Karlsruhe Institute of Technology

The Maxwell–Landau–Lifshitz–Gilbert System: Mathematical Theory and Numerical Approximation

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der KIT-Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

DISSERTATION

von

Jan Adam Bohn

Tag der mündlichen Prüfung: 5. Mai 2021

1. Referent: Prof. Dr. Willy Dörfler
2. Referent: Prof. Dr. Michael Feischl
3. Referent: Prof. Dr. Thanh Tran

Acknowledgment

I gratefully acknowledge funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

Deep gratitude is owed to Prof. Dr. Thanh Tran for his willingness to serve as a referee on this thesis and to write a report. I would also like to thank him a lot for the helpful remarks concerning this thesis and especially for attending the oral exam which took place at 10pm local time for him. I continue in German.

Während meiner Dissertationszeit hatte ich das große Glück, stets zwei hervorragende Betreuer zur Verfügung zu haben. Herrn Prof. Dr. Dörfler möchte ich besonders danken für das sehr vertrauensvolle Verhältnis und dass er mir bei Fragen und Problemen jederzeit zur Seite stand. Eine Besprechung mit Herrn Dörfler versorgte mich stets mit neuer Inspiration und Motivation. Bedanken möchte ich mich für die viele Zeit die er mit der Betreuung aufgenommen hat und vielen hilfreichen Tipps und Anmerkungen zu dieser Arbeit. Herzlich möchte ich mich bedanken für die Initiative mich mit Lehre zu betrauen, worauf ich von alleine vielleicht nicht gekommen wäre. Die vielen schönen Ereignisse und Eindrücke in der Lehre, bei denen ich von Herrn Dörfler sehr viel lernen konnte, würde ich sicherlich missen. Sehr verbunden bin ich für sämtliche Unterstützungen und all die Freiheiten und Privilegien die er für mich möglich gemacht hat.

Obwohl Prof. Dr. Feischl eigentlich nach einem halben Jahr schon wieder weg aus Karlsruhe war, kam es mir doch nie so vor. Ich erinnere mich gerne zurück an die ersten Besprechungen im Büro und die später folgenden Telefonate, es war immer eine große Freude und Inspiration für mich Mathematik zusammen zu betreiben. Die verschiedenen Forschungsthemen und Anreize die Michael vorgeschlagen hat empfinde ich als sehr bereichernd. Größter Dank gebührt Michael für die unzähligen hilfreichen Kommentare zu dieser Arbeit und die unendliche Geduld beim Korrekturlesen. Für die tollen Konferenzerfahrungen und alle anderen Vorteile des Doktorandenseins die er mir nahe gebracht hat bin ich ihm zutiefst dankbar. Michael hat mir gleichzeitig sämtliche Freiheiten gelassen und doch Unterstützung in jeder denkbaren Form geleistet, wann immer diese nötig war. Dafür bin ich Michael zutiefst dankbar. Ohne dich wäre diese Zeit bei weitem nicht so schön, erfolgreich, interessant und lehrreich geworden. Bessere PhD-Betreuer kann man sich nicht wünschen!

Außerordentlicher Dank gebührt jenen, die sämtlichen organisatorischen Aufwand übernommen haben. Besonders hervorheben möchte ich Henriette, Laurette und Sonja für ihre umfangreiche Hilfe bei allen Fragen und Anliegen, die exzellente IT-Betreuung von Christian und Mathias und den Vorstand des SFB für sein außerordentliches Engagement.

Größter Dank gebührt allen weiteren Menschen, die diese Zeit so schön und unterhaltsam gestaltet haben. Meiner AG, der 12:30 Mittagsessensrunde, der erweiterten Skullkingrunde, M3nß4 4 L1f3, der Bempp Arbeitsgruppe, K.V.S, meinen Schul- und Studienfreunden, meinen (ehemaligen) Kellerkollegen und meinem Mentor Herrn Dr. Köhler möchte ich aufrichtig für die vielen schönen Stunden und die Freundschaft danken. Ich hoffe doch noch eine Gelegenheit zu finden mich ein wenig revanchieren zu können.

Allen Mitarbeitern des SFBs und am Institut denen ich bisher begegnen durfte möchte ich meinen Dank und meine Bewunderung aussprechen. Seitdem ich hier angefangen habe bin ich zutiefst beeindruckt von der freundschaftlichen und sehr harmonischen

Atmosphäre zu der jeder Einzelne beiträgt. Ich werde immer mit Freude und Wehmut an euch und meine Doktorandenzeit zurückdenken.

Ich danke meiner Familie für die unendliche Liebe und bedingungslose Unterstützung die ich seit jeher erfahren durfte. Die tiefe Zufriedenheit, Freude und Dankbarkeit die ich empfinde wenn ich daran denke dass ich euch als Familie habe lässt sich kaum in Worte fassen.

Das größte Glück dass mir bisher über den Weg gelaufen ist, ist meine Freundin Larissa. Danke für deine Liebe, dein Verständnis und deine Unterstützung und unsere gemeinsame Zukunft auf die ich mich sehr freue.

Abstract

This thesis deals with the mathematical theory and numerical approximation of the Landau–Lifshitz–Gilbert equation coupled to the Maxwell equations without artificial boundary conditions.

As a starting point, the physical equations are stated on the unbounded three dimensional space and reformulated in a mathematically precise way to a coupled partial differential – boundary integral system.

We derive a weak form of the whole coupled system, state the relation to the strong form and show uniqueness of the Maxwell part of the solution. A numerical algorithm is proposed based on the tangent plane scheme for the LLG part and using a finite element and boundary element coupling as spatial discretization and the backward Euler method and Convolution Quadrature as time discretization for the interior Maxwell part and the boundary, respectively. Under minimal assumptions on the regularity of solutions, we present well-posedness and convergence of the numerical algorithm.

For the pure Maxwell equations without the coupling to the LLG equation, we are able to show stronger results than in the coupled case. We derive a weak form for the Maxwell transmission problem and demonstrate existence and uniqueness of the weak solutions as well as equivalence with a strong solution. The proposed algorithm of finite-element/boundary-element coupling via Convolution Quadrature converges with only minimal assumptions on the regularity of the input data.

Again for the full Maxwell–LLG system, we show a-priori error bounds in the situation of a sufficiently regular solution. This is done by a combination of the known linearly implicit backward difference formula time discretizations with higher order non-conforming finite element space discretizations for the LLG equation and the leapfrog and Convolution Quadrature time discretization with higher order discontinuous Galerkin elements and continuous boundary elements for the boundary integral formulation of Maxwell's equations. The precise method of coupling allows us to solve the system at the cost of the individual parts, with the same convergence rates under the same regularity assumptions and the same CFL conditions as for an uncoupled examination.

Numerical experiments illustrate and expand on the theoretical results and demonstrate the applicability of the methods.

For the formulation of the boundary integral equations, the study of the Laplace transform is inevitable. We collect and extend the properties of the Laplace transform from literature. In the suitable functional analytic setting, we give extensive proofs in a self contained way of all the required properties.

Contents

1. Introduction

1.1. Motivation

The understanding of magnetization dynamics is overly important for various current technical developments: Magnetization phenomena occur, e.g., in magnetic sensors, generators, electric motors or magnetic storage devices. In a magnetic storage device, for example the magnet stripe of a common credit card, a hard drive disk of a personal computer or a magnetic tape storage (used for long time storage in data in archives [124, 136], or big data centers (e.g. in the project CASTOR at CERN [50])) the encoding of the data is achieved via the direction of the magnetization.

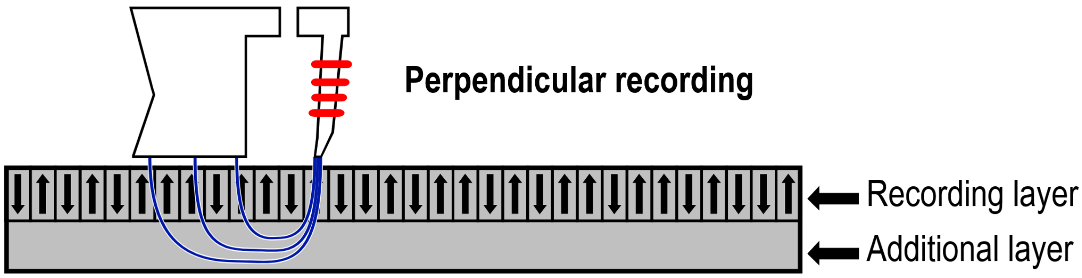

Figure 1.1.1.: Perpendicular recording: Writing head on a hard disk from [49].

Figure 1.1.1 depicts how a magnetic storage device is working: The electromagnet ("writing head") introduces a magnetic field and therefore the magnetization in the actual storage cell in the recording layer gets aligned to the intended direction. Whether the magnetization is pointing upwards or downwards, the block encodes a "1" or a "0", so each cell corresponds to one bit.

For the practical relevance, several aspects come into place: The smaller each of the tiny cells can be chosen (without influencing the neighboring bits during the writing process), the higher is the storage capacity of the device. The time it takes to align a bit is directly related to the writing speed. Further aspects comprise the energy efficiency of the writing process, the possibility and speed of reading the data and the manufacturing costs (e.g. due to the used materials). For the long time safety of the data it is important how stable the writing procedure is, especially with respect to exterior impacts like external (earth) magnetic fields or physical effects like hits or heat.

Effects like these can be simulated with the algorithms and models considered in this thesis, but also with reduced models, see e.g. [55, 141, 64].

In the development of even smaller, faster and more efficient devices, also cutting edge writing techniques are explored: The control of magnetization by optical means [31], especially using circularly polarized light seems to be a promising ansatz [147]. In this emerging field of research, femtosecond laser pulses are used to switch the magnetization of ferromagnetic materials in order to improve the speed, density, and stability of magnetic hard drives, with possible implications for the field of spintronics [75].

To cover all those physical aspects, it is essential to consider the coupling to the full Maxwell system, like we do it in this thesis in contrast to earlier approaches (e.g. [63, 104, 105]), which deal with a quasi static approximation of Maxwell's equations. Yet another application that is being considered is the production of radiation from a magnetic material. The interest comes from the observation that one can build THz emitters in this way, which are otherwise hard to achieve [129]. Such THz emitters are important for a broad range of technical applications ranging from chemistry and medicine to physics and material sciences [120, 150]: information and communications technology, spectroscopy and imaging, nondestructive evaluation (material and circuitry diagnosis), security (detection of drugs and explosives), global environmental monitoring, ultrafast computing and astrophysics. Despite all that, the THz region of the electromagnetic spectrum is still an comparatively unexplored region due to the lack of strong and broadband THz emission sources and sensitive detectors.

Altogether, scientific research and industrial application require a deep understanding of the physical ongoings in the technical devices and further this is demanded for different materials, different shapes (of parts) of the devices and their respective arrangement. A very helpful approach for this understanding is the numerical simulation for visualization and for computing concrete figures (that otherwise would have to be measured in a physical experiment).

This dissertation considers the analysis of algorithms for the Maxwell–Landau–Lifshitz–Gilbert system in two complementary attempts: We propose algorithms that, almost independent of the regularity of the solution, give reliable approximations. On the other hand, if the problem is good-natured in a way (the solution is regular enough), we show that the approximation and exact solution only differ by an a-prioiri known tolerance, and that we can predict how the error reduces in terms of the discretization parameters.

1.2. Physical Derivation

In this section we want to introduce the Maxwell–Landau–Lifshitz–Gilbert system from a physical point of view. This section is based on [141], [121] and [106].

1.2.1. The Landau–Lifshitz–Gilbert equation

Micromagnetism describes the study of magnetic phenomena on a length scale from a few nanometers up to several micrometers. The scale is large enough to consider continuous quantities above the atomic structures but small enough to observe magnetic structures like domain walls or vortices. We consider the situation of a ferromagnetic body whose volume is described by the three dimensional domain $\Omega \subset \mathbb{R}^3$. The physical quantity of magnetization is described in every point $x \in \Omega$ (and at time $t \in \mathbb{R}$) by a three dimensional vector $M(t,x) \in \mathbb{R}^3$ that represents the magnetic moment per unit volume. As long as the temperature stays constant and sufficiently low, the absolute value of the magnetization remains constant (see [141]), i.e.

$$|M| = M_S > 0,$$

where M_S is called saturation magnetization.

Static micromagnetics

Following the micromagnetic theory, the magnetization always aligns itself such that a state of minimum energy is reached. This means that the final state is a minimizer of the so-called total magnetic Gibbs free energy, which comprises many individual energy contributions. For simplicity however, we consider only a simplified model in this thesis, we refer to [141] for a more detailed presentation and to Section 7.2 for the extensibility of the mathematical results to cover further physical effects.

Due to quantum mechanical effects, neighboring magnetic moments strive to be aligned in the same direction. Therefore, the exchange energy includes the term

$$\mathcal{E}_{\mathrm{ex}}(M) = \frac{A}{M_S^2} \int_\Omega |\nabla M|^2 \,\mathrm{d}x$$

which penalizes unequal directions in the magnetic field. Here $A > 0$ denotes the exchange stiffness constant.

If an external magnetic field H is applied, the magnetization aligns with it. Accordingly, Zeeman's energy penalizes deviations of the magnetization from this direction, i.e.

$$\mathcal{E}_{\mathrm{ze}}(M) = -\int_\Omega \mu H \cdot M \,\mathrm{d}x.$$

For simplicity we assume in this section that the magnetic permeability $\mu \in \mathbb{R}_+$ is scalar and constant, see Section 7.2 for a discussion.

Taken together, for the energy $\mathcal{E}(M) = \mathcal{E}_{\mathrm{ex}}(M) + \mathcal{E}_{\mathrm{ze}}(M)$, we consider the minimization problem

$$\min_{|M|=M_S} \mathcal{E}(M).$$

If we define the effective magnetic field as

$$\mu H_{\mathrm{eff}} = -\frac{\partial \mathcal{E}}{\partial M} = \frac{2A}{M_S^2}\Delta M + \mu H,$$

the corresponding Euler Lagrange equations can be derived as

$$\begin{aligned} m \times H_{\mathrm{eff}} &= 0 \quad \text{in } \Omega, \\ \partial_n M &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Therefore a minimizer of the energy is aligned parallel to the effective field.

Dynamic micromagnetics

We will now study the physical equations that deal with how the magnetization aligns itself from an unstable initial state to the final static state.

In [103], the physicists Lew Dawidowitsch Landau (1908–1968) and Jewgeni Michailowitsch Lifschitz (1915–1985) proposed the phenomenological model

$$\partial_t M = -\gamma_0 M \times H_{\mathrm{eff}} - \frac{\gamma_0 \lambda}{M_S} M \times (M \times H_{\mathrm{eff}}), \tag{1.1}$$

with damping parameter $\lambda > 0$ and the (rescaled) gyromagnetic ratio γ_0. This describes the motion as a damped precision, i.e. the magnetization rotates around while simultaneously being damped towards the effective field.

In [69], Thomas Lewis Gilbert (born 1922) proposed to add a different damping term, resulting in

$$\partial_t M = -\gamma_0 M \times H_{\mathrm{eff}} - \frac{\alpha}{M_S} M \times \partial_t M \tag{1.2}$$

with the (dimensionless) Gilbert damping parameter $\alpha > 0$. Up to rescaling of the constants, the equations (1.1) and (1.2) are equivalent, see Section 2.1.1. We restate (1.2) as the Landau–Lifshitz–Gilbert (LLG) equation

$$\partial_t M = -\frac{\gamma_0}{1+\alpha^2} M \times H_{\mathrm{eff}} - \frac{\alpha\gamma_0}{(1+\alpha^2)M_S} M \times (M \times H_{\mathrm{eff}}). \tag{1.3}$$

This equation is equipped with initial data $M(0) = M^0$ and we use the boundary condition from the static case $\partial_n M = 0$, see [133] for a justification.

Taking the scalar product with M in (1.3), we obtain

$$\partial_t |M|^2 = 2\partial_t M \cdot M = 0, \tag{1.4}$$

i.e. the modulus of the magnetization stays constant, $|M| = M_S$ is contained in the LLG equation as long as this is fulfilled for the initial data.

1.2.2. The Maxwell equations

In [117], James Clerk Maxwell (1831–1879) collected a set of partly experimentally known laws and put them together in a coherent set of differential equations: Maxwell's equations. They describe the phenomena of classical electrodynamics and therefore form the theoretical basis of optics and electrical engineering. The four Maxwell equations read:

Gauss's law: The charge ρ is the source of the electric displacement field D.

$$\nabla \cdot D = \rho. \tag{1.5}$$

Gauss's magnetic law: The magnetic flux density B has no sources: Magnetic monopoles do not exist.

$$\nabla \cdot B = 0. \tag{1.6}$$

Faraday's law: Changes in magnetic flux density lead to an electric vortex field E.

$$\partial_t B = -\nabla \times E. \tag{1.7}$$

Ampères law: Electric currents J_e, including the displacement current $\partial_t D$, lead to a magnetic vortex field H.

$$J_e + \partial_t D = \nabla \times H. \tag{1.8}$$

Simplification

With the conservation law of the electric charge,

$$\partial_t \rho + \nabla \cdot J_e = 0, \tag{1.9}$$

we see that Gauss's law (1.5) and Gauss's magnetic law (1.6) are true at any time, as far as they are satisfied at a particular time point t_0, i.e.

$$(\nabla \cdot B)(t_0) = 0 \quad \text{and} \quad (\nabla \cdot D)(t_0) = \rho(t_0)$$

imply

$$(\nabla \cdot B)(t) = 0 \quad \text{and} \quad (\nabla \cdot D)(t) = \rho(t)$$

for any time t by taking the divergence in (1.8) and (1.7), respectively, and noting that the divergence of the curl is always zero.

Constitutive equations

In the case of linear materials, the magnetic flux density and the magnetic field, and the electric displacement field and the electric field, respectively are linked by the constitutive relations

$$D = \varepsilon E \quad \text{and} \quad B = \mu(H + M) \tag{1.10}$$

with the electric and magnetic permeabilities $\varepsilon, \mu \in \mathbb{R}^{3\times 3}$ (uniformly positive definite and symmetric matrices).

In vacuum, the permeabilities are positive constants, i.e. it holds $\varepsilon = \varepsilon_0 \mathrm{Id}$, $\mu = \mu_0 \mathrm{Id}$ with Id the 3×3 unit matrix and the magnetic and electric vacuum permeabilitiees $\varepsilon_0, \mu_0 \in \mathbb{R}_+$.

In the case of a conductive material, Ohm's law says

$$J_e = \sigma E + J$$

for the conductivity $\sigma \in \mathbb{R}^{3\times 3}$ and the applied current J.

Body surrounded by vacuum

The above equations hold in the full space, i.e. in the whole $\mathbb{R}^3$. We consider the situation of a bounded body surrounded by vacuum, e.g. the magnet, the magnetic storage device, the generator. We assume that the volume of the body defines the domain $\Omega \subset \mathbb{R}^3$. Therefore, electric current, the conductivity and charge are supported inside of Ω and in the exterior domain $\mathbb{R}^3 \setminus \overline{\Omega}$, the electric and magnetic permeabilites are positive scalars. Furthermore, we assume that at starting time $t = 0$ electric and magnetic field are supported inside of Ω, i.e.

$$H(0,x) = E(0,x) = 0 \quad \text{for all } x \in \mathbb{R}^3 \setminus \overline{\Omega}. \tag{1.11}$$

This corresponds to the situation, when at the beginning of the experiment everything is at rest in the exterior domain, compare Section 7.2.

Transmission conditions

The above setting can be translated to a coupled problem with interface conditions on $\partial\Omega$, also see [121, Section 1.2.2] and Remark 2.32. On the boundary of the magnetic body Ω, with possibly jumping material parameters ε^+, μ^+ inside and ε^-, μ^- outside, Maxwell's equations do not hold in classical form: As either εE or E are not continuous over the boundary (except $E = 0$), $\nabla \cdot (\epsilon E)$ or $\nabla \times E$ are not defined in the classical sense. To have the respective differential operators well-defined at least in a weak sense, we require

$$n \times E^+ = n \times E^- \quad \text{and} \quad n \cdot (\mu^+ H^+) = n \cdot (\mu^- H^-),$$

where n is the exterior normal vector on the boundary of Ω. For the remaining boundary conditions, we have to take the equations (1.5) and (1.8) into account. Again with some formal arguments, considering the equations on an arbitrary domain covering a part of the boundary, integrating over the respective domain, we obtain by Stokes' theorem

$$n \times (H^+ - H^-) = J_S \quad \text{and} \quad n \cdot (\mu^+ H^+ - \mu^- H^-) = \rho_S,$$

with the surface current density J_S and the surface charge density ρ_S. In most instances, (except having strongly growing singularities towards the boundary), those are negligible and we can assume that $J_S = 0$ and $\rho_S = 0$.

Altogether, after a suitable non-dimensionalization (see [141, Section 3.1] for the details, defining $m = M/M_S$ which yields $|m| = 1$), we obtain the following system.

1.3. The Maxwell–Landau–Lifshitz–Gilbert System

Let $\Omega \subset \mathbb{R}^3$ not necessarily convex be a bounded, connected, open and Lipschitz domain with connected, piecewise smooth boundary Γ (or a finite collection of such domains). By $\mathbb{S}^2$ we denote the unit sphere in $\mathbb{R}^3$, and by $T > 0$ we denote the final time. We denote the space-time cylinders by $\Omega_T := (0,T) \times \Omega$, $\overline{\Omega}_T^{\mathrm{c}} := (0,T) \times \overline{\Omega}^{\mathrm{c}}$ and $\Gamma_T := [0,T] \times \Gamma$.

We will often refer to Ω as the interior domain, and to $\overline{\Omega}^{\mathrm{c}}$ as the exterior domain. We seek a magnetization

$$m : [0, T] \times \Omega \to \mathbb{S}^2$$

and electric and magnetic fields

$$E, H : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$$

that satisfy the Maxwell–Landau–Lifshitz–Gilbert (MLLG) equations: in the interior domain

$$\partial_t m = -m \times H_{\text{eff}} - \alpha m \times (m \times H_{\text{eff}}) \quad \text{in } \Omega_T, \tag{1.12a}$$
$$\varepsilon \partial_t E = \nabla \times H - (\sigma E + J) \quad \text{in } \Omega_T, \tag{1.12b}$$
$$\mu \partial_t H = -\nabla \times E - \mu \partial_t m \quad \text{in } \Omega_T, \tag{1.12c}$$

with $H_{\text{eff}} = \frac{1}{1+\alpha^2}(C_e \Delta m + H)$ and in the exterior domain

$$\varepsilon_0 \partial_t E = \nabla \times H \quad \text{in } \overline{\Omega}^{\mathrm{c}}_T, \tag{1.12d}$$
$$\mu_0 \partial_t H = -\nabla \times E \quad \text{in } \overline{\Omega}^{\mathrm{c}}_T, \tag{1.12e}$$

with the boundary condition for the magnetization

$$\partial_n m = 0 \quad \text{on } \mathit{\Gamma}_T, \tag{1.12f}$$

the transmission conditions (for n being the outward pointing normal vector to $\partial\Omega$)

$$E^{\text{int}} \times n = E^{\text{ext}} \times n \text{ and } H^{\text{int}} \times n = H^{\text{ext}} \times n \quad \text{on } \mathit{\Gamma}_T, \tag{1.12g}$$

and the initial conditions

$$m(0) = m^0, \quad E(0) = E^0, \quad H(0) = H^0 \quad \text{in } \Omega, \tag{1.12h}$$

and

$$E(0) = 0, \quad H(0) = 0 \quad \text{in } \overline{\Omega}^{\mathrm{c}}. \tag{1.12i}$$

The applied current density $J : [0, T] \times \Omega \to \mathbb{R}^3$, the electric and magnetic permeability matrices $\varepsilon, \mu : \Omega \to \mathbb{R}^{3\times 3}$ and the conductivity of the ferromagnetic domain $\sigma : \Omega \to \mathbb{R}^{3\times 3}$ are considered given data. The damping parameter α and the exchange constant C_e are positive constants. Outside of the domain Ω, the material parameters are assumed to be scalar and constant:

$$\mu = \mu_0, \quad \varepsilon = \varepsilon_0, \quad \sigma = 0.$$

We assume the given initial data satisfies

$$|m^0| = 1, \quad \operatorname{div}(\mu H^0 + \mu m^0) = 0 \quad \text{and } \operatorname{div}(\varepsilon E^0) = \rho(0) \quad \text{in } \Omega,$$

therefore (1.4) implies $|m(t, \cdot)| = 1$ for all $0 \le t \le T$ and (1.9) yields $\operatorname{div}(\mu H(t) + \mu m(t)) = 0$ and $\operatorname{div}(\varepsilon E(t)) = \rho(t)$ in Ω and $\operatorname{div}(H) = \operatorname{div}(E) = 0$ in Ω^{c} for all time $0 \le t \le T$. The transmission condition and consistency with the exterior initial data demand $\gamma_T E^0 = \gamma_T H^0 = 0$.

1.4. Literature Overview

The LLG equation serves as an important practical tool and as a valid model for micromagnetic phenomena occurring in, e.g., magnetic sensors, recording heads, and magneto-resistive storage devices [69, 103, 132]. Classical results concerning existence and non-uniqueness of solutions can be found in [17, 153]. In a ferro-magnetic material, magnetization is created or affected by external electro-magnetic fields. It is therefore necessary to augment the LLG equations with the Maxwell system, see e.g. [55, 100, 153]. Existence, regularity and local uniqueness for the MLLG equations are studied in [54, 47].

While in many applications, the quasi-static approximation of the Maxwell system, i.e. the eddy-current equations yield sufficiently accurate results, recent breakthroughs in ultrafast magnetism will require the full Maxwell system to be modeled correctly [31, 147, 129].

Numerical approximation methods are known for many variants of simpler versions of the MLLG system, i.e. for the LLG, ELLG (eddy-current LLG) equations [14, 16, 15, 29, 30, 55, 104, 105] (the list is not exhaustive), and even with the full Maxwell system on bounded domains [24, 25].

There is a rich literature on numerical methods for the Landau–Lifshitz(–Gilbert) equations, for the numerical literature up to 2007 we refer to the review [55]. Linear finite element discretizations in space and linearly implicit backward Euler in time for the LLG equation are proposed in [14, 15] and, using a discrete energy inequality and compactness arguments, the convergence without rates towards nonsmooth weak solutions is proved. Convergence of this type was previously shown in [30] for fully implicit methods that are based on the Landau–Lifshitz equation. In [16], convergence without rates towards weak solutions is shown for a method (formally) of "almost" second order in time, for the LLG equation with a more general type of the effective magnetic field.

For the ELLG system, originating from the seminal work [14], the recent works [104, 105] consider a similar numeric integrator for a bounded domain. While the numerical integrator of [105] treats LLG and eddy current simultaneously per time step, [104] adapts an idea of [25] and decouples the time-steps for LLG and the eddy current equation. The recent work [63] considers a finite element/boundary element coupling discretization for the full space ELLG system and even derives strong error estimates additionally to the weak convergence of the approximations.

In this thesis we study the full MLLG equations on the whole $\mathbb{R}^3$. In Chapter 3, we build on the tangent plane scheme introduced in [14] to propose a numerical algorithm which couples finite elements in the magnetic domain with Convolution Quadrature boundary elements for the unbounded exterior domain. This is inspired by [25] and the work [99], which derives a coupling based on Convolution Quadrature in the exterior domain for the Maxwell equations.

The heart of Chapter 3 is to show that Convolution Quadrature coupled to the nonlinear LLG equations can be reformulated in a weak sense with minimal assumptions on the regularity of the data. This inspires a numerical algorithm which is shown to converge towards a weak solution in a weak sense.

In Chapter 4, we concentrate on the Maxwell equations on the whole three-dimensional space without the coupling to the LLG equation.

As in [99], we use the discretization via finite element/boundary element coupling which has the advantage that there are minimal restrictions on the shape of the interior domain (especially convexity is not needed). Other methods such as nonlocal boundary conditions on balls [76, 77], local absorbing boundary conditions [61, 78], perfectly matched layers [32], need particular geometries, e.g., because waves may leave and reenter a non-convex domain. The inclusion of a non-convex domain in a larger convex domain is computationally undesirable in situations such as a cavity or an antenna-like structure or a far-spread non-connected collection of small domains.

With the reformulation of Chapter 2, this leads to transparent boundary conditions, which yield the restriction of the solution to the domain and which are integral equations in space and time. The coercivity of the arising Calderon operator for Maxwell's equations is proven in [99]: The continuous-time and discrete-time coercivity is obtained from the Laplace-domain coercivity using the (operator valued) Herglotz theorem [81] from [27] and the properties of Convolution Quadrature [113, 114, 116] (see also [119], where Convolution Quadrature is analyzed in the time-domain in a variational setting).

In Chapter 4, we remove the quite extensive regularity assumptions posed on the exact solution from [99] and show existence of a unique weak solution of the reformulated system. As a byproduct, we obtain a numerical algorithm which converges weakly to the weak solution.

In Chapter 5, we show that the algorithms also exhibit strong convergence behavior with a-priori known higher order error rates in the case of sufficiently regular solutions for the full MLLG system. Furthermore, these methods satisfy an energy inequality irrespective of the solution regularity, which is an important robustness factor, see Chapter 3.

This is based on recent higher order convergence results for the LLG equations from [4] and the Maxwell system from [99]. In [99], second order in time and first order in space results for the full space Maxwell system are considered. Based on linearly implicit BDF-methods [5, 6, 57], [4] supplies up to 5-th order in time and arbitrary order in space convergence methods for the LLG equation. Up to this work, so far only first order schemes or higher order schemes without rigorous convergence proof were considered in the literature: A first-order error bound for a linearly implicit time discretization of the Landau–Lifshitz equation was proved in [52] and for the eddy current Maxwell–LLG system in [53]. Optimal-order error bounds for linearly implicit time discretizations based on the backward Euler and Crank–Nicolson methods combined with finite element full discretizations for a different version of the Landau–Lifshitz equation were obtained under sufficient regularity assumptions in [67] and [18], respectively. In contrast to [14, 15, 16, 30], these methods do not satisfy an energy inequality irrespective of the solution regularity.

Numerical discretizations for the coupled system of the LLG equation with the eddy current approximation of the Maxwell equations are studied in [63], with first-order error bounds in space and time under sufficient regularity assumptions.

There are several methods for the LLG equations that are of formal order 2 in time (though only of order 1 in space), e.g., [130, 16, 58], but none of them comes with an error analysis. Fully implicit BDF time discretizations for LLG equations have been used successfully in the computational physics literature [148], though without giving any error analysis.

We summarize with the classification of this thesis in the most closely related literature context in Table 1.4.

	weak convergence	convergence with rates
LLG	Alouges in [14]	Akrivis et al. in [4] (higher order)
full space Maxwell	**Chapter 4**	Kovács & Lubich in [99] (higher order)
bounded domain MLLG	Banas et al. in [25]	
full space ELLG	Feischl & Tran in [63]	Feischl & Tran in [63] (first order)
full space MLLG	**Chapter 3**	**Chapter 5** (higher order)

Table 1.4.1.: Overview of the results of the dissertation in the literature context.

1.5. Contributions and Outline of the Dissertation

In Chapter 2, the system on the unbounded domain is rewritten into a coupled interior – boundary integral system. At first, this is done in a formal, motivating way and thereafter the precise dependency between the solutions of the two systems is investigated (similar formal derivations may be found, e.g. in [99, 22]; rigorous results which do not depend on additional regularity assumptions seem to be unavailable in literature; related considerations can be found in [149, 143]). This includes the precise definitions of solutions to the respective systems and the study of the time harmonic systems combined with the properties of the Laplace transform.

For the numerical treatment of the LLG equation, there are typically two approaches:

1) Weak convergence: Under low regularity assumptions on the input data (initial data, external fields), we show the existence of a weak solution and the weak convergence of subsequences towards the solution(s).

2) Convergence with rates: Under the assumption of sufficiently smooth solutions, we show the convergence with rates of the approximations.

We proceed in a similar way in this thesis:

In Chapter 3, the weak convergence of the MLLG system is treated. We use the corresponding results for the MLLG system on bounded domains (cf. [25]) and combine them with the new results from the boundary integral Maxwell system from Chapter 4. Suitable notions of solutions are introduced, that especially guarantee the well-definedness of traces and the boundary integral operator. Equivalence between the solution that arises from the boundary integral formulation and the weak solution one actually obtains convergence towards is shown. Therefore the projection property of the time harmonic Calderon operator together with careful applications of the Laplace transform properties are employed. Furthermore uniqueness of the system is discussed. We propose an algorithm (based on [25] and [99]) and show boundedness of the approximations. Therefore weakly convergent subsequences can be extracted and their limit functions are identified as weak solutions of the MLLG system.

In Chapter 4 we propose a similar program as in Chapter 3 for the full space Maxwell system without the coupling to the LLG equation, i.e. we consider a weak convergence version of [99]. In comparison to Chapter 3, we are able to introduce stronger notions of solutions. We propose a non-symmetric algorithm with favorable properties in comparison to the symmetric approach. The weak convergence of the approximations is shown for the whole sequence (and not only subsequences) and to a solution that can be extended to the infinite time interval $[0, \infty)$.

In Chapter 5, we combine and extend the convergence results with rates for the LLG equation from [4] with the results for the boundary integral Maxwell system from [99] to obtain convergence with rates for the coupled MLLG system. Despite the coupling, the precise method allows us to solve the system at the cost of the individual parts. Furthermore the same convergence rates are obtained under the same regularity assumptions and the same CFL conditions as in the uncoupled case. Also a discrete energy inequality remains true independently of the solution regularity.

The proposed algorithms are implemented in FEniCS [8] and Bempp [145] and numerical results that confirm the theoretical findings can be found in Chapter 6.

For the relations between the different notions of solutions of the MLLG system, the study of the (vector valued) Laplace transform is inevitable. The required properties are derived in a suitable functional analytic setting in Chapter B in the Appendix. In the literature, similar results without proof can be found in [115], a few results in a more general setting in [19] and for the setting with vector valued distributions, see [143].

We conclude the thesis with an outlook in Chapter 7.

1.6. Notation

Before turning to the main part of the dissertation, we give a few notes on the used notation and a list of symbols in this section.

For a function $f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x)$ (and also more generally for other functions) we write $f(x)$ for simplicity, although we consider the mapping $(x \mapsto f(x))$ and not the evaluation at $x \in \mathbb{R}$. Especially when dealing with the Laplace transform $\mathcal{L}$, it is more convenient to write $\mathcal{L}(f(x))$ (also for more complex examples) instead of $\mathcal{L}(x \mapsto f(x))$.

For a function E depending on space and time, we write (for a time point t) $E(t)$ in place of the space-dependent function $E(t,\cdot) : x \mapsto E(t,x)$.

When dealing with Convolution Quadrature, we do not consistently distinguish between a sequence and the evaluation of a (time dependent) function on the time grid points. Given a step size $\tau > 0$ and sequence $(\varphi^i)_{i\in\mathbb{N}_0}$, we write (for $t_i = i\tau$) $\phi(t_i)$ instead of ϕ^i and vice versa.

Norms in function spaces are denoted by $\|\cdot\|$ and the absolute value (of finite dimensional vectors) by $|\cdot|$.

List of Symbols

Reformulation and function spaces

γ_T	tangential trace operator
$\mathcal{S}(s)$, $\mathcal{D}(s)$	time harmonic electric single and double layer potentials
$B(s)$, $B(\partial_t)$	time harmonic and time domain Calderon operator
$L^p(D)$	Lebesgue spaces for $p \in [0,\infty]$ on domain D
$H^k(D)$	space of k-times weakly differentiable functions in $L^2(D)$
$H(\mathrm{curl},\Omega)$	space of functions with existing curl in $L^2(D)$
$H(\partial_t,\mathrm{curl},\Omega_T)$	space of functions with existing time derivative and curl in space in $L^2(D)$
$H(\mathrm{curl},\Omega_T)$	space of functions with existing curl in space in $L^2(D)$
$H^k_{0,*}([0,T])$	spaces of k times weakly differentiable functions in $L^2(D)$ with vanishing initial condition, respectively
$H^k_{*,0}([0,T])$	spaces of k times weakly differentiable functions in $L^2(D)$ with vanishing end condition, respectively
$C^\infty(D)$	space of infinitely differentiable functions
H_Γ	trace space
∂_t^{-1}	integration in time operator
$\mathcal{L}$, $\mathcal{L}^{-1}$	(inverse) Laplace transform
B_m	inverse Laplace transform of $B(s)s^{-m}$
$[\cdot,\cdot]_X$	Hilbert space scalar product on Hilbert space X
$[\cdot,\cdot]_D$	$L^2(D)$ scalar product for domain D
$\langle\cdot,\cdot\rangle_\Gamma$	anti symmetric pairing on $\mathcal{H}_\Gamma$
$[\cdot,\cdot]_{\mathcal{H}_\Gamma}$	Hilbert space scalar product on $\mathcal{H}_\Gamma$
$[\cdot,\cdot]_\Gamma$	$L^2(\Gamma)$ scalar product

Weak convergence for the MLLG system

φ	trace variable for $\mu_0\gamma_T H$
ψ	trace variable for $-\gamma_T E$
τ, h	time step size and mesh width
$\mathcal{S}^1(\mathcal{T}_h)$	$\mathcal{P}^1$-FEM space of globally continuous and piecewise affine functions
$\mathrm{I}_h^{\mathcal{S}} u$	interpolation onto $\mathcal{S}^1(\mathcal{T}_h)$
$\mathcal{K}_{m_h}$	discrete, nodewise tangent space
$\mathcal{X}_h$	Nédélec's $H(\mathrm{curl}, \Omega)$-conforming ansatz space
$\mathrm{I}_h^{\mathcal{X}}$	interpolation onto $\mathcal{X}_h$
$\mathrm{P}_h^{\mathcal{X}}$	L^2-projection onto $\mathcal{X}_h$

Weak convergence for the pure Maxwell system

$\mathcal{Y}_h$	approximation space of piecewise constant functions
$\mathrm{I}_h^{\mathcal{Y}}$	interpolation onto $\mathcal{Y}_h$

Convergence with rates for the MLLG system

$\mathrm{P}(m)$	continuous orthogonal projection onto the tangent plane at m
$\mathcal{T}(m)$	continuous tangent space at m
$\mathcal{S}_h^r$	Lagrange finite element space of continuous, piecewise polynomial functions of degree r
R_h	Ritz projection onto $\mathcal{S}_h^r$
$\mathcal{T}_h(m)$	discrete tangent space at m
$P_h(m)$	$L^2(\Omega)$-projection onto the discrete tangent space at m
$\mathcal{W}_h^r$	discontinuous Galerkin space of elementwise polynomial functions of degree r
$\mathrm{I}_h^{\mathcal{W}}$	finite element interpolation onto $\mathcal{S}_h^r$
Ψ_h^r	boundary element space of continuous, piecewise polynomial functions of degree r
I_h^{Ψ}	boundary element interpolation onto Ψ_h^r
$\dot{m}_h^n$	time derivative approximation at t_n
$\widehat{m}_h^n$	normalized extrapolation at t_n

Numerics

$\boldsymbol{E}(\mathcal{V}_h)$	Coefficient vector of E_h in the basis of $\mathcal{V}_h$
$\boldsymbol{\phi}(\mathcal{V}_h)$	Vector of basis functions of $\mathcal{V}_h$
$\mathcal{X}_h, N1$	first order Nédélec space
$\mathcal{Y}_h, N0$	piecewise constant space
$S^1(\mathcal{T}_h, \mathbb{R}), S^1(\mathcal{T}_h, \mathbb{R}^3), S1$	space of (vector valued) linear elements
$\mathcal{V}_h^{RT}, RT$	Raviart–Thomas space
$\mathcal{V}_h^{NC}, NC$	Nédélec space
$\mathcal{V}_h^{RWG}, RWG$	Rao–Wilton–Glisson space (scaled RT space)
$\mathcal{V}_h^{SNC}, SNC$	scaled Nédélec space
$BRT, BNC, BRWG, BSNC$	spaces of barycentrically refined grid functions
$\mathcal{V}_h^{BC}, BC$	Buffa–Christiansen space
$\mathcal{V}_h^{RBC}, RBC$	rotated Buffa–Christiansen space
$\phi_{\lvert \phi^j = 0}$	sequence with the j-th entry set to zero
$(B(\widetilde{\partial}_t^\tau)\phi)$	approximated Convolution Quadrature operator
${}_{DSRS}F_{DS}$	weak form of F
$F_{DS \to RS}$	strong form of F

The Laplace transform

Scalar valued Laplace transform and differential operators

$\mathcal{L}, \mathcal{L}^{-1}$	(inverse) Laplace transform
$\mathcal{F}, \mathcal{F}^{-1}$	(inverse) Fourier transform
$L^2_c[0,\infty)$	e^{-ct}-weighted $L^2[0,\infty)$ spaces for $c \in \mathbb{R}$
$L^2_*[0,\infty)$	union of $L^2_c[0,\infty)$ spaces
$\mathcal{H}(\sigma_0)$	Hardy space on $\{\Re s > \sigma_0\}$ for $\sigma_0 \in \mathbb{R}$
$\mathcal{H}$	union of $\mathcal{H}(\sigma_0)$ spaces
$B(\partial_t)f$	Laplace differential operator on $[0,\infty)$ defined as $\mathcal{L}^{-1}(B(s)\mathcal{L}(f)(s))$
$H^m_{0,c}[0,\infty)$	exponentially weighted spaces with zero condition at $t = 0$ for $c \in \mathbb{R}$
$H^m_{0,*}[0,\infty)$	union of $H^m_{0,c}[0,\infty)$ spaces
$C^\infty(0,\infty)$	smooth and compactly supported functions on $(0,\infty)$
$\mathcal{H}_m(\sigma_0)$	space of analytic and by s^m bounded functions on $\{\Re s > \sigma_0\}$
$\mathcal{H}_m$	union of $\mathcal{H}_m(\sigma_0)$ spaces
$A * b$	convolution between functions A and b
$B(\partial_t)f$	Laplace differential operator on $[0,T]$ defined for suitable $m \in \mathbb{N}$ as $\partial_t^m \mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}(f)(s))$
$H^m_{0,*}[0,T]$	space of m-times differentiable functions with initial condition zero
$H^m_{*,0}[0,T]$	space of m-times differentiable functions with end condition zero

Vector valued Laplace transform and differential operators

$\mathcal{L}, \mathcal{L}^{-1}$	vector valued (inverse) Laplace transform
$\mathcal{F}, \mathcal{F}^{-1}$	vector valued (inverse) Fourier transform
$X, [\,\cdot\,,\,\cdot\,]_X$	Hilbert space and Hilbert space scalar product
$L(X)$	space of linear, bounded operators $X \to X$
$L^2_c([0,\infty), X)$	Hilbert space valued e^{-ct}-weighted $L^2[0,\infty)$ space
$L^2_*([0,\infty), X)$	union of $L^2_c([0,\infty), X)$ spaces
$\mathcal{H}(\sigma_0, X)$	Hilbert space valued Hardy space on $\{\Re s > \sigma_0\}$
$\mathcal{H}(X)$	union of $\mathcal{H}(\sigma_0, X)$ spaces
$B(\partial_t)f$	Laplace differential operator on $[0,\infty)$ defined as $\mathcal{L}^{-1}(B(s)\mathcal{L}(f)(s))$
$H^m_{0,c}([0,\infty), X)$	exponentially weighted spaces with homogeneous initial condition
$H^m_{0,*}([0,\infty), X)$	union of $H^m_{0,c}([0,\infty), X)$ spaces
$\mathcal{H}_m(\sigma_0)$	space of analytic and by s^m bounded operators $X \to X$ on $\{\Re s > \sigma_0\}$
$\mathcal{H}_m$	union of $\mathcal{H}_m(\sigma_0)$ spaces
$A * b$	convolution of operator family $A(t) \in L(X)$ with $b(t) \in X$
$L^1_c([0,\infty), X)$	exponentially weighted $L^1([0,\infty), X)$ space
$L^\infty_c([0,\infty), X)$	exponentially weighted $L^\infty([0,\infty), X)$ space
$B(\partial_t)f$	Laplace differential operator on $[0,T]$ defined for suitable $m \in \mathbb{N}$ as $\partial_t^m \mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}(f)(s))$
$H^m_{0,*}([0,T], X)$	space of m-times differentiable functions with initial condition zero
$H^m_{*,0}([0,T], X)$	space of m-times differentiable functions with end condition zero

2. Reformulation and Function Spaces

In this chapter, we reformulate the Maxwell–LLG system (1.12) into a system that has advantageous properties concerning numerical approximation and analytic considerations. Related studies for the acoustic wave equation can be found in [143, 101] and for the Maxwell equations in [149] and [143, Appendix A]. We start with a formal derivation in Section 2.1, introduce the relevant function spaces and operators in Section 2.2 and Section 2.3 and consider precise arguments in Section 2.4.

2.1. Reformulation of the System

Let us recall the Maxwell–LLG system (1.12). The interior problem reads as :

$$\begin{aligned} \partial_t m &= -m \times H_{\text{eff}} - \alpha m \times (m \times H_{\text{eff}}) && \text{in } \Omega_T, \\ \varepsilon \partial_t E &= \nabla \times H - (\sigma E + J) && \text{in } \Omega_T, \\ \mu \partial_t H &= -\nabla \times E - \mu \partial_t m && \text{in } \Omega_T, \end{aligned}$$

with $H_{\text{eff}} = \frac{1}{1+\alpha^2}(C_e \Delta m + H)$, while the exterior problem reads as

$$\begin{aligned} \varepsilon_0 \partial_t E &= \nabla \times H && \text{in } \overline{\Omega}^{\text{c}}_T, \\ \mu_0 \partial_t H &= -\nabla \times E && \text{in } \overline{\Omega}^{\text{c}}_T, \end{aligned}$$

coupled by the transmission conditions

$$\gamma_T E^{\text{int}} = \gamma_T E^{\text{ext}} \quad \text{and} \quad \gamma_T H^{\text{int}} = \gamma_T H^{\text{ext}} \quad \text{on } [0, T] \times \Gamma,$$

where the tangential trace operator is given by $\gamma_T u = u \times n$ for the outward pointing normal vector n. Note that the notation E^{int}, E^{ext} means that the function (or the limit to the boundary) from the inside or the outside, respectively, is taken, but n always points from the interior domain to the exterior domain. The initial data and boundary condition for the LLG equation stay the same as in (1.12).

While the interior Maxwell equations remains unchanged, we present different versions of the LLG equation in Subsection 2.1.1, and in Subsection 2.1.2 the exterior Maxwell equations are rewritten into an integral equation on the boundary. At the end of the section, using this coupled formulation, the whole Maxwell–LLG system is rewritten. Most of the analysis is presented in a fairly formal way, while rigorous arguments can be found in Section 2.4.

2.1.1. Reformulation of the LLG equation

In this section we present the different versions of the LLG equation. We start with the formulation arising from the physical modeling, the so-called Landau–Lifshitz form

$$\partial_t m = -\frac{1}{1+\alpha^2} m \times h_{\text{eff}} - \frac{\alpha}{1+\alpha^2} m \times (m \times h_{\text{eff}}), \tag{2.1}$$

with $h_{\text{eff}} = C_e \Delta m + H$. By scalar product with m, we obtain $\partial_t m \cdot m = 0$, i.e. $\partial_t |m|^2 = 2\partial_t m \cdot m = 0$, so the absolute value of the magnetization stays constant. Due to the property of the initial condition $|m_0(x)| = 1$, we obtain $|m(t,x)| = 1$ for all time and space points.

Application of $m \times (\,\cdot\,)$ to (2.1) and using the vector identity for $a, b, c \in \mathbb{R}^3$

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c \tag{2.2}$$

gives together with $|m| = 1$ that

$$m \times \partial_t m = -\frac{1}{1+\alpha^2} m \times (m \times h_{\mathrm{eff}}) + \frac{\alpha}{1+\alpha^2} m \times h_{\mathrm{eff}}.$$

Multiplying this outcome with α and subtracting it from (2.1), yields the Gilbert form

$$\partial_t m - \alpha m \times \partial_t m = -m \times h_{\mathrm{eff}}. \tag{2.3}$$

The Gilbert form will be used in the definition of the weak solution of the MLLG system in Chapter 3.

In Chapter 5 and for the approximation of the LLG equation, the following alternative form plays a crucial role. By applying $m \times \cdot$ to (2.3), again with the vector identity (2.2) and $|m| = 1$ we obtain

$$\alpha \partial_t m + m \times \partial_t m = h_{\mathrm{eff}} - (m \cdot h_{\mathrm{eff}})m. \tag{2.4}$$

Under the condition $|m| = 1$, all of the versions (2.1), (2.3) and (2.4) are equivalent. For the Landau–Lifshitz and the Gilbert form, $|m| = 1$ already follows, if the initial data fulfills $|m^0| = 1$. Similar considerations and a more detailed proof of the equivalence can be found in [141, Proposition 3.1.1] and [72, Lemma 1.2.1], respectively.

2.1.2. Reformulation of the exterior Maxwell equations

In this section we present the reformulation of the exterior Maxwell equations to a boundary integral equation in a formal way.

As the Maxwell equations are formulated on the whole space $\mathbb{R}^3$, we are not able to apply a standard finite element discretization to discretize the problem in space. Similarly to [99, Section 2 and 4.2], we transform the interior–exterior Maxwell equations with the transmission conditions into a boundary integral equation on the boundary Γ coupled to the Maxwell equations in Ω.

We start with a formal derivation and return to the precise smoothness requirements later in Section 2.4. All identities below can be shown to hold true in a reasonable sense for functions that are smooth enough and the functions and there derivatives are integrable enough.

As the reformulation is independent of what happens in the interior, we only consider the exterior problem, where we assume to have given boundary values $\gamma_T E^{\mathrm{int}}$, $\gamma_T H^{\mathrm{int}}$ from inside. The problem is considered on the time interval $[0,\infty)$ instead of $[0,T]$. Given the exterior part $E, H : [0,\infty) \times \overline{\Omega}^{\mathrm{c}} \to \mathbb{R}^3$ of a solution of (1.12),

$$\begin{aligned}
\varepsilon_0 \partial_t E - \nabla \times H &= 0 && \text{in } (0,\infty) \times \overline{\Omega}^{\mathrm{c}}, \\
\mu_0 \partial_t H + \nabla \times E &= 0 && \text{in } (0,\infty) \times \overline{\Omega}^{\mathrm{c}}, \\
E(0) &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\
H(0) &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\
\gamma_T E &= \gamma_T E^{\mathrm{int}} && \text{in } [0,\infty) \times \Gamma, \\
\gamma_T H &= \gamma_T H^{\mathrm{int}} && \text{in } [0,\infty) \times \Gamma,
\end{aligned}$$

we differentiate the first equation in time and eliminate the magnetic field variable H by inserting the second equation. We therefore obtain the second order problem

$$\begin{aligned} \varepsilon_0\mu_0\partial_t^2 E + \nabla\times(\nabla\times E) &= 0 && \text{in } (0,\infty)\times\overline{\Omega}^{\mathrm{c}}, \\ E(0) &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \partial_t E(0) &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \gamma_T E &= \gamma_T E^{\mathrm{int}} && \text{in } [0,\infty)\times\mathit{\Gamma}, \\ \int_0^t \gamma_T(\nabla\times E)(r,\cdot)\,\mathrm{d}r &= -\mu_0\gamma_T H^{\mathrm{int}}(t,\cdot) && \text{in } [0,\infty)\times\mathit{\Gamma}. \end{aligned}$$

The next step is to apply the Laplace transform $\mathcal{L}$ to this system. We set $U := \mathcal{L}(E)$, where $\mathcal{L}$ is the Laplace transform given by

$$V(s) := \mathcal{L}(v)(s) := \int_0^\infty v(t)e^{-st}\,\mathrm{d}t, \qquad \text{for } s\in\mathbb{C}.$$

For suitable functions, the Laplace transform $\mathcal{L}$ is invertible by the inverse Laplace transform defined for any $\varepsilon > 0$ as

$$\mathcal{L}^{-1}(V)(t) := \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{st}V(s)\mathrm{d}s = v(t), \qquad \text{for } t\in[0,\infty).$$

One can show that the definition of the inverse Laplace transform does not depend on the parameter $\varepsilon > 0$ (cf. Lemma B.53).

With the properties of the Laplace transform from Example B.61, we have for a function v with $v(0) = 0$ that $\mathcal{L}(\partial_t v)(s) = s\mathcal{L}(v)(s)$, i.e. derivative in time on v transforms to multiplication with s of V. Similarly, $\mathcal{L}(\int_0^t v(r)\,\mathrm{d}r)(s) = (1/s)\mathcal{L}(v)(s)$, i.e. integration of v over time corresponds to a multiplication with $1/s$ of V. We set

$$\left(\partial_t^{-1}E\right)(t) := \int_0^t E(r)\,\mathrm{d}r = \mathcal{L}^{-1}\left(\frac{1}{s}(\mathcal{L}E)(s)\right)(t).$$

With the differential properties of the Laplace transform and the initial conditions for E, we obtain $\mathcal{L}(\partial_t^2 E)(s) = s^2\mathcal{L}(E)(s)$ and the equation for $U = \mathcal{L}E$ reads

$$\begin{aligned} \varepsilon_0\mu_0 s^2 U(s) + \nabla\times\nabla\times U(s) &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}} \text{ for } s\in\mathbb{C}, \\ \mathcal{L}^{-1}(U)(0) &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \partial_t\mathcal{L}^{-1}(U)(0) &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \gamma_T U(s) &= \mathcal{L}(\gamma_T E^{\mathrm{int}})(s) && \text{in } \mathit{\Gamma} \text{ for } s\in\mathbb{C}, \\ \frac{1}{s}\gamma_T(\nabla\times U(s)) &= -\mu_0\mathcal{L}(\gamma_T H^{\mathrm{int}})(s) && \text{in } \mathit{\Gamma} \text{ for } s\in\mathbb{C}. \end{aligned}$$

We fix $s\in\mathbb{C}$ and look at the corresponding time-harmonic equation for U, i.e.

$$\begin{aligned} \varepsilon_0\mu_0 s^2 U + \nabla\times\nabla\times U &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \gamma_T U &= A_T && \text{in } \mathit{\Gamma}, \\ \gamma_N U &= A_N && \text{in } \mathit{\Gamma}, \end{aligned} \tag{2.5}$$

with the Neumann trace operator $\gamma_N U := s^{-1}\gamma_T(\nabla\times U)$ and the abbreviations $A_T = \mathcal{L}(\gamma_T E^{\mathrm{int}})$ and $A_N = \mathcal{L}(-\mu_0\gamma_T H^{\mathrm{int}})$. This problem can be solved analytically. There is a representation formula for the solution U which will hold under a compatibility condition on the traces A_T and A_N. Conversely, this compatibility condition is equivalent to the solvability of the problem. To state the result, we introduce the following operators: For $x\in\mathbb{R}^3\setminus\mathit{\Gamma}$, the electric single layer potential is given by

$$\left(\mathcal{S}(s)\varphi\right)(x) := s\int_{\mathit{\Gamma}} G(s,x-y)\varphi(y)\mathrm{d}y - s^{-1}\frac{1}{\epsilon_0\mu_0}\nabla\int_{\mathit{\Gamma}} G(s,x-y)\mathrm{div}_{\mathit{\Gamma}}\varphi(y)\mathrm{d}y$$

and the electric double layer potential, for $x \in \mathbb{R}^3 \setminus \Gamma$, by

$$(\mathcal{D}(s)\varphi)(x) = \nabla \times \int_\Gamma G(s, x-y)\varphi(y) \, \mathrm{d}y,$$

see [46] for more details, where the fundamental solution $G(s, z)$ is given for $z \in \mathbb{R}^3 \setminus \{0\}$, as

$$G(s, z) = \frac{e^{-s\sqrt{\varepsilon_0\mu_0}|z|}}{4\pi|z|}.$$

We use the Calderon operator $B(s)$, cf. [99, Section 3],

$$B(s) := \mu_0^{-1} \begin{pmatrix} (i\sqrt{\mu_0\varepsilon_0})^{-1}V(s) & K(s) \\ -K(s) & -i\sqrt{\mu_0\varepsilon_0}V(s) \end{pmatrix} \tag{2.6}$$

with the boundary integral operators (cf. [46, Equation (30)] for the identities)

$$\begin{aligned} V(s) &= i\sqrt{\mu_0\varepsilon_0}\{\!\{\gamma_T \circ \mathcal{S}(s)\}\!\} = (i\sqrt{\mu_0\varepsilon_0})^{-1}\{\!\{\gamma_N \circ \mathcal{D}(s)\}\!\}, \\ K(s) &= \{\!\{\gamma_T \circ \mathcal{D}(s)\}\!\} = \{\!\{\gamma_N \circ \mathcal{S}(s)\}\!\}, \end{aligned} \tag{2.7}$$

where $\{\!\{\cdot\}\!\}$ denotes the average

$$\{\!\{\gamma_T u\}\!\} := \frac{\gamma_T u^{\mathrm{int}} + \gamma_T u^{\mathrm{ext}}}{2}.$$

With these operators, we are able to rephrase (2.5).

The result [46, Theorem 8] shows that U is a solution to (2.5) (that fulfills the Silver–Müller radiation condition (see (2.18) for the definition)) if and only if

$$B(s)\begin{pmatrix} -A_N \\ -A_T \end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix} A_T \\ -A_N \end{pmatrix} \tag{2.8a}$$

and

$$U = \mathcal{S}(s)(-A_N) + \mathcal{D}(s)(-A_T). \tag{2.8b}$$

Equation (2.8a) corresponds to a compatibility condition for the traces A_T, A_N and in the case this compatibility condition is fulfilled, we may represent the solution of (2.5) with equation (2.8b). Conversely, every solution of (2.5) can be represented by its traces with (2.8b) and its traces fulfill the compatibility condition (2.8a).

With this result, the time harmonic problem is solved, we can replace the exterior problem by the compatibility condition and to get back to a problem in space and time, we apply the inverse Laplace transform.

Applying the inverse Laplace transform to (2.8), and inserting $A_T = \mathcal{L}(\gamma_T E^{\mathrm{int}})$, $A_N = \mathcal{L}(-\mu_0\gamma_T H^{\mathrm{int}})$, we obtain that (E, H) is the exterior part of a solution of (1.12) if and only if

$$\begin{aligned} E &= \mathcal{S}(\partial_t)(\mu_0\gamma_T H^{\mathrm{int}}) + \mathcal{D}(\partial_t)(-\gamma_T E^{\mathrm{int}}) && \text{in } (0,\infty) \times \overline{\Omega}^{\mathrm{c}}, \\ H &= -\frac{1}{\mu_0}\partial_t^{-1}\nabla \times E && \text{in } (0,\infty) \times \overline{\Omega}^{\mathrm{c}}, \\ B(\partial_t)\begin{pmatrix} \mu_0\gamma_T H^{\mathrm{int}} \\ -\gamma_T E^{\mathrm{int}} \end{pmatrix} &= \frac{1}{2\mu_0}\begin{pmatrix} \gamma_T E^{\mathrm{int}} \\ \mu_0\gamma_T H^{\mathrm{int}} \end{pmatrix} && \text{in } [0,\infty) \times \Gamma. \end{aligned} \tag{2.9}$$

Here $\mathcal{S}(\partial_t)$, $\mathcal{D}(\partial_t)$, $\mathcal{B}(\partial_t)$ are defined via

$$\mathcal{B}(\partial_t)u := \mathcal{L}^{-1}(B(s)\mathcal{L}u) := \mathcal{L}^{-1}(s \mapsto B(s)\mathcal{L}(u)(s)).$$

Note that the first two formulas in (2.9) are representation formulas for the exterior solution and the last one is a compatibility condition for $\gamma_T E^{\text{int}}$ and $\gamma_T H^{\text{int}}$. Consistency with the exterior solution of (1.12) demands $\gamma_T E^{\text{int}}(0,x) = \gamma_T E^{\text{ext}}(0,x) = 0$ and $\gamma_T H^{\text{int}}(0,x) = \gamma_T H^{\text{ext}}(0,x) = 0$ for all $x \in \Gamma$.

The derivation so far showed how we can replace the exterior equations by the compatibility condition for the boundary functions and how we obtain the new system. Before restating the full MLLG system, we consider the operator $B(\partial_t)$ in more detail. The following questions may be posed.

1) How can we define $B(\partial_t)$ without deeper knowledge of the Laplace transform, and for which classes of functions is it defined?

2) The Laplace transform as well as the inverse Laplace transform act on functions that are defined on the whole positive line or an unbounded complex line, respectively. So the obtained boundary integral equation is now on the bounded set Γ in space, but in time, the integration is performed at least over the whole unbounded line $\mathbb{R}_+$, which still poses problems in numerical approximations. How can we come back to a problem that is posed on the finite time interval $[0,T]$?

The answers to these questions will be given in the following:

Step 1: The properties of the Laplace transform (Lemma B.22) show for a family of analytic and suitably bounded operators $A(s)$ and functions ϕ

$$\mathcal{L}^{-1}\left(A(s)\mathcal{L}\phi\right) = \mathcal{L}^{-1}(A(s)) * \phi, \tag{2.10}$$

where $*$ denotes convolution

$$(C * w)(t) := \int_0^t C(r)w(t-r)\,\mathrm{d}r = \int_0^t C(t-r)w(r)\,\mathrm{d}r.$$

Step 2: As we only have $\|B(s)\| \le Cs^2$ (cf. Lemma 2.13), we cannot conclude that $\mathcal{L}^{-1}(B(s))$ exists. However for $m > 3$,

$$B_m(t) := \mathcal{L}^{-1}(B(s)s^{-m})(t) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{st}s^{-m}B(s)\mathrm{d}s$$

exists for all $t \ge 0$ and is a continuous and bounded function on $[0,T]$ (cf. Lemma B.66). For $m \in \mathbb{N}$ and $\phi \in C^m([0,\infty),\mathcal{H}_\Gamma^2)$ with $\phi(0) = \partial_t\phi(0) = \cdots = \partial_t^{m-1}\phi(0) = 0$ we have

$$\mathcal{L}^{-1}(s^m\mathcal{L}(\phi)(s)) = \partial_t^m\phi.$$

Step 3: Altogether, it holds for $m \in \mathbb{N}$, $m > 3$,

$$\begin{aligned}
(B(\partial_t)\phi)(t) &= \mathcal{L}^{-1}(B(s)\mathcal{L}(\phi)(s))(t)\\
&= \mathcal{L}^{-1}(B(s)s^{-m}s^m\mathcal{L}(\phi)(s))(t)\\
&= \left(\mathcal{L}^{-1}(B(s)s^{-m}) * \mathcal{L}^{-1}(\mathcal{L}(\phi)(s)s^m)\right)(t)\\
&= \int_0^t B_m(r)(\partial_t^m\phi)(t-r)\,\mathrm{d}r\\
&= -B_m(t)\partial_t^{m-1}\phi(0) + \partial_t\int_0^t B_m(r)\partial_t^{m-1}(\phi(t-r))\,\mathrm{d}r\\
&= \partial_t\int_0^t B_m(r)\partial_t^{m-1}(\phi(t-r))\,\mathrm{d}r\\
&= \cdots = \partial_t^m\int_0^t B_m(r)\phi(t-r)\,\mathrm{d}r,
\end{aligned}$$

which states

$$B(\partial_t)\phi = \partial_t^m(B_m * \phi). \tag{2.11}$$

From this formula we can see that $B(\partial_t)\phi(\bar{t})$ only depends on $\phi(t)$ for $0 \le t \le \bar{t}$, which is also known as *Causality*. This is why we can drop the unbounded time interval $[0, \infty)$ and deal with $[0, T]$. Furthermore the above formula shows that we can define $B(\partial_t)\phi$ even for functions $\phi \in L^2([0, T])$, at least as long as $B_m * \phi$ is m times differentiable in time. With this, we obtain the final formulation of the MLLG system. As in Chapter B we define the convolution operator $B(\partial_t)$ following 2.11 as $B(\partial_t) \cdot = \partial_t^m (B_m * \cdot)$, as long as we work on finite time intervals. In the following, $m \in \mathbb{N}$, $m > 3$ shall be fixed. As in Section B.1.3, we show that the definition does not depend on the choice of $m \in \mathbb{N}$. Hopefully, the reader might not get confused because of the naming of the variable m, as it stays on the one hand for the magnetization $m : [0, T] \times \Omega \to \mathbb{R}^3$ and on the other hand for the natural number $m \in \mathbb{N}$ in the definition of the Calderon operator.

We summarize the reformulation in the following subsection.

2.1.3. The resulting system

The coupled Maxwell–Landau–Lifshitz–Gilbert equation (1.12) is rewritten into the following system *only* in the interior domain Ω and on its boundary Γ. Find the functions m, E and $H : [0, T] \times \Omega \to \mathbb{R}^3$ which satisfy the following coupled system: in the interior domain

$$\begin{aligned} \partial_t m - \alpha m \times \partial_t m &= -m \times (C_e \Delta m + H) && \text{in } \Omega_T, && (2.12a)\\ \varepsilon \partial_t E - \nabla \times H &= -(J + \sigma E) && \text{in } \Omega_T, && (2.12b)\\ \mu \partial_t H + \nabla \times E &= -\mu \partial_t m && \text{in } \Omega_T, && (2.12c) \end{aligned}$$

coupled to the boundary integral equations

$$B(\partial_t) \begin{pmatrix} \mu_0 \gamma_T H \\ -\gamma_T E \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mu_0^{-1} \gamma_T E \\ \gamma_T H \end{pmatrix} \qquad \text{on } [0, T] \times \partial\Omega, \tag{2.12d}$$

and where m satisfies the boundary condition as before

$$\partial_n m = 0 \qquad \text{on } [0, T] \times \partial\Omega, \tag{2.12e}$$

and with the same initial conditions for the problems in Ω as in (1.12)

$$m(0) = m^0, \quad E(0) = E^0, \quad H(0) = H^0 \quad \text{in } \Omega. \tag{2.12f}$$

2.2. Function Spaces

In this section we introduce all the needed function spaces. We recall that $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain, and $T > 0$.

2.2.1. Lebesgue and Sobolev spaces

We shortly repeat the definitions of the most important function spaces required in the following. Let $D \subset \mathbb{R}^n$ and $m, n \in \mathbb{N}$. We define the Lebesgue spaces for $p \in [1, \infty]$ as

$$L^p(D) := L^p(D, \mathbb{C}^m) := \{v : D \to \mathbb{C}^m \mid v \text{ measurable and } \int_D |v(x)|^p \, \mathrm{d}x < \infty\},$$

together with the norm

$$\|v\|_{L^p(D)} = \begin{cases} \left(\int_D |v(x)|^p \, \mathrm{d}x\right)^{1/p}, & \text{for } p \in [1, \infty) \\ \operatorname{ess\,sup}_{x \in D} |v(x)|, & \text{for } p = \infty \end{cases}.$$

For $p = 2$, the $L^2(\Omega)$-space is a Hilbert space together with the scalar product

$$[u, v]_{L^2(D)} = \int_D \overline{u}(x) \cdot v(x) \, \mathrm{d}x.$$

We conterminously write $\|\cdot\|_{L^2}$, $\|\cdot\|_D$, $[\cdot,\cdot]_{L^2}$ or $[\cdot,\cdot]_D$ for the $L^2(D)$-norms and $L^2(D)$-products, if the domain or $p = 2$ are clear from the context. For $k \in \mathbb{N}$ we define the space of k-times weakly differentiable functions $H^k(D)$ via

$$H^k(D) = \{v \in L^2(D) \,|\, \partial_x^\alpha v \in L^2(D) \text{ for all multiindices } |\alpha|_1 \leq k\}.$$

This is a Hilbert space with respect to the scalar product

$$[u, v]_{H^k(D)} = \sum_{\alpha \in \mathbb{N}_0^n, |\alpha|_1 \leq k} [\partial_x^\alpha u, \partial_x^\alpha v]_{L^2(D)}$$

and the induced norm $\|v\|^2_{H^k(D)} = [v, v]_{H^k(D)}$. For the Maxwell equations in three dimensions (so $m = n = 3$), the natural space contains functions with weak curl, we define

$$H(\mathrm{curl}, \Omega) = \{v \in L^2(\Omega) \,|\, \nabla \times v \in L^2(\Omega)\},$$

which is a Hilbert space with the scalar product

$$[u, v]_{H(\mathrm{curl},\Omega)} = [u, v]_{L^2(\Omega)} + [\nabla \times u, \nabla \times v]_{L^2(\Omega)}$$

and the induced norm $\|v\|^2_{H(\mathrm{curl},\Omega)} = [v, v]_{H(\mathrm{curl},\Omega)}$. For space and time dependent functions, we may consider $\Omega_T \subset \mathbb{R}^4$ as four dimensional domain and the above definitions cover, e.g. the definition of the spaces $L^2(\Omega_T)$ and $H^1(\Omega_T)$. We define the anisotropic Hilbert spaces for $k, l \in \mathbb{N}_0$

$$H^{k,l}(\Omega_T) := \{v \in L^2(\Omega_T) \,|\, \partial_t^i v, \partial_x^\alpha v \in L^2(\Omega_T) \text{ for all } 0 \leq i \leq k \text{ and all } |\alpha|_1 \leq l\}$$

and

$$H^{1,\mathrm{curl}}(\Omega_T) = H(\partial_t, \mathrm{curl}, \Omega_T) = \{v \in L^2(\Omega_T) \,|\, \partial_t v, \nabla_x \times v \in L^2(\Omega_T)\},$$
$$H^{0,\mathrm{curl}}(\Omega_T) = H(\mathrm{curl}, \Omega_T) = \{v \in L^2(\Omega_T) \,|\, \nabla_x \times v \in L^2(\Omega_T)\}.$$

Again, those spaces are Hilbert spaces with their standard norms and inner products. For $k \in \mathbb{N}$ and $\phi \in H^k([0,T])$, the traces $\phi(0), \ldots, \phi^{k-1}(0)$ and $\phi(T), \ldots, \phi^{k-1}(T)$ are well defined, so we may define the spaces with vanishing initial condition or end condition, respectively, as

$$H^k_{0,*}([0,T]) := \{\varphi \in H^k([0,T]) \,|\, \varphi(0) = \cdots = \partial_t^{k-1}\varphi(0) = 0\}$$

and

$$H^k_{*,0}([0,T]) := \{\varphi \in H^k([0,T]) \,|\, \varphi(T) = \cdots = \partial_t^{k-1}\varphi(T) = 0\}.$$

As closed subspaces of the Hilbert space $H^k([0,T])$, those are Hilbert spaces, too. In Subsection 2.2.4, the definitions will be generalized to Hilbert space valued functions $\varphi : [0,T] \to X$ for a Hilbert space X.

For $s \in (0,1)$ we define the interpolation spaces $H^s(D) := \{L^2(D), H^1(D)\}_s$ as enclosure of $C^\infty(D)$ (space of infinitely differentiable functions) with respect to the norm

$$\|u\|^2_{\{L^2(D),H^1(D)\}_s} := \int_0^\infty t^{-2s-1} \inf_{v \in H^1(D)} (\|u - v\|_{L^2(D)} + t\|v\|_{H^1(D)})^2 \, \mathrm{d}t.$$

Similarly on the boundary ∂D, denoting $H^1(\partial D)$ the functions with existing surface gradient in $L^2(\partial D)$, we define by interpolation $H^{1/2}(\partial D) := \{L^2(D\partial), H^1(\partial D)\}_{1/2}$. As dual space with respect to the $[\,\cdot\,,\,\cdot\,]_{\partial D}$-product, we denote $H^{-1/2}(\partial D) := (H^{1/2}(\partial D))'$. Those spaces are Hilbert spaces and the trace operator $\gamma : L^2(D) \to H^{-1/2}(\partial D)$ is continuous. We refer to [137, Section 2.3] for an overview and the references given therein for the details.

2.2.2. Hilbert spaces

In this section we collect properties of Hilbert spaces as needed in the following chapters based on [154]. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 2.1. *A $\mathbb{K}$-vector space X with a scalar product $[\cdot,\cdot]_X : X \times X \to \mathbb{K}$ is called Hilbert space if X is complete with respect to the norm $\|\cdot\|_X = ([\cdot,\cdot]_X)^{1/2}$.*

Lemma 2.2 (Fréchet–Riesz representation theorem)**.** *We denote the dual space of X by X', i.e. the space of linear, continuous mappings $X \to \mathbb{K}$, endowed with the usual supremum norm $\|\cdot\|_{X'}$. For every $x' \in X'$ there exists exactly one $y \in X$ such that $x'(x) = [x, y]_X$ for all $x \in X$ and it holds $\|x'\|_{X'} = \|y\|_X$.*

Lemma 2.3. *If X is a separable Hilbert space, there exists a orthonormal basis B satisfying for each $x \in X$*

$$x = \sum_{e \in B} [x, e]_X e \quad \text{and} \quad \|x\|_X^2 = \sum_{e \in B} |[e, x]_X|^2.$$

Definition 2.4. *For a sequence $(v_n)_{n\in\mathbb{N}} \subset X$ we say v_n converges weakly to $v \in \mathcal{H}_\Gamma$ with respect to a pairing $\langle\cdot,\cdot\rangle_X$,*

$$v_n \rightharpoonup v \text{ w.r.t. } \langle\cdot,\cdot\rangle_X,$$

if and only if

$$\langle v_n, u\rangle_X \to \langle v, u\rangle_X \text{ for all } u \in X.$$

The following Lemma will be central in the subsequent chapters.

Lemma 2.5. *Let X be an Hilbert space. Let $(v_n)_{n\in\mathbb{N}} \subset X$ be a bounded sequence in X. Then there exists a subsequence $(n_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ and a limit function $v \in X$ such that v_n converges weakly to v with respect to the Hilbert space product, i.e.*

$$v_{n_j} \rightharpoonup v \quad \text{w.r.t. } [\cdot,\cdot]_X, \text{ for } j \to \infty.$$

We also write $v_n \overset{\text{sub}}{\rightharpoonup} v$ in that case.

2.2.3. The trace space for boundary integral formulation

For the boundary integral formulation and the positivity of the Calderon operator, we require a particular trace space from [99, Section 2.1], for more details we refer to [46, 43, 44, 45]. We keep the formal definition short and focus on the properties.

We recall the definition of the tangential trace on the boundary Γ for $w \in C(\overline{\Omega})$ as

$$\gamma_T w = w \times n,$$

where n is the outward pointing normal vector on Γ. Note that this definition can be extended continuously to $H(\text{curl}, \Omega)$ via the formula

$$[\nabla \times v, w]_\Omega - [v, \nabla \times w]_\Omega = \int_\Gamma (w \times n) \cdot v \, \mathrm{d}\sigma.$$

The actual image space $\mathcal{H}_\Gamma$ of $\gamma_T : H(\text{curl}, \Omega) \to \mathcal{H}_\Gamma$ is introduced in the following, for a more detailed overview we refer to [46, Section 2.2]. If Ω has smooth boundary, we introduce the differential surface operator div_Γ using local charts and transformations, see, e.g., [123, Section 2.5.2] or [121, Section 3.4]. In the case of piecewise smooth boundary, the definition is more involved, yielding a piecewise definition inside and with distributional jumps across the smooth boundary segments, see [46, Section 2.2]. In both cases the following properties hold true, where we refer to [123, Section 5.4.1] and [121, Section 3.5] for the smooth case and [43, 44, 45] for the piecewise smooth or Lipschitz case.

Definition 2.6 (Trace space, [45]). *The trace space is given by*

$$\mathcal{H}_\Gamma := \{w \in \gamma_T(H^1(\Omega))' \,|\, \mathrm{div}_\Gamma w \in H^{-1/2}(\Gamma)\}$$

with the norm

$$\|w\|^2_{\mathcal{H}_\Gamma} = \|w\|^2_{\gamma_T(H^1(\Omega))'} + \|\mathrm{div}_\Gamma w\|^2_{H^{-1/2}(\Gamma)}.$$

The following properties hold true.

- *The trace operator $\gamma_T : H(\mathrm{curl}, \Omega) \to \mathcal{H}_\Gamma$ is continuous and surjective, see [45, Theorem 4.1].*
- *The anti-symmetric pairing*

  $$\langle w, v\rangle_\Gamma := \int_\Gamma (w \times n) \cdot v \,\mathrm{d}\sigma = \int_\Gamma -(w \times v) \cdot n \,\mathrm{d}\sigma$$

  for $w, v \in L^2(\Gamma)^3$ can be extended to a continuous, anti-symmetric bilinear form on $\mathcal{H}_\Gamma$. The boundary space $\mathcal{H}_\Gamma$ is its own dual with respect to $\langle\cdot,\cdot\rangle_\Gamma$, see [46, Theorem 2].
- *For $w, v \in H(\mathrm{curl}, \Omega)$ Green's formula holds*

  $$[\nabla \times v, w]_\Omega - [v, \nabla \times w]_\Omega = -\langle \gamma_T v, \gamma_T w\rangle_\Gamma. \tag{2.13}$$

Proof. The proofs can be found in the respective references cited above. We only show the validity of Green's formula. For $v, w \in \mathbb{C}^\infty(\Omega)$, we see with integration by parts

$$[\nabla \times v, w]_\Omega = [v, \nabla \times w]_\Omega - \langle v, w\rangle_\Gamma.$$

By using the identity for vectors $a, b, n \in \mathbb{R}^3$ with $|n| = 1$

$$((a \times n) \times (b \times n)) \cdot n = (a \times b) \cdot n,$$

we obtain

$$\langle v, w\rangle_\Gamma = \langle \gamma_T v, \gamma_T w\rangle_\Gamma.$$

The facts that $\mathbb{C}^\infty(\Omega)$ is dense in $H(\mathrm{curl}, \Omega)$ and that $\gamma_T : H(\mathrm{curl}, \Omega) \to \mathcal{H}_\Gamma$ is continuous conclude the assertion. □

Note that the anti-symmetric pairing $\langle\cdot,\cdot\rangle_\Gamma$ is not the Hilbert space scalar product on $\mathcal{H}_\Gamma$, which we denote by $[\cdot,\cdot]_{\mathcal{H}_\Gamma}$ (compare also the $L^2(\Gamma)$-product $[\cdot,\cdot]_\Gamma$). However we may define the corresponding adjoint T^* of an operator $T\colon \mathcal{H}_\Gamma \to \mathcal{H}_\Gamma$ as well as weak convergence with respect to $\langle\cdot,\cdot\rangle_\Gamma$ (which coincides with ordinary weak convergence in $\mathcal{H}_\Gamma$). This is proven by the following lemmas.

We have that $\mathcal{H}_\Gamma$ is its own dual with respect to $\langle\cdot,\cdot\rangle_\Gamma$, so the operator

$$\Phi : \mathcal{H}_\Gamma \to {\mathcal{H}_\Gamma}', v \mapsto \langle v, \cdot\rangle_\Gamma$$

is continuous and continuously invertible. By Hilbert space theory, we have that

$$\Psi : \mathcal{H}_\Gamma \to {\mathcal{H}_\Gamma}', v \mapsto [v, \cdot]_{\mathcal{H}_\Gamma}$$

is an isometric isomorphism, where $[\cdot,\cdot]_{\mathcal{H}_\Gamma}$ denotes the Hilbert space scalar product. By use of these isomorphisms, we are able to prove the following lemmas.

Definition 2.7. *For a sequence $(v_n)_{n\in\mathbb{N}} \subset \mathcal{H}_\Gamma$ we say v_n* converges weakly to $v \in \mathcal{H}_\Gamma$ with respect to $\langle\cdot,\cdot\rangle_\Gamma$,

$$v_n \rightharpoonup v \text{ w.r.t. } \langle\cdot,\cdot\rangle_\Gamma,$$

if and only if

$$\langle v_n, u\rangle_\Gamma \to \langle v, u\rangle_\Gamma \text{ for all } u \in \mathcal{H}_\Gamma.$$

Lemma 2.8. *For a sequence $(v_n)_{n\in\mathbb{N}} \subset \mathcal{H}_\Gamma$, it holds*

$$v_n \rightharpoonup v \text{ w.r.t. } [\cdot,\cdot]_{\mathcal{H}_\Gamma} \quad \Longleftrightarrow \quad v_n \rightharpoonup v \text{ w.r.t. } \langle\cdot,\cdot\rangle_\Gamma.$$

Proof. Let $(v_n)_{n\in\mathbb{N}} \subset \mathcal{H}_\Gamma$ with $v_n \rightharpoonup v$ w.r.t. $[\cdot,\cdot]_{\mathcal{H}_\Gamma}$. As $\Psi^{-1}\Phi$ is a bounded operator, we have $\Psi^{-1}\Phi(v_n) \rightharpoonup \Psi^{-1}\Phi(v_n) =: \widetilde{v}$ with respect to $[\cdot,\cdot]_{\mathcal{H}_\Gamma}$. For arbitrary $w \in \mathcal{H}_\Gamma$ it holds

$$\begin{aligned}
\langle v_n, w\rangle_\Gamma &= \Phi(v_n)(w) \\
&= [\Psi^{-1}\Phi(v_n), w]_{\mathcal{H}_\Gamma} \\
&\to [\widetilde{v}, w]_{\mathcal{H}_\Gamma} \\
&= \Psi(\widetilde{v})(w) \\
&= \langle \Phi^{-1}\Psi\widetilde{v}, w\rangle_\Gamma.
\end{aligned}$$

Thus $v_n \rightharpoonup \Phi^{-1}(\Psi\widetilde{v}) = v$ with respect to the anti-symmetric pairing $\langle\cdot,\cdot\rangle_\Gamma$.

Conversely, choose an arbitrary sequence $(v_n)_{n\in\mathbb{N}} \subset \mathcal{H}_\Gamma$ with $v_n \rightharpoonup v$ w.r.t. $\langle\cdot,\cdot\rangle_\Gamma$. With the same computations as above, we obtain for arbitrary $w \in \mathcal{H}_\Gamma$

$$[\Psi^{-1}\Phi(v_n), w]_{\mathcal{H}_\Gamma} = \langle v_n, w\rangle_\Gamma \to \langle v, w\rangle_\Gamma = [\Psi^{-1}\Phi(v), w]_{\mathcal{H}_\Gamma}.$$

Thus $\Psi^{-1}\Phi(v_n) \rightharpoonup \Psi^{-1}\Phi(v)$ w.r.t. $[\cdot,\cdot]_{\mathcal{H}_\Gamma}$, and as $\Phi^{-1}\Psi$ is a bounded operator, we conclude $v_n \rightharpoonup v$ w.r.t. $[\cdot,\cdot]_{\mathcal{H}_\Gamma}$. □

In the following lemma, we show that we can define for an operator $T : \mathcal{H}_\Gamma \to \mathcal{H}_\Gamma$ the $\langle\cdot,\cdot\rangle_\Gamma$-adjoint operator T^* completely analogous to the adjoint operator with respect to the Hilbert space scalar product on $\mathcal{H}_\Gamma$.

Lemma 2.9. *For a continuous linear operator $T : \mathcal{H}_\Gamma \to \mathcal{H}_\Gamma$ there exists a unique linear continuous operator $T^* : \mathcal{H}_\Gamma \to \mathcal{H}_\Gamma$ that satisfies*

$$\langle T^*v, w\rangle_\Gamma = \langle v, Tw\rangle_\Gamma \qquad \text{for all } w, v \in \mathcal{H}_\Gamma.$$

Moreover it holds $(T^*)^* = T$ and there exist constants $c, C > 0$ independent of T such that

$$c\|T\| \leq \|T^*\| \leq C\|T\|.$$

Proof. We set $B = \Psi^{-1}\Phi$ satisfying

$$\langle v, w\rangle_\Gamma = [Bv, w]_{\mathcal{H}_\Gamma}.$$

Then we define $T^* := B^{-1}T'B$, where T' denotes the adjoint of T with respect to $[\cdot,\cdot]_{\mathcal{H}_\Gamma}$. Therefore it holds

$$\begin{aligned}
\langle T^*v, w\rangle_\Gamma &= [T'Bv, w]_{\mathcal{H}_\Gamma} \\
&= [Bv, Tw]_{\mathcal{H}_\Gamma} \\
&= \langle v, Tw\rangle_\Gamma.
\end{aligned}$$

Continuity and uniqueness of T^* follow from the continuity and uniqueness of T and B. □

Remark 2.10. *It holds $\|T\|_\Gamma = \|T^*\|_\Gamma$ with respect to the (equivalent) operator norm*

$$\|T\|_\Gamma = \sup_{v,w\in\mathcal{H}_\Gamma} \frac{|\langle Tv, w\rangle_\Gamma|}{\|v\|_{\mathcal{H}_\Gamma}\|w\|_{\mathcal{H}_\Gamma}}.$$

2.2.4. Bochner spaces

For functions depending on space and time, we need to introduce the following function spaces mapping into Banach or Hilbert spaces. The precise definitions of these spaces and the proofs of the properties can be found, e.g. in [19, 62, 88].

For a Banach space X and an interval $I \subset \mathbb{R}$, we define the Bochner-Lebesgue spaces for $p \in [1, \infty]$ as

$$L^p(I, X) = \{v : I \to X \,|\, v \text{ measurable and } \int_I \|v(x)\|_X^p \, \mathrm{d}x < \infty\},$$

together with the norm

$$\|v\|_{L^p(I,X)} = \begin{cases} \left(\int_D \|v(x)\|_X^p \, \mathrm{d}x\right)^{1/p}, & \text{for } p \in [1, \infty), \\ \text{ess sup}_{x \in D} \|v(x)\|_X, & \text{for } p = \infty. \end{cases}$$

For $p = 2$ and if X is a Hilbert space, this is a Hilbert space together with the scalar product

$$[u, v]_{L^2(D,X)} = \int_D [u(x), v(x)]_X \, \mathrm{d}x.$$

For $k \in \mathbb{N}$ we define the space of k- times weakly differentiable functions $H^k(I, X)$ via

$$H^k(I, X) = \{v \in L^2(I, X) \,|\, \partial_t^j v \in L^2(I, X) \text{ for } 1 \leq j \leq k\}.$$

If X is a Hilbert space, this is again a Hilbert space with respect to the scalar product

$$[u, v]_{H^k(I,X)} = \sum_{0 \leq j \leq k} [\partial_t^j u, \partial_t^j v]_{L^2(I,X)}.$$

For $k \in \mathbb{N}$ and $\phi \in H^k([0, T], X)$, the traces $\phi(0), \dots, \phi^{k-1}(0)$ and $\phi(T), \dots, \phi^{k-1}(T)$ are well defined, and we define the spaces with vanishing initial condition or end condition, respectively, as

$$H^k_{0,*}([0, T], X) = \{\varphi \in H^k([0, T], X) \,|\, \varphi(0) = \dots = \partial_t^{k-1} \varphi(0) = 0\} \tag{2.14}$$

and

$$H^k_{*,0}([0, T], X) = \{\varphi \in H^k([0, T], X) \,|\, \varphi(T) = \dots = \partial_t^{k-1} \varphi(T) = 0\}. \tag{2.15}$$

Similarly, we define the spaces of vector valued functions depending continuously on time, i.e.

$$\begin{aligned} C(I, X) &= \{v : I \to X \,|\, v \text{ is continuous } \}, \\ C^k(I, X) &= \{v : I \to X \,|\, \partial_t^j v \text{ exists and is continuous for } 0 \leq j \leq k\}, \\ C^k_{0,*}([0, T], X) &= \{\varphi \in C^k([0, T], X) \,|\, \varphi(0) = \dots = \partial_t^{k-1} \varphi(0) = 0\} \\ C^k_{*,0}([0, T], X) &= \{\varphi \in C^k([0, T], X) \,|\, \varphi(T) = \dots = \partial_t^{k-1} \varphi(T) = 0\}. \end{aligned}$$

endowed with the corresponding supremum norms over the time interval.

We proof the following assertion that will be crucial in the following chapters. We introduce the anti-symmetric pairing in $L^2([0, T], \mathcal{H}_\Gamma)$ as

$$\langle \varphi, \psi \rangle_{\Gamma_T} = \int_0^T \langle \varphi(t), \psi(t) \rangle_\Gamma \, \mathrm{d}t.$$

Lemma 2.11. *Let $(\phi^n)_{n \in \mathbb{N}}$ be a bounded sequence in $L^2([0, T], \mathcal{H}_\Gamma)$. Then there exists a subsequence $(n_j)_{j \in \mathbb{N}} \subset \mathbb{N}$ and a limit function $\phi \in L^2([0, T], \mathcal{H}_\Gamma)$, such that ϕ^{n_j} converges weakly to ϕ with respect to the anti-symmetric pairing $\langle \cdot, \cdot \rangle_{\Gamma_T}$.*

Proof. As $L^2([0, T], \mathcal{H}_\Gamma)$ is an Hilbert space, there exists a $\phi \in L^2([0, T], \mathcal{H}_\Gamma)$ and a subsequence $(n_j)_{j \in \mathbb{N}}$ with $\phi_{n_j} \rightharpoonup \phi$ in $L^2([0, T], \mathcal{H}_\Gamma)$ for $j \to \infty$. For any $w \in L^2([0, T], \mathcal{H}_\Gamma)$ the mapping $\langle \cdot, w \rangle_{\Gamma_T} : L^2([0, T], \mathcal{H}_\Gamma) \to \mathbb{R}$ is continuous and therefore it holds $\langle \phi_{n_j}, w \rangle_{\Gamma_T} \rightharpoonup \langle \phi, w \rangle_{\Gamma_T}$ in $\mathbb{R}$. This is equivalent to $\langle \phi_{n_j}, w \rangle_{\Gamma_T} \to \langle \phi, w \rangle_{\Gamma_T}$, which concludes the assertion as $w \in L^2([0, T], \mathcal{H}_\Gamma)$ was chosen arbitrarily. ☐

2.3. The Calderon Operator

2.3.1. The time harmonic Calderon operator

We collect three of the most important properties of the time harmonic Calderon operator in this section. We consider the coercivity, the boundedness and the projection property of the operator. As we will see in the following section, these properties can be transferred to the space and time dependent Calderon operator $B(\partial_t)$ in an adequate way.

We recall the definition of the Calderon operator from (2.6), (also compare Section A.3 in the appendix)

$$B(s) = \mu_0^{-1} \begin{pmatrix} (i\sqrt{\mu_0\varepsilon_0})^{-1}V(s) & K(s) \\ -K(s) & -i\sqrt{\mu_0\varepsilon_0}V(s) \end{pmatrix}$$

with the boundary integral operators

$$\begin{aligned} V(s) &= i\sqrt{\mu_0\varepsilon_0}\{\!\{\gamma_T \circ \mathcal{S}(s)\}\!\} = (i\sqrt{\mu_0\varepsilon_0})^{-1}\{\!\{\gamma_N \circ \mathcal{D}(s)\}\!\}, \\ K(s) &= \{\!\{\gamma_T \circ \mathcal{D}(s)\}\!\} = \{\!\{\gamma_N \circ \mathcal{S}(s)\}\!\}, \end{aligned}$$

with the single layer potential $\mathcal{S}(s)$ and the double layer potential $\mathcal{S}(s)$.

In (numerical) analysis of PDEs, it is very useful to deal with a coercive and continuous bilinear form. In the following chapters, the coercivity of the Calderon operator with respect to $\langle \cdot, \cdot \rangle_\Gamma$ will play a crucial role. The coercivity is, e.g. used for the energy estimates of the MLLG system, the uniqueness of the Maxwell part of the solutions and in combination with Convolution Quadrature it is preserved for the time discretized operators.

Lemma 2.12 (Coercivity Lemma, cf. [99, Lemma 3.1])**.** *There exists $\beta > 0$ such that the Calderon operator satisfies*

$$\Re \left\langle \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, B(s) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_\Gamma \geq \beta g(s) \left((\varepsilon_0\mu_0)^{-1} \|s^{-1}\varphi\|_{\mathcal{H}_\Gamma}^2 + \|s^{-1}\psi\|_{\mathcal{H}_\Gamma}^2 \right)$$

for $\Re s > 0$ and all $\varphi, \psi \in \mathcal{H}_\Gamma$, with $g(s) = \min(1, |s|^2\varepsilon_0\mu_0)\Re s$.

The Calderon operator is a bounded operator and therefore the above bilinear form is well defined indeed for $\varphi, \psi \in \mathcal{H}_\Gamma$. For better readability we write $\mathcal{H}_\Gamma$ instead of $\mathcal{H}_\Gamma^2 := (\mathcal{H}_\Gamma)^2 := \mathcal{H}_\Gamma \times \mathcal{H}_\Gamma$.

Lemma 2.13 (Boundedness, cf. [99, Lemma 2.3])**.** *For $\Re(s) \geq \epsilon > 0$ the Calderon operator*

$$B(s) : (\mathcal{H}_\Gamma)^2 \to (\mathcal{H}_\Gamma)^2$$

satisfies

$$\|B(s)\phi\|_{\mathcal{H}_\Gamma} \leq C(\epsilon) \left\| s^2\phi \right\|_{\mathcal{H}_\Gamma}$$

for $\phi \in \mathcal{H}_\Gamma^2$.

Another important property of the Calderon operator is the following projection identity. Suitable arrangement of the building blocks of the Calderon operator results in a projection. From this projection property we deduce that the image of this projection is the space of suitable exterior data. The Calderon operator considered here is a linear transformation of its building blocks and therefore this property stays valid in a certain manner. This plays an important role in the equivalence of solutions. Without that property, one would obtain convergence to a solution which would not correspond any more to the solution of the original system.

Lemma 2.14 (Projection property, cf. [46, Equation (35)]). *The operator*

$$\frac{1}{2} - \begin{pmatrix} K(s) & V(s) \\ V(s) & K(s) \end{pmatrix}$$

is a projection, the so-called projection on suitable exterior data.

Remark 2.15. *The operator*

$$\frac{1}{2} + \begin{pmatrix} K(s) & V(s) \\ V(s) & K(s) \end{pmatrix}$$

is a projection, the so-called projection on suitable interior data.

2.3.2. The time dependent Calderon operator

As already introduced in Section 2.1.2, we will use the notation for Laplace differential operators, see Chapter B:

$$B(\partial_t)w = \partial_t^m(\mathcal{L}^{-1}(B(s)s^{-m}) * w), \tag{2.16}$$

where $\mathcal{L}^{-1}$ is the inverse Laplace transform.

With this definition at hand, the above lemmas for the time harmonic Calderon operator translate to the time domain operator as well. For the coercivity, this yields the following result.

Lemma 2.16 (Time domain coercivity, cf. [99, Lemma 4.2]). *For arbitrary $T > 0$ and all $\varphi, \psi \in H^2_{0,*}([0,T], \mathcal{H}_\Gamma)$ we have that*

$$\begin{aligned} \int_0^T e^{-2t/T} \left\langle \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix}, B(\partial_t) \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix} \right\rangle_\Gamma \mathrm{d}t \\ \geq c \int_0^T e^{-2t/T} \left(\|\partial_t^{-1}\varphi(t)\|^2_{\mathcal{H}_\Gamma} + \|\partial_t^{-1}\psi(t)\|^2_{\mathcal{H}_\Gamma} \right) \mathrm{d}t, \end{aligned}$$

where the constant $c > 0$ only depends on T (and on $\beta > 0$ from Lemma 2.12). Furthermore it holds

$$\int_0^T \left\langle \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix}, B(\partial_t) \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix} \right\rangle_\Gamma \mathrm{d}t \geq 0.$$

Proof. The first assertion follows with the operator valued Herglotz Theorem B.83 (with $\sigma = \tau/T$), where, in contrast to [99, Lemma 4.2], an additional density argument is applied to reduce the regularity assumptions on the functions. The second assertion follows similarly considering the limit $\sigma, \sigma_0 \to 0$ in Theorem B.83. □

Similarly, the boundedness of the time harmonic Calderon operator can be transferred to the time dependent case.

Lemma 2.17 (Time domain boundedness, cf. [99, Lemma 6.5]). *For $\phi \in H^2_{0,*}([0,T], \mathcal{H}_\Gamma)$ it holds*

$$\|B(\partial_t)\phi\|_{H^2_{0,*}([0,T],\mathcal{H}_\Gamma)} \leq C\|\phi\|_{L^2([0,T],\mathcal{H}_\Gamma)},$$

where the constant $C > 0$ depends on T.

Proof. As in [99, Lemma 6.5], this follows from the time harmonic estimate from Lemma 2.13, and Causality (and Plancherel's formula combined with density arguments, see Chapter B) concludes the assertion (compare Remark B.41). □

With

$$\widehat{B}(s) = \begin{pmatrix} K(s) & V(s) \\ V(s) & K(s) \end{pmatrix},$$

the previous lemma suggests that $P = (1/2\,\mathrm{Id} - \widehat{B}(\partial_t))$ can be defined as an operator from $H^m_{0,*}([0,T], \mathcal{H}_\Gamma)$ to $L^2([0,T], \mathcal{H}_\Gamma)$. If the projection property can be transferred to time dependent operators, at first sight, $P^2\phi = P\phi$ would be only valid for functions $\phi \in H^{2m}_{0,*}([0,T], \mathcal{H}_\Gamma)$. Due to the refined analysis developed in Lemma 3.6 and Theorem B.81, it can be shown that $P^2\phi = P\phi$ even is valid for functions $\phi \in L^2([0,T], \mathcal{H}_\Gamma)$, as long as $P\phi$ exists.

Lemma 2.18 (Time domain projection property)**.** *For $\phi \in L^2([0,T], \mathcal{H}_\Gamma)$ such that $(1/2\,\mathrm{Id} - \widehat{B}(\partial_t))\phi$ exists, it holds*

$$\left(\frac{1}{2}\mathrm{Id} - \widehat{B}(\partial_t)\right)^2 \phi = \left(\frac{1}{2}\mathrm{Id} - \widehat{B}(\partial_t)\right)\phi.$$

Proof. Similar techniques are applied in Lemma 3.6 to show the equivalence between the considered solutions without additional regularity requirements. □

2.4. Rigorous Reformulation

In this section, we want to carry out the formal reformulation from Section 2.1 in a mathematically rigorous way. First of all, we define in a precise way, what we understand as solutions of the MLLG system on the whole space (1.12) and of the reformulated boundary integral MLLG system (2.12). Then we show the connection between the two systems: If the solutions are smooth enough (and extendable to the whole time interval $[0,\infty)$ in the exterior/on the boundary), then the solutions coincide.

Before being able to consider equivalence of the systems, we need some properties of the time harmonic Maxwell system which we derive in the following section.

2.4.1. The time harmonic Maxwell system

The time harmonic Maxwell equations arise out of applying the Laplace transform to the time dependent Maxwell system as in Section 2.1.2, or out of inserting a time harmonic ansatz $E(t,x) = e^{i\omega t}\widehat{E}(x)$, $H(t,x) = e^{i\omega t}\widehat{H}(x)$ in the time dependent Maxwell system. We consider two systems and the possibilities to solve them from literature, i.e. the version from [46] and the version from [152, Section 3.4]. We show that, under certain conditions, their solutions coincide and therefore we can use all the properties that were shown for the systems independently of each other.

The systems read:

- (THME,a): In [46], we find the following problem:

  Find $e, h \in H_{\mathrm{loc}}(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}})$ that fulfill the Silver-Müller radiation condition (2.18) and

$$\begin{aligned} -\varepsilon_0 i\omega e + \nabla \times h &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \mu_0 i\omega h + \nabla \times e &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \gamma_T(e) &= \gamma_T e^{\mathrm{int}} && \text{on } \Gamma, \\ \gamma_T(h) &= \gamma_T h^{\mathrm{int}} && \text{on } \Gamma, \end{aligned}$$

  where $(\gamma_T e^{\mathrm{int}}, \mu_0\gamma_T h^{\mathrm{int}}) \in \mathcal{H}_\Gamma \times \mathcal{H}_\Gamma$ are suitable exterior data and $\Im\omega > 0$.

  We say that $(a,b) \in \mathcal{H}_\Gamma \times \mathcal{H}_\Gamma$ *is suitable exterior data* if and only if the traces fulfill

$$B(-i\omega)\begin{pmatrix} -b \\ -a \end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix} a \\ -b \end{pmatrix}, \tag{2.17}$$

where B is the Calderon operator. We also write (a, b, ω) *suitable*, to indicate the dependency on ω, if this is not clear from the context.

- (THME,b): In [152, Section 3.4] we find the following problem:
  Find $e, h \in H(\text{curl}, \overline{\Omega}^{\text{c}})$ that fulfill

$$\begin{aligned} -\varepsilon_0 i\omega e + \nabla \times h &= 0 && \text{in } \overline{\Omega}^{\text{c}}, \\ \mu_0 i\omega h + \nabla \times e &= 0 && \text{in } \overline{\Omega}^{\text{c}}, \\ \gamma_T(e) &= n \times e^{\text{int}} && \text{on } \Gamma, \end{aligned}$$

  where $e^{\text{int}} \in H^{-1/2}(\text{curl}_\Gamma, \Gamma)$ and $\Im\omega < 0$.

Remark 2.19. *For (THME,a) the Silver-Müller radiation conditions are imposed at ∞, i.e. it holds*

$$\int_{\partial B_r} |\sqrt{\mu_0}\gamma_T h \times n + \sqrt{\varepsilon_0}\gamma e|^2 \, \mathrm{d}\sigma \to 0 \quad \textit{for} \quad r \to \infty. \tag{2.18}$$

Remark 2.20. *The restriction on $\Im\omega > 0$ in (THME,a) says, how the Silver-Müller radiation condition* (2.18) *and the compatibility condition* (2.17) *are understood, in fact with respect to ω such that $\Im\omega > 0$. If $\Im\omega < 0$, we may interchange e and h, formulate the Silver-Müller radiation* (2.18) *and the compatibility condition* (2.17) *with respect to $-\omega$ and obtain uniqueness and existence. How this can be done in detail is presented in the following paragraphs.*

In (THME,b) the restriction $\Im\omega < 0$ can be weakened to $\Im\omega \neq 0$, as we will see in Lemma 2.23.

Existence

In this paragraph, we rephrase the existence results of the problems from the underlying papers.

Lemma 2.21. *For $\gamma_T e^{\text{int}}, \gamma_T h^{\text{int}} \in \mathcal{H}_\Gamma$, if $(\gamma_T e^{\text{int}}, \mu_0\gamma_T h^{\text{int}}, \omega)$ is suitable and $\Im\omega > 0$, we set $s := -i\omega$, and the solution of problem (THME,a) is given via*

$$\begin{aligned} e &= \mathcal{D}(s)(-\gamma_T e^{\text{int}}) + \mathcal{S}(s)(-\mu_0\gamma_T h^{\text{int}}), \\ h &= \frac{1}{-i\omega\mu_0}\nabla \times e = \mathcal{S}(s)(\varepsilon_0\gamma_T e^{\text{int}}) + \mathcal{D}(s)(-\gamma_T h^{\text{int}}) \end{aligned} \tag{2.19}$$

and e, h fulfill the Silver-Müller radiation condition.

Conversely, if we have such a solution, then $(\gamma_T e^{\text{int}}, \mu_0\gamma_T h^{\text{int}})$ is suitable exterior data.

Proof. In [46] originally the second order formulation of (THME,a) is considered. By rewriting this as a first order system, we directly obtain from [46, Definition 3 and Theorem 8] the following statement (also compare Section A.3): If (a, b, ω) is suitable, the problem

$$\begin{aligned} -\varepsilon_0\mu_0 i\omega A + \nabla \times B &= 0 && \text{in } \overline{\Omega}^{\text{c}}, \\ i\omega B + \nabla \times A &= 0 && \text{in } \overline{\Omega}^{\text{c}}, \\ \gamma_T(A) &= a && \text{on } \Gamma, \\ \gamma_T(B) &= b && \text{on } \Gamma \end{aligned}$$

has a unique solution (A, B) that fulfills the Silver-Müller radiation condition.

Conversely, if we have such a solution (A, B), then $(\gamma_T A, \gamma_T B) = (a, b)$ is suitable exterior data.

Furthermore, there is the representation formula (2.19) given for the solution.

Attention has to be paid, because the definition of the integral operators in this manuscript differs from the ones used in [46] by a multiplicative factor. (This is also mentioned in the Erratum [125] and we refer to an overview of the different rescalings in Section A.3 in the Appendix.) If we denote by $\widetilde{\mathcal{S}}$, $\widetilde{\mathcal{D}}$, the single and double layer potentials from [46] and by $\widetilde{\gamma}_N$ the Neumann trace operator from [46], then the following relations hold for $\kappa = \omega\sqrt{\mu_0\varepsilon_0}$, $s = -i\omega$,

$$\begin{aligned} \frac{1}{i\sqrt{\mu_0\varepsilon_0}}\widetilde{\mathcal{S}}(\kappa) &= \mathcal{S}(s), \\ \widetilde{\mathcal{D}}(\kappa) &= \mathcal{D}(s), \\ i\sqrt{\mu_0\varepsilon_0}\widetilde{\gamma}_N &= \gamma_N. \end{aligned}$$

Now we can apply [46, Theorem 8] and obtain in terms of the operators used in this manuscript that

$$\begin{aligned} A &= \mathcal{D}(s)(-a) + \mathcal{S}(s)(-b), \\ B &= \frac{1}{-i\omega}\nabla \times A \\ &= -\varepsilon_0\mu_0\mathcal{S}(s)(-a) + \mathcal{D}(s)(-b). \end{aligned}$$

Similarly, the property that traces (a, b) are suitable is rewritten with the operators used in this manuscript as

$$B(-i\omega)\begin{pmatrix} -b \\ -a \end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix} a \\ -b \end{pmatrix}.$$

Now we insert $A = e$, $B = \mu_0 h$, $a = \gamma_T e^{\text{int}}$ and $b = \mu_0\gamma_T h^{\text{int}}$ in those formulas and obtain all the assertions in the statement of this lemma. □

Remark 2.22. *(THME,b) in [152, Section 3.4] is originally formulated for smooth Ω, and not for piecewise smooth or Lipschitz Ω as we use it in this section. Nevertheless, in the suitable functional analytic setting, the properties of the system (the system we have after applying Lemma 2.23) that we use from [152, Section 3.4] can also be proven for piecewise smooth or Lipschitz Ω. This is due to the results from [43, 44, 45] introducing the Maxwell trace spaces on non-smooth boundaries. Related arguments can be found in [143, Section A.2].*

Without using the claim that the results from [152, Section 3.4] stay valid, the results of this section nevertheless hold true for smoothly bounded $\overline{\Omega}^{\text{c}}$.

Lemma 2.23. *The second version of the time harmonic Maxwell equations (THME,b) is uniquely solvable for $\Im\omega \neq 0$. Additionally, the boundary condition*

$$\gamma_T(e) = n \times e^{\text{int}} \quad \text{for } e^{\text{int}} \in H^{-1/2}(\text{curl}_\Gamma, \Gamma) \tag{2.20}$$

can be exchanged by the boundary condition

$$\gamma_T(e) = \gamma_T e^{\text{int}} \quad \text{for } \gamma_T e^{\text{int}} \in \mathcal{H}_\Gamma, \tag{2.21}$$

i.e. the first assertion in this lemma and also Lemma 2.27 stay valid with (2.21) *instead of* (2.20).

Proof. If $\Im\omega < 0$, the problem is uniquely solvable due to [152, Section 3.4]. If we have $\Im\omega > 0$, we again state the corresponding system

$$\begin{aligned} -\varepsilon_0 i\omega e + \nabla \times h &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \mu_0 i\omega h + \nabla \times e &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \gamma_T(e) &= n \times e^{\mathrm{int}} && \text{on } \varGamma. \end{aligned}$$

We may multiply the first equation by -1 and rearrange the formulas as

$$\begin{aligned} -\varepsilon_0 i(-\omega)e + \nabla \times (-h) &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \mu_0 i(-\omega)(-h) + \nabla \times e &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \gamma_T(e) &= n \times e^{\mathrm{int}} && \text{on } \varGamma. \end{aligned}$$

So we see that $(e, -h, -\omega)$ solves (THME,b) with $\Im(-\omega) < 0$ and therefore its existence and uniqueness is satisfied.

Concerning the boundary condition, if $\overline{\Omega}^{\mathrm{c}}$ is a smoothly bounded domain, the following arguments show that the spaces $n \times H^{-1/2}(\mathrm{curl}_\varGamma, \varGamma)$ and $\mathcal{H}_\varGamma$ coincide. As stated in [152, Section 3.4] and [123], every $e^{\mathrm{int}} \in H^{-1/2}(\mathrm{curl}_\varGamma, \varGamma)$ can be lifted to $H(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}})$, i.e. there exists $\widetilde{e}^{\mathrm{int}} \in H(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}})$ with $\gamma\widetilde{e}^{\mathrm{int}} = e^{\mathrm{int}}$ on the boundary. As $\gamma_T : H(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}}) \to \mathcal{H}_\varGamma$ is continuous, it follows $\gamma_T\widetilde{e}^{\mathrm{int}} = n \times \gamma\widetilde{e}^{\mathrm{int}} \in \mathcal{H}_\varGamma$. The reverse direction follows as $\gamma_T : H(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}}) \to \mathcal{H}_\varGamma$ is surjective and $\gamma : H(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}}) \to H^{-1/2}(\mathrm{curl}_\varGamma, \varGamma)$ is continuous (cf. [152, Theorem 3.1] or [123]).

If $\overline{\Omega}^{\mathrm{c}}$ is not smoothly bounded, but only piecewise smooth or Lipschitz bounded, we refer to Remark 2.22. In this case, we have to formulate (THME,b) directly with the boundary condition $\gamma_T(e) = \gamma_T e^{\mathrm{int}}$ for $\gamma_T e^{\mathrm{int}} \in \mathcal{H}_\varGamma$ and repeat the proofs from [152, Section 3.4]. All used statements can be transferred and stay valid. Similar arguments are also considered in [143, Section A.2]. □

Properties

In the following, we collect some properties of the problems (THME,a) and (THME,b) that are shown in the references [46] and [152, Section 3.4].

Lemma 2.24. *Let $\alpha, \omega \in \mathbb{C}$, $\Im\omega > 0$ and $a, b, c, d \in \mathcal{H}_\varGamma$. If (a, b, ω) and (c, d, ω) are suitable exterior data then also $(a + \alpha c, b + \alpha d, \omega)$ is suitable exterior data and*

$$(a, b, \omega) \text{ suitable } \Leftrightarrow (-b, \mu_0\varepsilon_0 a, \omega) \text{ suitable} \Leftrightarrow (-\mu_0^{-1} b, \varepsilon_0 a, \omega) \text{ suitable.}$$

Proof. Suitability is a linear property, as the set of suitable functions is the kernel of a linear operator, i.e. if (a, b, ω) and (c, d, ω) are suitable exterior data and $\alpha \in \mathbb{C}$, then also $(a + \alpha c, b + \alpha d, \omega)$ is suitable exterior data and

$$(-b, \mu_0\varepsilon_0 a, \omega) \text{ suitable} \Leftrightarrow (-\mu_0^{-1} b, \varepsilon_0 a, \omega) \text{ suitable.}$$

Moreover, by inserting the formulas

$$B(s) = \mu_0^{-1} \begin{pmatrix} (i\sqrt{\mu_0\varepsilon_0})^{-1} V(s) & K(s) \\ -K(s) & -i\sqrt{\mu_0\varepsilon_0} V(s) \end{pmatrix},$$

we compute

$$\begin{aligned} (a, b, \omega) \text{ suitable} \Leftrightarrow B(s) \begin{pmatrix} -b \\ -a \end{pmatrix} &= \frac{1}{2\mu_0} \begin{pmatrix} a \\ -b \end{pmatrix} \\ \Leftrightarrow \begin{pmatrix} -(i\sqrt{\mu_0\varepsilon_0})^{-1} V(s) b - K(s) a \\ K(s) b + i\sqrt{\mu_0\varepsilon_0} V(s) a \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} a \\ -b \end{pmatrix} \end{aligned}$$

and

$$(-\mu_0^{-1}b, \varepsilon_0 a, \omega) \text{ suitable } \Leftrightarrow \qquad B(s)\begin{pmatrix} -\varepsilon_0 a \\ \mu_0^{-1} b \end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix} -\mu_0^{-1} b \\ -\varepsilon_0 a \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -(i\sqrt{\mu_0\varepsilon_0})^{-1}V(s)\varepsilon_0 a + K(s)\mu_0^{-1} b \\ K(s)\varepsilon_0 a - i\sqrt{\mu_0\varepsilon_0}V(s)\mu_0^{-1} b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\mu_0^{-1} b \\ -\varepsilon_0 a \end{pmatrix}.$$

Multiplying the first line of the last expression with μ_0 and by dividing the second line by $-\varepsilon_0$, yields (a, b, ω) suitable $\Leftrightarrow$ $(-\mu_0^{-1}b, \varepsilon_0 a, \omega)$ suitable. □

The following remark may not be needed directly in the following but explains some connections between the Maxwell equations and the compatibility condition.

Remark 2.25 (Consistency with respect to rearrangement)**.** *By various transformations, the system (THME,a) may be rearranged into a new system, that is again similar to the old one just with permuted variables (e.g. interchange the roles of e and h). The previous lemma shows, that suitability is a consistent property under such transformations. We may rearrange*

$$\begin{aligned} -\varepsilon_0\mu_0 i\omega A + \nabla \times B &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ i\omega B + \nabla \times A &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \gamma_T(A) &= a && \text{on } \Gamma, \\ \gamma_T(B) &= b && \text{on } \Gamma, \end{aligned}$$

by a multiplication with $\varepsilon_0\mu_0$ in the second line as

$$\begin{aligned} -\varepsilon_0\mu_0 i\omega(-B) + \nabla \times (\varepsilon_0\mu_0 A) &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ i\omega(\varepsilon_0\mu_0 A) + \nabla \times (-B) &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \gamma_T(-B) &= -b && \text{on } \Gamma, \\ \gamma_T(\varepsilon_0\mu_0 A) &= \varepsilon_0\mu_0 a && \text{on } \Gamma. \end{aligned}$$

Both versions differ just by A replaced with $-B$ and B replaced with $\varepsilon_0\mu_0 A$. But, as seen in the above Lemma, suitability is preserved similarly: For a suitable data set (a, b), also $(-b, \varepsilon_0\mu_0 a)$ is suitable and vice versa. We do not discuss further rearrangements like changing the roles of ε_0 and μ_0, scaling the wave number or further manipulations, here.

Remark 2.26. *In terms of traces $\gamma_T e^{\mathrm{int}}$, $\gamma_T h^{\mathrm{int}}$ the above statements take the form*

$$(\gamma_T e^{\mathrm{int}}, \mu_0\gamma_T h^{\mathrm{int}}) \text{ suitable } \Leftrightarrow (-\gamma_T h^{\mathrm{int}}, \varepsilon_0\gamma_T e^{\mathrm{int}}) \text{ suitable.}$$

The precise equations are

$$B(-i\omega)\begin{pmatrix} -\mu_0\gamma_T h^{\mathrm{int}} \\ -\gamma_T e^{\mathrm{int}} \end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix} \gamma_T e^{\mathrm{int}} \\ -\mu_0\gamma_T h^{\mathrm{int}} \end{pmatrix} \Leftrightarrow$$

$$B(-i\omega)\begin{pmatrix} -\varepsilon_0\gamma_T e^{\mathrm{int}} \\ \gamma_T h^{\mathrm{int}} \end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix} -\gamma_T h^{\mathrm{int}} \\ -\varepsilon_0\gamma_T e^{\mathrm{int}} \end{pmatrix}.$$

This equivalence corresponds to the rewrite of

$$\begin{aligned} -\varepsilon_0 i\omega e + \nabla \times h &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \mu_0 i\omega h + \nabla \times e &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \gamma_T(e) &= \gamma_T e^{\mathrm{int}} && \text{on } \Gamma, \\ \gamma_T(h) &= \gamma_T h^{\mathrm{int}} && \text{on } \Gamma, \end{aligned}$$

as

$$\begin{aligned} -\varepsilon_0 i\omega(-h) + \nabla \times (\varepsilon_0 \mu_0^{-1} e) &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \mu_0 i\omega(\varepsilon_0 \mu_0^{-1} e) + \nabla \times (-h) &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \gamma_T(-h) &= -\gamma_T h^{\mathrm{int}} && \text{on } \Gamma, \\ \gamma_T(\varepsilon_0 \mu_0^{-1} e) &= \varepsilon_0 \mu_0^{-1} \gamma_T e^{\mathrm{int}} && \text{on } \Gamma. \end{aligned}$$

It is $(\gamma_T e^{\mathrm{int}}, \mu_0 \gamma_T h^{\mathrm{int}})$ *suitable if and only if the pair of traces* $(-\gamma_T h^{\mathrm{int}}, \mu_0 \varepsilon_0 \mu_0^{-1} \gamma_T e^{\mathrm{int}}) = (-\gamma_T h^{\mathrm{int}}, \varepsilon_0 \gamma_T e^{\mathrm{int}})$ *is suitable. This again illustrates the symmetry between* E *and* H *in the Maxwell equations. We do not discuss further rearrangements like changing the roles of* ε_0 *and* μ_0*, scaling the wave number or further manipulations, here.*

Lemma 2.27. *For the solution of (THME,b) it holds for* $s = -i\omega$*, with* $\Re(s) \geq \sigma_0 > 0$

$$\|\nabla \times e\|_{\overline{\Omega}^{\mathrm{c}}} + |s| \|e\|_{\overline{\Omega}^{\mathrm{c}}} \leq C(\Gamma, \mu_0, \varepsilon_0, \sigma_0) |s|^2 \|\gamma_T e^{\mathrm{int}}\|_{\mathcal{H}_\Gamma}$$

and similarly

$$\|\nabla \times h\|_{\overline{\Omega}^{\mathrm{c}}} + |s| \|h\|_{\overline{\Omega}^{\mathrm{c}}} \leq C(\Gamma, \mu_0, \varepsilon_0, \sigma_0) |s|^2 \|\gamma_T h\|_{\mathcal{H}_\Gamma}.$$

Proof. This can be proven by using [46, Theorem 1], to reformulate the statements of [152, Theorem 3.1, Corollary 3.2, Lemma 3.3] for functions in $\mathcal{H}_\Gamma = H_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ for Lipschitz or piecewise continuous domains instead for functions in $H^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ on smooth domains. All proofs hold in an analogous way (compare Remark 2.22). Similar arguments can be found in [143, Proposistion A.3.1]. The assertion for h then follows by interchanging the roles of e and h. □

Equivalence

In this paragraph we show that the solutions of both problems coincide (for $\Im\omega > 0$), thus a solution of (THME,a) is a solution of (THME,b) and vice versa. Therefore all the properties stated in the previous paragraph are fulfilled for both solutions.

Lemma 2.28. *Let* $\Im\omega > 0$ *and given boundary data* $\gamma_T e^{\mathrm{int}} \in \mathcal{H}_\Gamma$*. The solutions of (THME,a) and (THME,b) exist and they coincide.*

Proof. The proof follows the following strategy. By Lemma 2.23, we obtain a solution (e, h) of (THME,b). With the Dirichlet-to-Neumann operator from [46, Section 10], we obtain boundary data $\mu_0 \gamma_T \widetilde{h}$ that together with $\gamma_T e^{\mathrm{int}}$ fulfills the compatibility condition (2.17), i.e. $(\gamma_T e^{\mathrm{int}}, \mu_0 \gamma_T \widetilde{h})$ is suitable exterior data. Therefore we can define a solution $(\widetilde{e}, \widetilde{h})$ of (THME,a). The last step is to show that $(\widetilde{e}, \widetilde{h})$ solves (THME,b). As $\gamma_T \widetilde{e} = \gamma_T e^{\mathrm{int}}$ is fulfilled by the above construction, it remains to show the regularity assumptions of (THME,b), i.e. that $\widetilde{e}, \widetilde{h} \in H(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}})$. Then, by the unique solvability of (THME,b) it follows that $(e, h) = (\widetilde{e}, \widetilde{h})$.

To show that $\widetilde{e}, \widetilde{h} \in H(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}})$, we show that the solution of problem (THME,a) (and its curl) decreases exponentially for $|x| \to \infty$. We therefore adapt the proofs of [22, Theorem 4.4 (c)] and [26, Lemma 7]. Together with $\widetilde{e}, \widetilde{h} \in H_{\mathrm{loc}}(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}})$ this then concludes the assertion.

For the convenience of the reader, we repeat the definitions of the integral operators. The electric single layer potential is given, for $x \in \mathbb{R}^3 \setminus \Gamma$, by

$$(\mathcal{S}(s)\varphi)(x) := s \int_\Gamma G(s, x-y)\varphi(y)\mathrm{d}y - \frac{1}{\epsilon_0 \mu_0 s} \nabla \int_\Gamma G(s, x-y)\mathrm{div}_\Gamma \varphi(y)\mathrm{d}y$$

and the electric double layer potential, for $x \in \mathbb{R}^3 \setminus \Gamma$,

$$(\mathcal{D}(s)\varphi)(x) = \nabla \times \int_\Gamma G(s, x-y)\varphi(y)\ \mathrm{d}y,$$

where the fundamental solution $G(s,z)$ is given for $z \in \mathbb{R}^3 \setminus \{0\}$, as

$$G(s,z) = \frac{e^{-s\sqrt{\varepsilon_0\mu_0}|z|}}{4\pi|z|}.$$

In the following, we will derive suitable bounds on $|(\mathcal{S}(s)\varphi)(x)|$ and $|(\mathcal{D}(s)\varphi)(x)|$ for $\varphi \in \mathcal{H}_\Gamma$.

As $\mathcal{H}_\Gamma$ is its own dual and as $\gamma_T : H(\mathrm{curl}, \Omega) \to \mathcal{H}_\Gamma$ is surjective and continuous, we have for $v \in \mathcal{H}_\Gamma$, the existence of a $\widetilde{v} \in H(\mathrm{curl}, \Omega)$ with $\gamma_T \widetilde{v} = v$ and therefore

$$\begin{aligned}
\|v\|_{\gamma_T(H^1(\Omega))'} &= \sup_{\phi \in H^1(\Omega)} \frac{\langle v, \gamma_T \phi\rangle_\Gamma}{(\|\phi\|_\Omega^2 + \|\nabla\phi\|_\Omega^2)^{1/2}} \\
&= \sup_{\phi \in C^\infty(\Omega)} \frac{[n \times \widetilde{v}, n \times n \times \phi]_\Gamma}{(\|\phi\|_\Omega^2 + \|\nabla\phi\|_\Omega^2)^{1/2}} \\
&= \sup_{\phi \in C^\infty(\Omega)} \frac{[n \times \widetilde{v}, \phi]_\Gamma}{(\|\phi\|_\Omega^2 + \|\nabla\phi\|_\Omega^2)^{1/2}} \\
&= \|v\|_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

From this, we immediately infer

$$\|v\|_{\mathcal{H}_\Gamma}^2 = \|v\|_{\gamma_T(H^1(\Omega))'}^2 + \|\mathrm{div}_\Gamma v\|_{H^{-1/2}(\Gamma)}^2 \geq \|v\|_{H^{-1/2}(\Gamma)}^2. \tag{2.22}$$

The next step is to choose a suitably big ball B_R, such that $\Omega \subset B_{R/2}$. For values in the complement $\mathbb{R}^3 \setminus B_R$, we are then able to suitably bound the fundamental solution G and then obtain the desired bounds on the integral operators. We have for $x \in \mathbb{R}^3 \setminus B_R$, $\Omega \subset B_{R/2}$, $s = -i\omega$, $\Re s \geq \sigma_0 > 0$, $y \in \Omega$

$$|x-y| \geq |x| - |y| \geq |x| - R/2 \geq R/2$$

and therefore using $|x-y| \geq |x| - R/2$ and $|x-y| \geq R/2$ we obtain

$$\begin{aligned}
\|G(s, x-\cdot)\|_{H^{1/2}(\Gamma)} &\leq \left\| \frac{e^{-s\sqrt{\varepsilon_0\mu_0}|x-\cdot|}}{4\pi|x-\cdot|} \right\|_{H^1(\Omega)} \\
&\leq C(\varepsilon_0, \mu_0) \left\| \left(1 + |s| + \frac{1}{|x-\cdot|}\right) \frac{e^{-s\sqrt{\varepsilon_0\mu_0}|x-\cdot|}}{4\pi|x-\cdot|} \right\|_{L^2(\Omega)} \\
&\leq C(s, \varepsilon_0, \mu_0, R, |\Omega|) \sup_{y\in\Omega} e^{-\Re s\sqrt{\varepsilon_0\mu_0}|x-y|} \\
&\leq C(s, \varepsilon_0, \mu_0, R) \sup_{y\in\Omega} e^{-\sigma_0\sqrt{\varepsilon_0\mu_0}|x-y|} \\
&\leq C(s, \varepsilon_0, \mu_0, R, \sigma_0) e^{-\sigma_0\sqrt{\varepsilon_0\mu_0}|x|}.
\end{aligned} \tag{2.23}$$

Similarly, one estimates

$$\|\nabla G(s, x-\cdot)\|_{H^{1/2}(\Gamma)} \leq C(s, \varepsilon_0, \mu_0, R, \sigma_0) e^{-\sigma_0\sqrt{\varepsilon_0\mu_0}|x|}. \tag{2.24}$$

Using (2.22), (2.23) and (2.24), we get for $v \in \mathcal{H}_\Gamma$, $|x| \geq R$,

$$\begin{aligned}
|(\mathcal{S}(s)v)(x)| &\leq C(s)\|G(s, x-\cdot)\|_{H^{1/2}(\Gamma)} \|v\|_{H^{-1/2}(\Gamma)} \\
&\quad + C(s, \mu_0, \varepsilon_0) \|\nabla G(s, x-\cdot)\|_{H^{1/2}(\Gamma)} \|\mathrm{div}_\Gamma v\|_{H^{-1/2}(\Gamma)} \\
&\leq C(\varepsilon_0, \mu_0, R, \sigma_0) e^{-\sigma_0\sqrt{\varepsilon_0\mu_0}|x|} \|v\|_{\mathcal{H}_\Gamma}.
\end{aligned}$$

By the definition of the single layer potential and analogical estimates for ∇G we have

$$\begin{aligned}|(\nabla \times \mathcal{S}(s)v)(x)| &\leq C(s,\varepsilon_0,\mu_0)|(\mathcal{D}(s)v)(x)| \\ &\leq \|\nabla G(s,x-\cdot)\|_{H^{1/2}(\Gamma)}\|v\|_{H^{-1/2}(\Gamma)} \\ &\leq C(\varepsilon_0,\mu_0,R,\sigma_0)e^{-\sigma_0\sqrt{\varepsilon_0\mu_0}|x|}\|v\|_{\mathcal{H}_\Gamma}.\end{aligned}$$

Similarly, we obtain bounds

$$|(\mathcal{D}(s)v)(x)| \leq C(\varepsilon_0,\mu_0,R,\sigma_0)e^{-\sigma_0\sqrt{\varepsilon_0\mu_0}|x|}\|v\|_{\mathcal{H}_\Gamma}$$

and using $|\nabla \times (\mathcal{D}(s)v)(x)| \leq C(s,\varepsilon_0,\mu_0)|(\mathcal{S}(s)v)(x)|$ (cf. [46, Lemma 5])

$$|(\nabla \times \mathcal{D}(s)v)(x)| \leq C(\varepsilon_0,\mu_0,R,\sigma_0)e^{-\sigma_0\sqrt{\varepsilon_0\mu_0}|x|}\|v\|_{\mathcal{H}_\Gamma}.$$

Thus, the solution is in $H(\mathrm{curl}, B_R^{\mathrm{c}})$ for a large enough ball B_R containing Ω. By [46], it is in $H_{\mathrm{loc}}(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}})$ (in the sense that for every bounded set B, the functions are in $H(\mathrm{curl}, B)$, cf. [46, Section 2]), so it is in $H(\mathrm{curl}, B_R \setminus \Omega)$. Altogether, the solution of problem (THME,a) is in $H(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}})$, fulfills the regularity assumptions of (THME,b) and therefore is a solution of (THME,b). By unique solvability of (THME,b) the two solutions coincide. □

Remark 2.29. *In [46], existence and uniqueness is also shown for $\Im\omega = 0$, $\omega > 0$. Lemma 2.21 can be shown to hold analogously, but in the proof of Lemma 2.28 we need $\Im\omega > 0$ to show exponential decay of the solution for $|x| \to \infty$. Thus we are not able to show with these arguments, that the solution is in $H(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}})$. This also indicates the reason why (THME,b) is only formulated for $\Im\omega \neq 0$, as it gives solutions in $H(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}})$ which may not occur for $\Im\omega = 0$.*

For the well-definiteness of the inverse Laplace transform, we need complex differentiable functions. In the following lemma we state that this is fulfilled for the integral operators.

Lemma 2.30. *For $s \in \mathbb{C}$, $\Re s > 0$, the family of single layer operators $\mathcal{S}(s) : \mathcal{H}_\Gamma \to H(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}})$, the family of double layer potentials $\mathcal{D}(s) : \mathcal{H}_\Gamma \to H(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}})$ and the family of the Calderon operators $B(s) : \mathcal{H}_\Gamma \times \mathcal{H}_\Gamma \to \mathcal{H}_\Gamma \times \mathcal{H}_\Gamma$ are complex differentiable in the sense of Definition B.49.*

Proof. The proof is sketched roughly, see [143] for similar arguments for the wave equation. We only consider the single layer potential $\mathcal{S}(s)$, the proof for the double layer potentials $\mathcal{D}(s)$ proceeds similarly and the assertion for the Calderon operators $B(s)$ follows by its definition as the composition of traces and averages of the single and double layer potentials.

For simplicity, we only look at a part of the first term in the definition of the single layer potential, i.e.

$$(\mathcal{S}_1(s)\varphi)(x) := \int_\Gamma G(s,x-y)\varphi(y)\mathrm{d}y.$$

The second term and the multiplication with s may be treated analogously. As the fundamental solution $G(s,z) = \frac{e^{-s\sqrt{\varepsilon_0\mu_0}|z|}}{4\pi|z|}$ is analytic and uniformly bounded for $|z| \geq c > 0$, we see for fixed $x \notin \Gamma$ and fixed φ, that

$$|\mathcal{S}_1(s+h)\varphi - \mathcal{S}_1(s)\varphi - h\partial_s\mathcal{S}_1(s)\varphi| \leq C(x,\varphi)h,$$

i.e. $s \mapsto (\mathcal{S}_1(s)\varphi)(x)$ is complex differentiable.

By similar arguments as in the previous Lemma, as the potential decays exponentially for $|x| \to \infty$ (and the constants there depend in a "harmless" way on s), we may obtain for a big enough ball B_R for $h \to 0$

$$\|\mathcal{S}_1(s+h)\varphi - \mathcal{S}_1(s)\varphi - h\partial_s \mathcal{S}_1(s)\varphi\|_{H(\mathrm{curl},\mathbb{R}^3 \setminus B_R)} \leq C(R)h\|\varphi\|_{\mathcal{H}_\Gamma}.$$

This concludes the complex differentiability on $H(\mathrm{curl}, \mathbb{R}^3 \setminus B_R)$.

For the bounded domain $B_R \setminus \Omega$ (and similarly for the bounded domain Ω), we note that the symbol of $(\mathcal{S}_1(s+h)\varphi - \mathcal{S}_1(s)\varphi - h\partial_s \mathcal{S}_1(s)\varphi)(x)$ is

$$\begin{aligned} &G(s+h,z) - G(s,z) - h\partial_s G(s,z) \\ &\qquad = \frac{e^{-(s+h)\sqrt{\varepsilon_0\mu_0}|z|} - e^{-s\sqrt{\varepsilon_0\mu_0}|z|} + h\sqrt{\varepsilon_0\mu_0}|z|e^{-s\sqrt{\varepsilon_0\mu_0}|z|}}{4\pi|z|}. \end{aligned}$$

The denominator can be bounded by Ch uniformly for $|z| \leq R$ for a constant $C > 0$. Then the bound follows similarly as the bound in the proof of the continuity of $\mathcal{S}_1(s) : \mathcal{H}_\Gamma \to H(\mathrm{curl}, B_R \setminus \Omega)$. This concludes, together with a more concrete formulation of the arguments, the proof. □

Preparations

In this paragraph, we rewrite the system such that the results obtained so far can be directly applied in the following sections.

Applying the Laplace transform to the time dependent Maxwell equations, we obtain for $\Re s > 0$

$$\begin{aligned} \varepsilon_0 s\widehat{E} - \nabla \times \widehat{H} &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \mu_0 s\widehat{H} + \nabla \times \widehat{E} &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \gamma_T(\widehat{E}) &= \gamma_T \widehat{E} && \text{on } \Gamma, \\ \gamma_T(\widehat{H}) &= \gamma_T \widehat{H} && \text{on } \Gamma. \end{aligned} \tag{2.25}$$

With $s = -i\omega$ it is $\Im\omega > 0$ and we rewrite the system as

$$\begin{aligned} \varepsilon_0 - i\omega\widehat{E} + \nabla \times (-\widehat{H} = 0) && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \mu_0 i\omega(-\widehat{H}) + \nabla \times \widehat{E} = 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \gamma_T(\widehat{E}) = \gamma_T \widehat{E} && \text{on } \Gamma, \\ \gamma_T(-\widehat{H}) = -\gamma_T \widehat{H} && \text{on } \Gamma. \end{aligned}$$

Thus, $(\gamma_T \widehat{E}, -\mu_0 \gamma_T \widehat{H}, \omega)$ is suitable exterior data, i.e. we obtain (see Remark 2.26 with h replaced by $-\widehat{H}$)

$$B(-i\omega)\begin{pmatrix} \mu_0 \gamma_T \widehat{H} \\ -\gamma_T \widehat{E} \end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix} \gamma_T \widehat{E} \\ \mu_0 \gamma_T \widehat{H} \end{pmatrix}. \tag{2.26}$$

Conversely, if $(\gamma_T \widehat{E}, -\mu_0 \gamma_T \widehat{H}, \omega)$ is suitable exterior data, we may represent the solution via

$$\begin{aligned} e &= \mathcal{D}(s)(-\gamma_T \widehat{E}) + \mathcal{S}(s)(\mu_0 \gamma_T \widehat{H}), \\ -h &= \frac{1}{-i\omega\mu_0} \nabla \times e \\ &= \mathcal{S}(s)(\varepsilon_0 \gamma_T \widehat{E}) + \mathcal{D}(s)(\gamma_T \widehat{H}). \end{aligned}$$

2.4.2. Definition of solutions

In this section we define solutions of the MLLG system posed on the full space (1.12) and the reformulated MLLG system (2.12).

Definition 2.31. *We call* (m, E, H) *a solution of the MLLG system* (1.12)*, if :*

- *The regularity assumptions for the magnetization*
$$m \in L^2([0,T], H^2(\Omega)) \cap H^1([0,T], L^2(\Omega))$$
and for the electric and magnetic field
$$E, H \in L^2([0,T], H(\mathrm{curl}, \mathbb{R}^3 \setminus \mathit{\Gamma})) \cap H^1([0,T], L^2(\mathbb{R}^3))$$
are fulfilled.
- *It holds* $|m| = 1$ *almost everywhere in* Ω_T*.*

The following equations hold:

- *The LLG equation* (1.12a) *in* $L^2([0,T] \times \Omega)$*,*
- *the interior Maxwell equations* (1.12b)–(1.12c) *in* $L^2([0,T] \times \Omega)$*,*
- *the exterior Maxwell equations in* (1.12d)–(1.12e) *in* $L^2([0,T] \times \overline{\Omega}^{\mathrm{c}})$*,*
- *the boundary condition* (1.12f) *in* $L^2([0,T], H^{1/2}(\mathit{\Gamma}))$*,*
- *the transmission condition* (1.12g) *in* $L^2([0,T], H_{\mathit{\Gamma}})$*,*
- *the initial data* (1.12h)–(1.12i) *in the sense of traces in* $L^2(\mathbb{R}^3)$*.*

Remark 2.32. *The regularity assumption* $E \in L^2([0,T], H(\mathrm{curl}, \mathbb{R}^3 \setminus \mathit{\Gamma}))$ *together with the transmission condition* $\gamma_T E^{\mathrm{int}} = \gamma_T E^{\mathrm{ext}}$ *in* $L^2([0,T], H_{\mathit{\Gamma}})$ *is equivalent to the regularity assumption* $E \in L^2([0,T], H(\mathrm{curl}, \mathbb{R}^3))$*.*

Definition 2.33. *We call* (m, E, H) *a solution of the MLLG system* (2.12)*, if :*

- *The regularity assumptions for the magnetization*
$$m \in L^2([0,T], H^2(\Omega)) \cap H^1([0,T], L^2(\Omega))$$
and for the electric and magnetic field
$$E, H \in H(\partial_t, \mathrm{curl}, \Omega_T)$$
are fulfilled.
- *It holds* $|m| = 1$ *almost everywhere in* Ω_T*.*

The following equations hold:

- *The LLG equation* (2.12a) *in* $L^2([0,T] \times \Omega)$*,*
- *the Maxwell equations* (2.12b)–(2.12c) *in* $L^2([0,T] \times \Omega)$*,*
- *the boundary integral equation* (2.12d) *in* $L^2([0,T], H_{\mathit{\Gamma}})$ *in the sense that* $B_m * \left(\begin{smallmatrix}\mu_0 \gamma_T(H) \\ -\gamma_T(E)\end{smallmatrix}\right) \in H^m_{0,*}([0,T], \mathcal{H}_{\mathit{\Gamma}})$ *and*
$$\partial_t^m B_m * \begin{pmatrix}\mu_0 \gamma_T H \\ -\gamma_T E\end{pmatrix} = \frac{1}{2\mu_0} \begin{pmatrix}\gamma_T E \\ \mu_0 \gamma_T H\end{pmatrix} \quad \text{in } L^2([0,T], \mathcal{H}_{\mathit{\Gamma}}),$$
- *the boundary condition* (2.12e) *in* $L^2([0,T], H^{1/2}(\mathit{\Gamma}))$*,*
- *the initial data* (2.12f) *in the sense of traces in* $L^2(\Omega)$*.*

Remark 2.34. *The condition* $|m| = 1$ *almost everywhere in* Ω_T *may be removed in both definitions as long as the initial data fulfills* $|m^0| = 1$ *almost everywhere in* Ω*. In this case, for both definitions one can show that* $|m| = 1$ *is satisfied for all times.*

2.4.3. Equivalence of the solutions

In this section we establish an equivalence between the exterior Maxwell system and the system on the boundary, in the sense that if the solutions are smooth enough (and extendable to the time interval $[0,\infty)$ in the exterior domain or on the boundary), they coincide.

Theorem 2.35. *A solution (m, E, H) in the sense of Definition 2.31, that*

- *can be extended to the time interval $[0,\infty)$ on the exterior domain, i.e. there exists a constant $c > 0$, such that*

$$(e^{-ct}\widetilde{E}, e^{-ct}\widetilde{H}) \in L^2([0,\infty), H(\mathrm{curl}, \overline{\Omega}^{\mathrm{c}})) \cap H^1([0,\infty), L^2(\overline{\Omega}^{\mathrm{c}}))$$

with

$$(\widetilde{E}, \widetilde{H}) = (E, H) \quad \text{in } L^2([0,T], \overline{\Omega}^{\mathrm{c}})$$

and $(\widetilde{E}, \widetilde{H})$ fulfills the exterior Maxwell equations (1.12d)–(1.12e) in an e^{-ct}-weighted $L^2([0,\infty) \times \overline{\Omega}^{\mathrm{c}})$ sense (i.e. (1.12d)–(1.12e) multiplied by e^{-ct} hold in $L^2([0,\infty) \times \overline{\Omega}^{\mathrm{c}})$),

is a solution in the sense of Definition 2.33.

Let (m, E, H) be a solution in the sense of Definition 2.33, that

- *satisfies the regularity assumptions of Definition 2.31 in Ω and*
- *the traces $\gamma_T E$, $\gamma_T H$ are in $H^2_{0,*}([0,T], \mathcal{H}_\Gamma)$ and*
- *can be extended to the time interval $[0,\infty)$ on the boundary, i.e. there exists a constant $c > 0$ such that*

$$e^{-ct}\gamma_T\widetilde{E}, e^{-ct}\gamma_T\widetilde{H} \in H^2_{0,*}([0,\infty), \mathcal{H}_\Gamma)$$

with

$$(\gamma_T\widetilde{E}, \gamma_T\widetilde{H}) = (\gamma_T E, \gamma_T H) \quad \text{in } L^2([0,T], \mathcal{H}_\Gamma)$$

and $(\gamma_T\widetilde{E}, \gamma_T\widetilde{H})$ fulfill the boundary integral equation (2.12d) in an e^{-ct}-weighted $L^2([0,\infty), \mathcal{H}_\Gamma)$ sense (i.e. (2.12d) multiplied by e^{-ct} holds in $L^2([0,\infty), \mathcal{H}_\Gamma)$).

Then

$$(E, H) := \begin{cases} (E, H) & \text{in } \Omega, \\ (S(\partial_t)\mu_0\gamma_T H - D(\partial_t)\gamma_T E, -S(\partial_t)\epsilon_0\gamma_T E - D(\partial_t)\gamma_T H) & \text{in } \overline{\Omega}^{\mathrm{c}}, \end{cases} \tag{2.27}$$

is a solution in the sense of Definition 2.31. The existence of the convolution operators in the second line of (2.27) holds in a matrix vector multiplication sense, see Remark 2.37.

Proof. The equivalence in the LLG-part can be shown exactly as in Section 2.1.1. As $|m| = 1$ is satisfied almost everywhere, the applications of $m \times \cdot$ are transformations from $L^2(\Omega_T) \to L^2(\Omega_T)$.

Concerning the interior Maxwell part, the equations and regularity assumptions are the same in both defintions, so there is nothing to show and all we have to consider are the equations in the exterior domain or the boundary integral equations, respectively.

Let (m, E, H) be a solution in the sense of Definition 2.31 that can be extended to $[0,\infty)$ on the exterior domain as stated in the theorem. We denote the extension by $(\widetilde{E}, \widetilde{H})$.

We apply the vector valued Laplace transform from Section B.2 (with respect to the Hilbert space $L^2(\overline{\Omega}^{\mathrm{c}})$) to the on $[0,\infty)$ extended version of (1.12d)–(1.12e). Due to $\widetilde{E}(0) = E(0) = 0$ we have $\mathcal{L}\partial_t \widetilde{E} = s\mathcal{L}\widetilde{E}$ and by interchanging the Laplace transform and space (differential) operators (see Lemma B.54 that this is possible) we obtain

$$\begin{aligned} \varepsilon_0 s\mathcal{L}\widetilde{E} - \nabla\times\mathcal{L}\widetilde{H} &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \mu_0 s\mathcal{L}\widetilde{H} + \nabla\times\mathcal{L}\widetilde{E} &= 0 && \text{in } \overline{\Omega}^{\mathrm{c}}, \\ \gamma_T(\mathcal{L}\widetilde{E}) &= \mathcal{L}\gamma_T\widetilde{E} && \text{on } \mathit{\Gamma}, \\ \gamma_T(\mathcal{L}\widetilde{H}) &= \mathcal{L}\gamma_T\widetilde{H} && \text{on } \mathit{\Gamma}, \end{aligned}$$

for $s\in\mathbb{C}$ with $\Re s \geq$ for $c>0$. Thus $(\mathcal{L}E, \mathcal{L}H)$ solve the time harmonic Maxwell equations (THME). More precisely, with $s=-i\omega$ it is $\Im\omega > 0$ and they solve the system (2.25). Therefore the traces are suitable exterior data and we have

$$B(s)\mathcal{L}\begin{pmatrix}\mu_0\gamma_T\widetilde{H}\\ -\gamma_T\widetilde{E}\end{pmatrix} = \frac{1}{2\mu_0}\mathcal{L}\begin{pmatrix}\gamma_T\widetilde{E}\\ \mu_0\gamma_T\widetilde{H}\end{pmatrix}. \tag{2.28}$$

The inverse Laplace transform of the right-hand side exists, thus the one of the left-hand side, too, so we obtain

$$B(\partial_t)\begin{pmatrix}\mu_0\gamma_T\widetilde{H}\\ -\gamma_T\widetilde{E}\end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix}\gamma_T\widetilde{E}\\ \mu_0\gamma_T\widetilde{H}\end{pmatrix}.$$

As $B(s)$ is a family of analytic operators (see Lemma 2.30) and $\|B(s)\| \leq Cs^2$ (see Lemma 2.13), it holds for $m\in\mathbb{N}$, $m>3$

$$\partial_t^{-m}B(\partial_t)\begin{pmatrix}\mu_0\gamma_T\widetilde{H}\\ -\gamma_T\widetilde{E}\end{pmatrix} = B_m * \begin{pmatrix}\mu_0\gamma_T\widetilde{H}\\ -\gamma_T\widetilde{E}\end{pmatrix}.$$

and

$$B(\partial_t)\begin{pmatrix}\mu_0\gamma_T\widetilde{H}\\ -\gamma_T\widetilde{E}\end{pmatrix} = \partial_t^m\partial_t^{-m}B(\partial_t)\begin{pmatrix}\mu_0\gamma_T\widetilde{H}\\ -\gamma_T\widetilde{E}\end{pmatrix} = \partial_t^m B_m * \begin{pmatrix}\mu_0\gamma_T\widetilde{H}\\ -\gamma_T\widetilde{E}\end{pmatrix}.$$

So we have $B_m * \binom{\mu_0\gamma_T(\widetilde{H})}{-\gamma_T(\widetilde{E})} \in H^m_{0,*}([0,T],\mathcal{H}_\mathit{\Gamma})$ and

$$\partial_t^m B_m * \begin{pmatrix}\mu_0\gamma_T\widetilde{H}\\ -\gamma_T\widetilde{E}\end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix}\gamma_T\widetilde{E}\\ \mu_0\gamma_T\widetilde{H}\end{pmatrix}.$$

By the Causality of $\partial_t^m B_m *$, it is immediate that $B_m * \binom{\mu_0\gamma_T(H)}{-\gamma_T(E)} \in H^m_{0,*}([0,T],\mathcal{H}_\mathit{\Gamma})$ and

$$\partial_t^m B_m * \begin{pmatrix}\mu_0\gamma_T H\\ -\gamma_T E\end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix}\gamma_T E\\ \mu_0\gamma_T H\end{pmatrix} \quad \text{in } L^2([0,T],\mathcal{H}_\mathit{\Gamma}).$$

So we have a solution in the sense of Definition 2.33. (Furthermore it can be continued to $[0,\infty)$ on the boundary in a e^{-ct}-weighted $L^2([0,\infty),\mathcal{H}_\mathit{\Gamma})$-sense.)

Now, let (E,H) be a solution in the sense of Definition 2.33 with all the properties stated in the theorem. We consider the boundary equation for the extensions $\gamma_T\widetilde{H}, \gamma_T\widetilde{E}$. We integrate the boundary equation m-times in time, and as $e^{-ct}B_m * \binom{\mu_0\gamma_T\widetilde{H}}{-\gamma_T\widetilde{E}} \in H^m_{0,*}([0,\infty),\mathcal{H}_\mathit{\Gamma})$, we obtain

$$B_m * \begin{pmatrix}\mu_0\gamma_T\widetilde{H}\\ -\gamma_T\widetilde{E}\end{pmatrix} = \frac{1}{2\mu_0}\partial_t^{-m}\begin{pmatrix}\gamma_T\widetilde{E}\\ \mu_0\gamma_T\widetilde{H}\end{pmatrix}. \tag{2.29}$$

We apply the vector valued Laplace transform from Section B.2 (with respect to the Hilbert space $\mathcal{H}_\Gamma$) to equation (2.29) which results in

$$s^{-m}B(s)\mathcal{L}\begin{pmatrix}\mu_0\gamma_T\widetilde{H}\\ -\gamma_T\widetilde{E}\end{pmatrix} = \frac{1}{2\mu_0}s^{-m}\mathcal{L}\begin{pmatrix}\gamma_T\widetilde{E}\\ \mu_0\gamma_T\widetilde{H}\end{pmatrix}$$

for $\Re s > 0$. Multiplying the outcome by s^m and setting $s = -i\omega$, we obtain suitable exterior data like stated in (2.26). The latter can therefore be represented as

$$\begin{aligned} h &= \mathcal{S}(s)\mathcal{L}(-\varepsilon_0\gamma_T\widetilde{E}) + \mathcal{D}(s)\mathcal{L}(-\gamma_T\widetilde{H}),\\ e &= \frac{1}{-i\omega\varepsilon_0}\nabla\times h = \mathcal{S}(s)(\mu_0\gamma_T\widetilde{H}) + \mathcal{D}(s)\mathcal{L}(-\gamma_T\widetilde{E}). \end{aligned}$$

By Lemma 2.27, we obtain

$$\|\nabla\times h\|_{\overline{\Omega}^{\mathrm{c}}} + |s|\|h\|_{\overline{\Omega}^{\mathrm{c}}} \le C(\Gamma,\mu_0,\varepsilon_0,\sigma_0)|s|^2\|\mathcal{L}(\gamma_T\widetilde{H})\|_{\mathcal{H}_\Gamma}$$

for $\Re(s) \ge \sigma_0 > 0$ and similarly

$$\|\nabla\times e\|_{\overline{\Omega}^{\mathrm{c}}} + |s|\|e\|_{\overline{\Omega}^{\mathrm{c}}} \le C(\Gamma,\mu_0,\varepsilon_0,\sigma_0)|s|^2\|\mathcal{L}(\gamma_T\widetilde{E})\|_{\mathcal{H}_\Gamma}.$$

Due to the regularity assumption of the extension, we have that $s^2\mathcal{L}(\gamma_T\widetilde{H})$, $s^2\mathcal{L}(\gamma_T\widetilde{E})$ are uniquely square integrable over each vertical line $\Re s \ge c > 0$ and as $\mathcal{S}$, $\mathcal{D}$ are both families of analytic operators (see Lemma 2.30), we have that e and h are suitably bounded and analytic. Therefore the inverse Laplace transform exists (cf. Definition B.55) and on $\overline{\Omega}^{\mathrm{c}}$ we define

$$H := \mathcal{L}^{-1}h, \quad E := \mathcal{L}^{-1}e,$$

which satisfy the regularity assumptions of Definition 2.31 in $[0,T]$ and are a extension to $[0,\infty)$ on $\overline{\Omega}^{\mathrm{c}}$.
The initial condition $H(0) = 0$ is fulfilled, as $\mathcal{L}^{-1}(sh(s))$ exists, thus $H = \partial_t^{-1}\mathcal{L}^{-1}(sh(s))$, which gives $H(0) = 0$. Similar arguments for E conclude $E(0) = 0$.
The transmission condition $\gamma_T E^{\mathrm{int}} = \gamma_T E^{\mathrm{ext}}$, $\gamma_T H^{\mathrm{int}} = \gamma_T H^{\mathrm{ext}}$ in $L^2([0,T], H_\Gamma)$ follows by the properties of the time harmonic operators (as all traces are suitable) and by interchanging the tangential trace operator and the Laplace transform. □

Remark 2.36. *Let us summarize: To get from Definition 2.31 to Definition 2.33, we only need extendability on the exterior domain. To get back, we need extendability on the boundary and we need additional regularity, in particular the traces have to be two times differentiable in time with vanishing derivatives at $t = 0$. The reason is that the single and double layer potentials are second order operators in time (see e.g. Lemma 2.27) and we need the regularity of the traces to ensure that they exist.*

The Calderon operator is a second order differential operator in time, too, nevertheless we do not need the additional regularity to get from Definition 2.31 to Definition 2.33. The existence of the inverse Laplace transform is inherently guaranteed by equation (2.28).

Remark 2.37. *The statement in the equation* (2.27) *of this theorem should be read with care. In the previous proof, we only showed that $\mathcal{L}^{-1}h$ (and similar statements for $\mathcal{L}^{-1}e$) exists, i.e.*

$$\mathcal{L}^{-1}\left(\mathcal{S}(s)\mathcal{L}(-\varepsilon_0\gamma_T E) + \mathcal{D}(s)\mathcal{L}(-\gamma_T H)\right)$$

exists. This means that in a matrix vector multiplication sense

$$\begin{pmatrix} S & D\\ -1/\mu_0 D & \epsilon_0 S\end{pmatrix}(\partial_t)\begin{pmatrix}\mu_0\gamma_T H\\ -\gamma_T E\end{pmatrix} = (S(\partial_t)\mu_0\gamma_T H - D(\partial_t)\gamma_T E, -S(\partial_t)\epsilon_0\gamma_T E - D(\partial_t)\gamma_T H)$$

exists. We did not show that $\mathcal{S}(\partial_t)(-\varepsilon_0\gamma_T E) = \mathcal{L}^{-1}(\mathcal{S}(s)\mathcal{L}(-\varepsilon_0\gamma_T E))$ *and* $\mathcal{D}(\partial_t)(-\gamma_T H) = \mathcal{L}^{-1}(\mathcal{D}(s)\mathcal{L}(-\gamma_T H))$ *exist themselves. This might be possible to show under suitable bounds* $\|\mathcal{S}(s)\| \le Cs^2$, $\|\mathcal{D}(s)\| \le Cs^2$. *A comparable bound for the single layer can be found in [143, Section A.4]. This is shown by using similar estimates as in Lemma 2.27 and inserting special traces such that* $\mathcal{D}(s)\gamma_T H$ *vanishes. Due to the refined analysis of the Laplace differential operators in Chapter B, we need less time regularity than in [143, Section A.4].*

Remark 2.38. *In the pure Maxwell case, i.e. without the LLG equation, under the assumptions of Chapter 4, a continuation of* $\gamma_T E$, $\gamma_T H$ *in a* e^{-ct}*-weighted* $L^2([0,\infty),\mathcal{H}_\Gamma)$ *sense can be shown to exist due our later result, Theorem 4.28, as the energy norm only grows polynomially in* T *and* J *can be extended to* $[0,\infty)$ *in a smooth way such that* $J(t) = 0$, $t > 2T$.

Remark 2.39. *If* $\gamma_T E$, $\gamma_T H$ *are sufficiently smooth, a continuation in* $H^2_{0,*}([0,\infty),\mathcal{H}_\Gamma)$ *can be shown to exist. The idea is to extend the functions to* $[0,\infty)$ *in a smooth enough way. Therefore, they may not any more fulfill the boundary integral equation for* $t > T$, *but if we apply the projection on suitable exterior data, the boundary integral equation again is fulfilled on* $[0,\infty)$. *As the projection operator is a second order differential operator in time, we need the functions to be four times differentiable in time to get an extension in* $H^2_{0,*}([0,\infty),\mathcal{H}_\Gamma)$.

For $\gamma_T E, \gamma_T H \in H^4_{0,*}([0,T],\mathcal{H}_\Gamma)$, *we set*

$$\begin{pmatrix} \gamma_T \widetilde{E} \\ \mu_0 \gamma_T \widetilde{H} \end{pmatrix} := \left(\frac{1}{2} + B(\partial_t)\mu_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \partial_t^{-4} \mathbb{1}_{[0,T]} \partial_t^4 \begin{pmatrix} \gamma_T E \\ \mu_0 \gamma_T H \end{pmatrix}.$$

We now want to show, that, indeed, $\gamma_T \widetilde{E}, \gamma_T \widetilde{H}$ *is a continuation in* $H^2_{0,*}([0,\infty),\mathcal{H}_\Gamma)$. *It is clear, that for every* $c > 0$,

$$e^{-ct} \partial_t^{-4} \mathbb{1}_{[0,T]} \partial_t^4 \begin{pmatrix} \mu_0 \gamma_T H \\ \gamma_T E \end{pmatrix} \in H^4_{0,*}([0,\infty),\mathcal{H}_\Gamma).$$

For $U : [0,T] \to \mathcal{H}_\Gamma$ *this is the Taylor polynomial continuation for* $t > T$ *by*

$$\partial_t^{-4} \mathbb{1}_{[0,T]} \partial_t^4 U(t) = U(T) + (t-T)U'(T) + \cdots + \frac{(t-T)^3}{3!} U^{(3)}(T).$$

As $\|B(s)\| \le C|s|^2$, *it follows from the properties of the Laplace transform that*

$$e^{-ct}\gamma_T \widetilde{H}, e^{-ct}\gamma_T \widetilde{E} \in H^2_{0,*}([0,\infty),\mathcal{H}_\Gamma)$$

for every $c > 0$. *Furthermore,* $(\gamma_T \widetilde{E}, \gamma_T \widetilde{H})$ *fulfills the boundary integral equation* (2.12d) *in an* e^{-ct}*-weighted* $L^2([0,\infty),\mathcal{H}_\Gamma)$ *sense by the projection property of the Calderon operator (see Lemma 2.14 and apply the same arguments as in the proof of Theorem 4.3). By* (2.12d) *and by the Causality property of* $B(\partial_t)$, *it holds*

$$(\gamma_T \widetilde{E}, \gamma_T \widetilde{H}) = (\gamma_T E, \gamma_T H)$$

on $[0,T]$.

3. Weak Convergence for the MLLG System

In this chapter, we consider weak convergence of the approximations towards the exact solution of the boundary integral Maxwell–LLG system. In Section 3.1, we introduce suitable notions of solutions and discuss the connection with the ones from Section 2.4 (see Remark 3.5). In Section 3.2, we present the approximation scheme. Finally, in Section 3.3, we show that the approximations converge (in a weak-subsequence sense) towards the exact solutions from Section 3.1.

3.1. Weak Solutions, Equivalence and Uniqueness

In this section we consider the boundary integral MLLG system. We introduce a reduced regularity and a weak solution (for an overview see Remark 3.5) and show equivalence and uniqueness of (a part of) these solutions.

3.1.1. Definition of a reduced regularity solution

Originating from the boundary integral MLLG system (2.12), we define a suitable solution that needs lower regularity assumptions for its definition than the one from Definition 2.33. Given sufficient regularity, both solutions coincide.

For the convenience of the reader, we repeat the MLLG system (2.12). We seek functions m, E and $H : [0,T] \times \Omega \to \mathbb{R}^3$ which satisfy: in the interior domain

$$\partial_t m - \alpha m \times \partial_t m = -m \times (C_e \Delta m + H) \qquad \text{in } \Omega_T, \tag{3.1a}$$
$$\varepsilon \partial_t E - \nabla \times H = -(J + \sigma E) \qquad \text{in } \Omega_T, \tag{3.1b}$$
$$\mu \partial_t H + \nabla \times E = -\mu \partial_t m \qquad \text{in } \Omega_T, \tag{3.1c}$$

coupled to the boundary integral equations

$$B(\partial_t) \begin{pmatrix} \mu_0 \gamma_T H \\ -\gamma_T E \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mu_0^{-1} \gamma_T E \\ \gamma_T H \end{pmatrix} \qquad \text{on } [0,T] \times \partial\Omega, \tag{3.1d}$$

where m satisfies the boundary condition

$$\partial_n m = 0 \qquad \text{on } [0,T] \times \partial\Omega, \tag{3.1e}$$

and with the initial conditions

$$m(0) = m^0, \quad E(0) = E^0, \quad H(0) = H^0 \quad \text{in } \Omega. \tag{3.1f}$$

We multiply the LLG equation (3.1a) with a smooth test function ρ and use $\partial_n m = 0$ on Γ to obtain for the Laplace term

$$\begin{aligned}[\Delta m \times m, \rho]_\Omega &= [\nabla m \times \rho, \nabla m]_\Omega - [\nabla m \times m, \nabla \rho]_\Omega + [m \times \rho, \partial_n m]_\Gamma \\ &= -[\nabla m \times m, \nabla \rho]_\Omega.\end{aligned}$$

For the Maxwell part, we integrate equation (3.1b) and (3.1c) once in time. We recall the operator

$$(\partial_t^{-1} v)(t) = \mathcal{L}^{-1}\left(s^{-1}\mathcal{L}(v)\right)(t) = \int_0^t v(s)\ \mathrm{d}s = (1 * v)(t)$$

and apply it to the equations (3.1b)–(3.1c). For a function $G \in C^1[0,T]$ it holds

$$\partial_t^{-1}\partial_t G(t) = \int_0^t \partial_t G(\tau)\ \mathrm{d}\tau = G(t) - G(0).$$

Similarly, the boundary integral equation is integrated once in time. From Lemma B.82, we obtain that the operator ∂_t^{-1} commutes with $B(\partial_t)$ for suitable functions ϕ (smooth enough with vanishing derivatives at $t = 0$) in the sense

$$\partial_t^{-1}\partial_t^m B_m * \phi = \partial_t^m \partial_t^{-1} B_m * \phi = \partial_t^m B_m * \partial_t^{-1}\phi.$$

We apply the stated modifications to the MLLG system (3.1) and arrive at the following definition.

The LLG part is given in weak form tested with smooth functions, while the Maxwell and boundary part are integrated once in time and given in strong form, i.e. without test functions.

Definition 3.1. *We consider a solution of the MLLG equations, i.e. (m, E, H) that satisfies*

- $m \in H^1(\Omega_T)$ *with* $|m| = 1$ *almost everywhere,* $m(0) = m^0$ *in the sense of traces, and for all* $\rho \in C^\infty(\Omega_T)$ *we have*

$$[\partial_t m, \rho]_{\Omega_T} - \alpha[m \times \partial_t m, \rho]_{\Omega_T} = -C_e[\nabla m \times m, \nabla\rho]_{\Omega_T} + [H \times m, \rho]_{\Omega_T}.$$

- $E, H \in L^2(\Omega_T)$ *such that* $\partial_t^{-1}E, \partial_t^{-1}H \in H(\mathrm{curl}, \Omega_T)$ *and*

$$\begin{aligned} \varepsilon(E - E^0) - \nabla \times (\partial_t^{-1}H) + \sigma\partial_t^{-1}E &= -\partial_t^{-1}J \qquad &&\text{in } L^2(\Omega_T),\\ \mu(H - H^0) + \nabla \times (\partial_t^{-1}E) &= -\mu(m - m^0) \qquad &&\text{in } L^2(\Omega_T), \end{aligned}$$

as well as $B_m * \binom{\mu_0\gamma_T(\partial_t^{-1}H)}{-\gamma_T(\partial_t^{-1}E)} \in H^m_{0,*}([0,T], \mathcal{H}_\Gamma)$ *with*

$$\partial_t^m B_m * \begin{pmatrix} \mu_0\gamma_T(\partial_t^{-1}H) \\ -\gamma_T(\partial_t^{-1}E) \end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix} \gamma_T(\partial_t^{-1}E) \\ \mu_0\gamma_T(\partial_t^{-1}H) \end{pmatrix} \qquad \text{in } L^2([0,T], \mathcal{H}_\Gamma).$$

Given sufficient regularity, the solutions in Definition 3.1 and Definition 2.33 coincide.

Theorem 3.2. *Every solution in the sense of Definition 2.33 is a solution in the sense of Definition 3.1. Conversely, a solution in the sense of Definition 3.1, that fulfills the regularity assumptions of Definition 2.33, is a solution in the sense of Definition 2.33.*

Proof. To get from Definition 2.33 to Definition 3.1, we apply the above modifications that hold under the stated regularity. Especially for the boundary integral equation, we refine the arguments (the regularity for ϕ from above is not needed). We note that B_m* and ∂_t^{-1} commute as they are smoothing operators in time, see Lemma B.82. As $B_m * \binom{\mu_0\gamma_T H}{-\gamma_T E} \in H^m_{0,*}([0,T], \mathcal{H}_\Gamma)$, the operators ∂_t^{-1} and ∂_t^m commute and it is

$$\partial_t^{-1}\partial_t^m B_m * \begin{pmatrix} \mu_0\gamma_T H \\ -\gamma_T E \end{pmatrix} = \partial_t^m \partial_t^{-1} B_m * \begin{pmatrix} \mu_0\gamma_T H \\ -\gamma_T E \end{pmatrix} = \partial_t^m B_m * \partial_t^{-1}\begin{pmatrix} \mu_0\gamma_T H \\ -\gamma_T E \end{pmatrix}.$$

To get from Definition 3.1 to Definition 2.33, assuming the regularity assumptions from Definition 2.33, we proceed in a similar way to reverse the modifications. For the LLG equation, integration by parts shows

$$[\nabla m \times m, \nabla \rho]_{\Omega_T} = -[\Delta m \times m, \rho]_{\Omega_T} + [m \times \rho, \partial_n m]_{\Gamma_T},$$

and by the use of cut off functions (as all the other terms in the LLG equation are bounded in $L^2(\Omega_T)$) we obtain

$$[\partial_n m \times m, \rho]_{\Gamma_T} = 0,$$

i.e. $\partial_n m \times m = 0$ on Γ_T. By $|m| = 1$ almost everywhere, we deduce $m \cdot \partial_n m = 0$ and finally conclude

$$\partial_n m = 0.$$

For the interior Maxwell part, (formally) inserting $t = 0$ into the equation gives $E(0) = E^0$ and $H(0) = H^0$ (note $(\partial_t^{-1} G)(0) = 0$ for any function G) and due to the given regularity assumptions, rigorous arguments similarly show that $E(0) = E^0$ and $H(0) = H^0$ hold in the sense of traces. The differential equations from Definition 3.1 follow by deriving in time and noting that $\partial_t \partial_t^{-1} G = G$ applies for any function G.

For the boundary integral equation, under the stated regularity assumptions, $\partial_t^m B_m *$ and ∂_t^{-1} commute as above, which together with deriving the equation in time, concludes the proof.

□

3.1.2. Definition of a weak solution

In this section, we motivate the definition of a weak solution of the MLLG system (3.1). In the following sections, we will show that such a solution exists and that the approximations converge (in a weak subsequence sense) towards that solution.

We recall the notations for the scalar products

$$\langle \varphi, \psi \rangle_{\Gamma_T} = \int_0^T \langle \varphi, \psi \rangle_{\Gamma} \, \mathrm{d}t$$

and

$$[v, w]_{\Omega_T} = \int_0^T \int_\Omega v \cdot w \, \mathrm{d}x \, \mathrm{d}t,$$

for suitable functions φ, ψ, v, w.

For the LLG part, we apply the same modifications as in Section 3.1.1 and therefore this part remains unchanged in comparison to Definition 3.1.

The Maxwell equations (3.1b)–(3.1c) are multiplied by smooth test functions ζ_E, ζ_H and the resulting equations are integrated over Ω_T. We state the following version of the integration by parts formula, for $a, b \in C^1([0, T])$ it holds

$$\int_0^T a(t) b(t) \, \mathrm{d}t = (\partial_t^{-1} a)(T) b(T) - \int_0^T (\partial_t^{-1} a)(t) \partial_t b(t) \, \mathrm{d}t.$$

Assuming $\zeta_E(T) = 0$ and using the integration by parts formula, we obtain

$$[\nabla \times E, \zeta_E]_{\Omega_T} = -[\nabla \times (\partial_t^{-1} E), \partial_t \zeta_E]_{\Omega_T}.$$

Similarly, it holds with integration by parts and $\zeta_E(T) = 0$

$$[\partial_t E, \zeta_E]_{\Omega_T} = -[E, \partial_t \zeta_E]_{\Omega_T} - [E(0), \zeta_E(0)]_\Omega.$$

For the boundary equation, we similarly multiply (3.1d) by smooth test functions and integrate over the respective space time domains (such that we obtain the anti-symmetric pairing, see Definition 2.6). For the left hand side, we use integration by parts yielding

$$\langle v, \gamma_T E\rangle_{\Gamma_T} = -\langle \gamma_T(\partial_t^{-1}E), \partial_t \gamma_T \zeta\rangle_{\Gamma_T}.$$

We introduce the abbreviations $\widetilde{\psi} := -\gamma_T(\partial_t^{-1}E)$ and $\widetilde{\varphi} := \mu_0\gamma_T(\partial_t^{-1}H)$ and rewrite

$$\gamma_T(\partial_t^{-1}E) = 2\gamma_T(\partial_t^{-1}E) + \widetilde{\psi} \quad \text{and} \quad \mu_0\gamma_T(\partial_t^{-1}H) = 2\mu_0\gamma_T(\partial_t^{-1}H) - \widetilde{\varphi}.$$

For the Calderon operator on the right hand side, we note for $\phi \in C^m([0,\infty), \mathcal{H}_\Gamma^2)$ with $\phi(0) = \partial_t\phi(0) = \cdots = \partial_t^{m-1}\phi(0) = 0$, that $\widetilde{\phi} := \partial_t^{-1}\phi \in C^{m+1}([0,\infty), \mathcal{H}_\Gamma^2)$ and $\widetilde{\phi}(0) = \cdots = \partial_t^m\widetilde{\phi}(0) = 0$. Furthermore it is $\mathcal{L}^{-1}(\mathcal{L}(\phi)s^m) = \mathcal{L}^{-1}(\mathcal{L}(\widetilde{\phi})s^{m+1})$ and

$$\begin{aligned}(B(\partial_t)\phi)(t) &= \mathcal{L}^{-1}(B(s)\mathcal{L}(\phi)(s))(t)\\ &= \left(\mathcal{L}^{-1}(B(s)s^{-m}) * \mathcal{L}^{-1}(\mathcal{L}(\widetilde{\phi})(s)s^{m+1})\right)(t)\\ &= \partial_t^{m+1}(B_m * \widetilde{\phi})(t)\end{aligned}$$

and

$$(B_m * \widetilde{\phi})(0) = \cdots = \partial_t^m(B_m * \widetilde{\phi})(0) = 0.$$

We therefore obtain for $v \in C^{m+1}([0,T], \mathcal{H}_\Gamma^2)$ with $v(T) = \partial_t v(T) = \cdots = \partial_t^m v(T) = 0$, by integrating $m+1$ times by parts in time, that

$$\begin{aligned}\langle v, \partial_t^m(B_m * \phi)\rangle_{\Gamma_T} &= \left\langle v, \partial_t^{m+1}(B_m * \widetilde{\phi})\right\rangle_{\Gamma_T}\\ &= -\left\langle \partial_t v, \partial_t^m(B_m * \widetilde{\phi})\right\rangle_{\Gamma_T} + \left[\left\langle v, \partial_t^m(B_m * \widetilde{\phi})\right\rangle_{\Gamma}\right]_0^T\\ &= -\left\langle \partial_t v, \partial_t^m(B_m * \widetilde{\phi})\right\rangle_{\Gamma_T}\\ &= \cdots = (-1)^{m+1}\left\langle \partial_t^{m+1}v, B_m * \widetilde{\phi}\right\rangle_{\Gamma_T}.\end{aligned}$$

The term on the right hand side is well defined for smooth v and for only (square) integrable $\widetilde{\phi}$ (in time).

In conclusion, we multiply the system (3.1) with suitable testfunctions, apply the above manipulations to the respective terms and end up with the definition of a weak solution.

Definition 3.3. *The functions $(m, E, H, \widetilde{\varphi}, \widetilde{\psi})$ are a weak solution of the MLLG equation if:*

- *$m \in H^1(\Omega_T)$ with $|m| = 1$ almost everywhere in Ω_T, $E, H \in L^2(\Omega_T)$ such that $\partial_t^{-1}E, \partial_t^{-1}H \in H(\mathrm{curl}, \Omega_T)$ and $\widetilde{\varphi}, \widetilde{\psi} \in L^2([0,T], \mathcal{H}_\Gamma)$.*
- *For all $\rho \in C^\infty(\overline{\Omega_T})$, all $\zeta_E, \zeta_H \in C^\infty(\overline{\Omega_T})$ with $\zeta_E(T) = \zeta_H(T) = 0$ and all $v, w \in \gamma_T(C^\infty(\overline{\Omega_T})) \cap H_{*,0}^{m+1}([0,T], \mathcal{H}_\Gamma)$ we have*

$$\begin{aligned}[\partial_t m, \rho]_{\Omega_T} - \alpha[m \times \partial_t m, \rho]_{\Omega_T} &= -C_e[\nabla m \times m, \nabla\rho]_{\Omega_T} + [H \times m, \rho]_{\Omega_T},\\ -[\varepsilon E, \partial_t\zeta_E]_{\Omega_T} - [\varepsilon E^0, \zeta_E(0)]_\Omega &= -[\nabla \times (\partial_t^{-1}H), \partial_t\zeta_E]_{\Omega_T} - [\sigma E + J, \zeta_E]_{\Omega_T},\\ -[\mu H, \partial_t\zeta_H]_{\Omega_T} - [\mu H^0, \zeta_H(0)]_\Omega &= [\nabla \times (\partial_t^{-1}E), \partial_t\zeta_H]_{\Omega_T} - [\mu\partial_t m, \zeta_H]_{\Omega_T},\\ (-1)^{m+1}\left\langle \partial_t^{m+1}\begin{pmatrix} v\\ w\end{pmatrix}, B_m * \begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix}\right\rangle_{\Gamma_T} &= -\frac{1}{2\mu_0}\left\langle \begin{pmatrix}\partial_t v\\ \partial_t w\end{pmatrix}, \begin{pmatrix} 2\gamma_T(\partial_t^{-1}E) + \widetilde{\psi}\\ 2\mu_0\gamma_T(\partial_t^{-1}H) - \widetilde{\varphi}\end{pmatrix}\right\rangle_{\Gamma_T}.\end{aligned} \tag{3.2}$$

- *It holds $m(0) = m^0$ in the sense of traces.*

Definition 3.4. *We say a solution of the MLLG system has bounded energy, if for almost all $t \in [0, T]$*

$$\|\nabla m(t)\|^2_{L^2(\Omega)} + \int_0^t \|\partial_t m(s)\|^2_{L^2(\Omega)} \, \mathrm{d}s + \|H(t)\|^2_{L^2(\Omega)} + \|E(t)\|^2_{L^2(\Omega)} \leq C,$$

where $C > 0$ is independent of t.

Remark 3.5 (Overview of the solutions.)**.** *So far, we considered four different definitions of solutions to the Maxwell–LLG system.*

We started with the strong solution on the full space in Definition 2.31, which is the mathematical rigorous formulation of the MLLG system that arises from the physical derivation (1.12)*.*

Given sufficient smoothness (and extensibility to the time interval $[0, \infty)$), Theorem 2.35 shows that the full space solution is equivalent to the boundary integral solution introduced in Definition 2.33.

In Section 3.1.1, we considered in Definition 3.1 a version of the boundary integral solution that needs lower regularity for its definition. Again, given sufficient regularity, it is equivalent to the boundary integral solution from Definition 2.33.

Finally, in this section, we introduced a weak solution in Definition 3.3. This is the solution for which we construct in Section 3.3 converging approximations. To close the gap between the reduced-regularity solution and the weak solution, we show in Theorem 3.6 in the following that they are equivalent (in some sense).

3.1.3. Equivalence of the solutions

In this section we show equivalence of the solutions from Definition 3.1 and Definition 3.3.

Theorem 3.6. *If (m, E, H) is a solution in the sense of Definition 3.1, then*

$$(m, E, H, \mu_0 \gamma_T \partial_t^{-1} H, -\gamma_T \partial_t^{-1} E)$$

is a solution in the sense of Definition 3.3. If $(m, E, H, \widetilde{\varphi}, \widetilde{\psi})$ is a solution in the sense of Definition 3.3, then (m, E, H) is a solution in the sense of Definition 3.1.

Proof. Step 1: Let (m, E, H) be a solution in the sense of Definition 3.1. We multiply the Maxwell part of Definition 3.1 with the respective test functions of Definition 3.3. Integration by parts in time shows

$$[\varepsilon(E - E^0), -\partial_t \xi]_{\Omega_T} = -[\varepsilon E, \partial_t \xi]_{\Omega_T} - [\varepsilon E^0, \xi(0)]_\Omega$$

and yields the equations stated in Definition 3.3.

We introduce the variable $\widetilde{\varphi} = \mu_0 \gamma_T \partial_t^{-1} H$ for the tangential trace of H as well as $\widetilde{\psi} = -\gamma_T \partial_t^{-1} E$ for the tangential trace of E. For $b = -\partial_t \binom{v}{w} \in H^m_{*,0}([0, T], \mathcal{H}^2_\Gamma)$ we integrate by parts m times in time to obtain with $a = \binom{\widetilde{\varphi}}{\widetilde{\psi}}$

$$\begin{aligned} \langle b, \partial_t^m (B_m * a) \rangle_{\Gamma_T} &= - \left\langle \partial_t b, \partial_t^{m-1} (B_m * a) \right\rangle_{\Gamma_T} + \left[\left\langle b, \partial_t^{m-1} (B_m * a) \right\rangle_\Gamma \right]_0^T \\ &= - \left\langle \partial_t b, \partial_t^{m-1} (B_m * a) \right\rangle_{\Gamma_T} = \cdots = (-1)^m \left\langle \partial_t^m b, (B_m * a) \right\rangle_{\Gamma_T}. \end{aligned}$$

Thus we have a solution in the sense of Definition 3.3.

Step 2: Now let $(m, E, H, \widetilde{\varphi}, \widetilde{\psi})$ be a solution in the sense of Definition 3.3. The interior Maxwell parts of the Definition 3.3 and Definition 3.1 are equivalent via integration by parts in time. We will prove below that the operator

$$Q(\partial_t) := \left(\frac{1}{2\mu_0} \begin{pmatrix} 0 & -\partial_t^{-m} \\ \partial_t^{-m} & 0 \end{pmatrix} + B_m * \right)$$

is almost a projection in the sense

$$Q(\partial_t) \begin{pmatrix} v \\ w \end{pmatrix} = \frac{1}{\mu_0} \partial_t^{-m} \begin{pmatrix} -w \\ v \end{pmatrix} \tag{3.3}$$

for all $v, w \in L^2([0,T], \mathcal{H}_\Gamma)$ such that there exist $v', w' \in L^2([0,T], \mathcal{H}_\Gamma)$ with

$$\frac{1}{\mu_0} \begin{pmatrix} -w \\ v \end{pmatrix} = Q(\partial_t) \begin{pmatrix} v' \\ w' \end{pmatrix}. \tag{3.4}$$

Integration by parts in the last equation of Definition 3.3 shows that

$$\begin{aligned} &(-1)^{m+1} \left\langle \partial_t^{m+1} \begin{pmatrix} v \\ w \end{pmatrix}, B_m * \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} \right\rangle_{\Gamma_T} \\ &\qquad = \frac{(-1)^{m+1}}{2\mu_0} \left\langle \begin{pmatrix} \partial_t^{m+1} v \\ \partial_t^{m+1} w \end{pmatrix}, \partial_t^{-m} \begin{pmatrix} 2\gamma_T(\partial_t^{-1} E) + \widetilde{\psi} \\ 2\mu_0 \gamma_T(\partial_t^{-1} H) - \widetilde{\varphi} \end{pmatrix} \right\rangle_{\Gamma_T}, \end{aligned}$$

i.e. $\frac{1}{\mu_0} \partial_t^{-m} \binom{\gamma_T \partial_t^{-1} E}{\mu_0 \gamma_T \partial_t^{-1} H}$ is in the range of $Q(\partial_t)$. Hence (3.3) implies

$$\begin{aligned} \frac{1}{\mu_0} \partial_t^{-2m} \begin{pmatrix} \gamma_T \partial_t^{-1} E \\ \mu_0 \gamma_T \partial_t^{-1} H \end{pmatrix} &= Q(\partial_t) \partial_t^{-m} \begin{pmatrix} \mu_0 \gamma_T \partial_t^{-1} H \\ -\gamma_T \partial_t^{-1} E \end{pmatrix} \\ &= \frac{1}{2\mu_0} \partial_t^{-2m} \begin{pmatrix} \gamma_T \partial_t^{-1} E \\ \mu_0 \gamma_T \partial_t^{-1} H \end{pmatrix} + B_m * \partial_t^{-m} \begin{pmatrix} \mu_0 \gamma_T \partial_t^{-1} H \\ -\gamma_T \partial_t^{-1} E \end{pmatrix}. \end{aligned}$$

This shows $B_m * \binom{\mu_0 \gamma_T (\partial_t^{-1} H)}{-\gamma_T (\partial_t^{-1} E)} \in H^m_{0,*}([0,T], \mathcal{H}_\Gamma)$ and further differentiation in time leads to the boundary integral equation in Definition 3.1.
It remains to show (3.3). To that end, we use the definition of Q and obtain for $v, w \in L^2([0,T], \mathcal{H}_\Gamma)$ and $\omega := \sqrt{\mu_0 \varepsilon_0}$

$$\begin{aligned} Q(\partial_t) \begin{pmatrix} v \\ w \end{pmatrix} &= \mathcal{L}^{-1} \left(\frac{1}{2\mu_0} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} s^{-m} + s^{-m} B(s) \right) \begin{pmatrix} \mathcal{L}v(s) \\ \mathcal{L}w(s) \end{pmatrix} \\ &= \mathcal{L}^{-1} \left(\frac{1}{2\mu_0} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} s^{-m} + s^{-m} \mu_0^{-1} \begin{pmatrix} (i\omega)^{-1} V(s) & K(s) \\ -K(s) & -i\omega V(s) \end{pmatrix} \right) \begin{pmatrix} \mathcal{L}v(s) \\ \mathcal{L}w(s) \end{pmatrix} \\ &= \mathcal{L}^{-1} \frac{1}{\mu_0 s^m} \left(\frac{1}{2} \begin{pmatrix} 0 & -1 \\ i\omega & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -i\omega & 0 \end{pmatrix} \begin{pmatrix} K(s) & V(s) \\ V(s) & K(s) \end{pmatrix} \right) \begin{pmatrix} (i\omega)^{-1} \mathcal{L}v(s) \\ \mathcal{L}w(s) \end{pmatrix} \\ &= \frac{1}{\mu_0} \begin{pmatrix} 0 & -1 \\ i\omega & 0 \end{pmatrix} \mathcal{L}^{-1} \frac{1}{s^m} \left(\frac{1}{2} - \begin{pmatrix} K(s) & V(s) \\ V(s) & K(s) \end{pmatrix} \right) \begin{pmatrix} (i\omega)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{L}v(s) \\ \mathcal{L}w(s) \end{pmatrix}. \end{aligned}$$

In [46, Equation (35)] it is shown that

$$\frac{1}{2} - \begin{pmatrix} K(s) & V(s) \\ V(s) & K(s) \end{pmatrix}$$

is a projection. Hence, the above together with the properties of the Laplace transform from Lemma B.79 and Lemma B.82 show (3.3). This concludes the proof. □

Remark 3.7 (Summary)**.** *The difference between Definition 3.1 and Definition 3.3 is mainly (up to multiplication with test functions and integration by parts in the interior) in the formulation of the boundary integral equation. There are two main differences.*

1) We define the (non-modified) time harmonic Calderon operator (compare Lemma 2.14)

$$\widehat{B}(s) = \begin{pmatrix} K(s) & V(s) \\ V(s) & K(s) \end{pmatrix}.$$

Ignoring time integration and the modification of B, the boundary integral equation in Definition 3.1 corresponds to (see Lemma 2.14 and (THME,a) in Section 2.4.1)

$$-\widehat{B}(s)\begin{pmatrix} \gamma_T \widehat{e} \\ \gamma_T \widehat{h} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \gamma_T \widehat{e} \\ \gamma_T \widehat{h} \end{pmatrix}, \tag{3.5}$$

i.e. $\gamma_T \widehat{e}$, $\gamma_T \widehat{h}$ are suitable exterior data. The boundary integral equation in Definition 3.3 corresponds to

$$\left(\frac{1}{2} - \widehat{B}(s)\right)\begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_T \widehat{e} \\ \gamma_T \widehat{h} \end{pmatrix}, \tag{3.6}$$

i.e. $\gamma_T \widehat{e}$, $\gamma_T \widehat{h}$ is in the image of $\frac{1}{2} - B(s)$. By Lemma 2.14, we have that $P := \frac{1}{2} - B(s)$ is a projection, and applying it to (3.6) *gives*

$$P\begin{pmatrix} \gamma_T \widehat{e} \\ \gamma_T \widehat{h} \end{pmatrix} = P^2\begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix} = P\begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_T \widehat{e} \\ \gamma_T \widehat{h} \end{pmatrix}.$$

Thus $\gamma_T \widehat{e}$, $\gamma_T \widehat{h}$ are suitable exterior data. It needs a careful analysis of the Laplace differential operators on $[0, T]$ (like it is presented in Section B.2.3), to show that these arguments can be applied in a similar way to the respective time differential operators.

2) Concerning the time integration, the boundary integral equation from Definition 3.3 is given in weak form, i.e. in the term $\langle \partial_t^{m+1} w, B_m * \widetilde{\phi} \rangle_{\Gamma_T}$, all derivatives in time are with respect to the smooth test function, such that $\widetilde{\phi} \in L^2([0, T], \mathcal{H}_\Gamma)$ suffices. Under consideration of the right hand side of that equation, we obtain for $f \in L^2([0, T], \mathcal{H}_\Gamma)$

$$B_m * \widetilde{\phi} = \partial_t^{-m} f.$$

Thus we can conclude, that $B_m * \widetilde{\phi} \in H^m_{0,*}([0, T], \mathcal{H}_\Gamma)$ and m derivatives in time exist on both sides. Hence, the weak formulation of the boundary integral equation in Definition 3.3, inherently includes regularity.

3.1.4. Uniqueness of the solutions

In this section, we consider uniqueness of the interior Maxwell part of the MLLG system. Uniqueness of all components only holds under additional regularity assumptions, compare Remark 3.9.

Theorem 3.8. *The interior Maxwell components of a solution in the sense of Definition 3.3 are unique, i.e. if there is a magnetization m such that $(m, E_1, H_1, \widetilde{\varphi}_1, \widetilde{\psi}_1)$ and $(m, E_2, H_2, \widetilde{\varphi}_2, \widetilde{\psi}_2)$ are both solutions in the sense of Definition 3.3, then it holds*

$$(E_1, H_1) = (E_2, H_2).$$

Proof. Assume, that there exist two solutions in the sense of Definition 3.3 as stated in the theorem. By Theorem 3.6, we have that $(m, E_1, H_1, \mu_0 \gamma_T \partial_t^{-1} H_1, -\gamma_T \partial_t^{-1} E_1)$ and

$(m, E_2, H_2, \mu_0\gamma_T\partial_t^{-1}H_2, -\gamma_T\partial_t^{-1}E_2)$ are solutions in the sense of Definition 3.3. The difference $U := \partial_t^{-1}(E^1 - E^2)$, $V := \partial_t^{-1}(H^1 - H^2)$ satisfies

$$(U, V) \in H^1(\mathrm{curl}, \Omega_T) \times H^1(\mathrm{curl}, \Omega_T)$$

and for all

$$\zeta_E, \zeta_H \in C^\infty(\overline{\Omega_T}) \text{ with } \zeta_E(T) = \zeta_H(T) = 0$$

and all

$$v, w \in \gamma_T(C^\infty(\overline{\Omega_T}) \cap H^{m+1}_{*,0}([0,T], \mathcal{H}_\Gamma)$$

it holds (using $E_1(0) = E_2(0)$ and $H_1(0) = H_2(0)$)

$$\begin{aligned} &[\varepsilon\partial_t U, \partial_t\zeta_E]_{\Omega_T} + [\mu\partial_t V, \partial_t\zeta_H]_{\Omega_T} + (-1)^{m+1}\left\langle \begin{pmatrix} \partial_t^{m+1}v \\ \partial_t^{m+1}w \end{pmatrix}, B_m * \begin{pmatrix} \mu_0\gamma_T V \\ -\gamma_T U \end{pmatrix} \right\rangle_{\Gamma_T} \\ &\quad = [\nabla\times V, \partial_t\zeta_E]_{\Omega_T} - [\sigma U, \partial_t\zeta_E]_{\Omega_T} - [\nabla\times U, \partial_t\zeta_H]_{\Omega_T} - \frac{1}{2\mu_0}\left\langle \begin{pmatrix} \partial_t v \\ \partial_t w \end{pmatrix}, \begin{pmatrix} \gamma_T U \\ \mu_0\gamma_T V \end{pmatrix} \right\rangle_{\Gamma_T}. \end{aligned} \tag{3.7}$$

Moreover it is $U(0) = 0$ and $V(0) = 0$ in $L^2(\Omega)$ in the sense of traces. By a density/limit argument, since all quantities are bounded in $L^2(\Omega_T)$ or $L^2([0,T], \mathcal{H}_\Gamma)$, respectively, we can weaken the C^∞-regularity assumptions for the test functions to "smooth enough" functions. We are able to test with

$$\partial_t\zeta_E := (\overline{\partial}_t)^{-m}\widehat{\zeta}_E, \qquad \partial_t\zeta_H := (\overline{\partial}_t)^{-m}\widehat{\zeta}_H, \qquad \partial_t v := (\overline{\partial}_t)^{-m}\widehat{v}, \qquad \partial_t w := (\overline{\partial}_t)^{-m}\widehat{w}, \tag{3.8}$$

where

$$(\overline{\partial}_t)^{-1}g(s) := \int_s^T g(r)\,\mathrm{d}r$$

for

$$(\widehat{\zeta}_E, \widehat{\zeta}_H, \widehat{v}, \widehat{w}) \in L^2(\Omega_T) \times L^2(\Omega_T) \times L^2([0,T], \mathcal{H}_\Gamma) \times L^2([0,T], \mathcal{H}_\Gamma).$$

For $g \in L^2([0,T])$ it holds $(\overline{\partial}_t)^{-m}g \in H^m_{*,0}([0,T])$ and it holds for $f \in L^2(0,T)$

$$\begin{aligned} [f, \overline{\partial}_t^{-1}g]_{(0,T)} &= \int_0^T f(s)\int_s^T g(r)\,\mathrm{d}r\,\mathrm{d}s \\ &= \int_0^T\int_0^r f(s)g(r)\,\mathrm{d}s\,\mathrm{d}r \\ &= [\partial_t^{-1}f, g]_{(0,T)}. \end{aligned} \tag{3.9}$$

Using the indicator function

$$\mathbb{1}_{[0,r]}(t) = \begin{cases} 1, & t \in [0,r] \\ 0, & \text{else,} \end{cases}$$

we test (3.7) according to (3.8) with

$$\widehat{\zeta}_E := \mathbb{1}_{[0,r]}\partial_t^{-m}U, \quad \widehat{\zeta}_H := \mathbb{1}_{[0,r]}\partial_t^{-m}V, \quad \widehat{v} := -\mu_0\mathbb{1}_{[0,r]}\partial_t^{-m}\gamma_T V, \quad \widehat{w} := \mathbb{1}_{[0,r]}\partial_t^{-m}\gamma_T U$$

for arbitrary $0 \le r \le T$. We obtain for

$$\widetilde{U} := \partial_t^{-m}U, \qquad \widetilde{V} := \partial_t^{-m}V,$$

by the use of integration by parts 2.13 (with $\Omega_r = (0,r)\times\Omega$ and $\Gamma_r = [0,r]\times\Gamma$)

$$\begin{aligned}
&[\varepsilon\partial_t\widetilde{U},\widetilde{U}]_{\Omega_r} + [\mu\partial_t\widetilde{V},\widetilde{V}]_{\Omega_r} + \left\langle \begin{pmatrix}\mu_0\gamma_T\widetilde{V}\\ -\gamma_T\widetilde{U}\end{pmatrix}, B_m * \begin{pmatrix}\mu_0\gamma_T V\\ -\gamma_T U\end{pmatrix}\right\rangle_{\Gamma_r}\\
&\quad= [\nabla\times V,\widetilde{U}]_{\Omega_r} - [\sigma\widetilde{U},\widetilde{U}]_{\Omega_r} - [\nabla\times\widetilde{U},\widetilde{V}]_{\Omega_r} + \frac{1}{2\mu_0}\left\langle \begin{pmatrix}\mu_0\gamma_T\widetilde{V}\\ -\gamma_T\widetilde{U}\end{pmatrix}, \begin{pmatrix}\gamma_T\widetilde{U}\\ \mu_0\gamma_T\widetilde{V}\end{pmatrix}\right\rangle_{\Gamma_r}\\
&\quad= -[\sigma\widetilde{U},\widetilde{U}]_{\Omega_r} + [\nabla\times\widetilde{V},\widetilde{U}]_{\Omega_r} - [\nabla\times\widetilde{U},\widetilde{V}]_{\Omega_r} + \langle\gamma_T\widetilde{V},\gamma_T\widetilde{U}\rangle_{\Gamma_r}\\
&\quad= -[\sigma\widetilde{U},\widetilde{U}]_{\Omega_r}.
\end{aligned}$$

By the positivity of the Calderon operator, (compare Lemma 2.16, i.e. using (2.11), Lemma 2.12 and considering the limit $\sigma,\sigma_0\to 0$ in Lemma B.83) we have

$$\left\langle \begin{pmatrix}\mu_0\gamma_T\widetilde{V}\\ -\gamma_T\widetilde{U}\end{pmatrix}, B_m * \begin{pmatrix}\mu_0\gamma_T V\\ -\gamma_T U\end{pmatrix}\right\rangle_{\Gamma_r} = \left\langle \begin{pmatrix}\mu_0\gamma_T\widetilde{V}\\ -\gamma_T\widetilde{U}\end{pmatrix}, B(\partial_t)\begin{pmatrix}\mu_0\gamma_T\widetilde{V}\\ -\gamma_T\widetilde{U}\end{pmatrix}\right\rangle_{\Gamma_r} \geq 0$$

and therefore

$$\begin{aligned}
0 \leq \frac{\varepsilon}{2}\|\widetilde{U}(r)\|_\Omega^2 + \frac{\mu}{2}\|\widetilde{V}(r)\|_\Omega^2 &\leq [\varepsilon\partial_t\widetilde{U},\widetilde{U}]_{\Omega_r} + [\mu\partial_t\widetilde{V},\widetilde{V}]_{\Omega_r}\\
&\quad + \left\langle \begin{pmatrix}\mu_0\gamma_T\widetilde{V}\\ -\gamma_T(\widetilde{U})\end{pmatrix}, B_m * \begin{pmatrix}\mu_0\gamma_T V\\ -\gamma_T U\end{pmatrix}\right\rangle_{\Gamma_r} + [\sigma\widetilde{U},\widetilde{U}]_{\Omega_r} = 0.
\end{aligned}$$

We conclude $\widetilde{U} = U = \widetilde{V} = V = 0$, which gives the desired result. □

Remark 3.9. *Under the stated regularity assumptions, uniqueness in the magnetization m is unclear and not expected in the literature. Assuming more regularity for the magnetization (see, e.g. [4, Lemma 4.2]), one can show uniqueness of (m, E, H) (see Lemma 5.16 for similar arguments for the MLLG system).*

The uniqueness with respect to $\widetilde{\varphi},\widetilde{\psi}$ is not true, as we ask that the projection on suitable exterior data applied to $\widetilde{\varphi},\widetilde{\psi}$ gives $\gamma_T\partial_t^{-1}H, \gamma_T\partial_t^{-1}E$. The projection on suitable exterior data is not injective, so the variables $\widetilde{\varphi},\widetilde{\psi}$ are only unique up to a difference of elements in the kernel of the projection, so by suitable interior data (cf. Lemma 2.14 and [46, Theorem 8]).
However, with any solution $(m, E, H, \widetilde{\varphi}, \widetilde{\psi})$ in the sense of Definition 3.3, we have that also the functions $(m, E, H, \mu_0\gamma_T\partial_t^{-1}H, -\gamma_T\partial_t^{-1}E)$ form a solution. Hence, in this sense, the last four components are unique.

3.2. Approximation

In this section, we illustrate the approximation scheme for the MLLG system. After giving some basic definitions concerning space and time discretization in Section 3.2.1, we present the tangent plane scheme used for the LLG part, the implicit Euler discretization for the interior Maxwell part and the Convolution Quadrature for the boundary integral equation. We conclude with the coupled algorithm in Section 3.2.7.

3.2.1. Preliminaries

In this section we present the basic definitions and spaces for the approximation.

For the time discretization we use a constant time step size $\tau := T/N$ for $N \in \mathbb{N}$ to approximate the solution on the time points $0 = t_0, \ldots, t_n = T$, $t_j = \tau j$. We assume that the step size is small enough, i.e. $\tau \leq \tau_0$ for some $\tau_0 > 0$.

For the spatial discretization (cf. [25]), let $\mathcal{T}_h$ be a regular triangulation of the polyhedral bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ into compact tetrahedra. By $\mathcal{S}^1(\mathcal{T}_h)$ we denote the standard $\mathcal{P}^1$-FEM space of globally continuous and piecewise affine functions from Ω to $\mathbb{R}^3$

$$\mathcal{S}^1(\mathcal{T}_h) := \{\phi_h \in C(\overline{\Omega}, \mathbb{R}^3) \,|\, \phi_{h|K} \in \mathcal{P}^1(K) \text{ for all } K \in \mathcal{T}_h\}.$$

By $\mathcal{N}_h$ we denote the set of nodes of the triangulation $\mathcal{T}_h$. As we have $|m(t,x)| = 1$ almost everywhere, we define the discrete space for the magnetization by

$$\mathcal{M}_h := \{\phi_h \in \mathcal{S}^1(\mathcal{T}_h) \,|\, |\phi_h(\gamma)| = 1 \text{ for all } \gamma \in \mathcal{N}_h\}.$$

By $|m(t,x)| = 1$ we get $\partial_t m(t,x) \cdot m(t,x) = 0$ and therefore we define the ansatz space for the time derivative of the magnetization

$$\mathcal{K}_{m_h} := \{\phi_h \in \mathcal{S}^1(\mathcal{T}_h) \,|\, m_h(\gamma) \cdot \phi_h(\gamma) = 0 \text{ for all } \gamma \in \mathcal{N}_h\}$$

for any $m_h \in \mathcal{M}_h$. We define the nodal interpolation operator for $u \in C(\overline{\Omega})$ (or $u \in H^{3/2+\epsilon}(\Omega)$ for $\epsilon > 0$)

$$\mathrm{I}_h^{\mathcal{S}} u := \sum_{\gamma \in \mathcal{N}_h} u(\gamma)\phi_\gamma,$$

where ϕ_γ for $\gamma \in \mathcal{N}_h$ is the elementwise linear hat function with $\phi_\gamma(\gamma') = \delta_{\gamma,\gamma'}$ for all $\gamma' \in \mathcal{N}_h$.
To discretize the Maxwell system in the interior, we use Nédélec's $H(\text{curl}, \Omega)$-conforming ansatz space (cf. [121]),

$$\mathcal{X}_h := \{\phi_h \in H(\text{curl}, \Omega) \,|\, \phi_{h|K} \in \mathcal{P}^1_{skw}(K) \text{ for all } K \in \mathcal{T}_h\},$$

where

$$\mathcal{P}^1_{skw}(K) := \{v : K \to \mathbb{R}^3, v(x) = a + Bx \,|\, a \in \mathbb{R}^3, B \in \mathbb{R}^{3\times 3}, B^T = -B\}.$$

We define the interpolation $\mathrm{I}_h^{\mathcal{X}} : C(\overline{\Omega}) \to \mathcal{X}_h$ by

$$\int_e u(s) \cdot \tau(s) \,\mathrm{d}s = \int_e (\mathrm{I}_h^{\mathcal{X}} u)(s) \cdot \tau(s) \,\mathrm{d}s$$

for all edges e of the triangulation and corresponding tangential vector τ. Here $e(s)$ is a normalized path on e and $\tau(s) := e'(s)$, $|\tau(s)| = 1$, the normalized tangential vector on e. The interpolation $\mathrm{I}_h^{\mathcal{X}}$ is well defined, there exists a basis ϕ^e of $\mathcal{X}_h$ satisfying

$$\int_e \phi^{e'} \cdot \tau(s) \,\mathrm{d}s = \delta_{ee'}$$

for all edges e, e' of the triangulation.

For the functions on the boundary, we use the approximation space $\gamma_T(\mathcal{X}_h)$, which results in the well known Raviart–Thomas space (cf. e.g. [68, Chapter 3]), together with the interpolation $\gamma_T \circ \mathrm{I}_h^{\mathcal{X}}$.

Lemma 3.10. *The following approximation properties hold true for sufficiently smooth functions for a constant $C > 0$*

$$\|\phi - \mathrm{I}_h^{\mathcal{S}}\phi\|_{L^2(\Omega)} + h\|\nabla(\phi - \mathrm{I}_h^{\mathcal{S}}\phi)\|_{L^2(\Omega)} \le Ch^2\|\phi\|_{H^2(\Omega)},$$
$$\|\phi - \mathrm{I}_h^{\mathcal{X}}\phi\|_{L^2(\Omega)} + \|\nabla \times (\phi - \mathrm{I}_h^{\mathcal{X}}\phi)\|_{L^2(\Omega)} \le Ch(\|\phi\|_{H^1(\Omega)} + \|\nabla \times \phi\|_{H^1(\Omega)}),$$
$$\|\gamma_T(\phi - \mathrm{I}_h^{\mathcal{X}}\phi)\|_{\mathcal{H}_\Gamma} \le Ch(\|\phi\|_{H^1(\Omega)} + \|\nabla \times \phi\|_{H^1(\Omega)}).$$

For a proof see, e.g. [39, 121] and use that $\gamma_T : H(\text{curl}, \Omega) \to \mathcal{H}_\Gamma$ is bounded.

For a sequence of space-dependent approximations $(G_h^j)_{j=0}^N$, $G_h^j : \Omega \to \mathbb{R}$ we define in the following the space and time dependent functions $G_{\tau,h}^-, G_{\tau,h}, G_{\tau,h}^+ : [0,T] \times \Omega \to \mathbb{R}$. For $t \in [t_j, t_j + 1)$ and $x \in \Omega$ we define the interval-wise constant functions

$$G_{\tau,h}^-(t,x) := (G_h^j)_{\tau,h}^-(t,x) := G_h^j(x), \qquad G_{\tau,h}^+(t,x) := (G_h^j)_{\tau,h}^+(t,x) := G_h^{j+1}(x) \quad (3.10)$$

and the interval-wise linear function

$$G_{\tau,h}(t,x) := (G_h^j)_{\tau,h}(t,x) := \frac{t_{j+1} - t}{\tau} G_h^j(x) + \frac{t - t_j}{\tau} G_h^{j+1}(x).$$

We use similar notations also for functions that are not necessarily approximations in space or time, i.e. for $J : \Omega_T \to \mathbb{R}$ the interval wise constant and interval wise linear functions $J_\tau^-, J_\tau, J_\tau^+ : [0,T] \times \Omega \to \mathbb{R}$ are defined with respect to the sequence $(J(t_j))_{j=0}^N$. Furthermore these notations are used for sequences $(g^j)_{j=0}^N \subset \mathbb{R}$ and then $g_\tau^-, g_\tau, g_\tau^+ : [0,T] \to \mathbb{R}$ form the corresponding time depended functions.

3.2.2. Convolution Quadrature

Following [99, Section 2.3] we give a short recap of Convolution Quadrature and introduce some notation. For more details see [113, 114, 115, 116] and [27].

Convolution Quadrature (CQ) discretizes the convolution $B(\partial_t)w(t)$ by the discrete convolution

$$(B(\partial_t^\tau)w)(n\tau) = \sum_{j=0}^{n} B_{n-j}^\tau w(j\tau), \quad (3.11)$$

where the weights B_n are defined as the coefficients of

$$B\Big(\frac{\delta(\zeta)}{\tau}\Big) = \sum_{n=0}^{\infty} B_n^\tau \zeta^n. \quad (3.12)$$

In the present chapter we choose

$$\delta(\zeta) = 1 - \zeta,$$

which corresponds to the first-order backward difference formula (i.e. the implicit Euler method).

From [115], it is known that the method is of first order, i.e.

$$\|(B(\partial_t)w)(t) - (B(\partial_t^\tau)w)(t)\| = O(\tau), \text{ uniformly in } t = n\tau \le T,$$

for functions w that are sufficiently smooth including their extension by 0 to negative values of t. An important property of this discretization is that it preserves the coercivity of the continuous-time convolution in the time discretization, see Lemma 3.18 and Lemma 3.21.

Remark 3.11. *We use the first order Convolution Quadrature $\delta(\zeta) = 1 - \zeta$. In this case $\partial_t^\tau \varphi$ and $(\partial_t^\tau)^{-1}\phi$ can be expressed in a simple and clear way. By the Neumann series formula we have for $|\zeta| < 1$*

$$\frac{1}{1-\zeta} = \sum_{n=0}^{\infty} \zeta^n$$

and for the first order Convolution Quadrature scheme $\delta(\zeta) = 1 - \zeta$, we obtain for a sequence $(\varphi^j)_j$

$$\left((\partial_t^\tau)^{-1}\varphi\right)(t_n) = \sum_{j=0}^{n} \tau\varphi^j.$$

Similarly we see that

$$(\partial_t^\tau \varphi)(t_n) = \frac{\varphi^n - \varphi^{n-1}}{\tau}$$

which allows us to use a consistent notation with regard to the implicit Euler discretisation (3.15)*.*

3.2.3. The discrete system

We write the MLLG system in the following form which serves as a starting point for the discretization. For the LLG equation, we use the alternative form (2.4)

$$\alpha\partial_t m + m \times \partial_t m = C_e\Delta m + H - (m \cdot (C_e\Delta m + H))m.$$

By $\partial_t m \cdot m = 0$, the terms on the left hand side and the right hand side are orthogonal on m, so it suffices to multiply this equation with a test function ρ that is orthogonal on m. Therefore the nonlinear term $(m \cdot (C_e\Delta m + H))m$ on the right hand side vanishes. We obtain by integration by parts and by using the boundary condition $\partial_n m = 0$ on Γ for all testfunctions ρ with $\rho \cdot m = 0$ that

$$[\alpha\partial_t m, \rho]_\Omega + [m \times \partial_t m, \rho]_\Omega = -[C_e\nabla m, \nabla\rho]_\Omega + [H, \rho]_\Omega. \tag{3.13}$$

For the Maxwell part, we symmetrize the differential operators by integration by parts in space. We introduce the variables φ, ψ for the traces $\varphi := \mu_0\gamma_T H$ and $\psi := -\gamma_T E$ and obtain

$$[\nabla \times H, \zeta_E]_\Omega = \frac{1}{2}[\nabla \times H, \zeta_E]_\Omega + \frac{1}{2}[H, \nabla \times \zeta_E]_\Omega - \frac{1}{2\mu_0}\langle\varphi, \gamma_T\zeta_E\rangle_\Gamma$$

and

$$[\nabla \times E, \zeta_H]_\Omega = \frac{1}{2}[\nabla \times E, \zeta_H]_\Omega + \frac{1}{2}[E, \nabla \times \zeta_H]_\Omega + \frac{1}{2}\langle\psi, \gamma_T\zeta_H\rangle_\Gamma.$$

Altogether, we use the system

$$\begin{aligned}
[\alpha\partial_t m, \rho]_\Omega + [m \times \partial_t m, \rho]_\Omega &= -[C_e\nabla m, \nabla\rho]_\Omega + [H, \rho]_\Omega, \\
[\varepsilon\partial_t E, \zeta_E]_\Omega &= \frac{1}{2}[\nabla \times H, \zeta_E]_\Omega + \frac{1}{2}[H, \nabla \times \zeta_E]_\Omega \\
&\quad - \frac{1}{2\mu_0}\langle\varphi, \gamma_T\zeta_E\rangle_\Gamma - [\sigma E + J, \zeta_E]_\Omega, \\
[\mu\partial_t H, \zeta_H]_\Omega &= -\frac{1}{2}[\nabla \times E, \zeta_H]_\Omega - \frac{1}{2}[E, \nabla \times \zeta_H]_\Omega \\
&\quad - \frac{1}{2}\langle\psi, \gamma_T\zeta_H\rangle_\Gamma - [\mu\partial_t m, \zeta_H]_\Omega, \\
\left\langle \begin{pmatrix} v_\varphi \\ v_\psi \end{pmatrix}, B(\partial_t)\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_\Gamma &= \frac{1}{2}\left\langle \begin{pmatrix} v_\varphi \\ v_\psi \end{pmatrix}, \begin{pmatrix} \mu_0^{-1}\gamma_T E \\ \gamma_T H \end{pmatrix} \right\rangle_\Gamma.
\end{aligned} \tag{3.14}$$

3.2.4. Tangent plane scheme for the LLG discretization

Using the discrete tangent space $\mathcal{K}_{m_h}$, we formulate a discrete version of (3.13): Given an approximation to the magnetization $m_h^j \approx m(t_j)$, we seek an approximation to the time derivative of the magnetization $w_h^j \approx \partial_t m(t_j)$. We define the function $w_h^j \in \mathcal{K}_{m_h^j}$ such that for all $\rho_h \in \mathcal{K}_{m_h^j}$

$$\alpha[w_h^j, \rho_h]_\Omega + \left[m_h^j \times w_h^j, \rho_h\right]_\Omega = -C_e\left[\nabla(m_h^j + \theta\tau w_h^j), \nabla\rho_h\right]_\Omega + \left[H_h^j, \rho_h\right]_\Omega.$$

The parameter $\theta \in [0, 1]$ determines how "implicit" the term $[\nabla m, \nabla\rho]_\Omega$ is treated. For $\theta \geq 1/2$ we obtain bounded approximations, see Lemma 3.20. For $\theta < 1/2$ one needs to add an additional CFL condition, compare [63]. The approximation m_h^{j+1} on the next time step is then computed via

$$m_h^{j+1} \approx m(t_{j+1}) \approx m(t_j) + \tau\partial_t m(t_j) \approx m_h^j + \tau w_h^j.$$

To comply with $|m| = 1$ at least in a node wise sense, we add a normalization and define m_h^{j+1} by

$$m_h^{j+1}(z) := \frac{m_h^j(z) + \tau w_h^j(z)}{|m_h^j(z) + \tau w_h^j(z)|} \quad \text{for all nodes } z \in \mathcal{N}_h.$$

The normalization step is unconditionally well defined, since by the node wise orthogonality it holds $|m_h^j(z) + \tau w_h^j(z)|^2 = |m_h^j(z)|^2 + \tau^2 |w_h^j(z)|^2 \geq 1$ for all nodes $z \in \mathcal{N}_h$. The normalization step is not necessary for the convergence and could also be skipped, see Remark 3.15.

This scheme is called *tangent plane scheme* due to the orthogonality condition in the test space and was first proposed by Alouges, cf. [14].

3.2.5. Implicit Euler method for the interior Maxwell discretization

For the interior Maxwell part, we replace the continuous equations in (3.14) by finite counterparts using the approximation spaces from Section 3.2.3. For the time discretization, we use the finite difference

$$\partial_t G(t_{j+1}) \approx \partial_t^\tau G^{j+1} := \frac{G^{j+1} - G^j}{\tau} \tag{3.15}$$

for $G \in \{E, H\}$. This is endowed with an implicit treatment of the terms on the right hand side, i.e. we seek $E_h^{j+1}, H_h^{j+1} \in \mathcal{X}_h$ such that for all $\zeta_h^E, \zeta_h^H \in \mathcal{X}_h$

$$\begin{aligned}
[\varepsilon \partial_t^\tau E_h^{j+1}, \zeta_h^E]_\Omega &= \frac{1}{2}[\nabla \times H_h^{j+1}, \zeta_h^E]_\Omega + \frac{1}{2}[H_h^{j+1}, \nabla \times \zeta_h^E]_\Omega \\
&\quad - \frac{1}{2\mu_0}\langle \varphi_h^{j+1}, \gamma_T \zeta_h^E \rangle_\Gamma - [\sigma E_h^{j+1} + J^{j+1}, \zeta_h^E]_\Omega, \\
[\mu \partial_t^\tau H_h^{j+1}, \zeta_h^H]_\Omega &= -\frac{1}{2}[\nabla \times E_h^{j+1}, \zeta_h^H]_\Omega - \frac{1}{2}[E_h^{j+1}, \nabla \times \zeta_h^H]_\Omega \\
&\quad - \frac{1}{2}\langle \psi_h^{j+1}, \gamma_T \zeta_h^H \rangle_\Gamma - [\mu w_h^j, \zeta_h^H]_\Omega.
\end{aligned}$$

This scheme results in an implicit coupling of the interior Maxwell equations to the boundary integral equation via the terms $\langle \varphi_h^{j+1}, \gamma_T \zeta_h^E \rangle_\Gamma$ and $\langle \psi_h^{j+1}, \gamma_T \zeta_h^H \rangle_\Gamma$ (see also boundary discretization below). The coupling with the LLG equation via the term $[\mu w_h^j, \zeta_h^H]_\Omega$ is of explicit manner, and therefore the computations can be performed independently of each other in every time step.

3.2.6. Convolution Quadrature for the boundary discretization

For the equation on the boundary, we use a Galerkin ansatz as above and replace the continuous equations in (3.14) by finite counterparts using the approximation spaces from Section 3.2.3. For the Calderon term $B(\partial_t)$, we use Convolution Quadrature (see Section 3.2.2)

$$\left\langle \begin{pmatrix} v^\varphi \\ v^\psi \end{pmatrix}, \left(B(\partial_t) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right)(t_{j+1}) \right\rangle_\Gamma \approx \left\langle \begin{pmatrix} v_h^\varphi \\ v_h^\psi \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} \right)(t_{j+1}) \right\rangle_\Gamma$$

together with an implicit treatment of the right hand side. Altogether we obtain: Compute $\varphi_h^{j+1}, \psi_h^{j+1} \in \gamma_T(\mathcal{X}_h)$ such that for all $v_h^\varphi, v_h^\psi \in \gamma_T(\mathcal{X}_h)$

$$\left\langle \begin{pmatrix} v_h^\varphi \\ v_h^\psi \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} \right)(t_{j+1}) \right\rangle_\Gamma = \frac{1}{2}\left(\langle v_h^\varphi, \mu_0^{-1} \gamma_T E_h^{j+1} \rangle_\Gamma + \langle v_h^\psi, \gamma_T H_h^{j+1} \rangle_\Gamma \right).$$

This scheme concludes the implicit coupling of the boundary integral equation to the interior Maxwell equations via the terms $\langle v_h^\varphi, \mu_0^{-1} \gamma_T E_h^{j+1} \rangle_\Gamma$ and $\langle v_h^\psi, \gamma_T H_h^{j+1} \rangle_\Gamma$.

3.2.7. Algorithm

We approximate the solution of the MLLG system by the following algorithm:

Algorithm 3.12. Input: Discretized initial data m_h^0, H_h^0, E_h^0, $\varphi_h^0 = 0$, $\psi_h^0 = 0$ and the parameter $\theta \in [0,1]$.
For $j = 0, 1, 2, \ldots, N-1$ we compute

- For given m_h^j, H_h^j we compute the unique solution $w_h^j \in \mathcal{K}_{m_h^j}$ such that we have for all $\rho_h \in \mathcal{K}_{m_h^j}$

$$\alpha[w_h^j, \rho_h]_\Omega + \left[m_h^j \times w_h^j, \rho_h\right]_\Omega = -C_e \left[\nabla(m_h^j + \theta\tau w_h^j), \nabla\rho_h\right]_\Omega + \left[H_h^j, \rho_h\right]_\Omega. \quad (3.16)$$

- We compute $E_h^{j+1}, H_h^{j+1} \in \mathcal{X}_h$ and $\varphi_h^{j+1}, \psi_h^{j+1} \in \gamma_T(\mathcal{X}_h)$ such that we have for all $\zeta_h^E, \zeta_h^H \in \mathcal{X}_h$ and $v_h^\varphi, v_h^\psi \in \gamma_T(\mathcal{X}_h)$

$$\begin{aligned}[\varepsilon\partial_t^\tau E_h^{j+1}, \zeta_h^E]_\Omega &= \frac{1}{2}[\nabla \times H_h^{j+1}, \zeta_h^E]_\Omega + \frac{1}{2}[H_h^{j+1}, \nabla \times \zeta_h^E]_\Omega \\ &\quad - \frac{1}{2\mu_0}\langle\varphi_h^{j+1}, \gamma_T\zeta_h^E\rangle_\Gamma - [\sigma E_h^{j+1} + J^{j+1}, \zeta_h^E]_\Omega,\end{aligned} \quad (3.17)$$

$$\begin{aligned}[\mu\partial_t^\tau H_h^{j+1}, \zeta_h^H]_\Omega &= -\frac{1}{2}[\nabla \times E_h^{j+1}, \zeta_h^H]_\Omega - \frac{1}{2}[E_h^{j+1}, \nabla \times \zeta_h^H]_\Omega \\ &\quad - \frac{1}{2}\langle\psi_h^{j+1}, \gamma_T\zeta_h^H\rangle_\Gamma - [\mu w_h^j, \zeta_h^H]_\Omega,\end{aligned} \quad (3.18)$$

$$\left\langle \begin{pmatrix} v_h^\varphi \\ v_h^\psi \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h \\ \psi_h\end{pmatrix}\right)(t_{j+1}) \right\rangle_\Gamma = \frac{1}{2}\left(\langle v_h^\varphi, \mu_0^{-1}\gamma_T E_h^{j+1}\rangle_\Gamma + \langle v_h^\psi, \gamma_T H_h^{j+1}\rangle_\Gamma\right). \quad (3.19)$$

- Define m_h^{j+1} by

$$m_h^{j+1}(z) := \frac{m_h^j(z) + \tau w_h^j(z)}{|m_h^j(z) + \tau w_h^j(z)|} \quad \text{for all nodes } z \in \mathcal{N}_h. \quad (3.20)$$

Output: Sequence of approximations $m_h^j, E_h^j, H_h^j, \varphi_h^j, \psi_h^j$ for $j = 0, 1, 2, \ldots, N$.

Lemma 3.13. *Algorithm 3.12 is well defined in the sense, that for every $j \geq 0$, there exist unique approximations m_h^{j+1}, E_h^{j+1}, H_h^{j+1}, φ_h^{j+1}, ψ_h^{j+1} that satisfy* (3.16)–(3.19).

Proof. The proof that the tangent plane scheme is well defined can be conducted as in [14] or [25]: We define the bilinear form $a^j(\cdot,\cdot)$ on $\mathcal{K}_{m_h^j}$ by

$$a(\Phi, \phi) := \alpha[\Phi, \phi]_\Omega + \left[m_h^j \times \Phi, \phi\right]_\Omega + C_e\theta\tau\,[\nabla\Phi, \nabla\phi]_\Omega \quad (3.21)$$

and the linear functional $L^j(\cdot)$ on $\mathcal{K}_{m_h^j}$ by

$$L^j(\phi) := -C_e\left[\nabla m_h^j, \nabla\phi\right]_\Omega + \left[H_h^j, \phi\right]_\Omega. \quad (3.22)$$

The equation (3.16) is equivalent to

$$a(w_h^j, \phi_h) = L^j(\phi_h)$$

for all $\phi_h \in \mathcal{K}_{m_h^j}$. Furthermore it is

$$a(\phi, \phi) = \alpha[\phi, \phi]_\Omega + \left[m_h^j \times \phi, \phi\right]_\Omega + C_e\theta\tau\,[\nabla\phi, \nabla\phi]_\Omega$$
$$= \alpha\|\phi\|_\Omega^2 + C_e\theta\tau\|\nabla\phi\|_\Omega^2$$

positive definite and therefore a unique solution w_h^j to (3.16) exists for all $j \geq 0$.

For the Maxwell part, we define the bilinear form $a(\cdot,\cdot)$ on $\mathcal{X}_h \times \mathcal{X}_h \times \gamma_T(\mathcal{X}_h) \times \gamma_T(\mathcal{X}_h)$ by

$$\begin{aligned}
&a((\Phi, \Psi, \Theta, \Upsilon), (\phi, \psi, \theta, \upsilon)) \\
&\quad := \frac{1}{\tau}[\varepsilon\Phi, \phi]_\Omega + \frac{1}{\tau}[\mu\Psi, \psi]_\Omega + \left\langle \begin{pmatrix}\theta\\ \upsilon\end{pmatrix}, B_0^\tau \begin{pmatrix}\Theta\\ \Upsilon\end{pmatrix}\right\rangle_\Gamma + [\sigma\Phi, \phi]_\Omega \\
&\quad\quad - \frac{1}{2}[\Psi, \nabla\times\phi]_\Omega - \frac{1}{2}[\nabla\times\Psi, \phi]_\Omega + \frac{1}{2}[\Phi, \nabla\times\psi]_\Omega + \frac{1}{2}[\nabla\times\Phi, \psi]_\Omega \\
&\quad\quad + \frac{1}{2}\langle\Upsilon, \gamma_T\psi\rangle_\Gamma + \frac{1}{2\mu_0}\langle\Theta, \gamma_T\phi\rangle_\Gamma - \frac{1}{2}\langle\theta, \mu_0^{-1}\gamma_T\Phi\rangle_\Gamma - \frac{1}{2}\langle\upsilon, \gamma_T\Psi\rangle_\Gamma
\end{aligned}$$

and the linear functional $L^j(\cdot)$ on $\mathcal{X}_h \times \mathcal{X}_h \times \gamma_T(\mathcal{X}_h) \times \gamma_T(\mathcal{X}_h)$ by

$$\begin{aligned}
L^j(\phi, \psi, \theta, \upsilon) := \frac{1}{\tau}[\varepsilon E_h^j, \phi]_\Omega + \frac{1}{\tau}[\mu H_h^j, \psi]_\Omega - [J^{j+1}, \phi]_\Omega - \mu[w_h^i, \phi]_\Omega \\
- \left\langle \begin{pmatrix}\theta\\ \upsilon\end{pmatrix}, \sum_{l=0}^{j} B_{j+1-l}^\tau \begin{pmatrix}\varphi_h^l\\ \psi_h^l\end{pmatrix}\right\rangle_\Gamma .
\end{aligned}$$

The equations (3.17)–(3.19) are equivalent to

$$a((E_h^{j+1}, H_h^{j+1}, \varphi_h^{j+1}, \phi_h^{j+1}), (\phi, \psi, \theta, \upsilon)) = L^j(\phi, \psi, \theta, \upsilon)$$

for all $(\phi, \psi, \theta, \upsilon) \in \mathcal{X}_h \times \mathcal{X}_h \times \gamma_T(\mathcal{X}_h) \times \gamma_T(\mathcal{X}_h)$. Next, we aim to show that the bilinear form $a(\cdot,\cdot)$ is positive definite on $\mathcal{X}_h \times \mathcal{X}_h \times \gamma_T(\mathcal{X}_h) \times \gamma_T(\mathcal{X}_h)$. We have $B_0^\tau = B(\tau^{-1})$ and by Lemma 2.12 for all $\zeta \in \mathcal{H}_\Gamma \times \mathcal{H}_\Gamma$ and $s > 0$

$$\langle\zeta, B(s)\zeta\rangle_\Gamma \geq C(s, \mu_0, \varepsilon_0)\|\zeta\|_{\mathcal{H}_\Gamma}^2.$$

Therefore

$$\begin{aligned}
&a((\Phi, \Psi, \Theta, \Upsilon), (\Phi, \Psi, \Theta, \Upsilon)) \\
&\quad = \frac{1}{\tau}[\varepsilon\Phi, \Phi]_\Omega + \frac{1}{\tau}[\mu\Psi, \Psi]_\Omega + \left\langle \begin{pmatrix}\Theta\\ \Upsilon\end{pmatrix}, B_0^\tau \begin{pmatrix}\Theta\\ \Upsilon\end{pmatrix}\right\rangle_\Gamma + [\sigma\Phi, \Phi]_\Omega \\
&\quad \geq C(\tau, \mu, \varepsilon)(\|\Phi\|_\Omega^2 + \|\Psi\|_\Omega^2 + \|\Theta\|_{\mathcal{H}_\Gamma}^2 + \|\Upsilon\|_{\mathcal{H}_\Gamma}^2)
\end{aligned}$$

is positive definite which yields the desired result. □

3.3. Convergence

In this section, we consider the convergence of the previously introduced algorithm. The proof is divided into three parts: In Section 3.3.1 we show the boundedness of the approximations in the respective Hilbert spaces. Therefore, we are able to extract weakly weakly convergent subsequences in Section 3.3.2 and in Section 3.3.3, we finally identify the limit functions as weak solutions of the MLLG system.

We collect some assumptions and general formulas, which we will need in the following.

Assumption 3.14.

- *The triangulations $\mathcal{T}_h$ are uniformly shape regular and satisfy the angle condition*

$$\int_\Omega \nabla\zeta(x) \cdot \nabla\xi(x) \, \mathrm{d}x \le 0$$

for all linear basis functions $\zeta, \xi \in \mathcal{S}^1(\mathcal{T}_h)$ with $\xi \neq \zeta$ (cf. [25, (5.1)-(5.7)]).

- $J^{\pm}_{\tau,h} \rightharpoonup J$ *in* $L^2(\Omega_T)$.
- $E^0_h \rightharpoonup E^0$ *and* $H^0_h \rightharpoonup H^0$ *in* $L^2(\Omega)$.
- $m^0_h \rightharpoonup m^0$ *in* $H^1(\Omega)$.

Remark 3.15. *The angle condition ensures, despite the normalization step* (3.20) *in Algorithm 3.12,*

$$\|\nabla m_h^{j+1}\|_\Omega \le \|\nabla(m_h^j + \tau w_h^j)\|_\Omega,$$

cf. [25, Remark 5.1]. The angle condition is fulfilled, if all dihedral angles of the tetrahedral mesh are smaller than or equal $90°$. *Alternatively, the Algorithm 3.12 could be formulated without the normalization step* (3.20) *and therefore the angle condition can be removed. In this case only a (globally) quasi-uniform family of triangulations is necessary for the convergence, see Remark 7, Lemma 8 and Theorem 9 in [2].*

Remark 3.16. *The propositions in the following sections hold for symmetric, coercive and bounded material tensors*

$$\varepsilon, \mu : \Omega \to \mathbb{R}^{3\times 3}$$

and bounded, non-negative

$$\sigma : \Omega \to \mathbb{R}^{3\times 3},$$

i.e. we have

- *Symmetry: For arbitrary $\zeta, \xi \in L^2(\Omega)$ it holds*

$$[\zeta, \mu\xi]_\Omega = [\mu\zeta, \xi]_\Omega$$

and

$$[\zeta, \varepsilon\xi]_\Omega = [\varepsilon\zeta, \xi]_\Omega.$$

- *Coercivity: There exist constants $\mu^-, \varepsilon^- > 0$ such that for arbitrary $\zeta \in L^2(\Omega)$*

$$\mu^- \|\zeta\|_\Omega^2 \le [\zeta, \mu\zeta]_\Omega$$

and

$$\varepsilon^- \|\zeta\|_\Omega^2 \le [\zeta, \varepsilon\zeta]_\Omega$$

and

$$0 \le [\zeta, \sigma\zeta]_\Omega.$$

- *Boundedness: There exist constants $\mu^+, \varepsilon^+, \sigma^+ > 0$, such that for arbitrary $\xi, \zeta \in L^2(\Omega)$*

$$[\xi, \mu\zeta]_\Omega \le \mu^+ \|\zeta\|_\Omega \|\xi\|_\Omega$$

and

$$[\xi, \varepsilon\zeta]_\Omega \le \varepsilon^+ \|\zeta\|_\Omega \|\xi\|_\Omega$$

and

$$[\xi, \sigma\zeta]_\Omega \le \sigma^+ \|\zeta\|_\Omega \|\xi\|_\Omega.$$

For the ease of presentation, some of the results in the following sections are formulated for scalar and constant material parameters $\varepsilon, \mu \in \mathbb{R}_{>0}$ and $\sigma \in \mathbb{R}_{\ge 0}$.

First, we recall the positivity of the time-discretized Calderon operator $B(\partial_t^\tau)$, which we will use at a later point.

Lemma 3.17 ([27, Lemma 2.3])**.** *It holds for $0 < \rho < 1$, $0 < \tau \le 1$ and sequences $(\varphi^i)_{i=0}^\infty$ and $(\psi^i)_{i=0}^\infty$ in $\mathcal{H}_\Gamma$ (with only finite many nonzero entries)*

$$\sum_{n=0}^\infty \rho^{2n} \Re \left\langle \begin{pmatrix} \varphi^n \\ \psi^n \end{pmatrix}, B\left(\partial_t^\tau\right) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} (t_n) \right\rangle_\Gamma$$
$$\ge C \min\left(\frac{1-\rho}{\tau}, \left(\frac{1-\rho}{\tau}\right)^3 \right) \sum_{n=0}^\infty \rho^{2n} \left(\|(\partial_t^\tau)^{-1}\varphi(t_n)\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^{-1}\psi(t_n)\|_{\mathcal{H}_\Gamma}^2 \right).$$

The constant $C > 0$ depends on ε_0, μ_0 and $\beta > 0$ from Lemma 2.12.

Proof. For $0 < \rho < 1$ and $|\xi| \le \rho$ we have

$$\left| \frac{\zeta(\xi)}{\tau} \right| \ge \Re\left(\frac{\zeta(\xi)}{\tau} \right) = \Re\left(\frac{1-\xi}{\tau} \right) \ge \frac{1-\rho}{\tau} > 0.$$

Therefore we have for $\varphi, \psi \in \mathcal{H}_\Gamma$ by Lemma 2.12

$$\Re \Big\langle \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, B\left(\frac{\zeta(\xi)}{\tau}\right) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \Big\rangle_\Gamma$$
$$\ge C \min\left(\frac{1-\rho}{\tau}, \left(\frac{1-\rho}{\tau}\right)^3 \right) \left(\|(\tfrac{\zeta(\xi)}{\tau})^{-1}\varphi\|_{\mathcal{H}_\Gamma}^2 + \|(\tfrac{\zeta(\xi)}{\tau})^{-1}\psi\|_{\mathcal{H}_\Gamma}^2 \right)$$

for $|\xi| \le \rho$. Now the assertion follows by the time-discrete operator-valued Herglotz theorem [99, Lemma 2.1]. □

3.3.1. Boundedness of the approximations

In this section, we use discrete energy estimates to show the boundedness of the approximations of Algorithm 3.12. We start with the non-negativity of the time discretized Calderon operator due to Convolution Quadrature properties.

Lemma 3.18. *It holds for $0 < \tau \le 1$ and $t_j \le T$ for arbitrary sequences $(\varphi^i)_{i=0}^j$ and $(\psi^i)_{i=0}^j$ in $\mathcal{H}_\Gamma$*

$$\sum_{i=0}^j \left\langle \begin{pmatrix} \varphi^i \\ \psi^i \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right) (t_i) \right\rangle_\Gamma \ge 0.$$

Proof. The proof follows by letting $\rho \to 1$ for fixed τ in Lemma 3.17. □

Remark 3.19. *The following lemma is formulated for space dependent material parameters $\varepsilon, \mu, \sigma : \Omega \to \mathbb{R}^{3\times 3}$. In this way it gets clear, where the conditions from Remark 3.16 come from and how they have to be applied. Similar arguments with scalar material parameters can be found in the proofs of Lemma 3.22 and Lemma 4.13.*

Lemma 3.20. *The approximations stay bounded for $\theta \ge 1/2$, i.e. we have for $0 < \tau \le \tau_0$ and $j \in \mathbb{N}_0$ with $t_j \le T$ that*

$$\mathcal{E}_h^j := \frac{\mu^-}{2} \|H_h^j\|_\Omega^2 + \frac{\varepsilon^-}{2} \|E_h^j\|_\Omega^2 + \mu^+ \frac{C_e}{2} \|\nabla m_h^j\|_\Omega^2 \le C_1$$

and additionally

$$\sum_{i=1}^j \|H_h^i - H_h^{i-1}\|_\Omega^2 + \sum_{i=1}^j \|E_h^i - E_h^{i-1}\|_\Omega^2 + \tau \sum_{i=1}^j \|w_h^{i-1}\|_\Omega^2$$
$$+ \sum_{i=1}^j \tau^2(\theta - 1/2) \|\nabla w_h^{i-1}\|_\Omega^2 + \tau \underbrace{\sum_{i=1}^j \left\langle \begin{pmatrix} \varphi_h^i \\ \psi_h^i \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} \right) (t_i) \right\rangle_\Gamma}_{\ge 0} \le C_2. \tag{3.23}$$

The constants C_1 and C_2 depend on T, τ_0, α, $\varepsilon^\pm$, $\mu^\pm$, J and $\mathcal{E}_h^0$, but are independent of h and τ.

Proof. We test in Algorithm 3.12 with $\zeta_h^E = E_h^{j+1}$, $\zeta_h^H = H_h^{j+1}$, $v_h^\varphi = \varphi_h^{j+1}$ and $v_h^\psi = \psi_h^{j+1}$ and add up (3.17)–(3.19) to obtain

$$[\varepsilon\partial_t^\tau E_h^{j+1}, E_h^{j+1}]_\Omega + [\mu\partial_t^\tau H_h^{j+1}, H_h^{j+1}]_\Omega + \left\langle \begin{pmatrix}\varphi^{j+1}\\ \psi^{j+1}\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h\\ \psi_h\end{pmatrix}\right)(t_{j+1})\right\rangle_\Gamma$$
$$= -[\sigma E_h^{j+1} + J_h^{j+1}, E_h^{j+1}]_\Omega - [\mu w_h^j, H_h^{j+1}]_\Omega.$$

Thus we have for all $i \geq 1$ (rewrite the above equation for $i := j+1$)

$$\frac{1}{\tau}[E_h^i - E_h^{i-1}, \varepsilon E_h^i]_\Omega + \frac{1}{\tau}[H_h^i - H_h^{i-1}, \mu H_h^i]_\Omega + \left\langle \begin{pmatrix}\varphi_h^i\\ \psi_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h\\ \psi_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma$$
$$= -[\sigma E_h^i + J_h^i, E_h^i]_\Omega - [\mu w_h^{i-1}, H_h^i]_\Omega. \tag{3.24}$$

To treat the terms $[E_h^i - E_h^{i-1}, \varepsilon E_h^i]_\Omega$ and $[H_h^i - H_h^{i-1}, \mu H_h^i]_\Omega$ we repeat Abel's summation by parts: For $u_i \in \mathbb{R}^n$ and $j \geq i \geq 1$, there holds by the third binomial formula and telescoping summation for a symmetric matrix A and $u \cdot_A v := u^T A v$, $|u|_A^2 := u^T A v$

$$\begin{aligned}\sum_{i=1}^j (u_i - u_{i-1}) \cdot_A u_i &= \frac{1}{2}\sum_{i=1}^j |u_i - u_{i-1}|_A^2 + \frac{1}{2}\sum_{i=1}^j (u_i - u_{i-1}) \cdot_A u_{i-1}\\ &\quad + \frac{1}{2}\sum_{i=1}^j (u_i - u_{i-1}) \cdot_A u_i\\ &= \frac{1}{2}\sum_{i=1}^j |u_i - u_{i-1}|_A^2 + \frac{1}{2}\sum_{i=1}^j |u_i|_A^2 - |u_{i-1}|_A^2\\ &= \frac{1}{2}\sum_{i=1}^j |u_i - u_{i-1}|_A^2 + \frac{1}{2}|u_j|_A^2 - \frac{1}{2}|u_0|_A^2.\end{aligned} \tag{3.25}$$

Summing up the equations (3.24) for $i = 1, \dots, j$, multiplying by τ and applying Abel's summation by parts to the respective terms we obtain

$$\begin{aligned}&\frac{\mu^-}{2}\left(\|H_h^j\|_\Omega^2 - \frac{\mu^+}{\mu^-}\|H_h^0\|_\Omega^2 + \sum_{i=1}^j \|H_h^i - H_h^{i-1}\|_\Omega^2\right)\\ &+\frac{\varepsilon^-}{2}\left(\|E_h^j\|_\Omega^2 - \frac{\varepsilon^+}{\varepsilon^-}\|E_h^0\|_\Omega^2 + \sum_{i=1}^j \|E_h^i - E_h^{i-1}\|_\Omega^2\right)\\ &+ \tau\sum_{i=1}^j \left\langle \begin{pmatrix}\varphi_h^i\\ \psi_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h\\ \psi_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma\\ &\qquad = -\tau\sigma^-\sum_{i=1}^j \|E_h^i\|_\Omega^2 - \tau\sum_{i=1}^j [J_h^i, E_h^i]_\Omega - \tau\sum_{i=1}^j [\mu w_h^{i-1}, H_h^i]_\Omega.\end{aligned} \tag{3.26}$$

We test in Algorithm 3.12 with $\rho = w_h^j$ for $j = i-1$ and receive

$$\alpha[w_h^{i-1}, w_h^{i-1}]_\Omega = -C_e[\nabla(m_h^{i-1} + \theta\tau w_h^{i-1}), \nabla w_h^{i-1}]_\Omega + [H_h^{i-1}, w_h^{i-1}]_\Omega.$$

With the mesh condition (Remark 3.15) we have $\|\nabla m_h^i\|_\Omega \leq \|\nabla(m_h^{i-1} + \tau w_h^{i-1})\|_\Omega$ and therefore get

$$\begin{aligned}\|\nabla m_h^i\|_\Omega^2 &\leq \|\nabla m_h^{i-1}\|_\Omega^2 + 2\tau[\nabla m_h^{i-1}, \nabla w_h^{i-1}]_\Omega + \tau^2\|\nabla w_h^{i-1}\|_\Omega^2\\ &= \|\nabla m_h^{i-1}\|_\Omega^2 + \frac{2\tau}{C_e}(-\alpha\|w_h^{i-1}\|_\Omega^2 + [H_h^{i-1}, w_h^{i-1}]_\Omega) - \tau^2(2\theta - 1)\|\nabla w_h^{i-1}\|_\Omega^2.\end{aligned}$$

We rewrite this as

$$\mu^+\frac{C_e}{2}\|\nabla m_h^i\|_\Omega^2 + \alpha\tau\mu^+\|w_h^{i-1}\|_\Omega^2 + C_e\mu^+\tau^2(\theta-1/2)\|\nabla w_h^{i-1}\|_\Omega^2 \\ \leq \mu^+\frac{C_e}{2}\|\nabla m_h^{i-1}\|_\Omega^2 + \mu^+\tau[H_h^{i-1}, w_h^{i-1}]_\Omega.$$

Summing from $i = 1, \ldots, j$ yields

$$\mu^+\frac{C_e}{2}\|\nabla m_h^j\|_\Omega^2 + \tau\alpha\mu^+\sum_{i=1}^j\|w_h^{i-1}\|_\Omega^2 + C_e\mu^+\tau^2(\theta-1/2)\sum_{i=1}^j\|\nabla w_h^{i-1}\|_\Omega^2 \\ \leq \mu^+\frac{C_e}{2}\|\nabla m_h^0\|_\Omega^2 + \mu^+\tau\sum_{i=1}^j[H_h^{i-1}, w_h^{i-1}]_\Omega,$$

and together with (3.26) finally results in

$$\frac{\mu^-}{2}\left(\|H_h^j\|_\Omega^2 + \sum_{i=1}^j\|H_h^i - H_h^{i-1}\|_\Omega^2\right) + \frac{\varepsilon^-}{2}\left(\|E_h^j\|_\Omega^2 + \sum_{i=1}^j\|E_h^i - E_h^{i-1}\|_\Omega^2\right) + \tau\sigma\sum_{i=1}^j\|E_h^i\|_\Omega^2 \\ + \mu^+\frac{C_e}{2}\|\nabla m_h^j\|_\Omega^2 + \tau\sum_{i=1}^j\mu^+\alpha\|w_h^{i-1}\|_\Omega^2 + \sum_{i=1}^j C_e\mu^+\tau^2(\theta-1/2)\|\nabla w_h^{i-1}\|_\Omega^2 \\ + \tau\sum_{i=1}^j\left\langle\begin{pmatrix}\varphi_h^i\\ \psi_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h\\ \psi_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma \\ \leq \frac{\mu^+}{2}\|H_h^0\|_\Omega^2 + \frac{\varepsilon^+}{2}\|E_h^0\|_\Omega^2 + \mu^+\frac{C_e}{2}\|\nabla m_h^0\|_\Omega^2 + \tau\sum_{i=1}^j\left(-[J_h^i, E_h^i]_\Omega\right) \\ + \mu^+\tau\sum_{i=1}^j[H_h^{i-1}, w_h^{i-1}]_\Omega - \tau\sum_{i=1}^j[H_h^i, \mu w_h^{i-1}]_\Omega.$$

We estimate the right hand side with Cauchy–Schwartz for arbitrary $\delta_1, \delta_2 > 0$

$$\tau\sum_{i=1}^j\left(-[J_h^i, E_h^i]_\Omega\right) + \mu^+\tau\sum_{i=1}^j[H_h^{i-1}, w_h^{i-1}]_\Omega - \tau\sum_{i=1}^j[H_h^i, \mu w_h^{i-1}]_\Omega \\ \leq \sum_{i=1}^j\frac{\tau}{2\delta_1}\|J_h^i\|_\Omega^2 + \sum_{i=1}^j\frac{\tau\delta_1}{2}\|E_h^i\|_\Omega^2 + \sum_{i=1}^j\frac{\mu^+\tau}{2\delta_2}\|H_h^i\|_\Omega^2 \\ + \sum_{i=1}^j\frac{\mu^+\tau}{2\delta_2}\|H_h^{i-1}\|_\Omega^2 + \sum_{i=1}^j\mu^+\tau\delta_2\|w_h^{i-1}\|_\Omega^2.$$

As the ferromagnetic domain may not be conductive (i.e. $\sigma = 0$ is possible), the term $\sum_{i=1}^j\frac{\tau\delta_1}{2}\|E_h^i\|_\Omega^2$ on the right hand side cannot be absorbed by the respective terms on the left hand side. Therefore we use

$$\sum_{i=1}^j\frac{\tau\delta_1}{2}\|E_h^i\|_\Omega^2 \leq \sum_{i=1}^j\tau\delta_1\|E_h^i - E_h^{i-1}\|_\Omega^2 + \sum_{i=1}^j\tau\delta_1\|E_h^{i-1}\|_\Omega^2$$

and similarly

$$\sum_{i=1}^j\frac{\tau\mu^+}{2\delta_2}\|H_h^i\|_\Omega^2 \leq \sum_{i=1}^j\frac{\tau\mu^+}{\delta_2}\|H_h^i - H_h^{i-1}\|_\Omega^2 + \sum_{i=1}^j\frac{\tau\mu^+}{\delta_2}\|H_h^{i-1}\|_\Omega^2.$$

We obtain with

$$\mathcal{E}_h^j := \frac{\mu^-}{2}\|H_h^j\|_\Omega^2 + \frac{\varepsilon^-}{2}\|E_h^j\|_\Omega^2 + \mu^+\frac{C_e}{2}\|\nabla m_h^j\|_\Omega^2$$

that

$$\begin{aligned}
&\mathcal{E}_h^j + \left(\frac{\mu^-}{2} - \frac{\tau\mu^+}{\delta_2}\right)\sum_{i=1}^j \|H_h^i - H_h^{i-1}\|_\Omega^2 + \left(\frac{\varepsilon^-}{2} - \tau\delta_1\right)\sum_{i=1}^j \|E_h^i - E_h^{i-1}\|_\Omega^2 + \tau\sigma^- \sum_{i=1}^j \|E_h^i\|_\Omega^2 \\
&\quad + \tau\sum_{i=1}^j \mu^+(\alpha-\delta_2)\|w_h^{i-1}\|_\Omega^2 + \sum_{i=1}^j C_e\mu^+\tau^2(\theta - 1/2)\|\nabla w_h^{i-1}\|_\Omega^2 \\
&\quad + \tau\sum_{i=1}^j \left\langle \begin{pmatrix}\varphi_h^i \\ \psi_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h \\ \psi_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma \\
&\quad \leq \max\left(\frac{\mu^+}{\mu^-}, \frac{\varepsilon^+}{\varepsilon^-}\right)\mathcal{E}_h^0 + \sum_{i=1}^j \frac{\tau}{2\delta_1}\|J_h^i\|_\Omega^2 + \sum_{i=1}^j \tau\delta_1\|E_h^{i-1}\|_\Omega^2 + \sum_{i=1}^j \frac{3\tau\mu^+}{2\delta_2}\|H_h^{i-1}\|_\Omega^2 \\
&\quad \leq \max\left(\frac{\mu^+}{\mu^-}, \frac{\varepsilon^+}{\varepsilon^-}\right)\mathcal{E}_h^0 + \sum_{i=1}^j \frac{\tau}{2\delta_1}\|J_h^i\|_\Omega^2 + \left(\frac{2\delta_1}{\varepsilon^-} + \frac{3\mu^+}{\mu^-\delta_2}\right)\tau\sum_{i=1}^j \mathcal{E}_h^{i-1}.
\end{aligned} \tag{3.27}$$

We have to ensure

$$\frac{\mu^-}{2} - \frac{\tau\mu^+}{\delta_2} > 0, \qquad \frac{\varepsilon^-}{2} - \tau\delta_1 > 0 \quad \text{and} \quad \alpha - \delta_2 > 0,$$

which is possible for $\delta_1, \delta_2 = O(1)$ and for small enough $\tau > 0$.
Moreover it holds (cf. Lemma 3.18)

$$\sum_{i=1}^j \left\langle \begin{pmatrix}\varphi_h^i \\ \psi_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h \\ \psi_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma \geq 0.$$

Thus equation (3.27) can be simplified to

$$\mathcal{E}_h^i \leq C + c\tau\sum_{i=1}^j \mathcal{E}_h^{i-1}$$

and the discrete Gronwall Lemma (Lemma A.2) gives $\mathcal{E}_h^i \leq \widetilde{C}$ for $i \leq j$. Thus we have

$$\frac{2\delta_1}{\varepsilon}\tau\sum_{i=1}^j \mathcal{E}_h^{i-1} \leq \widehat{C},$$

which concludes the assertion together with (3.27). □

We now look at the boundary functions. In Lemma 3.20 we used

$$\sum_{i=1}^j \left\langle \begin{pmatrix}\varphi^i \\ \psi^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi \\ \psi\end{pmatrix}\right)(t_i)\right\rangle_\Gamma \geq 0.$$

By the following result, we modify Lemma 3.20 to obtain a statement about the boundedness of φ^j, ψ^j.

Lemma 3.21 ([99, Lemma 5.3]). *It holds for $0 < \tau \leq \tau_0$ and $t_j \leq T$ and any sequences $(\varphi(t_i))_{i=0}^j$ and $(\psi(t_i))_{i=0}^j$ in $\mathcal{H}_\Gamma$*

$$\sum_{i=0}^j e^{-2t_i/T}\left\langle \begin{pmatrix}\varphi^i \\ \psi^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi \\ \psi\end{pmatrix}\right)(t_i)\right\rangle_\Gamma \geq C\left(\sum_{i=0}^j \|(\partial_t^\tau)^{-1}\varphi(t_i)\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^{-1}\psi(t_i)\|_{\mathcal{H}_\Gamma}^2\right).$$

The constant $C > 0$ depends on $T, \tau_0, \varepsilon_0, \mu_0$ and $\beta > 0$ of Lemma 2.12.

Proof. The proof proceeds analogously as the one of Lemma 3.18 by setting $\rho := e^{-\tau/T} < 1$ and using $\frac{1-e^{-x}}{x} = \int_0^1 e^{-xr}\,\mathrm{d}r \geq e^{-x} \geq e^{-1}$ for $x \in [0,1]$, instead of letting $\rho \to 1$ for fixed τ. □

The following lemma provides energy bounds for the quantities on the boundary. It is a modification of Lemma 3.20 with the missing factors $e^{-t_i/T}$ that show up in Lemma 3.21.

Lemma 3.22. *For $\theta > 1/2$ and $0 < \tau \leq \tau_0$, $j \in \mathbb{N}_0$ with $t_j \leq T$ it holds*

$$\tau \sum_{i=0}^{j} \Big(\|(\partial_t^\tau)^{-1}\varphi(t_i,\cdot)\|^2_{\mathcal{H}_\Gamma} + \|(\partial_t^\tau)^{-1}\psi(t_i,\cdot)\|^2_{\mathcal{H}_\Gamma} \Big) \leq C$$

for a constant $C > 0$ depending on $T, \varepsilon, \varepsilon_0, \mu, \mu_0$, β, τ_0, α, J and $\mathcal{E}_h^0$, but independent of h and τ.

Proof. The proof works analogously as the one of Lemma 3.20, by inserting the missing factors $e^{-t_i/T}$. We test in Algorithm 3.12 with $\zeta_h^E = E_h^{j+1}$, $\zeta_h^H = H_h^{j+1}$, $v_h^\varphi = \varphi_h^{j+1}$ and $v_h^\psi = \psi_h^{j+1}$ and add up (3.17)–(3.19) to obtain

$$\begin{aligned} \varepsilon[\partial_t^\tau E_h^{j+1}, E_h^{j+1}]_\Omega + \mu[\partial_t^\tau H_h^{j+1}, H_h^{j+1}]_\Omega + \left\langle \begin{pmatrix} \varphi_h^{j+1} \\ \psi_h^{j+1} \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} \right)(t_{j+1}) \right\rangle_\Gamma \\ = -[\sigma E_h^{j+1} + J_h^{j+1}, E_h^{j+1}]_\Omega - \mu[w_h^j, H_h^{j+1}]_\Omega. \end{aligned}$$

By rewriting the above equation for $i := j+1$, multiplying it by $e^{-2t_i/T}$, and by using the abbreviations

$$\widetilde{E}_h^i := e^{-t_i/T} E_h^i, \quad \widetilde{H}_h^i := e^{-t_i/T} H_h^i, \quad \widetilde{w}_h^i := e^{-t_i/T} w_h^i, \text{ and } \widetilde{J}_h^i := e^{-t_i/T} J_h^i,$$

we have for all $i \geq 1$

$$\begin{aligned} &\frac{\varepsilon}{\tau}[\widetilde{E}_h^i - e^{-\tau/T}\widetilde{E}_h^{i-1}, \widetilde{E}_h^i]_\Omega + \frac{\mu}{\tau}[\widetilde{H}_h^i - e^{-\tau/T}\widetilde{H}_h^{i-1}, \widetilde{H}_h^i]_\Omega \\ &\qquad + e^{-2t_i/T} \left\langle \begin{pmatrix} \varphi_h^i \\ \psi_h^i \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} \right)(t_i) \right\rangle_\Gamma \\ &\qquad = -[\sigma\widetilde{E}_h^i + \widetilde{J}_h^i, \widetilde{E}_h^i]_\Omega - \mu[\widetilde{w}_h^{i-1}, e^{-\tau/T}\widetilde{H}_h^i]_\Omega. \end{aligned} \tag{3.28}$$

To treat the terms $[\widetilde{E}_h^i - e^{-\tau/T}\widetilde{E}_h^{i-1}, \widetilde{E}_h^i]_\Omega$ and $[\widetilde{H}_h^i - e^{-\tau/T}\widetilde{H}_h^{i-1}, \widetilde{H}_h^i]_\Omega$ we modify Abel's summation by parts. For $u_i \in \mathbb{R}^n$ and $j \geq i \geq 1$, there holds

$$\begin{aligned} \sum_{i=1}^{j} (u_i - e^{-\tau/T}u_{i-1}) \cdot u_i &= \frac{1}{2}\sum_{i=1}^{j} |u_i - e^{-\tau/T}u_{i-1}|^2 + \frac{1}{2}\sum_{i=1}^{j}(u_i - e^{-\tau/T}u_{i-1}) \cdot e^{-\tau/T}u_{i-1} \\ &\quad + \frac{1}{2}\sum_{i=1}^{j}(u_i - e^{-\tau/T}u_{i-1}) \cdot u_i \\ &= \frac{1}{2}\sum_{i=1}^{j} |u_i - e^{-\tau/T}u_{i-1}|^2 + \frac{1}{2}\sum_{i=1}^{j} |u_i|^2 - e^{-2\tau/T}|u_{i-1}|^2 \\ &\geq \frac{1}{2}\sum_{i=1}^{j} |u_i - e^{-\tau/T}u_{i-1}|^2 + \frac{1}{2}\sum_{i=1}^{j} |u_i|^2 - |u_{i-1}|^2 \\ &= \frac{1}{2}\sum_{i=1}^{j} |u_i - e^{-\tau/T}u_{i-1}|^2 + \frac{1}{2}|u_j|^2 - \frac{1}{2}|u_0|^2. \end{aligned} \tag{3.29}$$

This result also stays true if we replace $\cdot$ by $\cdot_A$ and $|\cdot|$ by $|\cdot|_A$, if A is a positive semi definite and symmetric matrix. Summing up the equations (3.28) for $i = 1, \ldots, j$, multiplying by τ and applying the modified summation by parts to $\widetilde{E}_h^i = e^{-t_i/T} E_h^i$ and $\widetilde{H}_h^i = e^{-t_i/T} H_h^i$ we obtain

$$\begin{aligned}
&\frac{\mu}{2}\left(\|\widetilde{H}_h^j\|_\Omega^2 - \|\widetilde{H}_h^0\|_\Omega^2 + \sum_{i=1}^{j}\|\widetilde{H}_h^i - e^{-\tau/T}\widetilde{H}_h^{i-1}\|_\Omega^2\right)\\
&+\frac{\varepsilon}{2}\left(\|\widetilde{E}_h^j\|_\Omega^2 - \|\widetilde{E}_h^0\|_\Omega^2 + \sum_{i=1}^{j}\|\widetilde{E}_h^i - e^{-\tau/T}\widetilde{E}_h^{i-1}\|_\Omega^2\right)\\
&+\tau\sum_{i=1}^{j} e^{-2t_i/T}\left\langle \begin{pmatrix}\varphi_h^i\\ \psi_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h\\ \psi_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma + \tau\sigma\sum_{i=1}^{j}\|\widetilde{E}_h^i\|_\Omega^2\\
&\leq -\tau\sum_{i=1}^{j}[\widetilde{J}_h^i, \widetilde{E}_h^i]_\Omega - \tau\sum_{i=1}^{j}\mu[\widetilde{w}_h^{i-1}, e^{-\tau/T}\widetilde{H}_h^i]_\Omega\\
&\leq \left(\tau\sum_{i=1}^{j}\|\widetilde{J}_h^i\|_\Omega^2\right)^{1/2}\left(\tau\sum_{i=1}^{j}\|\widetilde{E}_h^i\|_\Omega^2\right)^{1/2} + \left(\tau\sum_{i=1}^{j}\|\widetilde{H}_h^i\|_\Omega^2\right)^{1/2}\left(\tau\sum_{i=1}^{j}\|\mu\widetilde{w}_h^i\|_\Omega^2\right)^{1/2}.
\end{aligned} \tag{3.30}$$

By Assumptions 3.14 and Lemma 3.20, we have

$$\tau\sum_{i=1}^{j}\|\widetilde{E}_h^i\|_\Omega^2 + \tau\sum_{i=1}^{j}\|\widetilde{H}_h^i\|_\Omega^2 + \tau\sum_{i=1}^{j}\|\widetilde{J}_h^i\|_\Omega^2 + \tau\sum_{i=1}^{j}\|\widetilde{w}_h^i\|_\Omega^2 \leq C.$$

As all other terms on the left hand side of (3.30) are positive and/or bounded, we have

$$\tau\sum_{i=0}^{j} e^{-2i\tau/T}\left\langle \begin{pmatrix}\varphi_h^i\\ \psi_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h\\ \psi_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma \leq C.$$

Therefore, by Lemma 3.21 for some constants $c, C > 0$

$$\begin{aligned}
&\tau\sum_{i=0}^{j}\|(\partial_t^\tau)^{-1}\varphi_h^i(t_i)\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^{-1}\psi_h(t_i)\|_{\mathcal{H}_\Gamma}^2\\
&\qquad\leq c\tau\sum_{i=0}^{j} e^{-2i\tau/T}\left\langle \begin{pmatrix}\varphi_h^i\\ \psi_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h\\ \psi_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma \leq C,
\end{aligned}$$

which yields the assertion.

□

Remark 3.23 (Alternative proof of Lemma 3.20 and Lemma 3.22). *In this remark we describe an alternative possibility to prove Lemma 3.20 and Lemma 3.22 at once.*

In Lemma 3.20 we use Lemma 3.18 for the Calderon term and obtain bounds for the quantities $\|\nabla m_h^j\|_\Omega$, $\|E_h^j\|_\Omega$, *and* $\|H_h^j\|_\Omega$ *(and some further quantities).*

In Lemma 3.22 we adapt the proof of Lemma 3.20 by inserting the missing factors $e^{-t_i/T}$ *to apply Lemma 3.21 for the Calderon term. For this purpose, we consider the quantities* $\widetilde{m}_h^i := e^{-t_i/T} m_h^i$, $\widetilde{E}_h^i := e^{-t_i/T} E_h^i$, $\widetilde{H}_h^i := e^{-t_i/T} H_h^i$. *Due to Lemma 3.20 and the estimate* $e^{-1} \leq e^{-t_i/T} \leq 1$, *these quantities are bounded and the proof of Lemma 3.22 can be concluded.*

The alternative idea of proving Lemma 3.22 and Lemma 3.20 at once follows the following lines: Instead of showing Lemma 3.20, we could also adapt the proof of Lemma 3.20 from the beginning to the end for the modified quantities and obtain the bounds

$$\frac{\mu_0}{2}\|\widetilde{H}^j\|_\Omega^2 + \frac{\varepsilon_0}{2}\|\widetilde{E}^j\|_\Omega^2 + \mu_0\frac{C_e}{2}\|\nabla\widetilde{m}^j\|_\Omega^2 \leq C$$

and

$$\sum_{i=1}^{j}\|\widetilde{H}^i - e^{-\tau/T}\widetilde{H}^{i-1}\|_\Omega^2 + \sum_{i=1}^{j}\|\widetilde{E}^i - e^{-\tau/T}\widetilde{E}^{i-1}\|_\Omega^2 + \tau\sigma\sum_{i=1}^{j}\|\widetilde{E}^i\|_\Omega^2 + \tau\sum_{i=1}^{j}\|\widetilde{w}^{i-1}\|_\Omega^2$$

$$+\sum_{i=1}^{j}\tau^2(\theta-1/2)\|\nabla\widetilde{w}^{i-1}\|_\Omega^2 + \tau\sum_{i=1}^{j}e^{-2t_i/T}\left\langle\begin{pmatrix}\varphi^i\\ \psi^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right)(t_i)\right\rangle_\Gamma \leq C$$

Due to Lemma 3.21 and $e^{-1} \leq e^{-t_i/T} \leq 1$, *this yields the assertions of Lemma 3.20 and Lemma 3.22. We refer to Lemma 4.13, where this alternative way of proof is executed in the pure Maxwell case.*

Remark 3.24. *In the following we use an unusual notation which we explain in this remark. For a sequence* $(\phi^j)_{j\in\mathbb{N}_0}$, *we consider the sum* $\sum_{j=1}^{k+1}\phi^j$ *and with the discrete time integration operator* $(\partial_t^\tau)^{-1}$ *it holds*

$$\sum_{j=1}^{k+1}\phi^j = \sum_{j=0}^{k}\phi^{j+1} = ((\partial_t^\tau)^{-1}(\phi^{j+1})_{j\in\mathbb{N}_0})(t_k).$$

If we write the sequence $(((\partial_t^\tau)^{-1}(\phi^{j+1})_{j\in\mathbb{N}_0})(t_k))_{k\in\mathbb{N}_0}$ *as a time dependent function like in* (3.10), *we use the notation*

$$((\partial_t^\tau)^{-1}(\phi^{j+1})_j)_\tau^\pm \quad \textit{instead of} \quad (((\partial_t^\tau)^{-1}(\phi^{j+1})_{j\in\mathbb{N}_0})(t))_\tau^\pm.$$

For $\phi^0 = 0$ *the above terms simplify to*

$$((\partial_t^\tau)^{-1}(\phi^{j+1})_{j\in\mathbb{N}_0})(t_k) = ((\partial_t^\tau)^{-1}\phi)(t_{k+1})$$

and

$$((\partial_t^\tau)^{-1}(\phi^{j+1})_j)_\tau^- = ((\partial_t^\tau)^{-1}\phi)_\tau^+.$$

Let P_h be the L^2-orthogonal projection on the closed (finite-dimensional) subspace $\mathcal{X}_h$, i.e.

$$\mathrm{P}_h^{\mathcal{X}} : L^2(\Omega) \to \mathcal{X}_h$$

is linear and it holds for every $v \in L^2(\Omega)$

$$[(1-\mathrm{P}_h^{\mathcal{X}})v, \xi_h]_\Omega = 0 \text{ for all } \xi_h \in \mathcal{X}_h.$$

We now consider boundedness of $\operatorname{curl} E$ and $\operatorname{curl} H$. We therefore collect the corresponding terms including the boundary terms on the right hand side of (3.17)–(3.18) and integrate in time in a discrete way. We define for $\xi \in L^2(\Omega_T)$

$$\begin{aligned}
f_{\tau,h}(\xi) &:= \frac{1}{2}[(\nabla\times(\partial_t^\tau)^{-1}(H_h^{j+1})_j)_{\tau,h}^-, \mathrm{P}_h^{\mathcal{X}}\xi]_{\Omega_T} + \frac{1}{2}[((\partial_t^\tau)^{-1}(H_h^{j+1})_j)_{\tau,h}^-, \nabla\times(\mathrm{P}_h^{\mathcal{X}}\xi)]_{\Omega_T}\\
&\quad - \frac{1}{2\mu_0}\left\langle((\partial_t^\tau)^{-1}(\varphi_h^{j+1})_j)_{\tau,h}^-, \gamma_T(\mathrm{P}_h^{\mathcal{X}}\xi)\right\rangle_{\Gamma_T},\\
g_{\tau,h}(\xi) &:= \frac{1}{2}[(\nabla\times(\partial_t^\tau)^{-1}(E_h^{j+1})_j)_{\tau,h}^-, \mathrm{P}_h^{\mathcal{X}}\xi]_{\Omega_T} + \frac{1}{2}[((\partial_t^\tau)^{-1}(E_h^{j+1})_j)_{\tau,h}^-, \nabla\times(\mathrm{P}_h^{\mathcal{X}}\xi)]_{\Omega_T}\\
&\quad + \frac{1}{2}\left\langle((\partial_t^\tau)^{-1}(\psi_h^{j+1})_j)_{\tau,h}^-, \gamma_T(\mathrm{P}_h^{\mathcal{X}}\xi)\right\rangle_{\Gamma_T}.
\end{aligned}$$

Remark 3.25. *We will need these identities in the following (see Lemma 3.30): Integration by parts shows*

$$\begin{aligned}
f_{\tau,h}(\xi) &= [(\nabla\times(\partial_t^\tau)^{-1}(H_h^{j+1})_j)_{\tau,h}^-, \mathrm{P}_h^{\mathcal{X}}\xi]_{\Omega_T}\\
&\quad - \frac{1}{2\mu_0}\left\langle((\partial_t^\tau)^{-1}(\varphi_h^{j+1})_j)_{\tau,h}^- - \mu_0(\gamma_T((\partial_t^\tau)^{-1}(H_h^{j+1})_j))_{\tau,h}^-, \gamma_T(\mathrm{P}_h^{\mathcal{X}}\xi)\right\rangle_{\Gamma_T}\\
&= [((\partial_t^\tau)^{-1}(H_h^{j+1})_j)_{\tau,k}^-, \nabla\times\mathrm{P}_h^{\mathcal{X}}\xi]_{\Omega_T}\\
&\quad - \frac{1}{2\mu_0}\left\langle((\partial_t^\tau)^{-1}(\varphi_h^{j+1})_j)_{\tau,h}^- + (\mu_0\gamma_T((\partial_t^\tau)^{-1}(H_h^{j+1})_j))_{\tau,h}^-, \gamma_T(\mathrm{P}_h^{\mathcal{X}}\xi)\right\rangle_{\Gamma_T}
\end{aligned}$$

and

$$
\begin{aligned}
g_{\tau,h}(\xi) &= [(\nabla \times (\partial_t^\tau)^{-1}(E_h^{j+1})_j)_{\tau,h}^-, \mathrm{P}_h^{\mathcal{X}}\xi]_{\Omega_T} \\
&\quad + \frac{1}{2}\left\langle ((\partial_t^\tau)^{-1}(\psi_h^{j+1})_j)_{\tau,h}^- + (\gamma_T((\partial_t^\tau)^{-1}(E_h^{j+1})_j))_{\tau,h}^-, \gamma_T(\mathrm{P}_h^{\mathcal{X}}\xi)\right\rangle_{\Gamma_T} \\
&= [((\partial_t^\tau)^{-1}(E_h^{j+1})_j)_{\tau,k}^-, \nabla \times \mathrm{P}_h^{\mathcal{X}}\xi]_{\Omega_T} \\
&\quad + \frac{1}{2}\left\langle ((\partial_t^\tau)^{-1}(\psi_h^{j+1})_j)_{\tau,h}^- - (\gamma_T((\partial_t^\tau)^{-1}(E_h^{j+1})_j))_{\tau,h}^-, \gamma_T(\mathrm{P}_h^{\mathcal{X}}\xi)\right\rangle_{\Gamma_T}.
\end{aligned}
$$

Lemma 3.26. *For $\xi \in L^2(\Omega_T)$ it holds*

$$|f_{\tau,h}(\xi)| \leq C\|\xi\|_{\Omega_T}$$

and

$$|g_{\tau,h}(\xi)| \leq C\|\xi\|_{\Omega_T}.$$

The constant $C > 0$ does not depend on h or τ. Therefore it is $f_{\tau,h} \in L^2(\Omega_T)'$ and we use the $L^2(\Omega_T)$ representation $f_{\tau,h} \in L^2(\Omega_T)$ such that $f_{\tau,h}(\xi) = [f_{\tau,h}, \xi]_{\Omega_T}$ for all $\xi \in L^2(\Omega_T)$. Similarly we identify $g_{\tau,h} \in L^2(\Omega_T)$ such that $g_{\tau,h}(\xi) = [g_{\tau,h}, \xi]_{\Omega_T}$ for all $\xi \in L^2(\Omega_T)$. Moreover $f_{\tau,h}$ and $g_{\tau,h}$ are bounded uniformly with respect to τ and h.

Proof. We test equation (3.17) with $\zeta_h \in \mathcal{X}_h$, multiply by τ and sum over $j = 0, \ldots, k$ to obtain

$$
\begin{aligned}
&[\varepsilon E_h^{k+1}, \zeta_h]_\Omega - [\varepsilon E_h^0, \zeta_h]_\Omega \\
&\qquad = \frac{1}{2}[(\nabla \times (\partial_t^\tau)^{-1}(H_h^{j+1})_j)(t_k), \zeta_h]_\Omega + \frac{1}{2}[((\partial_t^\tau)^{-1}(H_h^{j+1})_j)(t_k), \nabla \times \zeta_h]_\Omega \\
&\qquad\quad - \frac{1}{2\mu_0}\langle((\partial_t^\tau)^{-1}(\varphi_h^{j+1})_j)(t_k), \gamma_T\zeta_h\rangle_\Gamma + [((\partial_t^\tau)^{-1}(\sigma E^{j+1} + J^{j+1})_j)(t_k), \zeta_h]_\Omega
\end{aligned}
$$

For ζ_h we insert $\mathrm{P}_h^{\mathcal{X}}\xi(t)$, integrate over $[t_k, t_{k+1}]$, sum up from $k = 0, \ldots, N-1$ and obtain

$$f_{\tau,h}(\xi) = [E_{\tau,h}^+ - E_h^0, \varepsilon\mathrm{P}_h^{\mathcal{X}}\xi]_{\Omega_T} - [((\partial_t^\tau)^{-1}(\sigma E^{j+1} + J^{j+1})_j)_{\tau,h}^-, \mathrm{P}_h^{\mathcal{X}}\xi]_{\Omega_T}.$$

With Lemma 3.20 and Assumption 3.14 we have

$$\|E_h^{k+1}\|_\Omega + \|E_h^0\|_\Omega + \|(\partial_t^\tau)^{-1}(\sigma E^{j+1} + J^{j+1})_j(t_k)\|_\Omega \leq C$$

and as $\mathrm{P}_h^{\mathcal{X}}$ is an L^2 orthogonal projection and therefore bounded, we have

$$|f_{\tau,h}(\xi)| \leq C\|\mathrm{P}_h^{\mathcal{X}}\xi\|_{\Omega_T} \leq C\|\xi\|_{\Omega_T},$$

which concludes the first assertion.

The assertion for $g_{\tau,h}(\xi)$ follows similarly by using

$$\|(\partial_t^\tau)^{-1}w_h(t_j)\|_\Omega \leq C,$$

which is again a consequence of Lemma 3.20. □

We summarize the results of this section in the following lemma.

Lemma 3.27. *There exists a constant $C > 0$ independent of τ and h such that*

$$\begin{aligned}
\|m_{\tau,h}\|_{H^1(\Omega_T)} + \|m^{\pm}_{\tau,h}\|_{L^2([0,T],H^1(\Omega))} &\le C,\\
\|w^-_{\tau,h}\|_{\Omega_T} &\le C,\\
\|E_{\tau,h}\|_{\Omega_T} + \|E^{\pm}_{\tau,h}\|_{\Omega_T} &\le C,\\
\|H_{\tau,h}\|_{\Omega_T} + \|H^{\pm}_{\tau,h}\|_{\Omega_T} &\le C,\\
\|((\partial_t^\tau)^{-1}\varphi_h)_{\tau,h}\|_{L^2([0,T],\mathcal{H}_\Gamma)} + \|((\partial_t^\tau)^{-1}\varphi_h)^{\pm}_{\tau,h}\|_{L^2([0,T],\mathcal{H}_\Gamma)} &\le C,\\
\|((\partial_t^\tau)^{-1}\psi_h)_{\tau,h}\|_{L^2([0,T],\mathcal{H}_\Gamma)} + \|((\partial_t^\tau)^{-1}\psi_h)^{\pm}_{\tau,h}\|_{L^2([0,T],\mathcal{H}_\Gamma)} &\le C,\\
\|f_{\tau,h}\|_{\Omega_T} + \|g_{\tau,h}\|_{\Omega_T} &\le C.
\end{aligned}$$

Proof. Most estimates follow directly from Lemmas 3.20 – 3.26. We only sketch the proof by showing two short identities, which conclude the bounds. We have for a sequence $(\phi_h^i)_{i=0}^N$

$$\begin{aligned}
\|\phi_{\tau,h}\|^2_{[0,T]} &= \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \left|\frac{t_{i+1}-s}{\tau}\phi_h^i + \frac{s-t_i}{\tau}\phi_h^{i+1}\right|^2 \,\mathrm{d}t\\
&\le 2\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \left|\frac{t_{i+1}-s}{\tau}\right|^2 |\phi_h^i|^2 + \left|\frac{s-t_i}{\tau}\right|^2 |\phi_h^{i+1}|^2 \,\mathrm{d}t\\
&\le 2\|\phi^+_{\tau,h}\|^2_\Omega + 2\|\phi^-_{\tau,h}\|^2_\Omega\\
&\le 4\tau\sum_{i=0}^{N}|\phi_h^i|^2.
\end{aligned}$$

By [72, Lemma 3.3.2] the discrete differential quotient in time of the magnetization is bounded by the discrete derivative, i.e. it is $\frac{1}{\tau}\|m_h^{j+1} - m_h^j\|_\Omega \le \|w_h^j\|_\Omega$ and therefore

$$\|\partial_t m_{\tau,h}\|_{\Omega_T} = \|(\partial_t^\tau m_h^j)^+_{\tau,h}\|_{\Omega_T} \le \|w^-_{\tau,h}\|_{\Omega_T} \le C.$$

□

3.3.2. Existence of weakly convergent subsequences

Due to the boundedness of the quantities in the respective Hilbert spaces (cf. Lemma 3.27), we are now able to extract weakly convergent subsequences (cf. Lemma 2.5).

Throughout the manuscript we do not (re-)name the sequences when passing to a subsequence, all convergence properties only hold for subsequences. We write $v_{\tau,h} \overset{\text{sub}}{\rightharpoonup} v$ for $\tau, h \to 0$, to denote that for any $(\tau_n, h_n) \to 0$ for $n \to \infty$ there exists a subsequence $(n_j)_{j\in\mathbb{N}}$, such that $v_{h_{n_j}} \rightharpoonup v$ for $j \to \infty$.

Lemma 3.28. *There exist functions*

$$(m, H, E, \widetilde{\varphi}, \widetilde{\psi}) \in H^1(\Omega_T, \mathbb{S}^2) \times L^2(\Omega_T) \times L^2(\Omega_T) \times L^2([0,T], \mathcal{H}_\Gamma) \times L^2([0,T], \mathcal{H}_\Gamma)$$

such that

$$
\begin{aligned}
m_{\tau,h} &\overset{\text{sub}}{\rightharpoonup} m && \text{in } H^1(\Omega_T),\\
m_{\tau,h}, m_{\tau,h}^{\pm} &\overset{\text{sub}}{\rightharpoonup} m && \text{in } L^2([0,T],H^1(\Omega)),\\
m_{\tau,h}, m_{\tau,h}^{\pm} &\overset{\text{sub}}{\to} m && \text{in } L^2(\Omega_T),\\
w_{\tau,h}^{-} &\overset{\text{sub}}{\rightharpoonup} \partial_t m && \text{in } L^2(\Omega_T),\\
H_{\tau,h}, H_{\tau,h}^{\pm} &\overset{\text{sub}}{\rightharpoonup} H && \text{in } L^2(\Omega_T),\\
E_{\tau,h}, E_{\tau,h}^{\pm} &\overset{\text{sub}}{\rightharpoonup} E && \text{in } L^2(\Omega_T),\\
((\partial_t^\tau)^{-1}\varphi_h)_{\tau,h}, ((\partial_t^\tau)^{-1}\varphi_h)_{\tau,h}^{\pm} &\overset{\text{sub}}{\rightharpoonup} \widetilde{\varphi} && \text{in } L^2([0,T],\mathcal{H}_\Gamma),\\
((\partial_t^\tau)^{-1}\psi_h)_{\tau,h}, ((\partial_t^\tau)^{-1}\psi_h)_{\tau,h}^{\pm} &\overset{\text{sub}}{\rightharpoonup} \widetilde{\psi} && \text{in } L^2([0,T],\mathcal{H}_\Gamma),
\end{aligned}
$$

where the subsequences are successively constructed, i.e. for arbitrary time step sizes $\tau \to 0$ and mesh sizes $h \to 0$ there exist subindices τ_{n_l}, h_{n_l} for which the above convergence properties are satisfied simultaneously.

Proof. The proof for the LLG part works analogously as in [25, Lemma 5.5, Lemma 5.6] and we therefore only sketch it: The uniform boundedness in the respective Hilbert spaces together with Lemma 2.5 gives weakly convergent subsequences. By the Rellich–Kondrachov theorem, the convergence holds strongly in $L^2(\Omega_T)$. It remains to show that the limit functions coincide, i.e.

$$\lim_{\tau,h \overset{\text{sub}}{\to} 0} m_{\tau,h} = \lim_{\tau,h \overset{\text{sub}}{\to} 0} m_{\tau,h}^{+} = \lim_{\tau,h \overset{\text{sub}}{\to} 0} m_{\tau,h}^{-}$$

and

$$\lim_{\tau,h \overset{\text{sub}}{\to} 0} w_{\tau,h}^{-} = \partial_t m.$$

For the first assertion, we refer to the arguments that are presented below for the Maxwell and boundary part. For the second assertion, we set $w := \lim_{\tau,h \overset{\text{sub}}{\to} 0} w_{\tau,h}^{-}$ (weakly in $L^2(\Omega_T)$ for a subsequence). As in [72, Lemma 3.3.13], one shows the inequality

$$\|\partial_t m_{\tau,h} - w_{\tau,h}^{-}\|_{L^1(\Omega_T)} \le C\tau \|w_{\tau,h}^{-}\|_{L^2(\Omega_T)}^2,$$

which gives together with the weak semicontinuity of the norm

$$\|\partial_t m - w\|_{L^1(\Omega_T)} \le \liminf_{t,h \overset{\text{sub}}{\to} 0} \|\partial_t m_{\tau,h} - w_{\tau,h}^{-}\|_{L^1(\Omega_T)} = 0,$$

i.e. $w = \partial_t m$.

For the Maxwell and the boundary part, by the uniform boundedness of the approximations in the respective Hilbert spaces (cf. Lemma 3.27) and uniqueness of weak limits, we have the existence of limit functions and the weak convergence of a (fixed) subsequence

$$(E_{\tau,h}, H_{\tau,h}, ((\partial_t^\tau)^{-1}\varphi_h)_{\tau,h}, ((\partial_t^\tau)^{-1}\psi_h)_{\tau,h}) \overset{\text{sub}}{\rightharpoonup} (E, H, \widetilde{\varphi}, \widetilde{\psi}) \in L^2(\Omega_T)^2 \times L^2([0,T],\mathcal{H}_\Gamma)^2.$$

It remains to show that $(E_{\tau,h}^{\pm}, H_{\tau,h}^{\pm}, ((\partial_t^\tau)^{-1}\varphi_h)_{\tau,h}^{\pm}, ((\partial_t^\tau)^{-1}\psi_h)_{\tau,h}^{\pm})$ converge to the same limit functions. We show exemplarily that $E_{\tau,h}^{-}$ converges to the same limit function as $E_{\tau,h}$. The proof can then be adapted for $E_{\tau,h}^{+}, H_{\tau,h}^{\pm}$ and the functions on the boundary.

It holds for $w \in C_0^1(\Omega_T)$

$$
\begin{aligned}
[E_{\tau,h} - E_{\tau,h}^-, w]_{\Omega_T} &= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} [E_h^j + \frac{t-t_j}{\tau}\left(E_h^{j+1} - E_h^j\right) - E_h^j, w(t)]_\Omega \, \mathrm{d}t \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \frac{t-t_j}{\tau}[E_h^{j+1} - E_h^j, w(t)]_\Omega \, \mathrm{d}t \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \frac{t-t_j}{\tau}[E_h^{j+1} - E_h^j, w(t_j)]_\Omega \, \mathrm{d}t \\
&\quad + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \frac{t-t_j}{\tau}[E_h^{j+1} - E_h^j, w(t) - w(t_j)]_\Omega \, \mathrm{d}t.
\end{aligned}
$$

By $w(T) = w(0) = 0$ we see

$$
\begin{aligned}
\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \frac{t-t_j}{\tau}[E_h^{j+1} - E_h^j, w(t_j)]_\Omega \, \mathrm{d}t &= \frac{\tau}{2} \sum_{j=0}^{N-1} [E_h^{j+1} - E_h^j, w(t_j)]_\Omega \\
&= -\frac{\tau}{2} \sum_{j=0}^{N-1} [E_h^{j+1}, w(t_{j+1}) - w(t_j)]_\Omega.
\end{aligned}
$$

Therefore we have by the boundedness of $E_{\tau,h}^{\pm}$

$$
\begin{aligned}
|[E_{\tau,h} - E_{\tau,h}^-, w]_{\Omega_T}| &\leq \frac{1}{2}\left(\tau \sum_{j=0}^{N-1} \|E_h^{j+1}\|_\Omega^2\right)^{1/2} \left(\tau \sum_{j=0}^{N-1} \|w(t_{j+1}) - w(t_j)\|_\Omega^2\right)^{1/2} \\
&\quad + \left(\tau \sum_{j=0}^{N-1} \|E_h^j - E_h^{j+1}\|_\Omega^2\right)^{1/2} \left(\tau \sum_{j=0}^{N-1} \Big\| \int_{t_j}^{t_{j+1}} \frac{w(t) - w(t_j)}{\tau} \, \mathrm{d}t \Big\|_\Omega^2\right)^{1/2} \\
&\leq C \|E_{\tau,h}^{\pm}\|_{L^2(\Omega_T)} \max_{j=0,\ldots,N-1} \max_{t \in [t_j, t_{j+1}]} \|w(t) - w(t_j)\|_\Omega \to 0.
\end{aligned}
$$

As $C_0^1(\Omega_T)$ is dense in $L^2(\Omega_T)$ and $E_{\tau,h}^-$ is uniformly bounded in $L^2(\Omega_T)$, it holds $E_{\tau,h}^- \overset{\text{sub}}{\rightharpoonup} E$. □

Lemma 3.29. *It holds*

$$((\partial_t^\tau)^{-1}(E_h^{j+1})_j)_{\tau,h}^- \overset{\text{sub}}{\rightharpoonup} (\partial_t)^{-1} E \text{ in } L^2(\Omega_T)$$

and

$$((\partial_t^\tau)^{-1}(H_h^{j+1})_j)_{\tau,h}^- \overset{\text{sub}}{\rightharpoonup} (\partial_t)^{-1} H \text{ in } L^2(\Omega_T)$$

for $h, \tau \to 0$.

Proof. Due to $\|((\partial_t^\tau)^{-1}(E_h^{j+1})_j)_{\tau,h}^-\|_{\Omega_T} \leq C$, there exists a function $\widetilde{E} \in L^2(\Omega_T)$ such that $((\partial_t^\tau)^{-1}(E_h^j)_{j+1})_{\tau,h}^- \overset{\text{sub}}{\rightharpoonup} \widetilde{E}$. For $v \in C^\infty(\overline{\Omega_T})$ we have

$$[\widetilde{E}, v]_{\Omega_T} \overset{\text{sub}}{\leftharpoonup} [((\partial_t^\tau)^{-1}(E_h^{j+1})_j)_{\tau,h}^-, v]_{\Omega_T}$$

and

$$
\begin{aligned}
[((\partial_t^\tau)^{-1}(E_h^{j+1})_j)_{\tau,h}^-, v]_{\Omega_T} &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} [((\partial_t^\tau)^{-1}(E_h^{j+1})_j)(t_n), v(t)]_\Omega \,\mathrm{d}t \\
&= \sum_{n=0}^{N-1} \tau \sum_{j=0}^{n} \left[E_h^{j+1}, \int_{t_n}^{t_{n+1}} v(t)\mathrm{d}t\right]_\Omega \\
&= \sum_{j=0}^{N-1} \tau \sum_{n=j}^{N-1} \left[E_h^{j+1}, \int_{t_n}^{t_{n+1}} v(t)\mathrm{d}t\right]_\Omega \\
&= \tau \sum_{j=0}^{N-1} \left[E_h^{j+1}, \int_{t_j}^{T} v(t)\mathrm{d}t\right]_\Omega \\
&= \int_0^T \left[(E)_{\tau,h}^+(s), \int_{\lfloor s/\tau \rfloor * \tau}^{T} v(t)\mathrm{d}t\right]_\Omega \mathrm{d}s.
\end{aligned}
$$

By the weak convergence $(E)_{\tau,h}^+(s) \overset{\text{sub}}{\rightharpoonup} E$ and Fubini's theorem (as $E \in L^2(\Omega_T)$, it holds $\partial_t^{-1}E \in L^2(\Omega_T)$) we deduce

$$
\begin{aligned}
\int_0^T \left[(E)_{\tau,h}^+(s), \int_{\lfloor s/\tau \rfloor * \tau}^{T} v(t)\mathrm{d}t\right]_\Omega \mathrm{d}s &\overset{\text{sub}}{\to} \int_0^T \left[E(s), \int_s^T v(t)\mathrm{d}t\right]_\Omega \mathrm{d}s \\
&= \int_0^T \left[\int_0^t E(s)\mathrm{d}s, v(t)\right]_\Omega \mathrm{d}t = [\partial_t^{-1}E, v]_{\Omega_T}.
\end{aligned}
$$

The fact that $C^\infty(\overline{\Omega_T}) \subset L^2(\Omega_T)$ is dense concludes the assertion for E. Similar arguments work for H. □

Theorem 3.30. *There exists a subsequence such that*

$$
\begin{aligned}
f_{\tau,h} &\overset{\text{sub}}{\rightharpoonup} (\nabla \times \partial_t^{-1}H) && \text{in } L^2(\Omega_T), \\
g_{\tau,h} &\overset{\text{sub}}{\rightharpoonup} (\nabla \times \partial_t^{-1}E) && \text{in } L^2(\Omega_T).
\end{aligned}
$$

For sufficiently smooth ξ, *it holds for* $\partial_t\xi_{\tau,h}^+ := (\partial_t^\tau \mathrm{I}_h^{\mathcal{X}}\xi)_{\tau,h}^+ \to \partial_t\xi$ *in* $H(\mathrm{curl}, \Omega_T)$ *and*

$$
\begin{aligned}
\frac{1}{2}\left\langle (\gamma_T((\partial_t^\tau)^{-1}(H_h^{j+1})_j))_{\tau,h}^-, \gamma_T(\partial_t\xi_{\tau,h}^+)\right\rangle_{\Gamma_T} &\overset{\text{sub}}{\to} \left\langle \gamma_T(\partial_t^{-1}H), \gamma_T(\partial_t\xi)\right\rangle_{\Gamma_T} \\
&\quad - \frac{1}{2\mu_0}\left\langle \widetilde{\varphi}, \gamma_T(\partial_t\xi)\right\rangle_{\Gamma_T}, \\
\frac{1}{2}\left\langle (\gamma_T((\partial_t^\tau)^{-1}(E_h^{j+1})_j))_{\tau,h}^-, \gamma_T(\partial_t\xi_{\tau,h}^+)\right\rangle_{\Gamma_T} &\overset{\text{sub}}{\to} \left\langle \gamma_T(\partial_t^{-1}E), \gamma_T(\partial_t\xi)\right\rangle_{\Gamma_T} \\
&\quad + \frac{1}{2}\langle \widetilde{\psi}, \gamma_T(\partial_t\xi)\rangle_{\Gamma_T}.
\end{aligned}
$$

Proof. As $f_{\tau,h}$ is bounded by Lemma 3.26, there exists a weakly convergent subsequence, such that $f_{\tau,h} \overset{\text{sub}}{\rightharpoonup} f$ in $L^2(\Omega_T)$. Now we show that $f = \nabla \times (\partial_t^{-1}H)$. Let $\zeta \in C_0^\infty(\Omega_T)$ and particularly $\gamma_T\zeta = 0$. It holds $\mathrm{I}_h^{\mathcal{X}}\zeta \to \zeta$ in $L^2(\Omega_T)$ (cf. Lemma 3.10). Therefore we have

$$
[f_{\tau,h}, \mathrm{I}_h^{\mathcal{X}}\zeta]_{\Omega_T} \to [f, \zeta]_{\Omega_T}.
$$

Moreover, we have $\mathrm{P}_h^{\mathcal{X}}\mathrm{I}_h^{\mathcal{X}}\zeta = \mathrm{I}_h^{\mathcal{X}}\zeta$ and $\nabla \times \mathrm{I}_h^{\mathcal{X}}\zeta \to \nabla \times \zeta$ in $L^2(\Omega_T)$ (cf. Lemma 3.10), $\gamma_T\mathrm{I}_h^{\mathcal{X}}\zeta = 0$ (cf. [121, Lemma 5.35]) and $((\partial_t^\tau)^{-1}(H_h^{j+1})_j)_{\tau,h}^- \overset{\text{sub}}{\rightharpoonup} \partial_t^{-1}H$ (see Lemma 3.29). This implies

$$
\begin{aligned}
[f_{\tau,h}, \mathrm{I}_h^{\mathcal{X}}\zeta]_{\Omega_T} &= [((\partial_t^\tau)^{-1}(H_h^{j+1})_j)_{\tau,h}^-, \nabla \times \mathrm{I}_h^{\mathcal{X}}\zeta]_{\Omega_T} \\
&\overset{\text{sub}}{\to} [\partial_t^{-1}H, \nabla \times \zeta]_{\Omega_T},
\end{aligned}
$$

which concludes $f = \nabla \times (\partial_t^{-1} H)$.

Now let ξ be sufficiently smooth. We have $((\partial_t^\tau)^{-1}(\varphi_h^{j+1})_j)^-_{\tau,h} \stackrel{\text{sub}}{\rightharpoonup} \widetilde{\varphi}$ in $L^2([0,T],\mathcal{H}_\Gamma)$ (cf. Lemma 3.28) as well as $\gamma_T \mathrm{I}_h^{\mathcal{X}} \xi \to \gamma_T \xi$ in $L^2([0,T],\mathcal{H}_\Gamma)$ (as $\gamma_T : H(\mathrm{curl},\Omega) \to \mathcal{H}_\Gamma$ is continuous) and therefore we obtain with Remark 3.25

$$
\begin{aligned}
&\frac{1}{2}[(\nabla \times (\partial_t^\tau)^{-1}(H_h^{j+1})_j)^-_{\tau,h}, \mathrm{I}_h^{\mathcal{X}}\xi]_{\Omega_T} \\
&\qquad = [f_{\tau,h}, \mathrm{I}_h^{\mathcal{X}}\xi]_{\Omega_T} - \frac{1}{2}[((\partial_t^\tau)^{-1}(H_h^{j+1})_j)^-_{\tau,h}, \nabla \times (\mathrm{P}_h^{\mathcal{X}}\mathrm{I}_h^{\mathcal{X}}\xi)]_{\Omega_T} \\
&\qquad\quad + \frac{1}{2\mu_0}\left\langle ((\partial_t^\tau)^{-1}(\varphi_h^{j+1})_j)^-_{\tau,h}, \gamma_T(\mathrm{P}_h^{\mathcal{X}}\mathrm{I}_h^{\mathcal{X}}\xi)\right\rangle_{\Gamma_T} \\
&\qquad \stackrel{\text{sub}}{\to} [\nabla \times \partial_t^{-1}H, \xi]_{\Omega_T} - \frac{1}{2}[\partial_t^{-1}H, \nabla \times \xi]_{\Omega_T} + \frac{1}{2\mu_0}\langle \widetilde{\varphi}, \gamma_T \xi\rangle_{\Gamma_T}.
\end{aligned}
$$

Furthermore we have

$$
\begin{aligned}
&-\frac{1}{2}\langle (\gamma_T((\partial_t^\tau)^{-1}(H_h^{j+1})_j))^-_{\tau,h}, \gamma_T(\mathrm{I}_h^{\mathcal{X}}\xi)\rangle_{\Gamma_T} \\
&\qquad = \frac{1}{2}[(\nabla \times (\partial_t^\tau)^{-1}(H_h^{j+1})_j)^-_{\tau,h}, \mathrm{I}_h^{\mathcal{X}}\xi]_{\Omega_T} \\
&\qquad\quad - \frac{1}{2}[((\partial_t^\tau)^{-1}(H_h^{j+1})_j)^-_{\tau,h}, \nabla \times \mathrm{I}_h^{\mathcal{X}}\xi]_{\Omega_T} \\
&\qquad \stackrel{\text{sub}}{\to} [\nabla \times \partial_t^{-1}H, \xi]_{\Omega_T} - [\partial_t^{-1}H, \nabla \times \xi]_{\Omega_T} + \frac{1}{2\mu_0}\langle \widetilde{\varphi}, \gamma_T \xi\rangle_{\Gamma_T} \\
&\qquad = -\left\langle \gamma_T(\partial_t^{-1}H), \gamma_T\xi\right\rangle_{\Gamma_T} + \frac{1}{2\mu_0}\langle \widetilde{\varphi}, \gamma_T \xi\rangle_{\Gamma_T}.
\end{aligned}
$$

The statement of the theorem now is shown by replacing ξ with $\partial_t^\tau \xi$ and using that $(\partial_t^\tau \mathrm{I}_h^{\mathcal{X}}\xi)^+_{\tau,h} \to \partial_t \xi$. Similar considerations for $g_{\tau,h}$ and $(\gamma_T((\partial_t^\tau)^{-1}(E_h^{j+1})_j))^-_{\tau,j}$ conclude the assertion. ☐

Remark 3.31. *Even for arbitrarily smooth test functions with non-vanishing boundary values, we are **not** able to show $\widetilde{\varphi} = \mu_0 \gamma_T(\partial_t^{-1}H)$ and therefore also **not***

$$[(\nabla \times (\partial_t^\tau)^{-1}(H_h^{j+1})_j)^-_{\tau,h}, \mathrm{I}_h^{\mathcal{X}}\xi]_{\Omega_T} \stackrel{\text{sub}}{\to} [\nabla \times \partial_t^{-1}H, \xi]_{\Omega_T}$$

*and **not***

$$\left\langle (\gamma_T((\partial_t^\tau)^{-1}(H_h^{j+1})_j))^-_{\tau,h}, \gamma_T(\mathrm{I}_h^{\mathcal{X}}\xi)\right\rangle_{\Gamma_T} \stackrel{\text{sub}}{\to} \left\langle \gamma_T(\partial_t^{-1}H), \gamma_T\xi\right\rangle_{\Gamma_T}.$$

But we will see, that we have convergence to a solution in the sense of Definition 3.3, thus E and H solve the MLLG equations in the interior and their boundary values are suitable exterior data. The projection of $\widetilde{\varphi}, \widetilde{\psi}$ on suitable exterior data gives $\mu_0\gamma_T H, \gamma_T E$. The equivalence between the solutions from Theorem 3.8 shows that this is a reasonable notion of solution.

3.3.3. Convergence towards the exact solution

In this section we show that the accumulation points of the previously constructed sequences indeed are solutions of the MLLG system.

Theorem 3.32. *Let $(m_{\tau,h}, E_{\tau,h}, H_{\tau,h}, \varphi_{\tau,h}, \psi_{\tau,h})$ be the approximations obtained by Algorithm 3.12 and assume that $\theta \in (1/2, 1]$ and the validity of Assumption 3.14. Then there exists for any $(\tau, h) \to 0$ a subsequence of $(m_{\tau,h}, E_{\tau,h}, H_{\tau,h}, \varphi_{\tau,h}, \psi_{\tau,h})$, such that*

$$(m_{\tau,h}, E_{\tau,h}, H_{\tau,h}, ((\partial_t^\tau)^{-1}\varphi_h)_{\tau,h}, ((\partial_t^\tau)^{-1}\psi_h)_{\tau,h})$$

converges weakly in

$$H^1(\Omega_T) \times L^2(\Omega_T)^2 \times L^2([0,T], \mathcal{H}_\Gamma)^2$$

to a weak solution of the MLLG system in the sense of Definition 3.3.

Proof. We have to show that weak limit functions from Lemma 3.28 are weak solutions in the sense of Definition 3.3. We choose test functions

$$\rho \in C^\infty(\overline{\Omega_T}), \quad \zeta_H, \zeta_E \in C^\infty(\overline{\Omega_T})$$

with $\zeta_H(T) = \zeta_E(T) = 0$ and

$$v, w \in \gamma_T(C^\infty(\overline{\Omega_T}))$$

with $v(T) = \partial_t v(T) = \cdots = \partial_t^m v(T) = 0 = w(T) = \cdots = \partial_t^m w(T)$. As discrete test functions we take

$$\rho_h(t,\cdot) := \mathrm{I}_h^{\mathcal{S}}(m_{\tau,h}^- \times \rho),$$

$$\zeta_{E,h}(t,\cdot) := \mathrm{I}_h^{\mathcal{X}} \zeta_E(t,\cdot), \qquad \zeta_{H,h}(t,\cdot) := \mathrm{I}_h^{\mathcal{X}} \zeta_H(t,\cdot),$$

and

$$v_h(t,\cdot) := \gamma_T(\mathrm{I}^{\mathcal{X}}\widehat{v})(t,\cdot) \text{ and } w_h(t,\cdot) := \gamma_T(\mathrm{I}^{\mathcal{X}}\widehat{w})(t,\cdot).$$

Here $\widehat{v}, \widehat{w} \in C^\infty(\overline{\Omega_T})$ with $\gamma_T \widehat{v} = v$ and $\gamma_T \widehat{w} = w$. The proof that the limit function m from Lemma 3.28 satisfies the LLG equation can be found in [25, Proof of Theorem 5.2] or [14]. We briefly sketch that the approximations converge to a weak solution, that it holds $m(0) = m_0$ in the sense of traces and that $|m| = 1$ is fulfilled almost everywhere. Equation (3.16) of Algorithm 3.12 implies

$$[\alpha w_{\tau,h}^-, \rho_h]_{\Omega_T} + [m_{\tau,h}^- \times w_{\tau,h}^-, \rho_h]_{\Omega_T} = -C_e[\nabla(m_{\tau,h}^- + \theta\tau v_{\tau,h}^-), \nabla\rho_h]_{\Omega_T} + [H_{\tau,h}^-, \rho_h]_{\Omega_T}.$$

Using the approximation properties of the nodal interpolation, the strong $L^2(\Omega_T)$ convergence of $m_{\tau,h}^- \times \rho$ to $m \times \rho$, the uniform bound $\|\sqrt{k} w_{\tau,h}^-\|_{\Omega_T}^2 \leq C$ from Lemma 3.20 together with the weak convergence properties from Lemma 3.28, we conclude

$$[\alpha\partial_t m + m \times \partial_t m, m \times \rho]_{\Omega_T} = -C_e[\nabla m, \nabla(m \times \rho)]_{\Omega_T} + [H, m \times \rho]_{\Omega_T}.$$

Now suitable vector identities for the scalar and cross product together with $|m = 1$ and $\partial_t m \cdot m = 0$ (this is shown below in an independent way) conclude that we obtain the LLG equation in Definition 3.3. The equality $m(0) = m^0$ follows from the weak convergence in $H^1(\Omega_T)$ and Assumption 3.14. The triangle inequality

$$\||m| - 1\|_{\Omega_T} \leq \||m| - |m_{\tau,h}^-|\|_{\Omega_T} + \||m_{\tau,h}^-| - 1\|_{\Omega_T}$$

together with

$$\||m_{\tau,h}^-(t)| - 1\|_\Omega \leq h \max_{j=0,\ldots,N} \|\nabla m_h^j\|$$

gives $|m| = 1$ and $\partial_t m \cdot m = 0$ almost everywhere.

We now consider the Maxwell equations in the interior, where we present the arguments only for one of the equations, while the second one can be treated analogously. For simplicity we write ζ instead of ζ_H. By testing with $\zeta_h(t_k)$ and summing up from $k = 0.\ldots, N-1$, Algorithm 3.12 gives

$$\begin{aligned}\mu[(\partial_t^\tau H)_{\tau,h}^+, \zeta_{\tau,h}^-]_{\Omega_T} = &-\frac{1}{2}[\nabla \times E_{\tau,h}^+, \zeta_{\tau,h}^-]_{\Omega_T} - \frac{1}{2}[E_{\tau,h}^+, \nabla \times \zeta_{\tau,h}^-]_{\Omega_T} \\ &- \frac{1}{2\mu}\langle \psi_{\tau,h}^+, \gamma_T \zeta_{\tau,h}^- \rangle_{\Gamma_T} - \mu[w_{\tau,h}^-, \zeta_{\tau,h}^-]_{\Omega_T}.\end{aligned}$$

We consider each of the terms separately. By discrete integration by parts (see Lemma A.3 in the Appendix) and $\zeta(T,\cdot)=0$ we obtain

$$\begin{aligned}\mu[(\partial_t^\tau H)^+_{\tau,h},\zeta^-_{\tau,h}]_{\Omega_T} &= -\mu[H^+_{\tau,h},(\partial_t^\tau\zeta)^+_{\tau,h}]_{\Omega_T}-\mu[H_h^0,\zeta_h(0)]_\Omega\\ &\overset{\text{sub}}{\to} -\mu[H,\partial_t\zeta]_{\Omega_T}-\mu[H^0,\zeta(0)]_\Omega,\end{aligned}$$

where we used the weak convergence of $H_h^0 \rightharpoonup H^0$ (cf. Assumption 3.14), $H^+_{\tau,h} \overset{\text{sub}}{\rightharpoonup} H$ (cf. Theorem 3.28), $\zeta_h(0)\to\zeta(0)$ in $L^2(\Omega)$ and $(\partial_t^\tau\zeta)^+_{\tau,h}\to\partial_t\zeta$ in $L^2(\Omega_T)$, as ζ is smooth. We have by $\zeta(T)=0$, discrete integration by parts and Theorem 3.30 that

$$\begin{aligned}-\frac{1}{2}[\nabla\times E^+_{\tau,h},\zeta^-_{\tau,h}]_{\Omega_T}-\frac{1}{2}[E^+_{\tau,h},\nabla\times\zeta^-_{\tau,h}]_{\Omega_T}-\frac{1}{2\mu_0}\langle\psi^+_{\tau,h},\gamma_T\zeta^-_{\tau,h}\rangle_{\Gamma_T}&\\ =\frac{1}{2}[(\nabla\times(\partial_t^\tau)^{-1}(E_h^{j+1})_j)^-_{\tau,h},(\partial_t^\tau\zeta_h)^+_{\tau,h}]_{\Omega_T}&\\ +\frac{1}{2}[((\partial_t^\tau)^{-1}(E_h^{j+1})_j)^-_{\tau,h},\nabla\times(\partial_t^\tau\zeta_h)^+_{\tau,h}]_{\Omega_T}&\\ +\frac{1}{2\mu_0}\langle((\partial_t^\tau)^{-1}(\psi_h^{j+1})_j)^-_{\tau,h},(\gamma_T\partial_t^\tau\zeta_h)^+_{\tau,h}\rangle_{\Gamma_T}&\\ =g_{\tau,h}((\partial_t^\tau\zeta_h)^+_{\tau,h})\overset{\text{sub}}{\to}[\nabla\times\partial_t^{-1}E,\partial_t\zeta]_{\Omega_T}.&\end{aligned}$$

The remaining term is a straightforward application of Lemma 3.28

$$-\mu[w^-_{\tau,h},\zeta^-_{\tau,h}]_{\Omega_T}\overset{\text{sub}}{\to}-\mu[w,\zeta]_{\Omega_T}.$$

This shows (together with similar arguments for the second Maxwell equation) that the interior equations in (3.2) are satisfied.

For the boundary equation in Definition 3.3, Algorithm 3.12 gives by testing with $\tau v_h(t_{k+1}),\tau w_h(t_{k+1})$ and summation from $k=0$ to $k=N-1$

$$\left\langle\begin{pmatrix}v^+_{\tau,h}\\ w^+_{\tau,h}\end{pmatrix},\left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h\\ \psi_h\end{pmatrix}\right)^+_{\tau,h}\right\rangle_{\Gamma_T}=\frac{1}{2}\left(\langle v^+_{\tau,h},\mu_0^{-1}\gamma_T E^+_{\tau,h}\rangle_{\Gamma_T}+\langle w^+_{\tau,h},\gamma_T H^+_{\tau,h}\rangle_{\Gamma_T}\right). \tag{3.31}$$

With discrete integration by parts like above, we see with Theorem 3.30 that

$$\langle v^+_{\tau,h},\gamma_T E^+_{\tau,h}\rangle_{\Gamma_T}\overset{\text{sub}}{\to}-\langle\partial_t v,2\gamma_T\partial_t^{-1}E+\widetilde{\psi}\rangle_{\Gamma_T}$$

and

$$\langle w^+_{\tau,h},\mu_0\gamma_T H^+_{\tau,h}\rangle_{\Gamma_T}\overset{\text{sub}}{\to}-\langle\partial_t w,2\mu_0\gamma_T\partial_t^{-1}H-\widetilde{\varphi}\rangle_{\Gamma_T}.$$

We now consider the term on the left hand side of (3.31). The strategy is to apply the adjoint operator of $B(\partial_t^\tau)$ to the test functions via integration by parts. By setting

$v_h^j := v_h(t_j)$, the $\langle\cdot,\cdot\rangle_\Gamma$-adjoint B^* of B, $\overline{v}_h^j := v_h^{N-j}$ and by using $\psi_h^0 = \varphi_h^0 = 0$ we have

$$\begin{aligned}
\mathrm{LHS}_h^\tau &:= \left\langle \begin{pmatrix} v_{\tau,h}^+ \\ w_{\tau,h}^+ \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix}\right)_{\tau,h}^+ \right\rangle_{\Gamma_T} \\
&= \tau \sum_{j=1}^N \left\langle \begin{pmatrix} v_h^j \\ w_h^j \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix}\right)(t_j) \right\rangle_\Gamma \\
&= \tau \sum_{j=1}^N \left\langle \begin{pmatrix} v_h^j \\ w_h^j \end{pmatrix}, \sum_{k=0}^j B_{j-k}\begin{pmatrix} \varphi_h^k \\ \psi_h^k \end{pmatrix} \right\rangle_\Gamma \\
&= \tau \sum_{k=0}^N \left\langle \sum_{j=\max(1,k)}^N B_{j-k}^* \begin{pmatrix} v_h^j \\ w_h^j \end{pmatrix}, \begin{pmatrix} \varphi_h^k \\ \psi_h^k \end{pmatrix} \right\rangle_\Gamma \\
&= \tau \sum_{k=0}^N \left\langle \sum_{j=0}^{N-\max(1,k)} B_{N-j-k}^* \begin{pmatrix} v_h^{N-j} \\ w_h^{N-j} \end{pmatrix}, \begin{pmatrix} \varphi_h^k \\ \psi_h^k \end{pmatrix} \right\rangle_\Gamma \\
&= \tau \sum_{k=0}^N \left\langle \sum_{j=0}^{N-\max(1,k)} B_{N-j-k}^* \begin{pmatrix} \overline{v}_h^j \\ \overline{w}_h^j \end{pmatrix}, \begin{pmatrix} \varphi_h^k \\ \psi_h^k \end{pmatrix} \right\rangle_\Gamma \\
&= \tau \sum_{k=1}^N \left\langle B^*(\partial_t^\tau)\begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix}(T-t_k), \begin{pmatrix} \varphi_h^k \\ \psi_h^k \end{pmatrix} \right\rangle_\Gamma .
\end{aligned}$$

Now we integrate by parts and obtain by using $v_h(T) = w_h(T) = 0$ (and the convention $B^*(\partial_t^\tau)\phi(t) = 0$ for $t < 0$)

$$\begin{aligned}
\mathrm{LHS}_h^\tau &= \sum_{k=1}^N \left\langle B^*(\partial_t^\tau)\begin{pmatrix} \overline{v}_h \\ \overline{w}_h \end{pmatrix}(T-t_k), (\partial_t^\tau)^{-1}\begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix}(t_k) - (\partial_t^\tau)^{-1}\begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix}(t_{k-1}) \right\rangle_\Gamma \\
&= \sum_{k=1}^N \left\langle B^*(\partial_t^\tau)\begin{pmatrix} \overline{v}_h \\ \overline{w}_h \end{pmatrix}(T-t_k) - B^*(\partial_t^\tau)\begin{pmatrix} \overline{v}_h \\ \overline{w}_h \end{pmatrix}(T-t_{k+1}), (\partial_t^\tau)^{-1}\begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix}(t_k) \right\rangle_\Gamma \\
&= \tau \sum_{k=1}^N \left\langle B^*(\partial_t^\tau)\begin{pmatrix} \partial_t^\tau \overline{v}_h \\ \partial_t^\tau \overline{w}_h \end{pmatrix}(T-t_k), (\partial_t^\tau)^{-1}\begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix}(t_k) \right\rangle_\Gamma \\
&= \tau \sum_{k=1}^N \left\langle (B^*(\partial_t^\tau)\partial_t^\tau)\begin{pmatrix} \overline{v}_h \\ \overline{w}_h \end{pmatrix}(T-t_k), (\partial_t^\tau)^{-1}\begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix}(t_k) \right\rangle_\Gamma .
\end{aligned}$$

Here we additionally used

$$(B^*(\partial_t^\tau)\partial_t^\tau)(\phi^j)(t_k) := \big(B^*(s)s\big)(\partial_t^\tau)(\phi^j)(t_k) = (B^*(\partial_t^\tau))(\partial_t^\tau\phi^j)(t_k).$$

In this situation, we are able to apply the weak convergence result Lemma 3.28 for the approximations and Convolution Quadrature convergence results of [116] for the smooth test functions. We apply a operator valued version of the CQ convergence result [115, Theorem 3.2], as done e.g. in [99] and [27]. Due to $\|B^*(s)s\|_{L(\mathcal{H}_\Gamma)} \leq Cs^3$ for $\Re s \geq \sigma > 0$ and $\overline{v}(0) = v(T) = 0$, $\partial_t\overline{v}(0) = -\partial_t v(T) = 0, \ldots, \partial_t^m\overline{v}(0) = 0$ and similarly $\overline{w}(0) = \cdots = \partial_t^m\overline{w}(0) = 0$ we have

$$(B^*(\partial_t^\tau)\partial_t^\tau)\begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix}(T-t_k) \to (B^*(\partial_t)\partial_t)\begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix}(T-t_k)$$

uniformly in $0 \leq t_k \leq T$, $t_k = \tau k$, $k \geq 1$. By the pointwise convergence and the boundedness of the first derivative of $B^*(\partial_t)\partial_t\binom{\overline{v}}{\overline{w}}(T-\cdot)$, dominated convergence

$$B^*(\partial_t^\tau)\partial_t^\tau\begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix}(T-\cdot)^+ \to B^*(\partial_t)\partial_t\begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix}(T-\cdot) \text{ in } L^2([0,T],\mathcal{H}_\Gamma).$$

Moreover the discrete Herglotz theorem (Theorem A.4) shows

$$\tau \sum_{k=1}^{N} \left\| (B^*(\partial_t^\tau)\partial_t^\tau) \begin{pmatrix} \overline{v}_h \\ \overline{w}_h \end{pmatrix} (T - t_k) - (B^*(\partial_t^\tau)\partial_t^\tau) \begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix} (T - t_k) \right\|_{\mathcal{H}_\Gamma}^2$$
$$\leq C\tau \sum_{k=1}^{N} \|(\partial_t^\tau)^3(\overline{v}_h - \overline{v})(T - t_k)\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^3(\overline{w}_h - \overline{w})(T - t_k)\|_{\mathcal{H}_\Gamma}^2 \to 0$$

for $(\tau, h) \to 0$. So, we obtain

$$\mathrm{LHS}_h^\tau \overset{\text{sub}}{\to} \left\langle (B^*(\partial_t)\partial_t) \begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix} (T - \cdot), \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} \right\rangle_{\Gamma_T} =: \mathrm{LHS}.$$

Now, in the continuous expression, we reverse the integration by parts and we obtain

$$\begin{aligned}
\mathrm{LHS} &= \int_0^T \left\langle \mathcal{L}^{-1}(B^*(r)r^{-m-1})(s) * \begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix} (T - s), \begin{pmatrix} \widetilde{\varphi}(s) \\ \widetilde{\psi}(s) \end{pmatrix} \right\rangle_\Gamma \mathrm{d}s \\
&= \int_0^T \left\langle \partial_x^{m+1} \int_0^x \mathcal{L}^{-1}(B^*(r)r^{-m})(s) \begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix} (x - s)\, \mathrm{d}s \Big|_{x=T-t}, \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} (t) \right\rangle_\Gamma \mathrm{d}t \\
&= \int_0^T \left\langle \partial_x^{m} \int_0^x B_m^*(s)\partial_x \begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix} (x - s)\, \mathrm{d}s \Big|_{x=T-t}, \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} (t) \right\rangle_\Gamma \mathrm{d}t \\
&= (-1)^{m+1} \int_0^T \left\langle \int_0^{T-t} B_m^*(s) \left(\partial_t^{m+1} \begin{pmatrix} v \\ w \end{pmatrix} \right) (T - (T - t - s))\, \mathrm{d}s, \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} (t) \right\rangle_\Gamma \mathrm{d}t \\
&= (-1)^{m+1} \int_0^T \left\langle \int_t^T B_m^*(s - t)\partial_t^{m+1} \begin{pmatrix} v \\ w \end{pmatrix} (s)\, \mathrm{d}s, \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} (t) \right\rangle_\Gamma \mathrm{d}t \\
&= (-1)^{m+1} \int_0^T \left\langle \partial_t^{m+1} \begin{pmatrix} v \\ w \end{pmatrix} (s), \int_0^s B_m(s - t) \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} (t)\, \mathrm{d}t \right\rangle_\Gamma \mathrm{d}s \\
&= (-1)^{m+1} \int_0^T \left\langle \partial_t^{m+1} \begin{pmatrix} v \\ w \end{pmatrix} (s), \left(B_m * \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} \right) (s) \right\rangle_\Gamma \mathrm{d}s.
\end{aligned}$$

This is exactly the term that shows up in the formulation of our weak solution in Definition 3.3. □

Theorem 3.33. *The solutions of Theorem 3.32 have bounded energy in the sense of Definition 3.4, i.e. for almost all $t \in [0, T]$*

$$\|\nabla m(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\partial_t m(s)\|_{L^2(\Omega)}^2\, \mathrm{d}s + \|H(t)\|_{L^2(\Omega)}^2 + \|E(t)\|_{L^2(\Omega)}^2 \leq C,$$

where $C > 0$ is independent of t.

Proof. The proof proceeds analogously as in [25]. From the discrete energy estimates in Lemma 3.20, we get for any $t' \in [0, T]$

$$\|\nabla m_{\tau,h}(t')\|_\Omega^2 + \int_0^{t'} \|v_{\tau,h}^-(s)\|_\Omega^2\, \mathrm{d}s + \|E_{\tau,h}(t')\|_\Omega^2 + \|H_{\tau,h}(t')\|_\Omega^2 \leq C,$$

where C only depends polynomially on T. Integration in time yields for any measurable set $\mathcal{A} \subset [0, T]$

$$\int_{\mathcal{A}} \|\nabla m_{\tau,h}(t')\|_\Omega^2 + \int_{\mathcal{A}} \|v_{\tau,h}^-\|_{\Omega_{t'}}^2 + \int_{\mathcal{A}} \|E_{\tau,h}(t')\|_\Omega^2 + \int_{\mathcal{A}} \|H_{\tau,h}(t')\|_\Omega^2 \leq \int_{\mathcal{A}} C,$$

whence weak lower semi-continuity leads to

$$\int_{\mathcal{A}} \|\nabla m(t')\|_{\Omega}^2 + \int_{\mathcal{A}} \|\partial_t m\|_{\Omega_{t'}}^2 + \int_{\mathcal{A}} \|E(t')\|_{\Omega}^2 + \int_{\mathcal{A}} \|H(t')\|_{\Omega}^2 \leq \int_{\mathcal{A}} C.$$

The desired result now follows from standard measure theory; see, e.g. [60, IV,Theorem 4.4]. □

4. Weak Convergence for the Pure Maxwell System

In this chapter we consider the weak convergence for the pure Maxwell system without the coupling to the LLG equation. In Section 4.1, we recall the strong boundary integral solution from Chapter 2 (without the LLG part), introduce a suitable weak solution and show equivalence and uniqueness of these solutions. In comparison to the coupled setting in Chapter 3, we introduce solutions with higher regularity and uniqueness and equivalence to the solution from Chapter 2 hold without any restriction. In Section 4.2, we propose a non-symmetric approximation scheme which seems to have advantageous properties concerning the boundedness and weak convergence of the approximations. We exploit this (together with the uniqueness of the pure Maxwell solution) in Section 4.3, where we show that the sequence of approximations converges weakly towards the exact solution, and not only subsequences.

4.1. Weak Solutions, Equivalence and Uniqueness

In this section we introduce a strong and a weak solution for the boundary integral Maxwell system and show equivalence and uniqueness of these solutions.

4.1.1. Definition of a strong solution

For the convenience of the reader, we recall the pure Maxwell system (without the coupling to the LLG equation) from (2.12), immediately written as coupled boundary integral system.

Find the functions E and $H : [0,T] \times \Omega \to \mathbb{R}^3$ which satisfy the following coupled system: in the interior domain

$$\varepsilon \partial_t E - \nabla \times H = -(J + \sigma E) \qquad \text{in } \Omega_T, \tag{4.1a}$$

$$\mu \partial_t H + \nabla \times E = 0 \qquad \text{in } \Omega_T, \tag{4.1b}$$

coupled to the boundary integral equation

$$B(\partial_t) \begin{pmatrix} \mu_0 \gamma_T H \\ -\gamma_T E \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mu_0^{-1} \gamma_T E \\ \gamma_T H \end{pmatrix} \qquad \text{on } [0,T] \times \partial\Omega \tag{4.1c}$$

and with the initial conditions

$$E(0) = E^0, \quad H(0) = H^0 \quad \text{in } \Omega. \tag{4.1d}$$

As in Section 2.4.2, with fixed $m \in \mathbb{N}$, $m > 3$, this leads to the following definition of a solution of the boundary integral Maxwell system (4.1).

Definition 4.1. *A pair of functions*

$$(E, H) \in H(\partial_t, \operatorname{curl}, \Omega_T) \times H(\partial_t, \operatorname{curl}, \Omega_T)$$

is called solution of Maxwell's equations, if and only if

$$\begin{aligned} \varepsilon \partial_t E - \nabla \times H + \sigma E = -J \qquad &\textit{in } L^2(\Omega_T), \\ \mu \partial_t H + \nabla \times E = 0 \qquad &\textit{in } L^2(\Omega_T) \end{aligned}$$

as well as $B_m * \binom{\mu_0\gamma_T H}{-\gamma_T E} \in H^m_{0,*}([0,T],\mathcal{H}_\Gamma)$ *with*

$$\partial_t^m \left(B_m * \begin{pmatrix} \mu_0\gamma_T H \\ -\gamma_T E \end{pmatrix} \right) = \frac{1}{2\mu_0} \begin{pmatrix} \gamma_T E \\ \mu_0\gamma_T H \end{pmatrix} \qquad \text{in } L^2([0,T],\mathcal{H}_\Gamma) \tag{4.2}$$

and $H(0) = H^0$, $E(0) = E^0$ *in* $L^2(\Omega)$ *in the sense of traces.*

Note that this is the same definition of a solution as in Definition 2.33 (just without the LLG part). In comparison to Definition 3.1 for the LLG-case, we do not have to weaken the regularity assumptions on the solution (compare Theorem 3.2).

To show existence of and convergence to a solution, we will derive an equivalent weak form.

4.1.2. Definition of a weak solution

In this section we introduce the definition of a weak solution of the Maxwell system. We recall the definition of the inner products for suitable functions φ, ψ, v, w

$$\langle \varphi, \psi \rangle_{\Gamma_T} = \int_0^T \langle \varphi, \psi \rangle_\Gamma \, \mathrm{d}t$$

and

$$[v, w]_{\Omega_T} = \int_0^T \int_\Omega v \cdot w \, \mathrm{d}x \, \mathrm{d}t.$$

We multiply the system (4.1) with test functions ζ_E, ζ_H, v, $-\gamma_T\zeta_E$, integrate over the respective space time cylinders and apply the following transformations: With φ for the tangential trace of H,

$$\mu_0\gamma_T H = \varphi,$$

integration by parts shows

$$\begin{aligned} [\nabla \times H, \zeta]_{\Omega_T} &= [H, \nabla \times \zeta]_{\Omega_T} - \langle \gamma_T H, \gamma_T \zeta \rangle_{\Gamma_T}, \\ &= [H, \nabla \times \zeta]_{\Omega_T} - \langle \mu_0^{-1}\varphi, \gamma_T \zeta \rangle_{\Gamma_T}. \end{aligned} \tag{4.3}$$

For $w \in C^m([0,T],\mathcal{H}^2_\Gamma)$ with $w(T) = \partial_t w(T) = \cdots = \partial_t^{m-1} w(T) = 0$, we integrate by parts m times to obtain

$$\begin{aligned} \langle w, \partial_t^m (B_m * v) \rangle_{\Gamma_T} &= -\left\langle \partial_t w, \partial_t^{m-1}(B_m * v) \right\rangle_{\Gamma_T} + \left[\left\langle w, \partial_t^{m-1}(B_m * v) \right\rangle_\Gamma \right]_0^T \\ &= -\left\langle \partial_t w, \partial_t^{m-1}(B_m * v) \right\rangle_{\Gamma_T} \\ &= \cdots = (-1)^m \langle \partial_t^m w, (B_m * v) \rangle_{\Gamma_T}. \end{aligned} \tag{4.4}$$

Adding up all the resulting equations, we obtain the following system.

Definition 4.2. *A triple of functions*

$$(E, H, \varphi) \in H(\mathrm{curl}, \Omega_T) \times H(\mathrm{curl}, \Omega_T) \times L^2([0,T],\mathcal{H}_\Gamma)$$

is called a weak solution of the Maxwell equations if all $\zeta_E, \zeta_H \in C^\infty(\overline{\Omega_T})$ *with* $\gamma_T\zeta_E \in H^m_{*,0}([0,T],\mathcal{H}_\Gamma)$ *and all* $v \in \gamma_T(C^\infty(\overline{\Omega_T})) \cap H^m_{*,0}([0,T],\mathcal{H}_\Gamma)$ *satisfy*

$$\begin{aligned} &[\varepsilon\partial_t E, \zeta_E]_{\Omega_T} + [\mu\partial_t H, \zeta_H]_{\Omega_T} + (-1)^m \left\langle \begin{pmatrix} \partial_t^m v \\ -\gamma_T(\partial_t^m \zeta_E) \end{pmatrix}, B_m * \begin{pmatrix} \varphi \\ -\gamma_T E \end{pmatrix} \right\rangle_{\Gamma_T} \\ &\quad = [H, \nabla \times \zeta_E]_{\Omega_T} - \frac{1}{2\mu_0} \langle \varphi, \gamma_T \zeta_E \rangle_{\Gamma_T} - [\sigma E + J, \zeta_E]_{\Omega_T} \\ &\qquad - [\nabla \times E, \zeta_H]_{\Omega_T} + \frac{1}{2\mu_0} \langle v, \gamma_T E \rangle_{\Gamma_T}. \end{aligned}$$

Moreover, we require $H(0) = H^0$ *and* $E(0) = E^0$ *in* $L^2(\Omega)$ *in the sense of traces.*

4.1.3. Equivalence of the solutions

In this section we show equivalence of the weak solution from Definition 4.2 to the strong solution from Definition 4.1.

Theorem 4.3. *If (E, H) is a solution in the sense of Definition 4.1, then $(E, H, \mu_0\gamma_T H)$ is a solution in the sense of Definition 4.2.*

Conversely, if (E, H, φ) is a solution in the sense of Definition 4.2, then (E, H) is a solution in the sense of Definition 4.1.

Proof. The proof is similar to the proof of Lemma 3.6, with some deviations due to the different coupling to the boundary integral equation in the definition of the weak solution. Furthermore the formulations in this section are integrated once in time in comparison to Section 3.1.1. For the convenience of the reader, we sketch the most important arguments and refer to Lemma 3.6 for a more detailed version.

Step 1: We multiply the system of Definition 4.1 with the respective test functions of Definition 4.2 and introduce the variable $\varphi = \mu_0\gamma_T H$ for the tangential trace of H. Then, integration by parts for the interior as in (4.3), together with the identities (4.4) for the boundary integral equation, and adding up the equations gives a solution in the sense of Definition 4.2.

Step 2: Now let (E, H, φ) be a solution in the sense of Definition 4.2. We choose arbitrary $v, w \in C^\infty(\Gamma_T) \cap H^m_{*,0}([0, T], \mathcal{H}_\Gamma)$ and $\zeta_E, \zeta_H \in C^\infty(\overline{\Omega_T})$ with $-\gamma_T\zeta_E = w$. By testing with these functions and integration by parts, we obtain

$$\begin{aligned}
&[\varepsilon\partial_t E, \zeta_E]_{\Omega_T} + [\mu\partial_t H, \zeta_H]_{\Omega_T} + (-1)^m \left\langle \begin{pmatrix}\partial_t^m v\\ \partial_t^m w\end{pmatrix}, B_m * \begin{pmatrix}\varphi\\ -\gamma_T E\end{pmatrix}\right\rangle_{\Gamma_T}\\
&\quad= [\nabla\times H, \zeta_E]_{\Omega_T} + \langle\gamma_T H, -w\rangle_{\Gamma_T} + \frac{1}{2\mu_0}\langle\varphi, w\rangle_{\Gamma_T} - [\sigma E + J, \zeta_E]_{\Omega_T}\\
&\qquad - [\nabla\times E, \zeta_H]_{\Omega_T} + \frac{1}{2\mu_0}\langle v, \gamma_T E\rangle_{\Gamma_T}.
\end{aligned} \tag{4.5}$$

By the use of cut-off functions, we let $\|\zeta_E\|_{\Omega_T}, \|\zeta_H\|_{\Omega_T} \to 0$, while $-\gamma_T\zeta_E = w$ is fixed, which gives

$$\begin{aligned}
&(-1)^m \left\langle \begin{pmatrix}\partial_t^m v\\ \partial_t^m w\end{pmatrix}, B_m * \begin{pmatrix}\varphi\\ -\gamma_T E\end{pmatrix}\right\rangle_{\Gamma_T}\\
&\quad= -\langle\gamma_T H, w\rangle_{\Gamma_T} + \frac{1}{2\mu_0}\langle\varphi, w\rangle_{\Gamma_T} + \frac{1}{2\mu_0}\langle v, \gamma_T E\rangle_{\Gamma_T}.
\end{aligned} \tag{4.6}$$

By subtracting this equation from (4.5), we obtain

$$\begin{aligned}
&[\varepsilon\partial_t E, \zeta_E]_{\Omega_T} + [\mu\partial_t H, \zeta_H]_{\Omega_T}\\
&\quad= [\nabla\times H, \zeta_E]_{\Omega_T} - [\sigma E + J, \zeta_E]_{\Omega_T} - [\nabla\times E, \zeta_H]_{\Omega_T},
\end{aligned}$$

i.e. the Maxwell equations in Ω_T are fulfilled (by density of $C^\infty(\overline{\Omega_T}) \subset L^2(\Omega_T)$). Coming back to (4.6), using integration by parts, the vanishing end conditions of the test functions, and the density of $C^\infty(\Gamma_T) \subset L^2([0, T], \mathcal{H}_\Gamma)$, we deduce

$$\left(\frac{1}{2\mu_0}\begin{pmatrix}0 & -\partial_t^{-m}\\ \partial_t^{-m} & 0\end{pmatrix} + B_m *\right)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \frac{1}{\mu_0}\partial_t^{-m}\begin{pmatrix}\gamma_T E\\ \mu_0\gamma_T H\end{pmatrix}. \tag{4.7}$$

Employing the fact that the operator

$$Q(\partial_t) := \left(\frac{1}{2\mu_0}\begin{pmatrix}0 & -\partial_t^{-m}\\ \partial_t^{-m} & 0\end{pmatrix} + B_m *\right)$$

is a modification of a projection, as in the proof of Theorem (3.6), we obtain

$$B_m * \begin{pmatrix} \mu_0 \gamma_T H \\ -\gamma_T E \end{pmatrix} = \frac{1}{2\mu_0} \partial_t^{-m} \begin{pmatrix} \gamma_T E \\ \mu_0 \gamma_T H \end{pmatrix}. \tag{4.8}$$

By $\gamma_T E, \mu_0 \gamma_T H \in L^2([0,T], \mathcal{H}_\Gamma)$ on the right hand side, we have

$$B_m * \begin{pmatrix} \mu_0 \gamma_T H \\ -\gamma_T E \end{pmatrix} \in H^m_{0,*}([0,T], \mathcal{H}_\Gamma).$$

Therefore another m derivatives in time exist on both sides of (4.8) and we obtain a solution in the sense of Definition 4.1. □

Remark 4.4. *Theorem 4.3 shows that we have equivalence of the Definitions 4.1 and 4.2, the only difference is that if we have a solution in the sense of Definition 4.2, we are not yet able to show $\varphi = \mu_0 \gamma_T H$. In the subsequent section, we will show the uniqueness of the solution of Definition 4.2, thus every weak solution in the sense of Definition 4.2 fulfills*

$$\varphi = \mu_0 \gamma_T H$$

and both solutions coincide and are unique. This is different in the LLG setting due to the symmetric formulation with two boundary variables in Definition 3.3 and due to the non-uniqueness of the LLG part.

4.1.4. Uniqueness of the solutions

In this section we show that both solutions of the Maxwell system are unique.

Theorem 4.5. *A solution in the sense of Definition 4.2 is unique, i.e. if there exist two solutions (E_1, H_1, φ_1) and (E_2, H_2, φ_2) in the sense of Definition 4.2, then it holds*

$$(E_1, H_1, \varphi_1) = (E_2, H_2, \varphi_2).$$

Proof. Again, the proof shares some similarities with the uniqueness proof for the Maxwell part of the Maxwell–LLG system from Theorem 3.8 and we therefore only sketch it. Assume, that there exist two solutions in the sense of Definition 4.2. The difference $U := E^1 - E^2$, $V := H^1 - H^2$, $\psi := \varphi^1 - \varphi^2$ satisfies

$$(U, V, \psi) \in H(\mathrm{curl}, \Omega_T) \times H(\mathrm{curl}, \Omega_T) \times L^2([0,T], \mathcal{H}_\Gamma)$$

and for all

$$\zeta_E, \zeta_H \in C^\infty(\overline{\Omega_T}) \text{ with } \gamma_T \zeta_E(T) = \cdots = \partial_t^{m-1} \gamma_T \zeta_E(T) = 0$$

and all

$$v \in \gamma_T(C^\infty(\overline{\Omega_T})) \text{ with } v(T) = \partial_t v(T) = \cdots = \partial_t^{m-1} v(T) = 0$$

it holds

$$[\varepsilon \partial_t U, \zeta_E]_{\Omega_T} + [\mu \partial_t V, \zeta_H]_{\Omega_T} + (-1)^m \left\langle \begin{pmatrix} \partial_t^m v \\ -\gamma_T(\partial_t^m \zeta_E) \end{pmatrix}, B_m * \begin{pmatrix} \psi \\ -\gamma_T U \end{pmatrix} \right\rangle_{\Gamma_T}$$
$$= [U, \nabla \times \zeta_E]_{\Omega_T} - \frac{1}{2\mu_0} \langle \psi, \gamma_T \zeta_E \rangle_{\Gamma_T} - [\sigma U, \zeta_E]_{\Omega_T} - [\nabla \times U, \zeta_H]_{\Omega_T} + \frac{1}{2\mu_0} \langle v, \gamma_T U \rangle_{\Gamma_T}. \tag{4.9}$$

Moreover it is $U(0) = 0$ and $V(0) = 0$ in $L^2(\Omega)$ in the sense of traces. As in the proof of Theorem 3.8, we choose special test functions and obtain for for arbitrary $0 \le r \le T$ and

$$\widetilde{U} := \partial_t^{-m} U, \qquad \widetilde{V} := \partial_t^{-m} V, \qquad \widetilde{\psi} := \partial_t^{-m} \psi$$

that

$$[\varepsilon\partial_t \widetilde{U}, \widetilde{U}]_{\Omega_r} + [\mu\partial_t \widetilde{V}, \widetilde{V}]_{\Omega_r} + \left\langle \begin{pmatrix} \widetilde{\psi} \\ -\gamma_T(\widetilde{U}) \end{pmatrix}, B_m * \begin{pmatrix} \psi \\ -\gamma_T U \end{pmatrix} \right\rangle_{\Gamma_r} + [\sigma\widetilde{U}, \widetilde{U}]_{\Omega_r} = 0.$$

By the positivity of the time dependent Calderon operator from Lemma 2.16 it holds

$$\left\langle \begin{pmatrix} \widetilde{\psi} \\ -\gamma_T(\widetilde{U}) \end{pmatrix}, B_m * \begin{pmatrix} \psi \\ -\gamma_T U \end{pmatrix} \right\rangle_{\Gamma_r} = \left\langle \begin{pmatrix} \widetilde{\psi} \\ -\gamma_T(\widetilde{U}) \end{pmatrix}, B(\partial_t) \begin{pmatrix} \widetilde{\psi} \\ -\gamma_T \widetilde{U} \end{pmatrix} \right\rangle_{\Gamma_r} \ge 0$$

and therefore

$$\begin{aligned} 0 \le \varepsilon_0 \|\widetilde{U}(r)\|_\Omega^2 + \mu_0 \|\widetilde{V}(r)\|_\Omega^2 &\le [\varepsilon\partial_t \widetilde{U}, \widetilde{U}]_{\Omega_r} + [\mu\partial_t \widetilde{V}, \widetilde{V}]_{\Omega_r} \\ &\quad + \left\langle \begin{pmatrix} \widetilde{\psi} \\ -\gamma_T(\widetilde{U}) \end{pmatrix}, B_m * \begin{pmatrix} \psi \\ -\gamma_T U \end{pmatrix} \right\rangle_{\Gamma_r} + [\sigma\widetilde{U}, \widetilde{U}]_{\Omega_r} = 0. \end{aligned}$$

Thus we have $\widetilde{U} = U = \widetilde{V} = V = 0$.

We test (4.9) again with $v := (\overline{\partial}_t)^{-m} e^{-2t/T} \partial_t^{-m} \phi$, $\zeta_E = \zeta_H = 0$ and get by the positivity of the time dependent Calderon operator from Lemma 2.16

$$\begin{aligned} 0 &= \left\langle \begin{pmatrix} e^{-2t/T} \partial_t^{-m} \psi \\ 0 \end{pmatrix}, B_m * \begin{pmatrix} \psi \\ 0 \end{pmatrix} \right\rangle_{\Gamma_T} \\ &= \left\langle \begin{pmatrix} e^{-2t/T} \partial_t^{-m} \psi \\ 0 \end{pmatrix}, B(\partial_t) \begin{pmatrix} \partial_t^{-m} \psi \\ 0 \end{pmatrix} \right\rangle_{\Gamma_T} \ge c \int_0^T \|\partial_t^{-m-1} \psi(t,\cdot)\|_{\mathcal{H}_\Gamma}^2 \,\mathrm{d}t \end{aligned}$$

for a $c > 0$. Thus $\partial_t^{-m-1} \psi = \psi = 0$, which gives the desired result. □

The following corollary follows immediately from Theorem 4.3 and Theorem 4.5.

Corollary 4.6. *For every solution in the sense of Definition 4.2 it holds $\varphi = \mu_0 \gamma_T H$. The solutions of Definition 4.1 and Definition 4.2 coincide and are unique.*

4.2. Approximation

In this section we illustrate the discretisation of the system. We start with a recall of the continuous equations such that they fit to the discrete system. We introduce the discrete spaces and time approximation schemes and we conclude with the resulting algorithm. In comparison to Section 3.2, we use the same time discretization schemes, but a non-symmetric space discretization which seems to have advantageous properties, see Remark 4.24.

Separating the equations belonging to the independent test functions ζ_E, ζ_H and v in Definition 4.2, we obtain

$$\begin{aligned} [\varepsilon\partial_t E, \zeta_E]_\Omega + \left\langle \begin{pmatrix} 0 \\ -\gamma_T \zeta_E \end{pmatrix}, B(\partial_t) \begin{pmatrix} \varphi \\ -\gamma_T E \end{pmatrix} \right\rangle_\Gamma &= [H, \nabla \times \zeta_E]_\Omega - \frac{1}{2\mu_0} \langle \varphi, \gamma_T \zeta_E \rangle_\Gamma \\ &\quad - [\sigma E + J, \zeta_E]_\Omega \\ [\mu\partial_t H, \zeta_H]_\Omega &= -[\nabla \times E, \zeta_H]_\Omega \\ \left\langle \begin{pmatrix} v_\varphi \\ 0 \end{pmatrix}, B(\partial_t) \begin{pmatrix} \varphi \\ -\gamma_T E \end{pmatrix} \right\rangle_\Gamma &= \frac{1}{2\mu_0} \langle v_\varphi, \gamma_T E \rangle_\Gamma. \end{aligned}$$

This serves as a basis for the following discretization. This approximation with non-symmetric space differential operators and one boundary variable has advantageous properties concerning the boundedness and the convergence. We will be able to bound $\nabla \times E$ in $L^2(\Omega)$ and every term in the discrete equation will converge to the corresponding continuous counterpart (compare Remark 3.31). In comparison to Chapter 3, the space discretization differs in using piecewise constant functions for the magnetic field.

4.2.1. Space discretization

For the spatial discretization, let $\mathcal{T}_h$ be a regular decomposition of the polyhedral bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ into compact tetrahedra. To discretize the electric part of the Maxwell system in the interior, we use, as before, Nédélec's first order $H(\mathrm{curl}, \Omega)$-conforming ansatz space (cf. e.g. [121]),

$$\mathcal{X}_h := \{\phi_h \in H(\mathrm{curl}, \Omega) \,|\, \phi_{h|K} \in \mathcal{P}^1_{skw}(K) \text{ for all } K \in \mathcal{T}_h\}.$$

As ansatz space for the magnetic field H, we use the piecewise constant functions

$$\mathcal{Y}_h := \{\phi_h \in L^2(\Omega) \,|\, \phi_h(x) = \phi_K, \text{ for all } x \in K \text{ for a } \phi_K \in \mathbb{R}, \text{ for all } K \in \mathcal{T}_h\}.$$

It holds $\nabla \times \mathcal{X}_h \subset \mathcal{Y}_h$. We denote the interpolation on $\mathcal{X}_h$ by $\mathrm{I}_h^{\mathcal{X}}$ and as interpolation on $\mathcal{Y}_h$ we use the L^2-orthogonal projection for $u \in L^2(\Omega)$

$$\mathrm{I}_h^{\mathcal{Y}} u := \sum_{T \in \mathcal{T}_h} \left(\frac{1}{|T|} \int_T u(x) \mathrm{d}x \right) \mathbb{1}_T.$$

For the functions on the boundary, we use the Raviart–Thomas space $\gamma_T(\mathcal{X}_h)$ and the projection $\gamma_T \circ \mathrm{I}_h^{\mathcal{X}}$.

Lemma 4.7. *We recall the approximation properties of the interpolations that hold true for smooth enough functions*

$$\begin{aligned} \|\phi - \mathrm{I}_h^{\mathcal{Y}} \phi\|_{L^2(\Omega)} &\leq Ch \|\phi\|_{H^1(\Omega)}, \\ \|\phi - \mathrm{I}_h^{\mathcal{X}} \phi\|_{L^2(\Omega)} + \|\nabla \times (\phi - \mathrm{I}_h^{\mathcal{X}} \phi)\|_{L^2(\Omega)} &\leq Ch (\|\phi\|_{H^1(\Omega)} + \|\nabla \times \phi\|_{H^1(\Omega)}), \\ \|\gamma_T (\phi - \mathrm{I}_h^{\mathcal{X}} \phi)\|_{\mathcal{H}_\Gamma} &\leq Ch (\|\phi\|_{H^1(\Omega)} + \|\nabla \times \phi\|_{H^1(\Omega)}). \end{aligned}$$

4.2.2. Time discretization

For the time discretization we use a constant time step size $\tau := T/N$ for $N \in \mathbb{N}$ to approximate the solution on the time points $0 = t_0, \dots, t_n = T$, $t_j = \tau j$. We assume that the step size is small enough, i.e. $\tau \leq \tau_0$ for some $\tau_0 > 0$.

For the interior Maxwell part, we use the first order differential quotient

$$\partial_t^\tau G^{j+1} := \frac{G^{j+1} - G^j}{\tau} \tag{4.10}$$

for $G \in \{E, H\}$ together with an implicit treatment of the remaining terms (i.e. evaluation at t_{j+1}).

To discretize $B(\partial_t)$, we use Convolution Quadrature

$$\left(B(\partial_t^\tau) w\right)((j+1)\tau) := \sum_{l=0}^{j+1} B^\tau_{j+1-l} w(l\tau), \tag{4.11}$$

where the weights B_n are defined as the coefficients of

$$B\left(\frac{\zeta(\xi)}{\tau}\right) = \sum_{n=0}^{\infty} B^\tau_n \xi^n. \tag{4.12}$$

As in Chapter 3, we use the first order Convolution Quadrature $\delta(\zeta) = 1 - \zeta$, cf. Remark 3.11 and Section 3.2.2.

4.2.3. Algorithm

We approximate the solution of the Maxwell system by the following algorithm:

Algorithm 4.8. Input: Discretized initial data H_h^0, E_h^0 and $\varphi_h^0 = 0$.
For $j = 0, 1, 2, \dots, N-1$

- Compute $(E_h^{j+1}, H_h^{j+1}) \in \mathcal{X}_h \times \mathcal{Y}_h$ and $\varphi_h^{j+1} \in \gamma_T(X_h)$ such that we have for all $(\zeta_h^E, \zeta_h^H) \in \mathcal{X}_h \times \mathcal{Y}_h$ and $v_h^\varphi \in \gamma_T(X_h)$

$$[\varepsilon \partial_t^\tau E_h^{j+1}, \zeta_h^E]_\Omega + \left\langle \begin{pmatrix} 0 \\ -\gamma_T \zeta_h^E \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix} \right)(t_{j+1}) \right\rangle_\Gamma = [H_h^{j+1}, \nabla \times \zeta_h^E]_\Omega - \frac{1}{2\mu_0} \langle \varphi_h^{j+1}, \gamma_T \zeta_h^E \rangle_\Gamma - [\sigma E_h^{j+1} + J^{j+1}, \zeta_h^E]_\Omega \tag{4.13}$$

$$[\mu \partial_t^\tau H_h^{j+1}, \zeta_h^H]_\Omega = -[\nabla \times E_h^{j+1}, \zeta_h^H]_\Omega \tag{4.14}$$

$$\frac{1}{2\mu_0} \langle v_h^\varphi, \gamma_T E_h^{j+1} \rangle_\Gamma = \left\langle \begin{pmatrix} v_h^\varphi \\ 0 \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix} \right)(t_{j+1}) \right\rangle_\Gamma . \tag{4.15}$$

Output: Sequence of approximations E_h^j, H_h^j, φ_h^j.

As before, for a sequence of space-dependent approximations $(G_h^j)_j$, $G_h^j : \Omega \to \mathbb{R}$, we will define the space and time dependent functions $G_{\tau,h}^-, G_{\tau,h}, G_{\tau,h}^+ : [0,T] \times \Omega \to \mathbb{R}$, see (3.10).

In the following lemma, we show that the approximations are well defined, indeed.

Lemma 4.9. *Algorithm 4.8 is well defined in the sense that for every $j \geq 0$ there exist unique approximations $E_h^{j+1}, H_h^{j+1}, \varphi_h^{j+1}$ that satisfy* (4.13)–(4.15).

Proof. The proof in the non-symmetric case is similar to the one of the symmetric approximation from Chapter 3, see Lemma 3.13.

We define the bilinear form $a(\cdot,\cdot)$ on $\mathcal{X}_h \times \mathcal{Y}_h \times \gamma_T(\mathcal{X}_h)$ by

$$\begin{aligned} a((\Phi, \Psi, \Theta), (\phi, \psi, \theta)) := {}& 1/\tau [\varepsilon \Phi, \phi]_\Omega + 1/\tau [\mu \Psi, \psi]_\Omega - [\Psi, \nabla \times \phi]_\Omega + [\sigma \Phi, \phi] \\ & + \left\langle \begin{pmatrix} \theta \\ -\gamma_T \phi \end{pmatrix}, B_0^\tau \begin{pmatrix} \Theta \\ -\gamma_T \Psi \end{pmatrix} \right\rangle_\Gamma + [\nabla \times \Phi, \psi]_\Omega \\ & + \frac{1}{2\mu_0} \langle \Theta, \gamma_T \phi \rangle_\Gamma - \frac{1}{2\mu_0} \langle \theta, \gamma_T \Phi \rangle_\Gamma \end{aligned}$$

and the linear functional $L^j(\cdot)$ on $\mathcal{X}_h \times \mathcal{Y}_h \times \gamma_T(\mathcal{X}_h)$ by

$$\begin{aligned} L^j(\phi, \psi, \theta) := {}& 1/\tau [\varepsilon E^j, \phi]_\Omega + [\mu 1/\tau H^j, \psi]_\Omega - [J^{j+1}, \phi]_\Omega \\ & - \left\langle \begin{pmatrix} \theta \\ -\gamma_T \phi \end{pmatrix}, \sum_{l=0}^{j} B_{j+1-l}^\tau \begin{pmatrix} \varphi^l \\ -\gamma_T E^l \end{pmatrix} \right\rangle_\Gamma . \end{aligned}$$

The equations (4.13)–(4.15) are equivalent to

$$a((E_h^{j+1}, H_h^{j+1}, \varphi_h^{j+1}), (\phi, \psi, \theta)) = L^j((\phi, \psi, \theta))$$

for all $(\phi, \psi, \theta) \in \mathcal{X}_h \times \mathcal{Y}_h \times \gamma_T(\mathcal{X}_h)$. Next, we aim to show that the bilinear form $a(\cdot,\cdot)$ is positive definite on $\mathcal{X}_h \times \mathcal{Y}_h \times \gamma_T(\mathcal{X}_h)$. We have $B_0^\tau = B(\tau^{-1})$ and by Lemma 2.12 for all $\zeta \in \mathcal{H}_\Gamma \times \mathcal{H}_\Gamma$ and $s > 0$

$$\langle \zeta, B(s) \zeta \rangle_\Gamma \geq C(s, \mu_0, \varepsilon_0) \|\zeta\|_\Gamma^2 .$$

Therefore

$$\begin{aligned}
a((\Phi,\Psi,\Theta),(\Phi,\Psi,\Theta)) &= 1/\tau[\varepsilon\Phi,\Phi]_\Omega + 1/\tau[\mu\Psi,\Psi]_\Omega - [\Psi,\nabla\times\Phi]_\Omega + [\sigma\Phi,\Phi] \\
&\quad + \left\langle \begin{pmatrix}\Theta \\ -\gamma_T\Phi\end{pmatrix}, B_0^\tau \begin{pmatrix}\Theta \\ -\gamma_T\Psi\end{pmatrix}\right\rangle_\Gamma + [\nabla\times\Phi,\Psi]_\Omega \\
&\quad + \frac{1}{2\mu_0}\langle\Theta,\gamma_T\Phi\rangle_\Gamma - \frac{1}{2\mu_0}\langle\Theta,\gamma_T\Phi\rangle_\Gamma \\
&= 1/\tau[\varepsilon\Phi,\Phi]_\Omega + 1/\tau[\mu\Psi,\Psi]_\Omega + [\sigma\Phi,\Phi] \\
&\quad + \left\langle \begin{pmatrix}\Theta \\ -\gamma_T\Phi\end{pmatrix}, B_0^\tau \begin{pmatrix}\Theta \\ -\gamma_T\Psi\end{pmatrix}\right\rangle_\Gamma \\
&\geq C(\tau,\mu,\varepsilon)(\|\Phi\|_\Omega^2 + \|\Psi\|_\Omega^2 + \|\Theta\|_{\mathcal{H}_\Gamma}^2)
\end{aligned}$$

is positive definite which yields the desired result. □

4.3. Convergence

In this section we show the convergence of the algorithm. At first we show bounds on the approximations in Section 4.3.1, secondly we extract weakly convergent subsequences in Section 4.3.2 and finally we identify the limits as weak solutions of our system in Section 4.3.3.

4.3.1. Boundedness of the approximations

We require the following natural assumptions:

Assumption 4.10.

- *The triangulations $\mathcal{T}_h$ are uniformly shape regular.*
- $J_{\tau,h}^{\pm} \rightharpoonup J$ *in* $L^2(\Omega_T)$.
- $E_h^0 \rightharpoonup E^0$ *and* $H_h^0 \rightharpoonup H^0$ *in* $L^2(\Omega)$.

Remark 4.11. *The results in this section are formulated for scalar material constants $\varepsilon,\mu \in \mathbb{R}_{>0}$ and $\sigma \in \mathbb{R}_{\geq 0}$ but hold verbatim for symmetric, coercive and bounded material tensors*

$$\varepsilon,\mu : \Omega \to \mathbb{R}^{3\times 3}$$

and bounded, non-negative $\sigma : \Omega \to \mathbb{R}^{3\times 3}$ (compare Remark 3.16).

For the variables on the boundary, we recall the following lemma (cf. Lemma 3.21).

Lemma 4.12 (cf. [99, Lemma 5.3])**.** *We have for $0 < \tau \leq 1$ and $t_j \leq T$*

$$\sum_{i=0}^{j} e^{-2t_i/T}\left\langle \begin{pmatrix}\varphi(t_i) \\ \psi(t_i)\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi \\ \psi\end{pmatrix}\right)(t_i)\right\rangle_\Gamma \geq C\left(\sum_{i=0}^{j} \|(\partial_t^\tau\varphi)^{-1}(t_i)\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau\psi)^{-1}(t_i)\|_{\mathcal{H}_\Gamma}^2\right)$$

for any finite sequences $(\varphi(t_i))_{i=0}^j$ and $(\psi(t_i))_{i=0}^j$ in $\mathcal{H}_\Gamma$. The constant $C > 0$ depends on T, ε_0, μ_0 and $\beta > 0$ from Lemma 2.12.

Lemma 4.13. *We have for $j \in \mathbb{N}_0$ and $\tau > 0$ with $\tau \leq \tau_0$, $t_j \leq T$ the boundedness of the discrete energy*

$$\mathcal{E}_h^j := \frac{\mu}{2}\|H_h^j\|_\Omega^2 + \frac{\varepsilon}{2}\|E_h^j\|_\Omega^2 \leq C_1$$

and

$$\sum_{i=1}^{j} \|H_h^i - H_h^{i-1}\|_\Omega^2 + \sum_{i=1}^{j} \|E_h^i - E_h^{i-1}\|_\Omega^2 \leq C_2$$

and for the boundary values

$$\tau \sum_{i=0}^{j} \Big(\|(\partial_t^\tau)^{-1}\varphi_h(t_i)\|_{\mathcal{H}_\Gamma}^2 + \|((\partial_t^\tau)^{-1}\gamma_T E_h)(t_i)\|_{\mathcal{H}_\Gamma}^2 \Big) \leq C_3.$$

The constants C_1, C_2 and C_3 depend on T, τ_0, ε, ε_0, μ, μ_0, J and $\mathcal{E}_h^0$, but are independent of h and τ.

Proof. The proof works analogously to the combination of the proofs of Lemma 3.20 and 3.22 (applied to the non-symmetric formulation (4.13)–(4.15)), compare Remark 3.23.

We test in Algorithm 4.8 with $\zeta_h^E = E_h^{j+1}$, $\zeta_h^H = H_h^{j+1}$, $v_h^\varphi = \varphi_h^{j+1}$ and add up the equations to obtain

$$\begin{aligned}
&[\varepsilon \partial_t^\tau E_h^{j+1}, E_h^{j+1}]_\Omega + [\mu \partial_t^\tau H_h^{j+1}, H_h^{j+1}]_\Omega + \left\langle \begin{pmatrix} \varphi_h^{j+1} \\ -\gamma_T E_h^{j+1} \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix} \right)(t_{j+1}) \right\rangle_\Gamma \\
&\qquad = [H_h^{j+1}, \nabla \times E_h^{j+1}]_\Omega - \frac{1}{2\mu_0}\langle \varphi_h^{j+1}, \gamma_T E_h^{j+1}\rangle_\Gamma - [\sigma E_h^{j+1} + J^{j+1}, E_h^{j+1}]_\Omega \\
&\qquad\quad - [\nabla \times E_h^{j+1}, H_h^{j+1}]_\Omega + \frac{1}{2\mu_0}\langle \varphi_h^{j+1}, \gamma_T E_h^{j+1}\rangle_\Gamma \\
&\qquad = -[\sigma E_h^{j+1} + J^{j+1}, E_h^{j+1}]_\Omega.
\end{aligned}$$

Thus we have for all $i \geq 1$

$$\begin{aligned}
\frac{\varepsilon_0}{\tau}[E_h^i - E_h^{i-1}, E_h^i]_\Omega + \frac{\mu_0}{\tau}[H_h^i - H_h^{i-1}, H_h^i]_\Omega + \left\langle \begin{pmatrix} \varphi_h^i \\ -\gamma_T E_h^i \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix} \right)(t_i) \right\rangle_\Gamma \\
= -[\sigma E_h^i + J^i, E_h^i]_\Omega.
\end{aligned} \tag{4.16}$$

We define

$$\widetilde{G}_h^i := e^{-t_i/T} G_h^i$$

for $G \in \{E, H, J\}$, multiply the above equation by $e^{-2t_i/T}$ and obtain for all $i \geq 1$

$$\begin{aligned}
&\frac{\varepsilon}{\tau}[\widetilde{E}_h^i - e^{-\tau/T}\widetilde{E}_h^{i-1}, \widetilde{E}_h^i]_\Omega + \frac{\mu}{\tau}[\widetilde{H}_h^i - e^{-\tau/T}\widetilde{H}_h^{i-1}, \widetilde{H}_h^i]_\Omega \\
&\quad + e^{-2t_i/T} \left\langle \begin{pmatrix} \varphi_h^i \\ -\gamma_T E_h^i \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix} \right)(t_i) \right\rangle_\Gamma = -[\sigma \widetilde{E}_h^i + \widetilde{J}_h^i, \widetilde{E}_h^i]_\Omega.
\end{aligned} \tag{4.17}$$

To treat the terms $[\widetilde{E}_h^i - e^{-\tau/T}\widetilde{E}_h^{i-1}, \widetilde{E}_h^i]_\Omega$ and $[\widetilde{H}_h^i - e^{-\tau/T}\widetilde{H}_h^{i-1}, \widetilde{H}_h^i]_\Omega$ we use the modified Abel's summation by parts (3.29): For $u_i \in \mathbb{R}^n$ and $j \geq i \geq 1$, there holds

$$\sum_{i=1}^{j} (u_i - e^{-\tau/T}u_{i-1}) \cdot u_i \geq \frac{1}{2}\sum_{i=1}^{j} |u_i - e^{-\tau/T}u_{i-1}|^2 + \frac{1}{2}|u_j|^2 - \frac{1}{2}|u_0|^2.$$

Summing up the equations (4.17) for $i = 1, \ldots, j$, multiplying by τ and applying the modified summation by parts to $\widetilde{E}_h^i = e^{-t_i/T} E_h^i$ and $\widetilde{H}_h^i = e^{-t_i/T} H_h^i$, we obtain

$$\begin{aligned}
&\frac{\mu}{2}\left(\|\widetilde{H}_h^j\|_\Omega^2 - \|\widetilde{H}_h^0\|_\Omega^2 + \sum_{i=1}^j \|\widetilde{H}_h^i - e^{-\tau/T}\widetilde{H}_h^{i-1}\|_\Omega^2\right)\\
&\quad + \frac{\varepsilon}{2}\left(\|\widetilde{E}_h^j\|_\Omega^2 - \|\widetilde{E}_h^0\|_\Omega^2 + \sum_{i=1}^j \|\widetilde{E}_h^i - e^{-\tau/T}\widetilde{E}_h^{i-1}\|_\Omega^2\right)\\
&\quad + \tau \sum_{i=1}^j e^{-2t_i/T} \left\langle \begin{pmatrix} \varphi_h^i \\ -\gamma_T E_h^i \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix}\right)(t_i) \right\rangle_\Gamma\\
&\qquad\qquad\qquad \leq -\tau\sigma \sum_{i=1}^j \|\widetilde{E}_h^i\|_\Omega^2 - \tau \sum_{i=1}^j [\widetilde{J}_h^i, \widetilde{E}_h^i]_\Omega.
\end{aligned} \tag{4.18}$$

We estimate with the Cauchy–Schwartz estimate for arbitrary $\delta_1 > 0$

$$-\tau \sum_{i=1}^j [\widetilde{J}_h^i, \widetilde{E}_h^i]_\Omega \leq \left(\tau \sum_{i=1}^j \|\widetilde{J}_h^i\|_\Omega^2\right)^{\frac{1}{2}} \left(\tau \sum_{i=1}^j \|\widetilde{E}_h^i\|_\Omega^2\right)^{\frac{1}{2}} \leq \sum_{i=1}^j \frac{\tau}{2\delta_1}\|\widetilde{J}_h^i\|_\Omega^2 + \sum_{i=1}^j \frac{\tau\delta_1}{2}\|\widetilde{E}_h^i\|_\Omega^2.$$

To absorb the term $\sum_{i=1}^j \frac{\tau\delta_1}{2}\|E_h^i\|_\Omega^2$ on the right hand side, we estimate

$$\sum_{i=1}^j \frac{\tau\delta_1}{2}\|\widetilde{E}_h^i\|_\Omega^2 \leq \sum_{i=1}^j \tau\delta_1 \|\widetilde{E}_h^i - e^{-\tau/T}\widetilde{E}_h^{i-1}\|_\Omega^2 + \sum_{i=1}^j \tau\delta_1 \|\widetilde{E}_h^{i-1}\|_\Omega^2$$

and obtain for the energy

$$\widetilde{\mathcal{E}}_h^j := \frac{\mu}{2}\|\widetilde{H}_h^j\|_\Omega^2 + \frac{\varepsilon}{2}\|\widetilde{E}_h^j\|_\Omega^2$$

that

$$\begin{aligned}
&\widetilde{\mathcal{E}}_h^j + \frac{\mu}{2}\sum_{i=1}^j \|\widetilde{H}_h^i - e^{-\tau/T}\widetilde{H}_h^{i-1}\|_\Omega^2 + \left(\frac{\varepsilon}{2} - \tau\delta_1\right)\sum_{i=1}^j \|\widetilde{E}_h^i - e^{-\tau/T}\widetilde{E}_h^{i-1}\|_\Omega^2\\
&\quad + \tau\sigma \sum_{i=1}^j \|\widetilde{E}_h^i\|_\Omega^2 + \tau \sum_{i=1}^j e^{-2t_i/T} \left\langle \begin{pmatrix} \varphi_h^i \\ -\gamma_T E_h^i \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix}\right)(t_i) \right\rangle_\Gamma\\
&\quad \leq \widetilde{\mathcal{E}}_h^0 + \sum_{i=1}^j \frac{\tau}{2\delta_1}\|J_h^i\|_\Omega^2 + \frac{2\delta_1}{\varepsilon}\tau \sum_{i=1}^j \widetilde{\mathcal{E}}_h^{i-1}.
\end{aligned}$$

Note that $\delta_1 < \frac{\varepsilon_0}{2\tau_0}$ implies $\left(\frac{\varepsilon_0}{2} - \tau\delta_1\right) > 0$ for arbitrary $\tau \leq \tau_0$. Moreover it holds by Lemma 4.12

$$\begin{aligned}
&\sum_{i=1}^j e^{-2t_i/T} \left\langle \begin{pmatrix} \varphi_h^i \\ -\gamma_T E_h^i \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix}\right)(t_i) \right\rangle_\Gamma\\
&\qquad \geq C \sum_{i=1}^j \|(\partial_t^\tau)^{-1}\varphi_h(t_i)\|_{\mathcal{H}_\Gamma} + \|(\partial_t^\tau)^{-1}\gamma_T E_h(t_i)\|_{\mathcal{H}_\Gamma}.
\end{aligned}$$

With the discrete version of Gronwall's lemma (see Lemma A.2 in the Appendix), we have the boundedness of $\widetilde{\mathcal{E}}_h^i$ which, together with the estimates $e^{-1} \leq e^{-t_i/T} \leq 1$, concludes the assertion.

□

The results obtained so far are already enough to show convergence to a weak solution as in Chapter 3. In the Maxwell case without the coupling to the LLG equation, with a few additional assumptions on the input data, we are able to show the boundedness of even more quantities and therefore obtain stronger convergence and regularity results of the solution. A direct transfer of the following results to the MLLG case is not possible to our knowledge, as it would require more regularity of the magnetization, i.e. the boundedness of the second derivative $\partial_t^2 m$ is missing, compare Remark 4.16 (Actually, it would need a stronger energy estimate for the LLG equation that gives bounds on $\partial_t^2 m$).

We need the following additional assumptions.

Assumption 4.14. *We assume the validity of the Assumption 4.10 and additionally*

- $\|\partial_t^\tau J_h^j\|_\Omega \leq C$ *for* $j \geq 1$.
- $\|\nabla \times E_h^0\|_\Omega \leq C$ *independently of* h, $\gamma_T E_h^0 = 0$ *and* $H^0 \in H(\mathrm{curl}, \Omega)$, $\gamma_T H^0 = 0$ *and* $H_h^0 := \mathrm{I}_h^{\mathcal{Y}} H^0$.

Remark 4.15. *The assumption* $H_h^0 := \mathrm{I}_h^{\mathcal{Y}} H^0$ *is needed for the bound*

$$[H_h^0, \nabla \times \zeta_h^H]_\Omega \leq C \|\zeta_h^H\|_\Omega$$

for all $\zeta_h^H \in \mathcal{X}_h$. *For* $H_h^0 := \mathrm{I}_h^{\mathcal{Y}} H^0$, *this follows from* $\nabla \times \mathcal{X}_h \subset \mathcal{Y}_h$, $\gamma_T H^0 = 0$, $H^0 \in H(\mathrm{curl}, \Omega)$ *and*

$$[\mathrm{I}_h^{\mathcal{Y}} H^0, \nabla \times \zeta_h^H]_\Omega = [H^0, \nabla \times \zeta_h^H]_\Omega = [\nabla \times H^0, \zeta_h^H]_\Omega.$$

Alternatively, starting values could be used that satisfy $\gamma_T H_h^0 = 0$ *and* $\|\nabla \times H_h^0\| \leq C$ *independently of* h. *If* H^0 *is smooth enough, this is, e.g., satisfied for Nédélec or higher order finite elements.*

Remark 4.16. *If we introduce a right hand side* G *for the second Maxwell equation, i.e. we consider*

$$\mu \partial_t H + \nabla \times E = G,$$

instead of

$$\mu \partial_t H + \nabla \times E = 0,$$

then the additional assumption

$$\|\partial_t^\tau G_h^j\|_\Omega \leq C$$

for $j \geq 1$ *is needed.*

Lemma 4.17. *Under the additional Assumption 4.14, we have for* $j \in \mathbb{N}_0$ *and* $\tau > 0$ *with* $\tau < \tau_0$, $t_j \leq T$ *the boundedness*

$$\frac{\varepsilon}{2} \|\partial_t^\tau E_h^j\|_\Omega^2 + \frac{\mu}{2} \|\partial_t^\tau H_h^j\|_\Omega^2 \leq C_1$$

and

$$\sum_{i=2}^{j} \|\partial_t^\tau E_h^i - \partial_t^\tau E_h^{i-1}\|_\Omega^2 + \sum_{i=2}^{j} \|\partial_t^\tau H_h^i - \partial_t^\tau H_h^{i-1}\|_\Omega^2 + \tau \sum_{i=2}^{j} \sigma \|E_h^i\|_\Omega^2 \leq C_2.$$

The constants C_1 *and* C_2 *depend on* T, τ_0, ε, ε_0, μ, μ_0, J, *and the bounds from Assumption 4.14, but are independent of* h *and* τ.

Proof. We add up the equations (4.13)–(4.15)

$$\begin{aligned}[]
[\varepsilon\partial_t^\tau E_h^{j+1},\zeta_h^E]_\Omega+[\mu\partial_t^\tau H_h^{j+1},\zeta_h^H]_\Omega + \left\langle \begin{pmatrix} v_h^\varphi \\ -\gamma_T\zeta_h^E \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix}\right)(t_{j+1}) \right\rangle_\Gamma \\
= [H_h^{j+1},\nabla\times\zeta_h^E]_\Omega - [\nabla\times E_h^{j+1},\zeta_h^H]_\Omega - \frac{1}{2\mu}\langle\varphi_h^{j+1},\gamma_T\zeta_h^E\rangle_\Gamma \\
+ \frac{1}{2\mu}\langle v_h^\varphi,\gamma_T E_h^{j+1}\rangle_\Gamma - [\sigma E_h^{j+1} + J^{j+1},\zeta_h^E]_\Omega.
\end{aligned} \tag{4.19}$$

We consider the difference of the equations for $j = i-1$ and $j = i-2$ for $i \geq 2$, divide by τ and obtain by $A(\partial_t^\tau)B(\partial_t^\tau) = AB(\partial_t^\tau)$ (for the CQ boundary term, cf., e.g., [116, Formula (17)]) the relation

$$\begin{aligned}[]
[\varepsilon(\partial_t^\tau)^2 E_h^i,\zeta_h^E]_\Omega+[\mu(\partial_t^\tau)^2 H_h^i,\zeta_h^H]_\Omega + \left\langle \begin{pmatrix} v_h^\varphi \\ -\gamma_T\zeta_h^E \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix} \partial_t^\tau\varphi_h \\ -\partial_t^\tau\gamma_T E_h \end{pmatrix}\right)(t_i) \right\rangle_\Gamma \\
= [\partial_t^\tau H_h^i,\nabla\times\zeta_h^E]_\Omega - [\partial_t^\tau\nabla\times E_h^i,\zeta_h^H]_\Omega - \frac{1}{2\mu}\langle\partial_t^\tau\varphi_h^i,\gamma_T\zeta_h^E\rangle_\Gamma \\
+ \frac{1}{2\mu}\langle v_h^\varphi,\partial_t^\tau\gamma_T E_h^i\rangle_\Gamma - [\sigma\partial_t^\tau E_h^i + \partial_t^\tau J^i,\zeta_h^E]_\Omega.
\end{aligned}$$

We test with $\zeta_h^E = \partial_t^\tau E_h^i$, $\zeta_h^H = \partial_t^\tau H_h^i$ and $v_h^\varphi := \partial_t^\tau\varphi_h^i$ and obtain for $i \geq 2$

$$\begin{aligned}[]
[\varepsilon(\partial_t^\tau)^2 E_h^i,\partial_t^\tau E_h^i]_\Omega+[\mu(\partial_t^\tau)^2 H_h^i,\partial_t^\tau H_h^i]_\Omega + \left\langle \begin{pmatrix} \partial_t^\tau\varphi_h^i \\ -\gamma_T\partial_t^\tau E_h^i \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix} \partial_t^\tau\varphi_h \\ -\partial_t^\tau\gamma_T E_h \end{pmatrix}\right)(t_i) \right\rangle_\Gamma \\
= -[\sigma\partial_t^\tau E_h^i + \partial_t^\tau J^i,\partial_t^\tau E_h^i]_\Omega.
\end{aligned}$$

Summing up from $i = 2,\dots,j$, multiplying by τ and applying Abel's summation by parts (3.25) gives

$$\begin{aligned}
&\frac{\varepsilon}{2}\left(\|\partial_t^\tau E_h^j\|_\Omega^2 - \|\partial_t^\tau E_h^1\|_\Omega^2 + \sum_{i=2}^j \|\partial_t^\tau E_h^i - \partial_t^\tau E_h^{i-1}\|_\Omega^2\right) \\
&+ \frac{\mu}{2}\left(\|\partial_t^\tau H_h^j\|_\Omega^2 - \|\partial_t^\tau H_h^1\|_\Omega^2 + \sum_{i=2}^j \|\partial_t^\tau H_h^i - \partial_t^\tau H_h^{i-1}\|_\Omega^2\right) \\
&+ \tau\sum_{i=2}^j \left\langle \begin{pmatrix} \partial_t^\tau\varphi_h^i \\ -\gamma_T\partial_t^\tau E_h^i \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix} \partial_t^\tau\varphi_h \\ -\partial_t^\tau\gamma_T E_h \end{pmatrix}\right)(t_i) \right\rangle_\Gamma = \tau\sum_{i=2}^j -[\sigma\partial_t^\tau E_h^i + \partial_t^\tau J^i,\partial_t^\tau E_h^i]_\Omega.
\end{aligned} \tag{4.20}$$

With $\varphi_h^0 = \gamma_T E_h^0 = \partial_t^\tau\varphi_h^0 = \partial_t^\tau\gamma_T E_h^0 = 0$ we have (inserting $\partial_t^\tau\phi(t_i) = (\phi^i - \phi^{i-1})/\tau$)

$$\begin{aligned}
\tau\sum_{i=0}^1 \left\langle \begin{pmatrix} \partial_t^\tau\varphi_h^i \\ -\gamma_T\partial_t^\tau E_h^i \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix} \partial_t^\tau\varphi_h \\ -\partial_t^\tau\gamma_T E_h \end{pmatrix}\right)(t_i) \right\rangle_\Gamma \\
= \frac{1}{\tau}\left\langle \begin{pmatrix} \varphi_h^1 \\ -\gamma_T E_h^1 \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix}\right)(t_1) \right\rangle_\Gamma
\end{aligned}$$

which we add on (4.20) to obtain

$$\begin{aligned}
&\frac{\varepsilon}{2}\|\partial_t^\tau E_h^j\|_\Omega^2 + \frac{\varepsilon}{2}\sum_{i=2}^{j}\|\partial_t^\tau E_h^i - \partial_t^\tau E_h^{i-1}\|_\Omega^2 + \frac{\mu}{2}\|\partial_t^\tau H_h^j\|_\Omega^2 + \frac{\mu}{2}\sum_{i=2}^{j}\|\partial_t^\tau H_h^i - \partial_t^\tau H_h^{i-1}\|_\Omega^2 \\
&\quad + \tau\sum_{i=2}^{j}\sigma\|E_h^i\|_\Omega^2 + \tau\sum_{i=0}^{j}\left\langle\begin{pmatrix}\partial_t^\tau\varphi_h^i\\ -\gamma_T\partial_t^\tau E_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\partial_t^\tau\varphi_h\\ -\partial_t^\tau\gamma_T E_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma \\
&\qquad = \frac{\varepsilon}{2}\|\partial_t^\tau E_h^1\|_\Omega^2 + \frac{\mu}{2}\|\partial_t^\tau H_h^1\|_\Omega^2 - \tau\sum_{i=2}^{j}[\partial_t^\tau J^i, \partial_t^\tau E_h^i]_\Omega \\
&\qquad\quad + \frac{1}{\tau}\left\langle\begin{pmatrix}\varphi_h^1\\ -\gamma_T E_h^1\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h\\ -\gamma_T E_h\end{pmatrix}\right)(t_1)\right\rangle_\Gamma .
\end{aligned} \tag{4.21}$$

We estimate with the Cauchy–Schwartz estimate for arbitrary $\delta_1 > 0$

$$\begin{aligned}
\tau\sum_{i=2}^{j} -[\partial_t^\tau J^i, \partial_t^\tau E_h^i]_\Omega &\le \left(\tau\sum_{i=2}^{j}\|\partial_t^\tau J^i\|_\Omega^2\right)^{\frac{1}{2}}\left(\tau\sum_{i=2}^{j}\|\partial_t^\tau E_h^i\|_\Omega^2\right)^{\frac{1}{2}} \\
&\le \frac{\tau}{2\delta_1}\sum_{i=2}^{j}\|\partial_t^\tau J^i\|_\Omega^2 + \frac{\tau\delta_1}{2}\sum_{i=2}^{j}\|\partial_t^\tau E_h^i\|_\Omega^2 \\
&\le \frac{\tau}{2\delta_1}\sum_{i=2}^{j}\|\partial_t^\tau J^i\|_\Omega^2 + \tau\delta_1\sum_{i=2}^{j}\|\partial_t^\tau E_h^i - \partial_t^\tau E^{i-1}\|_\Omega^2 \\
&\quad + \tau\delta_1\sum_{i=1}^{j-1}\|\partial_t^\tau E_h^i\|_\Omega^2 .
\end{aligned} \tag{4.22}$$

We test (4.19) for $j = 0$ with $\zeta_h^E = \partial_t^\tau E_h^1$, $\zeta_h^H = \partial_t^\tau H_h^1$, $v_h^\varphi = \partial_t^\tau \varphi_h^1$, use $\gamma_T E_h^0 = \varphi_h^0 = 0$ and obtain

$$\begin{aligned}
&[\varepsilon\partial_t^\tau E_h^1, \partial_t^\tau E_h^1]_\Omega + [\mu\partial_t^\tau H_h^1, \partial_t^\tau H_h^1]_\Omega + \left\langle\begin{pmatrix}\partial_t^\tau\varphi_h^1\\ -\gamma_T\partial_t^\tau E_h^1\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h\\ -\gamma_T E_h\end{pmatrix}\right)(t_1)\right\rangle_\Gamma \\
&\quad = [H_h^1, \nabla\times\partial_t^\tau E_h^1]_\Omega - [\nabla\times E_h^1, \partial_t^\tau H_h^1]_\Omega - \frac{1}{2\mu}\langle\varphi_h^1, \gamma_T\partial_t^\tau E_h^1\rangle_\Gamma \\
&\qquad + \frac{1}{2\mu}\langle\partial_t^\tau\varphi_h^1, \gamma_T E_h^1\rangle_\Gamma - [\sigma E_h^1 + J^1, \partial_t^\tau E_h^1]_\Omega \\
&\quad = [H_h^1, \nabla\times\partial_t^\tau E_h^1]_\Omega - [\nabla\times E_h^1, \partial_t^\tau H_h^1]_\Omega - [\sigma E_h^1 + J^1, \partial_t^\tau E_h^1]_\Omega .
\end{aligned} \tag{4.23}$$

By $H_h^0 = \mathrm{I}_h^{\mathcal{Y}} H^0$ (cf. Assumption 4.14) and $\gamma_T H^0 = 0$ we have

$$\begin{aligned}
[H_h^0, \nabla\times\partial_t^\tau E_h^1]_\Omega &= [H^0, \nabla\times\partial_t^\tau E_h^1]_\Omega \\
&= [\nabla\times H^0, \partial_t^\tau E_h^1]_\Omega .
\end{aligned}$$

We use this outcome and estimate for $\delta_2 > 0$

$$\begin{aligned}[H_h^1, \nabla\times\partial_t^\tau E_h^1]_\Omega - [\nabla\times E_h^1, \partial_t^\tau H_h^1]_\Omega &= [\partial_t^\tau H_h^1, \nabla\times(E_h^1 - E_h^0)]_\Omega + [H_h^0, \nabla\times\partial_t^\tau E_h^1]_\Omega \\ &\quad - [\nabla\times E_h^1, \partial_t^\tau H_h^1]_\Omega \\ &= -[\partial_t^\tau H_h^1, \nabla\times E_h^0]_\Omega + [H_h^0, \nabla\times\partial_t^\tau E_h^1]_\Omega \\ &= -[\partial_t^\tau H_h^1, \nabla\times E_h^0]_\Omega + [\nabla\times H^0, \partial_t^\tau E_h^1]_\Omega \\ &\leq \frac{\delta_2}{2}\left(\|\partial_t^\tau E_h^1\|_\Omega^2 + \|\partial_t^\tau H_h^1\|_\Omega^2\right) \\ &\quad + \frac{1}{2\delta_2}\left(\|\nabla\times E_h^0\|_\Omega^2 + \|\nabla\times H^0\|_\Omega^2\right).\end{aligned} \tag{4.24}$$

Furthermore we estimate for $\delta_3 > 0$

$$-[\sigma E_h^1 + J^1, \partial_t^\tau E_h^1]_\Omega \leq \frac{\delta_3}{2}\|\partial_t^\tau E_h^1\|_\Omega^2 + \frac{1}{2\delta_3}\|\sigma E_h^1 + J^1\|_\Omega^2,$$

which we combine with (4.23) and (4.24) to obtain

$$\begin{aligned}&\left(\varepsilon - \frac{\delta_2+\delta_3}{2}\right)\|\partial_t^\tau E_h^1\|_\Omega^2 + \left(\mu - \frac{\delta_2}{2}\right)\|\partial_t^\tau H_h^1\|_\Omega^2 \\ &\quad + \frac{1}{\tau}\left\langle\begin{pmatrix}\varphi_h^1 \\ -\gamma_T E_h^1\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h \\ -\gamma_T E_h\end{pmatrix}\right)(t_1)\right\rangle_\Gamma \\ &\quad \leq \frac{1}{2\delta_2}\left(\|\nabla\times E_h^0\|_\Omega^2 + \|\nabla\times H^0\|_\Omega^2\right) + \frac{1}{2\delta_3}\|\sigma E_h^1 + J^1\|_\Omega^2.\end{aligned}$$

We choose $\delta_2 \leq \mu$ and $\delta_2 + \delta_3 \leq \varepsilon$ which gives

$$\begin{aligned}&\frac{\varepsilon}{2}\|\partial_t^\tau E_h^1\|_\Omega^2 + \frac{\mu}{2}\|\partial_t^\tau H_h^1\|_\Omega^2 + \frac{1}{\tau}\left\langle\begin{pmatrix}\varphi_h^1 \\ -\gamma_T E_h^1\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h \\ -\gamma_T E_h\end{pmatrix}\right)(t_1)\right\rangle_\Gamma \\ &\quad \leq \frac{1}{2\delta_2}\left(\|\nabla\times E_h^0\|_\Omega^2 + \|\nabla\times H^0\|_\Omega^2\right) + \frac{1}{2\delta_3}\|\sigma E_h^1 + J_h^1\|_\Omega^2.\end{aligned} \tag{4.25}$$

We combine (4.21), (4.22) and (4.25) to obtain

$$\begin{aligned}&\frac{\varepsilon}{2}\|\partial_t^\tau E_h^j\|_\Omega^2 + \left(\frac{\varepsilon}{2} - \tau\delta_1\right)\sum_{i=2}^{j}\|\partial_t^\tau E_h^i - \partial_t^\tau E_h^{i-1}\|_\Omega^2 + \frac{\mu}{2}\|\partial_t^\tau H_h^j\|_\Omega^2 + \frac{\mu}{2}\sum_{i=2}^{j}\|\partial_t^\tau H_h^i - \partial_t^\tau H_h^{i-1}\|_\Omega^2 \\ &\quad + \tau\sum_{i=2}^{j}\sigma\|E_h^i\|_\Omega^2 + \tau\sum_{i=0}^{j}\left\langle\begin{pmatrix}\partial_t^\tau\varphi_h^i \\ -\gamma_T\partial_t^\tau E_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\partial_t^\tau\varphi_h \\ -\partial_t^\tau\gamma_T E_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma \\ &\quad \leq \frac{1}{2\delta_2}\left(\|\nabla\times E_h^0\|_\Omega^2 + \|\nabla\times H^0\|_\Omega^2\right) + \frac{1}{2\delta_3}\|\sigma E_h^1 + J^1\|_\Omega^2 \\ &\qquad + \frac{\tau}{2\delta_1}\sum_{i=2}^{j}\|\partial_t^\tau J^i\|_\Omega^2 + \tau\delta_1\sum_{i=1}^{j-1}\|\partial_t^\tau E_h^i\|_\Omega^2.\end{aligned} \tag{4.26}$$

With this outcome, Assumption 4.14, Lemma 4.13 and Lemma 3.18, we have for

$$\delta_1 \leq \frac{\varepsilon}{2\tau_0}$$

that

$$\frac{\varepsilon}{2}\|\partial_t^\tau E_h^j\|_\Omega^2 \leq C + \tau\delta_1\sum_{i=1}^{j-1}\|\partial_t^\tau E_h^i\|_\Omega^2$$

for $j \geq 1$ and therefore by the discrete Gronwall Lemma A.2

$$\frac{\varepsilon}{2}\|\partial_t^\tau E_h^j\|_\Omega^2 \leq C.$$

Together with (4.26), this yields the assertion. □

Again, the following lemma provides energy bounds for the quantities on the boundary.

Lemma 4.18 (cf. [99, Lemma 5.3]). *We have for $0 < \tau \leq 1$ and $t_j \leq T$*

$$\sum_{i=0}^{j} e^{-2t_i/T}\left\langle \begin{pmatrix}\partial_t^\tau \varphi(t_i)\\ \partial_t^\tau \psi(t_i)\end{pmatrix}, \left(B(\partial_t^\tau)\partial_t^\tau \begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right)(t_i)\right\rangle_\Gamma \geq C\left(\sum_{i=0}^{j} \|\varphi(t_i)\|_{\mathcal{H}_\Gamma}^2 + \|\psi(t_i)\|_{\mathcal{H}_\Gamma}^2\right)$$

for any finite sequences $(\varphi(t_i))_{i=0}^j$ and $(\psi(t_i))_{i=0}^j$ in $\mathcal{H}_\Gamma$ with $\varphi^0 = \psi^0 = 0$. The constant $C > 0$ depends on T, ε_0, μ_0 and $\beta > 0$ from Lemma 2.12.

Proof. For sequences $(\phi^i)_{i\in\mathbb{N}_0}$ with $\phi^0 = 0$ it holds $((\partial_t^\tau)^{-1}\partial_t^\tau \phi)(t_i) = \phi^i$ for $i \geq 0$ and the assertion follows from Lemma 4.12. □

By a modification of Lemma 4.17 with the factors $e^{-t_i/T}$ we are able to show the following result.

Lemma 4.19. *Under the additional Assumptions 4.14, we have for $j \in \mathbb{N}_0$ and $\tau > 0$ with $\tau < \tau_0$, $t_j \leq T$ the boundedness*

$$\tau \sum_{i=0}^{j}\left(\|\varphi_h^i\|_{\mathcal{H}_\Gamma}^2 + \|\gamma_T E_h^i\|_{\mathcal{H}_\Gamma}^2\right) \leq C,$$

where the constants $C > 0$ depends on T, τ_0, ε, ε_0, μ, μ_0, J and the bounds from Assumption 4.14, but is independent of h and τ.

Proof. Just as in the proof of Lemma 4.17, we arrive at

$$[\varepsilon(\partial_t^\tau)^2 E_h^i, \partial_t^\tau E_h^i]_\Omega + [\mu(\partial_t^\tau)^2 H_h^i, \partial_t^\tau H_h^i]_\Omega + \left\langle \begin{pmatrix}\partial_t^\tau \varphi_h^i\\ -\gamma_T \partial_t^\tau E_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\partial_t^\tau \varphi_h\\ -\partial_t^\tau \gamma_T E_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma = -[\sigma\partial_t^\tau E_h^i + \partial_t^\tau J_h^i, \partial_t^\tau E_h^i]_\Omega \tag{4.27}$$

for $i \geq 2$. We multiply the equation by $e^{-2i\tau/T}$ and define

$$\widetilde{\partial}_t^\tau E_h^i := e^{-i\tau/T}\partial_t^\tau E_h^i, \widetilde{\partial}_t^\tau H_h^i := e^{-i\tau/T}\partial_t^\tau H_h^i, \widetilde{\partial}_t^\tau J^i := e^{-\tau/T}\partial_t^\tau J^i, \ldots$$

and

$$\widetilde{E}_h^i := e^{-i\tau/T}E_h^i, \widetilde{H}_h^i := e^{-i\tau/T}H_h^i, \widetilde{J}^i := e^{-i\tau/T}J^i, \ldots$$

We recall the adapted Abel's summation by parts (3.29). For $u_i \in \mathbb{R}^n$ and $j \geq i \geq 1$, there holds

$$\sum_{i=1}^{j}(u_i - e^{-\tau/T}u_{i-1})\cdot u_i \geq \frac{1}{2}\sum_{i=1}^{j}|u_i - e^{-\tau/T}u_{i-1}|^2 + \frac{1}{2}|u_j|^2 - \frac{1}{2}|u_0|^2. \tag{4.28}$$

Summing up from $i = 2, \dots, j$, multiplying by τ and applying the modified Abel's summation by parts (4.28) gives

$$\begin{aligned}
&\frac{\varepsilon_0}{2}\left(\|\widetilde{\partial}_t^\tau E_h^j\|_\Omega^2 - \|\widetilde{\partial}_t^\tau E_h^1\|_\Omega^2 + \sum_{i=2}^{j}\|\widetilde{\partial}_t^\tau E_h^i - e^{-\tau/T}\widetilde{\partial}_t^\tau E_h^{i-1}\|_\Omega^2\right)\\
&+\frac{\mu_0}{2}\left(\|\widetilde{\partial}_t^\tau H_h^j\|_\Omega^2 - \|\widetilde{\partial}_t^\tau H_h^1\|_\Omega^2 + \sum_{i=2}^{j}\|\widetilde{\partial}_t^\tau H_h^i - e^{-\tau/T}\widetilde{\partial}_t^\tau H_h^{i-1}\|_\Omega^2\right)\\
&+\tau\sum_{i=2}^{j} e^{-2i\tau/T}\left\langle\begin{pmatrix}\partial_t^\tau\varphi_h^i\\ -\gamma_T\partial_t^\tau E_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\partial_t^\tau\varphi_h\\ -\partial_t^\tau\gamma_T E_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma\\
&\qquad\qquad = \tau\sum_{i=2}^{j} -[\sigma\widetilde{\partial}_t^\tau E_h^i + \widetilde{\partial}_t^\tau J_h^i, \widetilde{\partial}_t^\tau E_h^i]_\Omega.
\end{aligned} \tag{4.29}$$

We test (4.19) for $j = 0$ with $\zeta_h^E = \widetilde{\partial}_t^\tau E_h^1$, $\zeta_h^H = \widetilde{\partial}_t^\tau H_h^1$, $v_h^\varphi = \widetilde{\partial}_t^\tau \varphi_h^1$, multiply by $e^{-\tau/T}$, use $\gamma_T E_h^0 = \varphi_h^0 = 0$ and obtain

$$\begin{aligned}
&[\varepsilon\widetilde{\partial}_t^\tau E_h^1, \widetilde{\partial}_t^\tau E_h^1]_\Omega + [\mu\widetilde{\partial}_t^\tau H_h^1, \widetilde{\partial}_t^\tau H_h^1]_\Omega + e^{-\tau/T}\left\langle\begin{pmatrix}\widetilde{\partial}_t^\tau\varphi^1\\ -\gamma_T\widetilde{\partial}_t^\tau E_h^1\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h\\ -\gamma_T E_h\end{pmatrix}\right)(t_1)\right\rangle_\Gamma\\
&\qquad = [\widetilde{H}_h^1, \nabla\times\widetilde{\partial}_t^\tau E_h^1]_\Omega - [\nabla\times\widetilde{E}_h^1, \widetilde{\partial}_t^\tau H_h^1]_\Omega - [\sigma\widetilde{E}_h^1 + \widetilde{J}_h^1, \widetilde{\partial}_t^\tau E_h^1]_\Omega.
\end{aligned} \tag{4.30}$$

With $\varphi_h^0 = \gamma_T E_h^0 = \partial_t^\tau\varphi_h^0 = \partial_t^\tau\gamma_T E_h^0 = 0$ we have

$$\begin{aligned}
&\tau\sum_{i=0}^{1} e^{-2i\tau/T}\left\langle\begin{pmatrix}\partial_t^\tau\varphi_h^i\\ -\gamma_T\partial_t^\tau E_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\partial_t^\tau\varphi_h\\ -\partial_t^\tau\gamma_T E_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma\\
&\qquad = \frac{e^{-2\tau/T}}{\tau}\left\langle\begin{pmatrix}\varphi_h^1\\ -\gamma_T E_h^1\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h\\ -\gamma_T E_h\end{pmatrix}\right)(t_1)\right\rangle_\Gamma\\
&\qquad = e^{-\tau/T}\left\langle\begin{pmatrix}\widetilde{\partial}_t^\tau\varphi_h^1\\ -\gamma_T\widetilde{\partial}_t^\tau E_h^1\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h\\ -\gamma_T E_h\end{pmatrix}\right)(t_1)\right\rangle_\Gamma .
\end{aligned}$$

As in the proof of of Lemma 4.17, we use $\widetilde{H}_h^0 = \mathrm{I}_h^{\mathcal{Y}} H^0$ (cf. Assumption 4.14) and $\gamma_T\widetilde{H}^0 = \gamma_T H^0 = 0$ to obtain

$$[\widetilde{H}_h^1, \nabla\times\widetilde{\partial}_t^\tau E_h^1]_\Omega - [\nabla\times\widetilde{E}_h^1, \widetilde{\partial}_t^\tau H_h^1]_\Omega = e^{-\tau/T}(-[\widetilde{\partial}_t^\tau H_h^1, \nabla\times E_h^0]_\Omega + [\nabla\times H^0, \widetilde{\partial}_t^\tau E_h^1]_\Omega) \tag{4.31}$$

Inserting all this in (4.30), we get

$$\begin{aligned}
&[\varepsilon\widetilde{\partial}_t^\tau E_h^1, \widetilde{\partial}_t^\tau E_h^1]_\Omega + [\mu\widetilde{\partial}_t^\tau H_h^1, \widetilde{\partial}_t^\tau H_h^1]_\Omega\\
&\qquad +\tau\sum_{i=0}^{1} e^{-2i\tau/T}\left\langle\begin{pmatrix}\partial_t^\tau\varphi_h^i\\ -\gamma_T\partial_t^\tau E_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\partial_t^\tau\varphi_h\\ -\partial_t^\tau\gamma_T E_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma\\
&= e^{-\tau/T}(-[\widetilde{\partial}_t^\tau H_h^1, \nabla\times E_h^0]_\Omega + [\nabla\times H^0, \widetilde{\partial}_t^\tau E_h^1]_\Omega) - [\sigma\widetilde{E}_h^1 + \widetilde{J}_h^1, \widetilde{\partial}_t^\tau E_h^1]_\Omega.
\end{aligned} \tag{4.32}$$

Adding this to (4.29) finally gives

$$\frac{\varepsilon_0}{2}\left(\|\widetilde{\partial}_t^\tau E_h^j\|_\Omega^2 + \|\widetilde{\partial}_t^\tau E_h^1\|_\Omega^2 + \sum_{i=2}^{j} \|\widetilde{\partial}_t^\tau E_h^i - e^{-\tau/T}\widetilde{\partial}_t^\tau E_h^{i-1}\|_\Omega^2\right)$$
$$+\frac{\mu_0}{2}\left(\|\widetilde{\partial}_t^\tau H_h^j\|_\Omega^2 + \|\widetilde{\partial}_t^\tau H_h^1\|_\Omega^2 + \sum_{i=2}^{j} \|\widetilde{\partial}_t^\tau H_h^i - e^{-\tau/T}\widetilde{\partial}_t^\tau H_h^{i-1}\|_\Omega^2\right)$$
$$+\tau\sum_{i=0}^{j} e^{-2i\tau/T}\left\langle\begin{pmatrix}\partial_t^\tau \varphi_h^i\\ -\gamma_T\partial_t^\tau E_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\partial_t^\tau\varphi_h\\ -\partial_t^\tau\gamma_T E_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma$$
$$= \tau\sum_{i=2}^{j} -[\sigma\widetilde{\partial}_t^\tau E_h^i + \widetilde{\partial}_t^\tau J_h^i, \widetilde{\partial}_t^\tau E_h^i]_\Omega$$
$$+ e^{-\tau/T}(-[\widetilde{\partial}_t^\tau H_h^1, \nabla\times E_h^0]_\Omega + [\nabla\times H^0, \widetilde{\partial}_t^\tau E_h^1]_\Omega) - [\sigma\widetilde{E}_h^1 + \widetilde{J}_h^1, \widetilde{\partial}_t^\tau E_h^1]_\Omega. \tag{4.33}$$

By Assumption 4.14, Lemma 4.17 and Cauchy–Schwartz, we have

$$e^{-\tau/T}(-[\widetilde{\partial}_t^\tau H_h^1, \nabla\times E_h^0]_\Omega + [\nabla\times H^0, \widetilde{\partial}_t^\tau E_h^1]_\Omega) - [\sigma\widetilde{E}_h^1 + \widetilde{J}_h^1, \widetilde{\partial}_t^\tau E_h^1]_\Omega \le C,$$
$$\tau\sum_{i=2}^{j} -[\sigma\widetilde{\partial}_t^\tau E_h^i + \widetilde{\partial}_t^\tau J_h^i, \widetilde{\partial}_t^\tau E_h^i]_\Omega \le C.$$

As all other terms on the left hand side of (4.33) are positive and/or bounded, we deduce

$$\tau\sum_{i=0}^{j} e^{-2i\tau/T}\left\langle\begin{pmatrix}\partial_t^\tau \varphi_h^i\\ -\gamma_T\partial_t^\tau E_h^i\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\partial_t^\tau\varphi_h\\ -\partial_t^\tau\gamma_T E_h\end{pmatrix}\right)(t_i)\right\rangle_\Gamma \le C.$$

Therefore, by Lemma 4.12 and $\gamma_T E_h^0 = \varphi_h^0 = 0$ for $c > 0$,

$$C \ge c\tau\sum_{i=1}^{j} \|(\partial_t^\tau)^{-1}\partial_t^\tau\varphi(t_i)\|_{\mathcal{H}_\Gamma} + \|(\partial_t^\tau)^{-1}\partial_t^\tau\gamma_T E(t_i)\|_{\mathcal{H}_\Gamma}$$
$$= c\tau\sum_{i=1}^{j} \|\varphi_h^i\|_{\mathcal{H}_\Gamma} + \|\gamma_T E_h^i\|_{\mathcal{H}_\Gamma}$$

which yields the assertion. □

The following lemma is a direct consequence of Lemma 4.17, Assumption 4.14, Algorithm 4.8 and $\nabla\times\mathcal{X}_h \subset \mathcal{Y}_h$.

Lemma 4.20. *We have for $j \ge 0$ the boundedness of*

$$\|\nabla\times E_h^j\|_\Omega \le C$$

and for $\zeta_h \in \mathcal{X}_h$, $\gamma_T\zeta_h = 0$ it holds

$$[H_h^j, \nabla\times\zeta_h] \le C\|\zeta_h\|_\Omega.$$

If we denote by $\mathrm{P}_{0,h}^{\mathcal{X}}$ the L^2-orthogonal-projection on $\mathcal{X}_{0,h} := \{\zeta\in\mathcal{X}_h | \gamma_T\zeta = 0\}$, then

$$[H_h^j, \nabla\times\mathrm{P}_{0,h}^{\mathcal{X}}\,\cdot\,]_\Omega : L^2(\Omega)\to\mathbb{R}, \zeta\mapsto [H_h^j, \nabla\times\mathrm{P}_{0,h}^{\mathcal{X}}\zeta]_\Omega$$

is bounded in $L^2(\Omega)'$.

Proof. The proof follows from Algorithm 4.8, as all other terms in the respective equations are bounded due to the Lemmas 4.13 – 4.19 and the Assumptions 4.10 and 4.14. □

Remark 4.21. *The constants in the energy estimates* (4.13)*,* (4.17)*,* (4.20) *grow exponentially in T, but similar arguments as in [99, Lemma 4.3, Lemma 7.1] would yield constants depending only polynomially on T. In this case the maximal time step size τ_0 depends on T.*

We sum up the obtained results in the following theorem.

Theorem 4.22. *There exists a constant $C > 0$ independent of τ and h such that*

$$\begin{aligned} \|E_{\tau,h}\|_{\Omega_T} + \|\partial_t E_{\tau,h}\|_{\Omega_T} + \|\nabla \times E_{\tau,h}\|_{\Omega_T} &\le C, \\ \|H_{\tau,h}\|_{\Omega_T} + \|\partial_t H_{\tau,h}\|_{\Omega_T} + \|[H_{\tau,h}, \nabla \times \mathrm{P}^{\mathcal{X}}_{0,h} \cdot\,]_{\Omega_T}\|_{L^2(\Omega_T)'} &\le C, \\ \|E^{\pm}_{\tau,h}\|_{\Omega_T} + \|\nabla \times E^{\pm}_{\tau,h}\|_{\Omega_T} &\le C, \\ \|H^{\pm}_{\tau,h}\|_{\Omega_T} &\le C, \\ \|\varphi_{\tau,h}\|_{L^2([0,T],\mathcal{H}_\Gamma)} + \|\varphi^{\pm}_{\tau,h}\|_{L^2([0,T],\mathcal{H}_\Gamma)} &\le C. \end{aligned}$$

Proof. Most statements follow directly from Lemmas 4.13 – 4.20, the proof can be concluded as in Lemma 3.27. □

4.3.2. Existence of weakly convergent subsequences

Due to the shown boundedness of the approximations, we are now able to extract weakly convergent subsequences. Again as in Chapter 3, we do not (re-)name the sequences when passing to a subsequence. We write $v_{\tau,h} \overset{\text{sub}}{\rightharpoonup} v$ for $\tau, h \to 0$, to denote that for any $(\tau_n, h_n) \to 0$ for $n \to \infty$ there exists a subsequence $(n_j)_{j\in\mathbb{N}}$, such that $v_{h_{n_j}} \rightharpoonup v$ for $j \to \infty$. We note that in the following, the convergences hold simultaneously for one subsequence.

Theorem 4.23. *There exist functions*

$$(E, H, \varphi) \in H^{1,\mathrm{curl}}(\Omega_T) \times H^{1,\mathrm{curl}}(\Omega_T) \times L^2([0,T], \mathcal{H}_\Gamma)$$

such that

$$\begin{aligned} E_{\tau,h} &\overset{\text{sub}}{\rightharpoonup} E && \text{in } H^{1,\mathrm{curl}}(\Omega_T), \\ E_{\tau,h}, E^{\pm}_{\tau,h} &\overset{\text{sub}}{\rightharpoonup} E && \text{in } H^{0,\mathrm{curl}}(\Omega_T) \\ \gamma_T E_{\tau,h}, \gamma_T E^{\pm}_{\tau,h} &\overset{\text{sub}}{\rightharpoonup} \gamma_T E && \text{in } L^2([0,T], \mathcal{H}_\Gamma), \\ H_{\tau,h} &\overset{\text{sub}}{\rightharpoonup} H && \text{in } H^{1,0}(\Omega_T), \\ H_{\tau,h}, H^{\pm}_{\tau,h} &\overset{\text{sub}}{\rightharpoonup} H && \text{in } L^2(\Omega_T), \\ \varphi_{\tau,h}, \varphi^{\pm}_{\tau,h} &\overset{\text{sub}}{\rightharpoonup} \varphi && \text{in } L^2([0,T], \mathcal{H}_\Gamma), \end{aligned}$$

where the subsequences are successively constructed, i.e., for arbitrary mesh sizes $h \to 0$ and time step sizes $\tau \to 0$ there exist subindices τ_{n_l}, h_{n_l} for which the above convergence properties are satisfied simultaneously.

Proof. By the uniform boundedness of the approximations in the respective Hilbert spaces (cf. Theorem 4.22) and uniqueness of weak limits, we have the existence of limit functions and the weak convergence of a (fixed) subsequence

$$(E_{\tau,h}, H_{\tau,h}, \varphi_{\tau,h}) \overset{\text{sub}}{\rightharpoonup} (E, H, \varphi) \in H(\mathrm{curl}, \Omega_T) \times H^{1,0}(\Omega_T) \times L^2([0,T], \mathcal{H}_\Gamma).$$

It remains to show that $(E^{\pm}_{\tau,h}, H^{\pm}_{\tau,h}, \varphi^{\pm}_{\tau,h})$ converges to the same limit functions and that $H \in H(\text{curl}, \Omega_T)$. Similarly to the proof of Lemma 3.29, we sketch that $\varphi^{-}_{\tau,h}$ converges to the same limit function as $\varphi_{\tau,h}$. It holds for $w \in C^1_0([0,T], \mathcal{H}_\Gamma)$

$$\begin{aligned}\langle \varphi_{\tau,h} - \varphi^{-}_{\tau,h}, w\rangle_{\Gamma_T} &= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \frac{t-t_j}{\tau} \langle \varphi_h^{j+1} - \varphi_h^j, w(t_j)\rangle \, \mathrm{d}t \\ &\quad + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \frac{t-t_j}{\tau} \langle \varphi_h^{j+1} - \varphi_h^j, w(t) - w(t_j)\rangle \, \mathrm{d}t.\end{aligned}$$

By $w(T) = w(0) = 0$ we see

$$\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \frac{t-t_j}{\tau} \langle \varphi_h^{j+1} - \varphi_h^j, w(t_j)\rangle \, \mathrm{d}t = -\frac{\tau}{2} \sum_{j=0}^{N-1} \langle \varphi_h^j, w(t_{j+1}) - w(t_j)\rangle.$$

Therefore we have by the boundedness of $\varphi^{\pm}_{\tau,h}$

$$\begin{aligned}|\langle \varphi_{\tau,h} - \varphi^{-}_{\tau,h}, w\rangle_{\Gamma_T}| &\le \frac{1}{2} \left(\tau \sum_{j=0}^{N-1} \|\varphi_h^j\|^2_\Omega\right)^{1/2} \left(\tau \sum_{j=0}^{N-1} \|w(t_{j+1}) - w(t_j)\|^2_\Omega\right)^{1/2} \\ &\quad + \left(\tau \sum_{j=0}^{N-1} \|\varphi_h^j - \varphi_h^{j+1}\|^2_\Omega\right)^{1/2} \left(\tau \sum_{j=0}^{N-1} \|\int_{t_j}^{t_{j+1}} w(t) - w(t_j) \, \mathrm{d}t\|^2_\Omega\right)^{1/2} \\ &\le C \max_{j=0,\dots,N-1} \max_{t\in[t_j,t_{j+1}]} \|w(t) - w(t_j)\|_{\mathcal{H}_\Gamma} \to 0.\end{aligned}$$

As $C^1_0([0,T], \mathcal{H}_\Gamma)$ is dense in $L^2([0,T], \mathcal{H}_\Gamma)$, and the functions $\varphi^{-}_{\tau,h}$ are uniformly bounded in $L^2([0,T], \mathcal{H}_\Gamma)$, it holds $\varphi^{-}_{\tau,h} \overset{\text{sub}}{\rightharpoonup} \varphi$.

It remains to show that $\nabla \times H \in L^2(\Omega_T)$ exists. By Theorem 4.22 we have the boundedness and therefore the weak convergence of

$$[G_{\tau,h}, \cdot\,]_{\Omega_T} := [H_{\tau,h}, \nabla \times \mathrm{P}^{\mathcal{X}}_{0,h} \cdot\,]_{\Omega_T},$$

i.e. there exists a $G \in L^2(\Omega_T)$ with $G_{\tau,h} \overset{\text{sub}}{\rightharpoonup} G$. To show $G = \nabla \times H$, we choose $\zeta \in C^\infty_0(\Omega_T)$. It holds $\mathrm{I}^{\mathcal{X}}_h \zeta \to \zeta$ in $L^2(\Omega_T)$ (cf. Lemma 4.7). Therefore we have

$$[G_{\tau,h}, \mathrm{I}^{\mathcal{X}}_h \zeta]_{\Omega_T} \overset{\text{sub}}{\to} [G, \zeta]_{\Omega_T}.$$

Moreover we have $\gamma_T \mathrm{I}^{\mathcal{X}}_h \zeta = 0$ (cf. [121, Lemma 5.35]), thus $\mathrm{P}^{\mathcal{X}}_{0,h} \mathrm{I}^{\mathcal{X}}_h \zeta = \mathrm{I}^{\mathcal{X}}_h \zeta$ and $\nabla \times \mathrm{I}^{\mathcal{X}}_h \zeta \to \nabla \times \zeta$ in $L^2(\Omega_T)$ (cf. Lemma 4.7) and $H_{\tau,h} \overset{\text{sub}}{\rightharpoonup} H$. This implies

$$\begin{aligned}[G_{\tau,h}, \mathrm{I}^{\mathcal{X}}_h \zeta]_{\Omega_T} &= [H_{\tau,h}, \nabla \times \mathrm{I}^{\mathcal{X}}_h \zeta]_{\Omega_T} \\ &\overset{\text{sub}}{\to} [H, \nabla \times \zeta]_{\Omega_T}\end{aligned}$$

and as $\zeta \in C^\infty_0(\Omega_T)$ was chosen arbitrarily, we have $G = \nabla \times H \in L^2(\Omega_T)$. □

Remark 4.24. *In comparison to the LLG case and the symmetric discretization of the curl operator, in the non-symmetric case, each discretized term converges towards the expected continuous term. For smooth enough test functions ζ, v it holds (compare Remark 3.31 for the MLLG system, where this is not the case)*

$$\begin{aligned}[\nabla \times E^{+}_{\tau,h}, \zeta^{+}_{\tau,h}]_{\Omega_T} &\overset{\text{sub}}{\to} [\nabla \times E, \zeta_H]_{\Omega_T} \\ [H^{+}_{\tau,h}, \nabla \times \zeta^{+}_{\tau,h}]_{\Omega_T} &\overset{\text{sub}}{\to} [H, \nabla \times \zeta]_{\Omega_T}\end{aligned}$$

and

$$\langle v^{+}_{\tau,h}, \gamma_T E^{+}_{\tau,h}\rangle_{\Gamma_T} \overset{\text{sub}}{\to} \langle v, \gamma_T E\rangle_{\Gamma_T}.$$

4.3.3. Convergence towards the exact solution

We show that the limit functions indeed are a solution of the Maxwell system in the sense of Definition 4.2. Afterwards, with the uniqueness of the solution, we conclude that the convergence holds for the whole sequence (and not only for subsequences) and we are able to extend the solution to the time interval $[0, \infty)$.

Theorem 4.25. *Let $(E_{\tau,h}, H_{\tau,h}, \varphi_{\tau,h})$ be the approximations obtained by Algorithm 4.8 and assume the validity of the Assumptions 4.10 and 4.14. Then there exists for any sequence $(\tau, h) \to 0$ a subsequence $(\tau_j, h_j)_{j\in\mathbb{N}_0}$, such that*

$$(E_{\tau_j,h_j}, H_{\tau_j,h_j}, \varphi_{\tau_j,h_j})$$

converges weakly in

$$H^{1,\mathrm{curl}}(\Omega_T) \times H^{1,0}(\Omega_T) \times L^2([0,T], \mathcal{H}_\Gamma)$$

to a weak solution of the Maxwell system in the sense of Definition 4.2.

Proof. We choose arbitrary test functions

$$\zeta_h^H, \zeta_h^E \in C^\infty(\overline{\Omega_T}), \qquad v \in \gamma_T(C^\infty(\overline{\Omega_T}))$$

with $v(T) = \partial_t v(T) = \cdots = \partial_t^{m-1} v(T) = 0 = \gamma_T E(T) = \cdots = \partial_t^{m-1}\gamma_T E(T)$. As discrete test functions we take

$$\zeta_h^{H,h}(t,\cdot) := \mathrm{I}_h^{\mathcal{X}} \zeta_h^H(t,\cdot), \qquad \zeta_h^{E,h}(t,\cdot) := \mathrm{I}_h^{\mathcal{X}} \zeta_h^E(t,\cdot)$$

and

$$v_h(t,\cdot) := \gamma_T(\mathrm{I}_h^{\mathcal{X}} \widehat{v})(t,\cdot),$$

where $\gamma_T \widehat{v} = v$.

We first look at the second Maxwell equation, where we write ζ_h instead of ζ_h^H for simplicity. Moreover, we use the notation

$$\zeta_{\tau,h}^{\pm} := (\zeta_h(t_k))_{\tau,h}^{\pm}.$$

Algorithm 4.8 gives by testing with $\zeta_h(t_{k+1})$ and summing up from $k = 0, \ldots, N-1$

$$[\varepsilon(\partial_t^\tau H)_{\tau,h}^+, \zeta_{\tau,h}^+]_{\Omega_T} = -[\nabla \times E_{\tau,h}^+, \zeta_{\tau,h}^+]_{\Omega_T}.$$

By Theorem 4.23 and Lemma 4.7 we get by the limit $\tau, h \to 0$

$$[\varepsilon \partial_t H, \zeta_H]_{\Omega_T} = -[\nabla \times E, \zeta_H]_{\Omega_T}.$$

For the first Maxwell equation and the boundary equations, we test with $\zeta_h(t_{k+1}) := \zeta_{E,h}(t_{k+1})$, $v_h(t_{k+1})$, sum up from $k = 0, \ldots, N-1$ and obtain

$$\begin{aligned}
[\varepsilon(\partial_t^\tau E)_{\tau,h}^+, \zeta_{\tau,h}^+]_{\Omega_T} &+ \left\langle \begin{pmatrix} v_{\tau,h}^+ \\ -\gamma_T \zeta_{\tau,h}^+ \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix} \right)_{\tau,h}^+ \right\rangle_{\Gamma_T} \\
&= [H_{\tau,h}^+, \nabla \times \zeta_{\tau,h}^+]_{\Omega_T} - \frac{1}{2\mu_0} \langle \varphi_{\tau,h}^+, \gamma_T \zeta_{\tau,h}^+ \rangle_{\Gamma_T} \\
&\quad - [\sigma E_{\tau,h}^+ + J_{\tau,h}^+, \zeta_{\tau,h}^+]_{\Omega_T} + \frac{1}{2\mu_0} \langle v_{\tau,h}^+, \gamma_T E_{\tau,h}^+ \rangle_{\Gamma_T}.
\end{aligned}$$

By Theorem 4.23 and Lemma 4.7 we get for $\tau, h \to 0$

$$\begin{aligned}
[\varepsilon(\partial_t^\tau E)_{\tau,h}^+, \zeta_{\tau,h}^+]_{\Omega_T} &\overset{\text{sub}}{\to} [\varepsilon \partial_t E, \zeta]_{\Omega_T}, \\
[H_{\tau,h}^+, \nabla \times \zeta_{\tau,h}^+]_{\Omega_T} &\overset{\text{sub}}{\to} [H, \nabla \times \zeta]_{\Omega_T} \\
\frac{1}{2\mu_0}\langle \varphi_{\tau,h}^+, \gamma_T \zeta_{\tau,h}^+\rangle_{\Gamma_T} &\overset{\text{sub}}{\to} \frac{1}{2\mu_0}\langle \varphi, \gamma_T \zeta\rangle_{\Gamma_T}, \\
[\sigma E_{\tau,h}^+ + J_{\tau,h}^+, \zeta_{\tau,h}^+]_{\Omega_T} &\overset{\text{sub}}{\to} [\sigma E + J, \zeta]_{\Omega_T}, \\
\frac{1}{2\mu_0}\langle v_{\tau,h}^+, \gamma_T E_{\tau,h}^+\rangle_{\Gamma_T} &\overset{\text{sub}}{\to} \frac{1}{2\mu_0}\langle v, \gamma_T E\rangle_{\Gamma_T}.
\end{aligned}$$

For the boundary functions, again the proof shares similarities with the proof of Theorem 3.32 and we therefore only repeat the main steps. For shorter formulas, we use the abbreviations $w_h^j := -\gamma_T \zeta_{E,h}(t_j)$ and $\psi_h^j := -\gamma_T E_h^j$. By setting $\overline{v}_h^j := v_h^{N-j}$, $\overline{w}_h^j := w_h^{N-j}$, the $\langle\cdot,\cdot\rangle_\Gamma$-adjoint B^* of B, and by using $\psi_h^0 = \varphi_h^0 = 0$ we have

$$\begin{aligned}
X_h^\tau &:= \left\langle \begin{pmatrix} v_{\tau,h}^+ \\ w_{\tau,h}^+ \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} \right)_{\tau,h}^+ \right\rangle_{\Gamma_T} \\
&= \tau \sum_{k=1}^N \left\langle B^*(\partial_t^\tau) \begin{pmatrix} \overline{v}_h \\ \overline{w}_h \end{pmatrix} (T - t_k), \begin{pmatrix} \varphi_h^k \\ \psi_h^k \end{pmatrix} \right\rangle_\Gamma .
\end{aligned}$$

Now we are able to apply the convergence result of [116], especially because

$$\overline{v}(0) = v(T) = 0, \quad \partial_t \overline{v}(0) = -\partial_t v(T) = 0, \quad \ldots, \quad \partial_t^{m-1}\overline{v}(0) = 0$$

and

$$\overline{w}(0) = \cdots = \partial_t^{m-1}\overline{w}(0) = 0,$$

we have

$$B^*(\partial_t^\tau) \begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix} (T - t_k) \to B^*(\partial_t) \begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix} (T - t_k)$$

uniformly in $0 \le t_k \le T$, $t_k = \tau k$, $k \ge 1$. Therefore and by the smoothness of $B^*(\partial_t)\binom{\overline{v}}{\overline{w}}(T - \cdot)$, dominated convergence implies

$$B^*(\partial_t^\tau) \begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix} (T - \cdot)^+ \to B^*(\partial_t) \begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix} (T - \cdot) \text{ in } L^2([0,T], \mathcal{H}_\Gamma).$$

Moreover, the discrete Herglotz theorem A.4 shows

$$\begin{aligned}
\tau \sum_{k=1}^N &\left\| B^*(\partial_t^\tau) \begin{pmatrix} \overline{v}_h \\ \overline{w}_h \end{pmatrix} (T - t_k) - B^*(\partial_t^\tau) \begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix} (T - t_k) \right\|_{\mathcal{H}_\Gamma}^2 \\
&\le C\tau \sum_{k=1}^N \|(\partial_t^\tau)^2 (\overline{v}_h - \overline{v})(T - t_k)\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^2 (\overline{w}_h - \overline{w})(T - t_k)\|_{\mathcal{H}_\Gamma}^2 \to 0
\end{aligned}$$

for $(\tau, h) \to 0$. All in all, we obtain

$$X_h^\tau \overset{\text{sub}}{\to} \left\langle B^*(\partial_t) \begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix} (T - \cdot), \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_{\Gamma_T} =: X.$$

Now we reverse the integration by parts and obtain

$$
\begin{aligned}
X &= \int_0^T \left\langle \partial_t^m \left(\mathcal{L}^{-1}(B^*(r)r^m) * \begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix} \right)(T-\cdot), \begin{pmatrix} \varphi \\ \psi \end{pmatrix}(t) \right\rangle_\Gamma \,\mathrm{d}t \\
&= (-1)^m \int_0^T \left\langle \int_t^T B_m^*(s-t)\partial_t^m \begin{pmatrix} v \\ w \end{pmatrix}(s)\,\mathrm{d}s, \begin{pmatrix} \varphi \\ \psi \end{pmatrix}(t) \right\rangle_\Gamma \,\mathrm{d}t \\
&= (-1)^m \int_0^T \left\langle \partial_t^m \begin{pmatrix} v \\ w \end{pmatrix}(s), \left(B_m * \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right)(s) \right\rangle_\Gamma \,\mathrm{d}s.
\end{aligned}
$$

This is exactly the term that shows up in the formulation of our weak solution in Definition 4.2.
The equalities $E(0) = E^0$ and $H(0) = H^0$ follow by Assumption 4.10 and the weak convergence in $H^{1,0}(\Omega_T)$. □

Corollary 4.26. *The solutions of Theorem 4.23 have bounded energy, i.e. for almost all $t \in (0,T)$*

$$
\|E(t)\|_\Omega^2 + \|H(t)\|_\Omega^2 + \|\partial_t E(t)\|_\Omega^2 + \|\partial_t H(t)\|_\Omega^2 + \|\nabla \times E(t)\|_\Omega^2 \leq C.
$$

Proof. The proof proceeds analogously as in [25]. From the discrete energy estimates Lemma 4.13, Lemma 4.17 and Lemma 4.20, we get for any $t' \in [0,T]$

$$
\|E_{\tau,h}(t')\|_\Omega^2 + \|H_{\tau,h}(t')\|_\Omega^2 + \|\partial_t E_{\tau,h}(t')\|_\Omega^2 + \|\partial_t H_{\tau,h}(t')\|_\Omega^2 + \|\nabla \times E_{\tau,h}(t')\|_\Omega^2 \leq C,
$$

where C only depends polynomially on T and the bounds of J^j, $\partial_t^\tau J^j$. Integration in time yields for any measurable set $\mathcal{A} \subset [0,T]$

$$
\begin{aligned}
&\int_\mathcal{A} \|E_{\tau,h}(t')\|_\Omega^2 + \int_\mathcal{A} \|H_{\tau,h}(t')\|_\Omega^2 + \int_\mathcal{A} \|\partial_t E_{\tau,h}(t')\|_\Omega^2 \\
&\qquad + \int_\mathcal{A} \|\partial_t H_{\tau,h}(t')\|_\Omega^2 + \int_\mathcal{A} \|\nabla \times E_{\tau,h}(t')\|_\Omega^2 \leq \int_\mathcal{A} C,
\end{aligned}
$$

whence weak lower semi-continuity leads to

$$
\begin{aligned}
&\int_\mathcal{A} \|E(t')\|_\Omega^2 + \int_\mathcal{A} \|H(t')\|_\Omega^2 + \int_\mathcal{A} \|\partial_t E(t')\|_\Omega^2 \\
&\qquad + \int_\mathcal{A} \|\partial_t H(t')\|_\Omega^2 + \int_\mathcal{A} \|\nabla \times E(t')\|_\Omega^2 \leq \int_\mathcal{A} C.
\end{aligned}
$$

The desired result now follows from standard measure theory, see, e.g. [60, IV,Theorem 4.4]. □

Due to the uniqueness in the pure Maxwell case, we are able to show the following further results.

Corollary 4.27. *Theorem 4.23 holds for the whole sequence and not only for subsequences.*

Proof. By Theorem 4.5, the solution (E, H, φ) from Definition 4.2 is unique, so for any subsequence the arguments of Theorem 4.23 can be repeated and there exists a subsubsequence converging to (E, H, φ). This already gives the convergence of the whole sequence by a contradiction argument. □

Theorem 4.28. *There exists a unique pair of functions*

$$
(E, H) : (0, \infty) \times \Omega \to \mathbb{R}^3
$$

that is a solution of Maxwell's equations in the sense of Definition 4.1 for arbitrary $T > 0$. *If* J *is smooth enough, it holds*

$$e^{-ct}E, e^{-ct}H \in H(\partial_t, \mathrm{curl}, (0, \infty) \times \Omega)$$

for every $c > 0$ *and therefore we have a solution in the sense of Definition 4.1 for* $T = \infty$, *where the equations hold in a* e^{-ct}*-weighted* $L^2([0, \infty) \times \Omega)$, *or* e^{-ct}*-weighted* $L^2([0, \infty), \mathcal{H}_\Gamma)$ *-sense, respectively.*

Proof. The existence and uniqueness follows by Theorem 4.3, Theorem 4.5 and Theorem 4.25 by considering the limit $T \to \infty$ and using the uniqueness on $[0, T]$. The constants in the energy estimates can be shown to only grow polynomially in T (compare [99]), thus it remains to show that $\nabla \times H$ is bounded in an e^{-ct}-weighted $L^2([0, \infty) \times \Omega)$ sense. This again follows from the fact that (E, H) is a solution in the sense of Definition 4.1 and all other quantities in the first equation of Definition 4.1 are bounded in an e^{-ct}-weighted $L^2([0, \infty) \times \Omega)$ sense. □

5. Convergence with Rates for the MLLG System

In this chapter, we consider convergence with rates of the Maxwell–Landau–Lifshitz–Gilbert system. We derive an algorithm for the approximation of the MLLG system that, provided the exact solution is smooth enough, converges to the solution with an a priori-know error ratio. The work is based on [4] and [99], where the convergence with rates of the LLG equation and the convergence with rates for the Maxwell system is considered, respectively.

5.1. Introduction

In this section, we recall the MLLG system from Chapter 2 and derive a weak form which serves as a basis for the following discretization.

5.1.1. Coupled boundary integral formulation

For the convenience of the reader, we recall the coupled formulation of the MLLG equation from (2.4) for the LLG part and (2.12) for the Maxwell part. We seek a magnetization

$$m : [0,T] \times \mathit{\Omega} \to \mathbb{S}^2$$

and electric and magnetic fields

$$E, H : [0,T] \times \mathit{\Omega} \to \mathbb{R}^3$$

that satisfy the coupled boundary integral formulation for the MLLG system

$$\begin{aligned} \alpha\, \partial_t m + m \times \partial_t m &= -\, m \times (m \times (\mathit{\Delta} m + H)), && \text{in } \mathit{\Omega}_T := (0,T) \times \mathit{\Omega}, && (5.1a)\\ \varepsilon \partial_t E - \nabla \times H &= -\,\sigma E - J && \text{in } \mathit{\Omega}_T, && (5.1b)\\ \mu \partial_t H + \nabla \times E &= -\,\mu \partial_t m && \text{in } \mathit{\Omega}_T, && (5.1c)\\ B(\partial_t) \begin{pmatrix} \mu_0 \gamma_T H \\ -\gamma_T E \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} \mu_0^{-1}\, \gamma_T\, E \\ \gamma_T\, H \end{pmatrix} && \text{on } \mathit{\Gamma}_T := [0,T] \times \mathit{\Gamma}, && (5.1d) \end{aligned}$$

endowed with the boundary condition

$$\partial_n m = 0 \qquad \text{on } [0,T] \times \mathit{\Gamma}, \qquad (5.1e)$$

and the initial conditions

$$m(0) = m^0, \quad E(0) = E^0, \quad H(0) = H^0 \qquad \text{in } \mathit{\Omega}. \qquad (5.1f)$$

5.1.2. Weak formulation

We will now derive a weak formulation for the coupled MLLG system (5.1) which serves as a basis for the approximation. It uses for the LLG part the tangent space approach similar to [4, Section 1.2] and [14, 15], for the Maxwell part the symmetric approach as in Chapter 3 and [99, Section 4.5] is used.

Concerning the LLG equation, the term on the right-hand side in (5.1a) can be rewritten as $\mathrm{P}(m)(\Delta m + H)$, where (with Id the 3×3 unit matrix)

$$\mathrm{P}(m) = \mathrm{Id} - mm^T = -m \times (m \times \cdot\,)$$

is the orthogonal projection onto the tangent plane to the unit sphere $\mathbb{S}^2$ at m.

We consider a weak formulation, first proposed by Alouges [14, 15], which is based on a formulation which makes use of the tangent space

$$\mathcal{T}(m) := \{\varphi \in L^2(\Omega) \,|\, m \cdot \varphi = 0 \;\text{ a.e.}\} = \{\varphi \in L^2(\Omega) \,|\, \mathrm{P}(m)\varphi = \varphi\},$$

and requiring that the time derivative $\partial_t m$ is a function in this tangent space.

For Maxwell's equations we use the symmetric variational formulation as in Chapter 3, see [99], motivated by the analogous formulation for the acoustic wave equation [1, 27]. The basis of this formulation is the following integration by parts formula from (2.13) (for sufficiently regular functions u and v):

$$[\nabla \times u, v]_\Omega = \frac{1}{2}[\nabla \times u, v]_\Omega + \frac{1}{2}[u, \nabla \times v]_\Omega - \frac{1}{2}\langle \gamma_T u, \gamma_T v\rangle_\Gamma.$$

Furthermore we introduce the abbreviations for the traces $\varphi = \mu_0 \gamma_T H$ and $\psi = -\gamma_T E$.

Altogether, the following weak formulation of the coupled MLLG system (5.1) will serve as the basis of the numerical method studied in this chapter. Find $m \in H^1(\Omega)$, with $\partial_t m \in T(m)$, $E, H \in H(\mathrm{curl}, \Omega)$ and $\varphi, \psi \in \mathcal{H}_\Gamma$ such that for all test functions $\rho \in \mathcal{T}(m) \cap H^1(\Omega)$, $\zeta^E, \zeta^H \in H(\mathrm{curl}, \Omega)$ and $v^\varphi, v^\psi \in \mathcal{H}_\Gamma$ they satisfy the coupled weak system:

$$\alpha[\partial_t m, \rho]_\Omega + [m \times \partial_t m, \rho]_\Omega = -[\nabla m, \nabla \rho]_\Omega + [H, \rho]_\Omega, \tag{5.2a}$$

$$\begin{aligned} \varepsilon[\partial_t E, \zeta^E]_\Omega &= \frac{1}{2}[\nabla \times H, \zeta^E]_\Omega + \frac{1}{2}[H, \nabla \times \zeta^E]_\Omega \\ &\quad - \frac{1}{2}\langle \mu_0^{-1}\varphi, \gamma_T \zeta^E\rangle_\Gamma - [J + \sigma E, \zeta^E]_\Omega, \end{aligned} \tag{5.2b}$$

$$\begin{aligned} \mu[\partial_t H, \zeta^H]_\Omega &= -\frac{1}{2}[\nabla \times E, \zeta^H]_\Omega - \frac{1}{2}[E, \nabla \times \zeta^H]_\Omega \\ &\quad - \frac{1}{2}\langle \psi, \gamma_T \zeta^H\rangle_\Gamma - \mu[\partial_t m, \zeta^H]_\Omega, \end{aligned} \tag{5.2c}$$

$$\left\langle \begin{pmatrix} v^\varphi \\ v^\psi \end{pmatrix}, B(\partial_t)\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_\Gamma = \frac{1}{2}\left\langle \begin{pmatrix} v^\varphi \\ v^\psi \end{pmatrix}, \begin{pmatrix} \mu_0^{-1}\gamma_T E \\ \gamma_T H \end{pmatrix} \right\rangle_\Gamma. \tag{5.2d}$$

5.2. Space and Time Discretisations

The space and time discretisations of the MLLG system (5.2) is done by a combination of the corresponding discretizations of the LLG equation from [4], and of the Maxwell system from [99].

For the space discretization of the LLG equation (5.2a) we use continuous finite elements (of degree r), while for the interior–exterior Maxwell problem (5.2b)–(5.2d) discontinuous Galerkin elements (of degree r) for the interior problem and boundary elements (of degree r) for the boundary equation are used.

For the time discretization, we use a two-step linearly implicit BDF method (of second order) for the LLG equation (5.2a), and a leapfrog discretization for the interior Maxwell part (5.2b)–(5.2c) which is coupled to a Convolution Quadrature (of second order) for the boundary integral equation (5.2d).

These discretisation methods are the same, but of a higher order in space, as those in [99].

The coupling terms in the equations contain $L^2(\Omega)$ products of functions from different approximation spaces. They are naturally evaluated as the $L^2(\Omega)$-product of a continuous finite element function and a discontinuous finite element function.

The coupling of the time discretization is slightly more involved: We first state the underlying algorithms, where we consider the coupling terms H and $\partial_t m$ as exact, given input data. Then we replace the exact data by suitable approximations, resulting in an implicit coupling. It turns out that this yields a stable second order scheme which can be evaluated at comparable cost to an uncoupled system.

After describing the general discretization setting, we briefly repeat the discretizations of the uncoupled equations from [4, 99] for exact, given input data, and then pay special attention to the additional terms due to the coupling.

5.2.1. General setting

For the discretizations in the following subsections, we assume the following general setting:

We triangulate the bounded polyhedral domain Ω by a family of simplicial triangulations $\mathcal{T}_h$, where h denotes the maximal element diameter. For our results we consider a quasi-uniform and contact-regular family of such triangulations with maximum mesh width $h \to 0$; see, e.g. [59] for these notions. For the discontinuous Galerkin method, we adopt the following notation from [84, Section 2.3]: The faces $\mathcal{F}_h$ of $\mathcal{T}_h$ are decomposed into boundary and interior faces, $\mathcal{F}_h = \mathcal{F}_h^{\mathrm{bnd}} \cup \mathcal{F}_h^{\mathrm{int}}$. The triangulation of the boundary Γ is therefore, naturally, given by the outer faces $\mathcal{F}_h^{\mathrm{bnd}}$ of $\mathcal{T}_h$.

For the time discretization, we let $t_n = n\tau,\ n = 0, \dots, N$, be a uniform partition of the interval $[0, T]$ with time step $\tau = T/N$.

5.2.2. Spatial discretisation

Concerning the LLG equation (5.2a), we consider the continuous Lagrange finite element space $\mathcal{S}_h^r \subset H^1(\Omega)$ of continuous, piecewise polynomial functions of degree r. With a function $m \in H^1(\Omega)$ that vanishes nowhere on Ω, we associate the discrete tangent space

$$\mathcal{T}_h(m) = \{\phi_h \in \mathcal{S}_h^r \,|\, (m \cdot \phi_h, v_h) = 0, \quad \forall\, v_h \in \mathcal{S}_h^r\}. \tag{5.3}$$

In comparison to the discrete tangent space $\mathcal{K}_m$ from Section 3.2, here the orthogonality is employed in a L^2-projection sense instead of a nodewise sense. Similarly to the continuous case, $\partial_t m_h$ will be required to be in this discrete tangent space.

Concerning Maxwell's equations (5.2b)–(5.2d), we use the central flux discontinuous Galerkin (dG) discretization from [99] (see also [59, 82, 84]) in the interior and continuous boundary elements on the surface.

The dG space of vector valued functions, which are elementwise polynomial functions of degree r, is defined as

$$\mathcal{W}_h^r = \{v_h \in L^2(\Omega) \,|\, v_h|_K \text{ is a polynomial of degree } r \text{ for all } K \in \mathcal{T}_h\}.$$

The boundary element space Ψ_h^r is defined as

$$\Psi_h^r = \{v_h \times n \,|\, v_h : \Gamma \to \mathbb{R}^3 \text{ is piecewise polynomial of degree } r \text{ and continuous}\}$$

Jumps and averages over faces $F \in \mathcal{F}_h^{\mathrm{int}}$ are denoted analogously as for trace operators on Γ,

$$[\![w]\!]_F = \gamma_F^+ w - \gamma_F^- w \quad \text{ and } \quad \{\!\{w\}\!\}_F = \frac{1}{2}(\gamma_F^+ w + \gamma_F^- w),$$

where γ_F is the usual trace onto the face F. We often omit the subscript as it will be clear from the context.

The discrete curl operator with centered fluxes was presented in [84, Section 2.3], for $u_h, w_h \in \mathcal{W}_h^r$,

$$[\mathrm{curl}_h\, u_h, w_h]_\Omega = \sum_{K\in\mathcal{T}_h} [\mathrm{curl}\, u_h, w_h]_K - \sum_{F\in\mathcal{F}_h^{\mathrm{int}}} \langle [\![u_h]\!], \{\!\{w_h\}\!\}\rangle_F .$$

Following the arguments of the proof of Lemma 2.2 in [84], we obtain that the discrete curl operator satisfies the discrete version of Green's formula for the curl operator:

$$[\mathrm{curl}_h\, u_h, w_h]_\Omega - [u_h, \mathrm{curl}_h\, w_h]_\Omega = -\langle \gamma_T u_h, \gamma_T w_h\rangle_\Gamma . \tag{5.4}$$

Using the above discrete tangent space $T_h(m_h)$ and the discrete operator curl_h, the (FEM–dG–BEM) semi-discrete coupled boundary integral formulation of the MLLG problem (5.2) reads as follows: Find the semi-discretization solutions $m_h \in \mathcal{S}_h$, with $\partial_t m_h \in T_h(m_h)$, and $E_h, H_h \in \mathcal{W}_h^r$ and $\varphi_h, \psi_h \in \Psi_h^r$ such that for all test functions $\rho_h \in \mathcal{T}_h(m_h)$, and $\zeta_h^E, \zeta_h^H \in \mathcal{W}_h^r$ and $v_h^\varphi, v_h^\psi \in \Psi_h^r$ they satisfy the problem

$$\alpha\big[\partial_t m_h, \rho_h\big]_\Omega + \big[m_h \times \partial_t m_h, \rho_h\big]_\Omega = -\big[\nabla m_h, \nabla \rho_h\big]_\Omega + \big[H_h, \rho_h\big]_\Omega, \tag{5.5a}$$

$$\begin{aligned}\varepsilon[\partial_t E_h, \zeta_h^E]_\Omega = {} & \frac{1}{2}[\mathrm{curl}_h\, H_h, \zeta_h^E]_\Omega + \frac{1}{2}[H_h, \mathrm{curl}_h\, \zeta_h^E]_\Omega \\ & - \frac{1}{2}\langle \mu_0^{-1}\varphi_h, \gamma_T\, \zeta_h^E\rangle_\Gamma - [J + \sigma E_h, \zeta_h^E]_\Omega,\end{aligned} \tag{5.5b}$$

$$\begin{aligned}\mu[\partial_t H_h, \zeta_h^H]_\Omega = {} & -\frac{1}{2}[\mathrm{curl}_h\, E_h, \zeta_h^H]_\Omega - \frac{1}{2}[E_h, \mathrm{curl}_h\, \zeta_h^H]_\Omega \\ & - \frac{1}{2}\langle \psi_h, \gamma_T\, \zeta_h^H\rangle_\Gamma - \mu[\partial_t m_h, \zeta_h^H]_\Omega,\end{aligned} \tag{5.5c}$$

$$\left\langle \begin{pmatrix} v_h^\varphi \\ v_h^\psi \end{pmatrix}, B(\partial_t)\begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix}\right\rangle_\Gamma = \frac{1}{2}\Big(\langle v_h^\varphi, \mu_0^{-1}\, \gamma_T\, E_h\rangle_\Gamma + \langle v_h^\psi, \gamma_T\, H_h\rangle_\Gamma\Big). \tag{5.5d}$$

We note here that the coupling terms contain the $L^2(\Omega)$ products of semi-discrete functions from different spaces.

We further point out, that the space discretization of the Maxwell part is quite flexible and could also be proven to converge for higher order Nédélec, Raviart–Thomas, or edge elements, and also for different (but compatible, compare (5.62)) spaces on the boundary, cf. [99, Remark 5.1].

5.2.3. Full discretisation

For the time discretisation of the semi-discrete MLLG system (5.5) we will use, again precisely as in [4] and [99], a two-step linearly implicit backward difference formula (BDF method) for the LLG equation, and the leapfrog (Störmer–Verlet) method and the Convolution Quadrature for, respectively, the interior and boundary integral equation of Maxwell's equations.

Recall of the underlying time discretizations

We shall discretize the LLG equation (5.5a) in time by the linearly implicit 2-step BDF method (see, e.g. [79]), described by the polynomials δ and γ

$$\delta(\zeta) = \frac{1}{2}\zeta^2 - 2\zeta + \frac{3}{2} = \sum_{j=0}^{2} \delta_j \zeta^j, \qquad \gamma(\zeta) = 2 - \zeta = \sum_{j=0}^{1} \gamma_j \zeta^j .$$

These generating polynomials define the time derivative approximation $\dot m_h^n$ and the normalized extrapolation $\widehat m_h^n$ at time t_n, for $n \geq 2$, by

$$\dot m_h^n = \frac{1}{\tau}\sum_{j=0}^{2} \delta_j m_h^{n-j}, \qquad \widehat m_h^n = \sum_{j=0}^{1} \gamma_j m_h^{n-j-1} \Big/ \Big|\sum_{j=0}^{1} \gamma_j m_h^{n-j-1}\Big| . \tag{5.6}$$

Before stating the fully discrete scheme let us make some remarks on the time discretisation of the LLG equation. We extrapolate the known values m^{n-2} and m^{n-1} to a preliminary normalized approximation $\widehat{m}_h^n$ at t_n by (5.6). To formally avoid potentially undefined quantities, we define $\widehat{m}_h^n$ to be an arbitrary fixed unit vector if the denominator in the above formula is zero. We show in the following, that this does not occur if τ and h are sufficiently small. The derivative approximation $\dot{m}_h^n$ and the solution approximation m_h^n are related by the backward difference formula

$$\dot{m}_h^n = \frac{1}{\tau}\sum_{j=0}^{2}\delta_j m_h^{n-j}, \quad \text{i.e.} \quad m_h^n = \frac{1}{\delta_0}\Big(-\sum_{j=1}^{2}\delta_j m_h^{n-j} + \tau\dot{m}_h^n\Big). \tag{5.7}$$

The Maxwell part is discretized, exactly as in [99], by the leapfrog or Störmer–Verlet scheme (see, e.g., [79]) in the interior, and Convolution Quadrature on the boundary (see Section 3.2.2). We need some auxiliary definitions here as well:

$$\overline{f}^{n+1/2} = \tfrac{1}{2}(f^{n+1} + f^n) \qquad \text{averaging in time,} \tag{5.8}$$

$$\text{and} \quad \dot{\psi}^{n+1/2} = \tfrac{1}{\tau}(\psi^{n+1} - \psi^n) \qquad \text{a second-order discrete time derivative.} \tag{5.9}$$

As in [99] will need a stabilizing term with a parameter $\beta > 0$ that guarantees the stability of our scheme. For convenience, we define the LIFT of a function $\psi_h \in \mathcal{V}_h$ to Ω via $\mathrm{LIFT}\psi_h = L_h \in \mathcal{W}_h^r$ satisfying

$$[L_h, v_h^L]_\Omega = \frac{1}{4}\langle \psi_h, \gamma_T v_h^L\rangle_\Gamma \quad \text{for all } v_h^L \in \mathcal{W}_h^r. \tag{5.10}$$

Note that this is uniquely defined, as there is a coercive bilinear form on the left hand side of (5.10). In [99], the vector product $\langle v_h^\psi, \gamma_T \mathrm{LIFT}(\psi_h)\rangle_\Gamma$ is written as multiplication of the coefficients with the matrix $\mathbf{C}_1^T\mathbf{M}^{-1}\mathbf{C}_1$ (compare Section 6.4.1).

Convolution Quadrature (CQ) discretizes the convolution $B(\partial_t)w(t)$, defined by (2.16), by the discrete convolution

$$(B(\partial_t^\tau)w)(n\tau) = \sum_{j=0}^{n} B_{n-j}w(j\tau),$$

where the weights B_n are defined as the coefficients of

$$B\Big(\frac{\delta(\zeta)}{\tau}\Big) = \sum_{n=0}^{\infty} B_n\zeta^n.$$

In the present chapter and differently to Chapter 3, we choose

$$\delta(\zeta) = (1-\zeta) + \frac{1}{2}(1-\zeta)^2 = \frac{1}{2}\zeta^2 - 2\zeta + \frac{3}{2},$$

which corresponds to the second-order backward difference formula. From [115], it is known that the method is of order two,

$$\|(B(\partial_t)w)(t_n) - (B(\partial_t^\tau)w)(t_n)\| = O(\tau^2), \text{ uniformly in } 0 \le t_n \le T,$$

for functions w that are sufficiently smooth including their extension by 0 to negative values of t. An important property of this discretization is that it preserves the coercivity of the continuous-time convolution in the time discretization.

Coupling

We now concentrate on the discretization of the additional terms due to the coupling. The rest of the respective equations is discretized as in [4] and [99] and we refer there for the details. In the end of this section, the full discretization is stated.

Concerning the LLG part, assuming given exact input data $H(t)$, the right-hand side term in the n-th step is given by $(H(t_n), \varphi_h)$. This is discretized via

$$(H(t_n), \varphi_h) \approx (H_h^n, \varphi_h).$$

Concerning the Maxwell part, assuming given exact input $G = -\mu\partial_t m$, the corresponding terms in the uncoupled discretization are given by $(\partial_t m(t_{n-1}), \zeta_h^H)$ and $(\partial_t m(t_n), \zeta_h^H)$. These terms are discretized via

$$(\partial_t m(t_{n-1}), \zeta_h^H) \approx (\dot{m}_h^{n-1}, \zeta_h^H), \quad (\partial_t m(t_n), \zeta_h^H) \approx (\dot{m}_h^n, \zeta_h^H).$$

We therefore also need to impose starting data $\dot{m}_h^0$, $\dot{m}_h^1$. In comparison to [99], we also include conductivity to our physical model (but this is not relevant for the analysis). This additional term is discretized via

$$(\sigma E(t_{n-1/2}), \zeta_h^E) \approx (\sigma \overline{E}_h^{n-1/2}, \zeta_h^E).$$

Taken together, we will see, that this still yields a stable scheme under the same CFL-condition, with the same convergence rates and with comparable cost to the separated execution of both schemes.

The fully discrete MLLG system

In summary, we determine the approximations to m, and to E, H, and φ, ψ by solving the following fully discrete system: Find, for $n \geq 2$, $\dot{m}_h^n \in \mathcal{T}_h(\widehat{m}_h^n)$ (where $\widehat{m}_h^n$ and $\dot{m}_h^n$ are related via (5.7)), $H_h^{n-1/2}, E_h^n, H_h^n \in \mathcal{W}_h^r$, and $\varphi_h^n, \psi_h^n \in \Psi_h^r$ such that, for all test functions $\rho_h \in \mathcal{T}_h(\widehat{m}_h^n)$, and $\zeta_h^{H,1/2}, \zeta_h^E, \zeta_h^{H,1} \in \mathcal{W}_h^r$ and $v_h^\varphi, v_h^\psi \in \Psi_h^r$, the following holds

$$\alpha\big[\dot{m}_h^n, \rho_h\big]_\Omega + \big[\widehat{m}_h^n \times \dot{m}_h^n, \rho_h\big]_\Omega + \big[\nabla m_h^n, \nabla\rho_h\big]_\Omega = \big[H_h^n, \rho_h\big]_\Omega, \tag{5.11a}$$

$$\begin{aligned}\mu[H_h^{n-1/2}, \zeta_h^{H,1/2}]_\Omega = \mu[H_h^{n-1}, \zeta_h^{H,1/2}]_\Omega - \frac{\tau}{4}[\mathrm{curl}_h\, E_h^{n-1}, \zeta_h^{H,1/2}]_\Omega - \frac{\tau}{4}[E_h^{n-1}, \mathrm{curl}_h\, \zeta_h^{H,1/2}]_\Omega \\ - \frac{\tau}{4}\langle \psi_h^{n-1}, \gamma_T \zeta_h^{H,1/2}\rangle_\Gamma - \frac{\tau}{2}\mu[\dot{m}_h^{n-1}, \zeta_h^{H,1/2}]_\Omega,\end{aligned} \tag{5.11b}$$

$$\begin{aligned}\varepsilon[E_h^n, \zeta_h^E]_\Omega = \varepsilon[E_h^{n-1}, \zeta_h^E]_\Omega + \frac{\tau}{2}[\mathrm{curl}_h\, H_h^{n-1/2}, \zeta_h^E]_\Omega + \frac{\tau}{2}[H_h^{n-1/2}, \mathrm{curl}_h\, \zeta_h^E]_\Omega \\ - \frac{\tau}{2}\langle \mu_0^{-1}\varphi_h^{n-1/2}, \gamma_T\zeta_h^E\rangle_\Gamma - \tau[\sigma\overline{E}_h^{n-1/2} + J^{n-1/2}, \zeta_h^E]_\Omega,\end{aligned} \tag{5.11c}$$

$$\begin{aligned}\mu[H_h^n, \zeta_h^{H,1}]_\Omega = \mu[H_h^{n-1/2}, \zeta_h^{H,1}]_\Omega - \frac{\tau}{4}[\mathrm{curl}\, E_h^n, \zeta_h^{H,1}]_\Omega - \frac{\tau}{4}[E_h^n, \mathrm{curl}\, \zeta_h^{H,1}]_\Omega \\ - \frac{\tau}{4}\langle \psi_h^n, \gamma_T\zeta_h^{H,1}\rangle_\Gamma - \frac{\tau}{2}\mu[\dot{m}_h^n, \zeta_h^{H,1}]_\Omega,\end{aligned} \tag{5.11d}$$

$$\begin{aligned}\left\langle \begin{pmatrix} v_h^\varphi \\ v_h^\psi \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h \\ \psi_h\end{pmatrix}\right)^{n-1/2} \right\rangle_\Gamma = \frac{1}{2}\left\langle \begin{pmatrix} v_h^\varphi \\ v_h^\psi \end{pmatrix}, \begin{pmatrix} \mu_0^{-1}\gamma_T\overline{E}_h^{n-1/2} \\ \gamma_T H_h^{n-1/2}\end{pmatrix} \right\rangle_\Gamma \\ - \beta\tau^2\mu_0^{-1}\langle v_h^\psi, \gamma_T \mathrm{LIFT}(\partial_\tau \psi_h^{n-1/2})\rangle_\Gamma.\end{aligned} \tag{5.11e}$$

We assume that the following starting values are given: m_h^0, $\dot{m}_h^0$, H_h^0, E_h^0, $\varphi_h^0 = 0$, $\psi_h^0 = 0$, and m_h^1, $\dot{m}_h^1$. We note here that the values to perform the first step $n = 2$ are obtained by making a single time step with the sub-system (5.11b)–(5.11e).

Although the LLG and Maxwell equations are coupled in an implicit way, the cost of one step of the coupled algorithm is as high as one step with both uncoupled algorithms. This favorable property is explored in detail in Remark 6.4. The role of the stabilising term in (5.11e) is analogous to [99].

5.3. Main Results: Error Estimates and Discrete Energy Inequality

We state the main theorem of this chapter and postpone the proof to Section 5.7.

Theorem 5.1. *Assume that the solution components of the Maxwell–Landau–Lifshitz–Gilbert system* (5.1) *are sufficiently smooth, the precise regularity conditions are stated in* (5.13).

Consider the full discretization (5.11) *of the MLLG system by finite elements of degree r / linearly implicit 2-step BDF method for the LLG equation, and discontinuous Galerkin elements and continuous boundary elements of degree r / leapfrog method and Convolution Quadrature discretization for the boundary integral formulation of Maxwell's equation.*

Then, for any stabilization parameter $\beta \geq 1$, there exist $\bar{c} > 0$, $\overline{\tau} > 0$ and $\overline{h} > 0$ such that for all $\tau \leq \overline{\tau}$, $h \leq \overline{h}$ with

$$\tau \leq \bar{c} h, \tag{5.12}$$

the errors are bounded by, provided that the errors of the starting values satisfy a similar bound,

$$\begin{aligned} \max_{0 \leq t_n \leq T} \|m_h^n - m(t_n, \cdot)\|_{H^1(\Omega)} &\leq C(h^r + \tau^2), \\ \max_{0 \leq t_n \leq T} \|E_h^n - E(t_n, \cdot)\|_{L^2(\Omega)} &\leq C(h^r + \tau^2), \\ \max_{0 \leq t_n \leq T} \|H_h^n - H(t_n, \cdot)\|_{L^2(\Omega)} &\leq C(h^r + \tau^2), \end{aligned}$$

where $C > 0$ is independent of h, τ and n, but depends on the material parameters, on the regularity of the solution components, and grows exponentially in T.

Sufficient regularity conditions are

$$\begin{aligned} &m \in C^3([0,T], H^1(\Omega)) \cap C^1([0,T], W^{r+1,\infty}(\Omega)) \quad \text{and} \\ &\Delta m + H \in C([0,T], W^{r+1,\infty}(\Omega)) \quad \text{and} \\ &E, H \in C^3([0,T], L^2(\Omega)) \cap C^1([0,T], H^{r+1}(\Omega)) \quad \text{and} \\ &\varphi, \psi \in C^6_{0,*}([0,T], \mathcal{H}_\Gamma) \cap C^2_{0,*}([0,T], H^{r+1/2}(\Gamma)), \end{aligned} \tag{5.13}$$

where we again point out the condition of vanishing derivatives at $t = 0$ for the boundary functions in (2.14). Also note that it holds $\varphi = \mu_0 \gamma_T H$ and $\psi = -\gamma_T E$ if Theorem 5.1 applies.

Remark 5.2. *Note that there is no bound given for the functions on the boundary. An estimate like*

$$\tau \sum_{j=0}^{n-1} \|(\partial_t^\tau)^{-1} \varphi_h^{j+1/2} - \partial_t^{-1} \varphi(t_{j+1/2})\|^2_{\mathcal{H}_\Gamma} \\ + \|(\partial_t^\tau)^{-1} \overline{\psi}_h^{j+1/2} - \partial_t^{-1} \psi(t_{j+1/2})\|^2_{\mathcal{H}_\Gamma} \leq C(h^r + \tau^2)^2$$

can be obtained by some modifications, compare Remark 5.18 and Remark 5.7. This is the only part, where the results in the coupled situation are weaker then for the uncoupled Maxwell system, compare [99, Theorem 7.1], where a similar estimate without the integration in time via ∂_t^{-1} is stated. We refer to Section 7.2 for a discussion.

The electric and magnetic fields can be evaluated in the exterior domain with the representation formula (2.8).

Remark 5.3. *Note that the CFL condition* (5.12) *is exactly the same as for the uncoupled algorithms.*

The LLG algorithm asks for (see [4, Section 2])

$$\tau^4 \leq ch \tag{5.14}$$

for a sufficiently small constant $c > 0$. *Having* $\tau \leq \bar{c}h$ *and* $\tau \leq \tau_0$ *we may choose* τ_0 *sufficiently small meet this requirement. Concerning the coupled exterior-interior algorithm, the CFL condition is given in [99, Section 7] for matrices* $\mathbf{D}$, $\mathbf{M}$ *as*

$$\tau \|\mathbf{M}^{-1/2}\mathbf{D}\mathbf{M}^{-1/2}\|_2 \leq \sqrt{\varepsilon\mu}. \tag{5.15}$$

One can show that this is equivalent to $\tau \leq Ch$ *for a constant* $C > 0$.

Remark 5.4. *We note here that the results translate, subject to some purely technical modifications, to variable and matrix-valued material parameters in the interior:* $\varepsilon, \mu : \Omega \to \mathbb{R}^{3\times 3}$ *(symmetric, coercive and bounded) and* $\sigma : \Omega \to \mathbb{R}^{3\times 3}$ *(non-negative and bounded), compare Remark 3.16.*

5.3.1. Discrete energy inequality

In this section, we state another favorable property of the approximation (5.11): Under the same CFL condition as before, the discrete energy remains bounded. This is an important robustness property for the numerical scheme, compare Chapter 3.

We first present the energy inequality for the exact solution (5.1) and then show that a similar discrete estimate that is satisfied by the approximation (5.11).

Testing the weak problem (5.2) with $(\rho, \zeta^E, \zeta^H, v^\varphi, v^\psi) = (\partial_t m, E, H, \varphi, \psi)$, and then multiplying the LLG equation (5.2a) by μ (in order to cancel the mixed coupling terms $[\partial_t m, H]_\Omega$ in (5.2a) and (5.2c)) and summing up the equations, we obtain

$$\begin{aligned} &\mu\alpha[\partial_t m, \partial_t m]_\Omega + \mu[\nabla m, \partial_t \nabla m]_\Omega \\ &\qquad + [\varepsilon\partial_t E, E]_\Omega + [\mu\partial_t H, H]_\Omega + \left\langle \begin{pmatrix}\varphi\\ \psi\end{pmatrix}, B(\partial_t)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\rangle_\Gamma = -[J + \sigma E, E]_\Omega. \end{aligned} \tag{5.16}$$

We exploit $\partial_t \|e\|_{L^2}^2 = 2(\partial_t e, e)$, integrate (5.16) in time and use the Cauchy–Schwarz and Young's inequalities to get

$$\begin{aligned} &\int_0^t \|\partial_t m(s)\|_{L^2}^2 \,\mathrm{d}s + \|\nabla m(t)\|_{L^2}^2 + \|E(t)\|_{L^2}^2 + \|H(t)\|_{L^2}^2 + \int_0^t \left\langle \begin{pmatrix}\varphi\\ \psi\end{pmatrix}, B(\partial_t)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\rangle_\Gamma \mathrm{d}s \\ &\quad \leq C\left(\|\nabla m(0)\|_{L^2}^2 + \|E(0)\|_{L^2}^2 + \|H(0)\|_{L^2}^2 + \int_0^t \|E(s)\|_{L^2}^2 + \|J(s)\|_{L^2}^2 \,\mathrm{d}s\right), \end{aligned}$$

for a constant $C > 0$ depending on α, μ, ε, and σ. The coercivity of the Calderon operator 2.16, and finally using Gronwall's inequality A.1, conclude the energy inequality

$$\begin{aligned} &\int_0^t \|\partial_t m(s)\|_{L^2}^2 \,\mathrm{d}s + \|\nabla m(t)\|_{L^2}^2 + \|E(t)\|_{L^2}^2 + \|H(t)\|_{L^2}^2 \\ &\qquad \leq C\left(\|\nabla m(0)\|_{L^2}^2 + \|E(0)\|_{L^2}^2 + \|H(0)\|_{L^2}^2 + \int_0^t \|J(s)\|_{L^2}^2 \,\mathrm{d}s\right). \end{aligned} \tag{5.17}$$

Remark 5.5. *Note that the multiplication of the LLG equation with the factor* μ *for the cancellation of the coupling term* $[\partial_t m, H]$ *(compared to* (5.16)*) is not necessary, the Cauchy–Schwarz and Young's inequalities give for arbitrary* $\delta > 0$

$$[\partial_t m, H] \leq \frac{\delta}{2}\|\partial_t m\|_{L^2}^2 + \frac{1}{2\delta}\|H\|_{L^2}^2.$$

Therefore the term $\|\partial_t m\|_{L^2}^2$ *can be absorbed and the term* $\|H\|_{L^2}^2$ *is treated by Gronwall's lemma A.1. This is how we proceed in the following.*

Similarly as above, we test with discrete counterparts in the approximation (5.11). Then we can prove the following discrete energy inequality, which holds under very weak regularity assumptions on the data.

Proposition 5.6 (Energy inequality)**.** *The numerical solution satisfies the following discrete energy inequality: under the CFL condition (5.12), for a stabilization parameter $\beta \geqslant 1$ and for $\tau \leq \tau_0$ small enough, we have for $0 \leq n\tau \leq T$ that the discrete combined Maxwell–LLG energy*

$$\mathcal{E}_h^n := \mathcal{E}_{\mathrm{Max},h}^n + \mathcal{E}_{\mathrm{LLG},h}^n := \|H_h^n\|_\Omega^2 + \|E_h^n\|_\Omega^2 + \|\nabla m_h^n\|_\Omega^2 + \tau \sum_{i=0}^{n} \|\dot{m}_h^i\|_\Omega^2$$

is bounded by

$$\mathcal{E}_h^n \leq C \left(\mathcal{E}_h^0 + \mathcal{E}_{\mathrm{LLG},h}^1 + \tau \sum_{i=0}^{n-1} \|J^{i+1/2}\|_\Omega^2 \right).$$

The constant C depends on T, α, ε, μ, σ but is independent of h and τ.

Proof. The proof of this lemma transfers the arguments of the continuous energy inequality (5.17) to the fully discrete situation, using time discrete energy estimates obtained by testing the approximation system (5.11), respectively, with

$$\begin{aligned} &\rho_h = \dot{m}_h^n, \\ &\zeta_h^{H,1/2} = \frac{1}{2}\overline{H}_h^{n-1}, \quad \zeta_h^E = \overline{E}_h^{n-1/2}, \quad \zeta_h^{H,1} = \frac{1}{2}\overline{H}_h^n, \\ &v_h^\varphi = \varphi_h^{n-1/2}, \quad v_h^\psi = \overline{\psi}_h^{n-1/2}, \end{aligned} \tag{5.18}$$

where we recall (5.8), i.e. that $\overline{e}^j = \frac{1}{2}(e^{j+1/2} + e^{j-1/2})$. We combine the energy estimates of [4, Lemma 3.2], [27, Lemma 8.1] and [99, Lemma 7.1] and pay special attention to the coupling terms. To help the reader we will follow the structure of the referenced proofs below:

Energy estimates for LLG. (cf. [4, Lemma 3.2])

For the LLG equation (5.11a), we have $\rho_h = \dot{m}_h^n \in T_h(\widehat{m}_h^n)$ and with $(\widehat{m}_h^n \times \dot{m}_h^n, \dot{m}_h^n) = 0$, we obtain for $n \geq 2$

$$\alpha \|\dot{m}_h^n\|_{L^2}^2 + (\nabla m_h^n, \nabla \dot{m}_h^n) = (H^n, \dot{m}_h^n).$$

Summing up from 2 to n we deduce

$$\tau \sum_{j=2}^{n} \alpha \|\dot{m}_h^j\|_{L^2}^2 + \tau \sum_{j=2}^{n} [\nabla m_h^j, \nabla \dot{m}_h^j]_\Omega = \tau \sum_{j=2}^{n} [H^j, \dot{m}_h^j]_\Omega. \tag{5.19}$$

The A-stability of the second-order BDF method via Dahlquist's G-stability theory (as explained in [4, Lemma 3.2 and Lemma 10.1]) gives for constants $\gamma^\pm > 0$

$$\tau \sum_{j=2}^{n} [\nabla m_h^j, \nabla \dot{m}_h^j]_\Omega \geq \gamma^- [\nabla m_h^n, \nabla m_h^n]_\Omega - \gamma^+ \left([\nabla m_h^0, \nabla m_h^0]_\Omega + [\nabla m_h^1, \nabla m_h^1]_\Omega \right). \tag{5.20}$$

Combining (5.19) and (5.20) concludes

$$\mathcal{E}_{\mathrm{LLG},h}^n := \|\nabla m_h^n\|_{L^2}^2 + \tau \sum_{j=2}^{n} \|\dot{m}_h^j\|_{L^2}^2 \leqslant C \sum_{i=0}^{1} \|\nabla m_h^i\|_{L^2}^2 + C\tau \sum_{j=2}^{n} [H^j, \dot{m}_h^j]_\Omega, \tag{5.21}$$

with $C > 0$ depending on α, but independent of h, τ and n.

Energy estimates for Maxwell. (cf. [27, Lemma 8.1] and [99, Lemma 7.1])

By testing the discrete equations (5.11b)–(5.11e) (where the leapfrog steps for H_h^j (5.11b) and (5.11d) are expressed using the midpoint values $H_h^{j\pm 1/2}$) with the above functions (5.18), and then taking the sum of the equations from weighting the boundary integral equation with τ, after collecting and rearranging the terms, we obtain

$$\begin{aligned}
&\frac{1}{4}\mu\Big(\|H_h^{n+1/2}\|_{L^2}^2 - \|H_h^{n-3/2}\|_{L^2}^2\Big) + \frac{1}{2}\varepsilon\Big(\|E_h^n\|_{L^2}^2 - \|E_h^{n-1}\|_{L^2}^2\Big)\\
&- \frac{1}{4}\tau\Big(- [\operatorname{curl}_h E_h^{n-1}, \overline{H}_h^{n-1}]_\Omega - [E_h^{n-1}, \operatorname{curl}_h \overline{H}_h^{n-1}]_\Omega - \langle \psi_h^{n-1}, \gamma_T \overline{H}_h^{n-1}\rangle_\Gamma\Big)\\
&- \frac{1}{4}\tau\Big(- [\operatorname{curl}_h E_h^{n}, \overline{H}_h^{n}]_\Omega - [E_h^{n}, \operatorname{curl}_h \overline{H}_h^{n}]_\Omega - \langle \psi_h^{n}, \gamma_T \overline{H}_h^{n}\rangle_\Gamma\Big)\\
&+ \frac{1}{2}\tau\Big(- [\operatorname{curl}_h \overline{E}_h^{n-1/2}, H_h^{n-1/2}]_\Omega - [\overline{E}_h^{n-1/2}, \operatorname{curl}_h H_h^{n-1/2}]_\Omega - \langle \overline{\psi}_h^{n-1/2}, \gamma_T H_h^{n-1/2}\rangle_\Gamma\Big)\\
&\qquad + \beta\tau^3\mu_0^{-1}\langle \gamma_T^* \overline{\psi}_h^{n-1/2}, \gamma_T^* \partial_\tau e_{\psi,h}^{n-1/2}\rangle_\Gamma\\
&+ \tau\left\langle \begin{pmatrix}\varphi_h^{n-1/2}\\ \overline{\psi}_h^{n-1/2}\end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix}\varphi_h\\ \overline{\psi}_h\end{pmatrix}\right)^{n-1/2}\right\rangle_\Gamma\\
= &-\tau\mu[\dot{m}_h^{n-1}, \frac{1}{2}\overline{H}_h^{n-1}]_\Omega - \tau\mu[\dot{m}_h^{n}, \frac{1}{2}\overline{H}_h^{n}]_\Omega - \tau\sigma\|\overline{E}_h^{n-1/2}\|_{L^2}^2 - \tau[J_h^{n-1/2}, \overline{E}_h^{n-1/2}]_\Omega,
\end{aligned} \tag{5.22}$$

where we used the notation

$$\langle \gamma_T^* \varphi_h, \gamma_T^* \psi_h\rangle_\Gamma := [\gamma_T \varphi_h, \gamma_T(\mathrm{LIFT}\psi_h)]_{\mathcal{H}_\Gamma}. \tag{5.23}$$

Note that the LIFT-operator (5.10) is anti-symmetric w.r.t. the anti-symmetric pairing such that the term on the right hand side of (5.23) is symmetric and γ_T^* is well-defined.

We now rewrite the terms in the second to fourth lines of (5.22). Using that $\dot{\zeta}^n = (\zeta^{n+1/2} - \zeta^{n-1/2})/\tau$ (cf. (5.9)) we obtain

$$\begin{aligned}
&- \frac{1}{4}\tau\Big(- [\operatorname{curl}_h E_h^{n-1}, \overline{H}_h^{n-1}]_\Omega - [E_h^{n-1}, \operatorname{curl}_h \overline{H}_h^{n-1}]_\Omega - \langle \psi_h^{n-1}, \gamma_T \overline{H}_h^{n-1}\rangle_\Gamma\Big)\\
&- \frac{1}{4}\tau\Big(- [\operatorname{curl}_h E_h^{n}, \overline{H}_h^{n}]_\Omega - [E_h^{n}, \operatorname{curl}_h \overline{H}_h^{n}]_\Omega - \langle \psi_h^{n}, \gamma_T \overline{H}_h^{n}\rangle_\Gamma\Big)\\
&+ \frac{1}{2}\tau\Big(- [\operatorname{curl}_h \overline{E}_h^{n-1/2}, H_h^{n-1/2}]_\Omega - [\overline{E}_h^{n-1/2}, \operatorname{curl}_h H_h^{n-1/2}]_\Omega - \langle \overline{\psi}_h^{n-1/2}, \gamma_T H_h^{n-1/2}\rangle_\Gamma\Big)\\
= &- \frac{1}{16}\tau^2\Big(- [\operatorname{curl}_h E_h^{n}, \dot{H}_h^{n}]_\Omega - [E_h^{n}, \operatorname{curl}_h \dot{H}_h^{n}]_\Omega - \langle \psi_h^{n}, \gamma_T \dot{H}_h^{n}\rangle_\Gamma\Big)\\
&+ \frac{1}{16}\tau^2\Big(- [\operatorname{curl}_h E_h^{n-1}, \dot{H}_h^{n-1}]_\Omega - [E_h^{n-1}, \operatorname{curl}_h \dot{H}_h^{n-1}]_\Omega - \langle \psi_h^{n-1}, \gamma_T \dot{H}_h^{n-1}\rangle_\Gamma\Big).
\end{aligned} \tag{5.24}$$

Similarly, with the symmetry of the product in (5.23), we obtain

$$\beta\tau^3\mu_0^{-1}[\gamma_T^* \overline{\psi}_h^{n-1/2}, \gamma_T^* \partial_\tau \psi_h^{n-1/2}]_{\mathcal{H}_\Gamma} = \beta\tau^2\mu_0^{-1}\Big(\frac{1}{2}\|\gamma_T^* \psi_h^n\|_{\mathcal{H}_\Gamma}^2 - \frac{1}{2}\|\gamma_T^* \psi_h^{n-1}\|_{\mathcal{H}_\Gamma}^2\Big). \tag{5.25}$$

Combining the above identities, and introducing the modified energy for the Maxwell part (for $n = 0$, we define $H_h^{-1/2}$ from $2H^0 = H^{-1/2} + H^{1/2}$)

$$\begin{aligned}
\widetilde{\mathcal{E}}_{\mathrm{Max},h}^n := &\frac{1}{4}\mu\|H_h^{n+1/2}\|_{L^2}^2 + \frac{1}{2}\varepsilon\|E_h^n\|_{L^2}^2 + \frac{1}{4}\mu\|H_h^{n-1/2}\|_{L^2}^2\\
&- \frac{1}{16}\tau^2\Big(- [\operatorname{curl}_h E_h^{n}, \dot{H}_h^{n}]_\Omega - [E_h^{n}, \operatorname{curl}_h \dot{H}_h^{n}]_\Omega - \langle \psi_h^{n}, \gamma_T \dot{H}_h^{n}\rangle_\Gamma\Big)\\
&+ \beta\tau^2\mu_0^{-1}\frac{1}{2}\|\gamma_T^* \psi_h^n\|_{\mathcal{H}_\Gamma}^2,
\end{aligned} \tag{5.26}$$

the energy equality (5.22) can then be written as

$$\begin{aligned} &\widetilde{\mathcal{E}}^n_{\mathrm{Max},h} - \widetilde{\mathcal{E}}^{n-1}_{\mathrm{Max},h} + \tau \left\langle \begin{pmatrix} \varphi_h^{n-1/2} \\ \overline{\psi}_h^{n-1/2} \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} \right)^{n-1/2} \right\rangle_\Gamma \\ &= -\frac{\tau\mu}{2}[\dot{m}_h^{n-1}, \overline{H}_h^{n-1}]_\Omega - \frac{\tau\mu}{2}[\dot{m}_h^n, \overline{H}_h^n]_\Omega - \tau\sigma\|\overline{E}_h^{n-1/2}\|_{L^2}^2 - \tau[J_h^{n-1/2}, \overline{E}_h^{n-1/2}]_\Omega. \end{aligned} \tag{5.27}$$

Summing up from 1 to n yields

$$\begin{aligned} &\widetilde{\mathcal{E}}^n_{\mathrm{Max},h} - \widetilde{\mathcal{E}}^0_{\mathrm{Max},h} + \tau \sum_{j=1}^n \left\langle \begin{pmatrix} \varphi_h^{j-1/2} \\ \overline{\psi}_h^{j-1/2} \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} \right)^{j-1/2} \right\rangle_\Gamma \\ &\quad \le \frac{\tau\mu}{2} \sum_{j=0}^n |[\dot{m}_h^j, \overline{H}_h^j]_\Omega| - \tau \sum_{j=1}^n \Big(\sigma \|\overline{E}_h^{j-1/2}\|_{L^2}^2 + \tau [J_h^{j-1/2}, \overline{E}_h^{j-1/2}]_\Omega \Big). \end{aligned} \tag{5.28}$$

As in the proof of Lemma 8.1 from [27], with the CFL condition (5.15) and the lower bound on the stabilization parameter $\beta \ge 1$ and using (5.11b), one shows that the modified Maxwell energy is bounded from below by (using $(H_h^{n-1/2} + H_h^{n+1/2})/2 = H_h^n$)

$$c\widetilde{\mathcal{E}}^n_{\mathrm{Max},h} \ge \tfrac12 \|H_h^{n-1/2}\|_{L^2}^2 + \tfrac12 \|H_h^{n+1/2}\|_{L^2}^2 + \|E_h^n\|_{L^2}^2 \ge \|H_h^n\|_{L^2}^2 + \|E_h^n\|_{L^2}^2, \tag{5.29}$$

for a constant $c > 0$ depending on ε and μ. Similarly, using $\psi(0) = 0$, we get for $C > 0$

$$\widetilde{\mathcal{E}}^0_{\mathrm{Max},h} \le C(\|H_h^0\|_{L^2}^2 + \|E_h^0\|_{L^2}^2). \tag{5.30}$$

Combining (5.28), (5.29) and (5.30) with the positivity of the Calderon operator, the Cauchy–Schwarz and Young's inequalities, and collecting the terms, yields

$$\begin{aligned} \|H_h^n\|_{L^2}^2 + \|E_h^n\|_{L^2}^2 \le C\Big(&\|H_h^0\|_{L^2}^2 + \|E_h^0\|_{L^2}^2 + \tau \sum_{j=0}^n |[\dot{m}_h^j, \overline{H}_h^j]_\Omega| \\ &+ \tau \sum_{j=0}^n \|E_h^j\|_{L^2}^2 + \tau \sum_{j=1}^n \|J_h^{j-1/2}\|_{L^2}^2 \Big) \end{aligned} \tag{5.31}$$

for a constant $C > 0$ depending on μ, ε and σ.

Combination.

We now combine the two energy estimates (5.21) and (5.31). We exploit Cauchy–Schwarz and Young's inequalities to get for arbitrary $\varrho > 0$ (note $\overline{H}_h^j = H_h^j$)

$$\tau \sum_{j=0}^n |[\dot{m}_h^j, \overline{H}_h^j]_\Omega| \le \tau \frac{\rho}{2} \sum_{j=0}^n \|\dot{m}_h^j\|_{L^2}^2 + \tau \frac{1}{2\rho} \sum_{j=0}^n \|H_h^j\|_{L^2}^2.$$

Choosing $\varrho > 0$ sufficiently small (independently of h and τ) to absorb the $\sum \|\dot{m}_h^j\|_{L^2}^2$ term, we altogether obtain that the combined discrete MLLG energy

$$\mathcal{E}_h^n := \|\nabla m_h^n\|_{L^2}^2 + \tau \sum_{j=0}^n \|\dot{m}_h^j\|_{L^2}^2 + \|H_h^n\|_{L^2}^2 + \|E_h^n\|_{L^2}^2 \tag{5.32}$$

satisfies the bound

$$\mathcal{E}_h^n \le C\Big(\mathcal{E}_h^0 + \mathcal{E}^1_{\mathrm{LLG},h} + \tau \sum_{j=0}^n \mathcal{E}_h^j + \tau \sum_{j=1}^n \|J_h^{j-1/2}\|_{L^2}^2 \Big) \tag{5.33}$$

for a constant $C > 0$ depending on α, μ, ε and σ. By applying the discrete Gronwall inequality (see Lemma A.2 in the Appendix) for sufficiently small $\tau > 0$ (i.e. absorbing $\tau C \mathcal{E}_h^n$), we obtain the stated energy estimate. □

Remark 5.7. *Under the same conditions as in Proposition 5.6, it holds*

$$\tau \sum_{i=0}^{n} \Big(\|(\partial_t^\tau)^{-1} \varphi(t_{i+1/2})\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^{-1} \overline{\psi}(t_{i+1/2})\|_{\mathcal{H}_\Gamma}^2 \Big) \leq C \left(\mathcal{E}_h^0 + \mathcal{E}_h^1 + \tau \sum_{i=0}^{n} \|J^{i+1/2}\|_{L^2}^2 \right). \tag{5.34}$$

The constant C depends on T, α, ε, ε_0, μ, μ_0, σ but can be chosen independently of h and τ.

Proof. The assertion can be shown by a modification with factors $e^{-2t/T}$ as in Lemma 3.20. □

Remark 5.8. *The above proof gives some insight in how we can combine the arguments from [4] and [99] to prove Proposition 5.6. In the following sections, for simplicity however, we choose a different approach. Instead of repeating the steps from [4] and [99] and applying similar arguments as in the proof of Proposition 5.6 for the coupling terms, we directly use the stability results from [4] and [99]. Therefore we treat the coupling terms as additional terms on the right hand side, apply the stability results of the uncoupled equations (which use a Gronwall estimate) and then use absorption techniques to get rid of the additional terms. A further application of Gronwall's lemma concludes the assertion. This results in theoretically larger constants, but reuses more of the arguments of the uncoupled systems.*

5.4. Preparations

In this section, we collect the main results of the underlying papers [4] and [99] in a way such that they can be applied as needed in the following. We sketch how the proofs in the respective references have to be modified. The results are collected in a brief version, for a comprehensive study and the modifications in the proofs, we refer to the respective references.

5.4.1. The LLG equation

A continuous perturbation result

As in Section 4 from [4], let $m(t)$ be a solution of

$$\alpha\, \partial_t m + m \times \partial_t m = \mathrm{P}(m)(\Delta m + H). \tag{5.35}$$

for $0 \leqslant t \leqslant T$, and let $m_\star(t)$, also of unit length, solve the same equation up to a defect $d(t)$:

$$\alpha \partial_t m_\star + m_\star \times \partial_t m_\star = \mathrm{P}(m_\star)(\Delta m_\star + H) + d. \tag{5.36}$$

Then, the following perturbation result holds.

Lemma 5.9 (cf. [4, Lemma 4.1])**.** *Suppose that for $0 \leqslant t \leqslant T$, we have*

$$\begin{aligned} \|m_\star(t)\|_{W^{1,\infty}(\Omega)} + \|\partial_t m_\star(t)\|_{W^{1,\infty}(\Omega)} &\leqslant R \\ \text{and} \quad \|\Delta m_\star(t) + H(t)\|_{L^\infty(\Omega)} &\leqslant K. \end{aligned} \tag{5.37}$$

Then, the error $e(t) = m(t) - m_\star(t)$ satisfies, for $0 \leqslant t \leqslant T$,

$$\|e(t)\|_{H^1(\Omega)}^2 + \int_0^t \|\partial_t e(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s \leqslant C\Big(\|e(0)\|_{H^1(\Omega)}^2 + \int_0^t \|d(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s \Big), \tag{5.38}$$

where the constant C depends only on α, R, K and T.

Proof. The proof is similar to the proof of [4, Lemma 4.1], with some slight modifications to obtain the additional bound on $\int_0^t \|\partial_t e(s)\|^2_{L^2(\Omega)}\, \mathrm{d}s$. This is sketched briefly in the following:

As in [4, Lemma 4.1], we obtain

$$\alpha \frac{1}{2}\|\partial_t e\|^2_{L^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla e\|^2_{L^2} \leqslant c\|e\|^2_{H^1} + c\|d\|^2_{L^2}.$$

With the estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2_{L^2} = (\partial_t e, e) \leqslant \tfrac{1}{2}\|\partial_t e\|^2_{L^2} + \tfrac{1}{2}\|e\|^2_{L^2}, \quad \text{i.e.} \quad \frac{\alpha}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2_{L^2} \leqslant \frac{\alpha}{4}\|\partial_t e\|^2_{L^2} + \frac{\alpha}{4}\|e\|^2_{L^2},$$

we obtain

$$\frac{\alpha}{4}\|\partial_t e\|^2_{L^2} + \min\Big(\frac{\alpha}{4}, \frac{1}{2}\Big)\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2_{H^1} \leqslant \Big(c + \frac{\alpha}{4}\Big)\|e\|^2_{H^1} + c\|d\|^2_{L^2}.$$

Integration in time and Gronwall's inequality (see Lemma A.1 in the Appendix) imply the stated error bound. □

Consistency analysis

As in Section 6.2 from [4], we define the Ritz projection R_h and the $L^2(\Omega)$-projection $P_h(m)$ onto the discrete tangent space at m. We insert the following quantities, which are related to the exact solution,

$$\begin{aligned} m^n_{\star,h} &= \mathrm{R}_h m(t_n), \\ \widehat{m}^n_{\star,h} &= \sum_{j=0}^{1} \gamma_j m^{n-j-1}_{\star,h} \Big/ \Big|\sum_{j=0}^{1} \gamma_j m^{n-j-1}_{\star,h}\Big|, \\ \dot{m}^n_{\star,h} &= \mathrm{P}_h(\widehat{m}^n_{\star,h}) \frac{1}{\tau}\sum_{j=0}^{2} \delta_j m^{n-j}_{\star,h} \in T_h(\widehat{m}^n_{\star,h}), \end{aligned} \tag{5.39}$$

into the linearly implicit 2-step BDF approximation: We obtain the defect $d^n_h \in T_h(\widehat{m}^n_{\star,h})$ from

$$\alpha(\dot{m}^n_{\star,h}, \varphi_h) + (\widehat{m}^n_{\star,h} \times \dot{m}^n_{\star,h}, \varphi_h) = -(\nabla m^n_{\star,h}, \nabla\varphi_h) + (H^n, \varphi_h) + (d^n_h, \varphi_h) \tag{5.40}$$

for all $\varphi_h \in T_h(\widehat{m}^n_{\star,h})$.

The consistency error is bounded as follows.

Lemma 5.10 (cf. [4, Lemma 6.2]). *If the solution of the LLG equation* (5.35) *has the regularity*

$$\begin{aligned} &m \in C^3([0,T], L^2(\Omega)) \cap C^1([0,T], W^{r+1,\infty}(\Omega)) \quad \textit{and} \\ &\Delta m + H \in C([0,T], W^{r+1,\infty}(\Omega)), \end{aligned}$$

then the consistency error in (5.40) *is bounded by*

$$\|d^n_h\|_{L^2(\Omega)} \leqslant C(\tau^2 + h^r)$$

for $n \geq 2$ *with* $n\tau \leqslant T$.

Proof. The assertion directly follows from [4, Lemma 6.2] for the second order scheme, i.e. inserting $k = 2$. □

Error equation

The error $e_h^n = m_h^n - m_{\star,h}^n$ satisfies the error equation that is obtained by subtracting (5.40) from the BDF approximation for the LLG equation (5.11a) (with H_h^n replaced by $H(t_n)$). We use the notations

$$\widehat{e}_{m,h}^n = \widehat{m}_h^n - \widehat{m}_{\star,h}^n, \tag{5.41}$$

$$\dot{e}_{m,h}^n = \dot{m}_h^n - \dot{m}_{\star,h}^n = \frac{1}{\tau}\sum_{j=0}^{2}\delta_j e_h^{n-j} + s_h^n, \tag{5.42}$$

$$\text{with} \qquad s_h^n = (\mathbf{I} - \mathrm{P}_h(\widehat{m}_{\star,h}^n))\frac{1}{\tau}\sum_{j=0}^{2}\delta_j m_{\star,h}^{n-j}.$$

We then have the error equation

$$\alpha[\dot{e}_{m,h}^n, \varphi_h]_\Omega + [\widehat{e}_{m,h}^n \times \dot{m}_{\star,h}^n, \varphi_h]_\Omega + [\widehat{m}_h^n \times \dot{e}_{m,h}^n, \varphi_h]_\Omega + [\nabla e_h^n, \nabla\varphi_h]_\Omega = -[r_{m,h}^n, \varphi_h]_\Omega, \tag{5.43}$$

for all $\varphi_h \in \mathcal{T}_h(\widehat{m}_h^n)$, where

$$r_{m,h}^n = -(\mathrm{P}_h(\widehat{m}_h^n) - \mathrm{P}_h(\widehat{m}_{\star,h}^n))(\Delta m_\star(t_n) + H(t_n)) + d_h^n. \tag{5.44}$$

We repeat the following bound for s_h^n from [4].

Lemma 5.11 (cf. [4, Lemma 6.3])**.** *Under the regularity assumptions*

$$m \in C^3([0,T], H^1(\Omega)) \cap C^1([0,T], W^{r+1,\infty}(\Omega))$$

we have

$$\|s_h^n\|_{H^1(\Omega)} \leqslant C(\tau^2 + h^r). \tag{5.45}$$

Stability analysis

Similarly as in Section 7 from [4], we derive the following stability estimate.

Lemma 5.12 (cf. [4, Lemma 7.1])**.** *Suppose that the exact solution $m(t)$ is bounded by* (5.37) *and that $h \leqslant \overline{h}$ and $\tau \leqslant \overline{\tau}$ are sufficiently small. Assume that the right-hand side in the following estimate* (5.46) *is bounded by $\widehat{c}h$ with a sufficiently small constant $\widehat{c} > 0$ (note that the right-hand side is of size $O((\tau^2 + h^r)^2)$ in the case of a sufficiently regular solution).*

Then, the error $e_h^n = m_h^n - m_{\star,h}^n$ from the error equation (5.43) *satisfies for $t_n \leqslant T$,*

$$\|e_h^n\|_{H^1(\Omega)}^2 + \tau\sum_{j=2}^{n}\|\dot{e}_{m,h}^j\|_{L^2(\Omega)}^2 \leqslant C\Big(\sum_{i=0}^{1}\|e_h^i\|_{H^1(\Omega)}^2 + \tau\sum_{j=2}^{n}\|d_h^j\|_{L^2(\Omega)}^2 + \tau\sum_{j=2}^{n}\|s_h^j\|_{H^1(\Omega)}^2\Big), \tag{5.46}$$

where the constant C is independent of h, τ and n, but depends on α, R, K, M and T.

Proof. The proof is similar to the proof of [4, Lemma 7.1], with some modifications one obtains the additional bound on $\tau\sum_{j=2}^{n}\|\dot{e}_{m,h}^j\|_{L^2(\Omega)}^2$. This is sketched briefly in the following: The arguments in part (a) *Preparations.* and (b) *Energy estimates.* from the proof of [4, Lemma 7.1] can be conducted until we obtain

$$\begin{aligned}\alpha\frac{1}{2}\tau\sum_{j=2}^{n}\|\dot{e}_{m,h}^j\|_{L^2}^2 + \|\nabla e_h^n\|_{L^2}^2 \leqslant c\tau\sum_{j=2}^{n}\|e_h^j\|_{H^1}^2 + c\tau\sum_{j=2}^{n}(\|d_h^j\|_{L^2}^2 + \|s_h^j\|_{H^1}^2)\\ + c\sum_{i=0}^{1}\|e_h^i\|_{H^1}^2,\end{aligned} \tag{5.47}$$

with a constant $c > 0$ depending on α. As in [4, Lemma 7.1], we derive for $C > 0$

$$\|e_h^n\|_{L^2}^2 \leqslant C\tau \sum_{j=2}^{n} \|\dot{e}_{m,h}^j\|_{L^2}^2 + C\tau \sum_{j=2}^{n} \|s_h^j\|_{L^2}^2 + C \sum_{i=0}^{1} \|e_h^i\|_{L^2}^2,$$

i.e.

$$\frac{\alpha}{4C}\|e_h^n\|_{L^2}^2 - \frac{\alpha}{4}\tau \sum_{j=2}^{n} \|s_h^j\|_{L^2}^2 - \frac{\alpha}{4} \sum_{i=0}^{1} \|e_h^i\|_{L^2}^2 \leqslant \frac{\alpha}{4}\tau \sum_{j=2}^{n} \|\dot{e}_{m,h}^j\|_{L^2}^2. \tag{5.48}$$

Combining (5.47) and (5.48) gives

$$\begin{aligned} &\alpha\frac{1}{4}\tau \sum_{j=2}^{n}\|\dot{e}_{m,h}^j\|_{L^2}^2 + \frac{\alpha}{4C}\|e_h^n\|_{L^2}^2 + \|\nabla e_h^n\|_{L^2}^2 \\ &\qquad \leqslant c\tau \sum_{j=2}^{n} \|e_h^j\|_{H^1}^2 + (c + \frac{\alpha}{4})\tau \sum_{j=2}^{n}(\|d_h^j\|_{L^2}^2 + \|s_h^j\|_{H^1}^2) + (c + \frac{\alpha}{4})\sum_{i=0}^{1} \|e_h^i\|_{H^1}^2 \end{aligned} \tag{5.49}$$

and the discrete version of Gronwall's Lemma A.2 concludes the assertion. □

5.4.2. The Maxwell equations

A continuous perturbation result

As in [99, Section 6.1], we consider the Maxwell system with inhomogeneities $j, g : [0, T] \to L^2(\Omega)$ and $\rho, \sigma : [0, T] \to \mathcal{H}_\Gamma$,

$$\begin{aligned} &[\varepsilon\partial_t E, \zeta^E]_\Omega = \frac{1}{2}[\nabla \times H, \zeta^E]_\Omega + \frac{1}{2}[H, \nabla \times \zeta^E]_\Omega - \frac{1}{2}\langle \mu^{-1}\varphi, \gamma_T\, \zeta^E\rangle_\Gamma + [j, \zeta^E]_\Omega, \\ &[\mu\partial_t H, \zeta^H]_\Omega = -\frac{1}{2}[\nabla \times E, \zeta^H]_\Omega - \frac{1}{2}[E, \nabla \times \zeta^H]_\Omega - \frac{1}{2}\langle \psi, \gamma_T\, \zeta^H\rangle_\Gamma + [g, \zeta^E]_\Omega, \\ &\left\langle \begin{pmatrix} v^\phi \\ v^\psi \end{pmatrix}, B(\partial_t)\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_\Gamma = \frac{1}{2}\Big(\langle v^\phi, \mu^{-1}\, \gamma_T\, E\rangle_\Gamma + \langle v^\psi, \gamma_T\, H\rangle_\Gamma\Big) \\ &\qquad\qquad\qquad\qquad + [v^\phi, \rho]_{\mathcal{H}_\Gamma} + [v^\psi, \sigma]_{\mathcal{H}_\Gamma}. \end{aligned} \tag{5.50}$$

The system satisfies the following continuous stability result.

Lemma 5.13 (cf. [99, Lemma 6.1])**.** *The Maxwell energy*

$$\mathcal{E}(t) = \frac{1}{2}\Big(\varepsilon\|E(t)\|_{L^2(\Omega)}^2 + \mu\|H(t)\|_{L^2(\Omega)}^2\Big),$$

satisfies the bound, for $0 \leq t \leq T$

$$\begin{aligned} \mathcal{E}(t) \leq C\Big(\mathcal{E}(0) &+ \int_0^t (\|j(s)\|_{L^2(\Omega)}^2 + \|g(s)\|_{L^2(\Omega)}^2)\ \mathrm{d}s \\ &+ \int_0^t (\|\partial_t^2\rho(s)\|_{\mathcal{H}_\Gamma}^2 + \|\partial_t^2\sigma(s)\|_{\mathcal{H}_\Gamma}^2)\ \mathrm{d}s\Big), \end{aligned}$$

provided that $\rho(0) = \partial_t\rho(0) = 0$ *and* $\sigma(0) = \partial_t\sigma(0) = 0$ *and where the constant* $C > 0$ *depends on* T, ε_0, μ_0.

Proof. The assertion holds, in comparison to [99, Lemma 6.1], also for the continuous quantities instead of the space-discretized quantities. Furthermore, by the arguments of Section 2.2.3, it does not matter whether the inhomogeneities on the boundary ρ, σ are introduced with respect to the anti-symmetric pairing $\langle\cdot,\cdot\rangle_\Gamma$ as in (5.50) or with respect to the Hilbert space product $[\cdot,\cdot]_{\mathcal{H}_\Gamma}$ as in [99, Equation (6.2)]. □

By linearity, the same estimates hold for the errors one obtains subtracting (5.50) from the non-perturbed equations (i.e. (5.50) without the inhomogenieties).

Consistency analysis

Similar to [99, Section 7] and [27, Section 9], we insert the following quantities, which are related to the exact solution (also compare Section 5.6.1),

$$\begin{aligned}
\widetilde{H}_{\star,h}^{n-1/2} &= \mathrm{I}_h^{\mathcal{W}}\big(H(t_{n-1/2}) - \frac{\tau^2}{8}\partial_t^2 H(t_{n-1/2})\big) \quad \text{and} \\
E_{\star,h}^{n} &= \mathrm{I}_h^{\mathcal{W}} E(t_n), \qquad H_{\star,h}^{n} = \mathrm{I}_h^{\mathcal{W}} H(t_n) \quad \text{and} \\
\varphi_{\star,h}^{n-1/2} &= \mathrm{I}_h^{\Psi}\varphi(t_{n-1/2}), \qquad \psi_{\star,h}^{n} = \mathrm{I}_h^{\Psi}\psi(t_n)
\end{aligned} \tag{5.51}$$

into the leapfrog–Convolution Quadrature approximation of the interior–exterior Maxwell system (5.11a)–(5.11e) (with exact right hand side $g(t_n)$ instead of $\dot{m}_h^n$). We obtain the defects $\widetilde{d}_{E,h}^{n+1/2}$, $\widetilde{d}_{H,h}^{n}$, $\widetilde{d}_{\varphi,h^r}^{j+1/2}$, $\widetilde{d}_{\psi,h}^{j+1/2}$ in the error equation

$$\begin{aligned}
[\mu\widetilde{H}_{\star,h}^{n-1/2}, \zeta_h^{H,1/2}]_\Omega &= [\mu H_{\star,h}^{n-1}, \zeta_h^{H,1/2}]_\Omega - \frac{\tau}{4}[\nabla\times E_{\star,h}^{n-1}, \zeta_h^{H,1/2}]_\Omega - \frac{\tau}{4}[E_{\star,h}^{n-1}, \nabla\times\zeta_h^{H,1/2}]_\Omega \\
&\quad - \frac{\tau}{4}\langle\psi_{\star,h}^{n-1}, \gamma_T\zeta_h^{H,1/2}\rangle_\Gamma - \frac{\tau}{2}[g(t_{n-1}), \zeta_h^{H,1/2}]_\Omega + \frac{\tau}{2}[\widetilde{d}_{H,h}^{n-1}, \zeta_h^{H,1/2}]_\Omega, \\
[\varepsilon E_{\star,h}^{n}, \zeta_h^{E}]_\Omega &= [\varepsilon E_{\star,h}^{n-1}, \zeta_h^{E}]_\Omega + \frac{\tau}{2}[\nabla\times\widetilde{H}_{\star,h}^{n-1/2}, \zeta_h^{E}]_\Omega + \frac{\tau}{2}[\widetilde{H}_{\star,h}^{n-1/2}, \nabla\times\zeta_h^{E}]_\Omega \\
&\quad - \frac{\tau}{2\mu_0}\langle\varphi_{\star,h}^{n-1/2}, \gamma_T\zeta_h^{E}\rangle_\Gamma - \tau[j(t_{n-1/2}) - \widetilde{d}_{E,h}^{n-1/2}, \zeta_h^{E}]_\Omega, \\
[\mu H_{\star,h}^{n}, \zeta_h^{H,1}]_\Omega &= [\mu\widetilde{H}_{\star,h}^{n-1/2}, \zeta_h^{H,1}]_\Omega - \frac{\tau}{4}[\nabla\times E_{\star,h}^{n}, \zeta_h^{H,1}]_\Omega - \frac{\tau}{4}[E_{\star,h}^{n}, \nabla\times\zeta_h^{H,1}]_\Omega \\
&\quad - \frac{\tau}{4}\langle\psi_{\star,h}^{n}, \gamma_T\zeta_h^{H,1}\rangle_\Gamma - \frac{\tau}{2}[g(t_n), \zeta_h^{H,1}]_\Omega + \frac{\tau}{2}[\widetilde{d}_{H,h}^{n}, \zeta_h^{H,1}]_\Omega, \\
\left\langle \begin{pmatrix} v_h^\varphi \\ v_h^\psi \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix} \varphi_{\star,h} \\ \overline{\psi}_{\star,h} \end{pmatrix}\right)^{n-1/2} \right\rangle_\Gamma &= \frac{1}{2}\left\langle \begin{pmatrix} v_h^\varphi \\ v_h^\psi \end{pmatrix}, \begin{pmatrix} \mu_0^{-1}\gamma_T\overline{E}_{\star,h}^{n-1/2} + 2\widetilde{d}_{\varphi,h}^{n-1/2} \\ \gamma_T H_{\star,h}^{n-1/2} + 2\widetilde{d}_{\psi,h}^{n-1/2} \end{pmatrix} \right\rangle_\Gamma \\
&\quad - \beta\frac{\tau^2}{\mu_0}\langle v_h^\psi, \gamma_T \mathrm{LIFT}(\dot{\psi}_{\star,h}^{n-1/2})\rangle_\Gamma.
\end{aligned} \tag{5.52}$$

The consistency errors are bounded as follows.

Lemma 5.14 (cf. [99, Theorem 7.1]). *If the solution of the Maxwell system has the regularity*

$$\begin{aligned}
&E, H \in C^3([0,T], L^2(\Omega)) \cap C^1([0,T], H^{r+1}(\Omega)) \quad \textit{and} \\
&\varphi, \psi \in C^6_{0,*}([0,T], \mathcal{H}_\Gamma) \cap C^2_{0,*}([0,T], H^{r+1/2}(\Gamma)),
\end{aligned}$$

then the defects in the error equation (5.52) *are bounded by*

$$\begin{aligned}
&\|\widetilde{d}_{E,h}^{n+1/2}\|_{L^2} \le C(\tau^2 + h^r), \quad \|\widetilde{d}_{H,h}^{n}\|_{L^2} \le C(\tau^2 + h^r) \qquad \text{and} \\
&\tau\sum_{j=0}^{n} \|(\partial_t^\tau)^2\widetilde{d}_{\varphi,h^r}^{j+1/2}\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^2\widetilde{d}_{\psi,h}^{j+1/2}\|_{\mathcal{H}_\Gamma}^2 \le C(\tau^2 + h^r)^2.
\end{aligned}$$

Proof. This can be shown using the higher order estimates from Section 5.6.1 and adapting the consistency results from [99, Lemma 6.6] for the space discretization. Then the error of the full discretization can be treated as in [27, Section 9]. Furthermore, the proof also works if there is an inhomogeneity in the second Maxwell equation (there is no inhomogeneity in [99] because of the physical model).

□

Stability analysis

We state the stability results from [99, Section 7.1] under the CFL condition (compare Remark 5.3)

$$\tau \|\mathbf{M}^{-1/2}\mathbf{D}\mathbf{M}^{-1/2}\|_2 \leq \sqrt{\varepsilon\mu}. \tag{5.53}$$

The fully discrete electric and magnetic field satisfy the following stability bound.

Lemma 5.15 (cf. [99, Lemma 7.1]). *Under the CFL condition* (5.53) *and for a stabilization parameter $\beta \geq 1$, the discrete energy*

$$\mathcal{E}_h^n = \frac{\varepsilon}{2}\|E_h^n\|_{L^2(\Omega)}^2 + \frac{\mu}{2}\|H_h^n\|_{L^2(\Omega)}^2$$

is bounded for $0 \leq t_n \leq T$ by

$$\begin{aligned}\mathcal{E}_h^n \leq C\Big(\mathcal{E}_h^0 + \tau \sum_{k=0}^{n} \Big(\|j_h^{k-1/2}\|_{L^2(\Omega)}^2 + \|g_h^k\|_{L^2(\Omega)}^2\Big) \\ + \tau \sum_{k=0}^{n-1} \Big(\|(\partial_t^\tau)^2 \rho_h^{k+1/2}\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^2 \sigma_h^{k+1/2}\|_{\mathcal{H}_\Gamma}^2\Big)\Big),\end{aligned} \tag{5.54}$$

where $C > 0$ depends on T, ε, ε_0, μ, μ_0, but is independent of h, τ and n.

Proof. By a precise retracing of the proof of [27, Lemma 8.1], one can see that the term $\tau\|j_h^{n+1/2}\|_{L^2(\Omega)}^2$ which appears in [99, Lemma 7.1], can be omitted. (And also the term $\tau\|(\partial_t^\tau)^2\rho_h^{n+1/2}\|_{\mathcal{H}_\Gamma}^2 + \tau\|(\partial_t^\tau)^2\sigma_h^{n+1/2}\|_{\mathcal{H}_\Gamma}^2$ can be omitted, but this is not needed in the following.) For simplicity of (5.54), we set $j_h^{k-1/2} = 0$. □

5.5. Continuous Perturbation Result

We will present here a continuous perturbation result for the MLLG system (5.2), which will be transferred to the discretised setting in Section 5.7 to prove stability, hence, it gives insight into the main ideas of the coupling and of the fully discrete stability proof. The perturbation result below is a combination of the energy estimate from Maxwell's equations (Lemma 5.13, cf. [99, Section 4.5]) and of the perturbation result for the LLG equation (Lemma 5.9, cf. [4, Section 4]).

Let us recall the weak form of the MLLG system (5.2) which is solved by (m, E, H, φ, ψ). Furthermore, let $(m_\star, E_\star, H_\star, \varphi_\star, \psi_\star)$, with $m_\star$ also of unit length, solve the same equation up to a defects $(d_m, d_E, d_H, d_\varphi, d_\psi)$, i.e. for all $0 \leq t \leq T$

$$\begin{aligned}\alpha\partial_t m_\star + m_\star \times \partial_t m_\star &= \mathrm{P}(m_\star)(\Delta m_\star + H_\star) + d_m = \mathrm{P}(m)(\Delta m_\star + H_\star) + r_m, \\ \varepsilon\partial_t E_\star - \nabla \times H_\star &= -(J + \sigma E_\star) + d_E, \\ \mu\partial_t H_\star + \nabla \times E_\star &= -\mu\partial_t m_\star + d_H, \\ B(\partial_t)\begin{pmatrix}\varphi_\star \\ \psi_\star\end{pmatrix} &= \frac{1}{2}\begin{pmatrix}\mu_0^{-1}\gamma_T E_\star \\ \gamma_T H_\star\end{pmatrix} + \begin{pmatrix}d_\varphi \\ d_\psi\end{pmatrix}\end{aligned} \tag{5.55}$$

with

$$r_m = -\big(\mathrm{P}(m) - \mathrm{P}(m_\star)\big)(\Delta m_\star + H_\star) + d_m.$$

The weak formulation of the perturbed problem is analogous to (5.2) with the additional defect terms. In particular, explaining the role of r_m, the perturbed magnetization $m_\star$ also solves the perturbed weak formulation:

$$\alpha[\partial_t m_\star, \rho]_\Omega + [m_\star \times \partial_t m_\star, \rho]_\Omega + [\nabla m_\star, \nabla\rho]_\Omega - [H_\star, \rho]_\Omega = [r_m, \rho]_\Omega \quad \forall \rho \in \mathcal{T}(m) \cap H^1(\Omega). \tag{5.56}$$

The errors between the perturbed an non-perturbed solutions are denoted by

$$e_m = m - m_\star, \quad e_E = E - E_\star, \quad e_H = H - H_\star, \quad e_\varphi = \varphi - \varphi_\star, \quad e_\psi = \psi - \psi_\star.$$

Subtracting the weak formulation of the perturbed problem (using (5.56) for the magnetization $m_\star$) from the weak formulation (5.2), we obtain that the errors satisfy the error equations, for all $(\rho, \zeta^E, \zeta^H, v^\varphi, v^\psi) \in (\mathcal{T}(m) \cap H^1(\Omega)) \times H(\mathrm{curl}, \Omega)^2 \times \mathcal{H}_\Gamma{}^2$,

$$\begin{aligned}
\alpha[\partial_t e_m, \rho]_\Omega + {} & [e_m \times \partial_t m_\star, \rho]_\Omega + [m \times \partial_t e_m, \rho]_\Omega + [\nabla e_m, \nabla \rho]_\Omega \\
&= [e_H, \rho]_\Omega - [r_m, \rho]_\Omega, && (5.57a)\\
\varepsilon[\partial_t e_E, \zeta^E]_\Omega &= \frac{1}{2}[\nabla \times e_H, \zeta^E]_\Omega + \frac{1}{2}[e_H, \nabla \times \zeta^E]_\Omega - \frac{1}{2\mu_0}\langle e_\varphi, \gamma_T \zeta^E\rangle_\Gamma \\
&\quad - \sigma[e_E, \zeta^E]_\Omega - [d_E, \zeta^E]_\Omega, && (5.57b)\\
\mu[\partial_t e_H, \zeta^H]_\Omega &= -\frac{1}{2}[\nabla \times e_E, \zeta^H]_\Omega - \frac{1}{2}[e_E, \nabla \times \zeta^H]_\Omega - \frac{1}{2}\langle e_\psi, \gamma_T \zeta^H\rangle_\Gamma \\
&\quad - \mu[\partial_t e_m, \zeta^H]_\Omega - [d_H, \zeta^H]_\Omega, && (5.57c)
\end{aligned}$$

$$\left\langle \begin{pmatrix} v^\varphi \\ v^\psi \end{pmatrix}, \left(B(\partial_t)\begin{pmatrix} e_\varphi \\ e_\psi \end{pmatrix}\right)\right\rangle_\Gamma = \frac{1}{2}\left\langle \begin{pmatrix} v^\varphi \\ v^\psi \end{pmatrix}, \begin{pmatrix} \mu_0^{-1}\gamma_T e_E \\ \gamma_T e_H \end{pmatrix}\right\rangle_\Gamma - \left\langle \begin{pmatrix} v^\varphi \\ v^\psi \end{pmatrix}, \begin{pmatrix} d_\varphi \\ d_\psi \end{pmatrix}\right\rangle_\Gamma. \quad (5.57d)$$

We have the following perturbation result.

Lemma 5.16. *Let $m(t)$ and $m_\star(t)$ be weak solutions of unit length of* (5.2) *and of the perturbed weak system corresponding to* (5.55)*, respectively, and suppose that, for $0 \le t \le T$, we have*

$$\begin{aligned}
\|m_\star(t)\|_{W^{1,\infty}(\Omega)} + \|\partial_t m_\star(t)\|_{W^{1,\infty}(\Omega)} &\le R \\
\text{and} \quad \|\Delta m_\star(t) + H_\star(t)\|_{L^\infty(\Omega)} &\le K.
\end{aligned} \quad (5.58)$$

Additionally, we assume the defects to satisfy $d_\varphi(0) = \partial_t d_\varphi(0) = d_\psi(0) = \partial_t d_\psi(0) = 0$.

Then, the combined error

$$\mathcal{E}(t) := \|e_m(t)\|_{H^1}^2 + \|e_E(t)\|_{L^2}^2 + \|e_H(t)\|_{L^2}^2$$

satisfies, for $0 \le t \le T$,

$$\begin{aligned}
\mathcal{E}(t) \le C\Big(\mathcal{E}(0) &+ \int_0^t \left(\|d_m(r)\|_{L^2}^2 + \|d_E(r)\|_{L^2}^2 + \|d_H(r)\|_{L^2}^2\right) \mathrm{d}r \\
&+ \int_0^t \left(\|\partial_t^2 d_\varphi(r)\|_{\mathcal{H}_\Gamma}^2 + \|\partial_t^2 d_\psi(r)\|_{\mathcal{H}_\Gamma}^2\right) \mathrm{d}r\Big),
\end{aligned} \quad (5.59)$$

where the constant $C > 0$ depends only on α, ε, μ, σ, R, K, and T.

Proof. For the LLG part, by $|m_\star| = 1$, $\mathrm{P}(m_\star)$ is an projection and $\|\mathrm{P}(m_\star)\| \le 1$. We apply Lemma 5.9 and obtain with $d = d_m - \mathrm{P}(m_\star)e_H$ for a constant $\widehat{C} > 0$

$$\begin{aligned}
\|e(t)\|_{H^1(\Omega)}^2 &+ \int_0^t \|\partial_t e(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s \\
&\leqslant \frac{\widehat{C}}{2}\Big(\|e(0)\|_{H^1(\Omega)}^2 + \int_0^t \|d(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s\Big), \\
&\leqslant \widehat{C}\Big(\|e(0)\|_{H^1(\Omega)}^2 + \int_0^t \|d_m(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s + \int_0^t \|e_H(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s\Big).
\end{aligned} \quad (5.60)$$

For the Maxwell part, we apply Lemma 5.13 by replacing the defect j with $\sigma e_E + d_E$ and the defect g with $\mu \partial_t e_m + d_H$ to obtain

$$
\begin{aligned}
&\|e_E(t)\|_{L^2}^2 + \|e_H(t)\|_{L^2}^2 \\
&\quad \le \frac{\widetilde{C}}{2}\Big(\|e_E(0)\|_{L^2}^2 + \|e_H(0)\|_{L^2}^2 + \int_0^t \|\sigma e_E + d_E\|_{L^2}^2 + \|\mu \partial_t e_m + d_H\|_{L^2}^2 \\
&\qquad + \int_0^t \|\partial_t^2 d_\varphi\|_{\mathcal{H}_\Gamma}^2 + \|\partial_t^2 d_\psi\|_{\mathcal{H}_\Gamma}^2 \Big) \\
&\quad \le \widetilde{C}\Big(\|e_E(0)\|_{L^2}^2 + \|e_H(0)\|_{L^2}^2 + \int_0^t \|\sigma e_E\|_{L^2}^2 + \|d_E\|_{L^2}^2 + \|\mu \partial_t e_m\|_{L^2}^2 + \|d_H\|_{L^2}^2 \\
&\qquad + \int_0^t \|\partial_t^2 d_\varphi\|_{\mathcal{H}_\Gamma}^2 + \|\partial_t^2 d_\psi\|_{\mathcal{H}_\Gamma}^2 \Big)
\end{aligned} \tag{5.61}
$$

with a constant $\widetilde{C} > 0$ depending on t, μ, μ_0, ε and ε_0 as long as $d_\varphi(0) = \partial_t d_\varphi(0) = d_\psi(0) = \partial_t d_\psi(0)$.

Combining the LLG estimate (5.60), with the Maxwell estimate (5.61), multiplying (5.60) with the constant $\mu^2 \widetilde{C}$ (without loss of generality $\mu^2 \widetilde{C} \ge 1$) and adding up the equations, we obtain for $\mathcal{E}(t) = \|e_m(t)\|_{H^1}^2 + \|e_E(t)\|_{L^2}^2 + \|e_H(t)\|_{L^2}^2$ that

$$
\begin{aligned}
\mathcal{E}(t) + \widetilde{C} \int_0^t \|\mu \partial_t e_m\|_{L^2}^2 \le \widetilde{C}\widehat{C}\Big(\mathcal{E}(0) + \int_0^t \|e_m\|_{H^1}^2 + \|d_m\|_{L^2}^2 + \|e_H\|_{L^2}^2 \Big) \\
+ \widetilde{C}\Big(\int_0^t \|\sigma e_E\|_{L^2}^2 + \|d_E\|_{L^2}^2 + \|\mu \partial_t e_m\|_{L^2}^2 + \|d_H\|_{L^2}^2 \\
+ \int_0^t \|\partial_t^2 d_\varphi\|_{\mathcal{H}_\Gamma}^2 + \|\partial_t^2 d_\psi\|_{\mathcal{H}_\Gamma}^2 \Big).
\end{aligned}
$$

Absorption of the term $\int_0^t \|\mu \partial_t e_m\|_{L^2}^2$ gives

$$
\begin{aligned}
\mathcal{E}(t) &\le C\Big(\mathcal{E}(0) + \int_0^t \|d_m\|_{L^2}^2 + \|d_E\|_{L^2}^2 + \|d_H\|_{L^2}^2 \\
&\qquad + \int_0^t \|\partial_t^2 d_\varphi\|_{\mathcal{H}_\Gamma}^2 + \|\partial_t^2 d_\psi\|_{\mathcal{H}_\Gamma}^2 + \int_0^t \mathcal{E}(s)\, \mathrm{d}s \Big) \\
&=: \mathrm{rhs}(t) + C \int_0^t \mathcal{E}(s)\, \mathrm{d}s
\end{aligned}
$$

for a generic constant $C > 0$ depending on σ, $\widetilde{C}$ and $\widehat{C}$ and the monotonically increasing right hand side $\mathrm{rhs}(t)$ from (5.59). Finally, Gronwall's lemma A.1 concludes

$$
\mathcal{E}(t) \le \mathrm{rhs}(t) + C \int_0^t \mathrm{rhs}(\tau)\, \mathrm{d}\tau \le C\, \mathrm{rhs}(t).
$$

□

Remark 5.17. *In the previous proof (compare Remark 5.8), we directly used the estimates of the underlying papers (recalled in Section 5.4), which use Gronwall's lemma to give bounds for m depending on H and to give bounds for E, H depending on $\partial_t m$ (via the right hand side). Then another absorption and Gronwall argument can bound the dependencies because of the coupling. This is not really necessary, one could also directly combine the arguments of the Lemmas (resulting in theoretically smaller constants), but then one would have to repeat all the details of the proofs for the uncoupled problems. For simplicity however, we directly apply the results from Section 5.4.*

Remark 5.18. *Similar to Lemma 5.16, one can obtain a bound for $\widetilde{\mathcal{E}}(t) := \mathcal{E}(t) + \int_0^t \|\partial_t^{-1} e_\varphi\|_{\mathcal{H}_\Gamma}^2 + \|\partial_t^{-1} e_\psi\|_{\mathcal{H}_\Gamma}^2$, by modification with factors $\mathrm{e}^{-t/T}$ and considering Laplace transformed quantities.*

5.6. Consistency Analysis and Error Equations

The goal of this section is to prove suitable bounds on the defects, i.e. the fully discrete residuals, upon inserting suitable finite element projections of the exact solutions into the numerical method. After stating the used interpolations and projections and their properties in Section 5.6.1, we introduce the defects and the corresponding error equation in Section 5.6.2. In Section 5.6.3, we show that these equations hold indeed and that the defects can be bounded suitably.

5.6.1. Preliminaries

The error equations arise from taking the equations for the numerical method (5.11) and subtracting similar equations where suitable projections and interpolations (here the Ritz and L^2 projections and finite element interpolation) into the finite element space of polynomial degree r of the exact solutions are inserted into the method. The projections of the exact solutions only fulfill the method up to some defects. In this subsection we state the interpolations and their properties, the following subsection will contain the consistency analysis where we will derive error equations and bound the defects.

For the LLG equation, following [4, Section 6.2], we introduce the Ritz projection $R_h \colon H^1(\Omega) \to \mathcal{S}_h^r$ corresponding to the Poisson–Neumann problem via

$$(\nabla R_h \varphi, \nabla \psi) + (R_h \varphi, 1)(\psi, 1) = (\nabla \varphi, \nabla \psi) + (\varphi, 1)(\psi, 1)$$

for all $\psi \in \mathcal{S}_h^r$, (recall that $\mathcal{S}_h^r$ denotes a finite element space of polynomial degree r). The following interpolation bound holds by Céa's lemma and standard interpolation results.

Lemma 5.19. *There exists a constant $C > 0$, independent of h, such that for all $v \in H^{r+1}(\Omega) \cap H_0^1(\Omega)$,*

$$\|v - \mathrm{R}_h v\|_{L^2(\Omega)} + h\|\nabla(v - \mathrm{R}_h v)\|_{L^2(\Omega)} \le C h^{r+1} \|v\|_{H^{r+1}(\Omega)}.$$

By $\mathrm{P}_h(m)$, we denote the $L^2(\Omega)$-orthogonal projection onto the discrete tangent space $T_h(m)$,

$$\mathrm{P}_h(m) \colon \mathcal{S}_h^r \to T_h(m).$$

For Maxwell's equations, following [99, Section 6.3], we will use the finite element interpolation $\mathrm{I}_h^{\mathcal{W}}$ in Ω and the boundary element interpolation I_h^{Ψ} on Γ (both of polynomial degree r).

For a suitable set of Lagrange points $\mathcal{N}_L^{\Omega} \subset \Omega$ and corresponding (higher order polynomial) Lagrange basis functions φ_l^{Ω} (see [39, Section 3.6] for further details)

$$\varphi_l^{\Omega}(x_k^{\Omega}) = \begin{cases} 1, & l = k, \\ 0, & \text{else}, \end{cases} \qquad \text{for all } x_k \in \mathcal{N}_L^{\Omega}$$

the interpolation in the interior $\mathrm{I}_h^{\mathcal{W}}$ on the discrete approximation space $\mathcal{W}_h^r$ is defined as

$$\mathrm{I}_h^{\mathcal{W}}(\zeta) = \sum_{k=1}^{|\mathcal{N}_L^{\Omega}|} \zeta(x_k^{\Omega}) \varphi_l^{\Omega}.$$

Similarly, the interpolation I_h^{Ψ} on the boundary space Ψ_h^r is defined for a suitable set of Lagrange points $\mathcal{N}_L^{\Gamma} \subset \Gamma$ and corresponding (higher order polynomial) Lagrange basis functions φ_l^{Γ} (see [142, Section 4.1.7] for further details) as

$$\mathrm{I}_h^{\Psi}(\zeta) = \sum_{k=1}^{|\mathcal{N}_L^{\Gamma}|} \zeta(x_k^{\Gamma}) \varphi_l^{\Gamma}.$$

For first order elements, the Lagrange points correspond to the nodes of the triangulations in the interior and on the boundary. We assume (as in Section 5.2.3), that each Lagrange point of the boundary mesh also is a Lagrange point of the interior mesh, i.e. $\mathcal{N}_L^{\Gamma} = \mathcal{N}_L^{\Omega} \cap \Gamma$. Therefore it holds

$$(\mathrm{I}_h^{\Psi} \gamma F) = (\gamma \mathrm{I}_h^{\mathcal{W}} F) \quad \text{on } \Gamma.$$

Since the normal vector n is constant on every face of Γ, we have

$$\mathrm{I}_h^{\Psi}(\chi \times n) = (\mathrm{I}_h^{\Psi} \chi) \times n \qquad \text{for} \ \ \chi \in C(\Gamma),$$

which implies that I_h^{Ψ} maps $\mathcal{H}_\Gamma \cap C(\Gamma)$ into $\mathcal{H}_\Gamma$. Moreover, this yields the very useful relation

$$\mathrm{I}_h^{\Psi} \gamma_T F = \gamma_T \mathrm{I}_h^{\mathcal{W}} F \qquad \text{for} \ \ F \in C(\overline{\Omega}), \tag{5.62}$$

as is seen by noting that

$$\mathrm{I}_h^{\Psi} \gamma_T F = \mathrm{I}_h^{\Psi}(\gamma F \times n) = (\mathrm{I}_h^{\Psi} \gamma F) \times n = (\gamma \mathrm{I}_h^{\mathcal{W}} F) \times n = \gamma_T \mathrm{I}_h^{\mathcal{W}} F.$$

It is because of (5.62) that we work in the following with interpolation operators rather than orthogonal projections. We recall the standard results for the interpolation errors, see, e.g., [39, Thm. 4.4.20].

Lemma 5.20. *There exists a constant $C > 0$, independent of h, such that for all $v \in H^{r+1}(\Omega)$,*

$$\|v - \mathrm{I}_h^{\mathcal{W}} v\|_{L^2(\Omega)} + h \|\nabla(v - \mathrm{I}_h^{\mathcal{W}} v)\|_{L^2(\Omega)} \le C h^{r+1} \|v\|_{H^{r+1}(\Omega)}.$$

The following interpolation error estimate is a standard result for boundary element approximations, see, e.g., [142, Theorem 4.1.50] or [123].

Lemma 5.21. *There exists a constant $C > 0$, independent of h, such that for all $\varphi \in H^{r+1/2}(\Gamma)$,*

$$\|\varphi - \mathrm{I}_h^{\Psi} \varphi\|_{H^{1/2}(\Gamma)} \le C h^r \|\varphi\|_{H^{r+1/2}(\Gamma)}.$$

We remark that for piecewise smooth boundaries just piecewise $H^{r+1/2}$ regularity is needed.

5.6.2. Error equations

In summary, the consistency analysis will use the following quantities:

- For the LLG equation:

$$m_{\star,h}^n = \mathrm{R}_h m(t_n), \tag{5.63a}$$

$$\widehat{m}_{\star,h}^n = \sum_{j=0}^{1} \gamma_j m_{\star,h}^{n-j-1} \Big/ \Big| \sum_{j=0}^{1} \gamma_j m_{\star,h}^{n-j-1} \Big|, \tag{5.63b}$$

$$\dot{m}_{\star,h}^n = \mathrm{P}_h(\widehat{m}_{\star,h}^n) \frac{1}{\tau} \sum_{j=0}^{2} \delta_j m_{\star,h}^{n-j} \in T_h(\widehat{m}_{\star,h}^n). \tag{5.63c}$$

- For the Maxwell equations:

$$\begin{aligned} &\widetilde{H}_{\star,h}^{n-1/2} = \mathrm{I}_h^{\mathcal{W}}\Big(H(t_{n-1/2}) - \frac{\tau^2}{8} \partial_t^2 H(t_{n-1/2})\Big) \quad \text{and} \\ &E_{\star,h}^n = \mathrm{I}_h^{\mathcal{W}} E(t_n), \qquad H_{\star,h}^n = \mathrm{I}_h^{\mathcal{W}} H(t_n) \quad \text{and} \\ &\varphi_{\star,h}^{n-1/2} = \mathrm{I}_h^{\Psi} \varphi(t_{n-1/2}), \qquad \psi_{\star,h}^n = \mathrm{I}_h^{\Psi} \psi(t_n). \end{aligned} \tag{5.64}$$

- The fully discrete errors, defined by

$$\begin{aligned}
&e^n_{m,h} = m^n_h - m^n_{\star,h}, \qquad e^{n-1/2}_{H,h} = H^{n-1/2}_h - \widetilde{H}^{n-1/2}_{\star,h} \quad \text{and} \\
&e^n_{E,h} = E^n_h - E^n_{\star,h}, \qquad e^n_{H,h} = H^n_h - H^n_{\star,h} \quad \text{and} \\
&e^{n-1/2}_{\varphi,h} = \varphi^{n-1/2}_h - \varphi^{n-1/2}_{\star,h}, \qquad e^n_{\psi,h} = \psi^n_h - \psi^n_{\star,h}.
\end{aligned} \tag{5.65}$$

Furthermore, by comparing (5.63b) and (5.63c) with their respective counterparts in (5.6), we obtain

$$\widehat{e}^n_{m,h} = \widehat{m}^n_h - \widehat{m}^n_{\star,h}, \qquad \dot{e}^n_{m,h} = \dot{m}^n_h - \dot{m}^n_{\star,h} = \frac{1}{\tau}\sum_{j=0}^{2} \delta_j e^{n-j}_{m,h} + s^n_h,$$

where $s^n_h = (\mathrm{Id} - \mathrm{P}_h(\widehat{m}^n_{\star,h}))\frac{1}{\tau}\sum_{j=0}^2 \delta_j m^{n-j}_{\star,h}$, cf. [4, equation (6.23)].

We show in the following that the above errors satisfy the MLLG error equation system below with suitably bounded defects $d^n_{m,h}$, $d^{n+1/2}_{E,h}$, $d^n_{H,h}$, $d^{j+1/2}_{\varphi,h}$ and $d^{j+1/2}_{\psi,h}$. The MLLG error equation system reads, for all $\rho_h \in \mathcal{T}_h(\widehat{m}^n_h)$, $\zeta^{H,1/2}_h, \zeta^E_h, \zeta^{H,1}_h \in \mathcal{W}^r_h$, and $v^\varphi_h, v^\psi_h \in \Psi^r_h$,

$$\begin{aligned}
\alpha(\dot{e}^n_{m,h}, \rho_h) + (\widehat{e}^n_{m,h} \times \dot{m}^n_{\star,h}, \rho_h) &+ (\widehat{m}^n_h \times \dot{e}^n_{m,h}, \rho_h) + (\nabla \dot{e}^n_{m,h}, \nabla \rho_h) \\
&= (e^n_{H,h}, \rho_h) - (r^n_{m,h}, \rho_h), \qquad (5.66a)\\
\mu[e^{n-1/2}_{H,h}, \zeta^{H,1/2}_h]_\Omega &= \mu[e^{n-1}_{H,h}, \zeta^{H,1/2}_h]_\Omega - \frac{\tau}{4}[\mathrm{curl}_h\, e^{n-1}_{E,h}, \zeta^{H,1/2}_h]_\Omega - \frac{\tau}{4}[e^{n-1}_{E,h}, \mathrm{curl}_h\, \zeta^{H,1/2}_h]_\Omega \\
&\quad - \frac{\tau}{4}\langle e^{n-1}_{\psi,h}, \gamma_T \zeta^{H,1/2}_h\rangle_\Gamma - \frac{\tau}{2}\mu[\dot{e}^{n-1}_{m,h}, \zeta^{H,1/2}_h]_\Omega - \frac{\tau}{2}[d^{n-1}_{H,h}, \zeta^{H,1/2}_h]_\Omega, \\
&\qquad (5.66b)\\
\varepsilon[e^n_{E,h}, \zeta^E_h]_\Omega &= \varepsilon[e^{n-1}_{E,h}, \zeta^E_h]_\Omega + \frac{\tau}{2}[\mathrm{curl}_h\, e^{n-1/2}_{H,h}, \zeta^E_h]_\Omega + \frac{\tau}{2}[e^{n-1/2}_{H,h}, \mathrm{curl}_h\, \zeta^E_h]_\Omega \\
&\quad - \frac{\tau}{2}\mu_0^{-1}\langle e^{n-1/2}_{\varphi,h}, \gamma_T \zeta^E_h\rangle_\Gamma - \tau[\sigma \overline{e}^{n-1/2}_{E,h} + d^{n-1/2}_{E,h}, \zeta^E_h]_\Omega, \qquad (5.66c)\\
\mu[e^n_{H,h}, \zeta^{H,1}_h]_\Omega &= \mu[e^{n-1/2}_{H,h}, \zeta^{H,1}_h]_\Omega - \frac{\tau}{4}[\mathrm{curl}_h\, e^n_{E,h}, \zeta^{H,1}_h]_\Omega - \frac{\tau}{4}[e^n_{E,h}, \mathrm{curl}_h\, \zeta^{H,1}_h]_\Omega \\
&\quad - \frac{\tau}{4}\langle e^n_{\psi,h}, \gamma_T \zeta^{H,1}_h\rangle_\Gamma - \frac{\tau}{2}\mu[\dot{e}^n_{m,h}, \zeta^{H,1}_h]_\Omega - \frac{\tau}{2}[d^n_{H,h}, \zeta^{H,1}_h]_\Omega, \qquad (5.66d)\\
\left\langle \begin{pmatrix} v^\varphi_h \\ v^\psi_h \end{pmatrix}, \left(B(\partial^\tau_t)\begin{pmatrix} e_{\varphi,h} \\ \overline{e}_{\psi,h}\end{pmatrix}\right)^{n-1/2} \right\rangle_\Gamma &= \left\langle \frac{1}{2}\begin{pmatrix} v^\varphi_h \\ v^\psi_h \end{pmatrix}, \begin{pmatrix} \mu_0^{-1}\gamma_T \overline{e}^{n-1/2}_{E,h} \\ \gamma_T e^{n-1/2}_{H,h}\end{pmatrix} - \begin{pmatrix} d^{n-1/2}_{\varphi,h} \\ d^{n-1/2}_{\psi,h}\end{pmatrix}\right\rangle_\Gamma \\
&\quad - \beta\frac{\tau^2}{\mu_0}\langle v^\psi_h, \gamma_T \mathrm{LIFT}(\partial_\tau e^{n-1/2}_{\psi,h})\rangle_\Gamma, \qquad (5.66e)
\end{aligned}$$

with

$$r^n_{m,h} = -(\mathrm{P}_h(\widehat{m}^n_h) - \mathrm{P}_h(\widehat{m}^n_{\star,h}))(\Delta m(t_n) + H(t_n)) + d^n_{m,h}. \tag{5.67}$$

5.6.3. Consistency error

We aim for a suitable bound for the defects $(d^n_{m,h}, d^{n-1/2}_{E,h}, d^n_{H,h}, d^{n-1/2}_{\varphi,h}, d^{n-1/2}_{\psi,h})$ that are given in the error equations (5.66).

Lemma 5.22. *If the solution of the MLLG system has the regularity*

$$\begin{aligned}
&m \in C^3([0,T], H^1(\Omega)) \cap C^1([0,T], W^{r+1,\infty}(\Omega)) \quad \textit{and} \\
&\Delta m + H \in C([0,T], W^{r+1,\infty}(\Omega)) \quad \textit{and} \\
&E, H \in C^3([0,T], L^2(\Omega)) \cap C^1([0,T], H^{r+1}(\Omega)) \quad \textit{and} \\
&\varphi, \psi \in C^6_{0,*}([0,T], \mathcal{H}_\Gamma) \cap C^2_{0,*}([0,T], H^{r+1/2}(\Gamma)),
\end{aligned}$$

then the MLLG error equations (5.66) *hold with defects that are bounded by*

$$\begin{aligned} \|d_{m,h}^n\|_{L^2(\Omega)} \le C(\tau^2 + h^r), \quad \|s_h^n\|_{H^1(\Omega)} \le C(\tau^2 + h^r) \qquad \text{and} \\ \|d_{E,h}^{n+1/2}\|_{L^2} \le C(\tau^2 + h^r), \quad \|d_{H,h}^n\|_{L^2} \le C(\tau^2 + h^r) \qquad \text{and} \\ \tau \sum_{j=0}^{n} \|(\partial_t^\tau)^2 d_{\varphi,h}^{j+1/2}\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^2 d_{\psi,h}^{j+1/2}\|_{\mathcal{H}_\Gamma}^2 \le C(\tau^2 + h^r)^2 \end{aligned}$$

for $n \ge 2$ *concerning the LLG defects (first line) and for* $n \in \mathbb{N}$ *concerning the Maxwell defects (second and third line).*

Proof. The proof is split into a part for the LLG equation and a part for the Maxwell system, which both have a similar structure. First, we repeat the bounds we obtain from the underlying papers [4] and [99] in the uncoupled situation, so for “exact”, given right hand sides $H(t)$ and $\partial_t m$, respectively. We denote the corresponding defects with a tilde and we obtain bounds on the defects covering all terms except the coupling terms. Then we replace the exact data in the coupling terms by their approximations through the exact solution, yielding additional errors. Taking together the already bounded defects and the coupling defects, we obtain the final error equation and under the stated regularity the final defects can be bounded suitably. So, in summary, we do not have to repeat the whole consistency analysis for the whole system, but it is almost enough to focus on the additional coupling errors. For the LLG equation attention has to be paid to apply the arguments in the right order due to the nonlinearity.

Concerning the LLG equation, as in [4, Section 6.3], we recall, from (5.11), the fully discrete problem with the linearly implicit BDF method: find $\dot{m}_h^n \in \mathcal{T}_h(\widehat{m}_h^n)$ such that for all $\varphi_h \in \mathcal{T}_h(\widehat{m}_h^n)$,

$$\alpha(\dot{m}_h^n, \varphi_h) + (\widehat{m}_h^n \times \dot{m}_h^n, \varphi_h) + (\nabla m_h^n, \nabla \varphi_h) = (H_h^n, \varphi_h). \tag{5.68}$$

By Lemma 5.10 ([4, Section 6.3]) we obtain for the exact $H(t_n)$ that for all $\varphi_h \in \mathcal{T}_h(\widehat{m}_h^n)$,

$$\alpha(\dot{m}_{\star,h}^n, \varphi_h) + (\widehat{m}_{\star,h}^n \times \dot{m}_{\star,h}^n, \varphi_h) + (\nabla m_{\star,h}^n, \nabla \varphi_h) = (H(t_n) + \widetilde{r}_{m,h}^n, \varphi_h) \tag{5.69}$$

with

$$\widetilde{r}_{m,h}^n = -(\mathrm{P}_h(\widehat{m}_h^n) - \mathrm{P}_h(\widehat{m}_{\star,h}^n))(\Delta m(t_n) + H(t_n)) + \widetilde{d}_{m,h}^n, \tag{5.70}$$

where

$$\widetilde{d}_{m,h}^n \le C(\tau^2 + h^r) \tag{5.71}$$

for $n \ge 2$. Subtracting (5.69) from (5.68), the errors $e_{m,h}^n = m_h^n - m_{\star,h}^n$, $e_{H,h}^n = H_h^n - H_{\star,h}^n$ satisfy the error equation

$$\begin{aligned} \alpha(\dot{e}_{m,h}^n, \varphi_h) + (\widehat{e}_{m,h}^n \times \dot{m}_{\star,h}^n, \varphi_h) + (\widehat{m}_h^n \times \dot{e}_{m,h}^n, \varphi_h) + (\nabla e_{m,h}^n, \nabla \varphi_h) \\ = (H_h^n - H(t_n), \varphi_h) - (\widetilde{r}_{m,h}^n, \varphi_h), \end{aligned} \tag{5.72}$$

for all $\varphi_h \in \mathcal{T}_h(\widehat{m}_h^n)$. With

$$d_{m,h}^n = \widetilde{d}_{m,h}^n - (H_{\star,h}^n - H(t_n))$$

and

$$\begin{aligned} r_{m,h}^n &= -(\mathrm{P}_h(\widehat{m}_h^n) - \mathrm{P}_h(\widehat{m}_{\star,h}^n))(\Delta m(t_n) + H(t_n)) + d_{m,h}^n \\ &= \widetilde{r}_{m,h}^n - (H_{\star,h}^n - H(t_n)), \end{aligned}$$

we rewrite the right hand side term as

$$(H_h^n - H(t_n) - \widetilde{r}_{m,h}^n, \rho_h) = (e_{H,h}^n - r_{m,h}^n, \rho_h).$$

Under the stated regularity assumptions, the coupling consistency error in H can be bounded (the estimate holds concerning the $L^2(\Omega)$-norm) by

$$\begin{aligned} H_h^n - H(t_n) &\leq H_h^n - H_{\star,h}^n + H_{\star,h}^n - H(t_n) \\ &= e_{H,h}^n + O(h^{r+1}). \end{aligned}$$

This yields $\|d_{m,h}^n\|_{L^2} \leq C(\tau^2 + h^r)$ and the bound $\|s_h^n\|_{H^1(\Omega)} \leq C(\tau^2 + h^r)$ from Lemma 5.11 (Lemma 6.3 from [4]) concludes the LLG part.

Concerning Maxwell's equations, we obtain from Lemma 5.14 ([99, Section 7] and [27, Section 9]), by inserting $j(t) = -\sigma E(t) - J$ and $g(t) = \partial_t m(t)$, the defects $\widetilde{d}_{E,h}^{n+1/2}$, $\widetilde{d}_{H,h}^n$, $\widetilde{d}_{\varphi,h^r}^{j+1/2}$, $\widetilde{d}_{\psi,h}^{j+1/2}$ satisfying

$$\begin{aligned} [\mu \widetilde{H}_{\star,h}^{n-1/2}, \zeta_H]_\Omega &= [\mu H_{\star,h}^{n-1}, \zeta_H]_\Omega - \frac{\tau}{4}[\nabla \times E_{\star,h}^{n-1}, \zeta_H]_\Omega - \frac{\tau}{4}[E_{\star,h}^{n-1}, \nabla \times \zeta_H]_\Omega \\ &\quad - \frac{\tau}{4}\langle \psi_{\star,h}^{n-1}, \gamma_T \zeta_H\rangle_\Gamma - \frac{\tau\mu}{2}[\partial_t m(t_{n-1}), \zeta_H]_\Omega + \frac{\tau}{2}[\widetilde{d}_{H,h}^{n-1}, \zeta_H]_\Omega, \\ [\varepsilon E_{\star,h}^n, \zeta_E]_\Omega &= [\varepsilon E_{\star,h}^{n-1}, \zeta_E]_\Omega + \frac{\tau}{2}[\nabla \times \widetilde{H}_{\star,h}^{n-1/2}, \zeta_E]_\Omega + \frac{\tau}{2}[\widetilde{H}_{\star,h}^{n-1/2}, \nabla \times \zeta_E]_\Omega \\ &\quad - \frac{\tau}{2\mu_0}\langle \varphi_{\star,h}^{n-1/2}, \gamma_T \zeta_E\rangle_\Gamma - \tau[\sigma E(t_{n-1/2}) + J(t_{n-1/2}) - \widetilde{d}_{E,h}^{n-1/2}, \zeta_E]_\Omega, \\ [\mu H_{\star,h}^n, \zeta_H]_\Omega &= [\mu \widetilde{H}_{\star,h}^{n-1/2}, \zeta_H]_\Omega - \frac{\tau}{4}[\nabla \times E_{\star,h}^n, \zeta_H]_\Omega - \frac{\tau}{4}[E_{\star,h}^n, \nabla \times \zeta_H]_\Omega \\ &\quad - \frac{\tau}{4}\langle \psi_{\star,h}^n, \gamma_T \zeta_H\rangle_\Gamma - \frac{\tau\mu}{2}[\partial_t m(t_n), \zeta_H]_\Omega + \frac{\tau}{2}[\widetilde{d}_{H,h}^n, \zeta_H]_\Omega, \\ \left\langle \begin{pmatrix} v_\varphi \\ v_\psi \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix} \varphi_{\star,h} \\ \overline{\psi}_{\star,h} \end{pmatrix}\right)^{n-1/2} \right\rangle_\Gamma &= \frac{1}{2}\left\langle \begin{pmatrix} v_\varphi \\ v_\psi \end{pmatrix}, \begin{pmatrix} \mu_0^{-1}\gamma_T \overline{E}_{\star,h}^{n-1/2} + 2\widetilde{d}_{\varphi,h}^{n-1/2} \\ \gamma_T \widetilde{H}_{\star,h}^{n-1/2} + 2\widetilde{d}_{\psi,h}^{n-1/2} \end{pmatrix} \right\rangle_\Gamma \\ &\quad - \beta\frac{\tau^2}{\mu_0}\langle v_\psi, \gamma_T \mathrm{LIFT}(\dot{\psi}_{\star,h}^{n-1/2})\rangle_\Gamma, \end{aligned} \tag{5.73}$$

with

$$\begin{aligned} \|\widetilde{d}_{E,h}^{n+1/2}\|_{L^2} \leq C(\tau^2 + h^r), \quad \|\widetilde{d}_{H,h}^n\|_{L^2} \leq C(\tau^2 + h^r) \qquad \text{and} \\ \tau \sum_{j=0}^n \|(\partial_t^\tau)^2 \widetilde{d}_{\varphi,h^r}^{j+1/2}\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^2 \widetilde{d}_{\psi,h}^{j+1/2}\|_{\mathcal{H}_\Gamma}^2 \leq C(\tau^2 + h^r)^2. \end{aligned}$$

Subtracting this from (5.11b)–(5.11e), we deduce

$$\begin{aligned} [\mu e_{H,h}^{n-1/2}, \zeta_H]_\Omega &= [\mu e_{H,h}^{n-1}, \zeta_H]_\Omega - \frac{\tau}{4}[\nabla \times e_{E,h}^{n-1}, \zeta_H]_\Omega - \frac{\tau}{4}[e_{E,h}^{n-1}, \nabla \times \zeta_H]_\Omega \\ &\quad - \frac{\tau}{4}\langle e_{\psi,h}^{n-1}, \gamma_T \zeta_H\rangle_\Gamma - \frac{\tau\mu}{2}[\dot{m}_h^{n-1} - \partial_t m(t_{n-1}), \zeta_H]_\Omega - \frac{\tau}{2}[\widetilde{d}_{H,h}^{n-1}, \zeta_H]_\Omega, \\ [\varepsilon e_{E,h}^n, \zeta_E]_\Omega &= [\varepsilon e_{E,h}^{n-1}, \zeta_E]_\Omega + \frac{\tau}{2}[\nabla \times e_{H,h}^{n-1/2}, \zeta_E]_\Omega + \frac{\tau}{2}[e_{H,h}^{n-1/2}, \nabla \times \zeta_E]_\Omega \\ &\quad - \frac{\tau}{2\mu_0}\langle e_{\varphi,h}^{n-1/2}, \gamma_T \zeta_E\rangle_\Gamma - \tau[\sigma(\overline{E}_h^{n-1/2} - E(t_{n-1/2})) + \widetilde{d}_{E,h}^{n-1/2}, \zeta_E]_\Omega, \\ [\mu e_{H,h}^n, \zeta_H]_\Omega &= [\mu e_{H,h}^{n-1/2}, \zeta_H]_\Omega - \frac{\tau}{4}[\nabla \times e_{E,h}^n, \zeta_H]_\Omega - \frac{\tau}{4}[e_{E,h}^n, \nabla \times \zeta_H]_\Omega \\ &\quad - \frac{\tau}{4}\langle e_{\psi,h}^n, \gamma_T \zeta_H\rangle_\Gamma - \frac{\tau\mu}{2}[\dot{m}_h^n - \partial_t m(t_n), \zeta_H]_\Omega - \frac{\tau}{2}[\widetilde{d}_{H,h}^n, \zeta_H]_\Omega, \\ \left\langle \begin{pmatrix} v_\varphi \\ v_\psi \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix} e_{\varphi,h} \\ \overline{e}_{\psi,h} \end{pmatrix}\right)^{n-1/2} \right\rangle_\Gamma &= \frac{1}{2}\left\langle \begin{pmatrix} v_\varphi \\ v_\psi \end{pmatrix}, \begin{pmatrix} \mu_0^{-1}\gamma_T \overline{e}_{E,h}^{n-1/2} - 2\widetilde{d}_{\varphi,h}^{n-1/2} \\ \gamma_T e_{H,h}^{n-1/2} - 2\widetilde{d}_{\psi,h}^{n-1/2} \end{pmatrix} \right\rangle_\Gamma \\ &\quad - \beta\frac{\tau^2}{\mu_0}\langle v_\psi, \gamma_T \mathrm{LIFT}(\dot{e}_{\psi,h}^{n-1/2})\rangle_\Gamma. \end{aligned} \tag{5.74}$$

Observing (concerning the L^2-norm)

$$\begin{aligned}\dot{m}_h^n - \partial_t m(t_n) &= \dot{m}_h^n - \dot{m}_{\star,h}^n + \dot{m}_{\star,h}^n - \partial_t m(t_n) \\ &= \dot{e}_{m,h}^n + O(\tau^2 + h^r)\end{aligned}$$

and

$$\begin{aligned}\overline{E}_h^{n-1/2} - E(t_{n-1/2}) &= \overline{E}_h^{n-1/2} - \overline{E}_{\star,h}^{n-1/2} + \overline{E}_{\star,h}^{n-1/2} - E(t_{n-1/2}) \\ &= \overline{e}_{E,h}^{n-1/2} + O(\tau^2 + h^r)\end{aligned}$$

together with setting

$$\begin{aligned}&d_{E,h}^{n-1/2} = \widetilde{d}_{E,h}^{n-1/2} + \sigma(\overline{E}_{\star,h}^{n-1/2} - E(t_{n-1/2})), \quad d_{H,h}^{n} = \widetilde{d}_{H,h}^{n} + \mu(\dot{m}_{\star,h}^n - \partial_t m(t_n)), \\ &d_{\varphi,h}^{n-1/2} = \widetilde{d}_{\varphi,h}^{n-1/2}, \quad d_{\psi,h}^{n-1/2} = \widetilde{d}_{\psi,h}^{n-1/2},\end{aligned}$$

concludes the assertion.

□

5.7. Stability

In this section we will prove stability of the fully discrete MLLG system, that is, we will prove that the errors at time step n are bounded in terms of the inital errors and the defects. Combining the stability result with the consistency estimates from Section 5.6, we are able to proof Theorem 5.1 at the end of this section.

The stability analysis is the fully discrete analogue of the continuous perturbation result Lemma 5.16. Its proof uses the same ideas – translated to the fully discrete setting – and it is based on the careful combination of Lemma 5.12 (based on [4, Lemma 7.1]) and Lemma 5.15 (based on [99, Lemma 7.1], [27, Lemma 8.1]), with paying particular attention to the coupling terms in the MLLG system.

For the LLG part we need sufficient regularity of the solution, a smallness estimate on the right hand side which results in a mild CFL condition (5.14), and for the Maxwell part the bound on the stabilization parameter $\beta \geq 1$ and the CFL condition (5.15). As stated in Remark 5.3, this is covered by the CFL condition (5.12), i.e. for a constant $C > 0$

$$\tau \leq Ch.$$

Lemma 5.23 (Stability)**.** *Let the errors* (5.65) *satisfy the error equation* (5.66) *and suppose that the exact solution is smooth enough (cf. Lemma 5.16). Furthermore assume that $\beta \geq 1$ and the CFL condition* (5.12)*.*

Then, for sufficiently small $h \leq \overline{h}$ and $\tau \leq \overline{\tau}$, the error satisfies the following bound,

$$\begin{aligned}&\|e_{m,h}^n\|_{H^1(\Omega)}^2 + \|e_{E,h}^n\|_{L^2} + \|e_{H,h}^n\|_{L^2} \\ &\quad \leq C\Big(\|e_{m,h}^0\|_{H^1}^2 + \|e_{m,h}^1\|_{H^1}^2 + \|e_{E,h}^0\|_{L^2} + \|e_{H,h}^0\|_{L^2} + \tau\|\dot{e}_{m,h}^0\|_{L^2}^2 + \tau\|\dot{e}_{m,h}^1\|_{L^2}^2 \\ &\quad + \tau\sum_{j=2}^{n}(\|d_{m,h}^j\|_{L^2(\Omega)}^2 + \|s_h^j\|_{H^1(\Omega)}^2) + \tau\sum_{j=0}^{n}(\|d_{E,h}^{j-1/2}\|_{L^2(\Omega)}^2 + \|d_{H,h}^j\|_{L^2(\Omega)}^2) \\ &\quad + \tau\sum_{j=0}^{n}(\|(\partial_t^\tau)^2 d_{\varphi,h}^{j-1/2}\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^2 d_{\psi,h}^{j-1/2}\|_{\mathcal{H}_\Gamma}^2)\Big),\end{aligned} \tag{5.75}$$

for $n \geq 0$, where the constant C is independent of h, τ and n, but depends on α, R, K, M, μ, ε, σ, and T. This estimate holds under the smallness condition that the right-hand side of this estimate is bounded by $\widehat{c}h$ with a sufficiently small constant $\widehat{c}$ (note that the right-hand side is of size $O((\tau^2 + h^r)^2)$ in the case of a sufficiently regular solution).

Proof. Again, we consider the LLG part and the Maxwell part separately and conclude with a combination of both.

For the LLG equation, as long as the right hand side in the following estimate (denoted by rhs_{LLG} in the following) is small enough, we obtain by Lemma 5.12 (based on [4, Lemma 7.1]) for $n \geq 2$

$$\begin{aligned} &\|e_{m,h}^n\|_{H^1(\Omega)}^2 + \tau \sum_{j=2}^n \|\dot{e}_{m,h}^j\|_{L^2}^2 \\ &\quad \leq \frac{\widehat{C}}{2}\Big(\sum_{i=0}^1 \|e_{m,h}^i\|_{H^1(\Omega)}^2 + \tau \sum_{j=2}^n \|d_{m,h}^j - e_{H,h}^j\|_{L^2(\Omega)}^2 + \tau \sum_{j=2}^n \|s_h^j\|_{H^1(\Omega)}^2 \Big), \\ &\quad \leq \widehat{C}\Big(\|e_{m,h}^0\|_{H^1}^2 + \|e_{m,h}^1\|_{H^1}^2 + \tau \sum_{j=2}^n (\|d_{m,h}^j\|_{L^2(\Omega)}^2 + \|e_{H,h}^j\|_{L^2(\Omega)}^2 + \|s_h^j\|_{H^1(\Omega)}^2) \Big) \\ &\qquad =: \text{rhs}_{LLG}(t). \end{aligned} \tag{5.76}$$

It holds for the right hand side $\text{rhs}(t)$ in (5.75) for a $C > 0$

$$\text{rhs}_{LLG}(t) \leq C \text{rhs}(t),$$

i.e. the smallness assumption for the right hand side of Lemma 5.12 is satisfied, if the right hand side in the assertion of this lemma is small enough. Furthermore, the estimate (5.76) holds also for $n = 0$ and $n = 1$.

For the Maxwell part, we apply Lemma 5.15 (based on [99, Lemma 7.1]) under the stated assumptions and obtain that

$$\mathcal{E}_{M,h}^n = \|e_{E,h}^n\|_{L^2(\Omega)}^2 + \|e_{H,h}^n\|_{L^2(\Omega)}^2$$

is bounded, at $t_n = n\tau$, by

$$\begin{aligned} \mathcal{E}_{M,h}^n \leq \frac{\widetilde{C}}{2}\bigg(&\mathcal{E}_{M,h}^0 + \tau \sum_{k=0}^n \Big(\|\sigma \overline{e}_{E,h}^{k-1/2} + d_{E,h}^{k-1/2}\|_{L^2(\Omega)}^2 + \|\mu \dot{e}_{m,h}^k + d_{H,h}^k\|_{L^2(\Omega)}^2 \Big) \\ &+ \tau \sum_{k=0}^n \Big(\|(\partial_t^\tau)^2 d_{\psi,h}^{k+1/2}\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^2 d_{\varphi,h}^{k+1/2}\|_{\mathcal{H}_\Gamma}^2 \Big) \bigg), \\ \leq \widetilde{C}\bigg(&\mathcal{E}_{M,h}^0 + \tau \sum_{k=0}^n \Big(\|\sigma \overline{e}_{E,h}^{k-1/2}\|_{L^2(\Omega)}^2 + \|d_{E,h}^{k-1/2}\|_{L^2(\Omega)}^2 + \|\mu \dot{e}_{m,h}^k\|_{L^2(\Omega)}^2 + \|d_{H,h}^k\|_{L^2(\Omega)}^2 \Big) \\ &+ \tau \sum_{k=0}^n \Big(\|(\partial_t^\tau)^2 d_{\psi,h}^{k+1/2}\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^2 d_{\varphi,h}^{k+1/2}\|_{\mathcal{H}_\Gamma}^2 \Big) \bigg), \end{aligned}$$

where $\widetilde{C} > 0$ is independent of h, τ and n.

Now a combination of the two estimates, absorption of the error terms and the discrete Gronwall Lemma A.2 conclude the assertion. Therefore we multiply the LLG-estimate by $\mu^2 \widetilde{C}$ (without loss of generality $\mu^2 \widetilde{C} \geq 1$) and add it to the Maxwell estimate to obtain for $\mathcal{E}_h^n = \|e_{m,h}^n\|_{H^1(\Omega)}^2 + \mathcal{E}_{M,h}^n$ and $n \geq 0$

$$\begin{aligned} \mathcal{E}_h^n + \widetilde{C} \sum_{j=2}^n \|\mu \dot{e}_{m,h}^j\|_{L^2(\Omega)}^2 \leq\ & C\bigg(\mathcal{E}_h^0 + \|e_{m,h}^n\|_{H^1(\Omega)}^2 + \tau \sum_{j=2}^n \Big(\|d_{m,h}^j\|_{L^2(\Omega)}^2 + \|s_h^j\|_{H^1(\Omega)}^2 \Big) \\ &+ \tau \sum_{j=0}^n \Big(\|d_{E,h}^{j-1/2}\|_{L^2(\Omega)}^2 + \|d_{H,h}^j\|_{L^2(\Omega)}^2 + \|(\partial_t^\tau)^2 d_{\psi,h}^{j+1/2}\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^2 d_{\varphi,h}^{j+1/2}\|_{\mathcal{H}_\Gamma}^2 \Big) \bigg) \\ &+ \widetilde{C} \sum_{j=0}^n \|\mu \dot{e}_{m,h}^j\|_{L^2(\Omega)}^2 + C\tau \sum_{j=0}^n \mathcal{E}_h^j, \end{aligned}$$

for a generic constant $C > 0$ depending on σ, $\widehat{C}$ and $\widetilde{C}$. Direct absorption of the term $\widetilde{C} \sum_{j=2}^{n} \|\mu \dot{e}_{m,h}^{j}\|_{L^2(\Omega)}^2$ and for small enough τ the absorption of $\tau C \mathcal{E}_h^n$ (i.e. $\tau C \leq 1/2$), gives

$$\begin{aligned}\mathcal{E}_h^n \leq\ & C\Big(\mathcal{E}_h^0 + \|e_{m,h}^1\|_{H^1}^2 + \tau\|\dot{e}_{m,h}^0\|_{L^2}^2 + \tau\|\dot{e}_{m,h}^1\|_{L^2}^2 + \tau\sum_{j=2}^{n}\Big(\|d_{m,h}^j\|_{L^2}^2 + \tau\sum_{j=2}^{n}\|s_h^j\|_{H^1}^2\Big) \\ & + \tau\sum_{j=0}^{n}\Big(\|d_{E,h}^{j-1/2}\|_{L^2(\Omega)}^2 + \|d_{H,h}^j\|_{L^2(\Omega)}^2 + \|(\partial_t^\tau)^2 d_{\psi,h}^{j+1/2}\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^2 d_{\varphi,h}^{j+1/2}\|_{\mathcal{H}_\Gamma}^2\Big)\Big) \\ & + C\tau\sum_{j=0}^{n-1}\mathcal{E}_h^j.\end{aligned}$$

The discrete version of Gronwall's lemma A.2 concludes the assertion. □

Proof of Theorem 5.1. A combination of the interpolation properties from Section 5.6.1 together with the consistency and stability results Lemma 5.22 and Lemma 5.23 yields the assertion: By the above Lemmas, we have for a sufficiently smooth solution

$$\|e_{m,h}^n\|_{H^1(\Omega)}^2 + \|e_{E,h}^n\|_{L^2} + \|e_{H,h}^n\|_{L^2} \leq C(\tau^2 + h^r).$$

Recalling (5.65) and employing

$$\begin{aligned}\|\mathrm{R}_h m(t_n) - m(t_n)\|_{H^1(\Omega)} &\leqslant Ch^r, \\ \|\mathrm{I}_h^{\mathcal{W}} H(t_n) - H(t_n)\|_{L^2(\Omega)} &\leqslant Ch^r, \\ \|\mathrm{I}_h^{\mathcal{W}} E(t_n) - E(t_n)\|_{L^2(\Omega)} &\leqslant Ch^r,\end{aligned}$$

implies the error bound (5.33).

The smallness condition imposed in Lemma 5.23 is satisfied under the very mild CFL condition (compare [4, Remark 3.1]), for a sufficiently small $\bar{c} > 0$ (independent of h, τ and n),

$$\tau^2 \leqslant \bar{c} h^{1/2}.$$

Compare Remark 5.3 that this is fulfilled under the CFL condition (5.12). Taken together, this proves Theorem 5.1. □

6. Numerics

In this chapter we consider numerical experiments for the algorithms proposed in the Chapters 3–5.

6.1. Preliminaries

6.1.1. Notation

In this section we present the relevant notation we use in the following for the implementation.

For a finite dimensional space $\mathcal{V}_h$ and a function $E_h \in \mathcal{V}_h$, we denote by $\boldsymbol{E}(\mathcal{V}_h)$ the vector of coefficients with respect to the basis $\boldsymbol{\phi}(\mathcal{V}_h)$ used in the respective software platform for $\mathcal{V}_h$. Therefore we can represent E_h as

$$E_h = \sum_{j=1}^{|\mathcal{V}_h|} E_j(\mathcal{V}_h)\phi_j(\mathcal{V}_h) = \boldsymbol{E}(\mathcal{V}_h) \cdot \boldsymbol{\phi}(\mathcal{V}_h),$$

where $E_j(\mathcal{V}_h)$ is the j-th coefficient corresponding to the j-th basis function $\phi_j(\mathcal{V}_h)$, and $\boldsymbol{E}(\mathcal{V}_h)$, $\boldsymbol{\phi}(\mathcal{V}_h)$ denote the corresponding vectors of coefficients and basis functions. The variable ϕ is exclusively reserved for basis functions and not for coefficients.

For the spaces in the domain Ω, we abbreviate the first order Nédélec space $\mathcal{X}_h$ by $N1$ and the piecewise constant space $\mathcal{Y}_h$ by $N0$. The space of scalar linear elements $S^1(\mathcal{T}_h, \mathbb{R})$ and the space of vector valued linear elements $S^1(\mathcal{T}_h, \mathbb{R}^3)$ are both abbreviated with the symbol $S1$.

A short introduction to the spaces on the boundary can be found in Section 6.1.4 (cf. [144] for further details). We abbreviate the Raviart–Thomas space $\mathcal{V}_h^{RT}$ by RT, the Nédélec space $\mathcal{V}_h^{NC}$ by NC, the Rao–Wilton–Glisson space $\mathcal{V}_h^{RWG}$ by RWG, the scaled Nédélec space $\mathcal{V}_h^{SNC}$ by SNC, the Buffa–Christiansen space $\mathcal{V}_h^{BC}$ by BC and the rotated Buffa–Christiansen space $\mathcal{V}_h^{RBC}$ by RBC. Due to implementational reasons, some of the spaces have mathematically identical counterparts on a (barycentrically) refined grid. We put a B in front of the abbreviation to underline that difference wherever necessary (this is only relevant for the implementation).

For a sequence $(\phi^j)_{j\in\mathbb{N}_0}$ we define the sequence with the j-th entry set to zero as

$$\phi_{|\phi^j=0} := (\phi^0, \dots, \phi^{j-1}, 0, \phi^{j+1}, \dots).$$

6.1.2. Tangent plane scheme

We present the implementation of the tangent plane scheme in FEniCS [8]. As in [4, Section 2], we build up the saddle point problem to implement the tangent space constraint.

Instead of computing the unique solution $w_h^j \in \mathcal{K}_{m_h^j}$ such that for all $\rho_h \in \mathcal{K}_{m_h^j}$

$$\alpha[w_h^j, \rho_h]_\Omega + \left[m_h^j \times w_h^j, \rho_h\right]_\Omega = -C_e \left[\nabla(m_h^j + \theta\tau w_h^j), \nabla\rho_h\right]_\Omega + \left[H_h^j, \rho_h\right]_\Omega,$$

we use a saddle point approach: We seek $(w_h^j, \lambda_h) \in \mathcal{S}^1(\mathcal{T}_h, \mathbb{R}^3) \times \mathcal{S}^1(\mathcal{T}_h, \mathbb{R})$ such that for all $(\rho_h, \xi_h) \in S^1(\mathcal{T}_h, \mathbb{R}^3) \times \mathcal{S}^1(\mathcal{T}_h, \mathbb{R})$

$$\alpha[w_h^j, \rho_h]_\Omega + \Big[m_h^j \times w_h^j, \rho_h\Big]_\Omega = -C_e \Big[\nabla(m_h^j + \theta\tau w_h^j), \nabla\rho_h\Big]_\Omega + \Big[H_h^j, \rho_h\Big]_\Omega, \\ + [\rho_h \cdot m_h^j, \lambda_h]_\Omega + [w_h^j \cdot m_h^j, \xi_h]_\Omega.$$

We update and normalize by computing $m_h^{j+1}(z) := \frac{m_h^j(z)+\tau w_h^j(z)}{|m_h^j(z)+\tau w_h^j(z)|}$, i.e. projecting the outcome to $S^1(\mathcal{T}_h, \mathbb{R}^3)$. There are other possibilities to implement the tangent plane scheme, e.g., one could directly parameterize the tangent space. For simplicity however, we stick with the present approach.

6.1.3. Convolution Quadrature

In this section we present the formula for the approximation of the Convolution Quadrature weights.

An exact formula for the convolution weight operators B_n^τ is

$$B_n^\tau = \frac{1}{2\pi i}\int_{|\zeta|=\rho} B\Big(\frac{\delta(\zeta)}{\tau}\Big)\zeta^{-n-1}\,\mathrm{d}\zeta$$

for $\rho < 1$. As in [115, Formula (3.10)], we approximate the integral by the trapezoidal rule

$$B_n^\tau \approx \frac{\rho^{-n}}{L}\sum_{l=0}^{L-1} B\Big(\frac{\delta(\zeta_l)}{\tau}\Big)e^{-2\pi i n l/L}, \quad n = 0, \dots, N, \tag{6.1}$$

with $L = 2N$ or $L = N$ evaluation points

$$\zeta_l = \rho e^{2\pi i l/L}, \quad l = 0, \dots, L-1$$

and radius of integration $\rho = \epsilon^{0.5/N}$ for a predefined tolerance $\epsilon > 0$. We compute

$$B_0^\tau = B\Big(\frac{\delta(0)}{\tau}\Big)$$

exactly. Like it is proposed in [115], as

$$\|B(s)\| \le C|s|^2,$$

we rewrite for $m \in \mathbb{N}_0$

$$B(\partial_t^\tau)\phi = (B(s)s^{-m})(\partial_t^\tau)(\partial_t^\tau)^m\phi$$

and compute $\psi = (\partial_t^\tau)^m\phi$ and $(B(s)s^{-m})(\partial_t^\tau)\psi$ independently of each other. The weights for $(\partial_t^\tau)^m$ are computed exactly (the corresponding function in equation (3.12) with $B(s) = s^m$ is a polynomial). The weights for $(B(s)s^{-m})(\partial_t^\tau)\psi$ are approximated via (6.1) and indicated by $B_n^{m,\tau}$ (again, $B_0^{m,\tau} = B(\delta(0)/\tau)\tau^m\delta(0)^{-m}$ is computed exactly). We denote the resulting operator by $B(\widetilde{\partial_t^\tau})$. For the first order scheme, it is $\delta(\xi) = 1 - \xi$ and

$$\partial_t^\tau\phi(t_j) = \frac{\phi^j - \phi^{j-1}}{\tau}.$$

To separate the unknown ϕ^{n+1}, we point out

$$\begin{aligned}(B(\widetilde{\partial_t^\tau})\phi)(t_{n+1}) &= B_0^{m,\tau}((\partial_t^\tau)^m\phi)(t_{n+1}) + \sum_{j=0}^{n} B_{n+1-j}^{m,\tau}((\partial_t^\tau)^m\phi)(t_j)\\ &= B_0^{m,\tau}\frac{\phi^{n+1}}{\tau^m} + B_0^{m,\tau}(\partial_t^\tau)^m(\phi_{|\phi^{n+1}=0})(t_{n+1}) + \sum_{j=0}^{n} B_{n+1-j}^{m,\tau}((\partial_t^\tau)^m\phi)(t_j)\\ &= B_0^{m,\tau}\frac{\phi^{n+1}}{\tau^m} + B(\widetilde{\partial_t^\tau})(\phi_{|\phi^{n+1}=0})(t_{n+1}),\end{aligned}$$

where we recall that $(\phi_{|\phi^{n+1}=0})$ is the sequence $(\phi^0, \phi^1, \dots, \phi^n, 0, \phi^{n+2}, \dots)$. For higher order schemes, the additional factor $\delta(0)$ (it is $\delta(0) = 1$ for the first order scheme) comes into place and the equation reads

$$(B(\widetilde{\partial}_t^\tau)\phi)(t_{n+1}) = B_0^{m,\tau}\delta(0)^m \frac{\phi^{n+1}}{\tau^m} + B(\widetilde{\partial}_t^\tau)(\phi_{|\phi^{n+1}=0})(t_{n+1}).$$

In both cases, the factor $\delta(0)^m/\tau^m$ cancels in the final discretization matrix on the left hand side, as $B_0^{m,\tau} = B_0^\tau \delta(0)^{-m}\tau^m$.

6.1.4. Boundary spaces

We present an overview of the basis functions on the boundary and the corresponding discrete spaces from Bempp [145]. The degrees of freedom correspond to the edges of the triangulation and each basis function is assigned to one edge. For the j-th edge with length l_j, we denote the two triangles adjacent to the edge by T_+ and T_- , A_+ and A_- are the areas of the triangles and p_+ and p_- are the opposite corners of the triangles, compare Figure 6.1.1.

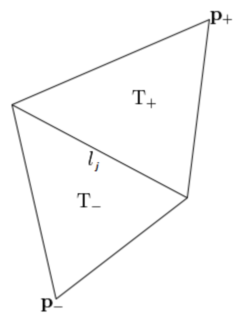

Figure 6.1.1.: Sketch of j-th edge with adjacent triangles from [90].

For the most of the standard basis functions, there are corresponding scaled basis functions (scaled by the edge length) implemented, which naturally generate the same discrete spaces. Even further, there are corresponding basis functions defined on the barycentrically refined grid, that are mathematically identical to the original basis functions (see also the following sketches, e.g. compare Figure 6.1.2 and Figure 6.1.4). Nevertheless, for the implementation a differentiation between all of them is necessary (inner products can only be computed if the functions are defined on the same grid). The barycentrically refined grid arises from adding the centroid on every cell (The centroid x_S of a cell is given by $x_S = (x_{C_1} + x_{C_2} + x_{C_3})/3$ for the corners x_{C_1}, x_{C_2}, x_{C_3} for a two dimensional grid) to the set of nodes and extend the set of edges by the connecting lines between the centroid and the corners of every cell, see Figure 6.1.4.

For the Raviart–Thomas (RT) space (compare Figure 6.1.2), the j-th basis function associated to the j-th edge of the mesh is defined as (compare Figure 6.1.1)

$$\phi_j(RT)(x) := \begin{cases} \frac{1}{2A_+}(x - p_+), & x \in T_+, \\ -\frac{1}{2A_-}(x - p_-), & x \in T_-. \\ 0, & \text{otherwise} \end{cases}$$

The Nédélec (NC) space (compare Figure 6.1.2) is generated by rotated RT functions, i.e. the j-th basis function is defined as $\phi_j(NC) = n \times \phi_j(RT)$ and it holds $\phi_j(RT) = \phi_j(NC) \times n$.

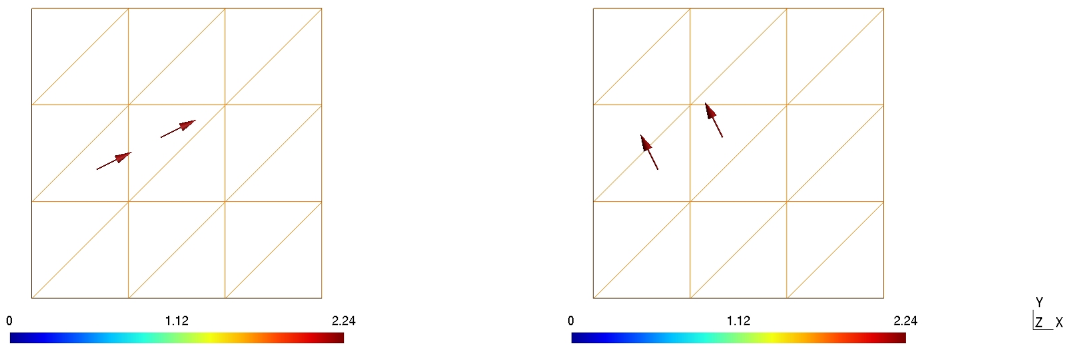

Figure 6.1.2.: Sketch of Raviart–Thomas (RT) element and Nédélec (NC) element.

The Rao–Wilton–Glisson (RWG) space (compare Figure 6.1.3) is identical to the Raviart–Thomas space, the basis functions are scaled with the edge length l_j of the associated edge, i.e. $\phi_j(RWG) = l_j\phi_j(RT)$. The basis functions of the scaled Nédélec space (SNC) (compare Figure 6.1.3), similarly, are scaled with the edge length, it holds $\phi_j(SNC) = l_j\phi_j(NC) = n \times \phi_j(RWG)$ and $\phi_j(RWG) = \phi_j(SNC) \times n$.

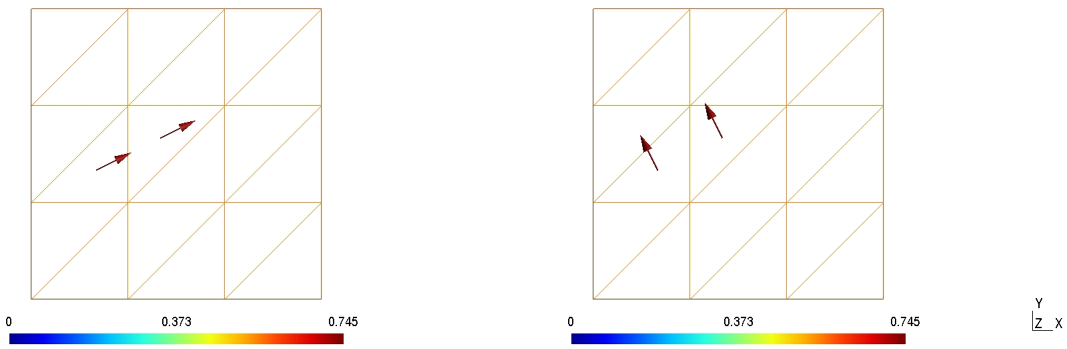

Figure 6.1.3.: Sketch of Rao–Wilton–Glisson (RWG) element and Scaled Nédélec (SNC) element.

A Raviart–Thomas basis function defined on the barycentrically refined grid (BRT) is in theory identical to the RT basis function (compare Figure 6.1.4), for the implementation we have to distinguish between RT and BRT. Similarly, the Nédélec basis function on the barycentric grid (BNC), the Rao–Wilton–Glisson element on the barycentric grid (BRWG) and the scaled Nédélec element on the barycentric grid (BSNC), are the correpsonding counterparts on the refined grid, compare Figure 6.1.5.

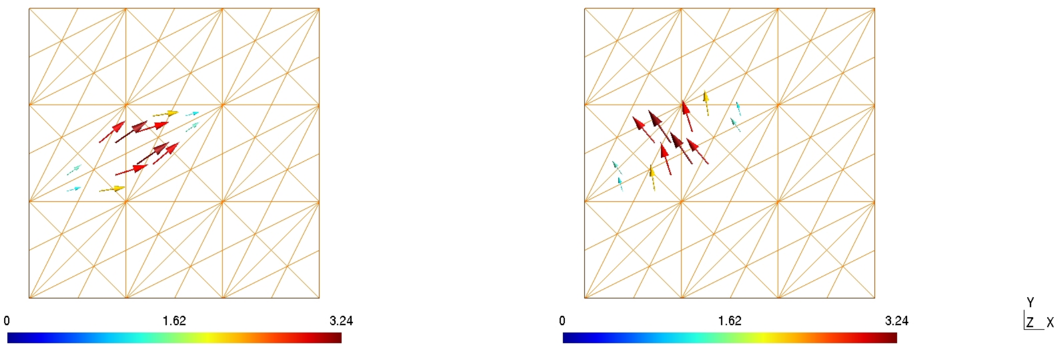

Figure 6.1.4.: Sketch of Raviart–Thomas element on barycentric grid (BRT) and Nédélec element on barycentric grid (BNC).

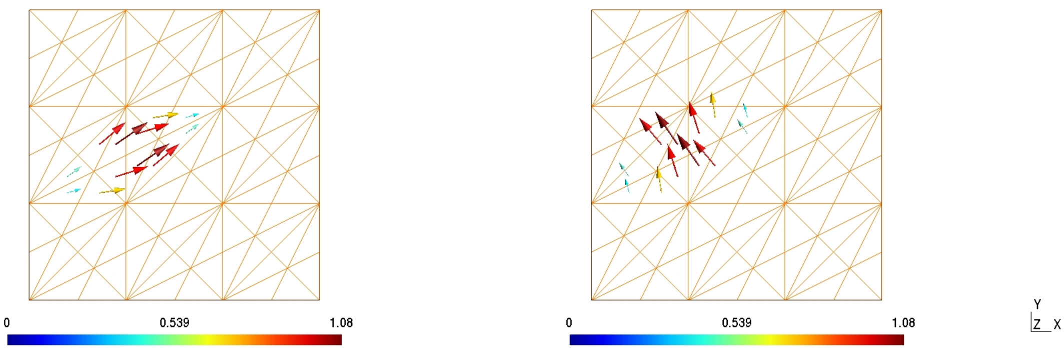

Figure 6.1.5.: Sketch of Rao–Wilton–Glisson element on barycentric grid (BRWG) and Scaled Nédélec element on barycentric grid (BSNC).

The Buffa–Christiansen (BC) space (compare Figure 6.1.6) contains basis functions defined on the barycentrically refined grid such that we have curl conforming vector fields on Γ and the space is L^2-dual to the Raviart–Thomas space, for details see [42]. The rotated Buffa–Christiansen (RBC) basis functions (compare Figure 6.1.6) are defined as rotations of the corresponding BC basis function, i.e. it holds $\phi_j(RBC) = n \times \phi_j(BC)$ and $\phi_j(BC) = \phi_j(RBC) \times n$.

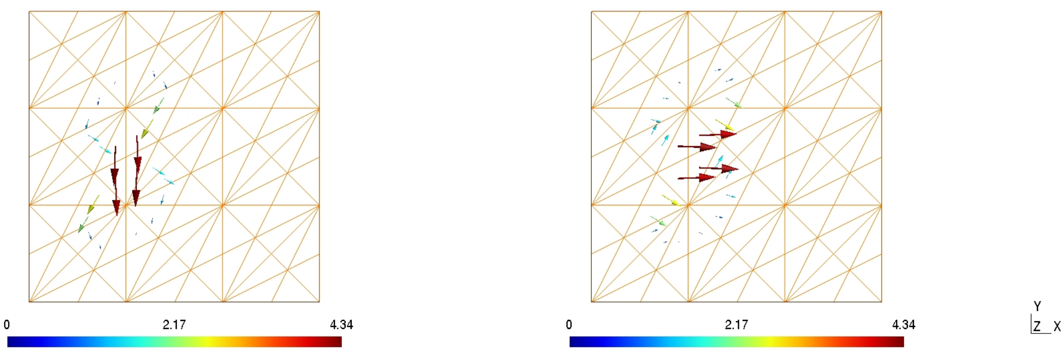

Figure 6.1.6.: Sketch of Buffa–Christiansen (BC) and Rotated Buffa–Christiansen (RBC) element.

6.1.5. Operator concept in Bempp

We briefly illuminate the operator concept in Bempp [145]. For discrete spaces DS (Domain Space), RS (Range Space), $DSRS$ (Dual Space to the Range Space), let F be a linear operator which we denote by

$$F : DS \xrightarrow{DSRS} RS.$$

We denote the weak form of F by ${}_{DSRS}F_{DS}$, it maps coefficients of a function u in the domain space DS to a weak form tested with functions of the dual space to the range space DSRS. It holds

$$[{}_{DSRS}F_{DS}]_{ij} = \int_\Gamma \phi_i(DSRS) \cdot F(\phi_j(DS)) \, \mathrm{d}s,$$

i.e. for all $\psi \in DSRS$

$$[\psi, F(u)]_\Gamma = \boldsymbol{\psi}(DSRS) \cdot {}_{DSRS}F_{DS} \cdot \boldsymbol{u}(DS).$$

We denote the strong form of F by $F_{DS\to RS} := F_{DS \xrightarrow{DSRS} RS}$, it maps coefficients of a function u in the domain space DS to coefficients of $F(u)$ given with respect to the range space RS. The coefficients of $F(u)$ are determined by testing with functions in the dual to range space DSRS, i.e. it holds for all $i = 1, \ldots, |DSRS|$

$$\int_\Gamma \phi_i(DSRS) \cdot \left(\sum_{j=1}^{|RS|} (F_{DS\to RS} \cdot \boldsymbol{u}(DS))_j \phi_j(RS) \right) \mathrm{d}s = \int_\Gamma \phi_i(DSRS) \cdot F(u) \, \mathrm{d}s.$$

Therefore it is for all $\psi \in DSRS$

$$[\psi, (F_{DS\to RS} \cdot \boldsymbol{u}(DS)) \cdot \phi(RS)]_\Gamma = [\psi, F(u)]_\Gamma$$

and we have

$$F_{DS\to RS} = ({}_{DSRS}\mathrm{Id}_{RS})^{-1} \, {}_{DSRS}F_{DS}.$$

We introduce the projection $P_{DSRS} : L^2(\Gamma) \to RS$, $u \mapsto P_{DSRS}u$ as the element $P_{DSRS}u \in RS$ satisfying for all $\psi \in DSRS$

$$[\psi, P_{DSRS}u]_\Gamma = [\psi, u]_\Gamma.$$

Therefore we can express the strong form as

$$(\boldsymbol{P_{DSRS}F(u)})(RS) = F_{DS\to RS} \cdot \boldsymbol{U}(DS).$$

6.1.6. Implemented operators

The trace mapping is implemented via the Bempp function

$$\gamma_T : N1 \xrightarrow{RT} RT,$$

i.e. $(\gamma_T)_{N1\to RT}$ is the matrix that maps coefficients with respect to $N1$ (abbreviation for $\mathcal{X}_h$) to coefficients given with respect to RT on the boundary. It holds for $E_h \in \mathcal{X}_h$

$$\gamma_T E_h = ((\gamma_T)_{N1\to RT} \cdot \boldsymbol{E}(N1)) \cdot \boldsymbol{\phi}(\mathrm{RT}),$$

i.e.

$$(\boldsymbol{\gamma_T E})(RT) = (\gamma_T)_{N1\to RT} \cdot \boldsymbol{E}(N1).$$

An application of $n \times \cdot$ can be implemented via a change of the basis functions, by $n \times \phi_j(RT) = \phi_j(NC)$, we have

$$n \times \gamma_T E_h = ((\gamma_T)_{N1\to RT} \cdot \boldsymbol{E}(\mathcal{X}_h)) \cdot \boldsymbol{\phi}(\text{NC}).$$

The Calderon operator is implemented in Bempp via

$$\widehat{B}(k) = \begin{pmatrix} \widehat{D} & \widehat{E} \\ \widehat{F} & \widehat{G} \end{pmatrix},$$

where

$$\begin{array}{c|c} \widehat{D} : BRWG \xrightarrow{RBC} BRWG & \widehat{E} : BC \xrightarrow{RBC} BRWG \\ \hline \widehat{F} : BRWG \xrightarrow{BSNC} BC & \widehat{G} : BC \xrightarrow{BSNC} BC \end{array}. \tag{6.2}$$

After rescaling this operator, i.e. setting

$$\widetilde{B}(k) := \begin{pmatrix} \widetilde{D} & \widetilde{E} \\ \widetilde{F} & \widetilde{G} \end{pmatrix} := \begin{pmatrix} \widehat{D} & \sqrt{\frac{\mu_0}{\varepsilon_0}}\widehat{E} \\ \sqrt{\frac{\varepsilon_0}{\mu_0}}\widehat{F} & \widehat{G} \end{pmatrix},$$

the condition for suitable exterior data of the time harmonic Maxwell equations can be rewritten as: The traces $\gamma_T E$, $\gamma_T H$ are suitable exterior data for the problem (with $\Im k > 0$, $s = -ik/\sqrt{\mu_0\varepsilon_0}$)

$$\begin{aligned} -s\varepsilon_0 u + \nabla \times v &= 0, \\ s\mu_0 v + \nabla \times u &= 0, \\ \gamma_T u &= \gamma_T E, \\ \gamma_T v &= \gamma_T H, \end{aligned}$$

if and only if

$$\left(\frac{1}{2} - \widetilde{B}(k)\right)\begin{pmatrix} \gamma_T E \\ \gamma_T H \end{pmatrix} = \begin{pmatrix} \gamma_T E \\ \gamma_T H \end{pmatrix}. \tag{6.3}$$

Comparing this with the compatibility condition (2.26), we rewrite (6.3) for $k = i\sqrt{\mu_0\varepsilon_0}s$ as

$$\frac{-1}{\mu_0}\begin{pmatrix} \mu_0^{-1}\widetilde{E} & -\widetilde{D} \\ \widetilde{G} & -\mu_0\widetilde{F} \end{pmatrix}(i\sqrt{\mu_0\varepsilon_0}s)\begin{pmatrix} \mu_0\gamma_T H \\ -\gamma_T E \end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix} \gamma_T E \\ \mu_0\gamma_T H \end{pmatrix}.$$

We deduce that the Calderon operator from Section 2.3 is given as

$$B(s) = \frac{-1}{\mu_0}\begin{pmatrix} \mu_0^{-1}\widetilde{E} & -\widetilde{D} \\ \widetilde{G} & -\mu_0\widetilde{F} \end{pmatrix}(i\sqrt{\mu_0\varepsilon_0}s). \tag{6.4}$$

The overview of the different rescalings in Section A.3 in the Appendix confirms this derivation, i.e. we have that $\widehat{B}(k) = B^{SB}(k)$, $\widetilde{B}(k) = B^{SV}(k)$ and $B(s) = B^{KL}(s)$.

6.1.7. Changes in Bempp-cl

The implementation details in Section 6.2.1 and Section 6.3.1 are with respect to the Bempp version 3.3.4. In the mean time, a newer version of Bempp, Bempp-cl has been published. For an overview of the changes (and the performance gains) we refer to [35]. We only state the few changes that are relevant for the algorithms in this thesis and especially for the implementation details in Section 6.4.1

The differentiation between scaled and unscaled basis functions has been removed, i.e. only the scaled versions *RWG* (instead of *RT*) and *SNC* (instead of *NC*) remain.

Furthermore, the mathematically irrelevant barycentrically refined spaces have been removed, i.e. we do not have to distinguish any more between RWG and BWRG, and SNC and BSNC basis functions (the latter have been removed).

The scaling by the factor $\sqrt{\mu_0/\varepsilon_0}$ in Section 6.1.6 to obtain the desired $\widetilde{B}$ is not necessary any more. This is done automatically by the implemented routines if we pass the magnetic and electric permeabilities μ_0 and ε_0 to them.

Even further, the Calderon operator can be generated with different domain, range and dual to range spaces than in (6.2), i.e. in Section 6.4.1 we choose all sub-operators to map

$$\widehat{D}, \widehat{E}, \widehat{F}, \widehat{G} : RWG \xrightarrow{SNC} RWG.$$

6.2. Weak Convergence for the MLLG System

6.2.1. Implementation details

In this section we present the implementation details of Algorithm 3.12 in FEniCS and Bempp. In comparison to Algorithm 3.12, the tangent plane scheme is formulated as a saddle point problem (cf. Section 6.1.2), the Convolution Quadrature weights are approximated by a quadrature rule (cf. Section 6.1.3) and the trace variable φ and the corresponding test functions are elements of the Buffa–Christiansen space instead of the Raviart–Thomas space.

Algorithm 6.1. Input: Discretized initial data m_h^0, H_h^0, E_h^0, $\varphi_h^0 = 0$, $\psi_h^0 = 0$ and parameter $\theta \in [0,1]$.
For $j = 0, 1, 2, \ldots, N-1$ we compute

- For given m_h^j, H_h^j we compute the unique solution $(w_h^j, \lambda_h) \in \mathcal{S}^1(\mathcal{T}_h, \mathbb{R}^3) \times \mathcal{S}^1(\mathcal{T}_h, \mathbb{R})$ such that for all $(\rho_h, \xi_h) \in S^1(\mathcal{T}_h, \mathbb{R}^3) \times \mathcal{S}^1(\mathcal{T}_h, \mathbb{R})$

$$\begin{aligned} \alpha[w_h^j, \rho_h]_\Omega + \left[m_h^j \times w_h^j, \rho_h\right]_\Omega = -C_e \left[\nabla(m_h^j + \theta\tau w_h^j), \nabla\rho_h\right]_\Omega + \left[H_h^j, \rho_h\right]_\Omega \\ + [\rho_h \cdot m_h^j, \lambda_h]_\Omega + [w_h^j \cdot m_h^j, \xi_h]_\Omega. \end{aligned} \tag{6.5}$$

- We compute $E_h^{j+1}, H_h^{j+1} \in \mathcal{X}_h$ and $\varphi_h^{j+1} \in \mathcal{V}_h^{BC}$, $\psi_h^{j+1} \in \gamma_T(\mathcal{X}_h)$ such that we have for all $\zeta_h^E, \zeta_h^H \in \mathcal{X}_h$ and $v_h^\varphi \in \mathcal{V}_h^{BC}$, $v_h^\psi \in \gamma_T(\mathcal{X}_h)$

$$\begin{aligned} [\varepsilon\partial_t^\tau E_h^{j+1}, \zeta_h^E]_\Omega &= \frac{1}{2}[\nabla \times H_h^{j+1}, \zeta_h^E]_\Omega + \frac{1}{2}[H_h^{j+1}, \nabla \times \zeta_h^E]_\Omega \\ &\quad - \frac{1}{2\mu_0}\langle\varphi_h^{j+1}, \gamma_T\zeta_h^E\rangle_\Gamma - [\sigma E_h^{j+1} + J^{j+1}, \zeta_h^E]_\Omega, \end{aligned} \tag{6.6}$$

$$\begin{aligned} [\mu\partial_t^\tau H_h^{j+1}, \zeta_h^H]_\Omega &= -\frac{1}{2}[\nabla \times E_h^{j+1}, \zeta_h^H]_\Omega - \frac{1}{2}[E_h^{j+1}, \nabla \times \zeta_h^H]_\Omega \\ &\quad - \frac{1}{2}\langle\psi_h^{j+1}, \gamma_T\zeta_h^H\rangle_\Gamma - [\mu w_h^j, \zeta_h^H]_\Omega, \end{aligned} \tag{6.7}$$

$$\left\langle \begin{pmatrix} v_h^\varphi \\ v_h^\psi \end{pmatrix}, \left(B(\widetilde{\partial}_t^\tau)\begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix}\right)(t_{j+1}) \right\rangle_\Gamma = \frac{1}{2}\left(\langle v_h^\varphi, \mu_0^{-1}\gamma_T E_h^{j+1}\rangle_\Gamma + \langle v_h^\psi, \gamma_T H_h^{j+1}\rangle_\Gamma\right). \tag{6.8}$$

- Define m_h^{j+1} by projecting

$$m_h^{j+1}(z) := \frac{m_h^j(z) + \tau w_h^j(z)}{|m_h^j(z) + \tau w_h^j(z)|}$$

to $S1$.

Output: Sequence of approximations $m_h^j, E_h^j, H_h^j, \varphi_h^j, \psi_h^j$ for $j = 0, 1, 2, \ldots, N$.

For the LLG part, after separating known and unknowns, we build the (bi)linear forms from equation (6.5) in FEniCS syntax and solve the equation with implemented routines.

For the Maxwell equations, the term $[\partial_t^\tau E_h^{j+1}, \zeta_h^E]_\Omega$ is a product of Nédélec functions, i.e. for the mass matrix

$$M_1 := {}_{N1}\mathrm{Id}_{N1},$$

it holds

$$[\partial_t^\tau E_h^{j+1}, \zeta_h^E]_\Omega = \frac{1}{\tau}\left(\boldsymbol{\zeta^E}(N1) \cdot M_1 \cdot \boldsymbol{E^{j+1}}(N1) - \boldsymbol{\zeta^E}(N1) \cdot M_1 \cdot \boldsymbol{E^j}(N1)\right).$$

We define the symmetric differential operator

$$D := \frac{1}{2}\, {}_{N1}(\nabla\times)_{N1} + \frac{1}{2}({}_{N1}(\nabla\times)_{N1})^T$$

and rewrite

$$\frac{1}{2}[\nabla \times H_h^{j+1}, \zeta_h^E]_\Omega + \frac{1}{2}[H_h^{j+1}, \nabla \times \zeta_h^E]_\Omega = \boldsymbol{\zeta^E}(N1) \cdot D \cdot \boldsymbol{H^{j+1}}(N1).$$

For the coupling to the boundary integral equation via the terms $\langle \varphi_h, \gamma_T \zeta_h^E \rangle_\Gamma$ and $\langle \psi_h, \gamma_T \zeta_h^H \rangle_\Gamma$, we express the anti-symmetric pairing $\langle \cdot, \cdot \rangle_\Gamma$ as $\langle \zeta, \xi \rangle_\Gamma = [\zeta \times n, \xi]_\Gamma$ and build up the respective terms for rotated basis functions with respect to the $L^2(\Gamma)$-product $[\cdot, \cdot]_\Gamma$. For the products $[\phi_j(BC) \times n, \phi_j(RT)]_\Gamma$ and $[\phi_j(RWG) \times n, \phi_j(RT)]_\Gamma$ we have to interchange the RT functions by the BRT functions (to compute the weak forms, the functions have to be defined on the same grid) and obtain

$$\begin{aligned}\langle \varphi_h, \gamma_T \zeta_h \rangle_\Gamma &= [\varphi_h \times n, \gamma_T \zeta]_\Gamma \\ &= -\boldsymbol{\varphi}(BC) \cdot {}_{RBC}\mathrm{Id}_{BRT} \cdot (\gamma_T)_{N1 \to RT} \cdot \boldsymbol{\zeta}(NC)\end{aligned}$$

and

$$\begin{aligned}\langle \psi_h, \gamma_T \zeta_h \rangle_\Gamma &= [\psi_h \times n, \gamma_T \zeta]_\Gamma \\ &= -\boldsymbol{\psi}(RWG) \cdot {}_{SNC}\mathrm{Id}_{BRT} \cdot (\gamma_T)_{N1 \to RT} \cdot \boldsymbol{\zeta}(NC).\end{aligned}$$

Most of the remaining terms in the first and second equation can be treated analogously, we have

$$\begin{aligned}-[\sigma E_h^{j+1} + J_h^{j+1}, \zeta_h^E]_\Omega &= -\boldsymbol{\zeta^E}(N1) \cdot M_1 \cdot \left(\sigma \boldsymbol{E^{j+1}}(N1) + \boldsymbol{J^{j+1}}(N1)\right), \\ [\partial_t^\tau H_h^{j+1}, \zeta_h^H]_\Omega &= \frac{1}{\tau}\left(\boldsymbol{\zeta^H}(N1) \cdot M_1 \cdot \boldsymbol{H^{j+1}}(N1) - \boldsymbol{\zeta^H}(N1) \cdot M_1 \cdot \boldsymbol{H^j}(N1)\right), \\ -[\mu w_h^j, \zeta_h^H]_\Omega &= -\mu \boldsymbol{\zeta^H}(N1) \cdot {}_{N1}\mathrm{Id}_{S1} \cdot \boldsymbol{w^j}(S1)\end{aligned}$$

and

$$-\frac{1}{2}[\nabla \times E_h^{j+1}, \zeta_h^H]_\Omega - \frac{1}{2}[E_h^{j+1}, \nabla \times \zeta_h^H]_\Omega = -\boldsymbol{\zeta^H}(N1) \cdot D \cdot \boldsymbol{E^{j+1}}(N1).$$

For the equation on the boundary, in Algorithm 6.1 the discrete test functions are RT and BC functions with respect to the anti-symmetric pairing $\langle \cdot, \cdot \rangle_\Gamma$. This can be equivalently reached by using $n \times \phi_j(RT) = \phi_j(NC)$ and $n \times \phi_j(BC) = \phi_j(RBC)$ functions with respect to the $L^2(\Gamma)$ scalar product $[\cdot, \cdot]_\Gamma$. For the NC functions, it is equivalent to test with the scaled Nédélec functions on the refined grid, instead. The discretized boundary equation takes the form

$${}_{\binom{RBC}{SNC}}B(\partial_t^\tau)_{\binom{BC}{RWG}}\begin{pmatrix}\boldsymbol{\varphi}(BC) \\ \boldsymbol{\psi}(RWG)\end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix}{}_{RBC}\mathrm{Id}_{BRT} & 0 \\ 0 & {}_{SNC}\mathrm{Id}_{BRT}\end{pmatrix}\begin{pmatrix}(\gamma_T)_{N1 \to RT}\boldsymbol{E}(N1) \\ \mu_0(\gamma_T)_{N1 \to RT}\boldsymbol{H}(N1)\end{pmatrix}.$$

We multiply this equation by -1, to obtain an overall positive definite discretization matrix, because ${}_{\binom{RBC}{SNC}}B(0)_{\binom{BC}{RWG}}$ is negative definite: It holds for $\zeta, \xi \in L^2(\Gamma)$

$$\langle \zeta, \xi \rangle_\Gamma = [\zeta \times n, \xi]_\Gamma$$

and for the basis functions we have $\phi_j(RBC) = n \times \phi_j(BC) = -\phi_j(BC) \times n$ and $\phi_j(SNC) = n \times \phi_j(RWG) = -\phi_j(RWG) \times n$. So it is

$$\begin{aligned}
\boldsymbol{\zeta}\begin{pmatrix} BC \\ RWG \end{pmatrix} {}_{\binom{RBC}{SNC}}B(\delta(0)/\tau)_{\binom{BC}{RWG}} \boldsymbol{\zeta}\begin{pmatrix} BC \\ RWG \end{pmatrix} &= [n \times \zeta, B(\delta(0)/\tau))\zeta] \\
&= -\langle \zeta, B(\delta(0)/\tau))\zeta \rangle \\
&\leq -c\|\zeta\|^2_{\mathcal{H}_\Gamma}
\end{aligned}$$

negative definite

We summarize and build up the resulting system. We recall the mass matrices

$$\begin{aligned}
M_1 &= {}_{N1}\mathrm{Id}_{N1}, \\
M_2 &= {}_{RBC}\mathrm{Id}_{BRT}(\gamma_T)_{N1 \to RT}, \\
M_3 &= {}_{BSNC}\mathrm{Id}_{BRT}(\gamma_T)_{N1 \to RT},
\end{aligned}$$

the symmetric, discrete differential operator

$$D = \frac{1}{2}\, {}_{N1}\nabla \times_{N1} + \frac{1}{2} ({}_{N1}\nabla \times_{N1})^T$$

and the Calderon sub operators

$$\begin{aligned}
B_{1,1} &= \frac{-1}{\mu_0^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, {}_{RBC}\widehat{E}_{BC}(i\sqrt{\mu_0\varepsilon_0}\delta(0)/\tau), \\
B_{1,2} &= \frac{1}{\mu_0}\, {}_{RBC}\widehat{D}_{BRWG}(i\sqrt{\mu_0\varepsilon_0}\delta(0)/\tau), \\
B_{2,1} &= \frac{-1}{\mu_0}\, {}_{BSNC}\widehat{G}_{BC}(i\sqrt{\mu_0\varepsilon_0}\delta(0)/\tau), \\
B_{2,2} &= \sqrt{\frac{\varepsilon_0}{\mu_0}}\, {}_{BSNC}\widehat{F}_{BRWG}(i\sqrt{\mu_0\varepsilon_0}\delta(0)/\tau).
\end{aligned}$$

The overall discretization matrix then is

$$Lhs := \begin{pmatrix}
(\frac{\varepsilon_0}{\tau} + \sigma)M_1 & -D & \frac{-1}{2\mu_0}M_2^T & 0 \\
D & \frac{\mu_0}{\tau}M_1 & 0 & \frac{-1}{2}M_3^T \\
\frac{1}{2\mu_0}M_2 & 0 & -B_{1,1} & -B_{1,2} \\
0 & \frac{1}{2}M_3 & -B_{2,1} & -B_{2,2}
\end{pmatrix}$$

with right hand side

$$Rhs^i := \begin{pmatrix}
\frac{\varepsilon_0}{\tau}M_1\boldsymbol{E^i}(N1) - M_1\boldsymbol{J^{i+1}}(N1) \\
\frac{\mu_0}{\tau}M_1\boldsymbol{H^i}(N1) - {}_{N1}\mathrm{Id}_{S1}\boldsymbol{w^i}(S1) \\
{}_{\binom{RBC}{BSNC}}B_{\binom{BC}{BRWG}}(\widetilde{\partial}_t^\tau)\begin{pmatrix} \boldsymbol{\varphi}(BC) \\ \boldsymbol{\psi}(BRWG) \end{pmatrix}_{|\binom{\boldsymbol{\varphi}(BC)}{\boldsymbol{\psi}(BRWG)}^{i+1}=0}(t_{i+1})
\end{pmatrix}$$

and the system to solve in the i-th time step is

$$Lhs \begin{pmatrix} \boldsymbol{E^{i+1}}(\mathcal{X}_h) \\ \boldsymbol{H^{i+1}}(\mathcal{X}_h) \\ \boldsymbol{\varphi^{i+1}}(BC) \\ \boldsymbol{\psi^{i+1}}(RWG) \end{pmatrix} = Rhs^i. \tag{6.9}$$

6.2.2. Numerical experiments

In this section we present some numerical experiments obtained with Algorithm 6.1.

Convergence in time

We consider an example on the three-dimensional unite cube

$$\Omega = [0,1]^3,$$

where we choose the observation time and the material parameters as

$$T = 0.125, \quad \varepsilon = \varepsilon_0 = 1.1, \quad \mu = \mu_0 = 1.2, \quad \sigma = 1.3, \quad \alpha = 1.4, \quad C_e = 1.5,$$

as well as the initial and input data

$$m^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad E^0 = H^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \varphi^0 = \psi^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad J(t) = (1 - t/T) \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix}.$$

Finally, the tolerance and restart parameter for the iterative solver (GMRES), the implicity parameter for the tangent plane scheme and the Convolution Quadrature parameters are set to

$$\text{tolgmres} = 10^{-8}, \quad \text{restart} = 20, \quad \theta = 1.0, \quad \rho_N = \text{tolgmres}^{1/(2N)}, \quad L = N.$$

As discretized initial data and input data we use L^2-projections of the exact data onto the respective approximation spaces. We look at the time discretization error on a fixed coarse mesh and compare the approximations to a reference solution computed on a fine time-grid.

We use time step sizes $\tau_i = T \cdot 2^{-i}$, for $i = 0, \ldots, 8$, and the reference solution is computed with $\tau_{\text{ref}} = \min(\tau_i)/2$. We compute the maximum over the time errors as

$$\text{err}_{G,i} = \max_{j=0,\ldots,N_i} \|G_h^j - G_h^{\text{ref}}(t_j)\| \quad \text{for } G \in \{m, E, H, \varphi, \psi\}$$

and obtain first order convergence results for E and H in the $L^2(\Omega)$-norm, see Figure 6.2.1.

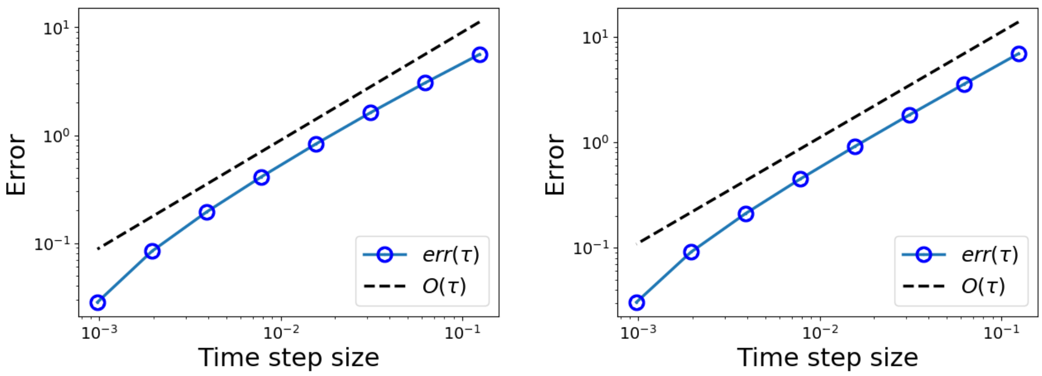

Figure 6.2.1.: Convergence in time for E (left) and H (right).

For the boundary variables φ and ψ, we obtain first order convergence results in $L^2(\Gamma)$, see Figure 6.2.2.

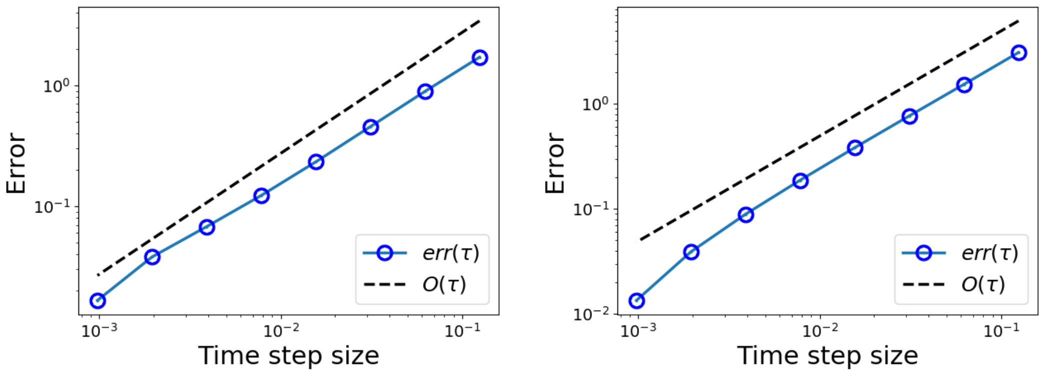

Figure 6.2.2.: Convergence in time for φ (left) and ψ (right).

Especially in this experiment, the convergence rate for the magnetization is slightly higher than 1 for small τ, as then the approximation is already near to the reference solution, see Figure 6.2.3.

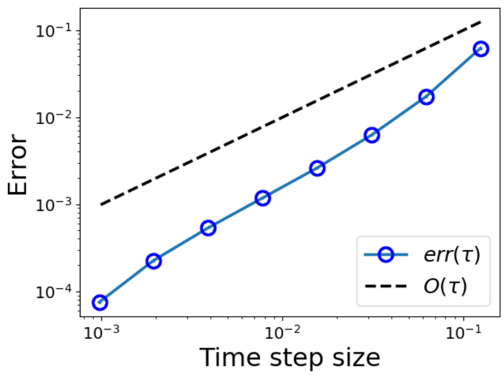

Figure 6.2.3.: Convergence in time for m.

Convergence in space

For the following experiment, we consider the three-dimensional unite cube

$$\Omega = [0,1]^3,$$

where we choose the observation time and the material parameters as

$$T = 0.25, \quad \varepsilon = \varepsilon_0 = 1.1, \quad \mu = \mu_0 = 1.2, \quad \sigma = 1.3, \quad \alpha = 1.4, \quad C_e = 1.5,$$

as well as the initial data fitting to the exact solutions below. The RHS-input data is chosen such that the exact solution is given by

$$m(x,t) := \begin{pmatrix} -(x_1^3 - 3x_1^2/2 + 1/4)\sin(3\pi t/(10T)) \\ \sqrt{1-(x_1^3 - 3x_1^2/2+1/4)^2} \\ -(x_1^3 - 3x_1^2/2 + 1/4)\cos(3\pi t/(10T)) \end{pmatrix}$$

and

$$E(t,x) = t^2 \begin{pmatrix} \sin(\pi x_1)^2 \sin(\pi x_2)^2 \sin(\pi x_3)^2 \\ 0 \\ 0 \end{pmatrix}, \quad H = -\partial_t^{-1} \nabla \times E/\mu, \quad \varphi = \psi = 0.$$

Finally, the tolerance and restart parameter for the iterative solver (GMRES), the stabilization parameter and the Convolution Quadrature parameters are set to

$$\text{tolgmres} = 10^{-8}, \quad \text{restart} = 20, \quad \beta = 1, \quad \rho_N = \text{tolgmres}^{1/(2N)}, \quad L = N.$$

As discretized initial data and input data we use interpolations to the respective spaces. We look at the space discretization error on a fixed time grid such that the time error is small enough ($N = 20$ for $T = 0.25$ is small enough) and compare the approximations with the exact solution.

We use mesh sizes $h = \sqrt{3} \approx 1.7, \sqrt{3}/2, \ldots, \sqrt{3}/12 \approx 0.14$ and compute the maximum space-errors (as norm we use $L^2(\Omega)$ and $H^1(\Omega)$ for m, $L^2(\Omega)$ for E, H and $L^2(\Gamma)$ for φ, ψ) as

$$\text{err}_{G,i} = \max_{j=0,\ldots,20} \|G^j_{h_i} - G(t_j)\|, \quad G \in \{m, E, H, \overline{\varphi}, \psi\}$$

We obtain for the magnetization second order convergence in the $L^2(\Omega)$-norm and first order convergence in the $H^1(\Omega)$-norm, see Figure 6.2.4.

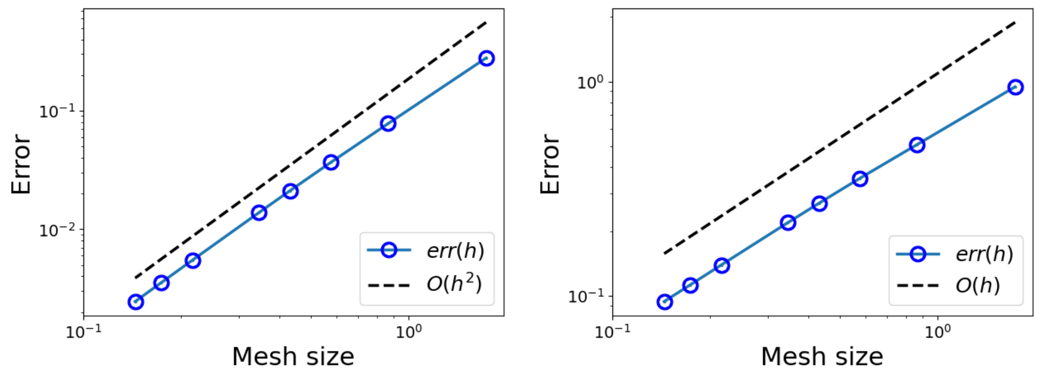

Figure 6.2.4.: Convergence in space for m in the $L^2(\Omega)$-norm (left) and $H^1(\Omega)$-norm (right).

We obtain first order convergence for the electric and magnetic fields in the $L^2(\Omega)$-norm, see Figure 6.2.5

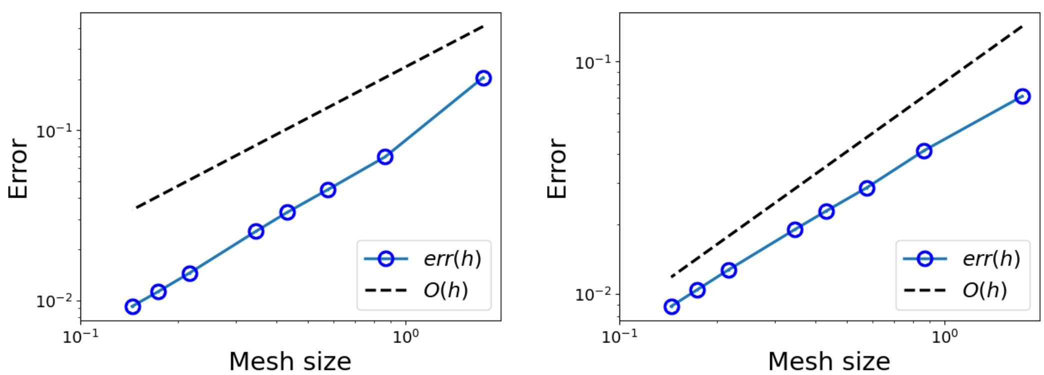

Figure 6.2.5.: Convergence in space in the $L^2(\Omega)$-norm for E (left) and H (right).

We obtain first order convergence for the boundary values in the $L^2(\Gamma)$-norm, see Figure 6.2.6:

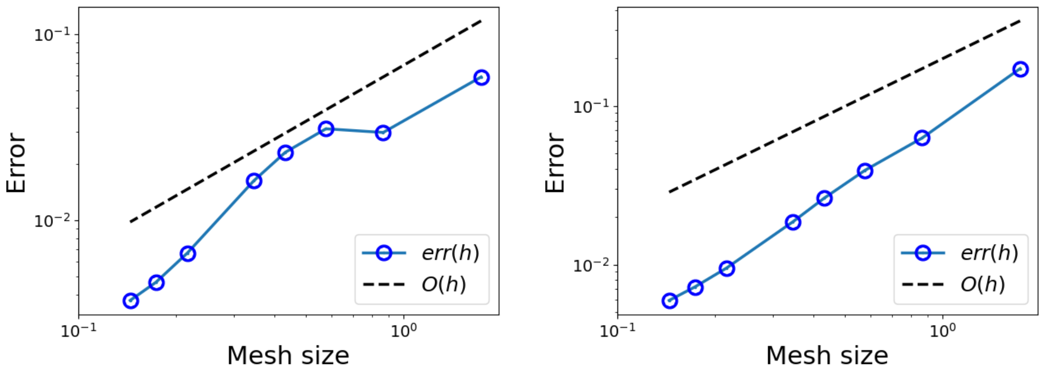

Figure 6.2.6.: Convergence in space in the $L^2(\Gamma)$-norm for φ (left) and ψ (right).

6.3. Weak Convergence for the Maxwell System

6.3.1. Implementation details

In this section we present the implementation details of Algorithm 4.8 used in Chapter 4.

The implemented algorithm differs from Algorithm 4.8 by using Buffa–Christiansen elements for the trace variable φ_h and the corresponding test functions v_h^φ (instead of RT functions). The Convolution Quadrature weights are approximated as in Section 6.1.3.

Algorithm 6.2. Input: Discretized initial data H_h^0, E_h^0 and $\varphi^0 = 0$.
For $j = 0, 1, 2, \dots, N-1$

- Compute $(E_h^{j+1}, H_h^{j+1}) \in \mathcal{X}_h \times \mathcal{Y}_h$ and $\varphi_h^{j+1} \in \mathcal{V}_h^{BC}$ such that we have for all $(\zeta_h^E, \zeta_h^H) \in \mathcal{X}_h \times \mathcal{Y}_h$ and $v_h^\varphi \in \mathcal{V}_h^{BC}$

$$[\varepsilon \partial_t^\tau E_h^{j+1}, \zeta_h^E]_\Omega + \left\langle \begin{pmatrix} 0 \\ -\gamma_T \zeta_h^E \end{pmatrix}, \left(B(\widetilde{\partial}_t^\tau) \begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix} \right) (t_{j+1}) \right\rangle_\Gamma$$
$$= [H_h^{j+1}, \nabla \times \zeta_h^E]_\Omega - \frac{1}{2\mu_0} \langle \varphi_h^{j+1}, \gamma_T \zeta_h^E \rangle_\Gamma - [\sigma E_h^{j+1} + J_h^{j+1}, \zeta_h^E]_\Omega \tag{6.10}$$

$$[\mu \partial_t^\tau H_h^{j+1}, \zeta_h^H]_\Omega = -[\nabla \times E_h^{j+1}, \zeta_h^H]_\Omega \tag{6.11}$$

$$\frac{1}{2\mu_0} \langle v_h^\varphi, \gamma_T E_h^{j+1} \rangle_\Gamma = \left\langle \begin{pmatrix} v_h^\varphi \\ 0 \end{pmatrix}, \left(B(\widetilde{\partial}_t^\tau) \begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix} \right) (t_{j+1}) \right\rangle_\Gamma . \tag{6.12}$$

Output: Sequence of approximations E_h^j, H_h^j, φ_h^j.

We separate knows and unknowns and rewrite (6.10)–(6.12) with

$$B_0^\tau = B(\delta(0)/\tau) = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}$$

as

$$[(\varepsilon/\tau + \sigma) E_h^{j+1}, \zeta_h^E]_\Omega + \langle \gamma_T \zeta_h^E, B_{2,2} \gamma_T E_h^{j+1} \rangle_\Gamma$$
$$- [H_h^{j+1}, \nabla \times \zeta_h^E]_\Omega + \langle \left(B_{2,1} + (2\mu_0)^{-1} \right) \varphi_h^{j+1}, \gamma_T \zeta_h^E \rangle_\Gamma$$
$$= [\varepsilon/\tau E_h^j, \zeta_h^E]_\Omega - [J_h^{j+1}, \zeta_h^E]_\Omega + \left\langle \begin{pmatrix} 0 \\ \gamma_T \zeta_h^E \end{pmatrix}, \left(B(\widetilde{\partial}_t^\tau) \begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix} \right)_{|\begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix}^{j+1} = 0} (t_{j+1}) \right\rangle_\Gamma ,$$

$$[\nabla \times E_h^{j+1}, \zeta_h^H]_\Omega + [\mu/\tau H_h^{j+1}, \zeta_h^H]_\Omega = [\mu/\tau H_h^j, \zeta_h^H]_\Omega$$

and

$$-\langle v_h^\varphi, \Big((2\mu_0)^{-1} + B_{1,2}\Big) \gamma_T E_h^{j+1}\rangle_\Gamma + \langle v_h^\varphi, B_{1,1}\varphi_h^{j+1}\rangle_\Gamma$$
$$= -\left\langle \begin{pmatrix} v_h^\varphi \\ 0 \end{pmatrix}, B(\widetilde{\partial}_t^\tau)\begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix}_{|\left(\begin{smallmatrix} \varphi_h \\ -\gamma_T E_h \end{smallmatrix}\right)^{j+1}=0} (t_{j+1}) \right\rangle_\Gamma .$$

We describe in the following how to discretize each of the terms in FEniCS and Bempp.

The term $[(\varepsilon/\tau + \sigma)E_h^{j+1}, \zeta_h^E]_\Omega$ is a product of Nédélec functions, i.e. for the mass matrix

$$M_1 := {}_{N1}\mathrm{Id}_{N1},$$

it holds

$$[(\varepsilon/\tau + \sigma)E_h^{j+1}, \zeta_h^E]_\Omega = (\varepsilon/\tau + \sigma)\boldsymbol{\zeta^E}(N1) \cdot M_1 \cdot \boldsymbol{E^{j+1}}(N1).$$

For the term $\langle \gamma_T \zeta_h^E, B_{2,2}\gamma_T E_h^{j+1}\rangle_\Gamma$, we express the anti-symmetric pairing $\langle \cdot, \cdot\rangle_\Gamma$ in terms of the $L^2(\Gamma)$-product $[\cdot, \cdot]_\Gamma$, i.e. it holds for $\zeta, \xi \in L^2(\Gamma)$

$$\langle \zeta, \xi\rangle_\Gamma = [\zeta \times n, \xi]_\Gamma.$$

The discretized Calderon sub operator $B_{2,2}^{dcd}$ is given as (compare (6.4))

$$B_{2,2}^{dcd} := {}_{BSNC}\widetilde{F}_{BRWG}(i\sqrt{\mu_0\varepsilon_0}),$$

so the coefficients of the input function should be given with respect to the BRWG space. If we apply the trace matrix $(\gamma_T)_{N1\to RT}$ to $\boldsymbol{E^{j+1}}(N1)$, we obtain coefficients of $\gamma_T E^{j+1}$ given with respect to RT-functions. This is why we additionally apply $(\mathrm{Id})_{BRT\to BRWG}$, to convert the coefficients. In fact, $(\mathrm{Id})_{BRT\to BRWG}$ is a diagonal matrix and the entries are the edge lengths to the power -1. Altogether, it holds

$$(\boldsymbol{\gamma_T E^{j+1}})(BRWG) = (\mathrm{Id})_{BRT\to BRWG} \cdot (\gamma_T)_{N1\to RT} \cdot \boldsymbol{E^{j+1}}(N1).$$

The coefficients of the test function should be given with respect to the $BSNC$ basis, by $\phi_j(BSNC) = n \times \phi_j(BRWG) = -\phi_j(BRWG) \times n$, we have

$$\gamma_T\zeta_h^E \times n = ((\mathrm{Id})_{BRT\to BRWG}(\gamma_T)_{N1\to RT}\boldsymbol{\zeta^E}(N1)) \cdot (\boldsymbol{\phi}(BRWG) \times n)$$
$$= (-(\mathrm{Id})_{BRT\to BRWG}(\gamma_T)_{N1\to RT}\boldsymbol{\zeta^E}(N1)) \cdot \boldsymbol{\phi}(BSNC).$$

All in all, we obtain

$$\langle \gamma_T\zeta_h^E, B_{2,2}\gamma_T E_h^{j+1}\rangle_\Gamma = \boldsymbol{\zeta^E}(N1) \cdot (-(\mathrm{Id})_{BRT\to BRWG} \cdot (\gamma_T)_{N1\to RT})^T \cdot$$
$$B_{2,2}^{dcd} \cdot (\mathrm{Id})_{BRT\to BRWG} \cdot (\gamma_T)_{N1\to RT} \cdot \boldsymbol{E^{j+1}}(N1).$$

To discretize $-[H_h^{j+1}, \nabla \times \zeta_h^E]_\Omega$, we define the differential operator

$$D := {}_{N0}(\nabla\times)_{N1}$$

and it holds

$$-[H_h^{j+1}, \nabla \times \zeta_h^E]_\Omega = \boldsymbol{\zeta^E}(N1) \cdot (-D)^T \cdot \boldsymbol{H^{j+1}}(N0).$$

We set (compare (6.4))

$$B_{2,1}^{dcd} := -\frac{1}{\mu_0\tau^m}\, {}_{BSNC}\widetilde{G}_{BC}(i\sqrt{\mu_0\varepsilon_0}\delta(0)/\tau)$$

and deduce

$$\left\langle \left(B_{2,1}+\frac{1}{2\mu_0}\right)\varphi^{j+1},\gamma_T\zeta_E\right\rangle_\Gamma = \left[n\times\gamma_T\zeta_E,\left(B_{2,1}+\frac{1}{2\mu_0}\right)\varphi^{j+1}\right]_\Gamma$$
$$= \boldsymbol{\zeta_E}(N1)((\mathrm{Id})_{BRT\to BRWG}(\gamma_T)_{N1\to RT})^T\cdot B_{2,1}^{dcd}\boldsymbol{\varphi^{j+1}}(BC)$$
$$+\frac{1}{2\mu_0}\boldsymbol{\zeta_E}(N1)(\gamma_T)_{N1\to RT})^T{}_{BNC}(\mathrm{Id})_{BC}\boldsymbol{\varphi^{j+1}}(BC).$$

By $B_{1,*}$ and $B_{2,*}$ we denote the first and second line of the Calderon operator, respectively. With the abbreviation

$$G_1 := (\gamma_T)_{N1\to BRWG} := (\mathrm{Id})_{BRT\to BRWG}\cdot(\gamma_T)_{N1\to RT},$$

we obtain

$$\left\langle\begin{pmatrix}0\\ \gamma_T\zeta_h^E\end{pmatrix},\left(B(\widetilde{\partial}_t^\tau)\begin{pmatrix}\varphi_h\\ -\gamma_T E_h\end{pmatrix}\right)_{|\binom{\varphi_h}{-\gamma_T E_h}^{j+1}=0}(t_{j+1})\right\rangle_\Gamma =$$
$$\boldsymbol{\zeta^E}(N1)(-G_1)^T{}_{BSNC}(B_{2,*})_{\binom{BC}{BRWG}}(\widetilde{\partial}_t^\tau)\begin{pmatrix}\boldsymbol{\varphi}(BC)\\ -G_1\boldsymbol{E}(N1)\end{pmatrix}_{|\binom{\boldsymbol{\varphi}(BC)}{-G_1\boldsymbol{E}(N1)}^{j+1}=0}(t_{j+1}).$$

Most of the remaining terms in the first and second equation can be treated analogously, we have

$$[\varepsilon/\tau E_h^j,\zeta_h^E]_\Omega = \varepsilon/\tau\boldsymbol{\zeta^E}(N1)\cdot M_1\cdot\boldsymbol{E^j}(N1),$$
$$-[J_h^{j+1},\zeta_h^E]_\Omega = -\boldsymbol{\zeta}(N1)\cdot M_1\cdot\boldsymbol{J^{j+1}}(N1),$$
$$[\nabla\times E_h^{j+1},\zeta_h^H]_\Omega = \boldsymbol{\zeta^H}(N0)\cdot D\cdot\boldsymbol{E^{j+1}}(N1),$$
$$[\mu/\tau H_h^{j+1},\zeta_h^H]_\Omega = \mu/\tau\boldsymbol{\zeta^H}(N0)\cdot{}_{N0}\mathrm{Id}_{N0}\cdot\boldsymbol{H^{j+1}}(N0),$$
$$[\mu/\tau H_h^{j},\zeta_h^H]_\Omega = \mu/\tau\boldsymbol{\zeta^H}(N0)\cdot{}_{N0}\mathrm{Id}_{N0}\cdot\boldsymbol{H^{j}}(N0).$$

For

$$B_{1,2}^{dcd} := \frac{1}{\mu_0}{}_{RBC}\widetilde{D}_{BRWG}(i\sqrt{\mu_0\varepsilon_0}\delta(0)/\tau),$$

it holds (by $\phi_j(RBC) = n\times\phi_j(BC) = -\phi_j(BC)\times n$) that

$$-\langle v_h^\varphi,\left((2\mu_0)^{-1}+B_{1,2}\right)\gamma_T E_h^{j+1}\rangle_\Gamma = [n\times v_h^\varphi,\left((2\mu_0)^{-1}+B_{1,2}\right)\gamma_T E_h^{j+1}]_\Gamma$$
$$=\frac{1}{2\mu_0}\boldsymbol{v^\varphi}(BC)\cdot{}_{RBC}\mathrm{Id}_{BRT}\cdot(\gamma_T)_{N1\to RT}\cdot\boldsymbol{E^{j+1}}(N1)$$
$$+\boldsymbol{v^\varphi}(BC)\cdot B_{1,2}^{dcd}\cdot(\gamma_T)_{N1\to BRWG}\cdot\boldsymbol{E^{j+1}}(N1).$$

Furthermore we define

$$B_{1,1}^{dcd} := \frac{-1}{\mu_0^2}{}_{RBC}\widetilde{E}_{BC}(i\sqrt{\mu_0\varepsilon_0}\delta(0)/\tau),$$

and write

$$\langle v_h^\varphi,B_{1,1}\varphi_h^{j+1}\rangle_\Gamma = [v_h^\varphi\times n,B_{1,1}\varphi_h^{j+1}]_\Gamma$$
$$= -\boldsymbol{v^\varphi}(BC)\cdot B_{1,1}^{dcd}\cdot\boldsymbol{\varphi^{j+1}}(BC).$$

The Convolution Quadrature right hand side term is computed in a similar way, we denote the first row of the Calderon operator (with $e_1 := (1,0)$) by

$$B_{1,*}(s) := e_1\cdot B(s)$$

and obtain

$$-\left\langle \begin{pmatrix} v_h^{\varphi} \\ 0 \end{pmatrix}, B(\widetilde{\partial}_t^{\tau}) \begin{pmatrix} \varphi_h \\ -\gamma_T E_h \end{pmatrix}_{|\left(\begin{smallmatrix} \varphi_h \\ -\gamma_T E_h \end{smallmatrix}\right)^{j+1}=0} (t_{j+1}) \right\rangle_{\Gamma} =$$

$$\boldsymbol{v}^{\boldsymbol{\varphi}}(BC) \; {}_{RBC}(B_{1,*})_{\left(\begin{smallmatrix} BC \\ BRWG \end{smallmatrix}\right)} (\widetilde{\partial}_t^{\tau}) \begin{pmatrix} \boldsymbol{\varphi}(BC) \\ -G_1 \boldsymbol{E}(N1) \end{pmatrix}_{|\left(\begin{smallmatrix} \boldsymbol{\varphi}(BC) \\ -G_1 \boldsymbol{E}(N1) \end{smallmatrix}\right)^{j+1}=0} (t_{j+1}).$$

We summarize and build up the full system. We recall the mass matrices

$$\begin{aligned} M_1 &= {}_{N1}\mathrm{Id}_{N1}, \\ M_0 &= {}_{N0}\mathrm{Id}_{N0}, \\ M_2 &= {}_{RBC}\mathrm{Id}_{BRT}(\gamma_T)_{N1\to RT} = -({}_{BNC}\mathrm{Id}_{BC})^T(\gamma_T)_{N1\to RT}, \end{aligned}$$

the discrete differential operator

$$D = {}_{NC}\nabla\times_{NC},$$

the trace operator

$$G_1 = (\mathrm{Id})_{BRT\to BRWG} \cdot (\gamma_T)_{N1\to RT}$$

and Calderon sub operators

$$\begin{aligned} B_{1,1}^{dcd} &= \frac{-1}{\mu_0^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\;{}_{RBC}\widehat{E}_{BC}(i\sqrt{\mu_0\varepsilon_0}\delta(0)/\tau), & B_{1,2}^{dcd} &= \frac{1}{\mu_0}\;{}_{RBC}\widehat{D}_{RWG}(i\sqrt{\mu_0\varepsilon_0}\delta(0)/\tau), \\ B_{2,1}^{dcd} &= \frac{-1}{\mu_0}\;{}_{SNC}\widehat{G}_{BC}(i\sqrt{\mu_0\varepsilon_0}\delta(0)/\tau), & B_{2,2}^{dcd} &= \sqrt{\frac{\varepsilon_0}{\mu_0}}\;{}_{SNC}\widehat{F}_{RWG}(i\sqrt{\mu_0\varepsilon_0}\delta(0)/\tau). \end{aligned}$$

The overall discretization matrix then is

$$Lhs := \begin{pmatrix} (\frac{\varepsilon_0}{\tau}+\sigma)M_1 - G_1^T B_{2,2}^{dcd} G_1 & -D^T & G_1^T B_{2,1}^{dcd} - \frac{1}{2\mu_0}(M_2)^T \\ D & \frac{\mu_0}{\tau}M_0 & 0 \\ \frac{1}{2\mu_0}M_2 + B_{1,2}^{dcd}G_1 & 0 & -B_{1,1}^{dcd} \end{pmatrix}$$

with right hand side

$$Rhs^j := \begin{pmatrix} \frac{\varepsilon_0}{\tau} M_1 \boldsymbol{E}^{\boldsymbol{j}}(N1) - M_1 \boldsymbol{J}^{\boldsymbol{j+1}}(N1) + R_1^j \\ \frac{\mu_0}{\tau} M_0 \boldsymbol{H}^{\boldsymbol{j}}(N0) \\ {}_{RBC}(B_{1,*})_{\left(\begin{smallmatrix} BC \\ BRWG \end{smallmatrix}\right)} (\widetilde{\partial}_t^{\tau}) \left(\begin{smallmatrix} \boldsymbol{\varphi}(BC) \\ -G_1\boldsymbol{E}(N1) \end{smallmatrix}\right)_{|\left(\begin{smallmatrix} \boldsymbol{\varphi}(BC) \\ -G_1\boldsymbol{E}(N1) \end{smallmatrix}\right)^{j+1}=0} (t_{j+1}) \end{pmatrix},$$

where

$$R_1^j = -(G_1)^T \; {}_{BSNC}(B_{2,*})_{\left(\begin{smallmatrix} BC \\ BRWG \end{smallmatrix}\right)} (\widetilde{\partial}_t^{\tau}) \begin{pmatrix} \boldsymbol{\varphi}(BC) \\ -G_1 \boldsymbol{E}(N1) \end{pmatrix}_{|\left(\begin{smallmatrix} \boldsymbol{\varphi}(BC) \\ -G_1\boldsymbol{E}(N1) \end{smallmatrix}\right)^{j+1}=0} (t_{j+1})$$

and the system to solve in the j-th time step is

$$Lhs \begin{pmatrix} \boldsymbol{E}^{\boldsymbol{j+1}}(\mathcal{X}_h) \\ \boldsymbol{H}^{\boldsymbol{j+1}}(\mathcal{Y}_h) \\ \boldsymbol{\varphi}^{\boldsymbol{j+1}}(BC) \end{pmatrix} = Rhs^j. \tag{6.13}$$

6.3.2. Numerical experiments

Convergence in time

We consider a simple example on the unite cube

$$\Omega = [0,1]^3,$$

where we choose observation time and the material parameters as

$$T = 0.25, \quad \mu_0 = 2.0, \quad \varepsilon_0 = 3.0, \quad \sigma = 0.1,$$

as well as the initial and input data

$$H^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad E^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \varphi^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad J(t) = t \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}.$$

The GMRES tolerance and Convolution Quadrature parameters are set to

$$\text{tolgmres} = 10^{-8}, \quad \rho_N = \text{tolgmres}^{N/2}, \quad L = N.$$

As discretized initial data and input data we use L^2-projections to the respective spaces. We look at the time discretization error on a fixed coarse mesh. The reference solution is computed on a fine time grid. For the time step sizes $\tau_i = T \cdot 2^{-i}$, for $i = 0, \dots, 7$, the reference solution is computed with $\tau_{\text{ref}} = \min(\tau_i)/2$. We consider the maximum $L^2(\Omega)$-error as

$$\text{err}_i = \max_{j=0,\dots,N_i} \|E_h^j - E_h^{\text{ref}}(t_j)\|_\Omega$$

and obtain first order convergence results for E and H in $L^2(\Omega)$, see Figure 6.3.1.

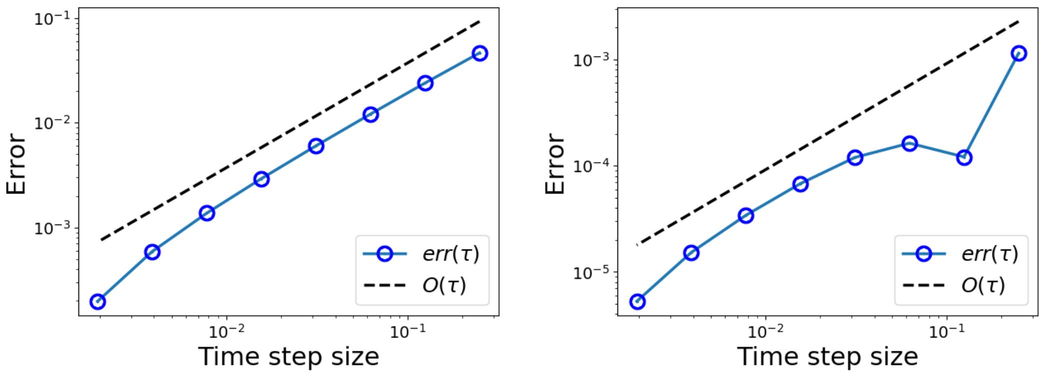

Figure 6.3.1.: Convergence in time for E (left) and H (right) in $L^2(\Omega)$.

We compute the maximum $L^2(\Gamma)$-error as

$$\text{err}_i = \max_{j=0,\dots,N_i} \|\varphi_h^j - \varphi_h^{\text{ref}}(t_j)\|_\Gamma$$

and obtain first order convergence results for φ and $\gamma_T E$ in $L^2(\Gamma)$, see Figure 6.3.2.

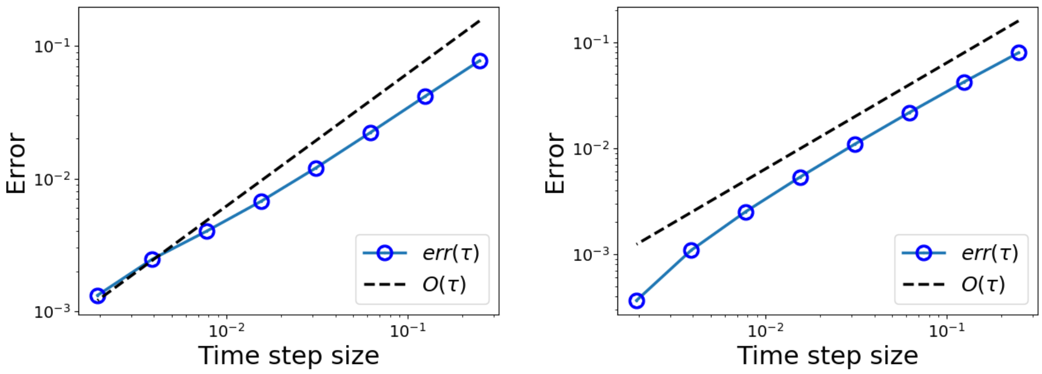

Figure 6.3.2.: Convergence in time for φ (left) and $\gamma_T E$ (right) in $L^2(\Gamma)$.

Convergence in space

For the following example, we consider the domain

$$\Omega = [0,1]^3,$$

where we choose the observation time and the material parameters as

$$T = 0.5, \quad \varepsilon = \varepsilon_0 = 1.0, \quad \mu = \mu_0 = 1.0, \quad \sigma = 1.3,$$

as well as the initial data fitting to the exact solutions below. The RHS-input data is chosen such that the exact solution is given by

$$E(t,x) = t^2 \begin{pmatrix} \sin(\pi x_1)^2 \sin(\pi x_2)^2 \sin(\pi x_3)^2 \\ 0 \\ 0 \end{pmatrix}, \quad H = -\partial_t^{-1} \nabla \times E/\mu, \quad \varphi = \psi = 0.$$

Finally, the tolerance and restart parameter for the iterative solver (GMRES), the stabilization parameter and the Convolution Quadrature parameters are set to

$$\text{tolgmres} = 10^{-8}, \quad \text{restart} = 20, \quad \beta = 1, \quad \rho_N = \text{tolgmres}^{1/(2N)}, \quad L = N.$$

As discretized initial data and input data we use interpolations to the respective spaces. We look at the space discretization error on a fixed time grid such that the time error is small enough ($N = 20$ for $T = 0.5$ is small enough) and compare the approximations with the exact solution.

For the mesh sizes $h = \sqrt{3} \approx 1.7, \sqrt{3}/2, \dots, \sqrt{3}/12 \approx 0.14$, we compute the maximum space-errors (as norm we use the norms of $H(\text{curl}, \Omega)$ and $L^2(\Omega)$ for E, $L^2(\Omega)$ for H and $L^2(\Gamma)$ for φ) as

$$\text{err}_{G,i} = \max_{j=0,\dots,20} \|G_{h_i}^j - G(t_j)\|, \quad G \in \{E, H, \varphi\}.$$

We obtain first order convergence results for E, see Figure 6.3.3.

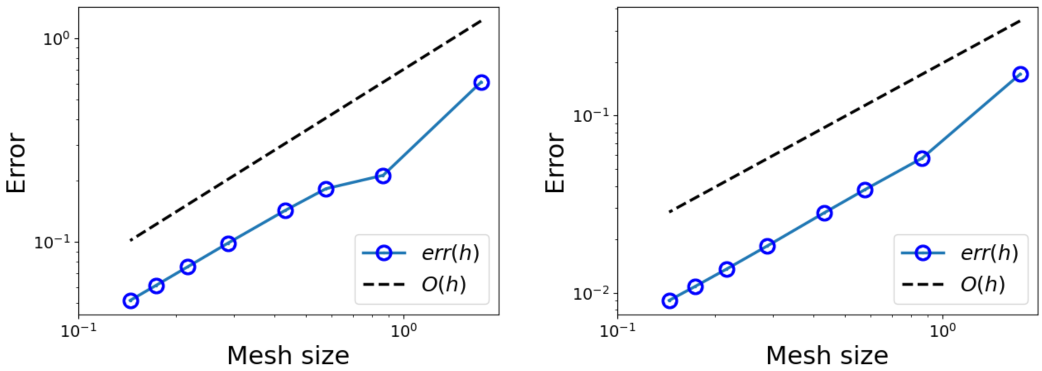

Figure 6.3.3.: Convergence in space for E in $H(\text{curl}, \Omega)$ (left) and $L^2(\Omega)$ (right).

We observe first order convergence results for H and φ, see Figure 6.3.4.

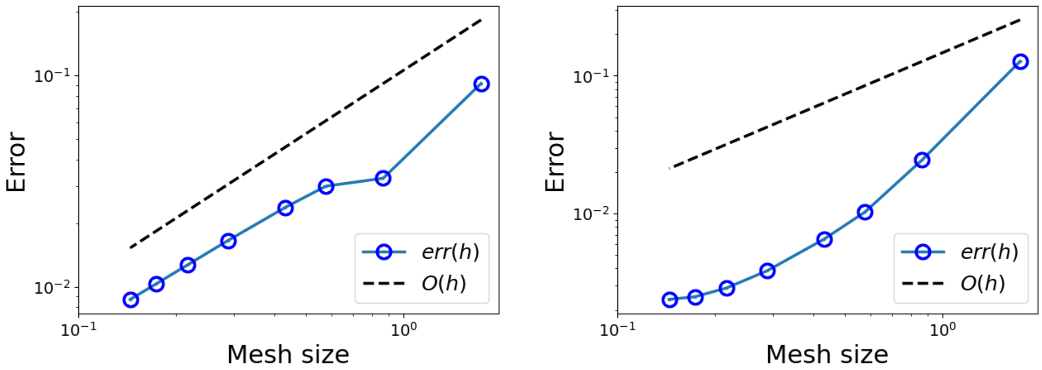

Figure 6.3.4.: Convergence in space for H in $L^2(\Omega)$ (left) and φ in $L^2(\Gamma)$ (right).

6.4. Convergence with Rates for the MLLG System

6.4.1. Implementation details

In this section we present the implementation details of Algorithm 5.11 used in Chapter 5.

In comparison to Algorithm 5.11, the tangent plane scheme is formulated as a saddle point problem (cf. Section 6.1.2) and the Convolution Quadrature weights are approximated by a quadrature rule (cf. Section 6.1.3).

Concerning the interior Maxwell space discretization, we use first order Nédélec elements $\mathcal{X}_h$ instead of the discontinuous Galerkin elements. For the boundary we use the trace space $\gamma_T(\mathcal{X}_h)$, i.e. the RT space. While there are higher order Nédélec- and discontinuous Galerkin-spaces implemented in FEniCS, in Bempp the coupling is only implemented for the first order Nédélec space and the first order continuous Galerkin space. Furthermore the Calderon operator and tangential trace operators are only available for the Maxwell boundary spaces which are of first order. For numerical experiments considering higher order convergence in space, an implementation of those spaces im Bempp would be necessary, which is beyond the scope of this thesis.

The proofs of Chapter 5 stay true for the implemented spaces in a similar way, compare Section 5.2.2 and [99, Remark 5.1].

Algorithm 6.3. Input: Discretized initial data m_h^0, H_h^0, E_h^0, $\varphi_h^0 = 0$, $\psi_h^0 = 0$ and starting values $\dot{m}_h^1$, m_h^1, H_h^1, E_h^1, φ_h^1, ψ_h^1.

For $n = 2, \dots, N$: For given values at t_{n-1}, determine the approximations for t_n by:

- Determine $(\dot{m}_h^n, \lambda_h) \in \mathcal{S}^1(\mathcal{T}_h, \mathbb{R}^3) \times \mathcal{S}^1(\mathcal{T}_h, \mathbb{R})$ such that we have for all $(\rho_h, \xi_h) \in S^1(\mathcal{T}_h, \mathbb{R}^3) \times \mathcal{S}^1(\mathcal{T}_h, \mathbb{R})$

$$\begin{aligned}\alpha[\dot{m}_h^n, \rho_h]_\Omega + [\widehat{m}_h^n \times \dot{m}_h^n, \rho_h]_\Omega = [\nabla m_h^n, \nabla \rho_h]_\Omega + [H_h^n, \rho_h]_\Omega \\ + [\rho_h \cdot \widehat{m}_h^n, \lambda_h]_\Omega + [\dot{m}_h^n \cdot \widehat{m}_h^n, \xi_h]_\Omega,\end{aligned} \tag{6.14}$$

where $\widehat{m}_h^n$ and $\dot{m}_h^n$ are related to m_h^{n-j} for $j = 0, 1, 2$ via

$$\dot{m}_h^n = \frac{1}{\tau} \sum_{j=0}^{2} \delta_j m_h^{n-j}, \qquad \widehat{m}_h^n = \sum_{j=0}^{1} \gamma_j m_h^{n-j-1} \Big/ \Big| \sum_{j=0}^{1} \gamma_j m_h^{n-j-1} \Big|. \tag{6.15}$$

- Determine $E_h^n, H_h^n \in \mathcal{X}_h$ and $\varphi_h^n, \psi_h^n \in \gamma_T(\mathcal{X}_h)$ such that we have for all $\zeta_h^{H,1/2}, \zeta_h^E, \zeta_h^{H,1} \in \mathcal{X}_h$ and $v_h^\varphi, v_h^\psi \in \gamma_T(\mathcal{X}_h)$

$$\begin{aligned}[\mu H_h^{n-1/2}, \zeta_h^{H,1/2}]_\Omega = [\mu H_h^{n-1}, \zeta_h^{H,1/2}]_\Omega - \frac{\tau}{4}[\nabla \times E_h^{n-1}, \zeta_h^{H,1/2}]_\Omega - \frac{\tau}{4}[E_h^{n-1}, \nabla \times \zeta_h^{H,1/2}]_\Omega \\ - \frac{\tau}{4}\langle \psi_h^{n-1}, \gamma_T \zeta_h^{H,1/2} \rangle_\Gamma - \frac{\tau\mu}{2}[\dot{m}_h^{n-1}, \zeta_h^{H,1/2}]_\Omega,\end{aligned} \tag{6.16}$$

$$\begin{aligned}[\varepsilon E_h^n, \zeta_h^E]_\Omega = [\varepsilon E_h^{n-1}, \zeta_h^E]_\Omega + \frac{\tau}{2}[\nabla \times H_h^{n-1/2}, \zeta_h^E]_\Omega + \frac{\tau}{2}[H_h^{n-1/2}, \nabla \times \zeta_h^E]_\Omega \\ - \frac{\tau}{2\mu_0}\langle \varphi_h^{n-1/2}, \gamma_T \zeta_h^E \rangle_\Gamma - \tau[\sigma \overline{E}_h^{n-1/2} + J^{n-1/2}, \zeta_h^E]_\Omega,\end{aligned} \tag{6.17}$$

$$\begin{aligned}[\mu H_h^n, \zeta_h^{H,1}]_\Omega = [\mu H_h^{n-1/2}, \zeta_h^{H,1}]_\Omega - \frac{\tau}{4}[\nabla \times E_h^n, \zeta_h^{H,1}]_\Omega - \frac{\tau}{4}[E_h^n, \nabla \times \zeta_h^{H,1}]_\Omega \\ - \frac{\tau}{4}\langle \psi_h^n, \gamma_T \zeta_h^{H,1} \rangle_\Gamma - \frac{\tau\mu}{2}[\dot{m}_h^n, \zeta_h^{H,1}]_\Omega,\end{aligned} \tag{6.18}$$

This is coupled with Convolution Quadrature on the boundary

$$\left\langle \begin{pmatrix} v_h^\varphi \\ v_h^\psi \end{pmatrix}, \left(B(\widetilde{\partial}_t^\tau) \begin{pmatrix} \varphi_h \\ \overline{\psi}_h \end{pmatrix} \right)^{n-1/2} \right\rangle_\Gamma = \frac{1}{2} \left\langle \begin{pmatrix} v_h^\varphi \\ v_h^\psi \end{pmatrix}, \begin{pmatrix} \mu_0^{-1} \gamma_T \overline{E}_h^{n-1/2} \\ \gamma_T H_h^{n-1/2} \end{pmatrix} \right\rangle_\Gamma \tag{6.19}$$

$$- \beta \frac{\tau^2}{\mu} \langle v_h^\psi, \gamma_T \mathrm{LIFT}(\dot{\psi}_h^{n-1/2}) \rangle_\Gamma, \tag{6.20}$$

where $\overline{f}^{n-1/2}$, $\dot{f}^{n-1/2}$ and LIFT are defined in (5.10).

Output: Sequence of approximations $m_h^n, E_h^n, H_h^n, \varphi_h^n, \psi_h^n$.

Remark 6.4. *Note that Algorithm 6.3 is equally expensive as the approximation for the uncoupled systems for the BDF-LLG-Approximation and the interior-exterior Maxwell-coupling:*

- *Given* H_h^{n-1}, E_h^{n-1}, φ_h^{n-1} *and* $\dot{m}_h^{n-1}$, *compute* $H_h^{n-1/2}$ *by* (6.16).
- *Insert* (6.17) *in* (6.19) *and solve for* $\varphi_h^{n-1/2}$ *and* ψ_h^n. *Then compute* E_h^n *with* (6.17).
- *Insert* (6.18) *in* (6.14) *and solve a linear system for* $\dot{m}_h^n$. *Then compute* H_h^n *with* (6.18) *and* m_h^n *with* (6.15).

We introduce the mass matrices

$$M_1 = {}_{N1}\mathrm{Id}_{N1}, \quad M_2 = {}_{SNC}\mathrm{Id}_{SNC}, \quad M_3 = {}_{N1}\mathrm{Id}_{S1}$$

and the differential operator

$$D = -\frac{1}{2}({}_{N1}\nabla\times_{N1} + ({}_{N1}\nabla\times_{N1})^T).$$

With the trace operator $(\gamma_T)_{N1\to RWG}$, we define

$$C_1 = -\frac{1}{2}(\gamma_T)^T_{N1\to RWG} M_2^T, \quad C_2 = \frac{1}{\mu_0} C_1$$

and initialize the time harmonic Calderon operator (compare Section 6.1.6)

$$B(s) = \begin{pmatrix} \widetilde{D} & \widetilde{E} \\ \widetilde{F} & \widetilde{G} \end{pmatrix}(s).$$

We build up the matrix

$$Lhs = \begin{pmatrix} (\epsilon + \sigma\tau/2)M_1 & 0 & \tau/2C_2 & 0 \\ 0 & \mu_0 M_1 & 0 & 2\beta\tau C_1 \\ -C_2^T & 0 & 1/\mu_0^2 \,{}_{RWG}\widetilde{E}_{RWG}(\delta(0)/\tau) & -1/\mu_0 \,{}_{RWG}\widetilde{D}_{RWG}(\delta(0)/\tau) \\ 0 & -C_1^T & 1/\mu_0 \,{}_{RWG}\widetilde{G}_{RWG}(\delta(0)/\tau) & -\,{}_{RWG}\widetilde{F}_{RWG}(\delta(0)/\tau) \end{pmatrix}.$$

Then, (6.16) takes the form

$$\begin{aligned} \mu M_1 \boldsymbol{H^{n-1/2}}(N1) = \mu M_1 \boldsymbol{H^{n-1}}(N1) + \frac{\tau}{2} D\boldsymbol{E^{n-1}}(N1) \\ - \frac{\tau}{2} C_1 \boldsymbol{\psi^{n-1}}(RWG) - \frac{\tau\mu}{2} M_3 \boldsymbol{\dot{m}^{n-1}}(S1). \end{aligned}$$

With

$$(\varepsilon+\tau\sigma/2)M_1\widetilde{\boldsymbol{E}}^{\boldsymbol{n}}(N1) = (\varepsilon-\tau\sigma/2)M_1\boldsymbol{E^{n-1}}(N1) - \tau D\boldsymbol{H^{n-1/2}}(N1) + \tau M_1 \boldsymbol{J^{n-1/2}}(NC),$$

inserting (6.17) in (6.19) gives

$$Lhs \begin{pmatrix} X \\ Y \\ \boldsymbol{\varphi^{n-1/2}}(RWG) \\ \overline{\boldsymbol{\psi}}^{\boldsymbol{n-1/2}}(RWG) \end{pmatrix} = \begin{pmatrix} (\varepsilon + \sigma\tau/2)M_1(\widetilde{\boldsymbol{E}}^{\boldsymbol{n}}(N1) + \boldsymbol{E^{n-1}}(N1))/2 \\ 2\beta\tau C_1 \boldsymbol{\psi^{n-1}}(RWG) \\ -B_{1,*}(\partial_t^\tau)\binom{\boldsymbol{\varphi}(RWG)}{\boldsymbol{\psi}(RWG)}_{|\binom{\varphi}{\psi}^{n-1/2}=0}(t_{n-1/2}) \\ -B_{2,*}(\partial_t^\tau)\binom{\boldsymbol{\varphi}(RWG)}{\boldsymbol{\psi}(RWG)}_{|\binom{\varphi}{\psi}^{n-1/2}=0}(t_{n-1/2}) + C_1^T \boldsymbol{H^{n-1/2}}(N1) \end{pmatrix}.$$

It follows $\psi_h^n = 2\overline{\psi}_h^{n-1/2} - \psi_h^{n-1}$ and (equivalent to the following is $\boldsymbol{E^n}(N1) = 2X - \boldsymbol{E^{n-1}}(N1)$)

$$(\varepsilon + \tau\sigma/2)M_1\boldsymbol{E^n}(N1) = (\varepsilon + \tau\sigma/2)M_1\widetilde{\boldsymbol{E}}^{\boldsymbol{n}}(N1) - \tau C_2 \boldsymbol{\varphi^{n-1/2}}(RWG).$$

Inserting (6.18) in (6.14) gives a linear system for $\dot{m}_h^n$ which is solved by separating known and unknowns and writing the resulting (bi)linear forms in FEniCS syntax. With the solution $\dot{m}_h^n$ at hand, we compute m_h^n with (6.15). Finally, H_h^n is computed with (6.18) via

$$\mu M_1 \boldsymbol{H^n}(N1) = \mu M_1 \boldsymbol{H^{n-1/2}}(N1) + \frac{\tau}{2} D\boldsymbol{E^n}(N1) - \frac{\tau}{2} C_1 \boldsymbol{\psi^n}(RWG) - \frac{\tau\mu}{2} M_3 \boldsymbol{\dot{m}^n}(S1).$$

6.4.2. Numerical experiments

Convergence in time

We consider a simple example on the three-dimensional unite cube

$$\Omega = [0,1]^3,$$

where we choose the observation time and the material parameters as

$$T = 0.25, \quad \varepsilon = \varepsilon_0 = 1.1, \quad \mu = \mu_0 = 1.2, \quad \sigma = 1.3, \quad \alpha = 1.4, \quad C_e = 1.5,$$

as well as the initial data

$$m^0 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad E^0 = \begin{pmatrix}0\\0\\0\end{pmatrix}, \quad H^0 = \begin{pmatrix}0\\0\\0\end{pmatrix}, \quad \varphi^0 = \begin{pmatrix}0\\0\\0\end{pmatrix}, \quad \psi^0 = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

The RHS-input data is chosen to guarantee that the solutions are smooth enough in time, i.e. that enough time derivatives vanish at $t = 0$ on the boundary:

$$J(t,x) = 100t^4 \sin(\pi x_1)^2 \sin(\pi x_2)^2 \sin(\pi x_3)^2 \begin{pmatrix}1\\0\\0\end{pmatrix}.$$

Finally, the tolerance and restart parameter for the iterative solver (GMRES), the stabilization parameter and the Convolution Quadrature parameters are set to

$$\text{tolgmres} = 10^{-8}, \quad \text{restart} = 20, \quad \beta = 1, \quad \rho_N = \text{tolgmres}^{1/(2N)}, \quad L = N.$$

As discretized initial data and input data we use interpolations to the respective spaces. We look at the time discretization error on a fixed coarse mesh and compare the approximations with a reference solution computed on a fine time-grid.

We use time step sizes $\tau_i = T \cdot 2^{-i}$, for $i = 0, \ldots, 6$, and the reference solution is computed with $\tau_{\text{ref}} = \min(\tau_i)/2$. We compute the maximum space-errors (as norm we use $H^1(\Omega)$ for m, $L^2(\Omega)$ for E, H and $L^2(\Gamma)$ for φ, ψ) as

$$\text{err}_{G,i} = \max_{j=0,\ldots,N_i} \|G_h^j - G_h^{\text{ref}}(t_j)\|, \quad G \in \{m, E, H, \overline{\varphi}, \psi\}$$

and obtain second order convergence results, see Figure 6.4.1–6.4.3.

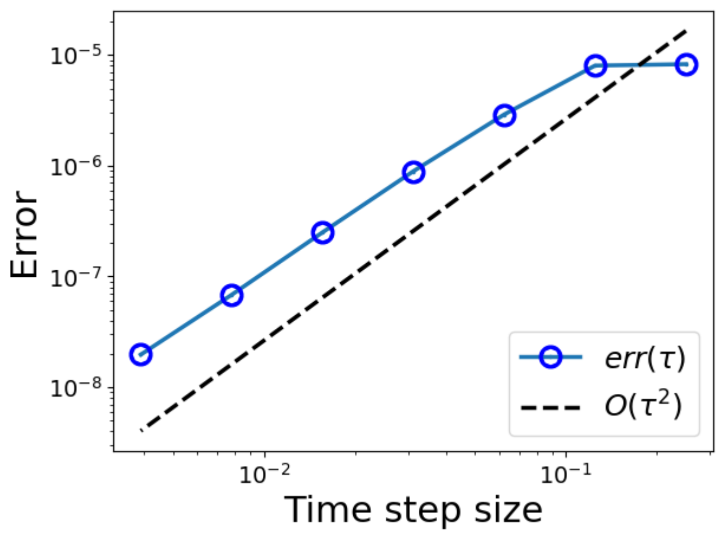

Figure 6.4.1.: Convergence in time for m.

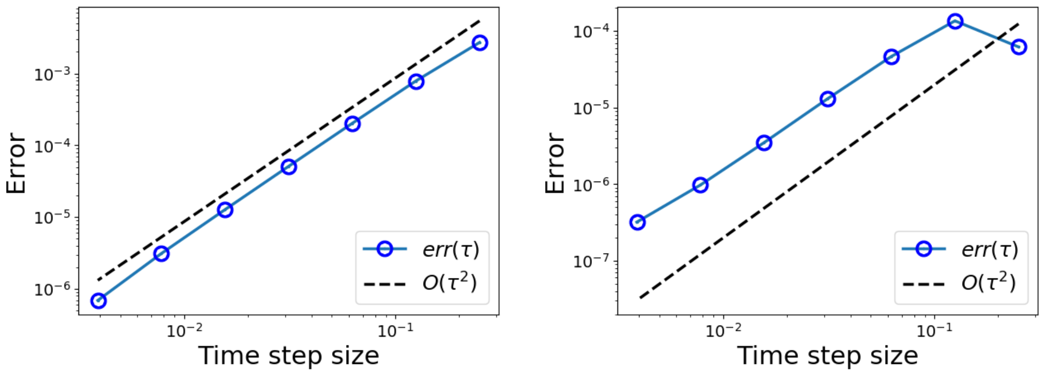

Figure 6.4.2.: Convergence in time for E (left) and H (right).

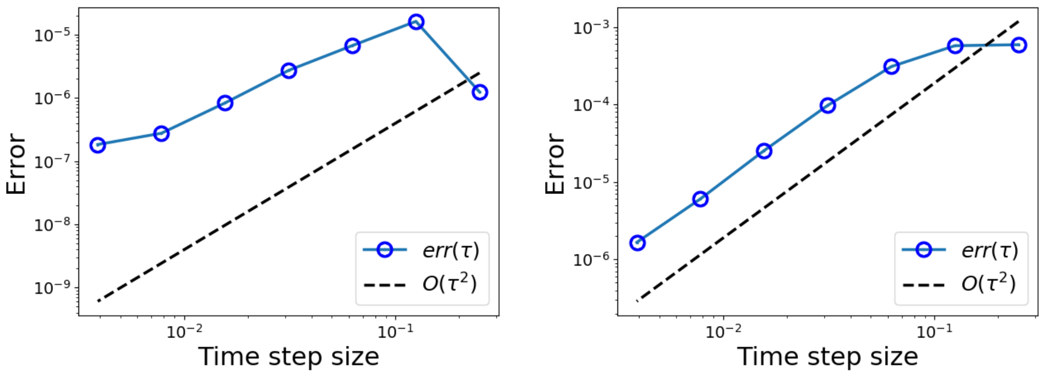

Figure 6.4.3.: Convergence in time for φ (left) and ψ (right).

Convergence in space versus exact solution

We consider an example on the three-dimensional unite cube

$$\Omega = [0,1]^3,$$

where we choose the observation time and the material parameters as

$$T = 0.25, \quad \varepsilon = \varepsilon_0 = 1.1, \quad \mu = \mu_0 = 1.2, \quad \sigma = 1.3, \quad \alpha = 1.4, \quad C_e = 1.5,$$

as well as the initial data fitting to the exact solutions below. The RHS-input data is chosen such that the exact solution is given by

$$m(x,t) := \begin{pmatrix} -(x_1^3 - 3x_1^2/2 + 1/4)\sin(3\pi t/(10T)) \\ \sqrt{1 - (x_1^3 - 3x_1^2/2 + 1/4)^2} \\ -(x_1^3 - 3x_1^2/2 + 1/4)\cos(3\pi t/(10T)) \end{pmatrix}$$

and

$$E(t,x) = t^2 \begin{pmatrix} \sin(\pi x_1)^2 \sin(\pi x_2)^2 \sin(\pi x_3)^2 \\ 0 \\ 0 \end{pmatrix}, \quad H = -\partial_t^{-1} \nabla \times E/\mu, \quad \varphi = \psi = 0.$$

Finally, the tolerance and restart parameter for the iterative solver (GMRES), the stabilization parameter and the Convolution Quadrature parameters are set to

$$\text{tolgmres} = 10^{-8}, \quad \text{restart} = 20, \quad \beta = 1, \quad \rho_N = \text{tolgmres}^{1/(2N)}, \quad L = N.$$

As discretized initial data and input data we use interpolations to the respective spaces. We look at the space discretization error on a fixed time grid such that the time error is small enough ($N = 20$ for $T = 0.25$ is small enough) and compare the approximations with the exact solution.

We use the mesh sizes $h = \sqrt{3} \approx 1.7, \sqrt{3}/2, \dots, \sqrt{3}/12 \approx 0.14$ and compute the maximum space-errors (we use the norm of the space $L^2(\Omega)$ or $H^1(\Omega)$ for m, $L^2(\Omega)$ for E, H and $L^2(\Gamma)$ for φ, ψ) as

$$\text{err}_{G,i} = \max_{j=0,\dots,20} \|G^j_{h_i} - G(t_j)\|, \quad G \in \{m, E, H, \overline{\varphi}, \psi\}$$

We obtain for m second order convergence in the L^2-norm and first order convergence in the H^1-norm, see Figure 6.4.4.

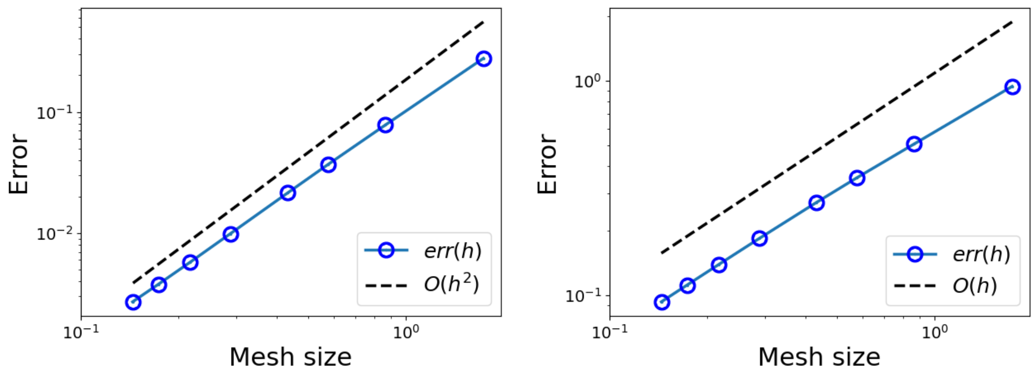

Figure 6.4.4.: Convergence in space versus exact solution for the magnetization in the $L^2(\Omega)$-norm (left) and $H^1(\Omega)$-norm (right).

We obtain first order convergence for E and H in the $L^2(\Omega)$-norm, see Figure 6.4.5.

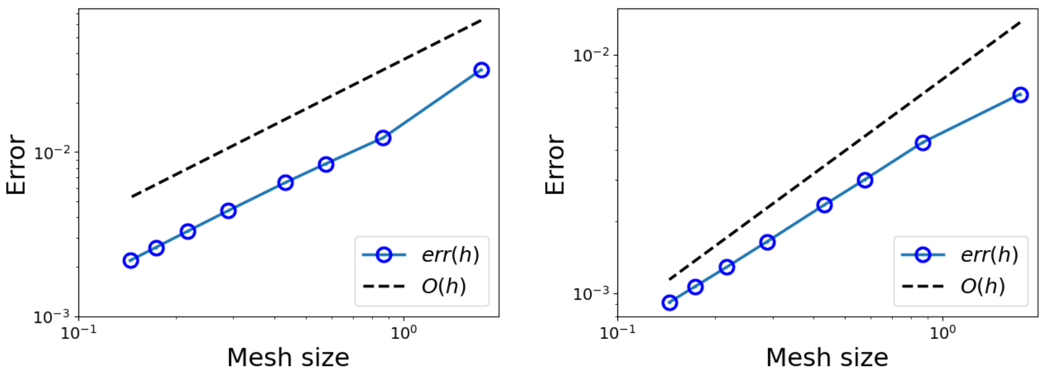

Figure 6.4.5.: Convergence in space versus exact solution in the $L^2(\Omega)$-norm for E and H.

We obtain first order convergence for φ and ψ in the $L^2(\Gamma)$-norm, see Figure 6.4.6.

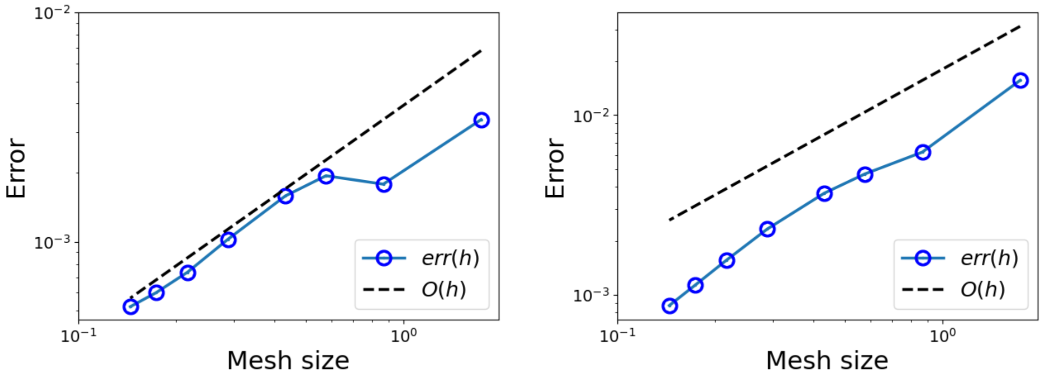

Figure 6.4.6.: Convergence in space versus exact solution in the $L^2(\Gamma)$-norm for φ and ψ.

Convergence in space versus reference solution

We show two more experiments where no exact solution is known, but we compare with a reference solution.

We consider an example on the three-dimensional unite cube and a second one on a thin plate domain

$$\Omega_1 = [0,1]^3, \quad \Omega_2 = [0,1] \times [0,1] \times [0,0.08],$$

where we choose the observation time and the material parameters as

$$T_1 = 0.25, \quad T_2 = 0.5, \quad \varepsilon = \varepsilon_0 = 1.1, \quad \mu = \mu_0 = 1.2, \quad \sigma = 1.3, \quad \alpha = 1.4, \quad C_e = 1.5.$$

With $g_1(x) := \Pi_{i=1}^3 \sin^4(\pi x_i)$ and $g_2(x) := \Pi_{i=1}^2 \sin^4(\pi x_i)$ we define the initial data for the first and second experiment, m_1^0 and m_2^0, respectively as

$$m_1^0(x) := \frac{1}{\sqrt{1+g_1(x)^2}} \begin{pmatrix} g_3(x) \\ 0 \\ 1 \end{pmatrix}, \quad m_2^0(x) := \frac{1}{\sqrt{1+(10g_2(x))^2}} \begin{pmatrix} 10g_2(x) \\ 0 \\ 1 \end{pmatrix}$$

and the Maxwell part is zero at starting time for both experiments

$$E_1^0 = E_2^0 = 0, \quad H_1^0 = H_2^0 = 0, \quad \varphi_1^0 = \varphi_2^0 = 0, \quad \psi_1^0 = \psi_2^0 = 0.$$

The RHS-input data is chosen to guarantee that the solutions are smooth enough in time, i.e. that enough time derivatives vanish at $t = 0$ on the boundary:

$$J_1(t,x) = t^3 \begin{pmatrix} 10g_1(x)^2 \\ 0 \\ 0 \end{pmatrix}, \quad J_2(t,x) = t^3 \begin{pmatrix} 10g_2(x)^2 \\ 0 \\ 0 \end{pmatrix}.$$

Finally, the tolerance and restart parameter for the iterative solver (GMRES), the stabilization parameter and the Convolution Quadrature parameters are set to

$$\text{tolgmres} = 10^{-8}, \quad \text{restart} = 20, \quad \beta = 1, \quad \rho_N = \text{tolgmres}^{1/(2N)}, \quad L = N.$$

As discretized initial data and input data we use interpolations to the respective spaces. We look at the space discretization error on a fixed time grid such that the time error is small enough ($N = 20$ is small enough).

We use mesh sizes $h = \sqrt{3} \approx 1.7, \sqrt{3}/2, \ldots, \sqrt{3}/12 \approx 0.14$ and $h_{\text{ref}} = \sqrt{3}/16$ for the first experiment and mesh sizes $h \approx \sqrt{2}/2, \ldots, \sqrt{2}/16$ and $h_{\text{ref}} \approx \sqrt{2}/24$ for the second experiment. We compute the maximum space-errors (we use the norm of the space $H^1(\Omega)$ for m, $L^2(\Omega)$ for E, H) via

$$\mathrm{err}_{G,i} = \max_{j=0,\dots,20} \| I_{h_{\mathrm{ref}}} G^j_{h_i} - G^j_{h_{\mathrm{ref}}} \|, \quad G \in \{m, E, H\},$$

where $I_{h_{\mathrm{ref}}}$ denotes the interpolation onto the respective approximation space corresponding to mesh size h_{ref}. Due to the additional interpolation of the approximation and as the reference solution may not be computed on a fine enough grid due to hardware restrictions, the rates of convergence are visibly slightly worse than compared to an exact solution. We obtain for m first order convergence in the $H^1(\Omega)$-norm in both experiments, see Figure 6.4.7.

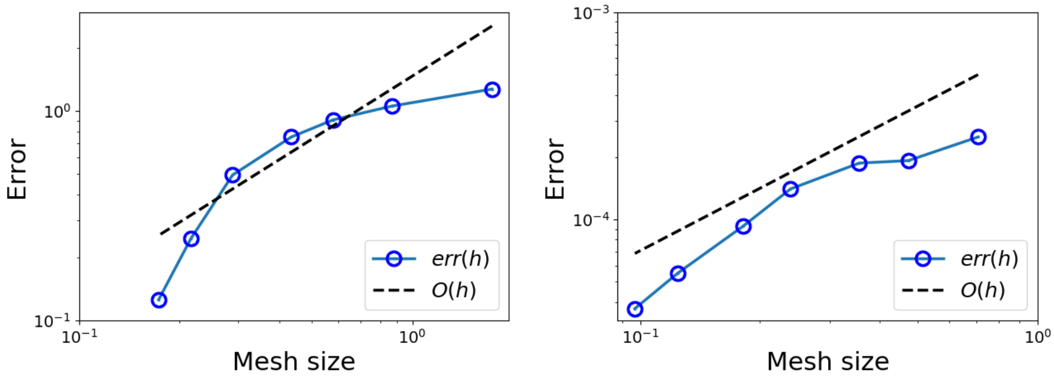

Figure 6.4.7.: Convergence in space versus reference solution for the magnetization in the $H^1(\Omega)$-norm for the first (left) and second (right) experiment.

We obtain first order convergence for E in the $L^2(\Omega)$-norm in both experiments, see Figure 6.4.8.

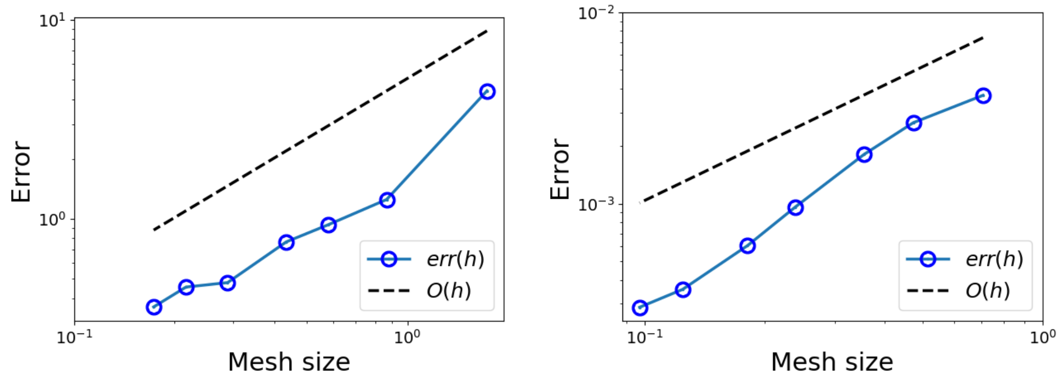

Figure 6.4.8.: Convergence in space versus reference solution in the $L^2(\Omega)$-norm for E for the first (left) and second (right) experiment.

We obtain first order convergence for H in the $L^2(\Omega)$-norm in both experiments, see Figure 6.4.9.

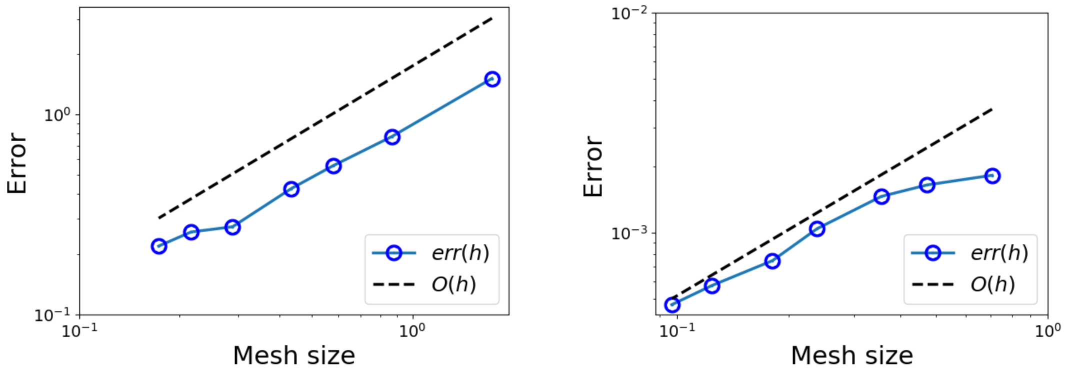

Figure 6.4.9.: Convergence in space versus reference solution in the $L^2(\Omega)$-norm for H for the first (left) and second (right) experiment.

7. Summary and Outlook

7.1. Summary

In this dissertation we have considered theory and numerical approximation of the coupling between the full space Maxwell system and the LLG equation which is relevant in various physical and industrial applications. After an introduction of the functional analytic foundations in Chapter 2, we have reformulated the system on the whole space in a rigorous way to a partial differential – boundary integral system. In Chapter 3, under minimal assumptions on the input data, the weak convergence of the approximations and the existence of a weak solution are shown. Without the coupling to the LLG equation, these results can even be improved in Chapter 4 for the full space Maxwell system. In Chapter 5 we show that the approximations with a priori known and higher order convergence rates in space and time converge to the exact solution, provided the latter is regular enough. Numerical experiments for all the considered algorithms are presented in Chapter 6 and underline the theoretical findings. The properties of the Laplace transform, which are essential for the definition and dependencies between the different solution terms of the MLLG system, are derived in Chapter B in the Appendix. We conclude the thesis with possible extensions of the theoretical results and indicate potential directions of future research in the following section.

7.2. Assumptions and Assertions

In this section we consider some of the assumptions we have made throughout the thesis and discuss their physical significance as well as possible generalizations of the theoretical results.

Shape of the magnetic device

The domain Ω that indicates the magnetic device is assumed to be open, connected, bounded with connected, piecewise smooth Lipschitz boundary, in Chapter 2, and in the following chapters, as far as we work with approximations, it is assumed to be the union of the elements of a regular triangulation. We recall that convexity is not needed and also finite collections of such domains can be treated.

Initial data supported inside of the magnetic body

In Section 1.2.2, we assume that at starting time $t = 0$ electric and magnetic field are supported inside of Ω, i.e.

$$H(0, x) = E(0, x) = 0 \quad \text{for all } x \in \mathbb{R}^3 \setminus \overline{\Omega}. \tag{7.1}$$

This corresponds to the situation when at the beginning of the experiment everything is at rest in the exterior domain:

Inserting $\partial_t B(0) = \partial_t D(0) = 0$ in (1.7) and (1.8) (no time evolution at starting time outside of Ω), the starting values satisfy

$$(\nabla \times E)(0, x) = (\nabla \times H)(0, x) = 0 \quad \text{for all } x \in \mathbb{R}^3 \setminus \overline{\Omega}.$$

The shifted variables $\overline{E} = E - E(0)$, $\overline{H} = H - H(0)$ (with any suitable extension inside) satisfy (7.1) and Faraday's (1.7) and Ampère's law (1.8) (with inhomogenieties, or even without additional inhomogenieties when choosing the extension such that $(\nabla \times E)(0, x) = (\nabla \times H)(0, x) = 0$ inside of Ω. By physical means, it is plausible that this exists.).

This also justifies to define the discrete starting values $\varphi_h^0 = 0$ and $\psi_h^0 = 0$ in the Algorithms 3.12 and 4.8. The regularity assumptions from Chapter 5 even ask for further derivatives of the boundary variables to vanish at $t = 0$.

Magnetic body surrounded by non-magnetic material and vacuum

Similar to [25] (on a bounded domain), we may consider the situation in which a magnetizable body is surrounded by other non-magnetizable objects, which in turn are surrounded by vacuum. This can be modeled by a domain $\Omega \subset \mathbb{R}^3$, comprising the whole experimental setup which is in vacuum and where $\omega \subset \Omega$ indicates the magnetizable parts. We thus obtain a coupled system, where on ω the coupled MLLG equations hold, on $\Omega \setminus \omega$ only the Maxwell equations hold, and on $\partial\Omega$ we obtain the usual boundary integral equation. The results of Chapter 3 are retained in an analogous manner. The results of Chapter 5 can be adapted in the same way, but the regularity requirements must be checked during application. If, for example, jumping material coefficients are located within the area of Ω (e.g. at object boundaries), then in general not enough regularity is at hand, see [155]. Refined analysis would have to show whether in such cases these could also be attributed to piecewise regularity assumptions (inside of Ω), if necessary.

Corner singularities

The conclusion remains similar with respect to possible boundary singularities: If one considers Maxwell's equations on domains with an reentrant corner, singularities can form there. This stays true also with the coupling to the LLG equation. Since in Chapter 3 and 4 analogies of the energy norms are bounded, the results already cover the case of those boundary singularities without modification of the proofs. For Chapter 5, on the other hand, it must be checked whether the regularity requirements remain fulfilled or whether adaptive methods can be applied to improve the numerical complexity, cf. Section 7.3.

Generalized effective magnetic field

If one considers in Section 1.2.1 (physical derivation of the LLG equation) further terms of the total magnetic Gibbs free energy, or drops the assumption $\mu \in \mathbb{R}_+$ there, a more complicated effective magnetic field $H_{\text{eff}} = \Delta m + H + \pi(t, m)$ comes into place. As in [25], the results of Chapter 3 stay valid, if the additional energy contributions $\pi(t, m)$ satisfy

$$\|\pi(t, m)\|_{L^2(\Omega)} \leq C$$

for any $m \in L^2(\Omega)$ with $|m| \leq 1$ almost everywhere and for $m_{\tau,h} \overset{\text{sub}}{\rightharpoonup} m$

$$\pi(\cdot, m_{\tau,h}) \overset{\text{sub}}{\rightharpoonup} \pi(\cdot, m) \quad \text{in } L^2(\Omega_T).$$

Also approximations π_h of π can be treated, see [41].

For the conservation of the strong convergence results from Chapter 5, we need a sufficiently good approximation π_h^n for the implementation and for the theory a suitable counterpart $\pi^n_{h,\star}$, which corresponds to the insertion of the exact solution into the approximation π^n_h. The precise requirements on π, π^n_h and $\pi^n_{h,\star}$ are the following (compare Section 5.6):

$$\|\pi^n_h - \pi^n_{\star,h}\|_{L^2(\Omega)} \leqslant L \sum_{j=0}^{2} \|\boldsymbol{e}^{n-j}_h\|_{H^1(\Omega)} + \frac{\alpha}{2}\|\dot{e}^n_{m,h}\|_{L^2(\Omega)} + c\|s^n_h\|_{H^1(\Omega)} \tag{7.2a}$$

and at the same time

$$\|\pi^n_{\star,h} - \pi(t_n, m(t_n))\|_{L^2(\Omega)} \le C(\tau^2 + h^r). \tag{7.2b}$$

In the case an approximate functional π_h is given, possible discretizations of the right hand side π^n_h are given by, e.g. $\pi_h(t_n, \widehat{m}^n_{\star,h})$ or $\pi_h(t_n, m^n_h)$. In both cases, we can ensure (7.2), if we have a Lipschitz type bound

$$\|\pi_h(t_n, u) - \pi_h(t_n, v)\|_{L^2(\Omega)} \leqslant L\|u - v\|_{H^1(\Omega)}$$

and an approximation error

$$\|\pi_h(t_n, m(t_n)) - \pi(t_n, m(t_n))\|_{L^2(\Omega)} \le C(\tau^2 + h^r).$$

The regularity assumptions take the form

$$\begin{aligned} &m \in C^3([0,T], H^1(\Omega)) \cap C^1([0,T], W^{r+1,\infty}(\Omega)), \\ &\Delta m + H + \pi(\cdot, m) \in C([0,T], W^{r+1,\infty}(\Omega)). \end{aligned} \tag{7.3}$$

Material parameters

In Section 1.2.2, we assume linear material laws, i.e. the electric and magnetic permeabilities ε and μ are symmetric and uniformly coercive matrices inside of Ω and positive scalars outside. This is also necessary for the analysis, the reformulation in Chapter 2 needs representation formulae which are only available for scalar material parameters in the exterior domain. In Chapters 3 to 5 we need that (at least suitable perturbations of) $[\,\cdot\,, \varepsilon\,\cdot\,]_\Omega$ and $[\,\cdot\,, \mu\,\cdot\,]_\Omega$ yield symmetric and coercive bilinear forms.

The analysis also covers the case of indefinite conductivity, i.e. it is enough to have that $[\,\cdot\,, \sigma\,\cdot\,]_\Omega$ is bounded. For simplified models like the ELLG system, it is essential that σ is strictly positive.

For the sake of completeness, we also note that constant, positive permeabilities can also be assumed for air in a good approximation, so that the experimental setup can alternatively be surrounded by air or other homogeneous gases.

Higher regularity convergence for the pure Maxwell system

Due to the linearity of Maxwell's equations, arguments similar to those in Chapter 4 possibly could be extended to higher derivatives. Assuming constant material parameters (scaled to one in the following), $k \in \mathbb{N}$ times differentiating Maxwell's equations results in

$$\begin{aligned} \partial_t^{k+1} E - \nabla \times \partial_t^k H &= \partial_t^k J \\ \partial_t^{k+1} H + \nabla \times \partial_t^k E &= \partial_t^k G \end{aligned} \tag{7.4}$$

Testing with $\partial_t^k E$, $\partial_t^k H$ gives (under neglect of boundary values integration by parts gives $[\nabla \times \partial_t^k H, \partial_t^k E]_\Omega = [\nabla \times \partial_t^k E, \partial_t^k H]_\Omega$)

$$\begin{aligned} \frac{1}{2}\partial_t\|\partial_t^k E\|^2_\Omega + \frac{1}{2}\partial_t\|\partial_t^k H\|^2_\Omega &= [\nabla \times \partial_t^k J, \partial_t^k E]_\Omega + [\nabla \times \partial_t^k G, \partial_t^k H]_\Omega \\ &\le \frac{1}{2}\left(\|\partial_t^k E\|^2_\Omega + \|\partial_t^k H\|^2_\Omega + \|\partial_t^k J\|^2_\Omega + \|\partial_t^k G\|^2_\Omega\right) \end{aligned}$$

Integration in time and Gronwall's Lemma A.1 conclude

$$\|\partial_t^k E(t)\|_\Omega^2 + \|\partial_t^k E(t)\|_\Omega^2 \le e^t \Big(\|\partial_t^k E(0)\|_\Omega^2 + \|\partial_t^k H(0)\|_\Omega^2 + \int_0^t \|\partial_t^k J(s)\|_\Omega^2 + \|\partial_t^k G(s)\|_\Omega^2 \, \mathrm{d}s \Big).$$

Consideration of (7.4) inductively gives (neglecting J and G for simplicity)

$$\nabla \times H = \partial_t E \in L^2, \quad \nabla \times \nabla \times H = \nabla \times \partial_t E = -\partial_t^2 H \in L^2, \ldots$$

and this also links $\partial_t^k E(0)$ and $\partial_t^k H(0)$ to regularity assumptions for the initial data, i.e. $(\nabla\times)^k E^0$ and $(\nabla\times)^k H^0$ have to exist.

Symmetric vs. non-symmetric approach for the Maxwell part

In Chapter 3 we considered a discretization symmetric in E and H, obtaining the symmetric differential operator $D = (\nabla \times + (\nabla\times)^T)/2$ and two trace variables φ and ψ. In Chapter 4, however, the discretizations differ in E (piecewise linear Nedelec elements) and H (piecewise constant elements) and in the differential operators $\nabla\times$ and $(\nabla\times)^T$, respectively. This results in differences in the convergence of certain points of the solution, compare Remark 3.31 and Remark 4.24. In the symmetric case we have (ignoring the integration in time)

$$\frac{1}{2}[\nabla \times E_h, \zeta_h]_\Omega + \frac{1}{2}[E_h, \nabla \times \zeta_h]_\Omega + \frac{1}{2}\langle \psi_h, \gamma_T \zeta_h \rangle_\Gamma \overset{\text{sub}}{\rightarrow} [\nabla \times E, \zeta]_\Omega \tag{7.5a}$$

and

$$\frac{1}{2} \langle\!\langle \gamma_T(E_h), \gamma_T \zeta_h \rangle\!\rangle_{\Gamma_T} \overset{\text{sub}}{\rightarrow} \langle \gamma_T E, \gamma_T \zeta \rangle_{\Gamma_T} + \frac{1}{2} \langle \psi, \gamma_T \zeta \rangle_{\Gamma_T}. \tag{7.5b}$$

Discretized terms may not correspond to continuous terms, while in the non-symmetric case the following holds:

$$[\nabla \times E_h, \zeta_h]_\Omega \overset{\text{sub}}{\rightarrow} [\nabla \times E, \zeta]_\Omega \quad \text{and} \quad [H_h, \nabla \times \zeta_h]_\Omega \overset{\text{sub}}{\rightarrow} [H, \nabla \times \zeta]_\Omega \tag{7.6a}$$

and

$$\langle\!\langle \gamma_T E_h, \gamma_T \zeta_h \rangle\!\rangle_{\Gamma_T} \overset{\text{sub}}{\rightarrow} \langle \gamma_T E, \gamma_T \zeta \rangle_{\Gamma_T} \quad \text{and} \quad \langle\!\langle \varphi_h, \gamma_T \zeta_h \rangle\!\rangle_{\Gamma_T} \overset{\text{sub}}{\rightarrow} \langle \varphi, \gamma_T \zeta \rangle_{\Gamma_T}. \tag{7.6b}$$

Nevertheless, in both cases, for the equivalence to the corresponding solution of Chapter 2, the projection property of the Calderon operator must be exploited, cf. Section 3.1 and Section 4.1.

One reason for the convergence in (7.5) is that in the discretized equation, traces and interior quantities cannot be separated: In the continuous equation, in the case of

$$\frac{1}{2}[\nabla \times E, \zeta]_\Omega + \frac{1}{2}[E, \nabla \times \zeta]_\Omega + \frac{1}{2}\langle \psi, \gamma_T \zeta \rangle_\Gamma \le C\|\zeta\|_\Omega$$

one can show by the use of cut-off functions that $\psi = -\gamma_T E$. This is not possible in the discretized setting, finally it cannot be shown that $\psi_h \overset{\text{sub}}{\rightarrow} -\gamma_T E$ holds. We further note (cf. Remark 5.2) that in Chapter 5 even (again ignoring time dependencies)

$$\|\psi_h + \gamma_T E\|_{\mathcal{H}_\Gamma} \le Ch^r$$

can be shown, but there in the consistency analysis the existence of a sufficiently smooth exact solution is assumed.

In the non-symmetric case of Chapter 4, the boundary integral equation implements a discrete Dirichlet–to–Neumann map for the first Maxwell equation, and in the second Maxwell equation

$$[\nabla \times E_h, \zeta_h]_\Omega \leq \|\zeta_h\|_\Omega$$

ensures the existence of $\nabla \times E_h \in L^2(\Omega)$. The question now arises to what extent these phenomena can also be used for a convergence analysis with rates. In [99], a symmetric method is used and in the stability analysis a term comparable to the one in (7.5a) is bounded. In the final result, however, no bound concerning the curl of the approximations is specified. Is this possible using the non-symmetric approach? Is it also possible to use other methods, e.g. such that work without the additional boundary variables ψ and φ, e.g. by using a Dirichlet–to–Neumann map or by including the conditions that $\gamma_T E_h$, $\gamma_T H_h$ are suitable exterior data in the respective approximation spaces?

Coupled vs. uncoupled Maxwell–LLG system

In Chapter 4, we transferred the results of Chapter 3 for the uncoupled Maxwell system and obtained stronger results. This is on the one hand due to the uniqueness of the linear Maxwell equations, on the other hand, however, due to the reason mentioned in Remark 4.15: The occurring phenomena are described in a continuous setting and boundary terms can be neglected (these are handled appropriately by the boundary element method). We recall Maxwell's equations with inhomogeneities J, G

$$\begin{aligned} \partial_t E - \nabla \times H &= J, \\ \partial_t H + \nabla \times E &= G, \end{aligned} \tag{7.7}$$

note that $G = \partial_t m$ in the case of MLLG coupling. We test with E and H, respectively and obtain (under neglect of the boundary terms, integration by parts yields $[\nabla \times H, E]_\Omega = [\nabla \times E, H]_\Omega$)

$$\frac{1}{2}\partial_t \|E\|_\Omega^2 + \frac{1}{2}\partial_t \|H\|_\Omega^2 = [J, E]_\Omega + [G, H]_\Omega \leq \frac{1}{2}\left(\|E\|_\Omega^2 + \|H\|_\Omega^2 + \|J\|_\Omega^2 + \|G\|_\Omega^2\right).$$

Gronwall's Lemma A.1 concludes for $t > 0$

$$\|E(t)\|_\Omega^2 + \|H(t)\|_\Omega^2 \leq e^t \left(\|E(0)\|_\Omega^2 + \|H(0)\|_\Omega^2 + \int_0^t \|J(s)\|_\Omega^2 + \|G(s)\|_\Omega^2 \,\mathrm{d}s\right).$$

In the coupled case, $G = \partial_t m$ can be bounded due to the LLG energy estimates.

For the further procedure in Chapter 4, we use discrete versions of the following arguments: Differentiating (7.7) in time, and testing with $\partial_t E$, $\partial_t H$ yields

$$\frac{1}{2}\partial_t \|\partial_t E\|_\Omega^2 + \frac{1}{2}\partial_t \|\partial_t H\|_\Omega^2 \leq \frac{1}{2}\left(\|\partial_t E\|_\Omega^2 + \|\partial_t H\|_\Omega^2 + \|\partial_t J\|_\Omega^2 + \|\partial_t G\|_\Omega^2\right).$$

Again Gronwall's lemma A.1 concludes (with $\partial_t E(0) = \nabla \times H^0$, $\partial_t H(0) = -\nabla \times E^0$)

$$\|\partial_t E(t)\|_\Omega^2 + \|\partial_t H(t)\|_\Omega^2 \leq e^t \left(\|\nabla \times H^0\|_\Omega^2 + \|\nabla \times E^0\|_\Omega^2 + \int_0^t \|\partial_t J(s)\|_\Omega^2 + \|\partial_t G(s)\|_\Omega^2 \,\mathrm{d}s\right).$$

These arguments cannot be applied in a similar way to the coupled case, since there is no stability result for $\partial_t G = \partial_t^2 m$.

With respect to (7.7) the question arises whether a stability result for $\partial_t^2 m$ is necessary at all (if $G = \partial_t m$ is bounded, why not $\partial_t H$ and $\nabla \times E$?) We sketch in the following why this question must be answered in the affirmative.

If we insert in (7.7) the special solutions $E(t,x) = (0, e(t,x_1), 0)^T$ and $H(t,x) = (0, 0, h(t,x_1))^T$ we get (with $x_1 \in \mathbb{R}$ again denoted by x)

$$\begin{aligned}\partial_t e + \partial_x h &= j,\\ \partial_t h + \partial_x e &= g.\end{aligned}$$

Assuming space periodic solutions $e(t,x) = \widehat{e}(t)\sin(kx)$, $h(t,x) = \widehat{h}(t)\cos(kx)$ for $k \in \mathbb{N}$ the problem simplifies to

$$\begin{aligned}\partial_t \widehat{e} - k\widehat{h} &= \widehat{j},\\ \partial_t \widehat{h} + k\widehat{e} &= \widehat{g}.\end{aligned}$$

A solution is given by (for simplicity $\widehat{j} = 0$)

$$\begin{aligned}\widehat{e}(t) &= \sin(kt)\int_0^t \cos(ks)\widehat{g}(s)\,\mathrm{d}s - \cos(kt)\int_0^t \sin(ks)\widehat{g}(s)\,\mathrm{d}s\\ \widehat{h}(t) &= \cos(kt)\int_0^t \cos(ks)\widehat{g}(s)\,\mathrm{d}s + \sin(kt)\int_0^t \sin(ks)\widehat{g}(s)\,\mathrm{d}s\end{aligned}$$

We recall that boundedness of $\nabla \times E$ corresponds to uniform boundedness of $k\widehat{e}$ and compute

$$\begin{aligned}\partial_t \widehat{h}(t) &= -k\sin(kt)\int_0^t \cos(ks)\widehat{g}(s) + k\cos(kt)\int_0^t \sin(ks)\widehat{g}(s)\,\mathrm{d}s + g(t)\\ &= g(0)\cos(kt) - \sin(kt)\int_0^t \sin(ks)\partial_t\widehat{g}(s)\,\mathrm{d}s - \cos(kt)\int_0^t \cos(ks)\partial_t\widehat{g}(s)\,\mathrm{d}s.\end{aligned}$$

Inserting special functions and further arguments show, that this cannot be bounded, if a uniform bound for $k\widehat{g}$ or $\partial_t g$ is unavailable. We will not go into further detail here. As $\partial_t\widehat{g}$ corresponds to $\partial_t^2 m$, $\partial_t\widehat{h}$ cannot be bounded. Similarly, we do not have a bound for $\nabla \times \partial_t m$ (corresponding to kg) so that $k\widehat{e}$ cannot be bounded uniformly. As a consequence $k\widehat{e}$ and $\partial_t\widehat{h}$ cannot be bounded independently of each other, but for the combination of both it naturally holds $\partial_t\widehat{h} + k\widehat{e} = \widehat{g}$, which is bounded again. We conclude with two resulting statements:

1) Possibly the arguments of Chapter 3 can be transferred to a space–time setting which considers the differential operator $\mathcal{M}$ and the associated Hilbert space $H(\mathcal{M}, \Omega_T)$, with

$$\mathcal{M} = \begin{pmatrix}\partial_t & -\nabla\times\\ \partial_t & \nabla\times\end{pmatrix}, \quad H(\mathcal{M},\Omega_T) = \left\{ \begin{pmatrix}E\\H\end{pmatrix} \in L^2(\Omega_T)^2 \,\middle|\, \mathcal{M}\begin{pmatrix}E\\H\end{pmatrix} \in L^2(\Omega_T)^2 \right\}.$$

This possibly could additionally show the weak convergence of

$$\mathcal{M}\begin{pmatrix}E_{\tau,h}\\H_{\tau,h}\end{pmatrix} \overset{\text{sub}}{\rightharpoonup} \mathcal{M}\begin{pmatrix}E\\H\end{pmatrix}.$$

The existence of $\mathcal{M}\binom{E}{H}$, however, already can be deduced from Definition 3.1 (rewriting the left hand side of Maxwell's equations in terms of $\mathcal{M}\binom{E}{H}$ and noticing that the right hand side is bounded).

2) To extend the results of Chapter 3 in a similar way as in Chapter 4, a stability result for $\partial_t^2 m$ or $\nabla \times \partial_t m$ from the LLG equation is needed first. If this furthermore can be applied in a suitable way to the discrete setting (and if the phenomena described in the previous paragraph for the non-symmetric Maxwell discretization can indeed be applied to strong convergence analysis), the results of Chapter 5 could be extended by the estimates

$$\|\nabla \times E(t_n) - \nabla \times E_h^n\|_{L^2(\Omega)}^2 + \sum_{j=0}^{n-1} \|\mu_0\gamma_T H(t_{j+1/2}) - \varphi_h^{j+1/2}\|_{\mathcal{H}_\Gamma}^2 \le C(\tau^2 + h^r)^2.$$

7.3. Outlook

We conclude the thesis with an outlook on possible future research projects.

In [23], error estimators for the LLG equation are presented with numerical results that underline their applicability. However, no rigorous error analysis is provided. How can efficient and reliable error estimators be constructed that ensure strong convergence of the solutions?

Another emerging field of research is to incorporate stochastic noise to the LLG equation. In [73, 74], similar to Chapter 3, almost sure weak convergence to weak martingale solutions of the (Maxwell–)LLG system is shown. Can the strong convergence results of Chapter 5 be extended in some way to the stochastic case?

In [66] the total helicity is explored as the sum of two terms: A term that measures the difference between the number of left-handed and right-handed photons of the free magnetic and electric fields, and another term that measures the screwiness of the magnetization density in matter. Without external input, between a static starting and a static final state, the total helicity stays constant, only a transfer between the handedness of the free fields and the chirality of the magnetization density can take place. Numerical simulations that could demonstrate the dynamics of the helicity transfer between the static states would be of great interest. Therefore, again it is necessary to consider the time evolution of the fully coupled Maxwell–LLG system and finally, this could lead to a numerical tool for a novel study of the all-optical switching.

In Chapter 5, strong regularity assumptions on the exact solution are formulated to ensure convergence with rates. In [63] it is shown that these are satisfied for the LLG equation, assuming sufficiently small initial data. Can we show the existence of sufficiently regular solutions for the coupled Maxwell–LLG system, which then would ultimately justify the assumptions on the exact solution in the numerical algorithm?

A. Appendix

A.1. Gronwall's Lemma, Integration by Parts and Discrete Counterparts

We state Gronwall's inequality and a discrete counterpart.

Lemma A.1 (Gronwall's lemma)**.** *Let $T > 0$ and $a : [0, T] \to \mathbb{R}_{\geq 0}$ be a continuous, non-negative function, that is bounded for all $t \in [0, T]$ by*

$$a(t) \leq g(t) + C \int_0^t a(s) \, \mathrm{d}s$$

for a constant $C > 0$ and a continuous, non-negative function $g : [0, T] \to \mathbb{R}_{\geq 0}$. Then a is bounded by

$$a(t) \leq g(t) + C \int_0^t g(s) e^{C(t-s)} \, \mathrm{d}s.$$

If g is monotonically increasing, we conclude

$$a(t) \leq \widetilde{C} g(t)$$

for $\widetilde{C} = e^{CT}$.

Lemma A.2 (Discrete version of Gronwall's lemma)**.** *Let $T > 0$, $N \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k \leq N$ and $\tau = T/N$. Let $(a^j)_{j=k}^N \subset \mathbb{R}_{\geq 0}$ be a real valued, non-negative sequence, that is bounded for all $j \in \{k, \dots, N\}$ by*

$$a^j \leq g^j + C\tau \sum_{i=k}^{j-1} a^i$$

for a constant $C > 0$ and a non-negative sequence $(g^j)_{j=k}^N \subset \mathbb{R}_{\geq 0}$. Then a is bounded for all $j \in \{k, \dots, N\}$ by

$$a^j \leq g^j + C\tau \sum_{i=k}^{j-1} g^i e^{C(t_{j-1} - t_i)} \, \mathrm{d}s.$$

If g is monotonically increasing, we conclude for all $j \in \{k, \dots, N\}$

$$a^j \leq \widetilde{C} g^j$$

for $\widetilde{C} = e^{CT}$.

Integration by parts shows for regular functions $a, b : [0, T] \to \mathbb{R}$

$$[\partial_t a, b]_{[0,T]} = a(T)b(T) - a(0)b(0) - [a, \partial_t b]_{[0,T]} \quad \text{and}$$
$$[a, b]_{[0,T]} = (\partial_t^{-1} a)(T)b(T) - [\partial_t^{-1} a, \partial_t b]_{[0,T]}.$$

Similarly, discrete counterparts hold true.

Lemma A.3 (Discrete Integration by Parts)**.** *For $N \in \mathbb{N}$ and sequences $(a^j)_{j=0,\dots,N}$, $(b^j)_{j=0,\dots,N}$ it holds*

$$[(\partial_t^\tau a)_\tau^+, b_\tau^-]_{[0,T]} = a^N b^N - a^0 b^0 - [a_\tau^+, (\partial_t^\tau b)_\tau^+]_{[0,T]},$$
$$[a_\tau^+, b_\tau^-]_{[0,T]} = ((\partial_t^\tau)^{-1}(a^{k+1})_k)(t_{N-1}) b^N - [((\partial_t^\tau)^{-1}(a^{k+1})_k)_\tau^-, (\partial_t^\tau b)_\tau^+]_{[0,T]}.$$

Proof. It holds

$$\begin{aligned}[(\partial_t^\tau a)_\tau^+, b_\tau^-]_{[0,T]} + [a_\tau^+, (\partial_t^\tau b)_\tau^+]_{[0,T]} &= \tau \sum_{j=0}^{N-1} \frac{a^{j+1} - a^j}{\tau} b^j + \tau \sum_{j=0}^{N-1} a^{j+1} \frac{b^{j+1} - b^j}{\tau} \\ &= \sum_{j=0}^{N-1} a^{j+1} b^{j+1} - a^j b^j \\ &= a^N b^N - a^0 b^0.\end{aligned}$$

The second assertion can be shown similarly, by introducing the sequence $c^0 := 0$, $c^j := ((\partial_t^\tau)^{-1}(a^{k+1})_k)(t_{j-1}) = \tau \sum_{k=0}^{j-1} a^{k+1}$ for $j = 1, \dots, N$ and using $(c^{j+1} - c^j)/\tau = a^{j+1}$ for $j = 0, \dots, N-1$:

$$\begin{aligned}[a_\tau^+, b_\tau^-]_{[0,T]} + [((\partial_t^\tau)^{-1}(a^{k+1})_k)_\tau^-, (\partial_t^\tau b)_\tau^+]_{[0,T]} &= \tau \sum_{j=0}^{N-1} \frac{c^{j+1} - c^j}{\tau} b^j + \tau \sum_{j=0}^{N-1} c^{j+1} \frac{b^{j+1} - b^j}{\tau} \\ &= c^N b^N - c^0 b^0 \\ &= ((\partial_t^\tau)^{-1}(a^{k+1})_k)(t_{N-1}) b^N.\end{aligned}$$

This is the version we apply in Chapter 3 and 4. The formula can also be rewritten in the clearer form

$$\begin{aligned}[a_\tau^+, b_\tau^-]_{[0,T]} &= ((\partial_t^\tau)^{-1} a)(N) b^N - \tau a^0 b^N \\ &\quad + \sum_{j=0}^{N-1} \tau a^0 (\partial_t^\tau b)(j+1) - [((\partial_t^\tau)^{-1}(a)_\tau^+, (\partial_t^\tau b)_\tau^+]_{[0,T]} \\ &= ((\partial_t^\tau)^{-1} a)(N) b^N - \tau a^0 b^N \\ &\quad + \sum_{j=0}^{N-1} a^0 (b^{j+1} - b^j) - [((\partial_t^\tau)^{-1}(a)_\tau^+, (\partial_t^\tau b)_\tau^+]_{[0,T]} \\ &= ((\partial_t^\tau)^{-1} a)(N) b^N - \tau a^0 b^0 - [((\partial_t^\tau)^{-1}(a)_\tau^+, (\partial_t^\tau b)_\tau^+]_{[0,T]}\end{aligned}$$

and it is $\tau a^0 = ((\partial_t^\tau)^{-1} a)(0)$. □

A.2. Discrete Herglotz Theorems

In this section we recall discrete versions of the Herglotz theorem, see Theorem B.83, [27, Lemma 2.1–2.3] and the original reference [81]. The notation is similar to Chapter B and the results are for the first order Convolution Quadrature scheme $\delta(\zeta) = 1 - \zeta$, see Section 3.2.2. Provided that the scheme is stable, the results can be extended to higher order schemes, see [99, Lemma 5.3].

Theorem 1 (Discrete Herglotz Theorem on $[0,T]$)**.** *Let $B, R \in \mathcal{H}_m(\sigma_0)$ for $\sigma_0 \in \mathbb{R}$. Let $a(\cdot,\cdot) : X \times X \to \mathbb{C}$ sesquilinear and continuous. If there exists a $c > 0$ such that for all $w \in \mathbb{C}$, all $\Re s > \sigma_0$*

$$\Re a(w, B(s)w) \geq c \|R(s)w\|_X^2,$$

then it holds for any sequence $(w^n)_{n=0,\dots,N} \subset X$ *and for all* $\sigma \geq \sigma_0$

$$\tau \sum_{j=0}^{N} e^{-2\sigma t_j} \Re a(w^j, (B(\partial_t^\tau)w)(t_j)) \geq c\tau \sum_{j=0}^{N} e^{-2\sigma t_j} \|R(\partial_t^\tau)w)(t_j)\|_X^2.$$

Proof. The assertion follows similarly to the proof of Lemma 3.17 and Lemma 3.21. □

Theorem A.4 (Discrete Herglotz Theorem on $[0,T]$)**.** *Let* $B \in \mathcal{H}_m(\sigma_0)$ *for* $\sigma_0 \in \mathbb{R}_+$*. For* $N \in \mathbb{N}$ *sufficiently large and a sequence* $(w^n)_{n=0,\dots,N} \subset X$*, it holds*

$$\tau \sum_{j=0}^{N} \|(B(\partial_t^\tau)w)(t_j)\|^2 \leq C\tau \sum_{j=0}^{N} \|((\partial_t^\tau)^m w)(t_j)\|^2.$$

The constant C *depends on* σ_0*,* T *and* B*, but not on* τ*.*

Proof. We extend w to a sequence $(w^n)_{n\in\mathbb{N}}$ such that $((\partial_t^\tau)^m w)(t_j) = 0$ for all $j > N$. This is always possible by an iterative procedure, as we can write $((\partial_t^\tau)^m w)(t_{k+1}) = w^{k+1}/\tau^m - f((w^n)_{n\leq k})$, where $f((w^n)_{n\leq k})$ does not depend on w^{k+1}. Now we compute iteratively w^{N+1}, such that $((\partial_t^\tau)^m w)(t_{N+1}) = 0$, w^{N+2} such that $((\partial_t^\tau)^m w)(t_{N+2}) = 0, \dots$.
Now we define the finite sequence $w_M^j := w^j$ for $j = 0, \dots, M$ and $w_M^j = 0$, $j > M$. As in Lemma 3.21 we have for $\rho = e^{-2\sigma_0\tau}$, $|\zeta| < \rho$ and sufficiently small τ

$$\Re\left(\frac{\delta(\zeta)}{\tau}\right) \geq \frac{1 - e^{-2\sigma_0\tau}}{\tau} = \int_0^{2\sigma_0} e^{-\tau r}\, \mathrm{d}r \geq 2\sigma_0 e^{-2\tau\sigma_0} > \sigma_0.$$

With similar arguments as in [27, Lemma 2.1, Lemma 2.3] we obtain

$$\tau \sum_{j=0}^{\infty} e^{-4\sigma_0 t_j} \|(B(\partial_t^\tau)w)(t_j)\|^2 \leq C\tau \sum_{j=0}^{\infty} e^{-4\sigma_0 t_j} \|((\partial_t^\tau)^m w_M)(t_j)\|^2.$$

For $j \geq M$, it is $w^j \leq C t_j^m$ (this can be shown by discrete integration) and therefore

$$\tau \sum_{j=0}^{\infty} e^{-4\sigma_0 t_j} \|((\partial_t^\tau)^m w_M)(t_j)\|^2 \leq \tau \sum_{j=0}^{N} e^{-4\sigma_0 t_j} \|((\partial_t^\tau)^m w_M)(t_j)\|^2 + C(\tau, m) e^{-4\sigma_0 t_M} t_M^m$$

and the limit $M \to \infty$ exists on the right hand side. We obtain by discrete Causality (i.e. $B(\partial_t^\tau)w(t_j)$ is independent of w^n, $n > j$) for $M > N$

$$\begin{aligned}\tau \sum_{j=0}^{N} e^{-4\sigma_0 t_j} \|(B(\partial_t^\tau)w)(t_j)\|^2 &= \tau \sum_{j=0}^{N} e^{-4\sigma_0 t_j} \|(B(\partial_t^\tau)w_M)(t_j)\|^2 \\ &\leq \tau \sum_{j=0}^{\infty} e^{-4\sigma_0 t_j} \|(B(\partial_t^\tau)w_M)(t_j)\|^2.\end{aligned}$$

Combining the previous estimates for the limit $M \to \infty$ gives

$$\tau \sum_{j=0}^{N} e^{-4\sigma_0 t_j} \|(B(\partial_t^\tau)w)(t_j)\|^2 \leq C\tau \sum_{j=0}^{N} e^{-4\sigma_0 t_j} \|((\partial_t^\tau)^m w)(t_j)\|^2.$$

Now the bounds $e^{-4\sigma_0 T} \leq e^{-4\sigma_0 t_j} \leq 1$ yield the assertion. □

A.3. Rescalings of the Calderon Operator

In this section, we give an overview of the different rescalings of the time harmonic Maxwell system and corresponding Calderon operators from literature. Especially for the implementation, it is crucial to deal with the right operators. The different versions vary in details, so it is necessary to give all the formulas in detail. The overview also supported the Erratum [125], in which similar content is considered. All problems are considered in the exterior domain $\overline{\Omega}^c$ of the open, bounded and Lipschitz domain Ω with piecewise smooth boundary Γ. We do not provide any proofs, however, in brackets we give some notes why the formulas are reasonable and coherent.

The first reformulation is from Buffa & Hiptmaier in [46]. The Calderon operator is symmetric, the formulation suits the second order formulation of Maxwells equations.

Buffa & Hiptmaier (BH) In [46] the tangential trace and the Neumann trace, respectively are introduced as

$$\begin{aligned}\gamma_T^{BH} u &= u \times n,\\ \gamma_N^{BH} u &= \frac{1}{k}\gamma_T(\nabla \times u)\end{aligned}$$

and the following second order formulation for given E^{BH}, H^{BH} is considered for $k > 0$

$$\begin{aligned}\nabla \times \nabla \times u - k^2 u &= 0 && \text{in } \overline{\Omega}^c,\\ \gamma_T^{BH} u &= E^{BH} && \text{on } \Gamma,\\ \gamma_N^{BH} u &= H^{BH} && \text{on } \Gamma.\end{aligned}$$

Defining $v := \frac{1}{k}\nabla \times u$, it is equivalent to solve

$$\begin{aligned}-ku + \nabla \times v &= 0,\\ -kv + \nabla \times u &= 0,\\ \gamma_T^{BH} u &= E^{BH},\\ \gamma_T^{BH} v &= H^{BH}.\end{aligned}$$

If E^{BH} and H^{BH} are suitable exterior data, the solution is given by

$$\begin{aligned}u &= -\mathcal{D}^{BH}(k)E^{BH} - \mathcal{S}^{BH}(k)H^{BH},\\ v &= -\mathcal{S}^{BH}(k)E^{BH} - \mathcal{D}^{BH}(k)H^{BH},\end{aligned} \tag{A.1}$$

with the electric single layer potential

$$\left(\mathcal{S}^{BH}(k)\varphi\right)(x) := k\int_\Gamma G^{BH}(k, x-y)\varphi(y)\mathrm{d}y + \frac{1}{k}\nabla\int_\Gamma G^{BH}(k, x-y)\mathrm{div}_\Gamma\varphi(y)\mathrm{d}y$$

and the electric double layer potential

$$(\mathcal{D}^{BH}(k)\varphi)(x) = \mathrm{curl}\int_\Gamma G^{BH}(k, x-y)\varphi(y)\,\mathrm{d}y,$$

where the fundamental solution $G^{BH}(k, z)$ is given as

$$G^{BH}(k, z) = \frac{e^{ik|z|}}{4\pi|z|}.$$

The boundary data E^{BH} and H^{BH} is suitable exterior data if and only if

$$\left(\frac{1}{2} - B^{BH}(k)\right)\begin{pmatrix}E^{BH}\\ H^{BH}\end{pmatrix} = \begin{pmatrix}E^{BH}\\ H^{BH}\end{pmatrix}.$$

Here the Calderon operator is defined as

$$B^{BH}(k) = \begin{pmatrix} W^{BH}(k) & V^{BH}(k) \\ V^{BH}(k) & W^{BH}(k) \end{pmatrix},$$

where

$$\begin{aligned} V^{BH}(k) &= \{\!\{\gamma_T^{BH} \circ \mathcal{S}^{BH}(k)\}\!\} = \{\!\{\gamma_N^{BH} \circ \mathcal{D}^{BH}(k)\}\!\}, \\ W^{BH}(k) &= \{\!\{\gamma_T^{BH} \circ \mathcal{D}^{BH}(k)\}\!\} = \{\!\{\gamma_N^{BH} \circ \mathcal{S}^{BH}(k)\}\!\}. \end{aligned}$$

(This is reasonable, by applying the averaged trace $\{\!\{\gamma_T G\}\!\} = (\gamma_T G^+ + \gamma_T G^-)/2$ to the representation formula (A.1) and inserting the zero solution inside of Ω, yielding $\{\!\{\gamma_T u\}\!\} = E^{BH}/2$).

The next reformulation is from Scroggs et al. from [144]. The Calderon operator is not as symmetric as before, therefore the formulation better fits to the first order Maxwell formulation. Nevertheless a scaling with the material parameters is necessary to arrive at the first order system we consider in Section 2.4.1. However, only the wavelength k is necessary to evaluate the Calderon operator, and not both of the material parameters ε_0 and μ_0. This is how the Calderon operator is implemented in Bempp 3.4.3. The formulation differs by a factor i and $-i$ in the definition of γ_N and $\mathcal{S}$, respectively in comparison to the operators from (BH).

Scroggs et al. (SA) In [144] the tangential trace and the Neumann trace are introduced, respectively as

$$\begin{aligned} \gamma_T^{SA} u &= u \times n, \\ \gamma_N^{SA} u &= \frac{1}{ik} \gamma_T (\nabla \times u) \end{aligned}$$

and for given E^{SA} and H^{SA} the following second order formulation is considered for $k > 0$

$$\begin{aligned} \nabla \times \nabla \times u - k^2 u &= 0 && \text{in } \overline{\Omega}^c, \\ \gamma_T^{SA} u &= E^{SA} && \text{on } \mathit{\Gamma}, \\ \gamma_N^{SA} u &= H^{SA} && \text{on } \mathit{\Gamma}. \end{aligned}$$

By defining $v := \frac{1}{ik} \nabla \times u$, it is equivalent to solve

$$\begin{aligned} iku + \nabla \times v &= 0, \\ -ikv + \nabla \times u &= 0, \\ \gamma_T^{SA} u &= E^{SA}, \\ \gamma_T^{SA} v &= H^{SA}. \end{aligned}$$

If E^{SA}, H^{SA} are suitable exterior data, the solution is given by

$$\begin{aligned} u &= -\mathcal{D}^{SA}(k) E^{SA} - \mathcal{S}^{SA}(k) H^{SA}, \\ v &= \mathcal{S}^{SA}(k) E^{SA} - \mathcal{D}^{SA}(k) H^{SA}, \end{aligned} \tag{A.2}$$

(Note that in view of (A.3), the first line of (A.2) corresponds to the first line of (A.1) and for the second line, changing the roles of u and v (i.e. $v, -u$ solve a similar system) yields the consistency of the formula) with the electric single layer potential

$$\left(\mathcal{S}^{SA}(k)\varphi\right)(x) := ik \int_{\mathit{\Gamma}} G^{SA}(k, x - y)\varphi(y)\mathrm{d}y - \frac{1}{ik} \nabla \int_{\mathit{\Gamma}} G^{SA}(k, x - y)\mathrm{div}_{\mathit{\Gamma}} \varphi(y)\mathrm{d}y$$

and the electric double layer potential

$$\left(\mathcal{D}^{SA}(k)\varphi\right)(x) = \operatorname{curl} \int_\Gamma G^{SA}(k, x-y)\varphi(y) \, \mathrm{d}y,$$

where the fundamental solution $G^{SA}(k,z)$ is given as

$$G^{SA}(k,z) = \frac{e^{ik|z|}}{4\pi|z|}.$$

The boundary data E^{SA}, H^{SA} is suitable exterior data if and only if

$$\left(\frac{1}{2} - B^{SA}(k)\right)\begin{pmatrix} E^{SA} \\ H^{SA} \end{pmatrix} = \begin{pmatrix} E^{SA} \\ H^{SA} \end{pmatrix}$$

(This is reasonable, by applying the averaged trace $\{\!\{\gamma_T \cdot\}\!\}$ to (A.2) and inserting the zero solution inside of Ω). Here the Calderon operator is defined as

$$B^{SA}(k) = \begin{pmatrix} W^{SA}(k) & V^{SA}(k) \\ -V^{SA}(k) & W^{SA}(k) \end{pmatrix},$$

where (the second equalities are reasonable by noticing $\gamma_N u = \gamma_T v$)

$$\begin{aligned} V^{SA}(k) &= \{\!\{\gamma_T^{SA} \circ \mathcal{S}^{SA}(k)\}\!\} = -\{\!\{\gamma_N^{SA} \circ \mathcal{D}^{SA}(k)\}\!\}, \\ W^{SA}(k) &= \{\!\{\gamma_T^{SA} \circ \mathcal{D}^{SA}(k)\}\!\} = \{\!\{\gamma_N^{SA} \circ \mathcal{S}^{SA}(k)\}\!\}. \end{aligned}$$

For an better overview, we express the different ways of definition with each other. It holds

$$\begin{aligned} \gamma_T^{SA} u &= \gamma_T^{BH} u, \\ \gamma_N^{SA} u &= -i\gamma_N^{BH} u, \\ G^{SA}(k,z) &= G^{BH}(k,z), \\ \mathcal{S}^{SA}(k)\varphi &= i\mathcal{S}^{BH}(k)\varphi, \\ \mathcal{D}^{SA}(k)\varphi &= \mathcal{D}^{BH}(k)\varphi, \\ B^{SA}(k) &= \begin{pmatrix} W^{BH}(k) & iV^{BH}(k) \\ -iV^{BH}(k) & W^{BH}(k) \end{pmatrix}. \end{aligned} \tag{A.3}$$

The following reformulation is an intermediate step between the reformulations from Scroggs et al. in [144] and Kovács & Lubich in [99]. It fits to the first order formulation that arises from the Laplace transformed, first order and time dependent Maxwell equations, which we consider in Section 2.4.1. In contrast to (SA) and (BH) it is a modification by the factor $\sqrt{\frac{\mu_0}{\varepsilon_0}}$ and $i\sqrt{\frac{\mu_0}{\varepsilon_0}}$, respectively, which employs the difference in the material parameters μ_0 and ε_0.

Ballani et al. (BA) In [22] similar conventions to the following ones are used. Throughout the following lines it holds $k = i\sqrt{\varepsilon_0\mu_0}s$ and the tangential trace and the Neumann trace are defined, respectively as

$$\begin{aligned} \gamma_T^{BA} u &= u \times n, \\ \gamma_N^{BA} u &= \frac{1}{ik}\sqrt{\frac{\varepsilon_0}{\mu_0}}\gamma_T(\nabla \times u) = \frac{-1}{\mu_0 s}\gamma_T(\nabla \times u). \end{aligned}$$

For $k > 0$ the second order formulation is considered

$$\begin{aligned} \nabla \times \nabla \times u - k^2 u = 0 \quad & \text{in } \overline{\Omega}^c, \\ \gamma_T^{BA} u = E^{BA} \quad & \text{on } \Gamma, \\ \gamma_N^{BA} u = H^{BA} \quad & \text{on } \Gamma. \end{aligned}$$

By defining $v := \frac{1}{ik}\sqrt{\frac{\varepsilon_0}{\mu_0}}\nabla \times u$, it is equivalent to solve

$$\begin{aligned} -s\varepsilon_0 u + \nabla \times v &= 0, \\ s\mu_0 v + \nabla \times u &= 0, \\ \gamma_T^{BA} u &= E^{BA}, \\ \gamma_T^{BA} v &= H^{BA}. \end{aligned}$$

If E^{BA}, H^{BA} are suitable exterior data, the solution is given by

$$\begin{aligned} u &= -\mathcal{D}^{BA}(k)E^{BA} - \mathcal{S}^{BA}(k)H^{BA}, \\ v &= \frac{\varepsilon_0}{\mu_0}\mathcal{S}^{BA}(k)E^{BA} - \mathcal{D}^{BA}(k)H^{BA} \end{aligned} \tag{A.4}$$

(Note that in view of (A.5), the first line of (A.4) corresponds to the first line of (A.1) and changing the roles of u and v (i.e. $v, -\varepsilon_0 u/\mu_0$ solve a similar system) yields the consistency of the second line.) with the electric single layer potential

$$\begin{aligned} \left(\mathcal{S}^{BA}(k)\varphi\right)(x) &= ik\sqrt{\frac{\mu_0}{\varepsilon_0}}\int_\Gamma G^{BA}(s, x-y)\varphi(y)\mathrm{d}y \\ &\quad - \frac{1}{ik}\sqrt{\frac{\mu_0}{\varepsilon_0}}\nabla\int_\Gamma G^{BA}(s, x-y)\operatorname{div}_\Gamma\varphi(y)\mathrm{d}y \\ &= -\mu_0 s\int_\Gamma G^{BA}(s, x-y)\varphi(y)\mathrm{d}y + \frac{1}{\epsilon s}\nabla\int_\Gamma G^{BA}(s, x-y)\operatorname{div}_\Gamma\varphi(y)\mathrm{d}y \end{aligned}$$

and the electric double layer potential

$$\left(\mathcal{D}^{BA}(k)\varphi\right)(x) = \operatorname{curl}\int_\Gamma G^{BA}(s, x-y)\varphi(y)\,\mathrm{d}y,$$

where the fundamental solution $G^{BA}(s, z)$ is given as

$$G^{BA}(s, z) = \frac{e^{-s\sqrt{\varepsilon_0\mu_0}|z|}}{4\pi|z|}.$$

The boundary data E^{BA}, H^{BA} is suitable exterior data if and only if (apply $\{\!\{\gamma_T\cdot\}\!\}$ to (A.4))

$$\left(\frac{1}{2} - B^{BA}(k)\right)\begin{pmatrix} E^{BA} \\ H^{BA} \end{pmatrix} = \begin{pmatrix} E^{BA} \\ H^{BA} \end{pmatrix}.$$

Here the Calderon operator is defined as

$$B^{BA}(k) = \begin{pmatrix} W^{BA}(k) & V^{BA}(k) \\ -\frac{\varepsilon_0}{\mu_0}V^{BA}(k) & W^{BA}(k) \end{pmatrix},$$

where (the second equalities are reasonable due to $\gamma_N u = \gamma_T v$)

$$\begin{aligned} V^{BA}(k) &= \{\!\{\gamma_T^{BA} \circ \mathcal{S}^{BA}(k)\}\!\} = -\frac{\mu_0}{\varepsilon_0}\{\!\{\gamma_N^{BA} \circ \mathcal{D}^{BA}(k)\}\!\}, \\ W^{BA}(k) &= \{\!\{\gamma_T^{BA} \circ \mathcal{D}^{BA}(k)\}\!\} = \{\!\{\gamma_N^{BA} \circ \mathcal{S}^{BA}(k)\}\!\}, \end{aligned}$$

For an better overview, we express the different ways of definition with each other. It holds

$$\begin{aligned}
\gamma_T^{BA} u &= \gamma_T^{SA} u = \gamma_T^{BH} u, \\
\gamma_N^{BA} u &= \sqrt{\frac{\varepsilon_0}{\mu_0}} \gamma_N^{SA} u = -i \sqrt{\frac{\varepsilon_0}{\mu_0}} \gamma_N^{BH} u, \\
G^{BA}(s,z) &= G^{SA}(k,z) = G^{BH}(k,z), \\
\mathcal{S}^{BA}(k)\varphi &= \sqrt{\frac{\mu_0}{\varepsilon_0}} \mathcal{S}^{SA}(k)\varphi = i \sqrt{\frac{\mu_0}{\varepsilon_0}} \mathcal{S}^{BH}(k)\varphi, \\
\mathcal{D}^{BA}(k)\varphi &= \mathcal{D}^{SA}(k)\varphi = \mathcal{D}^{BH}(k)\varphi, \\
B^{BA}(k) &= \begin{pmatrix} W^{SA}(k) & \sqrt{\frac{\mu_0}{\varepsilon_0}} V^{SA}(k) \\ -\sqrt{\frac{\varepsilon_0}{\mu_0}} V^{SA}(k) & W^{SA}(k) \end{pmatrix} = \begin{pmatrix} W^{BH}(k) & i\sqrt{\frac{\mu_0}{\varepsilon_0}} V^{BH}(k) \\ -i\sqrt{\frac{\varepsilon_0}{\mu_0}} V^{BH}(k) & W^{BH}(k) \end{pmatrix}.
\end{aligned} \tag{A.5}$$

The next reformulation is used by Kovács & Lubich in [99]. In the Chapters 2–5 we rely on that formulation, as the Calderon operator has the coercivity property stated in Lemma 2.12. The operators are comparable to the ones from (BA), adapted to the formulation for the trace variables $\psi = -\gamma_T E$ and $\varphi = \mu_0 \gamma_T H$, which yields the important coercivity property. The following formulations are based on the first version of [99], before the Erratum [125], but are without the sign error noticed in [125]. However, the operators here slightly differ from the ones introduced in [125], but the overall representation formulae coincide. As described in [125], the theory of [99] remains true without restriction and the following formulas even hold without the scaling of time proposed in [125], so for possibly $\varepsilon_0 \mu_0 \neq 1$.

Kovács & Lubich (KL) In [99] the tangential trace and Neumann trace operators, respectively are defined for $k = i\sqrt{\varepsilon_0 \mu_0} s$ as

$$\begin{aligned}
\gamma_T^{KL} u &= u \times n, \\
\gamma_N^{KL} u &= \frac{-1}{ik} \sqrt{\varepsilon_0 \mu_0} \gamma_T (\nabla \times u) = \frac{1}{s} \gamma_T (\nabla \times u)
\end{aligned}$$

and for $\Re s > 0$ the following problem is considered

$$\begin{aligned}
\times \nabla \times u + s^2 \varepsilon_0 \mu_0 u &= 0 && \text{in } \overline{\Omega}^c, \\
\gamma_T^{KL} u &= E^{KL} && \text{on } \mathit{\Gamma}, \\
\gamma_N^{KL} u &= -\mu_0 H^{KL} && \text{on } \mathit{\Gamma}.
\end{aligned}$$

By defining $v := -\frac{1}{ik}\sqrt{\varepsilon_0\mu_0} \nabla \times u = \frac{1}{s} \nabla \times u$, this is equivalent to solve

$$\begin{aligned}
s\mu_0 \varepsilon_0 u + \nabla \times v &= 0, \\
-sv + \nabla \times u &= 0, \\
\gamma_T^{KL} u &= E^{KL}, \\
\gamma_T^{KL} v &= -\mu_0 H^{KL}.
\end{aligned}$$

As the boundary data is scaled, this is again the same system as in (BA), if we define $\widetilde{v} = (-\mu_0)^{-1} v$, we arrive at

$$\begin{aligned}
-s\varepsilon_0 u + \nabla \times \widetilde{v} &= 0, \\
s\mu_0 \widetilde{v} + \nabla \times u &= 0, \\
\gamma_T^{KL} u &= E^{KL}, \\
\gamma_T^{KL} \widetilde{v} &= H^{KL}.
\end{aligned}$$

If E^{KL}, H^{KL} are suitable, the solution is given by

$$\begin{aligned} u &= -\mathcal{D}^{KL}(s)(E^{KL}) - \mathcal{S}^{KL}(s)(-\mu_0 H^{KL}), \\ v &= \varepsilon_0\mu_0\mathcal{S}^{KL}(s)(E^{KL}) - \mathcal{D}^{KL}(s)(-\mu_0 H^{KL}), \\ \widetilde{v} &= -\mathcal{S}^{KL}(s)(\varepsilon_0 E^{KL}) - \mathcal{D}^{KL}(s)(H^{KL}), \end{aligned} \tag{A.6}$$

with the electric single layer potential

$$\begin{aligned} \left(\mathcal{S}^{KL}(s)\varphi\right)(x) &= s\int_\Gamma G^{KL}(s, x-y)\varphi(y)\mathrm{d}y - s^{-1}\frac{1}{\varepsilon_0\mu_0}\nabla\int_\Gamma G^{KL}(s, x-y)\mathrm{div}_\Gamma\varphi(y)\mathrm{d}y \\ &= \frac{-ik}{\sqrt{\varepsilon_0\mu_0}}\int_\Gamma G^{KL}(s, x-y)\varphi(y)\mathrm{d}y \\ &\quad + \frac{1}{ik\sqrt{\varepsilon_0\mu_0}}\nabla\int_\Gamma G^{KL}(s, x-y)\mathrm{div}_\Gamma\varphi(y)\mathrm{d}y \end{aligned}$$

and the electric double layer potential

$$\left(\mathcal{D}^{KL}(s)\varphi\right)(x) = \mathrm{curl}\int_\Gamma G^{KL}(s, x-y)\varphi(y)\ \mathrm{d}y,$$

where the fundamental solution $G^{KL}(s, z)$ is given as

$$G^{KL}(s, z) = \frac{e^{-s\sqrt{\varepsilon_0\mu_0}|z|}}{4\pi|z|}.$$

The boundary data $E^{KL}, -\mu_0 H^{KL}$ is suitable exterior data if and only if

$$\left(\frac{1}{2} - \widehat{B}^{KL}(s)\right)\begin{pmatrix} E^{KL} \\ -\mu_0 H^{KL} \end{pmatrix} = \begin{pmatrix} E^{KL} \\ -\mu_0 H^{KL} \end{pmatrix}.$$

Here the Calderon operator is defined as

$$\widehat{B}^{KL}(s) = \begin{pmatrix} W^{KL}(s) & V^{KL}(s) \\ -\varepsilon_0\mu_0 V^{KL}(s) & W^{KL}(s) \end{pmatrix},$$

where

$$\begin{aligned} V^{KL}(s) &= \{\!\{\gamma_T^{KL}\circ\mathcal{S}^{KL}(s)\}\!\} = -(\varepsilon_0\mu_0)^{-1}\{\!\{\gamma_N^{KL}\circ\mathcal{D}^{KL}(s)\}\!\}, \\ W^{KL}(s) &= \{\!\{\gamma_T^{KL}\circ\mathcal{D}^{KL}(s)\}\!\} = \{\!\{\gamma_N^{KL}\circ\mathcal{S}^{KL}(s)\}\!\}. \end{aligned}$$

To obtain the desired coercivity, the suitability condition is rewritten as

$$B^{KL}(s)\begin{pmatrix} \mu_0 H^{KL} \\ -E^{KL} \end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix} E^{KL} \\ \mu_0 H^{KL} \end{pmatrix},$$

where

$$B^{KL}(s) = \frac{1}{\mu_0}\begin{pmatrix} V^{KL}(s) & W^{KL}(s) \\ -W^{KL}(s) & \varepsilon_0\mu_0 V^{KL}(s) \end{pmatrix}.$$

For an better overview, we express the different ways of definition with each other. It holds

$$
\begin{aligned}
\gamma_T^{KL} u &= \gamma_T^{BA} u = \gamma_T^{SA} u = \gamma_T^{BH} u,\\
\gamma_N^{KL} u &= -\mu_0 \gamma_N^{BA} u = -\sqrt{\varepsilon_0\mu_0}\gamma_N^{BH} u = i\sqrt{\varepsilon_0\mu_0}\gamma_N^{BH} u\\
G^{KL}(k,z) &= G^{BA}(s,z) = G^{SA}(k,z) = G^{BH}(k,z),\\
\mathcal{S}^{KL}(s)\varphi &= (-\mu_0)^{-1}\mathcal{S}^{BA}(k)\varphi = (-\sqrt{\varepsilon_0\mu_0})^{-1}\mathcal{S}^{SA}(k)\varphi = (i\sqrt{\varepsilon_0\mu_0})^{-1}\mathcal{S}^{BH}(k)\varphi,\\
\mathcal{D}^{KL}(s)\varphi &= \mathcal{D}^{BA}(k)\varphi = \mathcal{D}^{SA}(k)\varphi = \mathcal{D}^{BH}(k)\varphi,\\
\widehat{B}^{KL}(s) &= \begin{pmatrix} W^{BA}(k) & \frac{-1}{\mu_0}V^{BA}(k) \\ \varepsilon_0 V^{BA}(k) & W^{BA}(k) \end{pmatrix} = \begin{pmatrix} W^{SA}(k) & \frac{-1}{\sqrt{\varepsilon_0\mu_0}}V^{SA}(k) \\ \sqrt{\mu_0\varepsilon_0}V^{SA}(k) & W^{SA}(k) \end{pmatrix}\\
&= \begin{pmatrix} W^{BH}(k) & \frac{-i}{\sqrt{\varepsilon_0\mu_0}}V^{BH}(k) \\ i\sqrt{\varepsilon_0\mu_0}V^{BH}(k) & W^{BH}(k) \end{pmatrix},\\
B^{KL}(s) &= \frac{1}{\mu_0}\begin{pmatrix} \frac{-1}{\mu_0}V^{BA}(k) & W^{BA}(k) \\ -W^{BA}(k) & -\varepsilon_0 V^{BA}(k) \end{pmatrix} = \frac{1}{\mu_0}\begin{pmatrix} \frac{-1}{\sqrt{\varepsilon_0\mu_0}}V^{SA}(k) & W^{SA}(k) \\ -W^{SA}(k) & -\sqrt{\varepsilon_0\mu_0}V^{SA}(k) \end{pmatrix}\\
&= \frac{1}{\mu_0}\begin{pmatrix} \frac{-i}{\sqrt{\varepsilon_0\mu_0}}V^{BH}(k) & W^{BH}(k) \\ -W^{BH}(k) & -i\sqrt{\varepsilon_0\mu_0}V^{BH}(k) \end{pmatrix}.
\end{aligned}
\tag{A.7}
$$

Remark A.5. *In the implementation details from Section 6.1.6, it holds*

$$
\widehat{B}(k) = \begin{pmatrix} \widehat{D} & \widehat{E} \\ \widehat{F} & \widehat{G} \end{pmatrix} = B^{SA}(k) = \begin{pmatrix} W^{SA}(k) & V^{SA}(k) \\ -V^{SA}(k) & W^{SA}(k) \end{pmatrix},
$$

so the assertions there (and especially the signs) are correct which can be seen from comparing (A.7) *and* (6.4).

B. The Laplace Transform

In this chapter we introduce the Laplace transform. We consider the scalar valued case in Section B.1 for functions $u(t) \in \mathbb{R}$ and then generalize the results to vector valued functions $u(t) \in X$ for an Hilbert space X in Section B.2. Although many of the results are well-known, most of the proofs can't be found in the literature. Similar results without prove are given in [115, Section 2.1], in [143, 102] a related setting for vector valued distributions is considered.

B.1. Scalar Valued Laplace Transform and Differential Operators

B.1.1. Laplace transform on $[0, \infty)$

The Laplace transform of a function $u\colon [0,\infty) \to \mathbb{R}$ is defined as

$$(\mathcal{L}u)(s) := \int_0^\infty u(t)e^{-st}\,\mathrm{d}t \quad \text{for } s \in \mathbb{C} \tag{B.1}$$

and the inverse Laplace transform for $U\colon \{\Re(s) > \sigma_0\} \to \mathbb{C}$ as

$$(\mathcal{L}^{-1}U)(t) := \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} e^{st}U(s)\,\mathrm{d}s \quad \text{for } t \in [0,\infty) \tag{B.2}$$

for a $\sigma > \sigma_0$. We see that the inverse Laplace transform is a priori not uniquely defined. It turns out that the choice of σ does not matter for certain function classes and hence the definition is valid. We require the following well-known property of the Fourier transform

Theorem B.1 ([140, Chapter 9])**.** *The Fourier transform*

$$\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R}) \cap C(\mathbb{R}), (\mathcal{F}f)(x) := \int_{\mathbb{R}} f(\xi)e^{-ix\xi}\,\mathrm{d}\xi \tag{B.3}$$

can be extended to a continuous and continuously invertible operator

$$\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

The inverse operator is given as the extension of the inverse Fourier transform

$$\mathcal{F}^{-1} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R}) \cap C(\mathbb{R}), (\mathcal{F}^{-1}f)(x) := \frac{1}{2\pi}\int_{\mathbb{R}} f(\xi)e^{ix\xi}\,\mathrm{d}\xi.$$

Proof. By Plancherel's formula for the Fourier transform, it holds for all $\phi \in C_0^\infty(\mathbb{R})$

$$\|\mathcal{F}\phi\|_{\mathbb{R}}^2 = 2\pi\|\phi\|_{\mathbb{R}}^2.$$

Now the Fourier transform can be defined for $f \in L^2(\mathbb{R})$ by a density argument as

$$\mathcal{F}f := L^2(\mathbb{R}) - \lim_{r\to\infty}\int_{|x|\le r} f(\xi)e^{-ix\xi}\,\mathrm{d}\xi,$$

where the convergence is understood in $L^2(\mathbb{R})$. For $f \in L^1(\mathbb{R})$, this definition coincides with (B.3). Similar statements hold for the inverse Fourier transform. The proof of

$$\mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = \mathrm{Id}_{L^2(\mathbb{R})\to L^2(\mathbb{R})}$$

can be found in [140, Theorem 9.13]. □

The similarities between the Laplace transform $\mathcal{L}$ and the Fourier transform $\mathcal{F}$ are expressed in the identity

$$(\mathcal{L}u)(\sigma + i\tau) = \mathcal{F}(u(\cdot)e^{-\sigma\cdot})(\tau) \quad \text{for all } \sigma, \tau \in \mathbb{R},$$

where we extended u by zero on $(-\infty, 0)$. This allows us to define a useful domain of definition for the Laplace transform.

Definition B.2. *For $c \in \mathbb{R}$, we consider the space*

$$L^2_c[0,\infty) := \{u\colon [0,\infty) \to \mathbb{R} \text{ measurable} \,|\, (x \mapsto e^{-cx}u(x)) \in L^2[0,\infty)\}$$

equipped with the norm

$$\|u\|_{L^2_c[0,\infty)} := \|e^{-c\cdot}u\|_{[0,\infty)}.$$

For $u \in L^2_c[0,\infty)$, we redefine the Laplace transform

$$\mathcal{L}u(s) := \mathcal{F}(x \mapsto u(x)\mathbb{1}_{[0,\infty)}(x)e^{-\Re(s)x})(\Im(s)). \tag{B.4}$$

for all $s \in \mathbb{C}$ with $\Re(s) \geq c$.

Furthermore, we define

$$L^2_*[0,\infty) := \bigcup_{c\in\mathbb{R}} L^2_c[0,\infty)$$

and observe that $L^2_c[0,\infty) \subset L^2_{c'}[0,\infty)$ for all $c' \geq c$.

Remark B.3. *The function $U := \mathcal{L}u$ is well-defined on $\{\Re s \geq c\}$ and thus depends on the growth of $u(t)$ for $t \to \infty$. Even exponential growth is admissible.*

For $\Re s > c$, it is $e^{-\Re s\,\cdot}\,u(\,\cdot\,) \in L^1[0,\infty)$ and formula (B.4) *can be interpreted in the classical sense* (B.3)*, yielding a continuous function on each vertical line in* (B.1)*.*

Theorem B.4 (Plancherel's formula)**.** *It holds for $u, v \in L^2_c[0,\infty)$ and for all $\sigma \geq c$*

$$\int_0^\infty e^{-2\sigma t}\overline{u(t)}v(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{\sigma+i\mathbb{R}} \overline{\mathcal{L}u(s)}\mathcal{L}v(s)\,\mathrm{d}s,$$

especially we have

$$\|\mathcal{L}u\|^2_{\sigma+i\mathbb{R}} = 2\pi\|u\|^2_{L^2_\sigma[0,\infty)}.$$

Proof. This is a direct consequence of Plancherels formula for the Fourier Transform which gives for $u, v \in L^2(\mathbb{R})$

$$[\mathcal{F}u, \mathcal{F}v]_{\mathbb{R}} = 2\pi[u,v]_{\mathbb{R}}.$$

□

We are not able to establish boundedness

$$\mathcal{L} : L^2_*[0,\infty) \to \operatorname{Im}(\mathcal{L}).$$

However, for $u \in L^2_c[0,\infty)$, Plancherel's formula yields for $\sigma \geq c$

$$\|\mathcal{L}u\|_{\sigma+i\mathbb{R}} = \sqrt{2\pi}\|x \mapsto e^{-\sigma x}u(x)\|_{[0,\infty)}$$

and it holds

$$\|x \mapsto e^{-cx}u(x)\|_{[0,\infty)} = \sup_{\sigma>c}\|x \mapsto e^{-\sigma x}u(x)\|_{[0,\infty)}.$$

Thus, for arbitrary, but fixed $c \in \mathbb{R}$

$$\mathcal{L} : (L^2_c[0,\infty), \|\cdot\|_{L^2_c[0,\infty)}) \to (\operatorname{Im}(\mathcal{L}), \|\cdot\|_{\operatorname{Im}(\mathcal{L})})$$

is bounded and with bounded inverse on its image $\operatorname{Im}(\mathcal{L})$, which we will determine in the following. A suitable choice of $\|\cdot\|_{\operatorname{Im}(\mathcal{L})}$ is

$$\|\cdot\|_{\operatorname{Im}(\mathcal{L})} := \sup_{\sigma>c}\|\cdot\|_{\sigma+i\mathbb{R}}.$$

Lemma B.5. *For $u \in L^2_c[0,\infty)$, $U := \mathcal{L}u$ is analytic on its domain.*

Proof. For $z \in \mathbb{C}$ it holds

$$\left|\frac{e^z - 1}{z}\right| = \left|\int_0^1 e^{z\omega}\,\mathrm{d}\omega\right| \leq \int_0^1 \max(e^{\Re z}, 1)\,\mathrm{d}\omega = e^{\max(0,\Re z)}. \tag{B.5}$$

For $\Re s_2 > c$, $\delta > 0$ such that $\Re s_2 > c + 2\delta$, and for all $s_1 \in B_\delta(s_2)$ ($B_\delta(s_2)$ denoting the open ball with radius δ around s_2) we have by the Cauchy Schwartz estimate and (B.5)

$$\begin{aligned}
|U(s_1) - U(s_2)| &\leq \int_0^\infty |u(t)| e^{-ct} e^{ct} |e^{-s_1 t} - e^{-s_2 t}|\,\mathrm{d}t \\
&\leq \left(\int_0^\infty |u(t)|^2 e^{-2ct}\,\mathrm{d}t\right)^{1/2} \left(\int_0^\infty e^{2(c-\Re(s_1))t} |1 - e^{(s_1-s_2)t}|^2\,\mathrm{d}t\right)^{1/2} \\
&\leq \|e^{-c\cdot} u(\cdot)\|_{[0,\infty)} |s_1 - s_2| \left(\int_0^\infty e^{2\max(c-\Re(s_1), c-\Re(s_2))t} t^2\,\mathrm{d}t\right)^{1/2} \\
&\leq C(u, c, \delta) |s_1 - s_2|,
\end{aligned}$$

since $\max(c - \Re(s_1), c - \Re(s_2)) < \delta$ by definition. Therefore an integrable majorant for the integrand of $|U(s_1) - U(s_2)|/|s_1 - s_2|$ exists, we can interchange limit and integral and see that $\partial_s U(s) = \mathcal{L}(-tu(t))$ exists. Thus U has a complex derivative and is therefore analytic, see [140, Chapter 10]. □

Lemma B.6. *For $u \in L^2_c[0,\infty)$, the L^2-norm over each vertical line of $U = \mathcal{L}u$ is uniformly bounded for $\sigma \geq c$ and it holds*

$$\sup_{\sigma > c} \|U\|_{\sigma + i\mathbb{R}} = \|U\|_{c + i\mathbb{R}} < \infty.$$

Proof. By Plancherel's formula and the theorem of monotone convergence we have

$$\begin{aligned}
\sup_{\sigma > c} \|x \mapsto U(\sigma + ix)\|^2_{\mathbb{R}} = \sup_{\sigma > c} \int_{\sigma + i\mathbb{R}} |U(s)|^2\,\mathrm{d}s &= 2\pi \sup_{\sigma > c} \|x \mapsto e^{-\sigma x} u(x)\|^2_{[0,\infty)} \\
&= 2\pi \|x \mapsto e^{-cx} u(x)\|^2_{[0,\infty)}.
\end{aligned}$$

Another application of Plancherel's formula concludes

$$\sup_{\sigma > c} \|x \mapsto U(\sigma + ix)\|^2_{\mathbb{R}} = 2\pi \|x \mapsto e^{-cx} u(x)\|^2_{[0,\infty)} = \|U(c + i\cdot)\|^2_{\mathbb{R}}.$$

□

The properties of $U := \mathcal{L}u$ shown in Lemma B.5 and Lemma B.6 are already enough to characterize the image of the Laplace transform.

Theorem B.7 (Paley–Wiener theorem, cf. [128, Theorem V.], [140, Theorem 19.2])**.** *If $U : s \mapsto U(s)$ is analytic for $\Re(s) > \sigma_0$ and*

$$\sup_{\widetilde{\sigma} > \sigma_0} \int_{\widetilde{\sigma} + i\mathbb{R}} |U(s)|^2\,\mathrm{d}s < \infty,$$

then there exists a function $u \in L^2_*[0,\infty)$, such that $U = \mathcal{L}u$.

The space of the functions with these properties – to be analytic and uniformly square integrable over each vertical line for high enough real part – is the right choice as domain of the inverse Laplace transform. By transformation of variables $s = \sigma + i\xi$, we can express the inverse Laplace transform in terms of the Fourier transform

$$(\mathcal{L}^{-1}U)(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{(\sigma + i\xi)t} U(\sigma + i\xi) i\,\mathrm{d}\xi = e^{\sigma t} \mathcal{F}^{-1}(U(\sigma + i\cdot))(t).$$

Definition B.8. *For $\sigma_0 \in \mathbb{R}$, we define the Hardy space*

$$\mathcal{H}(\sigma_0) := \mathcal{H}^2(\sigma_0) := \{B \mid B(s) : \{\Re s > \sigma_0\} \to \mathbb{C} \text{ is analytic} \\ \text{and } \sup_{\sigma > \sigma_0} \int_{\sigma + i\mathbb{R}} |U(s)|^2 \, \mathrm{d}s < \infty\},$$

equipped with the norm

$$\|u\|^2_{\mathcal{H}^2(\sigma_0)} := \sup_{\sigma > \sigma_0} \int_{\sigma + i\mathbb{R}} |U(s)|^2 \, \mathrm{d}s.$$

We define the inverse Laplace transform for $U \in \mathcal{H}(\sigma_0)$ as

$$(\mathcal{L}^{-1}U)(t) := e^{\sigma t} \mathcal{F}^{-1}(U(\sigma + i \cdot))(t)$$

for a $\sigma > \sigma_0$. We collect the Laplace invertible functions

$$\mathcal{H} := \bigcup_{\sigma_0 \in \mathbb{R}} \mathcal{H}(\sigma_0) = \{B \,|\, \text{For a } \sigma_0 \in \mathbb{R},\ B(s) : \{\Re s > \sigma_0\} \to \mathbb{C} \text{ is analytic} \\ \text{and } \sup_{\sigma > \sigma_0} \int_{\sigma + i\mathbb{R}} |U(s)|^2 \, \mathrm{d}s < \infty\}.$$

We summarize the properties and the welldefinedness of the inverse Laplace transform. Especially we have a one-to-one identity through the Laplace transform between $L^2_*[0,\infty)$ and $\mathcal{H}$ and between $L^2_c[0,\infty)$ and $\mathcal{H}(c)$.

Theorem B.9. *For $\sigma_0 \in \mathbb{R}$ and $U \in \mathcal{H}(\sigma_0)$, there exists a unique $u \in L^2_*(\mathbb{R}_+)$, such that $U = \mathcal{L}u$. It holds $u \in L^2_{\sigma_0}[0,\infty)$ and*

$$\mathcal{L}^{-1}U = u.$$

The function U can be extended by the $L^2 - \lim_{\sigma \to \sigma_0}$ to the vertical line $\{\Re s = \sigma_0\}$ and $U|_{\{\Re s = \sigma_0\}} \in L^2(\sigma_0 + i\mathbb{R})$. There holds

$$2\pi \|e^{-\sigma_0 \cdot} u\|^2_{[0,\infty)} = \sup_{\widetilde{\sigma} > \sigma_0} \int_{\widetilde{\sigma} + i\mathbb{R}} |U(s)|^2 \, \mathrm{d}s = \int_{\sigma_0 + i\mathbb{R}} |U(s)|^2 \, \mathrm{d}s.$$

Proof. Existence follows from Theorem B.7 and Uniqueness of $u \in L^2_*[0,\infty)$ can be deduced with Plancherel's formula: Let $u_1 \in L^2_{\sigma_1}[0,\infty)$, $u_2 \in L^2_{\sigma_2}[0,\infty)$, $\mathcal{L}u_1 = U$, $\mathcal{L}u_2 = U$, then it holds by Plancherel's formula and linearity of the Laplace transform

$$\begin{aligned} 0 &= \sup_{\widetilde{\sigma} > \sigma_0} \int_{\widetilde{\sigma} + i\mathbb{R}} |U(s) - U(s)|^2 \, \mathrm{d}s \\ &\geq \sup_{\widetilde{\sigma} > \max(\sigma_1, \sigma_2)} \int_{\widetilde{\sigma} + i\mathbb{R}} |U(s) - U(s)|^2 \, \mathrm{d}s \\ &= \sup_{\widetilde{\sigma} > \max(\sigma_1, \sigma_2)} 2\pi \|e^{-\widetilde{\sigma} \cdot} (u_1 - u_2)\|^2_{[0,\infty)} \\ &\geq 0. \end{aligned}$$

Thus $u_1 = u_2$. By Plancherel's formula and the monotone convergence theorem we deduce

$$\begin{aligned} \sup_{\widetilde{\sigma} > \sigma_0} \int_{\widetilde{\sigma} + i\mathbb{R}} |U(s)|^2 \, \mathrm{d}s &= \sup_{\widetilde{\sigma} > \sigma_0} 2\pi \|e^{-\widetilde{\sigma} \cdot} u\|^2_{[0,\infty)} \\ &= 2\pi \|e^{-\sigma_0 \cdot} u\|^2_{[0,\infty)}, \end{aligned}$$

i.e. $u \in L^2_{\sigma_0}[0,\infty)$. It holds for every $\sigma > \sigma_0$ and almost every $t \in \mathbb{R}$

$$\begin{aligned}(\mathcal{L}^{-1}U)(t) &= e^{\sigma t}\mathcal{F}^{-1}\big((\mathcal{L}u)(\sigma + i\,\cdot\,)\big)(t)\\ &= e^{\sigma t}\mathcal{F}^{-1}\Big(\big(s \mapsto \mathcal{F}(u(\,\cdot\,)\mathbb{1}_{[0,\infty)}(\,\cdot\,)e^{-\Re s\,\cdot\,})(\Im s)\big)(\sigma + i\,\cdot\,)\Big)(t)\\ &= e^{\sigma t}u(t)\mathbb{1}_{[0,\infty)}(t)e^{-\sigma t}\\ &= u(t)\mathbb{1}_{[0,\infty)}(t).\end{aligned}$$

As $U = \mathcal{L}u$ on $\{\Re s > \sigma_0\}$ and $u \in L^2_{\sigma_0}[0,\infty)$, by Lebesgue's theorem we have

$$\begin{aligned}\|U(\sigma_1 + i\,\cdot\,) - U(\sigma_0 + i\,\cdot\,)\|^2_{\mathbb{R}} &= 2\pi\int_0^\infty |(e^{-\sigma_1 t} - e^{-\sigma_0 t})u(t)|^2\,\mathrm{d}t\\ &\to 0 \quad \text{for } \sigma_1 \to \sigma_0,\end{aligned}$$

so

$$U(\sigma_0 + i\,\cdot\,) := L^2(\mathbb{R}) - \lim_{\sigma\to\sigma_0} U(\sigma + i\,\cdot\,)$$

exists and it holds

$$\begin{aligned}\sup_{\widetilde{\sigma}>\sigma_0}\int_{\widetilde{\sigma}+i\mathbb{R}} |U(s)|^2\,\mathrm{d}s &= \sup_{\widetilde{\sigma}>\sigma_0} 2\pi\|e^{-\widetilde{\sigma}\,\cdot}\,u\|^2_{[0,\infty)}\\ &= 2\pi\|e^{-\sigma_0\,\cdot}\,u\|^2_{[0,\infty)}\\ &= \int_{\sigma_0+i\mathbb{R}} |U(s)|^2\,\mathrm{d}s.\end{aligned}$$

□

B.1.2. Laplace differential operators on $[0,\infty)$

For a function $B : \{\Re s > \sigma_0\} \to \mathbb{C}$ it is natural to define the Laplace differential operator $B(\partial_t)f$ via the Laplace transform as $\mathcal{L}^{-1}(B(s)\mathcal{L}(f)(s))$.

We start with a first, most general definition and in the following we give sufficient conditions for the welldefinedness and state the resulting properties of the operators.

Definition B.10. *For a function $B(s) : \{\Re s > \sigma_0\} \to \mathbb{C}$ for a $\sigma_0 \in \mathbb{R}$ and $f \in L^2_*[0,\infty)$, such that*

$$B(s)\mathcal{L}f \in \mathcal{H}, \tag{B.6}$$

we define $B(\partial_t)f$ as

$$B(\partial_t)f := \mathcal{L}^{-1}(B(s)\mathcal{L}(f)(s)). \tag{B.7}$$

To determine the properties of $B(\partial_t)$, we require function spaces with homogeneous initial conditions.

Definition B.11. *We define for $m \in \mathbb{N}$ the spaces with homogeneous initial condition up to order $m-1$, i.e.,*

$$H^m_0[0,\infty) := \{\phi \in H^m[0,\infty) \mid \phi(0) = \cdots = \phi^{(m-1)}(0) = 0\}, \quad H^0_0[0,\infty) := L^2[0,\infty)$$

equipped with the $H^m[0,\infty)$-norm and the exponentially weighted spaces

$$H^m_*[0,\infty) := \{\phi \mid x \mapsto e^{-cx}\phi(x) \in H^m[0,\infty) \text{ for some } c \in \mathbb{R}\}, \quad H^0_*[0,\infty) := L^2_*[0,\infty),$$

the exponentially weighted spaces with zero condition at $t = 0$

$$H^m_{0,*}[0,\infty) := \{\phi \mid x \mapsto e^{-cx}\phi(x) \in H^m_0[0,\infty) \text{ for some } c \in \mathbb{R}\}, \quad H^0_{0,*}[0,\infty) := L^2_*[0,\infty).$$

Furthermore, we define for fixed damping parameter $c \in \mathbb{R}$

$$H_c^m[0,\infty) := \{\phi \mid x \mapsto e^{-cx}\phi(x) \in H^m[0,\infty)\}, \quad H_c^0[0,\infty) := L_c^2[0,\infty)$$

and

$$H_{0,c}^m[0,\infty) := \{\phi \mid x \mapsto e^{-cx}\phi(x) \in H_0^m[0,\infty)\}, \quad H_{0,c}^0[0,\infty) := L_c^2[0,\infty).$$

We equip these spaces with the norms

$$\|u\|_{H^m[0,\infty)} := \|u\|_{H_0^m[0,\infty)} := \left(\sum_{j=0}^m \|\partial_t^j u\|_{[0,\infty)}^2\right)^{1/2}$$

and

$$\|u\|_{H_c^m[0,\infty)} := \|u\|_{H_{0,c}^m[0,\infty)} := \|e^{-c\cdot}u\|_{H^m[0,\infty)}.$$

Note that functions in $H_0^m[0,\infty)$ may have infinite support (in comparison to $C_0^m[0,\infty)$, where we collect m times differentiable functions with finite support, see Definition B.16 below), the sub index 0 illustrates the zero condition at $t = 0$. There is a possible ambiguity of notation between H_c^m for $c = 0$ and H_0^m denoting the space with zero initial condition. This, however, will not be a problem in the following, as the variable c will never be substituted with particular values.

Remark B.12. *It is equivalent for* $c \in \mathbb{R}$

$$\begin{aligned} f \in H_c^m[0,\infty) &\Leftrightarrow e^{-c\cdot} f \in H^m[0,\infty) \\ &\Leftrightarrow e^{-c\cdot} f, \dots, e^{-c\cdot}\partial_t^m f \in L^2[0,\infty) \\ &\Leftrightarrow f, \dots, \partial_t^m f \in L_c^2[0,\infty). \end{aligned}$$

The norms $\|\cdot\|_{H_c^m[0,\infty)}$ *and* $\|\cdot\|_{H_{0,c}^m[0,\infty)}$ *are equivalent (depending on c) to the norm*

$$\|f\|^2 := \sum_{k=0}^m \|\partial_t^j f\|_{L_c^2[0,\infty)}^2.$$

Theorem B.13. *For* $c \in \mathbb{R}$ *and* $m \in \mathbb{N}$ *the spaces* $H^m[0,\infty)$, $H^m[0,\infty)$, $H_c^m[0,\infty)$, $H_{0,c}^m[0,\infty)$ *and* $\mathcal{H}(c)$ *are Hilbert spaces together with the scalar products*

$$\begin{aligned} [u,v]_{H^m[0,\infty)} &:= \sum_{j=0}^m [\partial_t^j u, \partial_t^j v]_{[0,\infty)}, \\ [u,v]_{H_0^m[0,\infty)} &:= \sum_{j=0}^m [\partial_t^j u, \partial_t^j v]_{[0,\infty)}, \\ [u,v]_{H_c^m[0,\infty)} &:= \sum_{j=0}^m [e^{-2c\cdot}\partial_t^j u, \partial_t^j v]_{[0,\infty)}, \\ [u,v]_{H_{0,c}^m[0,\infty)} &:= \sum_{j=0}^m [e^{-2c\cdot}\partial_t^j u, \partial_t^j v]_{[0,\infty)}, \\ [U,V]_{\mathcal{H}(c)} &:= \frac{1}{2\pi}[U,V]_{c+i\mathbb{R}}. \end{aligned}$$

Proof. The space $H^m[0,\infty)$ is a Hilbert space due to [140], and $H_0^m[0,\infty)$ is a Hilbert space, as the trace mapping is continuous.

The space $H_c^m[0,\infty)$ is a Prehilbert space, as the scalar product is positive definite, linear and symmetric and the induced norm is a norm. For the completeness, let v_n be

a Cauchy sequence in $H_c^m[0,\infty)$. Therefore $e^{-c\cdot}v_n$ is a Cauchy sequence in $H^m[0,\infty)$ and converges to $\widetilde{v} \in H^m[0,\infty)$. This is equivalent to the convergence of $v_n \to e^{c\cdot}\widetilde{v}$ in $H_c^m[0,\infty)$.
To show that the subspace $H_{0,c}^m[0,\infty) \subset H_c^m[0,\infty)$ is a Hilbert space, it remains to show completeness. For a sequence $v_n \to v$ in $H_{0,c}^m[0,\infty)$, it holds for $T > 0$

$$v_n|_{[0,T]} \to v|_{[0,T]} \text{ in } H^m[0,T].$$

As the trace mapping is continuous, it holds $v^{(j)} = 0$, $j = 0, \dots m-1$ and $v \in H_{0,c}^m[0,\infty)$. The space $\mathcal{H}(\sigma_0)$ is a Hilbert space by the one to one identity of the Laplace transform from $\mathcal{L} : L_c^2[0,\infty) \to \mathcal{H}(\sigma_0)$, which is an invertible isometry by Plancherel's formula. □

Example B.14. *a) For the operator $B(s) = s$, $f \in H_{0,*}^1[0,\infty)$, it holds*

$$s(\mathcal{L}f)(s) \in \mathcal{H}$$

and we have

$$B(\partial_t)f = \partial_t f.$$

Thus the Laplace differential operator ∂_t coincides with the weak derivative ∂_t, if f is weakly differentiable and $f(0) = 0$.

b) For the operator $B(s) = s^{-1}$, $f \in L_*^2[0,\infty)$ it holds

$$s^{-1}(\mathcal{L}f)(s) \in \mathcal{H}$$

and we have

$$B(\partial_t) = \partial_t^{-1} f := \int_0^t f(\tau)\, \mathrm{d}\tau.$$

Thus the Laplace differential operator ∂_t^{-1} coincides with the integration over time $\int_0^t \,\mathrm{d}\tau$.

The condition to be zero at $t = 0$ comes from the fact that we extend the functions to zero on $(-\infty, 0)$ and otherwise the function regarded on $(-\infty,\infty)$ would have a singularity at $t = 0$.

Proof. a) Let $\sigma_0 \in \mathbb{R}$ and $f \in H_{0,\sigma_0}^1[0,\infty)$. For $\Re s > \sigma_0$ it holds

$$\begin{aligned}
|f(t)e^{-st}| &= |f(t)e^{-\sigma_0 t} \cdot e^{-(s-\sigma_0)t}| \\
&\le \int_0^t |f'(r)e^{-\sigma_0 r} - \sigma_0 f(r)e^{-\sigma_0 r}|\, \mathrm{d}r \cdot e^{-(\Re s-\sigma_0)t} \\
&\le e^{-t(\Re s-\sigma_0)} t^{1/2} (\|f'\|_{L^2_{\sigma_0}[0,\infty)} + |\sigma_0| \|f\|_{L^2_{\sigma_0}[0,\infty)}) \\
&\to 0 \quad \text{for } t \to \infty.
\end{aligned}$$

Using this outcome and integration by parts we obtain

$$\begin{aligned}
\mathcal{L}f(s) &= \int_0^\infty f(t)e^{-st}\, \mathrm{d}t \\
&= -\int_0^\infty f'(t)e^{-st}(-s)^{-1}\, \mathrm{d}t + \left[f(t)e^{-st}(-s)^{-1}\right]_0^\infty \\
&= \frac{1}{s}\mathcal{L}f'(s).
\end{aligned}$$

Thus (the case $s = 0$ follows from $\sigma_0 < 0$ and $f(0) = \lim_{t\to\infty} f(t) = 0$)

$$s(\mathcal{L}f)(s) = \mathcal{L}f'(s) \in \mathcal{H}(\sigma_0)$$

and

$$\mathcal{L}^{-1}(s(\mathcal{L}f)) = \partial_t f.$$

b) Let $\sigma_0 \in \mathbb{R}$ and $f \in L^2_{\sigma_0}[0,\infty)$. It is for $\epsilon > 0$

$$s^{-1}(\mathcal{L}f)(s) \in \mathcal{H}(\max(\epsilon,\sigma_0)) \tag{B.8}$$

and if $\partial_t^{-1} f(t) := \int_0^t f(\tau)\,\mathrm{d}\tau \in H^1_{0,*}[0,\infty)$, the same calculations as in a) show

$$\mathcal{L}^{-1}(s\partial_t^{-1} f) = f,$$

i.e.

$$\mathcal{L}^{-1}(s^{-1}\mathcal{L}f)(t) = \int_0^t f(\tau)\,\mathrm{d}\tau.$$

As $\partial_t^{-1} f(0) = 0$ and $\partial_t \partial_t^{-1} f = f \in L^2_*[0,\infty)$, it remains to show $\partial_t^{-1} f \in L^2_*[0,\infty)$. By the estimate for $z \in \mathbb{C}$

$$\left|\frac{e^z - 1}{z}\right| = \left|\int_0^1 e^{z\omega}\,\mathrm{d}\omega\right| \leq \int_0^1 \max(e^{\Re z}, 1)\,\mathrm{d}\omega = e^{\max(0,\Re z)},$$

we have

$$\begin{aligned}\left|\int_0^t f(\tau)\,\mathrm{d}\tau\right|^2 &\leq \int_0^t e^{2\sigma_0\tau}\,\mathrm{d}\tau \int_0^t e^{-2\sigma_0\tau}|f(\tau)|^2\,\mathrm{d}\tau \\ &\leq \frac{e^{2\sigma_0 t} - 1}{2\sigma_0}\|f\|^2_{L^2_{\sigma_0}[0,\infty)} \\ &\leq te^{\max(0,2\sigma_0 t)}\|f\|^2_{L^2_{\sigma_0}[0,\infty)}.\end{aligned}$$

Thus for every $\epsilon > 0$

$$\begin{aligned}\|\partial_t^{-1} f\|^2_{L^2_{\max(0,\sigma_0)+\epsilon}[0,\infty)} &= \int_0^\infty e^{-\max(0,2\sigma_0 t) - 2\epsilon t}|\partial_t^{-1} f|^2\,\mathrm{d}t \\ &\leq \int_0^\infty te^{-2\epsilon t}\,\mathrm{d}t\|f\|^2_{L^2_{\sigma_0}[0,\infty)} < \infty.\end{aligned} \tag{B.9}$$

□

Remark B.15. *If we compare* (B.9) *to* (B.8)*, we expect that this estimate is not optimal for $\sigma_0 > 0$, (in this case $\epsilon = 0$ should be possible). Indeed, we can improve our estimates for $\sigma_0 > 0$: By the previous estimate it holds for $\sigma > \sigma_0 > 0$*

$$\begin{aligned}\|\partial_t^{-1} f\|^2_{L^2_\sigma[0,\infty)} &= \int_0^\infty e^{-2\sigma t}|\partial_t^{-1} f(t)|^2\,\mathrm{d}t \\ &= \int_0^\infty e^{-2\sigma t}(\partial_t^{-1} f(t))^2\,\mathrm{d}t \\ &= \left[e^{-2\sigma t}(-2\sigma)^{-1}(\partial_t^{-1} f(t))^2\right]_0^\infty - \int_0^\infty e^{-2\sigma t}(-2\sigma)^{-1}2(\partial_t^{-1} f(t))f(t)\,\mathrm{d}t \\ &= \frac{1}{\sigma}\int_0^\infty e^{-\sigma t}\partial_t^{-1} f(t)e^{-\sigma t} f(t)\,\mathrm{d}t \\ &\leq \frac{1}{\sigma}\|f\|_{L^2_\sigma[0,\infty)}\|\partial_t^{-1} f\|_{L^2_\sigma[0,\infty)}.\end{aligned}$$

Division by $\|\partial_t^{-1} f\|_{L^2_\sigma[0,\infty)}$, $\sigma \to \sigma_0$ and monotone convergence yield

$$\|\partial_t^{-1} f\|_{L^2_{\sigma_0}[0,\infty)} \leq \frac{1}{\sigma_0}\|f\|_{L^2_{\sigma_0}[0,\infty)}.$$

This additionally shows continuity of the operator

$$\partial_t^{-1} : L^2_{\sigma_0}[0,\infty) \to L^2_{\max(\epsilon,\sigma_0)}[0,\infty).$$

We define the following spaces for a domain $\Omega \subset \mathbb{R}^m$, $m \in \mathbb{N}$.

Definition B.16. *The space of k-times continuously differentiable functions*

$$C^k(\Omega) := C^k(\Omega, \mathbb{R}^n) := \{f : \Omega \to \mathbb{R}^n \mid f \text{ is } k \text{ times continuously differentiable in } \Omega^\circ \text{ and continuously extendable to } \Omega\}.$$

The infinitely differentiable functions are defined as

$$C^\infty(\Omega) := C^\infty(\Omega, \mathbb{R}^n) := \bigcap_{k \in \mathbb{N}_0} C^k(\Omega, \mathbb{R}^n).$$

For a function $f : \mathbb{R}^m \to \mathbb{R}^n$ we define the support as

$$\operatorname{supp}(f) := \overline{\{x | f(x) \neq 0\}}$$

and the compactly supported functions

$$C_0^k(\Omega) := C_0^k(\Omega, \mathbb{R}^n) := \{f \in C^k(\Omega, \mathbb{R}^n) \mid \operatorname{supp}(f) \text{ is compact and } \operatorname{supp}(f) \subset \Omega\}$$

and

$$C_0^\infty(\Omega) := C_0^\infty(\Omega, \mathbb{R}^n) := \bigcap_{k \in \mathbb{N}_0} C_0^k(\Omega, \mathbb{R}^n).$$

Lemma B.17. *In the situation of Definition B.10, if there exists an $m \in \mathbb{N}_0$, $\sigma_1, \sigma_2 \in \mathbb{R}$ and a constant $C > 0$, such that for every $\phi \in C^\infty(0, \infty)$*

$$B(\partial_t)\phi = \mathcal{L}^{-1}(B(s)\mathcal{L}(\phi)(s)) \in L^2_*[0, \infty)$$

and

$$\|B(\partial_t)\phi\|_{L^2_{\sigma_1}[0,\infty)} \leq C\|\phi\|_{H^m_{0,\sigma_2}[0,\infty)},$$

then $B(\partial_t)f$ exists for every $f \in H^m_{0,\sigma_2}[0, \infty)$ and it holds

$$B(\partial_t)f = \lim_{\phi \to f \text{ in } H^m_{0,\sigma_2}[0,\infty)} B(\partial_t)\phi,$$

where the convergence is understood in $L^2_{\sigma_1}[0, \infty)$ and the limit $\phi \to f$ is in $H^m_{0,\sigma_2}[0, \infty)$. Under these assumptions, we can define $B(\partial_t)$ as a continuous operator

$$B(\partial_t) : H^m_{0,\sigma_2}[0, \infty) \to L^2_{\sigma_1}[0, \infty).$$

The assertion holds for any space that is dense in $H^m_{0,\sigma_2}[0, \infty)$ instead of $C_0^\infty(0, \infty)$.

Proof. Let $f \in H^m_{0,\sigma_2}[0, \infty)$. By Plancherel's formula, we have for $\phi \in C_0^\infty(0, \infty)$ that

$$\begin{aligned}\|B(s)\mathcal{L}\phi\|_{\mathcal{H}(\sigma_1)} &= \|B(\partial_t)\phi\|_{L^2_{0,\sigma_1}[0,\infty)} \\ &\leq C\|\phi\|_{H^m_{\sigma_2}[0,\infty)}.\end{aligned}$$

Thus, for a sequence $\phi \to f$ in $H^m_{0,\sigma_2}[0, \infty)$, $B(s)\mathcal{L}\phi$ is a Cauchy sequence and convergences in the Banach space $\mathcal{H}(\sigma_1)$ (see Theorem B.13). Especially the limit function is analytic, and on $\{\Re s > \max(\sigma_1, \sigma_2)\}$, it equals $B(s)\mathcal{L}f$, because on $\{\Re s > \sigma_2\}$, we have pointwise convergence of $\mathcal{L}\phi \to \mathcal{L}f$

$$\begin{aligned}|B(s)\mathcal{L}f(s) - B(s)\mathcal{L}\phi(s)| &\leq |B(s)| \|e^{-\Re s \cdot}(f - \phi)\|_{L^1[0,\infty)} \\ &\leq C(B, s, \sigma_2)\|f - \phi\|_{L^2_{\sigma_2}[0,\infty)} \to 0.\end{aligned}$$

Thus $B(s)\mathcal{L}f \in \mathcal{H}(\max(\sigma_1, \sigma_2))$ and $B(\partial_t)f$ exists.

If $\sigma_1 \geq \sigma_2$, we have by Plancherel's formula and the previously shown

$$B(\partial_t)f = L^2_{\sigma_1}[0,\infty) - \lim_{\phi \to f \text{ in } H^m_{\sigma_2}[0,\infty)} B(\partial_t)\phi.$$

If $\sigma_1 < \sigma_2$, $B(s)\mathcal{L}f$ has an extension in $\mathcal{H}(\sigma_1)$, i.e. there exists $g \in L^2_{\sigma_1}[0,\infty)$ such that $\mathcal{L}g = B(s)\mathcal{L}f$ on $\{\Re s > \sigma_2\}$. As the inverse Laplace transform is uniquely defined by one vertical line, this already shows $B(\partial_t)f \in L^2_{\sigma_1}[0,\infty)$ and

$$B(\partial_t)f = L^2_{\sigma_1}[0,\infty) - \lim_{\phi \to f \text{ in } H^m_{0,\sigma_2}[0,\infty)} B(\partial_t)\phi.$$

The operator

$$B(\partial_t) : H^m_{0,\sigma_2}[0,\infty) \to L^2_{\sigma_1}[0,\infty)$$

is continuous, as the approximating sequence can be chosen such that

$$\|\phi\|_{H^m_{0,\sigma_2}[0,\infty)} \leq 2\|f\|_{H^m_{0,\sigma_2}[0,\infty)}$$

and therefore by continuity of the norm

$$\begin{aligned}
\|B(\partial_t)f\|_{L^2_{\sigma_1}[0,\infty)} &\leq \lim_{\phi \to f \text{ in } H^m_{0,\sigma_2}[0,\infty)} \|B(\partial_t)\phi\|_{L^2_{\sigma_1}[0,\infty)} \\
&\leq C\|\phi\|_{H^m_{0,\sigma_2}[0,\infty)} \\
&\leq 2C\|f\|_{H^m_{0,\sigma_2}[0,\infty)}.
\end{aligned}$$

□

Example B.18. *a) For $\sigma_0 \in \mathbb{R}$ and every $\phi \in H^\infty_{0,*}[0,\infty) \cap C^\infty_0[0,\infty)$, it is by Example B.14*

$$s\mathcal{L}\phi(s) \in \mathcal{H}$$

and

$$\|\partial_t\phi\|_{L^2_{\sigma_0}[0,\infty)} \leq \|\phi\|_{H^1_{0,\sigma_0}[0,\infty)}.$$

For $f \in H^1_{0,\sigma_0}[0,\infty)$ we have

$$L^2_{\sigma_0}[0,\infty) - \lim_{\phi \to f \text{ in } H^1_{0,\sigma_0}[0,\infty)} \partial_t\phi = \partial_t f$$

and

$$\partial_t : H^1_{0,\sigma_0}[0,\infty) \to L^2_{\sigma_0}[0,\infty)$$

is continuous.
b) For $\sigma_0 \in \mathbb{R}$, $\epsilon > 0$ and every $\phi \in C^\infty_0(0,\infty)$, it is by Example B.14

$$s^{-1}\mathcal{L}\phi(s) \in \mathcal{H}$$

and

$$\|\partial_t^{-1}\phi\|_{L^2_{\max(\epsilon,\sigma_0)}[0,\infty)} \leq \|\phi\|_{L^2_{\sigma_0}[0,\infty)}.$$

For $f \in L^2_{\sigma_0}[0,\infty)$ we have

$$L^2_{\max(\epsilon,\sigma_0)}[0,\infty) - \lim_{\phi \to f \text{ in } L^2_{\sigma_0}[0,\infty)} \partial_t^{-1}\phi = \partial_t^{-1}f$$

and

$$\partial_t^{-1} : L^2_{\sigma_0}[0,\infty) \to L^2_{\max(\epsilon,\sigma_0)}[0,\infty)$$

is continuous.

In the following lemma, we give a more concrete condition that ensures the existence of $B(\partial_t)f$.

Lemma B.19. *In the setting of Definition B.10, if there exists an $m \in \mathbb{N}_0$, $\sigma_1 \in \mathbb{R}$ and a constant $C > 0$, such that B is analytic on its domain and*

$$|B(s)| \leq C|s|^m \text{ for all } \Re s > \sigma_1,$$

then $B(\partial_t)f$ exists for every $f \in H^m_{0,*}[0,\infty)$ and it holds

$$B(\partial_t)f = \partial_t^m \mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f) = \mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}\partial_t^m f).$$

Under these assumptions, the assumptions of Lemma B.17 are satisfied and we can define $B(\partial_t)$ as a continuous operator for $\sigma_2 \in \mathbb{R}$

$$B(\partial_t) : H^m_{0,\sigma_2}[0,\infty) \to L^2_{\max(\sigma_1,\sigma_2)}[0,\infty).$$

Proof. Let $f \in H^m_{0,\sigma_2}[0,\infty)$. Then it is $s^m\mathcal{L}f \in \mathcal{H}(\sigma_2)$ and

$$B(s)\mathcal{L}f \in \mathcal{H}(\max(\sigma_1,\sigma_2))$$

and $B(\partial_t)f$ exists. By Example B.14, it is

$$B(\partial_t)f = \partial_t^m\partial_t^{-m}B(\partial_t)f = \partial_t^m\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f)$$

and as $f \in H^m_{0,\sigma_2}[0,\infty)$

$$B(\partial_t)f = B(\partial_t)\partial_t^{-m}\partial_t^m f = \mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}(\partial_t^m f)).$$

The assumptions of Lemma B.17 are satisfied, we have for $\phi \in C_0^\infty(0,\infty)$, by Plancherel's formula and Example B.14

$$\begin{aligned}
\|B(\partial_t)\phi\|_{L^2_{\max(\sigma_1,\sigma_2)}[0,\infty)} &= \frac{1}{2\pi}\|B(s)\mathcal{L}\phi\|_{\mathcal{H}(\max(\sigma_1,\sigma_2))} \\
&\leq C\|s^m\mathcal{L}\phi\|_{\mathcal{H}(\max(\sigma_1,\sigma_2))} \\
&= C\|\partial_t^m\phi\|_{L^2_{\max(\sigma_1,\sigma_2)}[0,\infty)} \\
&\leq \|\phi\|_{H^m_{0,\sigma_2}[0,\infty)}.
\end{aligned}$$

□

Definition B.20. *We define for $m \in \mathbb{N}_0$ and $\sigma_0 \in \mathbb{R}$*

$$\mathcal{H}_m(\sigma_0) := \left\{B : \{\Re s > \sigma_0\} \to \mathbb{C} \text{ analytic} \;\middle|\; |B(s)| \leq C|s|^m \text{ for all } \Re s > \sigma_0\right\}$$

and

$$\mathcal{H}_m := \bigcup_{\sigma_0\in\mathbb{R}} \mathcal{H}_m(\sigma_0).$$

We call $B \in \mathcal{H}_0$ a smoothing operator.

We illustrate the statements of the previous theorem with the simple differential operators ∂_t and ∂_t^{-1}.

Example B.21. *a) It holds for every $\sigma_1 \in \mathbb{R}$ that $s \mapsto s \in \mathcal{H}_1(\sigma_1)$ as*

$$|s| \leq |s|^1 \text{ on } \{\Re s > \sigma_1\}$$

and for $\sigma_2 \in \mathbb{R}$, $f \in H^1_{0,\sigma_2}[0,\infty)$ *we have*

$$\partial_t f = \partial_t \mathcal{L}^{-1}(\mathcal{L}f) = \mathcal{L}^{-1}(\mathcal{L}\partial_t f) \text{ in } L^2_{\sigma_2}[0,\infty).$$

b) It holds for every $\sigma_1 > 0$

$$|s^{-1}| \leq \frac{1}{\sigma_1} \text{ on } \{\Re s > \sigma_1\}$$

and for every $\sigma_2 \in \mathbb{R}$ $f \in L^2_{\sigma_2}[0,\infty)$

$$\partial_t^{-1} f = \mathcal{L}^{-1}(s\mathcal{L}f) \text{ in } L^2_{\max(\sigma_1,\sigma_2)}[0,\infty)$$

and ∂_t^{-1} *is a smoothing operator.*

We define the convolution of $a, b \in L^2_*[0,\infty)$ as

$$(a * b)(t) := \int_0^t a(\tau)b(t-\tau)\, \mathrm{d}\tau = \int_0^t a(t-\tau)b(\tau)\, \mathrm{d}\tau = (b * a)(t)$$

and summarize some of the properties in the following lemma.

Lemma B.22. *It holds*

$$L^2_*[0,\infty) * L^2_*[0,\infty) \subset L^2_*[0,\infty).$$

More precise it is for $\sigma_1, \sigma_2 \in \mathbb{R}$, $a \in L^2_{\sigma_1}[0,\infty)$, $b \in L^2_{\sigma_2}[0,\infty)$ *for* $\sigma_1 \neq \sigma_2$

$$a * b \in L^2_{\max(\sigma_1,\sigma_2)}[0,\infty).$$

For $\sigma_1 = \sigma_2$, *in general we only have for some* $\epsilon > 0$

$$a * b \in L^2_{\sigma_1+\epsilon}[0,\infty).$$

Furthermore it is

$$\mathcal{L}(a * b) = \mathcal{L}a \cdot \mathcal{L}b \text{ on } \{\Re s > \max(\sigma_1,\sigma_2)\}$$

and the functions lie in $\mathcal{H}(\max(\sigma_1,\sigma_2))$ *(or* $\mathcal{H}(\sigma_1+\epsilon)$, *in the case* $\sigma_1 = \sigma_2$*). For* $A \in \mathcal{H}(\sigma_1)$, $B \in \mathcal{H}(\sigma_2)$

$$\mathcal{L}^{-1}A * \mathcal{L}^{-1}B = \mathcal{L}^{-1}(AB) \text{ almost everywhere in } [0,\infty)$$

and the functions are in $L^2_{\max(\sigma_1,\sigma_2)}[0,\infty)$ *(or* $L^2_{\sigma_1+\epsilon}[0,\infty)$, *in the case* $\sigma_1 = \sigma_2$*).*

Proof. For $1 \leq p,q,r \leq \infty$ witch $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ Young's inequality gives for $\phi \in L^p(\mathbb{R})$, $\psi \in L^q(\mathbb{R})$

$$\|\phi *_{\mathbb{R}} \psi\|_{L^r(\mathbb{R})} \leq \|\phi\|_{L^p(\mathbb{R})}\|\psi\|_{L^q(\mathbb{R})},$$

where $*_{\mathbb{R}}$ denotes the convolution over $\mathbb{R}$,

$$\phi *_{\mathbb{R}} \psi(t) := \int_{\mathbb{R}} \phi(\tau) * \psi(t-\tau)\, \mathrm{d}\tau.$$

If $\operatorname{supp}(\phi)$, $\operatorname{supp}(\psi) \subset [0,\infty)$, it holds

$$\phi *_{\mathbb{R}} \psi = \phi * \psi$$

almost everywhere on $\mathbb{R}$. Therefore we have for $r = p = 2$, $q = 1$ and without loss of generality $\sigma_1 < \sigma_2$

$$\begin{aligned}
\|a * b\|^2_{L^2_{\max(\sigma_1,\sigma_2)}[0,\infty)} &= \int_0^\infty e^{-2\max(\sigma_1,\sigma_2)t} |\int_0^t a(\tau)b(t-\tau)\ \mathrm{d}\tau|^2\ \mathrm{d}t \\
&= \int_0^\infty |\int_0^t e^{-\sigma_2\tau} a(\tau)|b(t-\tau)|e^{-\sigma_2(t-\tau)}|a(\tau)|\ \mathrm{d}\tau|^2\ \mathrm{d}t \\
&= \|e^{-\sigma_2\cdot} a * e^{-\sigma_2\cdot} b\|^2_{[0,\infty)} \\
&\le \|e^{-\sigma_2\cdot} a\|^2_{L^1[0,\infty)} \|e^{-\sigma_2\cdot} b\|^2_{[0,\infty)} \\
&\le C(\sigma_2 - \sigma_1)\|a\|^2_{L^2_{\sigma_1}[0,\infty)} \|b\|^2_{L^2_{\sigma_2}[0,\infty)} < \infty.
\end{aligned}$$

For $\sigma_1 = \sigma_2$, the same estimate for $\sigma_1 := \sigma_1$, $\sigma_2 := \sigma_1 + \epsilon$ gives

$$\begin{aligned}
\|a * b\|^2_{L^2_{\max(\sigma_1,\sigma_1+\epsilon)}[0,\infty)} &\le \|e^{-(\sigma_1+\epsilon)\cdot} a\|^2_{L^1[0,\infty)} \|e^{-(\sigma_1+\epsilon)\cdot} b\|^2_{[0,\infty)} \\
&\le \|e^{-\epsilon\cdot}\|^2_{[0,\infty)} \|a\|^2_{L^2_{\sigma_1}[0,\infty)} \|b\|^2_{L^2_{\sigma_1}[0,\infty)} < \infty.
\end{aligned}$$

We have by the Fourier convolution law for $c, d \in L^2(\mathbb{R})$, if $c *_{\mathbb{R}} d \in L^2(\mathbb{R})$, that

$$\mathcal{F}(c *_{\mathbb{R}} d) = \mathcal{F}(c)\ \cdot\ \mathcal{F}(d).$$

Similarly, for $C, D \in L^2(\mathbb{R})$, if $CD \in L^2(\mathbb{R})$, it holds

$$\mathcal{F}^{-1}C *_{\mathbb{R}} \mathcal{F}^{-1}D = \mathcal{F}^{-1}(CD).$$

These relations translate to the Laplace transform in a similar way, it holds for $s \in \mathbb{C}$, $\Re s > \max(\sigma_1, \sigma_2)$

$$(e^{-\Re s\cdot} a) * (e^{-\Re s\cdot} b) \in L^2[0,\infty)$$

and

$$\begin{aligned}
\mathcal{L}(a * b)(s) &= \mathcal{F}(e^{-\Re s\cdot} (a * b)(\,\cdot\,))(\Im(s)) \\
&= \mathcal{F}((e^{-\Re s\cdot} a) * (e^{-\Re s\cdot} b))(\Im(s)) \\
&= \mathcal{F}(e^{-\Re s\cdot} a)(\Im(s))(e^{-\Re s\cdot} b)(\Im(s)) \\
&= \mathcal{L}(a)(s)\ \cdot\ \mathcal{L}(b)(s).
\end{aligned}$$

Similarly we have for almost all $t \in [0,\infty)$ and $\sigma > \max(\sigma_1, \sigma_2)$

$$\begin{aligned}
\mathcal{L}^{-1}A * \mathcal{L}^{-1}B(t) &= \int_0^t e^{\sigma(t-s)} \mathcal{F}^{-1}(A(\sigma + i\,\cdot))(t-s) e^{\sigma s} \mathcal{F}^{-1}(B(\sigma + i\,\cdot))(s)\ \mathrm{d}s \\
&= e^{\sigma t} \mathcal{F}^{-1}(A(\sigma + i\,\cdot)) *_{\mathbb{R}} \mathcal{F}^{-1}(B(\sigma + i\,\cdot))(t) \\
&= e^{\sigma t} \mathcal{F}^{-1}(A(\sigma + i\,\cdot)B(\sigma + i\,\cdot))(t) \\
&= \mathcal{L}^{-1}AB(t).
\end{aligned}$$

□

Remark B.23. *Aa example that $\epsilon > 0$ is necessary in the previous Lemma B.22 for $\sigma_1 = \sigma_2$, can be constructed by*

$$A(s) := \frac{1}{s^{1/4}} \frac{1}{1+s}.$$

The function A is analytic for $\Re s > 0$ and $A \in L^2(i\mathbb{R}) \cap L^2(\sigma + i\mathbb{R})$ for all $\sigma > 0$, so

$$A \in \mathcal{H}(0).$$

Thus there exists $a \in L^2[0,\infty)$, such that $\mathcal{L}a = A$. By what we have already shown in Lemma B.22, it is

$$\mathcal{L}(a * a) = A^2 \quad \text{on } \{\Re s > 0\}.$$

However, as $A^2 \notin L^2(i\mathbb{R})$, it holds

$$\mathcal{L}(a * a) = A^2 \notin \mathcal{H}(0)$$

and thus

$$a * a \notin L^2[0,\infty).$$

For every $\epsilon > 0$ we have

$$a * a \in L^2_\epsilon[0,\infty)$$

and

$$A^2 \in \mathcal{H}(\epsilon).$$

Lemma B.24. *In the setting of Lemma B.19, it is for every $\epsilon > 0$*

$$B(s)s^{-(m+1)} \in \mathcal{H}(\max(\epsilon, \sigma_1)),$$

so $\mathcal{L}^{-1}(B(s)s^{-(m+1)})$ exists. Similarly $\mathcal{L}^{-1}(B(s)s^{-(m+2)})$ exists and is continuous regarded as function on $\mathbb{R}$. For f such that $B(\partial_t)f$ exists (e.g. $f \in H^m_{0,*}[0,\infty)$), it holds

$$B(\partial_t)f = \partial_t^{m+1}\mathcal{L}^{-1}(B(s)s^{-(m+1)}) * f = \partial_t^{m+2}\mathcal{L}^{-1}(B(s)s^{-m+2}) * f.$$

For $f \in H^{m+1}_{0,*}[0,\infty)$ we have

$$B(\partial_t)f = \mathcal{L}^{-1}(B(s)s^{-(m+1)}) * \partial_t^{m+1} f$$

and for $f \in H^{m+2}_{0,*}[0,\infty)$

$$B(\partial_t)f = \mathcal{L}^{-1}(B(s)s^{-m+2}) * \partial_t^{m+2} f.$$

Proof. It is $B(s)s^{-(m+1)}$ analytic for $\Re s > \max(\sigma_1, 0)$ and by

$$|B(s)s^{-(m+1)}| \le |s|^{-1} \quad \text{on } \Re s > \max(\sigma_1, 0),$$

we obtain

$$B(s)s^{-(m+1)} \in \mathcal{H}(\max(\epsilon, \sigma_1)).$$

The function $\mathcal{L}^{-1}(B(s)s^{-(m+2)})$ is continuous regarded as function on $\mathbb{R}$, as

$$B(s)s^{-(m+2)} \in L^1(\sigma + i\mathbb{R})$$

for $\sigma > \max(\sigma_1, \epsilon)$. By the Fourier transform properties from Lemma B.1 it is continuous on $[0,\infty)$ and

$$\mathcal{L}^{-1}(B(s)s^{-(m+2)})(0) = \partial_t^{-1}\mathcal{L}^{-1}(B(s)s^{-(m+1)})(0) = 0.$$

For $\sigma_2 \in \mathbb{R}$, $g \in L^2_{\sigma_2}[0,\infty)$ it holds with Lemma B.22

$$\mathcal{L}^{-1}(B(s)s^{-(m+1)}\mathcal{L}g) = \mathcal{L}^{-1}(B(s)s^{-(m+1)}) * g.$$

The assertion follows from

$$\partial_t^m \partial_t^{-m} = \mathrm{Id}_{L^2_*[0,\infty) \to L^2_*[0,\infty)}, \quad \partial_t^{-m}\partial_t^m = \mathrm{Id}_{H^m_{0,*}[0,\infty) \to H^m_{0,*}[0,\infty)}$$

and

$$\mathcal{L}^{-1}(B(s)f) = \partial_t^{m+1}\partial_t^{-m-1}\mathcal{L}^{-1}(B(s)f) = \partial_t^{m+1}\mathcal{L}^{-1}(s^{-m-1}B(s)f),$$

as well as

$$\mathcal{L}^{-1}(B(s)f) = \mathcal{L}^{-1}(B(s)(\partial_t^{-m-1}\partial_t^{m+1}f)) = \mathcal{L}^{-1}(s^{-m-1}B(s)(\partial_t^{m+1}f)).$$

□

Again, we illustrate the previous lemma for the simple differential operators ∂_t and ∂_t^{-1}.

Example B.25. *a) It holds for* $f \in H^1_{0,*}[0,\infty)$ *that*

$$\partial_t f = \partial_t^2 \int_0^t f(\tau)\ \mathrm{d}\tau = \partial_t^2 \partial_t^{-1} f$$

and

$$\partial_t f = \partial_t^3 \int_0^t (t-\tau) f(\tau)\ \mathrm{d}\tau = \partial_t^3 \partial_t^{-2} f.$$

For $f \in H^2_{0,*}[0,\infty)$ *we have*

$$\partial_t f = \int_0^t \partial_t^2 f(\tau)\ \mathrm{d}\tau = \partial_t^{-1} \partial_t^2 f$$

and if $f \in H^3_{0,*}[0,\infty)$*, even*

$$\partial_t f = \int_0^t (t-\tau) \partial_t^3 f(\tau)\ \mathrm{d}\tau = \partial_t^{-2} \partial_t^3 f.$$

The function $\mathcal{L}^{-1}(1/s^2) = \mathbb{1}_{[0,\infty)}(t)t$ *is continuous over* $\mathbb{R}$.
b) It is

$$\mathcal{L}\left(\mathbb{1}_{[0,\infty)}(t)t\right) = \frac{1}{s^2}$$

and

$$\mathcal{L}\left(\mathbb{1}_{[0,\infty)}(t)\frac{t^2}{2}\right) = \frac{1}{s^3}.$$

It holds for $f \in L^2_*[0,\infty)$ *that*

$$\partial_t^{-1} f = \partial_t \int_0^t (t-\tau) f(\tau)\ \mathrm{d}\tau = \partial_t \partial_t^{-2} f$$

and

$$\partial_t^{-1} f = \partial_t^2 \int_0^t \frac{(t-\tau)^2}{2} f(\tau)\ \mathrm{d}\tau = \partial_t^2 \partial_t^{-3} f.$$

For $f \in H^1_{0,*}[0,\infty)$ *we have*

$$\partial_t^{-1} f = \partial_t \int_0^t \frac{(t-\tau)^2}{2} \partial_t f(\tau)\ \mathrm{d}\tau = \partial_t^1 \partial_t^{-2} \partial_t f.$$

and if even $f \in H^2_{0,*}[0,\infty)$*,*

$$\partial_t^{-1} f = \int_0^t \frac{(t-\tau)^2}{2} \partial_t^2 f(\tau)\ \mathrm{d}\tau = \partial_t^{-3} \partial_t^2 f.$$

Remark B.26. *The formulas in Lemma B.19 and Lemma B.24, that do not need differentiabillity of* f*, also hold for* $f \in L^2_*[0,\infty)$*, if* $B(\partial_t)f$ *exists (i.e.* $B(s)Lf \in \mathcal{H}$*). In this case it is for* $m \in \mathbb{N}_0$

$$B(\partial_t)f = \partial_t^m \mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f)$$

and for high enough $m \in \mathbb{N}$

$$B(\partial_t)f = \partial_t^m \mathcal{L}^{-1}(B(s)s^{-m}) * f.$$

The last equality only holds if $\mathcal{L}^{-1}(B(s)s^{-m})$ *exists.*

We summarize some further properties concerning the concatenation of Laplace differential operators.

Theorem B.27. *Let $f \in L^2_*[0,\infty)$ and functions $A(s)$, $B(s)$. If $B(\partial_t)f$ and $(AB)(\partial_t)f$ exist, then $A(\partial_t)B(\partial_t)f$ exists and equals*

$$(AB)(\partial_t)f = A(\partial_t)B(\partial_t)f.$$

If furthermore $A(\partial_t)f$ exists, it holds

$$(AB)(\partial_t)f = A(\partial_t)B(\partial_t)f = B(\partial_t)A(\partial_t)f.$$

Corollary B.28. *For $A \in \mathcal{H}_m$, $B \in \mathcal{H}_n$, $AB \in \mathcal{H}_p$ and $f \in H^{\max(m,n,p)}_{0,*}[0,\infty)$ it is*

$$(AB)(\partial_t)f = A(\partial_t)B(\partial_t)f = B(\partial_t)A(\partial_t)f.$$

Example B.29. *For $f \in L^2_*[0,\infty)$, $\partial_t^{-1}f$ exists, $\mathcal{L}^{-1}(\mathcal{L}f)$ exists, so*

$$f = \partial_t\partial_t^{-1}f,$$

but $\partial_t f$ exists only for $f \in H^1_{0,*}[0,\infty)$.

The following Herglotz theorem, originally from [81], is an important result for the connection between positivity and boundedness of time dependent and time harmonic operators.

Theorem B.30 (Herglotz theorem, cf. [27, Lemma 2.2])**.** *Let $B, R \in \mathcal{H}^m(\sigma_0)$ for $\sigma_0 \in \mathbb{R}$. Then the following statements are equivalent:*

- *There exists $c > 0$ such that for all $w \in \mathbb{C}$, all $\Re s > \sigma_0$*

$$\Re(wB(s)w) \geq c|R(s)w|^2.$$

- *There exists $c > 0$ such that for all $w \in H^m_{0,*}[0,\infty)$, for all $\sigma \geq \sigma_0$*

$$\int_0^\infty e^{-2\sigma t}\Re(w(t)B(\partial_t)w(t))\ \mathrm{d}t \geq c\|R(\partial_t)w\|^2_{L^2_\sigma[0,\infty)}.$$

Additionally, the following statements are equivalent:

- *There exists $C > 0$ such that for all $\Re s > \sigma_0$*

$$|B(s)|^2 \leq C|R(s)|^2.$$

- *There exists $C > 0$ such that for all $w \in H^m_{0,*}[0,\infty)$, for all $\sigma \geq \sigma_0$*

$$\|B(\partial_t)w\|^2_{L^2_\sigma[0,\infty)} \leq C\|R(\partial_t)w\|^2_{L^2_\sigma[0,\infty)}.$$

Proof. The execution follows immediately by Plancherel's formula, and the reverse direction can be shown by localizing around arbitrary values by special sequences, cf. [27, Lemma 2.2]. $\square$

B.1.3. Laplace transform and differential operators on $[0, T]$

As we will mainly work on bounded time intervals, we want to define the Laplace transform and Laplace differential operators for functions with domain $[0, T]$, so e.g. for $f \in L^2([0, T])$. The Laplace transform can easily be defined by extending f to zero outside of $[0, T]$:

For $f \in L^2([0, T])$, it is $e^{-c\cdot} f \mathbb{1}_{[0,T]} \in L^2[0, \infty)$ for all $c \in \mathbb{R}$, thus

$$\mathcal{L}f := \mathcal{L}(f \mathbb{1}_{[0,T]})$$

exists and is analytic in the whole complex plane. Also $\mathcal{L}^{-1}$ of $\mathcal{L}f$ is well defined and returns a function with support in $[0, T]$. For general functions $B \in \mathcal{H}$, we can ensure $\operatorname{supp}(\mathcal{L}^{-1}B) \subset [0, T]$ by setting

$$\mathcal{L}^{-1} := \mathbb{1}_{[0,T]} \mathcal{L}^{-1}.$$

It should then be taken into account, that in general $\mathcal{L}^{-1}$ is only a right inverse, i.e. it is

$$\mathcal{L}^{-1}\mathcal{L} = \mathrm{Id}_{L^2([0,T]) \to L^2([0,T])},$$

but

$$\mathcal{L}\mathcal{L}^{-1} \neq \mathrm{Id}_{\mathcal{H} \to \mathcal{H}}.$$

The definition of Laplace differential operators cannot be done straightforward by restricting the function to $[0, T]$, as thereby an artificial singularity is set at $t = T$, if $f(T) \neq 0$. This can be seen by the slow decay of $\mathcal{L}f$, even if $f \in C^\infty[0, T]$ it is

$$\begin{aligned}(\mathcal{L}f)(s) &= \int_0^T e^{-st} f(t)\ \mathrm{d}t \\ &= \frac{1}{s} \int_0^T e^{-st} f'(t)\ \mathrm{d}t + \frac{1}{s} f(0) - \frac{e^{-sT}}{s} f(T) \\ &= \frac{1}{s} (\mathcal{L}f')(s) + \frac{1}{s} f(0) - \frac{e^{-sT}}{s} f(T).\end{aligned}$$

It holds $\mathcal{L}f' \in \mathcal{H}$, so $\frac{1}{s}(\mathcal{L}f')(s)$ is decaying fast enough to apply an differential operator of order one. But, as $f(0) \neq 0 \neq f(T)$, we only have

$$|\mathcal{L}f(s)| \leq |s|^{-1},$$

so for fixed $\Re s > 0$

$$|\mathcal{L}f(s)| \leq C(\Re s)(1 + |\Im s|)^{-1}.$$

Thus we are in general only able to apply differential operators $B(\partial_t)$, where

$$|B(s)| \leq C|s|^{1/2-\epsilon}$$

for $\epsilon > 0$ to ensure that $B(\partial_t)f$ exists.

To demand $f(T) = f(0) = 0$ is not an option, as this will not be satisfied by the considered functions and this would be a to strict restriction if we want to apply several Laplace differential operators successively.

A possibility to overcome this issue would be, to extend f on $[T, \infty)$ in a smooth way, such that $f(t) = 0$ for $t > 2T$, to apply the operator $B(\partial_t)$ on $[0, \infty)$ and to ensure, that $\mathbb{1}_{[0,T]} B(\partial_t) f$ does not depend on the arbitrarily chosen extension. We will go a slightly different way and define $B(\partial_t)f$ by choosing a special extension, which turns the definition in a more explicit and more handy form. The property, that the definition is invariant under any smooth enough extension of f to $[0, \infty)$ will be satisfied under weak assumptions. We motivate the approach in the following example, where we again consider the differential operators ∂_t and ∂_t^{-1}.

Example B.31. *a) If we consider*

$$(\mathcal{L}f)(s) = \int_0^T e^{-st} f(t)\ \mathrm{d}t$$
$$= \frac{1}{s}(\mathcal{L}f')(s) + \frac{1}{s} f(0) - \frac{e^{-sT}}{s} f(T).$$

in a distributional sense, we see that

$$\mathcal{L}^{-1}(s\mathcal{L}f) = \partial_t f + \delta_0 f(0) - \delta_T f(T),$$

where δ_x is the Dirac Delta distribution for $x \geq 0$ (zero everywhere, except $\delta_x(x) = \infty$ such that $\int_{\{x\}} \delta_x = 1$)

$$(\mathcal{L}\delta_x)(s) = \int_0^\infty e^{-st}\delta_x(t)\ \mathrm{d}t = e^{-sx}.$$

This again underlines that $\mathcal{L}^{-1}(s\mathcal{L}f)$ is the derivative of f over the whole $\mathbb{R}$. If we would smoothly extend f on $[0,\infty)$ and assume $f(0) = 0$, we would obtain the expected

$$\mathcal{L}^{-1}(s\mathcal{L}f) = \partial_t f \text{ on } [0,T].$$

b) The smoothing operator ∂_t^{-1} can be applied to $f \in L^2[0,T]$, as $\mathbb{1}_{[0,T]} f \in L^2_*[0,\infty)$ and the outcome is

$$\mathcal{L}^{-1}(s^{-1}\mathcal{L}f)(t) = \begin{cases} \int_0^t f(\tau)\ \mathrm{d}\tau, & \textit{for } t \in [0,T], \\ \int_0^T f(\tau)\ \mathrm{d}\tau, & \textit{for } t \in [T,\infty), \end{cases}.$$

Again, restriction to $[0,T]$ gives the expected, no matter which extension (instead of extension by zero) is chosen on $[T,\infty)$.

If we regard the last example, we see, that for $f \in L^2[0,T]$, $\partial_t^{-1} f \in H^1_{0,*}[0,\infty)$. In the following, ∂_t always stands for the weak derivative, and not for the Laplace differential operator $\mathcal{L}^{-1}(s\mathcal{L}\,\cdot)$. Thus, if we take $f \in H^m[0,T]$, it is $\partial_t^m f \in L^2[0,T]$ and if

$$f(0) = \cdots = f^{(m-1)}(0) = 0,$$

it holds

$$f = \partial_t^{-m}\partial_t^m f \text{ in } [0,T]$$

and for $t > T$ we obtain the extension

$$\partial_t^{-m}\partial_t^m f(t) = f(T) + (t-T)f'(T) + \cdots + \frac{(t-T)^{m-1}}{(m-1)!} f^{(m-1)}(T) \quad \text{for } t > T.$$

It holds $\partial_t^{-m}\partial_t^m f \in H^m_{0,*}[0,\infty)$ and we can apply $B(\partial_t)$ for $B \in \mathcal{H}_m$ to it. This yields that

$$\mathcal{L}^{-1}(s^{-m}B(s)\mathcal{L}(\partial_t^m f))$$

exists. As this definition would still require $\partial_t^m f$ to exist, we rewrite it as

$$\mathcal{L}^{-1}(s^{-m}B(s)\mathcal{L}(\partial_t^m f)) = \partial_t^m \mathcal{L}^{-1}(s^{-m}B(s)\mathcal{L}(\partial_t^{-m}\partial_t^m f)).$$

The term on the right hand side does not depend on the values of $\partial_t^{-m}\partial_t^m f$ on (T,∞) by definition of convolution

$$\begin{aligned} \mathbb{1}_{[0,T]}\partial_t^m \mathcal{L}^{-1}(s^{-m}B(s)\mathcal{L}(\partial_t^{-m}\partial_t^m f)) &= \mathbb{1}_{[0,T]}\partial_t^{m+1}\mathcal{L}^{-1}(s^{-m-1}B(s)\mathcal{L}(\partial_t^{-m}\partial_t^m f)) \\ &= \mathbb{1}_{[0,T]}\partial_t^{m+1}\mathcal{L}^{-1}(s^{-m-1}B(s)) * (\partial_t^{-m}\partial_t^m f) \\ &= \mathbb{1}_{[0,T]}\partial_t^{m+1}\mathcal{L}^{-1}(s^{-m-1}B(s)) * f. \end{aligned}$$

Thus we can replace $\partial_t^{-m}\partial_t^m f$ by f and define

Definition B.32. *Let $f \in L^2[0,T]$. Whenever there is an $m \in N_0$ such that the expression $\partial_t^m \mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f)$ exists (i.e. $\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f)$ exists and $\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f) \in H^m([0,T])$), we define*

$$B(\partial_t)f := \partial_t^m \mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f).$$

We call the function $B(s)$ or the mapping $B(\partial_t)$ causal, if for every f (and arbitrary $T>0$), such that $B(\partial_t)f$ exists, $B(\partial_t)f$ does not depend on an arbitrarily chosen extension of f in $L^2_*[0,\infty)$, i.e. for every $\widetilde{f} \in L^2_*[0,\infty)$,

$$f = \mathbb{1}_{[0,T]}\widetilde{f} \text{ in } L^2[0,T]$$

it holds

$$B(\partial_t)f = \mathbb{1}_{[0,T]}B(\partial_t)\widetilde{f} \text{ in } L^2[0,T].$$

Attention, this is a new definition of $B(\partial_t)$, that does not coincides in general with the one on $[0,\infty)$ of the previous subsection. The definition is well defined in the sense that it does not depend on the selection of $m \in \mathbb{N}$. If $B(\partial_t)f$ exists for $m_0 \in \mathbb{N}_0$ then for all $m > m_0$:

$$\partial_t^{m_0}\mathcal{L}^{-1}(B(s)s^{-m_0}\mathcal{L}f) = \partial_t^m \mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f).$$

We note, that if we work on the finite time interval $[0,T]$, the Laplace differential operators have to be understood following Definition B.32. Thus the derivative ∂_t is not understood as $\mathcal{L}^{-1}(s\mathcal{L}\,\cdot\,)$, but as the classical derivative, which is equivalent for smooth enough extended functions.

Example B.33. *a) The derivation Laplace operator $\mathcal{L}^{-1}(s\mathcal{L}\,\cdot\,)$ is not causal, but ∂_t interpreted as weak derivative is causal, again (although this is not a Laplace differential operator).*
b) The integration operator ∂_t^{-1} is causal.

In the following, we give sufficient conditions for the existence.

Definition B.34. *We define for $m \in \mathbb{N}_0$ the space of $m-$times weakly differentiable functions with initial condition zero as*

$$H^m_{0,*}[0,T] := \{\phi \in H^m[0,T] \mid f(0) = \cdots = f^{(m-1)}(0) = 0\}.$$

With the induced norm

$$\|\cdot\|_{H^m_{0,*}[0,T]} := \|\cdot\|_{H^m[0,T]} = \sqrt{\langle\,\cdot\,,\,\cdot\,\rangle_{H^m[0,T]}},$$

this is a Hilbert space.

Attention, the sub index $0,*$ in $H^m_{0,*}[0,T]$ has the meaning 0 at $t=0$ and arbitrary value at $t=T$, as we also use

$$H^m_{*,0}[0,T] := \{\phi \in H^m[0,T] \mid f(T) = \cdots = f^{(m-1)}(T) = 0\},$$

Lemma B.35. *For $f \in H^m_{0,*}[0,T]$ and $B \in \mathcal{H}_m$, we have that $B(\partial_t)f$ exists, it holds $\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f) \in H^m_{0,*}[0,T]$ and*

$$B(\partial_t)f = \mathbb{1}_{[0,T]}\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}(\partial_t^m f)) = \mathbb{1}_{[0,T]}\partial_t^{m+1}(\mathcal{L}^{-1}(B(s)s^{-(m+1)}) * f).$$

We can define $B(\partial_t)$ as a continuous operator

$$B(\partial_t) : H^m_{0,*}[0,T] \to L^2([0,T]).$$

Every $B \in \mathcal{H}_m$ is causal and for every smooth enough extension of f on $[0,\infty)$ it holds

$$B(\partial_t)f = \mathbb{1}_{[0,T]}\mathcal{L}^{-1}(B(s)\mathcal{L}f).$$

Proof. For an extension $\widetilde{f} \in H^m_{0,*}[0,\infty)$, it holds

$$|B(s)s^{-m}\mathcal{L}\widetilde{f}| \leq |\mathcal{L}\widetilde{f}|$$

and therefore by Lemma B.19

$$B(\partial_t)\widetilde{f} = \mathbb{1}_{[0,T]}\mathcal{L}^{-1}(B(s)\mathcal{L}\widetilde{f}).$$

For an arbitrary extension $\widetilde{f} \in L^2_*[0,\infty)$, it exists $\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}\widetilde{f})$ and it holds

$$\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}\widetilde{f}) = \partial_t^2\mathcal{L}^{-1}(B(s)s^{-m-2}) * \widetilde{f},$$

which does not depend on the extension. So $B(\partial_t)f$ does neither.

We extend $f \in H^m_{0,*}[0,T]$ by $\widetilde{f} \in H^m_{0,*}[0,\infty)$ and by Causality we conclude

$$\mathbb{1}_{[0,T]}\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f) = \mathbb{1}_{[0,T]}\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}\widetilde{f}) \in H^m_{0,*}[0,T].$$

The further assertions follow due to previous computations. □

Remark B.36. *In view of the previous lemma one may ask whether the assertions also hold for $f \in H^m[0,T]$ (without the homogeneous initial conditions). It holds*

$$\mathbb{1}_{[0,T]}\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}(\partial_t^m f)) = \mathbb{1}_{[0,T]}\mathcal{L}^{-1}(B(s)\mathcal{L}(\partial_t^{-m}\partial_t^m f)) = B(\partial_t)(\partial_t^{-m}\partial_t^m f),$$

i.e. this is just $B(\partial_t)$ applied to $\partial_t^{-m}\partial_t^m f$, which again fulfills the homogeneous initial conditions.

Lemma B.37. *In the situation of Lemma B.35, it holds for $f \in H^{m+1}_{0,*}[0,T]$*

$$B(\partial_t)f = \mathcal{L}^{-1}(B(s)s^{-(m+1)}) * \partial_t^{m+1}f.$$

Furthermore $\mathcal{L}^{-1}(B(s)s^{-(m+2)})$ is continuous and we have for $f \in H^{(m+2)}_{0,*}[0,T]$

$$B(\partial_t)f = \mathcal{L}^{-1}(B(s)s^{-m+2}) * \partial_t^{m+2}f.$$

Proof. The assertion follows due to Causality and Lemma B.24. □

We collect the following properties in analogue to the case on $[0,\infty)$.

Theorem B.38. *Let $f \in L^2_*[0,T]$ and functions $B(s)$ and causal $A(s)$. If $B(\partial_t)f$ and $A(\partial_t)B(\partial_t)f$ exist, then $(AB)(\partial_t)f$ exists and it equals*

$$(AB)(\partial_t)f = A(\partial_t)B(\partial_t)f.$$

If furthermore $A(\partial_t)f$ and $B(\partial_t)A(\partial_t)f$ exist and B is causal, it holds

$$(AB)(\partial_t)f = A(\partial_t)B(\partial_t)f = B(\partial_t)A(\partial_t)f.$$

Proof. Let $f \in L^2_*[0,\infty)$ and $n,m,p \in \mathbb{N}$ be integers for the existence of $B(\partial_t)f$, $A(\partial_t)B(\partial_t)f$ and $(AB)(\partial_t)f$. It holds for $g := B(\partial_t)f$, that $A(\partial_t)g$ exists and

$$\begin{aligned} A(\partial_t)g &= \partial_t^m\mathcal{L}^{-1}(A(s)s^{-m}\mathcal{L}g) \\ &= \partial_t^m\partial_t^n\partial_t^{-n}\mathcal{L}^{-1}(A(s)s^{-m}\mathcal{L}g) \\ &= \partial_t^{m+n}\mathcal{L}^{-1}(A(s)s^{-m}s^{-n}\mathcal{L}g) \\ &= \partial_t^{m+n}\mathcal{L}^{-1}(A(s)s^{-m}\mathcal{L}(\partial_t^{-n}g)). \end{aligned}$$

So we obtain that $\partial_t^n A(\partial_t)\partial_t^{-n} g$ exists, and we can use the Causality property of A. As $B(\partial_t)f$ exists, it is

$$B(s)s^{-n}\mathcal{L}(\mathbb{1}_{[0,T]}f) \in \mathcal{H}$$

and $\mathcal{L}^{-1}(B(s)s^{-n}\mathcal{L}f)$ is a function in $[0,\infty)$, which we will use in the following as extension of $\partial_t^{-n} g$ to $[0,\infty)$. Thus, by the Causality of A, we have

$$\begin{aligned} A(\partial_t)g &= \partial_t^{m+n}\mathcal{L}^{-1}(A(s)s^{-m}\mathcal{L}\partial_t^{-n}g) \\ &= \partial_t^{m+n}\mathcal{L}^{-1}(A(s)B(s)s^{-m-n}\mathcal{L}f). \end{aligned}$$

This already shows that $AB(\partial_t)$ exists and that p can be chosen smaller or equal than $p \leq m+n$.
If furthermore $A(\partial_t)f$ and $B(\partial_t)A(\partial_t)f$ exist and B is causal, then the same calculations give $(BA)(\partial_t)f = B(\partial_t)A(\partial_t)f$ and $(BA)(\partial_t)f = (AB)(\partial_t)f$ concludes the assertion.

□

Corollary B.39. *For $A \in \mathcal{H}_m$, $B \in \mathcal{H}_n$, $AB \in \mathcal{H}_p$, $f \in H_{0,*}^{\max(m,n,p)}[0,T]$ it holds*

$$(AB)(\partial_t)f = A(\partial_t)B(\partial_t)f = B(\partial_t)A(\partial_t)f.$$

Theorem B.40 (Herglotz theorem on $[0,T]$, cf. [27, Lemma 2.2])**.** *Let $B, R \in \mathcal{H}_m(\sigma_0)$ for $\sigma_0 \in \mathbb{R}$. If there exists a $c > 0$ such that for all $w \in \mathbb{C}$, all $\Re s > \sigma_0$*

$$\Re(\overline{w}B(s)w) \geq c|R(s)w|^2,$$

then it holds for all $w \in H_{0,*}^m[0,T]$, for all $\sigma \geq \sigma_0$

$$\int_0^T e^{-2\sigma t}\Re(\overline{w(t)}B(\partial_t)w(t))\,\mathrm{d}t \geq c\|e^{-\sigma \cdot}R(\partial_t)w\|_{L^2[0,T]}^2.$$

Proof. a) For B being a smoothing operator, this follows immediately by the Herglotz theorem on $[0,\infty)$ and Causality, as the function can be approximated by a sequence converging to 0 on $[T,\infty)$. For non-smoothing operators this is in general not possible, as they depend on derivatives, which explode, if we approximate the function in that way. We will show the result by the discrete Herglotz theorem and the convergence of Convolution Quadrature. Therefore, at first, the higher regularity $m+4$ is needed, which can be eliminated later by a density argument.

Let $w \in C^{m+4}[0,T]$ and $w(0) = \cdots = w^{(m+3)}(0) = 0$. If we use Convolution Quadrature with the second order backward difference formula (cf. [27, Chapter 2.3]), it holds by the discrete Herglotz theorem (cf. [27, Lemma 2.3]) for a $\rho = e^{-\sigma\tau} + O(\tau^2)$ and every function $v : [0,\infty) \to \mathbb{R}$ with finite support

$$\sum_{n=0}^{\infty} \rho^{2n}\Re\overline{v(t_n)}B(\partial_t^\tau)v(t_n) \geq c\sum_{n=0}^{\infty}\rho^{2n}\|R(\partial_t^\tau)v(t_n)\|^2$$

Therefore, for a smooth enough extension w_N of w, with $w_N(t_n) = 0$ for $n > N$ we obtain by discrete Causality

$$\begin{aligned} \sum_{n=0}^{N} \rho^{2n}\Re\overline{w(t_n)}B(\partial_t^\tau)w(t_n) &= \sum_{n=0}^{\infty}\rho^{2n}\Re\overline{w_N(t_n)}B(\partial_t^\tau)w_N(t_n) \\ &\geq c\sum_{n=0}^{\infty}\rho^{2n}|R(\partial_t^\tau)w_N(t_n)|^2 \\ &\geq c\sum_{n=0}^{N}\rho^{2n}|R(\partial_t^\tau)w_N(t_n)|^2 \\ &= \sum_{n=0}^{N}\rho^{2n}|R(\partial_t^\tau)w(t_n)|^2. \end{aligned}$$

As $wB(\partial_t)w$ and $|R(\partial_t)w|^2$ are differentiable, with continuous and therefore bounded derivative, it holds for $\tau \to 0$

$$\begin{aligned}\left|\tau \sum_{n=0}^{N} e^{-2\sigma t_n} \Re(\overline{w(t_n)} B(\partial_t) w(t_n)) - \int_0^T e^{-2\sigma\tau} \Re(\overline{w(\tau)} B(\partial_t) w(\tau))\, \mathrm{d}\tau\right| \\ \leq \tau \sum_{n=1}^{N} \sup_{\zeta\in[t_{n-1},t_n]} |e^{-2\sigma t_n} \Re(\overline{w(t_n)} B(\partial_t) w(t_n)) - e^{-2\sigma(\zeta)} \Re(\overline{w(\zeta)} B(\partial_t) w(\zeta))| \\ \leq \tau^2 \sum_{n=1}^{N} \sup_{\zeta\in[t_{n-1},t_n]} |\partial_\zeta e^{-2\sigma\zeta} (\overline{w(\zeta)} B(\partial_t) w(\zeta))| \to 0.\end{aligned}$$

and similar statements for $|R(\partial_t^\tau)w|^2$

By [116, Theorem 2.2 and following remarks] or [115, Theorem 3.1], the approximations converge uniformly in $0 \leq t_n \leq T$, for $\tau \to 0$,

$$|B(\partial_t^\tau)w(t_n) - B(\partial_t)w(t_n)| \leq C\tau^2$$

and

$$|R(\partial_t^\tau)w(t_n) - R(\partial_t)w(t_n)| \leq C\tau^2.$$

As w is continuous, it is bounded on $[0, T]$ and we have

$$\begin{aligned}\left|\tau \sum_{n=0}^{N} e^{-2\sigma t_n} \overline{w(t_n)} B(\partial_t^\tau) w(t_n) - e^{-2\sigma t_n} w(t_n) B(\partial_t) w(t_n)\right| &\leq C\tau^3 \sum_{n=0}^{N} |e^{-2\sigma t_n} w(t_n)| \\ &\leq C(w) \max(T, e^{-\sigma_0 T})\tau^2 \\ &\leq C(w, \sigma_0, T)\tau^2 \to 0.\end{aligned}$$

It is $R(\partial_t)w$ continuous and therefore pointwise bounded, and by the convergence, also $R(\partial_t^\tau)w(t_n)$ is uniformly bounded for $0 \leq t_n \leq T$. Therefore it holds

$$\begin{aligned}\left|\tau \sum_{n=0}^{N} e^{-2\sigma t_n} |R(\partial_t^\tau) w(t_n)|^2 - e^{-2\sigma t_n} |R(\partial_t) w(t_n)|^2\right| \\ \leq C\tau^3 \sum_{n=0}^{N} |e^{-2\sigma t_n} (R(\partial_t^\tau) w(t_n) + R(\partial_t) w(t_n))| \\ \leq C(R(\partial_t)w) \max(T, e^{-\sigma_0 T})\tau^2 \\ \leq C(R(\partial_t)w, \sigma_0, T)\tau^2 \to 0.\end{aligned}$$

As both, $\overline{w(t_n)} B(\partial_t^\tau) w(t_n)$ and $|R(\partial_t^\tau) w(t_n)|^2$ are uniformly convergent to the continuous, bounded $\overline{w} B(\partial_t)w$ and $|R(\partial_t)w|^2$, they are uniformly bounded and it holds

$$\left|\tau \sum_{n=0}^{N} e^{-2\sigma t_n} \Re(\overline{w(t_n)} B(\partial_t^\tau) w(t_n)) - \tau \sum_{n=0}^{N} \rho^{2n} \Re(w(t_n) B(\partial_t^\tau) w(t_n))\right| = O(\tau^2) \to 0$$

and

$$\left|\tau \sum_{n=0}^{N} e^{-2\sigma t_n} |R(\partial_t^\tau) w(t_n)|^2 - \tau \sum_{n=0}^{N} \rho^{2n} |R(\partial_t^\tau) w(t_n)|^2\right| = O(\tau^2) \to 0.$$

All in all, we have by the limit $\tau \to 0$ for $w \in C^{m+4}[0, T]$ with

$$w(0) = \cdots = w^{(m+1)}(0) = 0,$$

that

$$\int_0^T e^{-2\sigma t}\Re(\overline{w(t)}B(\partial_t)w(t))\ \mathrm{d}t \geq c\int_0^T e^{-2\sigma\tau}|R(\partial_t)w(\tau)|^2\ \mathrm{d}\tau.$$

For arbitrary $w \in H_0^m[0,T]$, the assertion follows by an approximating sequence in $w_n \in C^{m+4}[0,T]$ with zero initial condition in derivatives, which converges in $H_{0,*}^m[0,T]$ to w. Due to the continuous dependency shown in Lemma B.35 we have the convergence

$$\begin{aligned} w_n &\to w && \text{in } L^2[0,T],\\ B(\partial_t)w_n &\to B(\partial_t)w && \text{in } L^2[0,T],\\ R(\partial_t)w_n &\to R(\partial_t)w && \text{in } L^2[0,T], \end{aligned}$$

which yields the assertion. □

If we compare this result to the one on $[0,\infty)$ from Theorem B.30, the second case is missing.

Remark B.41. *In the case*

$$|B(s)| \leq C|R(s)|,$$

with the same ideas of proof, we can not get an estimate which is valid in the finite interval setting, because the R-term is on the other side of the estimate.

From Lemma B.30 we would obtain

$$\int_0^T e^{-2\sigma t}|B(\partial_t)w(t)|^2\ \mathrm{d}t \leq \int_0^\infty e^{-2\sigma t}|B(\partial_t)w(t)|^2 \leq C\|R(\partial_t)w\|^2_{L^2_\sigma[0,\infty)},$$

and the dependency of the last term on the right hand side on $[0,\infty)$ cannot be removed in general. I the case $R(s) = s^m$, one can obtain by Causality and a density argument (approximating $\partial_t^m w_N \to \mathbb{1}_{[0,T]}\partial_t^m w$ for $N \to \infty$)

$$\int_0^T e^{-2\sigma t}|B(\partial_t)w(t)|^2\ \mathrm{d}t \leq C(T,\sigma)\|\partial_t^m w\|^2_{L^2[0,T]},$$

which is again only on $[0,T]$. Similar arguments work for an operator R that satisfies $\|R(\partial_t)w\|^2_{L^2_\sigma(T,\infty)} \leq C\|\partial_t^m w\|^2_{L^2_\sigma(T,\infty)}$.

B.2. Vector Valued Laplace Transform and Differential Operators

In the following, for a Hilbert space X, we want to generalize the Laplace transform to Hilbert space valued functions $[0,\infty) \ni t \mapsto u(t) \in X$. For a family of operators $B(s) : X \to X$, we will define the corresponding convolution operator, with domain spaces living on $([0,\infty), X)$ and on $([0,T], X)$, respectively. This is done in a componentwise definition by using an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of X.

B.2.1. Vector valued Laplace transform on $[0,\infty)$

For a complex, separable Hilbert space X, we want to define the Laplace transform of a function $u : [0,\infty) \to X$ as

$$(\mathcal{L}u)(s) := \int_0^\infty u(t)e^{-st}\,\mathrm{d}t \qquad \text{for } s \in \mathbb{C}$$

and the inverse Laplace transform for $U : \{\Re(s) > \sigma_0\} \to X$ as

$$(\mathcal{L}^{-1}U)(t) := \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} e^{st}U(s)\,\mathrm{d}s \qquad \text{for } t \in [0,\infty)$$

for a $\sigma \geq \sigma_0$. We will transfer the results of the scalar case where $X = \mathbb{C}$, by defining $\mathcal{L}u$ as the element satisfying

$$[\mathcal{L}(u)(s), v]_X = \mathcal{L}([u,v]_X)(s)$$

for all $v \in X$. Therefore, let $(e_j)_{j\in\mathbb{N}}$ be a orthonormal basis of X, i.e.

$$[e_i, e_j]_X = \delta_{ij}$$

and for every $v \in X$ it holds

$$v = \sum_{j=1}^{\infty} [e_j, v]_X e_j.$$

Lemma B.42. *It is $u = \sum_{j\in\mathbb{N}} u_j e_j \in X$ if and only if $\sum_{j\in\mathbb{N}} |u_j|^2 < \infty$. It holds $u^k = \sum_{j\in\mathbb{N}} u_j^k e_j \to u = \sum_{j\in\mathbb{N}} u_j e_j$ in X if and only if $\sum_{j\in\mathbb{N}} |u_j^k - u_j|^2 \to 0$ for $k \to \infty$.*

Definition B.43. *For an interval $I \subset \mathbb{R}$, we define*

$$L^2(I,X) := \{u : I \to X \text{ measurable} \mid \int_I \|u(t)\|_X^2\,\mathrm{d}t < \infty\}.$$

Lemma B.44. *We have $u \in L^2(I,X)$ if and only if for all $j \in \mathbb{N}$ the coefficients $[e_j,u]_X$ are measurable and $\sum_{j\in\mathbb{N}} \|[e_j,u]_X\|_{L^2(I)}^2 < \infty$. It holds $\|u\|_{L^2(I,X)}^2 = \|\|u\|_X\|_{L^2(I)}^2 = \|\|u\|_{L^2(I)}\|_X^2$.*

Proof. This follows from $L^2(I,X) \cong L^2(I) \otimes X$ and Hilbert space theory $X \cong l^2(\mathbb{N}) \cong L^2(I)$ for the space of square summable sequences $l^2(\mathbb{N})$. ☐

Definition B.45. *For $c \in \mathbb{R}$, we define the spaces*

$$L_c^2([0,\infty),X) := \{u : [0,\infty) \to X \text{ measurable} \mid e^{-c\,\cdot}\,u(\,\cdot\,) \in L^2([0,\infty),X)\},$$

equipped with the norm

$$\|u\|_{L_c^2([0,\infty),X)} := \|e^{-c\,\cdot}\,u\|_{L^2([0,\infty),X)}$$

and

$$L^2_*([0,\infty),X) := \{u : [0,\infty) \to X \text{ measurable} \mid e^{-c\cdot} u(\,\cdot\,) \in L^2([0,\infty),X) \text{ for a } c \in \mathbb{R}\}.$$

We define the Laplace transform for $s \in \mathbb{C}$, $\Re s \geq c$ and $u \in L^2_c([0,\infty),X)$

$$\mathcal{L}u(s) := \sum_{j=1}^{\infty} \mathcal{L}([e_j,u]_X)(s) e_j. \tag{B.10}$$

Lemma B.46. *The Laplace transform from Definition B.45 is welldefined for $\Re s \geq c$ in the sense that $\mathcal{L}u \in L^2(\sigma + i\mathbb{R}, X)$ for all $\sigma \geq c$ and it holds for all $v \in X$ and all $\Re s > c$*

$$[\mathcal{L}(u)(s), v]_X = \mathcal{L}([u,v]_X)(s). \tag{B.11}$$

Proof. For the welldefinedness we note that $t \mapsto [e_j, u(t)]_X$ is measurable and by the Cauchy–Schwartz estimate we have $[e_j, u(t)]_X \leq \|u(t)\|_X$, so $(t \mapsto [e_j, u(t)]_X) \in L^2_*[0,\infty)$. With Fubini's theorem we have

$$\begin{aligned}
\sum_{j\in\mathbb{N}} \|\mathcal{L}[e_j,u]_X\|^2_{L^2(\sigma+i\mathbb{R})} &= \sum_{j\in\mathbb{N}} \|[e_j,u]_X\|^2_{L^2_\sigma[0,\infty)} \\
&= \int_0^\infty \sum_{j\in\mathbb{N}} e^{-\sigma t} |[e_j,u(t)]_X|^2 \,\mathrm{d}t \\
&= \|u\|_{L^2_\sigma([0,\infty),X)} < \infty
\end{aligned}$$

and Lemma B.44 shows the welldefinedness for $\sigma \geq c$.

By the real valued estimates (cf. Lemma B.1), we have for $c \in \mathbb{R}$, $u \in L^2_c([0,\infty),X)$, $\Re s > c$

$$|\mathcal{L}([e_j,u]_X)(s)| \leq \|e^{-s\cdot}[e_j,u(\cdot)]_X\|_{L^1[0,\infty)} \leq C(s,c)\|[e_j,u]_X\|_{L^2_c[0,\infty)}$$

and therefore by Fubini's theorem

$$\begin{aligned}
\|\mathcal{L}u(s)\|^2_X &= \sum_{j\in\mathbb{N}} |\mathcal{L}[e_j,u](s)|^2 \\
&\leq C(s,c) \sum_{j=1}^{\infty} \|[e_j,u]\|^2_{L^2_c[0,\infty)} \\
&= C(s,c) \sum_{j=1}^{\infty} \int_0^\infty |[e_j,u](t)|^2 \,\mathrm{d}t \\
&= C(s,c) \int_0^\infty e^{-ct} \sum_{j\in\mathbb{N}} |[e_j,u(t)]|^2 \,\mathrm{d}t \\
&= C(s,c)\|u\|^2_{L^2_c([0,\infty),X)} < \infty.
\end{aligned}$$

Thus the defining sum is a Cauchy sequence in X for each $s \in \{\Re s > c\}$ and we have the welldefinedness of (B.10). By the pointwise boundedness of $\mathcal{L}u(s)$, it is for $\Re s > c$ (summation and Laplace transform can be interchanged, see the following estimates)

$$\begin{aligned}
[\mathcal{L}u(s), v]_X &= \sum_{j\in\mathbb{N}} [\mathcal{L}u(s), e_j]_X [e_j, v]_X \\
&= \sum_{j\in\mathbb{N}} \mathcal{L}([u,e_j]_X)(s)[e_j,v]_X \\
&= \mathcal{L}(\sum_{j\in\mathbb{N}} [u,e_j]_X [e_j,v]_X)(s) \\
&= \mathcal{L}([u,v]_X)(s).
\end{aligned}$$

The sum and the Laplace transform can be interchanged by Fubini's theorem: It holds with Lemma B.44 and again Fubini's theorem

$$\begin{aligned}\sum_{j\in\mathbb{N}}\int_0^\infty |e^{-st}[u(s),e_j]_X[e_j,v]_X|\ \mathrm{d}t &= \sum_{j\in\mathbb{N}}|[e_j,v]_X|\int_0^\infty e^{-\Re st}|[u(t),e_j]_X|\ \mathrm{d}t\\ &\le \sum_{j\in\mathbb{N}}|[e_j,v]_X|\|[u,e_j]_X\|_{L^1_{\Re s}[0,\infty)}\\ &\le C(s,c)\sum_{j\in\mathbb{N}}|[e_j,v]_X|\|[u,e_j]_X\|_{L^2_c[0,\infty)}\\ &\le C(s,c)\|v\|_X\left(\sum_{j\in\mathbb{N}}\|[u,e_j]_X\|^2_{L^2_c[0,\infty)}\right)^{1/2}\\ &\le C(s,c)\|v\|_X\|u\|_{L^2_c([0,\infty),X)}<\infty.\end{aligned}$$

□

Theorem B.47 (Plancherel's formula, cf. [19, Theorem 1.8.2]). *It holds for $u,v\in L^2_c[0,\infty)$ for all $\sigma\ge c$*

$$\int_0^\infty e^{-2\sigma t}[u(t),v(t)]_X\ \mathrm{d}t=\frac{1}{2\pi}\int_{\sigma+i\mathbb{R}}[\mathcal{L}u(s),\mathcal{L}v(s)]_X\ \mathrm{d}s,$$

especially we have

$$\|\mathcal{L}u\|_{\sigma+i\mathbb{R},X}=\sqrt{2\pi}\|u\|_{L^2_\sigma[0,\infty),X}.$$

Instead of the Hilbert space scalar product $[\cdot,\cdot]_X$, the result also holds for any continuous and sesquilinear product on X.

Proof. Let $a[\cdot,\cdot]$ be a continuous, sesqilinear product on X with continuity constant $C>0$.

The assertion is a direct consequence of Plancherel's formula from the scalar case (see Theorem B.4), by using the component wise definition of the Laplace transform and interchanging sum and integral (which is possible due to the majorants $C\|\mathcal{L}u(s)\|_X\|\mathcal{L}v(s)\|_X$ and $Ce^{-2\sigma t}\|u(t)\|_X\|v(t)\|_X$)

$$\begin{aligned}a[\mathcal{L}u,\mathcal{L}v]_{\sigma+i\mathbb{R},X}&=\int_{\sigma+i\mathbb{R}}a[\mathcal{L}u(s),\mathcal{L}v(s)]\ \mathrm{d}s\\ &=\int_{\sigma+i\mathbb{R}}\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\overline{\mathcal{L}[e_i,u]_X(s)}\mathcal{L}[e_j,v]_X(s)a[e_i,e_j]\ \mathrm{d}s\\ &=\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\int_{\sigma+i\mathbb{R}}\overline{\mathcal{L}[e_i,u]_X(s)}\mathcal{L}[e_j,v]_X(s)a[e_i,e_j]\ \mathrm{d}s\\ &=\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}2\pi\int_0^\infty e^{-2\sigma t}\overline{[e_i,u(t)]_X}[e_j,v(t)]_Xa[e_i,e_j]\ \mathrm{d}t\\ &=2\pi\int_0^\infty e^{-2\sigma t}a[u(t),v(t)]\ \mathrm{d}t.\end{aligned}$$

□

Remark B.48. *Due to the dependency of the growth parameter $c\in\mathbb{R}$, we are not able to establish continuity*

$$\mathcal{L}:L^2_*([0,\infty),X)\to\operatorname{Im}(\mathcal{L}).$$

However, for $u\in L^2_c([0,\infty),X)$, Plancherel's formula yields $\sigma\ge c$

$$\|\mathcal{L}u\|_{L^2(\sigma+i\mathbb{R},X)}=\sqrt{2\pi}\|e^{-\sigma\,\cdot}u\|_{L^2([0,\infty),X)}$$

and it holds

$$\|e^{-c\cdot}u\|_{([0,\infty),X)} = \sup_{\sigma>c} \|e^{-\sigma\cdot}u\|_{L^2([0,\infty),X)}.$$

Thus, for arbitrary, but fixed $c \in \mathbb{R}$

$$\mathcal{L} : (L^2_c([0,\infty),X), \|\cdot\|_{L^2_c([0,\infty),X)}) \to (\mathrm{Im}(\mathcal{L}), \|\cdot\|_{\mathrm{Im}(\mathcal{L})})$$

is continuous and with bounded inverse on its image $\mathrm{Im}(\mathcal{L})$*, which we will determine in the following. A suitable choice of* $\|\cdot\|_{\mathrm{Im}(\mathcal{L})}$ *is*

$$\|\cdot\|_{\mathrm{Im}(\mathcal{L})} := \sup_{\sigma>c} \|\cdot\|_{L^2(\sigma+i\mathbb{R},X)}.$$

Definition B.49. *Let* $D \subset \mathbb{C}$ *open. We call* $U : D \to X$ *complex differentiable (holomorphic) in* $s_0 \in D$ *if*

$$\partial_s U(s_0) = \lim_{s\to s_0} \frac{U(s)-U(s_0)}{s-s_0}$$

exists, i.e.

$$\lim_{s\to s_0} \|\frac{U(s)-U(s_0)}{s-s_0} - \partial_s U(s_0)\|_X = 0.$$

Lemma B.50. *For* $u \in L^2_c([0,\infty),X)$*,* $U := \mathcal{L}u$ *is complex differentiable on* $\{\Re s > 0\}$*.*

Proof. We show that $\partial_s U(s) = \mathcal{L}(-tu(t))(s)$. The existence of $\mathcal{L}(-tu(t))(s)$ for $\Re s > c$ follows as in the scalar case. Let $\Re s_1, \Re s_2 > c + \delta$, it holds by Lemma B.46

$$\begin{aligned}
&\frac{\|U(s_2)-U(s_1)-(s_2-s_1)\partial_s U(s_1)\|_X}{|s_2-s_1|} \\
&\qquad = \sup_{v\in X,\|v\|_X=1} \frac{|[U(s_2)-U(s_1)-(s_2-s_1)\partial_s U(s_1), v]_X|}{|s_2-s_1|} \\
&\qquad = \sup_{v\in X,\|v\|_X=1} \frac{|\int_0^\infty e^{-s_2t}-e^{-s_1t}-(s_2-s_1)te^{-s_2t}[u(t),v]_X \,\mathrm{d}t|}{|s_2-s_1|} \\
&\qquad \le \int_0^\infty \frac{|e^{-s_2t}-e^{-s_1t}-(s_2-s_1)te^{-s_2t}|}{|s_1-s_2|}\|u(t)\|_X \,\mathrm{d}t
\end{aligned}$$

As in the scalar case, an integrable majorant exists and we can interchange integral and limit $s_1 \to s_2$. Choosing $\delta > 0$ arbitrary small, we conclude see that U is complex differentiable on $\{\Re s > 0\}$. □

Lemma B.51. *For* $u \in L^2_c([0,\infty),X)$ *the* L^2*-norm over each vertical line of* $U := \mathcal{L}u$ *is uniformly bounded for* $\sigma \ge c$ *and it holds*

$$\|U\|_{L^2(c+i\mathbb{R},X)} = \sup_{\sigma>c} \|U\|_{L^2(\sigma+i\mathbb{R},X)} < \infty.$$

Proof. We have by Plancherel's formula (integral and summation can be exchanged, as all terms are bounded, cf. Lemma B.44)

$$\begin{aligned}
\sup_{\sigma>c} \|U(\sigma+i\,\cdot\,)\|^2_{L^2((-\infty,\infty),X)} &= \sup_{\sigma>c} \int_{\sigma+i\mathbb{R}} \|U(s)\|_X^2 \,\mathrm{d}s \\
&= \sup_{\sigma>c} \int_{\sigma+i\mathbb{R}} \sum_{j\in\mathbb{N}} |\mathcal{L}([e_j,u]_X)(s)|^2 \,\mathrm{d}s \\
&= \sup_{\sigma>c} \sum_{j\in\mathbb{N}} \int_{\sigma+i\mathbb{R}} |\mathcal{L}([e_j,u]_X)(s)|^2 \,\mathrm{d}s \\
&= \sup_{\sigma>c} \sum_{j\in\mathbb{N}} 2\pi \int_0^\infty e^{-\sigma t}|[e_j,u(t)]|^2 \,\mathrm{d}t \\
&= \sup_{\sigma>c} 2\pi \|e^{-\sigma\cdot}u\|^2_{L^2([0,\infty),X)} \\
&\le 2\pi \|e^{-c\cdot}u\|^2_{L^2([0,\infty),X)}.
\end{aligned}$$

By Lebeque's theorem it holds

$$\sup_{\sigma>c} \|e^{-\sigma\cdot} u\|^2_{L^2([0,\infty),X)} = \lim_{\sigma\to c} \|e^{-\sigma\cdot} u\|^2_{L^2([0,\infty),X)} = \|e^{-c\cdot} u\|^2_{L^2([0,\infty),X)}$$

and therefore by Plancherel's formula

$$\sup_{\sigma>c} \|U(\sigma + i\cdot)\|^2_{L^2(\mathbb{R},X)} = 2\pi\|e^{-c\cdot} u\|^2_{L^2([0,\infty),X)} = \|U(c + i\cdot)\|^2_{L^2(\mathbb{R},X)}.$$

□

The properties of $U := \mathcal{L}u$ shown in Lemma B.50 and Lemma B.51 are already enough, to apply the inverse Laplace transform.

The space of the functions with these properties, to be complex differentiable and uniformly square integrable over each vertical line for high enough real part, is the right choice as domain space of the inverse Laplace transform.

Again we use a componentwise approach and the properties derived in the scalar case to define the inverse Laplace transform as

$$\mathcal{L}^{-1}U := \sum_{j\in\mathbb{N}} \mathcal{L}^{-1}([e_j, U]_X)e_j.$$

Definition B.52. *We collect the inverse Laplace transformable functions in the Hardy space*

$$U \in \mathcal{H}(X) := \{U \mid \text{For a } \sigma_0 \in \mathbb{R},\ U(s) : \{\Re s > \sigma_0\} \to X \text{ is holomorph and } \sup_{\sigma>\sigma_0} \int_{\sigma+i\mathbb{R}} \|U(s)\|^2_X \,\mathrm{d}s < \infty\}$$

and define the inverse Laplace transform as

$$\mathcal{L}^{-1}U := \sum_{j\in\mathbb{N}} \mathcal{L}^{-1}([e_j, U]_X)e_j.$$

For $\sigma_0 \in \mathbb{R}$, we introduce the space

$$\mathcal{H}(\sigma_0, X) := \{U \mid U(s) : \{\Re s > \sigma_0\} \to X \text{ is analytic and } \sup_{\sigma>\sigma_0} \int_{\sigma+i\mathbb{R}} \|U(s)\|^2_X \,\mathrm{d}s < \infty\},$$

equipped with the norm

$$\|u\|^2_{\mathcal{H}(\sigma_0)} := \sup_{\sigma>\sigma_0} \int_{\sigma+i\mathbb{R}} \|U(s)\|^2_X \,\mathrm{d}s.$$

We summarize the properties and the welldefinedness of the inverse Laplace transform.

Theorem B.53. *The inverse Laplace transform is a well defined operator*

$$\mathcal{L}^{-1} : \mathcal{H}(\sigma_0, X) \to L^2_{\sigma_0}([0,\infty), X).$$

Furthermore, $\mathcal{L}^{-1}$ is the inverse of the Laplace transform, i.e. for $U \in \mathcal{H}(\sigma_0, X)$ there exists exactly one $u \in L^2_*([0,\infty), X)$, such that $U = \mathcal{L}u$ and u is given through

$$\mathcal{L}^{-1}U = u.$$

Each $U \in \mathcal{H}(\sigma_0, X)$ is extendable to $\{\Re s = \sigma_0\}$ in $L^2(\sigma_0 \times i\mathbb{R})$ by the L^2- limit $\sigma \to \sigma_0$ and it holds

$$2\pi\|e^{-\sigma_0\cdot} u\|^2_{L^2([0,\infty),X)} = \sup_{\widetilde{\sigma}>\sigma_0} \int_{\widetilde{\sigma}+i\mathbb{R}} \|U(s)\|^2_X \,\mathrm{d}s = \int_{\sigma_0+i\mathbb{R}} \|U(s)\|^2_X \,\mathrm{d}s.$$

This gives a one-to-one identity through the Laplace transform between $L^2_*([0,\infty), X)$ and $\mathcal{H}(X)$ and between $L^2_c([0,\infty), X)$ and $\mathcal{H}(c, X)$.

Proof. As U is holomorph, each of the components is holomorph:

$$\lim_{h\to 0} \frac{|[e_j, U]_X(s+h) - [e_j, U]_X(s) - h[e_j, U']_X(s)|}{h} \leq \lim_{h\to 0} \frac{\|U(s+h) - U(s) - hU'(s)\|}{h} = 0.$$

Similarly, the $\mathcal{H}$-boundedness follows for $\sigma > \sigma_0$ for each component

$$\int_{\sigma+i\mathbb{R}} |[e_j, U(s)]_X|^2 \,\mathrm{d}s \leq \int_{\sigma+i\mathbb{R}} \|U(s)\|_X^2 \,\mathrm{d}s = \|U\|_{L^2(\sigma+i\mathbb{R},X)} < \infty.$$

We conclude $[e_j, U]_X \in \mathcal{H}(\sigma_0)$ for all $j \in \mathbb{N}$ and the inverse Laplace transform of each of the components exists. The sum over the components converges in $L^2_{\sigma_0}([0,\infty), X)$: By Fatou's lemma and the scalar version of Plancherel's formula from Theorem B.4 we deduce

$$\begin{aligned}
\|\mathcal{L}^{-1}U\|^2_{L^2_{\sigma_0}([0,\infty),X)} &= \int_0^\infty e^{-2\sigma_0 t} \|\mathcal{L}^{-1}U(t)\|_X^2 \,\mathrm{d}t \\
&= \int_0^\infty e^{-2\sigma_0 t} \sum_{j\in\mathbb{N}} |[e_j, \mathcal{L}^{-1}U(t)]_X|^2 \,\mathrm{d}t \\
&\leq \sum_{j\in\mathbb{N}} \int_0^\infty e^{-2\sigma_0 t} |\mathcal{L}^{-1}[e_j, U]_X(t)|^2 \,\mathrm{d}t.
\end{aligned}$$

With Fubini's theorem we conclude

$$\begin{aligned}
\|\mathcal{L}^{-1}U\|^2_{L^2_{\sigma_0}([0,\infty),X)} &= \frac{1}{2\pi} \sum_{j\in\mathbb{N}} \int_{\sigma_0+i\mathbb{R}} |[e_j, U]_X(s)|^2 \,\mathrm{d}s \\
&= \frac{1}{2\pi} \int_{\sigma_0+i\mathbb{R}} \sum_{j\in\mathbb{N}} |[e_j, U]_X(s)|^2 \,\mathrm{d}s \\
&= \frac{1}{2\pi} \int_{\sigma_0+i\mathbb{R}} \|U(s)\|^2 \,\mathrm{d}s \\
&= \frac{1}{2\pi} \|U\|_{\mathcal{H}(\sigma_0,X)} < \infty
\end{aligned}$$

and the welldefinedness follows from Lemma B.44.

Each of the coefficients $[e_j, U]_X$ can be written as $\mathcal{L}u_j$ for $u_j \in L^2_{\sigma_0}[0,\infty)$ and by the coefficientwise definitions it follows that $U = \mathcal{L}u = \mathcal{L}\left(\sum_{j\in\mathbb{N}} u_j e_j\right)$ holds.

Uniqueness of $u \in L^2_*([0,\infty), X)$ follows by Plancherel's formula: Let $u_1 \in L^2_{\sigma_1}$, $u_2 \in L^2_{\sigma_2}$, $\mathcal{L}u_1 = U$, $\mathcal{L}u_2 = U$, then it holds by Plancherel's formula and linearity of the Laplace transform

$$\begin{aligned}
0 &= \sup_{\widetilde{\sigma}>\sigma_0} \int_{\widetilde{\sigma}+i\mathbb{R}} \|U(s) - U(s)\|_X^2 \,\mathrm{d}s \\
&\geq \sup_{\widetilde{\sigma}>\max(\sigma_1,\sigma_2)} \int_{\widetilde{\sigma}+i\mathbb{R}} \|U(s) - U(s)\|_X^2 \,\mathrm{d}s \\
&= \sup_{\widetilde{\sigma}>\max(\sigma_1,\sigma_2)} 2\pi \|e^{-\widetilde{\sigma}\cdot}(u_1 - u_2)\|^2_{L^2([0,\infty),X)} \\
&\geq 0,
\end{aligned}$$

thus $u_1 = u_2$. With Plancherel's formula and the monotone convergence theorem we deduce

$$\begin{aligned}
\sup_{\widetilde{\sigma}>\sigma_0} \int_{\widetilde{\sigma}+i\mathbb{R}} \|U(s)\|_X^2 \,\mathrm{d}s &= \sup_{\widetilde{\sigma}>\sigma_0} 2\pi \|e^{-\widetilde{\sigma}\cdot} u\|^2_{L^2([0,\infty),X)} \\
&= 2\pi \|e^{-\sigma_0\cdot} u\|^2_{L^2([0,\infty),X)},
\end{aligned}$$

so $u \in L^2_{\sigma_0}[0,\infty)$. As $U = \mathcal{L}u$ on $\{\Re s > \sigma_0\}$ and $u \in L^2_{\sigma_0}([0,\infty), X)$, $U(\sigma + i\,\cdot\,)$, $\sigma \to \sigma_0$ is a Cauchy sequence, i.e. by Lebesgue theorem we have

$$\begin{aligned}\|U(\sigma_1 + i\,\cdot\,) - U(\sigma_2 + i\,\cdot\,)\|^2_{L^2(\mathbb{R},X)} &= 2\pi \int_0^\infty \|(e^{-\sigma_1 t} - e^{-\sigma_2 t})u(t)\|^2_X \,\mathrm{d}t \\ &\to 0 \quad \text{for } \sigma_1, \sigma_2 \to \sigma_0,\end{aligned}$$

so by the completeness of $L^2(\mathbb{R}, X)$

$$U(\sigma_0 + i\,\cdot\,) := L^2(\mathbb{R}, X) - \lim_{\sigma \to \sigma_0} U(\sigma + i\,\cdot\,)$$

exists and it holds

$$\begin{aligned}\sup_{\widetilde{\sigma} > \sigma_0} \int_{\widetilde{\sigma} + i\mathbb{R}} \|U(s)\|^2 \,\mathrm{d}s &= \sup_{\widetilde{\sigma} > \sigma_0} 2\pi \|e^{-\widetilde{\sigma}\,\cdot}\, u\|^2_{L^2([0,\infty),X)} \\ &= \lim_{\widetilde{\sigma} \to \sigma_0} 2\pi \|e^{-\widetilde{\sigma}\,\cdot}\, u\|^2_{L^2([0,\infty),X)} \\ &= \int_{\sigma_0 + i\mathbb{R}} \|U(s)\|^2 \,\mathrm{d}s.\end{aligned}$$

□

Lemma B.54. *For $c \in \mathbb{R}$, a continuous operator $B : X \to X$ and $u \in L^2_c[0,\infty)$ and $U \in \mathcal{H}(c, X)$ it holds for $\Re s > c$*

$$\mathcal{L}(Bu)(s) = B(\mathcal{L}U)(s)$$

and in $L^2_c([0,\infty), X)$

$$\mathcal{L}^{-1}(BU) = B\mathcal{L}^{-1}(U).$$

Similar statements hold for $B : X \to Y$ for a Hilbert space Y.

Proof. From Lemma B.46 we obtain for $v \in X$ and $\Re s > c$

$$[\mathcal{L}(Bu)(s), v]_X = \mathcal{L}([Bu, v]_X)(s) = \mathcal{L}([u, B'v]_X)(s) = [\mathcal{L}(u)(s), B'v]_X = [B\mathcal{L}(u)(s), v]_X.$$

From Theorem B.53, we conclude

$$\mathcal{L}^{-1}(BU) = \mathcal{L}^{-1}(B\mathcal{L}u) = \mathcal{L}^{-1}(\mathcal{L}Bu) = Bu = B\mathcal{L}^{-1}(U).$$

□

B.2.2. Vector valued Laplace differential operators on $[0,\infty)$

We denote by $L(X)$ the linear, bounded operators $X \to X$. For a function $B : \{\Re s > \sigma_0\} \to L(X)$ we want to define $B(\partial_t)f$ as $\mathcal{L}^{-1}(B(s)\mathcal{L}(f)(s))$.
The following definition is very general and not practical and will be refined in the following.

Definition B.55. *For a function $B(s) : \{\Re s > \sigma_0\} \to L(X)$ for a $\sigma_0 \in \mathbb{R}$ and $f \in L^2_*([0,\infty), X)$, such that*

$$B(s)\mathcal{L}f \in \mathcal{H}, \tag{B.12}$$

we say that $B(\partial_t)f$ exists and we define $B(\partial_t)f$ as

$$B(\partial_t)f := \mathcal{L}^{-1}(B(s)\mathcal{L}(f)(s)). \tag{B.13}$$

Remark B.56. *By the definition of the Laplace transforms, it is with* $B_{i,j}(s) := [e_i, B(s)e_j]_X$ *and* $f_j := [e_j, f]_X$

$$B(s)\mathcal{L}f(s) = \sum_{i\in\mathbb{N}}[e_i, B(s)\sum_{j\in\mathbb{N}}\mathcal{L}[e_j, f]_X(s)e_j]_X e_i = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} B_{i,j}(s)\mathcal{L}f_j(s)e_i$$

and

$$B(\partial_t)f = \sum_{i\in\mathbb{N}}\mathcal{L}^{-1}\left(\sum_{j\in\mathbb{N}} B_{i,j}\mathcal{L}f_j\right)e_i.$$

For sufficiently bounded functions we can interchange Laplace transform and summation and get

$$B(\partial_t)f = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\mathcal{L}^{-1}\left(B_{i,j}\mathcal{L}f_j\right)e_i.$$

We require the following spaces of functions with homogeneous initial condition.

Definition B.57. *We define for* $m \in \mathbb{N}$ *the spaces with homogeneous initial condition up to order* m

$$\begin{aligned} H_0^m([0,\infty),X) &:= \{\phi \in H^m([0,\infty),X) \mid \phi(0) = \cdots = \phi^{(m-1)}(0) = 0\},\\ H_0^0([0,\infty),X) &:= L^2([0,\infty),X) \end{aligned}$$

equipped with the $H^m([0,\infty),X)$*-norm and the exponentially weighted spaces*

$$\begin{aligned} H_*^m([0,\infty),X) &:= \{\phi \mid e^{-c\cdot}\phi \in H^m([0,\infty),X) \text{ for a } c\in\mathbb{R}\},\\ H_*^0([0,\infty),X) &:= L_*^2([0,\infty),X) \end{aligned}$$

the exponentially weighted spaces with homogeneous initial condition

$$\begin{aligned} H_{0,*}^m([0,\infty),X) &:= \{\phi \mid e^{-c\cdot}\phi \in H_0^m([0,\infty),X) \text{ for a } c\in\mathbb{R}\},\\ H_{0,*}^0([0,\infty),X) &:= L_*^2([0,\infty),X). \end{aligned}$$

Furthermore, we define for fixed damping parameter $c \in \mathbb{R}$

$$\begin{aligned} H_c^m([0,\infty),X) &:= \{\phi \mid e^{-c\cdot}\phi \in H^m([0,\infty),X)\},\\ H_c^0([0,\infty),X) &:= L_c^2([0,\infty),X) \end{aligned}$$

and

$$\begin{aligned} H_{0,c}^m([0,\infty),X) &:= \{\phi \mid e^{-c\cdot}\phi \in H_0^m([0,\infty),X)\},\\ H_{0,c}^0([0,\infty),X) &:= L_c^2([0,\infty),X). \end{aligned}$$

We equip these spaces with the norms

$$\|u\|_{H^m([0,\infty),X)} := \|u\|_{H_0^m([0,\infty),X)} := \left(\sum_{j=0}^m \|\partial_t^j u\|^2_{L^2([0,\infty),X)}\right)^{1/2}$$

and

$$\|u\|_{H_c^m([0,\infty),X)} := \|u\|_{H_{0,c}^m([0,\infty),X)} := \|e^{-c\cdot}u\|_{H^m([0,\infty),X)}.$$

Remark B.58. *As in Remark B.12 we note for* $c \in \mathbb{R}$

$$f \in H_c^m([0,\infty),X) \Leftrightarrow f,\dots,\partial_t^m f \in L_c^2([0,\infty),X)$$

and the norm $\|\cdot\|_{H_c^m([0,\infty),X)}$ *(and* $\|\cdot\|_{H_{0,c}^m([0,\infty),X)}$*) is equivalent to the norm*

$$\|f\|^2 := \sum_{k=0}^m \|\partial_t^j f\|^2_{L_c^2([0,\infty),X)}.$$

Theorem B.59. *For $c \in \mathbb{R}$ and $m \in \mathbb{N}$ the spaces $H^m([0,\infty),X)$, $H^m([0,\infty),X)$, $H_c^m([0,\infty),X)$, $H_{0,c}^m([0,\infty),X)$ and $\mathcal{H}(c,X)$ are Hilbert spaces together with the scalar products*

$$[u,v]_{H^m([0,\infty),X)} := \sum_{j=0}^m [\partial_t^j u, \partial_t^j v]_{L^2([0,\infty),X)},$$
$$[u,v]_{H_0^m([0,\infty),X)} := \sum_{j=0}^m [\partial_t^j u, \partial_t^j v]_{L^2([0,\infty),X)},$$
$$[u,v]_{H_c^m([0,\infty),X)} := \sum_{j=0}^m [e^{-2c\cdot}\partial_t^j u, \partial_t^j v]_{L^2([0,\infty),X)},$$
$$[u,v]_{H_{0,c}^m([0,\infty),X)} := \sum_{j=0}^m [e^{-2c\cdot}\partial_t^j u, \partial_t^j v]_{L^2([0,\infty),X)},$$
$$[U,V]_{\mathcal{H}(c)} := \frac{1}{2\pi}[U,V]_{L^2(c+i\mathbb{R},X)}.$$

Proof. The proof can be generalized from the scalar setting to the Hilbert space valued one, by the use of the following properties. For Hilbert spaces X_1, X_2, and continuous linear operators $B_1 : X_1 \to X_1$, $B_2 : X_2 \to X_2$, we have that $X_1 \otimes X_2$ is again a Hilbert space and $B_1 \otimes B_2$ is continuous.

The space $\mathcal{H}(c)$ is a Hilbert space by the one to one identity of the Laplace transform from $\mathcal{L} : L_c^2([0,\infty),X) \to \mathcal{H}(c)$, which is an invertible isometry by Plancherel's formula. □

Remark B.60. *We consider $H^m([0,\infty),X) \cong H^m([0,\infty),\mathbb{R}) \otimes X$ and the weak derivative in time is a continuous operator*

$$\partial_t \otimes \mathrm{Id} : H^m([0,\infty),\mathbb{R}) \otimes X \to H^{m-1}([0,\infty),\mathbb{R}) \otimes X.$$

This yields for $f \in H_\star^m([0,\infty),X)$ the weak derivative ∂_t

$$\partial_t f := \sum_{j\in\mathbb{N}} \partial_t [f,e_j]_X e_j.$$

and for the integration operator ∂_t^{-1}

$$\partial_t^{-1} f = \sum_{j\in\mathbb{N}} \partial_t^{-1} [f,e_j]_X e_j.$$

Example B.61. *a) For the operator $B(s) = s$, $f \in H_{0,*}^1([0,\infty),X)$, it holds $s(\mathcal{L}f)(s) \in \mathcal{H}$ and we have*

$$B(\partial_t)f = \partial_t f.$$

Thus the Laplace differential operator ∂_t coincides with the weak derivative ∂_t, if f is weakly differentiable and $f(0) = 0$.
b) For the operator $B(s) = s^{-1}$, $f \in L_*^2([0,\infty),X)$ it holds $s^{-1}(\mathcal{L}f)(s) \in \mathcal{H}$ and we have

$$B(\partial_t) = \partial_t^{-1} f := \int_0^t f(\tau)\ \mathrm{d}\tau.$$

Thus the Laplace differential operator ∂_t^{-1} coincides with the integration over time $\int_0^t\ \mathrm{d}\tau$.

Proof. b) Let $f \in L_\sigma^2([0,\infty,X))$, then $s^{-1}\mathcal{L}f(s) \in \mathcal{H}(\max(\sigma,\varepsilon))$ for $\varepsilon > 0$. Furthermore it holds for $\Re s > \max(\sigma,\varepsilon)$ that $r \mapsto \frac{1}{\Re s} e^{-\Re s r} f(r) \in L^1([0,\infty),X)$ and therefore we have by Fubini's Theorem

$$
\begin{aligned}
\mathcal{L}(\partial_t^{-1} f)(s) &= \int_0^\infty e^{-st} \int_0^t f(r) \,\mathrm{d}r \,\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty \mathbb{1}_{r \le t} e^{-st} f(r) \,\mathrm{d}r \,\mathrm{d}t \\
&= \int_0^\infty \int_r^\infty e^{-st} \,\mathrm{d}t f(r) \,\mathrm{d}r \\
&= \int_0^\infty \frac{1}{s} e^{-sr} f(r) \,\mathrm{d}r \\
&= \frac{1}{s} \mathcal{L} f(s).
\end{aligned}
$$

As $s^{-1}\mathcal{L}f(s) \in \mathcal{H}$, it holds $\partial_t^{-1} f = \mathcal{L}^{-1} s^{-1} \mathcal{L} f(s)$.
a) Let $f \in H^1_{0,\sigma}([0,\infty), X)$ for $\sigma \in \mathbb{R}$. It is $\partial_t^{-1}\partial_t f = f$ and therefore by *b*) for $\Re s \geq \max(\sigma, \varepsilon) > 0$

$$
\frac{1}{s}\mathcal{L}(\partial_t f)(s) = \mathcal{L} f(s).
$$

As $\mathcal{L}(\partial_t f) \in \mathcal{H}$, it holds $\partial_t f = \mathcal{L}^{-1}(s\mathcal{L}f)$. □

With Example B.61 we are able to state concrete conditions for the existence in Definition B.55.

Lemma B.62. *In the setting of Definition B.55, if there exists an $m \in \mathbb{N}_0$, $\sigma_1 \in \mathbb{R}$ and a constant $C > 0$, such that B is holomorphic inside of its definition regime and*

$$
\|B(s)\|_{L(X)} \leq C|s|^m \text{ for all } \Re s > \sigma_1,
$$

then $B(\partial_t)f$ exists for every $f \in H^m_{0,*}([0,\infty), X)$ and it holds

$$
B(\partial_t) f = \partial_t^m \mathcal{L}^{-1}(B(s) s^{-m} \mathcal{L} f) = \mathcal{L}^{-1}(B(s) s^{-m} \mathcal{L}(\partial_t^m f)).
$$

We can define $B(\partial_t)$ as a continuous operator for $\sigma_2 \in \mathbb{R}$

$$
B(\partial_t) : H^m_{0,\sigma_2}([0,\infty), X) \to L^2_{\max(\sigma_1,\sigma_2)}([0,\infty), X).
$$

Proof. Let $f \in H^m_{0,\sigma_2}([0,\infty), X)$. It follows from Example B.61 that $s^m \mathcal{L} f \in \mathcal{H}(\sigma_2)$ and

$$
B(s)\mathcal{L} f \in \mathcal{H}(\max(\sigma_1, \sigma_2)),
$$

i.e. $B(\partial_t)f$ exists. By Example B.61, it is

$$
B(\partial_t) f = \partial_t^m \partial_t^{-m} B(\partial_t) f = \partial_t^m \mathcal{L}^{-1}(B(s) s^{-m} \mathcal{L} f)
$$

and as $f \in H^m_{0,\sigma_2}([0,\infty), X)$ and the coefficientwise definitions

$$
B(\partial_t) f = B(\partial_t) \partial_t^{-m} \partial_t^m f = \mathcal{L}^{-1}(B(s) s^{-m} \mathcal{L}(\partial_t^m f)).
$$

We have for $\phi \in H^m_{0,\sigma_2}([0,\infty), X)$, by Plancherel's formula and Example B.61

$$
\begin{aligned}
\|B(\partial_t)\phi\|_{L^2_{\max(\sigma_1,\sigma_2)}([0,\infty),X)} &= \frac{1}{2\pi} \|B(s)\mathcal{L}\phi\|_{\mathcal{H}(\max(\sigma_1,\sigma_2))} \\
&\leq C \|s^m \mathcal{L}\phi\|_{\mathcal{H}(\max(\sigma_1,\sigma_2))} \\
&= C \|\partial_t^m \phi\|_{L^2_{\max(\sigma_1,\sigma_2)}([0,\infty),X)} \\
&\leq \|\phi\|_{H^m_{0,\sigma_2}([0,\infty),X)}.
\end{aligned}
$$

□

Definition B.63. *We define for* $m \in \mathbb{N}_0$

$$\mathcal{H}_m := \{B \mid \text{There exists a } \sigma_0 \in \mathbb{R} \text{ such that } B : \{\Re s > \sigma_0\} \to L(X) \text{ is holomorphic and } \|B(s)\|_{L(X)} \leq C|s|^m \text{ for all } \Re s > \sigma_0\}$$

and for $\sigma_0 \in \mathbb{R}$

$$\mathcal{H}_m(\sigma_0) := \{B : \{\Re s > \sigma_0\} \to L(X) \text{ holomorphic} \mid \|B(s)\|_{L(X)} \leq C|s|^m \text{ for all } \Re s > \sigma_0\}.$$

We call $B \in \mathcal{H}_0$ *a smoothing operator.*

We want to apply the (inverse) Laplace transform to operators $B(s) : X \to X$ and convolute the outcome with functions $f(t) \in X$. Again, we define the respective operations in a componentwise way.

Definition B.64. *For a family of bounded linear operators* $A(t) : X \to X, t \in [0, \infty)$ *we define the convolution with* $b(t) \in X$ *as*

$$(A * b)(t) := \int_0^t A(\tau)b(t-\tau)\ \mathrm{d}\tau := \sum_{i\in\mathbb{N}} \left(\sum_{k\in\mathbb{N}} \int_0^t [e_i, A(\tau)e_k]_X [e_k, b(t-\tau)]_X\ \mathrm{d}\tau \right) e_i$$

and the Laplace transformed operator as

$$(\mathcal{L}A)(s)b := \sum_{i\in\mathbb{N}} \left(\sum_{k\in\mathbb{N}} \mathcal{L}([e_i, A(\cdot)e_k]_X)(s)[e_k, b]_X \right) e_i.$$

Similarly we define the inverse Laplace transform of an operator family $B(s) : X \to X$, $s \in \Re s > \sigma_0$ *entrywise as*

$$(\mathcal{L}^{-1}B)(t)b := \sum_{i\in\mathbb{N}} \left(\sum_{k\in\mathbb{N}} \mathcal{L}^{-1}([e_i, B(\cdot)e_k]_X)(s)[e_k, b]_X \right) e_i.$$

In the following we give some conditions, when this is welldefined.

The natural norm for the operator space is the induced norm $\|B\|_{L(X)} := \sup_{\|v\|_X=1} \|Bv\|_X$. The differency compared to the scalar case is now, with the induced norm, L(X) is no Hilbert space, but only a Banach space (and to use the Frobenius norm is not an option). Plancherel's formula does not hold in general.

In the following we elaborate the workaround, which uses estimates that hold in Banach spaces and that correspond to $\mathcal{L}^{-1} : L^1(i\mathbb{R}) \to L^\infty([0, \infty))$ (Plancherel's formula allowed us to consider $\mathcal{L}^{-1} : L^2(i\mathbb{R}) \to L^2([0, \infty))$ in the scalar case). After defining the (inverse) Laplace transform and convolution in a componentwise setting, we consider the operator applied to functions. The resulting function again lies in a Hilbert space and we can recover some estimates of Plancherel's formula type.

We start with some estimates for the (inverse) Laplace transform and the convolution.

Lemma B.65. *For* $e^{-\sigma_0\cdot}u \in L^1([0, \infty), X)$ *the Laplace integral exists for* $\sigma \geq \sigma_0$ *and is in* $L^\infty([\sigma_0, \infty) \times \mathbb{R}, X) \cap C([\sigma_0, \infty) \times \mathbb{R}, X)$. *It holds*

$$\|\mathcal{L}u\|_{L^\infty([\sigma_0,\infty)\times\mathbb{R},X)} \leq \|e^{-\sigma_0\cdot}u\|_{L^1([0,\infty),X)},$$

Proof. By the triangle inequality, it holds for arbitrary $s = \sigma + ir \in \{\Re \geq \sigma_0\}$

$$\begin{aligned} \|\mathcal{L}u(s)\|_X &\leq \int_0^\infty \|e^{-st}u(t)\|_X\ \mathrm{d}t \\ &= \|e^{-\sigma\cdot}u\|_{L^1([0,\infty),X)} \\ &\leq \|e^{-\sigma_0\cdot}u\|_{L^1([0,\infty),X)}. \end{aligned}$$

Continuity follows similarly by Lebesgue's theorem. □

Lemma B.66. *For $B \in L^1(\sigma_0 + i\mathbb{R}, L(X))$ the inverse Laplace integral exists for that σ_0 and is in $e^{-\sigma_0 \cdot}\mathcal{L}^{-1}B \in L^\infty([0,\infty), L(X)) \cap C([0,\infty), L(X))$.*
If B is complex differentiable in $\Re s > \sigma_0$ and uniformly square integrable over each vertical line $\Re s = \sigma$, $\sigma > \sigma_0$ then the definition of the inverse Laplace transform is welldefined, in the sense that it does not depend on $\sigma \in [\sigma_0, \infty)$ and is supported in $[0,\infty)$. It holds

$$\|e^{-\sigma \cdot}\mathcal{L}^{-1}B\|_{L^\infty([0,\infty),L(X))} \leq \frac{1}{2\pi}\|B\|_{L^1(\sigma+i\mathbb{R},L(X))},$$

where $\sigma \in [\sigma_0, \infty)$ is arbitrary.

Proof. By the triangle inequality it holds for arbitrary $t \in [0,\infty)$

$$\begin{aligned}\|\mathcal{L}^{-1}B(t)\|_{L(X)} &\leq \frac{1}{2\pi}\int_{\sigma+i\mathbb{R}} \|e^{st}B(s)\|_{L(X)}\ \mathrm{d}s \\ &= \frac{1}{2\pi}e^{\sigma t}\|B(\sigma + i\cdot)\|_{L^1(\mathbb{R},L(X))}.\end{aligned}$$

If $B \in \mathcal{H}(\sigma_0)$, componentwise arguments conclude that the Definition does not depend on $\sigma \in [\sigma_0, \infty)$ and $\mathcal{L}^{-1}B(t) = 0$ for $t < 0$. □

Lemma B.67. *For $p \in [1,\infty)$ it holds*

$$\|B * v\|_{L^p([0,\infty),X)} \leq \|B\|_{L^p([0,\infty),L(X))}\|v\|_{L^1([0,\infty),X)}$$

and for $c \in \mathbb{R}$

$$\|e^{-c\cdot}(B * v)\|_{L^p([0,\infty),X)} \leq \|e^{-c\cdot}B\|_{L^p([0,\infty),L(X))}\|e^{-c\cdot}v\|_{L^1([0,\infty),X)}.$$

So

$$* : L^p_c([0,\infty), L(X)) \times L^1_c([0,\infty), X) \to L^p_c([0,\infty), X).$$

is continuous with continuity constant 1.

Proof. We have (the first equality follows by Fubini's theorem)

$$\begin{aligned}\|(B * v)(t)\|_X &= \left\|\int_0^t B(t-\tau)v(\tau)\ \mathrm{d}\tau\right\|_X \\ &\leq \int_0^t \left\|B(t-\tau)\|_{L(X)}\|v(\tau)\right\|_X\ \mathrm{d}\tau \\ &= (\|B\|_{L(X)} * \|v\|_X)(t).\end{aligned}$$

By Young's inequality for $1 \leq p,q,r \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and $\phi \in L^p([0,\infty))$, $\psi \in L^q[0,\infty)$ we deduce

$$\|\phi *_{\mathbb{R}} \psi\|_{L^r[0,\infty)} \leq \|\phi\|_{L^p[0,\infty)}\|\psi\|_{L^q[0,\infty)}$$

and as in the scalar case for $c \in \mathbb{R}$

$$e^{-ct}(B * v)(t) = (e^{-c\cdot}B * e^{-c\cdot}v)(t).$$

We conclude for $p \in [1,\infty)$ (the case $p = \infty$ is similar)

$$\begin{aligned}\|e^{-c\cdot}B * v\|_{L^p([0,\infty),X)} &= \left(\int_0^\infty \|e^{-ct}B * v(t)\|_X^p\ \mathrm{d}t\right)^{1/p} \\ &= \left(\int_0^\infty \|(e^{-c\cdot}B) * (e^{-c\cdot}v)(t)\|_X^p\ \mathrm{d}t\right)^{1/p} \\ &\leq \left(\int_0^\infty (\|e^{-c\cdot}B\|_{L(X)} * \|e^{-c\cdot}v\|_X)^p(t)\ \mathrm{d}t\right)^{1/p} \\ &= \left\|\|e^{-c\cdot}B\|_{L(X)} * \|e^{-c\cdot}v\|_X\right\|_{L^p([0,\infty),\mathbb{R})} \\ &\leq \left\|\|e^{-c\cdot}B\|_{L(X)}\right\|_{L^p([0,\infty),\mathbb{R})}\left\|\|e^{-c\cdot}v\|_X\right\|_{L^1([0,\infty),\mathbb{R})} \\ &= \|e^{-c\cdot}B\|_{L^p([0,\infty),L(X))}\|e^{-c\cdot}v\|_{L^1([0,\infty),X)}.\end{aligned}$$

Inserting $c = 0$ yields the first assertion. □

So far, we have estimates which are with respect to $L^1(i\mathbb{R})$ and $L^\infty([0,\infty))$. To come back to estimates that act on the used L^2-spaces, we insert a factor $e^{-\delta\cdot}$, to come back to estimates in $L^2(i\mathbb{R})$ and $L^1(i\mathbb{R})$. This is elaborated in the following Lemma.

Lemma B.68. *It holds for $p, r \in [1,\infty)$, $r \le p$ for any $\delta > 0$ and $v \in L^p[0,\infty)$*

$$\|e^{-\delta\cdot} v\|_{L^r[0,\infty)} \le C(\delta, p, r)\|v\|_{L^p[0,\infty)}.$$

Similar estimates are possible on a bounded domain, where the constant depends on the size of the domain. Here, on the unbounded domain, the decay of $e^{-\delta\cdot}$ towards infinity takes this role.

Proof. We want to apply Hölder's inequality

$$\|ab\|_{L^r[0,\infty)} \le \|a\|_{L^q[0,\infty)}\|b\|_{L^q[0,\infty)}$$

for $p, q, r \in [1,\infty)$ with $1/r = 1/q + 1/p$. First assume $r < p$. This gives for $r = r$, $p = p$ and $q = (1/r - 1/p)^{-1} \in [1,\infty)$

$$\begin{aligned}\|e^{-\delta\cdot} v\|_{L^r[0,\infty)} &\le \|e^{-\delta\cdot}\|_{L^q[0,\infty)}\|v\|_{L^p[0,\infty)}\\ &\le \left([e^{-\delta q\cdot}/(-\delta q)]_0^\infty\right)^{1/q}\|v\|_{L^p[0,\infty)}\\ &= (\delta q)^{1/p-1/r}\|v\|_{L^p[0,\infty)}.\end{aligned}$$

For $p = r$, it is $\|e^{-\delta\cdot}\|_{L^\infty[0,\infty)} = 1$ and the assertion follows similarly. □

We will use these estimates for the cases $r = 2$, $p = \infty$, i.e.

$$\|b\|_{L^2_{\sigma_0+2\varepsilon}([0,\infty),X)} \le C(\varepsilon)\|e^{-(\sigma_0+\varepsilon)\cdot} b\|_{L^\infty([0,\infty),X)}$$

and $r = 1$, $p = 2$, i.e.

$$\|e^{-(\sigma_0+\varepsilon)\cdot} u\|_{L^1([0,\infty),X)} \le C(\varepsilon)\|u\|_{L^2_{\sigma_0}([0,\infty),X)}.$$

We are now able to put together the previous lemmas. Similarly, as $L^2_c([0,\infty),X)$ we denote $L^1_c([0,\infty),X) := e^{c\cdot}L^1([0,\infty),X)$ and $L^\infty_c([0,\infty),X) := e^{c\cdot}L^\infty([0,\infty),X)$ with the respective $e^{-c\cdot}$-weighted norms.

Lemma B.69. *For $B \in L^1(\sigma_0 + i\mathbb{R}, L(X)) \cap \mathcal{H}(\sigma_0)$ the convolution with the inverse Laplace transform gives for every $\delta > 0$ a welldefined and continuous operator*

$$\mathcal{L}^{-1}B* : L^2_{\sigma_0}([0,\infty),X) \to L^2_{\sigma_0+\delta}([0,\infty),X)$$

and it holds

$$\|\mathcal{L}^{-1}B * u\|_{L^2_{\sigma_0+\delta}([0,\infty),X)} \le C(\delta)\|B\|_{L^1(\sigma_0+i\mathbb{R},L(X))}\|u\|_{L^2_{\sigma_0}([0,\infty),X)}.$$

Proof. The concatenation $(B \mapsto B*\cdot)\circ\mathcal{L}^{-1}$ is well-defined, as we have from Lemma B.66 the continuous mappings

$$\mathcal{L}^{-1} : L^1(\sigma_0 + i\mathbb{R}, L(X)) \cap \mathcal{H}(\sigma_0) \to L^\infty_{\sigma_0}([0,\infty), L(X))$$

and from Lemma B.67 the continuous

$$* : L^\infty_{\sigma_0}([0,\infty), L(X)) \times L^1_{\sigma_0}([0,\infty), X) \to L^\infty_{\sigma_0}([0,\infty), X).$$

By the use of Lemma B.68, it holds for $\varepsilon > 0$ that

$$\begin{aligned}\|\mathcal{L}^{-1}B * u\|_{L^2_{\sigma_0+2\varepsilon}([0,\infty),X)} &\leq C(\varepsilon)\|e^{-(\sigma_0+\varepsilon)\cdot}\mathcal{L}^{-1}B * u\|_{L^\infty([0,\infty),X)} \\ &\leq C(\varepsilon)\|e^{-(\sigma_0+\varepsilon)\cdot}\mathcal{L}^{-1}B\|_{L^\infty([0,\infty),L(X))}\|e^{-(\sigma_0+\varepsilon)\cdot}u\|_{L^1([0,\infty),X)} \\ &\leq \inf_{\sigma\in[\sigma_0,\sigma_0+\varepsilon]}\|B\|_{L^1(\sigma+i\mathbb{R},L(X))}C(\varepsilon)\|u\|_{L^2_{\sigma_0}([0,\infty),X)}.\end{aligned}$$

□

So far we considered inverse Laplace transform applied to an operator and convolution with the outcome in an independent way. In the following we consider the concatenation more closely, as therefore Plancherel's formula type arguments are again possible.

Lemma B.70. *For $B \in H(\sigma_0)$, we have continuity of*

$$\mathcal{L}^{-1}B* : L^2_{\sigma_0-\delta} \to L^2_{\sigma_0}, f \mapsto \mathcal{L}^{-1}B * f$$

for every $\delta > 0$ and it holds

$$\mathcal{L}^{-1}B * f = B(\partial_t)f$$

for all $f \in L^2_{\sigma_0-\delta}$.

Proof. For $B \in L^1(\sigma_0 + i\mathbb{R}, L(X)) \cap H(\sigma_0)$, $v \in L^2_{\sigma_0-\delta}([0,\infty), X)$ and by arguments of the scalar valued case, we have

$$\begin{aligned}\mathcal{L}^{-1}B * v &= \sum_{i\in\mathbb{N}}\left(\sum_{j\in\mathbb{N}}\mathcal{L}^{-1}([e_i, Be_j]_X) * [e_j, v]\right)e_i \\ &= \sum_{i\in\mathbb{N}}\left(\sum_{j\in\mathbb{N}}\mathcal{L}^{-1}([e_i, Be_j]_X\mathcal{L}[e_j, v])\right)e_i \\ &= \mathcal{L}^{-1}\left(B(s)\mathcal{L}v\right).\end{aligned} \tag{B.14}$$

The last equality holds true, as we can interchange integral and summation by Fubini's theorem: For sequences $A = (A_{ij})_{i,j\in\mathbb{N}}$, $b = (b_j)_{j\in\mathbb{N}}$, we set (for fixed $i \in \mathbb{N}$) $A_{i\cdot} := (A_{ij})_{j\in\mathbb{N}}$ and $x = (x_j)_{j\in\mathbb{N}}$ with $x_j := \frac{\overline{A_{ij}}}{\|A_{i\cdot}\|_{l^2}}$. It holds with $\|x\|_{l^2} = 1$

$$\begin{aligned}\|A_{i\cdot}\|^2_{l^2} :&= \sum_{j\in\mathbb{N}}|A_{ij}|^2 \\ &= \|A_{i\cdot}\|_{l^2}\sum_{j\in\mathbb{N}}A_{ij}\cdot\frac{\overline{A_{ij}}}{\|A_{i\cdot}\|_{l^2}} \\ &\leq \|A_{i\cdot}\|_{l^2}\left(\sum_{k\in\mathbb{N}}\left|\sum_{j\in\mathbb{N}}A_{kj}\cdot\frac{\overline{A_{ij}}}{\|A_{i\cdot}\|_{l^2}}\right|^2\right)^{1/2} \\ &= \|A_{i\cdot}\|_{l^2}\|Ax\|_{l^2} \\ &\leq \|A_{i\cdot}\|_{l^2}\|A\|,\end{aligned}$$

so $\|A_{i\cdot}\|_{l^2} \leq \|A\|$. Therefore we have

$$\begin{aligned}\sum_{j\in\mathbb{N}}|A_{ij}b_j| &\leq \left(\sum_{j\in\mathbb{N}}|A_{ij}|^2\right)^{1/2}\left(\sum_{j\in\mathbb{N}}|b_j|^2\right)^{1/2} \\ &= \|A_{i\cdot}\|_{l^2}\|b\|_{l^2} \\ &\leq \|A\|\|b\|_{l^2}.\end{aligned}$$

Using this estimate and $\|([e_i, Be_j])_{ij}\| = \|B\|_{L(X)}$ and $\|([v, e_j])_j\|_{l^2} = \|v\|_X$ we deduce

$$\begin{aligned}
\int_{\sigma_0+i\mathbb{R}} \left(\sum_{j\in\mathbb{N}} e^{\sigma_0 t} |[e_i, B(s)e_j]_X \mathcal{L}[e_j, v](s)| \right) \, \mathrm{d}s &\leq e^{\sigma_0 t} \int_{\sigma_0+i\mathbb{R}} \Big(\|B(s)\|_{L(X)} \|\mathcal{L}v(s)\|_X \Big) \, \mathrm{d}s \\
&\leq e^{\sigma_0 t} \int_{\sigma_0+i\mathbb{R}} \Big(\|B(s)\|_{L(X)} \|\mathcal{L}v(s)\|_X \Big) \, \mathrm{d}s \\
&\leq e^{\sigma_0 t} \|B(s)\|_{L^2(\sigma_0+i\mathbb{R}, L(X))} \|\mathcal{L}v(s)\|_{L^2(\sigma_0+i\mathbb{R}, X)} \\
&= e^{\sigma_0 t} \|B(s)\|_{L^2(\sigma_0+i\mathbb{R}, L(X))} \|v(s)\|_{L^2_{\sigma_0}([0,\infty), X)} \\
&< \infty.
\end{aligned}$$

By Plancherel's formula and (B.14) it follows

$$\begin{aligned}
\|\mathcal{L}^{-1} B * v\|_{L^2_{\sigma_0}} &= \frac{1}{\sqrt{2\pi}} \|B\mathcal{L}v\|_{L^2(\sigma_0+i\mathbb{R}, X)} \\
&\leq \frac{1}{\sqrt{2\pi}} \|B\|_{L^2(\sigma_0+i\mathbb{R}, L(X))} \|\mathcal{L}v\|_{L^\infty(\sigma_0+i\mathbb{R}, X)} \\
&\leq C \|B\|_{L^2(\sigma_0+i\mathbb{R}, L(X))} \|e^{-\sigma_0 t} v\|_{L^1([0,\infty), X)} \\
&\leq C(\delta) \|B\|_{L^2(\sigma_0+i\mathbb{R}, L(X))} \|v\|_{L^2_{\sigma_0-\delta}([0,\infty), X)}.
\end{aligned}$$

By the density of $L^1(\sigma_0 + i\mathbb{R}, L(X))$ in $L^2(\sigma_0 + i\mathbb{R}, L(X))$, we can define $v \mapsto \mathcal{L}^{-1} B * v$ for $B \in H(\sigma_0, L(X))$. Note, that $\mathcal{L}^{-1} B$ is not welldefined itself in general, as Plancherel's formula does not hold in the Banach space $L(X)$. □

The previous result corresponds to [143, Proposistion 3.2.2.]. We use similar arguments as in the previous lemma to improve the mapping properties of $\mathcal{L}^{-1} B*$ for L^∞-bounded operators on a real line. So far, for $B(s) \in \mathcal{H}(\sigma_0)$, the convolution operators map from $L^2_{\sigma_0-\delta}[0,\infty) \to L^2_{\sigma_0}[0,\infty)$, but under additional assumptions on the operator, we will get more time regularity.

Lemma B.71. *Let $\sigma_0 > 0$. For a family of operators B such that*

$$B(s) \leq C \qquad \text{for } \Re s \geq \sigma_0,$$

it is $B(s)/s \in H(\sigma_0)$ and it is continuous

$$\partial_t \mathcal{L}^{-1}(B(s)/s) * : L^2_{\sigma_0-\delta}[0,\infty) \to L^2_{\sigma_0}[0,\infty), f \mapsto \partial_t(\mathcal{L}^{-1}(B(s)/s) * f).$$

It holds for $f \in L^2_{\sigma_0-\delta}$

$$B(\partial_t) f = \partial_t(\mathcal{L}^{-1}(B(s)/s) * f).$$

Furthermore it is $B(s)/s^2 \in H(\sigma_0) \cap L^1(\sigma + i\mathbb{R})$ for all $\sigma \geq \sigma_0$ and similarly

$$\partial_t^2 \mathcal{L}^{-1}(B(s)/s^2) : L^2_{\sigma_0}[0,\infty) \to L^2_{\sigma_0+\delta}[0,\infty), f \mapsto \partial_t^2(\mathcal{L}^{-1}(B(s)/s^2) * f)$$

is continuous. The convolution kernel $\mathcal{L}^{-1}(B(s)/s^2)$ is bounded, continuous and it holds for $f \in L^2_{\sigma_0}$

$$B(\partial_t) f = \partial_t^2 \mathcal{L}^{-1}(B(s)/s^2) * f.$$

Proof. As $\sigma_0 > 0$, it follows $B(s)/s \in H(\sigma_0)$, so the convolution operator is well defined. Let $f \in L^2_{\sigma_0-\delta}[0,\infty)$. By Lemma B.70 it is $\mathcal{L}^{-1}(B(s)/s) * f = \mathcal{L}^{-1}(B(s)/s\mathcal{L}f)$, by Lemma B.62 $B(\partial_t)f$ exists and by Example B.61 we have $\partial_t^{-1} B(\partial_t) f = \mathcal{L}^{-1}(B(s)/s\mathcal{L}f)$. By derivation in time we obtain

$$B(\partial_t) f = \partial_t \mathcal{L}^{-1}(B(s)/s) * f.$$

As $\sigma_0 > 0$, it holds $B(s)/s^2 \in H(\sigma_0) \cap L^1(\sigma_0 + i\mathbb{R})$ and the inverse Laplace transform and convolution are well defined. We can see, by similar estimates as in the proof of Lemma B.70, that we can interchange integral and sum for $f \in L^2_{\sigma_0}[0,\infty)$ (using similar operator estimates and the assertions of Lemma B.69) and therefore it holds

$$\mathcal{L}^{-1}(B(s)/s^2) * f = \mathcal{L}^{-1}(B(s)/s^2\mathcal{L}f).$$

By Lemma B.62 it follows that $B(\partial_t)f$ exists (for $f \in L^2_*[0,\infty)$) and it is by Example B.61

$$\mathcal{L}^{-1}(B(s)/s^2\mathcal{L}f) = \partial_t^{-2}B(\partial_t)f.$$

By derivation in time we obtain $B(\partial_t)f = \partial_t^2\mathcal{L}^{-2}(B(s)/s) * f$.

□

We now can rewrite the previous lemma for $B \leq Cs^m$ in analogy to the scalar valued case.

Lemma B.72. *In the setting of Lemma B.62, every $\epsilon > 0$ satisfies*

$$B(s)s^{-(m+1)} \in \mathcal{H}(\max(\epsilon, \sigma_1))$$

and it holds for every $f \in H^m_{0,*}[0,\infty)$

$$B(\partial_t)f = \partial_t^{m+1}\mathcal{L}^{-1}(B(s)s^{-(m+1)}) * f$$

and every $f \in H^{m+1}_{0,*}[0,\infty)$

$$B(\partial_t)f = \mathcal{L}^{-1}(B(s)s^{-(m+1)}) * \partial_t^{m+1}f.$$

Furthermore every $\epsilon > 0$ satisfies

$$B(s)s^{-(m+2)} \in \mathcal{H}(\max(\epsilon, \sigma_1)) \cap L^1(\max(\epsilon, \sigma_1) + i\mathbb{R}, L(X)),$$

thus $\mathcal{L}^{-1}(B(s)s^{-(m+2)})$ is continuous and we have for $f \in H^m_{0,*}[0,\infty)$

$$B(\partial_t)f = \partial_t^{m+2}\mathcal{L}^{-1}(B(s)s^{-m+2}) * f$$

and for $f \in H^{(m+2)}_{0,*}[0,\infty)$

$$B(\partial_t)f = \mathcal{L}^{-1}(B(s)s^{-m+2}) * \partial_t^{m+2}f.$$

Proof. It is $B(s)s^{-(m+1)}$ analytic for $\Re s > \max(\sigma_1, 0)$ and by

$$|B(s)s^{-(m+1)}| \leq |s|^{-1} \quad \text{on } \Re s > \max(\sigma_1, 0),$$

we obtain

$$B(s)s^{-(m+1)} \in \mathcal{H}(\max(\epsilon, \sigma_1)).$$

For $\sigma_2 \in \mathbb{R}$, $g \in L^2_{\sigma_2}[0,\infty)$ it is by Lemma B.70 (for $\sigma_0 = \max(\sigma_1, \sigma_2)$)

$$\mathcal{L}^{-1}(B(s)s^{-(m+1)}\mathcal{L}g) = \mathcal{L}^{-1}(B(s)s^{-(m+1)}) * g.$$

Setting $g := \partial_t^m f$, the first assertions follow as in Lemma B.71 from

$$\partial_t^m\partial_t^{-m} = \mathrm{Id}_{L^2_*([0,\infty),X)\to L^2_*([0,\infty),X)}$$

and

$$\partial_t^{-m}\partial_t^m = \mathrm{Id}_{H^m_{0,*}([0,\infty),X)\to H^m_{0,*}([0,\infty),X)}.$$

To show that the function $\mathcal{L}^{-1}(B(s)s^{-(m+2)})$ is continuous regarded as function on $\mathbb{R}$ we consider

$$B(s)s^{-(m+2)} \in L^1(\sigma + i\mathbb{R}, L(X))$$

for $\sigma > \max(\sigma_1, \epsilon)$. By the properties of the Fourier transform it is continuous on $[0, \infty)$ and by the componentwise definition and interchanging limit and integral (with similar arguments as in Lemma B.70 we have for $v \in X$

$$\begin{aligned}
\|\mathcal{L}^{-1}(B(s)s^{-m-2})(0)v\|_X^2 &= \lim_{t\to 0} \|\mathcal{L}^{-1}(B(s)s^{-m-2})(t)v\|_X^2 \\
&= \lim_{t\to 0} \sum_{i\in\mathbb{N}} \left| \sum_{j\in\mathbb{N}} \mathcal{L}^{-1}(B_{i,j}(s)s^{-m-2})(t)v_j \right|^2 \\
&= \sum_{i\in\mathbb{N}} \lim_{t\to 0} \left| \sum_{j\in\mathbb{N}} \mathcal{L}^{-1}(B_{i,j}(s)s^{-m-2})(t)v_j \right|^2 \\
&= \sum_{i\in\mathbb{N}} \left| \sum_{j\in\mathbb{N}} \lim_{t\to 0} \mathcal{L}^{-1}(B_{i,j}(s)s^{-m-2})(t)v_j \right|^2 = 0.
\end{aligned}$$

Here integration and sum can be interchanged in the second and third line because of similar arguments as in Lemma B.70 and

$$\left| \sum_{j\in\mathbb{N}} \mathcal{L}^{-1}(B_{i,j}(s)s^{-m-2})(t)v_j \right|^2 \le \|\mathcal{L}^{-1}(s^{-m-2}B(s))(t)v\|_X \le C.$$

□

Remark B.73. *The formulas in Lemma B.62 and Lemma B.72, that do not need differentiabillity of f, also hold for $f \in L^2_*[0, \infty)$, if $B(\partial_t)f$ exists. In this case it is for $m \in \mathbb{N}_0$*

$$B(\partial_t)f = \partial_t^m \mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f)$$

and for high enough $m \in \mathbb{N}$

$$B(\partial_t)f = \partial_t^m \mathcal{L}^{-1}(B(s)s^{-m}) * f.$$

The last equality only holds if $\mathcal{L}^{-1}(B(s)s^{-m})*$ exists (e.g. if $B(s)s^{-m} \in \mathcal{H}$).

Proof. The first assertion follows by the scalar results and the componentvise definitions. The second one follows from Lemma B.70 for $f \in L^2_{\sigma_2}([0, \infty), X)$, $B(s)s^{-m} \in \mathcal{H}(\sigma_1)$ and $\sigma_0 = \max(\sigma_2 + \delta, \sigma_1)$ □

We summarize some further properties.

Theorem B.74. *Let $f \in L^2_*([0, \infty), X)$ and families of operators $A(s)$, $B(s) \in L(X)$. If $B(\partial_t)f$ and $(AB)(\partial_t)f$ exist, then $A(\partial_t)B(\partial_t)f$ exists and equals*

$$(AB)(\partial_t)f = A(\partial_t)B(\partial_t)f.$$

If furthermore $A(\partial_t)f$ exists and $AB\mathcal{L}f = BA\mathcal{L}f$ on a complex line with high enough real part, it holds

$$(AB)(\partial_t)f = A(\partial_t)B(\partial_t)f = B(\partial_t)A(\partial_t)f.$$

Proof. The proof follows directly by the scalar results and the componentvise definitions.

□

Corollary B.75. *For $A \in \mathcal{H}_m$, $B \in \mathcal{H}_n$, $AB \in \mathcal{H}_p$ $f \in H_{0,*}^{\max(m,n,p)}[0,\infty)$ it is*

$$(AB)(\partial_t)f = A(\partial_t)B(\partial_t)f.$$

If furthermore $A(s)B(s) = B(s)A(s)$ on a complex line with high enough real part, it is

$$(AB)(\partial_t)f = B(\partial_t)A(\partial_t)f.$$

Theorem B.76 (Herglotz Theorem, cf. [27, Lemma 2.2])**.** *Let $B, R \in \mathcal{H}^m(\sigma_0)$ for $\sigma_0 \in \mathbb{R}$ and $a(\cdot,\cdot) : X \times X \to \mathbb{C}$ sesquilinear and continuous. Then the following statements are equivalent:*

- *There exists $c > 0$ such that for all $w \in X$, all $\Re s > \sigma_0$*

$$\Re a(w, B(s)w) \geq c\|R(s)w\|_X^2.$$

- *There exists $c > 0$ such that for all $w \in H_{0,*}^m([0,\infty), X)$, for all $\sigma \geq \sigma_0$*

$$\int_0^\infty e^{-2\sigma t}\Re a(w(t), B(\partial_t)w(t))\ \mathrm{d}t \geq c\|R(\partial_t)w\|^2_{L^2_\sigma([0,\infty),X)}.$$

Additionally, the following statements are equivalent:

- *There exists $C > 0$ such that for all $w \in X$ and all $\Re s > \sigma_0$*

$$\|B(s)w\|_X^2 \leq C\|R(s)w\|_X^2.$$

- *There exists $C > 0$ such that for all $w \in H_{0,*}^m([0,\infty, X))$, for all $\sigma \geq \sigma_0$*

$$\|B(\partial_t)w\|^2_{L^2_\sigma([0,\infty),X)} \leq C\|R(\partial_t)w\|^2_{L^2_\sigma([0,\infty),X)}.$$

Proof. The execution follows from by Plancherel's formula, and the rear direction can be shown by localizing around arbitrary values by special sequences, cf. [27, Lemma 2.2]. ☐

B.2.3. Vector valued Laplace transform and differential operators on $[0,T]$

As in the scalar case, we want to define the Laplace transform and Laplace differential operators for functions defined on bounded domains, so e.g. for $f \in L^2([0,T], X)$. Similar results can be found in [115, Section 2.1].

The Laplace transform can easily be defined by extending f to zero outside of $[0,T]$: For $f \in L^2([0,T], X)$, it is $e^{-c\cdot} f\mathbb{1}_{[0,T]} \in L^2([0,\infty), X)$ for all $c \in \mathbb{R}$, thus

$$\mathcal{L}f := \mathcal{L}(f\mathbb{1}_{[0,T]})$$

exists and is holomorphic in the whole complex plane. Also $\mathcal{L}^{-1}$ of $\mathcal{L}f$ is well defined and gives back a function with support in $[0,T]$. For general functions $B \in \mathcal{H}$, we can ensure $\operatorname{supp}(\mathcal{L}^{-1}B) \subset [0,T]$ by setting

$$\mathcal{L}^{-1} := \mathbb{1}_{[0,T]}\mathcal{L}^{-1}.$$

It should be taken into account, that in general it is

$$\mathcal{L}^{-1}\mathcal{L} = \mathrm{Id}_{L^2([0,T],X)\to L^2([0,T],X)},$$

but

$$\mathcal{L}\mathcal{L}^{-1} \neq \mathrm{Id}_{\mathcal{H}\to\mathcal{H}}.$$

As in the scalar case, we apply some modifications to the Laplace differential operators to define them for functions on $[0,T]$. A possibility to do this would be, to extend f on $[T,\infty)$ in a smooth way, such that $f(t)=0$ for $t>2T$, to apply the operator $B(\partial_t)$ on $[0,\infty)$ and to ensure, that $\mathbb{1}_{[0,T]}B(\partial_t)f$ does not depend on the arbitrarily chosen extension. We will go the other way around: To define $B(\partial_t)f$, we reformulate the operator until we arrive at a formulation that suits $f\in L^2([0,T],X)$ and then show that the property, that the definition is invariant under any smooth enough extension of f to $[0,\infty)$ will be satisfied under weak assumptions.

Definition B.77. *Let $B(s)\in L(X)$ be a family of operators and $f\in L^2([0,T],X)$. Whenever there is an $m\in N_0$ such that*

$$B(s)s^{-m}\mathcal{L}f\in\mathcal{H} \text{ and } \mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f)\in H^m([0,T],X),$$

we say that $B(\partial_t)f$ exists and we set

$$B(\partial_t)f:=\partial_t^m\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f).$$

We call the function $B(s)$ or the mapping $B(\partial_t)$ causal, if for every f (and every $T>0$), such that $B(\partial_t)f$ exists, $B(\partial_t)f$ does not depend on an arbitrarily chosen extension of f in $L^2_*([0,\infty),X)$, i.e. for every $\widetilde{f}\in L^2_*([0,\infty),X)$,

$$f=\mathbb{1}_{[0,T]}\widetilde{f} \text{ in } L^2([0,T],X)$$

it holds that $B(\partial_t)f$ exists and

$$B(\partial_t)f=\mathbb{1}_{[0,T]}B(\partial_t)\widetilde{f} \text{ in } L^2([0,T],X).$$

We call it Causality that $B(\partial_t)f(T)$ only depends on values of f for $t<T$.

This is a new definition of $B(\partial_t)$, that does not coincide in general with the one on $[0,\infty)$ of the previous subsection. The definition is well defined in the sense that it does not depend on the selection of $m\in\mathbb{N}$. If existent for $m_0\in\mathbb{N}_0$ then it follows by Example B.61 for all $m>m_0$:

$$\partial_t^{m_0}\mathcal{L}^{-1}(B(s)s^{-m_0}\mathcal{L}f)=\partial_t^m\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f).$$

In the following, we define the suitable function spaces and give sufficient conditions for the existence.

Definition B.78. *We define for $m\in\mathbb{N}_0$ the space of $m-$times weakly differentiable functions with initial condition zero as*

$$H^m_{0,*}([0,T],X):=\{f\in H^m([0,T],X)\mid f(0)=\dots=f^{(m-1)}(0)=0\}.$$

With the induced norm

$$\|\cdot\|_{H^m_{0,*}([0,T],X)}:=\|\cdot\|_{H^m([0,T],X)}=\sqrt{\langle\cdot,\cdot\rangle_{H^m([0,T],X)}},$$

this is a Hilbert space.
The sub index $0,*$ in $H^m_{0,*}([0,T],X)$ has the meaning 0 at $t=0$ and arbitrary value at $t=T$, and we also define

$$H^m_{*,0}([0,T],X):=\{\phi\in H^m([0,T],X)\mid f(T)=\dots=f^{(m-1)}(T)=0\}.$$

Lemma B.79. *Let $m \in \mathbb{N}_0$. For*

$$f \in H^m_{0,*}([0,T],X)$$

and

$$B \in \mathcal{H}_m,$$

$B(\partial_t)f$ exists and it holds $\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f) \in H^m_{0,*}([0,T],X)$ and

$$B(\partial_t)f = \mathbb{1}_{[0,T]}\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}(\partial_t^m f))$$

We can define $B(\partial_t)$ as a continuous operator

$$B(\partial_t) : H^m_{0,*}([0,T],X) \to L^2([0,T],X).$$

Every $B \in \mathcal{H}_m$ is causal and for every smooth enough extension $\widetilde{f}$ of f on $[0,\infty)$ it holds

$$B(\partial_t)f = \mathbb{1}_{[0,T]}\mathcal{L}^{-1}(B(s)\mathcal{L}f).$$

Proof. As B is holomorphic and by

$$|B(s)s^{-m}\mathcal{L}f| \leq |\mathcal{L}f|$$

it follows $B(s)s^{-m}\mathcal{L}f \in \mathcal{H}$. For an extension $\widetilde{f} \in H^m_{0,*}([0,\infty),X)$ of f it holds by Lemma B.69 that

$$\begin{aligned}\mathcal{L}^{-1}(s^{-m}B(s)\mathcal{L}f) &= \partial_t^2\mathcal{L}^{-1}(B(s)s^{-m-2}) * f \\ &= \partial_t^2\mathcal{L}^{-1}(B(s)s^{-m-2}) * \widetilde{f} \\ &= \mathcal{L}^{-1}(s^{-m}B(s)\mathcal{L}\widetilde{f}).\end{aligned}$$

Inserting $\widetilde{f} = \partial_t^{-m}\partial_t^m f \in H^m_{0,*}([0,\infty),X)$, we see by Lemma B.62 that $\mathcal{L}^{-1}(s^{-m}B(s)\mathcal{L}\widetilde{f}) \in H^m_{0,*}([0,\infty),X)$, so it holds $\mathcal{L}^{-1}(s^{-m}B(s)\mathcal{L}f) \in H^m_{0,*}([0,T],X)$. By Lemma B.62 it follows $B(\partial_t)f = \mathbb{1}_{[0,T]}\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}(\partial_t^m f))$. By Plancherel's formula, we see for high enough $\sigma \in \mathbb{R}$

$$\begin{aligned}\|B(\partial_t)f\|_{L^2([0,T],X)} &\leq C(\sigma,T)\|\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}(\partial_t^m f))\|_{L^2_\sigma([0,\infty),X)} \\ &= C(\sigma)\|B(s)s^{-m}\mathcal{L}(\partial_t^m f))\|_{\mathcal{H}(\sigma)} \\ &\leq C(\sigma)\|\mathcal{L}(\partial_t^m f))\|_{\mathcal{H}(\sigma)} \\ &= \|e^{\sigma\cdot}\partial_t^m f\|_{L^2([0,T],X)} \\ &\leq \|f\|_{H^m_{0,*}([0,T],X)}.\end{aligned}$$

The Causality can be shown by the following argument. For an arbitrary extension $\widetilde{f} \in L^2_*([0,\infty),X)$, it exists $\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}\widetilde{f})$ and it holds

$$\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}\widetilde{f}) = \partial_t^2\mathcal{L}^{-1}(B(s)s^{-m-2}) * \widetilde{f},$$

which does not depend on the extension. So $B(\partial_t)f$ does neither. □

Lemma B.80. *In the setting of Lemma B.79, $f \in H^m_{0,*}([0,T],X)$*

$$B(\partial_t)f = \partial_t^{m+1}\mathcal{L}^{-1}(B(s)s^{-(m+1)}) * f = \partial_t^1\mathcal{L}^{-1}(B(s)s^{-(m+1)}) * \partial_t^m f$$

and every $f \in H^{m+1}_{0,*}([0,T],X)$

$$B(\partial_t)f = \mathcal{L}^{-1}(B(s)s^{-(m+1)}) * \partial_t^{m+1} f.$$

Furthermore $\mathcal{L}^{-1}(B(s)s^{-(m+2)})$ *is continuous and we have for* $f \in H^m_{0,*}([0,T],X)$

$$B(\partial_t)f = \partial_t^{m+2}\mathcal{L}^{-1}(B(s)s^{-m+2}) * f = \partial_t^2\mathcal{L}^{-1}(B(s)s^{-(m+2)}) * \partial_t^m f$$

and for $f \in H^{(m+2)}_{0,*}([0,T],X)$

$$B(\partial_t)f = \mathcal{L}^{-1}(B(s)s^{-m+2}) * \partial_t^{m+2} f.$$

Proof. The assertion follows as in the scalar case with Lemma B.69 and Lemma B.70. □

We collect the following properties in analogue to the case on $[0,\infty)$.

Theorem B.81. *Let* $f \in L^2_*([0,T],X)$ *and functions* $B(s)$ *and causal* $A(s)$*. If* $B(\partial_t)f$ *and* $A(\partial_t)B(\partial_t)f$ *exist, then* $(AB)(\partial_t)f$ *exists and it equals*

$$(AB)(\partial_t)f = A(\partial_t)B(\partial_t)f.$$

If furthermore $A(\partial_t)f$ *and* $B(\partial_t)A(\partial_t)f$ *exist and* B *is causal and* $A(s)B(s) = B(s)A(s)$ *on an imaginary line* $\sigma + i\mathbb{R}$ *for* σ *large enough, it holds*

$$(AB)(\partial_t)f = A(\partial_t)B(\partial_t)f = B(\partial_t)A(\partial_t)f.$$

Proof. The proof works analogous to the scalar case. Let $f \in L^2_*[0,\infty)$ and $n, m, p \in \mathbb{N}$ be integers for the existence of $B(\partial_t)f$, $A(\partial_t)B(\partial_t)f$ and $(AB)(\partial_t)f$. It holds for $g := B(\partial_t)f$, that $A(\partial_t)g$ exists and

$$A(\partial_t)g = \partial_t^{m+n}\mathcal{L}^{-1}(A(s)s^{-m}\mathcal{L}(\partial_t^{-n}g)).$$

So $\partial_t^n A(\partial_t)\partial_t^{-n}g$ exists, and we can use the Causality property of A with $\mathcal{L}^{-1}(B(s)s^{-n}\mathcal{L}f)$ as extension of $\partial_t^{-n}g$ to $[0,\infty)$. Therefore we have

$$A(\partial_t)g = \partial_t^{m+n}\mathcal{L}^{-1}(A(s)B(s)s^{-m-n}\mathcal{L}f).$$

This shows that $(AB)(\partial_t)$ exists an that p can be chosen smaller or equal than $p \le m+n$. If additionally B is causal, we get $(BA)(\partial_t)f = B(\partial_t)A(\partial_t)f$. Now let p be the maximum of the integers for the existence of $(AB)(\partial_t)f$ and $(BA)(\partial_t)f$.

Then $A(s)B(s) = B(s)A(s)$ on a line $\sigma + i\mathbb{R}$ with σ large enough such that it holds $s^{-p}A(s)B(s)\mathcal{L}f, s^{-p}B(s)A(s)\mathcal{L}f \in \mathcal{H}(\sigma)$, yields

$$A(\partial_t)B(\partial_t)f = (AB)(\partial_t)f = (BA)(\partial_t)f = B(\partial_t)A(\partial_t)f.$$

□

Corollary B.82. *For* $A \in \mathcal{H}_m(\sigma_1)$*,* $B \in \mathcal{H}_n(\sigma_2)$*,* $AB \in \mathcal{H}_p$*,* $f \in H^{\max(m,n,p)}_{0,*}([0,T],X)$ *it holds*

$$(AB)(\partial_t)f = A(\partial_t)B(\partial_t)f$$

and if A(s)B(s)=B(s)A(s) on a line $\sigma + i\mathbb{R}$ *with* $\sigma \ge \max(\sigma_1,\sigma_2)$ *it holds*

$$(AB)(\partial_t)f = A(\partial_t)B(\partial_t)f = B(\partial_t)A(\partial_t)f.$$

Proof. For $\sigma \ge \max(\sigma_1,\sigma_2)$ it holds

$$A(s)B(s)s^{-m-n} \le C,$$

so as $\mathcal{L}f$ is complex differentiable on the whole complex plane,

$$s^{-m-n}A(s)B(s)\mathcal{L}f, s^{-m-n}B(s)A(s)\mathcal{L}f \in \mathcal{H}(\sigma)$$

and all conditions are satisfied that where needed for σ in the proof the previous theorem. □

Theorem B.83 (Herglotz Theorem on $[0, T]$, cf. [27, Lemma 2.2])**.** *Let* $B, R \in \mathcal{H}_m(\sigma_0)$ *for* $\sigma_0 \in \mathbb{R}$*. Let* $a(\cdot, \cdot) : X \times X \to \mathbb{C}$ *sesquilinear and continuous. If there exists a* $c > 0$ *such that for all* $w \in \mathbb{C}$*, all* $\Re s > \sigma_0$

$$\Re a(w, B(s)w) \geq c\|R(s)w\|_X^2,$$

then it holds for all $w \in H_{0,*}^m([0, T], X)$*, for all* $\sigma \geq \sigma_0$

$$\int_0^T e^{-2\sigma t} \Re a(w(t), B(\partial_t)w(t)) \, \mathrm{d}t \geq c e^{-2\sigma T} \|R(\partial_t)w\|_{L^2([0,T],X)}^2.$$

Proof. The proof follows the lines of the scalar case, using the respective Hilbert space valued arguments. □

Bibliography

[1] Toufic Abboud, Patrick Joly, Jerónimo Rodríguez, and Isabelle Terrasse. Coupling discontinuous Galerkin methods and retarded potentials for transient wave propagation on unbounded domains. *J. Comput. Phys.*, 230(15):5877–5907, 2011.

[2] Claas Abert, Gino Hrkac, Markus Page, Dirk Praetorius, Michele Ruggeri, and Dieter Süss. Spin-polarized transport in ferromagnetic multilayers: An unconditionally convergent FEM integrator. *Comput. Math. Appl.*, 68:639–654, 2014.

[3] Robert A. Adams. *Sobolev Spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[4] Georgios Akrivis, Michael Feischl, Balázs Kovács, and Christian Lubich. Higher-order linearly implicit full discretization of the Landau–Lifshitz–Gilbert equation. *Math. Comp.*, 90(329):995–1038, 2021.

[5] Georgios Akrivis and Emmanuil Katsoprinakis. Backward difference formulae: new multipliers and stability properties for parabolic equations. *Math. Comp.*, 85(301):2195–2216, 2016.

[6] Georgios Akrivis, Buyang Li, and Christian Lubich. Combining maximal regularity and energy estimates for time discretizations of quasilinear parabolic equations. *Math. Comp.*, 86(306):1527–1552, 2017.

[7] Martin S. Alnæs. *UFL: a Finite Element Form Language*, chapter 17. Springer, 2012.

[8] Martin S. Alnæs, Jan Blechta, Johan Hake, August Johansson, Benjamin Kehlet, Anders Logg, Chris Richardson, Johannes Ring, Marie E. Rognes, and Garth N. Wells. The fenics project version 1.5. *Archive of Numerical Software*, 3(100), 2015.

[9] Martin S. Alnæs, Anders Logg, and Kent-Andre Mardal. *UFC: a Finite Element Code Generation Interface*, chapter 16. Springer, 2012.

[10] Martin S. Alnæs, Anders Logg, Kent-Andre Mardal, Ola Skavhaug, and Hans Petter Langtangen. Unified framework for finite element assembly. *International Journal of Computational Science and Engineering*, 4(4):231–244, 2009.

[11] Martin S. Alnæs, Anders Logg, Kristian B. Ølgaard, Marie E. Rognes, and Garth N. Wells. Unified form language: A domain-specific language for weak formulations of partial differential equations. *ACM Transactions on Mathematical Software*, 40(2), 2014.

[12] Martin S. Alnæs and Kent-Andre Mardal. On the efficiency of symbolic computations combined with code generation for finite element methods. *ACM Transactions on Mathematical Software*, 37(1), 2010.

[13] Martin S. Alnæs and Kent-Andre Mardal. *SyFi and SFC: Symbolic Finite Elements and Form Compilation*, chapter 15. Springer, 2012.

[14] François Alouges. A new finite element scheme for Landau-Lifchitz equations. *Discrete Contin. Dyn. Syst. Ser. S*, 1(2):187–196, 2008.

[15] François Alouges and Pascal Jaisson. Convergence of a finite element discretization for the Landau-Lifshitz equations in micromagnetism. *Math. Models Methods Appl. Sci.*, 16(2):299–316, 2006.

[16] Francois Alouges, Evaggelos Kritsikis, Jutta Steiner, and Jean-Christophe Toussaint. A convergent and precise finite element scheme for Landau-Lifschitz-Gilbert equation. *Numer. Math.*, 128(3):407–430, 2014.

[17] François Alouges and Alain Soyeur. On global weak solutions for Landau-Lifshitz equations: existence and nonuniqueness. *Nonlinear Anal.*, 18(11):1071–1084, 1992.

[18] Rong An. Optimal error estimates of linearized Crank-Nicolson Galerkin method for Landau-Lifshitz equation. *J. Sci. Comput.*, 69(1):1–27, 2016.

[19] Wolfgang Arendt, Charles J. K. Batty, Matthias Hieber, and Frank Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, volume 96 of *Monographs in Mathematics*. Birkhäuser/Springer Basel AG, Basel, second edition, 2011.

[20] Markus Aurada, Michael Feischl, Thomas Führer, Michael Karkulik, Jens Markus Melenk, and Dirk Praetorius. Classical FEM-BEM coupling methods: nonlinearities, well-posedness, and adaptivity. *Comput. Mech.*, 51(4):399–419, 2013.

[21] Markus Aurada, Michael Feischl, and Dirk Praetorius. Convergence of some adaptive FEM-BEM coupling for elliptic but possibly nonlinear interface problems. *ESAIM Math. Model. Numer. Anal.*, 46(5):1147–1173, 2012.

[22] Jonas Ballani, Lehel Banjai, Stefan Sauter, and Alexander Veit. Numerical solution of exterior Maxwell problems by Galerkin BEM and Runge-Kutta convolution quadrature. *Numer. Math.*, 123(4):643–670, 2013.

[23] Lubomir Baňas. Adaptive techniques for landau–lifshitz–gilbert equation with magnetostriction. *Journal of Computational and Applied Mathematics*, 215(2):304–310, 2008. Proceedings of the Third International Conference on Advanced Computational Methods in Engineering (ACOMEN 2005).

[24] Lubomir Baňas, Sören Bartels, and Andreas Prohl. A convergent implicit finite element discretization of the Maxwell–Landau–Lifshitz–Gilbert equation. *SIAM J. Numer. Anal.*, 46:1399–1422, 2008.

[25] Lubomir Baňas, Markus Page, and Dirk Praetorius. A convergent linear finite element scheme for the Maxwell-Landau-Lifshitz-Gilbert equations. *Electron. Trans. Numer. Anal.*, 44:250–270, 2015.

[26] Lehel Banjai, Christian Lubich, and Jens Markus Melenk. Runge-Kutta convolution quadrature for operators arising in wave propagation. *Numer. Math.*, 119(1):1–20, 2011.

[27] Lehel Banjai, Christian Lubich, and Francisco-Javier Sayas. Stable numerical coupling of exterior and interior problems for the wave equation. *Numer. Math.*, 129(4):611–646, 2015.

[28] Sören Bartels. Projection-free approximation of geometrically constrained partial differential equations. *Math. Comp.*, 2015.

[29] Sören Bartels, Joy Ko, and Andreas Prohl. Numerical analysis of an explicit approximation scheme for the Landau-Lifshitz-Gilbert equation. *Math. Comp.*, 77(262):773–788, 2008.

[30] Sören Bartels and Andreas Prohl. Convergence of an implicit finite element method for the Landau-Lifshitz-Gilbert equation. *SIAM J. Numer. Anal.*, 44(4):1405–1419 (electronic), 2006.

[31] Eric Beaurepaire, J.-C. Merle, A. Daunois, and Jean-Yves Bigot. Ultrafast spin dynamics in ferromagnetic nickel. *Phys. Rev. Lett.*, 76:4250–4253, May 1996.

[32] Jean-Pierre Berenger. A perfectly matched layer for the absorption of electromagnetic waves. *J. Comput. Phys.*, 114(2):185–200, 1994.

[33] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[34] Timo Betcke, Eric Burman, and Matthew W. Scroggs. Boundary element methods with weakly imposed boundary conditions. *SIAM Journal on Scientific Computing*, 41(3):A1357–A1384, 2019.

[35] Timo Betcke and Matthew Scroggs. The Bempp Handbook. `http://bempp.com/handbook/intro.html`, 2020. Accessed: 2021-02-04.

[36] Timo Betcke, Matthew W. Scroggs, and Wojciech Śmigaj. Product algebras for Galerkin discretizations of boundary integral operators and their applications. *ACM Transactions on Mathematical Software*, 46(1):4:1–4:22, 2020.

[37] Timo Betcke, Elwin van 't Wout, and Pierre Gélat. Computationally efficient boundary element methods for high-frequency Helmholtz problems in unbounded domains. In D. Lahaye, J. Tang, and K. Vuik, editors, *Modern Solvers for Helmholtz Problems*, pages 215–243. Springer International Publishing, 2017.

[38] Alain Bossavit. Two dual formulations of the 3-D eddy-currents problem. *COMPEL*, 4(2):103 – 116, 1985.

[39] Susanne C. Brenner and L. Ridgway Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[40] Franco Brezzi and Michel Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.

[41] Florian Bruckner, Dieter Süss, Michael Feischl, Thomas Führer, Petra Goldenits, Markus Page, Dirk Praetorius, and Michele Ruggeri. Multiscale modeling in micromagnetics: existence of solutions and numerical integration. *Math. Models Methods Appl. Sci.*, 24(13):2627–2662, 2014.

[42] Annalisa Buffa and Snorre H. Christiansen. A dual finite element complex on the barycentric refinement. *Math. Comp.*, 76(260):1743–1769, 2007.

[43] Annalisa Buffa and Patrick Ciarlet Jr. On traces for functional spaces related to Maxwell's equations. I. An integration by parts formula in Lipschitz polyhedra. *Math. Methods Appl. Sci.*, 24(1):9–30, 2001.

[44] Annalisa Buffa and Patrick Ciarlet Jr. On traces for functional spaces related to Maxwell's equations. II. Hodge decompositions on the boundary of Lipschitz polyhedra and applications. *Math. Methods Appl. Sci.*, 24(1):31–48, 2001.

[45] Annalisa Buffa, Martin Costabel, and Dongwoo Sheen. On traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains. *J. Math. Anal. Appl.*, 276(2):845–867, 2002.

[46] Annalisa Buffa and Ralf Hiptmair. Galerkin boundary element methods for electromagnetic scattering. In *Topics in computational wave propagation*, volume 31 of *Lect. Notes Comput. Sci. Eng.*, pages 83–124. Springer, Berlin, 2003.

[47] Gilles Carbou and Pierre Fabrie. Time average in micromagnetism. *J. Differential Equations*, 147(2):383–409, 1998.

[48] Karel Carva, Pavel Baláž, and Ilie Radu. *Laser-Induced Ultrafast Magnetic Phenomena.* 01 2017.

[49] Luca Cassioli. Perpendicular recording. `https://en.wikipedia.org/wiki/Hard_disk_drive`, 2005. Accessed: 2021-03-05.

[50] CERN. The CERN Advanced STORage manager. `https://castor.web.cern.ch/castor/`. Accessed: 2021-03-08.

[51] Patrick Ciarlet Jr. and Jun Zou. Fully discrete finite element approaches for time-dependent Maxwell's equations. *Numer. Math.*, 82(2):193–219, 1999.

[52] Ivan Cimrák. Error estimates for a semi-implicit numerical scheme solving the Landau-Lifshitz equation with an exchange field. *IMA J. Numer. Anal.*, 25(3):611–634, 2005.

[53] Ivan Cimrák. Error analysis of a numerical scheme for 3D Maxwell–Landau–Lifshitz system. *Math. Methods Appl. Sci.*, 30:1667–1683, 2007.

[54] Ivan Cimrák. Existence, regularity and local uniqueness of the solutions to the Maxwell–Landau–Lifshitz system in three dimensions. *J. Math. Anal. Appl.*, 329:1080–1093, 2007.

[55] Ivan Cimrák. A survey on the numerics and computations for the Landau-Lifshitz equation of micromagnetism. *Arch. Comput. Methods Eng.*, 15(3):277–309, 2008.

[56] David Colton and Rainer Kress. *Inverse Acoustic and Electromagnetic Scattering Theory.* Springer, Berlin, 1998.

[57] Germund Dahlquist. G-stability is equivalent to A-stability. *BIT*, 18(4):384–401, 1978.

[58] Giovanni Di Fratta, Carl-Martin Pfeiler, Dirk Praetorius, Michele Ruggeri, and Bernhard Stiftner. Linear second-order IMEX-type integrator for the (eddy current) Landau-Lifshitz-Gilbert equation. *IMA J. Numer. Anal.*, 40(4):2802–2838, 2020.

[59] Daniele Antonio Di Pietro and Alexandre Ern. *Mathematical aspects of discontinuous Galerkin methods*, volume 69 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Heidelberg, 2012.

[60] Jürgen Elstrodt. *Maß- und Integrationstheorie.* Springer-Lehrbuch. [Springer Textbook]. Springer-Verlag, Berlin, fourth edition, 2005. Grundwissen Mathematik. [Basic Knowledge in Mathematics].

[61] Bjorn Engquist and Andrew Majda. Absorbing boundary conditions for the numerical simulation of waves. *Math. Comp.*, 31(139):629–651, 1977.

[62] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[63] Michael Feischl and Thanh Tran. The eddy current–LLG equations: FEM-BEM coupling and a priori error estimates. *SIAM J. Numer. Anal.*, 55(4):1786–1819, 2017.

[64] Michael Feischl and Thanh Tran. Existence of regular solutions of the Landau-Lifshitz-Gilbert equation in 3D with natural boundary conditions. *SIAM J. Math. Anal.*, 49(6):4470–4490, 2017.

[65] Felix Hagemann and Tilo Arens and Timo Betcke and Frank Hettlich. Solving inverse electromagnetic scattering problems via domain derivatives. *Inverse Problems*, 35(8), 2019.

[66] Ivan Fernandez-Corbaton. Total helicity of electromagnetic fields and matter. *Physical review. B, Condensed matter*, 103:054406, 02 2021.

[67] Huadong Gao. Optimal error estimates of a linearized backward Euler FEM for the Landau-Lifshitz equation. *SIAM J. Numer. Anal.*, 52(5):2574–2593, 2014.

[68] Gabriel N. Gatica. *A simple introduction to the mixed finite element method.* SpringerBriefs in Mathematics. Springer, Cham, 2014. Theory and applications.

[69] Thomas Gilbert. A Lagrangian formulation of the gyromagnetic equation of the magnetic field. *Phys Rev*, 100:1243–1255, 1955.

[70] Thomas L. Gilbert. A phenomenological theory of damping in ferromagnetic materials. *IEEE Transactions on Magnetics*, 40(6):3443–3449, 2004.

[71] Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.

[72] Petra Goldenits. *Konvergente numerische Integration der Landau-Lifshitz-Gilbert Gleichung*. PhD thesis, Institue for Analysis and Scientific Computing, Vienna Univerity of Technology, 11 2012. https://repositum.tuwien.at/cris/rp/rp11356.

[73] Beniamin Goldys, Kim-Ngan Le, and Thanh Tran. A finite element approximation for the stochastic landau–lifshitz–gilbert equation. *Journal of Differential Equations*, 260(2):937–970, 2016.

[74] Beniamin Goldys, Kim-Ngan Le, and Thanh Tran. A finite element approximation for the stochastic maxwell–landau–lifshitz–gilbert system, 2017.

[75] Jon Gorchon, Charles-Henri Lambert, Yang Yang, Akshay Pattabi, Richard B. Wilson, Sayeef Salahuddin, and Jeffrey Bokor. Single shot ultrafast all optical magnetization switching of ferromagnetic co/pt multilayers. *Applied Physics Letters*, 111(4):042401, 2017.

[76] Marcus J. Grote and Joseph B. Keller. Nonreflecting boundary conditions for time-dependent scattering. *J. Comput. Phys.*, 127(1):52–65, 1996.

[77] Thomas Hagstrom. Radiation boundary conditions for the numerical simulation of waves. In *Acta numerica, 1999*, volume 8 of *Acta Numer.*, pages 47–106. Cambridge Univ. Press, Cambridge, 1999.

[78] Thomas Hagstrom, Assaf Mar-Or, and Dan Givoli. High-order local absorbing conditions for the wave equation: extensions and improvements. *J. Comput. Phys.*, 227(6):3322–3357, 2008.

[79] Ernst Hairer, Christian Lubich, and Gerhard Wanner. *Geometric numerical integration*, volume 31 of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2010. Structure-preserving algorithms for ordinary differential equations, Reprint of the second (2006) edition.

[80] Bo He and Fernando L. Teixeira. Differential forms, Galerkin duality, and sparse inverse approximations in finite element solutions of Maxwell equations. *IEEE Trans. Antennas and Propagation*, 55(5):1359–1368, 2007.

[81] Gustav Herglotz. Über potenzreihen mit positivem, reellen teil im einheitskreis. *Leipziger Berichte*, 63(501-511), 1911.

[82] Jan S. Hesthaven and Tim Warburton. *Nodal discontinuous Galerkin methods*, volume 54 of *Texts in Applied Mathematics*. Springer, New York, 2008. Algorithms, analysis, and applications.

[83] Ralf Hiptmair. Symmetric coupling for eddy current problems. *SIAM J. Numer. Anal.*, 40(1):41–65, 2002.

[84] Marlis Hochbruck and Andreas Sturm. Error analysis of a second-order locally implicit method for linear Maxwell's equations. *SIAM J. Numer. Anal.*, 54(5):3167–3191, 2016.

[85] Johan Hoffman, Johan Jansson, Rodrigo V. de Abreu, Cem Degirmenci, Niclas Jansson, Kaspar Müller, Murtazo Nazarov, and Jeanette H. Spühler. Unicorn: Parallel adaptive finite element simulation of turbulent flow and fluid-structure interaction for deforming domains and complex geometry. *Computer and Fluids*, in press, 2012.

[86] Johan Hoffman, Johan Jansson, Cem Degirmenci, Niclas Jansson, and Murtazo Nazarov. *Unicorn: a Unified Continuum Mechanics Solver*, chapter 18. Springer, 2012.

[87] Johan Hoffman, Johan Jansson, Niclas Jansson, C. Johnson, and Rodrigo V. de Abreu. *Turbulent Flow and Fluid–structure Interaction*, chapter 28. Springer, 2012.

[88] Tuomas Hytönen, Jan van Neerven, Mark Veraar, and Lutz Weis. *Analysis in Banach spaces. Vol. I. Martingales and Littlewood-Paley theory*, volume 63 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, 2016.

[89] Niclas Jansson, Johan Jansson, and Johan Hoffman. Framework for massively parallel adaptive finite element computational fluid dynamics on tetrahedral meshes. *SIAM Journal on Scientific Computing*, 34(1):C24–C41, 2012.

[90] JuPitEer. Raviart thomas labelled. `https://en.wikipedia.org/wiki/Raviart%E2%80%93Thomas_basis_functions`, 2015. Accessed: 2021-02-02.

[91] Robert C. Kirby. Algorithm 839: Fiat, a new paradigm for computing finite element basis functions. *ACM Transactions on Mathematical Software*, 30(4):502–516, 2004.

[92] Robert C. Kirby. *FIAT: Numerical Construction of Finite Element Basis Functions,*, chapter 13. Springer, 2012.

[93] Robert C. Kirby, Matthew G. Knepley, Anders Logg, and L. Ridgway Scott. Optimizing the evaluation of finite element matrices. *SIAM Journal on Scientific Computing*, 27(3):741–758, 2005.

[94] Robert C. Kirby and Anders Logg. A compiler for variational forms. *ACM Transactions on Mathematical Software*, 32(3), 2006.

[95] Robert C. Kirby and Anders Logg. Efficient compilation of a class of variational forms. *ACM Transactions on Mathematical Software*, 33(3), 2007.

[96] Robert C. Kirby and Anders Logg. Benchmarking domain-specific compiler optimizations for variational forms. *ACM Transactions on Mathematical Software*, 35(2):1–18, 2008.

[97] Robert C. Kirby, Anders Logg, L. Ridgway Scott, and Andy R. Terrel. Topological optimization of the evaluation of finite element matrices. *SIAM Journal on Scientific Computing*, 28(1):224–240, 2006.

[98] Robert C. Kirby and L. Ridgway Scott. Geometric optimization of the evaluation of finite element matrices. *SIAM Journal on Scientific Computing*, 29(2):827–841, 2007.

[99] Balázs Kovács and Christian Lubich. Stable and convergent fully discrete interior-exterior coupling of Maxwell's equations. *Numer. Math.*, 137(1):91–117, 2017.

[100] Martin Kružík and Andreas Prohl. Recent developments in the modeling, analysis, and numerics of ferromagnetism. *SIAM Rev.*, 48(3):439–483, 2006.

[101] Antonio R. Laliena and Francisco-Javier Sayas. A distributional version of Kirchhoff's formula. *J. Math. Anal. Appl.*, 359(1):197–208, 2009.

[102] Antonio R. Laliena and Francisco-Javier Sayas. Theoretical aspects of the application of convolution quadrature to scattering of acoustic waves. *Numer. Math.*, 112(4):637–678, 2009.

[103] Lew Landau and Evgeny Lifschitz. On the theory of the dispersion of magnetic permeability in ferromagnetic bodies. *Phys Z Sowjetunion*, 8:153–168, 1935.

[104] Kim-Ngan Le, Marcus Page, Dirk Praetorius, and Thanh Tran. On a decoupled linear FEM integrator for eddy-current-LLG. *Appl. Anal.*, 94(5):1051–1067, 2015.

[105] Kim-Ngan Le and Thanh Tran. A convergent finite element approximation for the quasi-static Maxwell-Landau-Lifshitz-Gilbert equations. *Comput. Math. Appl.*, 66(8):1389–1402, 2013.

[106] Günther Lehner. *Electromagnetic Field Theory for Engineers and Physicists.* 01 2010.

[107] Jacques Louis Lions. *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires.* Dunod Gauthier-Villars, Paris, 1969.

[108] Anders Logg. Automating the finite element method. *Archives of Computational Methods in Engineering*, 14(2):93–138, 2007.

[109] Anders Logg. Efficient representation of computational meshes. *International Journal of Computational Science and Engineering*, 4(4):283–295, 2009.

[110] Anders Logg, Kent-Andre Mardal, Garth N. Wells, et al. *Automated Solution of Differential Equations by the Finite Element Method.* Springer, 2012.

[111] Anders Logg, Kristian B. Ølgaard, Marie E. Rognes, and Garth N. Wells. *FFC: the FEniCS Form Compiler*, chapter 11. Springer, 2012.

[112] Anders Logg and Garth N. Wells. Dolfin: Automated finite element computing. *ACM Transactions on Mathematical Software*, 37(2), 2010.

[113] Christian Lubich. Convolution quadrature and discretized operational calculus. I. *Numer. Math.*, 52(2):129–145, 1988.

[114] Christian Lubich. Convolution quadrature and discretized operational calculus. II. *Numer. Math.*, 52(4):413–425, 1988.

[115] Christian Lubich. On the multistep time discretization of linear initial-boundary value problems and their boundary integral equations. *Numer. Math.*, 67(3):365–389, 1994.

[116] Christian Lubich. Convolution quadrature revisited. *BIT*, 44(3):503–514, 2004.

[117] James Clerk Maxwell. *A treatise on electricity and magnetism.* Dover Publications, Inc., New York, 1954. 3d ed, Two volumes bound as one.

[118] William McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, Cambridge, 2000.

[119] Jens Markus Melenk and Alexander Rieder. Runge-Kutta convolution quadrature and FEM-BEM coupling for the time-dependent linear Schrödinger equation. *J. Integral Equations Appl.*, 29(1):189–250, 2017.

[120] Daniel M. Mittleman. Perspective: Terahertz science and technology. *Journal of Applied Physics*, 122(23):230901, 2017.

[121] Peter Monk. *Finite element methods for Maxwell's equations.* Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003.

[122] Jean-Claude Nédélec. Mixed finite elements in $\mathbf{R}^3$. *Numer. Math.*, 35(3):315–341, 1980.

[123] Jean-Claude Nédélec. *Acoustic and electromagnetic equations*, volume 144 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 2001. Integral representations for harmonic problems.

[124] Heike Neuroth, Achim Oßwald, Regine Scheffel, Stefan Strathmann, and Mathias Jehn, editors. *nestor Handbuch: Eine kleine Enzyklopädie der digitalen Langzeitarchivierung, Version 2.3.* Göttingen, 2014.

[125] Jörg Nick, Balázs Kovács, and Christian Lubich. Erratum: Stable and convergent fully discrete interior-exterior coupling of Maxwell's equations. *arXiv:1605.04086*, 2020.

[126] Kristian B. Ølgaard, Anders Logg, and Garth N. Wells. Automated code generation for discontinuous galerkin methods. *SIAM Journal on Scientific Computing*, 31(2):849–864, 2008.

[127] Kristian B. Ølgaard and Garth N. Wells. Optimisations for quadrature representations of finite element tensors through automated code generation. *ACM Transactions on Mathematical Software*, 37, 2010.

[128] Raymond E. A. C. Paley and Norbert Wiener. *Fourier transforms in the complex domain*, volume 19 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 1987. Reprint of the 1934 original.

[129] Evangelos Th. Papaioannou and René Beigang. Thz spintronic emitters: a review on achievements and future challenges. *Nanophotonics*, 10(4):1243–1257, 2021.

[130] Dirk Praetorius, Michele Ruggeri, and Bernhard Stiftner. Convergence of an implicit-explicit midpoint scheme for computational micromagnetics. *Comput. Math. Appl.*, 75(5):1719–1738, 2018.

[131] Ricardo A. Prato Torres, Ernst P. Stephan, and Florian Leydecker. A FE/BE coupling for the 3D time-dependent eddy current problem. Part I: a priori error estimates. *Computing*, 88(3-4):131–154, 2010.

[132] Andreas Prohl. *Computational Micromagnetism.* Advances in Numerical Mathematics. B. G. Teubner, Stuttgart, 2001.

[133] George T. Rado and Johannes R. Weertman. Spin-wave resonance in a ferromagnetic metal. *Journal of Physics and Chemistry of Solids*, 11(3):315–333, 1959.

[134] Victor Raposo, Rodrigo Guedas, Felipe García-Sánchez, M. Auxiliadora Hernández, Marcelino Zazo, and Eduardo Martínez. Micromagnetic modeling of all optical switching of ferromagnetic thin films: The role of inverse faraday effect and magnetic circular dichroism. *Applied Sciences*, 10(4), 2020.

[135] Victor Raposo, Eduardo Martínez, Auxiliadora Hernández, and Marcelino Zazo. Micromagnetic modeling of all-optical switching. *IEEE Transactions on Magnetics*, 55(7):1–6, 2019.

[136] Ulf Rathje. *Technisches Konzept für die Datenarchivierung im Bundesarchiv.* Der Archivar, H. 2, Jahrgang 55,, 2002.

[137] Alexander Rieder. *Convolution Quadrature and Boundary Element Methods in wave propagation: a time domain point of view.* PhD thesis, 05 2017.

[138] Marie E. Rognes, David A. Ham, Colin J. Cotter, and Andrew T. T. McRae. Automating the solution of pdes on the sphere and other manifolds in fenics 1.2. *Geoscientific Model Development*, 6:2099–2119, 2013.

[139] Marie E. Rognes, Robert C. Kirby, and Anders Logg. Efficient assembly of h(div) and h(curl) conforming finite elements. *SIAM Journal on Scientific Computing*, 31(6):4130–4151, 2009.

[140] Walter Rudin. *Real and complex analysis.* McGraw-Hill Book Co., New York, third edition, 1987.

[141] Michele Ruggeri. *Coupling and numerical integration of the Landau-Lifshitz-Gilbert equation.* PhD thesis, Institue for Analysis and Scientific Computing, Vienna Univerity of Technology, 11 2016. https://repositum.tuwien.at/handle/20.500.12708/8012.

[142] Stefan A. Sauter and Christoph Schwab. *Boundary element methods*, volume 39 of *Springer Series in Computational Mathematics.* Springer-Verlag, Berlin, 2011. Translated and expanded from the 2004 German original.

[143] Francisco-Javier Sayas. *Retarded potentials and time domain boundary integral equations*, volume 50 of *Springer Series in Computational Mathematics.* Springer, [Cham], 2016. A road map.

[144] Matthew W. Scroggs, Timo Betcke, Erik Burman, Wojciech Śmigaj, and Elwin van 't Wout. Software frameworks for integral equations in electromagnetic scattering based on Calderón identities. *Comput. Math. Appl.*, 74(11):2897–2914, 2017.

[145] Wojciech Śmigaj, Timo Betcke, Simon Arridge, Joel Phillips, and Martin Schweiger. Solving boundary integral problems with BEM++. *ACM Transactions on Mathematical Software*, 41(2):6:1–6:40, 2015.

[146] Claudiu D. Stanciu. *Laser Induced Femtosecond magnetic Recording.* PhD thesis, Radboud Universiteit Nijmegen, 2008.

[147] Claudiu D. Stanciu, Fredrik Hansteen, Alexey V. Kimel, Andrey Kirilyuk, Arata Tsukamoto, Achioshi Itoh, and Theo Rasing. All-optical magnetic recording with circularly polarized light. *Phys. Rev. Lett.*, 99:047601, Jul 2007.

[148] Dieter Suess, Vasilios Tsiantos, Thomas Schrefl, Josef Fidler, Werner Scholz, Herrmann Forster, Rok Dittrich, and James J. Miles. Time resolved micromagnetics using a preconditioned time integration method. *Journal of Magnetism and Magnetic Materials*, 248(2):298–311, 2002.

[149] Isabelle Terrasse. *Resolution mathematique et numerique des equations de maxwell instationnaires par une methode de potentiels retardes.* PhD thesis, 1993. Thèse de doctorat dirigée par Nédélec, Jean-Claude Physique Palaiseau, Ecole polytechnique 1993.

[150] Masayoshi Tonouchi. Cutting-edge thz technology. *Nat. Photonics*, 1:97–105, 02 2007.

[151] Kerstin Vater, Timo Betcke, and Boris Dilba. Simple and efficient gpu parallelization of existing h-matrix accelerated bem code. *arXiv preprint arXiv:1711.01897*, 2017.

[152] Alexander Veit. Masterarbeit convolution quadrature for time-dependent maxwell equations. 2011.

[153] Augusto Visintin. On Landau-Lifshitz' equations for ferromagnetism. *Japan J. Appl. Math.*, 2(1):69–84, 1985.

[154] Dirk Werner. *Funktionalanalysis.* Springer-Verlag, Berlin, extended edition, 2000.

[155] Konstantin Zerulla. A uniformly exponentially stable ADI scheme for Maxwell equations. *J. Math. Anal. Appl.*, 492(1):124442, 48, 2020.