Karlsruhe Institute of Technology

Wave phenomena
analysis and numerics

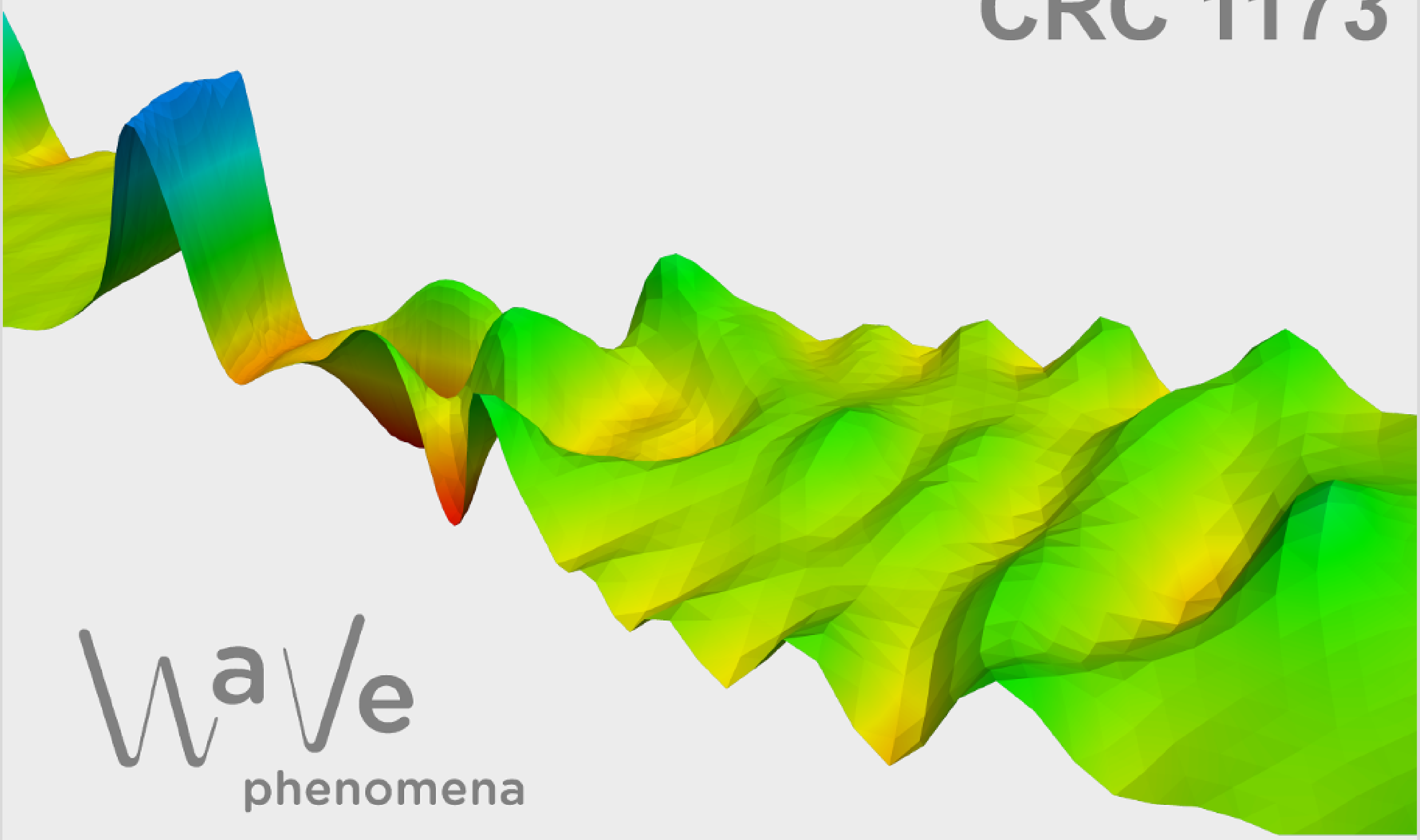
On smoothing estimates in modulation spaces and the NLS with slowly decaying initial data

Robert Schippa

CRC Preprint 2021/19, May 2021

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173



Wave phenomena

Participating universities

Universität Stuttgart

EBERHARD KARLS UNIVERSITÄT TÜBINGEN

TU WIEN

Funded by

DFG

ISSN 2365-662X

ON SMOOTHING ESTIMATES IN MODULATION SPACES AND THE NLS WITH SLOWLY DECAYING INITIAL DATA

ROBERT SCHIPPA

Abstract. We show new local L^p-smoothing estimates for the Schrödinger equation with initial data in modulation spaces via decoupling inequalities. Furthermore, we probe necessary conditions by Knapp-type examples for space-time estimates of solutions with initial data in modulation and L^p-spaces. The examples show sharpness of the smoothing estimates up to the endpoint regularity in a certain range. Moreover, the examples rule out global Strichartz estimates for initial data in $L^p(\mathbb{R}^d)$ for $d \geq 1$ and $p > 2$, which was previously known for $d \geq 2$. The estimates are applied to show new local and global well-posedness results for the cubic nonlinear Schrödinger equation on the line. Lastly, we show ℓ^2-decoupling inequalities for variable-coefficient versions of elliptic and non-elliptic Schrödinger phase functions.

1. Introduction

In this article we show new space-time estimates for the Schrödinger equation with slowly decaying data outside L^2-based Sobolev spaces:

$$\begin{cases} i\partial_t u + \Delta u &= 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0) &= u_0 \in X. \end{cases} \tag{1}$$

In the present work we consider modulation spaces $X = M^s_{p,q}(\mathbb{R}^d)$, $2 \leq p < \infty$, which are compared to initial data in L^p-based Sobolev spaces $X = L^p_\alpha(\mathbb{R}^d) = \langle D \rangle^{-\alpha} L^p(\mathbb{R}^d)$, $2 \leq p < \infty$. The latter initial data were recently considered by Dodson–Soffer–Spencer[20] (see also [21]) and by R. Mandel [41].

Modulation spaces are likewise used to model slowly decaying initial data. Feichtinger introduced modulation spaces in [23]. He provided also a more recent account [24] emphasizing the role of modulation spaces in signal processing; see also the textbook by Gröchenig [27]. As the body of literature is huge, we refer to the PhD thesis of L. Chaichenets [13] and references therein for a more exhaustive account on modulation spaces in the context of Schrödinger equations.

For the definition of modulation spaces, consider the Fourier multipliers

$$(\square_k f)\hat{}(\xi) = \sigma_k(\xi) \hat{f}(\xi), \quad k \in \mathbb{Z}^d,$$

with $(\sigma_k)_{k \in \mathbb{Z}^d} \subseteq C^\infty_c(\mathbb{R}^d)$ a smooth partition of unity, adapted to the translated unit cubes $Q_k = k + [-\frac{1}{2}, \frac{1}{2})^d$. For the precise definition, we refer to [13, Section 2.1]. We can suppose that $\sigma_k(\xi) = \sigma_0(\xi - k)$. The norm is defined by

$$\| f \|_{M^s_{p,q}} = \big(\sum_{k \in \mathbb{Z}^d} \langle k \rangle^{qs} \| \square_k f \|^q_{L^p(\mathbb{R}^d)} \big)^{\frac{1}{q}}.$$

If $s = 0$, we write $M_{p,q}$. Modulation spaces are closely related with L^p-spaces. For illustration, we collect embedding properties: By the embedding of ℓ^p-spaces and Bernstein's inequality, we have

$$M^s_{p,q_1} \hookrightarrow M^s_{p,q_2} \quad (q_1 \leq q_2), \tag{2}$$

$$M^s_{p_1,q} \hookrightarrow M^s_{p_2,q} \quad (p_1 \leq p_2). \tag{3}$$

2020 *Mathematics Subject Classification.* Primary: 35B45, 35Q55, Secondary: 42B37.

Key words and phrases. Smoothing estimates, Strichartz estimates, modulation spaces, ℓ^2-decoupling, nonlinear Schrödinger equation.

Rubio de Francia's inequality ([49]) and duality yield

$$M_{p,p'} \hookrightarrow L^p \hookrightarrow M_{p,p} \quad (2 \le p \le \infty), \tag{4}$$

$$M_{p,p} \hookrightarrow L^p \hookrightarrow M_{p,p'} \quad (1 \le p \le 2). \tag{5}$$

Furthermore, we can trade regularity for summability as

$$M^{s_1}_{p,q_1}(\mathbb{R}^d) \hookrightarrow M^{s_2}_{p,q_2}(\mathbb{R}^d) \qquad s_1 - s_2 > d\big(\frac{1}{q_2} - \frac{1}{q_1}\big) > 0 \tag{6}$$

by applying Hölder's inequality (cf. [13, Proposition 2.31]). By Plancherel's theorem, $M_{2,2} \sim L^2$.

Let $U(t) = e^{it\Delta}$ denote the propagator of (1), and let Uf denote the free solution for $f \in \mathcal{S}'(\mathbb{R}^d)$. In this paper we show new smoothing estimates

$$\|Uf\|_{L^p([-1,1]\times\mathbb{R}^d)} \lesssim \|f\|_{M^s_{p,q}(\mathbb{R}^d)} \tag{7}$$

via ℓ^2-decoupling. We prove the following:

Theorem 1.1. *Suppose that $d \ge 1$, $p \ge 2$, and $1 \le q \le \infty$.*

(A) *If $2 \le p \le \frac{2(d+2)}{d}$, then (7) holds true provided that $s > \max\big(0, \frac{d}{2} - \frac{d}{q}\big)$.*
(B) *If $\frac{2(d+2)}{d} \le p \le \infty$ and $2 \le q \le \infty$, then (7) holds true provided that $s > d - \frac{d+2}{p} - \frac{d}{q}$.*
(C) *If $\frac{2(d+2)}{d} \le p \le \infty$ and $1 \le q \le 2$, then (7) holds true provided that $s > 2\big(1 - \frac{1}{q}\big)\big(\frac{d}{2} - \frac{d+2}{p}\big)$.*
(D) *If $q = 1$, then (7) holds true with $s = 0$.*

The key argument in the proof of the estimates for $q \ge 2$ are the ℓ^2-decoupling inequalities for the paraboloid due to Bourgain–Demeter (cf. [9]). It turns out that after localization in space and parabolic rescaling the ℓ^2-decoupling inequality yields Strichartz estimates in modulation spaces by a kernel estimate. Originally, Wolff [61] brought up decoupling for the cone to analyze L^p-smoothing estimates for the (half-)wave equation:

$$\|e^{it\sqrt{-\Delta}} f\|_{L^p([1,2]\times\mathbb{R}^d)} \lesssim \|f\|_{L^p_s(\mathbb{R}^d)}.$$

We refer to [43, 44] and [52, Chapter 8] for further reading.

Regarding Strichartz estimates in modulation spaces, it seems the above space-time estimates were previously not investigated in the literature. Space-time estimates with an additional window decomposition were shown by B. Wang *et al.* [58, 57, 1]; see also Zhang [62]. We also refer to the surveys by Wang–Huo–Hao–Guo [59] and Ruzhansky–Sugimoto–Wang [50]. These estimates were further applied to prove well-posedness for nonlinear equations. For context we refer to S. Guo's work [29], in which he proved local well-posedness of the NLS in modulation spaces $M_{2,p}$ for $2 < p < \infty$. Oh–Wang [45] globalized this using the complete integrability. Below we discuss well-posedness of the cubic NLS outside L^2-based Sobolev spaces in greater detail.

We shall also compare Strichartz estimates in modulation spaces with L^p-smoothing estimates for Schrödinger equations, which were first discussed by Rogers [47]:

$$\|Uf\|_{L^p(I\times\mathbb{R}^d)} \lesssim \|f\|_{L^p_\alpha(\mathbb{R}^d)} \tag{8}$$

Rogers showed that the validity of (8) for some α is equivalent to validity of the adjoint Fourier restriction estimate $R^*(p \to p)$. We refer to [48, 40] for further discussion. The decoupling inequality can serve as a common base for (7) and (8).

Secondly, we give new necessary conditions for estimates of the kinds

$$\|Uf\|_{L^p([-1,1]\times\mathbb{R}^d)} \lesssim \|f\|_{M^s_{p,q}(\mathbb{R}^d)} \tag{9}$$

and

$$\|Uf\|_{L^p_t(I;L^q_x(\mathbb{R}^d))} \lesssim \|f\|_{L^r_s(\mathbb{R}^d)} \tag{10}$$

for $I \in \{[-1,1], \mathbb{R}\}$. We prove the following necessary conditions:

Proposition 1.2. *Let $p \geq 2$. Necessary for* (9) *to hold true is*

$$s \geq \max\big(0, d - \frac{d+2}{p} - \frac{d}{q}\big). \tag{11}$$

Necessary for (10) *to hold for $I = [-1,1]$ is*

$$s \geq \max\big(0, d - \frac{d}{q} - \frac{2}{p} - \frac{d}{r}\big), \quad q \geq r. \tag{12}$$

If $I = \mathbb{R}$, we have the additional conditions

$$\frac{2}{p} + \frac{d}{q} \leq \frac{d}{r}. \tag{13}$$

We refer to Section 2 for the discussion of further conditions.

Proposition 1.2 shows that the estimates in Theorem 1.1 are sharp up to the endpoint regularity for $2 \leq p \leq \frac{2(d+2)}{d}$ and for $(\frac{2(d+2)}{d} \leq p \leq \infty$ and $2 \leq q \leq \infty)$. Moreover, the examined examples show that global estimates (10) for $I = \mathbb{R}$ and $s = 0$ are impossible for $p > 2$. Mandel [41] previously showed this for $d \geq 2$ with a more explicit example.

Corollary 1.3. *Suppose that $d \geq 1$, $p, q \in [1, \infty]$, and $r \in (2, \infty]$. Then, there is no C such that the estimate*

$$\|Uf\|_{L_t^p(\mathbb{R}, L_x^q(\mathbb{R}^d))} \leq C \|f\|_{L^r(\mathbb{R}^d)} \tag{14}$$

holds true for any $f \in L^r_{rad}(\mathbb{R}^d)$.

A major difference to L^p-based Sobolev spaces, $p \neq 2$, is that the propagator $U(t)$ is bounded on modulation spaces. Already by Bényi *et al.* [4] was proved the bound for $2 \leq p \leq \infty$:

$$\|U(t)\|_{M^s_{p,q}(\mathbb{R}^d) \to M^s_{p,q}(\mathbb{R}^d)} \lesssim \langle t \rangle^{d \left| \frac{1}{2} - \frac{1}{p} \right|}. \tag{15}$$

Chaichenets [13, Section 3.1] observed by duality that the same bound holds for $1 \leq p \leq 2$, and sharpness was initially shown by Cordero–Nicola [18]. In [18] Gaussians were used as window functions in the definition of modulation spaces. Then the sharpness follows by computation of the kernel for a window. In Section 2 we recover the sharpness of (15) by the discussed examples.

Corollary 1.4 ([13, Section 3.1])**.** *The estimate* (15) *is sharp for $1 \leq p, q \leq \infty$ and any $s, t \in \mathbb{R}$.*

We think that this gives a more robust proof of sharpness. In fact, all the results proved in Sections 2 and 3 have straight-forward counterparts for fractional Schrödinger equations

$$\left\{ \begin{array}{ll} i\partial_t u + (-\Delta)^{\alpha/2} u &= 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0) &= u_0 \in M^s_{p,q}(\mathbb{R}^d) \end{array} \right. \tag{16}$$

for $\alpha > 1$ or non-elliptic Schrödinger equations, e.g.,

$$\left\{ \begin{array}{ll} i\partial_t u + (\partial_1^2 - \partial_2^2) u &= 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ u(0) &= u_0 \in M^s_{p,q}(\mathbb{R}^2). \end{array} \right. \tag{17}$$

The latter follow by considering the decoupling inequalities from [10]. Also generalizations to variable coefficients seem possible under additional assumptions. As decoupling inequalities for variable coefficient versions of (17) are not explicit in the literature, we take the opportunity to prove them here. As the results are more technical to state, we refer to Subsection 5.1. Section 5 is based on Chapter 7 of the author's PhD thesis [51].

We point out that the examples from Section 2 rely on (non-)stationary phase estimates, for which the precise form of the phase function is not important, as long as the characteristic surface has non-vanishing Gaussian curvature.

We apply the Strichartz estimates to the cubic nonlinear Schrödinger equation:

$$\left\{ \begin{array}{ll} i\partial_t u + \Delta u &= |u|^2 u \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ \quad u(0) &= f \in D, \end{array} \right. \tag{18}$$

where $D \in \{L^p_\alpha(\mathbb{R}), M^s_{p,2}(\mathbb{R}) + L^2(\mathbb{R})\}$. We prefer to work in these slightly larger spaces as the Duhamel term is in $L^2(\mathbb{R})$. Denote the solution space by S. For the proof of local well-posedness we apply the L^p-smoothing estimates to the homogeneous equation and use the usual L^2-based inhomogeneous Strichartz estimates for the Duhamel integral. We shall see that using modulation spaces allows to save Sobolev regularity: (18) has a local solution if $f \in M^s_{6,2}$ for any $s > 0$, but considering $f \in L^6_s(\mathbb{R})$ requires $s > \frac{1}{3}$.

Theorem 1.5. *Let $\varepsilon > 0$. (18) is analytically well-posed in the spaces $D \in \{L^4_\varepsilon(\mathbb{R}), M^\varepsilon_{4,2}(\mathbb{R}) + L^2(\mathbb{R})\}$, $S_T = L_t^{\frac{24}{7}}([0,T], L^4(\mathbb{R}))$, and in the spaces $D \in \{L^6_{\frac{1}{3}+\varepsilon}(\mathbb{R}), M^\varepsilon_{6,2}(\mathbb{R}) + L^2(\mathbb{R})\}$, $S_T = L^3_t([0,T], L^6(\mathbb{R}))$, i.e., there is $T = T(\|f\|_D)$ such that there is a unique solution in S_T.*

Furthermore, for $D \in \{M^\varepsilon_{4,2}(\mathbb{R}) + L^2(\mathbb{R}), M^\varepsilon_{6,2}(\mathbb{R}) + L^2(\mathbb{R})\}$, we have $u \in C([0,T], D)$ with continuous dependence on the initial data.

We remark that one could also add regularity in L^p-spaces to bound the propagator in $L^p_x(\mathbb{R})$ via fixed-time estimates in order to obtain continuous curves in $L^p(\mathbb{R}) + L^2(\mathbb{R})$. However, one always leaves the space of initial values as the Schrödinger propagator is unbounded in L^p for $p \neq 2$.

As the NLS is one of the most prominent nonlinear dispersive equations, the body of literature on its well-posedness is vast. To put our results into context, we only mention few results and also refer to the references therein. Tsutsumi [55] applied classical L^2-based Strichartz estimates to prove global well-posedness in $L^2(\mathbb{R})$. This is the limit of analytic well-posedness [17, 38] in L^2-based Sobolev spaces. Recently, Harrop-Griffiths–Killip–Vişan [33] proved sharp global well-posedness in $H^s(\mathbb{R})$, $s > -1/2$, using complete integrability. Outside L^2-based Sobolev spaces, we mention the early works by Vargas–Vega [56] and Grünrock [28] in Fourier Lebesgue spaces. Hyakuna [35] proved well-posedness results in L^p-spaces for some $1 < p < 2$, and Correia [19] considered generalized energy spaces $\dot{H}^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for $p > 2$. Chaichenets *et al.* [14] showed the first global results in modulation spaces $M_{p,p'}$ for p sufficiently close to 2 without smallness assumption on the initial data; see also [15, 16].

In the recent work [20] Dodson–Soffer–Spencer (see also [21]) used fixed-time L^p-estimates

$$\|e^{it\Delta} f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p_\alpha(\mathbb{R}^d)} \tag{19}$$

and Picard iteration to prove well-posedness of (18) with $X_0 = L^p_s(\mathbb{R})$ for $2 < p < \infty$. We remark that the sharp derivative loss $\alpha = 2d\big|\frac{1}{2} - \frac{1}{p}\big|$ for L^p-estimates (19) is known since the work of Fefferman–Stein [22] and Miyachi [42]. However, the sharp estimates were not used in [20], which resulted in high regularity. We show how the sharp fixed time and smoothing estimates improve the results in [20] for $p = 4n + 2$, $n \geq 2$, and also show global results with arguments due to Dodson *et al.* [20]. We contend that the splitting method applied in [56, 14], further reaching back to Bourgain's seminal contribution [6], to be related with the present approach to prove the global result.

Theorem 1.6. *Let $T > 0$ and $s > \frac{3}{2}$. If $f \in M^s_{4,2}(\mathbb{R})$, then there exists a unique solution $u \in L_t^{\frac{24}{7}}([0,T], L^4) \cap C([0,T], M^s_{4,2}(\mathbb{R}) + L^2(\mathbb{R}))$ to (42), which depends continuously on the initial data, i.e., for any $T > 0$ and $f_n \to f \in M^s_{4,2}(\mathbb{R})$, we have*

$$\|u_n - u\|_{L_t^{\frac{24}{7}}([0,T],L^4) \cap C([0,T], M^s_{4,2}(\mathbb{R}) + L^2(\mathbb{R}))} \to 0.$$

If $f \in M^s_{6,2}(\mathbb{R})$, then there exists a unique solution $u \in L^3_t([0,T], L^6) \cap C([0,T], M^s_{6,2}(\mathbb{R}) + L^2(\mathbb{R}))$ with continuous data-to-solution mapping $f \mapsto u$ as above.

Outline of the paper. In Section 2 we give necessary conditions for L^p-smoothing estimates in modulation spaces and Strichartz estimates in L^p-based Sobolev spaces. These rule out global Strichartz estimates for initial data in $L^r(\mathbb{R}^d)$, $r > 2$. In Section 3 we show Theorem 1.1 via ℓ^2-decoupling. In

Section 4 the estimates are applied to show new local and global well-posedness for the NLS. In Section 5 we show decoupling inequalities for variable coefficient versions of Schrödinger equations.

2. Necessary conditions

The purpose of this section is to collect necessary conditions to find the following estimates to hold:

$$\|Uf\|_{L_t^p([-1,1],L^q(\mathbb{R}^d))} \lesssim \|f\|_{M^s_{r,t}(\mathbb{R}^d)}, \tag{20}$$

and

$$\|Uf\|_{L_t^p(I;L_x^q(\mathbb{R}^d))} \lesssim \|f\|_{L^r_s(\mathbb{R}^d)}. \tag{21}$$

We shall use three Knapp-type examples, which are essentially well-known in the literature [54, 48]. However, it seems that these have not been examined in the above contexts.

We start with the *anisotropic Knapp example at unit frequencies*, which was previously used to determine the range of integrability coefficients for the L^2-based Strichartz estimate (21) (cf. [54, 53]). Consider

$$\hat{g}_\varepsilon(\xi) = \chi_{(1-\varepsilon,1+\varepsilon)}(\xi_1)\chi_{(-\varepsilon,\varepsilon)}(\xi_2)\dots\chi_{(-\varepsilon,\varepsilon)}(\xi_d). \tag{22}$$

We compute

$$\|f\|_{L^r_s(\mathbb{R}^d)} \sim \varepsilon^{d-\frac{d}{r}}, \text{ and } \|g_\varepsilon\|_{M^s_{r,t}(\mathbb{R}^d)} \sim \varepsilon^{d-\frac{d}{r}} \tag{23}$$

for any $s \in \mathbb{R}$, $1 \leq r,t \leq \infty$.
We observe

$$\begin{aligned} Ug_\varepsilon(x,t) &= C_d \int_{\mathbb{R}^d} e^{i(x.\xi+t|\xi|^2)}\chi_{(1-\varepsilon,1+\varepsilon)}(\xi_1)\chi_{(-\varepsilon,\varepsilon)}(\xi_2)\dots\chi_{(-\varepsilon,\varepsilon)}(\xi_d)d\xi \\ &= C_d e^{ix_1}e^{it}\int_{\mathbb{R}^d} e^{i((x+2te_1).\xi+t|\xi|^2)}\chi_{(-\varepsilon,\varepsilon)}(\xi_1)\dots\chi_{(-\varepsilon,\varepsilon)}(\xi_d)d\xi. \end{aligned}$$

Hence, $|Uf(x,t)| \gtrsim \varepsilon^d$ provided that $t \in [-\varepsilon^{-2},\varepsilon^{-2}]$ and $|x+2te_1| \lesssim \varepsilon^{-1}$.

Suppose that (21) holds true with $I = \mathbb{R}$. Then,

$$\varepsilon^{d-\frac{2}{p}-\frac{d}{q}} \lesssim \|Uf\|_{L_t^p(\mathbb{R};L_x^q(\mathbb{R}^d))} \lesssim \|f\|_{M^s_{r,t}(\mathbb{R}^d)} \sim \|f\|_{L^r_s(\mathbb{R}^d)} \lesssim \varepsilon^{d-\frac{d}{r}},$$

which requires

$$\frac{2}{p} + \frac{d}{q} \leq \frac{d}{r}. \tag{24}$$

For $I = [-1,1]$, we find

$$\varepsilon^{d-\frac{d}{q}} \lesssim \|Uf\|_{L_t^p(I;L_x^q(\mathbb{R}^d))} \lesssim \|f\|_{M^s_{r,t}(\mathbb{R}^d)} \sim \|f\|_{L^r_s(\mathbb{R}^d)} \lesssim \varepsilon^{d-\frac{d}{r}}.$$

As $\varepsilon \to 0$, this yields $q \geq r$.

The variant of the *anisotropic Knapp-example at high frequencies* rules out gain of derivatives: For $\lambda \in 2^{\mathbb{N}}$ consider

$$\hat{f}_\lambda(\xi) = \chi_{(\lambda-1,\lambda+1)}(\xi_1)\chi_{(-1,1)}(\xi_2)\dots\chi_{(-1,1)}(\xi_d). \tag{25}$$

We note that

$$\|f_\lambda\|_{M^s_{r,t}(\mathbb{R}^d)} \sim \|f_\lambda\|_{L^r_s(\mathbb{R}^d)} \sim \lambda^s. \tag{26}$$

Setting $\varepsilon = \lambda^{-1}$, we find

$$f_\lambda(x) = \lambda^d g_\varepsilon(\lambda x).$$

Furthermore, we find by change of variables and the computation for the unit frequency anisotropic Knapp-example:

$$\|Ug_\varepsilon(\lambda\cdot)\|_{L_t^p([-1,1];L_x^q(\mathbb{R}^d))} = \lambda^{-\frac{2}{p}-\frac{d}{q}}\|Ug_\varepsilon\|_{L_t^p([-\lambda^2;\lambda^2];L_x^q(\mathbb{R}^d))} \gtrsim \lambda^{-d}.$$

By (26) the validity of (21) or (20) requires $s \geq 0$.

Finally, we examine the *isotropic Knapp-example*, previously inspected in [48]. Let $\theta : \mathbb{R}^n \to \mathbb{R}$ denote a radial function, supported in $\{2^{-2} \leq |\xi| \leq 4\}$ and equal to 1 on $\{2^{-1} \leq |\xi| \leq 2\}$. We consider the radially symmetric functions

$$f_\lambda(x) = \big(\frac{\lambda}{2\pi}\big)^n \int \theta(\xi) e^{i(\lambda \langle x,\xi \rangle - \lambda^2 |\xi|^2)} d\xi. \tag{27}$$

By stationary phase, it was computed in [48, p. 50] that

$$|f_\lambda(x)| \lesssim 1, \quad |x| \gg \lambda : \; |f_\lambda(x)| \leq C_N (\lambda^{-1}|x|)^{-N}.$$

Hence,

$$\| f_\lambda \|_{L^r_s(\mathbb{R}^d)} \lesssim \lambda^{s+\frac{d}{r}}.$$

Moreover,

$$\| f_\lambda \|_{M^s_{r,t}(\mathbb{R}^d)} \lesssim \lambda^{s+\frac{d}{t}}.$$

The latter estimate follows as there are $\sim \lambda^d$ unit cubes in the λ-annulus and all of them give rise to a comparable L^p-norm.

Again by (non-)stationary phase, we find the lower bound (cf. [48, p. 50])

$$\big(\int_{1-\lambda^{-2}/10}^1 \| e^{it\Delta} f_\lambda \|^p_{L^q(\mathbb{R}^d)} \big)^{1/p} \gtrsim \lambda^{d - \frac{d}{q} - \frac{2}{p}}. \tag{28}$$

Hence, we find for (21) to hold:

$$\lambda^{d-\frac{d}{q}-\frac{2}{p}} \lesssim \| U f_\lambda \|_{L^p_t([-1,1];L^q_x(\mathbb{R}^d))} \lesssim \| f_\lambda \|_{L^r_s(\mathbb{R}^d)} \lesssim \lambda^{\frac{d}{r}+s}.$$

As $\lambda \to \infty$, we find

$$d - \frac{d}{q} - \frac{2}{p} \leq \frac{d}{r} + s. \tag{29}$$

We find the following for (20) to be true:

$$\lambda^{d-\frac{d}{q}-\frac{2}{p}} \lesssim \| U f_\lambda \|_{L^p_t([-1,1];L^q_x(\mathbb{R}^d))} \lesssim \| f_\lambda \|_{M^s_{r,t}(\mathbb{R}^d)} \lesssim \lambda^{\frac{d}{t}+s}.$$

Taking $\lambda \to \infty$, we find

$$d - \frac{d}{q} - \frac{2}{p} \leq \frac{d}{t} + s. \tag{30}$$

We are ready for the proof of Proposition 1.2:

Proof of Proposition 1.2. The claim (11) follows from the anisotropic Knapp-example at high frequencies and the isotropic Knapp-example (30).
Likewise, (12) follows. The condition $q \geq r$ follows from considering the anisotropic Knapp-example at low frequencies for finite times; the additional integrability condition (13) follows from considering the anisotropic Knapp-example globally in time. The proof is complete. □

We give the proof of Corollary 1.3, which asserts non-existence of global Strichartz estimates

$$\| U f \|_{L^p_t(\mathbb{R}; L^q_x(\mathbb{R}^d))} \lesssim \| f \|_{L^r(\mathbb{R}^d)}. \tag{31}$$

Proof of Corollary 1.3. In addition to the examples from above, we note the scaling condition

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{r}. \tag{32}$$

By considering the isotropic Knapp-example (29), using (32), and assuming (31), we find

$$\lambda^{d-\frac{2}{p}-\frac{d}{q}} = \lambda^{d-\frac{d}{r}} \lesssim \lambda^{\frac{d}{r}}.$$

For $r > 2$ and $\lambda \to \infty$ this is impossible. □

Lastly, we show Corollary 1.4, which asserts the sharp time-dependence in the fixed time estimate in modulation spaces.

Proof of Corollary 1.4. For $|t| \lesssim 1$ there is nothing to prove. For $|t| \gg 1$, we consider again a non-trivial Schwartz initial data, radially symmetric, with supp$\hat{f} \subseteq B(0,1)$. For this we find by non-stationary phase

$$|\int e^{ix.\xi} e^{it|\xi|^2} \hat{f}(\xi) d\xi| \lesssim_N (1+|x|)^{-N}$$

for $|x| \gg |t|$. Moreover,

$$|\int e^{ix.\xi} e^{it|\xi|^2} \hat{f}(\xi) d\xi| \gtrsim (1+|t|)^{-\frac{d}{2}}$$

for $|x| \lesssim |t|$ by [34, Theorem 7.7.5]. Hence, for $|t| \geq 1$,

$$\|U(t)f\|_{L^p(\mathbb{R}^d)} \gtrsim |t|^{-\frac{d}{2}} |t|^{\frac{d}{p}}.$$

This shows

$$\|U(t)\|_{M^s_{p,q} \to M^s_{p,q}} \gtrsim \langle t \rangle^{d|\frac{1}{2}-\frac{1}{p}|}$$

for $1 \leq p \leq 2$. By duality, we find the bound for $2 \leq p \leq \infty$. □

3. ℓ^2-decoupling implies Strichartz in modulation spaces

In this section we show Theorem 1.1. In the remainder of the section let $I = [0,1]$. Define

$$s(p,d) = \begin{cases} 0, & 2 \leq p \leq \frac{2(d+2)}{d}, \\ \frac{d}{2} - \frac{d+2}{p}, & \frac{2(d+2)}{d} < p \leq \infty. \end{cases} \tag{33}$$

To conclude Theorem 1.1, it is enough to prove the estimates

$$\|Uf\|_{L^p(I \times \mathbb{R}^d)} \lesssim \|f\|_{M^{s(p,d)+\varepsilon}_{p,2}(\mathbb{R}^d)}, \tag{34}$$

$$\|Uf\|_{L^p(I \times \mathbb{R}^d)} \lesssim \|f\|_{M_{p,1}(\mathbb{R}^d)}. \tag{35}$$

The remaining estimates follow after frequency localization and Hölder's inequality in the ℓ^q-spaces.

(34) is a consequence of ℓ^2-decoupling. After decoupling, this follows via a kernel estimate. The proof of (35) also invokes the kernel estimate. We set

$$\mathcal{E}f(x,t) = \int_{\mathbb{R}^d} e^{i(x.\xi + t|\xi|^2)} f(\xi) d\xi.$$

Recall the ℓ^2-decoupling theorem due to Bourgain–Demeter [9]:

Theorem 3.1 (ℓ^2-decoupling for the paraboloid)**.** *Let supp$(f) \subseteq \{\xi : |\xi| \leq 1\}$. Then, for any $R \geq 1$, we find the following estimate to hold:*

$$\|\mathcal{E}f\|_{L^p(B_{d+1}(0,R))} \lesssim_\varepsilon R^\varepsilon R^{s(p,d)} \big(\sum_{\square : R^{-\frac{1}{2}}-cube} \|\mathcal{E}f_\square\|^2_{L^p(w(B(0,R)))} \big)^{\frac{1}{2}}.$$

In the above display $w(B(0,R))$ denotes a smooth version of the indicator function on $B(0,R)$ with high polynomial decay off $B(0,R)$; see Subsection 5.1 for further explanation. We show that Theorem 3.1 implies Strichartz estimates in modulation spaces firstly for frequency localized functions:

Proposition 3.2. *Let supp$(\hat{f}) \subseteq \{\xi : \frac{\lambda}{4} \leq |\xi| \leq 4\lambda\}$. Then, we find the following estimate to hold*

$$\|Uf\|_{L^p(\mathbb{R}^d) \times I)} \lesssim_\varepsilon \lambda^{\varepsilon + s(p,d)} \|f\|_{M_{p,2}}. \tag{36}$$

for any $\varepsilon > 0$.

Proof. It is enough to consider for $B = B_d(0,\lambda)$

$$\|Uf\|_{L^p(B\times I)} \lesssim_\varepsilon \lambda^{s(p,d)+\varepsilon} \|f\|_{M_{p,2}(w_B)} \tag{37}$$

because we can write

$$\|Uf\|^p_{L^p(\mathbb{R}^d\times I)} = \sum_{B':\lambda\text{-ball}} \|Uf\|^p_{L^p(B'\times I)}.$$

Then, by translation invariance and Minkowski's inequality, we deduce from (37)

$$\begin{aligned} \|Uf\|_{L^p(\mathbb{R}^d\times I)} &= \big(\sum_{B':\lambda\text{-ball}} \|Uf\|^p_{L^p(B'\times I)} \big)^{\frac{1}{p}} \lesssim_\varepsilon \lambda^{s(p,d)+\varepsilon} \big(\sum_{B':\lambda\text{-ball}} \|f\|^p_{M_{p,2}(w_{B'})} \big)^{\frac{1}{p}} \\ &\lesssim_\varepsilon \lambda^{s(p,d)+\varepsilon} \big(\sum_{B':\lambda\text{-ball}} \big(\sum_k \|\Box_k f\|^2_{L^p(w_{B'})} \big)^{\frac{p}{2}} \big)^{\frac{1}{p}} \\ &\lesssim_\varepsilon \lambda^{s(p,d)+\varepsilon} \big(\sum_k \big(\sum_{B':\lambda\text{-ball}} \|\Box_k f\|^p_{L^p(w_{B'})} \big)^{\frac{2}{p}} \big)^{\frac{1}{2}} \\ &\lesssim_\varepsilon \lambda^{s(p,d)+\varepsilon} \big(\sum_k \|\Box_k f\|^2_{L^p(\mathbb{R}^d)} \big)^{\frac{1}{2}}. \end{aligned}$$

The ultimate estimate follows from summing the rapidly decaying weights. We turn to estimate $\|Uf\|_{L^p(B\times I)}$. Via scaling we reduce to unit frequencies:

$$\|Uf\|_{L^p(I\times B_d(0,\lambda))} = \lambda^{\frac{d+2}{p}} \|Ug\|_{L^p(B_{d+1}(0,\lambda^2))} \tag{38}$$

with $g(x) = f(x/\lambda)$.

We observe by rescaling and ℓ^2-decoupling:

$$\begin{aligned} &\| \int e^{i(x.\xi+t|\xi|^2)} \hat{f}(\xi) d\xi \|_{L^p(I\times B_d(0,\lambda))} = \lambda^{-\frac{d+2}{p}} \| \int e^{i(x'.\xi'+t'|\xi'|^2)} \hat{g}(\xi') d\xi' \|_{L^p(B_{d+1}(0,\lambda^2))} \\ &\lesssim_\varepsilon \lambda^{s(p,d)+\varepsilon-\frac{d+2}{p}} \big(\sum_{\Box:\lambda^{-1}\text{cube}} \|Ug_\Box\|^2_{L^p(w_{B_{d+1}(0,\lambda^2)})} \big)^{\frac{1}{2}}. \end{aligned}$$

The claim follows by a fixed-time kernel estimate. We compute the kernel with a_λ denoting the indicator function of the λ^{-1}-box centered at ξ_0:

$$K(x,t) = \int e^{i(x.\xi+t|\xi|^2)} a_\lambda(\xi) d\xi$$

Via a change of variables and Galilean symmetry, we find

$$K(x,t) = \lambda^{-d} e^{ix.\xi_0} e^{i\frac{t|\xi_0|^2}{\lambda^2}} \int e^{i\big(\frac{x.\xi'}{\lambda}+\frac{2\xi_0.\xi'}{\lambda}t\big)} e^{it\frac{|\xi|^2}{\lambda^2}} a(\xi) d\xi = \lambda^{-d} e^{ix.\xi_0} e^{i\frac{t|\xi_0|^2}{\lambda^2}} \int e^{i\frac{\xi'.y}{\lambda}} e^{it\frac{|\xi'|^2}{\lambda^2}} a(\xi') d\xi'.$$

By non-stationary phase, we find the following estimate:

$$|K(x,t)| \lesssim \lambda^{-d} \big(1 + \lambda^{-1}|y| + \lambda^{-2}|t|\big)^{-N}$$

Hence, a fixed time kernel estimate gives for $|t| \leq \lambda^2$

$$\|Ug_\Box(t)\|_{L^p(\mathbb{R}^d)} \lesssim \|g_\Box\|_{L^p(\mathbb{R}^d)}.$$

Integration in time gives

$$\|Ug_\Box\|_{L^p(w_{B_{d+1}(0,\lambda^2)})} \lesssim \lambda^{\frac{2}{p}} \|g_\Box\|_{L^p(\mathbb{R}^d)}.$$

We conclude the proof by inverting the change of variables:

$$\begin{aligned} &\lambda^{s(p,d)+\varepsilon-\frac{d+2}{p}} \big(\sum_{\Box:\lambda^{-1}\text{cube}} \|Ug_\Box\|^2_{L^p(w_{B_{d+1}(0,\lambda^2)})} \big)^{\frac{1}{2}} \\ &\lesssim \lambda^\varepsilon \lambda^{s(p,d)} \lambda^{-\frac{d}{p}} \big(\sum_\Box \|g_\Box\|^2_{L^p} \big)^{\frac{1}{2}} \lesssim \lambda^{\varepsilon+s(p,d)} \big(\sum_k \|\Box_k f\|^2_{L^p} \big)^{\frac{1}{2}}. \end{aligned}$$

$\Box$

By Galilean invariance and a related kernel estimate we prove the following:

Proposition 3.3. *Let $2 \leq p \leq \infty$. Then, we find the following estimate to hold:*

$$\|Uf\|_{L^p([-1,1]\times\mathbb{R}^d)} \lesssim \|f\|_{M_{p,1}}. \tag{39}$$

Proof. By Minkowski's inequality, it suffices to show

$$\|U\square_k f\|_{L^p([-1,1]\times\mathbb{R})} \lesssim \|\square_k f\|_{L^p(\mathbb{R}^d)}. \tag{40}$$

By Galilean invariance, we observe with $\hat{g}(\xi) = \hat{f}(\xi + k)$

$$\|U\square_k f(t)\|_{L^p} = \|U\square_0 g(t)\|_{L^p}.$$

Let $\chi \in C_c^\infty(\mathbb{R}^d)$. Clearly, $K(x,t) = \int_{\mathbb{R}^d} \chi(\xi) e^{i(x.\xi + t|\xi|^2)} d\xi$ is uniformly in $L^1(\mathbb{R}^d)$ for $|t| \leq 1$. Thus,

$$\|U\square_0 g\|_{L^p([-1,1]\times\mathbb{R}^d)} \lesssim \|\square_0 g\|_{L^p(\mathbb{R}^d)} \lesssim \|\square_k f\|_{L^p(\mathbb{R}^d)}.$$

$\square$

We can conclude the proof of Theorem 1.1:

Proof of Theorem 1.1. (D) is Proposition 3.3. Next, we show (A), (B), and (C) for $q = 2$. By Stein's square function estimate, Minkowski's inequality, and (36), we find

$$\begin{aligned} \|Uf\|_{L^p([-1,1]\times\mathbb{R}^d)} &\lesssim \big\| \big(\sum_N |P_N U f|^2 \big)^{\frac{1}{2}} \big\|_{L^p([-1,1]\times\mathbb{R}^d)} \lesssim \big(\sum_N \|P_N U f\|^2_{L^p([-1,1]\times\mathbb{R}^d)} \big)^{\frac{1}{2}} \\ &\lesssim_\varepsilon \big(\sum_N N^{2(s(p,d)+\varepsilon)} \|P_N f\|^2_{M_{p,2}} \big)^{\frac{1}{2}} \lesssim \|f\|_{M_{p,2}^{s(p,d)+\varepsilon}}. \end{aligned} \tag{41}$$

For $1 \leq q \leq 2$, (A) follows from (41) and interpolating with (D) and for $q \geq 2$, we use the embedding (6). Likewise, (B) follows for $q \geq 2$ via (6). (C) follows from interpolating (B) for $q = 2$ with (D).

$\square$

4. Solving the nonlinear Schrödinger equation with slowly decaying initial data

In the following we solve the nonlinear Schrödinger equation

$$\left\{ \begin{array}{ll} i\partial_t u + \Delta u &= |u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0) &= u_0 \in D \end{array} \right. \tag{42}$$

outside L^2-based Sobolev spaces.

4.1. **Local well-posedness of the cubic NLS for slowly decaying initial data.** In this subsection we prove new local well-posedness results. The local results do not take advantage of the defocusing effect, and the results in this section also hold for the focusing equation:

$$\left\{ \begin{array}{ll} i\partial_t u + \Delta u &= -|u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0) &= u_0 \in D. \end{array} \right.$$

The smoothing estimates are the key ingredient to estimate the homogeneous solution. In the following we use the terminology due to Bejenaru–Tao [2, Section 3]. For further reference, we write (42) as abstract evolution equation

$$u = L(f) + N_3(u,u,u), \tag{43}$$

where u takes values in some solution space S, $L : D \to S$ is a densely defined linear operator, and the trilinear operator $N_3 : S \times S \times S \to S$ is likewise densely defined. As in [2, Section 3], we refer to (43) as quantitatively well-posed in the spaces X, S, if the estimates

$$\|Lf\|_S \leq C\|f\|_D, \tag{44}$$

$$\|N_3(u_1,u_2,u_3)\|_S \leq C\|u_1\|_S \|u_2\|_S \|u_3\|_S \tag{45}$$

hold true for all $f \in D$, and $u_1, u_2, u_3 \in S$ and some constant C. This implies analytic well-posedness (cf. [2, Theorem 3]) and an expression of the solution in terms of its Picard iterates: We define the nonlinear maps $A_m : D \to S$ for $m = 1, 2, \ldots$ by the recursive formulae

$$\begin{aligned} A_1 f &= Lf, \\ A_m f &= \sum_{\substack{m_1, m_2, m_3 \geq 1, \\ m_1 + m_2 + m_3 = m}} N_3(A_{m_1} f, A_{m_2} f, A_{m_3} f) \text{ for } m > 1. \end{aligned}$$

Then we have the homogeneity property

$$A_m(\lambda f) = \lambda^m A_m(f) \text{ for all } \lambda \in \mathbb{R},\ m \geq 1 \text{ and } f \in D,$$

and the Lipschitz bound derived from (44) and (45)

$$\| A_m(f) - A_m(g) \|_S \leq \| f - g \|_D C_1^m \big(\| f \|_D + \| g \|_D \big)^{m-1}.$$

Furthermore, we have the absolutely convergent (in S) power series expansion

$$u[f] = \sum_{m=1}^{\infty} A_m(f)$$

for all $f \in B_D(0, \varepsilon_0)$. In case of (42), we have $A_1 f = Lf = (U(t)f)_{t \in \mathbb{R}}$ and for $m > 1$, $A_m = 0$ if not $m = 2j + 1$ for some $j \in \mathbb{N}$. A_{2j+1} admits expansion into ternary trees of depth j with $2j + 1$ nodes.

To show the linear estimate in Theorem 1.5 for initial data in $L^p_s(\mathbb{R})$, we use the following Schrödinger smoothing estimates due to Rogers in the special case of one dimension:

Theorem 4.1 ([47, Theorem 1])**.** *Let $p \geq 4$. Then, we find the following estimate to hold*

$$\| e^{it\partial_{xx}} u_0 \|_{L^p([-1,1] \times \mathbb{R})} \lesssim \| u_0 \|_{L^p_\alpha(\mathbb{R})} \tag{46}$$

provided that $\alpha > 2\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{2}{p}$.

Note that the case $p = 4$ is not mentioned in [47, Theorem 1], but follows by interpolating estimates for $p > 4$ with the energy estimate

$$\| e^{it\partial_{xx}} u_0 \|_{L^2([-1,1] \times \mathbb{R})} \lesssim \| e^{it\partial_{xx}} u_0 \|_{L^\infty([-1,1], L^2(\mathbb{R}))} \lesssim \| u_0 \|_{L^2(\mathbb{R})}.$$

The linear estimate for initial data in modulation spaces follows from Theorem 1.1. To show the trilinear estimate, we use inhomogeneous Strichartz estimates. Recall the following inhomogeneous Strichartz estimates for the one-dimensional Schrödinger equation (cf. [37, 26]):

Theorem 4.2. *Let q_i, $p_i \geq 2$ for $i = 1, 2$ and $\frac{2}{p_i} + \frac{1}{q_i} \leq \frac{1}{2}$. Then, we find the following estimate to hold:*

$$\| u \|_{L_t^{p_1}([0,T], L_x^{q_1}(\mathbb{R}))} \lesssim \| u(0) \|_{L^2(\mathbb{R})} + \| (i\partial_t + \partial_x^2) u \|_{L_t^{\tilde{p}_2'}([0,T], L_x^{\tilde{q}_2'}(\mathbb{R}))}.$$

We are ready for the proof of Theorem 1.5:

Proof of Theorem 1.5. In the following we consider $0 < T \leq 1$. The claim follows from [2, Theorem 3] once the linear and trilinear estimate are proved. For the linear estimate in L^p-spaces, it suffices to prove for $f \in L^4_s(\mathbb{R})$ or $f \in L^6_s(\mathbb{R})$

$$\| Lf \|_{L_t^{\frac{24}{7}}([0,T], L^4(\mathbb{R}))} \lesssim T^{\frac{1}{24}} \| f \|_{L^4_\varepsilon(\mathbb{R})}, \tag{47}$$

$$\text{and } \| Lf \|_{L_t^3([0,T], L^6(\mathbb{R}))} \lesssim T^{\frac{1}{6}} \| f \|_{L^6_{\frac{1}{3}+s}(\mathbb{R})}. \tag{48}$$

These estimates follow after Hölder in time from the L^p-smoothing estimate (46).

For the linear estimates in modulation spaces, we decompose $f = f_1 + f_2$, $f_1 \in M^s_{4,2}(\mathbb{R})$ or $f_1 \in M^s_{6,2}(\mathbb{R})$, and $f_2 \in L^2(\mathbb{R})$. It suffices to show

$$\|Lf\|_{L_t^{\frac{24}{7}}([0,T],L^4_x(\mathbb{R}))} \lesssim T^{\frac{1}{24}}(\|f_1\|_{M^s_{4,2}(\mathbb{R})} + \|f_2\|_{L^2(\mathbb{R})}), \tag{49}$$

$$\text{and } \|Lf\|_{L^3_t([0,T],L^6_x(\mathbb{R}))} \lesssim T^{\frac{1}{6}}(\|f_1\|_{M^s_{6,2}(\mathbb{R})} + \|f_2\|_{L^2(\mathbb{R})}). \tag{50}$$

Both estimates hold true by Theorem 1.1 applied to Lf_1 and Strichartz estimates applied to Lf_2. The trilinear estimate follows from the estimates

$$\| \int_0^t e^{i(t-s)\partial_x^2} F(s) ds \|_{L^8_t([0,T],L^4_x(\mathbb{R}))} \lesssim \|F\|_{L_t^{\frac{8}{7}}([0,T],L_x^{\frac{4}{3}}(\mathbb{R}))}$$

and

$$\| \int_0^t e^{i(t-s)\partial_x^2} F(s) ds \|_{L^6_t([0,T],L^6_x(\mathbb{R}))} \lesssim \|F\|_{L^1_t([0,1],L^2_x(\mathbb{R}))},$$

which are both covered by Theorem 4.2, and applying Hölder's inequality. Hence, choosing $T = T(\|f\|_D)$, we can apply the contraction mapping principle in $L^r_t([0,T], L^p(\mathbb{R}))$ with r as above.

To prove $u \in C([0,T], M^\varepsilon_{p,2} + L^2)$, it suffices to show $Lf \in C([0,T], M^\varepsilon_{p,2} + L^2)$ and $N_3(u,u,u) \in C([0,T], M^\varepsilon_{p,2} + L^2)$. Let $f = f_1 + f_2$ with $f_1 \in M^\varepsilon_{p,2}$ and $f_2 \in L^2$. By Minkowski's inequality, we find by $U(t)M^\varepsilon_{p,2} = M^\varepsilon_{p,2}$ and $U(t)L^2 = L^2$ that

$$\lim_{t\to 0} \|(Uf)(t) - f\|_{M^\varepsilon_{p,2}+L^2} \leq \limsup_{t\to 0} \|Uf_1(t) - f_1\|_{M^\varepsilon_{p,2}} + \limsup_{t\to 0} \|Uf_2(t) - f_2\|_{L^2} = 0.$$

The continuity in $M^\varepsilon_{p,2}$ and L^2 is a consequence of $(U(t))_{t\in\mathbb{R}}$ a C_0-group in both spaces.

For $N_3(u,u,u)$, it suffices to show continuity in L^2. By Strichartz estimates, we find

$$\|N_3(u,u,u)\|_{L^\infty_t([0,T],L^2)} \lesssim \|u\|^3_{L^r_t([0,T],L^p)} \tag{51}$$

and

$$\begin{aligned} &\big\| \int_0^t e^{i(t-s)\Delta}(|u|^2u)(s) ds - \int_0^{t+\delta} e^{i((t+\delta)-s)\Delta}(|u|^2u)(s) ds \big\|_{L^2} \\ &\leq \big\| (e^{i\delta\Delta} - 1) \int_0^t e^{i(t-s)\Delta}(|u|^2u)(s) ds \big\|_{L^2} + \big\| \int_t^{t+\delta} e^{i((t+\delta)-s)\Delta}(|u|^2u)(s) ds \big\|_{L^\infty_{\delta\in I} L^2}. \end{aligned}$$

For the first term, the limit is zero as $N_3(u,u,u)(t) \in L^2$ by (51) and $(U(t))_{t\in\mathbb{R}}$ a C_0-group in L^2. For the second term, we use again Strichartz estimates to find

$$\| \int_t^{t+\delta} e^{i((t+\delta)-s)\Delta}(|u|^2u)(s) ds \|_{L^\infty_{\delta\in I} L^2} \lesssim \|u\|^3_{L^r_t(I,L^p)}.$$

By multilinearity, we see by similar arguments that for differences of solutions

$$\|u - \tilde{u}\|_{C([0,T],M^\varepsilon_{p,2}+L^2)} \to 0$$

for $\|u(0) - \tilde{u}(0)\|_{M^\varepsilon_{p,2}+L^2} \to 0$ provided that $T = T(\|u(0)\|_{M^\varepsilon_{p,2}+L^2}, \|\tilde{u}(0)\|_{M^\varepsilon_{p,2}+L^2})$ is chosen small enough, according to the local existence time in $L^r_t([0,T], L^p)$. The proof is complete.

□

We remark that we have some flexibility in the solution space $S = L^p_t([0,T], L^p_x(\mathbb{R}))$. For small initial data, we can likewise iterate in $L^4_{t,x}([0,1] \times \mathbb{R})$ or $L^6_{t,x}([0,1] \times \mathbb{R})$. For $p > 6$, although the sharp linear estimates are still at disposal, it is not clear how to apply inhomogeneous Strichartz estimates as directly as above.

4.2. **Improved local results for slowly decaying data.** In the following we apply the iteration worked out by Dodson *et al.* [20] with the sharp fixed time and smoothing estimate. For simplicity, we focus on the small data case with $T = 1$. We recall the sharp fixed-time estimate for the linear propagation due to Fefferman–Stein [22] and Miyachi [42] for convenience.

Theorem 4.3 ([22, 42]). *Let $1 < p < \infty$, $d \geq 1$. Then, we find the following estimate to hold*

$$\|e^{it\Delta}u_0\|_{L^p(\mathbb{R}^d)} \lesssim \|u_0\|_{L^p_\alpha(\mathbb{R}^d)} \tag{52}$$

for $\alpha \geq 2d\big|\frac{1}{2} - \frac{1}{p}\big|$.

Let $n \geq 2$. The idea in [20] to solve (42) with initial data in $L^{4n+2}_{s(n)}$ is to split the expansion

$$u[f] = \sum_{m \geq 1} A_m(f) = Lf + N_3(u,u,u)$$

not into linear and nonlinear part, but to consider higher Picard iterates

$$\begin{aligned} u^0(t) &= Lf, \\ u^1(t) &= N_3(u^0,u^0,u^0), \dots \\ u^j(t) &= N_3\Big(\sum_{k=0}^{j-1} u^k, \sum_{k=0}^{j-1} u^k, \sum_{k=0}^{j-1} u^k\Big) - u^{j-1}(t). \end{aligned} \tag{53}$$

and to prove existence of $v \in S^0([-1,1]\times\mathbb{R}) = L^\infty_t L^2_x \cap L^4_t L^\infty_x$, which solves

$$v = u - \sum_{j=0}^{n-1} u^j.$$

This is equivalent to

$$v = N_3(u,u,u) - \sum_{j=1}^{n-1} u^j = N_3\Big(v + \sum_{j=0}^{n-1} u^j, v + \sum_{j=0}^{n-1} u^j, v + \sum_{j=0}^{n-1} u^j\Big) - \sum_{j=1}^{n-1} u^j. \tag{54}$$

This approach requires to prove estimates for $A_m(f)$ directly. We have the following:

Lemma 4.4. *Let $n \geq 2$ and $m \in \{1,\dots,2n-1\}$. With the above notations, we find*

$$\|A_m f\|_{L^\infty_t L^{\frac{4n+2}{m}}} \lesssim \|f\|^m_{L^{4n+2}_\alpha} \tag{55}$$

for $\alpha > (n-1)\big(2 - \frac{6}{4n+2}\big) + \frac{4n-2}{4n+2}$.
Furthermore, the estimate

$$\|A_m f\|_{L^\infty_t L^{\frac{4n+2}{m}}} \lesssim \|f\|^m_{M^\beta_{4n+2,2}} \tag{56}$$

holds true with $\beta > (n-1)\big(2 - \frac{6}{4n+2}\big) + \frac{2n-2}{4n+2}$.

Proof. In the following we estimate $A_m f$ in $L^{\frac{4n+2}{m}}$ recursively. Let

$$\alpha_1 = 2\Big(\frac{1}{2} - \frac{1}{4n+2}\Big) - \frac{2}{4n+2} = 1 - \frac{4}{4n+2} = \frac{4n-2}{4n+2} \tag{57}$$

such that by L^p-smoothing estimates in Theorem 4.1

$$\|A_1 f\|_{L^{4n+2}_{t,x}} \lesssim \|\langle \partial_x \rangle^{\alpha_1+\varepsilon} f\|_{L^{4n+2}}.$$

For $m > 1$ let α_m denote the number of derivatives such that

$$\|A_m f\|_{L^\infty_t L_x^{\frac{4n+2}{m}}} \lesssim \|\langle \partial_x \rangle^{\alpha_m} f\|^m_{L^{4n+2}}$$

and $\alpha_m \geq \alpha_{m-2}$. By

$$A_m f = \sum_{\substack{m_1,m_2,m_3\geq 1:\\ m_1+m_2+m_3=m}} \int_0^t e^{i(t-s)\partial_x^2}(A_{m_1}f)(s)\overline{(A_{m_2}f)(s)}(A_{m_3}f)(s)ds,$$

we can use Minkowski's inequality, the fixed time estimate with $\delta_m = 2\big(\frac{1}{2} - \frac{m}{4n+2}\big)$, and Leibniz's rule to conclude (again we do not keep track of complex conjugates as the estimates are invariant)

$$\begin{aligned}
\|A_m f\|_{L^{\frac{4n+2}{m}}} &\leq \sum_{\substack{m_1,m_2,m_3\geq 1:\\ m_1+m_2+m_3=m}} \|\langle\partial_x\rangle^{\delta_m}\big[(A_{m_1}f)(A_{m_2}f)(A_{m_3}f)\big]\|_{L_t^1 L_x^{\frac{4n+2}{m}}} \\
&\lesssim \sum_{\substack{m_1,m_2,m_3\geq 1:\\ m_1+m_2+m_3=m}} \|\langle\partial_x\rangle^{\delta_m} A_{m_1}f\|_{L_t^3 L_x^{\frac{4n+2}{m_1}}} \|\langle\partial_x\rangle^{\delta_m} A_{m_2}f\|_{L_t^3 L_x^{\frac{4n+2}{m_2}}} \|\langle\partial_x\rangle^{\delta_m} A_{m_3}f\|_{L_t^3 L_x^{\frac{4n+2}{m_3}}}.
\end{aligned}$$

Hence, $\alpha_m \leq \delta_m + \alpha_{m-2}$. Iterating the argument yields

$$\alpha_m \leq \delta_m + \delta_{m-2} + \ldots + \delta_3 + \alpha_1 + \varepsilon = \sum_{j=0}^{j^*} \delta_{m-2j} + \alpha_1 + \varepsilon. \tag{58}$$

We compute α_m for $m = 2n-1$. In this case $m - 2j^* = 3$, $j^* = n-2$. We evaluate the sum as

$$\begin{aligned}
\sum_{j=0}^{j^*} \delta_{m-2j} &= 2(j^*+1) - \frac{2}{4n+2}\sum_{j=0}^{j^*}(m-2j) \\
&= 2(j^*+1) - \frac{2}{4n+2}\big((j^*+1)m - 2(j^*+1)j^*\big) \\
&= (n-1)(2 - \frac{6}{4n+2}).
\end{aligned}$$

Hence, the above display yields (55) by (58) and (57). For the proof of (56) let

$$\beta_1 = \frac{1}{2} - \frac{3}{4n+2} = \frac{2n-2}{4n+2}$$

such that by L^p-smoothing in modulation spaces

$$\|A_1 f\|_{L^{4n+2}_{t,x}} \lesssim \|f\|_{M^{\beta_1+\varepsilon}_{4n+2,\varepsilon}}.$$

For $m > 1$ let β_m denote the number of derivatives such that

$$\|A_m f\|_{L_t^\infty L_x^{\frac{4n+2}{m}}} \lesssim \|f\|^m_{M^{\beta_m}_{4n+2,2}}$$

and $\beta_m \geq \beta_{m-2}$. By the same argument as above,

$$\beta_m \leq \delta_m + \delta_{m-2} + \ldots + \delta_3 + \beta_1 + \varepsilon = \sum_{j=0}^{j^*} \delta_{m-2j} + \beta_1 + \varepsilon.$$

For $m = 2n-1$ this proves (56).

□

In the following, for fixed $n \geq 2$, let

$$\alpha = (n-1)\big(2 - \frac{6}{4n+2}\big) + \frac{4n-2}{4n+2}, \quad \beta = (n-1)\big(2 - \frac{6}{4n+2}\big) + \frac{2n-2}{4n+2}.$$

A variant of the argument proves the following:

Lemma 4.5. *Let $n \geq 2$, $0 \leq j \leq n-1$, and u^j as in (53). Then, for $\varepsilon > 0$, there are $\varepsilon_n \leq 1$ and $\tilde{\varepsilon}_n \leq 1$ such that*

$$\|u^j\|_{L^\infty_{t,x}} + \|u^j\|_{L_t^\infty L^{\frac{4n+2}{2j+1}}} \lesssim \|\langle \partial_x \rangle^{\alpha+\frac{1}{2}+\varepsilon} f\|_{L^{4n+2}}$$

holds true provided that $\|\langle \partial_x \rangle^{\alpha+\frac{1}{2}+\varepsilon} f\|_{L^{4n+2}} \leq \varepsilon_n$, and

$$\|u^j\|_{L^\infty_{t,x}} + \|u^j\|_{L_t^\infty L^{\frac{4n+2}{2j+1}}} \lesssim \|f\|_{M^{\beta+\frac{1}{2}+\varepsilon}_{4n+2,2}}$$

holds true provided that $\|f\|_{M^{\beta+\frac{1}{2}+\varepsilon}_{4n+2,2}} \leq \tilde{\varepsilon}_n$.

The iteration is the same as in the proof of Lemma 4.4, with additional terms estimated with u^j also in L^∞_x. We can prove existence of v with the above estimates at hand:

Proposition 4.6. *Let $\varepsilon > 0$, $n \geq 2$, and ε_n, $\tilde{\varepsilon}_n \leq 1$ as in Lemma 4.5. Then, there is a unique $v \in S^0$ satisfying (54).*

Proof. We rewrite (54) modulo order of the arguments in N_3 as

$$\begin{aligned} v = N_3(v,v,v) + 3N_3(v,v,\sum_{j=0}^{n-1} u^j) + 3N_3(v, \sum_{j=0}^{n-1} u^j, \sum_{j=0}^{n-1} u^j) \\ + N_3(\sum_{j=0}^{n-1} u^j, \sum_{j=0}^{n-1} u^j, \sum_{j=0}^{n-1} u^j) - \sum_{j=1}^{n-1} u^j. \end{aligned}$$

By Theorem 4.2 and $u^j \in L^\infty_{t,x}$ by Lemma 4.5, we find

$$\|N_3(v,v,v)\|_{S^0} \lesssim \|v\|^3_{S^0}, \; \|N_3(v,v,\sum_{j=0}^{n-1} u^j)\|_{S^0} \lesssim \varepsilon \|v\|^2_{S^0}, \; \|N_3(v, \sum_{j=0}^{n-1} u^j, \sum_{j=0}^{n-1} u^j)\|_{S^0} \lesssim \varepsilon^2 \|v\|_{S^0}.$$

We rewrite the last term as

$$N_3(\sum_{j=0}^{n-1} u^j, \sum_{j=0}^{n-1} u^j, \sum_{j=0}^{n-1} u^j) - N_3(\sum_{j=0}^{n-2} u^j, \sum_{j=0}^{n-2} u^j, \sum_{j=0}^{n-2} u^j)$$

and estimate again via Strichartz estimates

$$\begin{aligned} &\|N_3(\sum_{j=0}^{n-1} u^j, \sum_{j=0}^{n-1} u^j, \sum_{j=0}^{n-1} u^j) - N_3(\sum_{j=0}^{n-2} u^j, \sum_{j=0}^{n-2} u^j, \sum_{j=0}^{n-2} u^j)\|_{S^0} \\ &\lesssim \|u^{n-1}\|_{L_t^\infty L^{\frac{4n+2}{2n-1}}} \big(\sum_{j=0}^{n-1} \|u^j\|_{L_t^\infty L_x^{4n+2}} \big)^2 \lesssim \varepsilon^{2n+1}. \end{aligned}$$

The claim follows from applying the contraction mapping principle. ☐

We have proved the following local well-posedness result for slowly decaying data:

Theorem 4.7. *Let $\varepsilon > 0$, $n \geq 2$, and f, ε_n, $\tilde{\varepsilon}_n$ as in Proposition 4.6. Let $\frac{2}{p} + \frac{1}{4n+2} = \frac{1}{2}$. Then, there is $u \in L^p_t([0,1], L^{4n+2}(\mathbb{R}))$, which satisfies (43). Furthermore, for $\|f_1\|_{L^{4n+2}_{\alpha+\frac{1}{2}+\varepsilon}} + \|f_2\|_{L^{4n+2}_{\alpha+\frac{1}{2}+\varepsilon}} \leq \varepsilon_n$ or $\|f_1\|_{M^{\beta+\frac{1}{2}+\varepsilon}_{4n+2,2}} + \|f_2\|_{M^{\beta+\frac{1}{2}+\varepsilon}_{4n+2,2}} \leq \tilde{\varepsilon}_n$, we have for the corresponding solutions $\|u_1 - u_2\|_{L^p([0,1],L^{4n+2})} \to 0$ as $\|f_1 - f_2\|_{L^{4n+2}_{\alpha+\frac{1}{2}+\varepsilon}} \to 0$ or $\|f_1 - f_2\|_{M^{\beta+\frac{1}{2}+\varepsilon}_{4n+2,2}} \to 0$, respectively.*

Proof. The claim follows as $v \in L^p_t([0,1], L^{4n+2})$ by Proposition 4.6 and $u^j \in L^\infty_t([0,1], L^{4n+2})$. Hence,

$$u = \sum_{j=0}^{n-1} u^j + v = Lf + N_3(u,u,u) \in L^p_t([0,1], L^{4n+2}),$$

and the continuity of the data-to-solution mapping follows from multilinearity. □

4.3. **Global well-posedness in modulation spaces.** Next, we show global results in modulation spaces stated in Theorem 1.6. We use the blow-up alternative due to Dodson–Soffer–Spencer [20, Section 3]. Since we work with initial data in modulation spaces, for which we have improved homogeneous Strichartz estimates, this requires less Sobolev regularity compared to the L^p-case.

Let $w(t) = U(t)f$ denote the first Picard iterate, and let $v(t) = u(t) - w(t)$. Then v satisfies

$$v(t) = N_3(v + w, v + w, v + w).$$

We have the following blow-up alternative:

Lemma 4.8 (Blow-up alternative). *Let $s > 0$, and $f \in M^s_{4,2}(\mathbb{R}) + L^2(\mathbb{R})$. If T^* is maximal such that $u \in L_t^{\frac{24}{7}}([0, T^*], L^4(\mathbb{R}))$ for $T < T^*$, but $u \notin L_t^{\frac{24}{7}}([0, T^*], L^4(\mathbb{R}))$, then*

$$\lim_{t\to T^*} \|v(t)\|_{L^2} = \infty.$$

The same blow-up alternative holds for $f \in M^s_{6,2}(\mathbb{R}) + L^2(\mathbb{R})$ in the solution space $L^3_t([0, T], L^6(\mathbb{R}))$.

Proof. We shall only look into the case $f \in M^s_{4,2}(\mathbb{R})+L^2(\mathbb{R})$ as the second claim follows *mutatis mutandis*. We argue by contradiction. Let $(t_n) \subseteq [0, T^*)$, $t_n \to T^*$, but

$$\|v(t_n)\|_{L^2(\mathbb{R})} \leq C. \tag{59}$$

Firstly, letting $f = f_1 + f_2$, $f_1 \in M^s_{4,2}(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R})$, we have

$$\|Lf_1(t)\|_{M^s_{4,2}(\mathbb{R})} \lesssim_{T^*} \|f_1\|_{M^s_{4,2}(\mathbb{R})}, \quad \|Lf_2(t)\|_{L^2(\mathbb{R})} \leq \|f_2\|_{L^2(\mathbb{R})}$$

Hence, $\|Lf(t)\|_{M^s_{4,2}(\mathbb{R})+L^2(\mathbb{R})} \lesssim_{T^*} \|f\|_{M^s_{4,2}(\mathbb{R})+L^2(\mathbb{R})}$. But together with (59), this implies that there is a sequence $t_n \to T^*$ with

$$\|u(t_n)\|_{M^s_{4,2}(\mathbb{R})+L^2(\mathbb{R})} \leq \tilde{C}.$$

Since the local existence time of solutions in $L_t^{\frac{24}{7}}([0, T], L^4_x(\mathbb{R}))$ only depends on $\|f\|_{M^s_{4,2}(\mathbb{R})+L^2(\mathbb{R})}$ by Theorem 1.5, we see that we can continue the solutions beyond T^*. This is a contradiction. □

Hence, for the proof of global well-posedness it suffices to show

$$\sup_{t\in[0,T]} \|v(t)\|_{L^2} \leq C(T)$$

for some non-decreasing function $C : [0, \infty) \to [0, \infty)$. Let

$$\begin{aligned}
M(v) &= \frac{1}{2}\int |v|^2 dx,\\
E(v) &= \int \frac{1}{2}|v_x|^2 + \frac{1}{4}|v|^4 dx,\\
\tilde{E}(v) &= \int \frac{1}{2}|v_x|^2 + \frac{1}{4}(|v + w|^4 - |w|^4) dx.
\end{aligned}$$

Note that a priori it is not clear that $E(v(t))$ is finite for $t \neq 0$. The following computations are carried out for initial data from a suitable a priori class, say $f \in \mathcal{S}(\mathbb{R})$. This ensures all quantities to be finite and allows to justify integration by parts arguments. Since we prove bounds depending only on $\|u_0\|_{M^s_{p,q}}$ for $p < \infty$, the arguments are a posteriori justified by density and well-posedness.

For the homogeneous solutions, we observe by the fixed time estimate in modulation spaces and their embedding properties,

$$\|w(t)\|_{M^s_{4,2}} \lesssim \langle t\rangle^{\frac{1}{8}} \|w(0)\|_{M^s_{4,2}}, \quad \|w(t)\|_{L^4} \lesssim \|w(t)\|_{M_{4,4/3}} \lesssim \|w(t)\|_{M^{\frac{1}{4}+\varepsilon}_{4,2}}.$$

By Sobolev embedding, we have for any $t \in [0,T]$

$$\|\langle \partial_x \rangle w(t)\|_{L^4} + \|\langle \partial_x \rangle w(t)\|_{L^\infty} \lesssim_T \|w(0)\|_{M^{\frac{3}{2}+\varepsilon}_{4,2}}. \tag{60}$$

For L^6-based modulation spaces, we find by the same embedding properties

$$\|\langle \partial_x \rangle w(t)\|_{L^\infty} + \|\langle \partial_x \rangle w(t)\|_{L^6} \lesssim_T \|w(0)\|_{M^{\frac{3}{2}+\varepsilon}_{6,2}}. \tag{61}$$

Hence, Hölder's inequality and the assumptions on the initial data imply

$$E \leq C(T)(\tilde{E} + M + 1).$$

By Lemma 4.8, solutions in $L^p([0,T], L^q)$ with p, q as in Theorem 1.6 for any $T > 0$ follow from the following:

Proposition 4.9. *Let $\varepsilon > 0$, $f \in M^{\frac{3}{2}+\varepsilon}_{4,2}$ or $f \in M^{\frac{3}{2}+\varepsilon}_{6,2}$. For all $T > 0$, we have*

$$\sup_{t \in [0,T]} M(v(t)) + E(v(t)) \lesssim_T 1.$$

Moreover, Theorem 1.6 follows from this proposition by piecing together the local solutions. We turn to its proof:

Proof of Proposition 4.9. Let

$$(f,g) = \Re \int f(x)\overline{g(x)} dx.$$

We aim to bound the quantity $M(v) + \tilde{E}(v) + 1$ by Grønwall's lemma. For the derivative of M, we compute

$$\begin{aligned} \partial_t M(v) = (v, v_t) &= (v, -i(v_{xx} + |v+w|^2(v+w)) \\ &= (v, 2|v|^2 w + v^2 \overline{w} + 2|w|^2 v + w^2 \overline{v} + |w|^2 w). \end{aligned}$$

By Hölder's inequality, we compute for $\|w(t)\|_{L^4} \lesssim_T 1$

$$\partial_t M(v) \lesssim_T E(v)^{\frac{3}{4}} + E(v)^{\frac{1}{2}} + E(v)^{\frac{1}{4}} \lesssim_T M(v) + E(v) + 1.$$

Similarly, for $\|w(t)\|_{L^6} + \|w(t)\|_{L^\infty} \lesssim_T 1$, we find

$$\partial_t M(v) \lesssim_T M(v)^{\frac{1}{2}} E(v)^{\frac{1}{2}} + M(v)^{\frac{1}{2}} + M(v)^{\frac{1}{2}} \lesssim_T M(v) + E(v) + 1.$$

For the derivative of $\tilde{E}$, we compute

$$\begin{aligned} \partial_t \int \frac{1}{2} |v_x|^2 dx = (-v_t, v_{xx}) &= -(v_t, -iv_t + |v+w|^2(v+w)) \\ &= -(v_t, |v+w|^2(v+w)) \end{aligned}$$

and

$$\partial_t \int \frac{1}{4}(|v+w|^4 - |w|^4) dx = (v_t + w_t, |v+w|^2(v+w)) - (w_t, |w|^2 w).$$

Hence,

$$\partial_t \tilde{E}(v) = (w_t, |v+w|^2(v+w) - |w|^2 w).$$

This expression has total homogeneity four in v, w. Let (h_v, h_w) denote the homogeneity in v, w. Then we have to estimate the cases $(1,3)$, $(2,2)$, $(3,1)$. Let the collected terms be denoted by $A_{(h_v,h_w)}$. If (60) holds true, then applications of Hölder's inequality give

$$\begin{aligned}|A_{(1,3)}| &= |(w_{xx}, 2|w|^2 v + \bar{v} w^2)| = |(w_x, 2(\overline{w}_x w v + \overline{w} w_x v + |w|^2 v_x + w_x w \overline{v}) + w^2 v_x)| \\ &\lesssim_T E(v)^{\frac{1}{4}} + E(v)^{\frac{1}{2}}, \\ |A_{(2,2)}| &= |(w_{xx}, 2|v|^2 w + \overline{w} v^2)| = |(w_x, 2(v \overline{v}_x w + \overline{v} v_x w + |v|^2 w_x + \overline{w} v v_x) + \overline{w}_x v^2)| \\ &\lesssim_T E(v)^{\frac{3}{4}} + E(v)^{\frac{1}{2}} M(v)^{\frac{1}{2}} + E(v)^{\frac{1}{2}}, \\ |A_{(3,1)}| &= |(w_{xx}, |v|^2 v)| = |(w_x, 2|v|^2 v_x + v^2 \overline{v}_x)| \lesssim_T E(v).\end{aligned}$$

If (61) holds true, then by another application of Hölder's inequality, we find

$$|A_{(1,3)}| \lesssim_T M(v)^{\frac{1}{2}} + E(v)^{\frac{1}{2}}, \quad |A_{(2,2)}| \lesssim_T M(v)^{\frac{1}{2}} E(v)^{\frac{1}{2}} + M(v)^{\frac{1}{2}}, \quad |A_{(3,1)}| \lesssim_T E(v).$$

Consequently,

$$\partial_t \tilde{E}(v) \lesssim_T (1 + M(v) + E(v)) \lesssim_T (1 + M(v) + \tilde{E}(v)).$$

Thus,

$$\partial_t (1 + M(v) + \tilde{E}(v)) \leq C(T)(1 + M(v) + \tilde{E}(v))$$

and Grønwall's inequality yields

$$1 + M(v) + \tilde{E}(v) \leq e^{\int_0^T C(t) dt}$$

with $C(t) = C(t, \|u_0\|_{M^{\frac{3}{2}+\varepsilon}_{4,2}})$, or $C(t) = C(t, \|u_0\|_{M^{\frac{3}{2}+\varepsilon}_{6,2}})$, respectively. Hence,

$$M(v) + E(v) \lesssim_T 1.$$

□

Furthermore, it seems possible with the sharp smoothing and fixed-time estimates to prove global results for initial data in L^{4n+2}_s or $M^s_{4n+2,2}$ using arguments due to Dodson *et al.* [20]; see also the previous subsection. As this approach still loses many derivatives as it does not use smoothing of the Duhamel term, the details are omitted. We plan to return to this problem in future work.

5. Variable-coefficient decoupling inequalities for non-elliptic Schrödinger equations

In this section we prove variable-coefficient decoupling inequalities for elliptic and hyperbolic phase functions. We start with describing the set-up in Subsection 5.1 and then carry out the proof in Subsection 5.2.

5.1. Variable-coefficient oscillatory integral operators.

We consider smooth functions $a \in C^\infty_c(\mathbb{R}^{n+1} \times \mathbb{R}^n)$, $a = a_1 \otimes a_2$, $0 \leq a_1, a_2 \leq 1$ and $\phi : B^{n+1}(0,1) \times B^n(0,1) \to \mathbb{R}$, which we shall refer to as amplitude and phase function.

We associate the oscillatory integral operator

$$Tf(t,x) = \int_{\mathbb{R}^n} e^{i\phi(t,x,\xi)} a(t,x,\xi) f(\xi) d\xi \tag{62}$$

and the rescaled versions

$$T^\lambda f(t,x) = \int_{\mathbb{R}^n} e^{i\lambda\phi(t/\lambda, x/\lambda, \xi)} a(t/\lambda, x/\lambda, \xi) f(\xi) d\xi \tag{63}$$

for different classes of phase functions.

Subject of discussion are variable-coefficient generalizations of the phase function

$$\phi_{hyp}(t,x;\xi) = \langle x, \xi \rangle + \frac{t\langle \xi, I^k_n \xi \rangle}{2}, \quad I^k_n = diag(1,\dots,1,\underbrace{-1,\dots,-1}_{k}), \quad 0 \leq k \leq n/2.$$

Set also $I_n = diag(1,\dots,1) \in \mathbb{R}^{n\times n}$.
In the following we shall always assume that there are at most as many negative eigenvalues as positive eigenvalues, which is no loss of generality since time reversal $t \to -t$ flips signs.

We define the Gauss map by

$$G : B^{n+1} \times B^n \to \mathbb{S}^n, \quad G(z;\xi) = \frac{G_0(z;\xi)}{|G_0(z;\xi)|}; \quad z = (t,x), \tag{64}$$

where $m \in \mathbb{N}$ and B_m denotes the unit ball in $\mathbb{R}^m$ and

$$G_0(z;\omega) = \bigwedge_{j=1}^{n} \partial_{\xi_j} \partial_z \phi(z;\xi) \tag{65}$$

with the standard identification $\bigwedge^n \mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$.

We impose the following conditions on the phase function:

$$\begin{aligned} &H1) \quad \text{rank } \partial^2_{\xi x} \phi(z;\xi) = n \quad \forall\, (z,\xi) \in B^{n+1} \times B^n, \\ &H2) \quad \partial^2_{\xi \xi} \, \langle \partial_z \phi(z;\xi), G(z;\xi_0) \rangle|_{\xi = \xi_0} \text{ is non-degenerate.} \end{aligned}$$

$H1)$ is a non-degeneracy condition, and $H2)$ implies that the constant coefficient approximation of ϕ is the adjoint Fourier restriction operator (i.e. extension operator) associated to a non-degenerate surface.

Contrary to the constant-coefficient case ϕ_{hyp}, rescaling $(t,x) \to (\lambda^2 t, \lambda x)$, $\xi \to \xi/\lambda$ yields no exact symmetry. Therefore, it is useful to quantify the conditions $H1)$ and $H2)$. Before doing so, we point out the following more precise versions of $H1)$ and $H2)$, which one may assume without loss of generality:

$$\begin{aligned} &H1') \quad \det \partial^2_{\xi x} \phi(z;\xi) \neq 0 \text{ for all } (z;\xi) \in T \times X \times \Xi = Z \times \Xi; \\ &H2'_{[k]}) \quad \partial_t \partial^2_{\xi\xi} \phi(z;\xi) \text{ is non-degenerate for all } (z;\xi) \in Z \times \Xi \\ &\qquad \text{and has exactly } k \text{ negative eigenvalues.} \end{aligned}$$

Here, T, X, Ξ denote balls of radius less or equal to one around the origin. To reduce from $H1)$ and $H2)$ to the conditions in the above display, one applies a rotation in space-time. This gives $G(0;0) = e_{n+1}$, and then one uses a partition of unity to suitably localize the support. Moreover, the implicit function theorem implies the existence of smooth functions Φ and Ψ taking values in X and Ω, respectively, such that

$$\partial_x \phi(z; \Psi(z;\xi)) = \xi \tag{66}$$

and

$$\partial_\xi \phi(t, \Phi(t,x;\xi);\xi) = x. \tag{67}$$

The first identity allows us to find a graph parametrization $\xi \mapsto (\partial_z \phi(z;\Psi(z;\xi))) = (\xi, (\partial_t \phi)(z;\Psi(z;\xi)))$ for a hypersurface Σ_z with non-vanishing curvature. From differentiating the second identity we find $\partial_x \Phi(0;0) = \partial^2_{x\xi} \phi(0;0)^{-1}$.
Later on, $H1')$ and $H2')$ are quantified. It turns out that one can perceive any phase function satisfying $H1')$ and $H2')$ after introducing a partition of unity and rescaling as small smooth perturbations of $\phi_{hyp} = \langle x, \xi \rangle + \frac{t \langle \xi, I^n_k \xi \rangle}{2}$.

For $h \in C^2(B^n(0,1), \mathbb{R})$ let the extension operator E_h be given by

$$E_h f(t,x) = \int_{B^n(0,1)} e^{i(x\xi + t h(\xi))} f(\xi) d\xi,$$

where $f \in L^2$, $\text{supp}(f) \subseteq B^n(0,1)$ and define a smooth weight function, which is essentially a characteristic function on some ball $B^{n+1}(\overline{z}, R)$, $\overline{z} = (\overline{t}, \overline{x})$:

$$w_{B(\overline{z},R)}(t,x) = (1 + R^{-1}|x - \overline{x}| + R^{-1}|t - \overline{t}|)^{-N}$$

for some large integer $N \in \mathbb{N}$, which is fixed later.

We define the decoupled L^p-norm for variable coefficient operators for
$1 \leq R \leq \lambda$. Let $\mathcal{T}_R$ denotes a finitely overlapping family of $R^{-1/2}$ balls covering $B^n(0,1)$. Set

$$\|T^\lambda f\|_{L^{p,R}_{dec}(S)} = \left(\sum_{\tau \in \mathcal{T}_R} \|T^\lambda f_\tau\|^2_{L^p(S)} \right)^{1/2}$$

for S measurable and

$$\alpha(p,k) = \begin{cases} k\left(\frac{1}{4} - \frac{1}{2p}\right), & 2 \leq p \leq \frac{2(n+2-k)}{n-k}, \\ \frac{n}{4} - \frac{n+2}{2p}, & \frac{2(n+2-k)}{n-k} \leq p < \infty. \end{cases} \tag{68}$$

We recall the constant-coefficient ℓ^2-decoupling theorem proved in [9, 10]:

Theorem 5.1. [10, Theorem 1.2, p. 280] *Let $R \geq 1$, $N \geq 10$, $2 \leq p < \infty, 0 \leq k \leq n/2$, $\alpha(p,k)$ as in* (68) *and $h : B^n(0,1) \to \mathbb{R}$ be a C^2-function with Hessian $\partial^2_{\xi\xi} h$ having modulus of eigenvalues in $[C^{-1}, C]$ for some $C > 0$. Then, we find for f with supp$(f) \subseteq B(0,1)$ the following estimate to hold:*

$$\|E_h f\|_{L^p(w_{B_R})} \lesssim_{C,N,\varepsilon} R^{\alpha(p,k)+\varepsilon} \|E_h f\|_{L^{p,R}_{dec}(w_{B_R})}$$

provided that $N \geq N(n,p)$.

Strictly speaking, this result was proved in [10] only for the hyperboloid $h(\xi) = \sum_{i=1}^n \alpha_i \xi_i^2$. However, the arguments from [46], which are illustrated in the context of elliptic surfaces in [9, Section 7], yield the more general translation invariant case in a straight-forward manner. See also the discussion below.

Originally, decoupling inequalities were studied for the cone by Wolff in [39, 61] to make progress on L^p-smoothing estimates (cf. [43, 44]) for the wave equation. These estimates were refined (cf. [25, 7]) until the breakthrough result of Bourgain-Demeter (cf. [9, 11]) where sharp decoupling inequalities for the paraboloid were proved. Subsequently, the result was generalized to hyperboloids (cf. [10]). These results also give estimates for exponential sums, in particular essentially sharp Strichartz estimates on irrational tori.

The theory was also extended to non-degenerate curves (cf. [12]). As already pointed out in Beltran-Hickman-Sogge [3], the decoupling theory seems to extend to the variable coefficient case sharply divergent from the $L^p - L^q$-estimates for oscillatory integral operators. In fact, it is well known that there are strictly less estimates admissible in the constant coefficient case due to Kakeya compression (cf. [5, 8, 60]).

Our first result is the following extension of Theorem 5.1:

Theorem 5.2. *Let $2 \leq p < \infty$, $n, M \in \mathbb{N}$, $0 \leq k \leq n/2$ and $\alpha(p,k)$ like in* (68)*. Suppose that (ϕ, a) satisfies $H1'$) and $H2'_{[k]}$). Then, we find the following estimate to hold:*

$$\|T^\lambda f\|_{L^p(\mathbb{R}^{n+1})} \lesssim_{\varepsilon,\phi,M,a} \lambda^{\alpha(p,k)+\varepsilon} \|T^\lambda f\|_{L^{p,\lambda}_{dec}(\mathbb{R}^{n+1})} + \lambda^{-M} \|f\|_2. \tag{69}$$

For variable-coefficient generalizations of the phase function $\phi_{cone}(t,x;\xi) = \langle x, \xi \rangle + t|\xi|$ associated to the adjoint Fourier restriction problem of the cone this was carried out in [3]. The proof of Theorem 5.2 adapts this general strategy from [3] to prove variable-coefficient decoupling from constant-coefficient decoupling: on small spatial scales the variable-coefficient oscillatory integral operator is well-approximated by a constant-coefficient operator. It is enough to make progress on this small scale because it extends to any scale by means of parabolic rescaling. Moreover, Iosevich–Liu–Xi [36] investigated decoupling inequalities for phase functions with a symmetric Carleson–Sjölin condition.

Already in the context of constant coefficients, approximating one surface by another on small scales and recovering arbitrary scales by rescaling was used to derive decoupling estimates for more general elliptic surfaces or the cone (cf. [9, Section 7, 8]), see also [46, 30].

It seems plausible that a similar approximation derives the variable-coefficient cone decoupling from the variable-coefficient paraboloid decoupling. Recently, in [32] was shown by the same approximation that broad-narrow considerations are also valid for the cone. We do not pursue this line of argument.

We recall different consequences of Theorem 5.2: The variable-coefficient ℓ^2-decoupling implies a stability theorem for exponential sums which is proved using the argument in [9] for the constant-coefficient case. Moreover, on small spatial scales the broad-narrow considerations from the constant-coefficient case extends to the variable- coefficient case. As used above, the decoupling theorem implies Strichartz and smoothing estimates without further arguments (e.g. dispersive estimates for the propagator).

5.2. **Variable-coefficient decoupling for hyperbolic phase functions.**

5.2.1. *Basic reductions.* Before we begin the proof of Theorem 5.2 in earnest, we carry out several reductions. Most importantly, we quantify the conditions $H1')$ and $H2'_{[k]})$. In dependence of ε, M and p from Theorem 5.2, we choose a small constant $0 < c_{par} \ll 1$ and a large integer $N = N_{\varepsilon,M,p}$ and define the following conditions which we will impose on the phase function for $A \geq 1$:

$H1_1)$ $|\partial^2_{x\xi}\phi(z;\xi) - I_n| \leq c_{par}$ for all $(z,\xi) \in Z \times \Xi$

$H2^1_{[k]})$ $|\partial_t \partial^2_\xi \phi - I_n| \leq c_{par}$ for all $(z,\xi) \in Z \times \Xi$

$D1^1_1)$ $\|\partial_{x_k}\partial^\beta_\xi \phi\|_{L^\infty(Z\times\Xi)} \leq c_{par}$ for $2 \leq |\beta| \leq N$

$D1^2_1)$ $\|\partial_t \partial^\beta_\xi \phi\|_{L^\infty(Z\times\Xi)} \leq c_{par}$ for $3 \leq |\beta| \leq N$

$D2_A)$ $\|\partial^2_z \partial^\beta_\xi \phi\|_{L^\infty} \leq \frac{c_{par}A}{100n}$ for $1 \leq |\beta| \leq 2N$

For technical reasons we also impose the following margin condition on the positional part a_1 of the amplitude a:

$M_A)$ $\mathrm{dist}(\mathrm{supp} a_1, \mathbb{R}^{n+1}\backslash Z) \geq 1/(4A)$

We already note the following consequence of $H2^1_{[k]})$:

$$|\partial_t \nabla_\xi \phi| \leq 2|\xi|. \tag{70}$$

In [31] it was shown that after introducing suitable partition of unities and performing changes of variables an elliptic phase function satisfying $H1')$ and $H2'_{[0]})$ reduces to the following normal form:

$$\phi(t,x;\xi) = \langle x,\xi\rangle + \frac{t|\xi|^2}{2} + \mathcal{E}(x,t;\xi) \tag{71}$$

with $\mathcal{E}$ being quadratic in (t,x) and ξ, to say

$$\partial^\alpha_{(x,t)}\partial^\beta_\xi \mathcal{E}(0;\xi) = 0 \quad \forall |\alpha| \leq 1, \ \beta \in \mathbb{N}^n_0.$$

The explicit representation (71) is not required for the following arguments. However, it is useful to keep it in mind stressing the nature of a small smooth perturbation to ϕ_{hyp}. We refer to data (ϕ, a) satisfying the above conditions for some $A \geq 1$ and $0 \leq k \leq n/2$ as type (A,k)-data. The notation and nomenclature is analogous to [3] to point out the similarity to the case of homogeneous variable-coefficient phase functions.

It turns out that these conditions are invariant under parabolic rescaling in a uniform sense, and this allows us to run the induction argument for normalized data. However, to reduce arbitrary hyperbolic phase functions, we have to do one rescaling which depends on the phase function. This gives rise to the dependence on ϕ in (69). If we confine ourselves in (69) to normalized data, there will be no explicit dependence on ϕ. We define the relevant constant as follows, where ε, M and p were fixed above and c_{par} and $N = N_{\varepsilon,M,p}$ in the definition of normalized data are chosen in dependence.

We denote by $\mathfrak{D}^\varepsilon_{A,k}(\lambda;R)$ the infimum over all $D \geq 0$ so that the estimate

$$\|T^\lambda f\|_{L^p(B_R)} \leq D R^{\alpha(p,k)+\varepsilon} \|T^\lambda f\|_{L^{p,R}_{dec}(w_{B_R})} + R^{2n}(\lambda/R)^{-N/8}\|f\|_{L^2}$$

holds true for all data (ϕ,a) of type (A,k), balls B_R of radius R contained in $B(0,\lambda)$ and $f \in L^2(B^n(0,1))$. For the weight function we take N as in $D2_A)$. The estimate

$$\mathfrak{D}^\varepsilon_{1,k}(\lambda;R) \leq C_\varepsilon \tag{72}$$

implies Theorem 5.2 since we can reduce to normal data. It turns out that it is enough to prove the following proposition:

Proposition 5.3. *Let $1 \leq R \leq \lambda^{1-\varepsilon/n}$. Then, we find the estimate* (72) *to hold true.*

In fact, we observe that for any $1 \leq \rho \leq R$ and $\rho^{-1/2}$-ball θ one may write

$$T^\lambda f_\theta = \sum_{\substack{\sigma \cap \tilde{\theta} \neq \emptyset, \\ \sigma : R^{-1/2}-ball}} T^\lambda f_\sigma,$$

where $\tilde{\theta}$ denotes the intersection of $\operatorname{supp}(f)$ and θ. We compute using Minkowski's and Cauchy-Schwarz inequality that for any weight w one has

$$\begin{aligned} \|T^\lambda f\|_{L^{p,\rho}_{dec}(w)} &= \Big(\sum_{\theta : \rho^{-1/2}-ball} \|T^\lambda f_\theta\|^2_{L^p(w)} \Big)^{1/2} = \Big(\sum_{\theta:\rho^{-1/2}-ball} \Big(\sum_{\substack{\sigma : R^{-1/2}-ball, \\ \sigma \cap \tilde{\theta} \neq \emptyset}} \|T^\lambda f_\theta\| \Big)^2 \Big)^{1/2} \\ &\leq \Big(\sum_{\theta:\rho^{-1/2}-ball} (R/\rho)^{n/2} \sum_{\substack{\sigma : R^{-1/2}-ball, \\ \sigma \cap \tilde{\theta} \neq \emptyset}} \|T^\lambda f_\sigma\|^2_{L^p} \Big)^{1/2} \lesssim (R/\rho)^{\frac{n}{4}} \|T^\lambda f\|_{L^{p,R}_{dec}(w)}. \end{aligned} \tag{73}$$

Since $\|T^\lambda f\|_{L^p(B_R)} \lesssim \|T^\lambda f\|_{L^{p,1}_{dec}(B_R)}$, from taking $\rho = 1$ in the above display it follows that

$$\mathfrak{D}^\varepsilon_{A,k}(\lambda; R) \lesssim R^{\frac{n}{4} - \alpha(p,k) - \varepsilon}, \tag{74}$$

which yields finiteness of $\mathfrak{D}^\varepsilon$. Moreover, we can reduce to

$$\mathfrak{D}^\varepsilon_{A,k}(\lambda; \lambda^{1-\frac{\varepsilon}{n}}) \leq C_\varepsilon. \tag{75}$$

Indeed, the support conditions of the amplitude a imply that the support of $T^\lambda f$ is always contained in $B(0,\lambda)$. We cover $B(0,\lambda)$ by an essentially disjoint family of $\lambda^{1-\frac{\varepsilon}{n}}$-balls

$$\|T^\lambda f\|^p_{L^p(B(0,\lambda))} \leq \sum_{B:\lambda^{1-\frac{\varepsilon}{n}}-balls} \|T^\lambda f\|^p_{L^p(B)},$$

and using Minkowski's inequality we find

$$\begin{aligned} \|T^\lambda f\|_{L^p(B)} &\lesssim \mathfrak{D}^\varepsilon_{A,k}(\lambda; \lambda^{1-\frac{\varepsilon}{n}}) \lambda^{\frac{\varepsilon}{4}} (\lambda^{1-\frac{\varepsilon}{n}})^{\alpha(p,k)+\varepsilon} \Big(\sum_{\theta:(\lambda^{1-\frac{\varepsilon}{n}})^{-1/2}-balls} \|T^\lambda f_\theta\|^2_{L^p(w_B)} \Big)^{1/2} \\ &\quad + (\lambda^{1-\varepsilon/n})^{2(n+1)} \lambda^{-\varepsilon N/8} \|f\|_2 \\ &\lesssim \mathfrak{D}^\varepsilon_{A,k}(\lambda; \lambda^{1-\frac{\varepsilon}{n}}) (\lambda^{1-\frac{\varepsilon}{n}})^{\alpha(p,k)+\varepsilon} \lambda^{\frac{\varepsilon}{4}} \Big(\sum_{\theta:\lambda^{-1/2}-balls} \|T^\lambda f_\theta\|^2_{L^p(w_{B(0,\lambda)})} \Big)^{1/2} \\ &\quad + \lambda^{2(n+1) - \frac{2\varepsilon(n+1)}{n}} \lambda^{-\varepsilon N/8} \|f\|_2. \end{aligned}$$

For N large enough in dependence of ε, n and M we find (69) to hold from (73) for normalized data.

5.2.2. *Rescaling of variable-coefficient phase functions.* We record the following trivial rescaling allowing us to reduce data of type A to data of type 1:

Lemma 5.4. *For any $A \geq 1$ we find the following estimate to hold:*

$$\mathfrak{D}^\varepsilon_{A,k}(\lambda; R) \lesssim_A \mathfrak{D}^\varepsilon_{1,k}(\lambda/A; R/A). \tag{76}$$

Proof. Let (ϕ, a) be a datum in A-normal form. We define $\tilde{\phi}(z;\xi) = A\phi(z/A;\xi)$ and amplitude $\tilde{a}(z;\xi) = a(z/A;\xi)$ and observe that $T^\lambda f = \tilde{T}^{\lambda/A} f$. Note the equivalent behaviour of ϕ and $\tilde{\phi}$ under one positional derivative. Hence, we find $(\tilde{\phi}, \tilde{a})$ to satisfy $H1_1$), $H2^1_{[k]}$), $D1^1_1$), $D1^2_1$), and the second derivative amounts to an additional factor of $1/A$. Hence, we find $D2_1$) to hold. The new margin of the new amplitude $\tilde{a}$ has been increased to size $1/4$ and we find M_1) to hold. This step might require the additional argument of decomposing the amplitude function through a partition of unity and translating each piece, if necessary, to adjust to the enlarged support $A\text{supp}(a)$. This involves a sum over $\mathcal{O}(A^{n+1})$ operators where each is associated to type 1-data.

Covering $B(0,R)$ with R/A-balls yields another factor of $\mathcal{O}(A^{n+1})$, but these pieces can be bounded by $\mathfrak{D}_{1,k}^{\varepsilon}(\lambda/A; R/A)$, and the proof is complete. Moreover, the form of the error term allows us to summarize the sum over $\mathcal{O}(A^{n+1})$ error terms again as error term. $\square$

Next, we show the following stability result for normalized phase functions under parabolic rescaling[1]. This allows us to properly run an induction argument.

Lemma 5.5. *[Parabolic rescaling for hyperbolic phase functions] Let $2 \leq p < \infty$, $1 \leq \rho \leq R \leq \lambda$, $0 \leq k \leq n/2$ and $\alpha(p,k)$ like in* (68)*. Suppose that (ϕ, a) satisfies $H1'$) and $H2'_{[k]}$) and let T^λ be the associated oscillatory integral operator. If g is supported in a ρ^{-1}-ball and ρ is sufficiently large, then there exists a constant $\overline{C}(\phi) \geq 1$ such that*

$$\|T^\lambda g\|_{L^p(w_{B_R})} \lesssim_{\varepsilon,N,\phi} \mathfrak{D}_{1,k}^{\varepsilon}(\lambda/\overline{C}\rho^2, R/\overline{C}\rho^2)(R/\rho^2)^{\alpha(p,k)+\varepsilon}\|T^\lambda g\|_{L^{p,R}_{dec}(w_{B_R})} + R^{2(n+1)}(\lambda/R)^{-N/8}\|g\|_2.$$

Proof of Lemma 5.5 for phase functions of type 1. Let $\xi_0 \in B^n(0,1)$ be the centre of ρ^{-1}-ball where g is supported. We perform the change of variables $\xi' = \rho(\xi - \xi_0)$ and we compute

$$T^\lambda g(z) = \int_{\mathbb{R}^n} e^{i\phi^\lambda(z;\xi)} a^\lambda(z;\xi) g(\xi) d\xi = \int_{\mathbb{R}^n} e^{i\phi^\lambda(z;\xi_0+\rho^{-1}\xi')} a^\lambda(z;\xi_0+\rho^{-1}\xi') \underbrace{\rho^{-n} g(\xi_0+\rho^{-1}\xi')}_{\tilde{g}(\xi')} d\xi'.$$

We expand ϕ to find

$$\phi(z;\xi_0+\xi'/\rho) = \phi(z;\xi_0) + [\nabla_\xi \phi(z;\xi_0)]\frac{\xi'}{\rho} + \rho^{-2}\int_0^1 (1-r)\langle \partial^2_{\xi\xi}\phi(z;\xi_0+r\xi'/\rho)\xi',\xi'\rangle dr.$$

Let $\Phi_{\xi_0}(t,x) = (t, \Phi(t,x;\xi_0))$; $\Phi^\lambda(t,x) = \lambda \Phi_{\xi_0}(t/\lambda, x/\lambda)$ and we introduce the dilations $D_\rho(t,x) = (\rho^2 t, \rho x)$ and $D'_{\rho^{-1}}(x) = \rho^{-1}x$. We find

$$e^{i\lambda\phi(\Phi_{\xi_0}(\rho^2 t/\lambda, \rho x/\lambda);\xi_0)} T^\lambda g \circ \Phi^\lambda_{\xi_0} \circ D_\rho = \tilde{T}^{\lambda/\rho^2}\tilde{g}, \tag{77}$$

where

$$\tilde{T}^{\lambda/\rho^2}\tilde{g}(t,x) = \int_{\mathbb{R}^n} e^{i\tilde{\phi}^{\lambda/\rho^2}(t,x;\xi)} \tilde{a}^{\lambda/\rho^2}(x;\xi)\tilde{g}(\xi) d\xi,$$

and the phase $\tilde{\phi}(t,x;\xi)$ is given by

$$\langle x, \xi\rangle + \int_0^1 (1-r)\langle \partial^2_{\xi\xi}\phi(\Phi_{\xi_0}(t, D'_{\rho^{-1}}x);\xi_0 + r\xi/\rho)\xi,\xi\rangle dr,$$

and the amplitude $\tilde{a}(y,t;\xi) = a(\Phi_{\xi_0}(t; D'_{\rho^{-1}}y);\xi_0+\xi/\rho)$.

We verify (77): From the definition

$$(\Phi^\lambda_{\xi_0} \circ D_\rho)(t,x) = \lambda \Phi_{\xi_0}(\rho^2 t/\lambda, \rho x/\lambda)$$

and

$$\begin{aligned}\phi^\lambda(\Phi^\lambda_{\xi_0}(D_\rho(t,x)),\xi_0+\xi/\rho) &= \lambda\phi(\Phi_{\xi_0}(\rho^2 t/\lambda, \rho x/\lambda);\xi_0+\xi/\rho) \\ \to \lambda\phi(\Phi_{\xi_0}(\rho^2 t/\lambda,\rho x/\lambda);\xi_0) &+ \lambda[\nabla_\xi\phi(\Phi_{\xi_0}(\rho^2 t/\lambda,\rho x/\lambda);\xi_0)]\frac{\xi}{\rho} \\ &+ \rho^{-2}\lambda\int_0^1 (1-r)\langle\partial^2_{\xi\xi}\phi(\Phi_{\xi_0}(\rho^2 t/\lambda,\rho x/\lambda);\xi_0+r\rho^{-1}\xi)\xi,\xi\rangle dr,\end{aligned}$$

[1] Here, the term parabolic refers to the rescaling of time by a quadratic factor compared to space and is not restricted to phase functions related to elliptic (parabolic) surfaces.

which proves (77). If ϕ is in normal form, then we can also write

$$\tilde{\phi}(t,x;\xi) = \langle x,\xi\rangle + \frac{t|\xi|^2}{2} + \int_0^1 (1-r)\langle \partial^2_{\xi\xi}\mathcal{E}(\Phi_{\xi_0}(t,D'_{\rho^{-1}}x),\xi_0 + r\xi/\rho)\xi,\xi\rangle, \tag{78}$$

and with $\tilde{g}$ being supported in $B^n(0,1)$ we can assume that $|\xi| \leq 1$.

A change of spatial variables gives

$$\|T^\lambda g\|_{L^p(B_R)} \lesssim_\phi \rho^{\frac{n+2}{p}} \|\tilde{T}^{\lambda/\rho^2}\tilde{g}\|_{L^p((\Phi^\lambda_{\xi_0}\circ D_\rho)^{-1}(B_R))},$$

where the implicit constant stems from the Jacobian of Φ_{ξ_0}, which is controlled by property $D1_1$). Note that the implicit constant can be chosen constant for data of type 1 provided that $c_{par} > 0$ is chosen small enough. We cover $(\Phi^\lambda_{\xi_0}\circ D_\rho)^{-1}(B_R)$ with essentially disjoint R/ρ^2-balls, $B_{R/\rho^2} \in \mathcal{B}_{R/\rho^2}$ and find

$$\|T^\lambda g\|_{L^p(B_R)} \lesssim_\phi \rho^{(n+2)/p}\big(\sum_{B_{R/\rho^2}\in\mathcal{B}_{R/\rho^2}} \|\tilde{T}^{\lambda/\rho^2}\tilde{g}\|^p_{L^p(B_{R/\rho^2})}\big)^{1/p}.$$

We argue below that

$$\begin{aligned}\|\tilde{T}^{\lambda/\rho^2}\tilde{g}\|_{L^p(B_{R/\rho^2})} &\lesssim_{\varepsilon,N} \mathfrak{D}^\varepsilon_{1,k}(\lambda/\overline{C}\rho^2, R/\overline{C}\rho^2)(R/\rho^2)^{\alpha(p,k)+\varepsilon}\|\tilde{T}^{\lambda/\rho^2}\tilde{g}\|_{L^{p,R/\rho^2}_{dec}(w_{B_{R/\rho^2}})} \\ &\quad + (R/\rho^2)^{2(n+1)}(\lambda/R)^{-N/8}\|g\|_{L^2(\mathbb{R}^n)}\end{aligned} \tag{79}$$

holds for each $B_{R/\rho^2}\in\mathcal{B}_{R/\rho^2}$ and some $\overline{C}\geq 1$.

If $(\tilde{\phi},\tilde{a})$ was a type-1 datum, this would be a consequence of the definitions. First, we show how to conclude the proof with (79): we can write

$$\cup_{B_{R/\rho^2}\in\mathcal{B}_{R/\rho^2}} B_{R/\rho^2} \subseteq (\Phi^\lambda_{\xi_0}\circ D_\rho)^{-1}(B_{C_\phi R}) = C_{R'},$$

where $B_{C_\phi R}$ is a ball concentric to B_R, but with enlarged radius $C_\phi R$ for some $C_\phi \geq 1$ because Φ_{ξ_0} is a diffeomorphism.
Hence, we find from summing the pth power on both sides over R/ρ^2 balls and inverting the change of variables

$$\begin{aligned}&\mathfrak{D}^\varepsilon_{1,k}(\lambda/\overline{C}\rho^2, R/\overline{C}\rho^2)(R/\rho^2)^{\alpha(p,k)+\varepsilon}\big(\sum_{B_{R/\rho^2}\in\mathcal{B}} \|\tilde{T}^{\lambda/\rho^2}\tilde{g}\|^p_{L^{p,R/\rho^2}_{dec}(w_{B_{R/\rho^2}})}\big)^{1/p} \\ &\leq \mathfrak{D}^\varepsilon_{1,k}(\lambda/\overline{C}\rho^2, R/\overline{C}\rho^2)(R/\rho^2)^{\alpha(p,k)+\varepsilon}\|\tilde{T}^{\lambda/\rho^2}\tilde{g}\|_{L^{p,R/\rho^2}_{dec}(w_{C_{R'}})}.\end{aligned}$$

Inverting the change of coordinates yields

$$\begin{aligned}\|T^\lambda g\|_{L^p(B_R)} &\lesssim_{\varepsilon,N,\phi} \mathfrak{D}^\varepsilon_{1,k}(\lambda/\overline{C}\rho^2, R/\overline{C}\rho^2)(R/\rho^2)^{\alpha(p,k)+\varepsilon} \\ &\quad \big(\sum_{\tilde{\theta}:(R/\rho^2)^{-1/2}-ball} \|T^\lambda g_\theta\|^p_{L^p(w_{B_R})}\big)^{1/p} + R^{2(n+1)}(\lambda/R)^{-N/8}\|g\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

It is straight-forward to check that the θ, which are the images of $\tilde{\theta}$ under the mapping $\xi\mapsto\rho(\xi-\xi_0)$, which inverts the change of variables in frequency space, form a cover of the supp$g$ with $R^{-1/2}$-balls. Note how the error term compensates the decomposition into R/ρ^2 balls. In fact, any R/ρ^2-ball contributes with $(R/\rho^2)^{2n}$ and there are roughly $\rho^{2(n+1)}$ R/ρ^2-balls.

It remains to prove (79) for each $B_{R/\rho^2}\in\mathcal{B}_{R/\rho^2}$. For this purpose record the following representations of $\tilde{\phi}_L = \tilde{\phi}(t,(L^{-1})^t x; L\xi)$:

$$\tilde{\phi}_L(t,x;\xi) = \langle x,\xi\rangle + \int_0^1 (1-r)\langle\partial^2_{\xi\xi}\phi(\Phi_{\xi_0}(t,D'_{\rho^{-1}}\circ L^{-1}x),\xi_0 + Lr\xi/\rho)L\xi, L\xi\rangle dr, \tag{80}$$

and from Taylor expansion we find (up to an irrelevant phase factor)

$$\tilde{\phi}_L(t,x;\xi) = \rho^2\phi(t,\Phi_{\xi_0}(t,D'_{\rho^{-1}}\circ L^{-1}x);\xi_0 + L\xi/\rho). \tag{81}$$

$\tilde{\phi}_L$ is an affinely changed version of $\tilde{\phi}$ for some invertible L, so that $\partial_t \partial^2_{\xi\xi} \tilde{\phi}_L(0,0;0) = I^k_n$. We perceive $L = diag(\sqrt{\mu_1}, \dots, \sqrt{\mu_n}) \cdot R$, where R is a rotation and $\mu_1, \dots, \mu_n$ are the eigenvalues of $\partial_t \partial^2_{\xi\xi} \tilde{\phi}$ which is already close to I^k_n quantified by property $H2^{[k]}_1$).

We verify $H1_1$) for $\tilde{\phi}$: Taking an x derivative of the integral term leads to an expression of the kind $\partial_x \partial^2_{\xi\xi} \phi \cdot \partial_x \Phi_{\xi_0} \cdot \rho^{-1}$.
$\partial_x \partial^2_{\xi\xi} \phi$ is controlled by property D^1_1) of ϕ. From the definition of Φ_{ξ_0} and the chain rule we find $\partial_x \Phi_{\xi_0} = (\partial_x \partial_\xi \phi)^{-1}$. Since $|\partial^2_{x\xi} \phi - I^k_n| \leq c_{par}$, we have $|\partial_x \Phi_{\xi_0}| \leq 2$ and we find the total expression to be of order c_{par}/ρ. Note that taking a frequency derivative does not magnify the size.
Likewise we verify $D1^1_1$) for $|\beta| = 2$. For higher derivatives in ξ we can argue with the representation (81) and observe that the bounds for large ρ become smaller and smaller since any derivative in ξ gives rise to a factor of ρ^{-1}. In this way one checks the validity of $D2_1$).

We check $H2^{[k]}_1$): For this purpose we write

$$\partial_t \partial^2_\xi \tilde{\phi}_L(t,x;\xi) - I^k_n = \partial_t \partial^2_\xi \tilde{\phi}_L(t,x;\xi) - \partial_t \partial^2_{\xi\xi} \tilde{\phi}_L(0,0;0)$$

and use the fundamental theorem of calculus. For an additional ξ derivative we find the contribution to be of size $\mathcal{O}(c_{par}\rho^{-1})$. For positional derivatives we use property $D2_1$) of ϕ to find this contribution to be also much smaller than c_{par}, and thus the claim follows.

The only cases of $D2_1$) which require additional reasoning to the above arguments are when there are two time derivatives and only one or two frequency derivatives. Else, the smallness is immediate from (81). In the case of two time derivatives and few frequency derivatives we have to consider combinations $\partial^2_{tt} \partial^2_{\xi\xi} \mathcal{E}$, $\partial_t \Phi_{\xi_0}$ and $\partial^2_{tt} \Phi_{\xi_0}$. $\partial^2_{tt} \partial^2_{\xi\xi} \phi$ and higher frequency derivatives are controlled by property $D2_1$) of ϕ and above we have seen that $\partial_t \Phi_{\xi_0}$ is controlled quantitatively through (70) of ϕ. The control over $\partial^2_{tt} \Phi_{\xi_0}$ follows from considering one further time derivative:

$$\begin{aligned} &\partial_{tt} \partial_\xi \phi^\lambda(\Phi^\lambda(t,x;\xi_0);\xi_0) + \partial_t \partial^2_{x\xi} \phi^\lambda(\Phi^\lambda(t,x;\xi_0)) \partial_t \Phi^\lambda \\ &+ \partial_t \partial^2_{x\xi} \phi^\lambda(\Phi^\lambda(t,x;\xi_0);\xi_0) \partial_t \Phi^\lambda(t,x;\xi_0) + \partial^2_{xx} \partial_\xi \phi^\lambda(\Phi^\lambda(t,x;\xi_0);\xi_0)(\partial_t \Phi^\lambda)^2 \\ &+ \partial^2_{x\xi} \phi^\lambda \partial^2_{tt} \Phi^\lambda = 0. \end{aligned} \tag{82}$$

Hence, we find $|\partial^2_{tt} \partial_\xi \tilde{\phi}_L|$, $|\partial^2_{tt} \partial^2_{\xi\xi} \tilde{\phi}_L| \leq C$ independent of ϕ with dependence only on the parameters in the definition of type 1 data. After invoking Lemma 5.4 with some constant independent of ϕ provided that ϕ is a datum of type 1, the proof is complete. ☐

Finally, we deal with the case of a general phase function. The proof is essentially a reprise of the proof of Lemma 5.5. However, the implicit constants are now allowed to depend on ϕ, and since we are not dealing with a normalized datum from the beginning, the constants may become arbitrarily large.

Proof. First, we use the trivial rescaling from Lemma 5.4 $\phi \to \phi^A = A\phi(z/A, \xi)$ to ensure that

$$\|\partial^2_z \partial^\beta_\xi \phi\|_{L^\infty} \leq \frac{c_{par}}{100nA} \quad \text{for } |\beta| = 1,2.$$

Later, we shall see how to choose $A = A(\phi)$. Next, we break the support of g into ρ^{-1}-balls, and again we will choose $\rho \geq 1$ later in dependence of ϕ.

We carry out the changes of coordinates from the proof of Lemma 5.5 and again arrive at the representations

$$\tilde{\phi}^A(t,x;\xi) = \langle x, \xi \rangle + \int_0^1 (1-r) \langle \partial^2_{\xi\xi} \phi^A(\Phi^A_{\xi_0}(t, D'_{\rho^{-1}} x); \xi_0 + r\xi/\rho)\xi, \xi \rangle dr, \tag{83}$$

$$\tilde{\phi}^A(t,x;\xi) = \rho^2 \phi^A(\Phi^A_{\xi_0}(t, D'_{\rho^{-1}} x); \xi_0 + \xi/\rho), \tag{84}$$

and we define $\tilde{\phi}^A_L$ analogous to the proof of Lemma 5.5. We check $H1_1$) from (83) which shows that

$$\partial^2_{x\xi} \tilde{\phi}^A_L = I_n + \mathcal{O}_\varphi(\rho^{-1}). \tag{85}$$

We also find $\|\partial_{x_k}\partial^2_{\xi\xi}\phi\|_{L^\infty} = \mathcal{O}_\phi(\rho^{-1})$ also follows from (84). Moreover, for higher order derivatives in ξ we get additional factors of ρ^{-1} which proves property $D1_1^1$).
Likewise, we verify $D1_1^2$) for sufficiently large ρ.

For the proof of $H2_1^{[k]}$) we write again

$$\partial_t \partial^2_{\xi\xi}\tilde{\phi}^A_L - I^k_n = \partial_t \partial^2_{\xi\xi}\tilde{\phi}^A_L(t,x;\xi) - \partial_t \partial^2_{\xi\xi}\tilde{\phi}^A_L(0,0;0) \tag{86}$$

and estimate the difference deploying the fundamental theorem of calculus. The above arguments already yield $\partial_t \partial^3_{\xi\xi\xi}\tilde{\phi}^A_L = \mathcal{O}_\phi(\rho^{-1})$, for positional derivatives we choose $A = A(\phi)$ large enough, so that $\partial_t\partial_z\partial^2_{\xi\xi}\tilde{\phi}^A_L \leq \frac{c_{par}}{100n}$ and we can also control this contribution. Note that here we also need $|\partial^2_{tt}\Phi^\lambda| = \mathcal{O}_\phi(A^{-1})$ which follows from (82).

We check $D2_1$) like in the proof of Lemma 5.5 after choosing $A = A(\phi)$ sufficiently large. □

5.2.3. *Approximation by extension operators.* Let (ϕ, a) be a datum of type 1 giving rise to the oscillatory integral operator T^λ and recall that we assume the amplitude function to be of product type: $a(z;\xi) = a_1(z)a_2(\xi)$. Further, recall that

$$\xi \mapsto (\nabla_{x,t}\phi^\lambda)(\overline{z};\Psi^\lambda(\overline{z};\xi))$$

is a graph parametrisation of a hypersurface $\Sigma_{\overline{z}}$. Thus, we have

$$\langle z, (\nabla_{x,t}\phi^\lambda(\overline{z};\Psi^\lambda(\overline{z};\xi))\rangle = \langle x,\xi\rangle + t h_{\overline{z}}(\xi) \tag{87}$$

for all $z = (x,t) \in \mathbb{R}^{n+1}$ with $z/\lambda \in Z$ where $h_{\overline{z}}(\xi) = (\partial_t\phi^\lambda(\overline{z};\Psi^\lambda(\overline{z};\xi)))$.

Moreover, from the definition of Ψ^λ we have

$$\begin{aligned} &\xi = \partial_x\phi^\lambda(\overline{z};\Psi^\lambda(\overline{z};\xi)),\\ &I_n = \partial^2_{x\xi}\phi^\lambda(\overline{z};\Psi^\lambda(\overline{z};\xi))(\partial_\xi\Psi^\lambda(\overline{z};\xi)),\\ &0 = \partial^3_{x\xi\xi}\phi(\overline{z};\Psi^\lambda(\xi))(\partial_\xi\Psi^\lambda(\overline{z};\xi))^2 + \partial^2_{x\xi}\phi^\lambda(\overline{z};\Psi^\lambda(\xi))\partial^2_{\xi\xi}\Psi^\lambda(\overline{z};\xi). \end{aligned} \tag{88}$$

And consequently, we find for 1-normalized data

$$\begin{aligned} \partial_\xi\Psi^\lambda(\overline{z};\xi) &\sim I_n,\\ |\partial^2_{\xi\xi}\Psi^\lambda(\overline{z};\xi)| &\ll 1. \end{aligned} \tag{89}$$

Let $E_{\overline{z}}$ denote the extension operator associated to $\Sigma_{\overline{z}}$ given by

$$E_{\overline{z}}g(x,t) = \int_{\mathbb{R}^n} e^{i(\langle x,\xi\rangle + t h_{\overline{z}}(\xi))} a_{\overline{z}}(\xi) g(\xi) d\xi,$$

where $a_{\overline{z}}(\xi) = a_2 \circ \Psi^\lambda(\overline{z};\xi)|\det \partial_\xi\Psi^\lambda(\overline{z};\xi)|$.
We shall see that on small spatial scales T^λ is effectively approximated by $E_{\overline{z}}$ and vice versa. We record the following consequence of dealing with 1-normalized data:

Lemma 5.6. *Let (ϕ, a) be a type 1 datum. Each eigenvalue μ of $\partial^2_{\xi\xi}h_{\overline{z}}$ satisfies $|\mu| \sim 1$ on supp$(a_{\overline{z}})$. For elliptic phase functions of type 1 we have $\mu \sim 1$ on supp$(a_{\overline{z}})$.*

Proof. From the definition of $h_{\overline{z}}$ we find

$$\begin{aligned} \partial_\xi h_{\overline{z}}(\xi) &= (\partial_t\partial_\xi\phi^\lambda(\overline{z};\Psi^\lambda(\overline{z};\xi))\partial_\xi\Psi^\lambda(\overline{z};\xi),\\ \partial^2_{\xi\xi}h_{\overline{z}}(\xi) &= (\partial_t\partial^2_{\xi\xi}\phi^\lambda(\overline{z};\Psi^\lambda(\overline{z};\xi))(\partial_\xi\Psi^\lambda(\overline{z};\xi))^2 + \partial_t\partial_\xi\phi^\lambda(\overline{z};\Psi^\lambda(\overline{z};\xi))\partial^2_{\xi\xi}\Psi^\lambda(\overline{z};\xi), \end{aligned}$$

and the claim follows from (89). □

This becomes needful when it comes to applying the constant-coefficient ℓ^2-decoupling theorem, which we repeated in Theorem 5.1, because Lemma 5.6 ensures uniformity of the constant from the decoupling inequality.

In the following we analyze $T^\lambda f(z)$ for $z \in B^{n+1}(\overline{z};K) \subseteq B(0,3\lambda/4)$ and $1 \leq K \leq \lambda^{1/2}$. The containment property can be assumed due to the margin condition. We see that the desired approximation identity holds on this spatial scale: we perform a change of variables $\xi = \Psi^\lambda(\overline{z};\tilde{\xi})$ and expand ϕ^λ around $\overline{z}$ to find

$$T^\lambda f(z) = \int_{\mathbb{R}^n} e^{i(\langle z-\overline{z}, \nabla_{x,t}\phi^\lambda(\overline{z};\Psi^\lambda(\overline{z};\tilde{\xi}))\rangle + \mathcal{E}^\lambda_{\overline{z}}(z-\overline{z};\tilde{\xi}))} a_1^\lambda(z) a_{\overline{z}}(\tilde{\xi}) f_{\overline{z}}(\tilde{\xi}) d\tilde{\xi},$$

where $f_{\overline{z}} = e^{i\phi^\lambda(\overline{z};\Psi^\lambda(\overline{z};\cdot))} f \circ \Psi^\lambda(\overline{z};\cdot)$ and

$$\mathcal{E}^\lambda_{\overline{z}}(v;\xi) = \frac{1}{\lambda}\int_0^1 (1-r)\langle(\partial^2_{zz}\phi)((\overline{z}+rv)/\lambda;\Psi^\lambda(\overline{z};\xi))v;v\rangle dr.$$

Lemma 5.7. *Let T^λ be an operator associated to a 1-normalized datum (ϕ,a), $0<\delta\leq 1/2$, $1\leq K\leq \lambda^{1/2-\delta}$ and $\overline{z}/\lambda \in Z$ so that $B(\overline{z};K)\subseteq B(0,3\lambda/4)$.*
Then, we find the estimates

$$\|T^\lambda f\|_{L^p(w_{B(\overline{z};K)})} \lesssim_N \|E_{\overline{z}} f_{\overline{z}}\|_{L^p(w_{B(0;K)})} + \lambda^{-\delta N/2}\|f\|_2, \tag{90}$$

$$\|E_{\overline{z}} f_{\overline{z}}\|_{L^p(w_{B(0;K)})} \lesssim_N \|T^\lambda f\|_{L^p(w_{B(\overline{z};K)})} + \lambda^{-\delta N/2}\|f\|_2. \tag{91}$$

to hold provided that N is chosen sufficiently large depending on n,δ and p. Here, the constant N is the same for the weight functions, the conditions on the derivatives $D1^1_1)$, $D1^2_1)$, $D2_1)$ and in the exponent of λ in the above estimates. Moreover, in case of sharp cutoff (90) *becomes*

$$\|T^\lambda f\|_{L^p(B(\overline{z};K))} \lesssim_N \|E_{\overline{z}} f_{\overline{z}}\|_{L^p(w_{B(0;K)})} + \lambda^{-\delta N/2}\|f\|_2. \tag{92}$$

Proof. We can replace f by $f\varphi$, where in view of the definition $a_{\overline{z}}$ and the fact that we are dealing with a datum of type 1, we can assume that φ is supported in $[0,2\pi]^n$. After performing a Fourier series decomposition of $e^{i\mathcal{E}^\lambda_{\overline{z}}(v,\xi)}\varphi(\xi)$, one may write

$$e^{i\mathcal{E}^\lambda_{\overline{z}}(v,\xi)}\varphi(\xi) = \sum_{k\in\mathbb{Z}^n} a_k(v) e^{i\langle k,\xi\rangle}, \tag{93}$$

where $a_k(v) = \int_{[0,2\pi]^n} e^{-i\langle k,\xi\rangle} e^{i\mathcal{E}^\lambda_{\overline{z}}(v;\xi)}\varphi(\xi)d\xi$.

Since $K\leq\lambda^{1/2}$ we find the favourable bound

$$\sup_{(v;\xi)\in B(0,K)\times \mathrm{supp} a_{\overline{z}}} |\partial^\beta_\xi \mathcal{E}^\lambda_{\overline{z}}(v;\xi)| \lesssim_N \frac{|v|^2}{\lambda}$$

as long as $\beta\in\mathbb{N}_0^n$ with $1\leq|\beta|\leq 2N$ by virtue of property $D2_1)$ and the computation in (88) showing that $|\partial^\beta_\xi\Psi^\lambda(\overline{z};\xi)|\lesssim 1$ as long as $1\leq|\beta|\leq 2N$. Consequently, integration by parts yields

$$|a_k(v)| \lesssim_N (1+|k|)^{-N},$$

whenever $|v|\leq 2\lambda^{1/2}$. We derive the following pointwise identity from (93):

$$|T^\lambda f(\overline{z}+v)| \leq \sum_{k\in\mathbb{Z}^n}|a_k(v)||E_{\overline{z}}(f_{\overline{z}}e^{i\langle k,\cdot\rangle})(v)| \lesssim \sum_{k\in\mathbb{Z}^n}(1+|k|)^{-N}|E_{\overline{z}}(f_{\overline{z}}e^{i\langle k,\cdot\rangle}(v)|.$$

We decompose further:

$$\|T^\lambda f\|_{L^p(w_{B(\overline{z};K)})} \leq \|(T^\lambda f)1_{B(\overline{z};2\lambda^{1/2})}\|_{L^p(w_{B(\overline{z};K)})} + \|(T^\lambda f)1_{\mathbb{R}^{n+1}\backslash B(\overline{z};2\lambda^{1/2})}\|_{L^p(w_{B(\overline{z};K)})}.$$

The second term leads to the error term, that is

$$\|(T^\lambda f)\chi_{\mathbb{R}^n\backslash B(\overline{z};2\lambda^{1/2})}\|_{L^p(w_{B(\overline{z};K)})} \lesssim \lambda^{\frac{n}{2p}-\delta(N-(n+2))}\|f\|_{L^2(\mathbb{R}^n)}. \tag{94}$$

In fact, we have $\|T^\lambda f\|_{L^\infty}\lesssim\|f\|_2$, and consequently,

$$\left(\int_{\mathbb{R}^{n+1}}(1+K^{-1}|x|)^{-(n+2)}|T^\lambda f|^p\right)^{1/p} \lesssim K^{n/p}\|f\|_{L^2} \lesssim \lambda^{\frac{1}{2p}}\|f\|_2,$$

and the factor $\lambda^{-\delta(N-(n+2))}$ stems from the additional decay of the weight $(1+K^{-1}|x|)^{-N}$ we are actually considering. This gives (94), and since the operator $E_{\overline{z}}$ is translation-invariant,

$$E_{\overline{z}}[e^{i\langle k,\cdot\rangle}g](t,x) = E_{\overline{z}}g(t,x+k) \quad \forall (t,x) \in \mathbb{R}^{n+1} \text{ and } k \in \mathbb{R}^n. \tag{95}$$

Minkowski's inequality yields

$$\|T^\lambda f 1_{B(\overline{z};2\lambda^{1/2})}\|_{L^p(w_{B(\overline{z};K)})} \lesssim_N \sum_{k\in\mathbb{Z}^n} (1+|k|)^{-N} \|E_{\overline{z}} f_{\overline{z}}\|_{L^p(w_{B((k,0),K)})}. \tag{96}$$

Next, observe that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n} (1+|k|)^{-N} \|E_{\overline{z}} f_{\overline{z}}\|_{L^p(w_{B((k,0),K)})} \\
&= \sum_{k\in\mathbb{Z}^n} (1+|k|)^{-\frac{N}{p}} (1+|k|)^{N(\frac{1}{p}-1)} \|E_{\overline{z}} f_{\overline{z}}\|_{L^p(w_{B((k,0),K)}} \\
&\leq \Big[\sum_{k\in\mathbb{Z}^n} (1+|k|)^{-N} \|E_{\overline{z}} f_{\overline{z}}\|^p_{L^p(w_{B((k,0),K)}} \Big]^{1/p} \Big(\sum_{k\in\mathbb{Z}^n} (1+|k|)^{N(\frac{1}{p}-1)p'} \Big)^{1/p'} \\
&= C(n,p,N) \Big(\int |E_{\overline{z}} f_{\overline{z}}|^p \sum_{k\in\mathbb{Z}^n} (1+|k|)^{-N} w_{B((k,0),K)} \Big)^{1/p} \\
&\lesssim_{n,p} \|E_{\overline{z}} f_{\overline{z}}\|_{L^p(w_{B(0,K)})}.
\end{aligned} \tag{97}$$

For the ultimate estimate one observes

$$\sum_{k\in\mathbb{Z}^n} (1+|k|)^{-N} w_{B((k,0),K)} \lesssim w_{B(0,K)}.$$

In order to prove (91), we write

$$E_{\overline{z}} f_{\overline{z}}(v) = \int_{\mathbb{R}^n} e^{i\phi^\lambda(\overline{z}+v;\Psi^\lambda(\overline{z};\xi)} e^{-\mathcal{E}^\lambda_{\overline{z}}(v;\xi)} a_{\overline{z}}(\xi) f \circ \Psi^\lambda(\overline{z};\xi) d\xi.$$

Again, we insert a smooth cutoff $\varphi(\xi)$ supported in $[0,2\pi]^n$ so that

$$e^{-i\mathcal{E}^\lambda_{\overline{z}}(v;\xi)} \varphi(\xi) = \sum_{k\in\mathbb{Z}^n} e^{i\langle k,\xi\rangle} b_k(v),$$

where $b_k(v) = \int_{[0,2\pi]^n} e^{-i\langle k,\xi\rangle} e^{-i\mathcal{E}^\lambda_{\overline{z}}(v;\xi)} \varphi(\xi) d\xi$. Once more, integration by parts yields the pointwise bound

$$|b_k(v)| \lesssim_N (1+|k|)^{-2N},$$

and inverting the change of variables gives

$$|E_{\overline{z}} f_{\overline{z}}(v)| \lesssim_N \sum_{k\in\mathbb{Z}^n} (1+|k|)^{-2N} |T^\lambda \underbrace{[e^{i\langle k,\partial_z \phi^\lambda(\overline{z},\cdot)\rangle} f]}_{\tilde{f}_k}(\overline{z},v)|.$$

From a similar argument to the one from the proof of (91), we have

$$\|E_{\overline{z}} f_{\overline{z}}\|_{L^p(w_{B(0,K)})} \lesssim_N \sum_{k\in\mathbb{Z}^n} (1+|k|)^{-2N} \|(T^\lambda \tilde{f}_k) \chi_{B(\overline{z},2\lambda^{1/2})}\|_{L^p(w_{B(\overline{z},K)})} + \lambda^{-\delta N/2} \|f\|_2. \tag{98}$$

The $k=0$ term is alright because it yields the desired quantity. For the higher order terms we use the estimate (90) and (97) to conclude

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n, k\neq 0} (1+|k|)^{-2N} \|(T^\lambda \tilde{f}_k)\chi_{B(\overline{z},2\lambda^{1/2})}\|_{L^p(w_{B(\overline{z};K)})} \\
&\lesssim_N 2^{-N} \sum_{k\in\mathbb{Z}^N, k\neq 0} (1+|k|)^{-N} \|E_{\overline{z}} f_{\overline{z}}\|_{L^p(w_{B((k,0),K)})} \\
&\lesssim_N 2^{-N} \|E_{\overline{z}} f_{\overline{z}}\|_{L^p(w_{B(0,K)})}.
\end{aligned}$$

Choosing N large enough depending on n and p this quantity can be absorbed into the lefthandside of (98) which yields the claim. □

5.2.4. *Conclusion of the proof.*

Proof of Proposition 5.3. To show Proposition 5.3 for fixed parameters n, ε and $N = N(n,\varepsilon)$, it is enough to prove that

$$\mathfrak{D}^{\varepsilon}_{1,k}(\lambda; R) \lesssim_\varepsilon 1 \text{ for all } 1 \leq R \leq \lambda^{1-\varepsilon/n}. \tag{99}$$

We perform an induction on the radius, and with the base case (small R) readily settled, we contend the following induction hypothesis:

There is a constant $\overline{C}_\varepsilon \geq 1$ such that $\mathfrak{D}^{\varepsilon}_{1,k}(\lambda'; R') \leq \overline{C}_\varepsilon$ holds for all $1 \leq R' \leq R/2$ and all λ' satisfying $R' \leq (\lambda')^{1-\varepsilon/n}$.

We use the approximation lemma on a small spatial scale and lift the resulting estimates to the correct spatial scales through parabolic rescaling: Let $\mathcal{B}_K$ denote a family of finitely-overlapping K-balls covering B_R for some $2 \leq K \leq \lambda^{1/4}$. After breaking B_R into $B(\overline{z};K)$-balls the estimate from Lemma 5.7 implies

$$\|T^\lambda f\|_{L^p(B_R)} \lesssim \big(\sum_{B(\overline{z};K)\in\mathcal{B}_K} \|T^\lambda f\|^p_{L^p(B(\overline{z};K))} \big)^{1/p} \lesssim \big(\sum_{B(\overline{z};K)\in\mathcal{B}_K} \|E_{\overline{z}} f_{\overline{z}}\|^p_{L^p(w_{B(0;K)})} \big)^{1/p}. \tag{100}$$

We apply the constant-coefficient decoupling theorem (Theorem 5.1) to each small scale and find after reverting the change of coordinates (again using that we are dealing with 1-normalized data):

$$\begin{aligned}
\|E_{\overline{z}} f_{\overline{z}}\|_{L^p(w_{B(0,K)})} &\lesssim_\varepsilon K^{\varepsilon/2+\alpha(p,k)} \|E_{\overline{z}} f_{\overline{z}}\|_{L^{p,K}_{dec}(w_{B(0,K)})} \\
&\lesssim K^{\alpha(p,k)+\varepsilon/2} \big(\sum_{\sigma: K^{-1/2}-ball} \|T^\lambda f_\sigma\|^2_{L^p(w_{B(\overline{z};K)})} \big)^{1/2} + \lambda^{-N/8} K^{2n} \|f\|_2.
\end{aligned} \tag{101}$$

Moreover, this estimate holds uniformly in $\overline{z}$ by virtue of the uniform estimates on the Hessian of $h_{\overline{z}}$ derived in Lemma 5.6. We plug (101) into (100) to find after using Minkowski's inequality:

$$\|T^\lambda f\|_{L^p(B_R)} \lesssim K^{\alpha(p,k)+\varepsilon/2} \left(\sum_{\sigma: K^{-1/2}-ball} \|T^\lambda f_\sigma\|^2_{L^p(w_{B_R})} \right)^{1/2} + \lambda^{-N/8} K^{2n} R^n \|f\|_{L^2}. \tag{102}$$

Next, apply Lemma 5.5 to each $T^\lambda f_\sigma$ which gives the estimate

$$\begin{aligned}
\|T^\lambda f_\sigma\|_{L^p(w_{B_R})} &\leq \mathfrak{D}^{\varepsilon}_{1,k}(\lambda/(\overline{C}K^2), R/(\overline{C}K^2)) (R/K^2)^{\alpha(p,k)+\varepsilon} \|T^\lambda f_\sigma\|_{L^{p,R}_{dec}(w_{B_R})} \\
&\quad + R^{2(n+1)} (\lambda/R)^{-N/8} \|f_\sigma\|_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{103}$$

We note that $\mathfrak{D}^{\varepsilon}_{1,k}(\lambda/(\overline{C}K^2), R/(\overline{C}K^2)) \lesssim_\varepsilon 1$ according to our induction hypothesis. Plugging (103) into (102) gives after applying orthogonality

$$\begin{aligned}\|T^\lambda f\|_{L^p(B_R)} &\leq C_\varepsilon \overline{C}_\varepsilon K^{\varepsilon/2} (R/K^2)^{\alpha(p,k)+\varepsilon} \Big(\sum_{\sigma: K^{-1/2}-ball} \|T^\lambda f_\sigma\|^2_{L^{p,R}_{dec}(w_{B_R})} \Big)^{1/2} \\ &\quad + R^{2(n+1)} (\lambda/R)^{-N/8} \|f\|_2 \\ &\leq C_\varepsilon \overline{C}_\varepsilon K^{-\varepsilon/2} R^{\alpha(p,k)+\varepsilon} \|T^\lambda f\|_{L^{p,R}_{dec}(w_{B_R})} + R^{2(n+1)} (\lambda/R)^{-N/8} \|f\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

and we see that induction closes. □

Proof of Theorem 5.2. To finish the proof of Theorem 5.2, we break the support of $f \in L^2(B^n(0,1))$ into ρ^{-1}-balls, $\rho = \rho(\phi)$, so that after parabolic rescaling we are dealing with a normalized phase function $\tilde{\phi}$. We can apply Proposition 5.3 to $\tilde{\phi}$, and the proof is completed using Lemma 5.5. □

Acknowledgements

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173. I would like to thank Peer Kunstmann and Friedrich Klaus for helpful discussions on modulation spaces, putting the results into context, and for a remark on proving the global result.

References

[1] Wang Baoxiang, Zhao Lifeng, and Guo Boling. Isometric decomposition operators, function spaces $E^\lambda_{p,q}$ and applications to nonlinear evolution equations. *J. Funct. Anal.*, 233(1):1–39, 2006.

[2] Ioan Bejenaru and Terence Tao. Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation. *J. Funct. Anal.*, 233(1):228–259, 2006.

[3] David Beltran, Jonathan Hickman, and Christopher D. Sogge. Variable coefficient Wolff-type inequalities and sharp local smoothing estimates for wave equations on manifolds. *Anal. PDE*, 13(2):403–433, 2020.

[4] Árpád Bényi, Karlheinz Gröchenig, Kasso A. Okoudjou, and Luke G. Rogers. Unimodular Fourier multipliers for modulation spaces. *J. Funct. Anal.*, 246(2):366–384, 2007.

[5] J. Bourgain. L^p-estimates for oscillatory integrals in several variables. *Geom. Funct. Anal.*, 1(4):321–374, 1991.

[6] J. Bourgain. Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity. *Internat. Math. Res. Notices*, (5):253–283, 1998.

[7] J. Bourgain. Moment inequalities for trigonometric polynomials with spectrum in curved hypersurfaces. *Israel J. Math.*, 193(1):441–458, 2013.

[8] Jean Bourgain. Some new estimates on oscillatory integrals. In *Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991)*, volume 42 of *Princeton Math. Ser.*, pages 83–112. Princeton Univ. Press, Princeton, NJ, 1995.

[9] Jean Bourgain and Ciprian Demeter. The proof of the l^2 decoupling conjecture. *Ann. of Math. (2)*, 182(1):351–389, 2015.

[10] Jean Bourgain and Ciprian Demeter. Decouplings for curves and hypersurfaces with nonzero Gaussian curvature. *J. Anal. Math.*, 133:279–311, 2017.

[11] Jean Bourgain and Ciprian Demeter. A study guide for the l^2 decoupling theorem. *Chin. Ann. Math. Ser. B*, 38(1):173–200, 2017.

[12] Jean Bourgain, Ciprian Demeter, and Larry Guth. Proof of the main conjecture in Vinogradov's mean value theorem for degrees higher than three. *Ann. of Math. (2)*, 184(2):633–682, 2016.

[13] Leonid Chaichenets. *Modulation spaces and nonlinear Schrödinger equationscs.* PhD thesis, Karlsruhe Institute of Technology (KIT), sep 2018.

[14] Leonid Chaichenets, Dirk Hundertmark, Peer Kunstmann, and Nikolaos Pattakos. On the existence of global solutions of the one-dimensional cubic NLS for initial data in the modulation space $M_{p,q}(\mathbb{R})$. *J. Differential Equations*, 263(8):4429–4441, 2017.

[15] Leonid Chaichenets, Dirk Hundertmark, Peer Kunstmann, and Nikolaos Pattakos. Nonlinear Schrödinger equation, differentiation by parts and modulation spaces. *J. Evol. Equ.*, 19(3):803–843, 2019.

[16] Leonid Chaichenets, Dirk Hundertmark, Peer Christian Kunstmann, and Nikolaos Pattakos. Local well-posedness for the nonlinear schrödinger equation in the intersection of modulation spaces. In Willy Dörfler, Marlis Hochbruck, Dirk Hundertmark, Wolfgang Reichel, Andreas Rieder, Roland Schnaubelt, and Birgit Schörkhuber, editors, *Mathematics of Wave Phenomena*, pages 89–107, Cham, 2020. Springer International Publishing.

[17] Michael Christ, James Colliander, and Terrence Tao. Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations. *Amer. J. Math.*, 125(6):1235–1293, 2003.
[18] Elena Cordero and Fabio Nicola. Sharpness of some properties of Wiener amalgam and modulation spaces. *Bull. Aust. Math. Soc.*, 80(1):105–116, 2009.
[19] Simão Correia. Local Cauchy theory for the nonlinear Schrödinger equation in spaces of infinite mass. *Rev. Mat. Complut.*, 31(2):449–465, 2018.
[20] Benjamin Dodson, Avraham Soffer, and Thomas Spencer. Global well-posedness for the cubic nonlinear Schr{ö}dinger equation with initial lying in L^p-based Sobolev spaces. *arXiv e-prints*, page arXiv:2012.14355, December 2020.
[21] Benjamin Dodson, Avraham Soffer, and Thomas Spencer. The nonlinear Schrödinger equation on Z and R with bounded initial data: examples and conjectures. *J. Stat. Phys.*, 180(1-6):910–934, 2020.
[22] C. Fefferman and E. M. Stein. H^p spaces of several variables. *Acta Math.*, 129(3-4):137–193, 1972.
[23] H. G. Feichtinger. Banach convolution algebras of Wiener type. In *Functions, series, operators, Vol. I, II (Budapest, 1980)*, volume 35 of *Colloq. Math. Soc. János Bolyai*, pages 509–524. North-Holland, Amsterdam, 1983.
[24] Hans G. Feichtinger. Modulation spaces: looking back and ahead. *Sampl. Theory Signal Image Process.*, 5(2):109–140, 2006.
[25] Gustavo Garrigós and Andreas Seeger. On plate decompositions of cone multipliers. *Proc. Edinb. Math. Soc. (2)*, 52(3):631–651, 2009.
[26] J. Ginibre and G. Velo. On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case. *J. Functional Analysis*, 32(1):1–32, 1979.
[27] Karlheinz Gröchenig. *Foundations of time-frequency analysis.* Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2001.
[28] Axel Grünrock. Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS. *Int. Math. Res. Not.*, (41):2525–2558, 2005.
[29] Shaoming Guo. On the 1D cubic nonlinear Schrödinger equation in an almost critical space. *J. Fourier Anal. Appl.*, 23(1):91–124, 2017.
[30] Shaoming Guo and Changkeun Oh. Remarks on Wolff's inequality for hypersurfaces. *Math. Proc. Cambridge Philos. Soc.*, 168(2):249–259, 2020.
[31] Larry Guth, Jonathan Hickman, and Marina Iliopoulou. Sharp estimates for oscillatory integral operators via polynomial partitioning. *Acta Math.*, 223(2):251–376, 2019.
[32] Terence L. J. Harris. Improved decay of conical averages of the Fourier transform. *Proc. Amer. Math. Soc.*, 147(11):4781–4796, 2019.
[33] Benjamin Harrop-Griffiths, Rowan Killip, and Monica Visan. Sharp well-posedness for the cubic NLS and mKdV in $H^s(\mathbb{R})$. *arXiv e-prints*, page arXiv:2003.05011, March 2020.
[34] Lars Hörmander. *The analysis of linear partial differential operators. I.* Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].
[35] Ryosuke Hyakuna. Local and global well-posedness, and $L^{p'}$-decay estimates for 1D nonlinear Schrödinger equations with Cauchy data in L^p. *J. Funct. Anal.*, 278(12):108511, 38, 2020.
[36] Alex Iosevich, Bochen Liu, and Yakun Xi. Microlocal decoupling inequalities and the distance problem on Riemannian manifolds. *arXiv e-prints*, page arXiv:1909.05171, September 2019.
[37] Markus Keel and Terence Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.
[38] Carlos E. Kenig, Gustavo Ponce, and Luis Vega. On the ill-posedness of some canonical dispersive equations. *Duke Math. J.*, 106(3):617–633, 2001.
[39] Izabella Ł aba and Thomas Wolff. A local smoothing estimate in higher dimensions. volume 88, pages 149–171. 2002. Dedicated to the memory of Tom Wolff.
[40] Sanghyuk Lee, Keith M. Rogers, and Andreas Seeger. On space-time estimates for the Schrödinger operator. *J. Math. Pures Appl. (9)*, 99(1):62–85, 2013.
[41] Rainer Mandel. Dispersive estimates, blow-up and failure of Strichartz estimates for the Schrödinger equation with slowly decaying initial data. *Pure Appl. Anal.*, 2(2):519–532, 2020.
[42] Akihiko Miyachi. On some singular Fourier multipliers. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 28(2):267–315, 1981.
[43] Gerd Mockenhaupt, Andreas Seeger, and Christopher D. Sogge. Wave front sets, local smoothing and Bourgain's circular maximal theorem. *Ann. of Math. (2)*, 136(1):207–218, 1992.
[44] Gerd Mockenhaupt, Andreas Seeger, and Christopher D. Sogge. Local smoothing of Fourier integral operators and Carleson-Sjölin estimates. *J. Amer. Math. Soc.*, 6(1):65–130, 1993.
[45] Tadahiro Oh and Yuzhao Wang. Global well-posedness of the one-dimensional cubic nonlinear Schrödinger equation in almost critical spaces. *J. Differential Equations*, 269(1):612–640, 2020.
[46] Malabika Pramanik and Andreas Seeger. L^p regularity of averages over curves and bounds for associated maximal operators. *Amer. J. Math.*, 129(1):61–103, 2007.
[47] Keith M. Rogers. A local smoothing estimate for the Schrödinger equation. *Adv. Math.*, 219(6):2105–2122, 2008.

[48] Keith M. Rogers and Andreas Seeger. Endpoint maximal and smoothing estimates for Schrödinger equations. *J. Reine Angew. Math.*, 640:47–66, 2010.
[49] José L. Rubio de Francia. A Littlewood-Paley inequality for arbitrary intervals. *Rev. Mat. Iberoamericana*, 1(2):1–14, 1985.
[50] Michael Ruzhansky, Mitsuru Sugimoto, and Baoxiang Wang. Modulation spaces and nonlinear evolution equations. In *Evolution equations of hyperbolic and Schrödinger type*, volume 301 of *Progr. Math.*, pages 267–283. Birkhäuser/Springer Basel AG, Basel, 2012.
[51] Robert Schippa. *Short-time Fourier transform restriction phenomena and applications to nonlinear dispersive equations*. PhD thesis, Universität Bielefeld, 2019.
[52] Christopher D. Sogge. *Fourier integrals in classical analysis*, volume 210 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, second edition, 2017.
[53] Robert S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44(3):705–714, 1977.
[54] Terence Tao. *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. Local and global analysis.
[55] Yoshio Tsutsumi. L^2-solutions for nonlinear Schrödinger equations and nonlinear groups. *Funkcial. Ekvac.*, 30(1):115–125, 1987.
[56] Ana Vargas and Luis Vega. Global wellposedness for 1D non-linear Schrödinger equation for data with an infinite L^2 norm. *J. Math. Pures Appl. (9)*, 80(10):1029–1044, 2001.
[57] Baoxiang Wang and Chunyan Huang. Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations. *J. Differential Equations*, 239(1):213–250, 2007.
[58] Baoxiang Wang and Henryk Hudzik. The global Cauchy problem for the NLS and NLKG with small rough data. *J. Differential Equations*, 232(1):36–73, 2007.
[59] Baoxiang Wang, Zhaohui Huo, Chengchun Hao, and Zihua Guo. *Harmonic analysis method for nonlinear evolution equations. I.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.
[60] L. Wisewell. Kakeya sets of curves. *Geom. Funct. Anal.*, 15(6):1319–1362, 2005.
[61] T. Wolff. Local smoothing type estimates on L^p for large p. *Geom. Funct. Anal.*, 10(5):1237–1288, 2000.
[62] Chunjie Zhang. Strichartz estimates in the frame of modulation spaces. *Nonlinear Anal.*, 78:156–167, 2013.

E-mail address: `robert.schippa@kit.edu`

Department of Mathematics, Karlsruhe Institute of Technology, Englerstrasse 2, 76131 Karlsruhe, Germany