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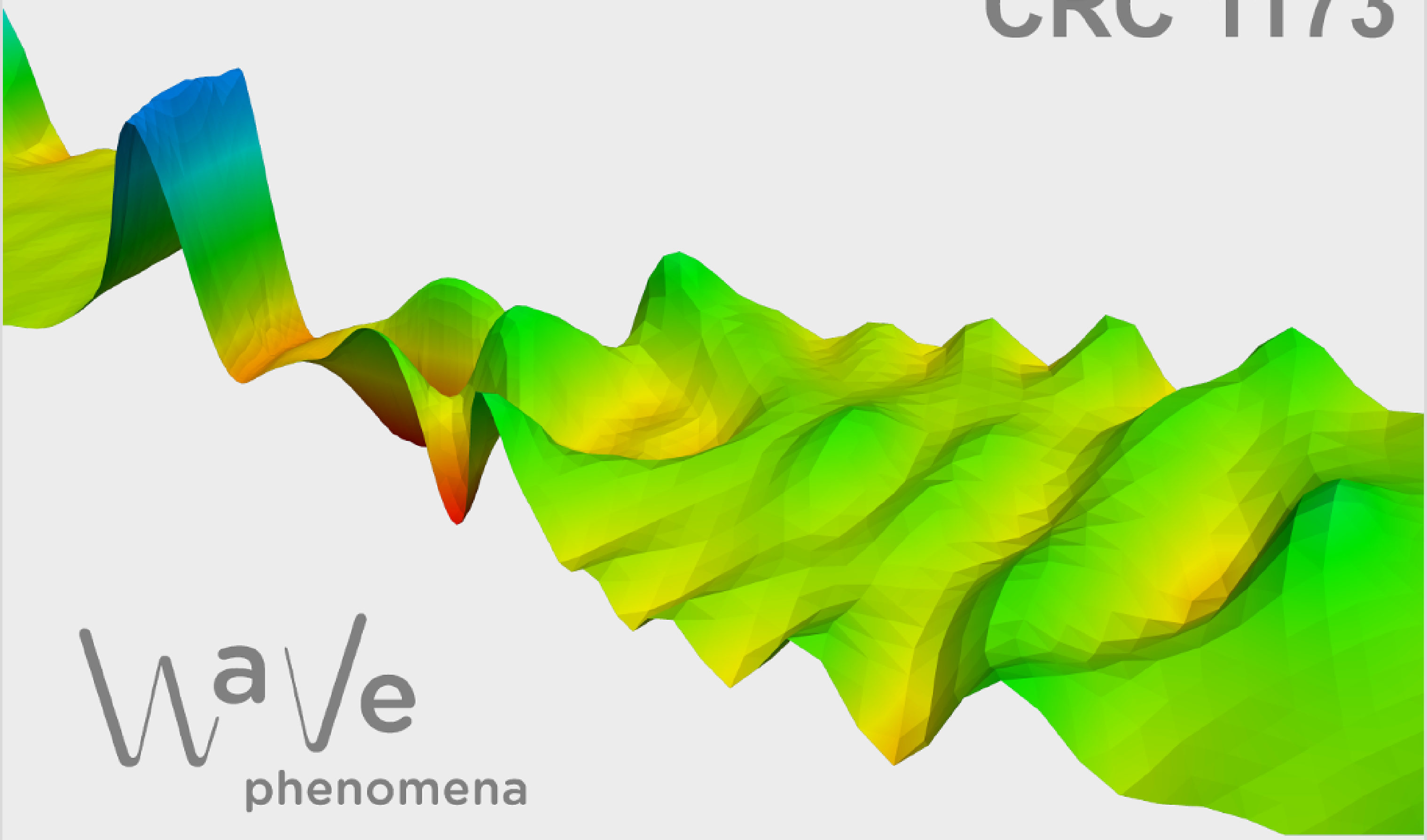

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TANGENTIAL CONE CONDITION AND LIPSCHITZ STABILITY FOR THE FULL WAVEFORM FORWARD OPERATOR IN THE ACOUSTIC REGIME

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Abstract. Time-domain full waveform inversion (FWI) in the acoustic regime comprises a parameter identification problem for the acoustic wave equation: Pressure waves are initiated by sources, get scattered by the earth's inner structure, and their reflected parts are picked up by receivers located on the surface. From these reflected wave fields the two parameters, density and sound speed, have to be reconstructed. Mathematically, FWI reduces to the solution of a nonlinear and ill-posed operator equation involving the parameter-to-solution map of the wave equation. Newton-like iterative regularization schemes are well suited and well analyzed to tackle this inverse problem. Their convergence results are often based on an assumption about the nonlinear map known as tangential cone condition. In this paper we verify this assumption for a semi-discrete version of FWI where the searched-for parameters are restricted to a finite dimensional space. As a byproduct we establish that the semi-discrete seismic inverse problem is Lipschitz stable, in particular, it is conditionally well-posed.

1. Introduction

The analysis of Newton-like methods for regularizing nonlinear ill-posed problems in Hilbert or Banach spaces often relies on a structural assumption which is known under the name *tangential cone condition* (TCC), see, e.g., [9, 11, 14, 16, 19]. It can be traced back to [17] and reads: Let $F\colon \mathsf{D}(F) \subset V \to W$ be the underlying nonlinear operator between Banach spaces which we assume to be Fréchet-differentiable (F-differentiable) with F-derivative F'. Then, F satisfies the TCC at $x^+ \in \mathrm{int}(\mathsf{D}(F))$ if

$$\|F(v) - F(w) - F'(w)(v-w)\|_W \le \eta \, \|F(v) - F(w)\|_W \quad \text{for all } v, w \in B_\rho(x^+)$$

for an $\eta < 1$ (sufficiently small) where $B_\rho(x^+)$ is the open ball in V of radius $\rho > 0$ about x^+ (sometimes the TCC is formulated in a ball with respect to a Bregman distance).

In the fully continuous (infinite dimensional) setting only a few academic examples are known where the TCC holds. However, in a semi-discrete setting the situation is more relaxed. For instance, a semi-discrete TCC has been derived for the inverse problem of the complete electrode model in 2D-electrical impedance tomography [13]. It turns out that injectivity of $F'(x^+)$ is essentially sufficient not only to yield a semi-discrete TCC but also a Lipschitz stability like

$$\|v - w\|_W \le c \|F(v) - F(w)\|_W \quad \text{for all } v, w \in B_\rho(x^+)$$

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where $c > 0$ is a constant. We will demonstrate this implication under rather general assumptions. Semi-discrete Lipschitz estimates and conditional well-posedness for various inverse problems have already been derived, e.g., in [1, 2, 3, 4, 5] and we add a variant of the seismic inverse problem to this list, namely time-domain full waveform inversion (FWI) in the acoustic regime. FWI entails the seismic inverse problem of recovering subsurface material parameters from partial measurements of reflected wave fields which are initiated by external sources. The searched-for parameters in the acoustic regime are pressure wave speed and bulk density. If we confine these parameters to suitable finite dimensional spaces, the F-derivative of the resulting parameter-to-solution map is in fact one-to-one.

The presentation of our findings is organized as follows. In the next section we set the stage by introducing the acoustic wave equation as a first order system and by recalling existence and uniqueness results. Then, in Section 3 we define our semi-discrete model where sources are fired in one part Σ of the propagation medium D and the resulting wave fields are recorded at a possibly different part Ω of D. Here, sound speed and bulk density are expressed as linear combinations of smooth basis functions which are locally independent in D: if a linear combination vanishes on an open subset of D it vanishes globally in D. For this model we formulate the seismic inverse problem and characterize the F-derivative of the forward map by a different but akin acoustic wave equation. For this wave equation we show a fundamental property in Proposition 3.1: there is a source supported in Σ such that the wave field of the F-derivative does not vanish identically on Ω. The proof is based on Holmgren's uniqeness theorem and the propagation of singularities along bicharacteristics of the wave operator. Finally, Section 4 presents the main result (Theorem 4.4) with all its preparatory work and with a remarkable uniqueness statement for semi-discrete FWI: the partial measurements of the reflected wave field determine uniquely both, density and sound speed, where only one single source has to be fired (Remark 4.5).

So as not to distract the reader from the overall picture we kept the main part of the paper rather short by moving technical and auxiliary material to three appendices: Appendix A contains the proof of Proposition 3.1. In Appendix B we prove Lipschitz continuity of the F-derivative of the forward map within an abstract setting. Therefore, Theorem B.2 covers other first order systems as well. The final Appendix C includes likewise an auxiliary statement which is nevertheless interesting in its own right: a semi-discrete mapping whose F-derivative is injective and continuous, satisfies the TCC and is Lipschitz stable (Lemma C.1).

2. The setting

We consider the acoustic wave equation as a first order system. Let $p\colon [0,\infty) \times D \to \mathbb{R}$ and $\mathbf{v}\colon [0,\infty) \times D \to \mathbb{R}^d$, $d \in \{2,3\}$, be the pressure and the velocity field, respectively, where $D \subset \mathbb{R}^d$ is a bounded, connected domain with a piecewise $\mathcal{C}^1$-boundary. Then,

$$c(x)\,\partial_t p(t,x) = \operatorname{div} \mathbf{v}(t,x) + f(t,x) \qquad \text{in } [0,\infty) \times D, \tag{2.1}$$

$$\varrho(x)\partial_t \mathbf{v}(t,x) = \nabla p(t,x) \qquad \text{in } [0,\infty) \times D, \tag{2.2}$$

with initial values $p(0,\cdot) = p_0$ and $\mathbf{v}(0,\cdot) = \mathbf{v}_0$. Here, $c, \varrho\colon D \to (0,\infty)$, $f\colon [0,\infty) \times D \to \mathbb{R}$. Note that $1/\sqrt{c\varrho}$ is the wave speed and ϱ denotes the bulk density. Further, (2.1)-(2.2) can be written as initial value problem

$$B\partial_t u = -Au + \widetilde{f}(t), \quad u(0) = \begin{pmatrix} p_0 \\ \mathbf{v}_0 \end{pmatrix} =: u_0, \tag{2.3}$$

where $u(t) = \big(p(t,\cdot), \mathbf{v}(t,\cdot)\big)$, $\widetilde{f}(t) = \big(f(t,\cdot), 0\big)$,

$$B = \begin{pmatrix} c & 0 \\ 0 & \varrho \mathbf{I}_3 \end{pmatrix}, \quad \text{and} \quad A = -\begin{pmatrix} 0 & \operatorname{div} \\ \nabla & 0 \end{pmatrix}. \tag{2.4}$$

Let us define the space $X = L^2(D) \times L^2(D, \mathbb{R}^d)$ and its subset

$$\mathsf{D}(A) := \big\{(p, \mathbf{v}) \in H^1(D) \times H^1(\operatorname{div}, D) : \ \mathbf{n} \cdot \mathbf{v}|_{\partial D_N} = 0, \ p|_{\partial D_D} = 0\big\} \tag{2.5}$$

with $\partial D = \partial D_D \,\dot{\cup}\, \partial D_N$. The operator $A \colon \mathsf{D}(A) \subset X \to X$ is maximal monotone, see [6, Chap. 7] for a definition.

If $(p_0, \mathbf{v}_0) \in \mathsf{D}(A)$, $f \in W^{1,1}\big([0,\infty), L^2(D)\big)$, and

$$c, \varrho \in \mathcal{P} := \{\lambda \in L^\infty(D) : 0 < \lambda_- < \lambda(\cdot) < \lambda_+ < \infty \text{ a.e.}\} \tag{2.6}$$

then (2.1)-(2.2) admit a unique classical solution $(p, \mathbf{v}) \in \mathcal{C}\big([0,\infty), \mathsf{D}(A)\big) \cap \mathcal{C}^1\big([0,\infty), X\big)$, see, e.g., [12].

If $(p_0, \mathbf{v}_0) \in X$, $f \in L^1_{\text{loc}}\big([0,\infty), L^2(D)\big)$ then (2.1)-(2.2) admit a unique mild/weak solution $u \in \mathcal{C}\big([0,\infty), X\big)$ which – in the notation of (2.3) – satisfies

$$Bu(t) = Bu_0 + A\int_0^t u(s)\mathrm{d}s + \int_0^t \widetilde{f}(s)\mathrm{d}s, \tag{2.7}$$

see, e.g., [18, Prop. 2.15].

3. The semi-discrete full waveform forward map

Let $(p_0, \mathbf{v}_0) \in \mathsf{D}(A)$ and $f \in W^{1,1}\big([0,T], L^2(\Sigma)\big)$ where $\Sigma \subset D$ is an open set where the sources can be initiated. As we can recover only finitely many degrees of freedom we restrict the parameters of (2.1)-(2.2) to a finite dimensional space. To this end we set

$$V := \operatorname{span}\{\varphi_j : j = 1, \dots, M\} \subset \mathcal{C}^1(\overline{D})$$

where the functions $\{\varphi_j : j = 1, \dots M\}$ are *locally independent* over D, that is, if a linear combination vanishes on a nonempty open subset Ω of D then the linear combination must be trivial:

$$\sum_{j=1}^M a_j \varphi_j|_\Omega = 0 \quad \Longrightarrow \quad a_j = 0, \ j = 1, \dots, M. \tag{3.1}$$

Concrete examples for V include:

(1) Polynomials: $V = \Pi_N(\mathbb{R}^d)$, the space of d-variate polynomials of total degree N. Here, the dimension of V is $M = \binom{N+d}{d}$.
(2) Real-analytic radial basis functions: Let $\varphi \colon \mathbb{R}^d \to \mathbb{R}$ be a positive definite and radially symmetric function, see, e.g., [23, Chap. 6]. For pairwise different knots $\xi_j \in D$, $j = 1, \dots, M$, the translates $\{\varphi(\cdot - \xi_j) : j = 1, \dots M\}$ are linear independent over D. If φ is additionally real-analytic then these translates are also locally independent. In fact, any linear combination of these translates is itself an analytic function and as such zero everywhere in D if it vanishes on a nonempty open subset.

For instance, the Gaussian $\varphi(x) = \exp(-\gamma|x|^2)$, $\gamma > 0$, and the multiquadrics $\varphi(x) = 1/(1+|x|^2)^\beta$, $\beta > 0$, have the required properties and are, moreover, positive.

With $V_+ := V \cap \mathcal{P}$ we define the parameter-to-solution (parameter-to-state) map by

$$F\colon V_+^2 \subset V^2 \to \mathcal{C}\big([0,T], X\big), \quad (c,\varrho) \mapsto (p, \mathbf{v}),$$

where $(p,\mathbf{v})$ solves (2.1)-(2.2). Note that F is well defined and F-differentiable. Its F-derivative $F'\colon V_+^2 \subset V^2 \to \mathcal{L}\big(V^2, \mathcal{C}([0,T],X)\big)$ is given by

$$F'(c,\varrho)[h_1,h_2] = (\overline{p}, \overline{\mathbf{v}})$$

where $(\overline{p}, \overline{\mathbf{v}}) \in \mathcal{C}\big([0,T],X\big)$ is the mild solution of

$$c(x)\,\partial_t \overline{p}(t,x) = \operatorname{div} \overline{\mathbf{v}}(t,x) - h_1(x)\,\partial_t p(t,x) \qquad \text{in } [0,T]\times D, \tag{3.2}$$

$$\varrho(x)\,\partial_t \overline{\mathbf{v}}(t,x) = \nabla \overline{p}(t,x) - h_2(x)\,\partial_t \mathbf{v}(t,x) \qquad \text{in } [0,T]\times D, \tag{3.3}$$

with $\overline{p}(0,\cdot) = 0$, $\overline{\mathbf{v}}(0,\cdot) = 0$ and $(p,\mathbf{v}) = F(c,\varrho)$, see, e.g., [12].

In seismic exploration only part of the wave field can be measured. To model this fact, we introduce the observation (restriction) operator $\Psi\colon \mathcal{C}\big([0,T],X\big) \to \mathcal{C}\big([0,T],X_\Omega\big)$, $X_\Omega := L^2(\Omega) \times L^2(\Omega,\mathbb{R}^d)$, $\Psi(p,\mathbf{v}) = (p|_\Omega, \mathbf{v}|_\Omega)$, where $\Omega \subset D$ is open, nonempty and connected.

The following property of the wave system (3.2)-(3.3) is fundamental for our main result in Theorem 4.4 below. Its technical and somewhat lengthy proof is content of Appendix A. For its formulation we introduce the space

$$W_0^{2,1} := \big\{ f \in W^{2,1}\big([0,T], L^2(\Sigma)\big) : f(0) = f'(0) = 0 \big\}.$$

Proposition 3.1. *Suppose that $h \in V^2 \backslash \{0\}$. If $T > 0$ is sufficiently large then there exists an $f \in W_0^{2,1}$ with $\operatorname{supp} f \subset (0,T) \times \Sigma$ such that the mild solution $(\overline{p}|_\Omega, \overline{\mathbf{v}}|_\Omega)$ of (3.2)-(3.3) is not identically zero in $(0,T)$. This f may depend on (c,ϱ) but not on h.*

We set $\Phi = \Psi \circ F$. Then, the semi-discrete inverse seismic problem in the acoustic regime consists in finding $(c,\varrho) \in V_+^2$ such that

$$\Phi(c,\varrho) \approx (\widetilde{p}, \widetilde{\mathbf{v}})$$

for the measured wave field $(\widetilde{p}, \widetilde{\mathbf{v}}) \in X_\Omega$. Note that

$$\Phi'(c,\varrho)[h_1,h_2] = \Psi F'(c,\varrho)[h_1,h_2] = (\overline{p}|_\Omega, \overline{\mathbf{v}}|_\Omega).$$

Remark 3.2. *From a numerical point of view, our choice of global basis functions for discretizing c and ϱ seems a bit far-fetched. The straightforward approach would be, for instance, to take indicator functions subordinate to the mesh of the used finite element discretization of (2.1)-(2.2). In Remark A.3 of Appendix A we will address this issue in greater detail.*

4. Injectivity, Lipschitz stability, and tangential cone condition

In a first step towards the tangential cone condition for Φ we verify injectivity of $\Phi'(c,\varrho)$. We cast the injectivity problem into an operator framework by setting

$$p_0 = 0 \quad \text{and} \quad \mathbf{v}_0 = 0 \tag{4.1}$$

(the environment is at rest before we fire the source). Then the map $f \mapsto (p, \mathbf{v})$ is linear and we redefine Φ by

$$\widetilde{\Phi}\colon V_+^2 \subset V^2 \to \underbrace{\mathcal{L}\big(W_0^{2,1}, \mathcal{C}\big([0,T], X_\Omega\big)\big)}_{=:\,\mathcal{W}}, \quad (c,\varrho) \mapsto \big(f \mapsto \Psi(p,\mathbf{v})\big),$$

that is, $\widetilde{\Phi}(c,\varrho)f = (p|_\Omega, \mathbf{v}|_\Omega)$ where $(p,\mathbf{v})$ solves (2.1)-(2.2). The F-derivative $\widetilde{\Phi}'(c,\varrho) \in \mathcal{L}(V^2, \mathcal{W})$ is still given via (3.2)-(3.3). Indeed,

$$\widetilde{\Phi}'(c,\varrho)[h]f = (\overline{p}|_\Omega, \overline{\mathbf{v}}|_\Omega) \tag{4.2}$$

where $h = (h_1, h_2) \in V^2$.

The injectivity of $\widetilde{\Phi}'$ is a direct consequence of Proposition 3.1.

Corollary 4.1. *The F-derivative $\widetilde{\Phi}'(c,\varrho) \in \mathcal{L}(V^2,\mathcal{W})$ is an injective mapping and we have that*

$$\min\big\{\|\widetilde{\Phi}'(c,\varrho)[h]\|_{\mathcal{W}} : h \in V^2,\ \|h\|_{V^2} = 1\big\} > 0. \tag{4.3}$$

Proof. Assume the minimum to be zero. As V^2 is finite dimensional and $\widetilde{\Phi}'(c,\varrho)$ is continuous, there is a normalized $h \in V^2$ such that $\widetilde{\Phi}'(c,\varrho)[h]f = 0$ for all $f \in W_0^{2,1}$. But then $h = 0$ by Proposition 3.1 contradicting $\|h\|_{V^2} = 1$. □

Theorem 4.2. *We have Lipschitz continuity of the F-derivative $V_+^2 \ni (c,\varrho) \mapsto \widetilde{\Phi}'(c,\varrho) \in \mathcal{L}(V^2,\mathcal{W})$, that is,*

$$\|\widetilde{\Phi}'(c_1,\varrho_1) - \widetilde{\Phi}'(c_2,\varrho_2)\|_{\mathcal{L}(V^2,\mathcal{W})} \lesssim \|(c_1,\varrho_1) - (c_2,\varrho_2)\|_{V^2}.[1] \tag{4.4}$$

The involved constant only depends on T, λ_-, and λ_+.

Proof. See Appendix B. □

Corollary 4.3. *For $(c^+,\varrho^+) \in V_+^2$ there exist an open ball $B_r(c^+,\varrho^+) \subset V_+^2$ with radius $r > 0$ and a positive constant $\mathrm{m} = \mathrm{m}(c^+,\varrho^+,r,T,\lambda_-,\lambda_+)$ such that*

$$\|\widetilde{\Phi}'(c,\varrho)[h]\|_{\mathcal{W}} \geq \mathrm{m}\,\|h\|_{V^2} \quad \textit{for all } h \in V^2 \textit{ and all } (c,\varrho) \in B_r(c^+,\varrho^+).$$

Proof. The assertion follows immediately from the previous corollary and theorem. Indeed, let $L = L(T,\lambda_-,\lambda_+)$ be the constant in (4.4) and let $\widetilde{\mathrm{m}} > 0$ be the minimum from (4.3). Choose $0 < r < \widetilde{\mathrm{m}}/(2L)$. Then, for all $(c,\varrho) \in B_r(c^+,\varrho^+)$ we have – using the reverse triangle inequality – that

$$\begin{aligned}\|\widetilde{\Phi}'(c,\varrho)[h]\|_{\mathcal{W}} &\geq \big|\|\widetilde{\Phi}'(c^+,\varrho^+)[h]\|_{\mathcal{W}} - \|\widetilde{\Phi}'(c^+,\varrho^+)[h] - \widetilde{\Phi}'(c,\varrho)[h]\|_{\mathcal{W}}\big| \\ &\geq \widetilde{\mathrm{m}}\|h\|_{V^2} - L\,r\|h\|_{V^2} \geq (\widetilde{\mathrm{m}}/2)\|h\|_{V^2}\end{aligned}$$

and $\mathrm{m} = \widetilde{\mathrm{m}}/2$ does the job. □

Lipschitz stability and the tangential cone condition for $\widetilde{\Phi}$ follow now immediately from Lemma C.1 of Appendix C.

Theorem 4.4. *For $(c^+,\varrho^+) \in V_+^2$ there exist an open ball $B_r(c^+,\varrho^+) \subset V_+^2$ such that*

$$\|(c_1,\varrho_1) - (c_2,\varrho_2)\|_{V^2} \lesssim \big\|\widetilde{\Phi}(c_1,\varrho_1) - \widetilde{\Phi}(c_2,\varrho_2)\big\|_{\mathcal{W}}$$

and

[1]The notation $A \lesssim B$ indicates the existence of a generic constant $c > 0$ such that $A \leq c\,B$.

$$\big\|\widetilde{\Phi}(c_1,\varrho_1)-\widetilde{\Phi}(c_2,\varrho_2)-\widetilde{\Phi}'(c_2,\varrho_2)[(c_1,\varrho_1)-(c_2,\varrho_2)]\big\|_{\mathcal{W}} \lesssim \|(c_1,\varrho_1)-(c_2,\varrho_2)\|_{V^2}\,\big\|\widetilde{\Phi}(c_1,\varrho_1)-\widetilde{\Phi}(c_2,\varrho_2)\big\|_{\mathcal{W}}$$

for all $(c_i,\varrho_i)\in B_r(c^+,\varrho^+)$, $i=1,2$.

Proof. Apply Lemma C.1 with $\Theta=\widetilde{\Phi}$, $\mathsf{D}(\Theta)=V^2_+$, $\mathcal{X}=V^2$, and $\mathcal{Y}=\mathcal{W}$. The necessary assumptions are satisfied according to Theorem B.2 and Corollary 4.3. Further, $\alpha=1$ due to (4.4). $\square$

Remark 4.5. *At the end of Section 3 we have introduced the parameter-to-solution map*

$$\Phi\colon V^2_+\subset V^2\to \mathcal{C}\big([0,T],X\big),\quad (c,\varrho)\mapsto (p|_\Omega,\mathbf{v}|_\Omega),$$

for one fixed source f in (2.1)-(2.2). In the language of the geophysical community, Φ models a one-shot experiment whereas $\widetilde{\Phi}$ describes a multi-shot experiment.

Theorem 4.4 holds accordingly for Φ provided the fired single source f coincides with one of those whose existence for (c^+,ϱ^+) is guaranteed by Proposition 3.1. To put it differently: for any dimension of V, the pair $(c^+,\varrho^+)\in V^2_+$ is in principle uniquely determined from a single shot experiment (when the 'right' source is fired).

Appendix A. Proof of Proposition 3.1

The following result will be crucial. It is a global version of Holmgren's uniqueness theorem for a hyperbolic equation with $\mathcal{C}^1$-coefficients.

Lemma A.1. *Suppose that $(p,\mathbf{v})\in L^2((0,T)\times D)$ is a weak solution to the homogeneous system*

$$\begin{aligned} c(x)\,\partial_t p(t,x) &= \operatorname{div}\mathbf{v}(t,x)\\ \varrho(x)\,\partial_t \mathbf{v}(t,x) &= \nabla p(t,x)\end{aligned}\qquad \text{in } [0,T]\times D,$$

with coefficients $c,\varrho\in V^2_+$. Let $E\subset D$ be open and nonempty. If $(p,\mathbf{v})\equiv 0$ in $(0,T)\times E$ for sufficiently large T, then there exists $T_1\in(0,T)$ such that

$$(p,\mathbf{v})\equiv 0 \ \text{ in } \left(\frac{T-T_1}{2},\frac{T+T_1}{2}\right)\times D.$$

This lemma is essentially the Holmgren-John-Tataru theorem. Note that the time T_1 can be made precise, see, e.g., [8, 15].

Even though, Holmgren's theorem applies also to systems, it will be advantageous to reduce the system (2.1)-(2.2) to two decoupled wave equations of second order:

$$\begin{aligned} c\,\partial_t^2 p &= \operatorname{div}\Big(\frac{1}{\varrho}\nabla p\Big)+\partial_t f,\\ \varrho\,\partial_t^2\mathbf{v} &= \nabla\Big(\frac{1}{c}\operatorname{div}\mathbf{v}\Big)+\nabla\frac{f}{c}.\end{aligned}\tag{A.1}$$

By (2.2) with zero initial data (4.1) we have that $\nabla\times(\varrho\mathbf{v})=0$. Hence,

$$\begin{aligned}\nabla\Big(\frac{1}{c}\operatorname{div}\mathbf{v}\Big) &= \nabla\Big(\frac{1}{c}\operatorname{div}\mathbf{v}\Big)-\frac{1}{c\varrho}\nabla\times(\nabla\times(\varrho\mathbf{v}))\\ &=\frac{1}{c}\Delta\mathbf{v}+\Big(\nabla\frac{1}{c}\Big)\operatorname{div}\mathbf{v}-\frac{1}{c\varrho}\big[\nabla\varrho\times(\nabla\times\mathbf{v})+\nabla\times(\nabla\varrho\times\mathbf{v})\big],\end{aligned}$$

which shows that the second-order system for $\mathbf{v}$ has a second-order hyperbolic operator as its principal part. This has the advantage that we can use also Tataru's uniqueness theorem which works only for scalar operators [20, 21]. Even though (A.1) is a second-order system, its principal part is a scalar second-order hyperbolic operator. With a small adjustment, Tataru's approach can be applied [7].

Remark A.2. *The classical Holmgren-John theorem applies only to operators with analytic coefficients. In the case of the wave equation Tataru proved that the conclusion of Holmgren-John is true for coefficients which are analytic in time and $\mathcal{C}^1$ in space or vice versa.*

The principal symbol of each equation in (A.1) is $q(x;\tau,\xi) = \tau^2 - |\xi|^2/(c\varrho)(x)$. Hence, we introduce a distance function (metric) in D by setting

$$\text{dist}(x,y) = \inf_{\gamma} \int_{\alpha}^{\beta} \sqrt{\varrho(\gamma(t))c(\gamma(t))}\,|\gamma'(t)|\,\mathrm{d}t \tag{A.2}$$

where the infimum is taken over all smooth curves γ in D satisfying $\gamma(\alpha) = x$ and $\gamma(\beta) = y$. Let

$$\text{dist}(x,E) = \inf_{y\in E} \text{dist}(x,y) \quad \text{and} \quad \text{dist}(D,E) = \sup_{x\in D} \text{dist}(x,E).$$

If $T > 2\,\text{dist}(D,E)$, then the function $(p,\mathbf{v})$ vanishes at 'half time', that is,

$$(p,\mathbf{v})(T/2,x) = 0 \quad \text{for all } x \in D,$$

see [8, Theorem 1.1]. In order to obtain a time interval where $(p,\mathbf{v})$ vanishes one needs to increase T. Let $T_1 \in (0,T)$. If $T > T_1 + 2\,\text{dist}(D,E)$, then

$$(p,\mathbf{v})(t,x) = 0 \quad \text{for all } (t,x) \in \left(\frac{T-T_1}{2}, \frac{T+T_1}{2}\right) \times D.$$

Now we turn to the actual proof of Proposition 3.1 which is devided into two steps.

First, we will show that there exists a forcing term f such that for sufficiently large $T > 0$, the wave field $u = (p,\mathbf{v})$ does not vanish in $(0,T)\times\Omega$. We argue by contradiction. Suppose that $u \equiv 0$ in $(0,T)\times\Omega$. Then, by Lemma A.1 there exists $T_1 \in (0,T)$ such that

$$u \equiv 0 \text{ in } \left(\frac{T-T_1}{2}, \frac{T+T_1}{2}\right) \times (D\backslash\Sigma). \tag{A.3}$$

Replacing the t variable by $t-(T-T_1)/2$, we work in the space time cylinder $(0,T_1)\times D$ instead of $\left(\frac{T-T_1}{2}, \frac{T+T_1}{2}\right) \times D$.

Let $f(t,x) = \lambda(t)g(x)$ where $\lambda \in \mathcal{C}_0^\infty(0,T_1)$ and $g \in H^1(D)$ with support in Σ, and consider the initial-boundary value problem

$$c\,\partial_t^2\widetilde{p} = \text{div}\left(\frac{1}{\varrho}\nabla\widetilde{p}\right) \qquad \text{in } (0,T_1)\times D, \tag{A.4}$$

with initial data $\widetilde{p}(0,x) = g(x)/c(x)$, $\partial_t\widetilde{p}(0,x) = 0$, and boundary data

$$\widetilde{p} = 0 \text{ on } (0,T_1)\times\partial D_D \quad \text{and} \quad \partial_\nu\widetilde{p} = 0 \text{ on } (0,T_1)\times\partial D_N.$$

The boundary data are inferred from (2.1)-(2.2). This problem has a unique solution $\widetilde{p} \in \mathcal{C}([0,T],H^1(D))$. Furthermore, we define

$$\widetilde{\mathbf{v}}(t,x) := \frac{1}{\varrho}\int_0^t \nabla\widetilde{p}(s,x)\,\mathrm{d}s. \tag{A.5}$$

Then $\widetilde{u} = (\widetilde{p},\widetilde{\mathbf{v}})$ satisfies system (2.1)-(2.2) with $f = 0$.

We will use propagation of singularities to establish that $\widetilde{u}$ is not zero in $(0,T)\times(D\backslash\Sigma)$. Indeed, the singularities of the initial data g will travel along the null bicharacteristics of the hyperbolic operator $\varrho c\,\partial_t^2-\Delta$ which is the principal part of (A.4), see, e.g., [22] or [10, Chap. 23].

The null bicharacteristics $\gamma\colon \mathbb{R}\to((0,T_1)\times D)\times(\mathbb{R}\times\mathbb{R}^d)$ are integral curves of the vector field $(\nabla_{\tau,\xi}q,-\nabla_{t,x}q)$ satisfying $q\circ\gamma=0$. Setting $\gamma(s)=(t(s),x(s);\tau(s),\xi(s))$, this gives

$$\frac{\mathrm{d}t}{\mathrm{d}s}=2\tau,\quad \frac{\mathrm{d}x}{\mathrm{d}s}=-\frac{2\xi}{c\varrho},\quad \frac{\mathrm{d}\tau}{\mathrm{d}s}=0,\quad \frac{\mathrm{d}\xi}{\mathrm{d}s}=-\nabla_x\frac{1}{c\varrho}|\xi|^2=\frac{\nabla_x(c\varrho)}{(c\varrho)^2}|\xi|^2,$$

so that $q(x,\tau,\xi)=\tau^2-|\xi|^2/[(c\varrho)(x)]=0$. Let

$$t(0)=0,\quad x(0)=\underline{x},\quad \tau(0)=\underline{\tau},\quad \xi(0)=\underline{\xi}.$$

From $q(\underline{x};\underline{\tau},\underline{\xi})=0$ we infer that $\underline{\tau}=\pm|\underline{\xi}|/\sqrt{(\varrho c)(\underline{x})}$. Hence, over each point $(\underline{x},\underline{\xi})$ at $t=0$ there are two bicharacteristics. Furthermore, from the ODE we infer that $\tau(s)=\underline{\tau}$ for all s and thus, $t=2\underline{\tau}s=\pm2|\underline{\xi}|s/\sqrt{c\varrho}$. So, in both bicharacteristics one can introduce t as a parameter, that is, $\gamma_\pm(t)=(t,x_\pm(t),\pm|\underline{\xi}|/\sqrt{(\varrho c)(\underline{x})},\xi_\pm(t))$. By the chain rule

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\mp\frac{\xi}{|\underline{\xi}|\sqrt{c\varrho}}\quad\text{and}\quad\frac{\mathrm{d}\xi}{\mathrm{d}t}=\pm\frac{\nabla_x(c\varrho)}{(c\varrho)^{3/2}}\frac{|\xi|^2}{|\underline{\xi}|}.$$

If $(\underline{x},\underline{\xi})$ is in the wave front set of g, then the segments of the two null bicharacteristics $\gamma_\pm$ in $(0,T_1)\times D$, with initial $(\underline{x},\underline{\xi})$ at $t=0$, will be in the wave front set of $\widetilde{u}$. The x component of the bicharacteristic is a geodesic of the metric (A.2).

There exist points $(\underline{x},\underline{\xi})\in\Sigma\times\mathbb{R}^d$ such that at least one of the two bicharacteristics with the initial data $(\underline{x},\underline{\xi})$ will satisfy $x_+(t)\in D\backslash\Sigma$ or $x_-(t)\in D\backslash\Sigma$ for some $t>0$. Suppose now that g is supported in Σ and that its wave front set contains such a point. Then the wave front set of the solution $\widetilde{u}=(\widetilde{p},\widetilde{\mathbf{v}})$ must contain the points $x_\pm(t)$ and thus, the solution cannot vanish in $(0,T)\times(D\backslash\Sigma)$.

The solution of (2.1)-(2.2) can now be expressed by Duhamel's principle via

$$u(t,x)=(p,\mathbf{v})(t,x)=\int_0^t\lambda(t-s)\widetilde{u}(s,x)\,\mathrm{d}s.$$

Indeed, one computes

$$\begin{aligned}\partial_t u(t,x)&=\lambda(0)\widetilde{u}(t,x)+\int_0^t\lambda'(t-s)\widetilde{u}(s,x)\,\mathrm{d}s\\&=\int_0^t\lambda(t-s)\partial_s\widetilde{u}(s,x)\,\mathrm{d}s+\lambda(t)\widetilde{u}(0,x)\end{aligned}$$

where we used that $\lambda(0)=0$. Moreover, in view of (A.4) and (A.5) we have that

$$c\,\partial_t\widetilde{p}=\int_0^t\operatorname{div}\left(\frac{1}{\varrho}\nabla\widetilde{p}(s,\cdot)\right)\mathrm{d}s=\int_0^t\operatorname{div}\partial_s\widetilde{\mathbf{v}}(s,\cdot)\,\mathrm{d}s=\operatorname{div}\widetilde{\mathbf{v}}\quad\text{and}\quad\varrho\,\partial_t\widetilde{\mathbf{v}}=\nabla\widetilde{p},$$

which yield

$$\partial_t p=\frac{1}{c}\operatorname{div}\mathbf{v}+\frac{1}{c}f\quad\text{and}\quad\partial_t\mathbf{v}=\frac{1}{\varrho}\nabla p.$$

Since $\widetilde{u}(t,x)$ is not identically zero for all $x\in D\backslash\Sigma$ and $t\in(0,T_1)$, there exists a function $\lambda\in\mathcal{C}_0^\infty(0,T_1)$ such that u will not vanish in $(0,T_1)\times(D\backslash\Sigma)$. This contradicts (A.3) and we have proved that u does not vanish in $(0,T)\times\Omega$.

In the final step of the proof we validate that the solution $(\overline{p}, \overline{\mathbf{v}})$ of (3.2)-(3.3) does not vanish identically on $(0,T) \times \Omega$. Assume the contrary. Then, it follows from (3.2)-(3.3) (or, more precisely, from its integrated version (2.7)) that

$$0 = h_1(x)\partial_t p(t,x) \quad \text{and} \quad 0 = h_2(x)\partial_t \mathbf{v}(t,x) \quad \text{in } [0,T] \times \Omega.$$

Thus, we must have that

$$\partial_t p(t,x) = 0 \quad \text{and} \quad \partial_t \mathbf{v}(t,x) = 0 \text{ in } [0,T] \times \Omega$$

since h_1 and h_2 cannot be identically zero restricted to Ω (otherwise they would vanish on D as well by our assumption (3.1) on the ansatz functions). Recalling the zero initial conditions (4.1) we must have $(p, \mathbf{v}) = 0$ in $[0,T] \times \Omega$ which contradicts our first finding.

Remark A.3. *We come back to the issue raised in Remark 3.2 of local vs. global basis functions for discretizing c and ϱ.*

Suppose we split D into open, connected subsets $\{D_j\}_j$ with piecewise $\mathcal{C}^1$-boundaries:

$$\overline{D} = \bigcup_{j=1}^{M} \overline{D}_j, \quad D_j \cap D_k = \emptyset, \ j \neq k.$$

Let $V_{\text{loc}} := \operatorname{span}\{p_j \chi_{D_j} : j = 1, \dots M\}$ where χ_{D_j} denotes the indicator function of D_j and p_j is a polynomial.

If we now represent c and ϱ in V_{loc}, we can prove, by a slight modification of our arguments from above, that for any $h \in V_{\text{loc}}^2$ there is a forcing term f and a time $T > 0$ such that the solution $(p, \mathbf{v})$ of (2.1)-(2.2) does not vanish in $(0,T) \times (\operatorname{supp} h_1 \cup \operatorname{supp} h_2)$. Thus, for each h we can guarantee that at least one of the forcing terms in (3.2)-(3.3) is active. This result is, however, not sufficient to carry over Proposition 3.1 (and hence Theorem 4.4) to V_{loc}. It remains to show that for each h there is one forcing term f for (2.1)-(2.2) such that the induced forcing terms in (3.2)-(3.3) guarantee $(\overline{p}, \overline{\mathbf{v}})$ not to vanish in $(0,T) \times \Omega$. We strongly conjecture this to be a fact, unfortunately, we are unable to give rigorous arguments at present.

Even if we succeed, Theorem 4.4 might hold only for the multi-shot operator $\widetilde{\Phi}$ since the applied source f depends on h and one source might not serve all $h \in V_{\text{loc}}^2$. This is then in contrast to the global ansatz functions where we could verify the TCC also for the one-shot operator Φ, see Remark 4.5.

Appendix B. A continuity result

In this appendix we verify that $\widetilde{\Phi}' \colon V_+^2 \subset V^2 \to \mathcal{L}(V^2, \mathcal{W})$ defined in (4.2) is a Hölder-continuous mapping. We first provide a result for the abstract evolution equation

$$Bu'(t) + Au(t) = f(t), \quad t \in [0,T], \quad u(0) = u_0, \tag{B.1}$$

in the spirit of [12]. The assumptions are $T > 0$, X Hilbert space,

- $B \in \mathcal{L}^*(X) = \{J \in \mathcal{L}(X) : J^* = J\}$ satisfying
$$\langle Bx, x\rangle_X = \langle x, Bx\rangle_X \geq \beta \|x\|_X^2$$
for some $\beta > 0$ and for all $x \in X$,
- $A \colon \mathsf{D}(A) \subset X \to X$ is *maximal monotone*: $\langle Ax, x\rangle_X \geq 0$ for all $x \in \mathsf{D}(A)$ and $I + A \colon \mathsf{D}(A) \to X$ is onto (I is the identity),
- $f \in L^1\big([0,T], X\big)$, $u_0 \in X$.

Using standard techniques one sees that (B.1) admits a unique mild solution $u \in \mathcal{C}([0,T],X)$ satisfying

$$\|u\|_{\mathcal{C}([0,T],X)} \lesssim \|u_0\|_X + \|f\|_{L^1([0,T],X)} \tag{B.2}$$

where the constant depends on T, $\|B\|$ and $\|B^{-1}\|$.

The following regularity result has been obtained in [12, Theorem 2.6] under more general assumptions on f and u_0.

Theorem B.1. *For some $k \in \mathbb{N}$, let $f \in W^{k,1}([0,T],X)$ with $f^{(\ell)}(0) = 0$, $\ell = 0,\dots,k-1$ (note that $f^{(\ell)}$ is continuous). Let u be the unique mild solution of (B.1) with $u_0 = 0$. Then $u \in \mathcal{C}^k([0,T],X) \cap \mathcal{C}^{k-1}([0,T],\mathsf{D}(A))$ and*

$$\|u\|_{\mathcal{C}^k([0,T],X)} \lesssim \|f\|_{W^{k,1}([0,T],X)} \tag{B.3}$$

where the constant depends on T, $\|B\|$, and $\|B^{-1}\|$.

From now on let $u_0 = 0$. We define the following parameter-to-source-to-solution map related to (B.1):

$$\widetilde{F} \colon \mathsf{D}(\widetilde{F}) \subset \mathcal{L}^*(X) \to \mathcal{S}, \quad B \mapsto (f \mapsto u), \tag{B.4}$$

where

$$\mathcal{S} := \mathcal{L}\big(W_0^{2,1}([0,T],X), \mathcal{C}([0,T],X)\big),$$

$$W_0^{2,1}([0,T],X) := \{f \in W^{2,1}([0,T],X) : f(0) = f'(0) = 0\},$$

and

$$\mathsf{D}(\widetilde{F}) := \{B \in \mathcal{L}^*(X) : \beta_- \|x\|_X^2 \le \langle Bx, x\rangle_X \le \beta_+ \|x\|_X^2\}$$

for given $0 < \beta_- < \beta_+ < \infty$.

Theorem B.2. *The map $\widetilde{F}$ is F-differentiable at $B \in \operatorname{int}(\mathsf{D}(\widetilde{F}))$ where*

$$\widetilde{F}'(B)[H]f = \overline{u} \quad \textit{for } H \in \mathcal{L}^*(X)$$

with $\overline{u} \in \mathcal{C}\big([0,T],X\big)$ being the mild (in fact the classical) solution of

$$B\overline{u}'(t) + A\overline{u}(t) = -Hu'(t), \ t \in [0,T], \quad \overline{u}(0) = 0, \tag{B.5}$$

where u is the classical solution of (B.1) with respect to f.

Moreover, $\widetilde{F}'$ is Lipschitz-continuous, that is,

$$\|\widetilde{F}'(B_1) - \widetilde{F}'(B_2)\|_{\mathcal{L}(\mathcal{L}^*(X),\mathcal{S})} \lesssim \|B_1 - B_2\|_{\mathcal{L}(X)}.$$

The involved constant only depends on T, β_-, and β_+.

Proof. We can be brief in proving F-differentiability as we will rely on results from [12]. A close inspection of the proofs of Lemma 3.3 and Theorem 3.6 of [12] yields, for H sufficiently small, that

$$\frac{1}{\|H\|_{\mathcal{L}(X)}} \|\widetilde{F}(B+H)f - \widetilde{F}(B)f - \widetilde{F}'(B)[H]f\|_{\mathcal{C}([0,T],X)} \lesssim \|H\|_{\mathcal{L}(X)}\, \|f\|_{W^{2,1}([0,T],X)}$$

which is the claimed differentiability.

Now we check the Lipschitz-continuity of $\widetilde{F}'$. To this end let $\overline{u} = \widetilde{F}'(B)[H]f$ and $\overline{v} = \widetilde{F}'(B + \delta B)[H]f$. By the regularity assumptions on f, $\overline{v}$ and $\overline{u}$ are the classical solutions of

$$(B + \delta B)\overline{v}'(t) + A\overline{v}(t) = -Hv'(t), \quad t \in (0,T), \quad \overline{v}(0) = 0,$$

$$B\overline{u}'(t) + A\overline{u}(t) = -Hu'(t), \quad t \in (0,T), \quad \overline{u}(0) = 0,$$

where u solves (B.1) and v solves (B.1) with B replaced by $B + \delta B$. Hence, $d = \overline{v} - \overline{u}$ mildly solves

$$Bd'(t) + Ad(t) = -H(v'(t) - u'(t)) - \delta B\overline{v}'(t), \quad t \in (0,T), \quad d(0) = 0.$$

By the continuous dependency of d on the right hand side, see (B.2), we get

$$\|d\|_{\mathcal{C}([0,T],X)} \lesssim \|H\|_{\mathcal{L}(X)} \|v - u\|_{\mathcal{C}^1([0,T],X)} + \|\delta B\|_{\mathcal{L}(X)} \|\overline{v}\|_{\mathcal{C}^1([0,T],X)}. \tag{B.6}$$

Next we apply the regularity estimate (B.3) to $v - u$ which solves

$$B(v'(t) - u'(t)) + A(v(t) - u(t)) = -\delta Bv'(t) \quad t \in (0,T), \quad v(0) - u(0) = 0.$$

Thus,

$$\|v - u\|_{\mathcal{C}^1([0,T],X)} \lesssim \|\delta B\|_{\mathcal{L}(X)} \|v\|_{\mathcal{C}^2([0,T],X)} \lesssim \|\delta B\|_{\mathcal{L}(X)} \|f\|_{W^{2,1}([0,T],X)}$$

where the right bound comes from the regularity of v. In a similar way we get

$$\|\overline{v}\|_{\mathcal{C}^1([0,T],X)} \lesssim \|H\|_{\mathcal{L}(X)} \|v\|_{\mathcal{C}^2([0,T],X)} \lesssim \|H\|_{\mathcal{L}(X)} \|f\|_{W^{2,1}([0,T],X)}.$$

Plugging these bounds into (B.6) we end up with

$$\sup_{H \in \mathcal{L}^*(X)} \sup_{f \in W_0^{2,1}([0,T],X)} \frac{\|\overline{v} - \overline{u}\|_{\mathcal{C}([0,T],X)}}{\|H\|_{\mathcal{L}(X)} \|f\|_{W^{2,1}([0,T],X)}} \lesssim \|\delta B\|_{\mathcal{L}(X)}$$

which is the claimed Lipschitz-continuity. □

To establish the connection of $\widetilde{\Phi}$ to $\widetilde{F}$ we return to the concrete settings of the previous sections for (2.3) where $X = L^2(D) \times L^2(D, \mathbb{R}^d)$ and B and A are given by (2.4) and (2.5). Now $\widetilde{\Phi} = \Psi \circ \widetilde{F} \circ P$ with the mapping

$$P\colon V_+^2 \subset V^2 \to \mathcal{L}^*(X), \quad (c, \varrho) \mapsto \begin{pmatrix} c & 0 \\ 0 & \varrho\,\mathbf{I}_3 \end{pmatrix}.$$

Note that the image of P is in $\mathsf{D}(\widetilde{F})$ by an appropriate choice of β_- and β_+ in terms of λ_- and λ_+ from (2.6).

Now, the Hölder-continuity (4.4) follows immediately from Theorem B.2 by the chain rule using $P'(c, \varrho)[h] = P(h_1, h_2)$.

Appendix C. Lipschitz stability and tangential cone condition in a semi-discrete setting

The following lemma is of interest independent of its use in this paper, since it provides elementary criteria that imply TCC and Lipschitz stability for semi-discrete mappings.

Lemma C.1. *Let $\Theta\colon \mathsf{D}(\Theta) \subset \mathcal{X} \to \mathcal{Y}$ be an F-differentiable mapping between Banach spaces where $\mathcal{X}$ is finite dimensional. Denote by x^+ an interior point of $\mathsf{D}(\Theta)$ and assume that $\Theta'(x^+)$ has a trivial null space.*

a) *If Θ' is continuous in $B_r(x^+)$ up to the boundary then there is a $\rho > 0$ such that Lipschitz stability holds, that is,*

$$\|v - w\|_{\mathcal{X}} \lesssim \|\Theta(v) - \Theta(w)\|_{\mathcal{Y}} \quad \textit{for all } v, w \in B_\rho(x^+). \tag{C.1}$$

Moreover, the TCC holds as well

$$\|\Theta(v) - \Theta(w) - \Theta'(w)(v - w)\|_{\mathcal{Y}} \le \eta(v, w)\, \|\Theta(v) - \Theta(w)\|_{\mathcal{Y}} \quad \textit{for all } v, w \in B_\rho(x^+) \tag{C.2}$$

where $\eta\colon B_r(x^+)\times B_r(x^+)\to[0,\infty)$ *is a continuous function which vanishes on the diagonal:* $\eta(w,w)=0$.

b) *If* Θ' *is even Hölder continuous of order* $\alpha\in(0,1]$, *i.e.,*

$$\|\Theta'(x)-\Theta'(y)\|_{\mathcal{L}(\mathfrak{X},\mathfrak{Y})}\le L\|x-y\|_{\mathfrak{X}}^{\alpha}\quad \text{for all } x,y\in B_r(x^+), \tag{C.3}$$

for one $L>0$, *then a stronger TCC holds*

$$\|\Theta(v)-\Theta(w)-\Theta'(w)(v-w)\|_{\mathfrak{Y}}\lesssim\|v-w\|_{\mathfrak{X}}^{\alpha}\,\|\Theta(v)-\Theta(w)\|_{\mathfrak{Y}}\quad \text{for all } v,w\in B_\rho(x^+). \tag{C.4}$$

c) *Conversely, if both,* (C.1) *and continuity of* Θ', *or* (C.2) *hold and* $\Theta(x^+)$ *is isolated, that is,* $\Theta(x^+)\notin\Theta\big(B_\rho(x^+)\backslash\{x^+\}\big)$, *then* $\Theta'(x^+)$ *has to have a trivial null space.*

Part c) is essentially known in the literature [11, Prop. 2.1].

Proof. a) By injectivity of $\Theta'(x^+)$, continuity of Θ', and finite-dimensionality of $\mathfrak{X}$ there is an $r_1\in(0,r]$ and an $m>0$ such that

$$\|\Theta'(x)v\|_{\mathfrak{Y}}\ge m\|v\|_{\mathfrak{X}}\ \text{ for all } x\in B_{r_1}(x^+) \text{ and all } v\in\mathfrak{X}.$$

For $E(v,w):=\Theta(v)-\Theta(w)-\Theta'(w)(v-w)$ we have that, for all $v,w\in B_r(x^+)$,

$$\|E(v,w)\|_{\mathfrak{Y}}=\Big\|\int_0^1\big(\Theta'(w+t(v-w))-\Theta'(w)\big)(v-w)\,\mathrm{d}t\Big\|_{\mathfrak{Y}}\le\sigma(v,w)\|v-w\|_{\mathfrak{X}}$$

with

$$\sigma(v,w)=\sup\{\|\Theta'(w+t(v-w))-\Theta'(w)\|:t\in[0,1]\}.$$

Choose $\rho\in(0,r_1]$ such that $\sigma(v,w)\le m/2$ for all $v,w\in B_\rho(x^+)$. We proceed – using the reverse triangle inequality – with

$$\begin{aligned}\|\Theta(v)-\Theta(w)\|_{\mathfrak{Y}}&=\|E(v,w)-\Theta'(w)(w-v)\|_{\mathfrak{Y}}\\&\ge\big|\|E(v,w)\|_{\mathfrak{Y}}-\|\Theta'(w)(w-v)\|_{\mathfrak{Y}}\big|\\&\ge m\,\|v-w\|_{\mathfrak{X}}-\sigma(v,w)\,\|v-w\|_{\mathfrak{X}}.\end{aligned}$$

Hence,

$$\|\Theta(v)-\Theta(w)\|_{\mathfrak{Y}}\ge\frac{m}{2}\,\|v-w\|_{\mathfrak{X}}\quad\text{for all } v,w\in B_\rho(x^+) \tag{C.5}$$

which is (C.1). Finally,

$$\|E(v,w)\|_{\mathfrak{Y}}\le\sigma(v,w)\,\|v-w\|_{\mathfrak{X}}\overset{(C.5)}{\le}\eta(v,w)\,\|\Theta(v)-\Theta(w)\|_{\mathfrak{Y}}$$

and (C.2) is verified with $\eta=2\sigma/m$.

b) Under (C.3) we estimate

$$\eta(v,w)\le\frac{2L}{m}\,\|v-w\|_{\mathfrak{X}}^{\alpha}$$

so that (C.2) yields (C.4).

c) As continuity and (C.1) together imply (C.2), it suffices to assume the latter condition. Suppose there is a $z\in\mathsf{N}(\Theta'(x^+))\backslash\{0\}$. Then, $v_\lambda:=x^++\lambda z\in B_\rho(x^+)$ for $0<\lambda<\varrho/\|z\|_{\mathfrak{X}}$ and

$$\begin{aligned}\|\Theta(v_\lambda)-\Theta(x^+)\|_{\mathfrak{Y}}&=\|\Theta(v_\lambda)-\Theta(x^+)-\lambda\Theta'(x^+)z\|_{\mathfrak{Y}}\\&\lesssim\eta(x^++\lambda z,x^+)\,\|\Theta(v_\lambda)-\Theta(x^+)\|_{\mathfrak{Y}}.\end{aligned}$$

Thus, $\Theta(v_\lambda) = \Theta(x^+)$ for $\lambda > 0$ small enough which contradicts the isolation of $\Theta(x^+)$. □

Finally, we want to emphasize that we can replace the injectivity assumption in the above lemma by Lipschitz stability. Indeed, continuity and (C.1) imply (C.2). Note that Lipschitz stability is known for a variety of semi-discrete inverse problems. We refer, e.g., to [1, 2, 3, 4, 5].

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