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Examples of noncompact nonpositively curved manifolds

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Abstract

We give a simple construction of new, complete, finite volume manifolds M of bounded, nonpositive curvature. These manifolds have ends that look like a mixture of locally symmetric ends of different ranks and their fundamental groups are not duality groups.

1. *Introduction*

The goal of this paper is to give a very simple construction of complete, finite volume, tame[†] n-manifolds M of bounded, nonpositive curvature. The manifolds obtained have interesting properties. For instance, the large scale geometry of their ends is a mixture of different types and their fundamental groups are not 'duality groups'[‡], in contrast with the typical examples of nonpositively curved manifolds such as locally symmetric spaces of noncompact type. If M is a locally symmetric manifold of noncompact type, then from a large-scale point of view M looks like a union of flat r-dimensional sectors, where r is the $\mathbb{Q}$-rank of M. So for (arithmetic[¶]) locally symmetric spaces, their large-scale geometry is determined by the rational structures of the spaces. Moreover, the fundamental group of M is a duality group, or in other words, the lift of the end of M to the universal cover $\widetilde{M}$ has homology concentrated in one dimension. This is a consequence of the fact that it is homotopy equivalent to the rational Tits building (of M), which is homotopy equivalent to a wedge of spheres of a single dimension.

In [**2**], we tried to capture the topology of the ends of general nonpositively curved, not necessarily locally symmetric, manifolds M from the geometry of M and $\widetilde{M}$, showing that many properties of locally symmetric manifolds that could be seen only by doing arithmetic before can actually be seen as purely nonpositive curvature phenomena. For example, we obtained that the lift of the end of M in $\widetilde{M}$ has homology only in dimension less than $n/2$. In other words, it is not an arithmetic phenomenon that the rational Tits building of a locally symmetric space has dimension less than half the dimension of the space. However, one cannot take this analogy too far and base all aspects of nonpositively curved manifolds on delicacies of locally symmetric spaces because there are still arithmetic things that are due to arithmetics, such as the rational Tits building being a building, and this is one of the main points of the examples in this paper.

Below, M is tame, so it is homeomorphic to the interior of a compact manifold-with-boundary, $(\overline{M}, \partial\overline{M})$ and its universal cover is a (noncompact) manifold-with-boundary $(\widetilde{\overline{M}}, \partial\widetilde{\overline{M}})$. We will abuse notation slightly and denote these manifolds-with-boundary by

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[†]'Tame' means that the manifold is homeomorphic to the interior of a compact manifold-with-boundary.

[‡]Here, being a *duality group* means that the preimage of the end of M in the universal cover $\widetilde{M}$ has (reduced) homology concentrated in only one dimension.

[¶]All irreducible higher rank locally symmetric spaces are arithmetic by Margulis' arithmeticity theorem [**6**].

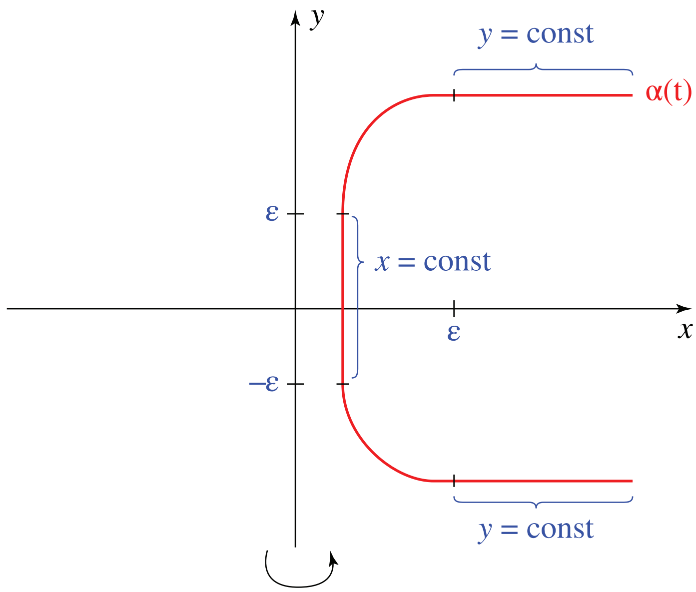

FIGURE 1 (colour online).

$(M, \partial M)$ and $(\widetilde{M}, \partial\widetilde{M})$, respectively. Note that $\partial\widetilde{M} \to \partial M$ is regular cover with covering group $\pi_1 M$, so we call it the $\pi_1 M$-cover of ∂M.

THEOREM 1. *For any $0 \leqslant i \leqslant j < \lfloor n/2 \rfloor$, there is a tame, complete, finite volume, Riemannian n-manifold M of bounded nonpositive curvature with the property that $\overline{H}_k(\partial\widetilde{M}) \neq 0$ if and only if $i \leqslant k \leqslant j$.*

In fact, in our examples, $\partial\widetilde{M}$ is homotopy equivalent to a union of wedges of spheres of dimensions ranging from i to j.

REMARK. One can show that

$$\overline{H}_*(\partial\widetilde{M}) \cong H^{n-1-*}(B\pi_1 M; \mathbb{Z}\pi_1 M),$$

so as an algebraic corollary, $\pi_1 M$ is not a duality group if $j > i$.

The construction is done inductively and the main idea is to assemble nonpositively curved spaces like products of hyperbolic manifolds with cusps via codimension 2 surgery along totally geodesic submanifolds. As usual, one needs to smooth out the metric around the places where surgery is done, but in this case, this is extremely easy.

The codimension 2 surgery involved at each step can be described as follows. We choose suitable manifolds M_1^k and M_2^k, each of which has an open set that is isometric to $\mathbb{T}^{k-2} \times \mathbb{D}^2$, where $\mathbb{T}^{k-2}$ is the flat square torus. Then we remove $\mathbb{T}^{k-2} \times \mathbb{D}^2_\varepsilon$ from each M_i and glue the resulting spaces together along the boundary preserving the product structure on $\mathbb{T}^{k-2} \times \partial\mathbb{D}^2_\varepsilon$ to obtain a new manifold M whose metric is singular on $\mathbb{T}^{k-2} \times \partial\mathbb{D}^2_\varepsilon$. Since the gluing is an isometry on the first factor, the singularity of the metric lies in the second factor, which is the double of $(\mathbb{D}^2 - \mathbb{D}^2_\varepsilon)$ along $\partial\mathbb{D}^2_\varepsilon$. To smooth out this singularity, replace this double by a ‘funnel’ that is the surface of revolution generated by the curve α in Figure 1, which clearly has nonpositive Gaussian curvature. Thus, we obtain a bounded nonpositively curved manifold M whose ends correspond to those of M_1 and M_2 and therefore have finite volume.

We illustrate a simple, nontrivial case here. The general case will be treated in the body of the paper.

An example. We construct M by taking two particular manifolds M_1 and M_2 (as described below) with isometric totally geodesic submanifolds T_1 and T_2 (respectively) and gluing the complement of an ε-neighborhood of T_1 to the complement of an ε-neighborhood of T_2.

Let $M_1 = S \times S$ be the product of two copies of a punctured torus endowed with a complete hyperbolic metric with finite area, and let a be a simple, closed geodesic in S. Modify the metric smoothly on a regular neighborhood of a in S and rescale it if necessary to make it a product $(-1, 1) \times \mathbb{S}^1$ without creating positive curvature on S. Give $M_1 = S \times S$, the product of the new metrics on S. Then $T_1 := a \times a$ is a flat, square 2-torus and has a neighborhood isometric to $\mathbb{D}^2 \times T_1$.

Let M_2 be obtained by taking a finite volume, complete, hyperbolic 4-manifold H with at least three torus-cusps C_1, C_2 and C_3, truncating C_2 and C_3 and gluing ∂C_2 to ∂C_3 via an affine diffeomorphism. Assume for simplicity that the cross-sections of each of these cusps are homothetic to the flat, square, 3-torus $\mathbb{T}^3$, so that the gluing can be done via an isometry and gives M_2 a bounded nonpositively curved metric. (This is standard but we will explain it in the next section.) One can make it so that the metric on M_2 is a product $(-1, 1) \times \mathbb{T}^3$ on a neighborhood of where the gluing takes place. Now, there is a square 2-torus T_2 factor in $\mathbb{T}^3$, so T_2 has a neighborhood isometric to the product $\mathbb{D}^2 \times T_2$.

Let M be obtained by gluing the complement of the ε-neighborhood of T_1 to the complement of the ε-neighborhood of T_2 along the boundaries. After smoothing out the metric as explained above, we obtain a finite volume, bounded nonpositively curved manifold M with two kinds of cusps, one corresponding to the end of M_1, and the other corresponding to the cusp C_1 of M_2.

In this example, $\partial\widetilde{M}$ is homotopically equivalent to a graph Σ with infinitely many components, each component either contractible or of infinite type (homotopy equivalent to an infinite wedge of circles). The first kind of cusp looks like a 2-dimensional flat sector from afar and is responsible for the infinite type components in Σ (see the product formula in Subsection 2.2). The second kind looks like a ray from afar and contributes the contractible components in Σ.

All the simplifying assumptions made above can be taken care of in general when no such assumptions are made. This is dealt with in the rest of the paper and is not difficult.

A simpler construction that gives a manifold very similar to the manifold M above can be obtained by taking $(M_1 - T_1)$ and stretching out the metric in a neighborhood of T_1 to make it complete and have finite volume without creating positive curvature. Since the metric on M_1 is a product $\mathbb{D}^2 \times T_1$, this can be achieved if one can stretch out the metric on $(\mathbb{D}^2 - \{0\})$ to obtain a complete, bounded nonpositively curved metric with finite area. This clearly can be done and is illustrated in Figure 2. This example is a good example but we did not discuss it above because it does not illustrate every step in the construction given in this paper. But we would like to note that this is a counterexample to a conjecture of Farb on geometric rank 1 manifolds and we will discuss this in Subsection 4.3.

2. *Proof of Theorem 1, part A - The construction*

2.1. *A special case*

The nontrivial part in proving Theorem 1 is proving the special case when n is even and $i = 0$ and $j = n/2 - 1$. This is done by inductively constructing manifolds M_n satisfying (1) in

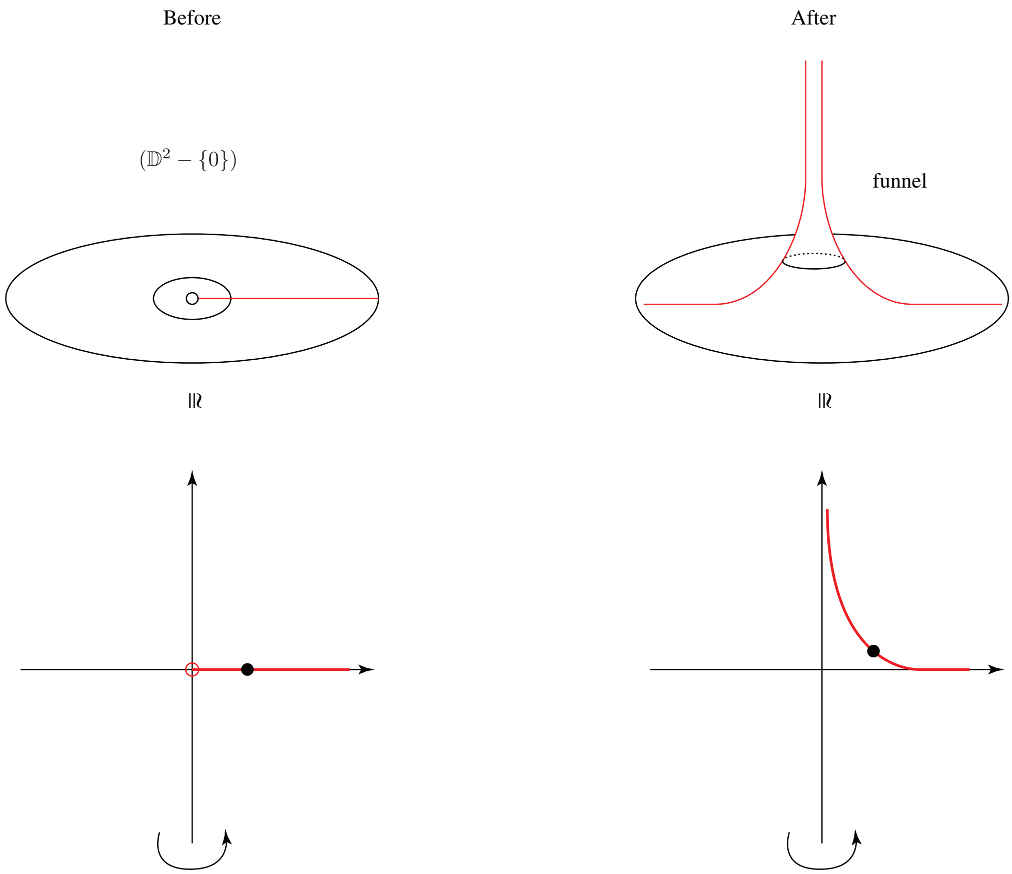

FIGURE 2 (colour online).

Proposition 2. In order to facilitate the induction, the manifolds M_n need to have the additional properties (2)–(4).

PROPOSITION 2. *For even n, there is a tame, complete, finite volume, n-manifold M_n of bounded non-positive curvature so that:*

1. $\overline{H}_k(\partial\widetilde{M_n}) \neq 0$ *for* $k < n/2$;
2. M_n *has at least two ends;*
3. M_n *contains an isometrically embedded* $T := \mathbb{T}^{n-1} \times (-1, 1)$, *where* $\mathbb{T}^{n-1} = (\mathbb{S}^1)^{n-1}$ *is a square flat torus of injectivity radius* 1; *and*
4. $M_n \setminus T$ *is connected.*

2.2. *The general case*

Theorem 1 follows from Proposition 2 by taking products with circles and non-compact surfaces. The key to showing this is the following product formula.

Product formula

If M and N are tame, aspherical manifolds, then one has the following product formula

$$\partial(\widetilde{M \times N}) \sim \partial\widetilde{M} * \partial\widetilde{N}, \tag{1}$$

where the symbol $\sim$ denotes homotopy equivalence.

Proof. This follows from

$$\begin{aligned} \partial\widetilde{M} * \partial\widetilde{N} &= \partial\widetilde{M} \times Cone(\partial\widetilde{N}) \bigcup_{\partial\widetilde{M}\times\partial\widetilde{N}} Cone(\partial\widetilde{M}) \times \partial\widetilde{N}, \\ \partial(\widetilde{M} \times \widetilde{N}) &= \partial\widetilde{M} \times \widetilde{N} \bigcup_{\partial\widetilde{M}\times\partial\widetilde{N}} \widetilde{M} \times \partial\widetilde{N}. \end{aligned}$$

The quantities on the right-hand side of the above two lines are homotopy equivalent since $\widetilde{N}$ and $\widetilde{M}$ are contractible and thus are, respectively, homotopy equivalent to the cones on their boundaries. Therefore,

$$\partial\widetilde{M} * \partial\widetilde{N} \sim \partial(\widetilde{M} \times \widetilde{N}).$$

Since $\widetilde{M \times N} = \widetilde{M} \times \widetilde{N}$, the above product formula follows. ☐

Shifting dimensions via products with circles and surfaces

Note that for a *non-compact* surface Σ the cover $\partial\widetilde{\Sigma}$ is homotopy equivalent to an infinite union of points, which we will write as $\partial\widetilde{\Sigma} \sim \vee_{i=1}^{\infty} S^0$. Therefore, $\partial(\widetilde{M \times \Sigma}) \sim \partial\widetilde{M} * (\vee_{i=1}^{\infty} S^0) \sim \vee_{i=1}^{\infty}(\partial\widetilde{M} * S^0)$. So

$$\overline{H}_*(\partial(\widetilde{M \times \Sigma})) \cong \bigoplus_{i=1}^{\infty} \overline{H}_{*-1}(\partial\widetilde{M}). \tag{2}$$

It is also clear that $\partial(\widetilde{M \times S^1}) \sim \partial\widetilde{M}$ so we have

$$\overline{H}_*(\partial(\widetilde{M \times S^1})) \cong \overline{H}_*(\partial\widetilde{M}). \tag{3}$$

Proof of Theorem 1 given Proposition 2

The proposition gives a $2(j-i+1)$-dimensional manifold $M_{2(j+i-1)}$ whose homology $\overline{H}_k(\partial\widetilde{M}_{2(j+1-i)})$ does not vanish precisely in the band of dimensions $0 \leqslant k \leqslant j-i$. Crossing with i noncompact surfaces shifts this band into the desired dimension range $i \leqslant k \leqslant j$ (by formula (2)) and then crossing with $n-2j-2$ circles raises the dimension of the manifold to n without affecting the band (by formula (3)). So, the resulting manifold

$$M = M_{2(j+1-i)} \times (\Sigma)^i \times (S^1)^{n-2j-2}, \tag{4}$$

satisfies the conclusions of Theorem 1.

2.3. *Proof of Proposition 2*

The manifolds M_n are constructed inductively, as follows.

Base case

Topologically, the base case M_2 is a twice-punctured torus. Start with a hyperbolic metric on M_2. In this metric, the two punctures appear as cusps. Let b be a geodesic[†] that starts in one cusp and ends in the other cusp, and a a nonseparating closed geodesic loop that does not intersect b. Let length$(a) \geqslant 2$ and modify the metric so that it is a flat cylinder on an

[†]It is not important that b is a geodesic. We could take any path.

1-neighborhood of a, hyperbolic outside of a compact set, and still nonpositively curved.† It is easy to see that M_2 with this metric satisfies the conditions in the proposition.

Before starting the inductive construction, we need to introduce a manifold that will be used in the inductive step. As mentioned in the introduction, the construction involves assembling nonpositively curved manifolds containing totally geodesic tori of low codimension. The following is one way to obtain such manifolds.

The building blocks N_n (Hyperbolic straightjackets)

Start with a complete, finite volume, connected, hyperbolic n-manifold H_n. After passing to a finite cover, if necessary, we may assume that H_n has at least three cusps, at least two of which (called C_+ and C_-) are homeomorphic to $\mathbb{T}^{n-1} \times (0, \infty)$. Then, the manifold $H_n \setminus (C_+ \cup C_-)$ has two boundary components $\partial C_+ \cong \mathbb{T}^{n-1} \cong \partial C_-$. Moreover, the induced metrics on ∂C_+ and ∂C_- are flat. Now, let $N_n = (H_n \setminus (C_+ \cup C_-))/\partial C_+ \sim \partial C_-$ be a manifold obtained by gluing the boundaries ∂C_+ and ∂C_- by an affine diffeomorphism.

Proposition 3. *For any $r > 0$, the manifold N_n has a complete, finite volume, Riemannian metric of bounded non-positive curvature in which a regular neighborhood of ∂C_+ is isometric to $\mathbb{T}^{n-1} \times (-r, r)$, where $\mathbb{T}^{n-1}$ is a square flat torus.*

First, note in the case when ∂C_+ and ∂C_- are square, flat tori and the affine diffeomorphism is an isometry, this is not hard. The hyperbolic metric near ∂C_+ or ∂C_- is a warped product and has the form

$$g_{\mathrm{hyp}} = e^{-2t} g_0 + dt^2,$$

where g_0 is a square, flat metric on $\mathbb{T}^{n-1}$ and for some a, the slice $t = a$ corresponds to where ∂C_+ or ∂C_- is. So around where ∂C_+ and ∂C_- are identified, the metric, after reparametrizing t via a shift by a, is

$$e^{2|t|-2a} g_0 + dt^2$$

on $\mathbb{T}^{n-1} \times [-1, 1]$, which is not smooth at $t = 0$. But one can replace the warping function $e^{2|t|-2a}$ by a smooth, convex function that, for some small enough ε, agrees with $e^{2|t|-2a}$ outside $(-2\varepsilon, 2\varepsilon)$ and that is equal to a positive constant on $(-\varepsilon, \varepsilon)$. Change the range‡ $(-\varepsilon, \varepsilon)$ of t-parameter to $(-r, r)$ but keep the metric otherwise the same to get a desired metric. The fact that the resulting metric has nonpositive curvature is a direct application of the Bishop-O'Neil formula [**3**].

In the general case, the main point is to first interpolate between the square flat metric g_0 on $\mathbb{T}^{n-1}$ and another flat metric g_1 on T^{n-1} so that the problem reduces to the above. That is, consider the following metric g on $\mathbb{T}^{n-1} \times [0, \infty)$.

$$g = e^{-2t}(h(t) g_0 + (1 - h(t)) g_1) + dt^2,$$

for some smooth function $h\colon [0, \infty) \to [0, 1]$ such that $h(t) = 0$ when t is close to 0 and $h(t) = 1$ when $t > l$, for some l. One can pick l large enough and an appropriate h so that g has nonpositive curvature as shown in [**1**, Lemma 2.2]. Truncate that cusp at $t = a > l$ and apply the above special case to get the desired metric.

Now we are ready for the inductive part of the construction.

†We can do this without changing the length of a.

‡In other words, we replace the cylinder $(-\varepsilon, \varepsilon) \times \mathbb{T}^{n-1}$ by the cylinder $(-r, r) \times \mathbb{T}^{n-1}$.

Inductive step

Suppose we have constructed M_{n-2}. We need to build M_n. Look at $M_{n-2} \times M_2$. It contains an isometrically embedded

$$\begin{aligned}\mathbb{T}^{n-3} \times (-1,1) \times a \times (-1,1) &\cong \mathbb{T}^{n-2} \times (-1,1)^2 \\ &\supset \mathbb{T}^{n-2} \times \mathbb{D}^2.\end{aligned}$$

On the other hand, suppose that N_n is an n-dimensional 'building block' described above, that is, a manifold obtained from a hyperbolic manifold by gluing a pair of cusps together so that they give an isometrically embedded copy of

$$\begin{aligned}\mathbb{T}^{n-1} \times (-1,3) \supset \mathbb{T}^{n-2} \times \mathbb{S}^1 \times \Big((-1,1) \coprod (1,3)\Big) \\ \supset \big(\mathbb{T}^{n-2} \times \mathbb{D}^2\big) \coprod \big(\mathbb{T}^{n-1} \times (1,3)\big).\end{aligned}$$

The '$\mathbb{T}^{n-2} \times \mathbb{D}^2$' is used in codimension 2 surgery, and the '$\mathbb{T}^{n-1} \times (1,3)$' implies that the resulting manifold M_n will have property (3), which let us continue the induction. Also recall that N_n has at least one cusp that is not glued to anything. We claim that the manifold

$$M_n := \big[N_n \setminus (\mathbb{T}^{n-2} \times \mathbb{D}^2)\big] \bigcup_{\mathbb{T}^{n-2} \times S^1} \big[(M_{n-2} \times M_2) \setminus (\mathbb{T}^{n-2} \times \mathbb{D}^2)\big] \tag{5}$$

obtained by taking the 'connect sum along $\mathbb{T}^{n-2}$' has a complete, finite volume metric of bounded nonpositive curvature. We explain this next.

Flat, codimension 2 surgery in nonpositive curvature

Suppose M and N are complete, finite volume manifolds of bounded nonpositive curvature and $S \subset M$ a totally geodesic submanifold. Suppose further that a regular neighborhood of S is *isometric* to $S \times \mathbb{D}^2$.

Cusps
The manifold $M \setminus S$ has a complete, finite volume, nonpositively curved metric of bounded nonpositive curvature obtained by replacing $S \times (\mathbb{D}^2 - \{0\})$ by $S \times$ funnel, where a *funnel* is defined as follows.

Definition 4 (Funnel). Let $f\colon (0,1] \to \mathbb{R}$ be a smooth, strictly convex, non-negative function that satisfies the following properties.

(i) $f(x) = 0$ when $x \geqslant 1/2$.
(ii) $f(x) \to \infty$ as $x \to 0$.
(iii) $\int_0^1 f(x)dx < \infty$.

Let funnel be the surface of revolution obtained by rotating the graph of $f(x)$ around the y-axis. Then it is diffeomorphic to $\mathbb{D}^2 - \{0\}$ but has negative Gaussian curvature (because $f(x)$ is strictly convex) and finite area (because of condition (iii) above). See Figure 2.

Remark. We use this in the alternative construction (Subsection 4.1) which works in special cases, but do not need it for the proof of Theorem 1.

Definition 5 (2-Sided Funnel). A 2-sided funnel is the surface of revolution obtained by rotating in the curve $\alpha(t)$, where $\alpha(t)$ is a smooth curve defined as in Figure 1, around the y-axis. A 2-sided funnel is diffeomorphic to $\mathbb{S}^1 \times (-1,1)$ but has negative Gaussian curvature.

Codimension two connect sum. If N also contains an isometrically embedded copy of $S \times \mathbb{D}^2$, then the *S-connect sum*

$$M \#_S N := \left[M \setminus (S \times \mathbb{D}^2)\right] \cup_{S \times \mathbb{S}^1} \left[N \setminus (S \times \mathbb{D}^2)\right] \tag{6}$$

has a complete, finite volume metric of bounded nonpositive curvature. After cutting out the regular neighborhoods $S \times \mathbb{D}^2$ from both manifolds, the metric is obtained by inserting a tube that looks topologically like $S \times (\mathbb{S}^1 \times (0,1))$ but metrically looks like $S \times \{\text{two sided funnel}\}$.

Remark. In the notation of equation (6),

$$M_n = N_n \#_{\mathbb{T}^{n-2}} (M_{n-2} \times M_2),$$

so M_n has a complete finite volume metric of bounded nonpositive curvature.

3. *Proof of Theorem 1, part B - Properties of the manifold M_n*

The manifold M_n contains the isometrically embedded $T := \mathbb{T}^{n-1} \times (1,3)$, which shows property (3). The space $N_n \setminus T$ is connected (because it is homotopy equivalent to the original connected hyperbolic manifold H_n we had before we glued two of its cusps together) and the product $M_{n-2} \times M_2$ is connected (the factors M_{n-2} and M_2 are connected because they satisfy property (4)) so the space

$$M_n \setminus T = [(N_n \setminus T) \setminus (\mathbb{T}^{n-2} \times \mathbb{D}^2)] \bigcup_{\mathbb{T}^{n-2} \times \mathbb{S}^1} [(M_{n-2} \times M_2) \setminus (\mathbb{T}^{n-2} \times \mathbb{D}^2)]$$

obtained via the codimension 2 surgery is also connected. This proves property (4).

Since both N_n and $M_{n-2} \times M_2$ have ends, the manifold M_n has at least two ends. This shows property (2). It also implies that $\partial \widetilde{M}_n$ has at least two components, so

$$\overline{H}_0(\partial \widetilde{M}_n) \neq 0.$$

It remains to establish the positive dimensional cases of property (1).

3.1. *Computing $H_{>1}(\partial \widetilde{M}_n)$*

Next, let z be a *connected* homology cycle representing a nontrivial homology class in $H_k(\partial \widetilde{M}_{n-2})$ for $0 < k < n/2 - 1$. Let $\tilde{b}$ be a lift of the path connecting the two ends of the twice punctured torus, and b^+ and b^- its endpoints. Look at the suspended cycle $\Sigma z = z * \{b^+, b^-\}$. Since z is connected, the suspended cycle Σz is simply connected. Therefore, a map $\Sigma z \to \partial \widetilde{M}_{n-2} * \partial \widetilde{M}_2 \sim \partial(\widetilde{M_{n-2} \times M_2})$ which represents the nontrivial $(k+1)$-homology class $[\Sigma z] \in H_{k+1}(\partial(\widetilde{M_{n-2} \times M_2}))$ lifts to a component of $\partial \widetilde{M}_n$. So, for $1 < k+1 < n/2$, we have

$$H_{k+1}(\partial \widetilde{M}_n) \neq 0.$$

3.2. *Computing $H_1(\partial \widetilde{M}_n)$*

Since M_{n-2} has two ends and $M_{n-2} \setminus T$ is connected, we can find a path $\beta : [0,1] \to M_{n-2} \setminus T$ connecting two different ends of M_{n-2}. Let $z = \partial \tilde{\beta} \in \overline{H}_0(\partial \widetilde{M}_{n-2})$ be the non-trivial zero cycle obtained as the boundary of a lift $\tilde{\beta}$ of β. Then, the image of $\Sigma z = \{\beta^+, \beta^-\} * \{b^+, b^-\}$ is contractible in M_n because it bounds $\beta \times b$. Therefore, in this case the nontrivial homology cycle $[\Sigma z] \in H_1(\partial(\widetilde{M_{n-2} \times M_2}))$ also lifts to a cycle in a component of $\partial \widetilde{M}_n$, showing that

$$H_1(\partial \widetilde{M}_n) \neq 0.$$

In summary, we have shown that $\overline{H}_k(\partial\widetilde{M}_n) \neq 0$ for $k < n/2$. This proves property (1), finishes the proof of Proposition 2, and thus also the proof of Theorem 1.

4. *Miscellaneous*

4.1. *A variant for narrow bands that only uses surfaces*

Note that the regular neighborhood of $a \times a$ inside $M_2 \times M_2$ is isometric to $a \times a \times D^2_\epsilon$. Replacing D^2_ϵ by a 'funnel' metric on $D^2_\epsilon \setminus \{0\}$, we get a complete, finite volume metric of bounded nonpositive curvature on

$$M'_4 := (M_2 \times M_2) \setminus (a \times a).$$

The arguments in the previous section apply to show that $\overline{H}_0(\partial\widetilde{M'_4}) \neq 0$ and $H_1(\partial\widetilde{M'_4}) \neq 0$. Taking products of the manifold M'_4 with itself and using the product formula (1), we get manifolds $(M'_4)^m$ of dimension $4m$ which have $\overline{H}_k(\partial\widetilde{(M'_4)^m}) \neq 0$ precisely when $m - 1 \leqslant k \leqslant 2m - 1$.

Remark. Taking products with circles S^1 and non-compact surfaces M_2, we get in this way manifolds $M := (M'_4)^m \times (M_2)^p \times (S^1)^q$ of dimension $\dim M = 4m + 2p + q$ for which $\overline{H}_k(\partial\widetilde{M})$ is non-zero in a band of dimensions $m - 1 + p \leqslant k \leqslant 2m - 1 + p$.

4.2. *Large scale geometry*

Denote by $[n]$ the set with n elements. It is easy to see that the main construction gives manifolds that on a large scale look like the Euclidean cone on a complex C_k, where C_k is defined inductively via $C_0 = [2], C_1 = ([2] * [2]) \coprod [n_4], \dots, C_k = (C_{k-1} * [2]) \coprod [n_{2k}]$ where n_{2k} is the number of ends of the $2k$ dimensional building block N_{2k} (see Subsection 2.3 for a description of N_{2k}).

4.3. *Geometric rank-1 manifolds with π_1 generated by a cusp*

Once upon a time, there was a conjecture that said the following.

Conjecture 6 (Farb). Let M be a tame, complete, finite volume n-manifold of bounded nonpositive curvature. Suppose M has geometric rank one. Then there is a loop in M that cannot be homotoped to leave every compact set.

This is known to be true in dimension $\leqslant 3$. We will first show that the manifold $(M_1 - T_1)$ from the introduction is a 4-dimensional counterexample to this conjecture, and then we will build higher dimensional counterexamples afterward.

A 4-dimensional counterexample

We will drop the index '1' in $(M_1 - T_1)$ as we no longer need it. First, note that the manifold $W := M - T$ has geometric rank 1 because it is neither a locally symmetric space, nor a product.[†] Thus, we only need to show that all loops in W can be homotoped to leave all compact sets. This is true because of the following lemma.

[†] W is not a product of two noncompact manifolds because it has more than one end. It is not a product of a non-compact manifold and a compact manifold, because its two ends do not have a common factor: The end cross sections are $\mathbb{T}^3$ and the irreducible graph manifold $((\mathbb{T}^2 - D^2) \times S^1) \bigcup_{S^1 \times S^1} (S^1 \times (\mathbb{T}^2 - D^2))$.

LEMMA 7. *Let $(S_1, \partial S_1)$ and $(S_2, \partial S_2)$ be compact, connected manifolds-with-boundary and pick basepoints $s_i \in \partial S_i$. Suppose that $T_i \subset (S_i - \partial S_i)$ are compact nonseparating hypersurfaces. Let $S_1 \vee S_2 = (S_1 \times \{s_2\}) \cup (\{s_1\} \times S_2)$. Then the composition*

$$(S_1 \vee S_2) \hookrightarrow \partial(S_1 \times S_2) \hookrightarrow (S_1 \times S_2) - (T_1 \times T_2),$$

is π_1-onto.

Proof. If $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is a loop in $(S_1 \times S_2) - (T_1 \times T_2)$, then the times at which γ_1 crosses T_1 are disjoint from the times at which γ_2 crosses T_2. So, one can decompose γ as concatenation $\gamma = \gamma^{(1)} \cdot \cdots \cdot \gamma^{(r)}$ of *paths* where for each $\gamma^{(k)}$ either the first coordinate path $\gamma_1^{(k)}$ never crosses T_1 or the second coordinate path $\gamma_2^{(k)}$ never crosses T_2. Using the fact that the T_i are nonseparating, we can homotope γ to be a concatenation of such *loops* (all based at (s_1, s_2)). Finally, each such loop $\gamma^{(k)}$ is homotopic to $\gamma_1^{(k)} \cdot \gamma_2^{(k)}$, so we are done. □

REMARK. Since T has codimension 2 in M, there is a loop γ in M that goes around T. One might wonder how γ can be a product of elements in $S_1 \vee S_2$. Let b_i be a loop in S_i based at s_i that intersects transversely with T_i precisely once. We claim that $\gamma = [b_1, b_2] = b_1 b_2 b_1^{-1} b_2^{-1}$. To see this, observe that $T' := b_1 \times b_2$ is an embedded torus in M that intersects T transversely at exactly one point p. So γ can be taken to be a loop in T' that goes around p. Removing T from M results in removing p from T'. Since $T' - \{p\}$ is a punctured torus, the loop γ, which goes around the puncture, must be the commutator of the generators b_1 and b_2.

Higher dimensional counterexamples can be constructed in a very similar manner. In dimension $n \geqslant 4$, let S_1 be the punctured torus as before, and let $T_1 = a_1$. Let S_2 be the building block N_{n-2} and let T_2 be $\mathbb{T}^{n-3}$, the square flat torus in N_{n-2} in Proposition 3. The manifold $W := (S_1 \times S_2) - (T_1 \times T_2)$ is an n-dimensional counterexample to Conjecture 6. It has geometric rank one for the same reasons as in the above example. To see that $\pi_1(W)$ is generated by loops coming from the end of M, we apply the above lemma. Thus, we have proved that Conjecture 6 is false for all $n \geqslant 4$.

PROPOSITION 8. *There is a counterexample to Conjecture 6 for each $n \geqslant 4$.*

4.4. *A thick-thin conjecture for nonpositively curved manifolds*

We would like to suggest the following replacement for Conjecture 6.

CONJECTURE 9. Let M be a tame, complete, finite volume n-manifold of bounded[†] nonpositive curvature. Then there is a compact subset $C \subset M$ that cannot be homotoped to leave every compact set.

Note that this conjecture makes sense (and is most easily stated) for general finite volume manifolds of bounded nonpositive curvature, not just those of geometric rank one. The conjecture is known to be true for locally symmetric manifolds M by a result of Pettet and Souto [**5**].[‡] Therefore, it is enough to understand it for geometric rank one manifolds.

Note that the examples in this paper are not counterexamples to Conjecture 9. To see this, pick an embedded loop b_i in S_i that intersects T_i transversely exactly once. This is possible

[†]The conjecture is not true without the lower curvature bound. There is a complete, finite volume, negatively curved metric on the product $\Sigma \times \mathbb{R}$, where Σ is a closed surface with genus $g \geqslant 2$ [**4**].

[‡]Such locally symmetric manifolds contain *maximal periodic flat tori* $\mathbb{T}^r \to M$, where r is the $\mathbb{R}$-rank of the locally symmetric space M. Pettet and Souto showed these tori cannot be homotoped into the end (even though loops in such a locally symmetric space can always be homotoped into the end whenever the $\mathbb{Q}$-rank is $\geqslant 2$).

since the hypersurfaces T_i are nonseparating. Now, let T_i' be a parallel copy of T_i. We pick it close to T_i, so that b_i intersects T_i' transversely at exactly one point $x_i = b_i \cap T_i'$. Then the closed submanifolds $T_1' \times b_2$ and $b_1 \times T_2'$ of W intersect transversely at a single point $x_1 \times x_2$. Therefore, the interior of W cannot be homotoped into its end, because if there was such a homotopy $h_t : W \to W$ with $h_0 = \mathrm{Id}_W$ and $h_1(W)$ contained in a sufficiently small neighborhood of the end of W, then we could move $T_1' \times b_2$ via the homotopy $h_t(T_1' \times b_2)$ to be disjoint from $b_1 \times T_2'$. This is a contradiction because intersection number is a homological invariant. Therefore, there is no such homotopy.

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