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Vladimir A. Osinov

Sufficient conditions for hyperbolicity and consistency of the dynamic equations for fluid-saturated solids

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Abstract Previous studies showed that the dynamic equations for a porous fluid-saturated solid may lose hyperbolicity and thus render the boundary-value problem ill-posed while the equations for the same but dry solid remain hyperbolic. This paper presents sufficient conditions for hyperbolicity in both dry and saturated states. Fluid-saturated solids are described by two different systems of equations depending on whether the permeability is zero or nonzero (locally undrained and drained conditions, respectively). The paper also introduces a notion of wave speed consistency between the two systems as a necessary condition which must be satisfied in order for the solution in the locally drained case to tend to the undrained solution as the permeability tends to zero. It is shown that the symmetry and positive definiteness of the acoustic tensor of the skeleton guarantee both hyperbolicity and the wave speed consistency of the equations.

Keywords Fluid-saturated solid · Hyperbolicity · Acoustic tensor

1 Motivation

According to the well-known definition, a boundary-value problem is said to be well-posed if a solution exists, is unique and depends continuously on the initial and boundary data. Well-posedness of a particular problem is determined by both the governing equations and the boundary conditions. Well-posedness is usually difficult to prove even for linear problems, not to mention nonlinear multidimensional cases. This is probably the main reason why issues related to well-posedness are not discussed or even mentioned in the majority of studies dealing with the numerical solution of boundary-value problems. On the other hand, it may be much easier to verify a necessary condition for well-posedness and thus to detect ill-posedness if this condition is violated. The present paper deals with dynamic problems. A necessary condition for well-posedness of dynamic problems for solids with rate-independent constitutive behaviour is hyperbolicity of the governing equations. This holds for both one-phase solids and porous fluid-saturated solids with rate-independent behaviour of the skeleton (except for the special case of incompressible constituents and the so-called u-p-approximation, see Sect. 3 for the details).

The requirement that the system of equations be hyperbolic imposes conditions on the eigenvalues and eigenvectors of the matrix of the system. In particular, the eigenvalues (the characteristic speeds) must be real. In most studies dealing with hyperbolicity for plastic solids, the objective is to find out whether and under what conditions the characteristic speeds may become complex numbers and hyperbolicity may thus be lost [1–7]. For one-phase solids, the squared characteristic speeds multiplied by the density are known to be the eigenvalues of the acoustic tensor. Real positive eigenvalues and the existence of a complete set of eigenvectors of the acoustic tensor are necessary and sufficient conditions for hyperbolicity [8]. For hyperelastic

V. A. Osinov (✉)
Institute of Soil Mechanics and Rock Mechanics, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
E-mail: vladimir.osinov@kit.edu

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materials, a conclusion on hyperbolicity can be drawn from the existence and properties (convexity) of a strain energy function. Hypoelasticity and plasticity models are more difficult to treat analytically and may require a numerical technique to calculate the eigenvalues.

Ill-posedness of a dynamic problem caused by the loss of hyperbolicity may be either a consequence of incorrect constitutive modelling or a manifestation of the real physical behaviour of the solid. The former is likely to be the case if there are two complex-conjugate eigenvalues of the acoustic tensor ('flutter instability' [9]), whereas the latter is believed to be the case if the acoustic tensor is singular ('stationary discontinuity' [9,10]) or has a negative eigenvalue, which is associated with the localization of deformation and shear band formation.

The effective stress principle originally established in soil mechanics provides a link between the constitutive description of a fluid-saturated solid under fully drained conditions with constant pore pressure (called 'dry solid' or 'skeleton' for brevity) and the description of the same solid under arbitrary drainage conditions with variable pore pressure. The effective stress principle states that the constitutive relations for the dry solid are valid for the saturated solid when written for the properly defined effective stresses. A question arising in this connection is whether the hyperbolicity of the dynamic equations for a dry solid guarantees hyperbolicity of the equations for the same but fluid-saturated solid. As shown in [7], the dynamic equations for the saturated solid may lose hyperbolicity while the equations for the dry solid remain hyperbolic. These observations motivate seeking sufficient conditions for hyperbolicity in both dry and saturated states.

The present paper addresses hyperbolicity of two systems of equations which describe fluid-saturated solids with zero and nonzero permeability (locally undrained and drained conditions, respectively). The paper also introduces a notion of wave speed consistency between the undrained and drained cases based on the argument that the drained solution should tend to the undrained solution as the permeability tends to zero. A proposition proved in Sect. 5 gives sufficient conditions for both hyperbolicity and the wave speed consistency for the drained and undrained cases. The proposition is valid for plastic as well as elastic solids.

2 Dynamic equations for fluid-saturated solids

2.1 Effective stress

Consider first a dry porous solid whose constitutive response is rate independent and incrementally linear. The constitutive relations in Cartesian coordinates x_1, x_2, x_3 can be written in rate form as

$$\frac{\partial \sigma_{ji}}{\partial t} = C_{jikl}\frac{\partial v_k}{\partial x_l}, \tag{1}$$

where $v_i, \sigma_{ji}, C_{jikl}$ are, respectively, the components of the velocity vector, the stress tensor and the stiffness tensor, and t is time. The summation convention for repeated indices will be used throughout this paper. The partial time derivatives will be written in all equations in place of the material derivatives neglecting the convective terms. The stiffness coefficients C_{jikl} in the constitutive relations will be treated as constants, which corresponds to a linearly elastic solid. The applicability of the results to plastic solids will be discussed in Sect. 7.

If the solid is saturated with a fluid, then σ_{ji} in (1) are the components of the *effective stress*, and v_k are the velocity components of the skeleton. The effective stress is defined here as a stress which depends on the macroscopic deformation of the skeleton and is not influenced by changes in the pore pressure. In the case of an isotropic elastic material of the skeleton (the solid phase), the effective stress components are [11]

$$\sigma_{ji} = \sigma_{ji}^{total} + \left(\delta_{ji} - \frac{C_{jikk}}{3K_s}\right)p_f, \tag{2}$$

where σ_{ji}^{total} are the total stress components, p_f is the pore pressure (positive for compression), K_s is the bulk modulus of the solid phase, and δ_{ji} is the Kronecker delta. If the stiffness tensor of the skeleton, C_{jikl}, is such that

$$C_{jikk} = 3K\delta_{ji} \tag{3}$$

with a scalar K, then the effective stresses (2) can be written as

$$\sigma_{ji} = \sigma_{ji}^{total} + \alpha p_f \delta_{ji}, \tag{4}$$

where

$$\alpha = 1 - \frac{K}{K_s}. \tag{5}$$

Condition (3) is satisfied, in particular, for an isotropic skeleton with the bulk modulus K. The parameter α is sometimes called Biot's effective stress coefficient. If $K_s \gg |C_{jikk}|$, then the solid phase may be considered incompressible, and the effective stresses (4) with $\alpha = 1$ are obtained in both isotropic and anisotropic cases, i.e. independently of whether (3) is satisfied or not.

The effective stress definition (2) in the general anisotropic case complicates the theory as compared to (4) as it gives rise to additional tensorial quantities in the equations of motion of the solid phase and in the constitutive equation for the pore pressure. In applications, however, it may be acceptable to use the effective stress definition (4) with a properly chosen $\alpha \neq 1$ even if the constitutive response of the skeleton is anisotropic and does not exactly obey (3). In such a case, the equations may be regarded as an approximation of the exact theory. We do not impose the condition (3) on the stiffness tensor but nevertheless define the effective stress by (4) with a given α in order to make the analysis applicable to the approximate theory. Further, we assume that α satisfies the inequality

$$\alpha > n, \tag{6}$$

where n is the porosity, which is justified for a porous solid with an elastic skeleton (for details, see [12–14] and references therein).

2.2 Zero permeability

If the skeleton permeability is zero (locally undrained conditions), then the velocity field is common to both the skeleton and the fluid. The equations of motion without mass forces are written for the total stress as

$$\frac{\partial \sigma_{ji}^{total}}{\partial x_j} = \varrho \frac{\partial v_i}{\partial t}, \tag{7}$$

where

$$\varrho = (1-n)\varrho_s + n\varrho_f \tag{8}$$

is the density of the medium, and ϱ_s, ϱ_f are the densities of the solid and fluid phases, respectively. The evolution equation for the pore pressure is [15]

$$\frac{\partial p_f}{\partial t} = -\alpha Q \frac{\partial v_k}{\partial x_k}, \tag{9}$$

where

$$Q = \left(\frac{n}{K_f} + \frac{\alpha - n}{K_s}\right)^{-1}, \tag{10}$$

and K_f is the pore fluid bulk modulus. Inequality (6) guarantees that $Q > 0$. Equations (1), (4), (9) give the constitutive equation for the total stress

$$\frac{\partial \sigma_{ji}^{total}}{\partial t} = \left(C_{jikl} + \alpha^2 Q \delta_{ji}\delta_{kl}\right) \frac{\partial v_k}{\partial x_l}. \tag{11}$$

The dynamic deformation of the saturated solid under locally undrained conditions is described by 9 scalar equations (7), (11) for 9 unknown functions v_i, σ_{ji}^{total}, where $\sigma_{ji}^{total} = \sigma_{ij}^{total}$.

2.3 Nonzero permeability

In a fluid-saturated solid with nonzero permeability (locally drained conditions), the solid skeleton and the pore fluid have in general different velocities. The velocity components for the solid and fluid phases will be denoted, respectively, by v_{si} and v_{fi}, where the first subscript stands for the phase and the second one indicates the Cartesian component. The equations of motion are written separately for the solid and fluid phases [15]:

$$\frac{\partial \sigma_{ji}}{\partial x_j} - (\alpha - n)\frac{\partial p_f}{\partial x_i} + \frac{\varrho_f g n^2}{k}(v_{fi} - v_{si}) = (1-n)\varrho_s \frac{\partial v_{si}}{\partial t}, \tag{12}$$

$$-n\frac{\partial p_f}{\partial x_i} - \frac{\varrho_f g n^2}{k}(v_{fi} - v_{si}) = n\varrho_f \frac{\partial v_{fi}}{\partial t}, \tag{13}$$

where k is the skeleton permeability (m/s) and g is the acceleration due to gravity. The constitutive relations (1) for the effective stresses are written with the skeleton velocity:

$$\frac{\partial \sigma_{ji}}{\partial t} = C_{jikl}\frac{\partial v_{sk}}{\partial x_l}. \tag{14}$$

The evolution equation for the pore pressure involves both the skeleton and the fluid velocities [15]:

$$\frac{\partial p_f}{\partial t} = -Q(\alpha - n)\frac{\partial v_{sk}}{\partial x_k} - Qn\frac{\partial v_{fk}}{\partial x_k}. \tag{15}$$

The dynamic deformation of the saturated solid under locally drained conditions is described by 13 scalar equations (12)–(15) for 13 unknown functions $v_{si}, v_{fi}, \sigma_{ji}, p_f$.

3 Definition of hyperbolicity

The dynamic equations of the previous section with the velocities and stresses as dependent variables are systems of first-order partial differential equations of the form

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{3} M_{ij}^{(k)}\frac{\partial u_j}{\partial x_k} = F_i(u_1, ..., u_N), \quad i = 1, ..., N, \tag{16}$$

where $u_1, ..., u_N$ are functions of Cartesian coordinates x_1, x_2, x_3 and time t, $M_{ij}^{(k)}$ are the components of real $N \times N$ matrices $M^{(k)}$, $k = 1, 2, 3$, and F_i are sufficiently smooth functions of their arguments.

Here we note that the two models of fluid-saturated solids mentioned in Sect. 1, namely the case of incompressible constituents and the u-p-approximation, are not covered by the analysis presented in this paper, as the governing equations of those models are not in the form (16). If both the solid and fluid phases are incompressible, then $Q \to \infty$ and the evolution equation for the pore pressure (15) reduces to the incompressibility condition imposed on the velocity fields. This condition does not contain any time derivatives and thus does not conform with (16). In the u-p-approximation widely used in the numerical modelling of fluid-saturated solids [15], the pore fluid is assumed to have the same acceleration as the solid skeleton, that is, $\partial v_{fi}/\partial t$ in (13) is replaced with $\partial v_{si}/\partial t$. This assumption yields two equations with the time derivative of the skeleton velocity and no equation with the time derivative of the fluid velocity, with the consequence that the system is again not in the form (16). The definition of hyperbolicity given below for systems (16) does not apply to these two cases.

Definition 1 (hyperbolicity). System (16) is called *hyperbolic* if for any real n_1, n_2, n_3 the matrix $M = \sum_{k=1}^{3} n_k M^{(k)}$ is diagonalizable by a real matrix ([16], Sect. 7.3.1).

Equivalently, the system is called hyperbolic if for any real n_1, n_2, n_3 the matrix M has N linearly independent real eigenvectors. Note that the notion of strict hyperbolicity (real and distinct eigenvalues of M) is a priori too strong for the equations studied here. Even in the simplest case of an isotropic solid in three-dimensional problems, the matrix M has a double eigenvalue that corresponds to transverse waves. Moreover, as will be seen below, there always exists a multiple eigenvalue equal to zero.

The definition of hyperbolicity leads to the eigenvalue problem

$$M_{ij}u_j^0 = cu_i^0, \quad i = 1, ..., N, \tag{17}$$

where M_{ij} are the components of the matrix $M = \sum_{k=1}^{3} n_k M^{(k)}$, and u_i^0 are the components of an eigenvector associated with an eigenvalue c. Without loss of generality, the factors n_1, n_2, n_3 in the definition of the matrix M will be taken to be the components of a unit vector $\mathbf{n}$. In this case, the eigenvalues c are referred to as the characteristic speeds since they coincide with the speeds of plane waves that propagate in the direction $\mathbf{n}$ and are solutions to a homogeneous system (16). This can be shown by considering a solution in the form of a plane wave propagating in the direction $\mathbf{n}$:

$$u_i(x_1, x_2, x_3, t) = u_i^0 f(y), \quad y = n_j x_j - ct, \quad i = 1, \ldots, N, \tag{18}$$

where $f(y)$ is a differentiable function, u_i^0 are amplitudes, and c is the wave speed. Substituting (18) into (16) with $F_i \equiv 0$ leads to the eigenvalue problem (17) for the amplitudes u_i^0 and the wave speed c. The verification of hyperbolicity amounts to the analysis of the wave speeds and the amplitude vectors $(u_1^0, ..., u_N^0)^T$ for all wave propagation directions $\mathbf{n}$.

4 Wave speed consistency

Suppose that the system (12)–(15) for nonzero permeability is hyperbolic, and we have solved a boundary-value problem with the impermeability condition on the boundary. Consider the system (7), (11) for zero permeability with the same values of the physical parameters $(C_{jikl}, K_f, K_s, \alpha, n, \varrho_s, \varrho_f)$. Suppose that this system is also hyperbolic, and we have solved the same boundary-value problem. It is reasonable to expect that the first solution will be close to the second one if the permeability is low enough and, furthermore, to expect that the difference between the two solutions will vanish as the permeability will tend to zero. This asymptotic property of the solutions may be regarded as a kind of consistency between the two systems of equations. However, this property does not follow directly from the equations for the drained case since they cannot be solved with $k = 0$: the equations degenerate and lead to the equality $\mathbf{v}_s = \mathbf{v}_f$.

A necessary condition for the asymptotic property to hold can be deduced from the fact that the domain of influence of initial and boundary data for hyperbolic equations is determined by the characteristic speeds. To be specific, consider the propagation of a plane wave induced by a prescribed disturbance on the boundary of a half-space. The asymptotic property can hold for an arbitrary boundary disturbance only if the largest characteristic speed in the drained case is not smaller than the largest characteristic speed in the undrained case. (Notice that the characteristic speeds in the drained case do not depend on the permeability). In other words, the drained solution cannot approach the undrained solution if the drained wave is unable to propagate as fast as the undrained wave. This argument suggests the following definition.

Definition 2 (wave speed consistency). Suppose that the two systems (7), (11) and (12)–(15) for the undrained and drained cases have the same values of the physical parameters and are both hyperbolic. The two systems are said to be *wave speed consistent* if for each direction $\mathbf{n}$ the largest characteristic speed in the drained case is not smaller than the largest characteristic speed in the undrained case for the same direction $\mathbf{n}$.

5 Hyperbolicity of the two systems

It is known that for a one-phase solid (in the present context—for a dry porous solid), the squared characteristic speeds multiplied by the density are the eigenvalues of the acoustic tensor

$$A_{ik} = C_{jikl} n_j n_l. \tag{19}$$

For the dynamic equations to be hyperbolic in the sense of the definition given in Sect. 3, it is necessary and sufficient that for each $\mathbf{n}$ the acoustic tensor has a complete set of real eigenvectors associated with positive eigenvalues [8]. As mentioned in Sect. 1, these necessary and sufficient conditions for hyperbolicity for a dry solid do not guarantee hyperbolicity for a fluid-saturated solid. It will be shown below that an additional condition which guarantees both hyperbolicity and the wave speed consistency is the symmetry of the acoustic tensor.

The condition of symmetry of the acoustic tensor for all directions $\mathbf{n}$ can be written in terms of the components of the stiffness tensor. The symmetry $A_{ik} = A_{ki}$, or $C_{jikl}n_jn_l = C_{jkil}n_jn_l$, for all $\mathbf{n}$ means that the matrix $L^{(ik)}$ with the components $L^{(ik)}_{jl} = C_{jikl} - C_{jkil}$ is skew-symmetric for all i, k, that is,

$$C_{jikl} - C_{jkil} = C_{lkij} - C_{likj} \tag{20}$$

for all i, k, j, l, or, equivalently, using the minor symmetry $C_{jikl} = C_{ijkl}$,

$$C_{ijkl} - C_{klij} = C_{kjil} - C_{ilkj}. \tag{21}$$

Equation (20) or (21) is the necessary and sufficient condition for the acoustic tensor to be symmetric for all directions $\mathbf{n}$. Notice that the major symmetry $C_{jikl} = C_{klji}$ is sufficient but not necessary for the symmetry of the acoustic tensor.

The following proposition gives sufficient conditions for both hyperbolicity and the wave speed consistency for fluid-saturated solids.

Proposition *If for each unit vector* $\mathbf{n}$ *the acoustic tensor (19) of the skeleton is symmetric and positive definite, then the systems (7), (11) and (12)–(15) for zero and nonzero permeability are hyperbolic and wave speed consistent.*

Proof The eigenvalue problem (17) obtained from system (7), (11) for zero permeability is

$$-\frac{1}{\varrho} n_j \sigma^0_{ji} = c v^0_i, \tag{22}$$

$$-\left(C_{jikl} + \alpha^2 Q \delta_{ji}\delta_{kl}\right) n_l v^0_k = c\sigma^0_{ji}, \tag{23}$$

where v^0_i, σ^0_{ji} are the velocity and total stress components. For $c \neq 0$, substituting σ^0_{ji} from (23) into (22) leads to an eigenvalue problem for the velocity components v^0_i:

$$\frac{1}{\varrho}\left(A_{ik} + \alpha^2 Q n_i n_k\right) v^0_k = c^2 v^0_i, \tag{24}$$

where A_{ik} are the components of the acoustic tensor (19) of the skeleton.

Equations (12)–(15) for nonzero permeability yield the eigenvalue problem

$$\frac{1}{(1-n)\varrho_s}\left[-n_j\sigma^0_{ji} + (\alpha - n) n_i p^0_f\right] = c v^0_{si}, \tag{25}$$

$$\frac{1}{\varrho_f} n_i p^0_f = c v^0_{fi}, \tag{26}$$

$$-C_{jikl} n_l v^0_{sk} = c\sigma^0_{ji}, \tag{27}$$

$$Q(\alpha - n) n_k v^0_{sk} + Q n n_k v^0_{fk} = c p^0_f, \tag{28}$$

where v^0_{si}, v^0_{fi}, σ^0_{ji}, p^0_f are the amplitudes of the skeleton velocities, fluid velocities, effective stresses and pore pressure, respectively. For $c \neq 0$, substituting σ^0_{ji} and p^0_f from (27), (28) into (25), (26) leads to an eigenvalue problem for the velocity components v^0_{si}, v^0_{fi}:

$$\frac{1}{(1-n)\varrho_s}\left[A_{ik} v^0_{sk} + Q(\alpha - n)^2 n_i n_k v^0_{sk} + Q(\alpha - n) n n_i n_k v^0_{fk}\right] = c^2 v^0_{si}, \tag{29}$$

$$\frac{1}{\varrho_f} Q n_i n_k\left[(\alpha - n) v^0_{sk} + n v^0_{fk}\right] = c^2 v^0_{fi}. \tag{30}$$

In the following, in order to make the proof easier, the equations will be written in a rotated coordinate system whose x_1-axis is parallel to the vector $\mathbf{n}$. The transition from the original to the rotated system is equivalent to the transformation of the coordinates and the dependent variables as vector and tensor components using the standard formulae with the matrix of the direction cosines. The matrix of the system after the rotation is connected with the original matrix through a similarity transformation which preserves the eigenvalues and diagonalizability.

Hyperbolicity for zero permeability. Equations (22), (23) written in the rotated system ($n_1 = 1, n_2 = n_3 = 0$) are

$$-\frac{1}{\varrho}\,\sigma^0_{1i} = cv^0_i, \tag{31}$$

$$-\left(C_{jik1} + \alpha^2 Q\delta_{ji}\delta_{k1}\right) v^0_k = c\sigma^0_{ji}. \tag{32}$$

We need to show that the eigenvalue problem (31), (32) yields 9 linearly independent real eigenvectors. The eigenvalue problem (24) for the velocity components can be written as

$$B_{ik}v^0_k = c^2 v^0_i \tag{33}$$

with the matrix

$$B = \frac{1}{\varrho}\begin{pmatrix} A_{11} + \alpha^2 Q & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}. \tag{34}$$

Since the acoustic tensor is symmetric and positive definite, this holds true for the matrix B as well. Hence, the matrix B has three linearly independent eigenvectors associated with positive eigenvalues. An eigenvector of B associated with an eigenvalue $\xi > 0$, when substituted into (32) with $c = \pm\sqrt{\xi}$, gives the components σ^0_{ji} of two linearly independent eigenvectors of (31), (32). Three linearly independent eigenvectors of B produce six linearly independent eigenvectors of (31), (32). Another three eigenvectors which complete the required set of 9 linearly independent eigenvectors are associated with $c = 0$. Each of them has only one nonzero component: either σ^0_{22}, σ^0_{33} or σ^0_{23}.

Hyperbolicity for nonzero permeability. Equations (25)–(28) in the rotated system become

$$\frac{1}{(1-n)\varrho_s}\left[-\sigma^0_{1i} + (\alpha - n)\delta_{1i}p^0_f\right] = cv^0_{si}, \tag{35}$$

$$\frac{1}{\varrho_f}\,\delta_{1i}p^0_f = cv^0_{fi}, \tag{36}$$

$$-C_{jik1}v^0_{sk} = c\sigma^0_{ji}, \tag{37}$$

$$Q(\alpha - n)v^0_{s1} + Qnv^0_{f1} = cp^0_f. \tag{38}$$

We need to show that the eigenvalue problem (35)–(38) yields 13 linearly independent real eigenvectors. Equations (29), (30) in the rotated system give the eigenvalue problem

$$Dw = c^2 w \tag{39}$$

with the column vector $w = (v^0_{s1}, v^0_{s2}, v^0_{s3}, v^0_{f1})^T$ and the matrix

$$D = \frac{1}{(1-n)\varrho_s}\begin{pmatrix} A_{11} & A_{12} & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & 0 \\ A_{31} & A_{32} & A_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \dfrac{(\alpha-n)^2 Q}{(1-n)\varrho_s} & 0 & 0 & \dfrac{(\alpha-n)nQ}{(1-n)\varrho_s} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{(\alpha-n)Q}{\varrho_f} & 0 & 0 & \dfrac{nQ}{\varrho_f} \end{pmatrix}. \tag{40}$$

Consider a symmetric matrix $\hat{D}$ with the same components as in D except for $\hat{D}_{14}$ and $\hat{D}_{41}$, which are

$$\hat{D}_{14} = \hat{D}_{41} = (\alpha - n)Q\sqrt{\frac{n}{(1-n)\varrho_s\varrho_f}}. \tag{41}$$

Taking into account that the acoustic tensor is positive definite, we have for any nonzero vector with components w_i, $i = 1, ..., 4$,

$$\begin{aligned}\hat{D}_{ik} w_i w_k &= \frac{1}{(1-n)\varrho_s} \sum_{i,k=1}^{3} A_{ik} w_i w_k \\ &\quad + Q \left[\frac{(\alpha - n)^2}{(1-n)\varrho_s} w_1^2 + 2(\alpha - n) \sqrt{\frac{n}{(1-n)\varrho_s \varrho_f}} w_1 w_4 + \frac{n}{\varrho_f} w_4^2 \right] \\ &= \frac{1}{(1-n)\varrho_s} \sum_{i,k=1}^{3} A_{ik} w_i w_k + Q \left(\frac{\alpha - n}{\sqrt{(1-n)\varrho_s}} w_1 + \sqrt{\frac{n}{\varrho_f}} w_4 \right)^2 > 0. \end{aligned} \tag{42}$$

This shows that $\hat{D}$ is positive definite and, because this matrix is symmetric, it has a complete set of linearly independent eigenvectors associated with real positive eigenvalues.

The matrices D and $\hat{D}$ are similar: $\hat{D} = T^{-1} D T$, where

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix}, \quad a = \sqrt{\frac{(1-n)\varrho_s}{n\varrho_f}}. \tag{43}$$

Since the matrix D is similar to $\hat{D}$, it has 4 linearly independent eigenvectors associated with the same real positive eigenvalues, say ξ. These 4 eigenvectors substituted into (37), (38) with $c = \pm\sqrt{\xi}$ give 8 linearly independent eigenvectors for (35)–(38). Another 5 eigenvectors which complete the set of 13 linearly independent eigenvectors are associated with $c = 0$ and have only one nonzero component v^0_{f2}, v^0_{f3}, σ^0_{22}, σ^0_{33} or σ^0_{23}.

Wave speed consistency. Now that we have proved that the two systems are hyperbolic, we will show that they are wave speed consistent. Let $\lambda_{max}(B)$, $\lambda_{max}(\hat{D})$ be the largest eigenvalues of the matrices B and $\hat{D}$ defined by (34), (40), (41). The condition for the wave speed consistency is $\lambda_{max}(\hat{D}) \geq \lambda_{max}(B)$. Since $\hat{D}$ and B are real and symmetric, we have ([17], Chap. 7)

$$\lambda_{max}(\hat{D}) = \max_{\|w\|=1} \hat{D}_{ij} w_i w_j, \quad \lambda_{max}(B) = \max_{\|u\|=1} B_{ij} u_i u_j, \tag{44}$$

where w_i, $i = 1, ..., 4$, and u_i, $i = 1, 2, 3$, are the components of vectors $w \in \mathbb{R}^4$ and $u \in \mathbb{R}^3$, and $\|w\| = \sqrt{w_i w_i}$, $\|u\| = \sqrt{u_i u_i}$ are the Euclidean norms of the vectors. Let $\mathcal{A}$ be a three-dimensional subspace of $\mathbb{R}^4$ comprised of vectors w such that

$$w_4 = \sqrt{\frac{n\varrho_f}{(1-n)\varrho_s}} w_1. \tag{45}$$

Then

$$\max_{\|w\|=1} \hat{D}_{ij} w_i w_j \geq \max_{w \in \mathcal{A},\ \|w\|=1} \hat{D}_{ij} w_i w_j, \tag{46}$$

because the variation of w on the right-hand side is more restrictive. Taking (42) into account, we see that for $w \in \mathcal{A}$

$$\hat{D}_{ij} w_i w_j = \frac{\varrho}{(1-n)\varrho_s} B_{ij} u_i u_j, \tag{47}$$

where $u \in \mathbb{R}^3$ is such that

$$u_1 = w_1, \ u_2 = w_2, \ u_3 = w_3. \tag{48}$$

Let $\mathcal{E}$ denote the set of vectors $u \in \mathbb{R}^3$ satisfying (48), where $w \in \mathcal{A}$, $\|w\| = 1$. The coordinates of the vectors $u \in \mathcal{E}$ lie on the ellipsoid

$$u_1^2 \left(1 + \frac{n\varrho_f}{(1-n)\varrho_s}\right) + u_2^2 + u_3^2 = 1. \tag{49}$$

The largest sphere enclosed in this ellipsoid has the radius

$$r = \sqrt{\frac{(1-n)\varrho_s}{\varrho}}. \tag{50}$$

Since the Euclidean norm of each vector $u \in \mathcal{E}$ is greater than or equal to r, we have

$$\max_{w\in\mathcal{A},\ \|w\|=1} \hat{D}_{ij} w_i w_j = \frac{\varrho}{(1-n)\varrho_s} \max_{u\in\mathcal{E}} B_{ij} u_i u_j \geq \frac{\varrho}{(1-n)\varrho_s} \max_{\|u\|=1} r^2 B_{ij} u_i u_j = \max_{\|u\|=1} B_{ij} u_i u_j. \tag{51}$$

From (44), (46), (51), it follows that

$$\lambda_{max}(\hat{D}) \geq \lambda_{max}(B). \tag{52}$$

□

6 Isotropic elastic solids

In the particular case of an isotropic elastic skeleton, the components of the acoustic tensor are

$$A_{ik} = (\lambda + \mu) n_i n_k + \mu \delta_{ik}, \tag{53}$$

where λ and μ are the Lamé constants. The matrices B and D in the rotated coordinate system ($n_1 = 1, n_2 = n_3 = 0$) become

$$B = \frac{1}{\varrho} \begin{pmatrix} \lambda + 2\mu + \alpha^2 Q & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \tag{54}$$

$$D = \frac{1}{(1-n)\varrho_s} \begin{pmatrix} \lambda + 2\mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \dfrac{(\alpha-n)^2 Q}{(1-n)\varrho_s} & 0 & 0 & \dfrac{(\alpha-n) n Q}{(1-n)\varrho_s} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{(\alpha-n) Q}{\varrho_f} & 0 & 0 & \dfrac{n Q}{\varrho_f} \end{pmatrix}. \tag{55}$$

As in a one-phase solid, there exist purely transverse and longitudinal waves. The squared transverse wave speeds in the undrained and drained cases are, respectively, μ/ϱ and $\mu/[(1-n)\varrho_s]$. The latter is the same as in the dry solid and is larger than in the undrained case. The squared longitudinal wave speed in the undrained case, denoted here by c_u^2, is

$$c_u^2 = \frac{1}{\varrho}(\lambda + 2\mu + \alpha^2 Q). \tag{56}$$

The longitudinal wave speeds and amplitudes in the drained case are determined by the equations

$$D_{11} v_{s1}^0 + D_{14} v_{f1}^0 = c^2 v_{s1}^0, \tag{57}$$

$$D_{41} v_{s1}^0 + D_{44} v_{f1}^0 = c^2 v_{f1}^0. \tag{58}$$

The eigenvalue problem (57), (58) leads to a quadratic equation for c^2:

$$(D_{11} - c^2)(D_{44} - c^2) - D_{14} D_{41} = 0. \tag{59}$$

It can be checked that the discriminant of (59) is always positive, so the two roots are different.

If the material parameters satisfy the so-called *dynamic compatibility* condition [18–20], the governing equations for longitudinal waves in the drained case admit travelling-wave solutions with the solid and fluid phases moving with the same velocity, as if the permeability were zero. The dynamic compatibility condition can easily be obtained by putting $v^0_{s1} = v^0_{f1}$ in (57), (58), which gives $D_{11} + D_{14} = D_{41} + D_{44}$ and, taking the components from (55), leads to the condition

$$(\lambda + 2\mu)\varrho_f = \alpha Q \left[(1 - n)\varrho_s - (\alpha - n)\varrho_f\right]. \tag{60}$$

If the dynamic compatibility condition (60) is satisfied, one of the two roots of (59) must be equal to c_u^2, since one of the two waves determined by (57), (58), namely the one with $v^0_{s1} = v^0_{f1}$, becomes the same as in the undrained case. It can be shown that the wave with $v^0_{s1} = v^0_{f1}$ is the faster of the two waves, and we therefore obtain equality in (52).

7 Plastic solids

So far it has been assumed that the skeleton of a fluid-saturated solid is elastic with a constant stiffness tensor C_{jikl}. A question of particular importance for applications is whether the proposition proved in Sect. 5 is also valid for plastic solids. Here we outline the results obtained in [8] for plastic fluid-saturated solids with an incompressible solid phase. The assumption of incompressibility of the solid phase leads to $\alpha = 1$ in the equation for the effective stress (4) and is justified, for instance, for soils. This assumption is crucial for the equations with a plastic skeleton whose stiffness tensor is not constant but depends on the current state (in particular, the stress state) and the direction of deformation (e.g. loading or unloading in elasto-plasticity). A variable stiffness tensor C_{jikl} makes the definition (2) of the effective stress inapplicable unless $K_s \to \infty$.

The plastic skeleton entails two changes to the equations as compared to the elastic skeleton. First, the constitutive relations and the equations of motion are supplemented with evolution equations for additional scalar or tensorial quantities involved in a particular plasticity model as dependent variables. We assume that the evolution of the new functions is determined by the deformation of the skeleton. This assumption covers a wide class of rate-independent plasticity models and can also be extended to certain rate-dependent visco-plasticity models. The second change caused by the plastic skeleton is that the coefficients of the equations are allowed to be functions of the dependent variables including the new ones. This especially concerns the stiffness tensor C_{jikl} which becomes a function of the current effective stress.

As shown in [8], the two systems of equations—the reduced system with constant coefficients consisting of the equations of motion and the constitutive relations, and the full system with variable coefficients and additional evolution equations—agree with each other from the viewpoint of hyperbolicity: they are either both hyperbolic or both non-hyperbolic. This result extends the validity of the proposition of Sect. 5 to fluid-saturated solids with a plastic skeleton.

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References

1. Loret, B., Prévost, J.H., Harireche, O.: Loss of hyperbolicity in elastic-plastic solids with deviatoric associativity. Eur. J. Mech. A/Solids **9**(3), 225–231 (1990)
2. Loret, B., Harireche, O.: Acceleration waves, flutter instabilities and stationary discontinuities in inelastic porous media. J. Mech. Phys. Solids **39**(5), 569–606 (1991)
3. An, L., Schaeffer, D.G.: The flutter instability in granular flow. J. Mech. Phys. Solids **40**(3), 683–698 (1992)

4. Bigoni, D., Zaccaria, D.: On the eigenvalues of the acoustic tensor in elastoplasticity. Eur. J. Mech. A/Solids **13**(5), 621–638 (1994)
5. Osinov, V.A.: Theoretical investigation of large-amplitude waves in granular soils. Soil Dyn. Earthq. Eng. **17**(1), 13–28 (1998)
6. Osinov, V.A., Gudehus, G.: Dynamics of hypoplastic materials: theory and numerical implementation. In: Hutter, K., Kirchner, N. (eds.) Dynamic Response of Granular and Porous Materials under Large and Catastrophic Deformations, pp. 265–284. Springer, Berlin (2003)
7. Osinov, V.A.: On well-posedness of the dynamic problem for an anisotropic fluid-saturated solid. Arch. Appl. Mech. **79**, 69–80 (2009)
8. Osinov, V. A.: On hyperbolicity of the dynamic equations for plastic fluid-saturated solids, *submitted*
9. Rice, J. R.: The localization of plastic deformation. In: Koiter, W.T. (ed.) Theoretical and Applied Mechanics. Proc. 14th IUTAM Congress, North-Holland, Amsterdam, pp. 207-220 (1976)
10. Hill, R.: Acceleration waves in solids. J. Mech. Phys. Solids **10**, 1–16 (1962)
11. Carroll, M.M.: An effective stress law for anisotropic elastic deformation. J. Geophys. Res. **84**(B13), 7510–7512 (1979)
12. Biot, M.A., Willis, D.G.: The elastic coefficients of the theory of consolidation. J. Appl. Mech. **24**, 594–601 (1957)
13. Zimmerman, R.W., Somerton, W.H., King, M.S.: Compressibility of porous rocks. J. Geophys. Res. **91**(B12), 12765–12777 (1986)
14. Berryman, J.G.: Effective stress for transport properties of inhomogeneous porous rock. J. Geophys. Res. **97**(B12), 17409–17424 (1992)
15. Zienkiewicz, O.C., Chan, A.H.C., Pastor, M., Schrefler, B.A., Shiomi, T.: Computational geomechanics with special reference to earthquake engineering. Wiley, Chichester (1999)
16. Evans, L.C.: Partial differential equations, 2nd edn. American Mathematical Society, Providence (2010)
17. Bellman, R.: Introduction to matrix analysis, 2nd edn. SIAM, Philadelphia (1997)
18. Biot, M. A.: Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range. J. Acoust. Soc. Am. **28**(2), 168-178 (1956)
19. Biot, M.A.: Mechanics of deformation and acoustic propagation in porous media. J. Appl. Physics **33**(4), 1482–1498 (1962)
20. Simon, B.R., Zienkiewicz, O.C., Paul, D.K.: An analytical solution for the transient response of saturated porous elastic solids. Int. J. Num. Anal. Meth. Geomech. **8**, 381–398 (1984)