragmentation model*, Stochastic Processes and their Applications **116** (2006), no. 3, 345–369.

[8] Florian Besau and Christoph Thäle, *Asymptotic normality for random polytopes in non-Euclidean geometries*, arXiv:1909.05607 (2019).

[9] Michel Bode, Nikolaos Fountoulakis, and Tobias Müller, *The probability of connectivity in a hyperbolic model of complex networks*, Random Structures & Algorithms **49** (2016), no. 1, 65–94.

[10] Vladimir I. Bogachev, *Measure Theory*, Springer-Verlag Berlin Heidelberg, Berlin, Heidelberg, 2007.

[11] Gilles Bonnet, Pierre Calka, and Matthias Reitzner, *Cells with many facets in a Poisson hyperplane tessellation*, Advances in Mathematics **324** (2018), 203–240.

[12] K. Böröczky, *Finite packing and covering*, Cambridge tracts in mathematics, vol. 154, Cambridge University Press, Cambridge, 2004.

[13] Leo Breiman, *Probability*, Addison-Wesley series in statistics, Addison-Wesley Publishing Company, Reading, Mass. and Menlo Park, Calif. and London and Don Mills, Ontario, 1968.

[14] E. M. Bronshteyn and L. D. Ivanov, *The approximation of convex sets by polyhedra*, Siberian Mathematical Journal **16** (1976), no. 5, 852–853.

[15] John E. Brothers, *Integral geometry in homogeneous spaces*, Transactions of the American Mathematical Society **124** (1966), no. 3, 480–517.

[16] Pierre Calka, *Asymptotic Methods for Random Tessellations*, Stochastic Geometry, Spatial Statistics and Random Fields: Asymptotic Methods (Evgeny Spodarev, ed.), Springer Berlin Heidelberg, Berlin, Heidelberg, 2013, pp. 183–204.

[17] Pierre Calka, Aurélie Chapron, and Nathanaël Enriquez, *Mean asymptotics for a Poisson-Voronoi cell on a Riemannian manifold*, arXiv:1807.09043 (2018).

[18] James W. Cannon, William J. Floyd, Richard Kenyon, Walter, and R. Parry, *Hyperbolic geometry*, Flavors of Geometry, Cambridge University Press, 1997, pp. 59–115.

[19] Min Chang, *Bonnesen-style inequalities on surfaces of constant curvature*, Journal of Inequalities and Applications **2018** (2018).

[20] Isaac Chavel, *Riemannian geometry: A modern introduction*, Cambridge tracts in mathematics, vol. 108, Cambridge Univ. Press, Cambridge, 1993.

[21] Nicolas Chenavier, *A general study of extremes of stationary tessellations with examples*, Stochastic Processes and their Applications **124** (2014), no. 9, 2917–2953.

[22] Nicolas Chenavier and Werner Nagel, *The largest order statistics for the inradius in an isotropic STIT tessellation*, Extremes **22** (2019), no. 4, 571–598.

[23] Thomas M. Cover and Bradley Efron, *Geometrical Probability and Random Points on a Hypersphere*, The Annals of Mathematical Statistics **38** (1967), no. 1, 213–220.

[24] D. S. Gentner, *Palm theory, mass-transports and ergodic theory for group-stationary processes*, PhD Thesis, Karlsruhe Institute of Technology (2011).

[25] M. H. A. Davis, *Markov Models and Optimization*, Chapman and Hall/CRC Monographs on Statistics and Applied Probability, vol. v.49, Routledge, Boca Raton, 2018.

[26] Christian Deuss, Julia Hörrmann, and Christoph Thäle, *A random cell splitting scheme on the sphere*, Stochastic Processes and their Applications **127** (2017), no. 5, 1544–1564.

[27] Peter Eichelsbacher and Christoph Thäle, *New Berry-Esseen bounds for non-linear functionals of Poisson random measures*, Electronic Journal of Probability **19** (2014), no. 102.

[28] Jürgen Elstrodt, *Maß- und Integrationstheorie*, 7., korr. und aktualisierte aufl. ed., Grundwissen Mathematik, Springer, Berlin, 2011.

[29] F. Gao, D. Hug, R. Schneider, *Intrinsic volumes and polar sets in spherical space*, Math. Notae **2001/02** (2003), no. 41, 159–176.

[30] Herbert Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, vol. Band 153, Springer-Verlag, Berlin, 1969.

[31] Nikolaos Fountoulakis and Joseph Yukich, *Limit theory for the number of isolated and extreme points in hyperbolic random geometric graphs*, 2019.

[32] Eduardo Gallego and Gil Solanes, *Integral geometry and geometric inequalities in hyperbolic space*, Differential Geometry and its Applications **22** (2005), no. 3, 315–325.

[33] Gilles Bonnet, *Poisson hyperplane tessellation: Asymptotic probabilities of the zero and typical cells*, PhD Thesis, Osnabrück (2016).

[34] C. Gloaguen, F. Fleischer, H. Schmidt, and V. Schmidt, *Fitting of stochastic telecommunication network models via distance measures and Monte–Carlo tests*, Telecommunication Systems **31** (2006), no. 4, 353–377.

[35] Paul Goodey and Ralph Howard, *Processes of flats induced by higher dimensional processes*, Advances in Mathematics **80** (1990), no. 1, 92–109.

[36] Lothar Heinrich, *Normal approximation for some mean-value estimates of absolutely regular tessellations*, Mathematical Methods of Statistics **3** (1994), 1–24.

[37] ———, *Central limit theorems for motion-invariant Poisson hyperplanes in expanding convex bodies*, Supplemento ai Rendiconti del Circolo Matematico di Palermo (2009), no. Serie II 81, 187–212.

[38] Lothar Heinrich and Lutz Muche, *Second-order properties of the point process of nodes in a stationary Voronoi tessellation*, Mathematische Nachrichten **281** (2008), no. 3, 350–375.

[39] Lothar Heinrich, Hendrik Schmidt, and Volker Schmidt, *Central limit theorems for Poisson hyperplane tessellations*, The Annals of Applied Probability **16** (2006), no. 2, 919–950.

[40] Sigurdur Helgason, *Differential geometry and symmetric spaces*, 1. aufl. ed., Pure and applied mathematics, vol. 12, Elsevier textbooks, s.l., 1962.

[41] Felix Herold, Daniel Hug, and Christoph Thäle, *Does a central limit theorem hold for the k-skeleton of Poisson hyperplanes in hyperbolic space?*, arXiv:1911.02120 (2019).

[42] Daniel Hug and Andreas Reichenbacher, *Geometric inequalities, stability results and Kendall's problem in spherical space*, arXiv:1709.06522 (2017).

[43] Daniel Hug, Matthias Reitzner, and Rolf Schneider, *Large Poisson-Voronoi cells and Crofton cells*, Advances in Applied Probability **36** (2004), no. 03, 667–690.

[44] Daniel Hug and Rolf Schneider, *Asymptotic Shapes of large Cells in Random Tessellations*, GAFA, Geom. funct. anal. **17** (2007), no. 1, 156–191.

[45] ______, *Faces of Poisson–Voronoi mosaics*, Probability Theory and Related Fields **151** (2011), no. 1-2, 125–151.

[46] ______, *Random Conical Tessellations*, Discrete & Computational Geometry **56** (2016), no. 2, 395–426.

[47] Daniel Hug and Christoph Thäle, *Splitting tessellations in spherical spaces*, Electronic Journal of Probability **24** (2019), no. 24, 60 pp.

[48] Daniel Hug, Christoph Thäle, and Wolfgang Weil, *Intersection and proximity of processes of flats*, Journal of Mathematical Analysis and Applications **426** (2015), no. 1, 1–42.

[49] J. Jonasson, I. Benjamini, O. Schramm, and J. Tykesson, *Visibility to infinity in the hyperbolic plane, despite obstacles*, Latin American Journal of Probability and Mathematical Statistics (2009), no. 6, 323–342.

[50] Zakhar Kabluchko, Alexander Marynych, Daniel Temesvari, and Christoph Thäle, *Cones generated by random points on half-spheres and convex hulls of Poisson point processes*, Probability Theory and Related Fields **175** (2019), no. 3-4, 1021–1061.

[51] Yan Y. Kagan, *Earthquake spatial distribution: the correlation dimension*, Geophysical Journal International **168** (2007), no. 3, 1175–1194.

[52] Olav Kallenberg, *On the structure of stationary flat processes*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **37** (1976), no. 2, 157–174.

[53] ______, *On the structure of stationary flat processes. II*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **52** (1980), no. 2, 127–147.

[54] ______, *Foundations of modern probability*, 2. ed. ed., Probability and Its Applications, Springer, New York, 2002.

[55] Ioannis Karatzas and Steven E. Shreve, *Brownian Motion and Stochastic Calculus*, vol. 113, Springer New York, New York, NY, 1998.

[56] Achim Klenke, *Probability Theory: A Comprehensive Course*, 2nd ed. 2014 ed., Universitext, Springer London, London and s.l., 2014.

[57] Igor N. Kovalenko, *A simplified proof of a conjecture of D. G. Kendall concerning shapes of random polygons*, Journal of Applied Mathematics and Stochastic Analysis **12** (1999), no. 4, 301–310.

[58] R. Lachièze-Rey, *Mixing properties for STIT tessellations*, Advances in Applied Probability **43** (2011), no. 1, 40–48.

[59] Raphael Lachieze-Rey and Giovanni Peccati, *Fine Gaussian fluctuations on the Poisson space, I: Contractions, cumulants and geometric random graphs*, Electronic Journal of Probability **18** (2013), no. 32, 32 pp.

[60] Raphaël Lachièze-Rey and Giovanni Peccati, *Fine Gaussian fluctuations on the Poisson space II: Rescaled kernels, marked processes and geometric U-statistics*, Stochastic Processes and their Applications **123** (2013), no. 12, 4186–4218.

[61] Günter Last, *Modern Random Measures: Palm Theory and Related Models*, New Perspectives in Stochastic Geometry (Wilfrid S. Kendall and Ilya Molchanov, eds.), Oxford University Press, 2009, pp. 77–110.

[62] ———, *Stationary Random Measures on Homogeneous Spaces*, Journal of Theoretical Probability **23** (2010), no. 2, 478–497.

[63] Günter Last and Andreas Brandt, *Market point processes on the real line: The dynamic approach*, Probability and Its Applications, Springer, New York and Berlin, 1995.

[64] Günter Last and Mathew Penrose, *Lectures on the Poisson Process*, Institute of Mathematical Statistics textbooks, vol. 7, Cambridge University Press, Cambridge, 2018.

[65] Günter Last, Mathew D. Penrose, Matthias Schulte, and Christoph Thäle, *Moments and Central Limit Theorems for Some Multivariate Poisson Functionals*, Advances in Applied Probability **46** (2014), no. 02, 348–364.

[66] Ming Liao, *Invariant Markov Processes Under Lie Group Actions*, SpringerLink Bücher, Springer, Cham, 2018.

[67] Thomas M. Liggett, *Interacting Particle Systems*, Grundlehren der mathematischen Wissenschaften, A Series of Comprehensive Studies in Mathematics, vol. 276, Springer, New York, NY, 1985.

[68] H. Maehara and H. Martini, *An analogue of Sylvester's four-point problem on the sphere*, Acta Mathematica Hungarica **155** (2018), no. 2, 479–488.

[69] Anatoliy Malyarenko, *Invariant Random Fields on Spaces with a Group Action*, Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.

[70] Domenico Marinucci and Giovanni Peccati, *Random Fields on the Sphere*, Cambridge University Press, Cambridge, 2011.

[71] S. Martínez and W. Nagel, *The β-mixing rate of STIT tessellations*, Stochastics **88** (2015), no. 3, 396–414.

[72] Georges Matheron, *Random sets and integral geometry*, Wiley series in probability and mathematical statistics, Wiley, New York, 1975.

[73] J. Mecke, *On the Intersection Density of Flat Processes*, Mathematische Nachrichten **151** (1991), no. 1, 69–74.

[74] J. Mecke, W. Nagel, and V. Weiß, *Some distributions for I-segments of planar random homogeneous STIT tessellations*, Mathematische Nachrichten **284** (2011), no. 11-12, 1483–1495.

[75] R. E. Miles, *Random Points, Sets and Tessellations on the Surface of a Sphere*, Sankhyā: The Indian Journal of Statistics, Series A (1961-2002) **33** (1971), no. 2, 145–174.

[76] ———, *A generalization of a formula of Steiner*, Probability Theory and Related Fields **61** (1982), no. 3, 375–378.

[77] Ilya Molchanov, *Theory of Random Sets*, 2nd ed. ed., Probability Theory and Stochastic Modelling Ser, vol. 87, Springer, London, 2017.

[78] Frank Morgan, *Geometric measure theory: A beginner's guide*, fifth edition ed., Academic Press, Amsterdam, 2016.

[79] Tobias Müller and Merlijn Staps, *The diameter of KPKVB random graphs*, Advances in Applied Probability **51** (2019), no. 2, 358–377.

[80] Werner Nagel, Joseph Mecke, Joachim Ohser, and Viola Weiss, *A tessellation model for crack patterns on surfaces*, Image Analysis & Stereology **27** (2008), no. 2, 73–78.

[81] Werner Nagel and Viola Weiss, *Crack STIT tessellations: Characterization of stationary random tessellations stable with respect to iteration*, Advances in Applied Probability **37** (2005), no. 4, 859–883.

[82] ———, *Mean values for homogeneous STIT tessellations in 3D*, Image Analysis & Stereology **27** (2008), no. 1, 29–37.

[83] Georges Nassif, Catherine Gloaguen, and Philippe Martins, *A Combined Stochastic and Physical Framework for Modeling Indoor 5G Millimeter Wave Propagation*, arXiv:2002.05162 (2020).

[84] Joachim Ohser and Frank Mücklich, *Statistical analysis of microstructures in materials science*, Statistics in practice, Wiley, Chichester, 2000.

[85] Atsuyuki Okabe, Barry N. Boots, Kokichi Sugihara, Sung Nok Chiu, and D. G. Kendall, *Spatial tessellations: Concepts and applications of Voronoi diagrams*, second edition ed.,

Wiley series in probability and statistics, John Wiley & Sons, Chichester and New York and Weinheim and Brisbane and Singapore and Toronto, 2008.

[86] Frank W. J. Olver, *The NIST handbook of mathematical functions*, Cambridge Univ. Press, Cambridge, 2010.

[87] Takashi Owada and D. Yogeshwaran, *Sub-tree counts on hyperbolic random geometric graphs*, arXiv:1802.06105 (2018).

[88] Giovanni Peccati and Matthias Reitzner (eds.), *Stochastic Analysis for Poisson Point Processes: Malliavin Calculus, Wiener-Itô Chaos Expansions and Stochastic Geometry*, Bocconi & Springer Series, Mathematics, Statistics, Finance and Economics, vol. 7, Springer, Cham, 2016.

[89] Giovanni Peccati and Cengbo Zheng, *Multi-Dimensional Gaussian Fluctuations on the Poisson Space*, Electronic Journal of Probability **15** (2010), no. 48, 1487–1527.

[90] Mathew D. Penrose and J. E. Yukich, *Limit theory for point processes in manifolds*, The Annals of Applied Probability **23** (2013), no. 6, 2161–2211.

[91] Sylvain Porret-Blanc, *Sur le caractère borné de la cellule de Crofton des mosaïques de géodésiques dans le plan hyperbolique*, Comptes Rendus Mathematique (2007), no. 344, 477–481.

[92] John G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, vol. 149, Springer New York, 2006.

[93] ———, *Foundations of Hyperbolic Manifolds*, vol. 149, Springer International Publishing, Cham, 2019.

[94] Claudia Redenbach, *Modelling foam structures using random tessellations*, In: Proceeding of the 10th European Conference of ISS, ESCULAPIO Pub. Co (2009).

[95] Andreas Reichenbacher, *Random tessellations: Stereological formulae in Euclidean space and Kendall's Problem in spherical space*, PhD Thesis, Karlsruhe Institute of Technology (2016).

[96] Matthias Reitzner and Matthias Schulte, *Central limit theorems for U-statistics of Poisson point processes*, The Annals of Probability **41** (2013), no. 6, 3879–3909.

[97] Matthias Reitzner, Matthias Schulte, and Christoph Thäle, *Limit theory for the Gilbert graph*, Adv. in Appl. Math. **88** (2017), 26–61.

[98] B. D. Ripley, *Modelling Spatial Patterns*, Journal of the Royal Statistical Society: Series B (Methodological) **39** (1977), no. 2, 172–192.

[99] W. Rother and M. Zähle, *Palm Distributions in Homogeneous Spaces*, Mathematische Nachrichten **149** (1990), no. 1, 255–263.

[100] L. A. Santaló and I. Yañez, *Averages for polygons formed by random lines in Euclidean and hyperbolic planes*, Journal of Applied Probability **9** (1972), no. 1, 140–157.

[101] Luis Antonio Santaló, *Integral geometry and geometric probability*, second edition ed., Cambridge mathematical library, Cambridge University Press, Cambridge, 2004.

[102] Rolf Schneider, *Convex bodies: The Brunn-Minkowski theory*, second edition ed., Encyclopedia of mathematics and its applications, vol. volume 151, Cambridge University Press, Cambridge, 2014.

[103] Rolf Schneider and Wolfgang Weil, *Stochastic and Integral Geometry*, Probability and Its Applications, Springer-Verlag Berlin Heidelberg, Berlin, Heidelberg, 2008.

[104] Tomasz Schreiber and Christoph Thäle, *Second-order theory for iteration stable tessellations*, Probability and Mathematical Statistics **2012**, no. 32, 281–300.

[105] ———, *Second-order properties and central limit theory for the vertex process of iteration infinitely divisible and iteration stable random tessellations in the plane*, Advances in Applied Probability **42** (2010), no. 4, 913–935.

[106] ———, *Geometry of iteration stable tessellations: Connection with Poisson hyperplanes*, Bernoulli **19** (2013), no. 5A, 1637–1654.

[107] ———, *Limit theorems for iteration stable tessellations*, The Annals of Probability **41** (2013), no. 3B, 2261–2278.

[108] Matthias Schulte, *Malliavin-Stein Method in Stochastic Geometry*, PhD Thesis, University of Osnabrück (2013).

[109] ———, *Normal Approximation of Poisson Functionals in Kolmogorov Distance*, Journal of Theoretical Probability **29** (2016), no. 1, 96–117.

[110] Gil Solanes, *Integral geometry and curvature integrals in hyperbolic space*, PhD Thesis, Universitat Autonoma de Barcelona, Spain (2003).

[111] Stefan Glasauer, *Integralgeometrie konvexer Körper im sphärischen Raum*, PhD Thesis, Karlsruhe Institute of Technology (1995).

[112] Dietrich Stoyan, Sung Nok Chiu, W. S. Kendall, and Joseph Mecke, *Stochastic geometry and its applications*, third edition ed., Wiley series in probability and statistics, John Wiley & Sons Inc, Chichester, West Sussex, United Kingdom, 2013.

[113] Murad S. Taqqu and Giovanni Peccati, *Wiener Chaos: Moments, Cumulants and Diagram Formulae: A survey with computer implementation (Bocconi Springer Series, Vol 1)*, 1. aufl. ed., Bocconi & Springer Series, 1, Springer Verlag Italia, s.l., 2011.

[114] Johan Tykesson and Pierre Calka, *Asymptotics of visibility in the hyperbolic plane*, Advances in Applied Probability **45** (2013), no. 2, 332–350.

[115] Guofang Wang and Chao Xia, *Isoperimetric type problems and Alexandrov-Fenchel type inequalities in the hyperbolic space*, Advances in Mathematics **259** (2014), 532–556.

[116] V. Yaskin, *The Busemann-Petty problem in hyperbolic and spherical spaces*, Advances in Mathematics **203** (2006), no. 2, 537–553.

[117] M. Zähle, *Random processes of Hausdorff rectifiable closet sets*, Mathematische Nachrichten **108** (1982), no. 1, 49–72.