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Ordinal Potential Differential Games to Model Human-Machine Interaction in Vehicle-Manipulators

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***Abstract*— Potential games have some useful characteristics related to the existence and computability of their Nash equilibria, which make their use attractive also in the context of modelling interactions. This paper presents the use of potential differential games to model human-machine interaction. We extend the definition of static ordinal potential games to differential games for modelling and analysing human-machine interaction in the control of large vehicle-manipulators. We provide sufficient and necessary conditions for the existence of a potential differential game. In addition, we present an optimization formulation finding a linear-quadratic (LQ) potential differential game to a original game. The suitability of the proposed modelling approach is verified using simulation examples.**

I. Introduction

A Vehicle-manipulator (VM) includes a mobile platform (vehicle) and a robotic arm (manipulator) that can be applied in numerous applications like teleoperated robots for nuclear waste cleaning [1], working vessels [2] or mobility assistants [3]. A further area in robotics is the application of large VMs. Large VMs are systems in which a large, hydraulically actuated manipulator is attached to a tractor or mid-size heavy-duty vehicle, see e.g. [4] or [5]. They are common in roadside maintenance or farming works. A human operator controlling the manipulator is inevitable due to its widespread use in complex and unstructured working environments. In addition, the applied sensors are unreliable [6] and the models of the hydraulic actuator are inaccurate due to their non-linearities [6]. Therefore, full automation of the system is not possible in the near future [7] and the operator has to control both the vehicle and the manipulator nowadays.

With promising research results in the field of autonomous vehicles and field robots [8], [9], it is conceivable to develop automated vehicles, leading to a lower workload of the operator. However, in contrast to autonomous field robot applications, the automation of the vehicle of a VM also has to take into account the motion of the human-controlled manipulator in order to fulfill the dedicated task with the manipulator. The challenge in this setup is the realization of a support for the operator without measuring the reference and the states of the manipulator.

To overcome this challenge, the so-called *limited information control* (LIC) method has been proposed by the authors [10], [11]. The benefit of the LIC is that it does not require the trajectory measurements of the manipulator, only the inputs of the human operator controlling the manipulator. However, the heuristic design of the LIC, which is suggested in [12], impedes a generalization of the concepts and requires a time intensive manual tuning of controller. For that reason, a systematic derivation of the parameters of the LIC is necessary. The basic idea is that first, an identification with full information is carried out for the VM, which serves as a basis for the design of the LIC parameters. Such an identification is also possible in practice, for example in a test area, where all the states and references are measurable. The systematic design has the following steps, see Fig. 1:

1. Design a cooperative shared controller with methods based on the theory of differential games, see e.g. [13] that satisfies higher-level requirements on the controller.
2. Modelling this cooperative setup as an ordinal potential game.
3. Designing of a LIC with help of the potential game

This paper presents the second step of this overall process. In [13], a systematic approach utilizing differential games is proposed to design a cooperative controller for human-machine interactions that enables a faster configuration and a better understanding of the emerged human-machine cooperative system. However, this method is not practical for VMs, since all system states must be measured to apply the feedback control law designed with the approach of [13]. Therefore, this is utilized as a baseline for the derivation, because [13] provides an automatic computation of the parameters of the cooperative controller with full information from high level requirements. The goal of the design of the LIC having such an overall behaviour as a full-information controller designed with the method of [13].

For that, a novel modelling approach of the cooperative setup by VMs is proposed. This modelling happens by the use of a special class of games, the *ordinal potential games*. The benefits of a potential game is its more compact representation of the original game and simpler computation of the equilibrium of the game. Furthermore, this modelling approach with potential games enables the systematic design of the LIC. In our framework, no uncertainties of the system states and trajectories can be taken into account, as they are not measurements with uncertainties. Therefore, the class of the robust games (see e.g. [14]) are not suitable for our applications.

Potential games are games in which all the players and their objective functions can be replaced by one player and

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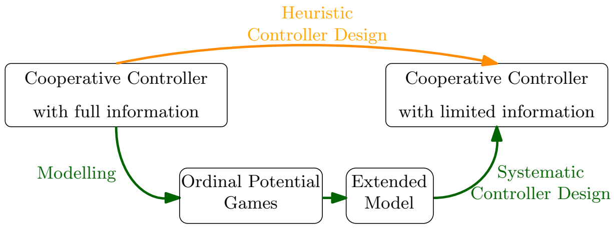


Fig. 1: The systematic design procedure for limited information cooperative controller [20]

one single objective function. The optimum of this single objective function yields the Nash equilibrium (NE) of the original N-player game. This means that there is no need for solving a coupled optimization problem to find a NE and therefore it is easier to determine.

The theory of the ordinal potential games for static games has intensively been investigated [15], [16] and used for technical applications like wireless network management [17] and energy optimization in smart grid system [18]. In [19], ordinal potential games are proposed with an underlying state space dynamics. However, the concept is not taken into consideration the differential games and the main results relate to *better reply games* with binary log-linear learning and *Markov games*. The authors did not provide any concept or results associated with differential games due to their application focus. To the best our knowledge, there is no contribution in the literature addressed to ordinal potential differential games. Therefore, this paper provides the extension of the ordinal potential games for differential games, which enables a broader use of the potential games in engineering applications. We provide the necessary and sufficient conditions for the existence of a linear-quadratic (LQ) potential differential game. Furthermore, an optimization method is given to find a potential game for a given a cooperative shared control setup defined by [13].

The remaining of the paper is structured as follows: Section II presents the state-of-the-art. Section III is addressed to potential differential games and the novel extension to *ordinal potential differential games*. The necessary and sufficient conditions for their existence are discussed in Section IV. A method finding an ordinal potential differential game is given in Section V. The application to two exemplary VMs is demonstrated with simulations in Section VI. Finally, a short summary and outlook is given in Section VII.

II. Differential Games for Human-Machine Interaction

In a cooperative control setup, both human and machine interact with the control system to achieve a common goal [21]. As mentioned in Section I, an established hypothesis is the use of optimal control theory for the description of human movements. In [22], it was shown that in a shared controller setup between two humans a NE is reached. A cooperative controller, in which both the human and the machine minimize their individual cost function and control one system can be analysed by means of game theory.

Generally, game theory is used to describe and model the interactions between several agents e.g. in economy or stock markets. In this paper, a strategic *game* is denoted by a tuple: $\Gamma = (\mathbb{P}, \mathbb{J}, \mathbb{U})$ with the set of N players $\mathbb{P} = \{1, 2, ..., N\}$ players. $\mathbb{J} = \left\{J^{(1)}, J^{(2)}, ..., J^{(N)}\right\}$ is denoted as the set of their cost functions and $\mathbb{U} = \{\mathcal{U}^{(1)} \times \mathcal{U}^{(2)} \times ... \times \mathcal{U}^{(N)}\}$ is the joint strategy set of the players. The optimal control strategies $\boldsymbol{u}^{(i)} \in \mathcal{U}^{(i)}$ are determined by the players as the result of minimizing their cost function $J^{(i)}$. As the costs of a player also depend on the actions of other players, a coupled optimization problem has to be solved to obtain the NE of the game. The widely used concept of the NE is defined as (see. e.g. in [23]):

Definition 1 (Nash-Equilibrium): The strategy $\boldsymbol{u}^{*(i)}$ of a strategic game Γ is called a Nash-Equilibrium if

$$J^{*(i)}(\boldsymbol{u}^{*(i)}, \boldsymbol{u}^{*(\neg i)}) = \min\ J^{(i)}(\boldsymbol{u}^{(i)}, \boldsymbol{u}^{*(\neg i)}) \qquad (1)$$

for all players $i \in \mathbb{P}$. The strategies of all players expect player i is denoted with $\boldsymbol{u}^{(\neg i)} = \left[\boldsymbol{u}^{(1)}, ..., \boldsymbol{u}^{(i-1)}, \boldsymbol{u}^{(i+1)}, ..., \boldsymbol{u}^{(N)}\right]$. A NE means that there is no incentive for a player to unilaterally deviate from his chosen strategy.

The theory of differential games was introduced by R. Isaacs in [24]. A differential game $\Gamma_d = (\mathbb{P}, \mathbb{J}, \mathbb{U}, \boldsymbol{f})$ is a extension of a static game, in which the control inputs[1] of the players $\left(\boldsymbol{u}^{(i)}\right)$ are determined by the optimization of the cost function $J^{(i)}(t, \tau, \boldsymbol{x}(t), \boldsymbol{u}(t))$ with respect to the dynamic system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t)), \quad t \in [0, \tau], \qquad (2)$$

Where the vector $\boldsymbol{u}$ is a combined input and defined as

$$\boldsymbol{u} = \left[\boldsymbol{u}, {}^{(1)}{}^T \boldsymbol{u}, {}^{(2)}{}^T ..., \boldsymbol{u}^{(N)}{}^T\right]^T. \qquad (3)$$

The dynamic system is defined for the time horizon $[0,\ \tau]$. In the following, the focus is set on linear time-invariant systems

$$\dot{\boldsymbol{x}}(t) = \mathbf{A}\boldsymbol{x}(t) + \sum_{i=1}^{N} \mathbf{B}^{(i)}\boldsymbol{u}^{(i)}(t), \qquad (4)$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

where $\mathbf{A}$ is the system matrix and $\mathbf{B}^{(i)}$ are the input matrices of the single players. Furthermore, it is assumed that the players have quadratic cost functions

$$J^{(i)} = \frac{1}{2}\int_0^\tau \boldsymbol{x}^T\mathbf{Q}^{(i)}\boldsymbol{x} + \boldsymbol{u}^T\mathbf{R}^{(i)}\boldsymbol{u}\ \mathrm{d}t, \qquad (5)$$

where the matrices $\mathbf{Q}^{(i)}$ are positive semi-definite, $\mathbf{R}^{(i)}$ are positive definite.

III. Potential Differential Games

The theory of potential games is introduced in [25]. The characteristics of the static *exact potential* and the static *ordinal potential games* are studied in [25], [15]. The advantage of the games lies in their simplified modelling: The game with many control strategy profiles $\mathcal{U}^{(i)}$ is reduced

[1] In this contribution, the terms "strategy" and "input" are used interchangeably.

to a single optimization of the potential function $J^{(p)}$, in which the optimum provides the NE of the original game that enables a simpler computation of that NE. In LQ-case, the potential function is

$$J^{(p)} = \frac{1}{2}\int_0^{\tau} \boldsymbol{x}^T\mathbf{Q}^{(p)}\boldsymbol{x} + \boldsymbol{u}^T\mathbf{R}^{(p)}\boldsymbol{u}\ \mathrm{d}t, \tag{6}$$

$\mathbf{Q}^{(p)}$ and $\mathbf{R}^{(p)}$ are positive semi-definite, positive definite respectively. There are some approaches in the literature how to extend exact potential games for dynamic systems (see [26], [27]). We extend their definitions for Potential LQ-differential Games.

Definition 2 (Exact Potential LQ-differential Game): A differential game Γ_d is called an *exact potential differential game* if a potential function $H^{(p)}$ exists such that

$$\frac{\partial H^{(p)}}{\partial \boldsymbol{u}^{(i)}} = \frac{\partial H^{(i)}}{\partial \boldsymbol{u}^{(i)}} \tag{7}$$

for $i \in \mathbb{P}$, where the Hamiltonian function of the player i is

$$H^{(i)} = \frac{1}{2}\boldsymbol{x}^T\mathbf{Q}^{(i)}\boldsymbol{x} + \frac{1}{2}\boldsymbol{u}^T\mathbf{R}^{(i)}\boldsymbol{u} + \boldsymbol{\lambda}^{(i)^T}\boldsymbol{f}(t), \tag{8}$$

where $\boldsymbol{\lambda}$ is the co-state variable.

A similar definition to the Definition 2 for special system classes is given in [26], in which the so-called *Hamiltonian potential* is introduced. According to their definition, $\frac{\partial H^{(p)}}{\partial x^{(i)}} = \frac{\partial H^{(i)}}{\partial x^{(i)}}$ must hold in addition to (7) in Definition 2, which means that the dynamic system can be decoupled with respect to the control dynamics and system states. A general use of the exact and the Hamiltonian potential games for engineering applications is limited as (7) requires a special structure of the game, see e.g. Chapter 5.2 in [28]. Therefore, it is helpful to use a less restrictive sub-class of potential games, the so-called *ordinal potential game* [25], which has a less restrictive definition than the exact potential games. An extension for differential games of the ordinal differential potential games does not exist. Therefore, a novel extension of the ordinal potential games for LQ-games is given. This definition requires neither the decoupled dynamics of the underlying system nor a rigorous equality of the derivates of the Hamiltonian functions, which enables a more general use of the potential games theory.

Definition 3 (Ordinal Potential Differential Game): The differential game Γ_d is called an *ordinal potential differential game* if a potential function $H^{(p)}$ exists such that

$$\operatorname{sign}\left(\frac{\partial H^{(p)}}{\partial \boldsymbol{u}^{(i)}}\right) = \operatorname{sign}\left(\frac{\partial H^{(i)}}{\partial \boldsymbol{u}^{(i)}}\right). \tag{9}$$

for all players $i \in \mathbb{P}$. If the players' Hamilton functions have a form as given in (8) and the potential function is quadratic as suggested in (6), then the game is an *ordinal potential LQ-differential game*.

IV. Necessary and Sufficient Conditions for an Ordinal Potential LQ-differential Game

The following derivation is restricted to the two-players case with scalar inputs, the cost function given in (5) and infinite optimization horizon ($\tau \to \infty$).

A. Computation of the NE in LQ-differential games

For the derivation of the sufficient and necessary conditions of ordinal potential differential games, some assumptions are taken into account. The matrices of the cost functions of the players have a diagonal structure $\mathbf{Q}^{(i)} = \mathrm{diag}[q_1^{(i)}, q_2^{(i)}, ..., q_n^{(i)}]$ and $\mathbf{R}^{(i)} = \mathrm{diag}[r_1^{(i)}, r_2^{(i)}]$. The resulting solution of the coupled optimization problem of the two-player case [23] is

$$\begin{aligned}\mathbf{0} = (&\mathbf{A}^T\mathbf{P}^{(i)} + \mathbf{P}^{(i)}\mathbf{A} + \mathbf{Q}^{(i)} - \mathbf{P}^{(i)}\mathbf{S}^{(i)} \\ &- \mathbf{P}^{(i)}\mathbf{S}^{(\neg i)}\mathbf{P}^{(\neg i)})\boldsymbol{x}, \\ \mathbf{S}^{(i)} = \ &\mathbf{B}^{(i)}\mathbf{R}^{(i)^{-1}}\mathbf{B}^{(i)^T}\end{aligned} \tag{10}$$

for $i \in 1, 2$. It can be solved for $\mathbf{P}^{(i)}$ according to [23]. Its result is the control law of the players

$$\boldsymbol{u}^{(i)} = -\mathbf{K}^{(i)}\boldsymbol{x}, \tag{11}$$

where $\mathbf{K}^{(i)} = \mathbf{R}^{(i)^{-1}}\mathbf{B}^{(i)^T}\mathbf{P}^{(i)}$ is the feedback gain of player i. The Hamiltonian of the potential game is

$$\begin{aligned}H^{(p)} = &\frac{1}{2}\left(\boldsymbol{x}^T\mathbf{Q}^{(p)}\boldsymbol{x} + \boldsymbol{u}^{(p)^T}\mathbf{R}^{(p)}\boldsymbol{u}^{(p)}\right) \\ &+\boldsymbol{\lambda}^{(p)^T}\left(\mathbf{A}\boldsymbol{x}(t) + \mathbf{B}^{(p)}\boldsymbol{u}^{(p)}(t)\right),\end{aligned} \tag{12}$$

where

$$\mathbf{Q}^{(p)} = \begin{bmatrix} q_{11}^{(p)} & \cdots & q_{1M}^{(p)} \\ \vdots & \ddots & \vdots \\ q_{1M}^{(p)} & \cdots & q_{MM}^{(p)} \end{bmatrix} \text{ and } \mathbf{R}^{(p)} = \begin{bmatrix} r_{11}^{(p)} & r_{21}^{(p)} \\ r_{21}^{(p)} & r_{22}^{(p)} \end{bmatrix}.$$

The matrix $\mathbf{Q}^{(p)}$ is positive semi-definite and $\mathbf{R}^{(p)}$ is positive definite. The control law of the potential game is

$$\boldsymbol{u}^{(p)} = -\mathbf{K}^{(p)}\boldsymbol{x},$$

with $\mathbf{K}^{(p)} = \mathbf{R}^{(p)^{-1}}\mathbf{B}^{(p)}\mathbf{P}^{(p)}$. The matrix $\mathbf{P}^{(p)}$ is obtained from the solution of the Ricatti equation [23]

$$\begin{aligned}&\mathbf{0} = \left(\mathbf{A}^T\mathbf{P}^{(p)} + \mathbf{P}^{(p)}\mathbf{A} + \mathbf{Q}^{(p)} - \mathbf{P}^{(p)}\mathbf{S}^{(p)}\right)\boldsymbol{x}, \\ &\mathbf{S}^{(p)} = \mathbf{B}^{(p)}\mathbf{R}^{(p)^{-1}}\mathbf{B}^{(p)^T}.\end{aligned} \tag{13}$$

Lemma 1: Necessary condition of an ordinal potential LQ-differential game

Let a potential function according to (12) be given. In addition, let $\boldsymbol{u}^{(1)^*}$, $\boldsymbol{u}^{(2)^*}$ denote a NE solution of a LQ-differential game, which stabilizes the system (4). If $H^{(p)}$ defines an ordinal potential differential game associated to the LQ problem then

$$[\boldsymbol{u}^{(1)^{*T}},\ \boldsymbol{u}^{(2)^{*T}}]^T \equiv \boldsymbol{u}^{(p)}. \tag{14}$$

holds.

Proof: As $[\boldsymbol{u}^{(1)^{*T}},\ \boldsymbol{u}^{(2)^{*T}}]^T$ stabilize the system dynamics the input trajectories must be bounded. According to Theorem 2.3 in [28], an ordinal potential game with bounded inputs and with a continuous potential function has at least one NE. The solution of (4) is obtained from the integral expression

$$\boldsymbol{x}^*(t) = e^{\mathbf{A}\cdot t}\cdot\boldsymbol{x}_0 + \sum_{i=1}^{2}\int_0^t e^{(t-s)\mathbf{A}}\mathbf{B}^{(i)}\boldsymbol{u}^{(i)^*}(s)\ \mathrm{d}s.$$

If the differential game is a potential game, the solution of (4) can be obtained from the optimization of the potential function (6) such that

$$\boldsymbol{x}^{(p)}(t) = e^{\mathbf{A}\cdot t}\cdot \boldsymbol{x}_0 + \int_0^t e^{(t-s)\mathbf{A}}\mathbf{B}^{(p)}\boldsymbol{u}^{(p)}(s)\ \mathrm{d}s$$

that lead to the same trajectories, which means

$$\left|\boldsymbol{x}^{(p)}(t) - \boldsymbol{x}^{*}(t)\right| = 0\ \ \forall t. \tag{15}$$

Substituting $\boldsymbol{x}^{(p)}$ and $\boldsymbol{x}^{*}$ in (15), the following is obtained

$$\left|\int_0^t e^{(t-s)\mathbf{A}}\mathbf{B}^{(p)}\boldsymbol{u}^{(p)}(s)\ \mathrm{d}s - \sum_{i=1}^{2}\int_0^t e^{(t-s)\mathbf{A}}\mathbf{B}^{(i)}\boldsymbol{u}^{(i)^*}(s)\ \mathrm{d}s\right| = 0. \tag{16}$$

From Definition 3,

$$\mathbf{B}^{(p)} = [\mathbf{B}^{(1)}, \mathbf{B}^{(2)}], \tag{17}$$

holds for a potential LQ-differential game, which can be shown straightforward from the definition of the matrix multiplication (see any linear algebra book, e.g. Chapter 2. in [23]):

$$\sum_{i=1}^{N}\mathbf{B}^{(i)}\cdot\boldsymbol{u}^{(i)} = \mathbf{B}^{(p)}\cdot\boldsymbol{u},$$

where $\boldsymbol{u}$ is given in (3). Substituting (17) in (16) leads to

$$\left|\int_0^t e^{(t-s)\mathbf{A}}[\mathbf{B}^{(1)}, \mathbf{B}^{(2)}]\boldsymbol{u}^{(p)}(s)\ \mathrm{d}s - \sum_{i=1}^{2}\int_0^t e^{(t-s)\mathbf{A}}\mathbf{B}^{(i)}\boldsymbol{u}^{(i)^*}(s)\ \mathrm{d}s\right| = 0, \tag{18}$$

which is true if $\left[\boldsymbol{u}^{(1)^{*T}},\ \boldsymbol{u}^{(2)^{*T}}\right]^T \equiv \boldsymbol{u}^{(p)}$ holds. ■

Lemma 2: Sufficient condition of an ordinal potential differential game
If for a two-player linear-quadratic game,

$$\left(\mathbf{B}^{(i)^T}\mathbf{P}^{(p)}\boldsymbol{x}\right)\cdot\left(\mathbf{B}^{(i)^T}\mathbf{P}^{(i)}\boldsymbol{x}\right) \geq 0 \tag{19}$$

holds for $i \in \mathbb{P}$ and $\forall \boldsymbol{x}$ then the game is an *ordinal potential differential game* with a Hamiltonian function given in (12).

Proof: For the proof, the Definition 3 is used:

$$\mathrm{sign}\left(\frac{\partial H^{(p)}}{\partial u^{(i)}}\right) = \mathrm{sign}\left(\frac{\partial H^{(i)}}{\partial u^{(i)}}\right). \tag{20}$$

Assuming a quadratic potential function (12) and a linear system (4) dynamics,

$$\frac{\partial H^{(p)}}{\partial \boldsymbol{u}^{(p)}} = \mathbf{R}^{(p)}\boldsymbol{u}^{(p)} + \mathbf{B}^{(p)^T}\boldsymbol{\lambda}^{(p)} \tag{21}$$

holds, where $\boldsymbol{\lambda}^{(p)} = \mathbf{P}^{(p)}\boldsymbol{x}$ can be applied. The control law of the potential game is obtained from the solution of (13) but it is modified to a sub-optimal solution. The reason for that is the following: An optimal control law means that $\frac{\partial H^{(p)}}{\partial \boldsymbol{u}^{(p)}} = 0$, which is not suitable for the analysis of the existence of a potential game as in that case (20) yields $0 = 0$. The sub-optimal control law of the potential game is

$$\boldsymbol{u}^{(p)} = -(1+\varepsilon)\mathbf{R}^{(p)^{-1}}\mathbf{B}^{(p)^T}\mathbf{P}^{(p)}\boldsymbol{x}, \tag{22}$$

in which $\varepsilon > 0$ is an arbitrary scalar. With $\varepsilon \to 0$, the optimal control law is obtained. Substituting (22) in (21) gives

$$\frac{\partial H^{(p)}}{\partial \boldsymbol{u}^{(p)}} = -\mathbf{R}^{(p)}(1+\varepsilon)\mathbf{R}^{(p)^{-1}}\mathbf{B}^{(p)^T}\mathbf{P}^{(p)}\boldsymbol{x} + \mathbf{B}^{(p)^T}\mathbf{P}^{(p)}\boldsymbol{x}, \tag{23}$$

which can be simplified with $\mathbf{R}^{(p)}$. From Lemma 1, it follows that the derivates of the potential function $H^{(p)}$ are

$$\frac{\partial H^{(p)}}{\partial \boldsymbol{u}^{(p)}} = \left[\frac{\partial H^{(p)}}{\partial u^{(1)}}, \frac{\partial H^{(p)}}{\partial u^{(2)}}\right]^T, \tag{24}$$

which is a vector with two scalar elements and therefore (23) can be split with (24) into

$$\frac{\partial H^{(p)}}{\partial u^{(i)}} = -\varepsilon\mathbf{B}^{(i)^T}\mathbf{P}^{(p)}\boldsymbol{x}, \tag{25}$$

for the two players $i \in \mathbb{P}$.

For the players of the original game, the derivates of the Hamiltonians are expressed as

$$\frac{\partial H^{(i)}}{\partial u^{(i)}} = \mathbf{R}^{(i)}u^{(i)} + \mathbf{B}^{(i)^T}\boldsymbol{\lambda}^{(i)} \tag{26}$$

holds for $i \in \mathbb{P}$ and $\boldsymbol{\lambda}^{(i)} = \mathbf{P}^{(i)}\boldsymbol{x}$ can be substituted. Analogously, an optimal control law of the players would mean $\frac{\partial H^{(i)}}{\partial \boldsymbol{u}^{(i)}} = 0$. The optimal control law is obtained with $\varepsilon \to 0$. Therefore, the applied sub-optimal control laws of the original game is

$$u^{(i)} = -(1+\varepsilon)\mathbf{R}^{(i)^{-1}}\mathbf{B}^{(i)^T}\mathbf{P}^{(i)}\boldsymbol{x}, \tag{27}$$

where $\varepsilon > 0$ is arbitrary and $i = \{1, 2\}$. The control law (27) yields the behaviour of players around the optimal solution. Substituting (27) in (26) gives

$$\frac{\partial H^{(i)}}{\partial u^{(i)}} = -\mathbf{R}^{(i)}(1+\varepsilon)\mathbf{R}^{(i)^{-1}}\mathbf{B}^{(i)^T}\mathbf{P}^{(i)}\boldsymbol{x} + \mathbf{B}^{(i)^T}\mathbf{P}^{(i)}\boldsymbol{x}, \tag{28}$$

which can be simplified to

$$\frac{\partial H^{(i)}}{\partial u^{(i)}} = -\varepsilon\mathbf{B}^{(i)^T}\mathbf{P}^{(i)}\boldsymbol{x}. \tag{29}$$

Substituting (25) and (29) in (20) and simplifying with ε yields

$$\mathrm{sign}\left(\mathbf{B}^{(i)^T}\mathbf{P}^{(p)}\boldsymbol{x}\right) = \mathrm{sign}\left(\mathbf{B}^{(i)^T}\mathbf{P}^{(i)}\boldsymbol{x}\right). \tag{30}$$

The equality of two sign can be reformed to the multiplication of the arguments of the sign functions such that

$$\left(\mathbf{B}^{(i)^T}\mathbf{P}^{(p)}\boldsymbol{x}\right)\cdot\left(\mathbf{B}^{(i)^T}\mathbf{P}^{(i)}\boldsymbol{x}\right) \geq 0, \tag{31}$$

for $i \in \mathbb{P}$, which proofs the lemma. ■

Remark 1: The sufficient condition requires solely a solution of the Ricatti equation (13) $\mathbf{P}^{(p)}$, which satisfies (19). This means that the pair $\left(\mathbf{Q}^{(p)}\text{ and }\mathbf{R}^{(p)}\right)$ is suitable for a potential game. Therefore, any further pairs $\left(\kappa\mathbf{Q}^{(p)}\text{ and }\kappa\mathbf{R}^{(p)}\right)$ with $\kappa > 0$ also yield an optimization problem that has the same solution.

V. Method for Finding an Ordinal Potential LQ-differential Game

For the derivation of an ordinal differential potential game, the deviation of the input of the potential games from the NE is defined such that

$$\boldsymbol{e}_u = \boldsymbol{u}^{(p)}(t, \boldsymbol{x}, \mathbf{Q}^{(p)}, \mathbf{R}^{(p)}) - \boldsymbol{u}(t, \boldsymbol{x}^*) \tag{32}$$

where $\boldsymbol{x}^*$ are the trajectories of the NE with the corresponding inputs of the original two-players games. To find a ordinal potential differential game of the original game, the deviation (32) is minimized, which is carried out with the following optimization:

$$\hat{\mathbf{Q}}^{(p)}, \hat{\mathbf{R}}^{(p)} = \arg \min_{\mathbf{Q}^{(p)}, \mathbf{R}^{(p)}} |\boldsymbol{e}_u|^2 \tag{33a}$$

$$\text{s.t. } \mathbf{A}^T\mathbf{P}^{(p)} + \mathbf{P}^{(p)}\mathbf{A} + \mathbf{Q}^{(p)} - \mathbf{P}^{(p)}\mathbf{S}^{(p)}\mathbf{P}^{(p)} = \mathbf{0} \tag{33b}$$

$$\left(\mathbf{B}^{(i)^T}\mathbf{P}^{(p)}\boldsymbol{x}\right) \cdot \left(\mathbf{B}^{(i)^T}\mathbf{P}^{(i)}\boldsymbol{x}\right) > 0, \tag{33c}$$

where $\mathbf{S}^{(p)} = \mathbf{B}^{(p)}\mathbf{R}^{(p)^{-1}}\mathbf{B}^{(p)^T}$. The minimization of (33a) ensures that the necessary condition for a ordinal potential game given in Lemma 1. The constraint (33b) ensures the minimization of the potential function $J^{(p)}$, which means that $\boldsymbol{u}^{(p)}$ is provided by the LQ-optimization of (6). The constraint (33c) guarantees the sufficient condition of Lemma 2.

Note that $|\boldsymbol{e}_u|^2$ can be also interpreted as a potential function since its minimum yields the NE of the original game. A similar consideration is suggested in [27], however the authors do not use this property for any further computations or identification. Here, we extend this idea and use it to find a LQ optimization problem that is a potential function of the original game. The optimization (33) is provides the weights $\left(\mathbf{Q}^{(p)}, \mathbf{R}^{(p)}\right)$ of the LQ potential function (6) that is suitable to model the original game. The optimizer is an interior-point optimizer algorithm provided by MATLAB [29].

VI. Applications to Simple Vehicle-Manipulator Systems

Two simplified vehicle-manipulator models are used to verify the Lemmas and the proposed method. The two examples are considered as generalizations of a holonomic VM and a non-holonomic VM modelled in Frénet Frame[2]: Some states represent the vehicle deviation from its reference ($\boldsymbol{x}_v$) and other states correspond to the manipulator deviation from its reference ($\boldsymbol{x}_m$). The operator can only control the manipulator and the automation has an impact on both the vehicle's and the manipulator's states. The motion of the vehicle has an impact on the motion of the manipulator, but the manipulator has no influence on the vehicle's dynamics. This characteristic can be observed in the structure of the matrices $\mathbf{A}, \mathbf{B}$. The zero elements in these matrices ensure the property described above. The VM states consist of the states of the manipulator and the vehicle, i.e. $\boldsymbol{x} = [\boldsymbol{x}_m, \boldsymbol{x}_v]^T$. The system dynamics is modelled as a linear time-invariant system such that

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}^{(h)}u^{(h)} + \mathbf{B}^{(a)}u^{(a)}. \tag{34}$$

[2]Systems modelled in Frénet Frame are characterized relative to the reference. For further details see Chapter 49. in [9].

where the first player $\square^{(h)}$ represents the human operator, who controls the manipulator, and the second player $\square^{(a)}$ is the automation, which controls the vehicle. Moreover, the parameters of $\mathbf{A}$ and $\mathbf{B}^{(i)}$ can be computed from the velocity of the vehicle and the actual configuration of the manipulator. Further details of the modelling VMs can be found e.g. in [5], [30].

To enable a detailed comparison, the ground truth game is defined for both examples. The root-mean-square error (RMSE) and the maximal absolute error (MAE) of the system state $\boldsymbol{x}$ are used as additional quantitative measures for the evaluation of the deviation from these ground truth game's trajectories.

A. First example

1) Dynamic System: The first example specifies a VM with holonomic vehicle modelled in a Frénet Frame. For more detail about the control models for holonomic wheeled robots, we refer to Chapter 49.3 in [9]. The first state represents the deviation of the manipulators from its reference. The second state describes the distance between the vehicle and its reference. The dynamic system is generated as a general example for holonomic VMs, which have the system state vector $\boldsymbol{x} = [x_m,\ x_v]^T$. The linear dynamic system of the first example has the system matrix

$$\mathbf{A} = \begin{bmatrix} 0.2 & 2 \\ 0 & 1.5 \end{bmatrix}$$

and its input matrices are $\mathbf{B}^{(h)} = \begin{bmatrix} 1.5 & 0 \end{bmatrix}^T$ and $\mathbf{B}^{(a)} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

2) Ground Truth Players: The game has two players with the following Hamiltonian function given in (5). The cost function matrices are

$$\mathbf{Q}^{(h)} = \text{diag}\begin{bmatrix} 5, & 0 \end{bmatrix},\ \mathbf{R}^{(h)} = \text{diag}\begin{bmatrix} 1.5, & 0.3 \end{bmatrix}$$
$$\mathbf{Q}^{(a)} = \text{diag}\begin{bmatrix} 5, & 1 \end{bmatrix},\ \mathbf{R}^{(a)} = \text{diag}\begin{bmatrix} 0, & 1.2 \end{bmatrix}.$$

The resulting cooperative feedback control law of the original game is

$$\mathbf{K}^{(h)} = [1.89,\ -0.47] \text{ and } \mathbf{K}^{(a)} = [0.59,\ 3.14],$$

which is obtained by numerically solving the coupled Riccati equation (10).

3) Results: Fig. 2 outlines that the trajectories of the ground truth game (GTG) are equivalent to the trajectories of the potential game (PG). The deviation from the original game is is in the numerical error range: The RMSE$(\boldsymbol{x}) = 10^{-5} \cdot [8.68,\ 4.86]$ and the MAE$(\boldsymbol{x}) = 10^{-6} \cdot [8.75,\ 5.03]$. The identified potential function has the weighting matrices

$$\mathbf{Q}^{(p)} = \begin{bmatrix} 2.27 & 0.06 \\ 0.06 & 0.77 \end{bmatrix} \text{ and } \mathbf{R}^{(p)} = \begin{bmatrix} 1.00 & 0.001 \\ 0.001 & 0.63 \end{bmatrix}.$$

Figures 3a and 3b show that the sufficient condition for a potential game is fulfilled, cf. (9), as the sign of the derivates of the cost functions are identical to the potential function, so in case of holonomic VMs (19) holds $\forall t$.

B. Second example

1) Dynamic System: This second example represents a VM with a non-holonomic vehicle modelled along its reference trajectory. The first state models again the manipulator's deviation x_m. The second state x_{dv} is the lateral deviation from the reference and the third state is the orientation error $x_{\Delta\theta}$. For more detailed about the modelling and control of non-holonomic vehicles, we refer to Chapter 49.4 in [9]. The system matrix of the VM is

$$A = \begin{bmatrix} 0.5 & 1.6 & 0 \\ 0 & 0 & 1.6 \\ 0 & 0 & 0 \end{bmatrix}$$

and the input matrices are $\mathbf{B}^{(1)} = \begin{bmatrix} 1.25 & 0 & 0 \end{bmatrix}^T$ and $\mathbf{B}^{(2)} = \begin{bmatrix} 0 & 0 & 0.85 \end{bmatrix}^T$. The parameters correspond to the geometry and the velocity of VM relative to its reference. The last row of $\mathbf{A}$ filled with 0-s is the result of the non-holonomic setup of the VM in Frénet Frame.

2) Ground Truth Players: In the second example, the matrices of the Hamiltonian functions (5) of the two players are

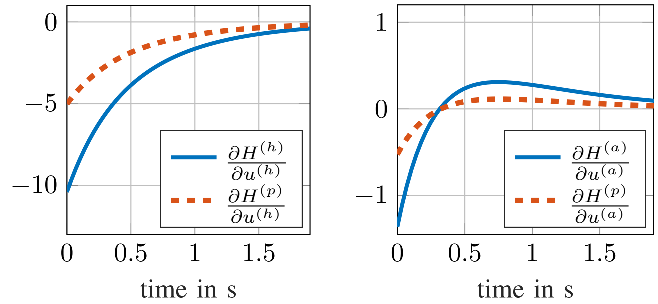
$$\mathbf{Q}^{(h)} = \text{diag}\left[8,\ 0,\ 0\right], \qquad \mathbf{R}^{(h)} = \text{diag}\left[0.8,\ 0\right]$$
$$\mathbf{Q}^{(a)} = \text{diag}\left[25,\ 4,\ 4\right], \qquad \mathbf{R}^{(a)} = \text{diag}\left[0.3,\ 1\right].$$

The resulting cooperative feedback control law of the original game is computed as in the first example:

$$\mathbf{K}^{(h)} = [3.59,\ 1.31,\ 0.30] \text{ and } \mathbf{K}^{(a)} = [-0.06,\ 2.12,\ 3.47].$$

3) Results: The matching trajectories of the ground truth game (GTG) of the potential game (PG) are given in Fig. 4. The deviations from the original game's trajectories is again small, which can be seen in the quantitative measures: The $\text{RMSE}(\boldsymbol{x}) = 10^{-2} \cdot [0.10,\ 0.86,\ 1.27]$ and the $\text{MAE}(\boldsymbol{x}) = 10^{-3} \cdot [0.21,\ 0.85,\ 1.83]$. The identified potential game has a Hamiltonian function with the weighting matrices

$$\mathbf{Q}^{(p)} = \begin{bmatrix} 8.94 & 0.04 & 0.04 \\ 0.04 & 7.01 & 0.33 \\ 0.04 & 0.33 & 8.04 \end{bmatrix} \text{ and } \mathbf{R}^{(p)} = \begin{bmatrix} 1.83 & 0.0 \\ 0.0 & 0.88 \end{bmatrix}.$$



(a) First player (b) Second player

Fig. 3: The dynamics of Hamiltonian functions

The Figures 5a and 5b show that the sufficient condition for a potential game is fulfilled, cf. (9), as the sign of the derivate of the cost functions are identical to the potential function, so (19) also holds $\forall t$ for a general, non-holonomic VM.

VII. Conclusion and Outlook

This paper presented an extension of the ordinal potential games for differential games to model human-machine interactions by the cooperative control of VMs. The concept of the potential games enables a constructive controller design of limited information controller, in which some system states and trajectories are not measurable. The necessary and sufficient condition for the existence of an ordinal potential differential game are derived. A method is given how to find an ordinal potential game to an original game. The correctness of the modelling with ordinal potential games is verified with the simulative applications of VMs.

This proposed method will be used for the systematic derivation of the limited information cooperative controller presented in [11] and [10], which enables a better understanding of the limited information cooperative controller and eliminates the heuristic tuning of the controller.

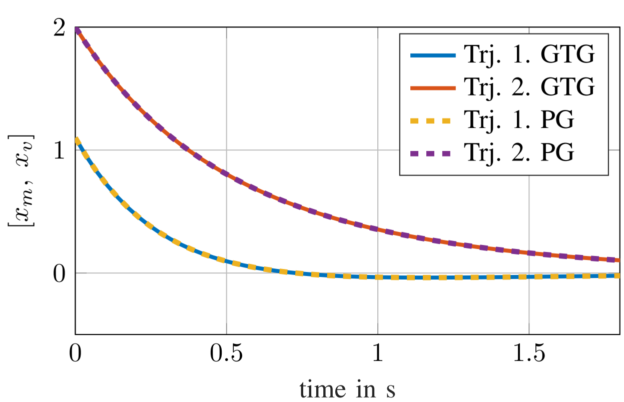


Fig. 2: Trajectories of the Ground Truth game (GTG) and of the potential game (PG)

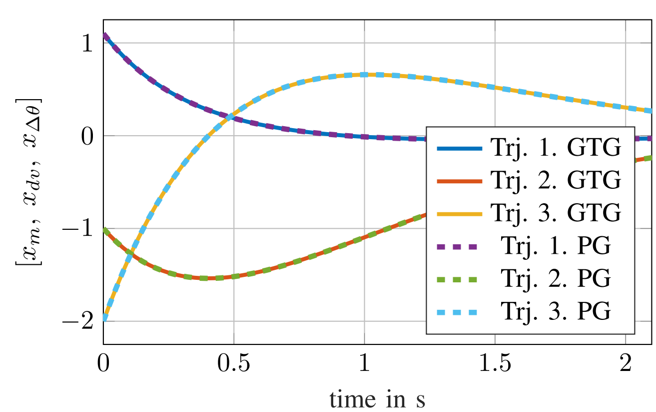
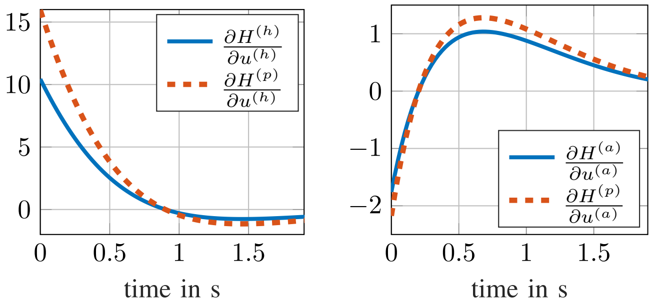


Fig. 4: Trajectories of the Ground Truth game (GTG) and of the potential game (PG)



(a) First player (b) Second player

Fig. 5: The dynamics of Hamiltonian functions

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