A necessary test for elliptical symmetry based on the uniform distribution over the Stiefel manifold

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Abstract. This paper provides a new procedure for testing the null hypothesis of multivariate elliptical symmetry. A test for uniformity over the Stiefel manifold based on modified degenerate V-statistics is employed since the test statistic proposed in this paper consists of independent random matrices, formed by the *scaled residuals* (or the *Studentized residuals*), which are uniformly distributed over the Stiefel manifold under the null hypothesis. Also, Monte Carlo simulation studies are carried out to evaluate the type I error and power of the test. Finally, the procedure is illustrated using the Iris data.

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§1. Introduction

The family of *elliptical contoured distributions* (or *elliptical distributions* for short) is a natural generalization of the multivariate normal distribution. The assumption of elliptical populations is frequently imposed in multivariate analysis. However, it is indispensable to test whether a sample comes from an elliptical population. Therefore, there exists a sizable literature on this subject (see Fang and Liang [8] for a survey). See also Manzotti et al. [18], Schott [22], Huffer and Park [14], Batsidis and Zografos [2] and the references therein.

Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ be iid p-dimensional random (column) vectors drawn from a population with mean vector $\boldsymbol{\mu}$ and covariance matrix $\Sigma = \Sigma' > 0$ (Σ' means transpose of Σ and $\Sigma > 0$ indicates that Σ is positive definite). Let $X = [\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N]$ be the $p \times N$ observation matrix. Then, the sample mean vector and covariance matrix can be expressed as

$$\bar{\boldsymbol{X}} = \frac{1}{N} X\mathbf{1},$$

$$S = \frac{1}{n} XQX', \qquad n = N - 1 \geq p, \tag{1.1}$$

respectively, where $\mathbf{1}$ is the vector of N ones,

$$Q = I_N - \frac{1}{N}\mathbf{1}\mathbf{1}', \tag{1.2}$$

I_d denotes the identity matrix of size d and the prime refers to transpose.

Some of the statistics for testing elliptical symmetry in the above-mentioned references consist of the so-called *scaled residuals* (or *Studentized residuals*)

$$\boldsymbol{W}_i = S^{-1/2}(\boldsymbol{X}_i - \bar{\boldsymbol{X}}), \qquad i = 1, \ldots, N,$$

where $S^{-1/2}$ indicates the inverse matrix of a symmetric square root of S. For instance, Manzotti et al. [18] considered the statistic $\boldsymbol{W}_i/||\boldsymbol{W}_i||$, where $||\cdot||$ stands for the Euclidean norm of a vector, which should approximately possess the uniform distribution over the unit sphere $\mathbb{S}^{p-1}$ on $\mathbb{R}^p$ when the distribution of $\boldsymbol{X}_i$ has elliptical symmetry, and they introduced the procedure of testing elliptical symmetry by using the limiting distribution of the average of some spherical harmonics over the $\boldsymbol{W}_i/||\boldsymbol{W}_i||$'s. Huffer and Park [14] provided Pearson's χ^2-statistic based on $||\boldsymbol{W}_i||^2$ with qc shells, obtained by dividing $\mathbb{R}^p$ into c spherical shells centered at the origin and q congruent sectors emanating from the origin. They also carried out numerical studies to compare the power of their test procedure with other tests for elliptical symmetry and multivariate normality under various alternatives. As pointed out by Fang and Liang [8] and Batsidis and Zografos [2], however, a downside in using the scaled residuals is that $\boldsymbol{W}_i$, $i = 1, \ldots, N$, are no longer independent, and their distribution is different from the distribution of $\Sigma^{-1/2}(\boldsymbol{X}_i - \boldsymbol{\mu})$, $i = 1, \ldots, N$.

In a recent paper, Iwashita and Klar [15] considered the (exact) joint distribution of $\{\boldsymbol{W}_i\}_{i=1}^N$, that is, the joint distribution of the $p \times N$ random matrix

$$W = [\boldsymbol{W}_1, \ldots, \boldsymbol{W}_N] = S^{-1/2}\,[\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N]\,Q = S^{-1/2}XQ,$$

under elliptical population. Note that, since Q is an $N \times N$ idempotent matrix with $\mathrm{rank}(Q) = n(= N - 1)$, there exists an $N \times n$ matrix K such that

$$KK' = Q, \quad K'K = I_n, \quad K'\mathbf{1} = \mathbf{0}, \tag{1.3}$$

where $\mathbf{0}$ is the column n-vector of zeroes.

The contribution of this paper is to show that $U = K'X'(nS)^{-1/2}$ possesses the uniform distribution over the Stiefel manifold, and then construct a procedure of a necessary test for elliptical symmetry. Here we note that the terminology "necessary test" has the same meaning as in Fang et al. [10].

The paper is organized as follows: In Section 2, we show that the proposed statistic U has the uniform distribution over the Stiefel manifold $\mathcal{O}(n,p)$ of orthonormal n-frames in $\mathbb{R}^p$ by applying the result in Iwashita and Klar [15]. In Section 3, we construct the procedure of testing elliptical symmetry by combining the method of Pycke [21] with the result in the Section 2. In Section 4, we conduct some numerical experiments to confirm the ability of our procedure. We also apply our test for the Iris data presented by Fisher [11].

§2. Preliminary

Let $\boldsymbol{X}$ be a p-dimensional random vector from an elliptical distribution with a location parameter $\boldsymbol{\mu} \in \mathbb{R}^p$ and a scale matrix Λ, a symmetric and positive definite matrix of order p, having a probability density function (pdf) of the form

$$f(\boldsymbol{x}) = c_p|\Lambda|^{-1/2}g((\boldsymbol{x}-\boldsymbol{\mu})'\Lambda^{-1}(\boldsymbol{x}-\boldsymbol{\mu})), \tag{2.1}$$

where g is a nonnegative function, and c_p is a normalizing constant (see, for example, Muirhead [19, Section 1.5], Fang and Zhang [9, Section 2.6.5]). Note that the characteristic function (cf) of $\boldsymbol{X}$ can be expressed as

$$\Psi(\boldsymbol{t}) = \exp(i\boldsymbol{t}'\boldsymbol{\mu})\psi(\boldsymbol{t}'\Lambda\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{R}^p, \quad i = \sqrt{-1}, \tag{2.2}$$

and, if they exist, $\mathrm{E}[\boldsymbol{X}] = \boldsymbol{\mu}$ and $\Sigma = \mathrm{Cov}[\boldsymbol{X}] = -2\psi'(0)\Lambda \equiv \gamma\Lambda > 0$.

Suppose X is a $k \times l$ random matrix and $H \in \mathcal{O}(k)$, where $\mathcal{O}(k)$ is the set of orthogonal matrices of order k. If $X \overset{d}{=} HX$ for every fixed H, where the notation "$\overset{d}{=}$" denotes equality in distribution, we call the distribution of X *left-spherical*. If X' is left-spherical, then X is *right-spherical*. When X is left- and right-spherical, we call X *spherical* (see Dawid [5]).

Throughout this paper, we assume the existence of the covariance matrix Σ and the pdf as given in (2.1) for nonsingularity of the sample covariance matrix (1.1) (see Balakrishnan et al. [1], Eaton and Perlman [6] and Okamoto [20]), and we will write $\boldsymbol{X} \sim \mathrm{EC}_p(\boldsymbol{\mu}, \Lambda; \psi)$ to indicate that $\boldsymbol{X}$ has an elliptical distribution whose cf has the form given in (2.2). In a similar way, if a $k \times l$ random matrix X has a left-spherical distribution with the cf $\phi_X(T)$ for $k \times l$ matrix T, then, we denote $X \sim \mathrm{LS}_{k\times l}(\phi_X)$, and $X \sim \mathrm{SS}_{k\times l}(\phi_X)$ means X has a *spherical distribution* with the cf $\phi_X(T)$ (see, in some detail, Fang and Zhang [9, Lemma 3.1.1 and Theorem 3.1.4]). Finally, $X \sim \mathcal{U}_{k,l}$ indicates that a $k \times l$ random matrix X is uniformly distributed over the Stiefel manifold $\mathcal{O}(k,l)$, i.e., X is left-spherical and $X'X = I_l$ (see Fang and Zhang [9, Definition 3.1.2]).

Suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ are independent random copies of $\boldsymbol{X} \sim \mathrm{EC}_p(\mathbf{0}, \Lambda; \psi)$ and

$$X = [\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N] = [\boldsymbol{X}_{(1)}, \ldots, \boldsymbol{X}_{(p)}]'. \tag{2.3}$$

We define the following subclass of the left-spherical distribution $\mathrm{LS}_{p\times N}(\phi_X)$,

$$\mathfrak{F}_{p\times N} = \{X(p \times N) \sim \mathrm{LS}_{p\times N}(\phi_X); \boldsymbol{X}_{(1)} \text{ is spherical}\} \tag{2.4}$$

(appeared in Fang and Zhang [9, p. 123] as $\mathfrak{F}_7$) and introduce a result related to $\mathfrak{F}_{p\times N}$; similar results appeared in Iwashita and Klar [15].

Lemma 2.1. *Let X be the observation matrix, defined by (2.3), based on independent random sample $\{\boldsymbol{X}_i\}_{i=1}^N$ from $\mathrm{EC}_p(\mathbf{0}, \Lambda; \psi)$. Then*

$$Y = S^{-1/2}X \sim \mathrm{LS}_{p\times N}(\phi_Y) \tag{2.5}$$

for the respective characteristic functions ϕ_Y, where S is defined in (1.1).

Proof. Iwashita and Klar [15] showed that if $\{\tilde{\boldsymbol{X}}_i\}_{i=1}^N$ is an iid sample from $\mathrm{EC}_p(\mathbf{0}, I_p; \psi)$, then $\tilde{Y} = \tilde{S}^{-1/2}\tilde{X} \sim \mathrm{LS}_{p\times N}(\phi_{\tilde{Y}})$, where $\tilde{X}$ and $\tilde{S}$ denote the $p \times N$ observation matrix and $p \times p$ covariance matrix based on $\{\tilde{\boldsymbol{X}}_i\}_{i=1}^N$, respectively. By straightforward manipulation, we have

$$\begin{aligned} Y = S^{-1/2}X &= (\Lambda^{1/2}\tilde{S}\Lambda^{1/2})^{-1/2}\Lambda^{1/2}\tilde{X} \\ &= [(\Lambda^{1/2}\tilde{S}\Lambda^{1/2})^{-1/2}\Lambda^{1/2}\tilde{S}^{1/2}]\tilde{S}^{-1/2}\tilde{X} = H_{\Lambda,\tilde{S}}\tilde{Y}, \end{aligned}$$

where $H_{\Lambda,\tilde{S}} = (\Lambda^{1/2}\tilde{S}\Lambda^{1/2})^{-1/2}\Lambda^{1/2}\tilde{S}^{1/2} \in \mathcal{O}(p)$ (see Balakrishnan et al. [1]). Note that the cf of Y can be expressed as, for $p \times N$ matrix T,

$$\begin{aligned} \phi_Y(T) &= \mathrm{E}[\mathrm{etr}(iT'H_{\Lambda,\tilde{S}}\tilde{Y})] \\ &= \mathrm{E}\left[\mathrm{etr}(iT'H_{\Lambda,\tilde{S}}\tilde{Y})\int_{\mathcal{O}(p)}(dH)\right] \\ &= \mathrm{E}\left[\int_{\mathcal{O}(p)}\mathrm{etr}(iT'H_{\Lambda,\tilde{S}}H\tilde{Y})(dH)\right] \quad (\text{use } \tilde{Y} \overset{d}{=} H\tilde{Y}, H \in \mathcal{O}(p)), \end{aligned}$$

where $\mathrm{etr}(*) = \exp(\mathrm{tr}(*))$, (dH) denotes the unit invariant Haar measure on $\mathcal{O}(p)$ (see, e.g., Muirhead [19, p.72]) and $i = \sqrt{-1}$. By straightforward calculations based on Muirhead [19, Theorem 7.4.1],

$$\begin{aligned}
\int_{\mathcal{O}(p)} \operatorname{etr}(iT'H_{\Lambda,\tilde{S}}H\tilde{Y})(dH) &= \int_{\mathcal{O}(p)} \operatorname{etr}(i\tilde{Y}T'H_{\Lambda,\tilde{S}}H)(dH) \\
&= {}_0F_1\left(\frac{p}{2}; -\frac{1}{4}\tilde{Y}T'H_{\Lambda,\tilde{S}}H'_{\Lambda,\tilde{S}}T\tilde{Y}'\right) \\
&= {}_0F_1\left(\frac{p}{2}; -\frac{1}{4}\tilde{Y}T'T\tilde{Y}'\right) \\
&\equiv \int_{\mathcal{O}(p)} \operatorname{etr}(iT'H\tilde{Y})(dH) \\
&= \operatorname{etr}(iT'\tilde{Y}) \int_{\mathcal{O}(p)} (dH) \quad (\text{use } H\tilde{Y} \overset{d}{=} \tilde{Y}) \\
&= \operatorname{etr}(iT'\tilde{Y}),
\end{aligned}$$

where ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; X)$ is the hypergeometric function of matrix argument (see, for example, Muirhead [19, Definition 7.3.1]). This implies $\phi_Y(T) = \phi_{\tilde{Y}}(T)$, the cf of $\tilde{Y}$, and, hence,

$$Y \overset{d}{=} \tilde{Y} \sim \mathrm{LS}_{p\times N}(\phi_{\tilde{Y}}),$$

which completes the proof. □

With the help of Lemma 2.1, we are able to obtain the following result.

Theorem 2.2. *Suppose* $\{\boldsymbol{X}_i\}_{i=1}^N$ *is an iid sample drawn from* $\mathrm{EC}_p(\mathbf{0}, \Lambda; \psi)$ *and* X *is the observation matrix defined in (2.3). Then,*

$$Y' \sim \mathrm{SS}_{N\times p}(\phi_{Y'}), \tag{2.6}$$

where Y *is defined in (2.5).*

Proof. Let $Y = [\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_N] = \left[\boldsymbol{Y}_{(1)}, \boldsymbol{Y}_{(2)}, \ldots, \boldsymbol{Y}_{(p)}\right]'$. Set $\boldsymbol{a} = (\boldsymbol{\alpha}'\boldsymbol{\alpha})^{-1/2}\boldsymbol{\alpha}$ for all $\boldsymbol{\alpha} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$. Then, by Theorem 2 in Iwashita and Klar [15],

$$Y\boldsymbol{\alpha} = S^{-1/2}X\boldsymbol{\alpha} \sim \mathrm{EC}_p(\mathbf{0}, (\boldsymbol{\alpha}'\boldsymbol{\alpha})I_p; \varphi),$$

where φ denotes the cf of $Y\boldsymbol{\alpha}$, which generally differs from ψ. Hence the distribution of $Y\boldsymbol{\alpha}$ depends on $\boldsymbol{\alpha}$ only through $\boldsymbol{\alpha}'\boldsymbol{\alpha}$. As $Y \sim \mathrm{LS}_{p\times N}(\phi_Y)$ by Lemma 2.1, using Fang and Zhang [9, Theorem 3.6.9], we obtain

$$Y = S^{-1/2}X \in \mathfrak{F}_{p\times N}.$$

As a side note, let $\mathcal{P}(p)$ denote the permutation group, a subgroup of $\mathcal{O}(p)$; that is, if a $p \times p$ matrix $H_{\mathcal{P}} \in \mathcal{P}(p)$, then $H'_{\mathcal{P}}H_{\mathcal{P}} = I_p$, and the elements of $H_{\mathcal{P}}$

are either 0 or 1 (see, Fang et al. [7, pp.5–6]). As $H_{\mathcal{P}} Y \overset{d}{=} Y \sim \mathrm{LS}_{p\times N}(\phi_Y)$, we have $[\boldsymbol{Y}_{(i_1)},\dots,\boldsymbol{Y}_{(i_p)}]' \overset{d}{=} Y \in \mathfrak{F}_{p\times N}$ for any permutation $(i_1,\dots,i_p)$ of $(1,\dots,p)$.

Taking into account that $\mathrm{rank}(YY') = \mathrm{rank}(Y) = p < N$, let $\lambda_1, \lambda_2, \dots, \lambda_p$ ($\lambda_i > 0$) be the eigenvalues of YY' and H_λ be an orthogonal matrix of the pertaining eigenvectors such that

$$H_\lambda YY'H'_\lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_p) \equiv \Lambda, \quad H_\lambda \in \mathcal{O}(p).$$

Here we note that H_λ is a random matrix on $\mathcal{O}(p)$. In a similar way as in the proof of Lemma 2.1, we obtain for $Y \sim \mathrm{LS}_{p\times N}(\phi_Y)$ and $p\times N$ matrix T,

$$\mathrm{E}[\mathrm{etr}(iT'H_\lambda Y)] = \mathrm{E}[\mathrm{etr}(iT'Y)].$$

This yields

$$H_\lambda Y \overset{d}{=} Y \sim \mathrm{LS}_{p\times N}(\phi_Y), \quad \Lambda = H_\lambda YY'H'_\lambda \overset{d}{=} YY' \sim \mathrm{SS}_{p\times p}(\phi_{YY'}).$$

Let $\mathcal{D}= (YY')^{1/2}$ be a symmetric square root of YY', i.e., $\mathcal{D}^2 = YY'$. Then $\mathcal{D} \overset{d}{=} \mathcal{D}' \sim \mathrm{SS}_{p\times p}(\phi_{\mathcal{D}})$, because the following fact holds for $H \in \mathcal{O}(p)$:
(2.7)

$$\mathcal{D} = (YY')^{1/2} \overset{d}{=} (HYY'H')^{1/2} = (H\mathcal{D}^2H')^{1/2} = (H\mathcal{D}H'H\mathcal{D}H)^{1/2} = H\mathcal{D}H'.$$

Hence, $\Lambda^{1/2} \overset{d}{=} (YY')^{1/2} = \mathcal{D} \sim \mathrm{SS}_{p\times p}(\phi_{\mathcal{D}})$, where $\Lambda^{1/2}$ is a symmetric square root of Λ.

If we set

$$\Delta = \mathrm{diag}(\pm\sqrt{\lambda_1}, \pm\sqrt{\lambda_2}, \dots, \pm\sqrt{\lambda_p}),$$

with an arbitrary choice of the sign in each component, it satisfies (2.7), therefore $\Delta \overset{d}{=} \Delta' \sim \mathrm{SS}_{p\times p}(\phi_\Delta)$. Applying Theorem A9.5 in Muirhead [19] to $YY' \overset{d}{=} \Delta^2$, there exists an $N \times p$ random matrix $U = [\boldsymbol{U}_1, \boldsymbol{U}_2, \dots, \boldsymbol{U}_p]$ with $U'U = I_p$, such that $Y \overset{d}{=} \Delta U'$. Since

$$Y \overset{d}{=} \Delta U' \overset{d}{=} \left[\pm\sqrt{\lambda_1}\boldsymbol{U}_1, \pm\sqrt{\lambda_2}\boldsymbol{U}_2, \dots, \pm\sqrt{\lambda_p}\boldsymbol{U}_p\right]' \in \mathfrak{F}_{p\times N},$$

it holds that $\boldsymbol{Y}_{(1)} \overset{d}{=} \pm\sqrt{\lambda_1}\boldsymbol{U}_1 = \sqrt{\lambda_1}(\pm\boldsymbol{U}_1) \sim \mathrm{EC}_N(\boldsymbol{0}, I_N; \varphi)$. Referring to Corollary and Theorem 2.3 in Fang et al. [7, p.30], we see that $||\boldsymbol{Y}_{(1)}|| \overset{d}{=} \sqrt{\lambda_1}$ and $\boldsymbol{Y}_{(1)}/||\boldsymbol{Y}_1|| \overset{d}{=} \boldsymbol{U}_1$ are independent, and $\boldsymbol{U}_1 \sim \mathcal{U}(\mathbb{S}^{N-1})$, where $\mathcal{U}(\mathbb{S}^{N-1})$ denotes the uniform distribution over the unit sphere in $\mathbb{R}^N$.

In the same way, we get, for $i = 1, \dots, p$, $\boldsymbol{Y}_{(i)} \overset{d}{=} \pm\sqrt{\lambda_i}\boldsymbol{U}_i$, where $\sqrt{\lambda_i}$ and $\boldsymbol{U}_i \sim \mathcal{U}(\mathbb{S}^{N-1})$ are independent. Thus, $U \sim \mathcal{U}_{N,p}$, independent of Δ, i.e., $Y' \overset{d}{=} U\Delta \sim \mathrm{SS}_{N\times p}(\phi_{Y'})$. □

According to Lemma 4 and its proof in Dawid [5], if $V \sim \mathrm{LS}_{k\times l}(\phi_V)$ and the fixed $q \times k$ matrix L satisfies $LL' = I_q$, then

$$LV \sim \mathrm{LS}_{q\times l}(\phi_{LV}). \tag{2.8}$$

Hence, using (1.3), (2.6) and (2.8), we obtain

$$Z \equiv K'Y' = K'X'S^{-1/2} \sim \mathrm{LS}_{n\times p}(\phi_Z),$$

and, actually, $Z \sim \mathrm{SS}_{n\times p}(\phi_Z)$. Referring to Fang and Zhang [9, p.101], and noting that $(n^{-1/2}Z)'(n^{-1/2}Z) = I_p$, we obtain

$$Z(Z'Z)^{-1/2} = n^{-1/2}Z \sim \mathcal{U}_{n,p}.$$

Summarizing the above yields the following result, which is the key to propose the test statistic for multivariate elliptical symmetry in the next section.

Corollary 2.3. *Let $X = [\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_N]$, where $\{\boldsymbol{X}_i\}_{i=1}^N$ is an iid sample drawn from $\mathrm{EC}_p(\boldsymbol{\mu}, \Lambda; \psi)$, and let S be the sample covariance matrix of (1.1). Then*

$$U = K'X'(nS)^{-1/2} \sim \mathcal{U}_{n,p}, \tag{2.9}$$

where K is an $N \times n$ matrix which satisfies the conditions of (1.3).

§3. Test of uniformity over Stiefel manifold $\mathcal{O}(n,p)$

In this section, we propose a new test procedure for uniformity over the Stiefel manifold $\mathcal{O}(n,p)$ as a generalization of tests proposed by Pycke [21]. Therein, he considered tests for uniformity of circular distributions against multimodal alternatives by making use of certain degenerate U- and V-statistics. Let $\{\Theta_i\}_{i=1}^m$ be an iid sample drawn from a distribution defined on the interval $[0, 2\pi]$. Pycke [21] identified the unit circle $\mathbb{S}^1$ with the interval $[0, 2\pi]$ in which the endpoints 0 and 2π are identified, and considered the degenerate U- and V-statistics

$$G = -\frac{2}{m-1}\sum_{i=2}^{m}\sum_{j=1}^{i-1}\log\{2 - 2\cos(\Theta_i - \Theta_j)\},$$

$$V_q = \frac{2}{m}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\cos(\Theta_i - \Theta_j) - q}{1 - 2q\cos(\Theta_i - \Theta_j) + q^2}, \quad q \in (0,1), \tag{3.1}$$

as test statistics for uniformity. Pycke [21] determined critical values for various significant levels and various sample sizes by Monte Carlo simulation.

Let $\boldsymbol{S}_1, \ldots, \boldsymbol{S}_m$ be independent d-dimensional random vectors drawn from a uniform distribution over the hypersphere $\mathbb{S}^{d-1}$ ($d \geq 2$), and let $\Theta_{ij} = \arccos(\boldsymbol{S}_i' \boldsymbol{S}_j)$ denote the enclosed angle between $\boldsymbol{S}_i$ and $\boldsymbol{S}_j$. For $d = 2$, the relation between Cartesian and polar coordinates yields $\Theta_i - \Theta_j = \Theta_{ij}$. Here, we consider

$$\text{(3.2)} \quad \tilde{V}_{\ell,d} = \frac{1}{m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} 2\cos\left(\ell \Theta_{ij}\right) = \frac{2}{m} + \frac{4}{m^2} \sum_{i=2}^{m} \sum_{j=1}^{i-1} \cos\left(\ell \arccos(\boldsymbol{S}_i' \boldsymbol{S}_j)\right),$$

$$\text{(3.3)} \quad V_{q,d} = \frac{2}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\cos(\Theta_{ij}) - q}{1 - 2q\cos(\Theta_{ij}) + q^2} = \frac{2}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\boldsymbol{S}_i' \boldsymbol{S}_j - q}{1 - 2q \boldsymbol{S}_i' \boldsymbol{S}_j + q^2},$$

where ℓ is a natural number and $q \in (0,1)$, as test statistics for uniformity over the Stiefel manifold. Clearly, (3.3) is a direct generalization of (3.1) for $d \geq 3$, whereas $\tilde{V}_{\ell,d}$ uses the individual components of the kernel function pertaining to V_q (see Pycke [21]). The statistics in (3.2) and (3.3) are V-statistics

$$V = \frac{1}{m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} h\left(\boldsymbol{S}_i' \boldsymbol{S}_j\right) = \frac{1}{m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} h\left(\Theta_{ij}\right),$$

with kernels $\tilde{h}_l(\theta) = 2\cos(l\theta)$, $l \geq 1$, and

$$h_q(\theta) = 2 \sum_{k=1}^{\infty} q^{k-1} \cos(k\theta) = \frac{2(\cos\theta - q)}{1 - 2q\cos\theta + q^2}, \quad q \in (0,1),$$

respectively. The distribution of Θ_{ij} under the hypotheses of uniformity can be obtained by direct computations, or one can resort to the distribution of the correlation coefficient under normality as done in Cai et al. [3, Lemma 12] (see also Cai and Jiang [4, Lemma 4.1]).

Proposition 3.1. *Let $d \geq 2$. Then, under the hypotheses of uniformity, Θ_{ij}, $1 \leq i < j \leq m$, are pairwise iid with the density function*

$$f(\theta) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(d/2)}{\Gamma((d-1)/2)} \cdot (\sin\theta)^{d-2}, \quad \theta \in [0, \pi].$$

Using Proposition 3.1, we obtain $\mathrm{E}[V] = \mathrm{E}\left[h\left(\Theta_{12}\right)\right] = \mu$ (say). This can explicitly be computed using, for $l, m = 0, 1, 2, \ldots$,

$$\int_0^\pi \cos\left((2m+1)x\right)(\sin x)^l \, dx = 0,$$

$$\int_0^\pi \cos(2mx)(\sin x)^{2l} \, dx = \begin{cases} \dfrac{(-1)^m}{2^{2l}} \dbinom{2l}{l-m} \pi, & l \geq m, \\ 0, & l < m, \end{cases}$$

$$\int_0^\pi \cos(2mx)(\sin x)^{2l+1} \, dx$$

$$= \begin{cases} \dfrac{(-1)^m}{2^{2l+1}} \dfrac{\Gamma(2l+2)}{\Gamma(3/2+l-m)\Gamma(3/2+l+m)} \pi, & l \geq m-1, \\ 0, & l < m-1, \end{cases}$$

(see Gradshteyn-Ryzhik [13, Section 3.631]; in some editions, the factor π in the second formula is missing).

Since $\boldsymbol{S}_i$ and $\boldsymbol{S}_j$ are uniformly distributed over the unit sphere, one may suppose that Proposition 3.1 remains valid if $\boldsymbol{S}_j$ is replaced by a fixed unit vector. This is indeed the case. To be specific, put $\Theta_i^{\boldsymbol{s}} = \arccos\left(\boldsymbol{s}'\boldsymbol{S}_i\right)$ with $\boldsymbol{s} \in \mathbb{R}^d, \|\boldsymbol{s}\| = 1$. Then, $\Theta_i^{\boldsymbol{s}}$ has the same distribution as Θ_{ij} (see Cai and Jiang [4], p. 31). As a consequence, $\mathrm{E}\left[h\left(\boldsymbol{s}, \boldsymbol{S}_1\right)\right] = \mu$, which shows that V is a degenerate V-statistic. Putting $\Phi(\boldsymbol{s}_1, \boldsymbol{s}_2) = h(\boldsymbol{s}_1, \boldsymbol{s}_2) - \mu$, we obtain

$$\begin{aligned} &\mathrm{E}\left[\Phi(\boldsymbol{S}_1, \boldsymbol{S}_2)\right] = 0, && \mathrm{E}\left[\Phi(\boldsymbol{s}, \boldsymbol{S}_1)\right] = 0, \\ &\mathrm{E}\left[\Phi^2(\boldsymbol{S}_1, \boldsymbol{S}_2)\right] < \infty, && \mathrm{E}\left[|\Phi(\boldsymbol{S}_1, \boldsymbol{S}_1)|\right] < \infty, \\ &\mathrm{E}\left[\Phi(\boldsymbol{S}_1, \boldsymbol{S}_1)\right] = \begin{cases} 2, & \text{if } h = \tilde{h}_\ell, \\ 2/(1-q), & \text{if } h = h_q. \end{cases} \end{aligned}$$

Then, the theory of V-statistics yields that $m(V-\mu)$ converges in distribution to a weighted sum of independent chi-squared random variables. In special cases, the weights can be obtained (see Proposition 1 in Pycke [21] for the circular case). However, we do not proceed in this direction, since, for small and medium sample sizes, it is preferable to use finite sample critical values obtained by simulation.

Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{mN}$ be iid p-dimensional random vectors drawn from $\mathrm{EC}_p(\mu, \Lambda; \psi)$. Partition this sample into m groups with equal size N, denoted by $\{\boldsymbol{X}_i^{(k)}\}_{k=1}^N$ for $k = 1, \ldots, m$. Next, based on (2.9), define m random matrices of size $n \times p$ by

$$U_k = \left[\boldsymbol{U}_1^{(k)}, \ldots, \boldsymbol{U}_p^{(k)}\right] = K' X'_{(k)} (nS_{(k)})^{-1/2}, \; n = N - 1 \geq p,$$

where

$$X_{(k)} = \left[\boldsymbol{X}_1^{(k)}, \ldots, \boldsymbol{X}_N^{(k)}\right], \quad nS_{(k)} = X_{(k)} Q X'_{(k)}, \quad k = 1, 2, \ldots, m.$$

Here, Q has appeared in (1.2), and K is an $N \times n$ constant matrix satisfying (1.3). Taking Corollary 2.3 into account, U_k's are independently and uniformly distributed over $\mathcal{O}(n,p)$. Hence, the p columns $\boldsymbol{U}_r^{(i)}, r = 1, \ldots, p$, of U_i are not independent, but each of $\boldsymbol{U}_r^{(i)}$ is uniformly distributed over the unit hypersphere $\mathbb{S}^{n-1}$, independent of $\boldsymbol{U}_r^{(j)}$ of U_j $(i \neq j)$. Hence we are able to construct a testing procedure based on $\{\boldsymbol{U}_r^{(k)}\}_{k=1}^m$ by making use of (3.2) and (3.3), which leads to the "necessary test procedure" for elliptical symmetry.

Remark 3.2. *The following reasoning explains the phrase "necessary test procedure". As a consequence of Corollary 2.3, when the distribution of the $\boldsymbol{X}_i$'s enjoys elliptical symmetry, the U_k's are independent and uniformly distributed over the Stiefel manifold. Therefore, if uniformity of the U_k's is not satisfied, we reject the hypothesis of elliptical symmetry. On the other hand, even if the U_k's have the uniformity over $\mathcal{O}(n,p)$, this does not imply elliptical symmetry of the $\boldsymbol{X}_i$'s – thus we use the terminology "necessary test procedure".*

§4. Some numerical experiments

In this section, we carry out Monte Carlo simulations to evaluate the type I errors and powers for the proposed tests, together with Rayleigh's test (Jupp [16]).

To evaluate the type I error, we consider the following three p-dimensional elliptical distributions as "null distribution":

(A1) the normal distribution $\mathscr{N}_p(\mathbf{0}, \Lambda)$ with pdf

$$f_{\mathscr{N}}(\boldsymbol{x}|\Lambda) = |2\pi\Lambda|^{-1/2} \exp(-\boldsymbol{x}'(2\Lambda)^{-1}\boldsymbol{x}),$$

(A2) the t-distribution with ν degrees of freedom $\mathscr{T}_p(\nu, \Lambda)$ with pdf

$$f_{\mathscr{T}}(\boldsymbol{x}|\Lambda) = T_p|\nu\pi\Lambda|^{-1/2} \left[1 + \nu^{-1}\boldsymbol{x}'\Lambda^{-1}\boldsymbol{x}\right]^{-(p+\nu)/2},$$

where $T_p = \Gamma\left[(p+\nu)/2\right]/\Gamma\left[\nu/2\right]$; we set $\nu = 3$,

(A3) the Kotz type distribution $\mathscr{K}_p(r, s, k, \Lambda)$ with pdf

$$f_{\mathscr{K}}(\boldsymbol{x}|\Lambda) = K_p|\pi\Lambda|^{-1/2}(\boldsymbol{x}'\Lambda^{-1}\boldsymbol{x})^{k-1} \exp\left(-r(\boldsymbol{x}'\Lambda^{-1}\boldsymbol{x})^s\right),$$

where $K_p = s\Gamma(p/2)/\Gamma((2k+p-2)/2s)r^{(2k+p-2)/2s}$ and $r, s > 0$, $2k+p > 2$ (see Fang et al. [7, Chapter 3]); we set $(r, s, k) = (1/2, 1, 2)$.

For each of these models, we used the scale matrices $\Lambda = \mathrm{diag}(4^2, 3^2, 2^2)$ $\mathrm{diag}(4^2, 3^2, 2^2, 1)$ for $p = 3, 4$, respectively, where $\mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ denotes a diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_p$.

We also examine four non-elliptical distributions to evaluate the power in the same manner as Liang et al. [17]:

(B1) the exponential distribution composed of *iid* univariate exponential distribution with pdf $f(x) = \exp(-x)$,

(B2) the exponential distribution composed of *iid* univariate exponential distribution with pdf $f(x) = (1/k^2)\exp\left(-(1/k^2)x\right)$, $k = 1, 2, \ldots, p$,

(B3) the multivariate chi-squared distribution composed of *iid* univariate χ_1^2, the chi-squared distribution with 1 degree of freedom,

(B4) the skew-normal distribution with pdf

$$f(\boldsymbol{x}|\boldsymbol{\alpha}, \Lambda) = 2f_{\mathscr{N}}(\boldsymbol{x}|\Lambda)\Phi(\boldsymbol{\alpha}'\Lambda^{-1/2}\boldsymbol{x}),$$

where $f_{\mathscr{N}}(\boldsymbol{x}|\Lambda)$ is defined in (A1) and $\Phi(*)$ denotes the standard normal cumulative distribution, with parameters, $\alpha = (2, -3, -1)', \Lambda = \mathrm{diag}(4^2, 3^2, 2^2)$ for $p = 3$, and $\alpha = (2, -3, -2, -5)', \Lambda = \mathrm{diag}(4^2, 3^2, 2^2, 1)$ for $p = 4$ (see, for instance, Genton [12, pp. 15, 16]).

For all distributions above, we choose the dimension $p = 3, 4$, the number of groups m and the sample size of each group N as $(m, N) = (5, 10), (10, 5)$, and set $\ell = 2, 3, 4, q = \sqrt{2/3}$ as used by Pycke [21]. By generating 10^6 samples from (A1), we obtain the critical values for every statistic based on the first column $\boldsymbol{U}_1^{(k)}$ of U_k for nominal levels $\alpha = 0.10, 0.05, 0.01$; they are summarized in Tables 1–4. "Rayleigh" means the modified Rayleigh statistic with error of order m^{-1} (see Jupp [16]) .

We evaluate the type I error rates for (A1)–(A3) and empirical powers for (B1)–(B4) based on Monte Carlo simulations with 10^5 iterations. Results are shown in Tables 5 and 6. From these tables, we observe that the type I error rates for $\tilde{V}_{\ell,d}$ and $V_{q,d}$ are in very good agreement with the nominal rate. The Rayleigh test, which is the score test of uniformity within the matrix von Mises-Fisher family, shows no power at all; this is not surprising in view of the empirical centering in the Studentized residuals which form the basis of the test procedure. Among the other statistics, $\tilde{V}_{3,d}$ shows the highest power, followed by $V_{\sqrt{2/3},d}$; generally, the power is rather low due to the small sample size m.

To conclude this section, we analyze the famous *Iris data*, which is presented by Fisher [11], to assess our test procedure. Results are shown in Tables 7–10. Here we note that in order to avoid the singularity of the sample covariance matrices for m groups of size $N = 5$, we modified the data set by swapping the first data set of Iris Setosa $(5.1, 3.5, 1.4, 0.2)$ with the seventh $(4.6, 3.4, 1.4, 0.3)$. We also give the values of the different statistics calculated using the rth column $\boldsymbol{U}_r^{(k)}$ of U_k, indicated by the index r in the tables, since the values of the statistics depend on the order of the elements of the vectors. In the case $p = 4, m = 10, N = 5$ at significant level 10%, the maximum for the values of the statistics $V_{\sqrt{2/3},4}, \tilde{V}_{2,4}$ indicates deviations to elliptical symmetry. Moreover, the proposed tests based on the 2-variates *petal* and *sepal* widths showed better performance for $m = 10, N = 5$ as using $m = 5, N = 10$.

§5. Conclusion

We have constructed a new test procedure for elliptical symmetry by making use of the uniform distribution over the Stiefel manifold. By simulation, the proposed test shows good performance of the type I error and power, compared to Rayleigh's test. Furthermore, the tests have been applied to the Iris data, raising doubts that this data set comes from an elliptical distribution.

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Table 1: Upper tail percentage points for $p = 3, N = 5$

α	$m \setminus$ Statistic	Rayleigh	$V_{\sqrt{2/3},3}$	$\tilde{V}_{2,3}$	$\tilde{V}_{3,3}$	$\tilde{V}_{4,3}$
0.10	5	18.36	12.58	4.66	4.25	4.27
	10	18.49	12.74	4.77	4.33	4.38
0.05	5	20.66	13.15	5.33	4.87	4.89
	10	20.95	13.21	5.48	5.02	5.05
0.01	5	25.38	14.60	6.84	6.01	6.04
	10	26.10	14.24	7.02	6.36	6.37

Table 2: Upper tail percentage points for $p = 3, N = 10$

α	$m \setminus$ Statistic	Rayleigh	$V_{\sqrt{2/3},3}$	$\tilde{V}_{2,3}$	$\tilde{V}_{3,3}$	$\tilde{V}_{4,3}$
0.10	5	36.52	7.36	−3.33	4.07	6.72
	10	36.68	2.67	−11.05	4.15	10.39
0.05	5	39.75	7.46	−2.99	4.51	7.16
	10	40.02	2.77	−10.72	4.60	10.88
0.01	5	46.24	7.72	−2.23	5.25	7.94
	10	46.72	3.00	−9.99	5.37	11.76

Table 3: Upper tail percentage points for $p = 4, N = 5$

α	$m \setminus$ Statistic	Rayleigh	$V_{\sqrt{2/3},4}$	$\tilde{V}_{2,4}$	$\tilde{V}_{3,4}$	$\tilde{V}_{4,4}$
0.10	5	23.32	12.57	4.65	4.25	4.27
	10	23.49	12.74	4.78	4.33	4.37
0.05	5	25.93	13.15	5.33	4.87	4.89
	10	26.21	13.20	5.48	5.01	5.04
0.01	5	31.27	14.61	6.84	6.02	6.04
	10	31.84	14.24	7.02	6.36	6.37

Table 4: Upper tail percentage points for $p = 4, N = 10$

α	$m \setminus$ Statistic	Rayleigh	$V_{\sqrt{2/3},4}$	$\tilde{V}_{2,4}$	$\tilde{V}_{3,4}$	$\tilde{V}_{4,4}$
0.10	5	46.98	7.36	−3.33	4.07	6.72
	10	47.12	2.67	−11.06	4.15	10.38
0.05	5	50.61	7.47	−2.99	4.51	7.16
	10	50.90	2.77	−10.72	4.60	10.88
0.01	5	57.86	7.72	−2.23	5.26	7.94
	10	58.38	3.00	−9.99	5.38	11.74

Table 5: Monte Carlo type I error rates and powers ($p = 3, N = 5, m = 10$)

α	Statistic	Normal	t	Kotz	Exp. 1	Exp. 2	χ^2	Skew Normal
0.10	Rayleigh	0.100	0.101	0.100	0.100	0.100	0.098	0.100
	$V_{\sqrt{2/3},3}$	0.101	0.101	0.101	0.161	0.130	0.228	0.101
	$\tilde{V}_{2,3}$	0.101	0.100	0.099	0.101	0.102	0.102	0.100
	$\tilde{V}_{3,3}$	0.100	0.100	0.099	0.226	0.159	0.351	0.099
	$\tilde{V}_{4,3}$	0.101	0.101	0.100	0.108	0.106	0.137	0.098
0.05	Rayleigh	0.050	0.050	0.049	0.050	0.049	0.048	0.049
	$V_{\sqrt{2/3},3}$	0.050	0.050	0.050	0.089	0.069	0.138	0.050
	$\tilde{V}_{2,3}$	0.051	0.049	0.050	0.050	0.052	0.050	0.050
	$\tilde{V}_{3,3}$	0.049	0.050	0.049	0.137	0.088	0.241	0.050
	$\tilde{V}_{4,3}$	0.050	0.050	0.050	0.056	0.054	0.074	0.049
0.01	Rayleigh	0.009	0.009	0.010	0.009	0.010	0.009	0.009
	$V_{\sqrt{2/3},3}$	0.010	0.010	0.010	0.023	0.016	0.044	0.010
	$\tilde{V}_{2,3}$	0.010	0.010	0.009	0.010	0.010	0.011	0.010
	$\tilde{V}_{3,3}$	0.010	0.009	0.010	0.042	0.022	0.094	0.009
	$\tilde{V}_{4,3}$	0.009	0.010	0.010	0.011	0.011	0.018	0.009

Table 6: Monte Carlo type I error rates and powers ($p = 4, N = 5, m = 10$)

α	Statistic	Normal	t	Kotz	Exp. 1	Exp. 2	χ^2	Skew Normal
0.10	Rayleigh	0.100	0.099	0.101	0.097	0.100	0.100	0.098
	$V_{\sqrt{2/3},4}$	0.100	0.102	0.101	0.138	0.110	0.173	0.099
	$\tilde{V}_{2,4}$	0.099	0.100	0.099	0.099	0.102	0.101	0.098
	$\tilde{V}_{3,4}$	0.101	0.100	0.100	0.173	0.110	0.238	0.101
	$\tilde{V}_{4,4}$	0.101	0.100	0.101	0.109	0.103	0.131	0.101
0.05	Rayleigh	0.050	0.049	0.050	0.048	0.050	0.049	0.049
	$V_{\sqrt{2/3},4}$	0.050	0.051	0.050	0.075	0.056	0.100	0.049
	$\tilde{V}_{2,4}$	0.049	0.050	0.049	0.050	0.051	0.051	0.049
	$\tilde{V}_{3,4}$	0.050	0.049	0.050	0.100	0.057	0.148	0.052
	$\tilde{V}_{4,4}$	0.049	0.050	0.051	0.056	0.052	0.070	0.052
0.01	Rayleigh	0.009	0.009	0.009	0.010	0.009	0.010	0.009
	$V_{\sqrt{2/3},4}$	0.009	0.010	0.010	0.017	0.012	0.028	0.009
	$\tilde{V}_{2,4}$	0.010	0.009	0.010	0.010	0.009	0.010	0.009
	$\tilde{V}_{3,4}$	0.010	0.009	0.010	0.026	0.012	0.046	0.010
	$\tilde{V}_{4,4}$	0.009	0.010	0.009	0.011	0.010	0.016	0.010

Table 7: Iris setosa data for $p = 4, m = 5, N = 10$

Statistic	Rayleigh	r	$V_{\sqrt{2/3},4}$	$\tilde{V}_{2,4}$	$\tilde{V}_{3,4}$	$\tilde{V}_{4,4}$
$\alpha\backslash$ Values	29.09	1	7.08	−4.11	3.51	4.17
		2	7.00	−3.40	3.61	2.83
		3	7.04	−4.03	3.26	4.17
		4	7.22	−4.14	0.16	4.05
0.10	46.98		7.36	−3.33	4.07	6.72
0.05	50.61		7.47	−2.99	4.51	7.16
0.01	57.86		7.72	−2.23	5.26	7.94

Table 8: Iris setosa data for $p = 4, m = 10, N = 5$

Statistic	Rayleigh	r	$V_{\sqrt{2/3},4}$	$\tilde{V}_{2,4}$	$\tilde{V}_{3,4}$	$\tilde{V}_{4,4}$
$\alpha\backslash$ Values	23.90	1	11.45	1.78	2.26	2.21
		2	10.80	3.15	2.15	1.06
		3	11.87	5.30	2.30	−0.65
		4	12.91	3.34	−4.34	0.11
0.10	23.49		12.74	4.78	4.33	4.37
0.05	26.21		13.20	5.48	5.01	5.04
0.01	31.84		14.24	7.02	6.36	6.37

Table 9: Iris setosa data with petal and sepal widths for $p = 2, m = 5, N = 10$

Statistic	Rayleigh	r	$V_{\sqrt{2/3},2}$	$\tilde{V}_{2,2}$	$\tilde{V}_{3,2}$	$\tilde{V}_{4,2}$
$\alpha\backslash$ Values	17.53	1	7.05	−2.86	3.10	2.79
		2	7.32	−3.97	0.04	4.32
0.10	25.78		7.36	−3.33	4.07	6.72
0.05	28.51		7.47	−2.98	4.51	7.16
0.01	34.05		7.73	−2.23	5.26	7.93

Table 10: Iris setosa data with petal and sepal widths for $p = 2, m = 10, N = 5$

Statistic	Rayleigh	r	$V_{\sqrt{2/3},2}$	$\tilde{V}_{2,2}$	$\tilde{V}_{3,2}$	$\tilde{V}_{4,2}$
$\alpha\backslash$ Values	16.96	1	11.14	3.65	3.53	0.31
		2	13.60	2.68	−3.20	4.48
0.10	13.33		12.75	4.77	4.33	4.37
0.05	15.44		13.21	5.48	5.02	5.05
0.01	19.90		14.25	7.01	6.36	6.36