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Tobias Lamm, Jan Metzger & Felix Schulze

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LOCAL FOLIATION OF MANIFOLDS BY SURFACES OF WILLMORE TYPE

by Tobias LAMM, Jan METZGER & Felix SCHULZE (*)

Abstract. — We show the existence of a local foliation of a three dimensional Riemannian manifold by critical points of the Willmore functional subject to a small area constraint around non-degenerate critical points of the scalar curvature. This adapts a method developed by Rugang Ye to construct foliations by surfaces of constant mean curvature.

Résumé. — Nour prouvons l'existence d'un feuilletage local d'une variété riemanienne de dimension trois autour des points critiques de la courbure scalaire par les points critiques non dégénérés de la fonctionnelle de Willmore sous la contrainte d'aire petite. On adapte une méthode développée par Rugang Ye pour construire un feuilletage par des surfaces à courbure moyenne constante.

1. Introduction

In this paper we consider the Willmore functional

$$\mathcal{F}(\Sigma) = \frac{1}{4}\int_\Sigma H^2 \,\mathrm{d}\mu$$

for surfaces Σ immersed in a 3-dimensional Riemannian manifold (M, g). Here $H = \lambda_1 + \lambda_2$ denotes the sum of the principal curvatures of Σ.

More precisely, we consider the variational problem

$$\inf\{\mathcal{F}(\Sigma) \mid \Sigma \hookrightarrow M \text{ with } |\Sigma| = a\} \tag{1.1}$$

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where $a \in (0,\infty)$ is a (small) prescribed constant and $|\Sigma|$ denotes the area of Σ with respect to the induced metric.

The Euler–Lagrange equation for this variational problem is

$$\Delta H + H|\mathring{A}|^2 + H \operatorname{Ric}(\nu,\nu) = \lambda H. \tag{1.2}$$

Here Δ denotes the Laplace–Beltrami operator on Σ, $\mathring{A}$ is the trace free part of the second fundamental form A of Σ and $\operatorname{Ric}(\nu,\nu)$ is the Ricci-curvature of (M,g) in direction of the normal ν to Σ. Note that the left hand side is two times the first variation of $\mathcal{F}$ and the right hand side of this expression is a Lagrange-parameter $\lambda \in \mathbb{R}$ multiplied with the first variation of the area functional.

In previous papers the first two authors have shown that if (M,g) is compact then there exists a small $a_0 \in (0,\infty)$ depending only on (M,g) such that the infimum in (1.1) is attained for all $a \in (0,a_0)$ on smooth surfaces Σ_a [6]. See [1] and [14] for alternative proofs and [11] for a recent parabolic approach. Existence and multiplicity results of Willmore surfaces in Riemannian manifolds have been studied previously in a perturbative setting in [12], [13] where the functionals $\mathcal{F}$ and the L^2-norm of $\mathring{A}$ are considered without a constraint.

For $a \to 0$ the surfaces Σ_a converge to critical points of the scalar curvature [5, 6, 9]. A similar result has been obtained previously for small isoperimetric surfaces by Druet [2]. This was later generalized by Laurain [8] to surfaces with constant mean curvature.

It is natural to ask about the precise structure of this family when a tends to zero. A similar situation was considered by Ye [15] for surfaces of constant mean curvature, which are critical for the isoperimetric problem, that is to minimize area subject to prescribed enclosed volume. He proves that given a non-degenerate critical point p of the scalar curvature one can find a pointed neighborhood $\dot{U} = U \setminus \{p\}$ which is foliated by hypersurfaces of constant mean curvature. That is $\dot{U} = \bigcup_{H\in(H_0,\infty)} \Sigma_H$ where Σ_H has constant mean curvature H. For $H \to \infty$ these surfaces become spherical and approach geodesic spheres $S_r(p)$ with radius $r \approx \frac{2}{H}$. Ye uses an implicit function argument to show that the Σ_H can be constructed as graphs over $S_r(0)$. The main difficulty is that the operator linearizing the mean curvature has an approximate kernel corresponding to translations. This approximate kernel can be dealt with by allowing a translation of the $S_r(0)$ and using the non-degeneracy of the second derivative of the scalar curvature. Our result in this paper is to adapt the method of Ye to the case of the Willmore functional. More precisely, we get the following result:

THEOREM 1.1. — *Let (M,g) be a smooth Riemannian manifold and let $p \in M$ be such that $\nabla \operatorname{Sc}(p) = 0$ and such that $\nabla^2 \operatorname{Sc}(p)$ is non-degenerate. Then there exists $a_0 \in (0,\infty)$, a neighborhood U of p and for each $a \in (0,a_0)$ a spherical surface Σ_a which satisfies* (1.2) *for some $\lambda \in \mathrm{R}$ and $|\Sigma_a| = a$. The Σ_a are mutually disjoint and $\bigcup_{(0,a_0)} \Sigma_a = U \setminus \{p\}$.*

More detailed information on the structure of this foliation can be found in Section 3 where the implicit function argument is carried out. In particular, we refer to Corollary 5.2 for some comments about the local uniqueness of the Σ_a.

The paper is organized as follows: In Section 2 we calculate the expansion of the Willmore functional on small geodesic spheres to set up the argument. In Section 3 we use the implicit function theorem to solve the equation in a very similar manner to Ye. First we solve the equation in the kernel of the linearized operator using the non-degeneracy condition on the scalar curvature and by a generic implicit function argument we solve perpendicular to the kernel. Proposition 4.4 in Section 4 establishes that the Σ_a indeed form a foliation as claimed. Finally in Section 5 we prove a local uniqueness reslut for the Σ_a as solutions to (1.2).

Remarks. — During the preparation of this manuscript, the authors learned that Norihisa Ikoma, Andrea Malchiodi and Andrea Mondino [4] have an independent proof of Theorem 1.1.

2. The Willmore operator on geodesic spheres

In this section we compute the basic geometric quantities and the Willmore operator of small geodesic spheres. We consider a setup similar as in [15], i.e. we consider a point $p \in M^3$ and an orthonormal basis $\{e_j\}_{j=1}^3$ of T_pM which we use to identify T_pM with $\mathbb{R}^3$. Furthermore, we consider the map

$$\phi : \mathbb{R}^3 \supset B_{\rho_p}(0) \to M : x \mapsto \exp_p(x^i e_i),$$

where $\rho_p > 0$ is the injectivity radius of p. Let $\widetilde{g}$ be the pulled back metric of M via ϕ, with $\langle \cdot,\cdot \rangle$ denoting the euclidean metric on $\mathbb{R}^3$. We consider the map $\Psi_\sigma : \mathbb{R}^3 \to \mathbb{R}^3 : x \mapsto \sigma x$ and denote $g := \sigma^{-2}\Psi_\sigma^* \widetilde{g}$.

We now compute the second fundamental form of $S_\rho(0) \subset T_pM$ for $0 < \rho < \sigma^{-1}\rho_p$. The normal to S_ρ w.r.t. g is given by $x/|x|$, and working w.l.o.g. at the north pole, i.e. e_i for $i = 1,2$ are tangent vectors and e_3 is parallel to the normal, we obtain

$$\begin{aligned} h_{ij} &= g\left(e_i, \nabla_{e_j}\nu\right) = g\left(e_i, \nabla_{e_j}\frac{x^l}{|x|}e_l\right) \\ &= g\left(e_i, \left(\frac{\delta^{jl}}{\rho} - \frac{x^l x^j}{\rho^3}\right)e_l\right) + \frac{x^l}{\rho} g\left(e_i, \Gamma^k_{jl} e_k\right) \\ &= \frac{1}{\rho}\left(g_{ij} - \frac{x^l x^j}{\rho^2} g_{il} + x^l \Gamma^k_{jl} g_{ik}\right). \end{aligned}$$

This yields since $x^l = 0$ for $l = 1, 2$

$$h^i_j = \frac{1}{\rho}\left(\delta^i_j + x^l \Gamma^i_{lj}\right). \tag{2.1}$$

Furthermore, we have

$$0 = g\left(\nu, p^\perp_\nu(e_i)\right) = \frac{x^k}{\rho} g_{ki} - \frac{x^i}{\rho},$$

where $p^\perp_\nu(\,\cdot\,) = e - \langle e_i, \nu\rangle\nu$ is the orthogonal projection onto the subspace perpendicular to ν. This gives

$$\delta_{im} = \frac{\partial}{\partial x^m}\left(x^k g_{ki}\right) = g_{mi} + x^k \frac{\partial}{\partial x^m} g_{ki}, \tag{2.2}$$

which we can use to compute

$$\begin{aligned} x^l \Gamma^i_{lj} &= \frac{1}{2} g^{ik}\left(x^l \frac{\partial}{\partial x^l} g_{jk} + x^l \frac{\partial}{\partial x^j} g_{lk} - x^l \frac{\partial}{\partial x^k} g_{lj}\right) \\ &= \frac{1}{2} g^{ik}\left(x^l \frac{\partial}{\partial x^l} g_{jk} + \delta_{jk} - g_{jk} - \delta_{kj} + g_{kj}\right) \\ &= \frac{1}{2} g^{ik} x^l \frac{\partial}{\partial x^l} g_{jk}. \end{aligned} \tag{2.3}$$

We denote partial derivatives with a semicolon, instead of a comma for covariant derivatives. From [10] and the definition of g we have the formula

$$\begin{aligned} g_{ij}(x) = g_{ij}(0) &+ \frac{\sigma^2}{3}\,\mathrm{R}_{ipqj}\, x^p x^q + \frac{\sigma^3}{6}\,\mathrm{R}_{ipqj,r}\, x^p x^q x^r \\ &+ \sigma^4\left(\frac{1}{20}\,\mathrm{R}_{ipqj,rs} + \frac{2}{45}\,\mathrm{R}_{ipqt}\,\mathrm{R}_{jrst}\right) x^p x^q x^r x^s + O(\sigma^5 |x|^5), \end{aligned} \tag{2.4}$$

where the curvature terms are all corresponding to g and are evaluated at 0. Since

$$\frac{\partial}{\partial x^m} g^{ij} = -g^{iv} g^{jw} \frac{\partial}{\partial x^m} g_{vw}$$

and differentiating further (using that the first derivatives of g_{ij} vanish at 0) we obtain:

$$g^{ij}(x) = g^{ij}(0) - \frac{\sigma^2}{3}\,\mathrm{R}^{i\ \ j}_{\ pq}\ x^p x^q - \frac{\sigma^3}{6}\,\mathrm{R}^{i\ \ j}_{\ pq,r}\, x^p x^q x^r + O(\sigma^4|x|^4). \tag{2.5}$$

Combining (2.4) and (2.5), we see

$$\begin{aligned}\frac{1}{2}g^{ik}x^l\frac{\partial}{\partial x^l}g_{jk} &= \frac{1}{2}g^{ik}\frac{\partial}{\partial s}\left(g_{kj}(sx)\right)\Big|_{s=1}\\ &= g^{ik}\Bigg(\frac{\sigma^2}{3}\,\mathrm{R}_{kpqj}\,x^p x^q + \frac{\sigma^3}{4}\,\mathrm{R}_{kpqj,r}\,x^p x^q x^r\\ &\qquad + \sigma^4\left(\frac{1}{10}\,\mathrm{R}_{kpqj,rs} + \frac{4}{45}\,\mathrm{R}_{kpqt}\,\mathrm{R}_{jrst}\right)x^p x^q x^r x^s\\ &\qquad + O(\sigma^5|x|^5)\Bigg)\\ &= \frac{1}{3}\sigma^2\,\mathrm{R}^i_{\ pqj}\,x^p x^q + \frac{1}{4}\sigma^3\,\mathrm{R}^i_{\ pqj,r}\,x^p x^q x^r\\ &\quad + \sigma^4\left(\frac{1}{10}\,\mathrm{R}^i_{\ pqj,rs} - \frac{1}{45}\,\mathrm{R}^i_{\ pqt}\,\mathrm{R}_{jrst}\right)x^p x^q x^r x^s\\ &\quad + O(\sigma^5|x|^5).\end{aligned} \tag{2.6}$$

Combining this with (2.1) and (2.3), we can thus write

$$\rho\cdot h^i_{\ j} = \delta^i_{\ j} + \frac{1}{2}g^{ik}x^l\frac{\partial}{\partial x^l}g_{ji} = \exp(a^i_{\ j})$$

where exp here is the exponential map on matrices and

$$\begin{aligned}a^i_{\ j} := \frac{1}{3}\sigma^2\,\mathrm{R}^i_{\ pqj}\,x^p x^q + \frac{1}{4}\sigma^3\,\mathrm{R}^i_{\ pqj,r}\,x^p x^q x^r\\ + \sigma^4\left(\frac{1}{10}\,\mathrm{R}^i_{\ pqj,rs} - \frac{7}{90}\,\mathrm{R}^i_{\ pqt}\,\mathrm{R}_{jrst}\right)x^p x^q x^r x^s + O(\sigma^5|x|^5).\end{aligned}$$

This yields

$$\begin{aligned}&\det{}_3\left(\delta^i_j + \frac{1}{2}g^{ik}x^l\frac{\partial}{\partial x^l}g_{jk}\right) = \exp(\mathrm{tr}(a^i_j))\\ &\quad = 1 - \frac{1}{3}\sigma^2\,\mathrm{R}_{pq}\,x^p x^q - \frac{1}{4}\sigma^3\,\mathrm{R}_{pq,r}\,x^p x^q x^r\\ &\qquad + \sigma^4\left(-\frac{1}{10}\,\mathrm{R}_{pq,rs} - \frac{7}{90}\,\mathrm{R}^k_{\ pqt}\,\mathrm{R}_{krst} + \frac{1}{18}\,\mathrm{R}_{pq}\,\mathrm{R}_{rs}\right)x^p x^q x^r x^s + O(\sigma^5|x|^5).\end{aligned}$$

Note that for the Gauss curvature we have from (2.1)

$$K_{S_\rho} = \frac{1}{\rho^2}\det{}_2\left(\delta^i_j + \frac{1}{2}g^{ik}x^l\frac{\partial}{\partial x^l}g_{jk}\right) = \frac{1}{\rho^2}\det{}_3\left(\delta^i_j + \frac{r}{2}g^{ik}\frac{\partial}{\partial r}g_{kj}\right),$$

since $\frac{\partial}{\partial r} g_{k3} = 0 \quad \forall\ k = 1,2,3$. Combining this with the above computation, this yields

$$\begin{aligned} &K_{S_\rho}(x) \\ &= \frac{1}{\rho^2}\Big(1 - \frac{1}{3}\sigma^2 \operatorname{Ric}_{pq} x^p x^q - \frac{1}{4}\sigma^3 \operatorname{Ric}_{pq,r} x^p x^q x^r \\ &\qquad - \sigma^4 \Big(\frac{\operatorname{Ric}_{pq,rs}}{10} + \frac{7}{90} \mathrm{R}^k{}_{pqt}\, \mathrm{R}_{krst} - \frac{\operatorname{Ric}_{pq} \operatorname{Ric}_{rs}}{18}\Big) x^p x^q x^r x^s\Big) \\ &\quad + \rho^{-2} O(\sigma^5 |x|^5). \end{aligned} \tag{2.7}$$

Combining (2.1), (2.3) and (2.6) we get for the mean curvature, using $\operatorname{Ric}_{pq} = -\mathrm{R}^i{}_{pqi}$,

$$\begin{aligned} H_{S_\rho} &= \frac{1}{\rho} \operatorname{tr}_2 \left(\delta^i{}_j + \frac{1}{2} g^{ik} x^l \frac{\partial}{\partial x^l} g_{jk}\right) = \frac{1}{\rho}\left(2 + \operatorname{tr}_3\left(\frac{1}{2} g^{ik} x^l \frac{\partial}{\partial x^l} g_{ji}\right)\right) \\ &= \frac{1}{\rho}\Big(2 - \frac{1}{3}\sigma^2 \operatorname{Ric}_{pq} x^p x^q - \frac{1}{4}\sigma^3 \operatorname{Ric}_{pq,r} x^p x^q x^r \\ &\qquad - \sigma^4 \left(\frac{1}{10} \operatorname{Ric}_{pq,rs} + \frac{1}{45} \mathrm{R}^i{}_{pqt}\, \mathrm{R}_{irst}\right) x^p x^q x^r x^s\Big) \\ &\quad + \rho^{-1} O(\sigma^5 |x|^5). \end{aligned} \tag{2.8}$$

For the norm squared of the traceless second fundamental form, we have $|\mathring{A}|^2 = \frac{1}{2}(H^2 - 4K)$, which yields

$$\begin{aligned} |\mathring{A}|^2 = \rho^{-2}\sigma^4 \left(\frac{1}{9} \mathrm{R}^i{}_{pq}{}^t\, \mathrm{R}_{irst} - \frac{1}{18} \operatorname{Ric}_{pq} \operatorname{Ric}_{rs}\right) x^p x^q x^r x^s \\ + \rho^{-2} O(\sigma^5 |x|^5) \end{aligned} \tag{2.9}$$

We now aim to compute the laplacian of H on S_1. Using formula (3.2) in [3], we have

$$\begin{aligned} \Delta^{S_1} H &= \Delta^g \widetilde{H} - \operatorname{Hess}(\widetilde{H})(\nu,\nu) + g(\nabla \widetilde{H}, \vec{H}) \\ &= \Delta^g \widetilde{H} - \operatorname{Hess}(\widetilde{H})(x,x) - H x^i \frac{\partial \widetilde{H}}{\partial x^i}, \end{aligned} \tag{2.10}$$

where $\vec{H} = -H\nu$ is the mean curvature vector of S_1, and $\widetilde{H}$ is any extension of H. We have

$$\Delta^g \widetilde{H} = g^{ij} \frac{\partial^2 \widetilde{H}}{\partial x^i \partial x^j} - g^{ij} \Gamma^k_{ij} \frac{\partial \widetilde{H}}{\partial x^k}. \tag{2.11}$$

From (2.4) and (2.5) we get

$$
\begin{aligned}
g^{ij}\Gamma^k_{ij} &= \frac{1}{2} g^{ij} g^{kl} \left(\frac{\partial}{\partial x^i} g_{lj} + \frac{\partial}{\partial x^j} g_{il} - \frac{\partial}{\partial x^l} g_{ij} \right) \\
&= g^{kl} g^{ij} \frac{\partial}{\partial x^i} g_{lj} - \frac{1}{2} g^{kl} g^{ij} \frac{\partial}{\partial x^l} g_{ij} \\
&= \left(g_\tau^{kl} + O(\sigma^2)\right) \left(g_\tau^{ij} + O(\sigma^2)\right) \\
&\qquad \times \left(\frac{\sigma^2}{3} \left(\mathrm{R}_{lipj} + \mathrm{R}_{lpij}\right) x^p + O(\sigma^3) \right) \\
&\quad - \frac{1}{2} \left(g_\tau^{kl} + O(\sigma^2)\right) \left(g_\tau^{ij} + O(\sigma^2)\right) \\
&\qquad \times \left(\frac{\sigma^2}{3} \left(\mathrm{R}_{ilpj} + \mathrm{R}_{iplj}\right) x^p + O(\sigma^3) \right) \\
&= \frac{2}{3} \sigma^2 \operatorname{Ric}^k{}_p\, x^p + O(\sigma^3).
\end{aligned} \tag{2.12}
$$

We choose an extension of the mean curvature H on S_1 via

$$\widetilde{H} := \rho H_{S_\rho}.$$

Combining this with (2.8) and (2.12), this yields

$$- g^{ij}\Gamma^k_{ij} \frac{\partial \widetilde{H}}{\partial x^k} = \frac{4}{9} \sigma^4 \operatorname{Ric}_p{}^k \operatorname{Ric}_{kq} x^p x^q + O(\sigma^5). \tag{2.13}$$

Similarly, using (2.5) and (2.8) we obtain

$$
\begin{aligned}
g^{ij} \frac{\partial^2 \widetilde{H}}{\partial x^i \partial x^j} &= \left(g^{ij} - \frac{\sigma^2}{3} \mathrm{R}^i_{pq}{}^j\, x^p x^q + O(\sigma^3) \right) \\
&\quad \cdot \Bigg(- \frac{2}{3} \sigma^2 \operatorname{Ric}_{ij} - \frac{\sigma^3}{2} \left(\operatorname{Ric}_{ij,p} + 2 \operatorname{Ric}_{ip,j} \right) x^p \\
&\qquad - \sigma^4 \bigg(\frac{1}{5} \operatorname{Ric}_{ij,pq} + \frac{2}{5} \operatorname{Ric}_{ip,jq} + \frac{2}{5} \operatorname{Ric}_{ip,qj} + \frac{1}{5} \operatorname{Ric}_{pq,ij} \\
&\qquad\quad + \frac{4}{45} \mathrm{R}^s{}_{ij}{}^t \mathrm{R}_{spqt} + \frac{8}{45} \mathrm{R}^s{}_{ip}{}^t \mathrm{R}_{sjqt} \bigg) x^p x^q + O(\sigma^5) \Bigg) \\
&= - \frac{2}{3} \sigma^2 \operatorname{Sc} - \sigma^3 \operatorname{Sc}_{,p} x^p \\
&\quad - \sigma^4 \bigg(\frac{3}{5} \operatorname{Sc}_{,pq} + \frac{1}{5} (\Delta \operatorname{Ric})_{pq} + \frac{2}{5} \operatorname{Ric}_p{}^s \operatorname{Ric}_{sq} \\
&\qquad + \frac{4}{45} \operatorname{Ric}^{st} \mathrm{R}_{spqt} + \frac{8}{45} \mathrm{R}^{si}{}_p{}^t \mathrm{R}_{siqt} \bigg) x^p x^q + O(\sigma^5),
\end{aligned} \tag{2.14}
$$

where we used that, due to the sign convention on the curvature tensor ($\mathrm{Ric}_{ij} = -\mathrm{R}^t{}_{ijt}$) and the second contracted Bianchi identity $2\,\mathrm{Ric}^t{}_{i,t} = \mathrm{Sc}_{,i}$, we have

$$\mathrm{Ric}_{ip,qj} = \mathrm{Ric}_{ip,jq} + \mathrm{R}_{qj}{}^s{}_i\,\mathrm{Ric}_{sp} + R_{qj}{}^s{}_p\,\mathrm{Ric}_{si}$$

and thus

$$g^{ij}\,\mathrm{Ric}_{ip,qj} = \frac{1}{2}\,\mathrm{Sc}_{,pq} + \mathrm{Ric}_q{}^s\,\mathrm{Ric}_{sp} + \mathrm{R}_q{}^{is}{}_p\,\mathrm{Ric}_{si}\,.$$

This yields, using (2.11) with (2.13) and (2.14) that

$$\begin{aligned}\Delta^g \widetilde{H} = &-\frac{2}{3}\sigma^2\,\mathrm{Sc} - \sigma^3\,\mathrm{Sc}_{,p}\,x^p \\ &- \sigma^4\Big(\frac{3}{5}\,\mathrm{Sc}_{,pq} + \frac{1}{5}(\Delta\,\mathrm{Ric})_{pq} - \frac{2}{45}\,\mathrm{Ric}_p{}^s\,\mathrm{Ric}_{sq} \\ &\qquad + \frac{4}{45}\,\mathrm{Ric}^{st}\,\mathrm{R}_{spqt} + \frac{8}{45}\,\mathrm{R}^{si}{}_p{}^t\,\mathrm{R}_{siqt}\Big)x^p x^q + O(\sigma^5).\end{aligned} \tag{2.15}$$

Furthermore, using (2.8) we see

$$\begin{aligned}-\frac{\partial \widetilde{H}}{\partial x^i \partial x^j}x^i x^j = &\frac{2}{3}\sigma^2\,\mathrm{Ric}_{pq}\,x^p x^q + \frac{3}{2}\sigma^3\,\mathrm{Ric}_{pq,r}\,x^p x^q x^r \\ &+ \sigma^4\Big(\frac{6}{5}\,\mathrm{Ric}_{pq,rs} + \frac{12}{45}\,\mathrm{R}^i{}_{pqt}\,\mathrm{R}_{irst}\Big)x^p x^q x^r x^s + O(\sigma^5)\end{aligned} \tag{2.16}$$

and combining (2.3) with (2.6) and (2.8)

$$\begin{aligned}x^i x^j \Gamma^k_{ij}\frac{\partial \widetilde{H}}{\partial x^k} &= \frac{1}{2}g^{kl}x^s x^j\,\frac{\partial}{\partial x^s}g_{jl}\,\frac{\partial \widetilde{H}}{\partial x^k} \\ &= -\frac{2}{9}\sigma^4\,\mathrm{R}^k{}_{pqr}\,\mathrm{Ric}_{ks}\,x^p x^q x^s x^r + O(\sigma^5) = O(\sigma^5),\end{aligned} \tag{2.17}$$

since $\mathrm{R}^k{}_{pqr}\,\mathrm{Ric}_{ks}\,x^p x^q x^r x^s = 0$ by symmetry considerations. Combining (2.16) and (2.17), this yields

$$\begin{aligned}-\mathrm{Hess}(\widetilde{H})(x,x) &= -\frac{\partial \widetilde{H}}{\partial x^i \partial x^j}x^i x^j + x^i x^j \Gamma^k_{ij}\frac{\partial \widetilde{H}}{\partial x^k} \\ &= \frac{2}{3}\sigma^2\,\mathrm{Ric}_{pq}\,x^p x^q + \frac{3}{2}\sigma^3\,\mathrm{Ric}_{pq,r}\,x^p x^q x^r \\ &\quad + \sigma^4\left(\frac{6}{5}\,\mathrm{Ric}_{pq,rs} + \frac{12}{45}\,\mathrm{R}^i{}_{pqt}\,\mathrm{R}_{irst}\right)x^p x^q x^r x^s \\ &\quad + O(\sigma^5).\end{aligned} \tag{2.18}$$

By (2.8) we have

$$
\begin{aligned}
(2.19)\ -Hx^i\frac{\partial\widetilde{H}}{\partial x^i} &= \frac{4}{3}\sigma^2\,\mathrm{Ric}_{pq}\,x^px^q + \frac{3}{2}\sigma^3\,\mathrm{Ric}_{pq,r}\,x^px^qx^r \\
&+\sigma^4\left(\frac{4}{5}\,\mathrm{Ric}_{pq,rs} + \frac{8}{45}\,\mathrm{R}^i{}_{pq}{}^t\,\mathrm{R}_{irst} - \frac{2}{9}\,\mathrm{Ric}_{pq}\,\mathrm{Ric}_{rs}\right)x^px^qx^rx^s + O(\sigma^5)
\end{aligned}
$$

Combining (2.10) with (2.15), (2.18) and (2.19) we arrive at

$$
\begin{aligned}
\Delta^{S_1}H &= \Delta^g\widetilde{H} - \mathrm{Hess}(\widetilde{H})(x,x) - Hx^i\frac{\partial\widetilde{H}}{\partial x^i} \\
&= -\frac{2}{3}\sigma^2\,\mathrm{Sc} + 2\sigma^2\,\mathrm{Ric}_{pq}\,x^px^q \\
&\quad - \sigma^3(\mathrm{Sc}_{,p}\,x^p - 3\,\mathrm{Ric}_{pq,r}\,x^px^qx^r) \\
&\quad - \sigma^4\left(\frac{3}{5}\,\mathrm{Sc}_{,pq} + \frac{1}{5}\Delta\,\mathrm{Ric}_{pq} - \frac{2}{45}\,\mathrm{Ric}_p{}^k\,\mathrm{Ric}_{kq}\right. \\
&\qquad\qquad \left. + \frac{4}{45}\,\mathrm{Ric}^{kl}\,\mathrm{R}_{kpql} + \frac{8}{45}\,\mathrm{R}^{kl}{}_p{}^m\,\mathrm{R}_{klqm}\right)x^px^q \\
&\quad + \sigma^4\left(2\,\mathrm{Ric}_{pq,rs} + \frac{4}{9}\,\mathrm{R}^k{}_{pq}{}^l\,\mathrm{R}_{krsl} - \frac{2}{9}\,\mathrm{Ric}_{pq}\,\mathrm{Ric}_{rs}\right)x^px^qx^rx^s \\
&\quad + O(\sigma^5)
\end{aligned}
\tag{2.20}
$$

We now aim to compute the area constrained Willmore equation on S_1, that is for $\lambda\in\mathbb{R}$ the quantity

$$
\mathcal{W}_{\sigma,\lambda} := \Delta^{S_1}H + H|\mathring{A}|^2 + H\,\mathrm{Ric}(\nu,\nu) + \sigma^2\lambda H. \tag{2.21}
$$

To deal with the Ricci term we do a Taylor expansion in normal coordinates on the original manifold around p. We get for the Ricci curvature of $\widetilde{g}$ that

$$
\mathrm{Ric}_{pq}(x) = \mathrm{Ric}_{pq}(0) + \mathrm{Ric}_{pq;r}(0)x^r + \frac{1}{2}\,\mathrm{Ric}_{pq;rs}(0)x^rx^s + O(|x|^3)\ .
$$

Rescaling as before via the map Ψ_σ, we obtain for the Ricci curvature of g

$$
\mathrm{Ric}_{pq}(x) = \sigma^2\,\mathrm{Ric}_{pq} + \sigma^3\,\mathrm{Ric}_{pq;r}\,x^r + \frac{\sigma^4}{2}\,\mathrm{Ric}_{pq;rs}\,x^rx^s + O(\sigma^5|x|^3). \tag{2.22}
$$

Recall that we denote partial derivatives with a semicolon, instead of a comma for covariant derivatives. Since the Christoffel symbols and derivatives thereof are of order at least σ^2 we see that we have on S^1:

$$
\mathrm{Ric}(\nu,\nu) = \sigma^2x^px^q\left(\mathrm{Ric}_{pq} + \sigma\,\mathrm{Ric}_{pq,r}\,x^r + \frac{\sigma^2}{2}\,\mathrm{Ric}_{pq,rs}\,x^rx^s\right) + O(\sigma^5)
$$

and thus, combining this with (2.8)

$$H\,\mathrm{Ric}(\nu,\nu) = 2\sigma^2\,\mathrm{Ric}_{pq}\,x^px^q + 2\sigma^3\,\mathrm{Ric}_{pq,r}\,x^px^qx^r + \sigma^4\left(\mathrm{Ric}_{pq,rs} - \frac{1}{3}\mathrm{Ric}_{pq}\mathrm{Ric}_{rs}\right)x^px^qx^rx^s + O(\sigma^5). \tag{2.23}$$

Combining this with (2.20), (2.8) and (2.9) we arrive at the following proposition, where we replace the radius ρ by r.

Proposition 2.1. — *Let $p\in M^3$ and $\{e_j\}_{j=1}^3$ be an orthonormal basis of T_pM, via which T_pM can be identified with $\mathbb{R}^3$. Furthermore, we consider the map*

$$\phi:\mathbb{R}^3\supset B_{\rho_p}(0)\to M : x\mapsto \exp_p(x^ie_i),$$

where $\rho_p>0$ is the injectivity radius of p. Let $\widetilde{g}$ be the pulled back metric of M via ϕ, and consider the map $\Psi_r:\mathbb{R}^3\to\mathbb{R}^3 : x\mapsto rx$ and the rescaled metric $g:=r^{-2}\Psi_r^*\widetilde{g}$. Then for $0<r<\rho_p$ one has the following expansion of the area constrained Willmore equation (2.21) on S_1:

$$\begin{aligned}\mathcal{W}_{r,\lambda}(S_1) = {}& 2r^2\left(\lambda - \frac{\mathrm{Sc}}{3} + 2\,\mathrm{Ric}_{pq}\,x^px^q\right) - r^3\left(\mathrm{Sc}_{,p}\,x^p + 5\,\mathrm{Ric}_{pq,s}\,x^px^qx^s\right)\\ &- r^4\bigg(\frac{\lambda}{3}\mathrm{Ric}_{pq} + \frac{3}{5}\mathrm{Sc}_{,pq} + \frac{1}{5}\Delta\,\mathrm{Ric}_{pq} - \frac{2}{45}\mathrm{Ric}_p{}^k\,\mathrm{Ric}_{kq}\\ &\qquad + \frac{4}{45}\mathrm{Ric}^{kl}\,\mathrm{R}_{kpql} + \frac{8}{45}\mathrm{R}^{kl}{}_p{}^m\,\mathrm{R}_{klqm}\bigg)x^px^q\\ &+ r^4\left(3\,\mathrm{Ric}_{pq,st} + \frac{2}{3}\mathrm{R}^k{}_{pq}{}^l\,\mathrm{R}_{kstl} - \frac{2}{3}\mathrm{Ric}_{pq}\mathrm{Ric}_{st}\right)x^px^qx^sx^t\\ &+ O(r^5).\end{aligned}$$

3. The equation

In this section we prove Theorem 1.1 via the implicit function theorem. We consider a setup similar to Ye [15]. Let (M,g) be given with injectivity radius $\rho>0$. Fix a base point $p\in M$ and an orthonormal frame $\{e_j\}_{j=1}^3$ for $T_p(M)$. Consider the map:

$$c:\mathbb{R}^3\supset B_\rho(0)\to M : \tau\mapsto \exp_p(\tau),$$

where $\exp_p : T_pM\to M$ denotes the exponential map of M at p. Let e_j^τ be the parallel transports of the e_j to $c(\tau)$ along the geodesic $t\mapsto c(t\tau)|_{t\in[0,1]}$. Define the map

$$F_\tau:\mathbb{R}^3\supset B_\rho(0)\to M : x\mapsto \exp_{c(\tau)}(x^ie_i^\tau).$$

Let $\Omega_1 := \{\varphi \in C^{4,\frac{1}{2}}(S_1) \mid \|\varphi\|_{C^{4,\frac{1}{2}}(S_1)} < 1\}$ and for $\varphi \in \Omega_1$ let $S_\varphi := \{(1+\varphi(x))x \mid x \in S_1\}$. For $\tau \in B_\rho \subset \mathbb{R}^3$ and $r \in (0,\rho/2)$ let $S(r,\tau,\varphi) = F_\tau(\Psi_r(S_\varphi))$ where Ψ_r denotes scaling by r as in Section 2. Define

$$\widetilde{\Phi} : (0,\rho/2) \times B_\rho(0) \times \Omega_1 \times \mathbb{R} \to C^{\frac{1}{2}}(S_1) : (r,\tau,\varphi,\lambda) \mapsto \widetilde{\Phi}(r,\tau,\varphi,\lambda)$$

where $\widetilde{\Phi}(r,\tau,\varphi,\lambda)$ is the function

$$\Delta H + H|\mathring{A}|^2 + H\operatorname{Ric}(\nu,\nu) + \lambda H \tag{3.1}$$

evaluated on $S(r,\tau,\varphi)$ with respect to the metric g and pulled back to S_1 via the parameterization $x \mapsto F_\tau(\Psi_r((1+\varphi(x))x))$.

Our goal is to find $r_0 \in (0,\rho/2)$ and a map

$$(0,r_0) \to B_\rho \times \Omega_1 \times \mathrm{R} : r \mapsto (\widetilde{\tau}(r), \widetilde{\varphi}(r), \widetilde{\lambda}(r))$$

so that

$$\widetilde{\Phi}(r, \widetilde{\tau}(r), \widetilde{\varphi}(r), \widetilde{\lambda}(r)) = 0.$$

Then for all $r \in (0,r_0)$ the surfaces $\Sigma_r := S(r,\widetilde{\tau}(r),\widetilde{\varphi}(r))$ solve the equation

$$\Delta H + H|\mathring{A}|^2 + H\operatorname{Ric}(\nu,\nu) + \widetilde{\lambda}(r)H = 0$$

as claimed. Up to reparameterization, the family $(\Sigma_r)_{r\in(0,r_0)}$ is the family of solutions as in Theorem 1.1, see Corollary 4.3 for details.

An equivalent way to define $\widetilde{\Phi}(r,\tau,\varphi,\lambda)$ is to evaluate the operator (3.1) on S_φ with respect to the metric $\widetilde{g}^{r,\tau} := (\phi_\tau \circ \Psi_r)^* g$. To get a uniform scale in r, we consider instead the rescaled metric $g^{r,\tau} := r^{-2}\widetilde{g}^{r,\tau}$ and define the rescaled function $\Phi(r,\tau,\varphi,\lambda)$ to be the operator

$$\Delta_{r,\tau} H_{r,\tau} + H_{r,\tau}|\mathring{A}_{r,\tau}|^2 + H_{r,\tau}\operatorname{Ric}_{r,\tau}(\nu,\nu) + r^2\lambda H_{r,\tau} \tag{3.2}$$

evaluated on S_φ with respect to $g^{r,\tau}$ and pulled back to S_1 via the parameterization $x \mapsto (1+\varphi(x))x$ of S_φ. From the scaling of the geometric quantities, we get

$$\Phi(r,\tau,\varphi,\lambda) = r^3\widetilde{\Phi}(r,\tau,\varphi,\lambda).$$

By definition

$$\Phi(r,\tau,0,\lambda) = \mathcal{W}_{r,\lambda}(S_1), \tag{3.3}$$

where $\mathcal{W}_{r,\lambda}(S_1)$ is from Proposition 2.1 and the geometric quantities in the expression for $\mathcal{W}_{r,\lambda}(S_1)$ are evaluated at $c(\tau)$. Note that after shifting by τ, the metric g in Proposition 2.1 corresponds to the metric $g^{r,\tau}$ here.

The linearization of the Willmore operator $\widetilde{\Phi}$ is denoted by W_λ. It was calculated in [7, Section 3]. For a variation of an arbitrary surface Σ with normal speed f it is given by

$$W_\lambda f = LLf + \frac{1}{2}\nabla^\star(H^2\nabla f) - 2\nabla^\star(H\mathring{A}(\nabla f,\cdot)) + \lambda Lf + fQ, \tag{3.4}$$

where $\nabla^\star = -\mathrm{div}$, $L = -\Delta - |A|^2 - \mathrm{Ric}(\nu,\nu)$, and

$$\begin{aligned} Q &= |\nabla H|^2 + 2\omega(\nabla H) + H\Delta H + 2\langle \nabla^2 H, \mathring{A}\rangle + 2H^2|\mathring{A}|^2 \\ &\quad + 2H\langle \mathring{A}, T\rangle - H\nabla\,\mathrm{Ric}(\nu,\nu,\nu) - \frac{1}{2}H^2|A|^2 + \frac{1}{2}H^2\,\mathrm{Ric}(\nu,\nu). \end{aligned} \tag{3.5}$$

Here $\omega = \mathrm{Ric}(\nu,\cdot\,)^T$ is the tangential projection of the 1-form $\mathrm{Ric}(\nu,\cdot\,)$ to Σ and $T = R(\,\cdot\,,\nu,\nu,\cdot\,)$. All the geometric quantities in W_λ are evaluated on Σ with respect to the corresponding ambient geometry. For given $f \in C^4(S_1)$ the family $t \mapsto S(r,\tau,tf)$ is a normal variation of $S(r,\tau,0)$ with normal speed rf, so that

$$\widetilde{\Phi}_\varphi(r,\tau,0,\lambda)f = rW_\lambda f. \tag{3.6}$$

Here we evaluate W_λ with respect to the metric g in M. Rescaling to the $g^{r,\tau}$ metric, we find that

$$\Phi_\varphi(r,\tau,0,\lambda)f = r^4 W_\lambda f = W_{r,\tau,\lambda}f, \tag{3.7}$$

where $W_{r,\tau,\lambda}$ is the linearized Willmore operator with respect to $g^{r,\tau}$:

$$\begin{aligned} W_{r,\tau,\lambda}f &= L_{r,\tau}L_{r,\tau}f + \frac{1}{2}\nabla^\star_{r,\tau}(H^2_{r,\tau}\nabla_{r,\tau}f) \\ &\quad - 2\nabla^\star_{r,\tau}(H_{r,\tau}\mathring{A}_{r,\tau}(\nabla_{r,\tau}f,\cdot\,) + r^2\lambda L_{r,\tau}f + Q_{r,\tau}f. \end{aligned} \tag{3.8}$$

Here we use the subscript ${}_{r,\tau}$ to denote quantities evaluated with respect to the metric $g^{r,\tau}$.

In the limit $r \to 0$ the metric $g^{r,\tau}$ converges to the Euclidean metric so that in the limit we have

$$W_{0,\tau,\lambda}f = L_0(L_0+2)f = (-\Delta)(-\Delta-2)f.$$

The kernel of this operator is given by

$$K := \ker W_{0,\tau,\lambda} = \mathrm{Span}\{1, x^1, x^2, x^3\},$$

where the x^i are the standard coordinate functions on S_1. We split this kernel into two parts:

$$K_0 := \mathrm{Span}\{1\} \quad \text{and} \quad K_1 := \mathrm{Span}\{x_1, x_2, x_3\}. \tag{3.9}$$

As in [15], the function space $C^{4,\frac{1}{2}}(S_1)$ splits as a direct sum into K and its L^2-orthogonal complement $K^\perp$. It is standard to verify that we have the direct sum decomposition of the target with respect to the L^2-scalar product:

$$C^{0,\frac{1}{2}} = K + W_{0,\tau,\lambda}(K^\perp).$$

Define the L^2-orthogonal projection maps

$$P_0 : C^{0,\frac{1}{2}}(S_1) \to K_0 \quad \text{and} \quad P_1 : C^{0,\frac{1}{2}}(S_1) \to K_1.$$

The maps $T_0 : K_0 \to \mathbb{R}$ and $T_1 : K_1 \to \mathbb{R}^3$ identify K_0 and K_1 with $\mathbb{R}$ and $\mathbb{R}^3$ according to the basis given in equation (3.9). Moreover, for $i \in \{0,1\}$ let $\widetilde{P}_i = T_i \circ P_i$. Denote by $\{e_1, e_2, e_3\}$ the standard basis of $\mathbb{R}^3$.

Lemma 3.1. — *We have*

$$\widetilde{P}_0(\Phi(r,\tau,\varphi,\lambda)) = 8\pi r^2 \left(\lambda + \frac{1}{3}\operatorname{Sc}(c(\tau))\right) + O(r^4) + \widetilde{P}_0\left(\int_0^1 \Phi_\varphi(r,\tau,t\varphi,\lambda)\varphi \,\mathrm{d}t\right)$$

and

$$\widetilde{P}_1(\Phi(r,\tau,\varphi,\lambda)) = \frac{4\pi}{3} r^3 \nabla_{e_i}\operatorname{Sc}(c(\tau)) e_i + O(r^5) + \widetilde{P}_1\left(\int_0^1 \Phi_\varphi(r,\tau,t\varphi,\lambda)\varphi \,\mathrm{d}t\right).$$

Proof. — Start by writing

$$\Phi(r,\tau,\varphi,\lambda) = \Phi(r,\tau,0,\lambda) + \int_0^1 \Phi_\varphi(r,\tau,t\varphi,\lambda)\varphi\,\mathrm{d}t.$$

By (3.3), for $i \in \{0,1\}$

$$\widetilde{P}_i(\Phi(r,\tau,0,\lambda)) = \widetilde{P}_i(\mathcal{W}_{r,\lambda}(S_1)),$$

Where $\mathcal{W}_{r,\lambda}(S_1)$ is evaluated at the base point $c(\tau)$. The right hand side can be calculated term by term from the expansion of $\mathcal{W}_{r,\lambda}(S_1)$ given in Proposition 2.1:

$$\widetilde{P}_0(\mathcal{W}_{r,\lambda}(S_1)) = 8\pi r^2\left(\lambda + \frac{1}{3}\operatorname{Sc}(c(\tau))\right) + O(r^4) \quad \text{and}$$
$$\widetilde{P}_1(\mathcal{W}_{r,\lambda}(S_1)) = \frac{4\pi}{3} r^3 \nabla_{e_i}\operatorname{Sc}(c(\tau)) e_i + O(r^5).$$

Note that all terms that contain an odd number of x^i-factors integrate to zero. For the other terms we used that $\int_{S_1} x^i x^p = \frac{4\pi}{3}\delta_{ip}$ and a similar expression for integrals involving four factors of components of x. □

Lemma 3.2. — *For every $\tau \in \mathbb{R}^3$ and every $\lambda \in \mathbb{R}$ we have that*

$$\Phi_{\varphi r}(0,\tau,0,\lambda) = \left.\frac{\partial}{\partial r}\right|_{r=0} W_{r,\tau,\lambda} = 0.$$

Proof. — For the proof, we have to calculate $\left.\frac{\partial}{\partial r}\right|_{r=0} W_{r,\tau,\lambda}$ from its expression (3.8) taking into account its definition (3.4) and (3.5). Since we compute $\frac{\partial}{\partial r} W_{r,\tau,\lambda}$ at $r = 0$ we see that all terms that are product of at least two quantities that vanish at $(r,\varphi)=(0,0)$ do not contribute to the derivative. In particular

$$\frac{\partial}{\partial r}\Big|_{r=0}\Big(|\nabla H_{r,\tau}|^2+2\omega_{r,\tau}(\nabla H_{r,\tau})+2\langle\nabla^2 H_{r,\tau},\mathring{A}_{r,\tau}\rangle \\ +2H_{r,\tau}^2|\mathring{A}_{r,\tau}|^2+2H_{r,\tau}\langle\mathring{A}_{r,\tau},T_{r,\tau}\rangle+r^2\lambda L_{r,\tau}\Big)=0. \tag{3.10}$$

From the proof of [15, Lemma 1.3], we quote equation (1.15) $\frac{\partial}{\partial r}\big|_{r=0}g^{r,\tau}=0$, its consequence $\frac{\partial}{\partial r}\big|_{r=0}\Delta_{r,\tau}=0$, equation (1.17) $\frac{\partial}{\partial r}\big|_{r=0}A_{r,\tau}=0$, $\frac{\partial}{\partial r}\big|_{r=0}\operatorname{Ric}_{r,\tau}=0$, and assertion (1), that is $\frac{\partial}{\partial r}\big|_{r=0}L_{r,\tau}=0$. These identities also imply that $\frac{\partial}{\partial r}\big|_{r=0}H_{r,\tau}=0$ and $\frac{\partial}{\partial r}\big|_{r=0}\mathring{A}_{r,\tau}=0$. From these formulas we find that

$$\frac{\partial}{\partial r}\Big|_{r=0}\Big(L_{r,\tau}L_{r,\tau}+2H_{r,\tau}\Delta_{r,\tau}H_{r,\tau}-\frac{1}{2}H_{r,\tau}^2|A_{r,\tau}|^2\Big)=0.$$

Since $\operatorname{Ric}_{r,\tau}=O(r^2)$ as in (2.22) and $\nabla_{r,\tau}\operatorname{Ric}_{r,\tau}(\nu_{r,\tau},\nu_{r,\tau},\nu_{r,\tau})=O(r^3)$ by a similar argument, also

$$\frac{\partial}{\partial r}\Big|_{r=0}\Big(-H_{r,\tau}\nabla_{r,\tau}\operatorname{Ric}_{r,\tau}(\nu_{r,\tau},\nu_{r,\tau},\nu_{r,\tau})+\frac{1}{2}H_{r,\tau}^2\operatorname{Ric}_{r,\tau}(\nu,\nu)\Big)=0.$$

To treat the final remaining terms in $W_{r,\tau,\lambda}$ rewrite it to

$$\begin{aligned}&\frac{1}{2}\nabla^*_{r,\tau}(H^2_{r,\tau}\nabla_{r,\tau}f)-2\nabla^*_{r,\tau}(H_{r,\tau}\mathring{A}_{r,\tau}(\nabla_{r,\tau}f,\cdot))\\ &=-H_{r,\tau}\langle\nabla H_{r,\tau},\nabla_{r,\tau}f\rangle-\frac{1}{2}H^2_{r,\tau}\Delta_{r,\tau}f+2\mathring{A}_{r,\tau}(\nabla H_{r,\tau},\nabla_{r,\tau}f)\\ &\quad+2H_{r,\tau}\langle\nabla^*_{r,\tau}\mathring{A}_{r,\tau},\nabla_{r,\tau}f\rangle+2H_{r,\tau}\langle\mathring{A}_{r,\tau},\nabla^2_{r,\tau}f\rangle\\ &=-\frac{1}{2}H^2_{r,\tau}\Delta_{r,\tau}f+2\mathring{A}_{r,\tau}(\nabla H_{r,\tau},\nabla_{r,\tau}f)+2H_{r,\tau}\langle\mathring{A}_{r,\tau},\nabla^2_{r,\tau}f\rangle\\ &\quad+2H_{r,\tau}\omega_{r,\tau}(\nabla_{r,\tau}f).\end{aligned} \tag{3.11}$$

In the last equality we used the Codazzi equation in the form $-\nabla^*\mathring{A}=\frac{1}{2}\nabla H+\omega$. By inspection we see that each term in this expression has vanishing derivative in r-direction. Hence

$$\frac{\partial}{\partial r}\Big|_{r=0}W_{r,\tau,\lambda}f=0$$

as claimed. □

Let φ_0 be the unique solution of the PDE

$$W_{0,\tau,\lambda}\varphi_0 = \left(-\frac{4}{3}\operatorname{Sc} + 4\operatorname{Ric}_{pq} x^p x^q\right)\Big|_{r=0}. \tag{3.12}$$

Note that the right hand side of this equation is an element of $K^\perp$ and hence $\varphi_0 \in K^\perp$ is indeed uniquely defined.

Lemma 3.3. — *Let (M,g) be a three dimensional manifold and $p \in M$ such that $\nabla \operatorname{Sc}(p) = 0$ and $\nabla^2 \operatorname{Sc}(p)$ is non-degenerate. For this base point there exists $r_0 \in (0,\infty)$, an open neighborhood $U \subset C^{4,\frac{1}{2}}(S_1)$ of φ_0, and functions*

$$\lambda : [0,r_0) \times U \to \mathbb{R} : (r,\varphi) \mapsto \lambda(r,\varphi)$$

and

$$\tau : [0,r_0) \times U \to \mathbb{R}^3 : (r,\varphi) \mapsto \tau(r,\varphi)$$

so that

$$\widetilde{P}_i\left(\Phi(r,\tau(r,\varphi),r^2\varphi,\lambda(r,\varphi))\right) = 0 \quad \textit{for} \quad i \in \{0,1\}, \tag{3.13}$$

$$\tau(0,\varphi_0) = 0, \qquad \textit{and} \qquad \lambda(0,\varphi_0) = -\frac{1}{3}\operatorname{Sc}(p). \tag{3.14}$$

Proof. — Calculate:

$$\begin{aligned}\Phi_\varphi(r,\tau,tr^2\varphi,\lambda) = \Phi_\varphi(0,\tau,0,\lambda) &+ r\int_0^1 \Phi_{\varphi r}(sr,\tau,str^2\varphi,\lambda)\,\mathrm{d}s \\ &+ tr^2\int_0^1 \Phi_{\varphi\varphi}(sr,\tau,str^2\varphi,\lambda)\varphi\,\mathrm{d}s.\end{aligned}$$

Moreover, we have

$$\begin{aligned}r\int_0^1 \Phi_{\varphi r}(sr,\tau,str^2\varphi,\lambda)\,\mathrm{d}s = r^2\int_0^1\int_0^1 & s\Phi_{\varphi rr}(usr,\tau,ustr^2\varphi,\lambda)\,\mathrm{d}u\,\mathrm{d}s \\ &+ r^3\int_0^1\int_0^1 st\Phi_{\varphi\varphi r}(usr,\tau,ustr^2\varphi,\lambda)\varphi\,\mathrm{d}u\,\mathrm{d}s \\ &+ r\Phi_{\varphi r}(0,\tau,0,\lambda).\end{aligned}$$

Hence

$$\begin{aligned}r^2\Phi_\varphi(r,\tau,tr^2\varphi,\lambda) = r^2\Phi_\varphi(0,\tau,0,\lambda) &+ r^3\Phi_{\varphi r}(0,\tau,0,\lambda) \\ &+ r^4\int_0^1 t\Phi_{\varphi\varphi}(sr,\tau,str^2\varphi,\lambda)\varphi\,\mathrm{d}s \\ &+ r^4\int_0^1\int_0^1 s\Phi_{\varphi rr}(usr,\tau,ustr^2\varphi,\lambda)\,\mathrm{d}u\,\mathrm{d}s + O(r^5).\end{aligned}$$

By equation (3.7) we have $\Phi_\varphi(0,\tau,0,\lambda) = W_{0,\tau,\lambda}$ and from Lemma 3.2 we get $\Phi_{\varphi r}(0,\tau,0,\lambda) = 0$. Therefore

$$\widetilde{P}_i\left(r^2\Phi_\varphi(r,\tau,tr^2\varphi,\lambda)\right) = O(r^4) \quad \text{for} \quad i \in \{0,1\}.$$

It follows from Lemma 3.1 that the system (3.13) is equivalent to

$$8\pi\left(\lambda + \frac{\mathrm{Sc}}{3}\right) = -r^{-2}\widetilde{P}_0\left(\int_0^1 \Phi_\varphi(r,\tau,tr^2\varphi,\lambda)\varphi\,\mathrm{d}t\right) + O(r^2) = O(r^2)$$

and

$$\begin{aligned}\frac{4\pi}{3}\nabla_{e_i}\mathrm{Sc}\,e_i &= r\widetilde{P}_1\left(\int_0^1\int_0^1 t\Phi_{\varphi\varphi}(sr,\tau,str^2\varphi,\lambda)\varphi\varphi\,\mathrm{d}s\,\mathrm{d}t\right)\\ &\quad + r\widetilde{P}_1\left(\int_0^1\int_0^1\int_0^1 s\Phi_{\varphi rr}(usr,\tau,ustr^2\varphi,\lambda)\varphi\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t\right)\\ &\quad + O(r^2)\\ &= O(r).\end{aligned}$$

By assumption $\nabla\,\mathrm{Sc}(p) = 0$. Hence, at $r = 0$, this system is satisfied for an arbitrary $\varphi_0 \in K^\perp$, if $\lambda|_{r=0} = -\frac{1}{3}\,\mathrm{Sc}(p)$ and $\tau|_{r=0} = 0$.

The derivative with respect to λ and τ at $r = 0$ of the left hand side of this system is given by the matrix

$$\begin{pmatrix} 8\pi & \frac{8\pi}{3}\nabla\,\mathrm{Sc}\,|_{r=0} \\ 0 & \frac{4\pi}{3}\nabla^2\,\mathrm{Sc}\,|_{r=0}\end{pmatrix} = \begin{pmatrix} 8\pi & 0 \\ 0 & \frac{4\pi}{3}\nabla^2\,\mathrm{Sc}\,|_{r=0}\end{pmatrix}.$$

By assumption $\nabla^2\,\mathrm{Sc}\,|_{r=0}$ is non-degenerate. Hence, it follows from the implicit function theorem that there exist functions $\lambda = \lambda(r,\varphi)$ and $\tau = \tau(r,\varphi)$ as claimed at least for (r,φ) in a neighborhood of $(0,\varphi_0) \in \mathbb{R} \times C^{\frac{1}{2}}(S_1)$. □

Lemma 3.4. — *Assume that (M,g), p, ϕ_0, r_0, U, λ and τ are as in Lemma 3.3. Then there exists $r_1 \in (0,r_0]$ and a function*

$$\varphi : [0,r_1) \to U : r \mapsto \varphi(r)$$

such that

$$\Phi(r,\tau(r,\varphi(r)),r^2\varphi(r),\lambda(r,\varphi(r))) = 0 \quad \textit{and} \quad \varphi(0) = \varphi_0.$$

In particular, for small enough r, we have constructed a surface of Willmore type with Lagrange multiplier $\lambda(r,\varphi(r))$.

Proof. — Consider the expansion

$$\begin{aligned}\Phi(r,\tau,r^2\varphi,\lambda) &= \Phi(r,\tau,0,\lambda) + r^2\int_0^1 \Phi_\varphi(r,\tau,tr^2\varphi,\lambda)\,\mathrm{d}t\\ &= r^2\left(2\lambda - \frac{2}{3}\operatorname{Sc} + 4\operatorname{Ric}_{pq} x^p x^q\right) + O(r^3) + r^2\Phi_\varphi(0,\tau,0,\lambda)\varphi\\ &\quad + r^4\int_0^1\int_0^1 t\Phi_{\varphi\varphi}(sr,\tau,str^2\varphi,\lambda)\varphi\varphi\,\mathrm{d}t\,\mathrm{d}s\\ &\quad + r^4\int_0^1\int_0^1\int_0^1 s\Phi_{\varphi rr}(usr,\tau,ustr^2\varphi,\lambda)\varphi\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t\\ &\quad + r^5\int_0^1\int_0^1\int_0^1 st\Phi_{\varphi\varphi r}(usr,\tau,ustr^2\varphi,\lambda)\varphi\varphi\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t,\end{aligned}$$

where we used the fact that $\Phi_{\varphi r}(0,\tau,0,\lambda) = 0$ from Lemma 3.2.

Since $\Phi_\varphi(0,\tau,0,\lambda) = W_{0,\tau,\lambda}$ as in equation (3.7), $\lambda(0,\varphi_0) = -\frac{1}{3}\operatorname{Sc}(p)$ and

$$W_{0,\tau,\lambda}\varphi_0 = \left(-\frac{4}{3}\operatorname{Sc} + 4\operatorname{Ric}_{pq} x^p x^q\right)\Big|_{r=0},$$

we conclude with the help of the implicit function theorem that, after dividing the above equation by r^2, there exists $r_1 \in (0, r_0]$ and solution $\varphi : [0, r_1) \to U$ as claimed. □

4. The foliation

In this section we show that the surfaces Σ_r indeed are a foliation of a pointed neighborhood of $p \in M$. The method used is very close to the arguments in [15, pp. 390–391]. We start with the following observation.

LEMMA 4.1. — *The operator $\Phi_{\varphi rr}(0,\tau,0,\lambda)$ maps even functions to even functions.*

Proof. — Note that $\Phi_{\varphi rr}(0,\tau,0,\lambda) = \frac{\partial^2}{\partial r^2}\big|_{r=0} W_{r,\tau,\lambda}$ where $W_{r,\tau,\lambda}$ is given by the expression in equation (3.8). To prove the claim we check this expression term by term as in the proof of Lemma 3.2.

We start by quoting from [15, Lemma 1.3] that $\frac{\partial^2}{\partial r^2}\big|_{r=0} L_{r,\tau}$ is an even operator. Hence, the claim follows from the facts that $\frac{\partial^2}{\partial r^2}\big|_{r=0} Q_{r,\tau}$ is an even function and in conjunction with equation (3.11) from the fact that the operator

$$\begin{aligned}f \mapsto -\frac{1}{2}H^2_{r,\tau}\Delta_{r,\tau} f + 2\mathring{A}_{r,\tau}(\nabla H_{r,\tau}, \nabla_{r,\tau} f)\\ + 2H_{r,\tau}\langle \mathring{A}_{r,\tau}, \nabla^2_{r,\tau} f\rangle + 2H_{r,\tau}\omega_{r,\tau}(\nabla_{r,\tau} f)\end{aligned} \tag{4.1}$$

maps even functions to even functions.

To show this, we quote from the proof of [15, Lemma 1.3] that the operator $\frac{\partial^2}{\partial r^2}\big|_{r=0}\Delta_{r,\tau}|_{r=0}$ is even and $\frac{\partial^2}{\partial r^2}\big|_{r=0}\operatorname{Ric}_{r,\tau}(\nu_{r,\tau},\nu_{r,\tau})$ resp. $\frac{\partial^2}{\partial r^2}\big|_{r=0}|A_{r,\tau}|^2$ are even functions. Note that [15, Equation (1.17)] implies that $\frac{\partial^2}{\partial r^2}\big|_{r=0}A_{r,\tau}$ is even, so that also $\frac{\partial^2}{\partial r^2}\big|_{r=0}H_{r,\tau}$ and $\frac{\partial^2}{\partial r^2}\big|_{r=0}\mathring{A}_{r,\tau}$ are even.

Using these, it is easy to check that

$$\begin{aligned}(4.2)\qquad \frac{\partial^2}{\partial r^2}\bigg|_{r=0}\Big(&H_{r,\tau}\Delta_{r,\tau}H_{r,\tau}+2\langle\nabla^2_{r,\tau}H_{r,\tau},\mathring{A}_{r,\tau}\rangle+2H_{r,\tau}\langle\mathring{A}_{r,\tau},T_{r,\tau}\rangle\\&-H_{r,\tau}\nabla_{r,\tau}\operatorname{Ric}_{r,\tau}(\nu_{r,\tau},\nu_{r,\tau},\nu_{r,\tau})+H^2_{r,\tau}|\mathring{A}_{r,\tau}|^2\\&+\frac{1}{2}H^2_{r,\tau}\operatorname{Ric}_{r,\tau}(\nu_{r,\tau},\nu_{r,\tau})\Big)\end{aligned}$$

is even. For example consider (we omit the subscript $_{r,\tau}$ for clarity in the notation:

$$\begin{aligned}\frac{\partial^2}{\partial r^2}\bigg|_{r=0}H\Delta H&=\left(\frac{\partial^2}{\partial r^2}\bigg|_{r=0}H\right)\Delta H_{0,\tau}+\left(\left(\frac{\partial}{\partial r}H\right)\left(\frac{\partial}{\partial r}\Delta H\right)\right)|_{r=0}\\&\quad+H_{0,\tau}\frac{\partial^2}{\partial r^2}\bigg|_{r=0}(\Delta H)\\&=H_{0,\tau}\left(2\left(\frac{\partial}{\partial r}\bigg|_{r=0}\Delta\right)\left(\frac{\partial}{\partial r}\bigg|_{r=0}H\right)+\left(\frac{\partial^2}{\partial r^2}\bigg|_{r=0}\Delta\right)H\right)\\&\quad+H_{0,\tau}\Delta_{0,\tau}\frac{\partial^2}{\partial r^2}\bigg|_{r=0}H\\&=H_{0,\tau}\left(\frac{\partial^2}{\partial r^2}\bigg|_{r=0}\Delta_{r,\tau}\right)H_{0,\tau}+H_{0,\tau}\Delta_{0,\tau}\left(\frac{\partial^2}{\partial r^2}\bigg|_{r=0}H_{r,\tau}\right).\end{aligned}$$

In the second and third equality we used from the proof of Lemma 3.2 that $\frac{\partial}{\partial r}\big|_{r=0}H_{r,\tau}=0$ and the fact that $H_{0,\tau}$ is constant. The right hand side is even, since $H_{0,\tau}$ is constant and thus even, since $\frac{\partial^2}{\partial r^2}\big|_{r=0}\Delta_{r,\tau}$ and $\Delta_{0,\tau}$ map even functions to even functions and since the product of even functions is even. The calculation for the other terms in (4.2) is similar. To treat the term $H_{r,\tau}\nabla_{r,\tau}\operatorname{Ric}_{r,\tau}(\nu_{r,\tau},\nu_{r,\tau},\nu_{r,\tau})$ use that $\nabla_{r,\tau}\operatorname{Ric}_{r,\tau}(\nu_{r,\tau},\nu_{r,\tau},\nu_{r,\tau})=O(r^3)$.

For the remaining terms in $Q_{r,\tau}$ note that the $\frac{\partial}{\partial r}\big|_{r=0}(\nabla_{r,\tau})$ is a first order differential operator that vanishes on constant functions. Hence,

$$\begin{aligned}\frac{1}{2}\frac{\partial^2}{\partial r^2}\bigg|_{r=0}|\nabla_{r,\tau}H_{r,\tau}|^2&=\left\langle\frac{\partial^2}{\partial r^2}\bigg|_{r=0}(\nabla_{r,\tau}H_{r,\tau}),\nabla_{0,\tau}H_{0,\tau}\right\rangle\\&\quad+\left|\left(\frac{\partial}{\partial r}\bigg|_{r=0}\nabla_{r,\tau}\right)H_{0,\tau}+\nabla_{0,\tau}\left(\frac{\partial}{\partial r}\bigg|_{r=0}H_{r,\tau}\right)\right|^2=0.\end{aligned}$$

This follows, since $H_{0,\tau}$ is constant, $\frac{\partial}{\partial r}\big|_{r=0}(\nabla_{r,\tau}) = H_{0,\tau}$ and $\nabla_{0,\tau}H_{0,\tau} = 0$ and since $\frac{\partial}{\partial r}\big|_{r=0}H_{r,\tau} = 0$ as in Lemma 3.2. A similar computation yields that $\frac{\partial^2}{\partial r^2}\big|_{r=0}\omega_{r,\tau}(\nabla_{r,\tau}H) = 0$. We established that $\frac{\partial^2}{\partial r^2}\big|_{r=0}Q_{r,\tau}$ is an even function.

It remains to consider the expression in (4.1). Note that the first term is even by reasoning as above. The second term satisfies

$$\frac{\partial^2}{\partial r^2}\bigg|_{r=0}\Big(\mathring{A}_{r,\tau}(\nabla H_{r,\tau}, \nabla_{r,\tau} f)\Big) = 0.$$

To treat the third term, use $\mathring{A}_{0,\tau} = 0$ and $\frac{\partial}{\partial r}\big|_{r=0}\mathring{A}_{0,\tau} = 0$ to compute

$$\frac{\partial^2}{\partial r^2}\bigg|_{r=0} H_{r,\tau}\langle \mathring{A}_{r,\tau}, \nabla^2_{r,\tau} f\rangle = H_{0,\tau}\left\langle \frac{\partial^2}{\partial r^2}\bigg|_{r=0}\mathring{A}_{r,\tau}, \nabla^2_{0,\tau} f\right\rangle.$$

Note that $\frac{\partial^2}{\partial r^2}\big|_{r=0}\mathring{A}_{r,\tau}$ is even and $\nabla^2_{0,\tau}$ maps even functions to even functions so that this operator also has the desired property.

For the last term from (4.1) we compute using $\omega_{0,\tau} = 0$, $\frac{\partial}{\partial r}\big|_{r=0}\omega_{r,\tau} = 0$ and $\frac{\partial}{\partial r}\big|_{r=0}H_{r,\tau} = 0$ that:

$$\frac{\partial^2}{\partial r^2}\bigg|_{r=0} H_{r,\tau}\omega_{r,\tau}(\nabla_{r,\tau} f) = H_{0,\tau}\left(\frac{\partial^2}{\partial r^2}\bigg|_{r=0}\omega_{r,\tau}\right)\nabla_{0,\tau} f.$$

Note that $\nabla_{0,\tau}$ is the tangential gradient on S^2 and maps even functions to odd vector fields. Furthermore, by equation (2.22) and the fact that $\nu_{r,\tau} = x + O(r^2)$ we have that

$$\omega_{r,\tau} = r^2 \operatorname{Ric}_{c(\tau)} x + O(r^3)$$

so that

$$\frac{\partial^2}{\partial r^2}\bigg|_{r=0}\omega_{r,\tau} = \operatorname{Ric}_{c(\tau)} x$$

and hence $\frac{\partial^2}{\partial r^2}\big|_{r=0}\omega_{r,\tau}$ is an odd one form. Consequently the function $H_{0,\tau}\big(\frac{\partial^2}{\partial r^2}\big|_{r=0}\omega_{r,\tau}\big)\nabla_{0,\tau} f$ is even whenever f is even. This concludes the proof. $\square$

Lemma 4.2. — *For $r \in (0, r_1)$ let $\Sigma_r := S(r, \tau(r, \varphi(r)), \varphi(r))$ be as in Lemma 3.4. Then $\tau(r) = O(r^2)$ as $r \to 0$.*

Proof. — It follows from the implicit function theorem that $\tau'(r) = O(r)$ if and only if

$$\widetilde{P}_1(\Phi_{\varphi\varphi}(0,\tau,0,\lambda)\varphi_0\varphi_0) = 0 \quad \text{and} \quad \widetilde{P}_1(\Phi_{\varphi rr}(0,\tau,0,\lambda)\varphi_0) = 0. \tag{4.3}$$

To establish the first identity, note that by the fact that equation (3.12) has unique solutions and since $W_{0,\tau,\lambda}$ is invariant under the reflection at the origin, it follows that φ_0 is an even function.

Furthermore, for every t in a neighborhood of 0 the euclidean Willmore operator $\Phi(0,\tau,t\varphi_0,\lambda)$ evaluates to an even function. Hence also

$$\Phi_{\varphi\varphi}(0,\tau,0,\lambda)\varphi_0\varphi_0 = \frac{\partial^2}{\partial t^2}\bigg|_{t=0} \Phi(0,\tau,t\varphi_0,\lambda)$$

is even. Since $\widetilde{P}_1$ vanishes on even functions, the first claim from (4.3) follows.

To prove the second identity, we note that by Lemma 4.1 the operator $\Phi_{\varphi rr}(0,\tau,0,\lambda)$ maps even functions to even functions and the claim follows in a similar manner. □

This lemma implies in particular that we can reparameterize the solutions that we found in Section 3 by their area.

Corollary 4.3. — *For $r \in (0,r_1)$ let $\Sigma_r := S(r,\tau(r,\varphi(r)),r^2\varphi(r))$ be as in Lemma 3.4. Consider the area of Σ_r in (M,g) as a function of r:*

$$a : (0,r_1) \to (0,\infty) : r \mapsto \int_{\Sigma_r} 1\,\mathrm{d}\mu_g.$$

Then there exists $r_2 \in (0,r_1]$ so that a is strictly increasing on $(0,r_2)$. In particular:

$$a(r) = 4\pi r^2 + O(r^4) \quad \textit{and} \quad a'(r) = 8\pi r + O(r^3).$$

Proof. — Note that a extends as a smooth function to $r = 0$ so that $a(0) = 0$ and hence the first claim follows from the second. We first note that

$$a'(r) = -\int_{\Sigma_r} g(\vec{H},X)\,\mathrm{d}\mu_g,$$

where X is the variation vector-field along this family. Note that X is not unique, whereas $X^\perp$ is well defined. Recall that from Lemma 3.4 we have that Σ_r is an exponential normal graph over $S_r(\tau(r))$ with height function $r^3\varphi(r)$ such that $\varphi(r) \to \varphi_0$ as $r \to 0$. Furthermore, by Lemma 4.2 we have that $\tau(r) = O(r^2)$ as $r \to 0$. This implies that

$$X^\perp\big|_{\Sigma_r} = \frac{\partial}{\partial r_\tau} + O(r^2)$$

where $r_\tau = d_g(\tau(r),\cdot\,)$. Furthermore, by the above and (2.8) we have that

$$H_{\Sigma_r} = H_{S_r(\tau(r))} + O(r^2) = \frac{2}{r} + O(r),$$

as well as

$$\nu_{\Sigma_r} = \nu_{S_r(\tau(r))} + O(r^3).$$

Also note that from (2.4) we have

$$\int_{S_r(\tau(r))} 1\,\mathrm{d}\mu_g = 4\pi r^2 + O(r^4).$$

This implies

$$a'(r) = -\int_{\Sigma_r} g(\vec{H}, X)\,\mathrm{d}\mu_g = 8\pi r + O(r^3). \qquad \square$$

Due to Corollary 4.3 there exists $a_0 \in (0,\infty)$ and a map $\widetilde{r} : (0, a_0) \to (0, r_2)$ such that $|\Sigma_{\tilde{r}(a)}| = a$. We slightly abuse notation by letting

$$\Sigma_a := \Sigma_{\tilde{r}(a)} \quad \text{for} \quad a \in (0, a_0) \tag{4.4}$$

This finishes the existence part of the proof of Theorem 1.1. To complete the proof, it remains to show the following:

Proposition 4.4. — *For $r \in (0, r_1)$ let $\Sigma_r := S(r, \tau(r, \varphi(r)), \varphi(r))$ be the surfaces from Lemma 3.4. Then there exist $r_2 \in (0, r_1]$ so that the family $\{\Sigma_r\}_{r\in(0,r_1)}$ is a foliation of a pointed neighborhood of p.*

Proof. — Define the maps

$$\begin{aligned} \Psi^r &:= \exp_p^{-1} \exp_{c(\tau(r))}, \\ \Psi(r,x) &:= \Psi^r(r(x + r^2\varphi(r)(x))) \quad \text{and} \\ \beta(r,x) &:= \frac{\Psi(r,x)}{|\Psi(r,x)|}. \end{aligned}$$

We claim that there exists $\widetilde{r} \in (0, r_1]$ such that $|\Psi(r,x)| \neq 0$ every $x \in S^2$ and such that $\beta(r,\cdot\,) : S^2 \to S^2$ is a family of diffeomorphisms which can be smoothly extended to $r = 0$ by the identity.

Indeed, this follows from the facts that

$$\frac{\partial \Psi}{\partial r} = (d_x \Psi^r)\left(x + r^2\varphi(r)(x) + r(r^2\varphi(r)(x))_r\right) + \left(\frac{\partial \Psi^r}{\partial r}\right)\left(r(x + r^2\varphi(r)(x))\right)$$

and

$$\left.\frac{\partial \Psi^r}{\partial r}\right|_{r=0} = \left.\frac{\partial}{\partial \tau^i}\left(\exp_p^{-1}\exp_{c(\tau(r))}\right)\right|_{\tau=0} \cdot \left.\frac{\partial \tau^i}{\partial r}\right|_{r=0} = 0$$

where we used Lemma 4.2 in the last equality. In combination

$$\frac{\partial \Psi}{\partial r}(0,x) = x.$$

Hence

$$\Psi(r,x) = rx + O(r^2) \quad \text{as} \quad r \to 0.$$

In particular, $\Psi(r,x) \neq 0$ for r small enough and

$$\beta(r,x) = \frac{x+O(r)}{|x+O(r)|}.$$

This establishes the claim.

Let $\eta(r,x) := |\Psi(r,\beta^{-1}(r,x))|$ and calculate

$$\frac{\partial\eta}{\partial r} = \frac{\Psi}{|\Psi|}\left(\frac{\partial\Psi}{\partial r} + (d_x\Psi)\left(\frac{\partial}{\partial r}\beta^{-1}\right)\right) = \frac{x+O(r)}{|x+O(r)|}\,(x+O(r)).$$

This yields

$$\left.\frac{\partial\eta}{\partial r}\right|_{r=0} = 1.$$

Consequently η is strictly increasing for r small enough which shows that all the surfaces are disjoint. □

5. Local Uniqueness

By inspecting the proof of Theorem 1.1 above and by the local uniqueness of solutions obtained via the implicit function theorem, we obtain a local uniqueness result for the Σ_a. To state this, we use the notation introduced at the beginning of Section 3.

Corollary 5.1. — *Fix $\alpha \in (0,1)$. Let (M,g) be a Riemannian 3-manifold and $p \in M$ be a non-degenerate critical point of the scalar curvature. Denote by $\varphi_0 \in C^\infty(S^2)$ the solution of (3.12) where* Ric *is evaluated at p. Then there exist $r_0 \in (0,\infty)$, a neigborhood $\Omega \subset C^{4,\alpha}(S^2)$ of φ_0, a neighborhood $U \subset \mathbb{R}^3$ of the origin, and an open interval $I \subset \mathbb{R}$ such that $-\frac{1}{3}\operatorname{Sc}(p) \in I$ with the following properties:*

Assume that $\Sigma \subset M$ is such that:

(1) *$\Sigma = S(r,\tau,r^2\varphi)$ up to reparameterization,*

(2) *On Σ we have that*

$$\Delta H + H|\mathring{A}|^2 + H\operatorname{Ric}(\nu,\nu) = \lambda H.$$

(3) *$(r,\tau,\varphi,\lambda) \in (0,r_0)\times U \times \Omega \times I$ and $P_i(\varphi) = 0$ for $i \in \{0,1\}$.*

Then $\Sigma = \Sigma_a$ where Σ_a is as in equation (4.4) *and such that $|\Sigma| = |\Sigma_a|$.*

Note that if $\Omega_b \subset C^{4,\alpha}(S^2)$ is any bounded subset and $\varphi \in \Omega_b$, then there exists a constant $C = C(\Omega_b)$ such that

$$\big|\mathcal{W}(S(r,\tau,r^2\varphi) - 4\pi\big| \leqslant Cr^2.$$

In [6] it was shown that there exists $\varepsilon = \varepsilon(M,g) > 0$ such that if Σ is a solution (1.2) with $|\Sigma| < \varepsilon$ and $\mathcal{W}(\Sigma) \leqslant 4\pi + \varepsilon$ also satisfies

$$\left|\lambda + \frac{1}{3}\operatorname{Sc}(p)\right| \leqslant Cr$$

for some constant $C = C(M,g)$. Hence, it follows that in the statement of Corollary 5.1 the condition on λ is in fact not needed.

Corollary 5.2. — *There exist $r_0' \in (0,\infty)$, a neigborhood $\Omega' \subset C^{4,\alpha}(S^2)$ of φ_0, a neighborhood $U' \subset \mathbb{R}^3$ of the origin with the following properties: Assume that $\Sigma \subset M$ is such that:*

(1) *$\Sigma = S(r,\tau,r^2\varphi)$ up to reparameterization,*
(2) *On Σ we have that*

$$\Delta H + H|\mathring{A}|^2 + H\operatorname{Ric}(\nu,\nu) = \lambda H.$$

(3) *$(r,\tau,\varphi) \in (0,r_0) \times U \times (\Omega \cap K)$ and $P_i(\varphi) = 0$ for $i \in \{0,1\}$.*

Then $\Sigma = \Sigma_a$ where Σ_a is as in equation (4.4) *and such that $|\Sigma| = |\Sigma_a|$.*

Note that this uniqueness applies to individual solutions of (1.2) and not to whole foliations. It is not difficult though, to prove a result similar to [15, Section 2] to deal with the uniqueness of foliations centered at p based on the a priori estimates on such surfaces in [5, 6, 9].

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Tobias LAMM
Karlsruhe Institute of Technology
Institute for Analysis
Englerstrasse 2
76131 Karlsruhe (Germany)
tobias.lamm@kit.edu

Jan METZGER
University of Potsdam
Institute for Mathematics
Karl-Liebknecht-Straße 24/25
14476 Potsdam (Germany)
jan.metzger@uni-potsdam.de

Felix SCHULZE
Department of Mathematics
University College London
25 Gordon St.
London WC1E 6BT (UK)
f.schulze@ucl.ac.uk