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Chapter 14

On Consistency of the Omega Ratio with Stochastic Dominance Rules

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Omega ratios have been introduced in [1] as a performance measure to compare the performance of different investment opportunities. It does not have some of the drawbacks of the famous Sharpe ratio. In particular, it is consistent with first order stochastic dominance. Omega ratios also have an interesting relation to expectiles, which found increasing interest recently as risk measures. There is some confusion in the literature about consistency with respect to second order stochastic dominance. In this paper, we clarify this and extend it to a consistency result with respect to stochastic dominance of order $1 + \gamma$ recently introduced in [2] and generalizing the classical concepts of stochastic dominance of first and second order. Several examples illustrate the usefulness of this result. Finally, some consistency results for even more general stochastic dominance rules are shown, including the concept of ϵ-almost stochastic dominance introduced in [3].

Keywords: omega ratio, stochastic dominance, expectiles, integrated distribution function.

1. Introduction

There is an ongoing debate on how to compare the performance of different investment opportunities. Very often one tries to use a performance measure that can be interpreted as a return-risk ratio. The most famous example is the *Sharpe ratio*, introduced in [4] to compare the performance of funds. It works well under the assumption that returns are normally distributed, but it has well known serious drawbacks if one dispenses from that unrealistic assumption. One of the problems with the Sharpe ratio is that it is not consistent with first order stochastic dominance (abbreviated

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as FSD from now on). This implies that an investor maximizing the Sharpe ratio may have a preference for an investment A over an investment B even though its returns are smaller for sure. This is clearly irrational behavior.

As an alternative, [1] introduced the *Omega ratio*, a concept that has been received with great interest. One of its advantages over the Sharpe ratio is its consistency with FSD. There is a bit of confusion in the literature whether or not it is also consistent with second order stochastic dominance (SSD). Wrong claims that this holds for all benchmarks can be found e.g. in [5] and [6]. An accurate statement showing that this depends on the used benchmark was recently published in [7]. There is also some recent interest in portfolio optimization problems using the Omega ratio. [8] introduce a linear programming algorithm to find an optimal portfolio maximizing the Omega ratio. [9] and [10] also discuss portfolio optimization problems using Omega ratio as a performance measure. In [9] it is shown that for some benchmarks this is an ill-posed problem in their setting, as the optimal Omega ratio may be infinite.

The increasing interest in the Omega ratio is also related to the fact that Omega ratios are strongly related to *expectiles*, which recently found a lot of attention as risk measures after it was shown that they are the only risk measures having the property of being coherent and elicitable at the same time, see e.g. [11] and [12].

This inspired us to reconsider the problem of consistency of the Omega ratio with stochastic dominance rules. In this contribution, we clarify the consistency properties with FSD and SSD and show that indeed these results can be unified and generalized by using the concept of fractional stochastic dominance of order $1 + \gamma$ recently introduced in [2]. In Example 2.10 of that paper, it was already observed that comparing a distribution with a degenerate one with respect to stochastic dominance of order $1+\gamma$ holds for all γ larger than the Omega ratio. Therefore it is not surprising that we can show a much more interesting result about consistency of Omega ratios with respect to this kind of stochastic dominance in Theorem 2.3 below.

The rest of the paper is organized as follows. In Section 2, we first introduce the main concepts used in this paper including the definition of Omega ratio and expectiles and the formal definitions of stochastic dominance rules. We then show our main result in Theorem 2.3 and illustrate its usefulness by several examples. In Section 3, these results are extended to more general combined convex and concave stochastic dominance rules, which generalize the well known concept of ϵ-almost stochastic dominance introduced in [3].

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2. Omega ratios and stochastic dominance

Let X be a real valued random variable with a finite mean EX, describing the return of an asset. We denote by $F_X(t) = P(X \leq t)$ its distribution function, and by

$$\phi_X(t) = \int_{-\infty}^{t} F_X(z)\,\mathrm{d}z = E(X-t)_-$$

the integrated distribution function of X, where here and in the following we use the abbreviations $x_+ := \max\{x, 0\}$ and $x_- := \max\{-x, 0\}$ for the positive and negative part of x. Note that $x = x_+ - x_-$.

It should be emphasized that

$$\bar{\delta}_X = \phi_X(EX) = E|X - EX|/2$$

is the absolute semideviation (from the mean). Its use as a risk measure is examined in [13].

[1] introduced the *Omega ratio* with benchmark t as

$$\Omega_X(t) = \frac{E\,(X-t)_+}{E\,(X-t)_-}. \tag{1}$$

The following properties are immediate. The function Ω_X is strictly positive, continuous and strictly decreasing from infinity to zero on its domain and $\Omega_X(EX) = 1$.

From

$$EX - t = E(X-t)_+ - E(X-t)_- \tag{2}$$

we can derive the following representation using the integrated distribution function:

$$\Omega_X(t) = \frac{EX - t + E(X-t)_-}{E(X-t)_-} = 1 + \frac{EX - t}{\phi_X(t)}. \tag{3}$$

Therefore we can also derive the integrated distribution function from the Omega ratio via

$$\phi_X(t) = \frac{EX - t}{\Omega_X(t) - 1}, \quad t \neq EX, \tag{4}$$

which can be continuously extended in $t = EX$. This shows that the Omega ratio determines the distribution. Taking the right derivative in (4)

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and taking into account that $EX = \Omega_X^{-1}(1)$ we get the following explicit expression for the distribution function in terms of the Omega ratios:

$$F_X(t) = \frac{1 - \Omega_X(t) + \Omega_X'^{+}(t) \cdot (t - \Omega_X^{-1}(1))}{(1 - \Omega_X(t))^2}, \quad t \neq \Omega_X^{-1}(1).$$

It is basically equivalent to a corresponding formula already mentioned in [14] as Theorem 1 (iv), where a very similar formula is stated for continuously differentiable distribution functions in the context of expectiles.

Recall that the expectiles $e_X(\alpha)$ of a random variable $X \in L^2$ have been defined by [14] as the minimizers of an asymmetric quadratic loss:

$$e_X(\alpha) = \arg\min_{t \in \mathbb{R}} \{E\ell_\alpha(X - t)\}, \tag{5}$$

where

$$\ell_\alpha(x) = \begin{cases} \alpha x^2 & \text{if } x \geq 0, \\ (1-\alpha)x^2 & \text{if } x < 0, \end{cases}$$

and $\alpha \in (0, 1)$. For $X \in L^1$, Equation (5) has to be modified (see [14]) to

$$e_X(\alpha) = \arg\min_{t \in \mathbb{R}} \{E[\ell_\alpha(X - t) - \ell_\alpha(X)]\}. \tag{6}$$

The minimizer in (5) or (6) is always unique and is identified by the first order condition

$$\alpha E(X - e_X(\alpha))_+ = (1 - \alpha) E(X - e_X(\alpha))_-. \tag{7}$$

From this equation, the one-to-one relation between expectiles and Omega ratios given below immediately follows, see [15]. It holds

$$e_X(\alpha) = \Omega_X^{-1}\left(\frac{1-\alpha}{\alpha}\right), \quad \Omega_X(t) = \frac{1 - e_X^{-1}(t)}{e_X^{-1}(t)}. \tag{8}$$

Next we recall the basic definitions of stochastic dominance. The well known concepts of first order stochastic dominance (FSD) and second order stochastic dominance (SSD) are defined as follows. We say that $X \leq_{FSD} Y$ if $Eu(X) \leq Eu(Y)$ for all increasing utility functions u, i.e. if all rational utility maximizers prefer Y to X. This holds if and only if $F_X(t) \geq F_Y(t)$ for all t. We say that $X \leq_{SSD} Y$ if $Eu(X) \leq Eu(Y)$ for all increasing and concave utility functions u, i.e. if all rational and risk averse utility maximizers prefer Y to X. This holds if and only if $\phi_X(t) \geq \phi_Y(t)$ for all t. In the mathematical literature, FSD is also known under the name *(usual) stochastic order* denoted by $X \leq_{st} Y$, and SSD is known as *increasing concave order* denoted by $X \leq_{icv} Y$, see e.g. [16] or [17]. In this paper,

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we stick to the notation FSD and SSD as usually used in the literature on finance and economics.

The following consistency result holds for the Omega ratio. We will give a simple proof of a more general result below in Theorem 2.3.

Theorem 2.1. *a) If* $X \leq_{FSD} Y$ *then* $\Omega_X(t) \leq \Omega_Y(t)$ *for all* t.
b) If $X \leq_{SSD} Y$ *then* $\Omega_X(t) \leq \Omega_Y(t)$ *for all* $t \leq EY$.

For $t > EY$ the Omega ratio is not consistent with SSD. Indeed, if $EX = EY$ and $X \leq_{SSD} Y$, then it follows immediately from (3) that $\Omega_X(t) \geq \Omega_Y(t)$ with strict inequality if $\phi_X(t) > \phi_Y(t)$.

[2] introduce a concept of generalized stochastic dominance with a real valued parameter $1+\gamma$ interpolating between FSD ($\gamma = 0$) and SSD ($\gamma = 1$). We repeat here the definitions and the main result.

Definition 2.1. For $0 \leq \gamma \leq 1$ let $\mathcal{U}_\gamma$ be the class of continuously differentiable functions u such that

$$0 \leq \gamma u'(y) \leq u'(x) \quad \text{for all } x \leq y. \tag{9}$$

Definition 2.2. For $0 \leq \gamma \leq 1$ we say that Y dominates X by $(1+\gamma)$-SD, denoted $X \leq_{(1+\gamma)-SD} Y$, if $Eu(X) \leq Eu(Y)$ for all functions $u \in \mathcal{U}_\gamma$.

Note that $u \in \mathcal{U}_0$ if and only if u is non-decreasing, and $u \in \mathcal{U}_1$ if and only if u is increasing and concave. Thus, $\gamma = 0$ corresponds to FSD and $\gamma = 1$ corresponds to SSD, with $0 < \gamma < 1$ corresponding to preference relations falling between FSD and SSD. The parameter γ provides a bound on how much marginal utility $u'(x)$ can decrease as x decreases, and its reciprocal $1/\gamma$ gives a bound on how much marginal utility can increase as x increases.

In Theorem 2.4 of [2], the following equivalence is shown, which yields a method to check this kind of stochastic dominance.

Theorem 2.2. *The following conditions are equivalent:*
a) $X \leq_{(1+\gamma)-SD} Y$,
b) $\int_{-\infty}^{t} (F_Y(z) - F_X(z))_+ \, \mathrm{d}z \leq \gamma \int_{-\infty}^{t} (F_X(z) - F_Y(z))_+ \, \mathrm{d}z$ *for all* $t \in \mathbb{R}$.

From this characterization we can deduce a condition for the comparison of a random variable X with a constant random variable c, see Example 2.10 in [2].

Corollary 2.1. *If* $\Omega_X(c) \leq \gamma$, *then we have* $X \leq_{(1+\gamma)-SD} c$ *for the constant random variable* c.

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We can now show the following result generalizing Theorem 2.1.

Theorem 2.3. *If* $X \leq_{(1+\gamma)-SD} Y$ *and* $\Omega_Y(t) \geq \gamma$ *then* $\Omega_X(t) \leq \Omega_Y(t)$.

Proof. Assume $\Omega_Y(t) = \delta \geq \gamma$. Then $Y \leq_{(1+\delta)-SD} t$ follows from Corollary 2.1. Since $\delta \geq \gamma$, we have $\mathcal{U}_\delta \subseteq \mathcal{U}_\gamma$, and thus

$$X \leq_{(1+\delta)-SD} Y \leq_{(1+\delta)-SD} t.$$

By transitivity, $X \leq_{(1+\delta)-SD} t$, which implies $\Omega_X(t) \leq \delta$, i.e. $\Omega_X(t) \leq \Omega_Y(t)$. □

Notice that Theorem 2.1 is just a special case of Theorem 2.3. Part a) of Theorem 2.1 follows by choosing $\gamma = 0$ and part b) by choosing $\gamma = 1$.

We will now give some illustrative examples showing the usefulness of this result. We start with a comparison of a normal and an exponential distribution, where we can compute the Omega ratios as well as the conditions for $(1 + \gamma)$-dominance explicitly.

Example 2.1. Consider a normally distributed random variable X with mean μ and standard deviation σ, denoted from now on as $X \sim N(\mu, \sigma^2)$, and an exponential random variable Y with mean $1/\lambda$, denoted as $Y \sim Exp(\lambda)$. The integrated distribution functions are given by

$$\phi_X(t) = (t - \mu)\Phi_{\mu,\sigma^2}(t) + \varphi_{\mu,\sigma^2}(t), \quad t \in \mathbb{R},$$
$$\phi_Y(t) = t - (1 - \exp(-\lambda t))/\lambda, \quad t > 0,$$

where Φ_{μ,σ^2} and φ_{μ,σ^2} denote the cumulative distribution function (cdf) and density of X, respectively. Using (3), the pertaining Omega ratios can be derived explicitly.

To be more specific, take $X \sim N(3/4, (3/2)^2)$ and $Y \sim Exp(1)$. The cdfs F and G of X and Y have two crossing points $x_1 = 0.633$ and $x_2 = 3.692$ with $F(x) \geq G(x)$ for $x \leq x_1$ and $x \geq x_2$ and $F(x) \leq G(x)$ for $x_1 \leq x \leq x_2$. Setting $x_0 = -\infty$ and $x_3 = \infty$, the areas

$$A_i = \int_{x_{i-1}}^{x_i} (F(x) - G(x))\, \mathrm{d}x, \quad i = 1, 2, 3,$$

are given by $A_1 = 0.378, A_2 = -0.138, A_3 = 0.0108$. By Corollary B.1 in [2], we have $X \leq_{(1+\gamma)-SD} Y$ if and only if

$$\gamma \geq \max\left\{\frac{-A_2}{A_1}, \frac{-A_2}{A_1 + A_3}\right\} = \frac{-A_2}{A_1} = 0.367 = \gamma_{\min}.$$

Now, $\Omega_Y(t) \geq \gamma_{\min}$ if $t \leq t_0 = 1.418$. Therefore, Theorem 2.3 yields $\Omega_X(t) \leq \Omega_Y(t)$ for $t \leq t_0$. Figure 1 shows the distribution functions and

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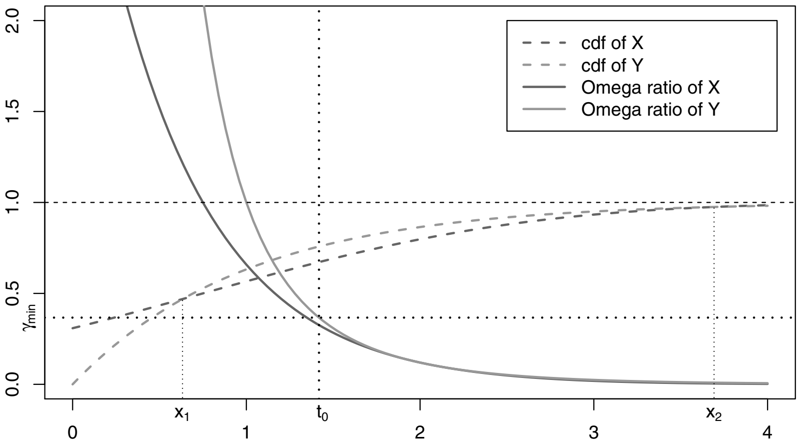

Fig. 1. Cdfs and Omega ratios of $X \sim N\left(3/4, (3/2)^2\right)$ and $Y \sim Exp(1)$.

Omega ratios of X and Y. We will reconsider this example later in Examples 3.1 and 3.2 where we derive further related inequalities.

The example shows that $\Omega_X(t) \le \Omega_Y(t)$ for $t \le t^*$ does not necessarily imply $X \le_{(1+\gamma^*)-SD} Y$, where $\gamma^* = \Omega_Y(t^*)$. The situation is different for the case of distributions from the same location-scale family that we consider in the following. To this end, assume

$$F(x) = H\left(\frac{x-\mu_1}{\sigma_1}\right) \quad \text{and} \quad G(x) = H\left(\frac{x-\mu_2}{\sigma_2}\right), \tag{10}$$

where H is the continuous cdf of a random variable with mean zero and standard deviation one. Then F and G have means μ_1 and μ_2 and standard deviations σ_1 and σ_2. Let $X \sim F$ and $Y \sim G$. A necessary and sufficient condition for $X \le_{SSD} Y$ is $\mu_1 \le \mu_2$ and $\sigma_1 \ge \sigma_2$. The cdfs are single-crossing at

$$x_1 = \frac{\mu_2\sigma_1 - \mu_1\sigma_2}{\sigma_1 - \sigma_2}. \tag{11}$$

If F single-crosses G from above, Corollary 2.5 in [2] shows that $X \le_{(1+\gamma)-SD} Y$ if and only if $\gamma A \ge B$, where

$$A = \int_{-\infty}^{x_1} (F(x) - G(x))\ \mathrm{d}x \quad \text{and} \quad B = \int_{x_1}^{\infty} (G(x) - F(x))\ \mathrm{d}x. \tag{12}$$

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We get the following result.

Theorem 2.4. *Let X and Y be from the same location-scale family as given in (10) with $\mu_1 \le \mu_2$ and $\sigma_1 \ge \sigma_2$. Define $\gamma^* = \Omega_X(x_1)$, where x_1 is the single crossing point of the cdfs of X and Y given in (11). Then, $X \le_{(1+\gamma^*)-SD} Y$. Furthermore, $\Omega_X(t) \le \Omega_Y(t)$ if and only if $t \le x_1$.*

Proof. First, we get

$$\Omega_X(x_1) \;=\; \frac{E(X-x_1)_+}{E(X-x_1)_-} \;=\; \frac{\int_{x_1}^{\infty}(1-H((x-\mu_1)/\sigma_1))\,\mathrm{d}x}{\int_{-\infty}^{x_1} H((x-\mu_1)/\sigma_1)\,\mathrm{d}x}$$

$$= \frac{\sigma_1 \int_{\frac{x_1-\mu_1}{\sigma_1}}^{\infty}(1-H(z))\,\mathrm{d}z}{\sigma_1 \int_{-\infty}^{\frac{x_1-\mu_1}{\sigma_1}} H(z)\,\mathrm{d}z} \;=:\; \frac{\sigma_1 \tilde{B}}{\sigma_1 \tilde{A}}.$$

Similarly,

$$\Omega_Y(x_1) = \frac{\sigma_2 \int_{\frac{x_1-\mu_2}{\sigma_2}}^{\infty}(1-H(z))\,\mathrm{d}z}{\sigma_2 \int_{-\infty}^{\frac{x_1-\mu_2}{\sigma_2}} H(z)\,\mathrm{d}z} \;=\; \frac{\sigma_2 \tilde{B}}{\sigma_2 \tilde{A}},$$

since

$$\frac{x_1-\mu_1}{\sigma_1} = \frac{x_1-\mu_2}{\sigma_2} = \frac{\mu_2-\mu_1}{\sigma_1-\sigma_2}.$$

Putting $\gamma^* = \Omega_X(x_1) = \Omega_Y(x_1)$, we get for the areas A and B defined in (12)

$$\frac{B}{A} = \frac{E(X-x_1)_+ - E(Y-x_1)_+}{E(X-x_1)_- - E(Y-x_1)_-} = \frac{(\sigma_1-\sigma_2)\tilde{B}}{(\sigma_1-\sigma_2)\tilde{A}} = \gamma^*.$$

Hence, $X \le_{(1+\gamma^*)-SD} Y$ by the remark preceding the theorem.

Now, assume $t \le x_1$. Then, $t_1 = (t-\mu_1)/\sigma_1 \ge t_2 = (t-\mu_2)/\sigma_2$ since $F(t) = H(t_1) \ge H(t_2) = G(t)$. Then, the same considerations as above yield $\Omega_X(t) \le \Omega_Y(t)$. The case $t \ge x_1$ follows by analogous reasoning. □

Theorem 2.4 yields simple explicit expressions for the important case of normally distributed random variables that is considered in the following example.

Example 2.2. As an illustration of Theorem 2.4, Figure 2 shows the cdfs and Omega ratios of $X \sim N(1,1)$ and $Y \sim N\left(3/2,(1/4)^2\right)$. Here, $x_1 = 5/3$, and $\Omega_X(5/3) = 0.185$. Hence, according to Theorem 2.4, $X \le_{(1+\gamma)-SD} Y$ with $\gamma = 0.185$. Further, $\Omega_X(t) \le \Omega_Y(t)$ on $(-\infty, x_1]$, and $\Omega_X(t) \ge \Omega_Y(t)$ on $[x_1, \infty)$.

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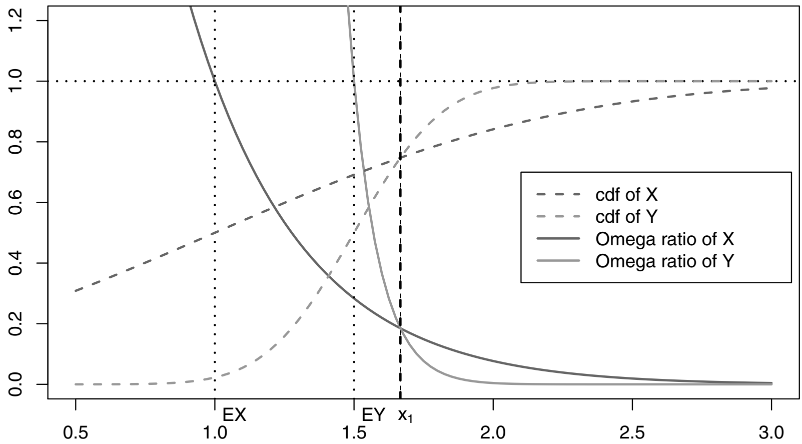

Fig. 2. Cdfs and Omega ratios of $X \sim N(1,1)$ and $Y \sim N\left(3/2, (1/4)^2\right)$.

We next consider an example where using the Sharpe ratio leads to irrational behavior, as it is not consistent with FSD. This example is quite realistic as it is simply obtained by comparing a normally distributed return with a truncated one, which is obtained by selling a call option at a price of zero. We also investigate in this case how the preference of selling such a call option for a fixed price depends on the benchmark that we use for the Omega ratio. The example shows that the chosen benchmark represents the risk aversion of the decision maker in a similar way as does the parameter γ in the generalized stochastic dominance rule.

Example 2.3. Consider an investment with an excess return over the risk free rate X with $X \sim N(\mu, \sigma)$ with $\mu = 2$ and $\sigma = 1$. Assume that we have the opportunity to sell a call option with strike price $K = 3$ for a price $C \geq 0$. If we give the call option away for free ($C = 0$) we get as remaining return Y a normal random variable right-censored at 3. No rational decision maker would do this, as $Y \leq_{FSD} X$. If we consider the Sharpe ratio, however, then it turns out that we should prefer Y to X: expected value and variance of Y are given by (see, e.g. [18], p. 763)

$$EY = \Phi(\alpha)(\mu + \sigma\lambda) + (1 - \Phi(\alpha))\, K,$$
$$V(Y) = \sigma^2 \Phi(\alpha) \left(\left(1 - \left(\lambda^2 - \lambda\alpha\right)\right) + (\alpha - \lambda)^2 \left(1 - \Phi(\alpha)\right) \right)$$

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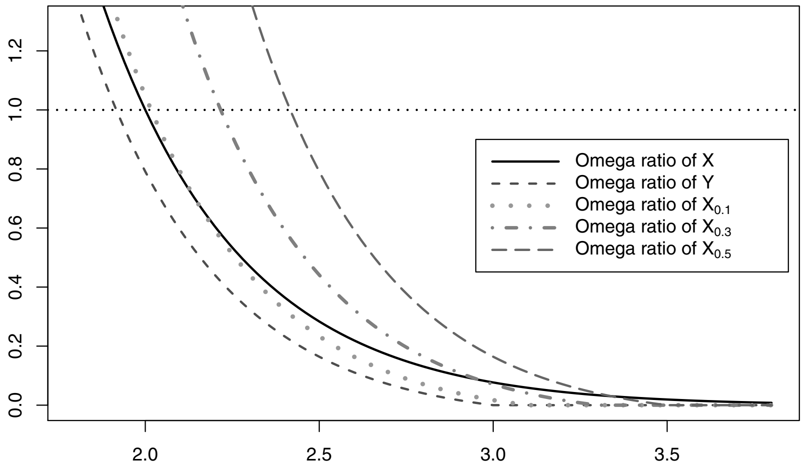

Fig. 3. Omega ratios of X, Y, and X_C for $C = 0.1, 0.3, 0.51$.

where

$$\alpha = \frac{K-\mu}{\sigma}, \quad \lambda = -\frac{\varphi(\alpha)}{\Phi(\alpha)}.$$

This yields $EY = 1.917, \sigma_Y = 0.867$, and $EY/\sigma_Y = 2.212$, whereas $EX/\sigma_X = 2$.

If we sell the call option for a price $C > 0$ then we get a return $X_C = Y + C$. The distribution function of X_C is given by

$$F_{X_C}(t) = F_X(t-C),\ t < K+C, \quad \text{and} \quad F_{X_C}(t) = 1,\ t \geq K+C,$$

where $F_X(t) = \Phi((t-\mu)/\sigma)$. Hence,

$$\phi_{X_C}(t) = \int_{-\infty}^{t} F_{X_C}(x)\mathrm{d}x = \begin{cases} \phi_X(t-C), & t < K+C, \\ \phi_X(K) + t - (K+C), & t \geq K+C. \end{cases}$$

Figure 3 shows the Omega ratios of X, Y, and X_C for $C = 0.1, 0.3, 0.5$.

We further obtain

$\Omega_{X_C}(1) \geq \Omega_X(1)$ if and only if $C \geq 0.029$,
$\Omega_{X_C}(2) \geq \Omega_X(2)$ if and only if $C \geq 0.083$,
$\Omega_{X_C}(3) \geq \Omega_X(3)$ if and only if $C \geq 0.32$,
$\Omega_{X_C}(4) \geq \Omega_X(4)$ if and only if $C \geq 1.03$.

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Note that $EX_C = EY + C$. Hence, $EX_{0.029} = 1.95, EX_{0.083} = 2, EX_{0.32} = 2.23$, and $EX_{1.03} = 2.94$. Thus we see that for a small benchmark we may accept an investment with a smaller mean than EX, and thus the decision is not too different from what we get when using the Sharpe ratio, as has been empirically observed by [19]. But in contrast to the Sharpe ratio our decisions are always consistent with FSD. We are never willing to give away an option for free. If we use a benchmark above the expected return of the investment, however, then even SSD dominance is not sufficient to choose X_C. Then we need a stronger dominance than SSD, meaning that our decisions are not consistent with SSD anymore.

We could also define a new stochastic dominance rule by requiring all Omega ratios for all benchmarks to be larger, or equivalently all expectiles to be larger. This concept is studied in detail in [15], where it is called *expectile ordering*. There it is shown that this is a strictly weaker condition than FSD and is not equivalent to any of the many known stochastic orders.

3. Omega ratios and combined concave and convex stochastic dominance

In [2], the combined concave and convex stochastic dominance of order $(1+\gamma_{cv}, 1+\gamma_{cx})$ is introduced, which generalizes the concept of $(1+\gamma)$-dominance and is defined as follows.

Definition 3.1. a) For $0 \leq \gamma_{cv}, \gamma_{cx} \leq 1$ let $\mathcal{U}_{\gamma_{cv},\gamma_{cx}}$ be the class of continuously differentiable functions u such that

$$0 \leq \gamma_{cv} u'(y) \leq u'(x) \leq \frac{1}{\gamma_{cx}} u'(y) \quad \text{for all } x \leq y.$$

If $\gamma_{cx} = 0$ this shall mean that the last inequality can be omitted.
b) For $0 \leq \gamma_{cv}, \gamma_{cx} \leq 1$ we say that Y dominates X by $(1+\gamma_{cv}, 1+\gamma_{cx})$-SD, denoted

$$X \leq_{(1+\gamma_{cv},1+\gamma_{cx})-SD} Y,$$

if $Eu(X) \leq Eu(Y)$ for all functions $u \in \mathcal{U}_{\gamma_{cv},\gamma_{cx}}$.

Note that $X \leq_{(1+\gamma_{cv},1+\gamma_{cx})-SD} Y$ implies $X \leq_{(1+\tilde{\gamma}_{cv},1+\tilde{\gamma}_{cx})-SD} Y$, where $\gamma_{cv} \leq \tilde{\gamma}_{cv} \leq 1, \gamma_{cx} \leq \tilde{\gamma}_{cx} \leq 1$. The case $\gamma_{cv} = \gamma, \gamma_{cx} = 0$ is concave $(1+\gamma)$-SD from Definition 2.2 and therefore we get a generalization of that concept. In the following, we consider two other interesting special cases:

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- The case $\gamma_{cv} = 0, \gamma_{cx} = \gamma$ is convex $(1+\gamma)$-SD, corresponding to preference relations falling between FSD and convex (risk-seeking) SSD (for the latter, see, e.g. [20], sec 3.11).
- The case $\gamma_{cv} = \gamma_{cx} = \gamma = \varepsilon/(1-\varepsilon)$ is ε-almost first degree stochastic dominance (ε-AFSD, [3]) with $\varepsilon = \gamma/(1+\gamma)$.

Theorem 4.3 in [2] provides an integral condition for $(1+\gamma_{cv}, 1+\gamma_{cx})$-SD. In particular, for $\gamma_{cv} = 0$, $X \leq_{(1,1+\gamma_{cx})-SD} Y$ if and only if

$$\int_t^{\infty} (F_Y(z) - F_X(z))_+ \, \mathrm{d}z \leq \gamma_{cx} \int_t^{\infty} (F_X(z) - F_Y(z))_+ \, \mathrm{d}z \; \forall\, t \in \mathbb{R}, \quad (13)$$

and for $\gamma_{cv} = \gamma_{cx} = \gamma$, $X \leq_{(1+\gamma,1+\gamma)-SD} Y$ if and only if

$$\int_{-\infty}^{\infty} (F_Y(z) - F_X(z))_+ \, \mathrm{d}z \leq \gamma \int_{-\infty}^{\infty} (F_X(z) - F_Y(z))_+ \, \mathrm{d}z. \quad (14)$$

For convex $(1+\gamma)$-SD, we have the following result.

Theorem 3.1. *If* $X \leq_{(1,1+\gamma_{cx})-SD} Y$ *and* $\Omega_X(t) \leq 1/\gamma_{cx}$ *then* $\Omega_X(t) \leq \Omega_Y(t)$.

Proof. First, note that $X \geq_{(1,1+\gamma_{cx})-SD} c$ if and only if $\Omega_X(c) \geq 1/\gamma_{cx}$ (see [2]). Now, assume $\Omega_X(t) = 1/\delta_{cx} \leq 1/\gamma_{cx}$. Then, $X \geq_{(1,1+\delta_{cx})-SD} t$. Since $\delta_{cx} \geq \gamma_{cx}$, we obtain

$$Y \;\geq_{(1,1+\delta_{cx})-SD}\; X \;\geq_{(1,1+\delta_{cx})-SD}\; t.$$

By transitivity, $Y \geq_{(1,1+\delta_{cx})-SD} t$, which in turn implies $\Omega_Y(t) \geq 1/\delta_{cx}$, i.e. $\Omega_Y(t) \geq \Omega_X(t)$. □

If the cdf F of X single-crosses the cdf G of Y from above, condition (13) entails that X dominates Y via convex $(1+\gamma_{cx})$-SD if and only if $B/A \geq 1/\gamma_{cx}$, where A and B are defined in (12). For this, it is necessary that $B - A = EX - EY \geq 0$. For location-scale families, we get the following result, which can be proven similar as Theorem 2.4.

Theorem 3.2. *Let* X *and* Y *be from the same location-scale family as given in (10) with* $\mu_1 \geq \mu_2$ *and* $\sigma_1 \geq \sigma_2$. *Define* $\gamma^* = 1/\Omega_X(x_1)$, *where* x_1 *is the single crossing point of the cdfs of* X *and* Y *given in (11). Then,* $X \geq_{(1,1+\gamma^*)-SD} Y$. *Furthermore,* $\Omega_X(t) \geq \Omega_Y(t)$ *if and only if* $t \geq x_1$.

The subsequent example considers distribution functions having two crossing points.

Example 3.1. In Example 2.1, the cdfs F and G of X and Y have two crossing points x_1 and x_2 with $F(x) \geq G(x)$ for $x \leq x_1$ and $x \geq x_2$ and

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$F(x) \leq G(x)$ for $x_1 \leq x \leq x_2$. Writing again

$$A_i = \int_{x_{i-1}}^{x_i} (F(x) - G(x))\ \mathrm{d}x, \quad i = 1, 2, 3,$$

it is easy to see that $X \leq_{(1,1+\gamma_{cx})-SD} Y$ if and only if

$$\gamma \geq \max\left\{\frac{-A_2}{A_3}, \frac{-A_2}{A_1 + A_3}\right\} = \frac{-A_2}{A_3} = \gamma_{cx}^{\min}.$$

In Example 2.1, we get $\gamma_{cx}^{\min} = 12.82$. Now, $\Omega_X(t) \leq 1/\gamma_{cx}^{\min} = 0.078$ if $t \geq t_1 = 2.99$. Therefore, Theorem 3.1 yields $\Omega_X(t) \leq \Omega_Y(t)$ for $t \geq t_1$.

Finally, we consider almost first degree stochastic dominance. If F single-crosses G from above, Equation (14) shows that $X \leq_{(1+\gamma,1+\gamma)-SD} Y$ if and only if $\gamma A \geq B$, which is the same condition as for concave $(1+\gamma)$-SD. In particular, $X \leq_{(1+\gamma,1+\gamma)-SD} c$ if $\Omega_X(c) \leq \gamma$. Using the same arguments as in the proof of Theorem 2.3 provides the following strengthening of Theorem 2.3.

Theorem 3.3. *If* $X \leq_{(1+\gamma,1+\gamma)-SD} Y$ *and* $\Omega_Y(t) \geq \gamma$ *then* $\Omega_X(t) \leq \Omega_Y(t)$.

Example 3.2. Again, we consider the situation of Example 2.1 with two crossing points. With a view to Equation (14), $X \leq_{(1+\gamma,1+\gamma)-SD} Y$ if and only if

$$\gamma \geq \frac{-A_2}{A_1 + A_3} = \tilde{\gamma}_{\min}.$$

For the distributions used in Example 2.1, $\tilde{\gamma}_{\min} = 0.356$, and $\Omega_Y(t) \geq \tilde{\gamma}_{\min}$ if $t \leq \tilde{t}_0 = 1.431$. Therefore, Theorem 3.3 yields $\Omega_X(t) \leq \Omega_Y(t)$ for $t \leq \tilde{t}_0$. Thus we get a slightly better result in comparison to Example 2.1 as we use a weaker stochastic dominance rule here.

References

1. C. Keating and W. F. Shadwick, A universal performance measure, *J. Perform. Measurement* **6**(3), 59–84 (2002).
2. A. Müller, M. Scarsini, I. Tsetlin and R. L. Winkler, Between first and second-order stochastic dominance, *Management Science* **63**(9), 2933–2947 (2017).
3. M. Leshno and H. Levy, Preferred by "all" and preferred by "most" decision makers: Almost stochastic dominance, *Management Science* **48**(8), 1074–1085 (2002).
4. W. F. Sharpe, Mutual fund performance, *The Journal of Business* **39**, 119–138 (1966).

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5. J.-L. Prigent, *Portfolio Optimization and Performance Analysis* (CRC Press, 2007).
6. W. M. Fong, Stochastic dominance and the omega ratio, *Finance Research Letters* **17**, 7–9 (2016).
7. S. Balder and N. Schweizer, Risk aversion vs. the omega ratio: Consistency results, *Finance Research Letters* **21**, 78–84 (2017).
8. H. Mausser, D. Saunders and L. Seco, Optimizing omega, *Risk*, 88–92 (2006).
9. C. Bernard, S. Vanduffel and J. Ye, Optimal strategies under omega ratio, Working paper (2018). Available at SSRN: https://ssrn.com/abstract=2947057 or http://dx.doi.org/10.2139/ssrn.2947057.
10. M. R. Metel, T. A. Pirvu and J. Wong, Risk management under omega measure, *Risks* **5**(2), 27 (2017). Available at https://doi.org/10.3390/risks5020027.
11. F. Bellini, B. Klar, A. Müller and E. R. Gianin, Generalized quantiles as risk measures, *Insurance: Mathematics and Economics* **54**, 41–48 (2014).
12. J. F. Ziegel, Coherence and elicitability, *Mathematical Finance* **26**, 901–918 (2016).
13. W. Ogryczak and A. Ruszczyński, From stochastic dominance to mean-risk models: Semideviations as risk measures, *European Journal of Operational Research* **116**, 33–50 (1999).
14. W. K. Newey and J. L. Powell, Asymmetric least squares estimation and testing, *Econometrica* **55**, 819–847 (1987).
15. F. Bellini, B. Klar and A. Müller, Expectiles, omega ratios and stochastic ordering, *Methodology and Computing in Applied Probability* , p. in print (2017).
16. A. Müller and D. Stoyan, Comparison methods for stochastic models and risks, *Wiley: New York* (2002).
17. M. Shaked and G. Shanthikumar, *Stochastic orders* (Springer, 2007).
18. W. H. Greene, *Econometric Analysis*, 5 edn. (Prentice Hall, 2003).
19. M. Eling and F. Schuhmacher, Does the choice of performance measure influence the evaluation of hedge funds?, *Journal of Banking & Finance* **31**, 2632–2647 (2007).
20. H. Levy, *Stochastic dominance: Investment decision making under uncertainty* (Springer, 2016).