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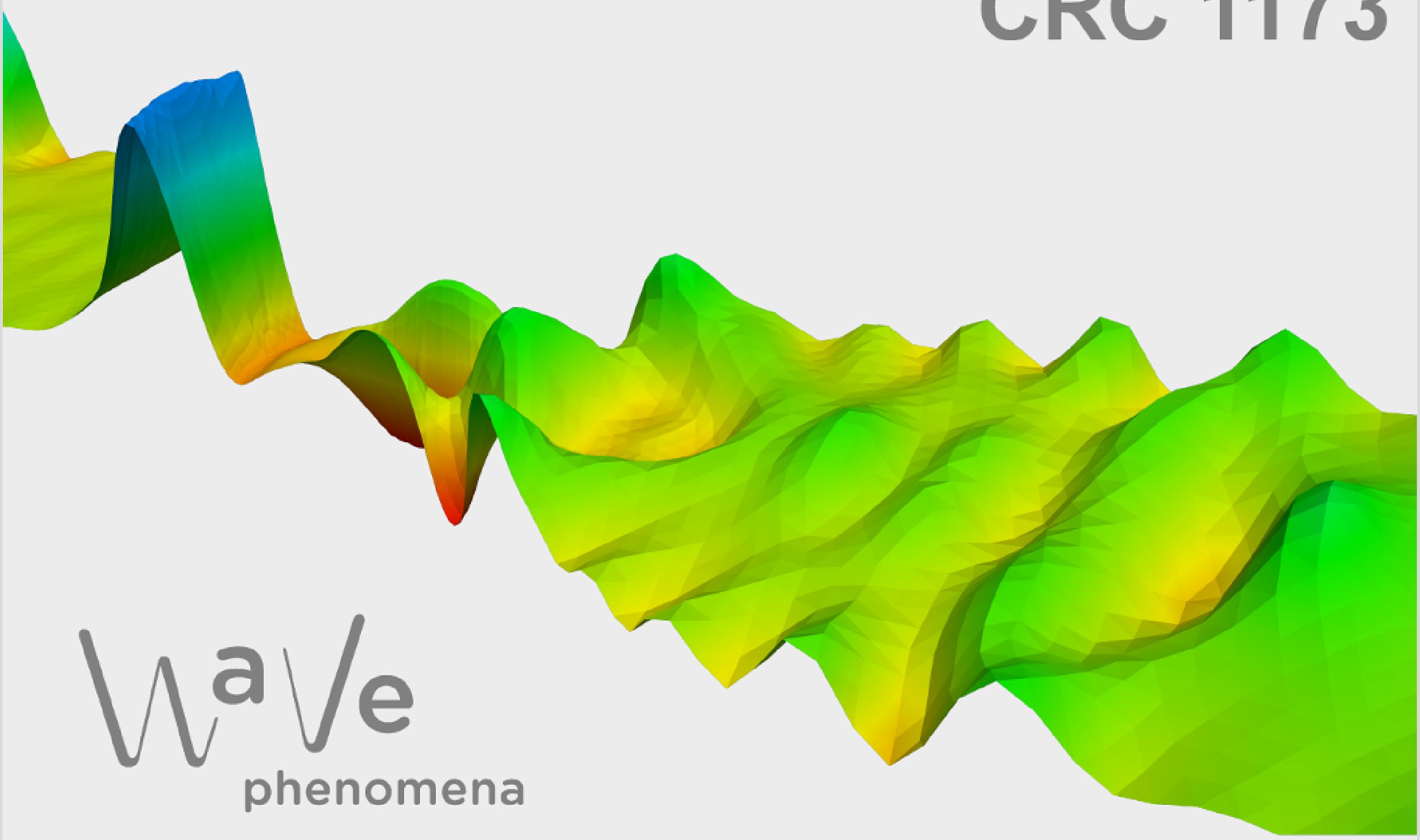

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ERROR ANALYSIS FOR FULL DISCRETIZATIONS OF QUASILINEAR WAVE-TYPE EQUATIONS WITH TWO VARIANTS OF THE IMPLICIT MIDPOINT RULE

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ABSTRACT. We study the full discretization of a general class of first- and second-order quasilinear wave-type problems with the implicit midpoint rule and a linearized variant thereof. Based on a proof by induction, we prove wellposedness and a rigorous error estimate for both schemes, combining energy techniques, inverse estimates, and a linearized fixed-point iteration for the analysis of the nonlinear scheme. To confirm the relevance of the general framework, we derive novel error esitmates for the full discretization of two prominent examples from nonlinear physics: the Westervelt equation and the Maxwell equations with Kerr nonlinearity.

1. INTRODUCTION

We study the full discretization of quasilinear wave-type problems of the form

$$\begin{cases} \Lambda(y(t))\partial_t y(t) = Ay(t) + F(t, y(t)), \quad t \in [0, T], \\ \qquad\qquad y(0) = y^0, \end{cases} \tag{1.1}$$

in a Hilbert space X, where the nonlinear operator Λ is bounded and locally Lipschitz continuous, whereas A is linear but unbounded in X. The nonlinear right-hand side F is assumed to be sufficiently regular. This is a very general framework, which covers both first- and second-order quasilinear wave-type problems. For instance, this includes the Westervelt equation and the Maxwell equations with Kerr nonlinearity, which are important applications in nonlinear acoustics and optics, respectively.

Despite the importance of quasilinear wave-type problems in physics, there are only very few rigorous convergence results concerning the full discretization of these equations. In particular, up to our knowledge the discretization of first-order problems was not analyzed before. Based on the method-of-lines and a conforming discretization in space, the full discretization of second-order quasilinear hyperbolic problems was studied by the following authors, using different discretizations in time: On the one hand, Ewing (1980) and Bales & Dougalis (1989) considered linearly implicit two-step schemes. On the other hand, various linearly implicit single-step schemes were studied by Bales (1986, 1988). Moreover, Makridakis (1993) analyzed a class of linearly implicit single-step schemes as well as a linearly and a fully implicit two-step scheme for quasilinear elastic wave equations. More recently, the full discretization of a specific class of quasilinear wave equations in 1D based on the Fourier spectral method and trigonometric integrators was studied by Gauckler *et al.* (2019).

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In the present paper, we also use the method-of-lines for the discretization of quasilinear wave-type problems of the general form (1.1). The corresponding space discretization was studied by Hochbruck & Maier (2021), including wellposedness and a rigorous error estimate. We now extend these results to the full discretization with two variants of the implicit midpoint rule. On the one hand, we study the classical (fully) implicit midpoint rule, which relies on the solution of nonlinear systems in every time step. On the other hand, we also analyze a linearized variant thereof, called the linearly impilcit midpoint rule, which was introduced by Kovács & Lubich (2018) for the time discretization of quasilinear wave-type problems on unbounded domains.

Contrary to the linear case, a major difficulty in the analysis of numerical schemes for quasilinear wave-type problems is that bounds on the numerical approximations in the energy norm in general do not suffice to ensure wellposedness of the numerical scheme. In particular, in our setting pointwise bounds for the approximations are crucial to ensure essential properties of the discrete counterpart of the nonlinear operator Λ.

Except Gauckler *et al.* (2019), who resolve this issue for the special case of the Fourier spectral method by deriving error estimates not only in the energy norm but also in stronger norms and finally using Sobolev's embedding, all previously cited results essentially rely on inverse estimates to obtain these bounds. As a consequence, these results depend on a step-size restriction $\tau < Ch^{\alpha}$ for some $\alpha > 0$ depending on the spatial dimension and the convergence order of the scheme. Here, $\tau, h > 0$ denote the time step and the space discretization parameter, respectively.

For the analysis of the linearly implicit midpoint rule, we follow a similar approach. First, we use that pointwise bounds on previous approximations are sufficient to analyze the next step of the linearized scheme. Hence, we prove wellposedness of the next step of the scheme and derive an error estimate in the energy norm. Subsequently, we apply inverse estimates to obtain pointwise bounds for the new approximations under a step-size restriction. Overall, using these arguments alternately we prove wellposedness and a rigorous error estimate for the linearly implicit midpoint rule by induction.

The analysis of the fully implicit midpoint rule is more involved, since existence and pointwise bounds of the next approximation are intertwined here. To resolve this dilemma, we approximate each step of the nonlinear scheme by a linearized fixed-point iteration. Since the analysis for the linearly implicit midpoint rule extends to these schemes, we directly obtain wellposedness and error estimates for all iterates. To conclude, we prove convergence of the iteration and provide a rigorous error estimate for the limit, which corresponds to the next approximation of the fully implicit midpoint rule.

Up to our knowledge, we present the first rigorous analysis for the full discretization of quasilinear wave-type equations with a nonlinear single-step scheme. Moreover, we emphasize that our abstract results yield novel results for prominent applications from physics. To strengthen this point, we briefly derive new error estimates for the full discretization of the Maxwell equations with Kerr nonlinearity as well as the Westervelt equation. For preliminary versions of these results with a more detailed discussion, we refer to the doctoral thesis (Maier, 2020).

Note that the space discretization of these applications was analyzed by Hochbruck & Maier (2021). Besides, the space discretization of the strongly damped Westervelt equation with continuous and discontinuous finite elements was studied by Nikolić & Wohlmuth (2019) and Antonietti *et al.* (2020), respectively.

Outline. We present the abstract framework in Section 2 and briefly recapitulate the corresponding space discretization from Hochbruck & Maier (2021) in Section 3. Based on these sections, we introduce the full discretization with two variants of the implicit midpoint rule in Section 4. Moreover, we state our main result, which yields wellposedness and a rigorous error estimate for both schemes within the abstract framework. We first present the corresponding proof for the linearly implicit midpoint rule in Section 5.

In Section 6, we extend the analysis also to the fully implicit scheme. Finally, we apply the abstract results to specific examples in Section 7. First, we focus on the Maxwell equations with Kerr nonlinearity in Section 7.1. We further study the Westervelt equation in Section 7.2.

Notation. Let X and Y be normed spaces. We denote the space of bounded linear operators mapping from X to Y by $\mathcal{L}(X,Y)$, equipped with the norm

$$\|A\|_{\mathcal{L}(X,Y)} \coloneqq \sup_{x\in X\setminus\{0\}} \frac{\|Ax\|_Y}{\|x\|_X}, \qquad A \in \mathcal{L}(X,Y).$$

We write $B_X(R)$ for the open ball of radius $R>0$ in X centered around 0. For the final time $T>0$ and the step size $\tau>0$, $N\in\mathbb{N}$ denotes the maximal number of time steps such that $N\tau\le T$ holds. We then write $t_r = r\tau$ for $r\le N$. Finally, $C>0$ is a generic constant, which may have different values on any occurrence.

2. Analytical setting

For Hilbert spaces $(X,(\cdot\mid\cdot)_X)$, $(Y,(\cdot\mid\cdot)_Y)$, $(Z_\partial,(\cdot\mid\cdot)_{Z_\partial})$, and $(Z,(\cdot\mid\cdot)_Z)$ with dense and continuous embeddings $Z\hookrightarrow Z_\partial\hookrightarrow Y\hookrightarrow X$, the induced norms are denoted by $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_{Z_\partial}$, and $\|\cdot\|_Z$, respectively. We further make use of a seminorm $|\cdot|_Y$ on Y, which satisfies for some constant $C_Y>0$

$$|\xi|_Y \le C_Y\|\xi\|_Y, \qquad \xi\in Y.$$

We now specify the abstract quasilinear wave-type problem (1.1). For specific examples satisfying these assumptions, we refer to Section 7.

ASSUMPTION 2.1 There exists a radius $R_Y>0$ such that the operators in (1.1) satisfy

- (Λ) $\{\Lambda(\xi)\mid \xi\in B_Y(R_Y)\}\subset\mathcal{L}(X)$ is a family of symmetric operators, which are uniformly positive definite and bounded, i.e., there are constants $c_\Lambda, C_\Lambda>0$ with

$$c_\Lambda\|\varphi\|_X^2 \le (\Lambda(\xi)\varphi\mid\varphi)_X, \qquad \|\Lambda(\xi)\|_{\mathcal{L}(X)}\le C_\Lambda, \qquad \varphi\in X,\ \xi\in B_Y(R_Y). \tag{2.1}$$

- (A) $A\in\mathcal{L}(D(A),X)$ with $Y\subset D(A)\subset X$, where $D(A)$ denotes the domain of A.
- (F) $F\colon[0,T]\times B_Y(R_Y)\to X$ is continuous in time and bounded, i.e., there is a constant $C_F>0$ with

$$\|F(t,\xi)\|_X\le C_F, \qquad t\in[0,T],\ \xi\in B_Y(R_Y).$$

Note that all our results are also valid for more general bounded domains instead of spheres $B_Y(R_Y)$. However, we refrain from this generalization for the sake of readability.

Due to (2.1) the operators

$$\mathcal{A}(\xi)\coloneqq\Lambda(\xi)^{-1}A, \qquad \mathcal{F}(t,\xi)\coloneqq\Lambda(\xi)^{-1}F(t,\xi), \qquad t\in[0,T],\ \xi\in B_Y(R_Y).$$

are well defined. Thus, we can rewrite (1.1) as

$$\begin{cases}\partial_t y(t)=\mathcal{A}(y(t))y(t)+\mathcal{F}(t,y(t)), & t\in[0,T],\\ \quad y(0)=y^0, \end{cases} \tag{2.2}$$

which is more convenient for our analysis.

Since there is no wellposedness result which applies to this quite general problem, we now assume wellposedness. Again, this is justified in Section 7 for the specific examples.

ASSUMPTION 2.2 Let $R_Y > 0$ such that Assumption 2.1 holds. The quasilinear Cauchy problem (2.2) has a unique solution y with maximal time of existence $t^*(y^0) > 0$, which satisfies

$$y \in C^3([0,T],X) \cap C^2([0,T],Y) \cap C^1([0,T],Z_\partial) \cap C([0,T],Z \cap B_Y(R_Y))$$

for $T < t^*(y^0)$. Moreover, there exist radii $R_{\partial_t}, R_{\mathcal{A}} > 0$ such that

$$\|\partial_t y(t)\|_Y < R_{\partial_t}, \qquad\qquad |\mathcal{A}(y(t))y(t)|_Y < R_{\mathcal{A}}$$

hold uniformly in $[0,T]$.

Moreover, we introduce for $\xi \in B_Y(R_Y)$ the state-dependent inner product

$$(\varphi \mid \psi)_{\Lambda(\xi)} \coloneqq (\Lambda(\xi)\varphi \mid \psi)_X\,, \qquad\qquad \varphi,\psi \in X, \tag{2.3}$$

and the induced state-dependent norm $\|\cdot\|_{\Lambda(\xi)}$, which is equivalent to the norm of X due to (2.1).

Finally, note that due to Assumption 2.2 the weak formulation of (1.1) on $(X, (\cdot \mid \cdot)_X)$ is equivalent to the weak formulation of (2.2) on $(X, (\cdot \mid \cdot)_{\Lambda(y)})$. Hence, it is sufficient to consider (2.2).

3. SPACE DISCRETIZATION

We briefly present the space discretization of (2.2). For further details including wellposedness as well as a rigorous error analysis, we refer to Hochbruck & Maier (2021).

Based on a finite-dimensional function space V_h we introduce spaces X_h and Y_h with

$$X_h = (V_h, (\cdot \mid \cdot)_X) \subset X, \qquad\qquad Y_h = (V_h, \|\cdot\|_{Y_h}), \tag{3.1}$$

where $\|\cdot\|_{Y_h}$ corresponds to the norm of Y. Furthermore, $|\cdot|_{Y_h}$ is a seminorm on Y_h corresponding to $|\cdot|_Y$, which satisfies for some constant $C_{Y_h} > 0$

$$|\xi_h|_{Y_h} \le C_{Y_h}\|\xi_h\|_{Y_h}, \qquad\qquad \xi_h \in Y_h. \tag{3.2}$$

For a sufficiently small space discretization parameter $h > 0$, we have the inverse estimates

$$\frac{1}{C_{X_h,Y_h}(h)}\|\xi_h\|_{X_h} \le \|\xi_h\|_{Y_h} \le C_{Y_h,X_h}(h)\|\xi_h\|_{X_h}, \qquad\qquad \xi_h \in Y_h, \tag{3.3}$$

with $C_{Y_h,X_h}(h), C_{X_h,Y_h}(h) > 0$ which may depend on h.

For the specific examples in Section 7 these constants are specified by $C_{Y_h,X_h}(h) \sim h^{-\frac{d}{2}}$, where $d \in \mathbb{N}$ is the spatial dimension. For the second constant, we obtain $C_{X_h,Y_h}(h) \sim h^{-1}$ for the Westervelt equation, whereas for the Maxwell equations $C_{X_h,Y_h}(h)$ is independent of h.

We emphasize that all our results can be generalized to more general nonconforming space discretizations including the case $X_h \not\subset X$, cf. (Maier, 2020). For instance, this includes the space discretization with isoparametric finite elements. This is relevant as many wellposedness results for quasilinear wave-type equations are only valid for spatial domains with smooth boundary.

The discrete quasilinear wave-type problem is for discretizations Λ_h, A_h, and F_h of Λ, A, and F, respectively, given by

$$\begin{cases} \Lambda_h(y_h(t))\partial_t y_h(t) = A_h y_h(t) + F_h(t, y_h(t)), & t \in [0,T], \\ \qquad\qquad y_h(0) = y_h^0, \end{cases} \tag{3.4}$$

where $y_h^0 \in X_h$ is the discrete initial value.

ASSUMPTION 3.1 There is $R_{Y_h} > 0$ such that the discrete operators in (3.4) satisfy uniformly in $h > 0$

(Λ_h) $\{\Lambda_h(\xi_h) \mid \xi_h \in B_{Y_h}(R_{Y_h})\} \subset \mathcal{L}(X_h)$ is a family of symmetric operators, which are uniformly positive definite and bounded, i.e., there are constants $c_{\Lambda_h}, C_{\Lambda_h} > 0$ such that

$$c_{\Lambda_h} \|\varphi_h\|_{X_h}^2 \leq (\Lambda_h(\xi_h)\varphi_h \mid \varphi_h)_{X_h}, \quad \|\Lambda_h(\xi_h)\|_{\mathcal{L}(X_h)} \leq C_{\Lambda_h}, \quad \varphi_h \in X_h,\, \xi_h \in B_{Y_h}(R_{Y_h}) \tag{3.5}$$

holds. Moreover, there are constants $L_{\Lambda_h}^{X_h}, L_{\Lambda_h}^{Y_h} > 0$ with

$$\begin{aligned} \|\Lambda_h(\varphi_h) - \Lambda_h(\psi_h)\|_{\mathcal{L}(X_h)} &\leq L_{\Lambda_h}^{X_h} \|\varphi_h - \psi_h\|_{Y_h}, && \varphi_h, \psi_h \in B_{Y_h}(R_{Y_h}), \\ \|\big(\Lambda_h(\varphi_h) - \Lambda_h(\psi_h)\big)\xi_h\|_X &\leq L_{\Lambda_h}^{Y_h} \|\varphi_h - \psi_h\|_X |\xi_h|_{Y_h}, && \varphi_h, \psi_h \in B_{Y_h}(R_{Y_h}),\, \xi_h \in Y_h. \end{aligned}$$

(A_h) $A_h \colon X_h \to X_h$ is dissipative in X_h, i.e., we have

$$(A_h \xi_h \mid \xi_h)_{X_h} \leq 0, \qquad \xi_h \in X_h. \tag{3.7}$$

(F_h) We have $F_h \colon [0,T] \times B_{Y_h}(R_{Y_h}) \to X_h$, which is continuous in time and bounded in Y_h, i.e., there is a constant $C_{F_h} > 0$ such that

$$|F_h(t, \xi_h)|_{Y_h} \leq C_{F_h}, \qquad t \in [0,T],\, \xi_h \in B_{Y_h}(R_{Y_h}) \tag{3.8}$$

holds. Furthermore, F_h is Lipschitz continuous in the second argument, i.e., there is a constant $L_{F_h} > 0$ with

$$\|F_h(t, \varphi_h) - F_h(t, \psi_h)\|_{X_h} \leq L_{F_h} \|\varphi_h - \psi_h\|_{X_h}, \qquad t \in [0,T],\, \varphi_h, \psi_h \in B_{Y_h}(R_{Y_h}).$$

As in the continuous setting, the discrete operators

$$\mathcal{A}_h(\xi_h) \coloneqq \Lambda_h(\xi_h)^{-1} A_h, \quad \mathcal{F}_h(t, \xi_h) \coloneqq \Lambda_h(\xi_h)^{-1} F_h(t, \xi_h), \quad t \in [0,T],\, \xi_h \in B_{Y_h}(R_{Y_h}), \tag{3.9}$$

are well defined due to (3.5). Thus, correspondingly to (2.2) we can rewrite (3.4) as

$$\begin{cases} \partial_t y_h(t) = \mathcal{A}_h(y_h(t)) y_h(t) + \mathcal{F}_h(t, y_h(t)), & t \in [0,T], \\ y_h(0) = y_h^0, \end{cases} \tag{3.10}$$

As shown in (Hochbruck & Maier, 2021, Lem. 3.3), the operators from (3.9) are again Lipschitz continuous, i.e., for $R_{Y_h} > 0$ as in Assumption 3.1 there are constants $L_{\mathcal{A}_h}, L_{\mathcal{F}_h} > 0$ such that for all $\varphi_h, \psi_h \in B_{Y_h}(R_{Y_h})$, we have

$$\|\big(\mathcal{A}_h(\varphi_h) - \mathcal{A}_h(\psi_h)\big)\xi_h\|_{X_h} \leq L_{\mathcal{A}_h} |\mathcal{A}_h(\varphi_h)\xi_h|_{Y_h} \|\varphi_h - \psi_h\|_{X_h}, \qquad \xi_h \in X_h, \tag{3.11}$$

$$\|\mathcal{F}_h(t, \varphi_h) - \mathcal{F}_h(t, \psi_h)\|_{X_h} \leq L_{\mathcal{F}_h} \|\varphi_h - \psi_h\|_{X_h}, \qquad t \in [0,T]. \tag{3.12}$$

Corresponding to (2.3) we introduce for $\xi_h \in B_{Y_h}(R_{Y_h})$ the discrete state-dependent inner product $(\cdot \mid \cdot)_{\Lambda_h(\xi_h)}$ as well as the induced discrete state-dependent norm $\|\cdot\|_{\Lambda_h(\xi_h)}$, which is again equivalent to the norm of X_h due to (3.5). More precisely, there are constants $c_{\Lambda_h}, C_{\Lambda_h} > 0$ with

$$c_{\Lambda_h} \|\xi_h\|_{X_h}^2 \leq \|\xi_h\|_{\Lambda_h(\zeta_h)}^2 \leq C_{\Lambda_h} \|\xi_h\|_{X_h}^2, \qquad \xi_h \in X_h. \tag{3.13}$$

Moreover, as shown in the following lemma the discrete state-dependent norm is continuous in time, cf. (Hochbruck & Maier, 2021, Lem. 3.2).

LEMMA 3.2 Let $R_{Y_h} > 0$ be the radius from Assumption 3.1. Moreover, let $\widehat{R}_{\partial_t} > 0$ and

$$z_h \in C^1([0,T], Y_h) \cap C([0,T], B_{Y_h}(R_{Y_h})),$$

with $\|\partial_t z_h\|_{Y_h} < \widehat{R}_{\partial_t}$. Then, it holds

$$\|\xi_h\|_{\Lambda_h(z_h(t))} \leq \mathrm{e}^{C'|t-s|} \|\xi_h\|_{\Lambda_h(z_h(s))}, \qquad s, t \in [0,T],\, \xi_h \in X_h \tag{3.14}$$

with the constant $C' = \frac{1}{2} L_{\Lambda_h}^{X_h} c_{\Lambda_h}^{-1} \widehat{R}_{\partial_t}$.

$$
\begin{tikzcd}
Z \arrow[r, "\hookrightarrow"] & Z_\partial \arrow[r, "\hookrightarrow"] & Y \arrow[rr, "\hookrightarrow"] \arrow[d, "I_h"'] \arrow[drr, "J_h"] & & X \arrow[d, shift left, "\Pi_h"] \\
& & Y_h \arrow[rr, leftrightarrow, "="] & & X_h \arrow[u, shift left, "\cup"]
\end{tikzcd}
$$

FIGURE 1. Overview of the relations of the discrete and continuous function spaces and the relating operators, cf. (Hochbruck & Maier, 2021, Fig. 1).

Thus, as in the continuous setting the weak form of (3.4) on $X_h = (V_h, (\cdot \mid \cdot)_X)$ is identical to the weak form of (3.10) on $(V_h, (\cdot \mid \cdot)_{\Lambda_h(y_h)})$ if the solution y_h of (3.10) is sufficiently regular.

Finally, we use the following operators relating the continuous and the discrete spaces. These relations are also shown in Figure 1.

(J_h) The reference operator $J_h : Y \to X_h$ is linear and bounded with

$$\|J_h \xi\|_{X_h} \leq C_{J_h} \|\xi\|_Y, \qquad \xi \in Y. \tag{3.15}$$

(I_h) The interpolation operator $I_h : Y \to Y_h$ is bounded with

$$\|I_h \xi\|_{Y_h} \leq C_{I_h} \|\xi\|_Y, \qquad |I_h \xi|_{Y_h} \leq C_{I_h} |\xi|_Y, \qquad \xi \in Y.$$

(Π_h) The projection $\Pi_h : X \to X_h$ satisfies

$$(\varphi_h \mid \psi)_X = (\varphi_h \mid \Pi_h \psi)_X, \qquad \varphi_h \in X_h,\ \psi \in X.$$

Preliminary to the analysis, we state the following assumptions. First, we fix the radii introduced above for the rest of this paper.

ASSUMPTION 3.3 Let $R_{Y_h} > 0$ be the radius from Assumption 3.1 and $y_h^0 \in B_{Y_h}(R_{Y_h})$. Furthermore, let $R_Y, R_{\partial_t}, R_{\mathcal{A}} > 0$ be chosen such that Assumption 2.2 and

$$C_{I_h} R_Y < R_{Y_h}$$

hold. Finally, we set $R_{\mathcal{A}_h} > 0$ with

$$\max\Big\{ C_{I_h} R_{\mathcal{A}},\ |\mathcal{A}_h(y_h^0) y_h^0|_{Y_h} \Big\} < R_{\mathcal{A}_h}.$$

Moreover, we assume consistency of the space discretization. To this end, we define the constant $C_{\mathcal{A}_h, Y_h, X_h}(h) > 0$ such that

$$|\mathcal{A}_h(\xi_h)\zeta_h|_{Y_h} \leq C_{\mathcal{A}_h, Y_h, X_h}(h) \|\zeta_h\|_X, \qquad \xi_h \in B_{Y_h}(R_{Y_h}),\ \zeta_h \in X_h, \tag{3.16}$$

holds for $h > 0$ sufficiently small. We further introduce the constant

$$C_{\max}(h) = \max\{1, C_{Y_h, X_h}(h), C_{\mathcal{A}_h, Y_h, X_h}(h)\}, \tag{3.17}$$

which in general deteriorates for $h \to 0$. Finally, we define the remainder terms

$$\mathcal{R}_\Lambda(\xi) \coloneqq \Lambda_h(I_h \xi) J_h - \Pi_h \Lambda(\xi), \qquad \xi \in B_Y(R_Y), \tag{3.18a}$$

$$\mathcal{R}_A \coloneqq A_h J_h - \Pi_h A, \tag{3.18b}$$

$$\mathcal{R}_F(t, \xi) \coloneqq F_h(t, I_h \xi) - \Pi_h F(t, \xi), \qquad t \in [0, T],\ \xi \in B_Y(R_Y), \tag{3.18c}$$

to relate the continuous operators and their discrete counterparts.

ASSUMPTION 3.4 Let Assumption 3.3 be satisfied. Moreover, for $h \to 0$ we have

(A_1) $\|(\mathrm{Id} - J_h)\zeta\|_X \to 0$,
(A_2) $C_{\max}(h)\|J_h y^0 - y_h^0\|_X \to 0$,
(A_3) $C_{\max}(h)\|(I_h - J_h)\zeta\|_X \to 0$,
(A_4) $C_{\max}(h)\|\mathcal{R}_A\zeta\|_X \to 0$,
(A_5) $C_{\max}(h)\|\mathcal{R}_\Lambda(\xi)\zeta_\partial\|_X \to 0$,
(A_6) $C_{\max}(h)\sup_{[0,T]}\|\mathcal{R}_F(\cdot,\xi)\|_X \to 0$,

uniformly for $\zeta_\partial \in Z_\partial$, $\zeta \in Z$, and $\xi \in Z \cap B_Y(R_Y)$.

Under these assumptions, Hochbruck & Maier (2021) prove wellposedness as well as a rigorous error estimate for the spatially discrete quasilinear wave-type problem (3.4).

4. FULL DISCRETIZATION

Following the method-of-lines approach, we apply two variants of the implicit midpoint rule with step size $\tau > 0$ to the spatially discrete quasilinear wave-type equation (3.10). On the one hand, we consider the fully implicit midpoint rule

$$y_h^{n+1} = y_h^n + \tau \mathcal{A}_h^{n+1/2} y_h^{n+1/2} + \tau \mathcal{F}_h^{n+1/2}, \qquad y_h^{n+1/2} = \frac{y_h^{n+1} + y_h^n}{2}, \tag{4.1}$$

with the short notation

$$\mathcal{A}_h^{n+1/2} := \mathcal{A}_h\Big(y_h^{n+1/2}\Big), \qquad \mathcal{F}_h^{n+1/2} := \mathcal{F}_h\Big(t_{n+1/2}, y_h^{n+1/2}\Big). \tag{4.2}$$

On the other hand, as proposed by Kovács & Lubich (2018) for quasilinear wave-type problems on unbounded domains, we also analyze the linearly implicit midpoint rule

$$y_h^{n+1} = y_h^n + \tau \underline{\mathcal{A}}_h^{n+1/2} y_h^{n+1/2} + \tau \underline{\mathcal{F}}_h^{n+1/2}, \qquad \underline{y}_h^{n+1/2} = \frac{3y_h^n - y_h^{n-1}}{2}, \tag{4.3}$$

where based on the notation $y_h^{-1} = y_h^0$ we use the approximation $\underline{y}_h^{1/2} = y_h^0$ in the first step. Moreover, we introduce the abbreviations

$$\underline{\mathcal{A}}_h^{n+1/2} := \mathcal{A}_h\Big(\underline{y}_h^{n+1/2}\Big), \qquad \underline{\mathcal{F}}_h^{n+1/2} := \mathcal{F}_h\Big(t_{n+1/2}, \underline{y}_h^{n+1/2}\Big).$$

Note that we use the same notation for the approximations obtained by either of the schemes for the sake of readability, as we state the main results in a unified fashion. Since the technical details of the analysis of these schemes are well separated in the following two sections, it is always clear from the context which scheme we refer to.

In our main result, we prove wellposedness as well as a rigorous error estimate for either of the implicit midpoint rules. This is based on the following step-size restriction: There exist constants $\varepsilon_0, C_0 > 0$ such that the discretization parameters $\tau, h > 0$ satisfy

$$\tau C_{\max}(h)^{\frac{1}{2}} \leq C_0 h^{\varepsilon_0}, \tag{4.4}$$

where $C_{\max}(h)$ is the constant defined in (3.17).

Although the implicit midpoint rule is in general unconditionally stable when applied to linear problems, we emphasize that the step-size restriction (4.4) is essential here, as the wellposedness of both schemes relies on bounds for the iterates in the stronger space Y_h. Hence, as stated in Makridakis (1993), this restriction is not induced by the techniques used for the analysis, but inherent in the problem itself.

We now state our main result, which is proven in Section 5 and Section 6.

THEOREM 4.1 Let Assumption 3.4 be true, $T < t^*(y^0)$, and $N \in \mathbb{N}$ with $\tau N \leq T$. Then, there exist $h_0, \tau_0 > 0$ such that for all $h < h_0$ and $\tau < \tau_0$ satisfying the step-size restriction (4.4), both the fully and the linearly implicit midpoint rule (4.1) and (4.3), respectively, are wellposed and satisfy for $n = 0, \dots, N$ the error estimate

$$\begin{aligned} \|y(t_n) - y_h^n\|_X \leq \|(\mathrm{Id} - J_h)y(t_n)\|_X + C(1+t_n)\mathrm{e}^{Ct_n}\Big(&\|J_h y^0 - y_h^0\|_X + \sup_{[0,t_n]} \|(I_h - J_h)y\|_X \\ &+ \tau^2 \big(\sup_{[0,t_n]} \|\partial_t y\|_Y + \sup_{[0,t_n]} \|\partial_t^2 y\|_Y + \sup_{[0,t_n]} \|\partial_t^3 y\|_X \big) \\ &+ \sup_{s_1,s_2\in[0,t_n]} \|\mathcal{R}_\Lambda(y(s_1))\partial_t y(s_2)\|_X + \sup_{[0,t_n]} \|\mathcal{R}_A y\|_X + \sup_{[0,t_n]} \|\mathcal{R}_F(\cdot,y)\|_X \Big), \end{aligned} \tag{4.5}$$

with a constant $C > 0$ independent of τ, h, n and T. The upper bound τ_0 depends on the constants ε_0, C_0 from the step-size restriction (4.4). Furthermore, we have

$$\|y(t_n) - y_h^n\|_X \to 0, \qquad n = 0, \dots, N, \tag{4.6}$$

for $\tau, h \to 0$ satisfying the step-size restriction (4.4).

In (Hochbruck & Maier, 2021, Sec. 5), a refined framework for the space discretization of quasilinear wave-type equations with nonlinearities Λ and F that are local in space is studied. In this special case, which is for convenience also briefly presented in Appendix A, we get the following refined version of our main result.

COROLLARY 4.1 Let (A_1)–(A_4) of Assumption 3.4 as well as Assumption A.1 be satisfied. Then, the statements of Theorem 4.1 hold. In particular, the approximations obtained by the linearly or fully implicit midpoint rule (4.1) or (4.3), respectively, satisfies for $n = 0, \dots, N$ the refined estimate

$$\begin{aligned} \|y(t_n) - y_h^n\|_X \leq \|(\mathrm{Id} - J_h)y(t_n)\|_X + C(1+t_n)\mathrm{e}^{Ct_n}\Big(&\|J_h y^0 - y_h^0\|_X + \sup_{[0,t_n]} \|(I_h - J_h)y\|_X \\ &+ \tau^2 \big(\sup_{[0,t_n]} \|\partial_t y\|_Y + \sup_{[0,t_n]} \|\partial_t^2 y\|_Y + \sup_{[0,t_n]} \|\partial_t^3 y\|_X \big) + \sup_{[0,t_n]} \|(I_h - J_h)\partial_t y\|_X \\ &+ \sup_{s_1,s_2\in[0,t_n]} \|(\mathrm{Id}-I_h)\Lambda(y(s_1))\partial_t y(s_2)\|_X + \sup_{[0,t_n]} \|\mathcal{R}_A y\|_X + \sup_{[0,t_n]} \|(\mathrm{Id}-I_h)F(\cdot,y)\|_X \Big), \end{aligned}$$

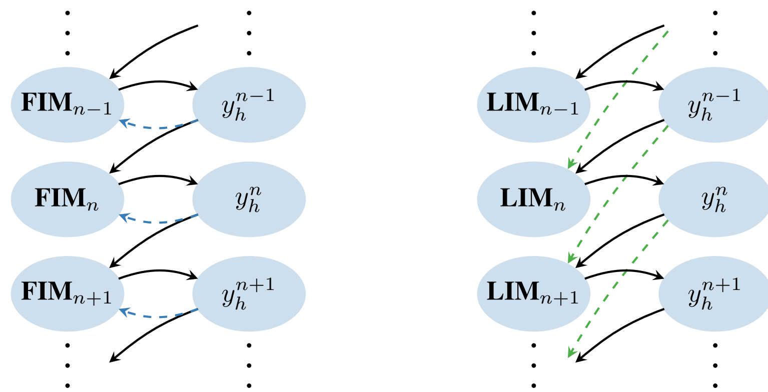
with a constant $C > 0$ independent of τ, h, n and T.

Proof. The result directly follows from Theorem 4.1 and Lemma A.2. □

REMARK 4.1 For the sake of readability, we only state our results for the time discretization with a constant step size. However, all results can also be generalized to variable step sizes $\tau_i \in [\tau_{\min}, \tau_{\max}]$, $i = 1, \dots, N$, for $0 < \tau_{\min} < \tau_{\max} < \tau_0$ fixed. For the linearly implicit midpoint rule (4.3), we then use the extrapolations

$$\underline{y}_h^{n+1/2} = y_h^n + \frac{\tau_{n+1}}{2\tau_n}\big(y_h^n - y_h^{n-1}\big), \qquad n = 1, \dots, N-1.$$

We conclude this section with a comparison of the two variants of the implicit midpoint rule. In Figure 2, the dependencies of both schemes are illustrated. On the bottom left of both subfigures, one step of either of the schemes given in (4.1) and (4.3) is indicated by $\mathbf{FIM}_{n+1}$ and $\mathbf{LIM}_{n+1}$, respectively. Previous steps of both schemes are denoted by the subscripts n and $n-1$. The corresponding approximations y_h^i, $i = n-1, n, n+1$, are shown on the right. Moreover, the black arrows indicate the usual dependencies



(A) Fully implicit midpoint rule.

(B) Linearly implicit midpoint rule.

FIGURE 2. $\mathbf{FIM}_{n+1}$, $\mathbf{LIM}_{n+1}$ correspond to (4.1), (4.3), respectively and $\mathbf{FIM}_{n}$, $\mathbf{LIM}_{n}$ denote (4.1), (4.3) with n replaced by $n-1$.

also known from linear problems; e.g., the current step $\mathbf{FIM}_{n+1}$ of the fully implicit midpoint rule depends on the previous approximation y_h^n to provide y_h^{n+1}.

For the fully implicit midpoint rule (4.1), the nonlinearities are evaluated at the implicitly given midpoint. Thus, as shown in Figure 2a on the left, characteristic properties of $\mathbf{FIM}_{n+1}$ also depend on y_h^{n+1}, which is currently unknown. This is indicated by the dashed blue arrows. In contrast, for the linearly implicit midpoint rule (4.3) we use the extrapolated approximation of the midpoint to linearize the scheme. Hence, characteristic properties of $\mathbf{LIM}_{n+1}$ do not rely on y_h^{n+1}, but only on y_h^n and y_h^{n-1}, as indicated by the dashed green arrows in Figure 2b.

The linearly implicit midpoint rule is appealing with respect to the computational efficiency, as only a linear system has to be solved in every step, whereas the fully implicit midpoint rule is nonlinear. Moreover, as illustrated in Figure 2 it is also more convenient for the analysis, since the characteristic properties of the current step do not depend on the unknown. In particular, proving the existence of a unique solution of (4.1) in the required spaces is significantly more involved as its counterpart for the linearized scheme (4.3). Hence, we first focus in Section 5 on the linearly implicit midpoint rule and extend these results in Section 6 to the fully implicit midpoint rule.

5. ERROR ANALYSIS FOR THE LINEARLY IMPLICIT MIDPOINT RULE

In this section we present the analysis of the linearly implicit midpoint rule (4.3). Usually, the analysis of fully discrete schemes consists of two steps, where we first show existence of all iterates and subsequently tackle the error analysis. However, as indicated in Figure 2 this approach is not feasible here as these proofs are intertwined. More precisely, even for the linearly implicit midpoint rule the wellposedness of the next step of the scheme relies on bounds on the previous iterates in the Y_h-norm. To resolve this dilemma, we proceed as follows.

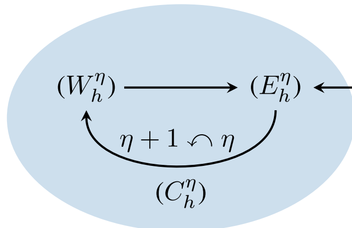


FIGURE 3. Relations between the main steps for the analysis of the full discretization with the linearly implicit midpoint rule.

Roadmap for the analysis of the linearly implicit midpoint rule.

- (W) Assumption 2.2 implies wellposedness of the continuous quasilinear Cauchy problem (2.2), i.e., for all $T < t^*(y^0)$ the bound $\|y\|_Y < R_Y$ holds uniformly on $[0,T]$.
- (W_h^η) Based on the assumption that the first $\eta \geq 0$ steps of the linarly implicit midpoint rule (4.3) are well defined, in Lemma 5.1 we prove existence of range of time steps $(0,\tau_0^\eta)$ such that the next iterate $y_h^{\eta+1}$ exists and satisfies $\|y_h^{\eta+1}\|_{Y_h} < R_{Y_h}$. More precisely, we first show that the next iterate $y_h^{\eta+1}$ is uniquely defined in X_h and subsequently introduce τ_0^η as the supremum over all time steps for which essential bounds in Y_h for the error analysis are satisfied. Finally, we ensure $\tau_0^\eta > 0$ using that Y_h is finite dimensional. However, at this point, this is based on a severe step-size restriction.
- (E_h^η) Using (W), (W_h^η), and energy techniques, we bound the error $e_h^{\eta+1} = J_h y(t_{\eta+1}) - y_h^{\eta+1}$ based on errors of the previous iterates e_h^η, $e_h^{\eta-1}$ in Lemma 5.2. As this is not suitable for the first step of the linearly implicit midpoint rule due to the different approximation of the midpoint $\underline{y}_h^{1/2} = y_h^0$, we further provide an alternative estimate in Lemma 5.3.
- (C_h^η) Based on an inverse estimate, the consistency from Assumption 3.4, and a uniform bound for the previous errors e_h^n with $n \leq \eta$, we prove $\|y_h^{\eta+1} - I_h y(t_{\eta+1})\|_{Y_h} \to 0$ for $\tau, h \to 0$ under the step-size restriction (4.4). From this we conclude that the current step $\mathbf{LIM}_{\eta+1}$ is wellposed under this relaxed step-size restriction and we may proceed with $(W_h^{\eta+1})$.

Overall, we show Theorem 4.1 for the linearly implicit midpoint rule by induction, as we alternately prove (W_h^η), (E_h^η), and (C_h^η). This is also illustrated in Figure 3, with the blue ellipse indicating the proof by induction.

Note that the overall structure of this approach is quite similar to the analysis for the space discretization of quasilinear wave-type problems, cf. (Hochbruck & Maier, 2021, Sec. 4). Moreover, we emphasize that the analysis of the linearly implicit midpoint rule for nonconforming space discretizations is discussed in detail in (Maier, 2020, Sec. 7.1.1).

The first step is to show wellposedness of a single step of the linearly implicit midpoint rule.

LEMMA 5.1 Let Assumption 3.3 be satisfied. For $0 \leq \eta < N$ fixed we assume

$$\|y_h^\eta\|_{Y_h} < \tfrac{1}{2}(R_{Y_h} + C_{I_h} R_Y), \qquad |\mathcal{A}_h^\eta y_h^\eta|_{Y_h} < \tfrac{1}{2}(R_{\mathcal{A}_h} + C_{I_h} R_{\mathcal{A}}), \qquad \|\underline{y}_h^{\eta+1/2}\|_{Y_h} < R_{Y_h}. \tag{5.1}$$

If $\eta > 0$, we further assume

$$\|y_h^{\eta-1/2}\|_{Y_h}, \|\underline{y}_h^{\eta-1/2}\|_{Y_h} < R_{Y_h}. \tag{5.2}$$

Then, the linearly implicit midpoint rule (4.3) has a unique solution $y_h^{\eta+1} \in X_h$. Moreover, there is a constant $\tau_0^\eta > 0$ which may depend on the space discretization parameter h such that for all $\tau < \tau_0^\eta$ we have the bounds

$$\|y_h^{\eta+1/2}\|_{Y_h}, \|y_h^{\eta+1}\|_{Y_h} < R_{Y_h}, \qquad |\underline{\mathcal{A}}_h^{\eta+1/2} y_h^{\eta+1/2}|_{Y_h} < R_{\mathcal{A}_h}.$$

Proof. The proof consists of two parts, where we first prove existence of the next iterate $y_h^{\eta+1}$ in X_h and subsequently derive the required bounds in Y_h.

(1) To show existence, we first observe that (4.3) is equivalent to

$$\big(\mathrm{Id} - \tfrac{\tau}{2}\underline{\mathcal{A}}_h^{\eta+1/2}\big) y_h^{\eta+1/2} = y_h^\eta + \tfrac{\tau}{2}\underline{\mathcal{F}}_h^{\eta+1/2}, \tag{5.3}$$

which is well defined due to (5.1) and Assumption 3.1. Since $\underline{\mathcal{A}}_h^{\eta+1/2}$ is a dissipative operator in $(X_h, \|\cdot\|_{\Lambda_h(\underline{y}_h^{\eta+1/2})})$, the norm equivalence (3.13) implies

$$\big(\mathrm{Id} - \frac{\tau}{2}\underline{\mathcal{A}}_h^{\eta+1/2}\big)^{-1} \in \mathcal{L}(X_h)$$

for all $\tau > 0$. Thus, $y_h^{\eta+1/2} \in X_h$ and hence also $y_h^{\eta+1} \in X_h$ is uniquely given by (5.3).

(2) To prove the bounds in Y_h, we introduce

$$\tau_0^\eta = \sup\{\tau_* \in [0,1] \,|\, \|y_h^{\eta+1/2}\|_{Y_h}, \|y_h^{\eta+1}\|_{Y_h} < R_{Y_h},\ |\underline{\mathcal{A}}_h^{\eta+1/2} y_h^{\eta+1/2}|_{Y_h} < R_{\mathcal{A}_h},\ \text{for all } \tau < \tau_*\}.$$

In the remainder of the proof, we show $\tau_0^\eta > 0$ using Banach's fixed-point theorem. However, we emphasize that this only yields a pessimistic lower bound, which strongly depends on the spatial discretization parameter h. Nevertheless, this is necessary and sufficient to prove wellposedness. Since we derive an improved restraint at the end of this section, we do not keep track of the exact value of the lower bound $\tau_*(h) > 0$ in the following, but use $\tau_*(h)$ as a monotonically decreasing generic constant for the sake of readability.

Preliminary to the application of the fixed-point theorem, we define

$$E_\tau^\eta = \big\{\xi_h \in X_h \ \mid\ \|\xi_h\|_{Y_h}, \|2\xi_h - y_h^\eta\|_{Y_h} < R_{Y_h},\ |\underline{\mathcal{A}}_h^{\eta+1/2}\xi_h|_{Y_h} < R_{\mathcal{A}_h}\big\}.$$

In order to prove that this space is non-empty, we first deduce from (3.2), (3.3), and (3.11)

$$\begin{aligned} |\underline{\mathcal{A}}_h^{\eta+1/2} y_h^\eta|_{Y_h} &\le C_{Y_h} C_{Y_h,X_h}(h) \|\big(\underline{\mathcal{A}}_h^{\eta+1/2} - \mathcal{A}_h^\eta\big) y_h^\eta\|_{X_h} + |\mathcal{A}_h^\eta y_h^\eta|_{Y_h} \\ &\le \big(1 + C_{Y_h} C_{Y_h,X_h}(h) L_{\mathcal{A}_h} \|\underline{y}_h^{\eta+1/2} - y_h^\eta\|_{X_h}\big) |\mathcal{A}_h^\eta y_h^\eta|_{Y_h}. \end{aligned}$$

On the one hand, for $\eta > 0$ we obtain from (3.8), (3.9), (4.3), and (5.2) the bound

$$\|\underline{y}_h^{\eta+1/2} - y_h^\eta\|_{X_h} = \tfrac{1}{2}\|y_h^\eta - y_h^{\eta-1}\|_{X_h} \le \tfrac{\tau}{2} c_{\Lambda_h}^{-1}\big(C_{A_h}(h) C_{X_h,Y_h}(h) R_{Y_h} + C_{X_h,Y_h}(h) C_{F_h}\big),$$

which for some constant $C_{\mathcal{A}_h y_h}(h) > 0$ implies

$$|\underline{\mathcal{A}}_h^{\eta+1/2} y_h^\eta|_{Y_h} \le (1 + \tau C_{\mathcal{A}_h y_h}(h)) |\mathcal{A}_h^\eta y_h^\eta|_{Y_h}. \tag{5.4}$$

Thus, we have $y_h^\eta \in E_\tau^\eta$ for $\tau < \tau_*(h)$. On the other hand, $y_h^0 \in E_\tau^0$ follows directly from $\underline{y}_h^{1/2} = y_h^0$ and (5.2).

We introduce the mapping

$$\Phi_\tau^\eta \colon E_\tau^\eta \to E_\tau^\eta, \qquad \Phi_\tau^\eta(\xi_h) = y_h^\eta + \tfrac{\tau}{2}\underline{\mathcal{A}}_h^{\eta+1/2}\xi_h + \tfrac{\tau}{2}\underline{\mathcal{F}}_h^{\eta+1/2}.$$

Since X_h is finite dimensional, there is $C_{A_h}(h) > 0$, which might deteriorate for $h \to 0$, such that

$$\|A_h\|_{\mathcal{L}(X_h)} \le C_{A_h}(h). \tag{5.5}$$

In particular, this yields that Φ_τ^η is contractive for $\tau < \tau_*(h)$.

Moreover, due to (3.2), (3.3), (3.5), and (3.8) there exists $C_{\mathcal{A}_h\Phi}(h) > 0$ with

$$|\underline{\mathcal{A}}_h^{\eta+1/2}\Phi_\tau^\eta(\xi_h)|_{Y_h} \leq |\underline{\mathcal{A}}_h^{\eta+1/2}y_h^\eta|_{Y_h} + \tau C_{\mathcal{A}_h\Phi}(h)$$

Thus, (5.4) yields $|\underline{\mathcal{A}}_h^{\eta+1/2}\Phi_\tau^\eta(\xi_h)|_{Y_h} < R_{\mathcal{A}_h}$ for $\tau < \tau_*(h)$. Since the other bounds necessary for $\Phi_\tau^\eta(\xi_h) \in E_\tau^\eta$ follow with similar arguments, this yields that Φ_τ^η is well defined.

Banach's fixed-point theorem and (5.3) imply $y_h^{\eta+1/2} \in E_\tau^\eta$, which concludes the proof. $\square$

Preliminary to the derivation of error estimates for a single step of the scheme, we derive an error recursion for the linearly implicit midpoint rule (4.3). To this end, based on the continuous solution y of (2.2) we introduce the short notation

$$\widetilde{y}^n = y(t_n), \quad \widetilde{y}^{n+1/2} = y(t_{n+1/2}), \quad \widetilde{\mathcal{A}}_{n+1/2} := \mathcal{A}(\widetilde{y}^{n+1/2}), \quad \widetilde{\mathcal{F}}_{n+1/2} := \mathcal{F}(t_{n+1/2}, \widetilde{y}^{n+1/2}), \tag{5.6}$$

such that the continuous solution satisfies

$$\widetilde{y}^{n+1} = \widetilde{y}^n + \tau\widetilde{\mathcal{A}}_{n+1/2}\widetilde{y}^{n+1/2} + \tau\widetilde{\mathcal{F}}_{n+1/2} + \delta^{n+1}. \tag{5.7}$$

This is a perturbed version of (4.3) with the additional defect δ^{n+1}.

We point out that, contrary to the notation for the fully discrete scheme (4.2), in the short notation (5.6) for the continuous solution the superscript $n + 1/2$ denotes the exact evaluation at $t_{n+1/2}$ instead of the implicit approximation of the midpoint introduced in (4.1). To take this into account in the error analysis, we further define the defect

$$\widehat{\delta}^{n+1/2} = \frac{\widetilde{y}^{n+1} + \widetilde{y}^n}{2} - \widetilde{y}^{n+1/2}. \tag{5.8}$$

Finally, we introduce the short notation

$$\widetilde{\Lambda}_h^{n+1/2} := \Lambda_h(I_h\widetilde{y}^{n+1/2}), \quad \widetilde{\mathcal{A}}_h^{n+1/2} := \mathcal{A}_h(I_h\widetilde{y}^{n+1/2}), \quad \widetilde{\mathcal{F}}_h^{n+1/2} := \mathcal{F}_h(t_{n+1/2}, I_h\widetilde{y}^{n+1/2}),$$

and the discrete errors

$$e_h^n := J_h\widetilde{y}^n - y_h^n, \qquad\qquad e_h^{n+1/2} := \frac{e_h^{n+1} + e_h^n}{2}. \tag{5.9}$$

Hence, we obtain from (4.3) and (5.7) the error recursion

$$e_h^{n+1} = e_h^n + \tau\widetilde{\mathcal{A}}_h^{n+1/2}e_h^{n+1/2} + \tau g_{h,\mathrm{LI}}^{n+1}, \tag{5.10}$$

with the right-hand side

$$\begin{aligned} g_{h,\mathrm{LI}}^{n+1} = {} & \big(\widetilde{\mathcal{A}}_h^{n+1/2} - \underline{\mathcal{A}}_h^{n+1/2}\big)y_h^{n+1/2} + \widetilde{\mathcal{F}}_h^{n+1/2} - \underline{\mathcal{F}}_h^{n+1/2} + \big(\widetilde{\Lambda}_h^{n+1/2}\big)^{-1}\Pi_h\Big(A\widehat{\delta}^{n+1/2} + \tfrac{1}{\tau}\delta^{n+1}\Big) \\ & + \big(\widetilde{\Lambda}_h^{n+1/2}\big)^{-1}\Big(\mathcal{R}_\Lambda(\widetilde{y}^{n+1/2})\big(\tfrac{1}{\tau}(\widetilde{y}^{n+1} - \widetilde{y}^n)\big) - \mathcal{R}_A\big(\tfrac{1}{2}(\widetilde{y}^{n+1} + \widetilde{y}^n)\big) - \mathcal{R}_F(t_{n+1/2}, \widetilde{y}^{n+1/2})\Big). \end{aligned} \tag{5.11}$$

In the next lemma, we provide an error estimate for one step of the linearly implicit midpoint rule. For the proof, we follow the ideas from (Hochbruck *et al.*, 2018, Lem. 5.1), where the time discretization of quasilinear wave-type equations with algebraically stable Runge–Kutta schemes is studied.

LEMMA 5.2 If the assumptions of Lemma 5.1 are satisfied for $0 \le \eta < N$ fixed, then for $\tau < \tau_0^\eta$ the error of the linearly implicit midpoint rule (4.3) satisfies the bound

$$\begin{aligned}\|e_h^{\eta+1}\|^2_{\widetilde{\Lambda}_h^{\eta+1}} \le \mathrm{e}^{C\tau}\|e_h^\eta\|^2_{\widetilde{\Lambda}_h^\eta} + \underline{C}\tau\|J_h\widetilde{y}^{\eta+1/2} - \underline{y}_h^{\eta+1/2}\|^2_{\widetilde{\Lambda}_h^{\eta+1/2}} + C\tau\Bigg(&\|(I_h - J_h)\widetilde{y}^{\eta+1/2}\|^2_{X_h} \\ &+ \sup_{s_1,s_2\in[t_\eta,t_{\eta+1}]}\|\mathcal{R}_\Lambda(y(s_1))\partial_t y(s_2)\|^2_{X_h} + \sup_{[t_\eta,t_{\eta+1}]}\|\mathcal{R}_A y\|^2_{X_h} + \sup_{[t_\eta,t_{\eta+1}]}\|\mathcal{R}_F(\cdot,y)\|^2_{X_h} \\ &+ \tau^4\Big(\sup_{[t_\eta,t_{\eta+1}]}\|\partial_t^3 y\|^2_X + \sup_{[t_\eta,t_{\eta+1}]}\|\partial_t^2 y\|^2_Y\Big)\Bigg),\end{aligned} \tag{5.12}$$

with constants $C, \underline{C} > 0$, which are independent of η, h and τ.

We point out that, as we do not exploit the specific definition of the extrapolated midpoint $\underline{y}_h^{n+1/2}$, this result is also valid for different linearized versions of the fully implicit midpoint rule (4.1). Thus, we trace the constants appearing with the linearization throughout the proof.

Proof of Lemma 5.2. First, we derive from (5.9) and (5.10)

$$\begin{aligned}\|e_h^{\eta+1}\|^2_{\widetilde{\Lambda}_h^{\eta+1/2}} - \|e_h^\eta\|^2_{\widetilde{\Lambda}_h^{\eta+1/2}} &= \left(e_h^{\eta+1} + e_h^\eta \mid e_h^{\eta+1} - e_h^\eta\right)_{\widetilde{\Lambda}_h^{\eta+1/2}} \\ &= 2\tau\left(e_h^{\eta+1/2} \mid \widetilde{\mathcal{A}}_h^{\eta+1/2} e_h^{\eta+1/2} + g_{h,\mathrm{LI}}^{\eta+1}\right)_{\widetilde{\Lambda}_h^{\eta+1/2}}.\end{aligned}$$

Hence, the dissipativity (3.7) of A_h and Young's inequality imply

$$\|e_h^{\eta+1}\|^2_{\widetilde{\Lambda}_h^{\eta+1/2}} - \|e_h^\eta\|^2_{\widetilde{\Lambda}_h^{\eta+1/2}} \le \tau\|e_h^{\eta+1/2}\|^2_{\widetilde{\Lambda}_h^{\eta+1/2}} + \tau\|g_{h,\mathrm{LI}}^{\eta+1}\|^2_{\widetilde{\Lambda}_h^{\eta+1/2}}, \tag{5.13}$$

To bound $e_h^{\eta+1/2}$, we use again (5.10), (3.7), and the Cauchy–Schwarz inequality to prove

$$\begin{aligned}\|e_h^{\eta+1/2}\|^2_{\widetilde{\Lambda}_h^{\eta+1/2}} &= \left(e_h^{\eta+1/2} \mid e_h^\eta\right)_{\widetilde{\Lambda}_h^{\eta+1/2}} + \tfrac{\tau}{2}\left(e_h^{\eta+1/2} \mid \widetilde{\mathcal{A}}_h^{\eta+1/2} e_h^{\eta+1/2} + g_{h,\mathrm{LI}}^{\eta+1}\right)_{\widetilde{\Lambda}_h^{\eta+1/2}} \\ &\le \|e_h^{\eta+1/2}\|_{\widetilde{\Lambda}_h^{\eta+1/2}}\Big(\|e_h^\eta\|_{\widetilde{\Lambda}_h^{\eta+1/2}} + \tfrac{\tau}{2}\|g_{h,\mathrm{LI}}^{\eta+1}\|_{\widetilde{\Lambda}_h^{\eta+1/2}}\Big).\end{aligned} \tag{5.14}$$

Hence, together with (5.13), the norm equivalence (3.14), and $\tau < \tau_0^\eta \le 1$ we conclude

$$\|e_h^{\eta+1}\|^2_{\widetilde{\Lambda}_h^{\eta+1}} \le \mathrm{e}^{C\tau}\|e_h^\eta\|^2_{\widetilde{\Lambda}_h^\eta} + \tfrac{3}{2}\mathrm{e}^{C'\tau}\tau\|g_{h,\mathrm{LI}}^{\eta+1}\|^2_{\widetilde{\Lambda}_h^{\eta+1/2}}. \tag{5.15}$$

In the remainder, we provide a bound for the right-hand side $g_{h,\mathrm{LI}}^{\eta+1}$. To begin with, (5.11), the triangle inequality, and $A \in \mathcal{L}(Y, X)$ imply

$$\begin{aligned}\|g_{h,\mathrm{LI}}^{\eta+1}\|_{\widetilde{\Lambda}_h^{\eta+1/2}} \le{}& \|\big(\widetilde{\mathcal{A}}_h^{\eta+1/2} - \underline{\mathcal{A}}_h^{\eta+1/2}\big)y_h^{\eta+1/2}\|_{\widetilde{\Lambda}_h^{\eta+1/2}} + \|\widetilde{\mathcal{F}}_h^{\eta+1/2} - \underline{\mathcal{F}}_h^{\eta+1/2}\|_{\widetilde{\Lambda}_h^{\eta+1/2}} \\ &+ \|\mathcal{R}_\Lambda(\widetilde{y}^{n+1/2})\big(\tfrac{1}{\tau}(\widetilde{y}^{n+1} - \widetilde{y}^n)\big)\|_{X_h} + \|\mathcal{R}_A\big(\tfrac{1}{2}(\widetilde{y}^{n+1} + \widetilde{y}^n)\big)\|_{X_h} \\ &+ \|\mathcal{R}_F(t_{n+1/2}, \widetilde{y}^{n+1/2})\|_{X_h} + \|\widehat{\delta}^{n+1/2}\|_Y + \|\tfrac{1}{\tau}\delta^{n+1}\|_X.\end{aligned} \tag{5.16}$$

For the first term, we have with (3.13), (3.11), and Lemma 5.1 the bound

$$\|\big(\widetilde{\mathcal{A}}_h^{\eta+1/2} - \underline{\mathcal{A}}_h^{\eta+1/2}\big)y_h^{\eta+1/2}\|_{\widetilde{\Lambda}_h^{\eta+1/2}} \le C_{\Lambda_h}^{\frac{1}{2}} L_{\mathcal{A}_h} R_{\mathcal{A}_h}\|I_h\widetilde{y}^{\eta+1/2} - \underline{y}_h^{\eta+1/2}\|_{X_h}.$$

Similarly, we derive for the second term with (3.13) and (3.12)

$$\|\widetilde{\mathcal{F}}_h^{\eta+1/2} - \underline{\mathcal{F}}_h^{\eta+1/2}\|_{\widetilde{\Lambda}_h^{\eta+1/2}} \le C_{\Lambda_h}^{\frac{1}{2}} L_{\mathcal{F}_h}\|I_h\widetilde{y}^{\eta+1/2} - \underline{y}_h^{\eta+1/2}\|_{X_h}.$$

Thus, due to

$$\|I_h\widetilde{y}^{\eta+1/2} - \underline{y}_h^{\eta+1/2}\|_{X_h} \le \|(I_h - J_h)\widetilde{y}^{\eta+1/2}\|_{X_h} + c_{\Lambda_h}^{-\frac{1}{2}}\|J_h\widetilde{y}^{\eta+1/2} - \underline{y}_h^{\eta+1/2}\|_{\widetilde{\Lambda}_h^{\eta+1/2}}$$

both terms are dominated by the right-hand side of (5.12) with

$$\underline{C} = \tfrac{3}{2}\mathrm{e}^{C'}C_{\Lambda_h}^{\frac{1}{2}}c_{\Lambda_h}^{-\frac{1}{2}}\big(L_{\mathcal{A}_h}R_{\mathcal{A}_h} + L_{\mathcal{F}_h}\big).$$

This is also the case for the terms depending on the three remainders $\mathcal{R}_\Lambda, \mathcal{R}_A, \mathcal{R}_F$, as we have

$$\tfrac{1}{\tau}(\widetilde{y}^{\eta+1} - \widetilde{y}^{\eta}) = \int_0^1 \partial_t y(t_\eta + s\tau)\,\mathrm{d}s.$$

Finally, for the two terms depending on the defects $\delta^{\eta+1}$ and $\widehat{\delta}^{\eta+1/2}$, we obtain with Taylor's theorem the bounds

$$\|\tfrac{1}{\tau}\delta^{\eta+1}\|_X \le C\tau^2 \sup_{[t_\eta,t_{\eta+1}]}\|\partial_t^3 y\|_X, \qquad \|\widehat{\delta}^{\eta+1/2}\|_Y \le C\tau^2 \sup_{[t_\eta,t_{\eta+1}]}\|\partial_t^2 y\|_Y. \tag{5.17}$$

Thus, the result follows from (5.15) and (5.16). □

In order to apply the error estimate from Lemma 5.2 for the error analysis of the linearly implicit midpoint rule (4.3), we further bound the error term depending on the extrapolated midpoint. More precisely, for $1 \le \eta < N$ we obtain with (3.13), (3.15), and Taylor's theorem

$$\begin{aligned}\|J_h\widetilde{y}^{\eta+1/2} - \underline{y}_h^{\eta+1/2}\|_{\widetilde{\Lambda}_h^{\eta+1/2}} &\le \tfrac{3}{2}\|J_h\widetilde{y}^{\eta} - y_h^{\eta}\|_{\widetilde{\Lambda}_h^{\eta+1/2}} + \tfrac{1}{2}\|J_h\widetilde{y}^{\eta-1} - y_h^{\eta-1}\|_{\widetilde{\Lambda}_h^{\eta+1/2}} \\ &\quad + \tfrac{1}{2}\|J_h(2\widetilde{y}^{\eta+1/2} - 3\widetilde{y}^{\eta} + \widetilde{y}^{\eta-1})\|_{\widetilde{\Lambda}_h^{\eta+1/2}} \\ &\le \tfrac{3}{2}\mathrm{e}^{C\tau}\|e_h^{\eta}\|_{\widetilde{\Lambda}_h^{\eta}} + \tfrac{1}{2}\mathrm{e}^{C\tau}\|e_h^{\eta-1}\|_{\widetilde{\Lambda}_h^{\eta-1}} + C\tau^2 \sup_{[t_{\eta-1},t_{\eta+1/2}]}\|\partial_t^2 y\|_Y.\end{aligned} \tag{5.18}$$

In particular, if the error terms depending on e_h^{η} and $e_h^{\eta-1}$ are of order τ^2, then this directly transfers to the error of the extrapolated midpoint. This is different for $\eta = 0$, as $\underline{y}_h^{1/2} = y_h^0$ yields

$$\begin{aligned}\|J_h\widetilde{y}^{1/2} - \underline{y}_h^{1/2}\|_{\widetilde{\Lambda}_h^{1/2}} &\le 2c_{\Lambda_h}^{-\frac{1}{2}}C_{J_h}\|\widetilde{y}^{1/2} - \widetilde{y}^{0}\|_Y + 2\|J_h\widetilde{y}^{0} - y_h^0\|_{\widetilde{\Lambda}_h^{1/2}} \\ &\le c_{\Lambda_h}^{-\frac{1}{2}}C_{J_h}\tau \sup_{[t_0,t_{1/2}]}\|\partial_t y\|_Y + 2\mathrm{e}^{C'\frac{\tau}{2}}\|e_h^0\|_{\widetilde{\Lambda}_h^0},\end{aligned} \tag{5.19}$$

i.e., this error contribution is at most of order τ.

To overcome this difficulty, we provide an alternative error estimate for the first step. In particular, if we apply Young's inequality with modified weights in the previous proof, we obtain for $\eta = 0$

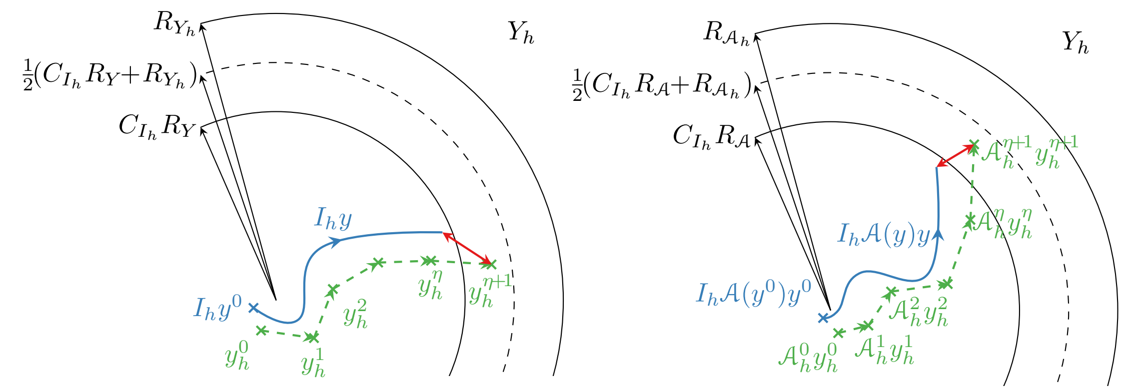
$$\|e_h^1\|_{\widetilde{\Lambda}_h^1}^2 \le C\|e_h^0\|_{\widetilde{\Lambda}_h^0}^2 + C\tau^2\|g_{h,\mathrm{LI}}^1\|_{\widetilde{\Lambda}_h^{1/2}}^2,$$

instead of (5.15). Based on a similar bound for $g_{h,\mathrm{LI}}^1$ as above and (5.19), we get the following result.

LEMMA 5.3 If the assumptions of Lemma 5.1 are satisfied for $0 \le \eta < N$ fixed, then for $\tau < \tau_0^{\eta}$ the error of the linearly implicit midpoint rule (4.3) satisfies

$$\begin{aligned}\|e_h^1\|_{\widetilde{\Lambda}_h^1}^2 \le C\|e_h^0\|_{\widetilde{\Lambda}_h^0}^2 + C\tau\bigg(&\|(I_h - J_h)\widetilde{y}^{1/2}\|_{X_h}^2 + \tau^4\Big(\sup_{[t_0,t_1]}\|\partial_t^3 y\|_X^2 + \sup_{[t_0,t_1]}\|\partial_t^2 y\|_Y^2 + \sup_{[t_0,t_{1/2}]}\|\partial_t y\|_Y^2\Big) \\ &+ \sup_{s_1,s_2\in[t_0,t_1]}\|\mathcal{R}_\Lambda(y(s_1))\partial_t y(s_2)\|_{X_h}^2 + \sup_{[t_0,t_1]}\|\mathcal{R}_A y\|_{X_h}^2 + \sup_{[t_0,t_1]}\|\mathcal{R}_F(\cdot,y)\|_{X_h}^2\bigg),\end{aligned}$$

with a constant $C > 0$ independent of h and τ.



(A) Balls for the solutions y and y_h.

(B) Balls for the differential operators $\mathcal{A}$ and $\mathcal{A}_h$ applied to the respective solution y and y_h.

FIGURE 4. Illustration of the different balls centered at the origin with the radii specified in Assumption 3.3. Additionally, the concept of the proof of Theorem 4.1 is depicted.

Using the preliminary lemmas alternately, we now prove Theorem 4.1 for the linearly implicit midpoint rule (4.3) by induction. The basic idea for the induction step $\eta \curvearrowright \eta+1$ is illustrated in Figure 4.

For $t \in [0, t_{\eta+1}]$ the interpolation $I_h y$ (blue) of the solution of (2.2) is shown in Figure 4a. Moreover, for $n \leq \eta$ the discrete approximations y_h^n (green) obtained by (4.3) are depicted. Correspondingly, the application of the differential operators $\mathcal{A}$ and $\mathcal{A}_h$ to the respective quantities are shown in Figure 4b. Due to the induction hypothesis, the discrete approximations are bounded by the intermediate radii.

Thus, Lemma 5.1 yields that both $y_h^{\eta+1}$ and $\mathcal{A}_h^{\eta+1} y_h^{\eta+1}$ exist and are contained in the balls centered at the origin with radii R_{Y_h} and $R_{\mathcal{A}_h}$, respectively. To conclude the induction step, we prove that these bounds are not sharp. More precisely, the error estimates from Lemma 5.2 and Lemma 5.3 together with the consistency from Assumption 3.4 and the step-size restriction (4.4) imply that the errors indicated by the red arrows are sufficiently small such that the approximations are also bounded by the intermediate radii. This finally proves the induction hypothesis for $\eta + 1$.

Proof of Theorem 4.1 *for the linearly implicit midpoint rule.* The proof consists of two parts. In the first part, we prove by induction that the linearly implicit midpoint rule is wellposed and that the approximations satisfy for $n = 0, \dots, N$ the bound

$$\begin{aligned} \mathrm{e}^{-Ct_n}\|e_h^n\|_{\widetilde{\Lambda}_h^n}^2 \leq C\|e_h^0\|_{\widetilde{\Lambda}_h^0}^2 &+ C\tau \sum_{r=0}^{n-1} \mathrm{e}^{-Ct_r}\|e_h^r\|_{\widetilde{\Lambda}_h^r}^2 + Ct_n \sup_{[0,t_n]} \|(I_h - J_h)y\|_{X_h}^2 \\ &+ Ct_n\tau^4\Big(\sup_{[0,t_n]}\|\partial_t y\|_Y^2 + \sup_{[0,t_n]}\|\partial_t^2 y\|_Y^2 + \sup_{[0,t_n]}\|\partial_t^3 y\|_X^2\Big) \\ &+ Ct_n\Big(\sup_{s_1,s_2\in[0,t_n]}\|\mathcal{R}_\Lambda(y(s_1))\partial_t y(s_2)\|_{X_h}^2 + \sup_{[0,t_n]}\|\mathcal{R}_A y\|_{X_h}^2 + \sup_{[0,t_n]}\|\mathcal{R}_F(\cdot,y)\|_{X_h}^2\Big). \end{aligned} \tag{5.20}$$

In the second part, we conclude the rigorous error estimate (4.5).

1. For $n = 0$, due to Assumption 3.4 there is a constant $\overline{h} > 0$ such that the initial value satisfies

$$\|y_h^0\|_{Y_h} < \tfrac{1}{2}(R_{Y_h} + C_{I_h} R_Y), \qquad |\mathcal{A}_h(y_h^0)y_h^0|_{Y_h} < \tfrac{1}{2}(R_{\mathcal{A}_h} + C_{I_h} R_{\mathcal{A}}),$$

for all $h < \overline{h}$. Hence, with $\underline{y}_h^{1/2} = y_h^0 \in B_{Y_h}(R_{Y_h})$ Lemma 5.1 and Lemma 5.3 imply wellposedness as well as (5.20) for the first step.

For the induction step, we assume that the assumptions of Lemma 5.1 and (5.20) are true up to some $n = \eta \in \{0, \ldots, N-1\}$ arbitrary but fixed. To close the induction argument, we show that this then transfers to $n = \eta + 1$.

First, due to Lemma 5.1 we get existence of

$$y_h^{\eta+1/2}, y_h^{\eta+1} \in B_{Y_h}(R_{Y_h}), \qquad \underline{\mathcal{A}}_h^{\eta+1/2} y_h^{\eta+1/2} \in B_{Y_h}(R_{\mathcal{A}_h}).$$

Thus, we obtain from (5.12) and (5.18)

$$\begin{aligned}\|e_h^{\eta+1}\|^2_{\widetilde{\Lambda}_h^{\eta+1}} \le \mathrm{e}^{C\tau}\|e_h^{\eta}\|^2_{\widetilde{\Lambda}_h^{\eta}} + C\tau\|e_h^{\eta-1}\|^2_{\widetilde{\Lambda}_h^{\eta-1}} + C\tau\Bigg(&\|(I_h - J_h)\widetilde{y}^{\,\eta+1/2}\|^2_{X_h} \\ &+ \sup_{s_1,s_2\in[t_\eta,t_{\eta+1}]} \|\mathcal{R}_\Lambda(y(s_1))\partial_t y(s_2)\|^2_{X_h} + \sup_{[t_\eta,t_{\eta+1}]}\|\mathcal{R}_A y\|^2_{X_h} + \sup_{[t_\eta,t_{\eta+1}]}\|\mathcal{R}_F(\cdot,y)\|^2_{X_h} \\ &+ \tau^4\Big(\sup_{[t_\eta,t_{\eta+1}]}\|\partial_t^3 y\|^2_X + \sup_{[t_{\eta-1},t_{\eta+1}]}\|\partial_t^2 y\|^2_Y\Big)\Bigg).\end{aligned}$$

Using the induction hypothesis to replace $\|e_h^\eta\|^2_{\widetilde{\Lambda}_h^\eta}$, this yields (5.20) for $n = \eta + 1$.

To conclude this part of the proof, we have to show that the step-size restriction (4.4) is sufficient to ensure that the assumptions of Lemma 5.1 also valid for $n = \eta+1$. Note that we only focus on the corresponding estimates for $y_h^{\eta+1}$ and $\mathcal{A}_h^{\eta+1} y_h^{\eta+1}$ here, as the bound for $\underline{y}_h^{\eta+3/2}$ follows with similar arguments. For all details, we refer to the proof of (Maier, 2020, Thm. 7.3).

With the discrete Gronwall inequality, (5.20) implies

$$\begin{aligned}\|e_h^{\eta+1}\|_{\widetilde{\Lambda}_h^{\eta+1}} \le C(1 + t_{\eta+1})\mathrm{e}^{Ct_{\eta+1}}\Bigg(&\|e_h^0\|_{\widetilde{\Lambda}_h^0} + \sup_{[0,t_{\eta+1}]}\|(I_h - J_h)y\|_X \\ &+ \tau^2\Big(\sup_{[0,t_{\eta+1}]}\|\partial_t y\|_Y + \sup_{[0,t_{\eta+1}]}\|\partial_t^2 y\|_Y + \sup_{[0,t_{\eta+1}]}\|\partial_t^3 y\|_X\Big) \\ &+ \sup_{s_1,s_2\in[0,t_{\eta+1}]}\|\mathcal{R}_\Lambda(y(s_1))\partial_t y(s_2)\|_X + \sup_{[0,t_{\eta+1}]}\|\mathcal{R}_A y\|_X + \sup_{[0,t_{\eta+1}]}\|\mathcal{R}_F(\cdot,y)\|_X\Bigg).\end{aligned} \tag{5.21}$$

Since this bound also holds for $t_{\eta+1}$ being replaced by T in the right-hand side, we get from Assumption 3.4

$$C_{\max}(h)\|e_h^{\eta+1}\|_{X_h} \to 0,$$

uniformly in η for $\tau, h \to 0$ satisfying the step-size restriction (4.4). Thus, (Hochbruck & Maier, 2021, Lem. 4.6) directly yields the existence of $\tau_0 > 0$ and $h_0 \le \overline{h}$ independent of η such that

$$\|y_h^{\eta+1}\|_{Y_h} < \tfrac{1}{2}(R_{Y_h} + C_{I_h} R_Y), \qquad |\mathcal{A}_h^{\eta+1} y_h^{\eta+1}|_{Y_h} < \tfrac{1}{2}(R_{\mathcal{A}_h} + C_{I_h} R_{\mathcal{A}})$$

holds for all $\tau < \tau_0$ and $h < h_0$ satisfying the step-size restriction (4.4).

2. To prove the error estimate (4.5), we first split the error into
$$\|y(t_n) - y_h^n\|_X \leq \|(\mathrm{Id} - J_h) y(t_n)\|_X + \|e_h^n\|_X,$$
where the first term already appears in the right-hand side of (4.5). The bound for the second term follows from the norm equivalence (3.13) and the bound for the discrete error (5.21). Finally, Assumption 3.4 yields (4.6). □

6. Error analysis for the fully implicit midpoint rule

Based on the results obtained in the previous section for the linearized scheme we now focus on the analysis of the fully implicit midpoint rule (4.1), which is more involved. In particular, a very important detail in the wellposedness result in Lemma 5.1 is that we only rely on the step size restriction for the derivation of the bounds, but not for the existence of the next approximation. This allows us to relax the step size restriction in the overall proof by induction, since the error estimates yield improved bounds. However, for the fully implicit midpoint rule this approach is not feasible, since existence of the next approximation is intertwined with the required bounds. Hence, if we apply Banach's fixed-point theorem to derive existence and the bounds simultaneously, relaxing the severe step size restriction using the error estimates is not an option.

To circumvent these analytical difficulties, we approximate the fully implicit scheme using a fixed-point iteration in every time step. This yields a sequence of linear problems, for which the results from the previous section are directly applicable. More precisely, for $\eta \in \{0, \dots, N-1\}$ we consider the sequence $\big(y_h^{\eta+1,k}\big)_{k\in\mathbb{N}_0}$ given by
$$y_h^{\eta+1,k+1} = y_h^\eta + \tau \mathcal{A}_h^{\eta+1/2,k} y_h^{\eta+1/2,k+1} + \tau \mathcal{F}_h^{\eta+1/2,k}, \qquad k \geq 0, \tag{6.1}$$
and $y_h^{\eta+1,0} = 2\underline{y}_h^{\eta+1/2} - y_h^\eta$. Here, $\underline{y}_h^{\eta+1/2}$ again denotes the extrapolated midpoint introduced in (4.3). Moreover, we use the notation
$$y_h^{\eta+1/2,k} = \frac{y_h^{\eta+1,k} + y_h^\eta}{2}, \qquad \mathcal{A}_h^{\eta+1/2,k} := \mathcal{A}_h(y_h^{\eta+1/2,k}), \qquad \mathcal{F}_h^{\eta+1/2,k} := \mathcal{F}_h(t_n, y_h^{\eta+1/2,k}).$$
Note that if the sequence given by (6.1) has a fixed point, this is the next approximation $y_h^{\eta+1}$ of the fully implicit midpoint rule.

Overall, we proceed as follows.

Roadmap for the analysis of the fully implicit midpoint rule.

- (W) Assumption 2.2 yields wellposedness of the continuous quasilinear Cauchy problem (2.2), i.e., for all $T < t^*(y^0)$ the bound $\|y\|_Y < R_Y$ holds uniformly on $[0, T]$.
- $(W_h^{\eta,k})$ Based on the assumption that the first $\eta \geq 0$ steps of the fully implicit midpoint rule (4.1) are well defined, we prove in Lemma 6.1 that there exists a range of time steps $(0, \tau_0^\eta)$ such that all approximations $y_h^{\eta+1,k}$ of the fixed-point iteration (6.1) exist with $\|y_h^{\eta+1,k}\|_{Y_h} < R_{Y_h}$. To do so, we apply the corresponding result for the linearly implicit midpoint rule from Lemma 5.1.
- $(E_h^{\eta,k})$ Using (W), $(W_h^{\eta,k})$, and Lemma 5.2, we bound the errors $e_h^{\eta+1,k} = J_h y(t_{\eta+1}) - y_h^{\eta+1,k}$ of the fixed-point iteration by the errors of the previous approximations $e_h^{\eta+1,k-1}$ and e_h^η in Lemma 6.2.
- $(C_h^{\eta,k})$ Based on an inverse estimate, the consistency from Assumption 3.4, and convergence of the previous errors e_h^η, $e_h^{\eta-1}$, we show in Lemma 6.3 that the step-size restriction (4.4) is sufficient to ensure improved bounds on all iterates.
- (L_h^η) In Lemma 6.4, we show that (6.1) defines a Cauchy sequence in X_h with limit $y_h^{\eta+1}$.

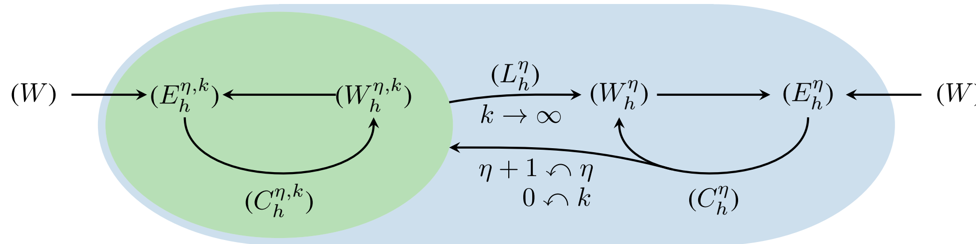


FIGURE 5. Relations between the main steps for the analysis of the full discretization with the fully implicit midpoint rule.

(W_h^η) Since we further obtain bounds for $y_h^{\eta+1}$ in Y_h under the step-size restriction (4.4), this yields wellposedness of the next step of the fully implicit midpoint rule.

(E_h^η) Using (W), $(W_h^{\eta,k})$, and Lemma 5.2, we bound the error $e_h^{\eta+1} = J_h y(t_{\eta+1}) - y_h^{\eta+1}$ of next step of the fully implicit midpoint rule based on e_h^η in Lemma 6.5.

(C_h^η) As for the linearly implicit midpoint rule, we conclude from an inverse estimate, the consistency from Assumption 3.4, and a uniform bound for the previous errors e_h^n with $n \le \eta$ that the current step $\mathbf{FIM}_{\eta+1}$ is wellposed under the step-size restriction (4.4) and we may proceed with $(W_h^{\eta+1})$.

Overall, we show Theorem 4.1 for the fully implicit midpoint rule by induction, as we alternately apply (W_h^η), (E_h^η), and (C_h^η). To prove (W_h^η), we approximate each step of the fully implicit midpoint rule by a fixed-point iteration, for which we again show wellposedness and an error esitmate, using $(W_h^{\eta,k})$, $(E_h^{\eta,k})$, and $(C_h^{\eta,k})$. Finally, we conclude convergence of the fixed-point iteration in (L_h^η). This approach is also illustrated in Figure 5, with the blue ellipse indicating the induction proof for the fully implicit midpoint rule. The analysis of the fixed-point iteration is characterized by the embedded green ellipse.

Again, we emphasize that the analysis of the fully implicit midpoint rule for nonconforming space discretizations is presented in detail in (Maier, 2020, Sec. 7.1.2).

Before we start with the analysis, we briefly discuss the relation to Banach's fixed-point theorem.

REMARK 6.1 With the introduction of the fixed-point iteration, our approach resembles the proof of Banach's fixed-point theorem. However, we emphasize that the theorem is not directly applicable here, as it does not allow for a differentiated treatment of the weaker space X_h and the stronger space Y_h. In particular, to apply the theorem we would require $y_h^{\eta+1,k} \mapsto y_h^{\eta+1,k+1}$ to be a contractive self-mapping on $B_{Y_h}(R_{Y_h})$. As shown in the proof of Lemma 5.1, this is only possible under a severe restriction on the time step, which in this case can not be relaxed afterwards as the fully implicit midpoint rule is a nonlinear scheme.

We now prove wellposedness of a single step of the fixed-point iteration. Note that we omit $k = 0$ here, as in this case the fixed-point iteration coincides with the linearly implicit midpoint rule and hence Lemma 5.1 is applicable.

LEMMA 6.1 Let Assumption 3.3 be satisfied. For $0 \le \eta < N$ fixed we assume

$$\|y_h^\eta\|_{Y_h} < \tfrac{1}{2}(R_{Y_h} + C_{I_h} R_Y), \qquad |\mathcal{A}_h^\eta y_h^\eta|_{Y_h} < \tfrac{1}{2}(R_{\mathcal{A}_h} + C_{I_h} R_{\mathcal{A}}), \qquad \|y_h^{\eta+1/2,1}\|_{Y_h} < R_{Y_h}.$$

Then, the fixed-point iteration (6.1) has a unique solution $y_h^{\eta+1,k+1} \in X_h$. Furthermore, there is a constant $\tau_0^\eta > 0$, which may depend on the space discretization parameter h, such that for all $\tau < \tau_0^\eta$ we have

$$\|y_h^{\eta+1/2,k+1}\|_{Y_h}, \|y_h^{\eta+1,k+1}\|_{Y_h} < R_{Y_h}, \qquad |\mathcal{A}_h^{\eta+1/2,k} y_h^{\eta+1/2,k+1}|_{Y_h} < R_{\mathcal{A}_h}. \tag{6.2}$$

Proof. The result follows from Lemma 5.1 by induction over $k \in \mathbb{N}$, with $y_h^{\eta+1/2,k}$ and $y_h^{\eta+1,k+1}$ instead of $\underline{y}_h^{\eta+1/2}$ and $y_h^{\eta+1}$, respectively. $\square$

With the notation

$$e_h^{\eta+1,k} := J_h \widetilde{y}^{\,\eta+1} - y_h^{\eta+1,k}, \qquad e_h^{\eta+1/2,k} := \frac{e_h^{\eta+1,k} + e_h^\eta}{2},$$

we derive as in (5.10) from (5.7) and (6.1) the error equation for the fixed-point iteration

$$e_h^{\eta+1,k+1} = e_h^\eta + \tau \widetilde{\mathcal{A}}_h^{\eta+1/2} e_h^{\eta+1/2,k+1} + \tau g_{h,\mathrm{FI}}^{\eta+1,k+1}, \qquad k \geq 0,$$

with right-hand side

$$\begin{aligned} g_{h,\mathrm{FI}}^{\eta+1,k+1} = {} & \big(\widetilde{\mathcal{A}}_h^{n+1/2} - \mathcal{A}_h^{n+1/2,k}\big) y_h^{n+1/2,k+1} + \widetilde{\mathcal{F}}_h^{n+1/2} - \mathcal{F}_h^{n+1/2,k} + \big(\widetilde{\Lambda}_h^{n+1/2}\big)^{-1} \Pi_h \Big(A \widehat{\delta}^{n+1/2} + \tfrac{1}{\tau}\delta^{n+1}\Big) \\ & + \big(\widetilde{\Lambda}_h^{n+1/2}\big)^{-1} \Big(\mathcal{R}_\Lambda(\widetilde{y}^{\,n+1/2})\big(\tfrac{1}{\tau}(\widetilde{y}^{\,n+1} - \widetilde{y}^{\,n})\big) - \mathcal{R}_A\big(\tfrac{1}{2}(\widetilde{y}^{\,n+1} + \widetilde{y}^{\,n})\big) - \mathcal{R}_F(t_{n+1/2}, \widetilde{y}^{\,n+1/2})\Big). \end{aligned}$$

In the next lemma we provide an error recursion for one step of the fixed-point iteration.

LEMMA 6.2 If the assumptions of Lemma 6.1 are satisfied for $0 \leq \eta < N$ fixed, then for $k \in \mathbb{N}$ and $\tau < \tau_0^\eta$ the error of the fixed-point iteration (6.1) satisfies the bound

$$\begin{aligned} \|e_h^{\eta+1,k+1}\|^2_{\widetilde{\Lambda}_h^{\eta+1}} \leq {} & \mathrm{e}^{C\tau} \|e_h^\eta\|^2_{\widetilde{\Lambda}_h^\eta} + \tfrac{1}{2}\underline{C}\tau \|e_h^{\eta+1,k}\|^2_{\widetilde{\Lambda}_h^{\eta+1}} + C\tau \Bigg(\|(I_h - J_h)\widetilde{y}^{\,\eta+1/2}\|^2_{X_h} \\ & + \sup_{s_1,s_2 \in [t_\eta, t_{\eta+1}]} \|\mathcal{R}_\Lambda(y(s_1))\partial_t y(s_2)\|^2_{X_h} + \sup_{[t_\eta,t_{\eta+1}]} \|\mathcal{R}_A y\|^2_{X_h} + \sup_{[t_\eta,t_{\eta+1}]} \|\mathcal{R}_F(\cdot, y)\|^2_{X_h} \\ & + \tau^4 \Big(\sup_{[t_\eta,t_{\eta+1}]} \|\partial_t^3 y\|^2_X + \sup_{[t_\eta,t_{\eta+1}]} \|\partial_t^2 y\|^2_Y \Big) \Bigg), \end{aligned}$$

with constants $C, \underline{C} > 0$, which are independent of η, k, h, and τ.

Proof. With similar arguments as in (5.18), we obtain

$$\begin{aligned} \|J_h \widetilde{y}^{\,\eta+1/2} - y_h^{\eta+1/2,k}\|_{\widetilde{\Lambda}_h^{\eta+1/2}} & \leq \tfrac{1}{2}\|J_h \widetilde{y}^{\,\eta+1} - y_h^{\eta+1,k}\|_{\widetilde{\Lambda}_h^{\eta+1/2}} + \tfrac{1}{2}\|J_h \widetilde{y}^{\,\eta} - y_h^\eta\|_{\widetilde{\Lambda}_h^{\eta+1/2}} \\ & \quad + \tfrac{1}{2}\|J_h(2\widetilde{y}^{\,\eta+1/2} - \widetilde{y}^{\,\eta+1} - \widetilde{y}^{\,\eta})\|_{\widetilde{\Lambda}_h^{\eta+1/2}} \\ & \leq \tfrac{1}{2}\|e_h^{\eta+1,k}\|_{\widetilde{\Lambda}_h^{\eta+1/2}} + \tfrac{1}{2}\|e_h^\eta\|_{\widetilde{\Lambda}_h^{\eta+1/2}} + C\tau^2 \sup_{[t_\eta,t_{\eta+1}]} \|\partial_t^2 y\|_Y. \end{aligned}$$

Thus, using Lemma 5.2 with the same replacements as in the previous proof, i.e., with $y_h^{\eta+1/2,k}$ and $y_h^{\eta+1,k+1}$ instead of $\underline{y}_h^{\eta+1/2}$ and $y_h^{\eta+1}$, respectively, yields the result. $\square$

Based on this error estimate, we now improve the wellposedness result from Lemma 6.1 for the fixed-point iteration for $0 \leq \eta < N$ fixed. To do so, we rely on wellposedness and convergence for the previous steps of the fully implicit midpoint rule. In particular, we assume for some $n \leq \eta$ that the errors e_h^n of the previous approximations obtained by the fully implicit midpoint rule (4.1) satisfy

$$C_{\max}(h)\|e_h^n\|_{X_h} \to 0 \tag{6.3}$$

uniformly in n for $h, \tau \to 0$ under the step-size restriction (4.4), where $C_{\max}(h)$ is given in (3.17). At the end of this section, we close this argument in the proof by induction for Theorem 4.1.

LEMMA 6.3 Let Assumption 3.4 and the assumptions of Lemma 5.1 be satisfied for $0 \leq \eta < N$ fixed. If (6.3) holds for $n \in \big\{\max\{0, \eta-1\}, \eta\big\}$, then there exist $\tau_0^\eta, h_0^\eta > 0$ such that the fixed-point iteration (6.1) is wellposed for all $h < h_0^\eta$ and $\tau < \tau_0^\eta$ under the step-size restriction (4.4). In particular, for $k \in \mathbb{N}$ the iterates satisfy

$$\|y_h^{\eta+1,k}\|_{Y_h} < \tfrac{1}{2}(R_{Y_h} + C_{I_h} R_Y), \qquad |\mathcal{A}_h^{\eta+1,k} y_h^{\eta+1,k}|_{Y_h} < \tfrac{1}{2}(R_{\mathcal{A}_h} + C_{I_h} R_{\mathcal{A}}). \tag{6.4}$$

Proof. Let $0 < \eta < N$ be fixed. For $\tau_0^\eta, h_0^\eta > 0$ fixed at the end of this proof, we define $\mathcal{E}^\eta\big(\tau_0^\eta, h_0^\eta\big) > 0$ as the minimal constant such that

$$\begin{aligned}\mathcal{E}^\eta\big(\tau_0^\eta, h_0^\eta\big) \geq C\|e_h^\eta\|^2_{\widetilde{\Lambda}_h^\eta} + C\tau\Bigg(&\|(I_h - J_h)\widetilde{y}^{\,\eta+1/2}\|^2_{X_h} + \tau^4\Big(\sup_{[t_\eta,t_{\eta+1}]}\|\partial_t^3 y\|_X^2 + \sup_{[t_\eta,t_{\eta+1}]}\|\partial_t^2 y\|_Y^2\Big) \\ &+ \sup_{s_1,s_2\in[t_\eta,t_{\eta+1}]}\|\mathcal{R}_\Lambda(y(s_1))\partial_t y(s_2)\|^2_{X_h} + \sup_{[t_\eta,t_{\eta+1}]}\|\mathcal{R}_A y\|^2_{X_h} + \sup_{[t_\eta,t_{\eta+1}]}\|\mathcal{R}_F(\cdot,y)\|^2_{X_h}\Bigg)\end{aligned}$$

holds for all $\tau < \tau_0^\eta$ and $h < h_0^\eta$ under the step-size restriction (4.4). This constant is well defined, since Assumption 3.4 and (6.3) imply

$$C_{\max}(h)\mathcal{E}^\eta\big(\tau_0^\eta, h_0^\eta\big) \to 0, \tag{6.5}$$

for $\tau_0^\eta, h_0^\eta \to 0$.

We first prove that all iterates satisfy the error estimate

$$\|e_h^{\eta+1,k}\|^2_{\widetilde{\Lambda}_h^{\eta+1}} \leq \big(\underline{C}\tau\big)^k\|J_h\widetilde{y}^{\,\eta+1/2} - \underline{y}_h^{\eta+1/2}\|^2 + \mathcal{E}^\eta\big(\tau_0^\eta, h_0^\eta\big)\sum_{i=1}^{k}\big(\underline{C}\tau\big)^{i-1}, \qquad k \in \mathbb{N}. \tag{6.6}$$

For $k = 1$, this bound follows directly from Lemma 5.2, as the fixed-point iteration (6.1) coincides with the linearly implicit midpoint rule (4.3) with a modified initial value. Moreover, Lemma 6.2 implies

$$\|e_h^{\eta+1,k+1}\|^2_{\widetilde{\Lambda}_h^{\eta+1}} \leq \underline{C}\tau\|e_h^{\eta+1,k}\|^2_{\widetilde{\Lambda}_h^{\eta+1}} + \mathcal{E}^\eta\big(\tau_0^\eta, h_0^\eta\big).$$

Thus, (6.6) follows with a proof by induction over $k \in \mathbb{N}$.

Since $\underline{C}$ is in particular independent of h, τ, and η, we get from (6.6) for $\tau < \frac{1}{\underline{C}}$

$$\|e_h^{\eta+1,k}\|^2_{\widetilde{\Lambda}_h^{\eta+1}} \leq \big(\underline{C}\tau\big)^k\|J_h\widetilde{y}^{\,\eta+1/2} - \underline{y}_h^{\eta+1/2}\|^2 + \frac{\mathcal{E}^\eta\big(\tau_0^\eta, h_0^\eta\big)}{1 - \underline{C}'\tau}.$$

Due to (5.18), (6.3), (6.5), and the norm equivalence (3.13) the assumptions of (Hochbruck & Maier, 2021, Lem. 4.6) are satisfied. This proves existence of $\tau_0^\eta < \frac{1}{\underline{C}}$ and $h_0^\eta > 0$ such that (6.4) holds.

Finally, as (6.4) follows for $\eta = 0$ with similar arguments using Lemma 5.3 instead of Lemma 5.2, this concludes the proof. □

We now turn towards the analysis of the fully implicit midpoint rule (4.1). To prove wellposedness, we show that the fixed-point iteration (6.1) is convergent in X_h. Moreover, we provide the necessary bounds for the limit in Y_h. This is illustrated in Figure 6, where the interpolation $I_h y$ (blue) of the solution of (2.2) and the last approximation y_h^η (green) of the fully implicit midpoint rule are depicted. In Lemma 6.4, we prove that the iterates of the fixed-point iterations $y_h^{\eta+1,k}$ (purple) converge to the limit $y_h^{\eta+1}$ (green), which is the next approximation of the fully implicit midpoint rule. Subsequently, in Lemma 6.5, we extend the error estimate for the iterates, indicated by the orange arrow, to an error estimate for the fully implicit midpoint rule (red). Finally, as indicated by the dashed green line on the right, we might proceed with the next step of the induction proof for Theorem 4.1.

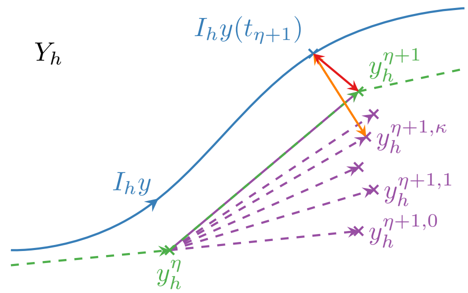


FIGURE 6. Visualization of the fixed-point iteration.

LEMMA 6.4 For $0 \le \eta < N$ fixed let the assumptions of Lemma 6.3 be satisfied. Then, there are $\tau_0^\eta, h_0^\eta > 0$ such that for all $h < h_0^\eta$ and $\tau < \tau_0^\eta$ under the step-size restriction (4.4) the sequence defined by (6.1) converges in X_h to the limit $y_h^{\eta+1} \in X_h$. Moreover, we have

$$\|y_h^{\eta+1}\|_{Y_h} < R_{Y_h}, \qquad |\mathcal{A}_h^{\eta+1} y_h^{\eta+1}|_{Y_h} < R_{\mathcal{A}_h}. \tag{6.7}$$

Proof. In the first part of the proof, we prove that (6.1) yields a Cauchy sequence in X_h. To this end, we get from (6.1) for $k \in \mathbb{N}$

$$\begin{aligned} &y_h^{\eta+1,k+1} - y_h^{\eta+1,k} - \tfrac{\tau}{2}\mathcal{A}_h^{\eta+1/2,k}\big(y_h^{\eta+1,k+1} - y_h^{\eta+1,k}\big) \\ &\quad = \tau\big(\mathcal{A}_h^{\eta+1/2,k} - \mathcal{A}_h^{\eta+1/2,k-1}\big) y_h^{\eta+1/2,k} + \tau\big(\mathcal{F}_h^{\eta+1/2,k} - \mathcal{F}_h^{\eta+1/2,k-1}\big). \end{aligned}$$

Thus, the weighted inner product with $y_h^{\eta+1,k+1} - y_h^{\eta+1,k}$ implies due to (3.7) and the Cauchy–Schwarz inequality

$$\begin{aligned} \|y_h^{\eta+1,k+1} - y_h^{\eta+1,k}\|_{\Lambda_h(y_h^{\eta+1/2,k+1})} &\le \tau \big\| \big(\mathcal{A}_h^{\eta+1/2,k} - \mathcal{A}_h^{\eta+1/2,k-1}\big) y_h^{\eta+1/2,k} \big\|_{\Lambda_h(y_h^{\eta+1/2,k+1})} \\ &\quad + \tau \|\mathcal{F}_h^{\eta+1/2,k} - \mathcal{F}_h^{\eta+1/2,k-1}\|_{\Lambda_h(y_h^{\eta+1/2,k+1})}. \end{aligned}$$

With (3.11), (3.12), (3.13), and (6.2), we deduce the existence of constants $\tau_0^\eta > 0$ and $\varepsilon_c \in (0,1)$ such that

$$\begin{aligned} \|y_h^{\eta+1,k+1} - y_h^{\eta+1,k}\|_{X_h} &\le \tau c_{\Lambda_h}^{-\frac12} C_{\Lambda_h}^{\frac12} (L_{\mathcal{A}_h} R_{\mathcal{A}_h} + L_{\mathcal{F}_h}) \|y_h^{\eta+1,k} - y_h^{\eta+1,k-1}\|_{X_h} \\ &\le \varepsilon_{\mathrm{c}} \|y_h^{\eta+1,k} - y_h^{\eta+1,k-1}\|_{X_h} \\ &\le \varepsilon_{\mathrm{c}}^k \|y_h^{\eta+1,1} - y_h^{\eta+1,0}\|_{X_h}, \end{aligned}$$

for all $\tau < \tau_0^\eta$, where we used the same argument iteratively in the last step. Thus, we conclude that $\big(y_h^{\eta+1,k}\big)_{k\in\mathbb{N}}$ is a Cauchy sequence in X_h, since for $\ell > m \ge 1$ we have

$$\begin{aligned} \|y_h^{\eta+1,\ell} - y_h^{\eta+1,m}\|_{X_h} &\le \|y_h^{\eta+1,\ell} - y_h^{\eta+1,\ell-1}\|_{X_h} + \cdots + \|y_h^{\eta+1,m+1} - y_h^{\eta+1,m}\|_{X_h} \\ &\le \varepsilon_{\mathrm{c}}^m \big(\varepsilon_{\mathrm{c}}^{\ell-m-1} + \cdots + \varepsilon_{\mathrm{c}} + 1\big) \|y_h^{\eta+1,1} - y_h^{\eta+1,0}\|_{X_h} \\ &\le \frac{\varepsilon_{\mathrm{c}}^m}{1-\varepsilon_{\mathrm{c}}} \|y_h^{\eta+1,1} - y_h^{\eta+1,0}\|_{X_h}. \end{aligned}$$

As X_h is a complete space, the Cauchy sequence converges to the limit $y_h^{\eta+1} \in X_h$.

In the second part of the proof, we focus on (6.7). First, there is k_0, which may depend on τ, h, and η such that

$$C_{Y_h,X_h}(h)\|y_h^{\eta+1} - y_h^{\eta+1,k}\|_{X_h} \leq \tfrac{1}{2}\big(R_{Y_h} - C_{I_h}R_Y\big),$$

for all $k \geq k_0$. Thus, (3.3) and (6.4) yield

$$\|y_h^{\eta+1}\|_{Y_h} \leq C_{Y_h,X_h}(h)\|y_h^{\eta+1} - y_h^{\eta+1,k}\|_{X_h} + \|y_h^{\eta+1,k}\|_{Y_h} < R_{Y_h}.$$

Since the second bound in (6.7) follows with similar arguments from (3.11) and (5.5), this concludes the proof. □

Moreover, we provide an error bound for a single step of the fully implicit midpoint rule. To do so, we first derive correspondingly to (5.10) and (5.11) the error recursion

$$e_h^{n+1} = e_h^n + \tau\widetilde{\mathcal{A}}_h^{n+1/2}e_h^{n+1/2} + \tau g_{h,\mathrm{FI}}^{n+1},$$

with the right-hand side

$$\begin{aligned} g_{h,\mathrm{FI}}^{\eta+1} &= \big(\widetilde{\mathcal{A}}_h^{n+1/2} - \mathcal{A}_h^{n+1/2}\big)y_h^{n+1/2} + \widetilde{\mathcal{F}}_h^{n+1/2} - \mathcal{F}_h^{n+1/2} + \big(\widetilde{\Lambda}_h^{n+1/2}\big)^{-1}\Pi_h\Big(A\widehat{\delta}^{n+1/2} + \tfrac{1}{\tau}\delta^{n+1}\Big) \\ &\quad + \big(\widetilde{\Lambda}_h^{n+1/2}\big)^{-1}\Big(\mathcal{R}_\Lambda(\widetilde{y}^{n+1/2})\big(\tfrac{1}{\tau}(\widetilde{y}^{n+1} - \widetilde{y}^n)\big) - \mathcal{R}_A\big(\tfrac{1}{2}(\widetilde{y}^{n+1} + \widetilde{y}^n)\big) - \mathcal{R}_F(t_{n+1/2}, \widetilde{y}^{n+1/2})\Big). \end{aligned} \tag{6.8}$$

LEMMA 6.5 If for $0 \leq \eta < N$ fixed the assumptions of Lemma 6.4 are true, then there exist $\tau_0^\eta, h_0^\eta > 0$ such that for all $h < h_0^\eta$ and $\tau < \tau_0^\eta$ under the step-size restriction (4.4) the error of the fully implicit midpoint rule (4.1) satisfies

$$\begin{aligned} \|e_h^{\eta+1}\|_{\widetilde{\Lambda}_h^{\eta+1}}^2 &\leq \mathrm{e}^{C\tau}\|e_h^\eta\|_{\widetilde{\Lambda}_h^\eta}^2 + C\tau\Bigg(\|(I_h - J_h)\widetilde{y}^{\eta+1/2}\|_{X_h}^2 + \tau^4\Big(\sup_{[t_\eta,t_{\eta+1}]}\|\partial_t^3 y\|_X^2 + \sup_{[t_\eta,t_{\eta+1}]}\|\partial_t^2 y\|_Y^2\Big) \\ &\quad + \sup_{[t_\eta,t_{\eta+1}]}\|\mathcal{R}_\Lambda(y)\partial_t y\|_{X_h}^2 + \sup_{[t_\eta,t_{\eta+1}]}\|\mathcal{R}_A y\|_{X_h}^2 + \sup_{[t_\eta,t_{\eta+1}]}\|\mathcal{R}_F(\cdot,y)\|_{X_h}^2\Bigg). \end{aligned}$$

Proof. We mainly apply the same ideas as in the proof of Lemma 5.2 with $g_{h,\mathrm{FI}}^{\eta+1}$ instead of $g_{h,\mathrm{LI}}^{\eta+1}$. For the right-hand side (6.8), this yields

$$\begin{aligned} \|g_{h,\mathrm{FI}}^{\eta+1}\|_{\widetilde{\Lambda}_h^{\eta+1/2}} &\leq \underline{C}'\|J_h\widetilde{y}^{\eta+1/2} - y_h^{\eta+1/2}\|_{\widetilde{\Lambda}_h^{\eta+1/2}} + C\Big(\|(I_h - J_h)\widetilde{y}^{\eta+1/2}\|_{X_h} + \|\widehat{\delta}^{\eta+1/2}\|_Y + \|\tfrac{1}{\tau}\delta^{\eta+1}\|_X \\ &\quad + \sup_{[t_\eta,t_{\eta+1}]}\|\mathcal{R}_\Lambda(\widetilde{y}^{\eta+1/2})\partial_t y\|_{X_h} + \|\mathcal{R}_A(\widetilde{y}^{\eta+1} + \widetilde{y}^\eta)\|_{X_h} + \|\mathcal{R}_F(t_{\eta+1/2}, \widetilde{y}^{\eta+1/2})\|_{X_h}\Big), \end{aligned}$$

with $\underline{C}' = \frac{2}{3}\mathrm{e}^{-C'}\underline{C}$. Furthermore, we derive with (3.13), (3.15), (5.8), and (5.14) the bound

$$\|J_h\widetilde{y}^{\eta+1/2} - y_h^{\eta+1/2}\|_{\widetilde{\Lambda}_h^{\eta+1/2}} \leq C_{\Lambda_h}^{\frac{1}{2}}C_{J_h}\|\widehat{\delta}^{\eta+1/2}\|_Y + \|e_h^\eta\|_{\widetilde{\Lambda}_h^{\eta+1/2}} + \tfrac{\tau}{2}\|g_{h,\mathrm{FI}}^{\eta+1}\|_{\widetilde{\Lambda}_h^{\eta+1/2}}.$$

Due to $\tau_0^\eta < \frac{1}{\underline{C}} < \frac{1}{\underline{C}'}$, this implies

$$\begin{aligned} \|g_{h,\mathrm{FI}}^{\eta+1}\|_{\widetilde{\Lambda}_h^{\eta+1/2}} &\leq (1 - \tfrac{\tau}{2}\underline{C}')^{-1}C\big(\|e_h^\eta\|_{\widetilde{\Lambda}_h^{\eta+1/2}} + \|(I_h - J_h)\widetilde{y}^{\eta+1/2}\|_{X_h} + \sup_{[t_\eta,t_{\eta+1}]}\|\mathcal{R}_\Lambda(\widetilde{y}^{\eta+1/2})\partial_t y\|_{X_h} \\ &\quad + \|\mathcal{R}_A\widetilde{y}^{\eta+1/2}\|_{X_h} + \|\mathcal{R}_F(t_{\eta+1/2}, \widetilde{y}^{\eta+1/2})\|_{X_h} + \|\widehat{\delta}^{n+1/2}\|_Y + \|\tfrac{1}{\tau}\delta^{\eta+1}\|_{X_h}\big), \end{aligned}$$

which together with (5.15) and (5.17) yields the result. □

We now conclude the main result for the fully implicit midpoint rule (4.1).

Proof of Theorem 4.1 *for the fully implicit midpoint rule.* As for the linearly implicit midpoint rule at the end of Section 5, we prove the result by induction, i.e., we alternately use Lemma 6.4 and Lemma 6.5 to prove existence of the next approximation and the error bound (5.20). Finally, we conclude that the step-size restriction (4.4) is sufficient to ensure that Lemma 6.4 is also applicable for the next step. $\square$

7. Application of the abstract result

Finally, we derive more specific bounds for the full discretization of two important classes of applications which fit into our abstract framework. More precisely, we first study the Maxwell equations with Kerr nonlinearity and subsequently discuss the full discretization of the Westervelt equation. We point out that the full discretization of these examples is also discussed in detail in Maier (2020), where Sections 3.3 and 8.2 are devoted to the Maxwell equations and Sections 3.2 and 8.1 to the Westervelt equation, respectively. Moreover, the corresponding space discretization of these examples is studied in (Hochbruck & Maier, 2021, Sec. 6). Thus, we only briefly discuss the space discretization and focus on the full discretization here.

7.1. **Maxwell equations.** We consider the Maxwell equations with Kerr nonlinearity, which state that the magnetic and electric fields $\mathcal{H}, \mathcal{E} \colon [0,T] \times \Omega \to \mathbb{R}$ satisfy

$$\begin{cases} \partial_t \mathcal{H} = -\nabla\times \mathcal{E}, & \text{on } [0,T]\times\Omega, \\ \big((1+|\mathcal{E}|^2\chi)\operatorname{Id} + 2(\mathcal{E}\otimes\mathcal{E})\chi\big)\partial_t\mathcal{E} = \nabla\times\mathcal{H}, & \text{on } [0,T]\times\Omega, \\ \mathcal{H}(0)=\mathcal{H}_0, \qquad \mathcal{E}(0)=\mathcal{E}_0 & \text{on } \Omega, \end{cases} \tag{7.1}$$

on a finite time interval $[0,T]$ and a bounded domain $\Omega \subset \mathbb{R}^3$, with initial values $\mathcal{H}_0, \mathcal{E}_0 \colon \Omega \to \mathbb{R}^3$ and subject to homogeneous perfectly conducting boundary conditions. Moreover, $\nabla\times$ and $\otimes$ are the curl operator and the Kronecker product, respectively, and $\chi \in L^\infty(\Omega)$ denotes the nonlinear susceptibility.

As discussed in (Hochbruck & Maier, 2021, Sec. 6.1) and in more detail in (Maier, 2020, Sec. 8.2), the problem (7.1) fits into the abstract framework (2.2) with the spaces

$$\begin{aligned} X &= L^2(\Omega)^3 \times L^2(\Omega)^3, & Y &= H^2(\Omega)^3 \times \{\varphi \in H^2(\Omega)^3 \mid \varphi\times\nu = 0\}, \\ Z_\partial &= H^p(\Omega)^3 \times \{\varphi\in H^p(\Omega)^3 \mid \varphi\times\nu=0\}, & Z &= H^{p+1}(\Omega)^3 \times \{\varphi \in H^{p+1}(\Omega)^3 \mid \varphi\times\nu = 0\}, \end{aligned}$$

for $p \geq 3$, equipped with the standard inner products and $|\cdot|_Y = \|\cdot\|_Y$. Here, $\times\nu$ is the cross product with the outer unit normal of Ω. For the discretization in space, we use the discontinuous Galerkin finite element method. In particular, the discrete spaces are given by (3.1) with $V_h \subset L^2(\Omega)^6$ consisting of piecewise polynomials of degree at most $p \in \mathbb{N}$ on an exact mesh of Ω, and

$$\|\cdot\|_{Y_h} = \|\cdot\|_{L^\infty(\Omega)^3\times L^\infty(\Omega)^3}, \qquad\qquad |\cdot|_{Y_h} = \|\cdot\|_{Y_h}.$$

Hence, the estimates (3.3) and (3.16) hold with constants

$$C_{X_h,Y_h}(h) = C, \qquad C_{Y_h,X_h}(h) = Ch^{-\frac{3}{2}}, \qquad C_{\mathcal{A}_h,Y_h,X_h}(h) = Ch^{-\frac{5}{2}}.$$

In particular, there exists $C_0 > 0$ such that for all $\varepsilon_0 > 0$ the step-size restriction (4.4) is a direct consequence of

$$\tau \leq C_0 h^{\frac{5}{4}+\varepsilon_0}. \tag{7.2}$$

Corollary 4.1 then yields the following result, which, up to our knowledge, is the first rigorous error estimate for the full discretization of the quasilinear Maxwell equations.

THEOREM 7.1 Let Assumption 3.3 be true, χ be sufficiently smooth, and $p \geq 3$. In particular, for $T > 0$ let the solution $(\mathcal{H}, \mathcal{E})$ of (7.1) satisfy

$$(\mathcal{H}, \mathcal{E}) \in C^3([0,T], X) \cap C^2([0,T], Y) \cap C^1([0,T], Z_\partial) \cap C([0,T], Z)$$

Then, there are $h_0, \tau_0 > 0$ such that for all $h < h_0$ and $\tau < \tau_0$ satisfying the step-size restriction (7.2), the approximations $(\mathcal{H}_h^n, \mathcal{E}_h^n)$ of both the fully and the linearly implicit midpoint rule (4.1) and (4.3) applied to (7.1) are well defined for $n = 0, \dots, N$. Moreover, we have the estimate

$$\|\mathcal{H}(t_n) - \mathcal{H}_h^n\|_{L^2(\Omega)^3} + \|\mathcal{E}(t_n) - \mathcal{E}_h^n\|_{L^2(\Omega)^3} \leq C_{\mathcal{H},\mathcal{E},\chi}(1 + t_n)\mathrm{e}^{Ct_n}\big(h^p + \tau^2\big),$$

where $C_{\mathcal{H},\mathcal{E},\chi}, C > 0$ are constants independent of h, t, and T, but $C_{\mathcal{H},\mathcal{E},\chi}$ depends on $\mathcal{H}$, $\mathcal{E}$, and χ, including their derivatives.

7.2. **Westervelt equation.** We further consider the Westervelt equation (Westervelt, 1963), which is a fundamental model in nonlinear acoustics. It states that on a finite time interval $[0, T]$ and a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, the pressure $u\colon [0, T] \times \Omega \to \mathbb{R}$ satisfies

$$\begin{cases} (1 - \kappa u)\partial_t^2 u = \Delta\, u + \kappa(\partial_t u)^2 & \text{on } [0,T] \times \Omega, \\ \qquad u(0) = u_0, \qquad \partial_t u(0) = v_0 & \text{on } \Omega, \end{cases} \tag{7.3}$$

with initial values $u_0, v_0 : \Omega \to \mathbb{R}$ and homogeneous Dirichlet boundary conditions. Here, $\kappa \in \mathbb{R}$ models the nonlinearity of the medium.

Introducing the spaces

$$\begin{aligned} X &= H_0^1(\Omega) \times L^2(\Omega), & Y &= \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times \big(H^2(\Omega) \cap H_0^1(\Omega)\big), \\ Z_\partial &= \big(H^p(\Omega) \cap H_0^1(\Omega)\big) \times \big(H^{p-1}(\Omega) \cap H_0^1(\Omega)\big), & Z &= \big(H^{p+1}(\Omega) \cap H_0^1(\Omega)\big) \times \big(H^p(\Omega) \cap H_0^1(\Omega)\big), \end{aligned}$$

for $p \geq 2$, equipped with the standard inner products and

$$|\xi|_Y = \|\xi^v\|_{H^2(\Omega)\cap H_0^1(\Omega)}, \qquad \xi = (\xi^u, \xi^v) \in Y,$$

the Westervelt equation (7.3) also fits into the abstract framework (2.2). The discrete spaces are given by (3.1), with $V_h \subset C(\Omega)^2$ being the Lagrangian finite element space of order p, and

$$\|\xi_h\|_{Y_h} = \|\xi_h\|_{L^\infty(\Omega)\times L^\infty(\Omega)}, \qquad |\xi_h|_{Y_h} = \|\psi_h\|_{L^3(\Omega)}, \qquad \xi_h = (\varphi_h, \psi_h) \in V_h.$$

As shown in (Hochbruck & Maier, 2021, Sec. 6.2), this yields the constants

$$C_{X_h,Y_h}(h) = Ch^{-1}, \qquad C_{Y_h,X_h}(h) = Ch^{-\frac{d}{2}}, \qquad C_{\mathcal{A}_h,Y_h,X_h}(h) = Ch^{-1-\frac{d}{6}}.$$

Thus, there is $C_0 > 0$ such that for all $\varepsilon_0 > 0$ the step-size restriction (4.4) follows from

$$\tau \leq C_0 h^{\frac{6+d}{12}+\varepsilon_0}. \tag{7.4}$$

The abstract result from Corollary 4.1 then yields the following.

THEOREM 7.2 Let Assumption 3.3 be true, $p \geq 2$, and $T > 0$. If the solution $(u, \partial_t u)$ of (7.3) satisfies

$$(u, \partial_t u) = y \in C^1([0,T], Z_\partial) \cap C([0,T], Z),$$

then there exist $h_0, \tau_0 > 0$ such that for all $h < h_0$ and $\tau < \tau_0$ satisfying the step-size restriction (7.4), the approximations (u_h^n, v_h^n) of both the fully and the linearly implicit midpoint rule (4.1) and (4.3) applied to (7.1) are well defined for $n = 0, \dots, N$. Furthermore, they satisfy

$$\|u(t_n) - u_h^n\|_{H_0^1(\Omega)} + \|\partial_t u(t) - v_h^n\|_{L^2(\Omega)} \leq C_u(1 + t)\mathrm{e}^{Ct}\big(h^p + \tau^2\big),$$

where $C_u, C > 0$ are constants independent of h, t, and T, but C_u depends on u including derivatives.

Note that compared to the analysis in Maier (2020), the introduction of the seminorms $|\cdot|_Y$ and $|\cdot|_{Y_h}$ allows for a more relaxed constant $C_{\max}(h)$ and thus also for a more relaxed step-size restriction (7.4). In particular, we obtain that the error estimate stated above is also valid for $p = 2$, whereas (Maier, 2020, Thm. 8.2) is restricted to $p \geq 3$.

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APPENDIX A. DISCRETIZATION OF LOCAL NONLINEARITIES

For the special case of nonlinearities Λ and F that are local in space, refined bounds for the nonlinear remainders $\mathcal{R}_\Lambda$ and $\mathcal{R}_F$ from (3.18a) and (3.18c), respectively, are shown in (Hochbruck & Maier, 2021, Sec. 5). Since these estimates are also useful for the full discretization considered here, we briefly recall the assumptions and the estimates.

In the following assumption, we narrow down the abstract framework to the special case of partial differential equations.

ASSUMPTION A.1 Let $d, d_r \in \mathbb{N}$ such that X, Y, Z, X_h, and Y_h are function spaces consisting of functions mapping from a bounded domain $\Omega \subset \mathbb{R}^d$ to $\mathbb{R}^{d_r}$. Moreover, the following properties hold.

(λf) For $\xi \in B_Y(R_Y)$ we have $\Lambda(\xi) \in \mathcal{L}(Y)$. Further, Λ and F are local in space, i.e., there exist $\lambda : \Omega \times \mathbb{R}^{d_r} \to \mathbb{R}^{d_r \times d_r}$ and $f : [0,T] \times \Omega \times \mathbb{R}^{d_r} \to \mathbb{R}^{d_r}$ such that for all $t \in [0,T]$, $\xi \in B_Y(R_Y)$, and $\varphi \in X$ it holds

$$\big(\Lambda(\xi)\varphi\big)(x) = \lambda(x, \xi(x))\varphi(x), \qquad \big(F(t,\xi)\big)(x) = f(t,x,\xi(x)), \qquad x \in \Omega.$$

(I_h) The operator I_h is a nodal interpolation operator, i.e., for some $M \in \mathbb{N}$ we have

$$I_h \xi = \sum_{m=0}^{M} \xi(\underline{x}^m)\, \phi_h^m, \qquad I_h\xi(\underline{x}) = \xi(\underline{x}), \qquad \xi \in Y,\ \underline{x} \in \Omega_{I_h},$$

with the interpolation points $\Omega_{I_h} = \{\underline{x}^0, \dots, \underline{x}^M\} \subset \Omega$ and the basis functions $\{\phi_h^0, \dots, \phi_h^M\} \subset Y_h$.

$(\Lambda_h F_h)$ For $t \in [0,T]$, $\xi_h \in B_{Y_h}(R_{Y_h})$, and $\psi_h \in X_h$ the discrete nonlinearities are given by

$$\Lambda_h(\xi_h)\psi_h = \sum_{m=0}^{M} \lambda(\underline{x}^m, \xi_h(\underline{x}^m))\psi_h(\underline{x}^m)\, \phi_h^m, \qquad F_h(t,\xi_h) = \sum_{m=0}^{M} f(t, \underline{x}^m, \xi_h(\underline{x}^m))\, \phi_h^m.$$

This yields the following estimate for the nonlinear remainder terms.

LEMMA A.2 ((Hochbruck & Maier, 2021, Lem. 5.2)) If Assumption 3.1 and Assumption A.1 hold, then we have for $t \in [0,T]$, $\xi \in Y$, and $\zeta \in B_Y(R_Y)$

$$\begin{aligned} \|\mathcal{R}_\Lambda(\zeta)\xi\|_X &\le \|(\mathrm{Id} - I_h)\Lambda(\zeta)\xi\|_X + C_{\Lambda_h}\|(I_h - J_h)\xi\|_X, \\ \|\mathcal{R}_F(t,\zeta)\|_X &\le \|(\mathrm{Id} - I_h)F(t,\zeta)\|_X. \end{aligned}$$