Application of the nested soft-collinear subtraction scheme to the description of deep inelastic scattering

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M.Sc. Konstantin Asteriadis
aus Karlsruhe

Referent:	Prof. Dr. Kirill Melnikov
Korreferent:	Prof. Dr. Matthias Steinhauser
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Abstract

Recently, remarkable progress was made in the understanding of how fully-differential next-to-next-to-leading order (NNLO) computations in perturbative Quantum Chromodynamics (QCD) for hadron collider processes can be performed. This progress includes development of promising subtraction schemes that allow us to treat infrared and collinear singularities efficiently. As the result of these developments, many phenomenologically important processes at hadron colliders have been computed with NNLO QCD accuracy. However, despite this progress, the search for the optimal subtraction scheme continues. In this thesis we discuss the recently proposed nested soft-collinear subtraction scheme and apply it to the description of deep inelastic scattering of an electron on a proton. Our results provide an important building block that will allow for description of more complex processes at hadronic colliders in a fully differential manner, using the nested soft-collinear subtraction scheme.

Contents

1. Introduction

The Standard Model of particle physics (SM) describes all known elementary particles and their interactions [1–3], except for the very weak force of gravity. Although predictions of the Standard Model are in good agreement with experimental observations, there are strong indications that physics beyond the Standard Model should exist. They include the existence of dark matter and dark energy, as well as the observed matter-anti-matter asymmetry. None of these phenomena can be explained with the Standard Model of particle physics. Since, contrary to earlier expectations, searches at the Large Hadron Collider (LHC) did not find any evidence for physics beyond the Standard Model, the current situation in particle physics is extremely puzzling and intriguing.

Since substantial further increase in the energy of colliding particles at the LHC and elsewhere is currently not feasible, in the next decade the focus of collider experiments will move towards higher experimental precision. This will allow us to refine existing measurements of the many SM parameters and to scrutinize Standard Model phenomena at the highest accessible energies [4]. To be able to use upcoming experimental measurements to stress-test the Standard Model and to search for New Physics, reliable theoretical predictions for hadron collider processes are needed. Precision of a few percent may be achieved if theoretical predictions include next-to-next-to-leading order (NNLO) corrections in perturbative Quantum Chromodynamics (QCD). However, calculations at this perturbative order are non-trivial and many challenges need to be addressed; they range from difficulties in computing two-loop amplitudes with several mass scales to an efficient treatment of real radiation to more conceptual issues such as a better understanding of non-perturbative effects in hard-scattering processes in hadron collisions.

Fully-differential computations allow for a comprehensive comparison of theory and experiment because rich physics information can be extracted from kinematic distributions rather than from fully inclusive observables. However, fully-differential descriptions of LHC processes are difficult since QCD amplitudes that describe real radiation possess infrared and collinear singularities that need to be treated with the utmost care. The goal of this thesis is to contribute to the developments of theoretical methods that will enable fully-differential NNLO QCD calculations.

From this perspective, the emerging understanding of how to treat infrared and collinear singularities in NNLO QCD computations without integrating over resolved phase space of final state particles, the so-called *subtraction* and *slicing methods*, is one of the most important recent advances in perturbative QCD and indeed in collider physics [5–20]. Thanks to these developments, many interesting processes at hadron colliders have been computed through

NNLO QCD precision [21–35]. However, existing methods typically obscure the physical origin of the singularities, and, as a result, their analytic structures and numerical implementations are complex and inefficient. For this reason, the search for the optimal subtraction scheme continues.

The recently proposed nested soft-collinear subtraction scheme [36], which we discuss in this thesis, possesses many features of a would-be optimal scheme. For example it is physically transparent, analytic, fully local, numerically efficient and highly modular.[1]

We use this modularity to first study subtractions for simpler processes and use the obtained results as building blocks for the more complex ones. Three basic processes need to be considered to obtain a complete set of building blocks sufficient for the application of the nested soft-collinear subtraction scheme to any process at hadron colliders with massless colour-charged particles. They are *(i)* production and decay of a colour-singlet particle; and *(ii)* a process with one colour-charged particle in the initial and one in the final state.

The nested soft-collinear subtraction scheme has so far been used to describe production and decay of vector bosons and Higgs boson [37,38] through NNLO QCD. In addition it was applied to deep-inelastic scattering of a proton on an electron [39]. The latter is the simplest process with colour-charged partons in the initial and final states. We note that, since partonic cross sections of these simple processes are known analytically, the subtraction formulas derived in the context of the soft-collinear subtraction scheme can be tested to a very high precision. Passing such test is an important prerequisite for applying them in a more general context.

The goal of this thesis is to discuss the application of the nested soft-collinear subtraction scheme to deep inelastic scattering process elaborating on Ref. [39]. We provide a detailed description of the nested soft-collinear subtraction scheme, including a step-by-step derivation of the subtraction terms in case of deep inelastic scattering. We hope that this thesis can serve as a useful reference for learning about this subject. We provide a detailed guide through this thesis in Chapter 2.

[1] A detailed discussion of these features is given in Section 3.5.

2. Organization of the thesis

In this chapter we briefly describe the organization of the thesis that may help to understand connections between its different parts. We begin with a short discussion of precision physics at hadron colliders in Chapter 3. In Chapter 4 we set up notations for the description of NNLO QCD corrections to the DIS process. In Chapter 5 we present a computation of the NLO partonic cross sections. We discuss quark-initiated contributions to the NNLO partonic cross section in Chapter 6. Gluon-initiated contributions to the NNLO partonic cross section with additional quark final-states are discussed in Chapter 7. We discuss numerical implementation of formulas in Chapter 8. In Chapter 9 we present analytic and numeric results. We conclude in Chapter 10. Many formulas are collected in appendices. Below we summarize parts of the thesis where information on specific topics can be found.

Nested soft-collinear subtractions

To understand the main idea of the nested-soft collinear subtraction scheme we recommend to read the LO and NLO discussions in Sections 4 and 5, respectively. The NNLO extension is discussed in Chapter 6. Appendix B contains description of operators that appear in the discussion of subtraction terms.

Partonic channels

In Table 2.1 we present all partonic processes that contribute to DIS through NNLO QCD and point to parts of the thesis where they are discussed. The *quark-initiated* process $q + e^- \to e^- + q + g + g$ is described in detail. Notation is set up in the introduction to Chapter 6. Subtraction terms are constructed in Section 6.1. Computation of counter terms is discussed in Sections 6.2 to 6.4. In particular, we discuss emissions of one or two soft gluons in Section 6.2; emissions collinear to partons in the initial-state in Section 6.3.1; emissions collinear to final-state partons in Section 6.3.2 and the emission of two partons that are collinear to each other in Section 6.3.3. Finally, subtraction terms for two partons that are emitted collinear to the same or to two different final-state parton(s) are discussed in Section 6.4.

We discuss *gluon-initiated* processes in Chapter 7, see Table 2.1 for more details. Since analytic computations are largely analogous to the quark-initiated processes we confine ourselves to showing results of the calculations but we do not go into details.

Contributing processes to DIS through NNLO QCD			
channel	tree-level	one-loop	two-loop
$q + e^- \to e^- + q$	Chapter 4	Section 5.3	Section 6.6
$q + e^- \to e^- + q + g$	Chapter 5	Section 6.5	—
$g + e^- \to e^- + q + \bar{q}$	Section 5.5	Section 7.4	—
$q + e^- \to e^- + q + g + g$	Chapter 6	—	—
$q + e^- \to e^- + q + q' + \bar{q}'$	Section 6.7	—	—
$g + e^- \to e^- + q + \bar{q} + g$	Chapter 7	—	—

Tab. 2.1.: In this table we list all partonic processes that contribute to DIS through NNLO QCD and point to places in this thesis where they are discussed. Note that anti-quark-initiated channels are not present in the table, since their computation is identical to the quark-initiated processes.

Functions describing differential cross sections		
function	definition	details
$F_{\rm LM}(1_q, 4_q)$	Eq. (4.5)	Chapter 4
$F_{\rm LV}(1_q, 4_q)$	Eq. (5.54)	Section 5.3
$F_{\rm LVV}(1_q, 4_q)$	Eq. (6.203)	Section 6.6
$F_{\rm LM}(1_q, 4_q \mid 5_g)$	Eq. (5.4)	Chapter 5
$F_{{\rm LM},g}(1_g, 4_q \mid 5_q)$	Eq. (5.78)	Section 5.5
$F_{\rm LV}(1_q, 4_q \mid 5_g)$	Eq. (6.185)	Section 6.5
$F_{{\rm LV},g}(1_g, 4_q \mid 5_q)$	Eq. (7.32)	Section 7.4
$F_{\rm LM}(1_q, 4_q \mid 5_g, 6_g)$	Eq. (6.4)	Chapter 6
$F_{\rm LM}^{\rm int}(1_q, 4_q, 5_q, 6_q)$	Eq. (6.220)	Section 6.7
$F_{\rm LM,ns}(1_q, 4_q \mid 5_q, 6_q)$	Eq. (6.222)	Section 6.7 and 6.7.1
$F_{\rm LM,s}(1_q, 4_q \mid 5_q, 6_q)$	Eq. (6.229)	Section 6.7 and 6.7.2
$F_{{\rm LM},g}(1_g, 4_q \mid 5_q, 6_g)$	Eq. (7.5)	Chapter 7

Tab. 2.2.: In this table we point to parts of this thesis where various functions that describe differential cross sections are discussed and defined.

Hard matrix elements/cross sections

Throughout this thesis, we use the various functions $F_{\rm LM}$, $F_{{\rm LM},g}$, $F_{\rm LV}$ etc. to describe partonic cross sections. In Table 2.2 we point to parts of the thesis where these functions are defined.

Analytic results for subtraction terms

We present finite remainders of the subtraction terms in Chapter 9, Sections 9.1 to 9.3. To understand them it is useful to read the NLO discussion in Chapter 5. In Tab. 2.2 we show where definitions and discussions of the many different functions that contain the matrix elements squared can be found.

Numerical implementation

We describe the numerical implementation of the subtraction scheme in Chapter 8. Required limits can be found in Appendix B. Discussion of the phase space parametrization is provided in Appendix F. To get familiar with notations used to describe partonic cross sections it is also advisable to read the beginning of Chapter 6.

3. Hard processes in hadron collisions and perturbative QCD

Man-made particle collisions with the highest energy currently occur at the Large Hadron Collider (LHC) at CERN. The overarching goal of the LHC is to discover physics beyond the Standard Model. Unfortunately, no new particles or interactions have been observed at the LHC so far. Since existing measurements can only rule out new particles as long as their masses are significantly smaller than the LHC center-of-mass energy of 13 TeV, such non-observation of new particles does not prove that they do not exist but only that they are heavy. However, although such heavy particles can not be produced directly at the LHC, they can affect physics there, if they are created but disappear back into a quantum vacuum in a short time interval. Such effects are small and observing them requires high precision, both in experimental measurements and in theoretical predictions. In general, high precision allows us to refine existing measurements of SM parameter and to explore ones that are currently beyond reach, especially properties of the recently discovered Higgs boson [40].

Achieving high precision on the theory side is complicated by the fact that hadrons are composite particles made of partons bound by the poorly understood non-perturbative strong force. Since, so far, it is not possible to fully describe properties of even a single proton from first principles, it is not obvious that a first-principles description of hadron collisions is possible.

To understand why this actually works, we note that hadrons colliding at high energies interact in various ways. Most of the time, such interactions happen through an elastic scattering processes where both hadrons stay intact, or processes of diffractive dissociation where one or both hadrons disintegrate into a small number of hadrons. However, with a much lower probability, individual partons in the colliding hadrons can get close to each other and interact by exchanging a large momentum, see Fig. 3.1. These rare processes are referred to as *hard scattering processes* and they are of great interest to modern particle physics. This is so because, thanks to a large momentum transfer and the phenomenon of asymptotic freedom, such processes can be accurately described in perturbative Quantum Chromodynamics (QCD) and because new heavy particles can be produced in such processes. A combination of these facts makes a detailed exploration of hard processes an excellent way to search for New Physics at the LHC.

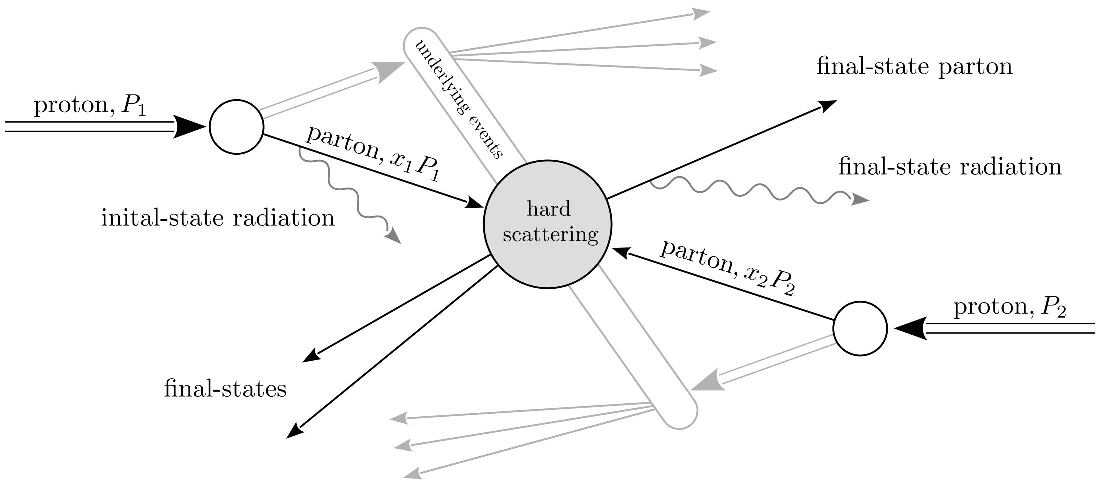


Fig. 3.1.: Schematic picture of a high-energy collision of two protons with momenta $P_{i=1,2}$. Because of short-distances and asymptotic freedom, scattering partons with momenta fractions $x_1 P_1$ and $x_2 P_2$ can be assumed to be free and there interaction (hard scattering) can be computed in perturbative QCD. Initial- and final state radiation needs to be included beyond LO. Underlying events of the proton remnants are shown in the background.

3.1. Hadronic cross sections

The foundation of theoretical predictions for hard scattering processes at hadron colliders is the factorization theorem [41]. It states that, up to power-suppressed terms, hadronic cross sections are described by the following formula [41]

$$\mathrm{d}\sigma_{\mathrm{H}} = \sum_{ij} \int_0^1 \mathrm{d}x_1 \, \mathrm{d}x_2 \, f_i(x_1) f_j(x_2) \, \mathrm{d}\hat{\sigma}_{ij}(x_1, x_2) \, \left[1 + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{Q} \right) \right] . \tag{3.1}$$

In Eq. (3.1) f_i are the so-called parton distribution functions and $\mathrm{d}\hat{\sigma}_{ij}$ are partonic cross sections that describe scattering of a parton i on a parton j. Parton distribution functions are non-perturbative and process-independent. For this reason, they can be extracted from a subset of experimental data and used to describe any process from a complementary dataset. All non-perturbative effects that go beyond the distribution functions are suppressed by powers of Λ_{QCD}/Q where $\Lambda_{\mathrm{QCD}} \sim 0.3\,\mathrm{GeV}$ is a non-perturbative QCD scale and $Q \gtrsim \mathcal{O}(10\,\mathrm{GeV})$ is a typical scale of a hard process.

The partonic cross sections $\mathrm{d}\hat{\sigma}_{ij}$ in Eq. (3.1) can be computed in QCD perturbation theory. Expanding $\mathrm{d}\hat{\sigma}_{ij}$ in powers of the strong coupling constant α_s, we write

$$\mathrm{d}\hat{\sigma}_{ij} = \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{lo}} + \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{nlo}} + \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{nnlo}} + \mathcal{O}(\alpha_s^3) . \tag{3.2}$$

In Eq. (3.2) contributions labeled with “lo” describe the leading order process, contributions labeled with “nlo” provide $\mathcal{O}(\alpha_s)$ corrections to $\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{lo}}$ and contributions labeled with “nnlo” provide $\mathcal{O}(\alpha_s^2)$ corrections. Computations of $\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{lo}}$ and $\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{nlo}}$ are well understood and largely

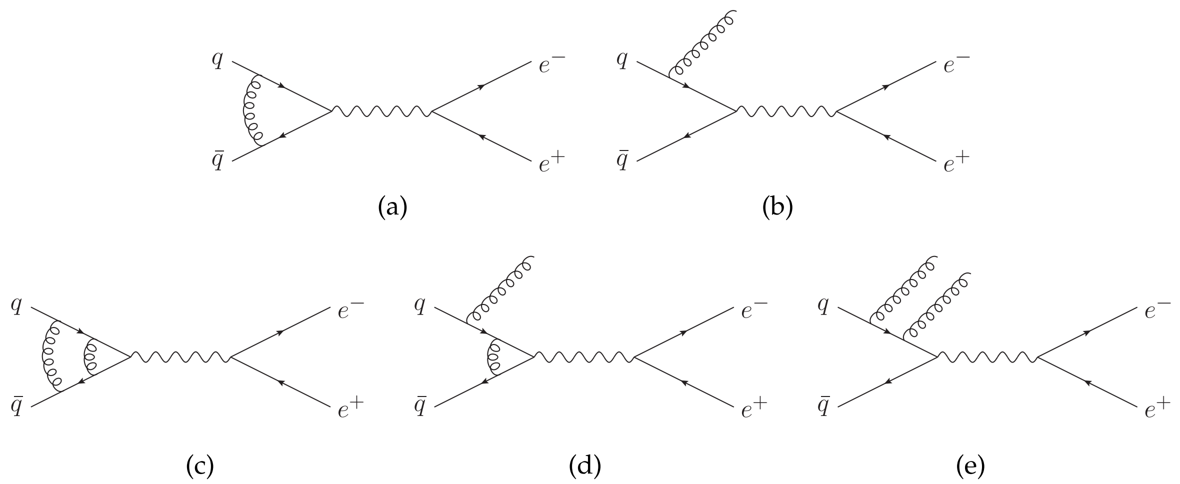


Fig. 3.2.: Examples for Feynman diagrams that contribute to the single-real (a), the single-virtual (b), the double-virtual (c), the real-virtual (d) and the double-real (e) contribution to the NLO and NNLO partonic cross section of the Drell-Yan process.

automated [42–48]. The goal of this thesis is to develop methods to compute $\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{nnlo}}$ at a fully differential level.

3.2. Infrared poles and their cancellation

In order to compute higher-order perturbative contributions to the partonic cross section $\mathrm{d}\hat{\sigma}_{ij}$ that describes a process $i + j \to X$, we need to include both *virtual* loops corrections to this process as well as *inelastic* processes $i + j \to X + g$, $i + j \to X + g + g$ etc with additional partons in the final state [49]. These two contributions are referred to as virtual and real corrections, respectively. Although these corrections are not infrared finite separately, upon combining them we obtain well-defined infrared-safe observables. Hence, we write the NLO (NNLO) QCD contribution to a partonic cross section as

$$\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{nlo}} = \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{v}} + \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{r}} + \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{pdf}}\,, \quad \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{nnlo}} = \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{vv}} + \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{rv}} + \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{rr}} + \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{pdf}}\,, \tag{3.3}$$

where $\mathrm{d}\hat{\sigma}_{\mathrm{v}}$ and $\mathrm{d}\hat{\sigma}_{\mathrm{vv}}$ describe one-loop and two-loop virtual corrections to the hard process $i + j \to X$, $\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{r}}$ describes a process with one additional parton in the final state $i + j \to X + f$ and $\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{rv}}$ describes the one-loop correction to $\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{r}}$, $\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{rr}}$ describes a process with two additional partons in the final state $i + j \to X + f_1 + f_2$ and $\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{pdf}}$ describes corrections to the partonic cross section caused by the collinear renormalization of parton distribution functions. Using the Drell-Yan process as an example, we show Feynman diagrams that contribute to the terms on the right-hand sides of Eqs. (3.3) in Fig. 3.2.[1]

The individual contributions on the right-hand side of Eq. (3.3) are not infrared-finite. Virtual corrections, present in $\mathrm{d}\hat{\sigma}_{\mathrm{vv}}$ and $\mathrm{d}\hat{\sigma}_{\mathrm{rv}}$, contain *explicit* infrared and collinear poles in the dimensional regularization parameter $\epsilon = (d-4)/2$ [50] that are known to be independent of

[1]More details on the collinear PDF renormalization can be found in Section 5.3 and Section 6.6.

the hard matrix elements [51–55].

As an example consider a process $q(p_1) + \bar{q}(p_2) \to X(p_X)$ where X is an arbitrary colour-singlet state (Z, $W^\pm$, γ, ZZ, W^+W^-, $\gamma\gamma$, ZZZ, etc.). According to Refs. [51–53], the singular structure of the UV-renormalized one-loop contribution to the cross section reads

$$
\begin{aligned}
2\Re\left(\mathcal{M}^\star_{\text{tree}} \cdot \mathcal{M}_{\text{1-loop}}\right)(p_1, p_2, p_X) = &-2C_F \left[\frac{\alpha_s(\mu)}{2\pi} \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\right] \left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon}\right] \\
&\times \cos(\pi\epsilon) \left(\frac{2p_1 \cdot p_2}{\mu^2}\right)^{-\epsilon} \times |\mathcal{M}_{\text{tree}}(p_1, p_2)|^2 + 2\Re\left(\mathcal{M}^\star_{\text{tree}} \cdot \mathcal{M}^{\text{fin}}_{\text{1-loop}}\right)(p_1, p_2, p_X) \,.
\end{aligned}
\tag{3.4}
$$

In Eq. (3.4) $\mathcal{M}^{\text{fin}}_{\text{1-loop}}$ is an infrared-finite remainder of a one-loop amplitude. We note that, in NNLO QCD computations also two-loop amplitudes are required. A formula similar to Eq. (3.4) is also known for the two-loop case [54, 55].

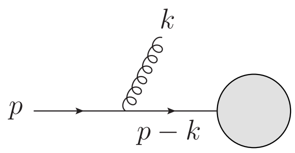
In contrast to explicit $1/\epsilon$ poles present in virtual correction, real emission contributions $\mathrm{d}\hat{\sigma}_{\text{rv}}$ and $\mathrm{d}\hat{\sigma}_{\text{rr}}$ contain kinematic singularities that become poles in $1/\epsilon$ *only* upon integrating over phase space of additional partons in the final state. However, we have to avoid such integration to keep partonic cross sections fully-differential. Hence, we need to develop a method that allows us to extract implicit $1/\epsilon$ poles from the real emission contributions without integrating over the resolved phase space.

We may hope to achieve that goal because in singular kinematic regions, responsible for the appearance of infrared and collinear poles, real emissions are always unresolved. Such kinematic configurations occur when a parton is emitted with vanishingly small energy (soft), or when the angle between the parton and another parton approaches zero (collinear). These unresolved real emissions develop singularities that produce $1/\epsilon$ poles that cancel the $1/\epsilon$ poles of virtual contributions. Expressing this statement in a language of well-defined mathematical formulas for the deep-inelastic scattering process at NNLO QCD, within the context of the nested soft-collinear scheme, is the goal of this thesis.

3.3. Singularities of real-emission contributions

Singularities of QCD amplitudes are related to kinematic limits where virtual intermediate particles become on-shell. In amplitudes with real emissions this can happen *(i)* when the energy of emitted gluons vanishes (soft singularity); or *(ii)* when gluons or (anti-)quarks are emitted in the direction of another parton (collinear singularity).

To illustrate this, consider a diagram that describes an emission of a gluon off an external incoming quark line. Considering soft and collinear limits, we find



$$
\sim \frac{1}{(p-k)^2} = \frac{1}{2E_pE_k(1-\cos\theta)} \underset{\substack{E_k \to 0 \\ \text{or} \\ \theta \to 0}}{\longrightarrow} \infty \,. \tag{3.5}
$$

As stated above, the reason for this divergence is that a virtual quark with momentum $(p-k)$ in this diagram becomes on-shell $(p-k)^2 \to 0$, in the soft $E_k \to 0$ and/or in the collinear $\theta \to 0$ limits.

In these limits, any QCD amplitude factorizes into a universal function that becomes singular in the limit and an amplitude of a lower multiplicity process [56]. To illustrate this statement, consider the tree-level amplitude of the process $q(p_1) + \bar{q}(p_2) \to X + g(k)$, where X is an arbitrary colour-singlet state, in soft and collinear limits. In the soft limit, where the energy of the gluon $g(k)$ vanishes, the amplitude squared reads [57]

$$|M_{\text{tree}}(\{p_1, p_2\}, k)|^2 \underset{E_k \to 0}{\approx} 2C_F\, g_{s,b}^2 \times \frac{p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} \times |M_{\text{tree}}(\{p_1, p_2\})|^2 \,. \tag{3.6}$$

In Eq. (3.6) $M_{\text{tree}}(\{p_1, p_2\})$ is the amplitude of the process $q(p_1) + \bar{q}(p_2) \to X$ without an additional gluon. As can be seen from the right-hand side of Eq. (3.6), soft singularities reside in an *eikonal function*

$$\text{Eik}(\{p_1, p_2\}, k) \equiv \frac{p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} \,. \tag{3.7}$$

The eikonal function Eq. (3.7) contains soft $E_k \to 0$ and collinear, $\vec{k} \parallel \vec{p}_1$ and $\vec{k} \parallel \vec{p}_2$, singularities.

As an example of a collinear singularity, we study the amplitude of the process $q(p_1) + \bar{q}(p_2) \to X + g(k)$ in the limit $\vec{k} \parallel \vec{p}_1$. The amplitude squared reads [57]

$$|M_{\text{tree}}(\{p_1, p_2\}, k)|^2 \underset{\vec{k} \parallel \vec{p}_1}{\approx} g_{s,b}^2 \times \frac{1}{p_1 \cdot k} P_{qq}(z) \times \frac{|M_{\text{tree}}(\{z \cdot p_1, p_2\})|^2}{z} \,, \tag{3.8}$$

where

$$z = \frac{E_1 - E_k}{E_1} \,, \tag{3.9}$$

and the function $P_{qq}(z)$ is the so-called *splitting function*. It reads [57]

$$P_{qq}(z) = C_F \left[\frac{1+z^2}{1-z} - \epsilon(1-z) \right] . \tag{3.10}$$

The matrix element on the right-hand side in Eq. (3.8) still depends on the gluon energy through the variable z, but this dependence is not singular. The collinear singularity resides in the overall factor $1/(p_1 \cdot k)$. Similarly to the soft case Eq. (3.6), collinear singularities are described by universal process-independent splitting functions.[2]

In NNLO QCD computations we have to consider tree-level amplitudes for processes with two additional partons compared to the Born process. Hence, in addition to the previously discussed single-soft emission and double-collinear limits, we also need to consider cases when

[2]Note that, collinear singularities described by Eq. (3.8) only depend on the four-momenta of the collinear gluon $g(k)$ and the collinear quark $q(p_1)$. Hence, the singular factor on the right-hand side of Eq. (3.8) is valid for an arbitrary processes with any number of external partons.

two gluons become soft (the double-soft limit) or two partons become collinear to another parton (the triple-collinear limit). It is well known [56] that also in these cases QCD amplitudes factorize into universal functions and amplitudes of lower multiplicities.

As an example, consider a process $q(p_1) + \bar{q}(p_2) \to X + g(k) + g(l)$, c.f. Fig. 3.2 (e). In the double-soft limit, $k \to 0$, $l \to 0$, $k \sim l$, we find[3]

$$|M_{\text{tree}}(\{p_1, p_2\}, k, l)|^2 \underset{E_k \sim E_l \to 0}{\sim} \text{Eik}(\{p_1, p_2\}, k, l) \times |M_{\text{tree}}(\{p_1, p_2\})|^2 \,. \tag{3.11}$$

Since the explicit formula for the double-soft eikonal function $\text{Eik}(\{p_1, p_2\}, k, l)$ in Eq. (3.11) is fairly complicated, we do not show it here. For the case of deep-inelastic scattering, that we need in this thesis, $\text{Eik}(\{p_1, p_2\}, k, l)$ is given in Appendix B.2. However, we want to emphasize that the double-soft limit Eq. (3.11) is structurally identical to the case of a single soft gluon in Eq. (3.6) in that all singularities factorize from the hard matrix element in terms of the double-soft eikonal function $\text{Eik}(\{p_1, p_2\}, k, l)$.

In the triple-collinear $\vec{k} \parallel \vec{l} \parallel \vec{p}_1$ limit the amplitude squared reads [56]

$$\begin{aligned}
&|M_{\text{tree}}(\{p_1, p_2\}, k, l)|^2 \\
&\underset{\vec{k}\parallel\vec{l}\parallel\vec{p}_1}{\sim} \frac{1}{((p_1 - k - l)^2)^2} \, P_{ggq}(p_1, k, l) \times |M_{\text{tree}}(\{z \cdot p_1, p_2\})|^2 \,,
\end{aligned} \tag{3.12}$$

where $z = (E_1 - E_k - E_l)/E_1$ and $P_{ggq}(p_1, k, l)$ is the splitting function that describes a collinear splitting $q \to q^\star + g + g$. Again, similar to the double-collinear case Eq. (3.8), the amplitude squared in Eq. (3.12) factorizes into a singular part and a regular part. Explicit formulas for P_{ggq} and other triple-collinear splitting functions are given in Appendix E.2.[4]

Finally, NNLO QCD corrections require us to include one-loop amplitudes to processes that contain an additional parton in the final state, c.f. Eq. (3.3). The singular behavior of real-virtual amplitudes was studied in Refs. [58–60]. It is similar to the behavior of tree-level amplitudes and we only show the soft limit as an example.[5] Considering the one-loop amplitude for the process $q(p_1) + \bar{q}(p_2) \to X + g(k)$ in the soft $E_k \to 0$ limit, we obtain [58–60]

$$\begin{aligned}
&2\Re\big(M^\star_{\text{tree}} \cdot M_{\text{1-loop}}\big)(p_1, p_2, k) \\
&\quad \underset{E_k \to 0}{\approx} 2C_F\, g_{s,b}^2 \times \frac{p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} \times 2\Re\big(M^\star_{\text{tree}} \cdot M_{\text{1-loop}}\big)(p_1, p_2) \\
&\qquad - 2C_F C_A \,\frac{g_{s,b}^4}{\epsilon^2}\, 2^{-\epsilon} \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right] \left[\frac{\Gamma^5(1-\epsilon)\Gamma^3(1+\epsilon)}{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)}\right] \\
&\qquad \times \left(\frac{p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)}\right)^{1+\epsilon} \times |M_{\text{tree}}(p_1, p_2)|^2 \,.
\end{aligned} \tag{3.13}$$

The squared amplitude in Eq. (3.13) contains two contributions. The first term on the right-hand

[3] For simplicity we do not show colour correlations in Eq. (3.11).

[4] We note that the arguments of the splitting function P_{ggq} in Appendix B.4 are slightly different and refer to Eq. (B.21) for the relation between the different arguments.

[5] For collinear limits we refer to the discussion in Section 6.5.

side of Eq. (3.13) is proportional to the one-loop correction to the hard process and the same *tree-level* eikonal function that appeared in the soft limit of a Born process $q(p_1) + \bar{q}(p_2) \to X + g$, c.f. Eq. (3.6). The second term on the right-hand side is proportional to the tree-level amplitude squared and the universal *one-loop corrected* eikonal function; note that this contribution is purely non-abelian.

Singular limits of the amplitudes that we just discussed can be used to extract $1/\epsilon$ divergences from real emission contributions without the need to integrate over resolved phase space. This is done with the help of subtraction methods that we now discuss.

3.4. The subtraction method

Real emission QCD amplitudes possess soft and collinear singularities. These singularities turn into poles in the dimensional regularization parameter $\epsilon = (d-4)/2$ upon phase space integration. To show this, we approximate integration over gluon momentum in the soft and collinear limits and obtain

$$\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2E_k}\,|M(\{p\},k)|^2 \underset{\substack{E_k \to 0\\ \theta \to 0}}{\sim} \int \frac{\mathrm{d}E_k}{E_k^{1+2\epsilon}}\,\frac{\mathrm{d}\theta}{\theta^{1+2\epsilon}} \times |M(\{p\})|^2 \sim \frac{1}{4\epsilon^2}\,. \tag{3.14}$$

We would like to extract $1/\epsilon$ poles and to regulate singularities in real matrix elements *without* integration over resolved phase space so that we can evaluate phase space integrals numerically for any infrared safe observable. This can be achieved with the subtraction method.

To illustrate the basic idea of this method, we consider the integral

$$I = \int_0^1 \mathrm{d}x\,\frac{1}{x^{1+\epsilon}}\,F(x)\,, \tag{3.15}$$

where $F(x)$ is an arbitrary function regular at $x = 0$. The integrand in Eq. (3.15) diverges at the lower integration boundary and the singularity is regulated by the parameter ϵ leading to a $1/\epsilon$ pole after integration. We want to extract this pole analytically and regulate the integral so that we can safely take the limit $\epsilon \to 0$.

To achieve this, we write $F(x) = \left[F(x) - F(0)\right] + F(0)$, use this expression in the integral Eq. (3.15) and find

$$I = \int_0^1 \mathrm{d}x\,\frac{1}{x^{1+\epsilon}}\left[F(x) - F(0)\right] + F(0)\int_0^1 \mathrm{d}x\,\frac{1}{x^{1+\epsilon}}\,. \tag{3.16}$$

In the second term on the right-hand side of Eq. (3.16) the function F decouples from the integral and we can integrate over x analytically. We obtain

$$I = -\frac{1}{\epsilon}F(0) + \int_0^1 \frac{\mathrm{d}x}{x}\,\left[F(x) - F(0)\right] + \mathcal{O}(\epsilon)\,. \tag{3.17}$$

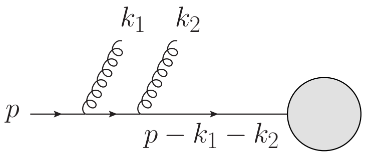
It follows from Eq. (3.17) that we have succeeded in isolating the $1/\epsilon$ pole in I and in regulating its integrand. The remaining integral in the second term on the right-hand side of Eq. (3.17) is finite and we have taken the $\epsilon \to 0$ limit. The integral in Eq. (3.17) can be computed numerically for any function $F(x)$.

3.5. An optimal subtraction scheme and nested soft-collinear subtractions

Singular limits of NLO QCD amplitudes and methods to use them to obtain NLO QCD cross section are well-known [42, 43, 61]. Moreover, all singular limits of QCD amplitudes required for computing NNLO QCD corrections have been known for about 20 years. Yet, it took quite some time before it was realized how to combine these NNLO limits and the ideas of NLO FKS subtraction [42, 43] to establish a valid subtraction scheme for NNLO computations. We describe one of the reasons for such a delay below.

Absence of entangled soft-collinear limits

The discussion of singularities in Section 3.3 applies to soft radiation at large angles and hard collinear radiation. But what happens if soft gluons also become collinear or collinear gluons also become soft or one gluon is soft and one gluon is collinear? Do new limits appear in these cases? Inspection of individual Feynman diagrams suggests that this is indeed the case. For example, considering a diagram



$$\sim \frac{1}{2p\cdot k_1 + 2p\cdot k_2 - 2k_1\cdot k_2} \times \frac{1}{2p\cdot k_1} \underset{\substack{\vec{k}_1 \| \vec{p} \\ \textbf{and} \\ k_2 \to 0}}{\longrightarrow} \infty\,, \tag{3.18}$$

we observe that an entangled soft-collinear singularity develops if one gluon becomes soft and the other becomes collinear. Such entangled soft-collinear limits of diagrams can not be analyzed in a process-independent way. However, it appears that this is not necessary. Indeed, individual Feynman diagrams are not physically quantities; they need to be combined into gauge-invariant scattering amplitudes and for these it can be checked explicitly that such entangled limits do not appear. It follows that remaining soft-collinear limits can be described by taking the known soft and collinear limits sequentially.

In Ref. [36] it was pointed out that this result is general thanks to the phenomenon known as *colour coherence*. This phenomenon is widely used to extend collinear parton showers to partially accommodate soft emissions. Colour coherence states that a soft parton does not resolve angles of a collinear parton because it has a large wavelength. As a result the known soft and collinear limits [56–60] are sufficient to describe and regulate all singularities in arbitrary real-emission NNLO QCD scattering amplitudes.

The optimal subtraction scheme

Given the importance of fully-differential NNLO QCD computations for the LHC physics program, many subtraction schemes that allow us to handle infrared and collinear singularities have been proposed [5–12] and used to compute important processes with NNLO QCD accuracy [21–35]. In spite of this success, the current subtraction methods are not fully satisfactory. In fact, upon reflection, it becomes clear that an optimal subtraction scheme should satisfy the following criteria:

- it should be *physically transparent* in a sense that it must only deal with *physical* singularities and there should be a clear mechanism of how different infrared poles cancel against each other;
- differential cross sections should be regulated *locally*, which means that such cross section can be evaluated at any resolved phase space point;
- infrared $1/\epsilon$ poles should be known *analytically* to establish their cancellation;
- a subtraction scheme should be *modular*, so that subtractions for complex processes can be built from subtractions derived for simpler processes;
- complex LHC processes require numerical integration over huge phase spaces. An *efficient* numerical evaluation and a scalable implementation of the subtraction scheme is therefore important;
- it should be applicable to all processes at the LHC.

Although none of the existing NNLO subtraction schemes satisfy all these criteria, this does not appear to be a problem so far, since phenomenological applications mainly focused on $2 \to 1$ and $2 \to 2$ processes. When considering more complicated $2 \to 3$ processes, the singularity structures become more complicated, and this increase in complexity may result in unfeasible computational times. The nested soft-collinear subtraction scheme which we describe in this thesis is a step in the direction of a more physically transparent and efficient subtraction method.

Nested soft-collinear subtractions

The so-called *nested soft-collinear subtraction scheme* was introduced in Ref. [36]. It is based on *(i)* the fact that the soft and collinear limits are independent and can thus be treated separately (a consequence of colour coherence), and, *(ii)* partitioning of radiative phase space into regions with defined structure of collinear singularities [6]. It utilizes known universal infrared and collinear limits of NNLO QCD amplitudes to both demonstrate the cancellation of infrared and collinear $1/\epsilon$ poles independent of the hard matrix element and to design analytic subtraction terms. It is a analytic and local, and, when applied to simple processes, it was shown to be efficient. Thus, it possesses many features of an optimal scheme discussed above. Although, in principle, there is no obstacle to making it fully general, this has yet to be done.

A completely general formulation of the scheme for hadronic processes should allow NNLO QCD calculations for a scattering process of $2 \to n$ coloured partons. Such a general framework can be constructed from simpler building blocks. Indeed, at NNLO QCD only two real partons can become unresolved at once. Since collinear singularities factorize on external legs [56], it is sufficient to study all possible triple-collinear initial- and final-state splittings for simple processes. Soft singularities depend on momenta and colour charges of *all* external partons [56]. However, it is well known [56] that non-trivial contributions depend on the momenta of two external partons *at most*. It is therefore useful to apply the subtraction scheme to simpler processes with only two external partons. To cover all kinematic cases we need to study initial-initial (colour-singlet production), final-final (colour-singlet decay), and initial-final (DIS-like) configurations. The results of such studies can then be used as building blocks to construct subtractions for more complicated hadron collider processes.

We note that since (inclusive) partonic cross sections of simple $2 \to 1$ (colour-singlet production), $1 \to 2$ (colour-singlet decay) and DIS processes are known analytically [62–66], subtraction formulas derived in the context of the soft-collinear subtraction scheme for these processes can be thoroughly tested. Passing such a test is an important prerequisite for applying obtained results to more complicated processes.

These tests were done for the production and decay of a Higgs boson and a vector boson in Refs. [37, 38]. This thesis describes the application of the nested soft-collinear subtraction scheme to deep inelastic scattering of an electron on a proton [39].

4. Deep inelastic scattering

In the remainder of this thesis we focus on deep inelastic scattering (DIS) of an electron on a proton, see Fig. 4.1. The goal is to study the application of the nested soft-collinear subtraction scheme to a processes that contains colour charged particles in the initial and final states. To this end, we consider the hadronic process

$$p(P_1) + e^-(p_2) \;\to\; e^-(p_3) + X(P_X)\,. \tag{4.1}$$

As discussed in Section 3.3, singular limits of QCD amplitudes are independent of hard matrix elements. Hence, to determine generic subtraction terms for DIS-like processes, it is sufficient to consider a process in Eq. (4.1) mediated by a t-channel photon exchange.

The differential cross section for the process in Eq. (4.1) is described by a convolution of partonic differential cross sections with parton distribution functions. We write

$$\mathrm{d}\sigma_H = \sum_i \int_0^1 \mathrm{d}x\, f_i(x)\, \mathrm{d}\hat{\sigma}_i(x)\,, \tag{4.2}$$

where f_i are parton distribution functions and $\mathrm{d}\hat{\sigma}_i$ are partonic cross sections that describe a photon-mediated scattering of a parton i on an electron.

The partonic cross sections can be computed in perturbative QCD. We expand $\mathrm{d}\hat{\sigma}_i$ in powers of the strong coupling constant α_s and write the partonic cross sections as

$$\mathrm{d}\hat{\sigma}_i = \mathrm{d}\hat{\sigma}_i^{\mathrm{lo}} + \mathrm{d}\hat{\sigma}_i^{\mathrm{nlo}} + \mathrm{d}\hat{\sigma}_i^{\mathrm{nnlo}} + \mathcal{O}(\alpha_s^3)\,. \tag{4.3}$$

In this chapter we discuss the leading order (LO) contribution $\mathrm{d}\hat{\sigma}_i^{\mathrm{lo}}$ to the differential partonic

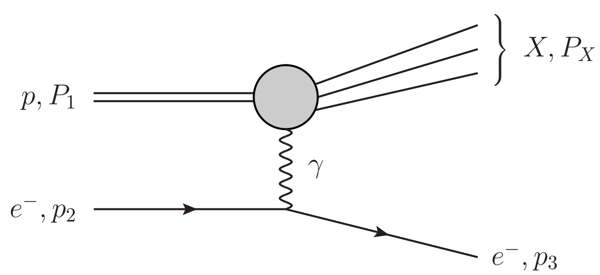


Fig. 4.1.: Schematic illustration of deep inelastic scattering of a proton on an electron that is mediated by a photon. The proton scatters into a number of hadronic jets X with accumulated momentum P_X while the electron stays intact.

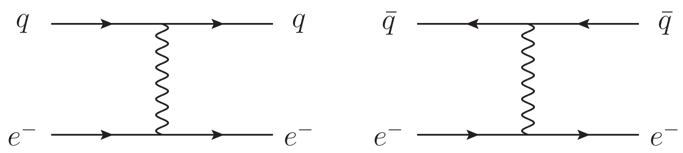


Fig. 4.2.: Feynman diagrams that contribute to the amplitudes of partonic quark-electron scattering process (left) and anti-quark-electron scattering process (right) at LO QCD.

cross section Eq. (4.3). We study next-to-leading order (NLO) contribution $\mathrm{d}\hat{\sigma}_i^{\mathrm{nlo}}$ in Chapter 5 and next-to-next-to-leading order (NNLO) contribution $\mathrm{d}\hat{\sigma}_i^{\mathrm{nnlo}}$ in Chapters 6 and 7.

At leading order in α_s, both quark-initiated process $q + e^- \to e^- + q$ and anti-quark-initiated process $\bar{q} + e^- \to e^- + \bar{q}$ contribute, see Fig. 4.2. All computations for quark-initiated and anti-quark-initiated processes are identical and we only consider the quark-initiated process $q(p_1) + e^-(p_2) \to e^-(p_3) + q(p_4)$ in what follows.

We write the cross section of the quark-electron scattering process as

$$2s \cdot \mathrm{d}\hat{\sigma}_{\mathrm{lo}} \equiv \int F_{\mathrm{LM}}(1_q, 4_q) \equiv \left\langle F_{\mathrm{LM}}\left(1_q, 4_q\right)\right\rangle_\delta \,, \tag{4.4}$$

where

$$\begin{aligned} F_{\mathrm{LM}}(1_q, 4_q) = \mathcal{N}\, [\mathrm{d}p_3][\mathrm{d}p_4]\, (2\pi)^d \delta^{(d)}\left(p_1 + p_2 - p_3 - p_4\right) \\ \times |M_{\mathrm{lo}}^{\mathrm{tree}}(p_1, p_2, p_3, p_4)|^2 \times \hat{\mathcal{O}}(p_3, p_4)\,, \end{aligned} \tag{4.5}$$

and

$$[\mathrm{d}p_i] = \frac{\mathrm{d}^{d-1} p_i}{(2\pi)^{d-1} 2E_i}\, \theta(E_{\max} - E_i)\,, \tag{4.6}$$

is the phase-space volume element of the parton i. We do not show the dependence on the electron momentum in F_{LM}. The quantity $E_{\max}$ is a sufficiently large but otherwise arbitrary[1] dimensionful parameter that provides an upper bound on energies of individual partons; its role will become clear later. The factor $\mathcal{N}$ in Eq. (4.5) includes all the relevant symmetry factors, $M_{\mathrm{lo}}^{\mathrm{tree}}$ is the matrix element described by the left-most Feynman diagrams in Fig. 4.2 and $\hat{\mathcal{O}}$ is an arbitrary infrared-safe observable. The notation $\langle F_{\mathrm{LM}}(1, 4)\rangle_\delta$ indicates that the corresponding cross section is fully-differential with respect to partonic momenta that appear as arguments of the function F_{LM}.

[1] More specifically, $E_{\max}$ should be greater than or equal to the maximal energy that a final state parton can have according to the momentum conservation constraint.

5. The NLO computation

We turn to a discussion of how to compute the infrared-finite partonic differential DIS cross section at NLO QCD using the nested soft-collinear subtraction scheme. We note that at this order in the perturbative expansion, the procedure is equivalent to the FKS subtraction scheme [42, 43]. Nevertheless, it is worth discussing it since many concepts and notation necessary for the NNLO computation can be illustrated and discussed already at NLO. Moreover, also in NNLO computations we have to deal with NLO amplitudes and their singularities so that understanding them is important.

At NLO, we obtain an infrared-finite partonic cross section by combining three contributions

$$\mathrm{d}\hat{\sigma}_{\mathrm{nlo}} = \mathrm{d}\hat{\sigma}_{\mathrm{v}} + \mathrm{d}\hat{\sigma}_{\mathrm{r}} + \mathrm{d}\hat{\sigma}_{\mathrm{pdf}}\,. \tag{5.1}$$

In Eq. (5.1) $\mathrm{d}\hat{\sigma}_{\mathrm{v}}$ describes the one-loop correction to a Born process, $\mathrm{d}\hat{\sigma}_{\mathrm{r}}$ describes the process with an additional parton in the final state and $\mathrm{d}\hat{\sigma}_{\mathrm{pdf}}$ originates from the collinear renormalization of parton distribution functions. At NLO both the quark-initiated and the gluon-initiated channels contribute.

We begin with the discussion of the quark-initiated channel and first focus on the real emission contribution[1]

$$q(p_1) + e^-(p_2) \to e^-(p_3) + q(p_4) + g(p_5)\,. \tag{5.2}$$

The relevant Feynman diagrams are shown in Fig. 5.1. In analogy to Eq. (4.4) we define[2]

$$2s \cdot \mathrm{d}\hat{\sigma}_{\mathrm{r}} \equiv \int [\mathrm{d}p_5]\;\; F_{\mathrm{LM}}(1_q, 4_q\,|\,5_g) \equiv \langle F_{\mathrm{LM}}(1_q, 4_q\,|\,5_g)\rangle_\delta\;, \tag{5.3}$$

where

$$\begin{aligned} F_{\mathrm{LM}}(1_q, 4_q\,|\,5_g) = \mathcal{N} \int [\mathrm{d}p_3][\mathrm{d}p_4]\; (2\pi)^d \delta^{(d)}\,(p_1 + p_2 - p_3 - p_4 - p_5) \\ \times |M^{\mathrm{tree}}_{\mathrm{nlo}}(p_1, p_2, p_3, p_4, p_5)|^2 \times \hat{\mathcal{O}}(p_3, p_4, p_5)\,. \end{aligned} \tag{5.4}$$

Phase-space volume elements $[\mathrm{d}p_{i=3,4,5}]$ are given in Eq. (4.6). The factor $\mathcal{N}$ in Eq. (5.4) includes all the relevant symmetry factors, $M^{\mathrm{tree}}_{\mathrm{nlo}}$ is the matrix element described by Feynman diagrams

[1] Computations for quark and anti-quark initiated processes are identical and we focus on the quark-initiated case.

[2] For simplicity we only show the momenta labels in function F_{LM}. For the very same reason we do not show the momenta labels of the electrons in function F_{LM} since they are not relevant for our discussion. Moreover we further simplify the notation during the discussion of the quark channel by not writing the subscripts defining the parton type. The bar in the argument of function $F_{\mathrm{LM}}(\cdot\,|\,\cdot)$ separates momenta of partonic emissions that develop singularities and momenta from partons present in the hard process.

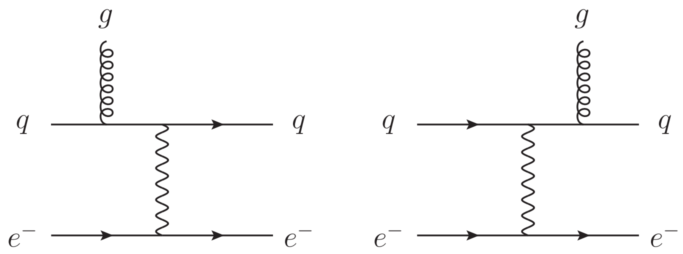


Fig. 5.1.: Feynman diagrams describing the single-real emission contribution to the quark initiated channel $q(p_1) + e^-(p_2) \to q(p_3) + e^-(p_4) + g(p_5)$ of deep-inelastic scattering in the NLO QCD computation.

shown in Fig. 5.1 and $\hat{\mathcal{O}}$ is an arbitrary infrared safe observable. Similar to the LO case, the notation $\langle F_{\mathrm{LM}}(1,4\,|\,5)\rangle_\delta$ indicates that the corresponding cross section is fully-differential with respect to partonic momenta that are shown as arguments of the function F_{LM}. We will proceed with the discussion of how infrared and collinear singularities can be extracted from the function $F_{\mathrm{LM}}(1,4\,|\,5)$ without integration over resolved phase space. This construction will provide subtraction terms for the real emission process.

5.1. Subtractions

To obtain the subtraction terms, we perform the iterative subtraction of soft and collinear singularities. We note that, since both singularities, soft and collinear, factorize in either the soft or a collinear limit, we can chose the order in which we regulate them. We begin by regulating soft singularities and introduce an operator S_5 that extracts the leading $E_5 \to 0$ singularity by acting on the function $F_{\mathrm{LM}}(1,4\,|\,5)$. Its action is defined as

$$\begin{aligned} S_5\, F_{\mathrm{LM}}(1,4\,|\,5) = {} & S_5 \bigg(\mathcal{N} \int [\mathrm{d}p_3][\mathrm{d}p_4] \\ & \times (2\pi)^d \delta^{(d)}(p_1 + p_2 - p_3 - p_4 - p_5)\, \hat{\mathcal{O}}(p_3, p_4, p_5)\, |M^{\mathrm{tree}}_{\mathrm{nlo}}(p_1, p_2, p_3, p_4, p_5)|^2 \bigg) \\ \equiv {} & \mathcal{N} \int [\mathrm{d}p_3][\mathrm{d}p_4]\, (2\pi)^d \delta^{(d)}\, (p_1 + p_2 - p_3 - p_4)\, \hat{\mathcal{O}}(p_3, p_4) \\ & \times \frac{1}{E_5^2} \lim_{E_5 \to 0} \Big[E_5^2\, |M^{\mathrm{tree}}_{\mathrm{nlo}}(p_1, p_2, p_3, p_4, p_5)|^2 \Big] . \end{aligned} \tag{5.5}$$

As we discussed earlier, in the soft limit singularities factorize from the leading-order matrix element. For the NLO QCD DIS matrix element squared we obtain [41]

$$\begin{aligned} & \frac{1}{E_5^2} \lim_{E_5 \to 0} \Big[E_5^2\, |M^{\mathrm{tree}}_{\mathrm{nlo}}(p_1, p_2, p_3, p_4, p_5)|^2 \Big] \\ & = 2C_F\, g_{s,b}^2 \times \frac{p_1 \cdot p_4}{(p_1 \cdot p_5)(p_4 \cdot p_5)} \times |M^{\mathrm{tree}}_{\mathrm{lo}}(p_1, p_2, p_3, p_4)|^2 , \end{aligned} \tag{5.6}$$

where $g_{s,b}$ is the bare QCD coupling and $C_F = 4/3$ is the colour factor. $M_{\rm lo}^{\rm tree}$ is the leading order amplitude introduced in Eq. (4.5). Inserting the limit Eq. (5.6) into Eq. (5.5) we obtain

$$S_5\, F_{\rm LM}(1,4\,|\,5) = 2C_F\, g_{s,b}^2 \times \frac{p_1 \cdot p_4}{(p_1 \cdot p_5)(p_4 \cdot p_5)} \times F_{\rm LM}(1,4)\,, \tag{5.7}$$

where $F_{\rm LM}(1,4)$ is the leading order differential cross section defined in Eq. (4.5).

According to Eq. (5.5) soft gluons factorize from the matrix element, the infrared safe observable and the energy-momentum conserving δ-function. To extract the soft singularity, we insert the identity operator $I = [I - S_5] + S_5$ into the phase space and obtain

$$\big\langle F_{\rm LM}(1,4\,|\,5)\big\rangle = \big\langle \big[I - S_5\big] F_{\rm LM}(1,4\,|\,5)\big\rangle + \big\langle S_5 F_{\rm LM}(1,4\,|\,5)\big\rangle\,. \tag{5.8}$$

In the first term on the right-hand side of Eq. (5.8) the soft singularity is regulated. In the second term on the right-hand side (subtraction term) we only need the function $F_{\rm LM}(1,4\,|\,5)$ in the soft limit Eq. (5.7). Since the soft gluon completely decouples from the hard process, we can analytically integrate over the phase space of the emitted gluons and compute the $1/\epsilon$ poles *independent* of the hard process. Note that, since the energy of the soft gluon is not bounded by energy conservation anymore, integration over E_5 extends up to $E_{\rm max}$ introduced in Eq. (4.6). Since the left-hand side of Eq. (5.8) is $E_{\rm max}$-independent, the explicit $E_{\rm max}$-dependence in the analytic subtraction term needs to cancel with an implicit dependence on $E_{\rm max}$ in the regulated term through the definition of the phase space; independence of the full result on $E_{\rm max}$ provides a useful check on the implementation of the subtraction scheme.

The soft-regulated term in Eq. (5.8) contains unregulated collinear singularities. We will now discuss how to regularize them. In the collinear limits, two different singular configurations exist if $\vec{p}_5 \parallel \vec{p}_1$ or $\vec{p}_5 \parallel \vec{p}_4$. We would like to deal with one collinear singularity at a time. To this end we introduce partition functions

$$1 = w^{51} + w^{54}\,. \tag{5.9}$$

The explicit form of the partition functions w^{5i} in Eq. (5.9) is not relevant as long as they possess the following property

$$C_{5i}\, w^{5j} = \delta_{ij}\,, \quad \text{for} \quad i,j \in \{1,4\}\,, \tag{5.10}$$

where C_{5i} are collinear operators that extract leading singularities in the collinear limits $\vec{p}_5 \parallel \vec{p}_1$ and $\vec{p}_5 \parallel \vec{p}_4$, respectively. These operators are introduced in analogy to the soft operator S_5 in Eq. (5.5).[3] The property Eq. (5.10) ensures that partition functions vanish in certain collinear limits and, therefore, cancel corresponding singularities. For instance the product $w^{51} F_{\rm LM}(1,4\,|\,5)$ is finite in the collinear $\vec{p}_5 \parallel \vec{p}_4$ limit. A possible choice of the partition functions

[3] Explicit formulas for the action of C_{5i}, with $i \in \{1,4\}$, on $F_{\rm LM}$ are given in the analytic computation in Section 5.2 and in Appendix B.

in Eq. (5.9) is

$$w^{51} = \frac{\rho_{54}}{\rho_{51}+\rho_{54}}\,, \quad w^{54} = \frac{\rho_{51}}{\rho_{51}+\rho_{54}}\,, \quad \text{with} \quad \rho_{5i} = 1 - \vec{n}_5 \cdot \vec{n}_i\,. \tag{5.11}$$

In Eq. (5.11) $\vec{n}_i$ are unit vectors that describe directions of momenta of the corresponding partons.

To discuss collinear regularization, we go back to Eq. (5.8). The first term on the right-hand side of Eq. (5.8) is regulated in the soft limit but still contains collinear singularities. We use Eq. (5.9) to rewrite it as

$$\left\langle \left[I - S_5\right] F_{\rm LM}(1,4\,|\,5) \right\rangle = \left\langle \left[I - S_5\right] w^{51}\, F_{\rm LM}(1,4\,|\,5) \right\rangle + \left\langle \left[I - S_5\right] w^{54}\, F_{\rm LM}(1,4\,|\,5) \right\rangle, \tag{5.12}$$

where, thanks to Eq. (5.10), the first term only contains the collinear singularity where $\vec{p}_5 \parallel \vec{p}_1$ and the second one the singularity where $\vec{p}_5 \parallel \vec{p}_4$. As an example, we consider the first term in Eq. (5.12) with the partition w^{51}. Using the collinear operator C_{51} we introduce yet another partition of unity $I = \left[I - C_{51}\right] + C_{51}$ and obtain

$$\begin{aligned} &\left\langle \left[I - S_5\right] w^{51} F_{\rm LM}(1,4\,|\,5) \right\rangle \\ &= \left\langle \left[I - C_{51}\right]\left[I - S_5\right] w^{51} F_{\rm LM}(1,4\,|\,5) \right\rangle_\delta + \left\langle C_{51}\left[I - S_5\right] w^{51} F_{\rm LM}(1,4\,|\,5) \right\rangle. \end{aligned} \tag{5.13}$$

The subscript δ in the first term on the right-hand side of Eq. (5.13) indicates that this contribution is now fully regulated and contains no singularities and is fully differential with respect to momenta p_4 and p_5. Repeating similar steps for the second partition w^{54} we arrive at the final result

$$\begin{aligned} \left\langle F_{\rm LM}(1,4\,|\,5) \right\rangle_\delta = \left\langle S_5 F_{\rm LM}(1,4\,|\,5) \right\rangle_\delta &+ \sum_{i\in\{1,4\}} \left\langle C_{5i}[1 - S_5] F_{\rm LM}(1,4\,|\,5) \right\rangle_\delta \\ &+ \sum_{i\in\{1,4\}} \langle \hat{O}^{(i)}_{\rm nlo} w^{5i} F_{\rm LM}(1,4\,|\,5) \rangle_\delta\,, \end{aligned} \tag{5.14}$$

where we introduced the notation

$$\hat{O}^{(i)}_{\rm nlo} \equiv [I - C_{5i}][I - S_5]\,. \tag{5.15}$$

The third term on the right-hand side of Eq. (5.14) is finite in four dimensions making it amenable to numerical calculation.[4] We want to emphasize that actions of all operators in Eq. (5.15) on the function $F_{\rm LM}$ are well-defined. Results can be found in Appendix B and are also given explicitly in the following discussion. The first and the second terms on the right-hand side of Eq. (5.14) are not finite in four dimensions and we continue with their computation.

[4]We demonstrate in Chapter 8 how to numerically evaluate such a formula.

5.2. Analytic integration of the subtraction terms

We now explain how to analytically integrate the subtraction terms on the right hand side of Eq. (5.14) and extract the $1/\epsilon$ poles explicitly.

5.2.1. Soft subtraction term

We begin with the soft subtraction term $\langle S_5 F_{\rm LM}(1,4\,|\,5)\rangle$. It contains both soft and collinear singularities that lead to a $1/\epsilon^2$ pole upon integration over gluon momentum p_5. The necessary limit has already been given in Eq. (5.7). As we mentioned there, the soft gluon $g(p_5)$ decouples from the function $F_{\rm LM}$ and we can integrate over its phase space without any reference to the hard matrix element. We find

$$\int [\mathrm{d}p_5]\,\frac{p_1\cdot p_4}{(p_1\cdot p_5)(p_4\cdot p_5)} = \int\limits_0^{E_{\rm max}} \frac{\mathrm{d}E_5}{E_5^{1+2\epsilon}} \times \int \frac{\mathrm{d}\Omega_5^{(d-1)}}{2(2\pi)^{d-1}}\,\frac{\rho_{14}}{\rho_{15}\,\rho_{45}}\,, \tag{5.16}$$

with $\rho_{ij} = 1-\vec{n}_i\cdot\vec{n}_j$ where we recall that $\vec{n}_{i,j}$ are unit vectors that describe directions of parton momenta p_i and p_j (so that $p_i\cdot p_j = E_iE_j\rho_{ij}$). The solid angle integral on the right-hand side of Eq. (5.16) is given by [36][5]

$$\int \frac{\mathrm{d}\Omega_i^{(d-1)}}{2(2\pi)^{d-1}}\,\frac{\rho_{ij}}{\rho_{ik}\,\rho_{jk}} = -\frac{2^{1-2\epsilon}}{\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]\eta_{ij}^{-\epsilon}K_{ij}\,, \tag{5.17}$$

where

$$K_{ij} = \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\eta_{ij}^{1+\epsilon}\,{}_2\mathrm{F}_1(1,1;1-\epsilon;1-\eta_{ij})\,. \tag{5.18}$$

In Eqs. (5.17, 5.18) $\eta_{ij} = \rho_{ij}/2$ and ${}_2\mathrm{F}_1$ is the Gauss hypergeometric function. An expansion of the function K_{ij} in the dimensional regularization parameter ϵ can be found in Eq. (A.22). The energy integral on the right-hand side of Eq. (5.16) evaluates to

$$\int\limits_0^{E_{\rm max}} \frac{\mathrm{d}E_5}{E_5^{1+2\epsilon}} = -\frac{1}{2\epsilon}E_{\rm max}^{-2\epsilon}\,. \tag{5.19}$$

Combining these, we obtain

$$\int [\mathrm{d}p_5]\,\frac{\rho_{14}}{E_5^2\,\rho_{15}\,\rho_{45}} = \frac{1}{\epsilon^2}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right](2E_{\rm max})^{-2\epsilon}\eta_{14}^{-\epsilon}K_{14}\,. \tag{5.20}$$

We combine Eqs. (5.7, 5.20) and write the soft subtraction term as

$$\langle S_5 F_{\rm LM}(1,4\,|\,5)\rangle = 2C_F\,\frac{[\alpha_{s,b}]}{\epsilon^2}\,(2E_{\rm max})^{-2\epsilon}\langle \eta_{14}^{-\epsilon}K_{14}F_{\rm LM}(1,4)\rangle_\delta\,. \tag{5.21}$$

[5] We discuss how to compute such a integral in Appendix G.3.

In Eq. (5.21) we introduced the notation

$$[\alpha_{s,b}] \equiv \left[\frac{g_{s,b}^2}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right], \tag{5.22}$$

that provides a convenient starting point for a transition to the $\overline{\text{MS}}$ coupling constant. Indeed, to leading order in α_s, the following relation holds[6]

$$[\alpha_{s,b}] = [\alpha_s]\,[1+\mathcal{O}(\alpha_s)]\,\mu^{2\epsilon}, \tag{5.23}$$

where the quantity $[\alpha_s]$ is defined through the renormalized coupling constant $\alpha_s(\mu)$

$$[\alpha_s] \equiv \left[\frac{\alpha_s(\mu)}{2\pi}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\right]. \tag{5.24}$$

The final result for the soft subtraction term reads

$$\left\langle S_5 F_{\text{LM}}(1,4\,|\,5)\right\rangle = 2C_F\,\frac{[\alpha_s]}{\epsilon^2}\left(\frac{4E_{\text{max}}^2}{\mu^2}\right)^{-\epsilon}\left\langle \eta_{14}^{-\epsilon} K_{14} F_{\text{LM}}(1,4)\right\rangle_\delta. \tag{5.25}$$

We note that higher-orders of α_s in Eq. (5.23) contribute (at least) to the NNLO correction of the partonic cross section and are not included in Eq. (5.25). These contributions reappear in the later NNLO discussion in the UV-renormalized virtual corrections.

5.2.2. Soft-regulated collinear subtraction term: initial state emission

As the next step, we study the soft-regulated collinear subtraction term

$$\left\langle C_{51}[1-S_5]F_{\text{LM}}(1,4\,|\,5)\right\rangle, \tag{5.26}$$

where the gluon momentum is taken in the collinear $\vec{p}_5 \parallel \vec{p}_1$ limit. In this subtraction term the soft singularity is already regulated. We therefore expect the highest pole to be $1/\epsilon$. Eq. (5.26) has two contributions: one where the function F_{LM} is taken in the collinear limit and another one where it is taken in the soft-collinear limit. It is convenient to consider the two contributions separately.

The singular soft-collinear limit can be easily obtained by considering Eq. (5.7) in the collinear $\vec{p}_5 \parallel \vec{p}_1$ limit. We obtain

$$C_{51}S_5F_{\text{LM}}(1,4\,|\,5) = \lim_{p_5\parallel p_1} S_5F_{\text{LM}}(1,4\,|\,5) \overset{(5.7)}{=} 2C_F\,g_{s,b}^2\,\times\frac{1}{E_5^2\,\rho_{15}}\times F_{\text{LM}}(1,4)\,. \tag{5.27}$$

The analytic integration can be done in full analogy to the calculation of the soft subtraction

[6] Further details on the UV renormalization can be found in App. A.

term discussed earlier. We find

$$\int \frac{\mathrm{d}\Omega_5^{d-1}}{2(2\pi)^{d-1}} \frac{1}{\rho_{5j}} = -\frac{2^{-2\epsilon}}{\epsilon} \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right] \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right] . \tag{5.28}$$

We use the above result to obtain the integrated soft-collinear subtraction term

$$\left\langle C_{51} S_5 F_{\mathrm{LM}}(1,4\,|\,5)\right\rangle = 2C_F \frac{[\alpha_{s,b}]}{2\epsilon^2} \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right] (2E_{\mathrm{max}})^{-2\epsilon} \left\langle F_{\mathrm{LM}}(1,4)\right\rangle_\delta . \tag{5.29}$$

We now consider the collinear subtraction term $\left\langle C_{51} F_{\mathrm{LM}}(1,4\,|\,5)\right\rangle$. Acting with C_{51} on the function F_{LM}, we obtain

$$C_{51} F_{\mathrm{LM}}(1,4\,|\,5) = g_{s,b}^2 \times \frac{1}{p_1 \cdot p_5} P_{qq}(z) \times \frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z} , \tag{5.30}$$

where

$$P_{qq}(z) = C_F \left[\frac{1+z^2}{1-z} - \epsilon(1-z)\right] , \tag{5.31}$$

is the splitting function describing a collinear splitting $q \to q^\star + g$. In the limit Eq. (5.29) the notation $F_{\mathrm{LM}}(z\cdot 1,4)$ stands for the leading order cross section where the initial-state momentum p_1 is rescaled by $z = (E_1 - E_5)/E_1$. The integration over the phase space of the gluon $g(p_5)$ reads then

$$\begin{aligned} &\int [\mathrm{d}p_5]\, \frac{1}{p_1 \cdot p_5} P_{qq}(z)\, \frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z} \\ &= \int \frac{\mathrm{d}\Omega_5^{(d-1)}}{2(2\pi)^{d-1}}\, \frac{1}{\rho_{15}} \times \int\limits_0^{E_{\mathrm{max}}} \mathrm{d}E_5\, E_5^{1-2\epsilon}\, \frac{1}{E_1 E_5} P_{qq}(z)\, \frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z} . \end{aligned} \tag{5.32}$$

The required angular integral is found in Eq. (5.28). The integration over E_5 can be simplified. We use $z = (E_1 - E_5)/E_1$ to write E_5 as $E_5 = (1-z)E_1$ and obtain

$$\int\limits_0^{E_{\mathrm{max}}} \mathrm{d}E_5\, E_5^{1-2\epsilon} \quad \to \quad \int\limits_{z_{\mathrm{min}}}^1 \mathrm{d}z\, E_1^{2-2\epsilon} (1-z)^{1-2\epsilon} , \tag{5.33}$$

where $z_{\mathrm{min}} = 1 - E_{\mathrm{max}}/E_1$. Putting everything together we obtain the following result

$$\begin{aligned} &\left\langle C_{51} F_{\mathrm{LM}}(1,4\,|\,5)\right\rangle \\ &= -\frac{[\alpha_{s,b}]}{\epsilon} \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right] (2E_1)^{-2\epsilon} \int\limits_{z_{\mathrm{min}}}^1 \mathrm{d}z\, (1-z)^{-2\epsilon} P_{qq}(z) \left\langle \frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z} \right\rangle_\delta . \end{aligned} \tag{5.34}$$

We note that, by construction, $E_{\mathrm{max}} \geq E_1$, so that $z_{\mathrm{min}} \leq 0$. For values $z \leq 0$ there is not enough energy to produce final state particles. This implies that the integrand in Eq. (5.34) vanishes for $z < z_{\min}$ because of the implicit energy-momentum conserving δ-function in $F_{\rm LM}(z \cdot 1, 4)$. We can therefore replace the lower integration boundary $z_{\min}$ in Eq. (5.34) with zero without changing the integral. Combining Eqs. (5.29, 5.34) we write

$$\begin{aligned}\left\langle C_{51}\left[I - S_5\right] F_{\rm LM}(1,4\,|\,5)\right\rangle &= -2C_F\,\frac{[\alpha_s]}{2\epsilon^2}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left(\frac{4E_{\max}^2}{\mu^2}\right)^{-\epsilon}\left\langle F_{\rm LM}(1,4)\right\rangle_\delta \\ &- \frac{[\alpha_s]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\int_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon} P_{qq}(z)\left\langle\frac{F_{\rm LM}(z\cdot 1,4)}{z}\right\rangle_\delta .\end{aligned} \tag{5.35}$$

In Eq. (5.35) we also expressed the coupling constant through the coupling constant renormalized in the $\overline{\rm MS}$ scheme using Eq. (5.24).

We now consider the soft $z \to 1$ singularity that is present in $P_{qq}(z)$ in Eq. (5.35). In the subtraction term Eq. (5.26) this soft singularity is regulated, so that the corresponding pole has to cancel between collinear contributions to Eq. (5.35), given in Eq. (5.34), and soft-collinear contributions, which are shown in Eq. (5.29). To extract the pole explicitly we divide the splitting function Eq. (5.31) into a singular and a non-singular term. We find

$$\begin{aligned}(1-z)^{-2\epsilon}P_{qq}(z) &= (1-z)^{-2\epsilon}C_F\left[\frac{1+z^2}{1-z} - \epsilon(1-z)\right] \\ &= 2C_F\,\frac{(1-z)^{-2\epsilon}}{1-z} + (1-z)^{-2\epsilon}P_{qq,\rm reg}(z)\,,\end{aligned} \tag{5.36}$$

where

$$P_{qq,\rm reg}(z) = -C_F\left[(1+z) + \epsilon(1-z)\right]. \tag{5.37}$$

The singular term in Eq. (5.36) is regulated using the plus prescription defined as

$$\int_0^1 \mathrm{d}x\,[f(x)]_+ \cdot g(x) \equiv \int_0^1 \mathrm{d}x\, f(x)[g(x) - g(1)]\,. \tag{5.38}$$

This allows us to rewrite the z integration in Eq. (5.34) as

$$\begin{aligned}&\int_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon}P_{qq}(z)\,\frac{F_{\rm LM}(z\cdot 1,4)}{z} \\ &= 2C_F\int_0^1 \mathrm{d}z\left[\frac{(1-z)^{-2\epsilon}}{1-z}\right]_+\frac{F_{\rm LM}(z\cdot 1,4)}{z} + 2C_F\underbrace{\int_0^1 \mathrm{d}z\,(1-z)^{-1-2\epsilon}}_{=-\frac{1}{2\epsilon}}\,F_{\rm LM}(1,4) \\ &+ \int_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon}P_{qq,\rm reg}(z)\,\frac{F_{\rm LM}(z\cdot 1,4)}{z}\,.\end{aligned} \tag{5.39}$$

In the second term on the right-hand side of Eq. (5.39) the $1/\epsilon$ pole corresponding to the soft singularity is explicit. The remaining z integrals are finite. We use Eq. (5.39) in Eq. (5.35) to obtain the fully-regulated collinear subtraction term. We find the following result

$$
\begin{aligned}
\left\langle C_{51}\left[I-S_5\right]F_{\mathrm{LM}}(1,4\,|\,5)\right\rangle = &-\frac{[\alpha_s]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon} \\
&\times\int_0^1 \mathrm{d}z\,\left(2C_F\left[\frac{(1-z)^{-2\epsilon}}{1-z}\right]_+ +(1-z)^{-2\epsilon}P_{qq,\mathrm{reg}}(z)\right)\left\langle\frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\right\rangle_\delta \\
&-2C_F\,\frac{[\alpha_s]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left[\frac{(4E_{\max}^2/\mu^2)^{-\epsilon}-(4E_1^2/\mu^2)^{-\epsilon}}{2\epsilon}\right]\left\langle F_{\mathrm{LM}}(1,4)\right\rangle_\delta .
\end{aligned}
\tag{5.40}
$$

Note that in Eq. (5.40) the $1/\epsilon^2$ pole has canceled between soft-collinear and collinear contributions in Eqs. (5.29, 5.35).

5.2.3. Soft-regulated collinear subtraction term: final state emission

We now study the soft regulated collinear subtraction term

$$
\left\langle C_{54}[1-S_5]F_{\mathrm{LM}}(1,4\,|\,5)\right\rangle , \tag{5.41}
$$

where the gluon momentum is taken in the collinear $\vec{p}_5 \parallel \vec{p}_4$ limit. We consider the two terms in Eq. (5.41) separately. Apart from the replacement $p_1 \to p_4$ where appropriate, the soft-collinear limit is identical to the previously discussed case of the initial-state emission, c.f. Eq. (5.27). Hence, the result for this contribution can be taken directly from Eq. (5.29). We find

$$
\left\langle C_{54}S_5F_{\mathrm{LM}}(1,4\,|\,5)\right\rangle = 2C_F\,\frac{[\alpha_{s,b}]}{2\epsilon^2}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right](2E_{\max})^{-2\epsilon}\left\langle F_{\mathrm{LM}}(1,4)\right\rangle_\delta . \tag{5.42}
$$

However, the collinear term in Eq. (5.41) is different. Indeed, the collinear limit reads

$$
C_{54}F_{\mathrm{LM}}(1,4\,|\,5) = g_{s,b}^2\times\frac{1}{p_4\cdot p_5}\,P_{qq}(z)\times F_{\mathrm{LM}}\left(1,\frac{1}{z}\cdot 4\right), \tag{5.43}
$$

where $P_{qq}(z)$ is given in Eq. (5.31). In the C_{54} limit Eq. (5.43) the notation $F_{\mathrm{LM}}(1,z^{-1}\cdot 4)$ stands for the leading order cross section where the energy of the final-state momentum p_4 is rescaled with $1/z$, where $z = E_4/(E_4+E_5)$. The integral over the phase space of the unresolved gluon reads

$$
\begin{aligned}
&\int [\mathrm{d}p_5]\,\frac{1}{p_4\cdot p_5}\,P_{qq}(z)\;F_{\mathrm{LM}}\left(1,\frac{1}{z}\cdot 4\right) \\
&=\int\frac{\mathrm{d}\Omega_5^{(d-1)}}{2(2\pi)^{d-1}}\,\frac{1}{\rho_{45}}\times\int_0^{E_{\max}}\frac{\mathrm{d}E_5}{E_4}\,E_5^{-2\epsilon}\,P_{qq}(z)\;F_{\mathrm{LM}}\left(1,\frac{1}{z}\cdot 4\right).
\end{aligned}
\tag{5.44}
$$

The required angular integral is similar to the previous case; it can be taken from Eq. (5.28). To compute the energy integral we use $z = E_4/(E_4+E_5)$ and write the gluon energy as

$$E_5 = E_4 \left(\frac{1-z}{z} \right) , \tag{5.45}$$

so that

$$\int_0^{E_{\max}} \frac{\mathrm{d}E_5}{E_4} \quad \to \quad \int_{z_{\min}}^{1} \frac{\mathrm{d}z}{z^2} , \quad z_{\min} = \frac{E_4}{E_4 + E_{\max}} . \tag{5.46}$$

Since $E_{\max} > 0$ we find for the lower integration bound $0 < z_{\min} < 1$. For values $z \in [0, z_{\min}]$ the energy of the outgoing quark is given by

$$E_{\text{out}} = \frac{1}{z} \cdot E_4 > \frac{1}{z_{\min}} \cdot E_4 = E_4 + E_{\max} \geq E_{\max} . \tag{5.47}$$

By construction, $E_{\max}$ is greater than the partonic center-of-mass energy of the collision, this implies that for values $z \in [0, z_{\min}]$ the integrand in Eq. (5.44) vanishes because of the energy-momentum conserving δ-function in $F_{\text{LM}}(1, z^{-1} \cdot 4)$. We can therefore set the lower integration boundary to zero without affecting the value of the integral. We further use the fact that we need to integrate over the phase space of the final-state quark $q(p_4)$ to absorb the factor $1/z$. We find

$$p_4 \to z \cdot p_4 \quad \Rightarrow \quad \int \frac{\mathrm{d}^{(d-1)} p_4}{(2\pi)^{(d-1)} 2E_4} \quad \to \quad z^{2-2\epsilon} \int \frac{\mathrm{d}^{(d-1)} p_4}{(2\pi)^{(d-1)} 2E_4} . \tag{5.48}$$

Putting everything together, we rewrite the energy integral in Eq. (5.44) as

$$\begin{aligned} &\int_0^{E_{\max}} \frac{\mathrm{d}E_5}{E_4} \, E_5^{-2\epsilon} \, P_{qq}(z) \, F_{\text{LM}}\left(1, \frac{1}{z} \cdot p_4\right) \\ &\quad \to \quad E_4^{-2\epsilon} \underbrace{\int_0^1 \mathrm{d}z \, z^{-2\epsilon} (1-z)^{-2\epsilon} \, P_{qq}(z)}_{= -\left[\frac{2C_F}{2\epsilon} + \gamma_{qq}^{22}\right]} \, F_{\text{LM}}(1, 4) . \end{aligned} \tag{5.49}$$

The anomalous dimension γ_{qq}^{22} that we introduced in Eq. (5.49) is a particular case of the following class of constants

$$\gamma_{qq}^{nk} \equiv - \int_0^1 \mathrm{d}z \left[z^{-n\epsilon} (1-z)^{-k\epsilon} P_{qq}(z) - 2C_F \frac{(1-z)^{-k\epsilon}}{1-z} \right] , \tag{5.50}$$

that we will use in what follows. An expansion of the constant γ_{qq}^{nk} in the dimensional regularization parameter ϵ can be found in Appendix E.6. Putting Eq. (5.44) and Eq. (5.49) together, we obtain the following result

$$\left\langle C_{54}F_{\rm LM}(1,4\,|\,5)\right\rangle = \frac{[\alpha_{s,b}]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left[\frac{2C_F}{2\epsilon}+\gamma_{qq}^{22}\right]\left\langle (2E_4)^{-2\epsilon}F_{\rm LM}(1,4)\right\rangle_\delta . \tag{5.51}$$

Finally, we combine Eq. (5.51) and the soft-collinear contribution Eq. (5.42) to obtain the soft-regulated collinear subtraction term

$$\begin{aligned}
&\left\langle C_{54}[1-S_5]F_{\rm LM}(1,4\,|\,5)\right\rangle \\
&= \frac{[\alpha_s]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\gamma_{qq}^{22}\left\langle\left(\frac{4E_4^2}{\mu^2}\right)^{-\epsilon}F_{\rm LM}(1,4)\right\rangle_\delta \\
&\quad - 2C_F\,\frac{[\alpha_s]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left\langle\frac{(4E_{\rm max}^2/\mu^2)^{-\epsilon}-(4E_4^2/\mu^2)^{-\epsilon}}{2\epsilon}\,F_{\rm LM}(1,4)\right\rangle_\delta .
\end{aligned} \tag{5.52}$$

Note that, as expected, the $1/\epsilon^2$ pole from the soft-collinear limit Eq. (5.42) cancels against the $1/\epsilon^2$ pole in Eq. (5.51), so that Eq. (5.52) is of order $1/\epsilon$.

We have computed all subtraction terms that appear in Eq. (5.14). We have therefore shown how the real emission term in Eq. (5.1) can be written as the sum of a regulated (finite) term that can be numerically integrated in four dimensions, and subtraction counterterms which have explicit poles in $1/\epsilon$. This concludes our discussion of the real emission contribution and we continue with the discussion of virtual corrections and collinear renormalization contributions to the differential cross section Eq. (5.1).

5.3. Virtual contribution and collinear renormalization

We now turn to the remaining terms in Eq. (5.1), beginning with $\mathrm{d}\hat{\sigma}_{\rm V}$. The pole structure of the virtual corrections to the DIS process can be obtained from general formulas given in references [51–53]; explicit calculation of the one-loop corrections is not required. For later reference we define

$$2s\cdot \mathrm{d}\hat{\sigma}_{\rm V} \equiv \int F_{\rm LV}(1_q,4_q) \equiv \left\langle F_{\rm LV}(1_q,4_q)\right\rangle_\delta , \tag{5.53}$$

with

$$\begin{aligned}
F_{\rm LV}(1_q,4_q) = \mathcal{N}\int [\mathrm{d}p_3][\mathrm{d}p_4]\,(2\pi)^d\delta^{(d)}\left(p_1+p_2-p_3-p_4\right) \\
\times 2\Re\left(M_{\rm lo}^{\rm tree\star}\cdot M_{\rm nlo}^{\rm 1\text{-}loop}\right)(p_1,p_2,p_3,p_4)\times \hat{\mathcal{O}}(p_3,p_4) .
\end{aligned} \tag{5.54}$$

Eq. (5.54) is defined in accordance with Eq. (4.5) but it now contains the 1-loop amplitude $M_{\rm nlo}^{\rm 1\text{-}loop}(p_1,p_2,p_3,p_4)$ that corresponds to the Feynman diagram shown in Fig. 5.2. According

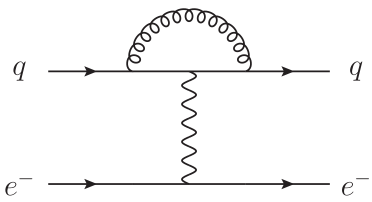


Fig. 5.2.: Feynman diagram describing virtual corrections to the quark initiated channel $q(p_1) + e^-(p_2) \to q(p_3) + e^-(p_4)$ of deep-inelastic scattering in the NLO QCD computation.

to references [51–53] we can write the singular structure of the function $F_{\rm LV}(1,4)$ as

$$F_{\rm LV}(1_q, 4_q) = \frac{\alpha_s(\mu)}{2\pi}\, 2 I_1(\epsilon)\, F_{\rm LM}(1,4) + F_{\rm LV}^{\rm fin}(1_q,4_q)\,, \tag{5.55}$$

where $I_1(\epsilon)$ reads

$$I_1(\epsilon) \equiv -C_F\, \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon}\right] \left(\frac{2p_1\cdot p_4}{\mu^2}\right)^{-\epsilon}. \tag{5.56}$$

The part $F_{\rm LV}^{\rm fin}(1,4)$ is finite and requires an explicit calculation of the quark form factor; the result is well-known, see e.g. Ref. [73].[7] We use Eqs. (5.55, 5.56) and write $\mathrm{d}\hat{\sigma}_{\rm V}$ as

$$2s \cdot \mathrm{d}\hat{\sigma}_{\rm V} = -2C_F\, \frac{[\alpha_s]}{\epsilon} \times \left[\frac{1}{\epsilon} + \frac{3}{2}\right] \left\langle \left(\frac{2p_1\cdot p_4}{\mu^2}\right)^{-\epsilon} F_{\rm LM}(1_q,4_q) \right\rangle_\delta + \left\langle F_{\rm LV}^{\rm fin}(1_q,4_q)\right\rangle_\delta . \tag{5.58}$$

We now move on to the final term of Eq. (5.1), $\mathrm{d}\hat{\sigma}_{\rm pdf}$, which is the collinear renormalization contribution to the cross section. Parton distribution functions $f_{i,b}$ in Eq. (4.2) are bare quantities that need to be renormalized. This is done with the help of the following equation

$$f_{i,b} \to \left[\delta_{ij} + \frac{\alpha_s(\mu)}{2\pi}\hat{P}_{ij}^{(0)} + \mathcal{O}(\alpha_s^2)\right] \otimes f_j(\mu)\,. \tag{5.59}$$

In Eq. (5.59) $\hat{P}_{ij}^{(0)}$ are the leading order Altarelli-Parisi splitting functions [74]; they are collected in Appendix E.4. The symbol $\otimes$ in Eq. (5.59) stands for the convolution

$$[f_1 \otimes f_2](z) \equiv \int\limits_0^1 \mathrm{d}x\, \mathrm{d}y\, f_1(x) f_2(y) \delta(z - xy)\,. \tag{5.60}$$

We insert Eq. (5.59) into Eq. (4.2) and rewrite it by separating the convolution with parton

[7] The finite part is not relevant for our discussion of IR poles. However, for completeness, we give it below

$$\left\langle F_{\rm LV}^{\rm fin}(1_q,4_q)\right\rangle_\delta = -\frac{\alpha_s(\mu)}{2\pi} \times 8C_F \times \left\langle F_{\rm LM}(1_q,4_q)\right\rangle_\delta . \tag{5.57}$$

distribution functions. To order $\mathcal{O}(\alpha_s)$ we obtain

$$\mathrm{d}\sigma_{\mathrm{H}} = \sum_i \int_0^1 \mathrm{d}x\, f_i(\mu;x)\, \mathrm{d}\hat{\sigma}_i + \sum_i \int_0^1 \mathrm{d}x\, f_i(\mu;x) \left(\frac{\alpha_s(\mu)}{2\pi\epsilon} \int_0^1 \mathrm{d}z\, \hat{P}_{ij}^{(0)}(z)\, \mathrm{d}\hat{\sigma}_j(z) \right) + \mathcal{O}(\alpha_s^2)\,, \tag{5.61}$$

where $\mathrm{d}\hat{\sigma}_i$ are the partonic cross sections that *only* contain virtual and real contributions. It follows from Eq. (5.61) that the contribution to the NLO cross section that arises because of renormalization of parton distribution functions reads

$$\mathrm{d}\hat{\sigma}_{\mathrm{pdf}} = \frac{\alpha_s(\mu)}{2\pi\epsilon} \int_0^1 \mathrm{d}z\, \hat{P}_{qq}^{(0)}(z)\, \mathrm{d}\hat{\sigma}_q^{\mathrm{lo}}(z)\,. \tag{5.62}$$

We rewrite Eq. (5.62) in terms of the function $F_{\mathrm{LM}}(1,4)$ and obtain

$$2s \cdot \mathrm{d}\hat{\sigma}_{\mathrm{pdf}} = \frac{\alpha_s(\mu)}{2\pi\epsilon} \int_0^1 \mathrm{d}z\, \hat{P}_{qq}^{(0)}(z) \left\langle \frac{F_{\mathrm{LM}}(z\cdot 1_q, 4_q)}{z} \right\rangle_\delta . \tag{5.63}$$

Note that the factor $1/z$ arises because of the flux factor in the definition of a cross section. Note also that the only other contribution that includes the boosted matrix element $F_{\mathrm{LM}}(z\cdot 1,4)$ and that, therefore, can cancel the pole from the collinear renormalization contribution Eq. (5.63) is in the collinear subtraction term Eq. (5.40). To further illustrate their similarity we note that

$$2C_F \left[\frac{(1-z)^{-2\epsilon}}{1-z} \right]_+ + (1-z)^{-2\epsilon} P_{qq,\mathrm{reg}}(z) = \hat{P}_{qq}^{(0)}(z) - \gamma_q\, \delta(1-z) + \mathcal{O}(\epsilon)\,, \tag{5.64}$$

where $\gamma_q = (3/2)C_F$ is the LO quark cusp anomalous dimension. Using Eq. (5.64) in the first term on the right-hand side of Eq. (5.40) one can see immediately that the $1/\epsilon$ pole of the collinear subtraction term cancels the collinear renormalization contribution Eq. (5.63).

5.4. Pole cancellation and the finite remainder

At NLO results are simple enough to explicitly demonstrate cancellation of $1/\epsilon$ poles and to derive a finite formula for the fully differential cross section of the quark-initiated channel Eq. (5.1). We first combine the real emission and the virtual contributions since we also need the combination of these two terms when discussing the NNLO computation in Ch. 6. We obtain

$$\begin{aligned} 2s \cdot (\mathrm{d}\hat{\sigma}_{\mathrm{r}} + \mathrm{d}\hat{\sigma}_{\mathrm{v}}) &= \sum_{i\in\{1,4\}} \langle \hat{\mathcal{O}}_{\mathrm{nlo}}^{(i)} w^{5i} F_{\mathrm{LM}}(1,4\,|\,5) \rangle_\delta + \langle F_{\mathrm{LV}}^{\mathrm{fin}}(1,4) \rangle_\delta \\ &+ 2C_F\, \frac{[\alpha_s]}{\epsilon} \left\langle \left[\frac{1}{\epsilon} \left(\frac{4E_{\mathrm{max}}^2}{\mu^2} \right)^{-\epsilon} \eta_{14}^{-\epsilon} K_{14} - \left(\frac{1}{\epsilon} + \frac{3}{2} \right) \left(\frac{2p_1\cdot p_4}{\mu^2} \right)^{-\epsilon} \right] F_{\mathrm{LM}}(1,4) \right\rangle_\delta \end{aligned}$$

$$
\begin{aligned}
&+\frac{[\alpha_s]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\gamma_{qq}^{22}\left\langle\left(\frac{4E_4^2}{\mu^2}\right)^{-\epsilon}F_{\rm LM}(1,4)\right\rangle_\delta \\
&-2C_F\,\frac{[\alpha_s]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left\langle\frac{2(4E_{\rm max}^2/\mu^2)^{-\epsilon}-(4E_1^2/\mu^2)^{-\epsilon}-(4E_4^2/\mu^2)^{-\epsilon}}{2\epsilon}\,F_{\rm LM}(1,4)\right\rangle_\delta \\
&-\frac{[\alpha_s]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon} \\
&\quad\times\int\limits_0^1{\rm d}z\left(2C_F\left[\frac{(1-z)^{-2\epsilon}}{1-z}\right]_+ +(1-z)^{-2\epsilon}P_{qq,{\rm reg}}(z)\right)\left\langle\frac{F_{\rm LM}(z\cdot 1,4)}{z}\right\rangle_\delta .
\end{aligned}
\tag{5.65}
$$

In Eq. (5.65) γ_{qq}^{22} is the anomalous dimension given in Eq. (5.50) and $P_{qq,\rm reg}(z)$ is the regular part of the splitting function $P_{qq}(z)$ given in Eq. (5.37). Eq. (5.65) only has $1/\epsilon$ poles that originate from the collinear $\vec{p}_5 \parallel \vec{p}_1$ singularity. The finite result is obtained upon including collinear renormalization Eq. (5.62). To this end, we expand Eq. (5.64) to first order in ϵ. We find

$$
\begin{aligned}
&2C_F\left[\frac{(1-z)^{-2\epsilon}}{1-z}\right]_+ +(1-z)^{-2\epsilon}P_{qq,{\rm reg}}(z) \\
&\qquad = \hat{P}_{qq}^{(0)}(z)-\gamma_q\,\delta(1-z)-\epsilon\mathcal{P}'_{qq}(z)+\mathcal{O}(\epsilon^2)\,,
\end{aligned}
\tag{5.66}
$$

where $\mathcal{P}'_{qq}(z)$ is a generalized splitting function that we defined as

$$
\mathcal{P}'_{qq}(z)\equiv C_F\left(4\left[\frac{\ln(1-z)}{1-z}\right]_+ +(1-z)-2(1+z)\ln(1-z)\right). \tag{5.67}
$$

Upon expansion of Eq. (5.65) in ϵ we obtain the finite NLO partonic cross section

$$
\begin{aligned}
2s\cdot{\rm d}\hat{\sigma}^q_{\rm nlo} &= \sum_{i\in\{1,4\}}\left\langle\hat{\mathcal{O}}^{(i)}_{\rm nlo}\,w^{5i}F_{\rm LM}(1_q,4_q\,|\,5_g)\right\rangle_\delta+\left\langle F_{\rm LV}^{\rm fin}(1_q,4_q)\right\rangle_\delta \\
&+\frac{\alpha_s(\mu)}{2\pi}\int\limits_0^1{\rm d}z\left\{\mathcal{P}'_{qq}(z)+\ln\left(\frac{4E_1^2}{\mu^2}\right)\hat{P}_{qq}^{(0)}(z)\right\}\left\langle\frac{F_{\rm LM}\left(z\cdot 1_q,4_q\right)}{z}\right\rangle_\delta \\
&+\frac{\alpha_s(\mu)}{2\pi}\left\langle\left\{2C_F\mathcal{S}_{14}^{E_{\rm max}}+\gamma'_q\right\}F_{\rm LM}(1_q,4_q)\right\rangle_\delta+\mathcal{O}(\epsilon)\,.
\end{aligned}
\tag{5.68}
$$

The first term on the right-hand side of Eq. (5.68) is the fully regulated real emission. This is the only piece with hard emission of a gluon in the final state. The second term is the finite remainder from one-loop amplitude. The third and fourth terms are finite subtraction counterterms. These terms are proportional to LO functions $F_{\rm LM}(1,4)$ and the "boosted" version $F_{\rm LM}(z\cdot 1,4)$. In Eq. (5.68) we defined the function $\mathcal{S}_{14}^{E_{\rm max}}$ as

$$
\mathcal{S}_{14}^{E_{\rm max}}\equiv{\rm Li}_2(1-\eta_{14})-\zeta_2+\frac{1}{2}\ln^2\left(\frac{E_1}{E_4}\right)-\ln\eta_{14}\,\ln\left(\frac{E_1E_4}{E_{\rm max}^2}\right)+\frac{\gamma_q}{C_F}\ln\eta_{14}\,, \tag{5.69}
$$

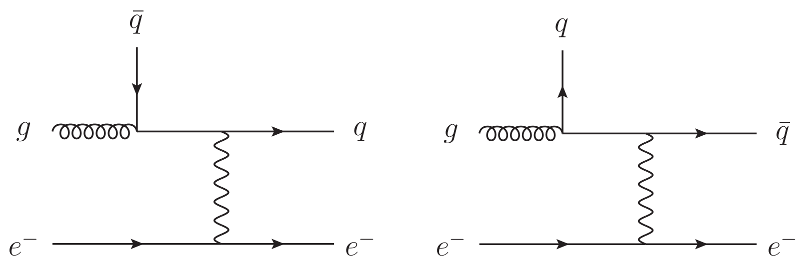


Fig. 5.3.: Feynman diagrams describing the gluon channel of deep-inelastic scattering. They first appear in the NLO QCD computation.

where $\gamma_q = (3/2)C_F$ is the LO quark cusp anomalous dimension. In Eq. (5.68) we further defined a generalized anomalous dimension γ'_q as

$$\gamma'_q \equiv C_F\left(\frac{13}{2} - \frac{2\pi^2}{3}\right). \tag{5.70}$$

5.5. Gluon channel

At next-to-leading order in perturbative QCD also the gluon-initiated channel

$$g(p_1) + e^-(p_2) \to e^-(p_3) + q(p_4) + \bar{q}(p_5)\,, \tag{5.71}$$

contributes to the DIS cross section. We will discuss this channel in what follows. Its analysis is simpler than what is required for the quark channel. This is so because single quark emissions do not develop soft singularities; for this reason we only have to regulate and extract collinear singularities in the process Eq. (5.71). In addition, since the gluon channel needs to be included at NLO QCD for the first time, no virtual contributions have to be considered. A glance at contributing Feynman diagrams Fig. 5.3 shows that only initial-state collinear singularities may appear. These collinear singularities get canceled by the collinear renormalization of parton distribution functions. Hence, for the gluon channel we write the cross section as

$$\mathrm{d}\hat{\sigma}_{\mathrm{nlo}} = \mathrm{d}\hat{\sigma}_{\mathrm{r}} + \mathrm{d}\hat{\sigma}_{\mathrm{pdf}}\,. \tag{5.72}$$

In Eq. (5.72) $\mathrm{d}\hat{\sigma}_{\mathrm{r}}$ describes the differential cross section of the process in Eq. (5.71) and $\mathrm{d}\hat{\sigma}_{\mathrm{pdf}}$ originates from the collinear renormalization of parton distribution functions.

We begin our discussion of the gluon-initiated channel with the analysis of the real emission contribution. As can be seen from diagrams Fig. 5.3 both quark and anti-quark can develop collinear singularities. It is convenient to rewrite the matrix element in such a way that only one singularity is present at a time. To accomplish this, we again introduce partition of unity

$$1 = w_g^{51} + w_g^{41}\,. \tag{5.73}$$

The partition functions w_g^{i1}, with $i \in \{4, 5\}$, in Eq. (5.73) need to possess the following property

$$C_{i1} w_g^{j1} = \delta_{ij}\,, \tag{5.74}$$

otherwise they are arbitrary. A possible choice is[8]

$$w_g^{51} \equiv \frac{\rho_{14}}{\rho_{14}+\rho_{15}}\,, \quad w_g^{41} \equiv \frac{\rho_{15}}{\rho_{14}+\rho_{15}}\,. \tag{5.75}$$

We rewrite the matrix element as

$$\begin{aligned} \left|M^{\rm tree}(1_g, 4_q, 5_{\bar q})\right|^2 &= w_g^{51}\left|M^{\rm tree}(1_g, 4_q, 5_{\bar q})\right|^2 + w_g^{41}\left|M^{\rm tree}(1_g, 4_q, 5_{\bar q})\right|^2 \\ &\Rightarrow w_g^{51}\left[\left|M^{\rm tree}(1_g, 4_q, 5_{\bar q})\right|^2 + \left|M^{\rm tree}(1_g, 5_q, 4_{\bar q})\right|^2\right], \end{aligned} \tag{5.76}$$

where in the last step we switched the momenta labeling of the quark and the anti-quark. We now only have to consider the collinear $\vec{p}_5 \parallel \vec{p}_1$ limit, which corresponds to a quark becoming collinear to the initial-state gluon in the first term, and an anti-quark becoming collinear to the initial-state gluon in the second term.[9] Following Eq. (5.3) we write

$$2s \cdot {\rm d}\hat\sigma_{\rm r} \equiv \int [{\rm d}p_5]\; w_g^{51} F_{{\rm LM},g}(1_g, 4_q\,|\,5_q) \equiv \left\langle w_g^{51} F_{{\rm LM},g}(1_g, 4_q\,|\,5_q)\right\rangle_\delta\,, \tag{5.77}$$

where

$$\begin{aligned} F_{{\rm LM},g}(1_g, 4_q\,|\,5_q) = \mathcal{N} \int & [{\rm d}p_3][{\rm d}p_4]\; (2\pi)^d \delta^{(d)}\,(p_1+p_2-p_3-p_4-p_5) \\ &\times \left[\left|M^{\rm tree}(1_g, 4_q, 5_{\bar q})\right|^2 + \left|M^{\rm tree}(1_g, 5_q, 4_{\bar q})\right|^2\right] \times \hat{\mathcal{O}}(p_3, p_4, p_5)\,. \end{aligned} \tag{5.78}$$

In Eq. (5.78) $[{\rm d}p_i]$ is the phase-space volume element of the parton i defined in Eq. (4.6). Note that, the mismatch between the actual $q\bar q$ final-state vs. labels of momenta p_4 and p_5 in $F_{{\rm LM},g}(1_g, 4_q\,|\,5_q)$ indicates the "averaging" over quark-anti-quark final states.

We continue with the construction of the subtraction scheme for the gluon-initiated process. Since we only have one collinear singularity to deal with, we write[10]

$$\left\langle w_g^{51} F_{{\rm LM},g}(1,4\,|\,5)\right\rangle = \left\langle C_{51}\, w_g^{51} F_{{\rm LM},g}(1,4\,|\,5)\right\rangle + \left\langle [I - C_{51}]\; w_g^{51} F_{{\rm LM},g}(1,4\,|\,5)\right\rangle_\delta\,. \tag{5.79}$$

The second term on the right-hand side is fully regulated and can be integrated in four dimensions. To simplify the subtraction term $\langle C_{51} w_g^{51} F_{{\rm LM},g}(1,4\,|\,5)\rangle$, we need the action of operator

[8] The partition functions are not well defined in the kinematic case where all partons are collinear to each other. However, this requires zero momentum transfer from the electron to the quark/gluon line and we do not consider this (pathological) case.

[9] With this procedure we lose the information of whether a parton is a quark or an anti-quark. This is not restrictive for any physics applications but makes the calculations easier.

[10] For simplicity we do not show the momenta labels of the electrons in the function $F_{{\rm LM},g}$ since they are not relevant for our discussion.

C_{51} on the function $F_{\mathrm{LM},g}(1,4\,|\,5)$. We find

$$C_{51}F_{\mathrm{LM},g}(1,4\,|\,5) = g_{s,b}^2 \times \frac{1}{p_1 \cdot p_5} \sum_{f\in\{q,\bar{q}\}} P_{fg}(z) \times \frac{F_{\mathrm{LM}}(z\cdot 1_f, 4_f)}{z}\,, \tag{5.80}$$

where $z = (E_1 - E_5)/E_1$ and the gluon-quark splitting function reads

$$P_{qg}(z) = P_{\bar{q}g}(z) = T_R\left[1 - \frac{2z(1-z)}{1-\epsilon}\right]. \tag{5.81}$$

In the splitting function Eq. (5.81) $T_R = 1/2$. The splitting function P_{qg} can be obtained from the splitting function P_{qq}. However, we note that the splitting function P_{qg}, which describes the collinear splitting of an initial-state gluon into a quark-anti-quark pair, possesses an additional ϵ-dependence. To compute P_{qg} we view the $g \to q\bar{q}$ splitting as

$$q(E) \to q(z'E) + g((1-z')E)\,, \quad z' = \frac{z}{z-1}\,, \tag{5.82}$$

where an incoming quark with energy $E = -E_1(1-z)$ splits into an outgoing quark with energy $z'E = zE_1$ and a gluon with energy $(1-z')E = -E_1$. In the situation described by Eq. (5.82) summing over final-state and averaging over initial-state colour and polarizations leads to the factor $C_F/2$. Doing the same for the splitting $g \to q\bar{q}$ we find $T_R/(d-2)$, where $d-2$ is the number of physical gluon polarizations in d-dimensional space time. In addition we have to add a relative minus sign that arises from the crossing of an odd number of fermions between initial and final state. Hence, using $P_{qq}(z')$ with corrected colour and polarization factors, we can describe this splitting as

$$\begin{aligned} &-\frac{T_R}{2(1-\epsilon)}\left(\frac{C_F}{2}\right)^{-1} \times \frac{P_{qq}(z')}{z'} \\ &= -\frac{T_R}{C_F(1-\epsilon)} \times \frac{z-1}{z}\, P_{qq}\left(\frac{z}{z-1}\right) \overset{(5.31)}{=} \frac{T_R}{z}\left[1 - \frac{2z(1-z)}{1-\epsilon}\right] \equiv \frac{P_{qg}(z)}{z}\,, \end{aligned} \tag{5.83}$$

which corresponds to our definition of the splitting function P_{qg} in Eq. (5.81).

After this small interlude we continue with the integration of Eq. (5.80) over the unresolved (anti-)quark momentum p_5. We repeat the same steps as in the analysis of the initial-state contribution to the quark channel in Section 5.2 and obtain

$$\begin{aligned} &\left\langle C_{51}F_{\mathrm{LM},g}(1,4\,|\,5)\right\rangle \\ &= -\frac{[\alpha_s]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right] \sum_{f\in\{q,\bar{q}\}} \int_0^1 \mathrm{d}z\, \left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon} (1-z)^{-2\epsilon} P_{fg}(z) \left\langle \frac{F_{\mathrm{LM}}(z\cdot 1_f, 4_f)}{z} \right\rangle_\delta . \end{aligned} \tag{5.84}$$

To account for collinear renormalization contributions, we use Eq. (5.61) and select terms where gluon parton distribution function appears. We obtain

$$2s \cdot \mathrm{d}\hat{\sigma}_{\mathrm{pdf}} = \frac{\alpha_s(\mu)}{2\pi\epsilon} \sum_{f \in \{q,\bar{q}\}} \int_0^1 \mathrm{d}z \, \hat{P}^{(0)}_{fg}(z) \left\langle \frac{F_{\mathrm{LM}}(z \cdot 1_f, 4_f)}{z} \right\rangle_\delta . \tag{5.85}$$

Finally, we combine Eq. (5.84) and Eq. (5.85) and obtain the final result for gluon-initiated channel

$$\begin{aligned} 2s \cdot \mathrm{d}\hat{\sigma}_{\mathrm{nlo}} = & \left\langle \hat{\mathcal{O}}_{\mathrm{nlo},g} F_{\mathrm{LM},g}(1_g, 4_q \,|\, 5_q) \right\rangle_\delta \\ & + \frac{\alpha_s(\mu)}{2\pi} \int_0^1 \mathrm{d}z \left[\mathcal{P}'_{qg}(z) + \ln\left(\frac{4E_1^2}{\mu^2}\right) \hat{P}^{(0)}_{qg}(z) \right] \sum_{f \in \{q,\bar{q}\}} \left\langle \frac{F_{\mathrm{LM}}(z \cdot 1_f, 4_f)}{z} \right\rangle_\delta . \end{aligned} \tag{5.86}$$

In writing Eq. (5.86) we used that $P_{\bar{q}g}(z) = P_{qg}(z)$ and defined

$$\hat{\mathcal{O}}_{\mathrm{nlo},g} \equiv \left[I - C_{51} \right] . \tag{5.87}$$

Splitting functions $\mathcal{P}'_{qg}$ and $\hat{P}^{(0)}_{qg}$ can be found in Eq. (E.24) and (E.20), respectively. The first term on the right-hand side of Eq. (5.86) is the fully regulated real emission. This is the only piece that contains the matrix element describing process $g + e^- \to e^- + q\bar{q}$. The second and third terms are finite subtraction counterterms that are proportional to the "boosted" LO function $\sum_{f \in \{q,\bar{q}\}} F_{\mathrm{LM}}(z \cdot 1_f, 4_f)$. Note that, due to the absence of a soft singularity in the gluon-initiated process, there is no contribution proportional to the LO function $F_{\mathrm{LM}}(1, 4)$. The result in Eq. (5.86) closes our discussion of NLO contributions to the differential cross section and we continue with the discussion of NNLO contributions.

6. The NNLO computation: quark-initiated channels

In this chapter, we discuss the computation of NNLO QCD corrections to the DIS partonic cross section using the nested soft-collinear subtraction scheme. As we already mentioned, at this order in the α_s expansion we need to combine four contributions to compute an infrared-finite cross section. We write

$$\mathrm{d}\hat{\sigma}_{\mathrm{nnlo}} = \mathrm{d}\hat{\sigma}_{\mathrm{vv}} + \mathrm{d}\hat{\sigma}_{\mathrm{rv}} + \mathrm{d}\hat{\sigma}_{\mathrm{rr}} + \mathrm{d}\hat{\sigma}_{\mathrm{pdf}}\,, \tag{6.1}$$

where $\mathrm{d}\hat{\sigma}_{\mathrm{vv}}$ describes the two-loop virtual corrections to the elastic process $q + e^- \to e^- + q$, $\mathrm{d}\hat{\sigma}_{\mathrm{rv}}$ describes a one-loop correction to a process with one additional parton in the final state (for example, $q + e^- \to e^- + q + g$), $\mathrm{d}\hat{\sigma}_{\mathrm{rr}}$ describes a process with two additional partons in the final state (for example, $q + e^- \to e^- + q + g + g$) and $\mathrm{d}\hat{\sigma}_{\mathrm{pdf}}$ describes corrections to the partonic cross section caused by the collinear renormalization of parton distribution functions.

We begin with the discussion of the double-real emission contribution $\mathrm{d}\hat{\sigma}_{\mathrm{rr}}$. Similar to the NLO QCD case discussed in Section 5, both quark-initiated processes $q/\bar{q} + e^- \to e^- + q/\bar{q} + g + g$ and $q/\bar{q} + e^- \to e^- + q/\bar{q} + q' + \bar{q}'$ and gluon-initiated process $g + e^- \to e^- + q + \bar{q} + g$ contribute at NNLO. We recall that at NNLO, two types of singularity arise which are not present at NLO: double soft and triple collinear singularities. A quark in the final state does not develop a soft singularity. Hence, only quark-initiated processes $q/\bar{q} + e^- \to e^- + q/\bar{q} + g + g$ contain genuine NNLO double-soft singularity in the limit when energies of both final-state gluons vanish. Gluon-initiated processes only posses triple-collinear singularities and their structure is much simpler. Indeed, the quark-initiated process with final state gluons $q + e^- \to e^- + q + g + g$ contains all possible singularities, with the other partonic channels containing a subset of these. For this reason, we focus on the partonic channel[1]

$$q(p_1) + e^-(p_2) \;\to\; e^-(p_3) + q(p_4) + g(p_5) + g(p_6)\,. \tag{6.2}$$

We outline the computation of the remaining partonic process $q + e^- \to e^- + q + q' + \bar{q}'$ in Section 6.7 and gluon-initiated processes in Chapter 7. A detailed discussion of the quark-initiated channel with gluons in the final state is sufficient to illustrate and discuss all the peculiarities of the subtraction scheme, as well as the analytic integration of the subtraction terms at NNLO QCD.

[1] Since computations for quark and anti-quark initiated channels are identical, we focus on the quark initiated channel.

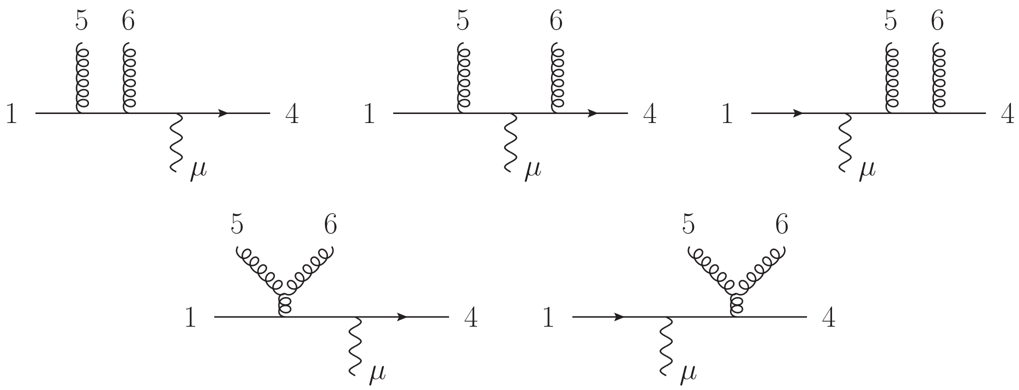


Fig. 6.1.: Partonic currents that contribute to the quark channel Eq. (6.2) of the double-real emission contribution of DIS. To obtain the complete Feynman diagrams for DIS they need to be contracted with the the leptonic current. We only show labels i of external momenta p_i. Abelian contributions in the first line also need to be included in the amplitude with momenta of the two gluon emissions exchanged $p_5 \leftrightarrow p_6$.

Feynman diagrams that contribute to the quark-initiated process Eq. (6.2) are shown in Fig. 6.1. Following the discussion in Section 5, we define[2]

$$2s \cdot \mathrm{d}\hat{\sigma}_{\mathrm{rr}} = \int [\mathrm{d}p_5][\mathrm{d}p_6]\, \theta(E_5 - E_6)\, F_{\mathrm{LM}}(1_q, 4_q \,|\, 5_g, 6_g) \equiv \langle F_{\mathrm{LM}}(1_q, 4_q \,|\, 5_g, 6_g) \rangle_\delta\,, \tag{6.3}$$

where

$$\begin{aligned} F_{\mathrm{LM}}(1_q, 4_q \,|\, 5_g, 6_g) = \mathcal{N} \int [\mathrm{d}p_3][\mathrm{d}p_4]\, (2\pi)^d \delta^{(d)}\, (p_1 + p_2 - p_3 - p_4 - p_5 - p_6) \\ \times |M^{\mathrm{tree}}_{\mathrm{nnlo}}(p_1, p_2, p_3, p_4, p_5, p_6)|^2 \times \hat{\mathcal{O}}(p_3, p_4, p_5, p_6)\,. \end{aligned} \tag{6.4}$$

The phase-space volume element $[\mathrm{d}p_i]$ of the parton i is defined in Eq. (4.6). E_{max} plays the same role as in the NLO QCD computation, see comments around Eq. (4.6) and the discussion of the soft subtraction term in Eq. (5.8). In Eq. (6.3) we order two gluons in energy by introducing $\theta(E_5 - E_6)$. Hence, the only single-soft singularity that needs to be regularized is $E_6 \to 0$ since $E_5 \to 0$ implies that both gluons $g(p_5)$ and $g(p_6)$ become soft. The factor $\mathcal{N}$ in Eq. (6.4) includes all the relevant symmetry factors, $[\mathrm{d}p_3][\mathrm{d}p_4]$ is the phase-space of the hard process, $M^{\mathrm{tree}}_{\mathrm{nnlo}}$ is the matrix element, composed of Feynman diagrams shown in Fig. 6.1, and $\hat{\mathcal{O}}$ is an arbitrary infrared safe observable. We will proceed with the discussion of how infrared and collinear singularities can be extracted from the function $F_{\mathrm{LM}}(1, 4 \,|\, 5, 6)$ without integration over resolved phase space.

6.1. Subtractions

In this section we employ the nested soft-collinear subtraction scheme to regularize singularities in the cross section Eq. (6.3) that describes partonic DIS process Eq. (6.2). Our goal is to extract

[2]For simplicity we do not show the momenta labels of the electrons in the function F_{LM}.

poles in the dimensional regularization parameter ϵ *without* integrating over the resolved phase space. We note that our construction follows the NLO discussion of quark-initiated processes in Chapter 5. We treat soft and collinear singularities of the amplitude iteratively, starting with soft ones.

Soft singularities

We begin by regulating the double-soft singularity. To this end we introduce an operator $\mathbb{S}$ that extracts the leading singularity in the limit $E_5 \sim E_6 \to 0$ by acting on the function $F_{\rm LM}(1,4\,|\,5,6)$. Similar to the single-soft operator discussed in the context of the NLO QCD computation, its action is defined by the following equation[3]

$$\begin{aligned}&\big\langle \mathbb{S} F_{\rm LM}(1,4\,|\,5,6)\big\rangle \\ &= \int [{\rm d}g_5][{\rm d}g_6]\,\theta(E_5-E_6)\,{\rm Eik}(p_1,p_4,p_5,p_6)\times\big\langle F_{\rm LM}(1,4)\big\rangle_\delta\,,\end{aligned} \tag{6.5}$$

where $\langle F_{\rm LM}(1,4)\rangle$ is the fully-differential cross section of the Born process $q+e^- \to e^- + q$. A complete definition of the double-soft limit, including the explicit form of the eikonal function ${\rm Eik}(p_1,p_4,p_5,p_6)$, is given in Appendix B.2. The essence of Eq. (6.5) is that soft gluons factorize from the hard matrix element squared [56], the infrared safe observable and the energy-momentum conserving δ-function. To extract the double-soft singularity we insert the unity operator decomposed as $I = \big[I-\mathbb{S}\big]+\mathbb{S}$ into the phase space and write

$$\big\langle F_{\rm LM}(1,4\,|\,5,6)\big\rangle = \big\langle \big[I-\mathbb{S}\big] F_{\rm LM}(1,4\,|\,5,6)\big\rangle + \big\langle \mathbb{S} F_{\rm LM}(1,4\,|\,5,6)\big\rangle\,. \tag{6.6}$$

In the first term on the right-hand side of Eq. (6.6) the double-soft singularity is regulated. In the second term the cross section is only needed in the double-soft limit Eq. (6.5). According to Eq. (6.5) soft gluons decouple from the hard process and the observable. For this reason, we can analytically integrate the double eikonal function over the phase space of two emitted gluons. This integration produces $1/\epsilon$ poles that are *independent* of the hard process and the observable. Further details of this integration will be given the next section. The double-soft regulated term $\big\langle \big[I-\mathbb{S}\big] F_{\rm LM}(1,4\,|\,5,6)\big\rangle$ still contains unregulated single-soft and collinear singularities. We will now discuss how to regularize them.

We begin with the single-soft singularity. Thanks to the energy ordering $E_5 > E_6$ there is only one single-soft singularity in the limit $E_6 \to 0$. To regularize it, we introduce an operator S_6 that extracts the leading single-soft singularity in the limit $E_6 \to 0$ and insert the identity operator $I = \big[I-S_6\big]+S_6$ into the first term on the right-hand side of Eq. (6.6). We find

$$\begin{aligned}&\big\langle \big[I-\mathbb{S}\big] F_{\rm LM}(1,4\,|\,5,6)\big\rangle \\ &= \big\langle \big[I-S_6\big]\big[I-\mathbb{S}\big] F_{\rm LM}(1,4\,|\,5,6)\big\rangle + \big\langle S_6\big[I-\mathbb{S}\big] F_{\rm LM}(1,4\,|\,5,6)\big\rangle\,.\end{aligned} \tag{6.7}$$

As we will see in Section 6.2, in the subtraction term the gluon $g(p_6)$ decouples from the

[3] In what follows we drop the subscripts of the momentum labels, specifying the parton kind.

function $F_{\mathrm{LM}}(1,4\,|\,5,6)$. We can analytically integrate over the unresolved phase space and extract the $1/\epsilon$ poles. The first term on the right-hand side of Eq. (6.7) is free of soft singularities, but it still contains collinear singularities. We will now discuss how to regularize them.

Collinear singularities

Even for a relatively simple process such as deep inelastic scattering, a large number of singular collinear kinematic configurations exist. To identify them we make use of the fact that in physical gauges collinear singularities factorize on external legs. As can be seen from diagrams Fig. 6.1 the amplitude possesses double-collinear singularities when $\vec{p}_5 \parallel \vec{p}_6$, $\vec{p}_{i=5,6} \parallel \vec{p}_1$ and $\vec{p}_{i=5,6} \parallel \vec{p}_4$, as well as the genuine NNLO triple-collinear singularities when $\vec{p}_5 \parallel \vec{p}_6 \parallel \vec{p}_{i=1,4}$.

The collinear singularities overlap, and we would like to split the phase space in such a way that in each region no overlapping singularities are present and we have to deal with *only* two singularities at a time. To achieve that, we introduce partition functions[4]

$$1 = w^{51,61} + w^{54,64} + w^{51,64} + w^{54,61}\,, \tag{6.8}$$

into the first term on the right-hand side of Eq. (6.7). The partition functions $w^{5i,6j}$ in Eq. (6.8) are designed to dampen all but a few collinear singularities. Similar to the partition functions introduced in the context of the NLO computation in Eq. (5.9), they should satisfy the following conditions

$$\lim_{5\parallel i} w^{5j,6k} \sim \delta_{ij}, \quad \lim_{6\parallel i} w^{5j,6k} \sim \delta_{ik}\,, \quad \lim_{5\parallel 6} w^{5j,6k} \sim \delta_{jk}\,, \quad \text{for} \quad i,j,k \in \{1,4\}\,. \tag{6.9}$$

Thanks to Eq. (6.9), the last two partition functions on the right-hand side of Eq. (6.8) only posses double-collinear singularities whose regularization is NLO-like and we can regulate these immediately.

As an example we consider the partitioning $51,64$. It contains two double-collinear singularities: one where $\vec{p}_5 \parallel \vec{p}_1$ and another where $\vec{p}_6 \parallel \vec{p}_4$. We introduce operators C_{51} and C_{64} that extract leading singularities in these limits. To regulate these singularities we rewrite the identity operator as

$$I = \left[I - C_{51}\right] + C_{51} = \left[I - C_{64}\right]\left[I - C_{51}\right] + \left[C_{51} + C_{64}\right] - C_{51}C_{64}\,, \tag{6.10}$$

and insert Eq. (6.10) into the first term on the right-hand side of Eq. (6.7). We obtain

$$\begin{aligned} &\left\langle \left[I - S_6\right]\left[I - \mathbb{S}\right]\, w^{51,64}\, F_{\mathrm{LM}}(1,4\,|\,5,6)\right\rangle \\ &= \left\langle \left[I - C_{64}\right]\left[I - C_{51}\right]\left[I - S_6\right]\left[I - \mathbb{S}\right] w^{51,64} F_{\mathrm{LM}}(1,4\,|\,5,6)\right\rangle_\delta \\ &\quad + \left\langle \left[C_{51} + C_{64}\right]\left[I - S_6\right]\left[I - \mathbb{S}\right] w^{51,64} F_{\mathrm{LM}}(1,4\,|\,5,6)\right\rangle \\ &\quad - \left\langle C_{51}C_{64}\left[I - S_6\right]\left[I - \mathbb{S}\right] w^{51,64} F_{\mathrm{LM}}(1,4\,|\,5,6)\right\rangle. \end{aligned} \tag{6.11}$$

[4]Explicit formulas for a possible choice of the partition functions $w^{5i,6j}$ can be found in Appendix A.3. They are taken from discussions of other processes in the context of the nested soft-collinear subtraction scheme [37,38] and are adopted from Ref. [6].

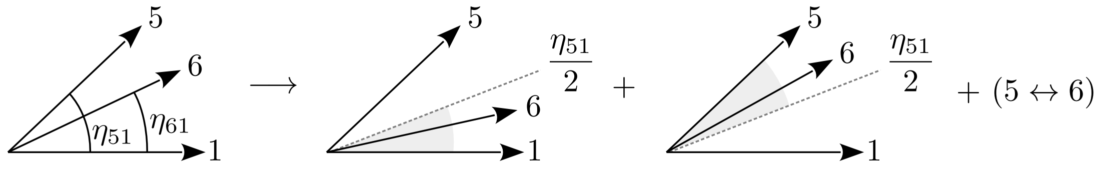


Fig. 6.2.: Splitting of the angular phase space of partition $w^{51,61}$ into regions with defined collinear singularities. First sector on the right-hand side corresponds to sector $\theta^{(a)}$ in Eq. (6.13). The second to $\theta^{(b)}$. Sectors $\theta^{(c)}$ and $\theta^{(d)}$ where $\eta_{61} > \eta_{51}$ are not shown explicitly. Schematically they are given by the displayed sectors but with direction of momenta 5 and 6 exchanged.

In the first term on the right-hand side of Eq. (6.11) all singularities are regulated and it can be integrated in 4 dimensions for arbitrary infrared safe observables. We can integrate over the unresolved phase space in the remaining two terms, and we present details of this in the following sections. The partition $54, 61$ is treated in an analogous way. However, the partitions $51, 61$ and $54, 64$ require more care.

To illustrate this, we consider partition $51, 61$. It contains two double-collinear singularities where $\vec{p}_5 \parallel \vec{p}_1$ and $\vec{p}_6 \parallel \vec{p}_1$, as well as the double-collinear singularity where $\vec{p}_5 \parallel \vec{p}_6$, and the triple-collinear singularity where $\vec{p}_5 \parallel \vec{p}_6 \parallel \vec{p}_1$. The various double-collinear singularities can be further separated by splitting the angular phase space of the two gluons into different regions. We do this by introducing yet another partition of unity[5]

$$\begin{aligned} 1 = \theta\Big(\rho_{16} < \frac{\rho_{15}}{2}\Big) + \theta\Big(\frac{\rho_{15}}{2} < \rho_{16} < \rho_{15}\Big) + \theta\Big(\rho_{15} < \frac{\rho_{16}}{2}\Big) \\ + \theta\Big(\frac{\rho_{16}}{2} < \rho_{15} < \rho_{16}\Big) \equiv \theta_1^{(a)} + \theta_1^{(b)} + \theta_1^{(c)} + \theta_1^{(d)}\,, \end{aligned} \tag{6.13}$$

where $\rho_{ij} = 1 - \vec{n}_i \cdot \vec{n}_j$ is defined in Eq. (5.11). We will refer to the four contributions shown in Eq. (6.13) as *sectors*; ordering of angles in the different sectors is shown schematically in Fig. 6.2 and the splitting of the angular phase space is shown in Fig. 6.3. In each partition and sector the possible collinear singularities are uniquely defined and no overlaps occur. We will use this observation to write down a fully-regulated double real contribution. We note that we use a parametrization of the phase space that naturally implements this sector decomposition and that was introduced in Ref. [6].

As an example, we consider sector (a). Thanks to Eq. (6.13) there are only two collinear singularities: a double-collinear one where $\vec{p}_6 \parallel \vec{p}_1$ and a triple-collinear one where $\vec{p}_5 \parallel \vec{p}_6 \parallel \vec{p}_1$. Introducing operators $\mathbb{C}_1$ and C_{61} that extract triple- and double-collinear singularities and

[5] In Eq. (6.13) we slightly abused notation of θ-functions to obtain a simpler formula. For instance the second term has to be understood as

$$\theta\Big(\frac{\rho_{51}}{2} < \rho_{16} < \rho_{15}\Big) = \theta\Big(\rho_{16} - \frac{\rho_{51}}{2}\Big)\theta(\rho_{15} - \rho_{16})\,. \tag{6.12}$$

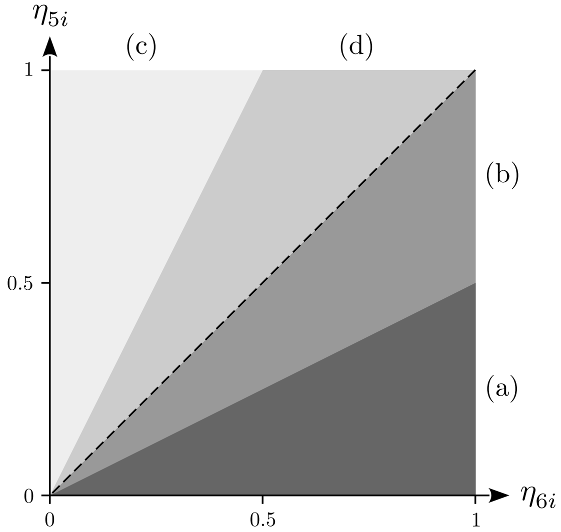


Fig. 6.3.: Sectors to isolate collinear singularities in the triple collinear case. Index $i = 1, 4$ corresponds to the index i in Eq. (6.51). Single double-collinear C_{5i} and C_{6i} along the coordinate axes, double-collinear C_{56} appear only along the bisecting (dashed line), triple collinear in the origin.

subtracting them iteratively, we obtain from the first term on the right-hand side of Eq. (6.7)

$$\begin{aligned}&\left\langle [I - S_6]\,[I - \mathbb{S}]\,\theta_1^{(a)} w^{51,61} F_{\rm LM}(1,4\,|\,5,6)\right\rangle \\ &= \left\langle [I - C_{61}]\,[I - \mathbb{C}_1]\,[I - S_6]\,[I - \mathbb{S}]\,\theta_1^{(a)} w^{51,61} F_{\rm LM}(1,4\,|\,5,6)\right\rangle \\ &\quad + \left\langle C_{61}\,[I - \mathbb{C}_1]\,[I - S_6]\,[I - \mathbb{S}]\,\theta_1^{(a)} w^{51,61} F_{\rm LM}(1,4\,|\,5,6)\right\rangle \\ &\quad + \left\langle \mathbb{C}_1\,[I - S_6]\,[I - \mathbb{S}]\,\theta_1^{(a)} w^{51,61} F_{\rm LM}(1,4\,|\,5,6)\right\rangle \,.\end{aligned} \tag{6.14}$$

In sector (a) of the partition $51, 61$ all singularities are regulated. We can now proceed in a similar way with the remaining partitions and sectors.

NNLO regulated differential cross section

To present a formula for the NNLO regulated differential cross section we need to introduce additional operators that extract various soft and collinear singularities. The complete list of such operators is presented below. It includes soft operators

$$\begin{aligned}&\mathbb{S} \quad \text{Double-soft:} \quad E_5 \sim E_6 \to 0\,, \\ &S_6 \quad \text{Single-soft:} \quad E_6 \to 0\,,\end{aligned} \tag{6.15}$$

and collinear operators

$$\begin{aligned}\mathbb{C}_i &\quad \text{Triple-collinear:} \quad \vec{p}_5 \parallel \vec{p}_6 \parallel \vec{p}_i\,, \\ C_{5i}\,, C_{6i} &\quad \text{Double-collinear:} \quad \vec{p}_5 \parallel \vec{p}_i\,,\ \vec{p}_6 \parallel \vec{p}_i\,, \\ C_{56} &\quad \text{Double-collinear:} \quad \vec{p}_5 \parallel \vec{p}_6\,,\end{aligned} \tag{6.16}$$

with $i,j \in \{1,4\}$. Full definition of these operators can be found in Appendix B. Using these operators, we write the fully-regulated contribution for the double-real emission cross section as

$$
\begin{aligned}
\left\langle F_{\rm LM}(1,4\,|\,5,6)\right\rangle_\delta &= \left\langle \mathbb{S} F_{\rm LM}(1,4\,|\,5,6)\right\rangle + \left\langle [I-\mathbb{S}]\, S_6 F_{\rm LM}(1,4\,|\,5,6)\right\rangle \\
&+ \sum_{\substack{i,j\in\{1,4\}\\ i\neq j}} \Big\langle [I-\mathbb{S}]\,[I-S_6]\left[C_{5i}w^{5i,6j} + C_{6i}w^{5j,6i} + \left(\theta_i^{(a)}C_{5i} + \theta_i^{(c)}C_{6i}\right)w^{5i,6i}\right] \\
&\qquad \times [{\rm d}p_5][{\rm d}p_6] F_{\rm LM}(1,4\,|\,5,6)\Big\rangle \\
&+ \sum_{i\in\{1,4\}} \Big\langle [I-\mathbb{S}]\,[I-S_6]\left[\theta_i^{(b)}C_{56} + \theta_i^{(d)}C_{56}\right][{\rm d}p_5][{\rm d}p_6]w^{5i,6i}F_{\rm LM}(1,4\,|\,5,6)\Big\rangle \\
&- \sum_{\substack{i,j\in\{1,4\}\\ i\neq j}} \Big\langle [I-\mathbb{S}]\,[I-S_6]\,C_{5i}C_{6j}[{\rm d}p_5][{\rm d}p_6]w^{5i,6j}F_{\rm LM}(1,4\,|\,5,6)\Big\rangle \\
&+ \sum_{i\in\{1,4\}} \Big\langle [I-\mathbb{S}]\,[I-S_6]\Big[\theta_i^{(a)}\mathbb{C}_i[I-C_{5i}] + \theta_i^{(b)}\mathbb{C}_i[I-C_{56}] + \theta_i^{(c)}\mathbb{C}_i[I-C_{6i}] \\
&\qquad\qquad + \theta_i^{(d)}\mathbb{C}_i[I-C_{56}]\Big][{\rm d}p_5][{\rm d}p_6]w^{5i,6i}F_{\rm LM}(1,4\,|\,5,6)\Big\rangle \\
&+ \sum_{\substack{i,j\in\{1,4\}\\ i\neq j}} \Big\langle \hat{\mathcal{O}}^{(i,j)}_{\rm nnlo}\,[{\rm d}p_5][{\rm d}p_6]w^{5i,6j}F_{\rm LM}(1,4\,|\,5,6)\Big\rangle_\delta \\
&+ \sum_{i\in\{1,4\}} \Big\langle \hat{\mathcal{O}}^{(i)}_{\rm nnlo}\,[{\rm d}p_5][{\rm d}p_6]w^{5i,6i}F_{\rm LM}(1,4\,|\,5,6)\Big\rangle_\delta .
\end{aligned}
\tag{6.17}
$$

In Eq. (6.17) we used the following notations

$$
\hat{\mathcal{O}}^{(i,j)}_{\rm nnlo} \equiv [I-\mathbb{S}]\,[I-S_6]\,[I-C_{6j}]\,[I-C_{5i}]\,, \tag{6.18}
$$

$$
\begin{aligned}
\hat{\mathcal{O}}^{(i)}_{\rm nnlo} \equiv [I-\mathbb{S}]\,[I-S_6]\,[I-\mathbb{C}_i]\Big(&\theta_i^{(a)}[I-C_{5i}] + \theta_i^{(b)}[I-C_{56}] \\
&+ \theta_i^{(c)}[I-C_{6i}] + \theta_i^{(d)}[I-C_{56}]\Big)\,.
\end{aligned}
\tag{6.19}
$$

We refer to the first two terms on the right-hand side of Eq. (6.17) as the "double-soft" (first) and the "double-soft-regulated single-soft" (second) subtraction terms. They are discussed in Section 6.2. The third and fourth terms on the right-hand side of Eq. (6.17) contain contributions where one of the gluons $g(p_5)$ or $g(p_6)$ is unresolved due to a collinear singularity. We refer to these terms as the "single-unresolved" collinear subtraction terms; we discuss them in Section 6.3. In the firth and the sixth term in Eq. (6.17) both gluons are unresolved. We refer to these contributions as "double-unresolved" collinear subtraction terms. We discuss them in Section 6.4. Finally, in the last two contributions in Eq. (6.17) all singularities are regulated. They can be computed numerically in four dimensions for arbitrary infrared safe observables. Further details of how this can be done are given in Chapter 8.

We continue with the discussion of the analytic computation of the subtraction terms; we begin with the soft subtraction terms given by the first and second terms on the right-hand side of Eq. (6.17).

6.2. Soft subtraction terms

In the regulated formula for the cross section Eq. (6.17) two soft subtraction terms are present: the double-soft subtraction term (first term on the right-hand side of Eq. (6.17)) and the double-soft regulated single soft subtraction term (second term on the right-hand side). The double-soft subtraction term $\left\langle \mathbb{S} F_{\rm LM}(1,4\,|\,5,6)\right\rangle$ was computed in Ref. [67].[6] It reads

$$\left\langle \mathbb{S} F_{\rm LM}(1,4\,|\,5,6)\right\rangle = [\alpha_s]^2 \left(\frac{4E_{\rm max}^2}{\mu^2}\right)^{-2\epsilon} \left\langle \left[C_F^2\, 2\eta_{14}^{-2\epsilon} K_{14}^2 + C_A C_F\, S_{gg}^{(\rm nab)}\right] F_{\rm LM}(1,4) \right\rangle_\delta, \tag{6.20}$$

where the formula for the non-abelian contribution $S_{gg}^{(\rm nab)}$ is provided in Appendix I.1. In what follows we focus on the double-soft regulated single-soft subtraction term

$$\left\langle \left[I - \mathbb{S}\right] S_6 F_{\rm LM}(1,4\,|\,5,6)\right\rangle. \tag{6.21}$$

Since $\mathbb{S}\, S_6\, F_{\rm LM} = S_5\, S_6\, F_{\rm LM}$, we can rewrite Eq. (6.21) as

$$\left\langle \left[I - \mathbb{S}\right] S_6 F_{\rm LM}(1,4\,|\,5,6)\right\rangle = \left\langle \left[I - S_5\right] S_6 F_{\rm LM}(1,4\,|\,5,6)\right\rangle. \tag{6.22}$$

The $E_6 \to 0$ limit is given by [57]

$$S_6\, F_{\rm LM}(1,4\,|\,5,6) = \frac{g_{s,b}^2}{E_6^2}\left[(2C_F - C_A)\frac{\rho_{14}}{\rho_{16}\,\rho_{46}} + C_A\left(\frac{\rho_{15}}{\rho_{16}\,\rho_{56}} + \frac{\rho_{45}}{\rho_{46}\,\rho_{56}}\right)\right] F_{\rm LM}(1,4\,|\,5)\,. \tag{6.23}$$

Since the hard function $F_{\rm LM}(1,4\,|\,5)$ is independent of the gluon momentum p_6, we can integrate over it. The required integral reads

$$\int [{\rm d}p_6] \frac{1}{E_6^2}\left[(2C_F - C_A)\frac{\rho_{14}}{\rho_{16}\,\rho_{46}} + C_A\left(\frac{\rho_{15}}{\rho_{16}\,\rho_{56}} + \frac{\rho_{45}}{\rho_{46}\,\rho_{56}}\right)\right]. \tag{6.24}$$

A similar integral has already appeared in the NLO computation, c.f. Eq. (5.17). The only difference is that energy integration now goes from zero to E_5. We find

$$\int_0^{E_5} \frac{{\rm d}E_6}{E_6^{1+2\epsilon}} = -\frac{E_5^{-2\epsilon}}{2\epsilon}\,. \tag{6.25}$$

Accounting for that, we write the integral Eq. (6.24) as

$$\int [{\rm d}p_6] S_6 F_{\rm LM}(1,4\,|\,5,6) = J_{145} F_{\rm LM}(1,4\,|\,5)\,, \tag{6.26}$$

where we introduced

$$J_{145} \equiv \frac{[\alpha_{s,b}]}{\epsilon^2}\left[(2C_F - C_A)\eta_{14}^{-\epsilon} K_{14} + C_A\left[\eta_{15}^{-\epsilon} K_{15} + \eta_{45}^{-\epsilon} K_{45}\right]\right](2E_5)^{-2\epsilon}\,. \tag{6.27}$$

[6]Note that, in Ref. [67] the double-soft subtraction term is computed for arbitrary number of external partons.

In Eq. (6.27) functions K_{ij} are given in Eq. (5.18). It follows from Eqs. (6.22, 6.26) that

$$\left\langle \left[I - \mathbb{S}\right] S_6 F_{\rm LM}(1,4\,|\,5,6) \right\rangle = \left\langle \left[I - S_5\right] J_{145} F_{\rm LM}(1,4\,|\,5) \right\rangle . \tag{6.28}$$

The function $F_{\rm LM}(1,4\,|\,5)$ describes the real emission process $q(p_1) + e^-(p_2) \to e^-(p_3) + q(p_4) + g(p_5)$. It contains both soft and collinear singularities but the soft singularity $E_5 \to 0$ has already been regulated.

Fully-regulated single-soft subtraction term

We continue with regulating remaining collinear singularities in Eq. (6.28). To this end, we follow the NLO procedure discussed in Chapter 5. To isolate collinear singularities we introduce partition of unity

$$1 = w^{51} + w^{54}\,, \tag{6.29}$$

and insert this expression into Eq. (6.28). The partition functions w^{5i} are defined in Eq. (5.11) We further insert the identity operator written as $I = \left[I - C_i\right] + C_i$ into the term that contains the partition function w^{5i}. We find

$$\begin{aligned} &\left\langle \left[I - S_5\right]\, J_{145}\, F_{\rm LM}(1,4\,|\,5) \right\rangle \\ &= \sum_{i\in\{1,4\}} \left\langle C_{5i}\left[I - S_5\right]\, J_{145}\, w^{5i}\, F_{\rm LM}(1,4\,|\,5) \right\rangle + \sum_{i\in\{1,4\}} \left\langle \hat{\mathcal{O}}^{(i)}_{\rm nlo}\, J_{145}\, w^{5i}\, F_{\rm LM}(1,4\,|\,5) \right\rangle . \end{aligned} \tag{6.30}$$

Note that collinear operators $C_{i=1,4}$ and the operators $\hat{\mathcal{O}}^{(i=1,4)}_{\rm nlo}$ in Eq. (6.30) have been already used in the NLO discussion in Chapter 5 but, as emphasized by writing them to the left of the function J_{145}, they now act on J_{145} as well. The second term on the right hand-side of Eq. (6.30) is free of singularities but it contains explicit $1/\epsilon$ poles, present in the function J_{145}. These poles need to cancel with similar NLO-like contributions that appear in e.g. the real-virtual corrections.[7] We find[8]

$$C_{5i} J_{145} \equiv \lim_{p_5 || p_i} J_{145} = \frac{[\alpha_{s,b}]}{\epsilon^2}\left[2C_F\, \eta_{14}^{-\epsilon} K_{14} + C_A\, \eta_{i5}^{-\epsilon}\frac{\Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\Gamma(1-2\epsilon)}\right](2E_5)^{-2\epsilon}\,, \tag{6.33}$$

[7] At first glance it may seem like that the second term on the right-hand side of Eq. (6.30) depends on the explicit form of the partition functions. However, since

$$\sum_{i=1,4} \hat{\mathcal{O}}^{(i)}_{\rm nlo} w^{5i} = \left[(w^{51} + w^{54}) - C_{51}w^{51} - C_{54}w^{54}\right]\left[I - S_5\right] = \left[I - C_{51} - C_{54}\right]\left[I - S_5\right], \tag{6.31}$$

this term is indeed independent of the explicit form of the partition functions as long as they are chosen to satisfy Eq. (5.10).

[8] To obtain the limit Eq. (6.33) we use

$$\lim_{p_5||p_i} K_{i5} = \lim_{\eta_{i5}\to 0}\left(\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\eta_{ij}^{1+\epsilon}\,{}_2{\rm F}_1\Big(1,1;1-\epsilon;1-\eta_{ij}\Big)\right) = \frac{\Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\Gamma(1-2\epsilon)}\,. \tag{6.32}$$

for $i = 1, 4$.

We now want to simplify the first term on the right-hand side of Eq. (6.30) and extract remaining $1/\epsilon$ poles by integrating over the unresolved phase space of $g(P_5)$. As can be seen from Eq. (6.33) in the collinear limits the function J_{145} provides additional ϵ-dependent powers of E_5, η_{15} and η_{45} but otherwise integration over the phase space of the unresolved gluon $g(p_5)$ is analogous to the NLO computation described in Section 5.2. For this reason in what follows we only sketch the computation. More details can be found in Section 5.2.

We begin with the collinear $\vec{p}_5 \parallel \vec{p}_1$ limit and consider collinear and soft-collinear contributions separately, starting with the latter. The required soft-collinear limit is known from the NLO discussion, see Eq. (5.27). It reads

$$C_{51}S_5F_{\rm LM}(1,4\,|\,5) = 2C_F\, g_{s,b}^2 \times \frac{1}{E_5^2\,\rho_{15}} \times F_{\rm LM}(1,4)\,. \tag{6.34}$$

Using Eq. (6.34) together with the limit in Eq. (6.33), we find the integral over the phase space of the unresolved gluon

$$\begin{aligned}\int [{\rm d}p_5]\; C_{51}S_5J_{145}\, w^{5i}F_{\rm LM}(1,4\,|\,5) &= 2C_F\,\frac{g_{s,b}^2[\alpha_{s,b}]}{\epsilon^2} \times 2^{-2\epsilon}\int\limits_0^{E_{\rm max}} \frac{{\rm d}E_5}{E_5^{1+4\epsilon}} \\ \times \int \frac{{\rm d}\Omega_5^{(d-1)}}{2(2\pi)^{d-1}} &\left[2C_F\,\eta_{14}^{-\epsilon}K_{14} + C_A\,\eta_{15}^{-\epsilon}\,\frac{\Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\frac{1}{\rho_{15}} \times F_{\rm LM}(1,4)\,.\end{aligned} \tag{6.35}$$

The energy integral is given in Eq. (G.1). The angular integral has two contributions. The first term in the square bracket in Eq. (6.35) is identical to the NLO case; the relevant result is given in Eq. (5.28). The integral of the second term reads

$$\int \frac{{\rm d}\Omega_5^{(d-1)}}{2(2\pi)^{d-1}}\,\eta_{15}^{-\epsilon}\,\frac{1}{\rho_{15}} = -\frac{2^{-2\epsilon}}{\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]\left[\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{2\Gamma(1-3\epsilon)}\right], \tag{6.36}$$

where we have used $\eta_{15} = \rho_{15}/2$. Combining the two results, we obtain

$$\begin{aligned}&\int \frac{{\rm d}\Omega_5^{(d-1)}}{2(2\pi)^{d-1}}\left[2C_F\,\eta_{14}^{-\epsilon}K_{14} + C_A\,\eta_{15}^{-\epsilon}\,\frac{\Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\frac{1}{\rho_{15}} \\ &= -\frac{2^{-2\epsilon}}{\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]\left[2C_F\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\eta_{14}^{-\epsilon}K_{14} + C_A\,\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\right]\,.\end{aligned} \tag{6.37}$$

Finally, we use this result to write the soft-collinear contribution as

$$\begin{aligned}\big\langle C_{51}\,S_5\,J_{145}\,w^{51}\,F_{\rm LM}(1,4\,|\,5)\big\rangle &= 2C_F\,\frac{[\alpha_{s,b}]^2}{4\epsilon^4}\,(2E_{\rm max})^{-4\epsilon} \\ \times \Big\langle &\left[2C_F\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\eta_{14}^{-\epsilon}K_{14} + C_A\,\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\right] F_{\rm LM}(1,4)\Big\rangle\,.\end{aligned} \tag{6.38}$$

We continue with the collinear contribution. To integrate over the phase space of the unresolved gluon we need the limit in Eq. (6.33) and the collinear $\vec{p}_5 \parallel \vec{p}_1$ limit of the function

$F_{\rm LM}(1,4\,|\,5)$. It reads

$$C_{51}F_{\rm LM}(1,4\,|\,5) = g_{s,b}^2 \times \frac{1}{p_1\cdot p_5}P_{qq}(z) \times \frac{F_{\rm LM}(z\cdot 1,4)}{z}\,, \tag{6.39}$$

where $z = (E_1 - E_5)/E_1$. The splitting function $P_{qq}(z)$ is defined in Eq. (5.31). Writing $E_5 = E_1(1-z)$, we obtain for the phase space integral[9]

$$\begin{aligned}\left\langle C_{51}w^{51}J_{145}F_{\rm LM}(1,4\,|\,5)\right\rangle = &-\frac{[\alpha_{s,b}]^2}{\epsilon^3}\,(2E_1)^{-4\epsilon}\int\limits_0^1 {\rm d}z\,(1-z)^{-4\epsilon}P_{qq}(z)\\ &\times\left\langle\left[2C_F\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\eta_{14}^{-\epsilon}K_{14} + C_A\,\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\right]\frac{F_{\rm LM}(z\cdot 1,4)}{z}\right\rangle,\end{aligned} \tag{6.40}$$

where we already integrated over the angular phase space using Eq. (6.37). Combining the result in Eq. (6.40) with the soft-collinear contribution Eq. (6.38) we find

$$\begin{aligned}&\left\langle C_{51}\left[I - S_5\right]J_{145}w^{51}F_{\rm LM}(1,4\,|\,5)\right\rangle\\ &= -\frac{[\alpha_s]^2}{\epsilon^3}\left\langle\left[2C_F\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\eta_{14}^{-\epsilon}K_{14} + C_A\,\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\right]\right.\\ &\quad\left.\times\left[\left(\frac{4E_1^2}{\mu^2}\right)^{-2\epsilon}\int\limits_0^1 {\rm d}z\,(1-z)^{-4\epsilon}P_{qq}(z)\,\frac{F_{\rm LM}(z\cdot 1,4)}{z} + \frac{2C_F}{4\epsilon}\left(\frac{4E_{\rm max}^2}{\mu^2}\right)^{-2\epsilon}F_{\rm LM}(1,4)\right]\right\rangle.\end{aligned} \tag{6.41}$$

Next, we consider the collinear $\vec{p}_5 \parallel \vec{p}_4$ limit described by the operator C_{54}. The relevant term in Eq. (6.30) reads

$$\left\langle C_{54}\left[I - S_5\right]w^{54}J_{145}F_{\rm LM}(1,4\,|\,5)\right\rangle. \tag{6.42}$$

Following the established procedure, we compute the two contributions in Eq. (6.42) separately. Thanks to the fact that J_{145} is symmetric under the exchange of p_1 and p_4, the soft-collinear contribution in this case is identical to Eq. (6.38). Hence, we write

$$\begin{aligned}\left\langle C_{54}S_5\,J_{145}\,w^{54}F_{\rm LM}(1,4\,|\,5)\right\rangle = &\,2C_F\,\frac{[\alpha_{s,b}]^2}{4\epsilon^4}\,(2E_{\rm max})^{-4\epsilon}\\ &\times\left\langle\left[2C_F\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\eta_{14}^{-\epsilon}K_{14} + C_A\,\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\right]F_{\rm LM}(1,4)\right\rangle.\end{aligned} \tag{6.43}$$

We continue with the first term on the right-hand side of Eq. (6.42) where we need to consider the collinear $\vec{p}_5 \parallel \vec{p}_4$ limit. It reads

$$C_{54}F_{\rm LM}(1,4\,|\,5) = g_{s,b}^2 \times \frac{1}{p_4\cdot p_5}\,P_{qq}\,(z)\,F_{\rm LM}\left(1,\frac{1}{z}\cdot 4\right), \tag{6.44}$$

[9] Additional insights into the derivation of Eq. (6.40) can be found in the NLO discussion around Eq. (5.34).

where $z = E_4/(E_4+E_5)$. The splitting function $P_{qq}(z)$ can be found in Eq. (5.31). Using Eq. (6.33) to construct the collinear limit of the function J_{145}, we obtain

$$\begin{aligned}&\left\langle C_{54}\, J_{145}\, w^{54} F_{\mathrm{LM}}(1,4\,|\,5)\right\rangle \\ &= \frac{[\alpha_{s,b}]g_{s,b}^2}{\epsilon^2}\int [\mathrm{d}p_5]\left\langle \left[2C_F\,\eta_{14}^{-\epsilon}K_{14} + C_A\,\eta_{45}^{-\epsilon}\,\frac{\Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\Gamma(1-2\epsilon)}\right](2E_5)^{-2\epsilon}\right. \\ &\qquad \left.\times\,\frac{1}{p_4\cdot p_5}P_{qq}(z)\times F_{\mathrm{LM}}\left(1,\frac{1}{z}\cdot 4\right)\right\rangle .\end{aligned} \tag{6.45}$$

Integration over the angular phase space of the gluon $g(p_5)$ is identical to the previous case; the result can be borrowed from Eq. (6.37). To compute the energy integral over E_5 we follow the NLO discussion in Section 5.2. We solve $z = E_4/(E_4+E_5)$ for E_5 and parameterize the gluon energy as $E_5 = E_4(1-z)/z$. Furthermore, we rescale the four-momentum of a quark $q(p_4)$ as $p_4 \to z\cdot p_4$. Additional details can be found in the discussion of the NLO QCD case starting at Eq. (5.45). With these manipulations we re-write Eq. (6.45) as

$$\begin{aligned}\left\langle C_{54}\, J_{145}\, w^{54} F_{\mathrm{LM}}(1,4\,|\,5)\right\rangle &= -\frac{[\alpha_{s,b}]^2}{\epsilon^3}\underbrace{\int_0^1 \mathrm{d}z\, z^{-2\epsilon}(1-z)^{-4\epsilon}P_{qq}(z)}_{\overset{(E.27)}{=}\;-\left[\frac{2C_F}{4\epsilon}+\gamma_{qq}^{24}\right]} \\ &\times\left\langle \left[2C_F\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\eta_{14}^{-\epsilon}K_{14} + C_A\,\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\right](2E_4)^{-4\epsilon}F_{\mathrm{LM}}(1,4)\right\rangle .\end{aligned} \tag{6.46}$$

Together with the soft-collinear contribution Eq. (6.43) we obtain

$$\begin{aligned}&\left\langle C_{54}\left[1-S_5\right] J_{145}\, w^{54} F_{\mathrm{LM}}(1,4\,|\,5)\right\rangle \\ &= -\frac{[\alpha_s]^2}{\epsilon^3}\left\langle \left[2C_F\left(\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right)\eta_{14}^{-\epsilon}K_{14} + C_A\left(\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\right)\right]\right. \\ &\qquad\left.\times\left[2C_F\,\frac{(4E_{\max}^2/\mu^2)^{-2\epsilon}-(4E_4^2/\mu^2)^{-2\epsilon}}{4\epsilon} - \gamma_{qq}^{24}\left(\frac{4E_4^2}{\mu^2}\right)^{-2\epsilon}\right]F_{\mathrm{LM}}(1,4)\right\rangle .\end{aligned} \tag{6.47}$$

Finally, we combine everything and find the following result for the double-soft regulated single-soft subtraction term

$$\begin{aligned}
&\left\langle S_6\left[1-\mathbb{S}\right]F_{\rm LM}(1,4\,|\,5,6)\right\rangle = \sum_{i\in\{1,4\}} \left\langle \hat{\mathcal{O}}^{(i)}_{\rm nlo}\, J_{145}\, w^{5i} F_{\rm LM}(1,4\,|\,5)\right\rangle \\
&\quad -\frac{[\alpha_s]^2}{\epsilon^3}\left\langle \left[-2C_F\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,\eta_{14}^{-\epsilon}K_{14} - C_A\,\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\right]\right. \\
&\quad\times\left\{2C_F\left(\frac{(4E_4^2/\mu^2)^{-2\epsilon}-(4E_{\rm max}^2/\mu^2)^{-2\epsilon}}{4\epsilon}+\frac{(4E_1^2/\mu^2)^{-2\epsilon}-(4E_{\rm max}^2/\mu^2)^{-2\epsilon}}{4\epsilon}\right)F_{\rm LM}(1,4)\right. \\
&\quad\left.\left.+\gamma_{qq}^{24}\left(\frac{4E_4^2}{\mu^2}\right)^{-2\epsilon}F_{\rm LM}(1,4)-\left(\frac{4E_1^2}{\mu^2}\right)^{-2\epsilon}\int_0^1 {\rm d}z\,\mathcal{P}_{qq,R4}(z)\frac{F_{\rm LM}(z\cdot 1,4)}{z}\right\}\right\rangle.
\end{aligned} \tag{6.48}$$

In Eq. (6.48) we introduced the following generalized splitting function

$$\mathcal{P}_{qq,Rn}(z) = 2C_F\left[\frac{(1-z)^{-n\epsilon}}{1-z}\right]_+ - C_F(1-z)^{-n\epsilon}\left[(1+z)+\epsilon(1-z)\right]. \tag{6.49}$$

NNLO cross sections receive contributions from final states with at most two additional partons, each of which can become soft and/or collinear. It follows that the highest pole in ϵ is $1/\epsilon^4$. In the subtraction term Eq. (6.48) the double-soft singularity is regularized. Hence, we expect the highest pole in Eq. (6.48) to be $1/\epsilon^3$. It is easy to see that this is indeed the case.

The only singularities in Eq. (6.48) are explicit poles in $1/\epsilon$. In the first term on the right-hand side of Eq. (6.48) the operator $\hat{\mathcal{O}}^{(i)}_{\rm nlo}$ regulates singularities of the NLO function $F_{\rm LM}(1,4\,|\,5)$, it is defined in Eq. (5.15). The $1/\epsilon$ poles that appear in other contributions to Eq. (6.48) will cancel with similar $1/\epsilon$ poles from double-virtual, real-virtual and collinear renormalization contributions. Note that this implies that the $1/\epsilon$ poles that are present in the factor J_{145} will cancel with similar contributions that multiply regulated NLO differential cross section.

6.3. Single-unresolved collinear subtraction terms

We continue with the study of single-collinear subtraction terms in Eq. (6.17) where one of the two gluons becomes collinear to another parton. We distinguish two such terms. In the first one a gluon becomes collinear to either initial of final state quark (third term on the right-hand side of Eq. (6.17)). It reads

$$\begin{aligned}
\sum_{\substack{i,j\in\{1,4\}\\ i\neq j}} &\left\langle \left[I-\mathbb{S}\right]\left[I-S_6\right]\left[C_{5i}w^{5i,6j}+C_{6i}w^{5j,6i}+\left(\theta^{(a)}C_{5i}+\theta^{(c)}C_{6i}\right)w^{5i,6i}\right]\right. \\
&\quad\left.\times\,[{\rm d}p_5][{\rm d}p_6]F_{\rm LM}(1,4\,|\,5,6)\right\rangle.
\end{aligned} \tag{6.50}$$

The second one is the contribution where the two gluons become collinear to each other (fourth term on the right-hand side of Eq. (6.17)). It reads

$$\sum_{i\in\{1,4\}}\left\langle \left[I-\mathbb{S}\right]\left[I-S_6\right]\left[\theta^{(b)}C_{56}+\theta^{(d)}C_{56}\right][{\rm d}p_5][{\rm d}p_6]w^{5i,6i}F_{\rm LM}(1,4\,|\,5,6)\right\rangle. \tag{6.51}$$

We consider the different contributions to Eq. (6.50) and Eq. (6.51) individually, starting with the initial-state splittings ($i = 1$ in Eq. (6.50)) in Section 6.3.1. We then discuss the final-state splitting ($i = 4$ in Eq. (6.50)) in Section 6.3.2. Finally, we discuss the gluon splitting Eq. (6.51) in Section 6.3.3.

6.3.1. Initial-state emission

We begin with discussing how to extract the $1/\epsilon$ poles from the subtraction term

$$\Big\langle [I - \mathbb{S}]\,[I - S_6]\Big[C_{51}w^{51,64} + C_{61}w^{54,61} + \Big(\theta^{(a)}C_{51} + \theta^{(c)}C_{61}\Big)w^{51,61}\Big] \\ \times [\mathrm{d}p_5][\mathrm{d}p_6]F_{\mathrm{LM}}(1,4\,|\,5,6)\Big\rangle\,, \tag{6.52}$$

where one of the two gluons is collinear to the initial-state quark $q(p_1)$. First, we note that in a collinear limit $C_{ij}F_{\mathrm{LM}}(1,4\,|\,5,6)$, with $i \in \{5,6\}$ and $j \in \{1,4\}$, no dependence on the sum of gluon energies $E_5 + E_6$ remains anywhere. Hence, after taking a limit where one of the gluons becomes collinear to an external quark, the double-soft operator $\mathbb{S}$ can be replaced with S_5S_6. This feature leads to simplifications in Eq. (6.52). Indeed, we find

$$\mathbb{S}\big[I - S_6\big]C_{ij}F_{\mathrm{LM}}(1,4\,|\,5,6) = \big[S_5S_6 - S_5S_6\big]C_{ij}F_{\mathrm{LM}}(1,4\,|\,5,6) = 0\,, \tag{6.53}$$

which means that we can omit all terms that involve the $\mathbb{S}$ operator in Eq. (6.52). We further note that, double-collinear operators at NNLO are defined in such a way that they also act on the phase space volume element. These limits are discussed in Appendix F.

We continue with the terms that contain double-collinear partitions $w^{51,64}$ and $w^{54,61}$

$$\Big\langle \big[I - S_6\big]\Big[C_{51}w^{51,64} + C_{61}w^{54,61}\Big][\mathrm{d}p_5][\mathrm{d}p_6]F_{\mathrm{LM}}(1,4\,|\,5,6)\Big\rangle\,. \tag{6.54}$$

Since collinear singularities factorize on external lines, this subtraction term is identical to the case of colour-singlet production. We refer to the discussion of this process in Ref. [37] for more details.[10] Both the collinear limits of $F_{\mathrm{LM}}(1,4\,|\,5,6)$ and the integration over momentum of the unresolved gluon follows the NLO discussion in Section 5.2 with additional constraints on its energy. We obtain

$$\begin{aligned}&\big\langle \big[I - S_6\big]C_{51}w^{51,64}[\mathrm{d}p_5][\mathrm{d}p_6]F_{\mathrm{LM}}(1,4\,|\,5,6)\big\rangle \\ &= -\frac{[\alpha_s]}{\epsilon}\int\limits_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon}P_{qq}(z)\,\Big\langle \theta(E_1(1-z) - E_6) \times w^{64}_{\mathrm{dc}}(2E_1)^{-2\epsilon}\big[1 - S_6\big] \\ &\quad \times \frac{F_{\mathrm{LM}}(z\cdot 1,4\,|\,6)}{z}\Big\rangle\,,\end{aligned} \tag{6.55}$$

[10] Note that in Ref. [37] the energy cut-off E_{max} was identified with the partonic center-of-mass energy. However, for the contribution in question, this is not relevant because for initial-state emission energy integration is bound by the energy-momentum conservation in the case of C_{51} and the energy ordering condition $E_6 < E_5$ in the case of C_{61}.

and

$$
\begin{aligned}
&\big\langle \big[I-S_6\big] C_{61} w^{54,61} [\mathrm{d}p_5][\mathrm{d}p_6] F_{\mathrm{LM}}(1,4\,|\,5,6) \big\rangle \\
&= -\frac{[\alpha_s]}{\epsilon} \times \int\limits_0^1 \mathrm{d}z\; (1-z)^{-2\epsilon} P_{qq}(z) \left\langle \theta(E_5 - E_1(1-z)) \times (2E_1)^{-2\epsilon} w_{\mathrm{dc}}^{54} \frac{F_{\mathrm{LM}}(z\cdot 1,4\,|\,5)}{z} \right\rangle \\
&\quad - \frac{[\alpha_s] C_F}{\epsilon^2} \times (2E_5)^{-2\epsilon} \big\langle w_{\mathrm{dc}}^{54} F_{\mathrm{LM}}(1,4\,|\,5) \big\rangle \,,
\end{aligned}
\tag{6.56}
$$

where w_{dc}^{54} and w_{dc}^{64} are the single collinear limits of the partition functions

$$
w_{\mathrm{dc}}^{54} = \lim_{p_6 || p_1} w^{54,61} \,, \quad w_{\mathrm{dc}}^{64} = \lim_{p_5 || p_1} w^{51,64} \,. \tag{6.57}
$$

Explicit expressions for w_{dc}^{54} and w_{dc}^{64} can be found in Eq. (A.14). We keep the soft subtraction implicit in Eq. (6.55) to enable easier extraction of remaining NLO singularities.

We combine Eqs. (6.55, 6.56) and simplify the result. To this end, we rename $6 \to 5$ in Eq. (6.55) and insert $I = \big[I - S_5\big] + S_5$ in Eq. (6.56). We obtain[11]

$$
\begin{aligned}
&\big\langle \big[I-S_6\big] \Big[C_{51} w^{51,64} + C_{61} w^{54,61}\Big] [\mathrm{d}p_5][\mathrm{d}p_6] F_{\mathrm{LM}}(1,4\,|\,5,6) \big\rangle \\
&= -\frac{[\alpha_{s,b}]}{\epsilon} \times \int\limits_0^1 \mathrm{d}z\; (1-z)^{-2\epsilon} P_{qq}(z) \left\langle w_{\mathrm{dc}}^{54} (2E_1)^{-2\epsilon} \big[1 - S_5\big] \frac{F_{\mathrm{LM}}(z\cdot 1,4\,|\,5)}{z} \right\rangle \\
&\quad - \frac{[\alpha_{s,b}]}{\epsilon} \times \int\limits_0^1 \mathrm{d}z\; (1-z)^{-2\epsilon} P_{qq}(z) \left\langle w_{\mathrm{dc}}^{54} (2E_1)^{-2\epsilon} \theta(E_5 - E_1(1-z)) S_5 \frac{F_{\mathrm{LM}}(z\cdot 1,4\,|\,5)}{z} \right\rangle \\
&\quad - \frac{[\alpha_{s,b}] C_F}{\epsilon^2} \times \big\langle w_{\mathrm{dc}}^{54} (2E_5)^{-2\epsilon} F_{\mathrm{LM}}(1,4\,|\,5) \big\rangle \,,
\end{aligned}
\tag{6.58}
$$

where we used $\theta(E_1(1-z) - E_5) + \theta(E_5 - E_1(1-z)) = 1$. In the second term on the right-hand side of Eq. (6.58) gluon $g(p_5)$ is taken in the soft limit. In this limit the gluon $g(p_5)$ decouples from the function F_{LM} and we can integrate over $[\mathrm{d}p_5]$ analytically. We obtain

$$
\begin{aligned}
&\int [\mathrm{d}p_5] \theta(E_5 - E_1(1-z))\, w_{\mathrm{dc}}^{54} \times S_5 \frac{F_{\mathrm{LM}}(z\cdot 1,4\,|\,5)}{z} \\
&= 2C_F\, g_{s,b}^2 \int [\mathrm{d}p_5] \theta(E_5 - E_1(1-z))\, w_{\mathrm{dc}}^{54} \times \frac{1}{E_5^2} \times \frac{\rho_{14}}{\rho_{15}\rho_{45}} \times \frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z} \\
&= 2C_F \frac{[\alpha_{s,b}]}{2\epsilon^2} \big\langle w_{\mathrm{dc}}^{54} \big\rangle_{S_5} \big[(2E_{\mathrm{max}})^{-2\epsilon} - (2E_1)^{-2\epsilon} (1-z)^{-2\epsilon} \big] \frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z} \,.
\end{aligned}
\tag{6.59}
$$

We note that we introduced the following notation to define the angular integral in Eq. (6.59)

$$
\langle \mathcal{O} \rangle_{S_5} \equiv \left(-\frac{2^{-2\epsilon}}{\epsilon} \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] \right)^{-1} \int \frac{\mathrm{d}\Omega_5^{(d-1)}}{2(2\pi)^{(d-1)}} \frac{\rho_{14}}{\rho_{15}\rho_{45}}\, \mathcal{O}(\Omega_5) \,, \tag{6.60}
$$

[11] Note that in order to combine the terms it is crucial that the partition functions $w^{51,64}$ and $w^{54,61}$ are defined in such a way that $w^{51,64}$ becomes $w^{54,61}$ upon exchanging p_5 with p_6.

where the function $\mathcal{O}$ has a residual dependence on the partition functions. The angular integral of the partition function $\langle w^{54}_{\rm dc}\rangle_{S_5}$ is computed in Appendix G. In Eq. (6.59) this function multiplies $1/\epsilon$ poles and it seems that this implies a dependence of the $1/\epsilon$ poles on the chosen partition function $w^{51,64}$. We want to emphasize that this is not the case. We refer to the discussion in Appendix H where we demonstrate this explicitly.

We use Eq. (6.59) to rewrite Eq. (6.58) in the following way

$$
\begin{aligned}
&\left\langle [I-\mathbb{S}]\,[I-S_6]\left[C_{51}w^{51,64}+C_{61}w^{54,61}\right][\mathrm{d}p_5][\mathrm{d}p_6]F_{\rm LM}(1,4\,|\,5,6)\right\rangle \\
&= -\frac{[\alpha_{s,b}]}{\epsilon}\times\int \mathrm{d}z\,(1-z)^{-2\epsilon}P_{qq}(z)\left\langle w^{54}_{\rm dc}(2E_1)^{-2\epsilon}\left[1-S_5\right]\frac{F_{\rm LM}(z\cdot 1,4\,|\,5)}{z}\right\rangle \\
&\quad -2C_F\,\frac{[\alpha_{s,b}]^2}{\epsilon^2}\times\int \mathrm{d}z\,(1-z)^{-2\epsilon}P_{qq}(z) \\
&\quad\times\left\langle\,\left[(2E_{\max})^{-2\epsilon}-(2E_1)^{-2\epsilon}(1-z)^{-2\epsilon}\right]\langle w^{54}_{\rm dc}\rangle_{S_5}\frac{F_{\rm LM}(z\cdot 1,4)}{z}\right\rangle_{\delta} \\
&\quad -\frac{[\alpha_{s,b}]C_F}{\epsilon^2}\times\left\langle w^{54}_{\rm dc}(2E_5)^{-2\epsilon}F_{\rm LM}(1,4\,|\,5)\right\rangle.
\end{aligned}
\tag{6.61}
$$

The first and the last terms on the right-hand side of Eq. (6.61) still contain unregulated singularities that arise when gluon $g(p_5)$ becomes soft or collinear. We need to regulate them, but before we do so, it is useful to combine Eq. (6.61) with the contributions from the triple-collinear partition $w^{51,61}$ in Eq. (6.52). We discuss the computation of these contributions below.

The relevant triple-collinear contributions are defined as

$$
\left\langle [I-\mathbb{S}]\,[I-S_6]\left[\theta^{(a)}C_{51}+\theta^{(c)}C_{61}\right][\mathrm{d}p_5][\mathrm{d}p_6]\,w^{51,61}\,F_{\rm LM}(1,4\,|\,5,6)\right\rangle. \tag{6.62}
$$

In addition to the partition functions $w^{51,61}$, the contribution Eq. (6.62) depends on sectors (a) and (c) that correspond to regions in the angular phase space of gluons $g(p_5)$ and $g(p_6)$ with definite double-collinear singularities. The different phase space regions are shown in Fig. 6.3. The double-collinear limits that appear in Eq. (6.62) coincide with the double-collinear limits in Eq. (6.54). The integration over the unresolved phase space is (almost) identical. The only difference is that instead of integrating over the full angular phase space we only integrate over a given sector. For instance for sector (a) we find

$$
\begin{aligned}
&\int (C_{51}[\mathrm{d}\Omega_5])\theta^{(a)}\,\frac{1}{\rho_{15}} \equiv \int (C_{51}[\mathrm{d}\Omega_5])\theta\left(\frac{\eta_{16}}{2}-\eta_{15}\right)\frac{1}{\rho_{15}} \\
&= -\frac{2^{-2\epsilon}}{\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\right]\left(\frac{\eta_{16}}{2}\right)^{-\epsilon} = \left(\frac{\eta_{16}}{2}\right)^{-\epsilon}\int (C_{51}[\mathrm{d}\Omega_5])\,\frac{1}{\rho_{15}}\,.
\end{aligned}
\tag{6.63}
$$

We obtain a similar result for sector (c)

$$\int \big(C_{61}[\mathrm{d}\Omega_6]\big)\theta_1^{(c)}\,\frac{1}{\rho_{16}} = \int \big(C_{61}[\mathrm{d}\Omega_6]\big)\theta\left(\eta_{16}-\frac{\eta_{15}}{2}\right)\frac{1}{\rho_{16}} \\ = \left(\frac{\eta_{15}}{2}\right)^{-\epsilon}\int \big(C_{51}[\mathrm{d}\Omega_5]\big)\,\frac{1}{\rho_{15}}\,. \tag{6.64}$$

Another difference to the discussion of double-collinear partitions in Eq. (6.54) is that the limits of the partition functions are now given by

$$w_{\mathrm{tc}}^{61} \equiv \lim_{p_5\|p_1} w^{51,61}\,, \quad w_{\mathrm{tc}}^{51} \equiv \lim_{p_6\|p_1} w^{51,61}\,. \tag{6.65}$$

The result for the triple-collinear partition Eq. (6.62) can therefore be obtained from the result for the double-collinear partition Eq. (6.61) with the replacement

$$w_{\mathrm{dc}}^{54} \quad\to\quad w_{\mathrm{tc}}^{51}\left(\frac{\rho_{15}}{4}\right)^{-\epsilon}\,. \tag{6.66}$$

Combining the results of this procedure with Eq. (6.61), we find an expression for Eq. (6.52)

$$\begin{aligned}
&\Big\langle [I-\mathbb{S}][I-S_6]\Big[C_{51}w^{51,64}+C_{61}w^{54,61}+\Big(\theta_1^{(a)}C_{51}+\theta_1^{(c)}C_{61}\Big)w^{51,61}\Big] \\
&\quad\times[\mathrm{d}p_5][\mathrm{d}p_6]F_{\mathrm{LM}}(1,4\,|\,5,6)\Big\rangle \\
&= -\frac{[\alpha_{s,b}]}{\epsilon}\int \mathrm{d}z\,(1-z)^{-2\epsilon}P_{qq}(z)\left\langle\left(w_{\mathrm{dc}}^{54}+w_{\mathrm{tc}}^{51}\left(\frac{\rho_{15}}{4}\right)^{-\epsilon}\right)(2E_1)^{-2\epsilon}\big[1-S_5\big]\frac{F_{\mathrm{LM}}(z\cdot 1,4\,|\,5)}{z}\right\rangle \\
&\quad -2C_F\,\frac{[\alpha_{s,b}]^2}{\epsilon^2}\times\int_0^1\mathrm{d}z\,(1-z)^{-2\epsilon}P_{qq}(z)\,\Big((2E_{\max})^{-2\epsilon}-(2E_1)^{-2\epsilon}(1-z)^{-2\epsilon}\Big) \\
&\qquad\times\left\langle\langle\Delta_{61}\rangle_{S_5}\frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\right\rangle_\delta \\
&\quad -C_F\,\frac{[\alpha_{s,b}]}{\epsilon^2}\times\left\langle\left(w_{\mathrm{dc}}^{54}+w_{\mathrm{tc}}^{51}\left(\frac{\rho_{15}}{4}\right)^{-\epsilon}\right)(2E_5)^{-2\epsilon}F_{\mathrm{LM}}(1,4\,|\,5)\right\rangle.
\end{aligned} \tag{6.67}$$

In writing Eq. (6.67) we introduced the function

$$\Delta_{61} \equiv w_{\mathrm{dc}}^{54}+\left(\frac{\rho_{15}}{4}\right)^{-\epsilon}w_{\mathrm{tc}}^{51}\,. \tag{6.68}$$

The first and the last terms on the right-hand side of Eq. (6.67) still contain unregulated collinear and soft singularities. We discuss how to extract them in the next section.

Fully-regulated initial-state emission

In this section we regulate remaining singularities that are present in the NLO-like single real emission contribution in the first and the last term on the right-hand side of Eq. (6.67). To regulate the NLO singularities we follow the discussion in Chapter 5. The collinear limits of the NNLO partition functions $w^{54}_{\rm dc} = C_{61}\, w^{54,61}$ and $w^{51}_{\rm tc} = C_{61}\, w^{51,61}$ possess all the properties that valid NLO partition functions should have. Therefore, they provide a proper partitioning of NLO collinear singularities and allow us to deal with one collinear singularity at a time.

We begin our discussion with the last term on the right-hand side of Eq. (6.67). In analogy to the NLO discussion, c.f. Eq. (5.14), we write

$$
\begin{aligned}
&\left\langle \left(w^{54}_{\rm dc} + w^{51}_{\rm tc} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \right) (2E_5)^{-2\epsilon} F_{\rm LM}(1,4\,|\,5) \right\rangle \\
&= \left\langle S_5 \left(w^{54}_{\rm dc} + w^{51}_{\rm tc} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \right) (2E_5)^{-2\epsilon} F_{\rm LM}(1,4\,|\,5) \right\rangle \\
&\quad + \left\langle [I - S_5] \left(C_{54}\, w^{54}_{\rm dc} + C_{51}\, w^{51}_{\rm tc} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \right) (2E_5)^{-2\epsilon} F_{\rm LM}(1,4\,|\,5) \right\rangle \\
&\quad + \left\langle \left[\hat{\mathcal{O}}^{(4)}_{\rm nlo}\, w^{54}_{\rm dc} + \hat{\mathcal{O}}^{(1)}_{\rm nlo}\, w^{51}_{\rm tc} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \right] (2E_5)^{-2\epsilon} F_{\rm LM}(1,4\,|\,5) \right\rangle_{\delta} .
\end{aligned}
\tag{6.69}
$$

The $\hat{\mathcal{O}}^{(i)}_{\rm nlo}$ operators are defined in Eq. (5.15). Computation of the subtraction terms is straightforward. For the soft subtraction term we obtain

$$
\begin{aligned}
&\int [\mathrm{d}p_5]\, S_5 \left(w^{54}_{\rm dc} + w^{51}_{\rm tc} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \right) (2E_5)^{-2\epsilon}\, F_{\rm LM}(1,4\,|\,5) \\
&\stackrel{(5.7)}{=} 2C_F\, g^2_{s,b} \times 2^{-2\epsilon} \underbrace{\int_0^{E_{\rm max}} \frac{\mathrm{d}E_5}{E_5^{1+4\epsilon}}}_{= -\frac{E_{\rm max}^{-4\epsilon}}{4\epsilon}} \times \underbrace{\int \frac{\mathrm{d}\Omega_5^{(d-1)}}{2(2\pi)^{d-1}} \frac{\rho_{14}}{\rho_{15}\rho_{45}} \left(w^{54}_{\rm dc} + w^{51}_{\rm tc} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \right)}_{\stackrel{(6.60)}{=} -\frac{2^{-2\epsilon}}{\epsilon} \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] \langle \Delta_{61} \rangle_{S_5}} F_{\rm LM}(1,4) \\
&= 2C_F\, \frac{[\alpha_{s,b}]}{4\epsilon^2}\, (2E_{\rm max})^{-4\epsilon} \langle \Delta_{61} \rangle_{S_5} F_{\rm LM}(1,4)\,,
\end{aligned}
\tag{6.70}
$$

where Δ_{61} was defined in Eq. (6.68). In case of the soft-regulated collinear subtraction term an additional factor $(2E_5)^{-2\epsilon}$ is present in comparison with to NLO computation. In case of the initial-state emission an additional factor $\rho_{15}^{-\epsilon}$ is present. Apart from this, the computation is similar to the NLO one. For this reason, we will not describe it and only provide one additional angular integral that is needed. It reads

$$
\int \frac{\mathrm{d}\Omega_5^{(d-1)}}{2(2\pi)^{d-1}} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \frac{1}{\rho_{15}} = -\frac{2^{-2\epsilon}}{\epsilon} \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] \left[\frac{2^\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} \right] . \tag{6.71}
$$

After following up the steps discussed in the context of the NLO computation, we find

$$
\begin{aligned}
&\left\langle C_{51}\,\left[I-S_5\right]\left(\frac{\rho_{15}}{4}\right)^{-\epsilon}(2E_5)^{-2\epsilon}\,w_{\rm tc}^{51}\,F_{\rm LM}(1,4\,|\,5)\right\rangle \\
&= -\frac{[\alpha_{s,b}]}{\epsilon}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right](2E_1)^{-4\epsilon}\int\limits_0^1 {\rm d}z\,(1-z)^{-4\epsilon}P_{qq}(z)\left\langle\frac{F_{\rm LM}(z\cdot 1,4)}{z}\right\rangle_\delta \\
&\quad -2C_F\,\frac{[\alpha_{s,b}]}{\epsilon}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]\left[\frac{(2E_{\rm max})^{-4\epsilon}-(2E_1)^{-4\epsilon}}{4\epsilon}\right]\left\langle F_{\rm LM}(1,4)\right\rangle_\delta ,
\end{aligned}
\tag{6.72}
$$

$$
\begin{aligned}
&\left\langle C_{54}\,\left[I-S_5\right](2E_5)^{-2\epsilon}\,w_{\rm dc}^{54}\,F_{\rm LM}(1,4\,|\,5)\right\rangle \\
&= \frac{[\alpha_{s,b}]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\gamma_{qq}^{24}\,\left\langle(2E_4)^{-4\epsilon}F_{\rm LM}(1,4)\right\rangle_\delta \\
&\quad -2C_F\,\frac{[\alpha_{s,b}]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left\langle\left[\frac{(2E_{\rm max})^{-4\epsilon}-(2E_4)^{-4\epsilon}}{4\epsilon}\right]F_{\rm LM}(1,4)\right\rangle_\delta .
\end{aligned}
\tag{6.73}
$$

This completes the extraction of the singularities in the last term of Eq. (6.67). Before combining these formulas, we proceed with the extraction of the remaining singularities in the first term on the right-hand side of Eq. (6.67). The soft singularity is already regulated and we only have to regulate the collinear singularities. The relevant formula reads

$$
\begin{aligned}
&\int\limits_0^1 {\rm d}z\,(1-z)^{-2\epsilon}P_{qq}(z)\left\langle\left[1-S_5\right]\left(w_{\rm dc}^{54}+w_{\rm tc}^{51}\left(\frac{\rho_{15}}{4}\right)^{-\epsilon}\right)(2E_1)^{-2\epsilon}\frac{F_{\rm LM}(z\cdot 1,4\,|\,5)}{z}\right\rangle \\
&= \int\limits_0^1 {\rm d}z\,(1-z)^{-2\epsilon}P_{qq}(z) \\
&\qquad\times\left\langle\left[1-S_5\right]\left(C_{54}\,w_{\rm dc}^{54}+C_{51}\,w_{\rm tc}^{51}\left(\frac{\rho_{15}}{4}\right)^{-\epsilon}\right)(2E_1)^{-2\epsilon}\frac{F_{\rm LM}(z\cdot 1,4\,|\,5)}{z}\right\rangle \\
&\quad+\int\limits_0^1 {\rm d}z\,(1-z)^{-2\epsilon}P_{qq}(z)\left\langle\left(\hat{\mathcal{O}}_{\rm nlo}^{(4)}w_{\rm dc}^{54}+\hat{\mathcal{O}}_{\rm nlo}^{(1)}w_{\rm tc}^{51}\left(\frac{\rho_{15}}{4}\right)^{-\epsilon}\right)(2E_1)^{-2\epsilon}\frac{F_{\rm LM}(z\cdot 1,4\,|\,5)}{z}\right\rangle_\delta .
\end{aligned}
\tag{6.74}
$$

To compute the subtraction term on the right-hand side of Eq. (6.74) we consider soft and soft-collinear contributions separately. We begin with the soft-collinear contribution.

The soft-collinear limit is obtained from Eq. (5.27) with the replacement $p_1 \to z\cdot p_1$. We find

$$
C_{5i}\,S_5\,F_{\rm LM}(z\cdot 1,4\,|\,5) = 2C_F\,g_{s,b}^2\,\times\frac{1}{E_5^2\,\rho_{i5}}\times F_{\rm LM}(z\cdot 1,4)\,,
\tag{6.75}
$$

with $i\in\{1,4\}$. The only dependence on z remains in the function $F_{\rm LM}$. Hence, we can integrate over the unresolved phase space of the gluon $g(p_5)$ in full analogy with the NLO computation. Using angular integral Eq. (6.71) for the initial-state emission and following steps that led to

Eqs. (5.29, 5.42), we obtain

$$\int_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon} P_{qq}(z) \left\langle S_5 \left(C_{54}\, w_{\mathrm{dc}}^{54} + C_{51}\, w_{\mathrm{tc}}^{51} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \right) (2E_1)^{-2\epsilon} \frac{F_{\mathrm{LM}}(z\cdot 1, 4\,|\,5)}{z} \right\rangle$$
$$= 2C_F \frac{[\alpha_{s,b}]}{2\epsilon^2} \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} + \frac{2^\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} \right] \int_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon} P_{qq}(z) \tag{6.76}$$
$$\times (2E_{\mathrm{max}})^{-2\epsilon} (2E_1)^{-2\epsilon} \left\langle \frac{F_{\mathrm{LM}}(z\cdot 1, 4)}{z} \right\rangle_\delta .$$

We continue with the collinear contributions to the first term on the right-hand side of Eq. (6.74). For final-state emissions, we can re-use the NLO result in Eq. (5.51). We find

$$\int_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon} P_{qq}(z) \left\langle C_{54}\, w_{\mathrm{dc}}^{54} (2E_1)^{-2\epsilon} \frac{F_{\mathrm{LM}}(z\cdot 1, 4\,|\,5)}{z} \right\rangle$$
$$= \frac{[\alpha_{s,b}]}{\epsilon} \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right] \left[\frac{2C_F}{2\epsilon} + \gamma_{qq}^{22} \right] \int_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon} P_{qq}(z) \tag{6.77}$$
$$\times \left\langle (2E_4)^{-2\epsilon} (2E_1)^{-2\epsilon} \frac{F_{\mathrm{LM}}(z\cdot 1, 4)}{z} \right\rangle_\delta .$$

For terms that describe initial-state emissions, we need to compute

$$\int_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon} P_{qq}(z) \left\langle C_{51}\, w_{\mathrm{tc}}^{51} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \frac{F_{\mathrm{LM}}(z\cdot 1, 4\,|\,5)}{z} \right\rangle . \tag{6.78}$$

Here we need to take a bit more care since we now have to deal with convolutions of splitting functions. The collinear limit of the cross section can be obtained from the NLO collinear limit given in Eq. (5.30) with the replacement $p_1 \to z \cdot p_1$. After straightforward manipulations, we write it as

$$C_{51}\, F_{\mathrm{LM}}(z\cdot 1, 4\,|\,5) = g_{s,b}^2 \times \frac{1}{p_1 \cdot p_5}\, P_{qq}\left(\frac{\tilde{z}}{z}\right) \times \frac{F_{\mathrm{LM}}(\tilde{z}\cdot 1, 4)}{\tilde{z}} , \tag{6.79}$$

where we introduced

$$\tilde{z} = \frac{zE_1 - E_5}{E_1} . \tag{6.80}$$

We solve the above equation for the energy of the unresolved gluon and obtain $E_5 = E_1(z - \tilde{z})$. Integration over E_5 becomes

$$\int_0^{E_{\mathrm{max}}} \mathrm{d}E_5\, E_5^{1-2\epsilon} \quad \to \quad \int_{\tilde{z}_{\mathrm{min}}}^{z} \mathrm{d}\tilde{z}\, E_1^{2-2\epsilon} (z - \tilde{z})^{1-2\epsilon} , \tag{6.81}$$

with $\tilde{z}_{\min} = (zE_1 - E_{\max})/E_1$. By construction $E_{\max} > E_1$ and $z \in [0,1]$, so that $\tilde{z}_{\min} \leq 0$. For values $\tilde{z} \leq 0$ there is not enough energy to produce final-state particles. This implies that the function $F_{\rm LM}(\tilde{z}\cdot 1, 4)$ vanishes because of the energy-momentum conserving δ-function inside it. We can therefore replace the lower integration boundary $\tilde{z}_{\min}$ with zero, without affecting the value of the integral as we have done previously. Using Eq. (6.71) to integrate over angular phase space of the unresolved gluon, we obtain

$$
\begin{aligned}
&\int_0^1 {\rm d}z\, (1-z)^{-2\epsilon} P_{qq}(z) \left\langle C_{51}\, w_{\rm tc}^{51} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \frac{F_{\rm LM}(z\cdot 1, 4\,|\,5)}{z} \right\rangle \\
&= -\frac{[\alpha_{s,b}]}{\epsilon} \left[\frac{2^\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right] (2E_1)^{-2\epsilon} \\
&\quad \times \int_0^1 \frac{{\rm d}z}{z}\, (1-z)^{-2\epsilon} P_{qq}(z) \int_0^z {\rm d}\tilde{z}\, (z-\tilde{z})^{-2\epsilon}\, P_{qq}\left(\frac{\tilde{z}}{z}\right) \times \left\langle \frac{F_{\rm LM}(\tilde{z}\cdot 1, 4)}{\tilde{z}} \right\rangle_\delta .
\end{aligned} \tag{6.82}
$$

We can further simplify this result by integrating over z since the function $F_{\rm LM}$ does not depend on it. We find

$$
\int_{\tilde{z}}^1 \frac{{\rm d}z}{z}\, (1-z)^{-2\epsilon} P_{qq}(z) \times (z-\tilde{z})^{-2\epsilon} P_{qq}\left(\frac{\tilde{z}}{z}\right) = \left[P_{qq}^{22} \otimes P_{qq}^{02}\right](\tilde{z})\,, \tag{6.83}
$$

where the convolution $\otimes$ is defined in Eq. (5.60) and generalized splitting functions read

$$
P_{qq}^{nk}(z) \equiv z^{-n\epsilon}(1-z)^{-k\epsilon} P_{qq}(z)\,. \tag{6.84}
$$

The result for the integral Eq. (6.83) is given in Appendix E as an expansion in the dimensional regularization parameter ϵ. Upon relabeling $\tilde{z} \to z$ in Eqs. (6.82, 6.83), we obtain

$$
\begin{aligned}
&\int_0^1 {\rm d}z\, (1-z)^{-2\epsilon} P_{qq}(z) \left\langle C_{51}\, w_{\rm tc}^{51} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \frac{F_{\rm LM}(z\cdot 1, 4\,|\,5)}{z} \right\rangle \\
&= -\frac{[\alpha_{s,b}]}{\epsilon} \left[\frac{2^\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right] (2E_1)^{-2\epsilon} \int_0^1 {\rm d}z\, \left[P_{qq}^{22} \otimes P_{qq}^{02}\right](z) \left\langle \frac{F_{\rm LM}(z\cdot 1, 4)}{z} \right\rangle_\delta .
\end{aligned} \tag{6.85}
$$

We insert Eqs. (6.70, 6.72, 6.73, 6.74, 6.76, 6.77, 6.85) into Eq. (6.69) and use the result to rewrite the double-collinear subtraction term Eq. (6.67) as

$$
\begin{aligned}
&\Big\langle [1-\mathbb{S}][1-S_6]\Big[C_{51}w^{51,64}+C_{61}w^{54,61}+\Big(\theta^{(a)}C_{51}+\theta^{(c)}C_{61}\Big)w^{51,61}\Big] \\
&\qquad\qquad \times [\mathrm{d}p_5][\mathrm{d}p_6]F_{\mathrm{LM}}(1,4\,|\,5,6)\Big\rangle \\
&= -\frac{[\alpha_s]}{\epsilon}\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\int_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon}P_{qq}(z)\,\Big\langle\Big[\hat{\mathcal{O}}^{(4)}_{\mathrm{nlo}}\,w^{54}_{\mathrm{dc}}+\hat{\mathcal{O}}^{(1)}_{\mathrm{nlo}}\,w^{51}_{\mathrm{tc}}\left(\frac{\rho_{15}}{4}\right)^{-\epsilon}\Big] \\
&\qquad \times\frac{F_{\mathrm{LM}}(z\cdot 1,4\,|\,5)}{z}\Big\rangle_\delta \\
&\quad -2C_F\,\frac{[\alpha_s]}{2\epsilon^2}\Big\langle\Big[\hat{\mathcal{O}}^{(4)}_{\mathrm{nlo}}\,w^{54}_{\mathrm{dc}}+\hat{\mathcal{O}}^{(1)}_{\mathrm{nlo}}\,w^{51}_{\mathrm{tc}}\left(\frac{\rho_{15}}{4}\right)^{-\epsilon}\Big]\left(\frac{4E_5^2}{\mu^2}\right)^{-\epsilon}F_{\mathrm{LM}}(1,4\,|\,5)\Big\rangle_\delta \\
&\quad +\frac{[\alpha_s]^2C_F}{\epsilon^3}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\left(\frac{4E_{\mathrm{max}}^2}{\mu^2}\right)^{-\epsilon} \\
&\qquad \times\int_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon}P_{qq}(z)\Big\langle\frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\Big\rangle_\delta \\
&\quad +\frac{[\alpha_s]^2}{\epsilon^2}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-2\epsilon}\int_0^1 \mathrm{d}z\,\left[P^{22}_{qq}\otimes P^{02}_{qq}\right](z)\Big\langle\frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\Big\rangle_\delta \\
&\quad -\frac{[\alpha_s]^2}{\epsilon^2}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\int_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon} \\
&\qquad \times P_{qq}(z)\Big\langle\Big[2C_F\frac{(4E_4/\mu^2)^{-\epsilon}-(4E_{\mathrm{max}}^2/\mu^2)^{-\epsilon}}{2\epsilon}+\gamma^{22}_{qq}\left(\frac{4E_4^2}{\mu^2}\right)^{-\epsilon}\Big]\frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\Big\rangle_\delta \\
&\quad -\frac{[\alpha_s]^2C_F}{\epsilon^3}\int_0^1 \mathrm{d}z\,\Big[\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\left(\frac{4E_{\mathrm{max}}^2}{\mu^2}\right)^{-\epsilon}-\left(\frac{4E_1^2}{\mu^2}\right)^{-2\epsilon}(1-z)^{-2\epsilon}\Big](1-z)^{-2\epsilon}P_{qq}(z) \\
&\qquad \times\Big\langle\langle\Delta_{61}\rangle_{S_5}\frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\Big\rangle \\
&\quad -\frac{[\alpha_s]^2C_F^2}{2\epsilon^4}\left(\frac{4E_{\mathrm{max}}^2}{\mu^2}\right)^{-2\epsilon}\Big\langle\Big[\langle\Delta_{61}\rangle_{S_5}-\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}-\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\Big]F_{\mathrm{LM}}(1,4)\Big\rangle_\delta \\
&\quad +2C_F\,\frac{[\alpha_s]^2}{2\epsilon^3}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-2\epsilon}\int_0^1 \mathrm{d}z\,(1-z)^{-4\epsilon}P_{qq}(z)\Big\langle\frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\Big\rangle_\delta \\
&\quad -2C_F\,\frac{[\alpha_s]^2}{2\epsilon^3}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left[\frac{2C_F}{4\epsilon}+\gamma^{24}_{qq}\right]\Big\langle\left(\frac{4E_4^2}{\mu^2}\right)^{-2\epsilon}F_{\mathrm{LM}}(1,4)\Big\rangle_\delta .
\end{aligned}
\tag{6.86}
$$

We note that the result presented in Eq. (6.86) contains implicit soft poles in the splitting functions. These poles can be conveniently extracted using plus prescriptions as soft regulators.

To illustrate this, consider

$$\begin{aligned}
&\int_0^1 \mathrm{d}z\, (1-z)^{-n\epsilon} P_{qq}(z)\, F(z) \\
&= \int_0^1 \mathrm{d}z \left[2C_F \frac{(1-z)^{-n\epsilon}}{1-z} - C_F (1-z)^{-2\epsilon} (1+z+(1-z)\epsilon) \right] F(z) \\
&= \int_0^1 \mathrm{d}z \left[2C_F \left[\frac{(1-z)^{-n\epsilon}}{1-z} \right]_+ - \frac{2C_F}{n\epsilon} \delta(1-z) - C_F (1-z)^{-n\epsilon} (1+z+(1-z)\epsilon) \right] F(z)\,,
\end{aligned} \tag{6.87}$$

where $F(z)$ represent an arbitrary function that is regular at $z = 1$. Thanks to Eq. (6.87) we write the splitting functions as

$$\begin{aligned}
(1-z)^{-n\epsilon} P_{qq}(z) = 2C_F \left[\frac{(1-z)^{-n\epsilon}}{1-z} \right]_+ - \frac{2C_F}{n\epsilon} \delta(1-z) \\
- C_F (1-z)^{-n\epsilon} [1+z+(1-z)\epsilon]\,.
\end{aligned} \tag{6.88}$$

The splitting function Eq. (6.88) contains three types of contributions: the first term on the right-hand side is regulated using the plus prescription. The second term contains soft $1/\epsilon$ pole explicitly. The third term is finite for all $z \in [0, 1]$.

Using Eq. (6.88) in Eq. (6.86) it is straightforward to extract all $1/\epsilon$ poles explicitly. Doing so, we end up with only $1/\epsilon^2$ poles since double-soft and single-soft singularities are regulated in Eq. (6.86). In what follows we often define various splitting functions; they can be found in Appendix E with all of their $1/\epsilon$ poles shown explicitly.

6.3.2. Final-state emission

We now turn to the discussion of the single unresolved subtraction term

$$\begin{aligned}
\Big\langle [I - \mathbb{S}]\, [I - S_6] \Big[C_{54} w^{54,61} + C_{64} w^{51,64} + \Big(\theta^{(a)} C_{54} + \theta^{(c)} C_{64} \Big) w^{54,64} \Big] \\
\times [\mathrm{d}p_5][\mathrm{d}p_6] F_{\mathrm{LM}}(1, 4\,|\,5, 6) \Big\rangle\,,
\end{aligned} \tag{6.89}$$

where one of the two gluons $g(p_5)$ or $g(p_6)$ is collinear to the final-state quark $q(p_4)$. The double-soft contribution in Eq. (6.89) vanishes, see the discussion at the beginning of Section 6.3.1. For this reason, all terms proportional to operator $\mathbb{S}$ in Eq. (6.89) can be omitted.

We begin the analysis of Eq. (6.89) by considering the following contribution from the double collinear partition

$$\big\langle [I - S_6]\, C_{54} [\mathrm{d}p_5][\mathrm{d}p_6] w^{54,61} F_{\mathrm{LM}}(1, 4\,|\,5, 6) \big\rangle\,. \tag{6.90}$$

The required collinear limit reads

$$C_{54}F_{\rm LM}(1,4\,|\,5,6) = g_{s,b}^2 \times \frac{1}{p_4\cdot p_5}\,P_{qq}\,(z) \times F_{\rm LM}\left(1,\frac{1}{z}\cdot 4\,|\,6\right), \tag{6.91}$$

where $z = E_4/(E_4+E_5)$. Further steps are analogous to the NLO computations discussed in Section 5.2. We remind that, as emphasized by writing the phase space measures of the two gluons to the right of the operator C_{54} in Eq. (6.90), collinear limit of the phase space must be taken. The only angular integral that is relevant in the collinear limit reads

$$\int (C_{64}[{\rm d}\Omega_6])\,\frac{1}{\rho_{64}} \overset{\rm (G.9)}{=} -\frac{1}{\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]2^{-2\epsilon}. \tag{6.92}$$

Following the NLO discussion and using Eq. (6.92) we obtain

$$\begin{aligned}&\big\langle\big[I-S_6\big]C_{54}[{\rm d}p_5][{\rm d}p_6]w^{54,61}F_{\rm LM}(1,4\,|\,5,6)\big\rangle\\ &= -\frac{[\alpha_{s,b}]}{\epsilon}\times\int\limits_0^1 {\rm d}z\,z^{-2\epsilon}(1-z)^{-2\epsilon}P_{qq}(z)\,\big\langle\theta((1-z)E_4-E_6)(2E_4)^{-2\epsilon}\\ &\quad\times w^{61}_{\rm dc}[1-S_6]F_{\rm LM}(1,4\,|\,6)\big\rangle,\end{aligned} \tag{6.93}$$

where $w^{61}_{\rm dc} = \lim_{\vec{p}_5\|\vec{p}_4} w^{54,61}$.

The second contribution in Eq. (6.89) that needs to be discussed is

$$\big\langle\big[I-S_6\big]C_{64}w^{51,64}[{\rm d}p_6]F_{\rm LM}(1,4\,|\,5,6)\big\rangle. \tag{6.94}$$

The required limits are

$$\begin{aligned}C_{64}F_{\rm LM}(1,4\,|\,5,6) &= g_{s,b}^2\times\frac{1}{p_4\cdot p_6}\,P_{qq}\,(z)\times F_{\rm LM}\left(1,\frac{1}{z}\cdot 4\,|\,5\right),\\ C_{64}S_6F_{\rm LM}(1,4\,|\,5,6) &= 2C_F\,g_{s,b}^2\;\times\frac{1}{E_6^2\,\rho_{46}}\times F_{\rm LM}(1,4\,|\,5),\end{aligned} \tag{6.95}$$

where $z = E_4/(E_4+E_6)$. Integration of limits in Eq. (6.95) over phase space of the gluon $g(p_6)$ is analogous to the NLO computation discussed in Section 5.2.3. Using Eq. (6.92), we obtain

$$\begin{aligned}&\big\langle\big[I-S_6\big]C_{64}w^{51,64}[{\rm d}p_6]F_{\rm LM}(1,4\,|\,5,6)\big\rangle = -\frac{[\alpha_{s,b}]}{\epsilon}\int\limits_0^1{\rm d}z\,z^{-2\epsilon}(1-z)^{-2\epsilon}P_{qq}(z)\\ &\quad\times\big\langle\theta(E_5-(1-z)E_4)(2E_4)^{-2\epsilon}w^{51}_{\rm dc}F_{\rm LM}(1,4\,|\,5)\big\rangle\\ &\quad-2C_F\,\frac{[\alpha_{s,b}]}{2\epsilon^2}\big\langle w^{51}_{\rm dc}(2E_5)^{-2\epsilon}F_{\rm LM}(1,4\,|\,5)\big\rangle.\end{aligned} \tag{6.96}$$

To combine contributions given in Eqs. (6.93, 6.96), we perform the same manipulations as in the case of initial-state emission, see the discussion that led to Eq. (6.58). That is, we rename $6\to 5$ in Eq. (6.93) and insert $I = \big[I-S_5\big]+S_5$ into Eq. (6.96). The sum of Eq. (6.93) and

Eq. (6.96) then reads

$$
\begin{aligned}
&\Big\langle [I - S_6] \Big[C_{54} w^{54,61} + C_{64} w^{51,64}\Big] [\mathrm{d}p_5][\mathrm{d}p_6] F_{\mathrm{LM}}(1,4\,|\,5,6)\Big\rangle \\
&= \frac{[\alpha_{s,b}]}{\epsilon} \left[\frac{2C_F}{2\epsilon} + \gamma_{qq}^{22}\right] \big\langle (2E_4)^{-2\epsilon} [1 - S_5]\, w_{\mathrm{dc}}^{51}\, F_{\mathrm{LM}}(1,4\,|\,5)\big\rangle \\
&\quad - \frac{[\alpha_{s,b}]}{\epsilon} \int_0^1 \mathrm{d}z\, z^{-2\epsilon} (1-z)^{-2\epsilon} P_{qq}(z) \big\langle \theta(E_5 - (1-z)E_4)(2E_4)^{-2\epsilon}\, S_5\, w_{\mathrm{dc}}^{51}\, F_{\mathrm{LM}}(1,4\,|\,5)\big\rangle \\
&\quad - \frac{[\alpha_{s,b}] C_F}{\epsilon^2} \big\langle (2E_5)^{-2\epsilon}\, w_{\mathrm{dc}}^{51}\, F_{\mathrm{LM}}(1,4\,|\,5)\big\rangle\,.
\end{aligned}
\tag{6.97}
$$

To arrive at Eq. (6.97) we used $1 = \theta(E_5 - (1-z)E_4) + \theta((1-z)E_4 - E_5)$ and integrated over z in the first term using Eq. (E.27). In the second term on the right-hand side of Eq. (6.97) the gluon $g(p_5)$ must be taken in the soft limit. Since soft gluons decouple from the function F_{LM} we can integrate analytically over p_5. We obtain

$$
\begin{aligned}
&\int [\mathrm{d}p_5]\, \theta(E_5 - (1-z)E_4)(2E_4)^{-2\epsilon}\, S_5\, w_{\mathrm{dc}}^{51}\, F_{\mathrm{LM}}(1,4\,|\,5) \\
&= 2C_F\, g_{s,b}^2 \times \underbrace{\int \frac{\mathrm{d}E_5}{E_5^{1+2\epsilon}}\, \theta(E_5 - (1-z)E_4)}_{= -\frac{(1-z)^{-2\epsilon} E_4^{-2\epsilon}}{2\epsilon}} \times \underbrace{\int \frac{\mathrm{d}\Omega_5^{(d-1)}}{2(2\pi)^{(d-1)}} \frac{\rho_{14}}{\rho_{15}\rho_{45}}\, w_{\mathrm{dc}}^{51}}_{\overset{(6.60)}{=} -\frac{1}{\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right] 2^{-2\epsilon} \langle w_{\mathrm{dc}}^{51}\rangle_{S_5}} \times (2E_4)^{-2\epsilon}\, F_{\mathrm{LM}}(1,4) \\
&= 2C_F\, \frac{[\alpha_{s,b}]}{2\epsilon^2} (1-z)^{-2\epsilon}\, \langle w_{\mathrm{dc}}^{51}\rangle_{S_5}\, (2E_4)^{-4\epsilon}\, F_{\mathrm{LM}}(1,4)\,,
\end{aligned}
\tag{6.98}
$$

where we used Eq. (5.7) to extract the soft limit. The remaining z integration is performed with the help of the following equation

$$
\int \mathrm{d}z\, z^{-2\epsilon} (1-z)^{-4\epsilon}\, P_{qq}(z) \overset{(\mathrm{E.27})}{=} -\left[\frac{2C_F}{4\epsilon} + \gamma_{qq}^{24}\right]. \tag{6.99}
$$

Finally, we use the results Eqs. (6.98, 6.99) in Eq. (6.97), and write double-collinear contributions to the subtraction term Eq. (6.89) as

$$
\begin{aligned}
&\Big\langle [I - \mathbb{S}]\, [I - S_6] \Big[C_{54} w^{54,61} + C_{64} w^{51,64}\Big] [\mathrm{d}p_5][\mathrm{d}p_6] F_{\mathrm{LM}}(1,4\,|\,5,6)\Big\rangle \\
&= 2C_F\, \frac{[\alpha_{s,b}]^2}{2\epsilon^3} \left[\frac{2C_F}{4\epsilon} + \gamma_{qq}^{24}\right] \big\langle (2E_4)^{-4\epsilon}\, \langle w_{\mathrm{dc}}^{51}\rangle_{S_5}\, F_{\mathrm{LM}}(1,4)\big\rangle_\delta \\
&\quad + \frac{[\alpha_{s,b}]}{\epsilon} \left[\frac{2C_F}{2\epsilon} + \gamma_{qq}^{22}\right] \big\langle (2E_4)^{-2\epsilon} [1 - S_5]\, w_{\mathrm{dc}}^{51}\, F_{\mathrm{LM}}(1,4\,|\,5)\big\rangle \\
&\quad - 2C_F\, \frac{[\alpha_{s,b}]}{2\epsilon^2} \big\langle (2E_5)^{-2\epsilon}\, w_{\mathrm{dc}}^{51}\, F_{\mathrm{LM}}(1,4\,|\,5)\big\rangle\,.
\end{aligned}
\tag{6.100}
$$

At this point we do not attempt to extract remaining NLO singularities in the second and the third term on the right-hand side of Eq. (6.100). We find it convenient to do this after the calculation of the contributions from the triple collinear partitions in Eq. (6.89), which depend on the partition function $w^{54,64}$. The triple-collinear contribution reads

$$\left\langle [I-\mathbb{S}]\,[I-S_6]\left[\theta_4^{(a)}C_{54}+\theta_4^{(c)}C_{64}\right][\mathrm{d}p_5][\mathrm{d}p_6]\; w^{54,64}\; F_{\rm LM}(1,4\,|\,5,6)\right\rangle . \tag{6.101}$$

In addition to the partition functions $w^{54,64}$ we now have to consider angular sectors (a) and (c). As discussed in the previous section, integration over phase space of unresolved gluons in triple-collinear contributions is almost identical to the integration in case of double-collinear contributions discussed earlier. The differences include restrictions of angular integrals to a given sector and the presence of new limits of the partition functions. Accounting for these differences, we obtain the triple-collinear contribution Eq. (6.101) from the double-collinear contribution Eq. (6.100) by a simple replacement

$$w^{51}_{\rm dc} \quad\to\quad \left(\frac{\rho_{45}}{4}\right)^{-\epsilon} w^{54}_{\rm tc}\,. \tag{6.102}$$

We define

$$\Delta_{64} \equiv w^{51}_{\rm dc} + \left(\frac{\rho_{45}}{4}\right)^{-\epsilon} w^{54}_{\rm tc}\,, \tag{6.103}$$

and write the full result as

$$\begin{aligned}
&\Big\langle [1-\mathbb{S}][1-S_6]\Big[C_{54}w^{54,61}+C_{64}w^{51,64}+\Big(\theta_4^{(a)}C_{54}+\theta_4^{(c)}C_{64}\Big)w^{54,64}\Big] \\
&\qquad\qquad \times [\mathrm{d}p_5][\mathrm{d}p_6]F_{\rm LM}(1,4\,|\,5,6)\Big\rangle \\
&= 2C_F\,\frac{[\alpha_{s,b}]^2}{2\epsilon^3}\left(\frac{2C_F}{4\epsilon}+\gamma^{24}_{qq}\right)\Big\langle (2E_4)^{-4\epsilon}\,\langle\Delta_{64}\rangle_{S_5}\,F_{\rm LM}(1,4)\Big\rangle_\delta \\
&\quad + \frac{[\alpha_{s,b}]}{\epsilon}\left(\frac{2C_F}{2\epsilon}+\gamma^{22}_{qq}\right)\left\langle (2E_4)^{-2\epsilon}[1-S_5]\left[w^{51}_{\rm dc}+\left(\frac{\rho_{45}}{4}\right)^{-\epsilon}w^{54}_{\rm tc}\right]F_{\rm LM}(1,4\,|\,5)\right\rangle \\
&\quad - 2C_F\frac{[\alpha_{s,b}]}{2\epsilon^2}\left\langle (2E_5)^{-2\epsilon}\left[w^{51}_{\rm dc}+\left(\frac{\rho_{45}}{4}\right)^{-\epsilon}w^{54}_{\rm tc}\right]F_{\rm LM}(1,4\,|\,5)\right\rangle .
\end{aligned} \tag{6.104}$$

Final-state emission fully regulated

In this section we extract the remaining $1/\epsilon$ singularities that are present in the NLO-like single real emission contributions in the second and the third term on the right-hand side of Eq. (6.104). We begin with the latter. It reads

$$-\frac{[\alpha_{s,b}]C_F}{\epsilon^2}\left\langle (2E_5)^{-2\epsilon}\left[w^{51}_{\rm dc}+w^{54}_{\rm tc}\left(\frac{\rho_{45}}{4}\right)^{-\epsilon}\right]F_{\rm LM}(1,4\,|\,5)\right\rangle . \tag{6.105}$$

Apart from the $1 \leftrightarrow 4$ replacement in the term in the square brackets in Eq. (6.105), it is identical to a similar term in Eq. (6.69). Following the discussion around Eq. (6.69), we write

$$\begin{aligned}
&\left\langle \left(w_{\rm dc}^{51} + w_{\rm tc}^{54} \left(\frac{\rho_{45}}{4}\right)^{-\epsilon} \right) (2E_5)^{-2\epsilon} F_{\rm LM}(1,4\,|\,5) \right\rangle \\
&= \left\langle S_5 \left(w_{\rm dc}^{51} + w_{\rm tc}^{54} \left(\frac{\rho_{45}}{4}\right)^{-\epsilon} \right) (2E_5)^{-2\epsilon} F_{\rm LM}(1,4\,|\,5) \right\rangle \\
&\quad + \left\langle [I - S_5] \left(C_{51}\, w_{\rm dc}^{51} + C_{54}\, w_{\rm tc}^{54} \left(\frac{\rho_{45}}{4}\right)^{-\epsilon} \right) (2E_5)^{-2\epsilon} F_{\rm LM}(1,4\,|\,5) \right\rangle \\
&\quad + \left\langle \left[\mathcal{O}_{\rm nlo}^{(1)}\, w_{\rm dc}^{51} + \mathcal{O}_{\rm nlo}^{(4)}\, w_{\rm tc}^{54} \left(\frac{\rho_{45}}{4}\right)^{-\epsilon} \right] (2E_5)^{-2\epsilon} F_{\rm LM}(1,4\,|\,5) \right\rangle_{\delta} .
\end{aligned} \tag{6.106}$$

Computation of the subtraction terms in Eq. (6.106) is analogous to the previously discussed case of the initial-state emission, see Section 6.3.1. The only new element is the factor $(\rho_{45}/4)^{-\epsilon}$. Repeating the calculation that led to Eq. (6.70) we find the soft subtraction term

$$\begin{aligned}
&\left\langle S_5 \left(w_{\rm dc}^{54} + w_{\rm tc}^{51} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \right) (2E_5)^{-2\epsilon}\, F_{\rm LM}(1,4\,|\,5) \right\rangle \\
&= 2C_F \, \frac{[\alpha_{s,b}]}{4\epsilon^2}\, (2E_{\rm max})^{-4\epsilon} \left\langle \langle \Delta_{64} \rangle_{S_5} F_{\rm LM}(1,4) \right\rangle_{\delta} ,
\end{aligned} \tag{6.107}$$

and, in analogy to Eqs. (6.72, 6.73), we obtain the collinear subtraction terms. They read

$$\begin{aligned}
&\left\langle C_{51}\, [I - S_5] (2E_5)^{-2\epsilon}\, w_{\rm dc}^{51}\, F_{\rm LM}(1,4\,|\,5) \right\rangle \\
&= -\frac{[\alpha_{s,b}]}{\epsilon} \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right] (2E_1)^{-4\epsilon} \times \int_0^1 {\rm d}z\, (1-z)^{-4\epsilon} P_{qq}(z) \left\langle \frac{F_{\rm LM}(z\cdot 1,4)}{z} \right\rangle_{\delta} \\
&\quad - 2C_F\, \frac{[\alpha_{s,b}]}{2\epsilon^2} \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right] (2E_{\rm max})^{-4\epsilon}\, \langle F_{\rm LM}(1,4) \rangle_{\delta} ,
\end{aligned} \tag{6.108}$$

$$\begin{aligned}
&\left\langle C_{54}\, [I - S_5] (2E_5)^{-2\epsilon} \left(\frac{\rho_{45}}{4}\right)^{-\epsilon} w_{\rm TC}^{54}\, F_{\rm LM}(1,4\,|\,5) \right\rangle \\
&= \frac{[\alpha_{s,b}]}{\epsilon} \left[\frac{2^{\epsilon}}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} \right] \gamma_{qq}^{24}\, \langle (2E_4)^{-4\epsilon} F_{\rm LM}(1,4) \rangle_{\delta} \\
&\quad - 2C_F\, \frac{[\alpha_{s,b}]}{\epsilon} \left[\frac{2^{\epsilon}}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} \right] \left\langle \left[\frac{(2E_{\rm max})^{-4\epsilon} - (2E_4)^{-4\epsilon}}{4\epsilon} \right] F_{\rm LM}(1,4) \right\rangle_{\delta} .
\end{aligned} \tag{6.109}$$

We continue with the discussion of the second term on the right-hand side of Eq. (6.104). The soft singularity is already regulated. To regulate remaining collinear singularities we write

$$\begin{aligned}
&\left\langle (2E_4)^{-2\epsilon} [1 - S_5] \left[w_{\rm dc}^{51} + w_{\rm tc}^{54} \left(\frac{\rho_{45}}{4}\right)^{-\epsilon} \right] F_{\rm LM}(1,4\,|\,5) \right\rangle \\
&= \left\langle (2E_4)^{-2\epsilon} [1 - S_5] \left[C_{51}\, w_{\rm dc}^{51} + C_{54}\, w_{\rm tc}^{54} \left(\frac{\rho_{45}}{4}\right)^{-\epsilon} \right] F_{\rm LM}(1,4\,|\,5) \right\rangle \\
&\quad + \left\langle \left[\mathcal{O}_{\rm nlo}^{(1)}\, w_{\rm dc}^{51} + \mathcal{O}_{\rm nlo}^{(4)}\, w_{\rm tc}^{54} \left(\frac{\rho_{45}}{4}\right)^{-\epsilon} \right] (2E_4)^{-2\epsilon} F_{\rm LM}(1,4\,|\,5) \right\rangle .
\end{aligned} \tag{6.110}$$

The computation of the subtraction terms is straightforward if we follow the NLO discussion in Section 5.2. We obtain

$$
\begin{aligned}
\big\langle (2E_4)^{-2\epsilon} C_{51}\,[I-S_5]\,w^{51}_{\rm dc}\,F_{\rm LM}(1,4\,|\,5)\big\rangle &= -\frac{[\alpha_{s,b}]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right](2E_1)^{-2\epsilon} \\
&\times \int\limits_0^1 {\rm d}z\,\left(2C_F\left[\frac{(1-z)^{-2\epsilon}}{1-z}\right]_+ + (1-z)^{-2\epsilon}P_{qq,{\rm reg}}(z)\right)\left\langle (2E_4)^{-2\epsilon}\frac{F_{\rm LM}(z\cdot 1,4)}{z}\right\rangle_\delta \\
&-2C_F\,\frac{[\alpha_{s,b}]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left[\frac{(2E_{\rm max})^{-2\epsilon}-(2E_1)^{-2\epsilon}}{2\epsilon}\right]\big\langle (2E_4)^{-2\epsilon}F_{\rm LM}(1,4)\big\rangle_\delta\,,
\end{aligned}
\tag{6.111}
$$

$$
\begin{aligned}
\left\langle (2E_4)^{-2\epsilon}C_{54}\,\left[1-S_5\right]\,\left(\frac{\rho_{45}}{4}\right)^{-\epsilon}w^{54}_{\rm dc}\,\frac{F_{\rm LM}(z\cdot 1,4\,|\,5)}{z}\right\rangle &= \frac{[\alpha_s]}{\epsilon}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right] \\
&\times \gamma^{42}_{qq}\left\langle (2E_4)^{-4\epsilon}\frac{F_{\rm LM}(z\cdot 1,4)}{z}\right\rangle_\delta + 2C_F\,\frac{[\alpha_s]}{\epsilon}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right] \\
&\times \left\langle (2E_4)^{-2\epsilon}\left[\frac{(2E_{\rm max})^{-2\epsilon}-(2E_4)^{-2\epsilon}}{2\epsilon}\right]\frac{F_{\rm LM}(z\cdot 1,4)}{z}\right\rangle_\delta .
\end{aligned}
\tag{6.112}
$$

Putting everything together we obtain a fully regulated version of the subtraction term in Eq. (6.104). It reads

$$
\boxed{
\begin{aligned}
&\Big\langle [I-\mathbb{S}]\,[I-S_6]\left[C_{54}w^{54,61}+C_{64}w^{51,64}+\left(\theta_4^{(a)}C_{54}+\theta_4^{(c)}C_{64}\right)w^{54,64}\right] \\
&\qquad\qquad \times [{\rm d}p_5][{\rm d}p_6]F_{\rm LM}(1,4\,|\,5,6)\Big\rangle \\
&= \frac{[\alpha_s]}{2\epsilon}\left\langle \left[2C_F\,\frac{(4E_4^2/\mu^2)^{-\epsilon}-(4E_5^2/\mu^2)^{-\epsilon}}{2\epsilon}+\gamma^{22}_{qq}\left(\frac{4E_4^2}{\mu^2}\right)^{-\epsilon}\right]\left[\hat{\mathcal{O}}^{(1)}_{\rm nlo}w^{51}_{\rm dc}+\hat{\mathcal{O}}^{(4)}_{\rm nlo}\left(\frac{\rho_{45}}{4}\right)^{-\epsilon}w^{54}_{\rm tc}\right]\right. \\
&\qquad \left.\times F_{\rm LM}(1,4\,|\,5)\right\rangle_\delta \\
&\quad + 2C_F\frac{[\alpha_s]^2}{2\epsilon^3}\left(\frac{4E_{\rm max}^2}{\mu^2}\right)^{-\epsilon}\left\langle \left[2C_F\,\frac{(4E_4^2/\mu^2)^{-\epsilon}-(4E_{\rm max}^2/\mu^2)^{-\epsilon}}{2\epsilon}+\gamma^{22}_{qq}\left(\frac{4E_4^2}{\mu^2}\right)^{-\epsilon}\right]\right. \\
&\qquad \left.\times\left[\langle\Delta_{64}\rangle_{S_5}-\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}-\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]F_{\rm LM}(1,4)\right\rangle_\delta \\
&\quad + \frac{[\alpha_s]^2}{\epsilon^2}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]\left[\frac{2C_F}{2\epsilon}+\gamma^{22}_{qq}\right]\left[\frac{2C_F}{2\epsilon}+\gamma^{42}_{qq}\right]\left\langle\left(\frac{4E_4^2}{\mu^2}\right)^{-2\epsilon}F_{\rm LM}(1,4)\right\rangle_\delta \\
&\quad - \frac{[\alpha_s]^2C_F}{\epsilon^3}\left[\frac{2C_F}{4\epsilon}+\gamma^{24}_{qq}\right]\left\langle\left[\langle\Delta_{64}\rangle_{S_5}+\left(\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right)\right]\left(\frac{4E_4^2}{\mu^2}\right)^{-2\epsilon}F_{\rm LM}(1,4)\right\rangle_\delta \\
&\quad - \frac{[\alpha_s]^2}{\epsilon^2}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\int\limits_0^1{\rm d}z\,(1-z)^{-2\epsilon}P_{qq}(z) \\
&\qquad \times\left\langle\left[\left(\frac{2C_F}{2\epsilon}+\gamma^{22}_{qq}\right)\left(\frac{4E_4^2}{\mu^2}\right)^{-\epsilon}-\frac{2C_F}{2\epsilon}\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}(1-z)^{-2\epsilon}\right]\frac{F_{\rm LM}(z\cdot 1,4)}{z}\right\rangle_\delta .
\end{aligned}
}
\tag{6.113}
$$

Although it is not immediately apparent, the above equation contains at most $1/\epsilon^2$ poles after replacing the splitting functions with regulated versions that given in Appendix E.

6.3.3. Double-collinear C_{56} sectors

We continue with the last missing contribution to the single-unresolved subtraction terms

$$\sum_{i\in\{1,4\}} \left\langle [I-\mathbb{S}]\,[I-S_6]\left[\theta_i^{(b)}C_{56}+\theta_i^{(d)}C_{56}\right][\mathrm{d}p_5][\mathrm{d}p_6]w^{5i,6i}F_{\mathrm{LM}}(1,4\,|\,5,6)\right\rangle . \tag{6.114}$$

We note that the collinear partons are always in the final state, in contrast to the scenarios discussed in the previous two subsections, where the limits involved initial and final state contributions. Hence, the double-collinear $\vec{p}_5 \parallel \vec{p}_6$ limit and the integration over the phase space of the unresolved gluon is identical for the initial state ($i=1$) and final state ($i=4$) partitions that contribute to Eq. (6.114).

For definiteness we discuss the partition 51, 61 and focus on sector (b) in what follows. The required double-collinear limit is given by

$$\begin{aligned} &C_{56}F_{\mathrm{LM}}(1,4\,|\,5,6)\\ &= g_{s,b}^2 \times \frac{1}{p_5\cdot p_6}\left[P_{gg}^{(0)}(z)\;F_{\mathrm{LM}}(1,4\,|\,5+6)+P_{gg}^{\perp}(z)\;\kappa_{\perp\mu}\kappa_{\perp\nu}F_{\mathrm{LM}}^{\mu\nu}(1,4\,|5+6)\right] , \end{aligned} \tag{6.115}$$

where $z=E_5/(E_5+E_6)$, so that gluon $g(p_6)$ becomes unresolved.[12] We will refer to the term in Eq. (6.115) that contains $\kappa_{\perp\mu}\kappa_{\perp\nu}F_{\mathrm{LM}}^{\mu\nu}(1,4\,|5+6)$ as the "spin-correlated" contribution. The last argument of the functions F_{LM} and $F_{\mathrm{LM}}^{\mu\nu}$ in Eq. (6.115) refers to an on-shell gluon that carries four-momentum $p_{56}=(E_5+E_6)\cdot n_5$ where $n_5^{\mu}=p_5^{\mu}/E_5$. The function $F_{\mathrm{LM}}^{\mu\nu}(1,4\,|\,5+6)$ describes the single-real emission contribution where the polarization vector of a gluon $g(p_{56})$ is removed from the matrix element. It is defined by the equation

$$F_{\mathrm{LM}}(1,4\,|\,5)=\sum_{\lambda=\pm}\varepsilon_{\mu}^{\lambda}(p_5)\varepsilon_{\nu}^{\lambda\star}(p_5)F_{\mathrm{LM}}^{\mu\nu}(1,4\,|\,5)\;, \tag{6.117}$$

where $\varepsilon_{\mu}^{\lambda}(p_5)$ is the polarization vector of a gluon with momentum p_5 and the sum over λ has do be understood in d dimensions. Hence, upon contracting $F_{\mathrm{LM}}^{\mu\nu}(1,4\,|\,5)$ with the (d-dimensional) metric tensor $g_{\mu\nu}$, we obtain

$$-g_{\mu\nu}\,F_{\mathrm{LM}}^{\mu\nu}(1,4\,|\,5)=F_{\mathrm{LM}}(1,4\,|\,5)\,. \tag{6.118}$$

[12] Note that this is just a choice. Equivalently we could think about the gluon $g(p_5)$ that becomes unresolved. In this case we have to choose $z=E_5/(E_5+E_6)\to E_6/(E_5+E_6)=1-z$. The splitting functions that appear in Eq. (6.115) are invariant under this transformation

$$P_{gg}^{(0)}(1-z)=P_{gg}^{(0)}(z)\,,\quad P_{gg}^{\perp}(1-z)=P_{gg}^{\perp}(z)\,. \tag{6.116}$$

The splitting functions in Eq. (6.115) read

$$P_{gg}^{(0)}(z) = 2C_A\left(\frac{z}{1-z}+\frac{1-z}{z}\right), \quad P_{gg}^{\perp}(z) = 4C_A(1-\epsilon)z(1-z)\,, \tag{6.119}$$

where $C_A = 3$ is the relevant colour factor. The (normalized) vector $\kappa_\perp = k_\perp/\sqrt{-k_\perp^2}$ in the limit Eq. (6.115) is defined by the Sudakov decomposition of p_6 in terms of p_5

$$p_6 = \alpha p_5 + \beta \bar{p}_5 + k_\perp\,, \tag{6.120}$$

where $\bar{p}_5 \equiv (E_5, -\vec{p}_5)$.

We begin with the integration over the unresolved phase space in the second term on the right-hand side of Eq. (6.115). The relevant contribution reads

$$\begin{aligned} g_{s,b}^2 \int \left[C_{56}[\mathrm{d}p_5][\mathrm{d}p_6]\theta_1^{(b)}\right] \theta(E_5-E_6) \\ \times \left[I-\mathbb{S}\right]\left[I-S_6\right] \times \frac{1}{p_5\cdot p_6} P_{gg}^{\perp}(z) \times w_{\rm tc}^1\, \kappa_{\perp\mu}\kappa_{\perp\nu} F_{\rm LM}^{\mu\nu}(1,4\,|5+6)\,, \end{aligned} \tag{6.121}$$

where we introduced the notation $w_{\rm tc}^1 \equiv \lim_{p_5\|p_6} w^{51,61}$. We find two factors in the integrand in Eq. (6.121) that depend on the direction of the momentum p_6. First, there is a factor $1/(p_5\cdot p_6)$ that contains the collinear $\vec{p}_5 \parallel \vec{p}_6$ singularity. It only depends on the component of p_6 in the direction of p_5. Second, there is a tensor $\kappa_{\perp\mu}\kappa_{\perp\nu}$ that only depends on components of p_6 transverse to p_5. These features allow us to average over directions of $\kappa_\mu^\perp$, if an appropriate phase space parametrization is used. A comprehensive discussion of the phase space parametrization that we employ and that has this property is given in Appendix F. Using that parametrization and integrating over directions of the momentum p_6, we obtain[13]

$$\begin{aligned} &-\frac{[\alpha_{s,b}]}{2\epsilon}\left[\frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)}\right] \times \int \frac{\mathrm{d}\Omega_5^{(d-1)}}{2(2\pi)^{d-1}} \left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon} w_{\rm tc}^1 \\ &\times \int_0^{E_{\rm max}} \mathrm{d}E_5\, E_5^{1-2\epsilon} \int_0^{E_{\rm max}} \mathrm{d}E_6\, E_6^{1-2\epsilon}\theta(E_5-E_6)\left[I-\mathbb{S}\right]\left[I-S_6\right] \\ &\times \frac{P_{gg}^{\perp}(z)}{E_5E_6}\left[\frac{1}{2}F_{\rm LM}(1,4\,|\,5+6) + \epsilon\, r_\mu^{(1)} r_\nu^{(1)} F_{\rm LM}^{\mu\nu}(1,4\,|\,5+6)\right], \end{aligned} \tag{6.122}$$

where $r_\mu^{(1)}$ is a vector that appears after averaging over directions of $\kappa_\mu^\perp$. The explicit expression for this vector can also be found in Appendix F.3.

We now consider the contribution of the first term on the right-hand side of Eq. (6.115) to the subtraction term. In this term the dependence on the direction of momentum p_6 factorizes

[13]A detailed discussion of the steps that lead to Eq. (6.122) is given in Appendix F.3.

entirely from the function $F_{\rm LM}$ and the angular integration becomes straightforward. We obtain

$$
\begin{aligned}
& g_{s,b}^2 \int \left[C_{56}[\mathrm{d}p_5][\mathrm{d}p_6]\theta_1^{(b)} \right] \\
& \quad \times \left[I - \mathbb{S} \right] \left[I - S_6 \right] \times \frac{1}{p_5 \cdot p_6} P_{gg}^{(0)}(z) \times w_{\rm tc}^1 \, F_{\rm LM}(1,4\,|\,5+6) \\
& = -\frac{[\alpha_{s,b}]}{2\epsilon} \left[\frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)} \right] \times \int [\mathrm{d}p_5] \left(\frac{\rho_{15}}{2} \right)^{-\epsilon} \left(1 - \frac{\rho_{15}}{2} \right)^{\epsilon} \, w_{\rm tc}^1 \\
& \quad \times \int_0^{E_{\rm max}} \mathrm{d}E_6 \, E_6^{1-2\epsilon} \theta(E_5 - E_6) \left[I - \mathbb{S} \right] \left[I - S_6 \right] \frac{P_{gg}^{(0)}(z)}{E_5 E_6} F_{\rm LM}(1,4\,|\,5+6)\,.
\end{aligned}
\tag{6.123}
$$

We use Eqs. (6.122, 6.123) and write Eq. (6.121) as follows

$$
\begin{aligned}
& \left\langle \left[I - \mathbb{S} \right] \left[I - S_6 \right] \theta_1^{(b)} C_{56}[\mathrm{d}p_5][\mathrm{d}p_6] w^{51,61} F_{\rm LM}(1,4\,|\,5,6) \right\rangle \\
& = -\frac{[\alpha_{s,b}]}{2\epsilon} \left[\frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)} \right] \times \int [\mathrm{d}p_5] \left(\frac{\rho_{15}}{2} \right)^{-\epsilon} \left(1 - \frac{\rho_{15}}{2} \right)^{\epsilon} \, w_{\rm tc}^1 \\
& \quad \times \int_0^{E_{\rm max}} \mathrm{d}E_6 \, E_6^{-1-2\epsilon} \theta(E_5 - E_6) \left[I - \mathbb{S} \right] \left[I - S_6 \right] \mathcal{P}_{56}(1,4,5,6)\,.
\end{aligned}
\tag{6.124}
$$

In Eq. (6.124) we introduced

$$
\mathcal{P}_{56}(1,4,5,6) \equiv \frac{E_6}{E_5} \left[P_{gg}(z,\epsilon) F_{\rm LM}(1,4\,|\,5+6) + \epsilon \, P_{gg}^{\perp}(z) \, r_\mu^{(1)} r_\nu^{(1)} F_{\rm LM}^{\mu\nu}(1,4|5+6) \right], \tag{6.125}
$$

where $z = E_5/(E_5+E_6)$ and

$$
P_{gg}(z,\epsilon) = P_{gg}^{(0)}(z) + \frac{1}{2} P_{gg}^{\perp}(z) = 2C_A \left(\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)(1-\epsilon) \right). \tag{6.126}
$$

Before continuing with the E_6 integration, we comment on the computation of the contribution of sector (d) in which gluons $g(p_5)$ and $g(p_6)$ switch their roles. In the angular phase space this is accounted for by a minor change in the parametrization, see Appendix F. The integration over the angular phase space of the unresolved gluon turns out to be identical to what we have discussed in the context of sector (b). The only difference is the energy ordering that becomes $\theta(E_6 - E_5)$, so that upon combining the contribution of sector (d) with the contribution of sector (b), the energy ordering $E_6 < E_5$ in Eq. (6.124) disappears and we obtain

$$
\begin{aligned}
& \left\langle \left[1 - \mathbb{S} \right] \left[1 - S_6 \right] \left[\theta_1^{(b)} + \theta_1^{(d)} \right] C_{56}[\mathrm{d}p_5][\mathrm{d}p_6] w^{51,61} F_{\rm LM}(1,4\,|\,5,6) \right\rangle \\
& = -\frac{[\alpha_{s,b}]}{2\epsilon} N_\epsilon \int [\mathrm{d}p_5] \left(\frac{\rho_{15}}{2} \right)^{-\epsilon} \left(1 - \frac{\rho_{15}}{2} \right)^{\epsilon} \, w_{\rm tc}^1 \int_0^{E_{\rm max}} \mathrm{d}E_6 \, E_6^{-1-2\epsilon} \\
& \quad \times \left[I - \mathbb{S} \right] \left[I - S_6 \right] \mathcal{P}_{56}(1,4,5,6)\,.
\end{aligned}
\tag{6.127}
$$

In Eq. (6.127) we defined

$$N_\epsilon \equiv \frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)}\,. \tag{6.128}$$

We now discuss how to integrate over E_6 in Eq. (6.127). Because of the operator $\left[I-\mathbb{S}\right]\left[I-S_6\right]$ we need to consider four terms with different soft limits of the function $\mathcal{P}_{56}(1,4,5,6)$. The required limits are discussed in Appendix C. The two limits that include single-soft operator S_6 read

$$\begin{aligned} S_6\mathcal{P}_{56}(1,4,5,6) &= 2C_A\, F_{\rm LM}(1,4\,|\,5)\,, \\ \mathbb{S}S_6\mathcal{P}_{56}(1,4,5,6) &= 2C_A\, S_5F_{\rm LM}(1,4\,|\,5)\,. \end{aligned} \tag{6.129}$$

Thanks to the above equations we can write

$$\left[I-\mathbb{S}\right]S_6\,\mathcal{P}_{56}(1,4,5,6) = 2C_A\left[I-S_5\right]F_{\rm LM}(1,4\,|\,5)\,. \tag{6.130}$$

Finally we also need the double-soft $\mathbb{S}$ limit of the function $\mathcal{P}_{56}(1,4,5,6)$. Note that the dependence of the function $F_{\rm LM}$ on energies E_5 and E_6 reads $F_{\rm LM}(1,4\,|\,5+6) = F_{\rm LM}(1,4\,|\,(E_5+E_6)\cdot n_5)$. Hence, taking the double-soft limit corresponds to the limit $E_5+E_6\to 0$. To emphasize this we write $S_{56}\equiv\mathbb{S}$. As we will see later it is convenient to write

$$r_\mu^{(1)}r_\nu^{(1)} = \left[r_\mu^{(1)}r_\nu^{(1)}+g_{\mu\nu}\right]-g_{\mu\nu}\,, \tag{6.131}$$

in the second term on the right-hand side of Eq. (6.125).

We contract Eq. (6.131) with $F_{\rm LM}^{\mu\nu}(1,4\,|\,5+6)$ using Eq. (6.118) and write the double-soft contribution as

$$\begin{aligned} \left[I-\mathbb{S}\right]\mathcal{P}_{56}(1,4,5,6) &= \frac{E_6}{E_5}\left[P_{gg}(z,\epsilon)+\epsilon\;P_{gg}^{\perp}(z)\right]\left[I-S_{56}\right]F_{\rm LM}(1,4\,|\,5+6) \\ &\quad+\epsilon\,\frac{E_6}{E_5}\;P_{gg}^{\perp}(z)\left[r_\mu^{(1)}r_\nu^{(1)}+g_{\mu\nu}\right]\left[I-S_{56}\right]F_{\rm LM}^{\mu\nu}(1,4|5+6)\right]. \end{aligned} \tag{6.132}$$

Combining Eq. (6.132) with contributions in Eq. (6.130) we obtain

$$\begin{aligned} &\left[I-\mathbb{S}\right]\left[I-S_6\right]\mathcal{P}_{56}(1,4,5,6) \\ &= \frac{E_6}{E_5}\left[P_{gg}(z,\epsilon)+\epsilon\,P_{gg}^{\perp}(z)\right]\left[I-S_{56}\right]F_{\rm LM}(1,4\,|\,5+6) \\ &\quad+\epsilon\,\frac{E_6}{E_5}\,P_{gg}^{\perp}(z)\left[r_\mu^{(1)}r_\nu^{(1)}+g_{\mu\nu}\right]\left[I-S_{56}\right]F_{\rm LM}^{\mu\nu}(1,4\,|\,5+6) \\ &\quad-2C_A\left[I-S_5\right]F_{\rm LM}(1,4\,|\,5)\,. \end{aligned} \tag{6.133}$$

We consider the three terms on the right-hand side of Eq. (6.133) separately, starting with the

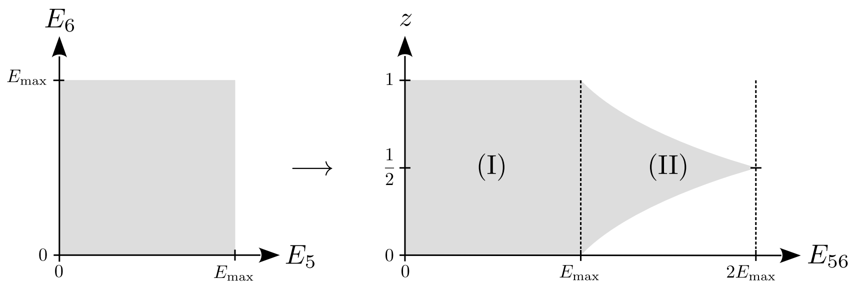


Fig. 6.4.: Energy phase in the E_5-E_6-plane (left) and after the substitution Eq. (6.135) in the E_{56}-z-plane (right) where $E_{56} = E_5 + E_6$ and $z = E_5/(E_5 + E_6)$. The energy phase space after the substitution is split into two regions (I) and (II) that are integrated separately, see Eq. (6.136).

last one. Inserting it into Eq. (6.127) we find

$$\begin{aligned} &C_A \frac{[\alpha_{s,b}]}{\epsilon} N_\epsilon \int [\mathrm{d}p_5] \left(\frac{\rho_{15}}{2}\right)^{-\epsilon} \left(1 - \frac{\rho_{15}}{2}\right)^{\epsilon} \int\limits_0^{E_{\max}} \mathrm{d}E_6\, E_6^{-1-2\epsilon} \left[I - S_5\right] w_{\mathrm{tc}}^1\, F_{\mathrm{LM}}(1,4\,|\,5) \\ &= -C_A \frac{[\alpha_{s,b}]}{2\epsilon^2} N_\epsilon \left\langle \left(\frac{\rho_{15}}{2}\right)^{-\epsilon} \left(1 - \frac{\rho_{15}}{2}\right)^{\epsilon} E_{\max}^{-2\epsilon} \left[I - S_5\right] w_{\mathrm{tc}}^1\, F_{\mathrm{LM}}(1,4\,|\,5) \right\rangle , \end{aligned} \tag{6.134}$$

where we used Eq. (G.1) for the integration over E_6. We keep the operator $\left[I - S_5\right]$ unexpanded to facilitate the extraction of the collinear singularities from this expression.

We continue with the discussion of the first and the second terms on the right-hand side of Eq. (6.133). We would like to integrate over energies of the unresolved gluon(s). However, we need to keep $E_5 + E_6$ fixed, but we can allow for arbitrary E_5 and E_6 otherwise. To accomplish this, we introduce $E_{56} \equiv E_5 + E_6$ and $z = E_5/E_{56}$ and write energies $E_{i=5,6}$ as

$$E_5 = zE_{56}\,, \quad E_6 = (1-z)E_{56}\,. \tag{6.135}$$

The integration domain over E_5 and E_6 splits into two regions (see Fig. 6.4)

$$\int\limits_0^{E_{\max}} \mathrm{d}E_5 \int\limits_0^{E_{\max}} \mathrm{d}E_6 \quad \to \quad \underbrace{\int\limits_0^{E_{\max}} \mathrm{d}E_{56}\, E_{56} \int\limits_0^1 \mathrm{d}z}_{\widehat{=}\ (\mathrm{I})} + \underbrace{\int\limits_{E_{\max}}^{2E_{\max}} \mathrm{d}E_{56}\, E_{56} \int\limits_{1-\frac{E_{\max}}{E_{56}}}^{\frac{E_{\max}}{E_{56}}} \mathrm{d}z}_{\widehat{=}\ (\mathrm{II})} \,. \tag{6.136}$$

We begin with the first term on the right-hand side of Eq. (6.133). Using the parametrization

Eq. (6.135) we write the contribution of region (I) as

$$
\begin{aligned}
&-\frac{[\alpha_{s,b}]}{2\epsilon}N_\epsilon \int\limits_{(\mathrm{I})}[\mathrm{d}p_5]\left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon}\mathrm{d}E_6\, E_6^{-1-2\epsilon}\times\frac{E_6}{E_5}\left[P_{gg}(z,\epsilon)+\epsilon\, P_{gg}^{\perp}(z)\right] \\
&\quad\times\left[I-S_{56}\right]\, w_{\mathrm{tc}}^{1}\, F_{\mathrm{LM}}(1,4\,|\,5+6) \\
&=-\frac{[\alpha_{s,b}]}{2\epsilon}N_\epsilon\int\limits_0^1 \mathrm{d}z\, z^{-2\epsilon}(1-z)^{-2\epsilon}\left[P_{gg}(z,\epsilon)+\epsilon\, P_{gg}^{\perp}(z)\right] \\
&\quad\times\int\frac{\mathrm{d}\Omega_5^{(d-1)}}{2(2\pi)^{(d-1)}}\left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon}\int\limits_0^{E_{\max}}\mathrm{d}E_{56}\, E_{56}^{1-4\epsilon}[1-S_{56}]\, w_{\mathrm{tc}}^{1}\, F_{\mathrm{LM}}(1,4\,|\,5+6)\,.
\end{aligned}
\tag{6.137}
$$

We rename $E_{56}\to E_5$, and rewrite the above equation as follows

$$
\frac{[\alpha_{s,b}]}{2\epsilon}N_\epsilon\left[\frac{2C_A}{\epsilon}+\gamma_{gg}^{22}\right]\left\langle\left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon}E_5^{-2\epsilon}\, w_{\mathrm{tc}}^{1}[1-S_5]F_{\mathrm{LM}}(1,4\,|\,5)\right\rangle. \tag{6.138}
$$

The anomalous dimension γ_{gg}^{22} reads[14]

$$
\gamma_{gg}^{22}\equiv-\int\limits_0^1\mathrm{d}z\left[z^{-2\epsilon}(1-z)^{-2\epsilon}\left[P_{gg}(z,\epsilon)+\epsilon P_{gg}^{\perp}(z)\right]-2C_A\left(\frac{z^{-2\epsilon}}{z}+\frac{(1-z)^{-2\epsilon}}{1-z}\right)\right]. \tag{6.139}
$$

An expansion of γ_{gg}^{22} in ϵ can be found in Appendix E.6.

We now discuss integration over region (II). In this region $E_{56}>E_{\max}$. Hence as long as the remaining gluon is resolved, the integrand vanishes because of the energy-momentum conservation. This implies that in region (II) we only have to consider terms that involve a single-soft limit where the gluon $g(p_{56})$ becomes unresolved. We find

$$
\begin{aligned}
&-\frac{[\alpha_{s,b}]}{2\epsilon}N_\epsilon\int\limits_{(\mathrm{II})}[\mathrm{d}p_5]\left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon}\int\limits_0^{E_{\max}}\mathrm{d}E_6\, E_6^{-1-2\epsilon}\frac{E_6}{E_5}\left[P_{gg}(z,\epsilon)+\epsilon\, P_{gg}^{\perp}(z)\right] \\
&\quad\times w_{\mathrm{tc}}^{1}\left[I-S_{56}\right]F_{\mathrm{LM}}(1,4\,|\,5+6) \\
&=2C_F\,\frac{[\alpha_{s,b}]g_{s,b}^2}{2\epsilon}N_\epsilon\times\int\frac{\mathrm{d}\Omega_5^{(d-1)}}{2(2\pi)^{(d-1)}}\,\frac{\rho_{14}}{\rho_{15}\,\rho_{45}}\,w_{\mathrm{tc}}^{1}\,\left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon} \\
&\quad\times\int\limits_{E_{\max}}^{2E_{\max}}\mathrm{d}E_{56}\, E_{56}^{-1-4\epsilon}\int\limits_{1-\frac{E_{\max}}{E_{56}}}^{\frac{E_{\max}}{E_{56}}}\mathrm{d}z\, z^{-2\epsilon}(1-z)^{-2\epsilon}\left[P_{gg}(z,\epsilon)+\epsilon P_{gg}^{\perp}(z)\right]F_{\mathrm{LM}}(1,4) \\
&=-2C_F\,\frac{[\alpha_{s,b}]^2}{\epsilon^2}\left[2^{2\epsilon}\delta_g(\epsilon)\right](2E_{\max})^{-4\epsilon}\langle\Delta_{56}^{1}\rangle_{S_5}\,F_{\mathrm{LM}}(1,4)\,,
\end{aligned}
\tag{6.140}
$$

[14]Note that in contrast to the anomalous dimension for quark splitting Eq. (5.50) the z integration over $P_{gg}(z,\epsilon)$ in Eq. (6.139) is divergent at the upper *and* lower integration bound. This is so because this functions contain both soft singularities for $E_5\to 0$ or $E_6\to 0$ corresponding to $z\to 0$ and $z\to 1$, respectively.

where we used Eq. (6.60) to integrate over directions of p_5 and defined

$$\Delta^1_{56} \equiv w^1_{\rm tc} \left(\frac{\rho_{15}}{2}\right)^{-\epsilon} \left(1-\frac{\rho_{15}}{2}\right)^{\epsilon} , \tag{6.141}$$

to write the result in a compact form. We also introduced a new quantity $\delta_g(\epsilon)$ in Eq. (6.140). It is defined as

$$\delta_g(\epsilon) \equiv \frac{N_\epsilon\, E_{\rm max}^{4\epsilon}}{2} \int\limits_{E_{\rm max}}^{2E_{\rm max}} {\rm d}E_{56}\, E_{56}^{-1-4\epsilon} \int\limits_{1-\frac{E_{\rm max}}{E_{56}}}^{\frac{E_{\rm max}}{E_{56}}} {\rm d}z\, z^{-2\epsilon}(1-z)^{-2\epsilon} \left[P_{gg}(z,\epsilon) + \epsilon\, P^{\perp}_{gg}(z)\right] . \tag{6.142}$$

The expansion of $\delta_g(\epsilon)$ in ϵ can be found in Appendix E.7. We combine contributions of region (I) Eq. (6.137) and region (II) Eq. (6.140) and obtain the following result

$$\begin{aligned} &-\frac{[\alpha_{s,b}]}{2\epsilon}\left[\frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)}\right] \times \int [{\rm d}p_5] \left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon} w^1_{\rm tc} \\ &\times \int\limits_0^{E_{\rm max}} {\rm d}E_6\, E_6^{-1-2\epsilon} \frac{E_6}{E_5}\left[P_{gg}(z,\epsilon) + \epsilon\, P^{\perp}_{gg}(z)\right] [I - S_{56}] F_{\rm LM}(1,4\,|\,5+6) \\ &= \frac{[\alpha_{s,b}]}{2\epsilon} N_\epsilon \left[\frac{2C_A}{\epsilon} + \gamma^{22}_{gg}\right] \left\langle \left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon} E_5^{-2\epsilon}\, w^1_{\rm tc}[1-S_5] F_{\rm LM}(1,4\,|\,5)\right\rangle \\ &\quad - 2C_F\, \frac{[\alpha_{s,b}]^2}{\epsilon^2}\left[2^{2\epsilon}\delta_g(\epsilon)\right](2E_{\rm max})^{-4\epsilon} \langle \Delta^1_{56}\rangle_{S_5}\, F_{\rm LM}(1,4)\,. \end{aligned} \tag{6.143}$$

We now discuss the second term on the right-hand side of Eq. (6.133), which is the spin-correlated contribution. It is given by the contribution of the second term on the right-hand side of Eq. (6.133) to Eq. (6.127). Integration over phase space region (I) is identical to the previous discussion. Using Eq. (6.137), we find

$$\begin{aligned} &-\frac{[\alpha_{s,b}]}{2\epsilon} N_\epsilon \int\limits_{\rm (I)} [{\rm d}p_5] \left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon} \int\limits_0^{E_{\rm max}} {\rm d}E_6\, E_6^{-1-2\epsilon} \times \frac{E_6}{E_5}\left[\epsilon P^{\perp}_{gg}(z)\right] \\ &\times \left[r^{(1)}_\mu r^{(1)}_\nu + g_{\mu\nu}\right] [I - S_{56}]\, w^1_{\rm tc}\, F^{\mu\nu}_{\rm LM}(1,4\,|\,5+6) \\ &= \frac{[\alpha_{s,b}]}{2} N_\epsilon \gamma^{\perp,22}_{gg} \left\langle \left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon} E_5^{-2\epsilon}\, w^1_{\rm tc}[I-S_5] \left[r^{(1)}_\mu r^{(1)}_\nu + g_{\mu\nu}\right] F^{\mu\nu}_{\rm LM}(1,4\,|\,5)\right\rangle , \end{aligned} \tag{6.144}$$

where we renamed $E_{56} \to E_5$. In Eq. (6.144) we used the anomalous dimension defined as

$$\gamma^{\perp,22}_{gg} = \int\limits_0^1 {\rm d}z\, z^{-2\epsilon}(1-z)^{-2\epsilon} P^{\perp}_{gg}(z)\,. \tag{6.145}$$

An expansion of $\gamma^{\perp,22}_{gg}$ in ϵ can be found in Appendix E.6.

We continue with the integration over region (II) defined in Eq. (6.136). As discussed previously, in this region only the term with the single-soft operator contributes. The soft limit is given by

$$
\begin{aligned}
&S_{56}\left[r_\mu^{(1)} r_\nu^{(1)} + g_{\mu\nu}\right] F_{\mathrm{LM}}^{\mu\nu}(1,4\,|\,5+6) \\
&= 2C_F\, g_{s,b}^2 \times \frac{1}{2E_{56}^2}\left[1 - \frac{2}{\rho_{45}} + \frac{(\rho_{14}-\rho_{15}(1-\rho_{45})-\rho_{45})^2}{\rho_{15}\rho_{45}^2(2-\rho_{15})}\right] \times F_{\mathrm{LM}}(1,4)\,.
\end{aligned}
\tag{6.146}
$$

This limit is particular because it depends on the chosen phase space parametrization through vector $r_\mu^{(1)}$. The derivation of Eq. (6.146) is discussed in Appendix C. However, pole cancellation should happen independent of the phase-space parametrization. We can verify this by noticing that the collinear C_{51} limit of Eq. (6.146)[15]

$$
1 - \frac{2}{\rho_{45}} + \frac{(\rho_{14}-\rho_{15}(1-\rho_{45})-\rho_{45})^2}{\rho_{15}\rho_{45}^2(2-\rho_{16})} \quad \underset{5\|1}{\longrightarrow} \quad 1 - \frac{2}{\rho_{14}}\,,
\tag{6.147}
$$

is finite. Note that in region (II) no soft singularity is present by construction since $E_{56} > E_{\mathrm{max}}$, see Fig. 6.4. Hence, there are no other $1/\epsilon$ factors in this contribution, so that this subtraction term only contributes to the finite part in the ϵ expansion. The phase space parametrization dependence of the finite part of the subtraction counterterm is not a problem, as this corresponds to a dependence of the fully-regulated term on the parametrization. This is analogous to the dependence on E_{max} that we discussed earlier.

Integration over region (II) is straightforward. We obtain

$$
\begin{aligned}
&-\frac{[\alpha_{s,b}]}{2\epsilon} N_\epsilon \int\limits_{\mathrm{(II)}} [\mathrm{d}p_5] \left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon} \int\limits_0^{E_{\mathrm{max}}} \mathrm{d}E_6\, E_6^{-1-2\epsilon} \times \frac{E_6}{E_5}\left[\epsilon P_{gg}^{\perp}(z)\right] \\
&\quad \times w_{\mathrm{tc}}^1 \left[I - S_{56}\right]\left[r_\mu^{(1)} r_\nu^{(1)} + g_{\mu\nu}\right] F_{\mathrm{LM}}^{\mu\nu}(1,4\,|\,5+6) \\
&= C_F\, [\alpha_{s,b}]^2 \delta_g^{\perp}(\epsilon) (2E_{\mathrm{max}})^{-4\epsilon} \left\langle r^{(1)\mu} r^{(1)\nu}\right\rangle_{\rho_5} F_{\mathrm{LM}}(1,4)\,,
\end{aligned}
\tag{6.148}
$$

where we introduced

$$
\delta_g^{\perp}(\epsilon) \equiv \frac{N_\epsilon (2E_{\mathrm{max}})^{4\epsilon}}{2} \int\limits_{E_{\mathrm{max}}}^{2E_{\mathrm{max}}} \mathrm{d}E_{67}\, E_{67}^{-1-4\epsilon} \int\limits_{1-\frac{E_{\mathrm{max}}}{E_{67}}}^{\frac{E_{\mathrm{max}}}{E_{67}}} \mathrm{d}z\, z^{-2\epsilon}(1-z)^{-2\epsilon} P_{gg}^{\perp}(z)\,,
\tag{6.149}
$$

in analogy to Eq. (6.142). An expansion of $\delta_g^{\perp}(\epsilon)$ in ϵ is given in Appendix E.7. To perform the angular integral in Eq. (6.148) we used

$$
\begin{aligned}
\left\langle r^{(1)\mu} r^{(1)\nu}\right\rangle_{\rho_5} &\equiv \left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]^{-1} \int \frac{\mathrm{d}\Omega_5^{(d-1)}}{2(2\pi)^{(d-1)}} w_{\mathrm{tc}}^1 \left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon} \\
&\quad \times \left[1 - \frac{2}{\rho_{64}} + \frac{(\rho_{14}-\rho_{16}(1-\rho_{46})-\rho_{46})^2}{\rho_{16}\rho_{46}^2(2-\rho_{16})}\right].
\end{aligned}
\tag{6.150}
$$

[15]To find the result in Eq. (6.147) was the reason for writing the identity in Eq. (6.131). Had we not done so, we would have obtained phase space parametrization dependent poles in multiple contributions whose cancellation, independent from other IR poles, is peculiar to show.

We present the required finite part in the ϵ-expansion of $\left\langle r^{(1)\mu} r^{(1)\nu} \right\rangle_{\rho_5}$ in combination with a similar contribution from partition 54, 64 in Appendix H.

We combine contributions of regions (I) and (II), given in Eq. (6.137) and Eq. (6.140), respectively, and obtain the following result for the spin-correlated contribution to the double-collinear subtraction term Eq. (6.114)

$$
\begin{aligned}
&-\frac{[\alpha_{s,b}]}{2\epsilon} N_\epsilon \times \int [\mathrm{d}p_5] \left(\frac{\rho_{15}}{2}\right)^{-\epsilon} \left(1-\frac{\rho_{15}}{2}\right)^{\epsilon} w^1_{\mathrm{tc}} \\
&\quad \times \int\limits_0^{E_{\max}} \mathrm{d}E_6\, E_6^{-1-2\epsilon} \epsilon\, \frac{E_6}{E_5}\, P^{\perp}_{gg}(z) \left[r^{(1)}_\mu r^{(1)}_\nu + g_{\mu\nu}\right] \left[I - S_{56}\right] F^{\mu\nu}_{\mathrm{LM}}(1,4\,|\,5+6) \\
&= \frac{[\alpha_{s,b}]}{2\epsilon} N_\epsilon \epsilon \gamma^{\perp,22}_{gg} \left\langle \left(\frac{\rho_{15}}{2}\right)^{-\epsilon} \left(1-\frac{\rho_{15}}{2}\right)^{\epsilon} E_5^{-2\epsilon}\, w^1_{\mathrm{tc}} \left[I - S_5\right] \left[r^{(1)}_\mu r^{(1)}_\nu + g_{\mu\nu}\right] F^{\mu\nu}_{\mathrm{LM}}(1,4\,|\,5) \right\rangle \\
&\quad + C_F\, [\alpha_{s,b}]^2 \delta^{\perp}_g(\epsilon) (2E_{\max})^{-4\epsilon} \left\langle r^{(1)\mu} r^{(1)\nu} \right\rangle_{\rho_5} F_{\mathrm{LM}}(1,4)\,.
\end{aligned}
\tag{6.151}
$$

Finally, we combine results shown in Eqs. (6.134, 6.143, 6.151) and obtain

$$
\begin{aligned}
&\left\langle \left[1 - \mathbb{S}\right] \left[1 - S_6\right] \left[\theta^{(b)}_1 C_{56} + \theta^{(d)}_1 C_{56}\right] [\mathrm{d}p_5][\mathrm{d}p_6] w^{51,61} F_{\mathrm{LM}}(1,4\,|\,5,6) \right\rangle \\
&= -C_A\, \frac{[\alpha_{s,b}]}{2\epsilon^2} N_\epsilon \left\langle \left(\frac{\rho_{15}}{2}\right)^{-\epsilon} \left(1-\frac{\rho_{15}}{2}\right)^{\epsilon} E_{\max}^{-2\epsilon} \left[I - S_5\right] w^1_{\mathrm{tc}}\, F_{\mathrm{LM}}(1,4\,|\,5) \right\rangle \\
&\quad + \frac{[\alpha_{s,b}]}{2\epsilon} N_\epsilon \left\langle \left(\frac{\rho_{15}}{2}\right)^{-\epsilon} \left(1-\frac{\rho_{15}}{2}\right)^{\epsilon} E_5^{-2\epsilon}\, w^1_{\mathrm{tc}} [1 - S_5] \right. \\
&\qquad \left. \times \left\{ \left[\frac{2C_A}{\epsilon} + \gamma^{22}_{gg}\right] F_{\mathrm{LM}}(1,4\,|\,5) + \epsilon \gamma^{\perp,22}_{gg} \left[r^{(1)}_\mu r^{(1)}_\nu + g_{\mu\nu}\right] F^{\mu\nu}_{\mathrm{LM}}(1,4\,|\,5) \right\} \right\rangle \\
&\quad - 2C_F\, \frac{[\alpha_{s,b}]^2}{2\epsilon^2} (2E_{\max})^{-4\epsilon} \left[2^{2\epsilon} \delta_g(\epsilon)\right] \left\langle \langle \Delta^1_{56} \rangle_{S_5} F_{\mathrm{LM}}(1,4) \right\rangle_\delta \\
&\quad + C_F\, [\alpha_{s,b}]^2\, \delta^{\perp}_g(\epsilon) (2E_{\max})^{-4\epsilon} \left\langle \left\langle r^{(1)\mu} r^{(1)\nu} \right\rangle_{\rho_5} F_{\mathrm{LM}}(1,4) \right\rangle_\delta .
\end{aligned}
\tag{6.152}
$$

Before discussing how to regulate the remaining singularities in Eq. (6.152), we provide a similar result for the contribution of partition 54, 64 in Eq. (6.114). The computation of the subtraction terms is identical to the case of partition 51, 61 that we just discussed. We can obtain the required result from Eq. (6.152) by simply replacing $1 \leftrightarrow 4$ everywhere. We then obtain the following result for the subtraction term defined in Eq. (6.114)

$$
\begin{aligned}
&\sum_{i\in\{1,4\}} \left\langle \left[I - \mathbb{S}\right] \left[I - S_6\right] \left[\theta^{(b)}_i C_{56} + \theta^{(d)}_i C_{56}\right] [\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6i} F_{\mathrm{LM}}(1,4\,|\,5,6) \right\rangle \\
&= -C_A\, \frac{[\alpha_{s,b}]}{2\epsilon^2} N_\epsilon \sum_{i\in\{1,4\}} \left\langle \left(\frac{\rho_{i5}}{2}\right)^{-\epsilon} \left(1-\frac{\rho_{i5}}{2}\right)^{\epsilon} E_{\max}^{-2\epsilon} \left[I - S_5\right] w^i_{\mathrm{tc}}\, F_{\mathrm{LM}}(1,4\,|\,5) \right\rangle \\
&\quad + \frac{[\alpha_{s,b}]}{2\epsilon} N_\epsilon \sum_{i\in\{1,4\}} \left\langle \left(\frac{\rho_{i5}}{2}\right)^{-\epsilon} \left(1-\frac{\rho_{i5}}{2}\right)^{\epsilon} E_5^{-2\epsilon}\, w^i_{\mathrm{tc}} [1 - S_5] \right.
\end{aligned}
$$

$$\times \left\{ \left[\frac{2C_A}{\epsilon} + \gamma_{gg}^{22} \right] F_{\rm LM}(1,4\,|\,5) + \epsilon \gamma_{gg}^{\perp,22} \left[r_\mu^{(i)} r_\nu^{(i)} + g_{\mu\nu} \right] F_{\rm LM}^{\mu\nu}(1,4\,|\,5) \right\} \Big\rangle$$
$$- 2C_F \, \frac{[\alpha_{s,b}]^2}{\epsilon^2} \times [2^{2\epsilon} \delta_g(\epsilon)] (2E_{\rm max})^{-4\epsilon} \Big\langle \langle \Delta_{65} \rangle_{S_5} \, F_{\rm LM}(1,4) \Big\rangle_\delta$$
$$+ C_F \, [\alpha_{s,b}]^2 \delta_g^\perp(\epsilon) (2E_{\rm max})^{-4\epsilon} \Big\langle \, \langle r^\mu r^\nu \rangle_{\rho_5} \, F_{\rm LM}(1,4) \Big\rangle_\delta . \tag{6.153}$$

In Eq. (6.153) we defined the following combined integrals

$$\langle r^\mu r^\nu \rangle_{\rho_5} \equiv \sum_{i \in \{1,4\}} \left\langle r^{(i)\mu} r^{(i)\nu} \right\rangle_{\rho_5} , \quad \langle \Delta_{65} \rangle_{S_5} = \sum_{i \in \{1,4\}} \langle \Delta_{65}^i \rangle_{S_5} . \tag{6.154}$$

We discuss them in Appendix H.

Fully-regulated double-collinear C_{56} sectors

We continue with the discussion of how to regulate singularities that are implicit in the first and second term on the right-hand side of Eq. (6.153) As we have seen previously, these singularities are of the NLO type. Thanks to the operator $\left[I - S_5 \right]$ the soft singularity is already regulated. Moreover, functions $w_{\rm tc}^{i=1,4}$ provide proper partitioning of the phase space such that uniquely defined collinear singularities appear in every sector.

We begin with the first term on the right-hand side of Eq. (6.153) Inserting $I = \left[I - C_{5i} \right] + C_{5i}$ to extract and regulate collinear singularities, we obtain

$$\begin{aligned}
&- C_A \, \frac{[\alpha_{s,b}]}{2\epsilon^2} N_\epsilon \sum_{i \in \{1,4\}} \left\langle \left(\frac{\rho_{i5}}{2} \right)^{-\epsilon} \left(1 - \frac{\rho_{i5}}{2} \right)^{\epsilon} E_{\rm max}^{-2\epsilon} \left[I - S_5 \right] w_{\rm tc}^i \, F_{\rm LM}(1,4\,|\,5) \right\rangle \\
&= -C_A \, \frac{[\alpha_{s,b}]}{2\epsilon^2} N_\epsilon E_{\rm max}^{-2\epsilon} \sum_{i \in \{1,4\}} \left\langle \hat{\mathcal{O}}_{\rm nlo}^{(i)} \left(\frac{\rho_{i5}}{2} \right)^{-\epsilon} \left(1 - \frac{\rho_{i5}}{2} \right)^{\epsilon} w_{\rm tc}^i \, F_{\rm LM}(1,4\,|\,5) \right\rangle_\delta \\
&\quad - C_A \, \frac{[\alpha_{s,b}]}{2\epsilon^2} N_\epsilon E_{\rm max}^{-2\epsilon} \sum_{i \in \{1,4\}} \left\langle C_{5i} \left[I - S_5 \right] \left(\frac{\rho_{i5}}{2} \right)^{-\epsilon} \left(1 - \frac{\rho_{i5}}{2} \right)^{\epsilon} w_{\rm tc}^i \, F_{\rm LM}(1,4\,|\,5) \right\rangle .
\end{aligned} \tag{6.155}$$

As before, the first term on the right-hand side of Eq. (6.155) is finite and the singularities are present in the subtraction term (second term in Eq. (6.155)). The latter can be computed with the help of previous discussion. We obtain

$$
\begin{aligned}
&- C_A\,\frac{[\alpha_{s,b}]}{2\epsilon^2} N_\epsilon E_{\max}^{-2\epsilon} \sum_{i\in\{1,4\}} \left\langle C_{5i}\big[I - S_5\big] \left(\frac{\rho_{i5}}{2}\right)^{-\epsilon} \left(1 - \frac{\rho_{i5}}{2}\right)^{\epsilon} w^{i}_{\rm tc}\, F_{\rm LM}(1,4\,|\,5) \right\rangle \\
&= -C_A\,\frac{[\alpha_{s,b}]^2}{2\epsilon^3} \left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right] N_\epsilon 2^\epsilon (2E_{\max})^{-2\epsilon} \Bigg\langle \Bigg[\gamma^{22}_{qq}(2E_4)^{-2\epsilon} \\
&\quad - \frac{2C_F}{2\epsilon}\Big(2(2E_{\max})^{-2\epsilon} + (2E_4)^{-2\epsilon}\Big)\Bigg] F_{\rm LM}(1,4) \Bigg\rangle_{\delta} \\
&\quad + C_A\,\frac{[\alpha_{s,b}]^2}{2\epsilon^3} \left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right] N_\epsilon 2^\epsilon (2E_{\max})^{-2\epsilon} (2E_1)^{-2\epsilon} \\
&\quad \times \int_0^1 {\rm d}z\;(1-z)^{-2\epsilon} P_{qq}(z) \left\langle \frac{F_{\rm LM}(z\cdot 1,4)}{z} \right\rangle_{\delta} .
\end{aligned}
\tag{6.156}
$$

We continue with the second on the right-hand side of Eq. (6.153). Similarly to Eq. (6.155), we obtain

$$
\begin{aligned}
&\frac{[\alpha_{s,b}]}{2\epsilon} N_\epsilon \sum_{i\in\{1,4\}} \Bigg\langle \left(\frac{\rho_{i5}}{2}\right)^{-\epsilon} \left(1 - \frac{\rho_{i5}}{2}\right)^{\epsilon} E_5^{-2\epsilon}\, w^{i}_{\rm tc}[1 - S_5] \\
&\quad \times \left\{ \left[\frac{2C_A}{\epsilon} + \gamma^{22}_{gg}\right] F_{\rm LM}(1,4\,|\,5) + \epsilon\gamma^{\perp,22}_{gg}\big[r^{(i)}_\mu r^{(i)}_\nu + g_{\mu\nu}\big] F^{\mu\nu}_{\rm LM}(1,4\,|\,5) \right\} \Bigg\rangle \\
&= \frac{[\alpha_{s,b}]}{2\epsilon} N_\epsilon \sum_{i\in\{1,4\}} \Bigg\langle \hat{\mathcal{O}}^{(i)}_{\rm nlo} \left(\frac{\rho_{i5}}{2}\right)^{-\epsilon} \left(1 - \frac{\rho_{i5}}{2}\right)^{\epsilon} E_5^{-2\epsilon}\, w^{i}_{\rm tc} \\
&\quad \times \left\{ \left[\frac{2C_A}{\epsilon} + \gamma^{22}_{gg}\right] F_{\rm LM}(1,4\,|\,5) + \epsilon\gamma^{\perp,22}_{gg}\big[r^{(i)}_\mu r^{(i)}_\nu + g_{\mu\nu}\big] F^{\mu\nu}_{\rm LM}(1,4\,|\,5) \right\} \Bigg\rangle \\
&\quad + \frac{[\alpha_{s,b}]}{2\epsilon} N_\epsilon \sum_{i\in\{1,4\}} \Bigg\langle C_{5i}\big[I - S_5\big] \left(\frac{\rho_{i5}}{2}\right)^{-\epsilon} \left(1 - \frac{\rho_{i5}}{2}\right)^{\epsilon} E_5^{-2\epsilon}\, w^{i}_{\rm tc} \\
&\quad \times \left\{ \left[\frac{2C_A}{\epsilon} + \gamma^{22}_{gg}\right] F_{\rm LM}(1,4\,|\,5) + \epsilon\gamma^{\perp,22}_{gg}\big[r^{(i)}_\mu r^{(i)}_\nu + g_{\mu\nu}\big] F^{\mu\nu}_{\rm LM}(1,4\,|\,5) \right\} \Bigg\rangle .
\end{aligned}
\tag{6.157}
$$

We now discuss how to simplify the second term on the right-hand side of Eq. (6.157). We begin with the soft-collinear contribution. The required $C_{5i}S_5$-limit of the spin-correlated amplitude $r^{(i)}_\mu r^{(i)}_\nu F^{\mu\nu}_{\rm LM}(1,4\,|\,5)$ is computed in Appendix C. It reads

$$
C_{5i}S_5\; r^{(i)}_\mu r^{(i)}_\nu F^{\mu\nu}_{\rm LM}(1,4\,|\,5) = C_{5i}S_5\; F_{\rm LM}(1,4\,|\,5)\,. \tag{6.158}
$$

It follows that the spin-correlated matrix element behaves like a regular NLO matrix element in this limit, which, together with $g_{\mu\nu}F^{\mu\nu}_{\rm LM}(1,4\,|\,5) = -F_{\rm LM}(1,4\,|\,5)$, implies that the second term in the curly brackets vanishes in the soft-collinear limit. To compute the first term in the curly brackets in the soft-collinear limit we follow the NLO discussion in Section 5.2 and obtain

$$\begin{aligned}
&-\frac{[\alpha_{s,b}]}{2\epsilon}N_\epsilon \sum_{i\in\{1,4\}} \left\langle C_{5i}S_5 \left(\frac{\rho_{i5}}{2}\right)^{-\epsilon}\left(1-\frac{\rho_{i5}}{2}\right)^{\epsilon} E_5^{-2\epsilon}\, w_{\rm tc}^{i}\left[\frac{2C_A}{\epsilon}+\gamma_{gg}^{22}\right]F_{\rm LM}(1,4\,|\,5)\right\rangle \\
&= -2C_F\,\frac{[\alpha_{s,b}]^2}{4\epsilon^3}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]\left[\frac{2C_A}{\epsilon}+\gamma_{gg}^{22}\right]N_\epsilon 2^\epsilon(2E_{\rm max})^{-4\epsilon}\langle F_{\rm LM}(1,4)\rangle_\delta\,.
\end{aligned} \tag{6.159}$$

We continue with the discussion of the collinear contributions to the subtraction term. We consider initial-state and final-state emissions separately. We begin with the initial-state emission. The required C_{51}-limit of the spin-correlated amplitude $r_\mu^{(1)}r_\nu^{(1)}F_{\rm LM}^{\mu\nu}(1,4\,|\,5)$ reads

$$C_{51}\, r_\mu^{(1)}r_\nu^{(1)}F_{\rm LM}^{\mu\nu}(1,4\,|\,5) = g_{s,b}^2\times\frac{1}{p_1\cdot p_5}P_{qq}^{\rm spin}(z)\times\frac{F_{\rm LM}(z\cdot 1,4)}{z}\,, \tag{6.160}$$

where $z=(E_1-E_5)/E_1$. The splitting function in Eq. (6.160) reads

$$P_{qq}^{\rm spin}(z) = \frac{C_F}{2}\,\frac{(1+z)^2}{1-z}\,. \tag{6.161}$$

Note that the difference between the collinear limit Eq. (6.160) and the collinear limit of $F_{\rm LM}(1,4\,|\,5)$ in Eq. (5.30) is only a different splitting function. Therefore, conceptually, the computation is identical to the NLO case discussed in Section (5.2). We obtain

$$\begin{aligned}
&\frac{[\alpha_{s,b}]}{2\epsilon}N_\epsilon\left\langle C_{51}\left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon}E_5^{-2\epsilon}\,w_{\rm tc}^1\right.\\
&\qquad\left.\times\left\{\left[\frac{2C_A}{\epsilon}+\gamma_{gg}^{22}\right]F_{\rm LM}(1,4\,|\,5)+\epsilon\gamma_{gg}^{\perp,22}\left[r_\mu^{(1)}r_\nu^{(1)}+g_{\mu\nu}\right]F_{\rm LM}^{\mu\nu}(1,4\,|\,5)\right\}\right\rangle\\
&= -\frac{[\alpha_{s,b}]^2}{2\epsilon^2}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]N_\epsilon 2^\epsilon(2E_1)^{-4\epsilon}\int_0^1 {\rm d}z(1-z)^{-2\epsilon}\\
&\qquad\times\left\{\left[\frac{2C_A}{\epsilon}+\gamma_{gg}^{22}-\epsilon\gamma_{gg}^{\perp,22}\right]P_{qq}(z)+\epsilon\gamma_{gg}^{\perp,22}P_{qq}^{\rm spin}(z)\right\}\left\langle\frac{F_{\rm LM}(z\cdot 1,4)}{z}\right\rangle_\delta\,.
\end{aligned} \tag{6.162}$$

We continue with the final-state emissions. The required C_{54}-limit of the spin-correlated amplitude $r_\mu^{(4)}r_\nu^{(4)}F_{\rm LM}^{\mu\nu}(1,4\,|\,5)$ reads

$$C_{54}\, r_\mu^{(4)}r_\nu^{(4)}F_{\rm LM}^{\mu\nu}(1,4\,|\,5) = g_{s,b}^2\times\frac{1}{p_4\cdot p_5}P_{qq}^{\rm spin}(z)\times F_{\rm LM}\left(1,\frac{1}{z}\cdot 4\right)\,, \tag{6.163}$$

where $z=E_4/(E_4+E_5)$. Apart from the different splitting function, the above limit is structurally identical to the collinear C_{54} limit of $F_{\rm LM}(1,4\,|\,5)$ in Eq. (5.43). Following the NLO calculation outlined in Section (5.2), we obtain

$$\begin{aligned}
&\frac{[\alpha_{s,b}]}{2\epsilon} N_\epsilon \Big\langle C_{54} \left(\frac{\rho_{45}}{2}\right)^{-\epsilon} \left(1-\frac{\rho_{45}}{2}\right)^{\epsilon} E_5^{-2\epsilon}\, w_{\rm tc}^4 \\
&\quad \times \left\{ \left[\frac{2C_A}{\epsilon} + \gamma_{gg}^{22}\right] F_{\rm LM}(1,4\,|\,5) + \epsilon \gamma_{gg}^{\perp,22} \left[r_\mu^{(4)} r_\nu^{(4)} + g_{\mu\nu}\right] F_{\rm LM}^{\mu\nu}(1,4\,|\,5) \right\} \Big\rangle \\
&= \frac{[\alpha_{s,b}]^2}{2\epsilon^2} \left[\frac{2^\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right] N_\epsilon 2^\epsilon \left\{ \left[\frac{2C_A}{\epsilon} + \gamma_{gg}^{22} - \epsilon \gamma_{gg}^{\perp,22}\right] \left[\frac{2C_F}{4\epsilon} + \gamma_{qq}^{24}\right] \right. \\
&\quad \left. + \left[\frac{2C_F}{4\epsilon} + \gamma_{qq,\rm spin}^{24}\right] \gamma_{gg}^{\perp,22} \right\} \left\langle (2E_4)^{-4\epsilon} F_{\rm LM}(1,4) \right\rangle_\delta .
\end{aligned} \tag{6.164}$$

In Eq. (6.164) we have used the anomalous dimension $\tilde{\gamma}_g(\epsilon)$ defined in Eq. (5.50) and, in addition, introduced its *spin-correlated* version

$$\gamma_{qq,\rm spin}^{24} \equiv -\int_0^1 {\rm d}z \left[z^{-2\epsilon}(1-z)^{-4\epsilon} P_{qq}^{\rm spin}(z) - 2C_F\, \frac{(1-z)^{-4\epsilon}}{1-z} \right]. \tag{6.165}$$

An expansion of $\gamma_{qq,\rm spin}^{24}$ in the dimensional regularization parameter ϵ can be found in Appendix E. Inserting subtraction terms in Eqs. (6.159, 6.162, 6.164) into Eq. (6.157) and combining with Eqs. (6.153, 6.155, 6.3.3 we obtain a fully-regulated result of the single-unresolved subtraction terms in Eq. (6.114) as

$$\begin{aligned}
&\sum_{i\in\{1,4\}} \Big\langle [1-\mathbb{S}][1-S_6]\left[\theta_i^{(b)} C_{56} + \theta_i^{(d)} C_{56}\right] [{\rm d}p_5][{\rm d}p_6] w^{5i,6i} F_{\rm LM}(1,4\,|\,5,6) \Big\rangle \\
&= -C_A \frac{[\alpha_s]}{2\epsilon^2} N_\epsilon E_{\rm max}^{-2\epsilon} \sum_{i\in\{1,4\}} \Big\langle \hat{\mathcal{O}}_{\rm nlo}^{(i)} \left(\frac{\rho_{i5}}{2}\right)^{-\epsilon} \left(1-\frac{\rho_{i5}}{2}\right)^{\epsilon} w_{\rm tc}^i\, F_{\rm LM}(1,4\,|\,5) \Big\rangle_\delta \\
&\quad + \frac{[\alpha_s]}{2\epsilon} N_\epsilon 2^{2\epsilon} \sum_{i\in\{1,4\}} \Big\langle \hat{\mathcal{O}}_{\rm nlo}^{(i)} \left(\frac{\rho_{i5}}{2}\right)^{-\epsilon} \left(1-\frac{\rho_{i5}}{2}\right)^{\epsilon} \left(\frac{4E_5^2}{\mu^2}\right)^{-\epsilon} w_{\rm tc}^i \\
&\qquad \times \left\{ \left[\frac{2C_A}{\epsilon} + \gamma_{gg}^{22}\right] F_{\rm LM}(1,4\,|\,5) + \epsilon \gamma_{gg}^{\perp,22} \left[r_\mu^{(i)} r_\nu^{(i)} + g_{\mu\nu}\right] F_{\rm LM}^{\mu\nu}(1,4\,|\,5) \right\} \Big\rangle \\
&\quad - C_A \frac{[\alpha_s]^2}{2\epsilon^3} \left[\frac{2^\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right] N_\epsilon 2^\epsilon \left(\frac{4E_{\rm max}^2}{\mu^2}\right)^{-\epsilon} \Big\langle \Big[\gamma_{qq}^{22} \left(\frac{4E_4^2}{\mu^2}\right)^{-\epsilon} \\
&\qquad - \frac{2C_F}{2\epsilon} \left\{ 2\left(\frac{4E_{\rm max}^2}{\mu^2}\right)^{-\epsilon} + \left(\frac{4E_4^2}{\mu^2}\right)^{-\epsilon} \right\} \Big] F_{\rm LM}(1,4) \Big\rangle_\delta \\
&\quad + C_A \frac{[\alpha_s]^2}{2\epsilon^3} \left[\frac{2^\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right] N_\epsilon 2^\epsilon \left(\frac{4E_{\rm max}^2}{\mu^2}\right)^{-\epsilon} \left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon} \\
&\qquad \times \int_0^1 {\rm d}z\, (1-z)^{-2\epsilon} P_{qq}(z) \Big\langle \frac{F_{\rm LM}(z\cdot 1,4)}{z} \Big\rangle_\delta \\
&\quad - \frac{[\alpha_s]^2}{2\epsilon^2} \left[\frac{2^\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right] N_\epsilon 2^\epsilon \left(\frac{4E_1^2}{\mu^2}\right)^{-2\epsilon} \int_0^1 {\rm d}z (1-z)^{-2\epsilon}
\end{aligned}$$

$$
\begin{aligned}
&\times \left\{ \left[\frac{2C_A}{\epsilon} + \gamma_{gg}^{22} - \epsilon\gamma_{gg}^{\perp,22} \right] P_{qq}(z) + \epsilon\gamma_{gg}^{\perp,22} P_{qq}^{\mathrm{spin}}(z) \right\} \left\langle \frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z} \right\rangle_\delta \\
&+ \frac{[\alpha_s]^2}{2\epsilon^2} \left[\frac{2^\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} \right] N_\epsilon 2^\epsilon \left\{ \left[\frac{2C_A}{\epsilon} + \gamma_{gg}^{22} - \epsilon\gamma_{gg}^{\perp,22} \right] \left[\frac{2C_F}{4\epsilon} + \gamma_{qq}^{24} \right] \right. \\
&+ \left. \left[\frac{2C_F}{4\epsilon} + \gamma_{qq,\mathrm{spin}}^{24} \right] \gamma_{gg}^{\perp,22} \right\} \left\langle \left(\frac{4E_4^2}{\mu^2} \right)^{-2\epsilon} F_{\mathrm{LM}}(1,4) \right\rangle_\delta \\
&- 2C_F \, \frac{[\alpha_s]^2}{4\epsilon^3} \left[\frac{2^\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} \right] \left[\frac{2C_A}{\epsilon} + \gamma_{gg}^{22} \right] N_\epsilon 2^\epsilon \left(\frac{4E_{\max}^2}{\mu^2} \right)^{-2\epsilon} \langle F_{\mathrm{LM}}(1,4) \rangle_\delta \\
&- 2C_F \, \frac{[\alpha_s]^2}{\epsilon^2} \times [2^{2\epsilon}\delta_g(\epsilon)] \left(\frac{4E_{\max}^2}{\mu^2} \right)^{-2\epsilon} \left\langle \langle \Delta_{65} \rangle_{S_5} F_{\mathrm{LM}}(1,4) \right\rangle_\delta \\
&+ C_F \, [\alpha_s]^2 \delta_g^{\perp}(\epsilon) \left(\frac{4E_{\max}^2}{\mu^2} \right)^{-2\epsilon} \left\langle \, \langle r^\mu r^\nu \rangle_{\rho_5} F_{\mathrm{LM}}(1,4) \right\rangle_\delta .
\end{aligned}
\tag{6.166}
$$

This equation, together with the similar Eqs. (6.86) and (6.104), completes the analysis of the single-collinear terms in Eq. (6.17). The only singularities are present as poles in at most $1/\epsilon^2$, and all implicit divergences in F_{LM} are regulated. We now proceed to the double-unresolved terms in Eq. (6.17) in the next section.

6.4. Double-unresolved collinear subtraction terms

We continue with the discussion of the subtraction terms in Eq. (6.17) where *both* gluons $g(p_5)$ and $g(p_6)$ are collinear to hard parton(s) and, therefore, are unresolved. There are two contributions to be discussed. First, there are subtraction terms where the two gluons are collinear to the *same* parton. They are described by the following subtraction terms

$$
\begin{aligned}
\sum_{i\in\{1,4\}} \Big\langle [I-\mathbb{S}]\,[I-S_6] \Big[&\theta_i^{(a)} \mathbb{C}_i [1 - C_{5i}] + \theta_i^{(b)} \mathbb{C}_i [I - C_{56}] + \theta_i^{(c)} \mathbb{C}_i [1 - C_{6i}] \\
&+ \theta_i^{(d)} \mathbb{C}_i [1 - C_{56}] \Big] [\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6i} F_{\mathrm{LM}}(1,4\,|\,5,6) \Big\rangle .
\end{aligned}
\tag{6.167}
$$

The corresponding contributions were computed in Ref. [68]. We provide the required formula in Appendix I.

The second contribution corresponds to kinematic configurations where each of the gluons is collinear to a different parton. The subtraction term reads

$$
-\sum_{\substack{i,j\in\{1,4\}\\ i\neq j}} \Big\langle [I-\mathbb{S}][I-S_6] C_{5i} C_{6j} [\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6j} F_{\mathrm{LM}}(1,4\,|\,5,6) \Big\rangle . \tag{6.168}
$$

We note that in this case the collinear limits and the integration over gluon angles are, effectively, NLO-like. We elaborate on this observation below.

As we pointed out at the beginning of Section 6.3, upon taking a limit where a gluon becomes collinear to an external hard quark, the double soft operator $\mathbb{S}$ becomes equivalent to the strongly ordered limit $S_5 S_6$. It follows that

$$\mathbb{S}\left[I - S_6\right] C_{5i} C_{6j} F_{\rm LM}(1,4\,|\,5,6) = \left[S_5 S_6 - S_5 S_6\right] C_{5i} C_{6j} F_{\rm LM}(1,4\,|\,5,6) = 0\,, \tag{6.169}$$

and we can drop the double-soft contribution to the subtraction term Eq. (6.168). Next, we rewrite the sum in Eq. (6.168) as follows

$$\sum_{\substack{i,j\in\{1,4\}\\ i\neq j}} C_{5i} C_{6j} w^{5i,6j} = \sum_{\substack{k,l\in\{5,6\}\\ k\neq l}} C_{k1} C_{l4} w^{k1,l4}\,. \tag{6.170}$$

The required collinear limit can be written for generic l and k in the following way

$$\begin{aligned} &C_{k1} C_{l4} F_{\rm LM}(1,4\,|\,5,6) \\ &= g_{s,b}^4 \times \frac{1}{p_1\cdot p_k} P_{qq}\left(z_k\right) \times \frac{1}{p_4\cdot p_l} P_{qq}\left(\bar{z}_l\right) \times \frac{F_{\rm LM}(1-k, 4+l)}{z_k}\,, \end{aligned} \tag{6.171}$$

where $z_k = (E_1 - E_k)/E_1$ and $\bar{z}_l = E_4/(E_4 + E_l)$. The splitting function $P_{qq}(z)$ is given in Eq. (5.31). In Eq. (6.171) the double-collinear limit to the initial-state and the double-collinear limit to the final-state factorize in terms of the known NLO double-collinear limits Eqs. (5.30, 5.43). Integrating over angles

$$\int \left[C_{k1} \frac{{\rm d}\Omega_k^{(d-1)}}{2(2\pi)^{d-1}}\right] \frac{1}{\rho_{1k}} \times \int \left[C_{l4} \frac{{\rm d}\Omega_l^{(d-1)}}{2(2\pi)^{d-1}}\right] \frac{1}{\rho_{4l}} \overset{\rm (G.9)}{=} \frac{2^{-4\epsilon}}{\epsilon^2} \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]^2\,, \tag{6.172}$$

and inserting the limit Eq. (5.30) into the subtraction term Eq. (6.168) we obtain

$$\begin{aligned} &-\sum_{\substack{k,l\in\{5,6\}\\ k\neq l}} \left[I - \mathbb{S}\right]\left[I - S_6\right] C_{k1} C_{l4} [{\rm d}p_k][{\rm d}p_l] w^{k1,l4} F_{\rm LM}(1,4\,|\,5,6) \\ &= -\frac{[\alpha_{s,b}]^2}{\epsilon^2} \sum_{\substack{k,l\in\{5,6\}\\ k\neq l}} 2^{-4\epsilon} \int_0^{E_{\rm max}} {\rm d}E_k\, E_k^{1-2\epsilon} \int_0^{E_{\rm max}} {\rm d}E_l\, E_l^{1-2\epsilon}\, \theta(E_5 - E_6) \\ &\quad \times [1 - S_6] \frac{1}{E_1 E_4 E_k E_l} P_{qq}\left(z_k\right) P_{qq}\left(z_l\right) \frac{F_{\rm LM}(1-k, 4+l)}{z_k}\,. \end{aligned} \tag{6.173}$$

To integrate over E_5 and E_6, we write the two terms of the sum in Eq. (6.173) explicitly. Upon renaming $5 \leftrightarrow 6$ in the term where $k = 6$ and $l = 5$, we obtain a formula where momenta in the function $F_{\rm LM}$ appear in a unique way

$$\begin{aligned} &-\frac{[\alpha_{s,b}]^2}{\epsilon^2} 2^{-4\epsilon} \int_0^{E_{\rm max}} \frac{{\rm d}E_5}{E_1} \frac{{\rm d}E_6}{E_4} E_5^{-2\epsilon} E_6^{-2\epsilon} \left[I - \theta(E_5 - E_6) S_6 - \theta(E_6 - E_5) S_5\right] \\ &\times P_{qq}\left(z_5\right) P_{qq}\left(\bar{z}_6\right) \frac{F_{\rm LM}(1-5, 4+6)}{z_5}\,. \end{aligned} \tag{6.174}$$

We proceed with the simplification of the three terms on the right-hand side of Eq. (6.174). We begin with the term without soft operators $S_{i=5,6}$. We note that in this term no energy ordering is present. Therefore integration over energies E_5 and E_6 is identical to the NLO case. For both integrals we follow steps described in Section 5.2 and obtain

$$
\begin{aligned}
&-\frac{[\alpha_{s,b}]^2}{\epsilon^2} 2^{-4\epsilon} \int\limits_0^{E_{\max}} \frac{\mathrm{d}E_5}{E_1} \frac{\mathrm{d}E_6}{E_4} E_5^{-2\epsilon} E_6^{-2\epsilon} P_{qq}(z_5)\, P_{qq}(\bar{z}_6)\, \frac{F_{\mathrm{LM}}(1-5,4+6)}{z_5} \\
&= \frac{[\alpha_{s,b}]^2}{\epsilon^2} \left[\frac{2C_F}{2\epsilon} + \gamma_{qq}^{22} \right] \int\limits_0^1 \mathrm{d}z\, (1-z)^{-2\epsilon} P_{qq}(z)\, (2E_1)^{-2\epsilon} (2E_4)^{-2\epsilon} \frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\,.
\end{aligned}
\tag{6.175}
$$

Next, we consider terms in Eq. (6.174) that contain the operator S_6. The required limits read

$$
S_6 F_{\mathrm{LM}}(1-5,4+6) = F_{\mathrm{LM}}(1-5,4)\,, \quad S_6 P_{qq}(\bar{z}_6) = 2C_F \frac{E_4}{E_6}\,. \tag{6.176}
$$

We rename $z_5 = z$ and use $E_5 = (1-z)E_1$ to trade E_5 integration for the z integration. We obtain

$$
\begin{aligned}
&2C_F \frac{[\alpha_{s,b}]^2}{2\epsilon^2} \times 2^{1-2\epsilon} \int\limits_0^{E_{\max}} \mathrm{d}E_6\, E_6^{-1-2\epsilon} \theta((1-z)E_1 - E_6) \\
&\times \int\limits_0^1 \mathrm{d}z\, (1-z)^{-2\epsilon} P_{qq}(z) (2E_1)^{-2\epsilon} \frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\,.
\end{aligned}
\tag{6.177}
$$

Since, by construction, $E_{\max} > (1-z)E_1$ for $z \in [0,1]$, the θ-function in Eq. (6.177) provides the upper bound for the E_6 integration for all values of $z \in [0,1]$. We obtain

$$
2^{1-2\epsilon} \int\limits_0^{(1-z)E_1} \mathrm{d}E_6\, E_6^{-1-2\epsilon} = -\frac{(1-z)^{-2\epsilon}(2E_1)^{-2\epsilon}}{\epsilon}\,. \tag{6.178}
$$

Using integral Eq. (6.178) in Eq. (6.177) we find

$$
\begin{aligned}
&\frac{[\alpha_{s,b}]^2}{\epsilon^2} 2^{-4\epsilon} \int\limits_0^{E_{\max}} \frac{\mathrm{d}E_5}{E_1} \frac{\mathrm{d}E_6}{E_4} E_5^{-2\epsilon} E_6^{-2\epsilon} \theta(E_5 - E_6) S_6 \times P_{qq}(z_5)\, P_{qq}(\bar{z}_6)\, \frac{F_{\mathrm{LM}}(1-5,4+6)}{z_5} \\
&= -2C_F \frac{[\alpha_{s,b}]^2}{2\epsilon^3} \int\limits_0^1 \mathrm{d}z\, (1-z)^{-4\epsilon} P_{qq}(z)\, (2E_1)^{-4\epsilon} \frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\,.
\end{aligned}
\tag{6.179}
$$

Finally we consider the term in Eq. (6.174) that contains the operator S_5. The required limits read

$$
S_5 F_{\mathrm{LM}}(1-5,4+6) = F_{\mathrm{LM}}(1,4+6)\,, \quad S_5 P_{qq}(z_5) = -2C_F \frac{E_1}{E_5}\,. \tag{6.180}
$$

According to Eq. (6.180) integration over E_5 factorizes from the function $F_{\rm LM}$. To integrate over E_6 we follow the NLO discussion. Then, writing $E_6 = E_4(1-z)/z$ and rescaling $E_4 \to z \cdot E_4$, we obtain

$$
\begin{aligned}
&\frac{[\alpha_{s,b}]^2}{\epsilon^2} 2^{-4\epsilon} \int_0^{E_{\max}} \frac{\mathrm{d}E_5}{E_1}\frac{\mathrm{d}E_6}{E_4} E_5^{-2\epsilon} E_6^{-2\epsilon} - \theta(E_6 - E_5) S_5 \times P_{qq}\,(z_5)\, P_{qq}\,(\bar{z}_6)\, \frac{F_{\rm LM}(1-5, 4+6)}{z_5} \\
&= 2C_F\, \frac{[\alpha_{s,b}]^2}{2\epsilon^2} \times \underbrace{\int_0^1 \mathrm{d}z\, z^{-2\epsilon}(1-z)^{-2\epsilon} P_{qq}(z)}_{= -\left[\frac{2C_F}{4\epsilon} + \gamma_{qq}^{24}\right]} \times \underbrace{2^{1-2\epsilon} \int_0^{E_{\max}} \frac{\mathrm{d}E_5}{E_5} E_5^{-2\epsilon} \theta((1-z)E_4 - E_5)}_{= -\frac{(1-z)^{-2\epsilon}(2E_4)^{-2\epsilon}}{\epsilon}} \\
&\quad \times (2E_4)^{-2\epsilon} F_{\rm LM}(1,4) \\
&= 2C_F\, \frac{[\alpha_{s,b}]^2}{2\epsilon^3} \left[\frac{2C_F}{4\epsilon} + \gamma_{qq}^{24}\right] (2E_4)^{-4\epsilon} F_{\rm LM}(1,4)\,.
\end{aligned}
\tag{6.181}
$$

Inserting Eqs. (6.175, 6.179, 6.181) into Eq. (6.174) we derive the following result for the subtraction term

$$
\boxed{
\begin{aligned}
&- \sum_{\substack{i,j\in\{1,4\}\\ i\neq j}} \left\langle [I - \mathbb{S}]\,[I - S_6]\, C_{5i} C_{6j} [\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6j} F_{\rm LM}(1,4\,|\,5,6)\right\rangle \\
&= 2C_F\, \frac{[\alpha_s]^2}{2\epsilon^3} \left[\frac{2C_F}{4\epsilon} + \gamma_{qq}^{24}\right] \left\langle \left(\frac{4E_4^2}{\mu^2}\right)^{-2\epsilon} F_{\rm LM}(1,4) \right\rangle_\delta \\
&\quad + \frac{[\alpha_s]^2}{\epsilon^2} \left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon} \int_0^1 \mathrm{d}z\, (1-z)^{-2\epsilon} P_{qq}(z) \\
&\qquad \times \left\langle \left[2C_F \frac{(4E_4^2/\mu^2)^{-\epsilon} - (1-z)^{-2\epsilon}(4E_1^2/\mu^2)^{-\epsilon}}{2\epsilon} + \gamma_{qq}^{22} (2E_4)^{-2\epsilon}\right] \frac{F_{\rm LM}(z\cdot 1, 4)}{z} \right\rangle_\delta .
\end{aligned}
}
\tag{6.182}
$$

We have now completed the regularization of IR singularities from the double-real corrections. We have written these corrections in terms of manifestly finite terms and subtraction counterterms where the divergences appear as explicit poles in $1/\epsilon$. These poles will cancel against the real-virtual, double-virtual, and collinear renormalization contributions. We will discuss these in the following two sections.

6.5. Real-virtual contribution

In this section we consider the real-virtual contributions $\mathrm{d}\hat{\sigma}_{\rm rv}$ to the partonic DIS cross section Eq. (6.1). We focus on the quark-initiated channel

$$
q(p_1) + e^-(p_2) \;\to\; e^-(p_3) + q(p_4) + g(p_5)\,. \tag{6.183}
$$

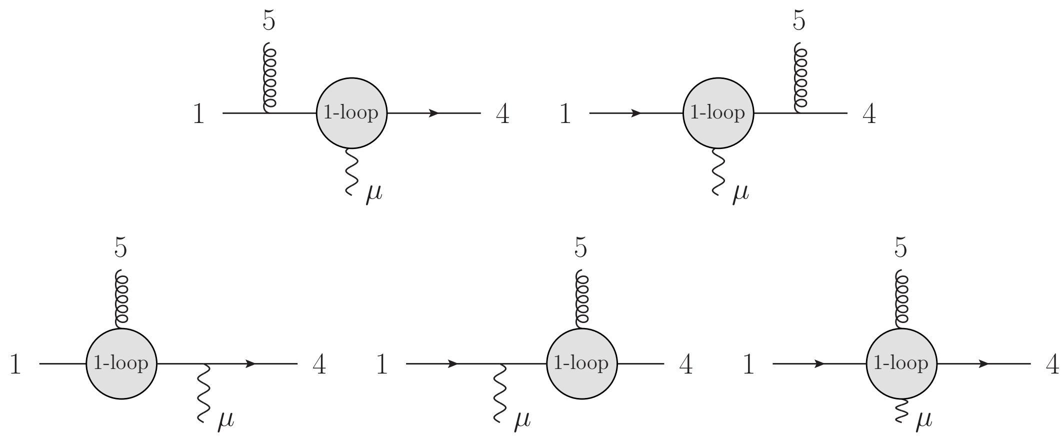


Fig. 6.5.: Partonic currents that contribute to the quark channel Eq. (6.183) of the real-virtual contribution of DIS. To obtain the complete Feynman diagrams for DIS they need to be contracted with the leptonic current. We only show labels i of external momenta p_i. Grey circles stand for all possible 1-loop subdiagrams.

The gluon channel $g + e^- \;\to\; e^- + q\bar{q} + g$ is discussed in Chapter 7. Feynman diagrams that contribute to the matrix element are schematically shown in Fig. 6.5.

In analogy to Eqs. (5.3, 5.4) we define a UV-renormalized contribution as

$$2s \cdot \mathrm{d}\hat{\sigma}_{\mathrm{rv}} \equiv \int [\mathrm{d}p_5] \;\; F_{\mathrm{LV}}\big(1_q, 4_q \,|\, 5_g\big) \equiv \big\langle F_{\mathrm{LV}}\big(1_q, 4_q \,|\, 5_g\big)\big\rangle_\delta \,, \tag{6.184}$$

where

$$\begin{aligned} F_{\mathrm{LV}}\big(1_q, 4_q \,|\, 5_g\big) = \mathcal{N} \int [\mathrm{d}p_3][\mathrm{d}p_4] \; (2\pi)^d \delta^{(d)} \left(p_1 + p_2 - p_3 - p_4 - p_5\right) \\ \times 2\Re\big(M_{\mathrm{nlo}}^{\mathrm{tree}} \cdot M_{\mathrm{nlo}}^{\text{1-loop}}\big)(p_1, p_2, p_3, p_4, p_5) \times \hat{\mathcal{O}}(p_3, p_4, p_5) \,, \end{aligned} \tag{6.185}$$

Quantities that appear on the right-hand side of Eq. (6.185) are defined as in the NLO case with the exception that the function F_{LV} is proportional to the interference of the tree-level amplitude $M_{\mathrm{nlo}}^{\mathrm{tree}}$ with the one-loop amplitude $M_{\mathrm{nlo}}^{\text{1-loop}}$. The latter is composed of Feynman diagrams shown in Fig. 6.5. Infrared and collinear $1/\epsilon$ poles that appear in the UV renormalized amplitude are given by Catani's formula [51–53]. Following these references, we decompose $F_{\mathrm{LV}}(1, 4 \,|\, 5)$ into a divergent and finite parts

$$F_{\mathrm{LV}}(1, 4 \,|\, 5) = \hat{I}_{145}^{\mathrm{rv}} F_{\mathrm{LM}}(1, 4 \,|\, 5) + F_{\mathrm{LV}}^{\mathrm{fin}}(1, 4 \,|\, 5) \,. \tag{6.186}$$

The operator $\hat{I}_{145}$ contains $1/\epsilon$ poles, $F_{\mathrm{LM}}(1, 4 \,|\, 5)$ is the Born cross section that can be found in Eq. (5.4) and $F_{\mathrm{LV}}^{\mathrm{fin}}(1, 4 \,|\, 5)$ is finite and contains no explicit $1/\epsilon$ poles. The operator $\hat{I}_{145}^{\mathrm{rv}}$ in

Eq. (6.186) reads

$$\hat{I}_{145}^{\mathrm{rv}} = \frac{[\alpha_s]}{\epsilon}\left[\left(\frac{1}{\epsilon}+\frac{3}{2}\right)(C_A-2C_F)(2E_1E_4\rho_{14})^{-\epsilon} - \left(\frac{1}{\epsilon}+\frac{3}{4}\right)C_A\Big((2E_1E_6\rho_{61})^{-\epsilon}+(2E_4E_6\rho_{64})^{-\epsilon}\cos(\pi\epsilon)\Big)\right]. \tag{6.187}$$

Additional singularities arise when we attempt to integrate the real-virtual contribution over momenta of final-state partons. To understand this, we note that the real-virtual amplitudes are singular in the same phase space regions as Born amplitudes. Hence, the needed subtractions are identical to the NLO calculation at the operator level. We write

$$\begin{aligned}\big\langle F_{\mathrm{LV}}(1,4\,|\,5)\big\rangle_\delta = \big\langle S_5F_{\mathrm{LV}}(1,4\,|\,5)\big\rangle_\delta + \sum_{i\in\{1,4\}}\big\langle C_{5i}[1-S_5]w^{5i}\,F_{\mathrm{LV}}(1,4\,|\,5)\big\rangle_\delta \\ + \sum_{i\in\{1,4\}}\langle\hat{\mathcal{O}}^{(i)}_{\mathrm{nlo}}w^{5i}F_{\mathrm{LV}}(1,4\,|\,5)\big\rangle_\delta\,.\end{aligned} \tag{6.188}$$

Operators $\hat{\mathcal{O}}^{(i)}_{\mathrm{nlo}}$ that regulate soft and collinear singularities are given in Eq. (5.15) and w^{5i} are the partition functions introduced in Eq. (5.9).

6.5.1. Soft subtraction term

We discuss how to simplify subtraction terms on the right-hand side of Eq. (6.188), starting with the first term that describes the soft subtraction. The required soft limit reads [58–60]

$$\begin{aligned}S_5\,F_{\mathrm{LV}}(1,4\,|\,5) &= 2C_F\,g_{s,b}^2\times\frac{p_1\cdot p_4}{(p_1\cdot p_5)(p_4\cdot p_5)}\times F_{\mathrm{LV}}(1,4)\\ &-2C_FC_A\,\frac{g_{s,b}^2[\alpha_{s,b}]}{\epsilon^2}\left[\frac{\Gamma^5(1-\epsilon)\Gamma^3(1+\epsilon)}{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)}\right]\times 2^{-\epsilon}\left(\frac{p_1\cdot p_4}{(p_1\cdot p_5)(p_4\cdot p_5)}\right)^{1+\epsilon}\\ &\times F_{\mathrm{LM}}(1,4)\,.\end{aligned} \tag{6.189}$$

We note that Eq. (6.189) has two contributions: the first term on the right-hand side of Eq. (6.189) contains one-loop hard matrix element and tree-level eikonal function. The second term on the right-hand side of Eq. (6.189) contains the tree-level hard matrix element and one-loop correction to the eikonal function. Note that this contribution is non-abelian.

Integration of the first term over gluon phase space is identical to the NLO case. To integrate the second term over the phase space of the unresolved gluon we use the following integral

$$\int\frac{\mathrm{d}\Omega_5^{(d-1)}}{2(2\pi)^{d-1}}\left(\frac{\rho_{14}}{\rho_{15}\rho_{45}}\right)^{1+\epsilon} = -\frac{2^{-3\epsilon}}{\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]\eta_{14}^{-2\epsilon}\tilde{K}_{14}\,, \tag{6.190}$$

where $\tilde{K}_{14}$ is defined through

$$\tilde{K}_{ij} \equiv \frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-4\epsilon)}\eta_{ij}^{1+3\epsilon}\,{}_2F_1(1+\epsilon,1+\epsilon,1-\epsilon,1-\eta_{ij})\,. \tag{6.191}$$

An expansion of $\tilde{K}_{14}$ in ϵ can be found in Eq. (A.23). Putting everything together, we determine the subtraction term

$$\begin{aligned}\left\langle S_5 F_{\rm LV}(1,4\,|\,5)\right\rangle &= 2C_F\,\frac{[\alpha_s]}{\epsilon^2}\,\left(\frac{4E_{\rm max}^2}{\mu^2}\right)^{-\epsilon}\,\left\langle \eta_{14}^{-\epsilon} K_{14} F_{\rm LV}(1,4)\right\rangle_\delta \\ &- 2C_F C_A\,\frac{[\alpha_s]^2}{4\epsilon^4}\left[\frac{\Gamma^5(1-\epsilon)\Gamma^3(1+\epsilon)}{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)}\right]\left(\frac{4E_{\rm max}^2}{\mu^2}\right)^{-2\epsilon}\,\left\langle \eta_{14}^{-2\epsilon}\tilde{K}_{14} F_{\rm LM}(1,4)\right\rangle .\end{aligned} \tag{6.192}$$

The function $F_{\rm LV}(1,4)$ that appears in the first term on the right-hand side of Eq. (6.192) corresponds to the one-loop cross section at NLO. It reads

$$F_{\rm LV}(1,4) = -2C_F\,\frac{[\alpha_s]}{\epsilon}\left(\frac{1}{\epsilon}+\frac{3}{2}\right)(4E_1E_4)^{-\epsilon}\eta_{14}^{-\epsilon}F_{\rm LM}(1,4) + F_{\rm LV}^{\rm fin}(1,4)\,. \tag{6.193}$$

6.5.2. Collinear subtraction terms

Next we discuss soft-regulated collinear subtraction terms. They read

$$\sum_{i\in\{1,4\}}\left\langle C_{5i}\left[I-S_5\right]w^{5i}\,F_{\rm LV}(1,4\,|\,5)\right\rangle_\delta . \tag{6.194}$$

We consider soft-collinear and collinear contributions individually and start with the soft-collinear contribution. Applying the soft S_5 limit in Eq. (6.189) to the cross section computed in the collinear $\vec{p}_5 \parallel \vec{p}_i$ limit we obtain

$$\begin{aligned} S_5 C_{5i}\,F_{\rm LV}(1,4\,|\,5) &= 2C_F\,g_{s,b}^2\times\frac{1}{E_5^2\,\rho_{i5}}\times F_{\rm LV}(1,4) \\ &- 2C_F C_A\,\frac{g_{s,b}^2[\alpha_{s,b}]}{\epsilon^2}\left[\frac{\Gamma^5(1-\epsilon)\Gamma^3(1+\epsilon)}{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)}\right]\times 2^{-\epsilon}\left(\frac{1}{E_5^2\,\rho_{i5}}\right)^{1+\epsilon}\times F_{\rm LM}(1,4)\,,\end{aligned} \tag{6.195}$$

for $i=1,4$. Integration over the unresolved phase space is straightforward. We find

$$\begin{aligned} -\sum_{i\in\{1,4\}}\left\langle S_5 C_{5i} F_{\rm LV}(1,4\,|\,5)\right\rangle_\delta &= -2C_F\,\frac{[\alpha_{s,b}]}{\epsilon^2}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right](2E_{\rm max})^{-2\epsilon}\left\langle F_{\rm LV}(1,4)\right\rangle_\delta \\ &+ C_F C_A\,\frac{[\alpha_{s,b}]^2}{2\epsilon^4}\left[\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]\left[\frac{\Gamma^5(1-\epsilon)\Gamma^3(1+\epsilon)}{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)}\right] \\ &\times (2E_{\rm max})^{-4\epsilon}\left\langle F_{\rm LM}(1,4)\right\rangle_\delta .\end{aligned} \tag{6.196}$$

We continue with the calculation of the collinear contribution $\left\langle\left[C_{51}w^{51}+C_{54}w^{54}\right]F_{\rm LV}(1,4\,|\,5)\right\rangle$

to the collinear subtraction term Eq. (6.194). The double-collinear limit C_{51} reads [58–60]

$$\begin{aligned} C_{51}F_{\rm LV}(1,4\,|\,5) &= g_{s,b}^2 \times \frac{1}{p_1\cdot p_5}P_{qq}(z) \times \frac{F_{\rm LV}(z\cdot 1,4)}{z} \\ &+ g_{s,b}^2[\alpha_{s,b}]\left[\frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\right] \times 2^{-\epsilon}\left(\frac{1}{p_1\cdot p_5}\right)^{1+\epsilon} P_{qq}^{\rm loop}(z) \times \frac{F_{\rm LM}(z\cdot 1,4)}{z}\,, \end{aligned} \tag{6.197}$$

where $z = (E_1 - E_5)/E_1$ and the one-loop splitting function $P_{qq}^{\rm loop}(z)$ is given in Eq. (E.17). The C_{54} limit reads

$$\begin{aligned} C_{54}F_{\rm LV}(1,4\,|\,5) &= g_{s,b}^2 \times \frac{1}{p_4\cdot p_5}P_{qq}(z) \times F_{\rm LV}\left(1,\frac{1}{z}\cdot 4\right) \\ &+ g_{s,b}^2[\alpha_{s,b}]\left[\frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\right] \times 2^{-\epsilon}\cos(\pi\epsilon)\left(\frac{1}{p_4\cdot p_5}\right)^{1+\epsilon} P_{qq}^{\rm loop}(z) \\ &\times F_{\rm LM}\left(1,\frac{1}{z}\cdot 4\right), \end{aligned} \tag{6.198}$$

where $z = E_4/(E_5+E_5)$. The factor $\cos(\pi\epsilon)$ comes from an analytical continuation of the loop integral to the kinematic region where both p_4 and p_5 are momenta of final-state particles.

Integration of Eqs. (6.197, 6.198) over the gluon phase space is analogous to the NLO case discussed in Section 5.2. To integrate the second term on the right-hand side of Eq. (6.198) over E_5, we define

$$\gamma_{qq,\rm loop}^{33} \equiv \int_0^1 {\rm d}z\, z^{-3\epsilon}(1-z)^{-3\epsilon}P_{qq}^{\rm loop}(z)\,. \tag{6.199}$$

We finally obtain

$$\begin{aligned} &\Big\langle [C_{51}+C_{54}]F_{\rm LV}(1,4\,|\,5)\Big\rangle \\ &= \frac{[\alpha_{s,b}]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left[\frac{2C_F}{2\epsilon}+\gamma_{qq}^{22}\right]\Big\langle (2E_4)^{-2\epsilon}F_{\rm LV}(1,4)\Big\rangle_\delta \\ &- \frac{[\alpha_{s,b}]^2}{\epsilon}\cos(\pi\epsilon)\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\gamma_{qq,\rm loop}^{33}\Big\langle_\delta (2E_4)^{-4\epsilon}F_{\rm LM}(1,4)\Big\rangle_\delta \\ &- \frac{[\alpha_{s,b}]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\int_0^1 {\rm d}z\,(1-z)^{-2\epsilon}P_{qq}(z)\Big\langle (2E_1)^{-2\epsilon}\frac{F_{\rm LV}(z\cdot 1,4)}{z}\Big\rangle_\delta \\ &- \frac{[\alpha_{s,b}]^2}{\epsilon}\left[\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\right]\int_0^1 {\rm d}z\,(1-z)^{-3\epsilon}P_{qq}^{\rm loop}(z)\Big\langle (2E_1)^{-4\epsilon}\frac{F_{\rm LM}(z\cdot 1,4)}{z}\Big\rangle_\delta\,. \end{aligned} \tag{6.200}$$

Putting Eq. (6.196) and Eq. (6.200) together we find the collinear subtraction term

$$
\begin{aligned}
&\sum_{i\in\{1,4\}} \left\langle C_{5i}\left[I-S_5\right] w^{5i}\, F_{\mathrm{LV}}(1,4\,|\,5)\right\rangle_\delta \\
&= \frac{[\alpha_s]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left[\frac{2C_F}{2\epsilon}+\gamma^{22}_{qq}\right]\left\langle \left(\frac{4E_4^2}{\mu^2}\right)^{-\epsilon} F_{\mathrm{LV}}(1,4)\right\rangle_\delta \\
&\quad - \frac{[\alpha_s]^2}{\epsilon}\cos(\pi\epsilon)\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\gamma^{33}_{qq,\mathrm{loop}}\left\langle \left(\frac{4E_4^2}{\mu^2}\right)^{-2\epsilon} F_{\mathrm{LM}}(1,4)\right\rangle_\delta \\
&\quad - \frac{[\alpha_s]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\int_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon}P_{qq}(z)\left\langle \frac{F_{\mathrm{LV}}(z\cdot 1,4)}{z}\right\rangle_\delta \\
&\quad - \frac{[\alpha_s]^2}{\epsilon}\left[\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-2\epsilon}\int_0^1 \mathrm{d}z\,(1-z)^{-3\epsilon}P^{\mathrm{loop}}_{qq}(z)\left\langle \frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\right\rangle_\delta \\
&\quad - 2C_F\,\frac{[\alpha_s]}{\epsilon^2}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left(\frac{4E_{\max}^2}{\mu^2}\right)^{-\epsilon}\left\langle F_{\mathrm{LV}}(1,4)\right\rangle_\delta \\
&\quad + C_F C_A\,\frac{[\alpha_s]^2}{2\epsilon^4}\left[\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]\left[\frac{\Gamma^5(1-\epsilon)\Gamma^3(1+\epsilon)}{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)}\right]\left(\frac{4E_{\max}^2}{\mu^2}\right)^{-2\epsilon}\left\langle F_{\mathrm{LM}}(1,4)\right\rangle_\delta .
\end{aligned}
\tag{6.201}
$$

The full real-virtual contribution is obtained by inserting the results for the subtraction terms Eqs. (6.192, 6.201) into Eq. (6.188).

6.6. Double-virtual contribution and collinear renormalization

In this section we describe the singular structure of the double-virtual contribution $\mathrm{d}\hat{\sigma}_{\mathrm{VV}}$. Similar to the case of one-loop QCD amplitudes, the singular structure of two-loop QCD amplitudes is known to be universal [54, 55]. In accordance with Eq. (5.53) we define the UV-renormalized contribution

$$
2s\cdot \mathrm{d}\hat{\sigma}_{\mathrm{VV}} \equiv \int F_{\mathrm{LVV}}(1_q,4_q) \equiv \left\langle F_{\mathrm{LVV}}(1_q,4_q)\right\rangle_\delta , \tag{6.202}
$$

where

$$
\begin{aligned}
F_{\mathrm{LVV}}(1_q,4_q) &= \mathcal{N}\int [\mathrm{d}p_3][\mathrm{d}p_4]\,(2\pi)^d\delta^{(d)}(p_1+p_2-p_3-p_4) \\
&\quad\times\left[\left|M^{\text{1-loop}}_{\mathrm{nlo}}\right|^2 + 2\Re\left(M^{\mathrm{tree}\star}_{\mathrm{lo}}\cdot M^{\text{2-loop}}_{\mathrm{nnlo}}\right)\right](p_1,p_2,p_3,p_4)\times\hat{\mathcal{O}}(p_3,p_4)\,.
\end{aligned}
\tag{6.203}
$$

In Eq. (6.203) $M^{\text{2-loop}}_{\mathrm{nnlo}}(p_1,p_2,p_3,p_4)$ is the 2-loop contribution to the DIS process. We isolate IR divergences in F_{LVV} using results of Refs. [54, 55] and write it as[16]

[16]For convenience we have split the non-singular finite part into two terms: function labeled with LV^2 corresponds to contributions from the 1-loop amplitude squared and the function labeled with LVV corresponds to contributions from the 2-loop amplitude multiplied with the tree-level amplitude. Their computation requires an explicit calculation of the quark form factor; it can be found in Ref. [73]. Also not relevant for our discussion of IR poles,

$$
\begin{aligned}
F_{\rm LVV}(1_q,4_q) &= \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \left(\frac{2p_1\cdot p_4}{\mu^2}\right)^{-2\epsilon} \Bigg\{ \left(\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\right)^2 C_F^2\left[\frac{2}{\epsilon^4}+\frac{6}{\epsilon^3}+\frac{9}{2\epsilon^2}\right] \\
&+\left(\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\right)\Bigg[\frac{C_F^2}{\epsilon}\left(-\frac{3}{8}+\frac{\pi^2}{2}-6\zeta_3\right) \\
&+C_AC_F\left(-\frac{11}{12\epsilon^3}-\frac{83}{18\epsilon^2}+\frac{\pi^2}{12\epsilon^2}-\frac{961}{216\epsilon}-\frac{11\pi^2}{48\epsilon}+\frac{12\zeta_3}{2\epsilon}\right)\Bigg]\Bigg\}F_{\rm LM}(1_q,4_q) \\
&+\frac{\alpha_s(\mu)}{2\pi}\left[2I_1(\epsilon)+\frac{\beta_0}{\epsilon}\right]F_{\rm LV}^{\rm fin}(1_q,4_q)+F_{\rm LV^2}^{\rm fin}(1_q,4_q)+F_{\rm LVV}^{\rm fin}(1_q,4_q)\,,
\end{aligned}
\tag{6.206}
$$

where $F_{\rm LV^2}^{\rm fin}$ is the finite remainder of the one-loop amplitude squared and $F_{\rm LVV}^{\rm fin}$ is the finite remainder of the interference between two-loop amplitude and tree-level amplitude. The operator $I_1(\epsilon)$ in Eq. (6.206) is given in Eq. (5.56). The one-loop coefficient of the QCD β-function, which appears in Eq. (6.206), reads

$$
\beta_0 = \frac{11}{6}C_A - \frac{2}{3}T_R N_f\,, \tag{6.207}
$$

where $T_R = 1/2$ and N_f is the number of massless quark flavors. Finally, note that the double-virtual contribution Eq. (6.206) contains $1/\epsilon$ poles that cancel against soft $1/\epsilon$ poles from both quark-initiated channels $q+e^-\to e^-+q+g+g$ and $q+e^-\to e^-+q+q'+\bar{q}'$.

We continue with the discussion of the collinear renormalization contribution to the partonic cross section Eq. (6.1). We find that renormalization of parton distribution functions leads to the following contribution to the NNLO partonic cross section of the quark-initiated channel

$$
\begin{aligned}
{\rm d}\hat\sigma_{\rm pdf} &= \frac{\alpha_s(\mu)}{2\pi\epsilon}\int_0^1 {\rm d}z\, \hat P_{qq}^{(0)}(z)\, {\rm d}\hat\Sigma_{\rm nlo}(z) \\
&+\left(\frac{\alpha_s(\mu)}{2\pi}\right)^2\int_0^1 {\rm d}z\left[\frac{[\hat P_{qq}^{(0)}\otimes\hat P_{qq}^{(0)}](z)-\beta_0\hat P_{qq}^{(0)}(z)}{2\epsilon^2}+\frac{\hat P_{qq}^{(1)}(z)}{2\epsilon}\right]{\rm d}\hat\sigma_{\rm lo}(z)\,.
\end{aligned}
\tag{6.208}
$$

In Eq. (6.208) we introduced the NLO partonic cross section ${\rm d}\hat\Sigma^{\rm nlo}$ that is composed of virtual

---

for completeness, we give them below

$$
\langle F_{\rm LV^2}^{\rm fin}(1_q,4_q)\rangle_\delta = \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2\times 16C_F^2\times\langle F_{\rm LM}(1,4)\rangle_\delta\,, \tag{6.204}
$$

$$
\begin{aligned}
\langle F_{\rm LVV}^{\rm fin}(1_q,4_q)\rangle_\delta &= \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2\times\Bigg[C_FC_A\left\{\frac{44}{3}\ln\left(\frac{2p_1\cdot p_4}{\mu^2}\right)-\frac{51157}{1296}+\frac{31\pi^4}{240}-\frac{107\pi^2}{72}+\frac{659}{36}\zeta_3\right\} \\
&+C_F^2\left\{\frac{255}{16}-\frac{11\pi^4}{90}+\frac{29\pi^2}{12}-15\zeta_3\right\}+C_FN_f\left\{\frac{4085}{648}+\frac{7\pi^2}{36}-\frac{1}{18}\zeta_3-\frac{8}{3}\ln\left(\frac{2p_1\cdot p_4}{\mu^2}\right)\right\}\Bigg]\times\langle F_{\rm LM}(1,4)\rangle_\delta\,.
\end{aligned}
\tag{6.205}
$$

and real contributions only, so that

$$d\hat{\sigma}_{\text{nlo}} = d\hat{\Sigma}_{\text{nlo}} + d\hat{\sigma}_{\text{pdf}}\,, \quad \text{with} \quad d\hat{\Sigma}_{\text{nlo}} = d\hat{\sigma}_{\text{v}} + d\hat{\sigma}_{\text{r}}\,. \tag{6.209}$$

We note that it is given in Eq. (5.65) in term of the function F_{LM}. The LO cross section $\hat{\sigma}_{\text{lo}}(z)$ is given in Eq. (4.4). The various splitting functions in Eq. (6.208) can be found in Appendix E.

We have derived all formulas relevant for the description of quark-initiated process $q + e^- \to e^- + q + g + g$ with NNLO accuracy. However, if we combine all the contributions discussed in this chapter, we will not obtain a finite formula. To obtain finite result for quark channels, processes with additional $q\bar{q}$-pair in the final state need to be considered. We discuss their computation in the next chapter.

6.7. Quark-anti-quark emission

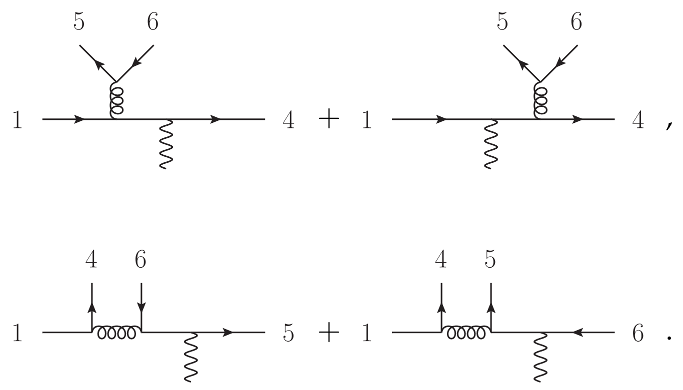
In this section we consider the partonic process

$$q(p_1) + e^-(p_2) \to e^-(p_3) + q(p_4) + q'(p_5) + \bar{q}'(p_6)\,, \tag{6.210}$$

which describes the emission of a quark-anti-quark pair. It appears for the first time at NNLO in the perturbative expansion of the partonic cross section. The major differences in dealing with the process Eq. (6.210) to earlier discussion of quark-initiated processes with two gluon emissions consists in *(i)* the fact that all three final state partons can carry "hard" momentum; and *(ii)* that the hard process must be split into different contributions with defined behavior in singular limits. We discuss consequences of these differences in the following.

To obtain the amplitude that describes the process Eq. (6.210), we need to sum over all massless quark flavours q' in the final states. To this end, we have to distinguish the case $q \neq q'$ and $q = q'$ and we found it convenient to first define two master amplitudes[17]

$\mathcal{A}_1(1_q, 4_q, 5_{q'}, 6_{\bar{q}'}) \equiv$, $\mathcal{A}_2(1_q, 4_q, 5_{q'}, 6_{\bar{q}'}) \equiv$. (6.211)



Note that the two amplitudes posses different sets of collinear singularities. $\mathcal{A}_1$ is singular if $\vec{p}_5 \parallel \vec{p}_6$ or $\vec{p}_5 \parallel \vec{p}_6 \parallel \vec{p}_i$, with $i \in \{1, 4\}$. $\mathcal{A}_2$ is singular if $\vec{p}_4 \parallel \vec{p}_1$ or $\vec{p}_j \parallel \vec{p}_4 \parallel \vec{p}_1$, with $j \in \{5, 6\}$.

[17]For simplicity we do not show the leptonic current in the Feynman diagrams and the sum over polarizations and colours is understood implicitly.

We continue with the construction of physical amplitudes from the master amplitudes Eq. (6.211). The amplitude for the process $q + e^- \to e^- + q + q' + \bar{q}'$ for $q \neq q'$ reads[18]

$$\mathcal{A}_{q\neq q'}(1,4,5,6) = \mathcal{A}_1\big(1_q, 4_q, 5_{q'}, 6_{\bar{q}'}\big) + \mathcal{A}_2\big(1_q, 4_q, 5_{q'}, 6_{\bar{q}'}\big)\,. \tag{6.212}$$

In case when $q = q'$ we have to take into account that we have two identical final-states and add contributes where we switch their momenta. We obtain

$$\mathcal{A}_{q=q'}(1,4,5,6) = \mathcal{A}_{q\neq q'}(1,4,5,6) + \mathcal{A}_{q\neq q'}(1,5,4,6)\,. \tag{6.213}$$

We use Eqs. (6.212, 6.213) to write the full amplitude squared and sum over all massless quark flavours in the final states. Upon straightforward re-labeling of parton momenta, we obtain the following expression

$$\begin{aligned}
|M^{q\bar{q}}_{\rm tree}(1,4,5,6)|^2 &= \sum_{q'\neq q} |\mathcal{A}_{q\neq q'}(1,4,5,6)|^2 + \frac{1}{2!}|\mathcal{A}_{q=q'}(1,4,5,6)|^2 \\
&\approx \sum_{q'} \Big[|\mathcal{A}_1(1,4,5,6)|^2 + |\mathcal{A}_2(1,5,4,6)|^2\Big] + \sum_{q'} 2\Re\Big[\mathcal{A}_1(1,4,5,6)\cdot\mathcal{A}_2^\star(1,4,5,6)\Big] \\
&\quad + \frac{1}{2!}2\Re\Big[\mathcal{A}_1(1,5,4,6)\mathcal{A}_1^\star(1,4,5,6) + \mathcal{A}_1(1,5,4,6)\mathcal{A}_2^\star(1,4,5,6) \\
&\qquad + \mathcal{A}_2(1,4,5,6)\mathcal{A}_1^\star(1,5,4,6) + \mathcal{A}_2(1,4,5,6)\mathcal{A}_2^\star(1,5,4,6)\Big]\,.
\end{aligned} \tag{6.214}$$

We emphasize that the sum over quark flavour q' in the first two terms on the right-hand side of Eq. (6.214) includes also the initial-state quark flavour q.

We split contributions to the matrix element squared $|M^{q\bar{q}}_{\rm tree}(1,4,5,6)|^2$ into the so-called "singlet", "non-singlet" and interference contributions. The latter are *finite* and do not posses any singularity. We define

$$\begin{aligned}
|M^{\rm tree}_{\rm ns}(1,4,5,6)|^2 &\equiv \sum_{q'} |\mathcal{A}_1(1,4,5,6)|^2 + \frac{1}{2!}2\Re\Big[\mathcal{A}_1(1,5,4,6)\mathcal{A}_1^\star(1,4,5,6) \\
&\qquad + \mathcal{A}_1(1,5,4,6)\mathcal{A}_2^\star(1,4,5,6) + \mathcal{A}_2(1,4,5,6)\mathcal{A}_1^\star(1,5,4,6) \\
&\qquad + \mathcal{A}_2(1,4,5,6)\mathcal{A}_2^\star(1,5,4,6)\Big]\,,
\end{aligned} \tag{6.215}$$

$$|M^{\rm tree}_{\rm s}(1,4,5,6)|^2 \equiv \sum_{q'} |\mathcal{A}_2(1,5,4,6)|^2\,, \tag{6.216}$$

$$|M^{\rm tree}_{\rm int}(1,4,5,6)|^2 \equiv \sum_{q'} 2\Re\Big[\mathcal{A}_1(1,4,5,6)\mathcal{A}_2^\star(1,4,5,6)\Big]\,. \tag{6.217}$$

The separation of terms in Eq. (6.214) into the non-singlet (Eq. (6.215)) and singlet (Eq. (6.216)) contributions is motivated by their behavior in singular limits. In these limits, *non-singlet* contributions are proportional to lower multiplicity matrix elements that describe processes with the original initial-state quark of a flavour q. In this sense, the tree-level process $q + e^- \to e^- + q + g + g$ and the one-loop corrected process $q + e^- \to e^- + q + g$, discussed in the

[18]Here, and in the following, we do not write quark flavours as subscript in the arguments of the amplitudes.

previous part of this chapter, are also classified as non-singlet contributions.

In contrast to this, singular limits of *singlet* contributions are proportional to matrix elements squared that are summed over all massless initial-state (anti-)quark flavours. Note that this behavior is similar to the gluon-initiated process to DIS, which we discussed in Section 5.5.

Since $1/\epsilon$ pole cancellation happens independent of the hard matrix element, we can present finite results for non-singlet contributions and singlet contributions separately. To this end, according to the splitting of the amplitude Eqs. (6.215, 6.217) we also split the double-real contribution $\mathrm{d}\hat{\sigma}_{\mathrm{rr}}$ to the partonic cross section and write

$$\mathrm{d}\hat{\sigma}_{\mathrm{rr}} = \mathrm{d}\hat{\sigma}_{\mathrm{rr}}^{\mathrm{ns}} + \mathrm{d}\hat{\sigma}_{\mathrm{rr}}^{\mathrm{s}} + \mathrm{d}\hat{\sigma}_{\mathrm{rr}}^{\mathrm{int}}\,. \tag{6.218}$$

We discuss the first two contributions on the right-hand side in Eq. (6.218) in the following sections. However, before that, we write the finite contribution as

$$2s\cdot \mathrm{d}\hat{\sigma}_{\mathrm{rr}}^{\mathrm{int}} = \int [\mathrm{d}p_5][\mathrm{d}p_6] F_{\mathrm{LM}}^{\mathrm{fin}}(1,4,5,6) \equiv \left\langle F_{\mathrm{LM}}^{\mathrm{int}}(1,4,5,6)\right\rangle, \tag{6.219}$$

where we defined

$$\begin{aligned} F_{\mathrm{LM}}^{\mathrm{int}}(1,4,5,6) = \mathcal{N}\int [\mathrm{d}p_3][\mathrm{d}p_4]\,(2\pi)^d\delta^{(d)}\left(p_1+p_2-p_3-p_4-p_5-p_6\right) \\ \times |M_{\mathrm{int}}^{\mathrm{tree}}(p_1,p_2,p_3,p_4,p_5,p_6)|^2 \times \hat{\mathcal{O}}(p_3,p_4,p_5,p_6)\,. \end{aligned} \tag{6.220}$$

We continue with non-singlet contributions in the next section.

6.7.1. Non-singlet contributions

We now discuss the non-singlet contribution to the cross section. It originates from Eq. (6.215). Following previous discussion we write

$$2s\cdot \mathrm{d}\hat{\sigma}_{\mathrm{rr}}^{\mathrm{ns}} = \int [\mathrm{d}p_5][\mathrm{d}p_6]\theta(E_5-E_6) F_{\mathrm{LM,ns}}(1,4\,|\,5,6) \equiv \left\langle F_{\mathrm{LM,ns}}(1,4\,|\,5,6)\right\rangle, \tag{6.221}$$

where we defined

$$\begin{aligned} F_{\mathrm{LM,ns}}(1,4\,|\,5,6) = \mathcal{N}\int [\mathrm{d}p_3][\mathrm{d}p_4]\,(2\pi)^d\delta^{(d)}\left(p_1+p_2-p_3-p_4-p_5-p_6\right) \\ \times |M_{\mathrm{ns}}^{\mathrm{tree}}(1,4,5,6)|^2 \times \hat{\mathcal{O}}(p_3,p_4,p_5,p_6)\,. \end{aligned} \tag{6.222}$$

The many terms that contribute to the matrix element $|M_{\mathrm{ns}}^{\mathrm{tree}}|^2$, given in Eq. (6.215), possess different singularities. The first term on the right-hand side of Eq. (6.215) is singular in the double-soft $p_5 \sim p_6 \to 0$ limit. It also possesses the double-collinear singularity when $\vec{p}_5 \parallel \vec{p}_6$ and/or triple-collinear singularities when $\vec{p}_5 \parallel \vec{p}_6 \parallel \vec{p}_1$ or $\vec{p}_5 \parallel \vec{p}_6 \parallel \vec{p}_4$. All limits of this term are proportional to N_f. Other contributions to the non-singlet matrix element in Eq. (6.215) only become singular in the triple-collinear limits when $\vec{p}_5 \parallel \vec{p}_6 \parallel \vec{p}_4$ and $\vec{p}_5 \parallel \vec{p}_6 \parallel \vec{p}_1$.

These singularities form a subset of singularities that the amplitude of the process $q + e^- \to e^- + q + gg$ possesses. For this reason, we can regulate them in full analogy to the previous discussion in Section 6.1 and only remove operators in Eq. (6.17) that correspond to singularities that are not present for the $q\bar{q}$-final state. We write

$$\begin{aligned}\big\langle F_{\rm LM,ns}(1,4\,|\,5,6)\big\rangle &= \big\langle \mathbb{S}F_{\rm LM,ns}(1,4\,|\,5,6)\big\rangle \\ &+ \sum_{i\in\{1,4\}} \Big\langle \big[I-\mathbb{S}\big]\Big[\theta_i^{(b)}C_{56}+\theta_i^{(d)}C_{56}\Big][{\rm d}p_5][{\rm d}p_6]w^{5i,6i}F_{\rm LM,ns}(1,4\,|\,5,6)\Big\rangle \\ &+ \sum_{i\in\{1,4\}} \Big\langle \big[I-\mathbb{S}\big]\mathbb{C}_i\Big[\theta_i^{(a)}+\Big(\theta_i^{(b)}+\theta_i^{(d)}\Big)\big[I-C_{56}\big]+\theta_i^{(d)}\Big][{\rm d}p_5][{\rm d}p_6] \\ &\qquad\qquad \times w^{5i,6i}F_{\rm LM,ns}(1,4\,|\,5,6)\Big\rangle \\ &+ \sum_{\substack{i,j\in\{1,4\}\\ i\neq j}} \big\langle \hat{\mathcal{O}}_{\rm nnlo}^{(i,j)}\,[{\rm d}p_5][{\rm d}p_6]w^{5i,6j}F_{\rm LM,ns}(1,4\,|\,5,6)\big\rangle_\delta \\ &+ \sum_{i=1,4} \big\langle \hat{\mathcal{O}}_{\rm nnlo}^{(i)}\,[{\rm d}p_5][{\rm d}p_6]w^{5i,6i}F_{\rm LM,ns}(1,4\,|\,5,6)\big\rangle_\delta\,,\end{aligned} \tag{6.223}$$

where $\hat{\mathcal{O}}_{\rm nnlo}^{(i)}$ and $\hat{\mathcal{O}}_{\rm nnlo}^{(i,j)}$ are defined in Eqs. (6.18, 6.19).[19]

The double-soft subtraction term (first term on the right-hand side in Eq. (6.223)) is computed in Ref. [67], the triple-collinear subtractions terms (fourth and fifth terms on the right-hand side of Eq. (6.223)) are computed in Ref. [68]. We collect results for these terms in the Appendix I. Finally, there is a double-collinear subtraction terms (second term on the right-hand side of Eq. (6.223)). The required limit is given in Appendix B. Integration over unresolved momenta is performed in analogy with the previous discussion. We obtain

$$\begin{aligned}&\sum_{i\in\{1,4\}} \Big\langle \big[I-\mathbb{S}\big]\Big[\theta_i^{(b)}C_{56}+\theta_i^{(d)}C_{56}\Big][{\rm d}p_5][{\rm d}p_6]w^{5i,6i}F_{\rm LM,ns}(1,4\,|\,5,6)\Big\rangle \\ &= -\frac{[\alpha_s]}{2\epsilon}N_\epsilon 2^{2\epsilon}\sum_{i\in\{1,4\}}\Big\langle \hat{\mathcal{O}}_{\rm nlo}^{(i)}\left(\frac{\rho_{i5}}{2}\right)^{-\epsilon}\left(1-\frac{\rho_{i5}}{2}\right)^{\epsilon}\left(\frac{4E_5^2}{\mu^2}\right)^{-\epsilon} w_{\rm tc}^{i} \\ &\qquad \times\Big\{\gamma_{gq}^{22}F_{\rm LM}(1,4\,|\,5)+\epsilon\gamma_{gq}^{\perp,22}\big[r_\mu^{(i)}r_\nu^{(i)}+g_{\mu\nu}\big]F_{\rm LM}^{\mu\nu}(1,4\,|\,5)\Big\}\Big\rangle \\ &\quad -\frac{[\alpha_s]^2}{2\epsilon^2}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]N_\epsilon 2^\epsilon\left(\frac{4E_1^2}{\mu^2}\right)^{-2\epsilon}\int_0^1{\rm d}z(1-z)^{-2\epsilon} \\ &\qquad \times\Big\{\Big[-\gamma_{gq}^{22}+\epsilon\gamma_{gq}^{\perp,22}\Big]P_{qq}(z)-\epsilon\gamma_{gq}^{\perp,22}P_{qq}^{\rm spin}(z)\Big\}\Big\langle\frac{F_{\rm LM}(z\cdot 1,4)}{z}\Big\rangle_\delta \\ &\quad +\frac{[\alpha_s]^2}{2\epsilon^2}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]N_\epsilon 2^\epsilon\Big\{\Big[-\gamma_{gq}^{22}+\epsilon\gamma_{gq}^{\perp,22}\Big]\Big[\frac{2C_F}{4\epsilon}+\gamma_{qq}^{24}\Big] \\ &\qquad -\Big[\frac{2C_F}{4\epsilon}+\gamma_{qq,\rm spin}^{24}\Big]\gamma_{gq}^{\perp,22}\Big\}\Big\langle\left(\frac{4E_4^2}{\mu^2}\right)^{-2\epsilon}F_{\rm LM}(1,4)\Big\rangle_\delta\end{aligned}$$

[19] We note that the action of some operators present in $\hat{\mathcal{O}}_{\rm nnlo}^{(i)}$ and $\hat{\mathcal{O}}_{\rm nnlo}^{(i,j)}$ is zero. For instance since no single soft singularity is present $S_6F_{\rm LM,ns}(1,4\,|\,5,6)=0$.

$$
\begin{aligned}
&+ 2C_F \, \frac{[\alpha_s]^2}{4\epsilon^3} \left[\frac{2^\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} \right] \gamma_{gq}^{22} N_\epsilon 2^\epsilon \left(\frac{4E_{\max}^2}{\mu^2} \right)^{-2\epsilon} \langle F_{\rm LM}(1,4) \rangle_\delta \\
&- 2C_F \, \frac{[\alpha_s]^2}{\epsilon^2} \times [2^{2\epsilon} \delta_q(\epsilon)] \left(\frac{4E_{\max}^2}{\mu^2} \right)^{-2\epsilon} \left\langle \langle \Delta_{65} \rangle_{S_5} F_{\rm LM}(1,4) \right\rangle_\delta \\
&- C_F \, [\alpha_s]^2 \delta_q^\perp(\epsilon) \left(\frac{4E_{\max}^2}{\mu^2} \right)^{-2\epsilon} \left\langle \langle r^\mu r^\nu \rangle_{\rho_5} F_{\rm LM}(1,4) \right\rangle_\delta .
\end{aligned}
\tag{6.224}
$$

In Eq. (6.224) we defined the functions $\delta_q(\epsilon)$, $\delta_q^\perp(\epsilon)$, γ_{gq}^{22} and $\gamma_{gq}^{\perp,22}$ which can all be found in Appendix E. After combining this result with other *non-singlet* contributions derived earlier in this chapter, all $1/\epsilon$ poles cancel out and the final result is obtained. We present it in Section 6.6.

6.7.2. Singlet contributions

The singlet channel can be computed independent of the remaining quark-initiated contributions. It has a simple singular structure. First, it contains no soft-singularities and, therefore, receives no double-virtual contributions. Second, only collinear singularities to the initial-state momentum p_1 are present that cancel with contributions from collinear renormalization of parton distribution functions. Since these poles have to be proportional to matrix elements squared summed over all massless initial-state quark and anti-quark flavours $q/\bar{q}$ we can isolate these terms in collinear renormalization contributions and obtain an IR finite result. We write the singlet contribution to the partonic cross section as

$$
{\rm d}\hat{\sigma}^{\rm s}_{\rm nnlo} = {\rm d}\hat{\sigma}^{\rm s}_{\rm rr} + {\rm d}\hat{\sigma}^{\rm s}_{\rm pdf} . \tag{6.225}
$$

We begin with the double-real contribution. The singlet contribution contains a double-collinear singularity that arises when $\vec{p}_5 \parallel \vec{p}_1$ and/or two triple-collinear singularities that appear when $\vec{p}_5 \parallel \vec{p}_4 \parallel \vec{p}_1$ or $\vec{p}_5 \parallel \vec{p}_6 \parallel \vec{p}_1$. Similar to the double-collinear singularities in the gluon-initiated channel, discussed in Section 5.5, the two triple-collinear singularities are physically equivalent and we can deal with them at once by introducing the partition of unity

$$
1 = w_{\rm s}^{41} + w_{\rm s}^{61} , \quad \text{with} \quad w_{\rm s}^{41} \equiv \frac{\rho_{16}}{\rho_{14}+\rho_{16}} , \quad w_{\rm s}^{61} \equiv \frac{\rho_{14}}{\rho_{14}+\rho_{16}} , \tag{6.226}
$$

and rewriting the singlet amplitude Eq. (6.216) in the following way

$$
\begin{aligned}
|M_{\rm s}^{\rm tree}(1,4,5,6)|^2 &= w_{\rm s}^{41} |M_{\rm s}^{\rm tree}(1,4,5,6)|^2 + w_{\rm s}^{61} |M_{\rm s}^{\rm tree}(1,4,5,6)|^2 \\
&\Rightarrow w_{\rm s}^{61} \left[|M_{\rm s}^{\rm tree}(1,6,5,4)|^2 + |M_{\rm s}^{\rm tree}(1,4,5,6)|^2 \right] .
\end{aligned}
\tag{6.227}
$$

In Eq. (6.227) in the last step we switched the momenta labeling of momenta p_4 and p_6. The contribution to the partonic cross section is then written as

$$2s \cdot \mathrm{d}\hat{\sigma}^{\mathrm{s}}_{\mathrm{rr}} = \int [\mathrm{d}p_5][\mathrm{d}p_6]\theta(E_5 - E_6) w^{61}_{\mathrm{s}} F_{\mathrm{LM,s}}(1,4\,|\,5,6) \equiv \left\langle w^{61}_{\mathrm{s}} F_{\mathrm{LM,s}}(1,4\,|\,5,6)\right\rangle , \tag{6.228}$$

where we defined

$$\begin{aligned} F_{\mathrm{LM,s}}(1,4\,|\,5,6) = \mathcal{N} \int [\mathrm{d}p_3][\mathrm{d}p_4]\, (2\pi)^d \delta^{(d)}\, (p_1 + p_2 - p_3 - p_4 - p_5 - p_6) \\ \times \left[|M^{\mathrm{tree}}_{\mathrm{s}}(1,6,5,4)|^2 + |M^{\mathrm{tree}}_{\mathrm{s}}(1,4,5,6)|^2 \right] \times \hat{\mathcal{O}}(p_3, p_4, p_5, p_6)\,. \end{aligned} \tag{6.229}$$

Note that, although no soft singularities are present, we found it convenient to keep energy ordering in Eq. (6.228). We also found it convenient to use the same partitioning and sectoring of the angular phase space, which can be found in Section 6.1, as well as the same phase space parametrization, see Appendix F. We follow the regularization procedure described in Section 6.1 and write

$$\begin{aligned} \left\langle w^{61}_{\mathrm{s}} F_{\mathrm{LM,s}}(1,4\,|\,5,6)\right\rangle &= \left\langle \left[C_{51} w^{51,64} + \theta^{(a)}_1 C_{51} w^{51,61} \right] [\mathrm{d}p_5][\mathrm{d}p_6] F_{\mathrm{LM}}(1,4\,|\,5,6) \right\rangle \\ &+ \left\langle \mathbb{C}_1 \left(\left[I - C_{51} \right] \theta^{(a)}_1 + \theta^{(b)}_1 + \theta^{(c)}_1 + \theta^{(d)}_1 \right) [\mathrm{d}p_5][\mathrm{d}p_6] w^{51,61} F_{\mathrm{LM}}(1,4\,|\,5,6) \right\rangle \\ &+ \sum_{\substack{i,j=1,4 \\ i\neq j}} \left\langle \hat{O}^{(i,j)}_{\mathrm{nnlo}}\, [\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6j} F_{\mathrm{LM}}(1,4\,|\,5,6) \right\rangle_\delta \\ &+ \sum_{i=1,4} \left\langle \hat{O}^{(i)}_{\mathrm{nnlo}}\, [\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6i} F_{\mathrm{LM}}(1,4\,|\,5,6) \right\rangle_\delta , \end{aligned} \tag{6.230}$$

where we use definitions of $\hat{\mathcal{O}}^{(i)}_{\mathrm{nnlo}}$ and $\hat{\mathcal{O}}^{(i,j)}_{\mathrm{nnlo}}$ as given in Eqs. (6.18, 6.19).

Required double-collinear and triple-collinear limits of the subtraction terms are given in Appendix B. Triple-collinear subtraction terms were computed in Ref. [68] and we present them result in Appendix I. In case of double-collinear subtraction terms, integration over unresolved momenta can be done following earlier discussions. We find[20]

$$\begin{aligned} &\left\langle \left[C_{51} w^{51,64} + \theta^{(a)}_1 C_{51} w^{51,61} \right] [\mathrm{d}p_5][\mathrm{d}p_6] F_{\mathrm{LM}}(1,4\,|\,5,6) \right\rangle \\ &= -\frac{[\alpha_s]}{\epsilon} \left(\frac{4E_1^2}{\mu^2} \right)^{-\epsilon} \int_0^1 \mathrm{d}z\, (1-z)^{-2\epsilon} P_{gq}(z) \left\langle \left[w^{54}_{\mathrm{dc}} + \hat{\mathcal{O}}^{(1)}_{\mathrm{nlo},g} w^{51}_{\mathrm{tc}} \left(\frac{\rho_{15}}{4} \right)^{-\epsilon} \right] \right. \\ &\quad \left. \times \frac{F_{\mathrm{LM},g}(z\cdot 1_g, 4_q\,|\,5_q)}{z} \right\rangle_\delta + \frac{[\alpha_s]^2}{\epsilon^2} \left[\frac{2^\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} \right] \left(\frac{4E_1^2}{\mu^2} \right)^{-2\epsilon} \\ &\quad \times \int_0^1 \mathrm{d}z \sum_{f\in\{q,\bar{q}\}} \left[P^{02}_{fg} \otimes P^{22}_{gq} \right](z) \left\langle \frac{F_{\mathrm{LM}}(z\cdot 1_f, 4_f)}{z} \right\rangle_\delta . \end{aligned} \tag{6.231}$$

[20]For details, see in particular the discussion around Eq. (6.82). In writing Eq. (6.231) we extended the definition in Eq. (6.84) to arbitrary splitting functions.

The collinear renormalization contribution is obtained from Eq. (6.208) by selecting terms that are proportional to the quark parton distribution function and the NLO cross section of the process $g + e^- \to e^- + q + \bar{q}$ or the LO cross section of the process $q/\bar{q} + e^- \to e^- + q/\bar{q}$ summed over all possible (anti-)quark flavours $q/\bar{q}$. The result reads

$$\mathrm{d}\hat{\sigma}^{\mathrm{s}}_{\mathrm{pdf}} = \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \int_0^1 \mathrm{d}z \sum_{f\in\{q,\bar{q}\}} \left[\frac{\hat{P}^{(1)}_{fq,\mathrm{s}}(z)}{2\epsilon} - \frac{\left[\hat{P}^{(0)}_{fg} \otimes \hat{P}^{(0)}_{gq}\right](z)}{2\epsilon^2}\right] \mathrm{d}\hat{\sigma}^{\mathrm{lo}}_{f}(z) + \frac{\alpha_s(\mu)}{2\pi} \int_0^1 \mathrm{d}z\, \hat{P}^{(0)}_{gq}(z)\, \mathrm{d}\hat{\sigma}^{\mathrm{r}}_{\mathrm{nlo},g}(z)\,. \tag{6.232}$$

Eq. (6.232) contains convolutions with quark-initiated and gluon-initiated cross sections. We labeled them accordingly. $\mathrm{d}\hat{\sigma}^{\mathrm{r}}_{\mathrm{nlo},g}$ is the real emission contribution to the gluon-initiated NLO cross section that is defined in Eq. (5.77). Upon combining Eqs. (6.231, 6.232) we obtain the final result that is presented Section 9.2.

7. The NNLO computation: gluon-initiated channel

In this chapter we consider gluon-initiated contributions to deep inelastic scattering. Such processes appear first at next-to-leading order in the perturbative expansion of the partonic cross sections. To obtain an infrared-finite contribution at NNLO we, therefore, do not need to consider double-virtual contributions for gluon-initiated processes. We write

$$\mathrm{d}\hat{\sigma}_{\mathrm{nnlo}} = \mathrm{d}\hat{\sigma}_{\mathrm{rv}} + \mathrm{d}\hat{\sigma}_{\mathrm{rr}} + \mathrm{d}\hat{\sigma}_{\mathrm{pdf}}\,, \tag{7.1}$$

where $\mathrm{d}\hat{\sigma}_{\mathrm{rv}}$ refers to a one-loop correction to process $g + e^- \to e^- + q + \bar{q}$, $\mathrm{d}\hat{\sigma}_{\mathrm{rr}}$ refers to a process $g + e^- \to e^- + q + \bar{q} + g$ and $\mathrm{d}\hat{\sigma}_{\mathrm{pdf}}$ contains corrections that originate in the collinear renormalization of parton distribution functions.

We begin with the discussion of the double-real contribution to the partonic cross section and note that only one partonic process

$$g(p_1) + e^-(p_2) \to e^-(p_3) + q(p_4) + \bar{q}(p_5) + g(p_6)\,, \tag{7.2}$$

needs to be considered. The amplitude describing process Eq. (7.2) is built from Feynman diagrams shown in Fig. 7.1. Both quark and anti-quark develop singularities if they become collinear to the initial-state gluon. In full analogy with the NLO discussion in Section 5.5 we rewrite the matrix element in such a way that only one of these collinear singularities is present at a time. To this end we introduce partition of unity $1 = w_g^{51} + w_g^{41}$, where partition functions w^{i1} are defined around Eq. (5.73), and write the matrix element squared as follows

$$\begin{aligned} \left|M^{\mathrm{tree}}(1_g, 4_q, 5_{\bar{q}}, 6_g)\right|^2 &= w_g^{51}\left|M^{\mathrm{tree}}(1_g, 4_q, 5_{\bar{q}}, 6_g)\right|^2 + w_g^{41}\left|M^{\mathrm{tree}}(1_g, 4_q, 5_{\bar{q}}, 6_g)\right|^2 \\ &\Rightarrow w_g^{51}\left[\left|M^{\mathrm{tree}}(1_g, 4_q, 5_{\bar{q}}, 6_g)\right|^2 + \left|M^{\mathrm{tree}}(1_g, 5_q, 4_{\bar{q}}, 6_g)\right|^2\right]. \end{aligned} \tag{7.3}$$

We note that in the last step we switched the momenta labeling of the quark and the anti-quark. In analogy to the discussion in previous sections we write[1]

$$2s \cdot \mathrm{d}\hat{\sigma}_{\mathrm{rr}} \equiv \int [\mathrm{d}p_5][\mathrm{d}p_6]\; w_g^{51} F_{\mathrm{LM},g}(1_g, 4_q\,|\,5_q, 6_g) \equiv \left\langle w_g^{51} F_{\mathrm{LM},g}(1_g, 4_q\,|\,5_q, 6_g)\right\rangle_\delta\,, \tag{7.4}$$

[1]Note that, the mismatch between the actual $q\bar{q}$ final-state vs. labels of momenta p_4 and p_5 in $F_{\mathrm{LM},g}(1_g, 4_q\,|\,5_q, 6_g)$ indicates the "averaging" over quark-anti-quark final states, see Eq. (7.3). For simplicity, we do not show these labels in the following computation.

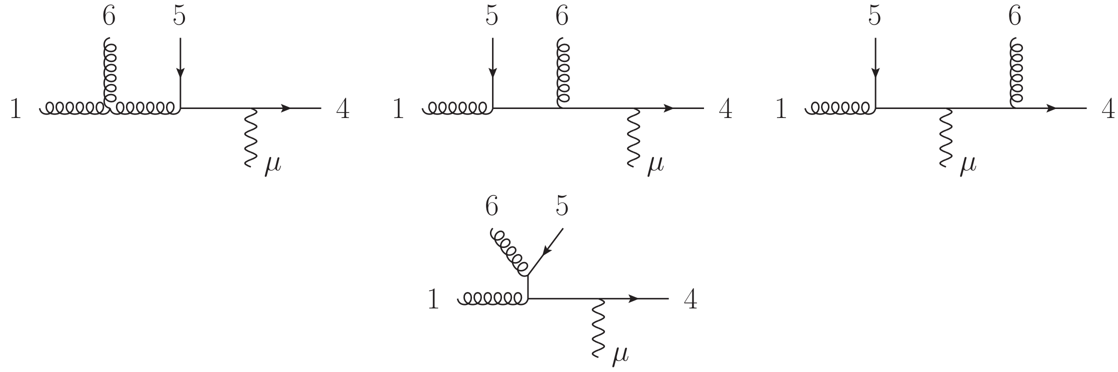


Fig. 7.1.: Partonic currents that contribute to the double-real emission contribution to the gluon-initiated cross section. The shown set is not complete, all Feynman diagrams also need to be included in the computation of the amplitude with inverted fermion line. To obtain the complete Feynman diagrams for DIS they need to be contracted with the the leptonic current. We only show labels i of external momenta p_i.

where we defined

$$F_{\mathrm{LM},g}(1_g, 4_q \,|\, 5_q, 6_g) = \mathcal{N} \int [\mathrm{d}p_3][\mathrm{d}p_4]\ (2\pi)^d \delta^{(d)}\left(p_1 + p_2 - p_3 - p_4 - p_5 - p_6\right) \\ \times \left[\left| M^{\mathrm{tree}}(1_g, 4_q, 5_{\bar{q}}, 6_g) \right|^2 + \left| M^{\mathrm{tree}}(1_g, 5_q, 4_{\bar{q}}, 6_g) \right|^2 \right] \times \hat{\mathcal{O}}(p_3, p_4, p_5, p_6)\,. \tag{7.5}$$

Note that for gluon-initiated contribution Eq. (7.4), we do *not* impose energy ordering for momenta p_5 and p_6. This is in contrast with the discussion of the process $q + e^- \to e^- + q + gg$ in Chapter 6. The reason is that the matrix element squared does not posses a single-soft singularity in the $E_5 \to 0$ limit.

The function $w_g^{51} F_{\mathrm{LM},g}(1,4\,|\,5,6)$ possesses the following singularities. A soft singularity is present when the energy of the gluon $g(p_6)$ vanishes. Double-collinear singularities develop when $\vec{p}_5 \parallel \vec{p}_1$ or $\vec{p}_6 \parallel \vec{p}_{i=1,4,5}$ and a triple-collinear singularity develops when $\vec{p}_5 \parallel \vec{p}_6 \parallel \vec{p}_1$. The singularities when $\vec{p}_4 \parallel \vec{p}_1$ is removed by the function w_g^{51}. These singularities form a subset of singularities that the amplitude of the process $q + e^- \to e^- + q + g + g$ possesses. For this reason, we can regulate them in full analogy with what has been discussed in Section 6.1. Removing all operators in Eq. (6.17) that do not lead to singular limits when considering the process in Eq. (7.2), we write

$$\left\langle w_g^{51} F_{\mathrm{LM},g}(1,4\,|\,5,6) \right\rangle_\delta = \left\langle S_6 w_g^{51} F_{\mathrm{LM},g}(1,4\,|\,5,6) \right\rangle + \left\langle F_{\mathrm{LM}}^{\mathrm{Sr,Cs}}(1,4\,|\,5,6) \right\rangle \\ + \left\langle F_{\mathrm{LM}}^{\mathrm{Sr,Cd}}(1,4\,|\,5,6) \right\rangle + \sum_{i\in\{1,4\}} \left\langle \hat{\mathcal{O}}^{(i)}_{\mathrm{nnlo},g}\ [\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6i} w_g^{51} F_{\mathrm{LM},g}(1,4\,|\,5,6) \right\rangle_\delta \\ + \sum_{\substack{i,j\in\{1,4\} \\ i\neq j}} \left\langle \hat{\mathcal{O}}^{(i,j)}_{\mathrm{nnlo},g}\ [\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6j} w_g^{51} F_{\mathrm{LM},g}(1,4\,|\,5,6) \right\rangle_\delta\,, \tag{7.6}$$

where we defined[2]

$$\hat{\mathcal{O}}^{(i,j)}_{\text{nnlo},g} \equiv \left[I - S_6\right]\left[I - C_{5i}\right]\left[I - C_{6j}\right], \tag{7.7}$$

$$\begin{aligned}\hat{\mathcal{O}}^{(i)}_{\text{nnlo},g} \equiv \left[I - S_6\right]\left[I - \mathbb{C}_i\right]\Big(&\theta^{(a)}\left[I - C_{5i}\right] + \theta^{(b)}\left[I - C_{65}\right] + \theta^{(c)}\left[I - C_{6i}\right] \\ &+ \theta^{(d)}\left[I - C_{65}\right]\Big).\end{aligned} \tag{7.8}$$

Partition function $w^{5i,6j}$ and angular sectors (a, b, c, d) in Eq. (7.6) are defined in Eq. (6.8) and Eq. (6.13), respectively. Double-collinear operators are defined to act on the phase space volume element that is parametrized in the same way as in the discussion of the process $q + e^- \to e^- + q + gg$, except for the fact that energies E_5 and E_6 are not ordered anymore.

The first three terms on the right-hand side in Eq. (7.6) describe various subtraction terms. The first term on the right-hand side in Eq. (7.6) is the single-soft subtraction term, we show results for this contribution in the following section. We discuss the second and the third term on the right-hand side in Eq. (7.6), corresponding to the soft-regulated (Sr) single-unresolved (Cs) and double-unresolved (Cd) collinear subtraction terms in Sections 7.2 and 7.3, respectively. The subtraction terms can be obtained in an analogous way to the discussion in Chapter 6 and we do not repeat the details here, but only show an outline and present the results.

7.1. Single-soft subtraction term

The first term on the right-hand side of Eq. (7.6) is the single-soft subtraction term. The required limit of the function $F_{\text{LM},g}$ is given in Appendix B. Integration over unresolved gluon momentum p_6 is performed in full analogy to process $q + e^- \to e^- + q + gg$ discussed in Section 6.2. The result reads[3]

$$\begin{aligned}\left\langle S_6 w_g^{51} F_{\text{LM},g}(1,4\,|\,5,6)\right\rangle &= \left\langle \left[I - C_{51}\right] J^g_{145} w_g^{51} F_{\text{LM},g}(1,4\,|\,5)\right\rangle_\delta \\ &- \frac{[\alpha_s]^2}{\epsilon^3}\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\left(\frac{4E_{\max}^2}{\mu^2}\right)^{-2\epsilon}\int\limits_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon} P_{qg}(z)\left\langle\Bigg[2C_F\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\eta_{14}^{-\epsilon} K_{14}\right. \\ &\left. + C_A\,\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\Bigg]\sum_{f\in\{q,\bar{q}\}}\frac{F_{\text{LM}}(z\cdot 1_f, 4_f)}{z}\right\rangle_\delta,\end{aligned} \tag{7.9}$$

where we defined

$$J^g_{145} \equiv \frac{[\alpha_s]}{\epsilon^2}\left[(2C_F - C_A)\eta_{45}^{-\epsilon} K_{45} + C_A\left[\eta_{14}^{-\epsilon} K_{14} + \eta_{15}^{-\epsilon} K_{15}\right]\right]\left(\frac{4E_{\max}^2}{\mu^2}\right)^{-\epsilon}. \tag{7.10}$$

[2]We note that in Eqs. (7.7, 7.8) the operator $\mathbb{C}_4$ and some of the double-collinear operators C_{ij} do not contribute. We kept these operators in Eqs. (7.7, 7.8) to retain the symmetric notation.

[3]In writing Eq. (7.9) we used $P_{\bar{q}g}(z) = P_{qg}(z)$.

In Eq. (7.9) the functions $F_{\mathrm{LM},g}(1,4\,|\,5)$ and $F_{\mathrm{LM}}(1,4)$ are given in Eqs. (5.78, 4.5), the splitting function P_{qg} can be found in Appendix E.1 and K_{ij} is defined in Eq. (5.18). We note that the splitting function P_{qg}, which describes the collinear splitting of an initial-state gluon into a quark-anti-quark pair, includes a factor $1/(1-\epsilon)$ that reflects the different number of gluon and quark polarizations. For more details, we refer to the NLO discussion in Section 5.5.

7.2. Single-unresolved collinear subtraction terms

The second term on the right-hand side of Eq. (7.6) contains all soft-regulated single-unresolved subtraction terms where one of the two emitted partons is collinear to another external parton. It reads

$$\begin{aligned}\langle F_{\mathrm{LM}}^{\mathrm{Sr,Cs}}(1,4\,|\,5,6)\rangle &= \Big\langle [I-S_6]\left[C_{51}w^{51,64} + C_{61}w^{54,61} + \left(\theta^{(a)}C_{51} + \theta^{(c)}C_{61}\right)w^{51,61}\right] \\ &\quad \times [\mathrm{d}p_5][\mathrm{d}p_6]w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle \\ &\quad + \Big\langle [I-S_6]\Big[C_{64}w^{51,64} + \theta_4^{(c)}C_{64}w^{54,64}\Big][\mathrm{d}p_5][\mathrm{d}p_6]w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle \\ &\quad + \sum_{i\in\{1,4\}}\Big\langle [I-S_6]\Big[\theta_i^{(b)}C_{56} + \theta_i^{(d)}C_{56}\Big][\mathrm{d}p_5][\mathrm{d}p_6]w^{5i,6i}w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle\,.\end{aligned} \tag{7.11}$$

The first term on the right-hand side in the above equation describes initial-state splitting, the second final-state splitting and the third the emission of two partons that are collinear to each other. We discuss the three terms separately, starting with the first one.

Initial-state emission

We begin with the following contribution to Eq. (7.11)

$$\Big\langle [I-S_6]C_{51}w^{51,64}[\mathrm{d}p_5][\mathrm{d}p_6]w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle\,. \tag{7.12}$$

Calculating the limits using Eq. (B.11) in the appendix and following the discussion of a similar limit at NLO given in Section 5.5, we find[4]

$$\begin{aligned}&\Big\langle [I-S_6]C_{51}w^{51,64}[\mathrm{d}p_5][\mathrm{d}p_6]w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle \\ &= -\frac{[\alpha_{s,b}]}{\epsilon}\sum_{f\in\{q,\bar{q}\}}\int_0^1 \mathrm{d}z\,(2E_1)^{-2\epsilon}(1-z)^{-2\epsilon}P_{fg}(z)\left\langle [I-S_5]w_{\mathrm{dc}}^{54}\frac{F_{\mathrm{LM}}(z\cdot 1_f,4_f\,|\,5_g)}{z}\right\rangle_\delta\,.\end{aligned} \tag{7.13}$$

We continue with the contribution of the angular sector (a) in the triple-collinear partition $51,61$ in Eq. (7.11). Apart from the angular ordering $\eta_{15} < \eta_{16}/2$ and a different partition function, this term is identical to Eq. (7.12). Consequences of these differences are discussed

[4] In this equation we renamed gluon momentum $p_6 \to p_5$ after integrating over unresolved phase space of the collinear (anti-)quark.

around Eq. (6.63). The result reads[4]

$$\begin{aligned}\Big\langle [I-S_6]\theta^{(a)}C_{51}w^{51,61}[\mathrm{d}p_5][\mathrm{d}p_6]w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle &= -\frac{[\alpha_{s,b}]}{\epsilon}\sum_{f\in\{q,\bar{q}\}}\int\limits_0^1 \mathrm{d}z\,\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon} \\ \times (1-z)^{-2\epsilon}P_{fg}(z)\,\Big\langle [I-S_5]\left(\frac{\eta_{16}}{2}\right)^{-\epsilon} w_{\mathrm{tc}}^{51}\frac{F_{\mathrm{LM}}(z\cdot 1_f,4_f\,|\,5_g)}{z}\Big\rangle_\delta .\end{aligned} \tag{7.14}$$

We now show results for the term proportional to C_{61} in Eq. (7.11); it describes the collinear splitting of the incoming gluon to two gluons. The computation is similar to the discussion of final state quark splitting in Section 5.2.2 but in case of gluons also spin correlations occur. We discussed in Section 6.3.3 how to deal with these. We note that the phase space parametrization used to describe $g\to gg$ splitting can be found in Appendix F. We obtain

$$\begin{aligned}&\Big\langle [1-S_6]C_{61}[\mathrm{d}p_5][\mathrm{d}p_6]\Big(w^{54,61}+\theta_1^{(c)}w^{51,61}\Big)w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle \\ &= \frac{[\alpha_{s,b}]}{\epsilon}\left[\frac{C_A}{\epsilon}\Big((2E_1)^{-2\epsilon}-(2E_{\max})^{-2\epsilon}\Big)+\gamma_g(2E_1)^{-2\epsilon}\right] \\ &\qquad\times\Big\langle\left[w_{\mathrm{dc}}^{54}+w_{\mathrm{tc}}^{51}\left(\frac{\rho_{51}}{4}\right)^{-\epsilon}\right]w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5)\Big\rangle \\ &\quad -\frac{[\alpha_{s,b}]}{\epsilon}(2E_1)^{-2\epsilon}\int\limits_0^1\mathrm{d}z\,\mathcal{P}^{\delta}_{gg,RR_2}(z)\Big\langle\left[w_{\mathrm{dc}}^{54}+w_{\mathrm{tc}}^{51}\left(\frac{\rho_{51}}{4}\right)^{-\epsilon}\right]w_g^{51}\frac{F_{\mathrm{LM},g}(z\cdot 1,4\,|\,5)}{z}\Big\rangle,\end{aligned} \tag{7.15}$$

where we introduced generalized splitting functions

$$\begin{aligned}\mathcal{P}^{\delta}_{gg,RR_2}(z) &= \mathcal{P}_{gg,RR_2}(z)+\beta_0\delta(1-z) = \hat{P}^{(0)}_{gg}(z)+\mathcal{O}(\epsilon), \\ \mathcal{P}_{gg,RR_2}(z) &= 2C_A\left\{\left[\frac{1}{(1-z)^{1+2\epsilon}}\right]_+ +(1-z)^{-2\epsilon}\left(\frac{1}{z}+z(1-z)-2\right)\right\},\end{aligned} \tag{7.16}$$

and γ_g is the LO gluon cusp anomalous dimension. Results shown in Eqs. (7.13, 7.14, 7.15) depend on NLO functions $F_{\mathrm{LM}}(1,4\,|\,5)$ and $F_{\mathrm{LM},g}(1,4\,|\,5)$ that possess collinear and soft singularities. We explain how to isolate them in what follows.

We begin by considering Eqs. (7.13, 7.14). The soft singularity of $F_{\mathrm{LM}}(1,4\,|\,5)$ is already regulated by $[I-S_5]$ and limits of partition functions w_{dc}^{54} and w_{tc}^{51} provide proper NLO partitioning. Hence, we only need to include the partition of unity $I=[I-C_{5i}]+C_{5i}$, with $i\in\{1,4\}$, to obtain a fully-regulated contribution. We compute the subtraction terms in full analogy to the NLO discussion in Chapter 5. We note that, small differences between these computations are already discussed in the context of the process $q+e^-\to e^-+q+gg$ in Chapter 6. The result reads

$$\begin{aligned}
&\Big\langle [I-S_6]\Big(C_{51}w^{51,64}+\theta_1^{(a)}C_{51}w^{51,61}\Big)[\mathrm{d}p_5][\mathrm{d}p_6]w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle \\
&= -\frac{[\alpha_s]}{\epsilon}\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\int\limits_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon}P_{qg}(z)\,\Big\langle\Big[\hat{\mathcal{O}}_{\mathrm{nlo}}^{(4)}w_{\mathrm{dc}}^{54}+\hat{\mathcal{O}}_{\mathrm{nlo}}^{(1)}\left(\frac{\eta_{15}}{2}\right)^{-\epsilon}w_{\mathrm{tc}}^{51}\Big] \\
&\qquad\times\sum_{f\in\{q,\bar{q}\}}\frac{F_{\mathrm{LM}}(z\cdot 1_f,4_f\,|\,5_g)}{z}\Big\rangle_\delta-\frac{[\alpha_s]^2}{\epsilon^2}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\int\limits_0^1\mathrm{d}z\,(1-z)^{-2\epsilon}P_{qg}(z) \\
&\qquad\times\Big\langle\Big[2C_F\frac{(4E_4^2/\mu^2)^{-\epsilon}-(4E_{\mathrm{max}}^2/\mu^2)^{-\epsilon}}{2\epsilon}+\left(\frac{4E_4^2}{\mu^2}\right)^{-\epsilon}\gamma_{qq}^{22}\Big]\sum_{f\in\{q,\bar{q}\}}\frac{F_{\mathrm{LM}}(z\cdot 1_f,4_f\,|\,5_g)}{z}\Big\rangle_\delta \\
&\quad+2C_F\frac{[\alpha_s]^2}{2\epsilon^2}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\left(\frac{4E_{\mathrm{max}}^2}{\mu^2}\right)^{-\epsilon}\int\limits_0^1\mathrm{d}z\,(1-z)^{-2\epsilon}P_{qg}(z) \\
&\qquad\times\sum_{f\in\{q,\bar{q}\}}\Big\langle\frac{F_{\mathrm{LM}}(z\cdot 1_f,4_f)}{z}\Big\rangle_\delta-\frac{[\alpha_s]^2}{\epsilon^2}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-2\epsilon} \\
&\qquad\times\int\limits_0^1\mathrm{d}z\,\left[P_{qg}^{22}\otimes P_{qq}^{02}\right](z)\sum_{f\in\{q,\bar{q}\}}\Big\langle\frac{F_{\mathrm{LM}}(z\cdot 1_f,4_f)}{z}\Big\rangle_\delta .
\end{aligned} \tag{7.17}$$

We note that operators $\hat{\mathcal{O}}_{\mathrm{nlo}}^{(i)}$ are defined in Eq. (5.15).

We continue with the contribution in Eq. (7.15). The function $F_{\mathrm{LM},g}(1,4\,|\,5)$ possesses only one, $\vec{p}_5 \parallel \vec{p}_1$, singularity. In terms proportional to the partition function w_{dc}^{54} this singularity is regulated. For other contributions to Eq. (7.15), the subtraction terms are constructed in analogy to the NLO discussion in Section 5.5. The major difference is that the momentum of an incoming gluon is z-dependent. We explained in Chapter 6 how to deal with this situation. We note that this gives rise to convolutions of splitting functions. We obtain

$$\begin{aligned}
&\Big\langle [1-S_6]C_{61}[\mathrm{d}p_5][\mathrm{d}p_6]\Big(w^{54,61}+\theta_1^{(c)}w^{51,61}\Big)w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle \\
&= \frac{[\alpha_s]}{\epsilon}\left[2C_A\frac{(4E_1^2/\mu^2)^{-\epsilon}-(4E_{\mathrm{max}}^2/\mu^2)^{-\epsilon}}{2\epsilon}+\gamma_g\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\right] \\
&\qquad\times\Big\langle\Big[w_{\mathrm{dc}}^{54}+\hat{\mathcal{O}}_{\mathrm{nlo},g}\,w_{\mathrm{tc}}^{51}\left(\frac{\rho_{51}}{4}\right)^{-\epsilon}\Big]w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5)\Big\rangle_\delta \\
&\quad-\frac{[\alpha_s]}{\epsilon}\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\int\limits_0^1\mathrm{d}z\,\mathcal{P}_{gg,RR_2}^{\delta}(z)\Big\langle\Big[w_{\mathrm{dc}}^{54}+\hat{\mathcal{O}}_{\mathrm{nlo},g}\,w_{\mathrm{tc}}^{51}\left(\frac{\rho_{51}}{4}\right)^{-\epsilon}\Big]w_g^{51}\frac{F_{\mathrm{LM},g}(z\cdot 1,4\,|\,5)}{z}\Big\rangle_\delta \\
&\quad-\frac{[\alpha_s]^2}{\epsilon^2}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\int\limits_0^1\mathrm{d}z\,(1-z)^{-2\epsilon}P_{qg}(z) \\
&\qquad\times\left[2C_A\frac{(4E_1^2/\mu^2)^{-\epsilon}-(4E_{\mathrm{max}}^2/\mu^2)^{-\epsilon}}{2\epsilon}+\beta_0\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\right]\Big\langle\frac{F_{LM,q}(z\cdot 1,4)}{z}\Big\rangle_\delta
\end{aligned}$$

$$+\frac{[\alpha_s]^2}{\epsilon^2}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-2\epsilon}\int_0^1 \mathrm{d}z\,[\mathcal{P}_{gg,RR_2}\otimes\mathcal{P}_{qg,R}](z)$$
$$\times\left\langle\frac{F_{LM,q}(z\cdot 1,4)}{z}\right\rangle_\delta . \tag{7.18}$$

The convolution of splitting functions in the last term on the right-hand side of Eq. (7.18) is given in Appendix E.1.

Final-state emission

We now consider the contribution

$$\left\langle [I-S_6]\left[C_{64}w^{51,64}+\theta_4^{(c)}C_{64}w^{54,64}\right][\mathrm{d}p_5][\mathrm{d}p_6]w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5,6)\right\rangle, \tag{7.19}$$

to Eq. (7.11), which describes the collinear splitting of a final-state (anti-)quark. We begin with terms proportional to $w^{51,64}$ in Eq. (7.19). We use the fact that functions P_{qq} and $P_{\bar{q}\bar{q}}$ are identical and write the collinear limit of the function $F_{\mathrm{LM},g}$ in the following way

$$C_{64}F_{\mathrm{LM},g}(1,4\,|\,5,6)=g_{s,b}^2\times\frac{1}{p_4\cdot p_6}\,P_{qq}\,(z)\times F_{\mathrm{LM},g}\left(1,\frac{1}{z}\cdot 4\,|\,5\right), \tag{7.20}$$

where $z=E_4/(E_4+E_6)$ and $P_{qq}(z)$ is given in Eq. (5.31). This limit was discussed in Section 5.2.3 (see Eq. (5.43)). Following the discussion there, we obtain

$$\left\langle [I-S_6]\left[C_{64}w^{51,64}+\theta_4^{(c)}C_{64}w^{54,64}\right][\mathrm{d}p_5][\mathrm{d}p_6]w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5,6)\right\rangle$$
$$=\frac{[\alpha_{s,b}]}{\epsilon}\left\langle\left[2C_F\frac{(2E_4)^{-2\epsilon}-(2E_{\max})^{-2\epsilon}}{2\epsilon}+\gamma_{qq}^{22}(2E_4)^{-2\epsilon}\right]\left[w_{\mathrm{dc}}^{51}+w_{\mathrm{tc}}^{54}\left(\frac{\rho_{45}}{4}\right)^{-\epsilon}\right]\right. \tag{7.21}$$
$$\left.\times\, w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5)\right\rangle.$$

To regulate the remaining collinear $\vec{p}_5 \parallel \vec{p}_1$ singularity in the function $F_{\mathrm{LM},g}(1,4\,|\,5)$ in Eq. (7.21) we insert the partition of unity $I=\left[I-C_{51}\right]+C_{51}$ to the right-hand side of Eq. (7.21). Looking at the subtraction term that contains the operator C_{51}, we note that $\lim_{\vec{p}_5\|\vec{p}_1} w_{\mathrm{tc}}^{51}w_g^{51}=1$, $C_{51}w_{\mathrm{dc}}^{54}=0$ and that the term in the square brackets only depends on the final-state momentum p_4. Hence, computation of this term is identical to the computation of the NLO subtraction term Eq. (5.84). We use the NLO result in Eq. (5.84) and write the fully-regulated contribution as

$$
\begin{aligned}
&\Big\langle [I-S_6]\Big[C_{64}w^{51,64}+\theta_4^{(c)}C_{64}w^{54,64}\Big][\mathrm{d}p_5][\mathrm{d}p_6]w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle\\
&=\frac{[\alpha_s]}{\epsilon}\Big\langle\Big[2C_F\frac{(4E_4^2/\mu^2)^{-\epsilon}-(4E_{\max}^2/\mu^2)^{-\epsilon}}{2\epsilon}+\left(\frac{4E_4^2}{\mu^2}\right)^{-\epsilon}\gamma_{qq}^{22}\Big]\\
&\quad\times\Big[\hat{\mathcal{O}}_{\mathrm{nlo},g}w_{\mathrm{dc}}^{51}+w_{\mathrm{tc}}^{54}\left(\frac{\rho_{45}}{4}\right)^{-\epsilon}\Big]w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5)\Big\rangle_\delta\\
&\quad-\frac{[\alpha_s]^2}{\epsilon^2}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}\sum_{f\in\{q,\bar{q}\}}\int_0^1\mathrm{d}z\,(1-z)^{-2\epsilon}P_{fg}(z)\\
&\quad\times\Big\langle\Big[2C_F\frac{(4E_4^2/\mu^2)^{-\epsilon}-(4E_{\max}^2/\mu^2)^{-\epsilon}}{2\epsilon}+\left(\frac{4E_4^2}{\mu^2}\right)^{-\epsilon}\gamma_{qq}^{22}\Big]\frac{F_{\mathrm{LM}}(z\cdot 1_f,4_f)}{z}\Big\rangle_\delta\,.
\end{aligned}
\tag{7.22}
$$

Double-collinear C_{56} sectors

The last missing contribution to the single-unresolved subtraction term in Eq. (7.11) describes a kinematic configuration where an emitted gluon and (anti-)quark are collinear to each other. It reads

$$
\sum_{i\in\{1,4\}}\Big\langle[I-S_6]\Big[\theta_i^{(b)}C_{56}+\theta_i^{(d)}C_{56}\Big][\mathrm{d}p_5][\mathrm{d}p_6]w^{5i,6i}w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle\,. \tag{7.23}
$$

The collinear limit reads

$$
C_{56}F_{\mathrm{LM},g}(1,4\,|\,5,6)=g_{s,b}^2\times\frac{1}{p_5\cdot p_6}\,P_{qq}\,(z)\times F_{\mathrm{LM},g}(1,4\,|\,5+6)\,, \tag{7.24}
$$

where $z=E_5/(E_5+E_6)$. Integration over angular phase space depends on the adopted phase space parametrization; it can be found in Appendix F. We obtain

$$
\begin{aligned}
&\int\Big[C_{56}[\mathrm{d}\Omega_6]\left(\theta_i^{(b)}+\theta_i^{(d)}\right)\Big]\frac{1}{\rho_{56}}\\
&=-\frac{1}{\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]\left[\frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)}\right]\eta_{i5}^{-\epsilon}(1-\eta_{i5})^\epsilon\,.
\end{aligned}
\tag{7.25}
$$

Note that the integral Eq. (7.25) depends on ρ_{i5} because of the angular ordering in sectors (b) and (d). We find

$$
\begin{aligned}
&\sum_{i\in\{1,4\}}\Big\langle[I-S_6]\Big[\theta_i^{(b)}C_{56}+\theta_i^{(d)}C_{56}\Big][\mathrm{d}p_5][\mathrm{d}p_6]w^{5i,6i}w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle\\
&=\frac{[\alpha_{s,b}]}{\epsilon}\left[\frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)}\right]2^{2\epsilon}\sum_{i\in\{1,4\}}\Big\langle\Big[2C_F\frac{(2E_5)^{-2\epsilon}-(2E_{\max})^{-2\epsilon}}{2\epsilon}+\gamma_{qq}^{22}(2E_5)^{-2\epsilon}\Big]\\
&\quad\times\eta_{i5}^{-\epsilon}(1-\eta_{i5})^\epsilon\,w_{\mathrm{tc}}^{i}w_g^{51}F_{\mathrm{LM},g}(1,4\,|\,5)\Big\rangle\,.
\end{aligned}
\tag{7.26}
$$

The function $F_{\mathrm{LM},g}(1,4\,|\,5)$ in Eq. (7.26) contains a singularity when $\vec{p}_5 \parallel \vec{p}_1$. However, for $i=4$ this singularity is regulated by w^4_{tc}, which vanishes in this limit, and we only have to regulate this singularity for $i=1$. This is done in full analogy to the NLO case discussed in Section 5.5. We obtain the final result

$$
\begin{aligned}
&\sum_{i\in\{1,4\}} \Big\langle [I-S_6]\Big[\theta_i^{(b)} C_{56} + \theta_i^{(d)} C_{56}\Big][\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6i} w_g^{51} F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle \\
&= \frac{[\alpha_s]}{\epsilon}\left[\frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)}\right] 2^{2\epsilon} \Big\langle \Big[2C_F \frac{(4E_5^2/\mu^2)^{-\epsilon} - (4E_{\max}^2/\mu^2)^{-\epsilon}}{2\epsilon} + \left(\frac{4E_5^2}{\mu^2}\right)^{-\epsilon} \gamma_{qq}^{22}\Big] \\
&\qquad \times \left(\hat{\mathcal{O}}_{\mathrm{nlo},g}\eta_{15}^{-\epsilon}(1-\eta_{15})^{\epsilon} w^1_{\mathrm{tc}} + \eta_{45}^{-\epsilon}(1-\eta_{45})^{\epsilon} w^4_{\mathrm{tc}}\right) w_g^{51} F_{\mathrm{LM},g}(1,4\,|\,5)\Big\rangle_\delta \\
&\quad + \frac{[\alpha_s]^2}{\epsilon}\left[\frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)}\right]\left[\frac{2^{\epsilon}}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right] 2^{2\epsilon}\int_0^1 \mathrm{d}z\,(1-z)^{-2\epsilon} P_{qg}(z) \\
&\qquad \times \Big\langle \Big[2C_F \frac{(4E_1^2/\mu^2)^{-2\epsilon}(1-z)^{-2\epsilon} - (4E_1^2/\mu^2)^{-\epsilon}(4E_{\max}^2/\mu^2)^{-\epsilon}}{4\epsilon} \\
&\qquad + \left(\frac{4E_1^2}{\mu^2}\right)^{-2\epsilon} \gamma_{qq}^{22}(1-z)^{-2\epsilon}\Big] \sum_{f\in\{q,\bar{q}\}} \frac{F_{\mathrm{LM}}(z\cdot 1_f, 4_f)}{z}\Big\rangle_\delta .
\end{aligned}
\tag{7.27}
$$

7.3. Double-unresolved collinear subtraction terms

The third term on the right-hand side in Eq. (7.6) refers to soft-regulated double-unresolved subtraction terms, where both emissions are collinear to another parton. It reads

$$
\begin{aligned}
\langle F_{\mathrm{LM}}^{\mathrm{Sr,Cd}}(1,4\,|\,5,6)\rangle &= \Big\langle [I-S_6]\Big[\theta_1^{(a)}\mathbb{C}_1[I-C_{61}] + \theta_1^{(b)}\mathbb{C}_1[I-C_{56}] + \theta_1^{(c)}\mathbb{C}_1[I-C_{51}] \\
&\quad + \theta_1^{(d)}\mathbb{C}_1[I-C_{56}]\Big][\mathrm{d}p_5][\mathrm{d}p_6] w^{51,61} w_g^{51} F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle \\
&\quad - \Big\langle [I-S_6] C_{51}C_{64}[\mathrm{d}p_5][\mathrm{d}p_6] w^{51,64} w_g^{51} F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle .
\end{aligned}
\tag{7.28}
$$

We note that contributions to Eq. (7.28) from the triple-collinear partition 51, 61 were computed in Ref. [68] and we only provide the required formula in Appendix I. In the following we consider the contribution from the double-collinear sector 51, 64, which is given by the following term

$$
-\Big\langle [I-S_6] C_{51}C_{64}[\mathrm{d}p_5][\mathrm{d}p_6] w^{51,64} w_g^{51} F_{\mathrm{LM},g}(1,4\,|\,5,6)\Big\rangle . \tag{7.29}
$$

Computing the corresponding limits and integrating over the phase space, we find

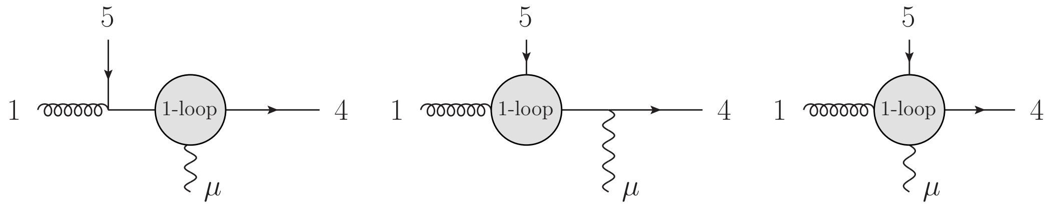


Fig. 7.2.: Partonic currents that describe real-virtual corrections to the gluon-initiated DIS cross section. The shown set is not complete, all Feynman diagrams also need to be included in the computation of the amplitude with inverted fermion line. To obtain the complete Feynman diagrams for DIS they need to be contracted with the the leptonic current. We only show labels i of external momenta p_i.

$$
\begin{aligned}
&-\Big\langle [I-S_6] C_{51} C_{64} [\mathrm{d}p_5][\mathrm{d}p_6] w^{51,64} w_g^{51} F_{\mathrm{LM},g}(1,4\,|\,5,6) \Big\rangle \\
&= \frac{[\alpha_s]^2}{\epsilon^2} \sum_{f\in\{q,\bar{q}\}} \int_0^1 \mathrm{d}z \, \left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon} (1-z)^{-2\epsilon} P_{fg}(z) \\
&\quad \times \Bigg\langle \Bigg[2C_F \frac{(4E_4^2/\mu^2)^{-\epsilon} - (4E_{\mathrm{max}}^2/\mu^2)^{-\epsilon}}{2\epsilon} + \left(\frac{4E_4^2}{\mu^2}\right)^{-\epsilon} \gamma_{qq}^{22} \Bigg] \frac{F_{\mathrm{LM}}(z\cdot 1_f, 4_f)}{z} \Bigg\rangle_\delta .
\end{aligned}
\tag{7.30}
$$

7.4. Real-virtual contribution

In this section we consider the one-loop corrections to the process $g + e^- \to e^- + q\bar{q}$. Feynman diagrams that describe this process are shown in Fig. 7.2. Following the discussion in Section 5.5 we define the UV-renormalized contribution as

$$
2s \cdot \mathrm{d}\hat{\sigma}_{\mathrm{rv}} \equiv \int [\mathrm{d}p_5] \; w_g^{51} F_{\mathrm{LV},g}(1,4\,|\,5) \equiv \Big\langle w_g^{51} F_{\mathrm{LV},g}(1,4\,|\,5) \Big\rangle_\delta , \tag{7.31}
$$

where

$$
\begin{aligned}
F_{\mathrm{LV},g}(1,4\,|\,5) &= \mathcal{N} \int [\mathrm{d}p_3][\mathrm{d}p_4] \; (2\pi)^d \delta^{(d)} \left(p_1 + p_2 - p_3 - p_4 - p_5\right) \\
&\times \Big[2\Re\big(M_{\mathrm{nlo}}^{\mathrm{tree}\star} \cdot M_{\mathrm{nlo}}^{\text{1-loop}}\big)(p_1,p_2,p_3,p_4,p_5) + 2\Re\big(M_{\mathrm{nlo}}^{\mathrm{tree}\star} \cdot M_{\mathrm{nlo}}^{\text{1-loop}}\big)(p_1,p_2,p_3,p_5,p_4) \Big] \\
&\times \hat{\mathcal{O}}(p_3,p_4,p_5) .
\end{aligned}
\tag{7.32}
$$

UV divergences of the one-loop amplitude $M_{\mathrm{nlo}}^{\text{1-loop}}$ follow from the Catani formula [51–53]. We use it to split $F_{\mathrm{LV},g}(1,4\,|\,5)$ into a part that contains explicit $1/\epsilon$ poles and a finite part. We write[5]

$$
F_{\mathrm{LV},g}(1,4\,|\,5) = \hat{I}_{145}^{\mathrm{rv},g} \, F_{\mathrm{LM},g}(1,4\,|\,5) + F_{\mathrm{LV},g}^{\mathrm{fin}}(1,4\,|\,5) , \tag{7.33}
$$

[5] Note that this is only possible because the UV $1/\epsilon$ poles are symmetric under the exchange of momenta between two final-state quarks or anti-quarks.

where

$$
\begin{aligned}
\hat{I}_{145}^{\text{rv,g}} = \frac{[\alpha_s]}{\epsilon}\Bigg[&\left(\frac{1}{\epsilon}+\frac{3}{2}\right)(C_A-2C_F)(2E_4E_6\rho_{46})^{-\epsilon}\cos(\pi\epsilon) \\
&- C_A\left(\frac{1}{\epsilon}+\frac{3}{4}+\frac{\gamma_g}{2}\right)\Big((2E_1E_4\rho_{14})^{-\epsilon}+(2E_1E_6\rho_{16})^{-\epsilon}\Big)\Bigg].
\end{aligned}
\tag{7.34}
$$

We now consider the IR singularities of the function $F_{\text{LV},g}(1,4\,|\,5)$. By construction, the only singularity is a collinear one that corresponds to the $\vec{p}_5 \parallel \vec{p}_1$ limit.[6] To regulate this singularity, we write

$$
\left\langle w_g^{51}F_{\text{LV},g}(1,4\,|\,5)\right\rangle_\delta = \left\langle [I-C_{51}]\,w_g^{51}F_{\text{LV},g}(1,4\,|\,5)\right\rangle_\delta + \left\langle C_{51}w_g^{51}F_{\text{LV},g}(1,4\,|\,5)\right\rangle. \tag{7.35}
$$

The collinear splitting $g \to q\bar{q}$ at one-loop order is described by the following formula [58–60]

$$
\begin{aligned}
C_{51}F_{\text{LM},g}(1,4\,|\,5) = {}& g_{s,b}^2 \times \frac{1}{p_1\cdot p_5}\sum_{f\in\{q,\bar{q}\}} P_{fg}(z)\times\frac{F_{\text{LV}}(z\cdot 1_f,4_f)}{z} + g_{s,b}^2[\alpha_{s,b}] \\
&\times\left[\frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\right]\times 2^{-\epsilon}\left(\frac{1}{p_1\cdot p_5}\right)^{1+\epsilon}\sum_{f\in\{q,\bar{q}\}} P_{fg}^{\text{loop}}(z)\times\frac{F_{\text{LM}}(z\cdot 1_f,4_f)}{z},
\end{aligned}
\tag{7.36}
$$

where $z = (E_1-E_5)/E_1$. Analogous to Eq. (6.197), the first term on the right-hand side of Eq. (7.36) contains the tree-level splitting functions $P_{fg}(z)$. Integrating over momenta p_5 of this contribution is performed analogously to the NLO discussion in Section 5.5 and the result can be taken from Eq. (5.84). The second term contains additional powers of the scalar product $\rho_{15}^{-\epsilon}$. The required integral is given in Eq. (6.71). It also contains additional powers of $E_5^{-\epsilon}$ that lead to additional powers of $(1-z)^{-\epsilon}$, after writing $E_5 = E_1(1-z)$. Apart from this, integration of the second term on the right-hand side of Eq. (7.36) over gluon momentum p_5 is again analogous to the NLO discussion. The result reads

$$
\begin{aligned}
&\left\langle C_{51}w_g^{51}F_{\text{LV},g}(1,4\,|\,5)\right\rangle \\
&= -\frac{[\alpha_s]}{\epsilon}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\int_0^1 \mathrm{d}z\,\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}(1-z)^{-2\epsilon}\sum_{f\in\{q,\bar{q}\}}P_{fg}(z)\left\langle\frac{F_{\text{LV}}(z\cdot 1_f,4_f)}{z}\right\rangle_\delta \\
&\quad -\frac{[\alpha_s]^2}{\epsilon}\left[\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\right]\int_0^1 \mathrm{d}z\,\left(\frac{4E_1^2}{\mu^2}\right)^{-2\epsilon}(1-z)^{-3\epsilon}\sum_{f\in\{q,\bar{q}\}}P_{fg}^{\text{loop}}(z)\left\langle\frac{F_{\text{LV}}(z\cdot 1_f,4_f)}{z}\right\rangle_\delta.
\end{aligned}
\tag{7.37}
$$

Note that the function F_{LV} also contains explicit loop-induced IR poles. We extract them using Eq. (7.33).

[6] We note that, individual diagrams shown on the very right in Fig. 7.2 are also singular in the collinear $\vec{p}_4 \parallel \vec{p}_5$ limit. However, to compute the amplitude squared such contributions are multiplied by the tree-level amplitude that is not singular in this collinear configuration. As a result, the divergence is not strong enough and can be integrated.

7.5. Collinear renormalization

Finally, we show results for the collinear renormalization contribution $\mathrm{d}\hat{\sigma}_{\mathrm{pdf}}$. Selecting terms in Eq. (6.208) that are proportional to the gluon parton distribution function we obtain

$$2s \cdot \mathrm{d}\hat{\sigma}_{\mathrm{pdf}} = \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \sum_{f\in\{q,\bar{q}\}} \int_0^1 \mathrm{d}z \left[\frac{\hat{P}^{(1)}_{fg}(z)}{2\epsilon} - \frac{\left[\hat{P}^{(0)}_{fg} \otimes \hat{P}^{(0)}_{gg}\right](z) + 2\left[\hat{P}^{(0)}_{fq} \otimes \hat{P}^{(0)}_{qg}\right](z)}{2\epsilon^2}\right] \mathrm{d}\hat{\sigma}^{\mathrm{lo}}_f$$
$$+ \frac{\alpha_s(\mu)}{2\pi} \int_0^1 \left[\hat{P}^0_{qg}(z)\, \mathrm{d}\hat{\sigma}^{\mathrm{nlo}}_q(z) + \hat{P}^{(0)}_{gg}(z)\, \mathrm{d}\hat{\sigma}^{\mathrm{nlo}}_g(z)\right]. \tag{7.38}$$

In writing Eq. (7.38) we used $\left[\hat{P}^{(0)}_{f\bar{q}} \otimes \hat{P}^{(0)}_{\bar{q}g}\right](z) = \left[\hat{P}^{(0)}_{fq} \otimes \hat{P}^{(0)}_{qg}\right](z)$. Eq. (7.38) is the last ingredient required to describe the gluon-initiated contributions to the NNLO QCD DIS cross section. Upon combining double-real contribution in Eq. (7.6) with real-virtual contributions in Eq. (7.35) and contribution from collinear renormalization in Eq. (6.208) we find that the poles cancel and we are left with a finite remainder of the subtraction terms and regulated cross sections. This result is presented in Section 9.3.

8. Numerical computation of regulated contributions

In Chapters 6 and 7 we explained how singular contributions to the NNLO partonic DIS cross sections can be regulated. We also discussed the analytic integration of the subtraction terms over *unresolved* phase space. In this chapter we would like to describe how regulated contributions derived in previous sections, can be integrated numerically over *resolved* phase space to obtain NNLO predictions for any infrared-safe observable in 4-dimensional space time.

As an example, consider the double-real emission process $q + e^- \to e^- + q + g + g$. The cross section of this process, written in terms of the regulated matrix element and subtraction terms, is shown in Eq. (6.17). The only regularized contribution that depends on the full matrix element of the above process reads

$$\begin{aligned}
&\sum_{\substack{i,j=1,4\\ i\neq j}} \left\langle [1-\mathbb{S}]\,[1-S_6]\,[1-C_{6j}]\,[1-C_{5i}]\,[\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6j} F_{\mathrm{LM}}(1_q,4_q\,|\,5_g,6_g)\right\rangle_\delta \\
&+\sum_{i=1,4}\Big\langle [1-\mathbb{S}]\,[1-S_6]\,[1-\mathbb{C}_i]\Big(\theta^{(a)}[1-C_{6i}]+\theta^{(b)}[1-C_{56}] \\
&\quad+\theta^{(c)}[1-C_{5i}]+\theta^{(d)}[1-C_{56}]\Big)[\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6i} F_{\mathrm{LM}}(1_q,4_q\,|\,5_g,6_g)\Big\rangle_\delta .
\end{aligned} \tag{8.1}$$

For the sake of clarity, we focus on the contribution of sector (a) to the second term on the right-hand side of Eq. (8.1). When written explicitly, this contribution reads

$$\begin{aligned}
&\Big\langle [I-\mathbb{S}]\,[I-S_6]\,[I-\mathbb{C}_1]\,[1-C_{61}]\,[\mathrm{d}p_5][\mathrm{d}p_6]\theta^{(a)}\, w^{51,61} F_{\mathrm{LM}}(1_q,4_q\,|\,5_g,6_g)\Big\rangle_\delta \\
&=\int [I-\mathbb{S}]\,[I-S_6]\,[I-\mathbb{C}_1]\,[1-C_{61}]\,[\mathrm{d}p_5][\mathrm{d}p_6]\theta(E_5-E_6)\theta^{(a)} \\
&\qquad\times w^{51,61} F_{\mathrm{LM}}(1_q,4_q\,|\,5_g,6_g)\,,
\end{aligned} \tag{8.2}$$

where

$$\begin{aligned}
F_{\mathrm{LM}}(1_q,4_q\,|\,5_g,6_g) = \mathcal{N}\int [\mathrm{d}p_3][\mathrm{d}p_4]\,(2\pi)^4\delta^{(4)}\,(p_1+p_2-p_3-p_4-p_5-p_6) \\
\times |M^{\mathrm{tree}}_{\mathrm{nnlo}}(p_1,p_2,p_3,p_4,p_5,p_6)|^2\times \hat{\mathcal{O}}(p_3,p_4,p_5,p_6)\,.
\end{aligned} \tag{8.3}$$

The contribution Eq. (8.2) is infrared-finite because of subtraction and partition functions. Hence, all objects including the phase space volume elements $[\mathrm{d}p_i]$, the matrix element $M^{\mathrm{tree}}_{\mathrm{nnlo}}$ and the observable $\hat{\mathcal{O}}$ can be computed in four dimensions. All required limits of the function

$F_{\rm LM}$ are collected in Appendix B. Partition function $w^{51,61}$ is given in Appendix A.3. To compute functions $F_{\rm LM}$ numerically we require matrix elements for three processes $q + e^- \to e^- + q + g + g$, $q + e^- \to e^- + q + g$ and $q + e^- \to e^- + q$. These matrix elements can be computed using formulas for vector boson currents $0 \to V^\star + q + \bar{q}$, $0 \to V^\star + q + \bar{q} + g$ and $0 \to V^\star + q + \bar{q} + g + g$ provided in Ref. [70–72].

8.1. Phase space parametrization in sector (a)

We begin by describing the parametrization of phase space that enables Monte Carlo integration. We found it convenient to work in the center-of-mass (COM) frame of the colliding quark $q(p_1)$ and the electron $e^-(p_2)$. Hence we first generate a variable $x \in [0,1]$ and write $s = x \cdot s_{\rm H}$ where $s_{\rm H}$ is the COM energy squared of the hadron-electron collision. The momenta p_1 and p_2 then read[1]

$$p_1 = \frac{\sqrt{s}}{2}(1,0,0,1)\,, \quad p_2 = \frac{\sqrt{s}}{2}(1,0,0,-1)\,. \tag{8.4}$$

Radiation phase space

We continue generating momenta of the two gluons. Therefore, we use a particular parametrization of the two gluon phase space for computing subtraction terms introduced in Ref. [7]. We show details of the parametrization in Appendix F.2. For the current discussion we need this parametrization in four dimensions.

We generate variables $x_1, x_2 \in [0,1]$ and write gluons energies E_5 and E_6 as [7]

$$E_5 = x_1 E_{\rm max}\,, \quad E_6 = x_1 x_2 E_{\rm max}\,. \tag{8.5}$$

We note that this automatically implements the energy ordering $E_5 > E_6$. The double-soft limit corresponds to $x_1 \to 0$ at fixed x_2 and the single-soft limit to $x_2 \to 0$ at fixed x_1. We continue to generate variables $x_3, x_4, \lambda \in [0,1]$ and write scalar products $\eta_{ij} = (1 - \vec{n}_i \cdot \vec{n}_j)/2$ in the following way [7][2]

$$\eta_{15} = x_3\,, \quad \eta_{16} = \frac{x_3 x_4}{2}\,, \quad \eta_{56} = \frac{x_3(1-x_4/2)^2}{N(x_3, x_4/2, \lambda)}\,. \tag{8.7}$$

The triple-collinear limit corresponds to $x_3 \to 0$ at fixed x_4 and the double-collinear limit to $x_4 \to 0$ at fixed x_3. We further generate an azimuthal angle $\varphi_5 \in [0, 2\pi]$. The gluon momenta

[1] We have chosen the beam axis along the z-direction.

[2] The function N in Eq. (8.7) reads

$$N(x_3, x_4, \lambda) = 1 + x_4(1-2x_3) - 2(1-2\lambda)\sqrt{x_4(1-x_3)(1-x_3 x_4)}\,. \tag{8.6}$$

p_5 and p_6 are then computed according to the following equations [7]

$$
\begin{aligned}
p_5 &= x_1 E_{\max} \cdot (1, \sin\theta_5 \cos\varphi_5, \sin\theta_5 \sin\varphi_5, \cos\theta_5)\,, \\
p_6 &= x_1 x_2 E_{\max} \cdot (1, \sin\theta_6 \cos(\varphi_5 + \varphi_{56}), \sin\theta_6 \sin(\varphi_5 + \varphi_{56}), \cos\theta_6)\,,
\end{aligned}
\tag{8.8}
$$

where

$$
\begin{aligned}
\cos\theta_5 &= 1 - 2x_3\,, \quad \sin\theta_5 = \sqrt{1-\cos^2\theta_5}\,, \\
\cos\theta_6 &= 1 - x_3 x_4\,, \quad \sin\theta_6 = \sqrt{1-\cos^2\theta_6}\,,
\end{aligned}
\tag{8.9}
$$

and

$$
\sin\varphi_{56} = \frac{2\sqrt{\lambda(1-\lambda)}(1-x_4/2)}{N(x_3, x_4/2, \lambda)}\,, \quad \cos\varphi_{56} = \pm\sqrt{1-\sin^2\varphi_{56}}\,. \tag{8.10}
$$

In Eq. (8.10) $\cos\varphi_{56}$ is chosen negative if the following condition is satisfied

$$
\frac{2(1-x_4/2)^2}{N(x_3, x_4/2, \lambda)} - \big(2 + x_4(1-2x_3)\big) > 0\,, \tag{8.11}
$$

and otherwise positive. The phase space volume element in this parametrization reads

$$
[\mathrm{d}p_5][\mathrm{d}p_6]\theta(E_5 - E_6)\theta^{(a)} = x_1^3 x_2 x_3 \mathcal{W}_{56}^{(a)}(x_3, x_4, \lambda) \times \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3\, \mathrm{d}x_4\, \mathrm{d}\lambda\, \mathrm{d}\varphi_5\,, \tag{8.12}
$$

where the function $\mathcal{W}_{56}^{(a)}$ is a weight given by

$$
\mathcal{W}_{56}^{(a)}(x_3, x_4, \lambda) = \frac{2E_{\max}^4 (1-x_4/2)}{(2\pi^2)\sqrt{\lambda(1-\lambda)}\,N(x_3, x_4/2, \lambda)}\,. \tag{8.13}
$$

The double-collinear operator C_{61} in Eq. (8.2) is defined such that it acts on the phase space volume element and therefore on the weight $\mathcal{W}_{56}(x_3, x_4, \lambda)$. Hence, in contributions to Eq. (8.2) that contain the operator C_{61}, we have to compute a weight given by $\lim_{x_4 \to 0} \mathcal{W}_{56}(x_3, x_4, \lambda) = \mathcal{W}_{56}(x_3, 0, \lambda)$.

Born phase space

Once the four-momenta of the two gluons are generated, it remains to generate the "Born phase space" element $[\mathrm{d}p_3][\mathrm{d}p_4] \times (2\pi)^4 \delta^{(4)}(Q - p_5 - p_6)$, where $Q = p_1 + p_2 - p_3 - p_4$. We remove the energy-momentum conserving δ-function by integration over the three-momentum of the outgoing electron $e^-(p_3)$ and the energy E_4 of outgoing quark $q(p_4)$. For the direction of momentum p_4 we generate a polar angle $\theta_4 \in [0, \pi]$ and an azimuthal angle $\varphi_4 \in [0, 2\pi]$. In this parametrization, momenta p_3 and p_4 are given by

$$
\begin{aligned}
p_3 &= p_1 + p_2 - p_4 - p_5 - p_6\,, \\
p_4 &= E_4 \cdot (1, \sin\theta_4 \cos\varphi_4, \sin\theta_4 \sin\varphi_4, \cos\theta_4)\,,
\end{aligned}
\tag{8.14}
$$

with

$$E_4 = \frac{s - 2\sqrt{s}\,(E_5 + E_6) + 2E_5E_6\rho_{56}}{2\sqrt{s} - 2(E_5\rho_{45} + E_6\rho_{46})}\,. \tag{8.15}$$

For the Born phase space element we find

$$[\mathrm{d}p_3][\mathrm{d}p_4] \times (2\pi)^4\delta^{(4)}(Q - p_5 - p_6) = \mathrm{d}\cos\theta_4\,\mathrm{d}\varphi_4 \times \mathcal{W}_{34}(E_4, E_5, E_6, \rho_{45}, \rho_{46})\,, \tag{8.16}$$

where the function $\mathcal{W}_{34}$ in Eq. (8.16) reads

$$\mathcal{W}_{34}(E_4, E_5, E_6, \rho_{45}, \rho_{46}) = \frac{1}{2(2\pi)} \times \frac{E_4}{|2\sqrt{s} - 2(E_5\rho_{45} + E_6\rho_{46})|}\,. \tag{8.17}$$

Note that p_3, E_4 and the weight $\mathcal{W}_{34}$ depend, through p_5 and p_6, on the integration variables $\{x_1, x_2, x_3, x_4\}$, which parametrize energies and angles of the emitted gluons. Soft and collinear operators in Eq. (8.2) act on the energy-momentum conservation condition. Hence, taking $F_{\mathrm{LM}}(1,4\,|\,5,6)$ in the double-soft $\mathbb{S}$ limit is equivalent to the computation of p_3, E_4 and weight $\mathcal{W}_{34}$ with $x_1 = 0$. Similarly, $F_{\mathrm{LM}}(1,4\,|\,5,6)$ in the double-collinear C_{61} limit is obtained by computing p_3, E_4 and weight $\mathcal{W}_{34}$ with $x_4 = 0$ etc. It is easy to see that, upon doing that, we obtain proper limits of the Born phase space. We elaborate on this in the next section.

8.2. Evaluation of the cross section

We now discuss how different contributions to the right-hand side of Eq. (8.2) are computed. We split Eq. (8.2) into 16 pieces, each describing a particular combination of soft and collinear limits. Therefore, we expand operator $\left[I - \mathbb{S}\right]\left[I - S_6\right]\left[I - \mathbb{C}_1\right]\left[1 - C_{61}\right]$ and find

$$\begin{aligned}
&\left[I - \mathbb{S}\right]\left[I - S_6\right]\left[I - \mathbb{C}_1\right]\left[1 - C_{61}\right] \\
&= 1 - \mathbb{S} - S_6 - \mathbb{C}_1 - C_{61} + \mathbb{S}S_6 + \mathbb{S}\mathbb{C}_1 + \mathbb{S}C_{61} + S_6\mathbb{C}_1 + S_6C_{61} + \mathbb{C}_1C_{61} \\
&\quad - \mathbb{S}S_6\mathbb{C}_1 - \mathbb{S}S_6C_{61} - \mathbb{S}\mathbb{C}_1C_{61} - S_6\mathbb{C}_1C_{61} + \mathbb{S}S_6\mathbb{C}_1C_{61}\,.
\end{aligned} \tag{8.18}$$

Upon using the parametrization of the phase space discussed in Section 8.1, all these limits are made explicit in terms of various limits in $\{x_1, x_2, x_3, x_4\}$ variables. Inserting the phase space parametrization shown in the previous section into Eq. (8.2) we obtain

$$\begin{aligned}
&\Big\langle \left[I - \mathbb{S}\right]\left[I - S_6\right]\left[I - \mathbb{C}_1\right]\left[1 - C_{61}\right][\mathrm{d}p_5][\mathrm{d}p_6]\theta^{(a)}\, w^{51,61} F_{\mathrm{LM}}(1,4\,|\,5,6)\Big\rangle \\
&= \int \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\,\mathrm{d}\lambda\,\mathrm{d}\cos\theta_4 \times x_1^3 x_2 x_3 \\
&\quad \times \Big\{ \left[I - \mathbb{S}\right]\left[I - S_6\right]\left[I - \mathbb{C}_1\right]\left[1 - C_{61}\right]\mathcal{W}_{34}(E_4, E_5, E_6, \rho_{45}, \rho_{46})\mathcal{W}_{56}^{(a)}(x_3, x_4, \lambda) \\
&\quad \times w^{51,61}(\rho_{15}, \rho_{45}, \rho_{16}, \rho_{46}, \rho_{56})|M^{\mathrm{tree}}_{\mathrm{nnlo}}(p_1, p_2, p_3, p_4, p_5, p_6)|^2\hat{\mathcal{O}}(p_3, p_4, p_5, p_6)\Big\}\,.
\end{aligned} \tag{8.19}$$

We note that the operators in Eq. (8.19) act on everything to the right of them, which includes weights, partition functions, amplitudes, and observable. We now discuss explicitly how some of these contributions are computed numerically.

We begin by considering the term in Eq. (8.18) that is proportional to the identity operator I. For each phase space point given by a set of eight generated variables $\{x_1, x_2, x_3, x_4, \lambda, \theta_4, \varphi_4, \varphi_5\}$ we first compute momenta p_5 and p_6 with Eq. (8.8) and direction of p_4 with Eq. (8.14). We then compute energy E_4 according to Eq. (8.15). If we find $E_4 < 0$, the current phase space point is not valid for the identity contribution; in this case we set Eq. (8.19) to zero and continue with remaining contributions shown in Eq. (8.18). If we find $E_4 > 0$, we compute p_3 with Eq. (8.14) and again check if $E_3 > 0$. In case we also pass the second test, we compute weights $\mathcal{W}_{34}$, $\mathcal{W}_{56}^{(a)}$, partition function $w^{51,61}$, observable $\hat{O}$ and matrix element squared $|M_{\text{nnlo}}^{\text{tree}}|^2$ numerically. The full contribution to Eq. (8.19) reads

$$
\begin{aligned}
I\,F_{\text{LM}}(1,4\,|\,5,6) \;\to\; x_1^3 x_2 x_3 \Big\{ & \mathcal{W}_{34}(E_4, E_5, E_6, \rho_{45}, \rho_{46}) \mathcal{W}_{56}^{(a)}(x_3, x_4, \lambda) \\
\times\; & w^{51,61}(\rho_{15}, \rho_{45}, \rho_{16}, \rho_{46}, \rho_{56}) \times |M_{\text{nnlo}}^{\text{tree}}(p_1, p_2, p_3, p_4, p_5, p_6)|^2 \\
\times\; & \hat{O}(p_3, p_4, p_5, p_6) \Big\}\,.
\end{aligned} \tag{8.20}
$$

Consider now a second contribution for the same phase space point. One of the terms that needs to be subtracted from Eq. (8.20) is $\mathbb{S}F_{\text{LM}}(1,4\,|\,5,6)$. Momenta p_5, p_6 and direction of p_4 are identical to the case $IF_{\text{LM}}(1,4\,|\,5,6)$. However, E_4 and p_3 need to be computed at $x_1 = 0$, which corresponds to the double-soft limit. We find

$$
\begin{aligned}
E_4^{\mathbb{S}} &= \left.\frac{s - 2\sqrt{s}\,(E_5 + E_6) + 2E_5E_6\rho_{56}}{2\sqrt{s} - 2(E_5\rho_{45} + E_6\rho_{46})}\right|_{x_1=0} = \frac{\sqrt{s}}{2} \\
\Rightarrow \quad & p_4^{\mathbb{S}} = E_4^{\mathbb{S}} \cdot (1, \sin\theta_4 \cos\varphi_4, \sin\theta_4 \sin\varphi_4, \cos\theta_4) \\
\Rightarrow \quad & p_3^{\mathbb{S}} = \Big(p_1 + p_2 - p_4^{\mathbb{S}} - p_5 - p_6\Big)\Big|_{x_1=0} = p_1 + p_2 - p_4^{\mathbb{S}}\,.
\end{aligned} \tag{8.21}
$$

Note that in the double-soft case the conditions $E_3 > 0$ and $E_4 > 0$ are always fulfilled and there is no need to check them explicitly.[3] We further compute weights $\mathcal{W}_{34}$, $\mathcal{W}_{56}^{(a)}$, partition function $w^{51,61}$, observable $\hat{O}$ with $x_1 = 0$ and p_1, p_2, $\vec{n}_5$, $\vec{n}_6$ and $p_3^{\mathbb{S}}$, $p_4^{\mathbb{S}}$. We show these formulas in the final result in Eq. (8.23). The required matrix element squared $|M_{\text{nnlo}}^{\text{tree}}|^2$ reads[4]

$$
\frac{1}{x_1^4} \lim_{x_1 \to 0} \Big(x_1^4\, |M_{\text{nnlo}}^{\text{tree}}|^2 \Big) \sim \text{Eik}\Big(p_1, p_4^{\mathbb{S}}, p_5, p_6\Big) \times \Big| M_{\text{lo}}^{\text{tree}}\Big(p_1, p_2, p_3^{\mathbb{S}}, p_4^{\mathbb{S}}\Big)\Big|^2\,. \tag{8.22}
$$

[3] It is easy to see from the independence of $p_3^{\mathbb{S}}$ and $p_4^{\mathbb{S}}$ in Eq. (8.21) on variables $\{x_2, x_3, x_4\}$ that this statement is true for all terms in Eq. (8.18) that contain the double-soft operator $\mathbb{S}$ in combination with arbitrary other operators.

[4] The double-soft eikonal function $\text{Eik}(p_1, p_4, p_5, p_6)$ in Eq. (8.22) can be found in Appendix B.2. We write a proportional sign because we neglect the strong coupling constant.

Hence, the double-soft contribution becomes

$$\begin{aligned} \$F_{\rm LM}(1,4\,|\,5,6) \;\to\; & x_1^3 x_2 x_3 \Big\{ \mathcal{W}_{34}(E_4^{\$},0,0,\rho_{45},\rho_{46}) \mathcal{W}_{56}(x_3,x_4,\lambda) \\ & \times\; w^{51,61}(\rho_{15},\rho_{45},\rho_{16},\rho_{46},\rho_{56}) \times \mathrm{Eik}\Big(p_1,p_4^{\$},p_5,p_6\Big) \left|M_{\rm lo}^{\rm tree}\Big(p_1,p_2,p_3^{\$},p_4^{\$}\Big)\right|^2 \\ & \times \hat{\mathcal{O}}\Big(p_3^{\$},p_4^{\$}\Big)\Big\}. \end{aligned} \tag{8.23}$$

As the last example, consider a term where both the double-soft operator $\$$ and the double-collinear operator C_{61} act on $F_{\rm LM}(1,4\,|\,5,6)$. Again, momenta p_5, p_6 and direction of p_4 are identical to the case $IF_{\rm LM}(1,4\,|\,5,6)$. To compute energy E_4 and p_3 we use Eqs. (8.14, 8.15) at $x_1 = x_4 = 0$ and obtain

$$\begin{aligned} E_4^{\$C_{61}} &= \left.\frac{s - 2\sqrt{s}\,(E_5+E_6) + 2E_5E_6\rho_{56}}{2\sqrt{s} - 2(E_5\rho_{45} + E_6\rho_{46})}\right|_{x_1=x_4=0} = \frac{\sqrt{s}}{2} \\ \Rightarrow\quad p_4^{\$C_{61}} &= E_4^{\$C_{61}} \cdot (1, \sin\theta_4\cos\varphi_4, \sin\theta_4\sin\varphi_4, \cos\theta_4) \\ \Rightarrow\quad p_3^{\$C_{61}} &= \Big(p_1+p_2-p_4^{\$C_{61}} - p_5 - p_6\Big)\Big|_{x_1=x_4=0} = p_1+p_2-p_4^{\$C_{61}}. \end{aligned} \tag{8.24}$$

Again there is not need to check $E_3 > 0$ and $E_4 > 0$. We then compute weights $\mathcal{W}_{34}$, $\mathcal{W}_{56}$, partition function $w^{51,61}$, observable $\hat{\mathcal{O}}$ at $x_1 = x_4 = 0$. The matrix element squared in the double-soft double-collinear limit is given by

$$\begin{aligned} & \frac{1}{x_1^4 x_4} \lim_{x_1,x_4\to 0} \Big(x_1^4 x_4 \,|M_{\rm nnlo}^{\rm tree}|^2 \Big) \\ & \quad\sim 4C_F^2\, \frac{p_1\cdot p_4^{\$C_{61}}}{E_6^2\rho_{16}\,(p_1\cdot p_5)(p_4^{\$C_{61}}\cdot p_5)} \times \left|M_{\rm lo}^{\rm tree}\Big(p_1,p_2,p_3^{\$C_{61}},p_4^{\$C_{61}}\Big)\right|^2. \end{aligned} \tag{8.25}$$

Hence, the double-soft double-collinear contribution to Eq. (8.19) reads

$$\begin{aligned} \$C_{61}F_{\rm LM}(1,4\,|\,5,6) \;\to\; & x_1^3 x_2 x_3 \Big\{ \mathcal{W}_{34}\Big(E_4^{\$C_{61}},0,0,\rho_{45},\rho_{14}\Big) \mathcal{W}_{56}^{(a)}(x_3,0,\lambda) \\ & \times\; w^{51,61}(\rho_{15},\rho_{45},0,\rho_{14},\rho_{15}) \times 4C_F^2\, \frac{p_1\cdot p_4^{\$C_{61}}}{E_6^2\rho_{16}\,(p_1\cdot p_5)(p_4^{\$C_{61}}\cdot p_5)} \\ & \times \left|M_{\rm lo}^{\rm tree}\Big(p_1,p_2,p_3^{\$C_{61}},p_4^{\$C_{61}}\Big)\right|^2 \times \hat{\mathcal{O}}\Big(p_3^{\$C_{61}},p_4^{\$C_{61}}\Big)\Big\}. \end{aligned} \tag{8.26}$$

The remaining 13 contributions, that appear in Eq. (8.18), are dealt with in the same way. Finally, we sum all contributions accounting for relative signs according to Eq. (8.18). The result is a numerical value of the differential cross section for a given phase space point. We repeat this procedure for all other sectors and partitions as well as for double-virtual and real-virtual contributions and subtraction terms. We present results from an numerical implementation that uses the Vegas algorithm [69] and follows the described procedure in the following chapter.

9. Results and their validation

In Chapters 5 - 7, we discussed the extraction of IR singularities from the double-real, real-virtual, double-virtual, and collinear renormalization contributions to the NNLO corrections, and presented analytic formulas for each of them. In this chapter, we will show the IR finite results that we obtained upon combining these formulas, and discuss how the analytic formulas for the subtraction terms were validated.

At NNLO we obtain an infrared finite result if we combine the quark-initiated processes $q + e^- \to e^- + q + gg$ and $q + e^- \to e^- + q + q'\bar{q}'$. We split this processes into finite, non-singlet and singlet contributions.[1] The finite contribution reads

$$2s \cdot \mathrm{d}\hat{\sigma}_{q,\mathrm{int}} = \left\langle F_{\mathrm{LM}}^{\mathrm{int}}\left(1_q, 4_q, 5_{q'}, 6_{\bar{q}'}\right)\right\rangle_{\delta}, \tag{9.1}$$

where $F_{\mathrm{LM}}^{\mathrm{int}}$ is defined in Eq. (6.220). Since remaining non-singlet and singlet contributions depend on different matrix elements, each of them is individually infrared finite. We present results for non-singlet (singlet) contributions in Section 9.1 (9.2) respectively. We also obtain an infrared finite result if we compute the gluon-initiated process $g + e^- \to e^- + q + \bar{q} + g$ and we show results for this process in Section 9.3. We discuss how we validated these results in Section 9.4.

9.1. Non-singlet contributions to the quark channel

We split the finite non-singlet cross section into terms with defined highest multiplicity in the final states and write

$$\mathrm{d}\hat{\sigma}_{q,\mathrm{ns}}^{\mathrm{nnlo}} = \mathrm{d}\hat{\sigma}_{q,\mathrm{ns},3\mathrm{j}}^{\mathrm{nnlo}} + \mathrm{d}\hat{\sigma}_{q,\mathrm{ns},2\mathrm{j}}^{\mathrm{nnlo}} + \mathrm{d}\hat{\sigma}_{q,\mathrm{ns},1\mathrm{j}}^{\mathrm{nnlo}}, \tag{9.2}$$

where $\mathrm{d}\hat{\sigma}_{q,\mathrm{ns},3\mathrm{j}}^{\mathrm{nnlo}}$ are contributions that contain matrix elements of the processes $q + e^- \to e^- + q + gg$ and $q + e^- \to e^- + q + q'\bar{q}'$, $\mathrm{d}\hat{\sigma}_{q,\mathrm{ns},2\mathrm{j}}^{\mathrm{nnlo}}$ are contributions that contain Born and one-loop matrix elements of the process $q + e^- \to e^- + q + g$ and $\mathrm{d}\hat{\sigma}_{q,\mathrm{ns},1\mathrm{j}}^{\mathrm{nnlo}}$ are contributions that contain Born, one-loop and two-loop matrix elements of the process $q + e^- \to e^- + q$. with up to two

[1] We refer to the discussion of the channel $q + e^- \to e^- + q + q' + \bar{q}'$ in Chapter 7 where we define singlet and non-singlet contributions. Note that, the channel $q + e^- \to e^- + q + g + g$ purely contributes to the non-singlet part.

loop corrections. The results read

$$
\begin{aligned}
2s \cdot \mathrm{d}\hat{\sigma}^{\mathrm{nnlo}}_{q,\mathrm{ns,3j}} = & \sum_{\substack{i,j \in \{1,4\} \\ i \neq j}} \Big\langle \hat{\mathcal{O}}^{(i,j)}_{\mathrm{nnlo}} \, [\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6j} \Big[F_{\mathrm{LM}}(1_q, 4_q \,|\, 5_g, 6_g) \\
& \qquad + F_{\mathrm{LM,ns}}(1_q, 4_q \,|\, 5_{q'\bar{q}'}, 6_{\bar{q}'q'}) \Big] \Big\rangle_\delta \\
+ & \sum_{i \in \{1,4\}} \Big\langle \hat{\mathcal{O}}^{(i)}_{\mathrm{nnlo}} \, [\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6i} \Big[F_{\mathrm{LM}}(1_q, 4_q \,|\, 5_g, 6_g) + F_{\mathrm{LM,ns}}(1_q, 4_q \,|\, 5_{q'\bar{q}'}, 6_{\bar{q}'q'}) \Big] \Big\rangle_\delta ,
\end{aligned}
\tag{9.3}
$$

$$
\begin{aligned}
2s \cdot \mathrm{d}\hat{\sigma}^{\mathrm{nnlo}}_{q,\mathrm{ns,2j}} = & \sum_{i \in \{1,4\}} \Big\langle \hat{\mathcal{O}}^{(i)}_{\mathrm{nlo}} w^{5i} F^{\mathrm{fin}}_{\mathrm{LV}}(1_q, 4_q \,|\, 5_g) \Big\rangle_\delta + \frac{\alpha_s(\mu)}{2\pi} \sum_{i \in \{1,4\}} \int_0^1 \mathrm{d}z \, \Big\langle \hat{\mathcal{O}}^{(i)}_{\mathrm{nlo}} w^{5i} \\
& \times \Big\{ \mathcal{P}'_{qq}(z) + \Big[\ln\Big(\frac{4E_1^2}{\mu^2}\Big) - \tilde{\Delta}'_{61} \Big] \hat{P}^{(0)}_{qq}(z) \Big\} \frac{F_{\mathrm{LM}}(z \cdot 1_q, 4_q \,|\, 5_q)}{z} \Big\rangle_\delta \\
+ & \frac{\alpha_s(\mu)}{2\pi} \sum_{i \in \{1,4\}} \Big\langle \hat{\mathcal{O}}^{(i)}_{\mathrm{nlo}} w^{5i} \Big\{ (2C_F - C_A) \mathcal{S}^{E_5}_{14} + C_A (\mathcal{S}^{E_5}_{15} + \mathcal{S}^{E_5}_{45}) + \gamma'_q + \gamma'_g \\
& + \sum_{j \in \{1,4,5\}} \tilde{\Delta}'_{6j} \Big[\gamma_j + 2C_F \ln\Big(\frac{E_5}{E_j}\Big) \Big] \Big\} F_{\mathrm{LM}}(1_q, 4_q \,|\, 5_g) \Big\rangle_\delta \\
+ & \frac{\alpha_s(\mu)}{2\pi} \, \gamma_{k_\perp,g} \sum_{i \in \{1,4\}} \Big\langle [I - S_5]\,[I - C_{5i}] \Big[r^{(i)}_\mu r^{(i)}_\nu + \frac{1}{2} g_{\mu\nu} \Big] w^i_{\mathrm{tc}} F^{\mu\nu}_{\mathrm{LM}}(1_q, 4_q \,|\, 5_g) \Big\rangle ,
\end{aligned}
\tag{9.4}
$$

$$
\begin{aligned}
2s \cdot \mathrm{d}\hat{\sigma}^{\mathrm{nnlo}}_{q,\mathrm{ns,1j}} = & \big\langle F^{\mathrm{fin}}_{\mathrm{LVV}}(1_q, 4_q) \big\rangle_\delta + \big\langle F^{\mathrm{fin}}_{\mathrm{LV}^2}(1_q, 4_q) \big\rangle_\delta \\
+ & \frac{\alpha_s(\mu)}{2\pi} \int_0^1 \mathrm{d}z \, \Big\{ \mathcal{P}'_{qq}(z) + \ln\Big(\frac{4E_1^2}{\mu^2}\Big) \hat{P}^{(0)}_{qq}(z) \Big\} \Big\langle \frac{F^{\mathrm{fin}}_{\mathrm{LV}}(z \cdot 1_q, 4_q)}{z} \Big\rangle_\delta \\
+ & \frac{\alpha_s(\mu)}{2\pi} \Big\{ 2C_F \, \mathcal{S}^{E_{\max}}_{14} + \gamma'_q \Big\} \big\langle F^{\mathrm{fin}}_{\mathrm{LV}}(1_q, 4_q) \big\rangle_\delta + \Big(\frac{\alpha_s(\mu)}{2\pi}\Big)^2 \Big\langle \Big\{ \Delta_{\mathrm{ns}}(E_1, E_4, E_{\max}, \eta_{14}) \\
& + C_F \big(\delta_{k_\perp,g} \langle r^\mu r^\nu \rangle_{\rho_5} - \delta_g \langle \Delta_{64} \rangle''_{S_5} + \widetilde{\gamma}_q(E_4, E_{\max}) \langle \Delta_{64} \rangle''_{S_5} \big) \Big\} F_{\mathrm{LM}}(1_q, 4_q) \Big\rangle_\delta \\
+ & \Big(\frac{\alpha_s(\mu)}{2\pi}\Big)^2 \int_0^1 \mathrm{d}z \, \Big\langle \big\{ C_F \, \widetilde{\mathcal{P}}_{qq}(E_1, E_{\max}, z) \langle \Delta_{61} \rangle''_{S_5} + \mathcal{T}_{\mathrm{ns}}(E_1, E_4, E_{\max}, \eta_{14}, z) \big\} \\
& \times \frac{F_{\mathrm{LM}}(z \cdot 1_q, 4_q)}{z} \Big\rangle_\delta .
\end{aligned}
\tag{9.5}
$$

The functions $\mathcal{T}_{\mathrm{ns}}$ and Δ_{ns} can be found in Appendix J.1. NNLO functions $F_{\mathrm{LM}}(1,4\,|\,5,6)$ and $F_{\mathrm{LM,ns}}(1,4\,|\,5,6)$ are defined in Eqs. (6.4, 6.222), NLO functions $F_{\mathrm{LM}}(1,4\,|\,5)$, $F^{\mu\nu}_{\mathrm{LM}}(1,4\,|\,5)$ are defined in Eqs. (5.4, 6.117), the finite remainder of the real-virtual contribution $F^{\mathrm{fin}}_{\mathrm{LV}}$ is defined in Eq. (6.186), the LO function $F_{\mathrm{LM}}(1,4)$ is defined in Eq. (4.5) and one-loop and two-loop finite remainders $F^{\mathrm{fin}}_{\mathrm{LV}}(1,4)$, $F^{\mathrm{fin}}_{\mathrm{LVV}}(1,4)$ and $F^{\mathrm{fin}}_{\mathrm{LV}^2}(1,4)$ are implicitly defined in Eqs. (5.58, 6.206). Explicit expressions for vectors $r^{(i)}$, with $i \in \{1,4\}$, can be found in Appendix F.3. NNLO operators $\hat{\mathcal{O}}^{(i)}_{\mathrm{nnlo}}$ and $\hat{\mathcal{O}}^{(i,j)}_{\mathrm{nnlo}}$ are defined in Eqs. (6.18, 6.19) and the NLO operator $\hat{\mathcal{O}}^{(i)}_{\mathrm{nlo}}$ is defined in Eq. (5.15). Partition functions $w^{5i,6j}$ can be found in Appendix A.3.

The Altarelli-Parisi splitting function $\hat{P}^{(0)}_{qq}$ and the generalized splitting function $\mathcal{P}'_{qq}$ can be found in Appendices E.4 and E.5. In Eq. (9.4) we used notations γ_i and C_i where $\gamma_i = \gamma_q(\gamma_g)$ and $C_i = C_F(C_A)$ if i labels a quark(gluon) where γ_q and γ_g are the LO quark and gluon cusp anomalous dimensions given by

$$\gamma_q = \frac{3}{2}C_F\,, \quad \gamma_g = \frac{11}{6}C_A - \frac{2}{3}T_R N_f\,. \tag{9.6}$$

The generalized anomalous dimension γ_q' can be found in Appendix E.

In the NLO infrared-finite result Eq. (5.68) we defined the function $\mathcal{S}_{14}^{E_{\max}}$, see Eq. (5.69). In the NNLO result in Eq. (9.4) we introduced a generalization $\mathcal{S}_{ij}^{E}$ of this function as

$$\begin{aligned}\mathcal{S}_{ij}^{E} \equiv \mathrm{Li}_2(1-\eta_{ij}) - \frac{\pi^2}{6} + \frac{\pi^2\lambda_{ij}}{2} + \frac{1}{2}\ln^2\left(\frac{E_i}{E_j}\right) - \ln\eta_{ij}\ln\left(\frac{E_iE_j}{E^2}\right) \\ + \frac{1}{2}\left[\frac{\gamma_i}{C_i}\ln\left(\frac{E_j\eta_{ji}}{E_i}\right) + \frac{\gamma_j}{C_j}\ln\left(\frac{E_i\eta_{ij}}{E_j}\right)\right],\end{aligned} \tag{9.7}$$

where $\lambda_{ij} = 1$ if partons i and j are both in the initial or both in the final state and $\lambda_{ij} = 0$ otherwise. γ_i and C_i in Eq. (9.7) are defined above Eq. (9.6). The quantities $\tilde{\Delta}_{ij}'$ are remainders of partitions functions defined as

$$\tilde{\Delta}_{6i}' \equiv -w_{\mathrm{tc}}^{5i}\ln\left(\frac{\eta_{i5}}{2}\right), \quad \tilde{\Delta}_{65}' \equiv -\sum_{j\in\{1,4\}} w_{\mathrm{tc}}^{j}\ln\left(\frac{\eta_{j5}}{4(1-\eta_{j5})}\right), \tag{9.8}$$

with $i \in \{1,4\}$. We discuss the computation of the only partition-dependent functions $\langle\Delta_{ij}\rangle_{S_5}''$ and $\langle r^\mu r^\nu\rangle_{\rho_5}$ in Appendix H. In addition we defined another generalized energy-dependent splitting function

$$\begin{aligned}\widetilde{\mathcal{P}}_{qq}(z,E_1,E_{\max}) = -C_F\Bigg\{2\mathcal{D}_1(z) - (1+z)\ln(1-z) + \ln\left(\frac{E_1}{E_{\max}}\right)\Bigg[2\mathcal{D}_0(z) - (1+z) \\ + \delta(1-z)\ln\left(\frac{E_1}{E_{\max}}\right)\Bigg]\Bigg\},\end{aligned} \tag{9.9}$$

and anomalous dimension

$$\widetilde{\gamma}_q(E_4,E_{\max}) = C_F\left[-\frac{7}{4} + \frac{3}{2}\ln\left(\frac{E_4}{E_{\max}}\right) - \ln^2\left(\frac{E_4}{E_{\max}}\right)\right]. \tag{9.10}$$

9.2. Singlet contributions to the quark channel

We continue with the results of the quark singlet contributions. As in the case of non-singlet contributions, we split the cross section as follows

$$\mathrm{d}\hat{\sigma}_{q,\mathrm{s}}^{\mathrm{nnlo}} = \mathrm{d}\hat{\sigma}_{q,\mathrm{s},3\mathrm{j}}^{\mathrm{nnlo}} + \mathrm{d}\hat{\sigma}_{q,\mathrm{s},2\mathrm{j}}^{\mathrm{nnlo}} + \mathrm{d}\hat{\sigma}_{q,\mathrm{s},1\mathrm{j}}^{\mathrm{nnlo}}\,. \tag{9.11}$$

We obtain for the contributions on the right-hand side of Eq. (9.11) the following results

$$
\begin{aligned}
2s \cdot \mathrm{d}\hat{\sigma}^{\mathrm{nnlo}}_{q,s,3\mathrm{j}} = & \sum_{i \in \{1,4\}} \Big\langle \Big[\theta^{(a)} \left[I - C_{6i} \right] + \theta^{(b)} I + \theta^{(c)} \left[I - C_{5i} \right] + \theta^{(d)} I \Big] \\
& \times [\mathrm{d}p_5][\mathrm{d}p_6] \left[I - \mathbb{C}_i \right] w^{5i,6i} F_{\mathrm{LM,s}}(1,4,5,6) \Big\rangle_\delta \\
& + \sum_{\substack{i,j \in \{1,4\} \\ i \neq j}} \left\langle \left[I - C_{5i} \right] \left[I - C_{6j} \right] [\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6j} F_{\mathrm{LM,s}}(1,4,5,6) \right\rangle_\delta ,
\end{aligned}
\tag{9.12}
$$

$$
\begin{aligned}
2s \cdot \mathrm{d}\hat{\sigma}^{\mathrm{nnlo}}_{q,s,2\mathrm{j}} = & \frac{\alpha_s(\mu)}{2\pi} \int_0^1 \mathrm{d}z \, \Big\langle \left[I - C_{51} \right] \Big\{ \mathcal{P}'_{gq}(z) + \Big[\ln\left(\frac{4E_1^2}{\mu^2} \right) - \tilde{\Delta}'_{61} \Big] \hat{P}^{(0)}_{gq}(z) \Big\} \\
& \times \frac{F_{\mathrm{LM}}(z \cdot 1_g, 4_{q\bar{q}} \,|\, 5_{\bar{q}q})}{z} \Big\rangle_\delta ,
\end{aligned}
\tag{9.13}
$$

$$
2s \cdot \mathrm{d}\hat{\sigma}^{\mathrm{nnlo}}_{q,s,1\mathrm{j}} = \left(\frac{\alpha_s(\mu)}{2\pi} \right)^2 \int_0^1 \mathrm{d}z \, \mathcal{T}_{\mathrm{s}}(E_1, z) \sum_{f \in \{q,\bar{q}\}} \left\langle \frac{F_{\mathrm{LM}}(z \cdot 1_f, 4_f)}{z} \right\rangle_\delta .
\tag{9.14}
$$

The function $\mathcal{T}_{\mathrm{s}}$ can be found in Appendix J.2. The NNLO function $F_{\mathrm{LM,s}}(1,4,5,6)$ is defined in Eq. (6.229), the NLO function $F_{\mathrm{LM},g}(1,4\,|\,5)$ is defined in Eq. (5.78) and the LO functions $F_{\mathrm{LM}}(1,4)$ is defined in Eq. (4.5). The splitting function $\hat{P}^{(0)}_{gq}$ and the generalized splitting function $\mathcal{P}'_{gq}$ can be found in Appendix E.4 and E.5. The functions $\tilde{\Delta}'_{ij}$ are shown in Eq. (9.8).

9.3. Gluon channel

We now show result for the contributions from the gluon-initiated process $g + e^- \to e^- + q\bar{q} + g$. Again, we write the cross section as follows

$$
\mathrm{d}\hat{\sigma}^{\mathrm{nnlo}}_g = \mathrm{d}\hat{\sigma}^{\mathrm{nnlo}}_{g,3\mathrm{j}} + \mathrm{d}\hat{\sigma}^{\mathrm{nnlo}}_{g,2\mathrm{j}} + \mathrm{d}\hat{\sigma}^{\mathrm{nnlo}}_{g,1\mathrm{j}} .
\tag{9.15}
$$

The three terms on the right-hand side read

$$
\begin{aligned}
2s \cdot \mathrm{d}\hat{\sigma}^{\mathrm{nnlo}}_{g,3\mathrm{j}} = & \sum_{\substack{i,j=1,4 \\ i \neq j}} \left\langle \hat{O}^{(i,j)}_{\mathrm{nnlo,g}} [\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6j} F_{\mathrm{LM}}(1_g, 4_{q\bar{q}} \,|\, 5_{\bar{q}q}, 6_g) \right\rangle_\delta \\
& + \sum_{i=1,4} \left\langle \hat{O}^{(i)}_{\mathrm{nnlo,g}} [\mathrm{d}p_5][\mathrm{d}p_6] w^{5i,6i} F_{\mathrm{LM}}(1_g, 4_{q\bar{q}} \,|\, 5_{\bar{q}q}, 6_g) \right\rangle_\delta ,
\end{aligned}
\tag{9.16}
$$

$$
\begin{aligned}
2s \cdot \mathrm{d}\hat{\sigma}^{\mathrm{nnlo}}_{g,2\mathrm{j}} = & \left\langle \left[I - C_{51} \right] F^{\mathrm{fin}}_{\mathrm{LV}}(1_g, 4_{q\bar{q}} \,|\, 5_{\bar{q}q}) \right\rangle_\delta + \frac{\alpha_s(\mu)}{2\pi} \sum_{i \in \{1,4\}} \int_0^1 \mathrm{d}z \Big\langle \hat{\mathcal{O}}^{(i)}_{\mathrm{nlo}} w^{5i} \\
& \times \Big\{ \mathcal{P}'_{qg}(z) + \Big[\ln\left(\frac{4E_1^2}{\mu^2} \right) - \tilde{\Delta}'_{61} \Big] \hat{P}^{(0)}_{qg}(z) \Big\} \sum_{f \in \{q,\bar{q}\}} \frac{F_{\mathrm{LM}}(z \cdot 1_f, 4_f \,|\, 5_g)}{z} \Big\rangle_\delta
\end{aligned}
$$

$$+\frac{\alpha_s(\mu)}{2\pi}\int_0^1 \mathrm{d}z\,\bigg\langle [I-C_{51}]\bigg\{\mathcal{P}'_{gg}(z)+\bigg[\ln\left(\frac{4E_1^2}{\mu^2}\right)-\tilde{\Delta}'_{61}\bigg]\hat{P}^{(0)}_{gg}(z)\bigg\} \qquad (9.17)$$
$$\times\frac{F_{\mathrm{LM}}(1_g,4_{q\bar{q}}\,|\,5_{\bar{q}q})}{z}\bigg\rangle_\delta+\frac{\alpha_s(\mu)}{2\pi}\bigg\langle [I-C_{51}]\bigg\{(2C_F-C_A)\,\mathcal{S}^{E_{\max}}_{45}$$
$$+C_A(\mathcal{S}^{E_{\max}}_{14}+\mathcal{S}^{E_{\max}}_{15})+2\gamma'_q+\sum_{i\in\{1,4,5\}}\tilde{\Delta}'_{6i}\bigg[\gamma_i+2C_i\ln\left(\frac{E_{\max}}{E_i}\right)\bigg]\bigg\}$$
$$\times F_{\mathrm{LM}}(1_g,4_{q\bar{q}}\,|\,5_{\bar{q}q})\bigg\rangle_\delta,$$

$$2s\cdot\mathrm{d}\hat{\sigma}^{\mathrm{nnlo}}_{g,1\mathrm{j}}=\frac{\alpha_s(\mu)}{2\pi}\int_0^1\mathrm{d}z\left\{\mathcal{P}'_{qg}(z)+\ln\left(\frac{4E_1^2}{\mu^2}\right)\hat{P}^{(0)}_{qg}(z)\right\}\sum_{f\in\{q,\bar{q}\}}\left\langle\frac{F^{\mathrm{fin}}_{\mathrm{LV}}(z\cdot 1_f,4_f)}{z}\right\rangle_\delta$$
$$+\left(\frac{\alpha_s(\mu)}{2\pi}\right)^2\int_0^1\left\langle\mathcal{T}_g(E_1,E_4,E_{\max},\eta_{14},z)\sum_{f\in\{q,\bar{q}\}}\frac{F_{\mathrm{LM}}(z\cdot 1_f,4_f)}{z}\right\rangle_\delta. \qquad (9.18)$$

The function $\mathcal{T}_g$ is given in Appendix J.3. The NNLO function $F_{\mathrm{LM},g}(1,4\,|\,5,6)$ is defined in Eq. (7.5), NLO functions $F_{\mathrm{LM}}(1,4\,|\,5)$ and $F_{\mathrm{LM},g}(1,4\,|\,5)$ are defined in Eqs. (5.4, 5.78) and the LO function $F_{\mathrm{LM}}(1,4)$ and the one-loop finite remainder $F^{\mathrm{fin}}_{\mathrm{LV}}(1,4)$ are defined in Eqs. (4.5, 5.54). NNLO operators $\hat{\mathcal{O}}^{(i)}_{\mathrm{nnlo},g}$ and $\hat{\mathcal{O}}^{(i,j)}_{\mathrm{nnlo},g}$ are defined in Eqs. (7.8, 7.7) and the NLO operator $\hat{\mathcal{O}}^{(i)}_{\mathrm{nlo}}$ is defined in Eq. (5.15). Partition functions $w_g^{5i,6j}$ are given in Appendix A.3. All the (generalized) splitting functions and anomalous dimensions can be found in Appendix E, functions $\tilde{\Delta}'_{ij}$ are shown in Eq. (9.8) and γ_i and C_i are understood as in the previous sections.

9.4. Numerical validation of the subtraction terms

In what follows we describe how we checked finite remainders of the subtraction terms presented in the previous sections. It is possible to check these terms numerically by comparing them to known NNLO QCD corrections to the inclusive cross sections for deep-inelastic scattering process $P+e^-\to e^-+X$. For comparison we use the program HOPPET [9,76,77] where analytic formulas for DIS NNLO QCD coefficient functions [64–66] are implemented. Since our goal is to check analytic results for the subtraction terms and not to discuss DIS phenomenology, we only implement the simplest setup of our fully differential description that allows a thorough check of the subtraction formulas. We describe this setup below.

We consider initial states that contain a single quark flavour and/or a gluon. For the sake of definiteness, we have chosen this quark to be an *up-quark*. In the final state, we allowed for contributions from 5 massless quark flavours (2 up, 3 down). We consider DIS process mediated by a virtual photon.

We use the following parameters for numerical evaluation. We chose the hadronic center-of-mass energy to be $\sqrt{s}=100\,\mathrm{GeV}$. To avoid on-shell photon exchange, we restrict momentum transfer $q^2=-Q^2$ from electron to proton to the interval $10\,\mathrm{GeV}<Q<100\,\mathrm{GeV}$. We use the NNPDF3.0 PDF set [78] as implemented in LHAPDF [79]. We use values of the strong coupling

NLO, quark		
μ (GeV)	numeric (pb)	analytic (pb)
50	59.1(2)	59.0
200	143.5(2)	143.3

NLO, gluon		
μ (GeV)	numeric (pb)	analytic (pb)
50	-222.6(1)	-222.6
200	-373.2(1)	-373.1

Tab. 9.1.: Results obtained for different choices of the factorization and renormalization scale, $\mu \in \{50\,\text{GeV}, 200\,\text{GeV}\}$. Both parton distribution functions and the strong coupling are still evaluated at $\mu = 100\,\text{GeV}$.

constant provided by NNPDF. We set the renormalization and factorization scales to a fixed value $\mu \equiv \mu_R = \mu_F = Q_{\max} = 100\,\text{GeV}$.

We write the inclusive partonic cross sections as

$$\sigma_{\text{nnlo}} = \sigma_{\text{lo}} + \sum_{i\in\{q,g\}} \left[\Delta\sigma_{\text{nlo},i} + \Delta\sigma_{\text{nnlo},i}\right] + \mathcal{O}(\alpha_s^3)\,, \tag{9.19}$$

and present results for LO, NLO and NNLO contributions separately.

At LO, we find[2]

$$\sigma_{\text{lo}}^{\text{num}} = 1418.89(1)\,\text{pb}\,, \quad \sigma_{\text{lo}}^{\text{an}} = 1418.89\,\text{pb}\,, \tag{9.20}$$

where the superscript *num* indicates results obtained numerically from the fully differential description using the nested soft-collinear subtraction scheme and the superscript *an* indicates the result obtained by using HOPPET. We note that the agreement between the numerical and the analytic LO results in Eq. (9.20) is prefect.

For the NLO contributions, we obtain

$$\Delta\sigma_{\text{nlo},q}^{\text{num}} = 101.16(4)\,\text{pb}\,, \quad \Delta\sigma_{\text{nlo},q}^{\text{an}} = 101.12\,\text{pb}\,, \quad \text{(quark-initiated)} \tag{9.21}$$

and

$$\Delta\sigma_{\text{nlo},g}^{\text{num}} = -297.90(1)\,\text{pb}\,, \quad \Delta\sigma_{\text{nlo},g}^{\text{an}} = -297.91\,\text{pb}\,. \quad \text{(gluon-initiated)} \tag{9.22}$$

We observe that the agreement is better than a permill, and within the Monte Carlo integration error which is of the same magnitude. In order to check the scale dependence of our NLO results, we also used different values for the renormalization and factorization scales $\mu \in \{50\,\text{GeV}, 200\,\text{GeV}\}$, for which we find a similar level of agreement, see Table 9.1.

We continue with the discussion of the NNLO contribution. Analytic results for quark-initiated channels are available for singlet and non-singlet contributions separately. To stress-test our formulas as much as possible, we split the fully differential calculation in the same

[2]Analytical results are obtained from a direct integration of analytic DIS coefficient functions. However, we do not show Monte Carlo errors of this computations because this error is always negligible.

NNLO, quark, singlet		
μ (GeV)	numeric (pb)	analytic (pb)
50	3.87(1)	3.86
200	16.47(2)	16.47

NNLO, quark, non-singlet		
μ (GeV)	numeric (pb)	analytic (pb)
50	9.16(3)	9.18
200	40.1(3)	40.2

NNLO, gluon		
μ (GeV)	numeric (pb)	analytic (pb)
50	-79.9(4)	-79.6
200	-225.2(4)	-224.8

Tab. 9.2.: Results obtained for different choices of the factorization and renormalization scale, $\mu \in \{50\,\text{GeV}, 200\,\text{GeV}\}$. We show results for quark-initiated and gluon-initiated contributions to the NNLO total cross section. Singlet and non-singlet quark-initiated contributions are shown individually. We use $N_f = 5$.

way[3]

$$\Delta\sigma_{\text{nnlo},q} = \Delta\sigma_{\text{nnlo},q,\text{ns}} + \Delta\sigma_{\text{nnlo},q,\text{s}}\,, \tag{9.23}$$

and compare the two contributions separately. To check the dependence of the non-singlet contribution on the number of light flavours N_f, we computed N_f-dependent and N_f-independent contributions separately. Our results read

$$\Delta\sigma^{\text{num}}_{\text{nnlo},q,\text{ns}} = \left[33.1(2) - 2.18(1)\cdot N_f\right]\text{pb}\,,\quad \Delta\sigma^{\text{an}}_{\text{nnlo},q,\text{ns}} = \left[33.1 - 2.17\cdot N_f\right]\text{pb}\,, \tag{9.24}$$

and for the singlet contribution

$$\Delta\sigma^{\text{num}}_{\text{nnlo},q,\text{s}} = 9.19(2)\,\text{pb}\,,\quad \Delta\sigma^{\text{an}}_{\text{nnlo},q,\text{s}} = 9.18\,\text{pb}\,. \tag{9.25}$$

For gluon-initiated process we find

$$\Delta\sigma^{\text{num}}_{\text{nnlo},g} = -142.4(4)\,\text{pb}\,,\quad \Delta\sigma^{\text{an}}_{\text{nnlo},g} = -142.7\,\text{pb}\,. \tag{9.26}$$

We note that the agreement is at the level of a few permill, and the numerical and analytic results are always compatible within the error. We also computed the contributions shown in Eqs. (9.24 - 9.26) for other choices of the scale μ and found a similar level of agreement. Some numerical results that illustrate these checks are collected in Table 9.2. However, to be certain that we do not miss deviations in contributions that are too small to be noticed in the full results shown in Eqs. (9.24 - 9.26) and Table 9.2, we have also computed the coefficients of $\ln^n(\mu^2)$ for

[3]We also defined a, by construction, finite contribution in Eq. (6.220). Note that this contribution vanishes when computing inclusive quantities and is, therefore, not further discussed.

NNLO, quark, singlet, $\ln^n(\mu^2/\mu_0^2)$		
n	numeric (pb)	analytic (pb)
0	9.16(2)	9.18
1	4.54(1)	4.55
2	0.514(1)	0.513

Tab. 9.3.: Results obtained for individual *coefficients* of $\ln^n(\mu^2/\mu_0^2)$, $n \in \{0,1,2\}$, for the choice $\mu_0 = 100\,\text{GeV}$, in the quark-singlet contribution. Note that, given this choice of μ_0, for $\mu = 100\,\text{GeV}$ the only non-vanishing logarithm is given for $n = 0$ and this coefficient should therefore coincide with the total result obtained in Eq. (9.25), which is indeed true.

NNLO, gluon, $E_{\max} = n \cdot \sqrt{s}$				
n	double-real	subtractions	real-virtual	total (pb)
1	-2.1(1)	-141.2(4)	0.898(1)	-142.4(4)
2	-8.7(2)	-135.0(4)		-142.7(4)
3	-12.7(2)	-131.0(4)		-142.8(4)
4	-15.3(2)	-128.1(4)		-142.5(4)
5	-17.4(2)	-126.0(4)		-142.5(4)

Tab. 9.4.: Results obtained for NNLO gluon-initiated contributions for different values of the parameter $E_{\max}$. We chose $E_{\max} = n \cdot \sqrt{s}$ to be a multiple $n = \{1,2,3,4,5\}$ of the partonic center-of-mass energy $\sqrt{s} = 100\,\text{GeV}$. For comparison, the HOPPET (analytic) value is given by $\Delta\sigma^{\text{an}}_{\text{nnlo},g} = -142.7\,\text{GeV}$. In the second, third and fourth column we show the results split into regularized double-real contributions, integrated subtractions and regularized real-virtual contributions. The $E_{\max}$-independent total result for $\Delta\sigma^{\text{num}}_{\text{nnlo},g}$ is shown in the last column. The real-virtual contribution is $E_{\max}$ independent and we only show one value for $n = 1$. However, it can be seen nicely how the double-real contribution, which implicitly depends on $E_{\max}$, decreases while the subtractions contribution, which explicitly depends on $E_{\max}$, increases by the same amount with growing $E_{\max}$.

$n \in \{0,1,2\}$ individually. We obtained permill agreement for all coefficients. We show such results in the case of quark singlet contribution, in Table 9.3.

We recall that, in the construction of the subtraction terms we introduced an explicit energy cut-off $E_{\max}$ into the phase-space volume element of final-state particles. Subtraction terms do explicitly depend on the parameter $E_{\max}$, but this dependence has to cancel with an implicit $E_{\max}$ dependence in the regulated resolved contributions that are computed numerically so that the physical result is $E_{\max}$-independent. To check that this is the case, we varied $E_{\max}$ in the numerical implementation and found a remarkably stable result. As an example, we show results for the gluon-initiated contribution for various values of $E_{\max}$ in Table 9.4.

We note that we also compared numerical and analytic results for coefficients of individual colour factors that appear in different partonic channels and found a permill agreement for all of them. Hence, we believe that extensive checks described above establish the validity of subtraction terms derived in this thesis.

Finally, we comment on the numerical efficiency of our implementation. All results presented in this chapter are computed to permill precision and required $\mathcal{O}(1000)$ CPU hours of running. It is certainly possible to improve on the numerical efficiency by, for example, optimizing the parametrization of the Born phase space. However, for phenomenology, the NNLO *contributions* to the partonic cross sections do not need to be known with permill precision. Since intended permill precision on the full NNLO total cross section Eq. (9.19) corresponds to a few percent precision on the NNLO contributions, the latter can be computed much faster. Indeed, we find that we can get permill precision on the total cross section Eq. (9.19) already after $\sim 50\,$CPU hours.

10. Conclusion

In this thesis we applied the nested soft-collinear subtraction scheme to the description of deep inelastic scattering process through NNLO in perturbative QCD. This is the first application of this subtraction scheme to a situation where colour charged particles appear both in initial and final states at leading order. As such, these results provide an important building block that will enable application of the nested subtraction scheme to arbitrary processes at the LHC.

As the name suggests, the nested soft-collinear subtraction is built on the premise that nested subtraction of soft and collinear limits is sufficient to regulate all singularities in real emission matrix elements that appear in NNLO QCD computations. Since soft and collinear limits of QCD amplitudes are universal, all singularities of real emission matrix elements can be described independent of hard matrix elements. Together with Catani's formula for virtual corrections, which likewise describes infrared poles of loops amplitudes using universal building blocks, results for infrared divergences of real emission processes allow for an explicit demonstration of the cancellation of infrared and collinear singularities independent of hard matrix elements. After infrared and collinear subtraction is done, real emission matrix elements become finite in four-dimensional space time and can be used to compute arbitrary infrared-safe observables. In this thesis this program was carried out for deep inelastic scattering but we expect that the results of this thesis can be used to explicitly demonstrate the cancellation of infrared and collinear singularities at NNLO QCD for arbitrary hard scattering processes at the LHC.

The two main results of this thesis are *i)* analytic formulas that provide integrated subtraction terms for the deep inelastic scattering process; and *ii)* a formula for regulated fully-differential partonic cross sections for DIS that admits straightforward numerical implementation. The analytic formulas were validated through a comparison of the results of our computation and the known formulas that describe inclusive NNLO QCD corrections for partonic cross sections in DIS. We have carried out such a comparison for all partonic channels and for different color factors, using different numerical values for input parameters, and found excellent agreement in all cases. This makes us confident that they are correct. We note that, for fully-differential descriptions of complex LHC processes with high multiplicities, efficient numerical evaluation is necessary and our implementation of the subtraction terms is quite promising. Indeed, we observed that we obtain permill precision on the NNLO total cross section σ_{nnlo}, which corresponds to a few percent precision on the NNLO QCD contribution $\Delta\sigma_{\text{nnlo}}$, after running for only $\mathcal{O}(50)$ CPU hours.

Thanks to the fact that singularities of QCD amplitudes are independent of hard matrix elements, the obtained analytic formulas for subtractions are, to a large extend, universal and can be used to construct subtractions for more complex processed. They can be directly applied to any DIS-like process with one parton in the initial state and one parton in the final state. In computations of processes with higher numbers of external momenta the results obtained in this thesis serve as important building blocks.

The nested soft-collinear subtraction scheme has already been used to describe production and decay of colour-singlet states [37,38] through NNLO QCD. At leading order these processes contain colour charged particles only in initial or final states. The subtraction terms for NNLO QCD corrections to deep-inelastic scattering, which are presented in this thesis, allow us to extend this subtraction scheme to processes that also involve partons both in initial and in final states. This is a crucial step in extending the nested soft-collinear subtraction scheme to arbitrary processes at the LHC.

A. Definitions and Notation

In this appendix we collect definitions and notations that are used in the main text.

A.1. Renormalized strong coupling constant

We often express intermediate results in terms of the following quantity

$$[\alpha_{s,b}] \equiv \left[\frac{\alpha_{s,b}}{2\pi}\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\right] = \left[\frac{g_{s,b}^2}{8\pi^2}\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\right], \tag{A.1}$$

where $\alpha_{s,b} = g_{s,b}^2/(4\pi)$ is the bare QCD coupling constant. The relation between bare and $\overline{\text{MS}}$ QCD coupling constants reads

$$\alpha_{s,b} S_{\epsilon} = \alpha_s(\mu)\mu^{2\epsilon}\left[1 - \frac{\alpha_s(\mu)}{2\pi}\frac{\beta_0}{\epsilon} + \mathcal{O}(\alpha_s^2)\right]. \tag{A.2}$$

In Eq. (A.2) μ is the renormalization scale. Also, we use

$$S_{\epsilon} = (4\pi)^{\epsilon} e^{-\epsilon\gamma_E}, \quad \beta_0 = \frac{11}{6}C_A - \frac{2}{3}T_R N_f, \tag{A.3}$$

$C_A = 3$, $T_R = 1/2$. N_F stands for the number of massless quark flavours. We also use

$$[\alpha_{s,b}] = [\alpha_s]\mu^{2\epsilon}\left[1 - \frac{\alpha_s(\mu)}{2\pi}\frac{\beta_0}{\epsilon} + \mathcal{O}(\alpha_s^2)\right], \tag{A.4}$$

where

$$[\alpha_s] \equiv \left[\frac{\alpha_s(\mu)}{2\pi}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\right]. \tag{A.5}$$

A.2. Four-momenta and scalar products

The momentum p_i of a massless parton i is written as

$$p_i^{\mu} = E_i\, n_i^{\mu}, \quad n_i = \begin{pmatrix} 1 \\ \vec{n}_i \end{pmatrix}, \quad \vec{n}_i^2 = 1. \tag{A.6}$$

The light-like vector n_i^μ can be written as

$$n_i^\mu = t^\mu + e_i^\mu\,, \quad t \equiv \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}\,, \quad e_i = \begin{pmatrix} 0 \\ \vec{n}_i \end{pmatrix}\,. \tag{A.7}$$

The product of two four-momenta p_i and p_j is written as

$$p_i \cdot p_j = E_i E_j \,(1 - \vec{n}_i \cdot \vec{n}_j) = E_i E_j \,\rho_{ij} = 2 E_i E_j \eta_{ij}\,, \tag{A.8}$$

with

$$\rho_{ij} = 1 - \vec{n}_i \cdot \vec{n}_j\,, \quad \eta_{ij} = \frac{\rho_{ij}}{2}\,. \tag{A.9}$$

A.3. Partition functions and angular sectors

In this appendix we collect various partition functions introduced in NLO and NNLO QCD computations.

A.3.1. NLO partition functions

At NLO we use

$$w^{51} = \frac{\rho_{45}}{\rho_{15} + \rho_{45}}\,, \quad w^{54} = \frac{\rho_{15}}{\rho_{15} + \rho_{45}}\,, \quad w_g^{51} = \frac{\rho_{14}}{\rho_{14} + \rho_{15}}\,, \quad w_g^{41} = \frac{\rho_{15}}{\rho_{14} + \rho_{15}}\,, \tag{A.10}$$

where ρ_{ij} are defined in Eq. (A.9).

A.3.2. NNLO partition functions

For NNLO calculation, we use

$$1 = w^{51,61} + w^{54,64} + w^{51,64} + w^{54,61}\,, \tag{A.11}$$

where

$$\begin{aligned} w^{51,61} &= \frac{\rho_{54}\rho_{64}}{d_5 d_6}\left(1 + \frac{\rho_{15}}{d_{5641}} + \frac{\rho_{16}}{d_{5614}}\right), \quad w^{54,64} = \frac{\rho_{15}\rho_{16}}{d_5 d_6}\left(1 + \frac{\rho_{46}}{d_{5641}} + \frac{\rho_{45}}{d_{5614}}\right), \\ w^{51,64} &= \frac{\rho_{45}\rho_{16}\rho_{56}}{d_5 d_6 d_{5614}}\,, \quad w^{54,61} = \frac{\rho_{15}\rho_{46}\rho_{56}}{d_5 d_6 d_{5641}}\,. \end{aligned} \tag{A.12}$$

In Eq. (A.12) we use the notations

$$d_{i=5,6} \equiv \rho_{1i} + \rho_{4i}\,, \quad d_{5614} \equiv \rho_{56} + \rho_{15} + \rho_{46}\,, \quad d_{5641} \equiv \rho_{56} + \rho_{45} + \rho_{16}\,. \tag{A.13}$$

Partition functions Eq. (A.12) evaluated in different double-collinear limits read

$$\begin{aligned} w_{\rm DC}^{64} &\equiv \lim_{p_5\|p_1} w^{51,64} = \left(\frac{\rho_{16}}{\rho_{16}+\rho_{46}}\right)^2, \quad w_{\rm DC}^{51} \equiv \lim_{p_6\|p_4} w^{51,64} = \left(\frac{\rho_{45}}{\rho_{15}+\rho_{45}}\right)^2, \\ w_{\rm DC}^{61} &\equiv \lim_{p_5\|p_4} w^{54,61} = \left(\frac{\rho_{46}}{\rho_{16}+\rho_{46}}\right)^2, \quad w_{\rm DC}^{54} \equiv \lim_{p_6\|p_1} w^{54,61} = \left(\frac{\rho_{15}}{\rho_{15}+\rho_{45}}\right)^2. \end{aligned} \tag{A.14}$$

For the required limits of the triple-collinear partition $51, 61$ we find

$$\begin{aligned} w_{\rm TC}^{61} &\equiv \lim_{p_5\|p_1} w^{51,61} = \frac{\rho_{46}}{\rho_{16}+\rho_{46}}\left(1+\frac{\rho_{16}}{\rho_{16}+\rho_{46}}\right), \\ w_{\rm TC}^{51} &\equiv \lim_{p_6\|p_1} w^{51,61} = \frac{\rho_{45}}{\rho_{15}+\rho_{45}}\left(1+\frac{\rho_{15}}{\rho_{15}+\rho_{45}}\right), \\ w_{\rm TC}^{1} &\equiv \lim_{p_5\|p_6} w^{51,61} = \left(\frac{\rho_{45}}{\rho_{15}+\rho_{45}}\right)^2\left(1+\frac{2\rho_{15}}{\rho_{15}+\rho_{45}}\right). \end{aligned} \tag{A.15}$$

For the triple-collinear partition $54, 54$ we obtain

$$\begin{aligned} w_{\rm TC}^{64} &\equiv \lim_{p_5\|p_4} w^{54,64} = \frac{\rho_{16}}{\rho_{16}+\rho_{46}}\left(1+\frac{\rho_{46}}{\rho_{16}+\rho_{46}}\right), \\ w_{\rm TC}^{54} &\equiv \lim_{p_6\|p_4} w^{54,64} = \frac{\rho_{15}}{\rho_{15}+\rho_{45}}\left(1+\frac{\rho_{45}}{\rho_{15}+\rho_{45}}\right), \\ w_{\rm TC}^{4} &\equiv \lim_{p_5\|p_6} w^{54,64} = \left(\frac{\rho_{15}}{\rho_{15}+\rho_{45}}\right)^2\left(1+\frac{2\rho_{45}}{\rho_{15}+\rho_{45}}\right). \end{aligned} \tag{A.16}$$

We also use

$$w_{\rm s}^{41} = \frac{\rho_{16}}{\rho_{14}+\rho_{16}}, \quad w_{\rm s}^{61} = \frac{\rho_{14}}{\rho_{14}+\rho_{16}}. \tag{A.17}$$

A.3.3. Triple-collinear angular sectors

In the triple-collinear partitions $5i, 6i$, $i \in \{1, 4\}$, we split the angular phase space into sectors (a) - (d). They are defined by the partition of unity

$$1 = \theta_i^{(a)} + \theta_i^{(b)} + \theta_i^{(c)} + \theta_i^{(d)}, \tag{A.18}$$

with

$$\begin{aligned} \theta_i^{(a)} &\equiv \theta\left(\frac{\eta_{i5}}{2} - \eta_{i6}\right), \quad \theta_i^{(b)} \equiv \theta\left(\eta_{i6} - \frac{\eta_{i5}}{2}\right)\theta\left(\eta_{i5} - \eta_{i6}\right), \\ \theta_i^{(c)} &\equiv \theta\left(\frac{\eta_{i6}}{2} - \eta_{i5}\right), \quad \theta_i^{(d)} \equiv \theta\left(\eta_{i5} - \frac{\eta_{i6}}{2}\right)\theta\left(\eta_{i6} - \eta_{i5}\right), \end{aligned} \tag{A.19}$$

where $i = 1, 4$ depending on the triple-collinear partition $w^{5i,6i}$ in which sectoring Eq. (A.18) is introduced.

A.4. Angular dependent functions K_{ij} and $\tilde{K}_{ij}$

We used the following functions

$$K_{ij} = \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right] \eta_{ij}^{1+\epsilon} {}_2F_1\big(1,1;1-\epsilon;1-\eta_{ij}\big)\,, \tag{A.20}$$

$$\tilde{K}_{ij} = \left[\frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-4\epsilon)}\right] \eta_{ij}^{1+3\epsilon} {}_2F_1\big(1+\epsilon,1+\epsilon;1-\epsilon;1-\eta_{ij}\big)\,. \tag{A.21}$$

in Eqs. (5.17, 6.190). The ϵ-expansions of K_{ij} and $\tilde{K}_{ij}$ read [75]

$$\begin{aligned} K_{ij} = 1 &+ \epsilon^2\left[-\frac{\pi^2}{6} + \mathrm{Li}_2\big(1-\eta_{ij}\big)\right] + \epsilon^3\left[-\frac{\pi^2}{6}\ln(\eta_{ij}) + \frac{\ln(1-\eta_{ij})\ln^2(\eta_{ij})}{2}\right. \\ &\left. + \ln(\eta_{ij})\mathrm{Li}_2\big(1-\eta_{ij}\big) + \mathrm{Li}_3\big(1-\eta_{ij}\big) + \mathrm{Li}_3\big(\eta_{ij}\big) - 3\zeta_3\right] + \epsilon^4\left[-\frac{\pi^4}{40}\right. \\ &- \frac{\pi^2}{6}\ln^2(\eta_{ij}) + \frac{\ln(1-\eta_{ij})\ln^3(\eta_{ij})}{2} - \frac{\ln^4(\eta_{ij})}{24} + \left(-\frac{\pi^2}{6} + \frac{\ln^2(\eta_{ij})}{2}\right) \\ &\times \mathrm{Li}_2\big(1-\eta_{ij}\big) + \ln(\eta_{ij})\mathrm{Li}_3\big(1-\eta_{ij}\big) + \ln(\eta_{ij})\mathrm{Li}_3\big(\eta_{ij}\big) \\ &\left. - \mathrm{Li}_4\left(1-\frac{1}{\eta_{ij}}\right) - \zeta_3\ln(\eta_{ij})\right] + \mathcal{O}(\epsilon^5)\,, \end{aligned} \tag{A.22}$$

$$\begin{aligned} \tilde{K}_{ij} = 1 &+ \epsilon^2\left[-\frac{2\pi^2}{3} + 4\mathrm{Li}_2\big(1-\eta_{ij}\big)\right] + \epsilon^3\Big[-2\pi^2\ln(\eta_{ij}) + 6\ln(1-\eta_{ij})\ln(\eta_{ij})^2 \\ &+ 12\ln(\eta_{ij})\mathrm{Li}_2\big(1-\eta_{ij}\big) + 4\mathrm{Li}_3\big(1-\eta_{ij}\big) + 12\mathrm{Li}_3\big(\eta_{ij}\big) - 28\zeta_3\Big] \\ &+ \epsilon^4\left[-\frac{2\pi^4}{9} - \frac{14\pi^2}{3}\ln^2(\eta_{ij}) + \frac{46\ln(1-\eta_{ij})\ln^3(\eta_{ij})}{3} - \frac{5\ln^4(\eta_{ij})}{6}\right. \\ &+ \left(-\frac{8\pi^2}{3} + 18\ln^2(\eta_{ij})\right)\mathrm{Li}_2\big(1-\eta_{ij}\big) + 2\mathrm{Li}_2^2\big(1-\eta_{ij}\big) \\ &+ 20\ln(\eta_{ij})\mathrm{Li}_3\big(1-\eta_{ij}\big) + 36\ln(\eta_{ij})\mathrm{Li}_3\big(\eta_{ij}\big) - 16\mathrm{Li}_4\big(1-\eta_{ij}\big) \\ &\left. - 20\mathrm{Li}_4\left(1-\frac{1}{\eta_{ij}}\right) - 16\mathrm{Li}_4\big(\eta_{ij}\big) - 20\zeta_3\ln(\eta_{ij})\right] + \mathcal{O}(\epsilon^5)\,. \end{aligned} \tag{A.23}$$

A.5. Plus-prescription $[\,\cdot\,]_+$ and convolution $\otimes$

The plus prescription $[\,\cdot\,]_+$ is defined as

$$\int_0^1 \mathrm{d}x\,[f(x)]_+ \cdot g(x) \equiv \int_0^1 \mathrm{d}x\, f(x)[g(x)-g(1)]\,. \tag{A.24}$$

We defined a class of functions regularized with the plus prescriptions as

$$\mathcal{D}_n(z) \equiv \left[\frac{\log^n(1-z)}{1-z}\right]_+ . \tag{A.25}$$

The symbol $\otimes$ is defined as the convolution

$$\left[f_1 \otimes f_2\right](z) \equiv \int_0^1 \mathrm{d}x\, \mathrm{d}y\, f_1(x) f_2(y) \delta(z - xy)\,. \tag{A.26}$$

B. Singular limits of tree-level functions F_{LM}

This appendix is a collection of all singular limits of the NLO and NNLO QCD tree-level amplitudes that contribute to the DIS partonic cross section.

B.1. Single-soft limit: S_6

The single-soft limits read

$$S_5 F_{\text{LM}}(1_q, 4_q \,|\, 5_g) = 2C_F\, g_{s,b}^2 \times \frac{p_1 \cdot p_4}{(p_1 \cdot p_5)(p_4 \cdot p_5)} \times F_{\text{LM}}(1_q, 4_q)\,, \tag{B.1}$$

$$\begin{aligned} S_6 F_{\text{LM}}(1_q, 4_q \,|\, 5_g, 6_g) &= g_{s,b}^2 \times \Big[(2C_F - C_A) S_{14}(6) + C_A \Big(S_{15}(6) + S_{45}(6)\Big)\Big] \\ &\quad \times F_{\text{LM}}(1_q, 4_q \,|\, 5_g)\,, \end{aligned} \tag{B.2}$$

$$\begin{aligned} S_6 F_{\text{LM},g}(1_g, 4_q \,|\, 5_q, 6_g) &= g_{s,b}^2 \times \Big[(2C_F - C_A) S_{14}(6) + C_A \Big(S_{15}(6) + S_{45}(6)\Big)\Big] \\ &\quad \times F_{\text{LM}}(1_g, 4_q \,|\, 5_q)\,, \end{aligned} \tag{B.3}$$

where

$$S_{ij}(k) = \frac{p_i \cdot p_j}{(p_i \cdot p_k)(p_j \cdot p_k)}\,. \tag{B.4}$$

B.2. Double-soft limit: $\mathbb{S}$

The required double-soft limit reads

$$\begin{aligned} \mathbb{S} F_{\text{LM}}(1_q, 4_q \,|\, 5_g, 6_g) &= g_{s,b}^4 \Big[4C_F^2 S_{14}(5) S_{14}(6) + C_A C_F \big[2S_{14}(5,6) - S_{11}(5,6) \\ &\quad - S_{44}(5,6)\big]\Big] \times F_{\text{LM}}(1_q, 4_q)\,, \end{aligned} \tag{B.5}$$

$$\mathbb{S} F_{\text{LM,ns}}(1_q, 4_q \,|\, 5_q, 6_q) = g_{s,b}^4 T_R C_F \big[2I_{14}(5,6) - I_{11}(5,6) - I_{44}(5,6)\big] \times F_{\text{LM}}(1_q, 4_q)\,, \tag{B.6}$$

where $S_{ij}(k)$ is defined in Eq. (B.4) and

$$\begin{aligned} S_{ij}(k,l) &= S_{ij}^{\text{so}}(k,l) - \frac{2 p_i \cdot p_j}{(p_k \cdot p_l)(p_i \cdot (p_k + p_l))(p_j \cdot (p_k + p_l))} \\ &\quad + \frac{(p_i \cdot p_k)(p_j \cdot p_l) + (p_i \cdot p_l)(p_j \cdot p_k)}{(p_i \cdot (p_k + p_l))(p_j \cdot (p_k + p_l))} \left(\frac{1-\epsilon}{(p_k \cdot p_l)^2} - \frac{1}{2} S_{ij}^{\text{so}}(k,l)\right)\,, \end{aligned} \tag{B.7}$$

$$S_{ij}^{\text{so}}(k,l) = \frac{p_i\cdot p_j}{p_k\cdot p_l}\left(\frac{1}{(p_i\cdot p_k)(p_j\cdot p_l)}+\frac{1}{(p_i\cdot p_l)(p_j\cdot p_k)}\right) - \frac{(p_i\cdot p_j)^2}{(p_i\cdot p_k)(p_j\cdot p_k)(p_i\cdot p_l)(p_j\cdot p_l)}, \tag{B.8}$$

$$I_{ij}(k,l) = \frac{(p_i\cdot p_k)(p_j\cdot p_l)+(p_i\cdot p_l)(p_j\cdot p_k)-(p_i\cdot p_j)(p_k\cdot p_l)}{(p_k\cdot p_l)^2(p_i\cdot(p_k+p_l))(p_j\cdot(p_k+p_l))}. \tag{B.9}$$

B.3. Double-collinear limit: C_{ji}

In this section we collect double-collinear limits. Note that in this appendix we do not show the splitting functions. These are collected in Appendix E. Further note that dots in the argument of functions F_{LM} and $F_{\text{LM},g}$ indicate further momenta that may be present.

Initial-state collinear radiation: C_{j1}

The required double-collinear limits to the initial-state momentum p_1 read

$$C_{51}F_{\text{LM}}(1_q,4_q\,|\,5_g,\dots) = g_{s,b}^2\times\frac{1}{p_1\cdot p_5}P_{qq}(z)\times\frac{F_{\text{LM}}(z\cdot 1_q,4_q\,|\,\dots)}{z}, \tag{B.10}$$

$$C_{51}F_{\text{LM},g}(1_g,4_q\,|\,5_q,\dots) = g_{s,b}^2\times\sum_{f\in\{q,\bar q\}}\frac{1}{p_1\cdot p_5}P_{fg}(z)\times\frac{F_{\text{LM}}(z\cdot 1_f,4_f\,|\,\dots)}{z}, \tag{B.11}$$

$$C_{51}F_{\text{LM,s}}(1_q,4_q\,|\,5_q,6_q) = g_{s,b}^2\times\frac{1}{p_1\cdot p_5}P_{gq}(z)\times\frac{F_{\text{LM},g}(z\cdot 1_g,4_q\,|\,5_q)}{z}, \tag{B.12}$$

with $z=(E_1-E_5)/E_1$ and

$$C_{61}F_{\text{LM}}(1_q,4_q\,|\,5_g,6_g) = g_{s,b}^2\times\frac{1}{p_1\cdot p_6}P_{qq}(z)\times\frac{F_{\text{LM}}(z\cdot 1_q,4_q\,|\,5_g)}{z}, \tag{B.13}$$

$$C_{61}F_{\text{LM},g}(1_g,4_q\,|\,5_q,6_g) = g_{s,b}^2\times\frac{1}{p_1\cdot p_6}P_{gg\,\mu\nu}(z)\times\frac{F_{\text{LM}}^{\mu\nu}(z\cdot 1_g,4_q\,|\,5_q)}{z}, \tag{B.14}$$

with $z=(E_1-E_6)/E_1$. All splitting functions can be found in Appendix E.1.

Final-state collinear radiation: C_{j4}

The required limits for final-state splitting read

$$C_{54}F_{\text{LM}}(1_q,4_q\,|\,5_g,\dots) = g_{s,b}^2\times\frac{1}{p_4\cdot p_5}\,P_{qq}(z)\times F_{\text{LM}}\left(1_q,\frac{1}{z}\cdot 4_q\,|\,\dots\right), \tag{B.15}$$

with $z=E_4/(E_4+E_5)$ and

$$C_{64}F_{\text{LM}}(1_q,4_q\,|\,5_g,6_g) = g_{s,b}^2\times\frac{1}{p_4\cdot p_6}\,P_{qq}(z)\times F_{\text{LM}}\left(1_q,\frac{1}{z}\cdot 4_q\,|\,5_g\right), \tag{B.16}$$

$$C_{64}F_{\text{LM},g}(1_g,4_q\,|\,5_q,6_g) = g_{s,b}^2\times\frac{1}{p_4\cdot p_6}\,P_{qq}(z)\times F_{\text{LM},g}\left(1_g,\frac{1}{z}\cdot 4_q\,|\,5_q\right), \tag{B.17}$$

with $z = E_4/(E_4 + E_6)$. The splitting function $P_{qq}(z)$ is given in Eq. (E.1).

Radiation of collinear partons: C_{56}

Finally, we also need the double-collinear limit where two radiated partons are collinear to each other. The required limits read

$$C_{56}F_{\text{LM}}(1_q, 4_q \,|\, 5_g, 6_g) = g_{s,b}^2 \times \frac{1}{p_5 \cdot p_6}\Big[P_{gg}^{(0)}(z)\; F_{\text{LM}}(1_q, 4_q \,|\, 5_g + 6_g) + P_{gg}^{\perp}(z)\; \kappa_{\perp\mu}\kappa_{\perp\nu}F_{\text{LM}}^{\mu\nu}(1_q, 4_q \,|\, 5_g + 6_g)\Big]\,, \tag{B.18}$$

$$C_{56}F_{\text{LM},g}(1_g, 4_q \,|\, 5_q, 6_g) = g_{s,b}^2 \times \frac{1}{p_5 \cdot p_6}\; P_{qq}\,(z) \times F_{\text{LM},g}\left(1_g, 4_q \,|\, \frac{1}{z} \cdot 5_q\right), \tag{B.19}$$

$$C_{56}F_{\text{LM,ns}}(1_q, 4_q \,|\, 5_q, 6_q) = g_{s,b}^2 \times \frac{1}{p_5 \cdot p_6}\; P_{gq\,\mu\nu}\,(z) \times F_{\text{LM}}^{\mu\nu}\,(1_g, 4_q \,|\, 5 + 6)\;, \tag{B.20}$$

with $z = E_5/(E_5 + E_6)$. The last arguments of the F_{LM} functions in Eq. (B.18) have to be understood as gluons that carry momentum $p_{56} = (E_5 + E_6) \cdot n_5$ where $n_5 = p_5/E_5$. The function $F_{\text{LM}}^{\mu\nu}(1_q, 4_q \,|\, 5_g)$ describes the single-real emission contribution $F_{\text{LM}}(1_q, 4_q \,|\, 5_g)$ where the polarization vector of the gluon $g(p_5)$ is removed from the matrix element. $\kappa_\perp$ is a vector parametrizing the transverse direction of p_6 with respect to the direction of p_5, it is defined in Eq. (6.120). The splitting functions $P_{gg}^{(0)}(z)$ and $P_{gg}^{\perp}(z)$ in Eq. (B.18) can be found in Appendix E.

B.4. Triple-collinear limit: $\mathbb{C}_i$

We now present limits where three partons become collinear to each other.

Initial-state collinear radiation: $\mathbb{C}_1$

For two gluons that are emitted collinear to initial-state quark $q(p_1)$, we obtain

$$\mathbb{C}_1 F_{\text{LM}}(1_q, 4_q \,|\, 5_g, 6_g) = g_{s,b}^2 \times \left(\frac{2}{s_{156}}\right)^2 P_{ggq}(z_5, z_6, z) \times F_{\text{LM}}(z \cdot 1_q, 4_q)\,. \tag{B.21}$$

In Eq. (B.21) we defined the (combined) scalar product

$$s_{156} = -2p_1 \cdot p_5 - 2p_1 \cdot p_6 + 2p_5 \cdot p_6\,, \tag{B.22}$$

and the momentum fractions

$$z = \frac{E_1}{E_1 - E_5 - E_6}\,, \quad z_{i=5,6} = \frac{E_i}{E_5 + E_6 - E_1}\,. \tag{B.23}$$

The triple collinear splitting function can be found in Appendix E.2.

Final-state collinear radiation: $\mathbb{C}_4$

For two gluons that are emitted collinear to final-state quark $q(p_4)$, we obtain

$$\mathbb{C}_4 F_{\rm LM}(1_q, 4_q \,|\, 5_g, 6_g) = g_{s,b}^2 \times \left(\frac{2}{s_{456}}\right)^2 P_{ggq}(z_5, z_6, z) \times F_{\rm LM}\left(1_q, \frac{1}{z} \cdot 4_q\right), \tag{B.24}$$

where momentum fractions z_5, z_6 and z are defined as

$$z = \frac{E_4}{E_4 + E_5 + E_6}, \quad z_{i=5,6} = \frac{E_i}{E_4 + E_5 + E_6}. \tag{B.25}$$

B.5. Strongly ordered double-soft limit: $S_6\mathbb{S}$

We find

$$\begin{aligned}\mathbb{S}S_6 F_{\rm LM}(1_q, 4_q \,|\, 5_g, 6_g) = 2C_F\, g_{s,b}^4 \times \big[(2C_F - C_A)S_{14}(6) + C_A\big(S_{15}(6) + S_{45}(6)\big)\big] S_{14}(5) \\ \times F_{\rm LM}(1_q, 4_q)\,,\end{aligned} \tag{B.26}$$

where $S_{ij}(k)$ can be found in Eq. (B.4).

B.6. Strongly ordered triple collinear limit: $C_{ji}\mathbb{C}_i$

Initial-state collinear radiation: $C_{j1}\mathbb{C}_1$

For the double-collinear limits to initial-state quark $q(p_1)$, we obtain

$$C_{j1}\mathbb{C}_1 F_{\rm LM}(1_q, 4_q \,|\, 5_g, 6_g) = g_{s,b}^4 \times \frac{1}{z(p_1 \cdot p_5)(p_1 \cdot p_6)} P_{qq}(z) P_{qq}(\bar{z}) \times \frac{F_{\rm LM}(z\bar{z} \cdot 1_q, 4_q)}{z\bar{z}}, \tag{B.27}$$

with $j \in \{5, 6\}$ and

$$\begin{aligned}C_{51}\mathbb{C}_1 F_{{\rm LM},g}(1_g, 4_q \,|\, 5_q, 6_g) = g_{s,b}^4 \times \sum_{f \in \{q, \bar{q}\}} \frac{1}{z(p_1 \cdot p_5)(p_1 \cdot p_6)} P_{fg}(z) P_{ff}(\bar{z}) \\ \times \frac{F_{\rm LM}(z\bar{z} \cdot 1_f, 4_f)}{z\bar{z}},\end{aligned} \tag{B.28}$$

The momentum fractions z and $\bar{z}$ read

$$z = \frac{E_1 - E_j}{E_1}, \quad \bar{z} = \frac{E_1 - E_5 - E_6}{E_1 - E_j}. \tag{B.29}$$

Final-state collinear radiation: $C_{j4}\mathbb{C}_4$

For the double-collinear limits to final-state quark $q(p_4)$, we obtain

$$C_{j4}\mathbb{C}_4 F_{\rm LM}(1_q, 4_q \,|\, 5_g, 6_g) = g_{s,b}^4 \times \frac{z}{(p_1 \cdot p_5)(p_1 \cdot p_6)} P_{qq}(z) P_{qq}(\bar{z}) \times F_{\rm LM}\left(1_q, \frac{1}{z\bar{z}} \cdot 4_q\right) \tag{B.30}$$

with $j \in \{5, 6\}$ and the momentum fractions

$$z = \frac{E_4}{E_4 + E_j}, \quad \bar{z} = \frac{E_4 + E_j}{E_4 + E_5 + E_6}. \tag{B.31}$$

The limit: $C_{56}\mathbb{C}_i$

Finally we also need the triple-collinear limit where the two gluons become collinear first. Starting with the limit Eq. (B.18) and approaching the triple-collinear limit, we obtain

$$\begin{aligned} C_{56}\mathbb{C}_1 F_{\text{LM}}(1_q, 4_q \,|\, 5_g, 6_g) &= g_{s,b}^4 \times \frac{1}{(p_5 \cdot p_6)(p_1 \cdot p_{56})} \times \Big\{ P_{gg}^{(0)}(z) P_{qq}(\bar{z}) \\ &+ \frac{1}{2} P_{gg}^{\perp}(z) \Big[- C_F \Big(1 - \bar{z} + \frac{4\bar{z}}{1-\bar{z}} \big[\kappa_\perp \cdot \bar{\kappa}_\perp\big]^2 \Big) + P_{qq}(\bar{z}) \Big] \Big\} \times \frac{F_{\text{LM}}(\bar{z} \cdot 1, 4)}{\bar{z}}, \end{aligned} \tag{B.32}$$

with momentum fractions

$$z = \frac{E_5}{E_5 + E_6}, \quad \bar{z} = \frac{E_1 - E_5 - E_6}{E_1}, \quad p_{56} = \frac{1}{z} \cdot p_5. \tag{B.33}$$

In Eq. (B.32) $\kappa_{\perp\mu}$ is the vector parametrizing the transverse direction of p_6 to the direction of p_5. Similarly, $\bar{\kappa}_{\perp\mu}$ is the vector parametrizing the transverse direction of p_{56} to the direction of p_1. In our phase space parametrization, c.f. Appendix F, we obtain $\big[\kappa_\perp \cdot \bar{\kappa}_\perp\big]^2 = \lambda$.

We find for the strongly ordered triple-collinear limit to the final-state momentum p_4

$$\begin{aligned} C_{56}\mathbb{C}_1 F_{\text{LM}}(1_q, 4_q \,|\, 5_g, 6_g) &= g_{s,b}^4 \times \frac{1}{(p_5 \cdot p_6)(p_1 \cdot p_{56})} \times \Big\{ P_{gg}^{(0)}(z) P_{qq}(\bar{z}) \\ &+ \frac{1}{2} P_{gg}^{\perp}(z) \Big[- C_F \Big(1 - \bar{z} + \frac{4\bar{z}}{1-\bar{z}} \big[\kappa_\perp \cdot \bar{\kappa}_\perp\big]^2 \Big) + P_{qq}(\bar{z}) \Big] \Big\} \times \frac{F_{\text{LM}}(\bar{z} \cdot 1, 4)}{\bar{z}}, \end{aligned} \tag{B.34}$$

with momentum fractions

$$z = \frac{E_5}{E_5 + E_6}, \quad \bar{z} = \frac{E_4}{E_4 + E_5 + E_6}, \quad p_{56} = \frac{1}{z} \cdot p_5. \tag{B.35}$$

B.7. Single-soft double-collinear limit: $S_6 C_{ji}$

The limit of a soft-collinear gluon reads

$$C_{5i} S_5 F_{\text{LM}}(1_q, 4_q \,|\, 5_g, \dots) = 2C_F \, g_{s,b}^2 \times \frac{1}{E_5^2 \rho_{i5}} \times F_{\text{LM}}(1_q, 4_q \,|\, \dots), \tag{B.36}$$

where $i \in \{1, 4\}$. If the soft and collinear gluons are different the limit is given by the product of the two NLO-like limits. We obtain

$$\begin{aligned} &C_{51} S_6 F_{\text{LM}}(1_q, 4_q \,|\, 5_g, 6_g) \\ &= 2C_F \, g_{s,b}^4 \times \frac{p_1 \cdot p_4}{(p_1 \cdot p_6)(p_4 \cdot p_6)} \times \frac{1}{p_1 \cdot p_5} P_{qq}(z) \times \frac{F_{\text{LM}}(z \cdot 1_q, 4_q)}{z}. \end{aligned} \tag{B.37}$$

with $z = (E_1 - E_5)/E_1$. The final state limit reads

$$
\begin{aligned}
&C_{54}S_6F_{\rm LM}(1_q,4_q\,|\,5_g,6_g) \\
&= 2C_F\, g_{s,b}^4 \times \frac{p_1\cdot p_4}{(p_1\cdot p_6)(p_4\cdot p_6)} \times \frac{1}{p_4\cdot p_5}\, P_{qq}(z) \times F_{\rm LM}\left(1_q, \frac{1}{z}\cdot 4_q\right).
\end{aligned}
\tag{B.38}
$$

with $z = E_4/(E_4+E_5)$. The splitting function $P_{qq}(z)$ in Eqs. (B.37, B.38) is defined in Eq. (E.1).

B.8. Single-soft triple-collinear limit: $S_6\mathbb{C}_i$

For initial-state emission we obtain

$$
\begin{aligned}
\mathbb{C}_1S_6F_{\rm LM}(1_q,4_q\,|\,5_g,6_g) = g_{s,b}^4 \times \frac{1}{E_6^2}&\left[\frac{2C_F-C_A}{\rho_{15}\rho_{16}} + \frac{C_A}{\rho_{15}\rho_{56}} + \frac{C_A}{\rho_{16}\rho_{56}}\right] \\
&\times \frac{1}{E_1E_5}\, P_{qq}(z) \times \frac{F_{\rm LM}(z\cdot 1_q, 4_q)}{z},
\end{aligned}
\tag{B.39}
$$

with $z = (E_1 - E_5)/E_1$. For final-state emission we find

$$
\begin{aligned}
\mathbb{C}_4S_6F_{\rm LM}(1_q,4_q\,|\,5_g,6_g) = g_{s,b}^4 \times \frac{1}{E_6^2}&\left[\frac{2C_F-C_A}{\rho_{45}\rho_{46}} + \frac{C_A}{\rho_{45}\rho_{56}} + \frac{C_A}{\rho_{46}\rho_{56}}\right] \\
&\times \frac{1}{E_4E_5}\, P_{qq}(z) \times F_{\rm LM}\left(1_q, \frac{1}{z}\cdot 4_q\right),
\end{aligned}
\tag{B.40}
$$

with $z = E_4/(E_4+E_5)$. The splitting function $P_{qq}(z)$ in Eqs. (B.39, B.40) is defined in Eq. (E.1).

B.9. Double-soft triple-collinear limit: $\mathbb{S}\mathbb{C}_i$

Taking the double-soft limit of the triple-collinear splitting function P_{ggq}, c.f. Eq. (B.21) and the explicit formulas around Eq. (E.13), we obtain

$$
\begin{aligned}
&\mathbb{S}\mathbb{C}_1F_{\rm LM}(1_q,4_q\,|\,5_g,6_g) \\
&= g_{s,b}^4 \times \left[C_F^2\, \mathbb{S}\left[(2/s_{156})^2P_{ggq}^{(\rm ab)}\right] + C_AC_F\mathbb{S}\left[(2/s_{156})^2P_{ggq}^{(\rm nab)}\right]\right] \times F_{\rm LM}(1_q,4_q),
\end{aligned}
\tag{B.41}
$$

where

$$
\mathbb{S}\left[(2/s_{156})^2P_{ggq}^{(\rm ab)}\right] = \frac{4}{E_5^2E_6^2\,\rho_{15}\rho_{16}},
\tag{B.42}
$$

$$
\begin{aligned}
\mathbb{S}\left[(2/s_{156})^2P_{ggq}^{(\rm nab)}\right] &= 4\Bigg\{(1-\epsilon)\frac{\tilde{t}_{45,1}^2}{2s_{56}^2\tilde{s}_{156}^2} + \frac{1}{s_{56}}\left[\frac{1}{z_6s_{15}} + \frac{1}{z_5s_{16}}\right] + \frac{1}{s_{56}z_{56}}\left[\frac{1}{s_{15}} + \frac{1}{s_{16}}\right] \\
&\quad - \frac{1}{s_{15}s_{16}z_5z_6} + \frac{1}{\tilde{s}_{156}s_{56}z_{56}}\left[\frac{z_5}{z_6} + \frac{z_6}{z_5} - 6\right] + \frac{1}{\tilde{s}_{156}z_{56}}\left[\frac{1}{z_6s_{15}} + \frac{1}{z_5s_{16}}\right] \\
&\quad - \frac{1}{\tilde{s}_{156}z_5z_6}\left[\frac{1}{s_{15}} + \frac{1}{s_{16}}\right]\Bigg\}.
\end{aligned}
\tag{B.43}
$$

Momentum fractions in Eq. (B.43) are defined as

$$z_{i=5,6} = \frac{E_i}{E_5 + E_6 - E_1}\,, \quad z_{56} \equiv z_5 + z_6\,. \tag{B.44}$$

The relevant scalar products read

$$s_{56} = 2E_5E_6\rho_{56}\,, \quad s_{15} = -2E_1E_5\rho_{15}\,, \quad s_{15} = -2E_1E_6\rho_{16}\,, \quad \tilde{s}_{156} \equiv s_{15} + s_{16}\,. \tag{B.45}$$

In Eq. (B.43) $\tilde{t}_{56,1}$ is the double-soft limit of the spin correlated structure $t_{56,1}$ given in Eq. (E.15). It reads

$$\tilde{t}_{56,1} \equiv 2\frac{z_5 s_{16} - z_6 s_{15}}{z_5 + z_6}\,. \tag{B.46}$$

B.10. Double-soft double-collinear limit: $\mathbb{S}C_{ji}$

These limits read

$$\mathbb{S}C_{6i}F_{\mathrm{LM}}(1_q,4_q\,|\,5_g,6_g) = 4C_F^2\, g_{s,b}^4 \times \frac{1}{E_6^2\,\rho_{i6}} \times \frac{p_1\cdot p_4}{(p_1\cdot p_5)(p_4\cdot p_5)} \times F_{\mathrm{LM}}(1_q,4_q)\,, \tag{B.47}$$

$$\mathbb{S}C_{5i}F_{\mathrm{LM}}(1_q,4_q\,|\,5_g,6_g) = 4C_F^2\, g_{s,b}^4 \times \frac{1}{E_5^2\,\rho_{i5}} \times \frac{p_1\cdot p_4}{(p_1\cdot p_6)(p_4\cdot p_6)} \times F_{\mathrm{LM}}(1_q,4_q)\,, \tag{B.48}$$

for $i \in \{1,4\}$. We also need the limit $\mathbb{S}C_{56}$ it is given in Section C.

B.11. Strongly ordered double-soft double-collinear limit: $\mathbb{S}S_6C_{ji}$

These limits read

$$\mathbb{S}S_6C_{6i}F_{\mathrm{LM}}(1_q,4_q\,|\,5_g,6_g) = 4C_F^2\, g_{s,b}^4 \times \frac{1}{E_6^2\,\rho_{i6}} \times \frac{p_1\cdot p_4}{(p_1\cdot p_5)(p_4\cdot p_5)} \times F_{\mathrm{LM}}(1_q,4_q)\,, \tag{B.49}$$

$$\mathbb{S}S_6C_{5i}F_{\mathrm{LM}}(1_q,4_q\,|\,5_g,6_g) = 4C_F^2\, g_{s,b}^4 \times \frac{1}{E_5^2\,\rho_{i5}} \times \frac{p_1\cdot p_4}{(p_1\cdot p_6)(p_4\cdot p_6)} \times F_{\mathrm{LM}}(1_q,4_q)\,, \tag{B.50}$$

for $i \in \{1,4\}$. We also need the limit $\mathbb{S}C_{56}$ it is given in Section C.

B.12. Strongly-ordered double-soft triple-collinear limit: $\mathbb{S}S_6\mathbb{C}_i$

The required strongly ordered double-soft triple-collinear limit follows immediately from the S_5 limit of Eqs. (B.39, B.40). For $i \in \{1,4\}$, we obtain

$$\begin{aligned} &S_6\mathbb{S}\mathbb{C}_iF_{\mathrm{LM}}(1_q,4_q\,|\,5_g,6_g) \\ &= 2C_F\, g_{s,b}^4 \times \frac{1}{E_5^2E_6^2}\left[\frac{2C_F - C_A}{\rho_{i5}\rho_{i6}} + \frac{C_A}{\rho_{i5}\rho_{56}} + \frac{C_A}{\rho_{i6}\rho_{56}}\right] \times F_{\mathrm{LM}}(1_q,4_q)\,. \end{aligned} \tag{B.51}$$

B.13. Single-soft strongly ordered triple-collinear limit: $S_6 C_{ji} \mathbb{C}_i$

Considering Eqs. (B.36, B.37) in the triple-collinear limits $\mathbb{C}_1$ we obtain

$$\begin{aligned}&\mathbb{C}_1 C_{i1} S_6 F_{\rm LM}(1_q, 4_q \,|\, 5_g, 6_g)\\ &= 2C_F\, g_{s,b}^4 \times \frac{1}{E_6^2 \rho_{16}} \times \frac{1}{p_1 \cdot p_5} P_{qq}(z) \times \frac{F_{\rm LM}(z \cdot 1_q, 4_q)}{z}\,, \quad i \in \{5,6\}\,,\end{aligned} \tag{B.52}$$

with $z = (E_1 - E_5)/E_1$. From the triple-collinear limit $\mathbb{C}_4$ of Eqs. (B.36, B.38) we obtain

$$\begin{aligned}&\mathbb{C}_4 C_{i4} S_6 F_{\rm LM}(1_q, 4_q \,|\, 5_g, 6_g)\\ &= 2C_F\, g_{s,b}^4 \times \frac{1}{E_6^2 \rho_{46}} \times \frac{1}{p_4 \cdot p_5} P_{qq}(z) \times F_{\rm LM}\left(1_q, \frac{1}{z} \cdot 4_q\right), \quad i \in \{5,6\}\,,\end{aligned} \tag{B.53}$$

with $z = E_4/(E_4 + E_5)$. The splitting function $P_{qq}(z)$ in Eqs. (B.52, B.53) is defined in Eq. (E.1).

B.14. Double-soft strongly-ordered triple-collinear limit: $\mathbb{S}\mathbb{C}_i C_{ji}$ and $\mathbb{S}\mathbb{C}_{5i} C_{6j}$

For the double-soft strongly ordered triple-collinear limit we find

$$\mathbb{S}\mathbb{C}_i C_{ji} F_{\rm LM}(1_q, 4_q \,|\, 5_g, 6_g) = 4C_F^2\, g_{s,b}^4 \times \frac{1}{E_5^2 E_6^2 \rho_{5i} \rho_{6i}} \times F_{\rm LM}(1_q, 4_q)\,, \tag{B.54}$$

for $i \in \{1,4\}$ and $j \in \{5,6\}$. In case that gluons $g(p_5)$ and $g(p_6)$ are collinear to different partons we obtain

$$\mathbb{S} C_{5i} C_{6i} F_{\rm LM}(1_q, 4_q \,|\, 5_g, 6_g) = 4C_F^2\, g_{s,b}^4 \times \frac{1}{E_5^2 E_6^2 \rho_{5i} \rho_{6j}} \times F_{\rm LM}(1_q, 4_q)\,, \tag{B.55}$$

for $i, j \in \{1,4\}$, with $i \neq j$.

B.15. Strongly-ordered double-soft strongly-ordered triple-collinear limit: $\mathbb{S} S_6 \mathbb{C}_i C_{ji}$

The strongly ordered double-soft strongly ordered triple-collinear limits of two gluon emissions read

$$S_6 \mathbb{S}\mathbb{C}_i C_{ji} F_{\rm LM}(1_q, 4_q \,|\, 5_g, 6_g) = 4C_F^2\, g_{s,b}^4 \times \frac{1}{E_5^2 E_6^2 \rho_{i5} \rho_{i6}} \times F_{\rm LM}(1_q, 4_q)\,, \tag{B.56}$$

$$S_6 \mathbb{S}\mathbb{C}_i C_{56} F_{\rm LM}(1_q, 4_q \,|\, 5_g, 6_g) = 4C_A C_F\, g_{s,b}^4 \times \frac{1}{E_5^2 E_6^2 \rho_{i5} \rho_{56}} \times F_{\rm LM}(1_q, 4_q)\,, \tag{B.57}$$

for $i \in \{1,4\}$ and $j \in \{5,6\}$. In addition we need for the double-collinear partitions

$$S_6\mathbb{S}C_{5i}C_{6j}F_{\mathrm{LM}}(1_q,4_q\,|\,5_g,6_g) = 4C_F^2\,g_{s,b}^4 \times \frac{1}{E_5^2E_6^2\rho_{i5}\rho_{j6}} \times F_{\mathrm{LM}}(1_q,4_q)\,, \tag{B.58}$$

for $i,j \in \{1,4\}$ with $i \neq j$.

C. Singular limits of tree-level functions $F_{\rm LM}^{\mu\nu}$

In this appendix we calculate the required single-soft and double-collinear limits of the spin-correlated amplitude $r_\mu^{(i)} r_\nu^{(i)} F_{\rm LM}^{\mu\nu}(1_q, 4_q \,|\, 5_g)$, with $i \in \{1,4\}$, for arbitrary angles between the hard emitters $q(p_1)$ and $q(p_4)$.

We first present the required limits. The single-soft limit reads

$$\begin{aligned} &S_5\, r_\mu^{(1)} r_\nu^{(1)} F_{\rm LM}^{\mu\nu}(1_q, 4_q \,|\, 5_g) \\ &= C_F\, g_{s,b}^2\, \frac{1}{E_5^2} \left(1 + 2\frac{\rho_{14}}{\rho_{15}\rho_{45}} - \frac{2}{\rho_{45}} + \frac{(\rho_{14} - \rho_{15}(1-\rho_{45}) - \rho_{45})^2}{\rho_{15}\rho_{45}^2(2-\rho_{15})} \right) F_{\rm LM}(1_q, 4_q)\,. \end{aligned} \tag{C.1}$$

We also use

$$\begin{aligned} &S_5 \left[r_\mu^{(1)} r_\nu^{(1)} + g_{\mu\nu} \right] F_{\rm LM}^{\mu\nu}(1_q, 4_q \,|\, 5_g) \\ &= C_F\, g_{s,b}^2\, \frac{1}{E_5^2} \left(1 - \frac{2}{\rho_{45}} + \frac{(\rho_{14} - \rho_{15}(1-\rho_{45}) - \rho_{45})^2}{\rho_{15}\rho_{45}^2(2-\rho_{15})} \right) F_{\rm LM}(1_q, 4_q)\,. \end{aligned} \tag{C.2}$$

To obtain the limit in Eq. (C.2) we used $-g_{\mu\nu} F_{\rm LM}^{\mu\nu}(1,4\,|\,5) = F_{\rm LM}(1,4\,|\,5)$ and the limit in Eq. (B.1). Note that, by construction, the eikonal function in Eq. (C.2) is not singular in the $\vec{p}_5 \parallel \vec{p}_1$ limit. The limits with $r^{(4)}$ are obtained by exchanging $1 \leftrightarrow 4$ in the eikonal factors on the right-hand side of Eqs. (C.1, C.2).

The collinear $\vec{p}_5 \parallel \vec{p}_1$ limit reads

$$C_{51} r_\mu r_\nu F_{\rm LM}^{\mu\nu}(1_q, 4_q \,|\, 5_g) = g_{s,b}^2 C_F \times \frac{1}{2E_1^2 \rho_{15}} \left(\frac{1+z}{1-z} \right)^2 \times \frac{F_{\rm LM}(z \cdot 1, 4)}{z}\,, \tag{C.3}$$

where $z = (E_1 - E_5)/E_1$ and for the collinear $\vec{p}_5 \parallel \vec{p}_4$ limit we obtain

$$C_{54} r_\mu r_\nu F_{\rm LM}^{\mu\nu}(1_q, 4_q \,|\, 5_g) = g_{s,b}^2 C_F \times \frac{1}{2E_4^2 \rho_{45}}\, z \left(\frac{1+z}{1-z} \right)^2 \times F_{\rm LM}\left(1_q, \frac{1}{z} \cdot 4_q \right), \tag{C.4}$$

where $z = E_4/(E_4 + E_5)$. Finally, considering Eq. (C.1) in a collinear $\vec{p}_5 \parallel \vec{p}_i$ limit we obtain

$$C_{5i} S_5\, r_\mu^{(i)} r_\nu^{(i)} F_{\rm LM}^{\mu\nu}(1,4\,|\,5) = 2C_F\, g_{s,b}^2 \times \frac{1}{E_5^2\, \rho_{i5}} \times F_{\rm LM}(1,4) = C_{5i} S_5\, F_{\rm LM}(1,4\,|\,5)\,. \tag{C.5}$$

In the following we explain how the above limits can be computed. We begin by calculating the single-soft limit Eq. (C.1) in Section C.1. We continue with the double-collinear $\vec{p}_5 \parallel \vec{p}_1$ limit Eq. (C.3) in Section C.2. We note that the computation of the double-collinear $\vec{p}_5 \parallel \vec{p}_4$ limit is analogous to the computation of the collinear $\vec{p}_5 \parallel \vec{p}_1$ limit and, therefore, it is not discussed.

C.1. Computation of the single-soft limit

In this section we compute the soft limit

$$S_5\, r_\mu^{(1)} r_\nu^{(1)} F_{\rm LM}^{\mu\nu}(1,4\,|\,5)\,. \tag{C.6}$$

The soft limit of the single gluon emission amplitude is determined by the eikonal current

$$J^\mu = \frac{p_4^\mu}{p_4\cdot p_5} - \frac{p_1^\mu}{p_1\cdot p_5}\,. \tag{C.7}$$

Therefore

$$S_5 F_{\rm LM}^{\mu\nu}(1,4\,|\,5) = C_F\, g_{s,b}^2 \frac{1}{E_5^2}\left[\frac{n_4^\mu}{\rho_{45}} - \frac{n_1^\mu}{\rho_{15}}\right]\left[\frac{n_4^\nu}{\rho_{45}} - \frac{n_1^\nu}{\rho_{15}}\right] F_{\rm LM}(1,4)\,, \tag{C.8}$$

where $n_i^\mu = p_i^\mu/E_i$. Upon contracting Eq. (C.8) with $r_\mu^{(1)} r_\nu^{(1)}$ we obtain

$$r_\mu^{(1)} r_\nu^{(1)}\; S_5 F_{\rm LM}^{\mu\nu}(1,4\,|\,5) = C_F\, g_{s,b}^2 \frac{1}{E_5^2}\left[\left(\frac{n_4\cdot r^{(1)}}{\rho_{45}} - \frac{n_1\cdot r^{(1)}}{\rho_{15}}\right)^2 - 2\frac{\rho_{14}}{\rho_{15}\rho_{45}}\right] F_{\rm LM}(1,4)\,. \tag{C.9}$$

To simplify the first term on the right-hand side of Eq. (C.9) we use explicit parametrization of vectors n_5^μ and $r^{(1)\mu}$. As explained in Appendix F, they read[1]

$$\begin{aligned} n_5^\mu &= t^\mu + \cos\theta_{15} n_3^\mu + \sin\theta_{15} b^\mu\,,\\ r^\mu &= \sin\theta_{15} e_1^\mu - \cos\theta_{15} b^\mu\,, \end{aligned} \tag{C.10}$$

where $\cos\theta_{15} = 1-\rho_{15}$ and vector b^μ, with $b^2=-1$, parametrizes components of p_5 that are transversal to momentum p_1; hence $b\cdot n_1 = b\cdot e_1 = 0$. We need to compute scalar products $(n_1\cdot r)$ and $(n_4\cdot r)$. For the latter we invert Eq. (C.10) and express b^μ in terms of n_5^μ and e_3^μ. We obtain

$$b^\mu = \frac{n_6^\mu - t^\mu - \cos\theta_{15} e_3^\mu}{\sin\theta_{15}} \quad\Rightarrow\quad n_4\cdot b = \frac{-(1-\rho_{45}) + \cos\theta_{15}(1-\rho_{14})}{\sin\theta_{15}}\,. \tag{C.11}$$

We then use

$$n_4\cdot e_1 = -(1-\rho_{14})\,,\quad n_1\cdot e_1 = -1\,,\quad n_1\cdot b = 0\,, \tag{C.12}$$

and write scalar products in the first term on the right-hand side of Eq. (C.9) in the following

[1] The used parametrization for vector $r^{(1)\mu}$ is the one of angular sector (b). However, the calculation with the parametrization that we find for sector (d) is identical, since they only differs by a global minus sign (and relabeling 5 with 6). From now on we do not show the superscript (1) of vector $r_\mu^{(1)}$ in the computation.

way

$$
\begin{aligned}
&\frac{n_4\cdot r}{\rho_{45}}-\frac{n_1\cdot r}{\rho_{15}}\\
&=\frac{\sin\theta_{15}(n_4\cdot e_3)-\cos\theta_{15}(n_3\cdot b)}{\rho_{45}}-\frac{\sin\theta_{15}(n_1\cdot e_3)-\cos\theta_{15}(n_1\cdot b)}{\rho_{15}}\\
&=-\frac{\sin\theta_{15}^2(1-\rho_{14})+\cos\theta_{15}(-(1-\rho_{45})+\cos\theta_{15}(1-\rho_{14}))}{\sin\theta_{15}\rho_{45}}+\frac{\sin\theta_{15}}{\rho_{15}}\\
&=-\frac{(1-\rho_{14})-\cos\theta_{15}(1-\rho_{45})}{\sin\theta_{15}\rho_{45}}+\frac{\sin\theta_{15}}{\rho_{15}}\,.
\end{aligned}
\tag{C.13}
$$

Squaring this expression, we find

$$
\left(\frac{n_4\cdot r}{\rho_{45}}-\frac{n_1\cdot r}{\rho_{15}}\right)^2=1+2\frac{\rho_{14}}{\rho_{15}\rho_{45}}-\frac{2}{\rho_{45}}+\frac{(\rho_{14}-\rho_{15}(1-\rho_{45})-\rho_{45})^2}{\rho_{15}\rho_{45}^2(2-\rho_{15})}\,. \tag{C.14}
$$

We use Eq. (C.14) in Eq. (C.9) and obtain the following result

$$
\begin{aligned}
&S_5\, r_\mu^{(1)} r_\nu^{(1)} F_{\rm LM}^{\mu\nu}(1_q,4_q\,|\,5_g)\\
&=C_F\, g_{s,b}^2\,\frac{1}{E_5^2}\left(1+2\frac{\rho_{14}}{\rho_{15}\rho_{45}}-\frac{2}{\rho_{45}}+\frac{(\rho_{14}-\rho_{15}(1-\rho_{45})-\rho_{45})^2}{\rho_{15}\rho_{45}^2(2-\rho_{15})}\right)F_{\rm LM}(1_q,4_q)\,.
\end{aligned}
\tag{C.15}
$$

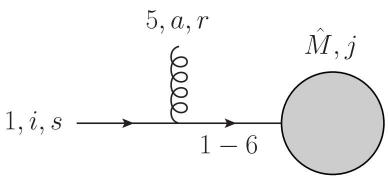
C.2. Computation of the double-collinear limits

In this section we compute the double-collinear limit $C_{51} r_\mu^{(1)} r_\nu^{(1)} F_{\rm LM}^{\mu\nu}(1,4\,|\,5)$. The vector $r^{(1)}$ possesses the following properties[2]

$$
r^2=-1\,,\quad r\cdot p_5=0\,, \tag{C.16}
$$

which follow from Eq. (C.10). It is well-known that for physical polarizations of gluons, only diagrams that describe emission of gluon $g(p_5)$ off the quark line with momentum p_1 develop collinear $\vec{p}_5\parallel\vec{p}_1$ singularity. Thanks to Eq. (C.16) r^μ can be considered a particular physical gluon polarization so the same holds.

We write the amplitude describing the radiation of gluon $g(p_5)$ of initial-state quark $q(p_1)$ as[3]

$A_{\rm sing} =$ 

$$
= \hat{M}^\dagger\,\frac{\hat{p}_1-\hat{p}_5}{(p_1-p_5)^2}(i g_{s,b}\gamma^\mu T^a_{ji})r_\mu u_s(1)\,. \tag{C.17}
$$

[2]From now on we do not show the superscript (1) of vector $r_\mu^{(1)}$ in the computation.

[3]We write the amplitude as $A_{\rm sing}$ to emphasize that it only contains singular contributions in the collinear ($\vec{p}_5\parallel\vec{p}_1$) limit.

In Eq. (C.17) i, j, a are colour labels, s, r are spin labels and $\hat{M}$ is a matrix in the Dirac directly related to the hard process. T^a are Gell-Mann matrices.

Upon squaring amplitude Eq. (C.17) and summing over colours and spins of external partons we obtain

$$\sum_{i,j,a,s} |A_{\text{sing}}|^2 = g_{s,b}^2 \times \frac{1}{((p_1 - p_5)^2)^2} \times \underbrace{\text{Tr}\left[T^a T^a\right]}_{= C_F} \times \text{Tr}\left[(\hat{p}_1 - \hat{p}_5)\hat{r}\hat{p}_1\hat{r}(\hat{p}_1 - \hat{p}_5)\hat{M}^\dagger \hat{M}\right]. \quad \text{(C.18)}$$

The structure inside the remaining trace can be simplified to[4]

$$\begin{aligned}
&(\hat{p}_1 - \hat{p}_5)\hat{r}\hat{p}_1\hat{r}(\hat{p}_1 - \hat{p}_5) \\
&= (\hat{p}_1 - \hat{p}_5)\hat{r}\left[2(p_1 \cdot r) - \hat{r}\hat{p}_1\right](\hat{p}_1 - \hat{p}_5) \\
&= 2(p_1 \cdot r)(\hat{p}_1 - \hat{p}_5)\hat{r}(\hat{p}_1 - \hat{p}_5) + (\hat{p}_1 - \hat{p}_5)\hat{p}_1(\hat{p}_1 - \hat{p}_5).
\end{aligned} \quad \text{(C.20)}$$

The second term on the right-hand side of Eq. (C.20) can be further simplified to

$$(\hat{p}_1 - \hat{p}_5)\hat{p}_1(\hat{p}_1 - \hat{p}_5) = \hat{p}_5\hat{p}_1\hat{p}_5 = \hat{p}_5\left[2(p_1 \cdot p_5) - \hat{p}_5\hat{p}_1\right] = 2(p_1 \cdot p_5)\hat{p}_5. \quad \text{(C.21)}$$

Making use of the transversality of the vector r^μ, we re-write the first term on the right-hand side of Eq. (C.20) as

$$\begin{aligned}
(\hat{p}_1 - \hat{p}_5)\hat{r}(\hat{p}_1 - \hat{p}_5) &= (\hat{p}_1 - \hat{p}_5)\left[2\underbrace{(r \cdot (p_1 - p_5))}_{= r \cdot p_1} - (\hat{p}_1 - \hat{p}_5)\hat{r}\right] \\
&= 2(r \cdot p_1)(\hat{p}_1 - \hat{p}_5) - (p_1 - p_5)^2\hat{r} \\
&= 2(r \cdot p_1)(\hat{p}_1 - \hat{p}_5) + 2(p_1 \cdot p_5)\hat{r},
\end{aligned} \quad \text{(C.22)}$$

Using Eqs. (C.21, C.22) in Eq. (C.20) we write the amplitude squared in Eq. (C.18) as

$$\begin{aligned}
\sum_{i,j,a,s} |A_{\text{sing}}|^2 = 2C_F g_{s,b}^2 \times \frac{1}{(2(p_1 \cdot p_5))^2} \times \Big[& 2(p_1 \cdot r)^2 \text{Tr}\left[(\hat{p}_1 - \hat{p}_5)\hat{M}^\dagger \hat{M}\right] \\
& + (p_1 \cdot p_5)(p_1 \cdot r)\text{Tr}\left[\hat{r}\hat{M}^\dagger \hat{M}\right] + (p_1 \cdot p_5)\text{Tr}\left[\hat{p}_5\hat{M}^\dagger \hat{M}\right] \Big].
\end{aligned} \quad \text{(C.23)}$$

We now consider the collinear ($\vec{p}_5 \parallel \vec{p}_1$) limit. Since the denominator scales like $(p_1 \cdot p_5)^2 \sim \rho_{51}^2$, only terms in the numerator that are proportional to ρ_{51} contribute to the limit. Terms proportional to ρ_{51}^n, with $n > 1$, are finite upon integration over the unresolved phase space and can be dropped. First we study the scalar product $p_1 \cdot r$. Using the explicit form of vector

[4]Using the anti-commutation relation of the γ matrices we write

$$\begin{aligned}
\hat{p}\hat{k} &= p_\mu k_\nu \gamma^\mu \gamma^\nu = p_\mu k_\nu \left[2g^{\mu\nu} - \gamma^\nu \gamma^\mu\right] = 2(p \cdot k) - \hat{k}\hat{p}, \\
\hat{p}\hat{p} &= (p \cdot p).
\end{aligned} \quad \text{(C.19)}$$

r^μ in Eq. (C.10) we find

$$(p_1 \cdot r) = E_1(e_3^\mu - t^\mu)(\sin\theta_{51} \times e_{3\mu} - \cos\theta_{51} \times b_\mu) = -E_1 \sin\theta_{51}\,. \tag{C.24}$$

We use the result in Eq. (C.24) and compute the required singular contribution of scalar products $(p_1 \cdot r)^2$ and $(p_1 \cdot p_5)(p_1 \cdot r)$ in Eq. (C.23). We obtain

$$(p_1 \cdot r)^2 = E_1^2 \sin^2\theta_{51} = E_1^2(1-\cos\theta_{51})(1+\cos\theta_{51}) = E_1^2\rho_{51}(2-\rho_{51}) \underset{p_5\|p_1}{\approx} 2E_1^2\rho_{51}\,, \tag{C.25}$$

and

$$(p_1 \cdot p_5)(p_1 \cdot r) = -E_1^2 E_5 \rho_{15} \sin\theta_{15} \underset{p_5\|p_1}{\sim} (\rho_{15})^{\frac{3}{2}}\,. \tag{C.26}$$

From Eq. (C.26) follows that the second term on the right-hand side of Eq. (C.23) does not contribute to the singular limit and we can neglect it. Using Eq. (C.25) in the first term on the right-hand side of Eq. (C.23) we obtain the singular limit of the amplitude

$$\begin{aligned}
&C_{51} \sum_{i,j,a,s} |A_{\text{sing}}|^2 \\
&= 2C_F g_{s,b}^2 \times \frac{1}{(2E_1E_5\rho_{51})^2} \times \left[4E_1^2\rho_{51}\left(\frac{E_1-E_5}{E_1}\right)\text{Tr}\left[\hat{p}_1\hat{M}^\dagger\hat{M}\right] + E_1E_5\rho_{51}\frac{E_5}{E_1}\text{Tr}\left[\hat{p}_1\hat{M}^\dagger\hat{M}\right]\right] \\
&= C_F g_{s,b}^2 \times \frac{1}{2E_1E_5\rho_{51}} \times \left[4\frac{E_1}{E_5} + \frac{E_5}{E_1-E_5}\right] \times \left(\frac{E_1-E_5}{E_1}\right)\text{Tr}\left[\hat{p}_1\hat{M}^\dagger\hat{M}\right].
\end{aligned} \tag{C.27}$$

The trace $\text{Tr}\left[\hat{p}_1\hat{M}^\dagger\hat{M}\right]$ in Eq. (C.27) is the colour and spin summed amplitude squared of the hard process. We introduce $E_5 = E_1(1-z)$ and use completeness relation

$$z \cdot \hat{p}_1 = \sum_i \bar{u}_i(z \cdot p_1) u_i(z \cdot p_1) \tag{C.28}$$

to write the trace as

$$\begin{aligned}
\text{Tr}\left[\hat{p}_1\hat{M}^\dagger\hat{M}\right] &= \frac{1}{z}\text{Tr}\left[(z\cdot\hat{p}_1)\hat{M}^\dagger\hat{M}\right] \\
&= \frac{1}{z}\sum_i [\hat{M}u_i(z\cdot p_1)]^\dagger[\hat{M}u_i(z\cdot p_1)] = \frac{1}{z}|M_{\text{lo}}^{\text{tree}}(z\cdot p_1, p_4)|^2\,,
\end{aligned} \tag{C.29}$$

In Eq. (C.29) $M_{\text{lo}}^{\text{tree}}$ is the colour and spin summed matrix element of the hard process. Finally, in F_{LM} notation Eq. (C.27) reads

$$C_{51} r_\mu^{(1)} r_\nu^{(1)} F_{\text{LM}}^{\mu\nu}(1_q, 4_q \,|\, 5_g) = 2C_F\, g_{s,b}^2 \times \frac{1}{4E_1^2\rho_{15}}\left(\frac{1+z}{1-z}\right)^2 \times \frac{F_{\text{LM}}(z\cdot 1_q, 4_q))}{z}\,. \tag{C.30}$$

D. Singular limits of one-loop functions F_{LV}

In this appendix we collect the singular limits one-loop amplitudes.

D.1. Single-soft limit: S_5

The single-soft limit reads

$$
\begin{aligned}
S_5\, F_{\text{LV}}(1_q, 4_q \,|\, 5_g) &= 2C_F\, g_{s,b}^2 \times \frac{p_1 \cdot p_4}{(p_1 \cdot p_5)(p_4 \cdot p_5)} \times F_{\text{LV}}(1_q, 4_q) \\
&- 2C_F C_A\, \frac{g_{s,b}^2 [\alpha_{s,b}]}{\epsilon^2} \left[\frac{\Gamma^5(1-\epsilon)\Gamma^3(1+\epsilon)}{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)} \right] \times 2^{-\epsilon} \left(\frac{p_1 \cdot p_4}{(p_1 \cdot p_5)(p_4 \cdot p_5)} \right)^{1+\epsilon} \\
&\times F_{\text{LM}}(1_q, 4_q)\,.
\end{aligned} \tag{D.1}
$$

D.2. Double-collinear limits: C_{5i}

The double-collinear limits to the initial read

$$
\begin{aligned}
C_{51} F_{\text{LV}}(1_q, 4_q \,|\, 5_g) &= g_{s,b}^2 \times \frac{1}{p_1 \cdot p_5} P_{qq}(z) \times \frac{F_{\text{LV}}(z \cdot 1_q, 4_q)}{z} \\
&+ g_{s,b}^2 [\alpha_{s,b}] \left[\frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \right] \times 2^{-\epsilon} \left(\frac{1}{p_1 \cdot p_5} \right)^{1+\epsilon} P_{qq}^{\text{loop}}(z) \times \frac{F_{\text{LM}}(z \cdot 1_q, 4_q)}{z}\,,
\end{aligned} \tag{D.2}
$$

where $z = (E_1 - E_5)/E_1$ and

$$
\begin{aligned}
C_{54} F_{\text{LV}}(1_q, 4_q \,|\, 5_g) &= g_{s,b}^2 \times \frac{1}{p_4 \cdot p_5} P_{qq}(z) \times F_{\text{LV}}\left(1_q, \frac{1}{z} \cdot 4_q\right) \\
&+ g_{s,b}^2 [\alpha_{s,b}] \times \left[\frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \right] \times 2^{-\epsilon} \cos(\pi\epsilon) \left(\frac{1}{p_4 \cdot p_5} \right)^{1+\epsilon} P_{qq}^{\text{loop}}(z) \\
&\times F_{\text{LM}}\left(1_q, \frac{1}{z} \cdot 4_q\right),
\end{aligned} \tag{D.3}
$$

where $z = E_4/(E_5 + E_5)$. The tree-level splitting function $P_{qq}(z)$ and the one-loop splitting function $P_{qq}^{\text{loop}}(z)$ are given in Eq. (E.1) and Eq. (E.17), respectively. For gluon-initiated contributions we find

$$C_{51} F_{\mathrm{LM},g}(1_g, 4_q \,|\, 5_q) = g_{s,b}^2 \times \frac{1}{p_1 \cdot p_5} \sum_{f \in \{q,\bar{q}\}} P_{fg}(z) \times \frac{F_{\mathrm{LV}}(z \cdot 1_f, 4_f)}{z} + g_{s,b}^2[\alpha_{s,b}] \times \left[\frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\right] \times 2^{-\epsilon} \left(\frac{1}{p_1 \cdot p_5}\right)^{1+\epsilon} \sum_{f \in \{q,\bar{q}\}} P_{fg}^{\mathrm{loop}}(z) \times \frac{F_{\mathrm{LM}}(z \cdot 1_f, 4_f)}{z}, \tag{D.4}$$

where $z = (E_1 - E_5)/E_1$. Splitting functions $P_{fg}(z)$ and $P_{fg}^{\mathrm{loop}}(z)$ are given in Eqs. (E.2, E.18).

D.3. Soft-collinear limits: $S_5 C_{5i}$

The required soft-collinear limit reads

$$S_5 C_{5i}\, F_{\mathrm{LV}}(1_q, 4_q \,|\, 5_g) = 2C_F\, g_{s,b}^2 \times \frac{1}{E_5^2\, \rho_{i5}} \times F_{\mathrm{LV}}(1_q, 4_q) - 2C_F C_A\, \frac{g_{s,b}^2[\alpha_{s,b}]}{\epsilon^2} \left[\frac{\Gamma^5(1-\epsilon)\Gamma^3(1+\epsilon)}{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)}\right] \times 2^{-\epsilon} \left(\frac{1}{E_5^2\, \rho_{i5}}\right)^{1+\epsilon} \times F_{\mathrm{LM}}(1_q, 4_q), \tag{D.5}$$

for $i \in \{1, 4\}$.

E. Splitting functions

In this appendix we collect various splitting functions for collinear limits in tree- and one-loop amplitudes, as well as Altarelli-Parisi splitting functions as arising in collinear renormalization of the parton distribution functions.

E.1. Double-collinear tree-level splitting functions

Here we collect required splitting functions for double-collinear splittings. In case of quarks they are given by

$$P_{qq}(z) = C_F\left[\frac{1+z^2}{1-z} - \epsilon(1-z)\right], \quad P_{qq}^{\text{spin}}(z) = \frac{C_F}{2}\,\frac{(1+z)^2}{1-z}, \tag{E.1}$$

$$P_{qg}(z) = T_R\left[1 - \frac{2z(1-z)}{1-\epsilon}\right], \quad P_{gq}(z) = C_F\left[\frac{1+(1-z)^2}{z} - \epsilon z\right], \tag{E.2}$$

Formulas for anti-quarks are identical, for instance $P_{\bar{q}g}(z) = P_{qg}(z)$. In case of splitting gluons spin-correlations occur. The required splitting functions read

$$P_{gg}^{\mu\nu}(z) = 2C_A\left[-g^{\mu\nu}\left(\frac{z}{1-z} + \frac{1-z}{z}\right) + 2(1-\epsilon)z(1-z)\kappa_\perp^\mu\kappa_\perp^\nu\right], \tag{E.3}$$

$$P_{gq}^{\mu\nu}(z) = T_R\left[-g^{\mu\nu} - 4z(1-z)\kappa_\perp^\mu\kappa_\perp^\nu\right], \tag{E.4}$$

where

$$\kappa_\perp^\mu = \frac{k_\perp^\mu}{\sqrt{-k_\perp^2}}, \quad k^\mu = \alpha p^\mu + \beta\bar{p}^\mu + k_\perp^\mu\,. \tag{E.5}$$

In Eqs. (E.3, E.4) $\kappa_\perp$ parametrizes the transverse momentum of a collinear gluon with momentum k to a gluon with momentum $p = (E_p, \vec{p})$, where $\bar{p} \equiv (E_p, -\vec{p})$. We also use the decomposition of the splitting function in Eqs. (E.3, E.4) that read

$$P_{gg}^{\mu\nu}(z) = -P_{gg}^{(0)}g^{\mu\nu} + P_{gg}^{\perp}(z)\kappa_\perp^\mu\kappa_\perp^\nu\,, \tag{E.6}$$

$$P_{gq}^{\mu\nu}(z) = -P_{gq}^{(0)}g^{\mu\nu} + P_{gq}^{\perp}(z)\kappa_\perp^\mu\kappa_\perp^\nu\,, \tag{E.7}$$

where

$$P_{gg}^{(0)} = 2C_A\left(\frac{z}{1-z} + \frac{1-z}{z}\right), \quad P_{gg}^{\perp}(z) = 4C_A(1-\epsilon)z(1-z)\,,$$
$$P_{gq}^{(0)} = T_R\,, \quad P_{gq}^{\perp}(z) = -4T_R z(1-z)\,. \tag{E.8}$$

Inside of an integral over $z \in [0,1]$ the following relations hold

$$(1-z)^{-n\epsilon}P_{qq}(z) = -\frac{2C_F}{n\epsilon}\delta(1-z) - C_F\big[(1+z)+\epsilon(1-z)\big] + 2C_F\left[\frac{(1-z)^{-n\epsilon}}{1-z}\right]_+, \tag{E.9}$$

$$(1-z)^{-4\epsilon}P_{qq}^{\text{loop}}(z) = C_F\Bigg[-\frac{\delta(1-z)}{2\epsilon} + \left(2\left[\frac{1}{1-z}\right]_+ - \frac{z+3}{2}\right)$$
$$-\epsilon\left(8\left[\frac{\ln(1-z)}{1-z}\right]_+ - 2(z+3)\ln(1-z)\right) + \mathcal{O}(\epsilon^2)\Bigg]. \tag{E.10}$$

Since Eq. (E.9) is finite, its expansion to arbitrary orders in ϵ is straightforward.

Convolutions of splitting functions

We use

$$\left[P_{qq}^{22}\otimes P_{qq}^{02}\right](z) = \int_z^1 \frac{\mathrm{d}\tilde z}{\tilde z}\,(1-\tilde z)^{-2\epsilon}P_{qq}(\tilde z)\times(\tilde z - z)^{-2\epsilon}P_{qq}\left(\frac{z}{\tilde z}\right)$$
$$= C_F^2\Bigg[\frac{\delta(1-z)}{\epsilon^2} + \frac{1}{\epsilon}2\big(1+z-2\mathcal{D}_0(z)\big) + 16\mathcal{D}_1(z) - \frac{2\pi^2}{3}\delta(1-z)$$
$$-(1+z)(8\ln(1-z)-\ln z) + \epsilon\bigg(\frac{8\pi^2\mathcal{D}_0(z)}{3} - 32\mathcal{D}_2(z) - (1+z)$$
$$\times\left\{2\pi^2 - 16\ln^2(1-z) + (\ln z - 2)\ln z - 4\mathrm{Li}_2(z)\right\}$$
$$-16\delta(1-z)\zeta_3\bigg) + \mathcal{O}(\epsilon^2)\Bigg], \tag{E.11}$$

$$[\mathcal{P}_{gg,RR_2}\otimes\mathcal{P}_{qg,R}](z) = T_R\beta_0\big[1-2z+2z^2+\epsilon\big(-1+2(-1+2z-2z^2)$$
$$\times\ln(1-z)\big) + \mathcal{O}(\epsilon^2)\big] + C_A T_R\Bigg[\frac{(1-z)(4+z(7+31z))}{z}$$
$$+2\big((1-2(1-z)z)\ln(1-z)+\ln z + 4z\ln z\big) + \epsilon\bigg(-\frac{1}{3} - \frac{2\pi^2}{3}$$
$$-\frac{2}{z} + \frac{46z}{3} - \frac{20\pi^2 z}{3} - 13z^2 + \frac{4\pi^2 z^2}{3} - 6\ln(1-z)$$
$$-\frac{16\ln(1-z)}{3z} - 32z\ln(1-z) + \frac{124z^2\ln(1-z)}{3} - 6\ln^2(1-z)$$
$$+12z\ln^2(1-z) - 10\ln z - \frac{8\ln z}{3z} - 20z\ln z - \frac{44z^2\ln z}{3}$$
$$-2\ln^2 z + (8+32z)\mathrm{Li}_2(z)\bigg) + \mathcal{O}(\epsilon^2)\Bigg]. \tag{E.12}$$

E.2. Triple-collinear tree-level splitting functions

The splitting function $P_{ggq}(z_1, z_2, z_3)$ for the $q \to q^\star + g + g$ splitting is taken from Ref. [56] to be

$$P_{ggq}(z_1, z_2, z_3) = C_F^2 P^{(\mathrm{ab})}_{g_1 g_2 q_3} + C_F C_A P^{(\mathrm{nab})}_{g_1 g_2 q_3}\,, \tag{E.13}$$

with the abelian part

$$\begin{aligned} P^{(\mathrm{ab})}_{g_1 g_2 q_3} = \Bigg\{ & \frac{s_{123}^2}{2 s_{13} s_{23}}\, z_3 \left[\frac{1+z_3^2}{z_1 z_2} - \epsilon \frac{z_1^2 + z_2^2}{z_1 z_2} - \epsilon(1+\epsilon) \right] \\ & + \frac{s_{123}}{s_{13}} \left[\frac{z_3(1-z_1) + (1-z_2)^3}{z_1 z_2} + \epsilon^2 (1+z_3) - \epsilon (z_1^2 + z_1 z_2 + z_2^2) \frac{1-z_2}{z_1 z_2} \right] \\ & + (1-\epsilon) \left[\epsilon - (1-\epsilon) \frac{s_{23}}{s_{13}} \right] \Bigg\} + (1 \leftrightarrow 2)\,, \end{aligned} \tag{E.14}$$

and the non-abelian part

$$\begin{aligned} P^{(\mathrm{nab})}_{g_1 g_2 q_3} = \Bigg\{ & (1-\epsilon) \left(\frac{t_{12,3}^2}{4 s_{12}^2} + \frac{1}{4} - \frac{\epsilon}{2} \right) + \frac{s_{123}^2}{2 s_{12} s_{13}} \left[\frac{(1-z_3)^2 (1-\epsilon) + 2 z_3}{z_2} \right. \\ & \left. + \frac{z_2^2 (1-\epsilon) + 2(1-z_2)}{1-z_3} \right] - \frac{s_{123}^2}{4 s_{13} s_{23}}\, z_3 \left[\frac{(1-z_3)^2 (1-\epsilon) + 2 z_3}{z_1 z_2} + \epsilon(1-\epsilon) \right] \\ & + \frac{s_{123}}{2 s_{12}} \left[(1-\epsilon) \frac{z_1 (2 - 2 z_1 + z_1^2) - z_2 (6 - 6 z_2 + z_2^2)}{z_2 (1-z_3)} + 2\epsilon \frac{z_3 (z_1 - 2 z_2) - z_2}{z_2 (1-z_3)} \right] \\ & + \frac{s_{123}}{2 s_{13}} \left[(1-\epsilon) \frac{(1-z_2)^3 + z_3^2 - z_2}{z_2 (1-z_3)} - \epsilon \left(\frac{2(1-z_2)(z_2 - z_3)}{z_2 (1-z_3)} - z_1 - z_2 \right) \right. \\ & \left. - \frac{z_3 (1-z_1) + (1-z_2)^3}{z_1 z_2} + \epsilon (1-z_2) \left(\frac{z_1^2 + z_2^2}{z_1 z_2} - \epsilon \right) \right] \Bigg\} + (1 \leftrightarrow 2)\,. \end{aligned} \tag{E.15}$$

The spin-correlated structure in Eq. (E.15) reads

$$t_{ij,k} \equiv 2\, \frac{z_i s_{jk} - z_j s_{ik}}{z_i + z_j} + \frac{z_i - z_j}{z_i + z_j}\, s_{ij}\,. \tag{E.16}$$

E.3. One-loop splitting functions

Required one-loop splitting functions read [58–60][1]

$$\begin{aligned} P^{\mathrm{loop}}_{qq}(z) = P_{qq}(z) \Bigg\{ & C_A \Bigg[-\frac{1}{\epsilon^2} + \frac{\ln(1-z)}{\epsilon} + \frac{1}{2} - \mathrm{Li}_2(z) - \frac{\ln^2(1-z)}{2} \\ & + \epsilon \left(1 + \ln(1-z) \left(\frac{\pi^2}{6} + \frac{\ln^2(1-z)}{6} - \frac{\ln(1-z) \ln z}{2} \right) - \mathrm{Li}_3(z) \right. \\ & \left. - \mathrm{Li}_3(1-z) + \zeta_3 \right) \Bigg] + (C_A - 2 C_F) \Bigg[-\frac{\ln z}{\epsilon} + \frac{1}{2} + \frac{\ln^2 z}{2} + \mathrm{Li}_2(1-z) \end{aligned} \tag{E.17}$$

[1] This formula is given in the tHV-regularization scheme [80].

$$
+\epsilon\Big(1-\frac{\pi^2}{6}\ln z-\frac{\ln^3 z}{6}+\frac{\ln(1-z)\ln^2 z}{2}+\mathrm{Li}_3(z)+\mathrm{Li}_3(1-z)-\zeta_3\Big)\Big]\Big\}
$$
$$
+P_{qq}^{\mathrm{new}}(z)\left\{\big(C_A+(C_A-2C_F)\big)\Big(-\frac{1}{2}-\epsilon\Big)\right\}+\mathcal{O}(\epsilon^2)\,,
$$

$$
\begin{aligned}
P_{qg}^{\mathrm{loop}}(z) = P_{qg}(z)\,&\Big\{C_A\Big[-\frac{1}{\epsilon^2}-\frac{\ln z}{\epsilon}+\mathrm{Li}_2(1-z)+\epsilon\mathrm{Li}_3(1-z)\Big] \\
&+(C_A-2C_F)\Big[\frac{\ln(1-z)-\ln(z)}{\epsilon}+\frac{\pi^2}{3}-\frac{\ln^2(1-z)}{2}-\mathrm{Li}_2(1-z) \\
&+\epsilon\Big(-\frac{\pi^2}{3}\ln(1-z)+\frac{\ln^3(1-z)}{6}+\mathrm{Li}_3(1-z)\Big)\Big]\Big\} \\
&+P_{qg}^{\mathrm{new}}(z)\Big\{\big(C_A+(C_A-2C_F)\big)\frac{1}{2(1-2\epsilon)}\Big\}+\mathcal{O}(\epsilon^2)\,,
\end{aligned}
\tag{E.18}
$$

where

$$
P_{qq}^{\mathrm{new}}=C_F\left[\frac{z(1+z)}{1-z}\right],\quad P_{qg}^{\mathrm{new}}=T_R\left[1-\frac{1-z}{1-\epsilon}\right]. \tag{E.19}
$$

E.4. Altarelli-Parisi splitting functions

In this appendix, all needed leading and next-to-leading order Altarelli-Parisi splitting functions, their convolutions and generalizations are collected. We further show LO cusp anomalous dimensions and generalizations.

Leading order splitting functions

The required leading order Altarelli-Parisi splitting functions are [74]

$$
\begin{aligned}
\hat{P}_{qq}^{(0)}(z) &= C_F\left[2\mathcal{D}_0(z)-(1+z)+\frac{3}{2}\,\delta(1-z)\right], \\
\hat{P}_{qg}^{(0)}(z) &= T_R\left[z^2+(1-z)^2\right], \\
\hat{P}_{gq}^{(0)}(z) &= C_F\left[\frac{1+(1-z)^2}{z}\right], \\
\hat{P}_{gg}^{(0)}(z) &= 2C_A\left[\mathcal{D}_0(z)+\frac{1}{z}+z(1-z)-2\right]+\beta_0\,\delta(1-z)\,,
\end{aligned}
\tag{E.20}
$$

where the function $\mathcal{D}_0(z)$ is defined in Eq. (A.25).

Next-to-leading order splitting functions

The required next-to-leading order Altarelli-Parisi splitting functions are [74]

$$
\begin{aligned}
\hat{P}^{(1)}_{qq,V}(z) = {} & C_A C_F \Bigg[\left(\frac{67}{9} - \frac{\pi^2}{3}\right) \mathcal{D}_0(z) + \left(-3\zeta_3 + \frac{17}{24} + \frac{11\pi^2}{18}\right) \delta(1-z) \\
& + \frac{\left(1+z^2\right) \ln^2(z)}{2(1-z)} + \frac{\left(5z^2+17\right) \ln(z)}{6(1-z)} + \frac{53-187z}{18} + (1+z)\zeta_2 \Bigg] \\
& - C_F T_R N_f \Bigg[\frac{20}{9}\mathcal{D}_0(z) + \left(\frac{1}{6} + \frac{2\pi^2}{9}\right) \delta(1-z) + \frac{2\left(1+z^2\right)\ln(z)}{3(1-z)} + \frac{4(1-z)}{3} \\
& - \frac{10(z+1)}{9} \Bigg] + C_F^2 \Bigg[\left(6\zeta_3 + \frac{3}{8} - \frac{\pi^2}{2}\right) \delta(1-z) - \frac{1}{2}(1+z)\ln^2(z) \\
& + \frac{\left(2z^2-2z-3\right)\ln(z)}{1-z} - \frac{2\left(1+z^2\right)\ln(1-z)\ln(z)}{1-z} - 5(1-z) \Bigg] , \\
\hat{P}^{(1)}_{q\bar{q},V}(z) = {} & \frac{C_F\left(2C_F - C_A\right)}{2} \Bigg[\frac{1+z^2}{1-z} \left(\ln^2(z) - 4\mathrm{Li}_2(-z) - 4\ln(z+1)\ln(z) - \frac{\pi^2}{3} \right) \\
& + 4(1-z) + 2(z+1)\ln(z) \Bigg] , \\
\hat{P}^{(1)}_{qq,s}(z) = {} & C_F T_R \Bigg[-\frac{56z^2}{9} + \left(\frac{8z^2}{3} + 5z + 1\right) \ln(z) + 6z + \frac{20}{9z} - (z+1)\ln^2(z) - 2 \Bigg] .
\end{aligned}
\tag{E.21}
$$

Convolutions of splitting functions

We also need various convolutions of splitting functions

$$
\begin{aligned}
\left[\hat{P}^{(0)}_{qq} \otimes \hat{P}^{(0)}_{qq}\right](z) = {} & C_F^2 \Bigg[8\mathcal{D}_1(z) + 6\mathcal{D}_0(z) + \left(\frac{9}{4} - \frac{2}{3}\pi^2\right) \delta(1-z) - \frac{(1+3z^2)}{1-z}\ln z \\
& - 4(1+z)\ln(1-z) - (5+z) \Bigg] , \\
\left[\hat{P}^{(0)}_{qg} \otimes \hat{P}^{(0)}_{gq}\right](z) = {} & C_F T_R \left[1 + \frac{4}{3z} - z - \frac{4}{3}z^2 + 2(1+z)\ln z \right] , \\
\left[\hat{P}^{(0)}_{qq} \otimes \hat{P}^{(0)}_{qg}\right](z) = {} & C_F T_R \left[2z - \frac{1}{2} + 2(1-2z+2z^2)\ln(1-z) - (1-2z+4z^2)\ln z \right] , \\
\left[\hat{P}^{(0)}_{qg} \otimes \hat{P}^{(0)}_{gg}\right](z) = {} & T_R \Bigg[\beta_0(1-2z+2z^2) + C_A \Bigg(-\frac{31z^2}{3} + 2(1-2z+2z^2)\ln(1-z) \\
& + 8z + \frac{4}{3z} + (2+8z)\ln z + 1 \Bigg) \Bigg] .
\end{aligned}
\tag{E.22}
$$

The sum of the last two convolutions in Eq. (E.22) reads

$$\sum_{x\in\{q,g\}} \left[\hat{P}^{(0)}_{qx} \otimes \hat{P}^{(0)}_{xg}\right](z) = \beta_0 T_R\big[z^2+(1-z)^2\big] + C_A T_R\bigg[1 - \frac{31z^2}{3} + 8z + \frac{4}{3z} + 2\big[z^2+(1-z)^2\big]\ln(1-z) + 2(4z+1)\ln(z)\bigg] + C_F T_R\bigg[-3z^2+5z-2 + 2\big[z^2+(1-z)^2\big]\ln(1-z) - \big[3z^2+(1-z)^2\big]\ln(z) + \frac{3}{2}\big[z^2+(1-z)^2\big]\bigg]. \tag{E.23}$$

E.5. Generalized splitting functions

We defined a number of generalized splitting functions as

$$\begin{aligned}
\mathcal{P}'_{qq}(z) &= C_F\big[4\mathcal{D}_1(z) + (1-z) - 2(1+z)\ln(1-z)\big]\,,\\
\mathcal{P}'_{qg}(z) &= T_R\big[2\big(z^2+(1-z)^2\big)\ln(1-z) + 2z(1-z)\big]\,,\\
\mathcal{P}'_{gq}(z) &= C_F\left[z + 2\left(\frac{1+(1-z)^2}{z}\right)\ln(1-z)\right]\,,\\
\mathcal{P}'_{gg}(z) &= C_A\left[4\mathcal{D}_1(z) + 4\left(\frac{1}{z} + z(1-z) - 2\right)\ln(1-z)\right]\,.
\end{aligned} \tag{E.24}$$

E.6. (Generalized) anomalous dimensions

The LO quark and gluon cusp anomalous dimensions read

$$\gamma_q = \frac{3}{2}C_F\,, \quad \gamma_g = \frac{11}{6}C_A - \frac{2}{3}T_R N_f\,. \tag{E.25}$$

We find it convenient to define various generalizations. We define

$$\begin{gathered}
\gamma'_q = C_F\left(\frac{13}{2} - \frac{2\pi^2}{3}\right), \quad \gamma'_g = C_A\left(\frac{67}{9} - \frac{2\pi^2}{3}\right) - \frac{23}{9}T_R N_f\,,\\
\gamma_{k_\perp,g} = -\frac{C_A}{3} + \frac{2}{3}T_R N_f\,,
\end{gathered} \tag{E.26}$$

and the ϵ dependent quantities

$$\begin{aligned}
\gamma^{nm}_{qq} &\equiv -\int_0^1 \mathrm{d}z\left[z^{-n\epsilon}(1-z)^{-m\epsilon}P_{qq}(z) - 2C_F\frac{(1-z)^{-m\epsilon}}{1-z}\right]\\
&= C_F\bigg[\frac{3}{2} + \epsilon\left(\frac{1}{2} + \frac{7m}{4} + \frac{5n}{4} - \frac{n\pi^2}{3}\right)\\
&\quad + \epsilon^2\left(\frac{m}{4} + \frac{15m^2}{8} + \frac{3n}{4} + 4mn + \frac{9n^2}{8} - \frac{mn\pi^2}{4} - 2n(m+n)\zeta_3\right) + \mathcal{O}(\epsilon^3)\bigg]\,,
\end{aligned} \tag{E.27}$$

$$
\begin{aligned}
\gamma^{24}_{qq,\mathrm{spin}} &= -\int_0^1 \mathrm{d}z \left[z^{-2\epsilon}(1-z)^{-4\epsilon} P^{\mathrm{spin}}_{qq}(z) - 2C_F \, \frac{(1-z)^{-4\epsilon}}{1-z} \right] \\
&= C_F \left[\frac{1}{4} + \epsilon \left(\frac{43}{4} - \frac{2\pi^2}{3} \right) + \epsilon^2 \left(\frac{261}{4} - \frac{7\pi^2}{3} - 24\zeta_3 \right) + \mathcal{O}(\epsilon^3) \right] ,
\end{aligned}
\tag{E.28}
$$

$$
\begin{aligned}
\gamma^{33}_{qq,\mathrm{loop}} &= \int_0^1 \mathrm{d}z \, z^{-3\epsilon}(1-z)^{-3\epsilon} P^{\mathrm{loop}}_{qq}(z) \\
&= C_A \left[-\frac{1}{2\epsilon^3} - \frac{3}{2\epsilon^2} + \frac{1}{\epsilon} \left(-10 + \frac{7\pi^2}{12} \right) - 63 + \frac{9\pi^2}{4} + 16\zeta_3 \right. \\
&\quad \left. + \epsilon \left(-\frac{781}{2} + \frac{45\pi^2}{3} + \frac{83\pi^4}{240} + 81\zeta_3 \right) + \mathcal{O}(\epsilon^2) \right] + C_F \left[\frac{1}{\epsilon} \left(\frac{-15}{6} + \frac{4\pi^2}{6} \right) \right. \\
&\quad \left. - 31 + \pi^2 + 44\zeta_3 + \epsilon \left(-\frac{575}{2} + \frac{39\pi^2}{4} + \frac{8\pi^4}{3} + 66\zeta_3 \right) + \mathcal{O}(\epsilon^2) \right] .
\end{aligned}
\tag{E.29}
$$

and

$$
\begin{aligned}
\gamma^{22}_{gg} &= -\int_0^1 \mathrm{d}z \, \left[z^{-2\epsilon}(1-z)^{-2\epsilon} \left[P_{gg}(z,\epsilon) + \epsilon P^{\perp}_{gg}(z) \right] - 2C_A \left(\frac{z^{-2\epsilon}}{z} + \frac{(1-z)^{-2\epsilon}}{1-z} \right) \right] \\
&= C_A \left[\frac{11}{3} + \epsilon \left(\frac{131}{9} - \frac{4\pi^2}{3} \right) + \epsilon^2 \left(\frac{1604}{27} - \frac{22\pi^2}{9} - 32\zeta_3 \right) + \mathcal{O}(\epsilon^3) \right] ,
\end{aligned}
\tag{E.30}
$$

$$
\gamma^{\perp,22}_{gg} = \int_0^1 \mathrm{d}z \, z^{-2\epsilon}(1-z)^{-2\epsilon} P^{\perp}_{gg}(z) = C_A \left[\frac{2}{3} + \epsilon \frac{14}{9} + \epsilon^2 \left(\frac{164}{27} - \frac{4\pi^2}{9} \right) + \mathcal{O}(\epsilon^3) \right] .
\tag{E.31}
$$

as well as

$$
\begin{aligned}
\gamma^{22}_{qg} &= \int_0^1 \mathrm{d}z \, z^{-2\epsilon}(1-z)^{-2\epsilon} \left[P_{gq}(z,\epsilon) + \epsilon P^{\perp}_{gq}(z) \right] \\
&= T_R \left[\frac{2}{3} + \epsilon \frac{26}{9} + \epsilon^2 \left(\frac{320}{27} - \frac{4\pi^2}{9} \right) + \mathcal{O}(\epsilon^3) \right] ,
\end{aligned}
\tag{E.32}
$$

$$
\gamma^{\perp,22}_{gq} = -\int_0^1 \mathrm{d}z \, z^{-2\epsilon}(1-z)^{-2\epsilon} P^{\perp}_{gq}(z) = C_A \left[\frac{2}{3} + \epsilon \frac{20}{9} + \epsilon^2 \left(\frac{224}{27} - \frac{4\pi^2}{9} \right) + \mathcal{O}(\epsilon^3) \right] .
\tag{E.33}
$$

E.7. Integrals over splitting functions

We use

$$\begin{aligned}\delta_g(\epsilon) &\equiv \frac{N_\epsilon\, E_{\max}^{4\epsilon}}{2} \int\limits_{E_{\max}}^{2E_{\max}} \mathrm{d}E_{56}\, E_{56}^{-1-4\epsilon} \int\limits_{1-\frac{E_{\max}}{E_{56}}}^{\frac{E_{\max}}{E_{56}}} \mathrm{d}z\, z^{-2\epsilon}(1-z)^{-2\epsilon} \left[P_{gg}(z,\epsilon) + \epsilon\, P_{gg}^{\perp}(z)\right] \\ &= C_A\left[-\frac{131}{72} + \frac{\pi^2}{6} + \frac{11}{6}\ln 2 + \epsilon\left(-\frac{1541}{216} + \frac{11\pi^2}{18} - \frac{\ln 2}{6} + 4\zeta_3\right)\right. \\ &\qquad \left. + \epsilon^2\left(-\frac{9607}{324} + \frac{125\pi^2}{216} + \frac{7\pi^4}{45} + \ln 2 + \frac{11\pi^2 \ln 2}{18} + \frac{77}{6}\zeta_3\right) + \mathcal{O}(\epsilon^3)\right],\end{aligned} \tag{E.34}$$

$$\begin{aligned}\delta_g^{\perp}(\epsilon) &\equiv \frac{N_\epsilon (2E_{\max})^{4\epsilon}}{2} \int\limits_{E_{\max}}^{2E_{\max}} \mathrm{d}E_{67}\, E_{67}^{-1-4\epsilon} \int\limits_{1-\frac{E_{\max}}{E_{67}}}^{\frac{E_{\max}}{E_{67}}} \mathrm{d}z\, z^{-2\epsilon}(1-z)^{-2\epsilon} P_{gg}^{\perp}(z) \\ &= C_A\left[\frac{13}{36} - \frac{\ln 2}{3} + \mathcal{O}(\epsilon)\right],\end{aligned} \tag{E.35}$$

and

$$\begin{aligned}\delta_q(\epsilon) &\equiv \frac{N_\epsilon\, E_{\max}^{4\epsilon}}{2} \int\limits_{E_{\max}}^{2E_{\max}} \mathrm{d}E_{56}\, E_{56}^{-1-4\epsilon} \int\limits_{1-\frac{E_{\max}}{E_{56}}}^{\frac{E_{\max}}{E_{56}}} \mathrm{d}z\, z^{-2\epsilon}(1-z)^{-2\epsilon} \left[P_{gq}(z,\epsilon) + \epsilon\, P_{gq}^{\perp}(z)\right] \\ &= T_R\left[\frac{23}{72} - \frac{\ln 2}{3} + \epsilon\left(\frac{103}{108} - \frac{\pi^2}{9} + \frac{35\ln 2}{36} - \frac{2\ln^2 2}{3}\right)\right. \\ &\qquad \left. + \epsilon^2\left(\frac{373}{81} - \frac{5\pi^2}{216} + \frac{67\ln 2}{54} - \frac{\pi^2 \ln 2}{3} + \frac{47\ln^2 2}{36} - \frac{2\ln^3 2}{3} - \frac{7}{3}\zeta_3\right) + \mathcal{O}(\epsilon^3)\right],\end{aligned} \tag{E.36}$$

$$\begin{aligned}\delta_q^{\perp}(\epsilon) &\equiv -\frac{N_\epsilon (2E_{\max})^{4\epsilon}}{2} \int\limits_{E_{\max}}^{2E_{\max}} \mathrm{d}E_{67}\, E_{67}^{-1-4\epsilon} \int\limits_{1-\frac{E_{\max}}{E_{67}}}^{\frac{E_{\max}}{E_{67}}} \mathrm{d}z\, z^{-2\epsilon}(1-z)^{-2\epsilon} P_{gq}^{\perp}(z) \\ &= T_R\left[\frac{13}{36} - \frac{\ln 2}{3} + \mathcal{O}(\epsilon)\right],\end{aligned} \tag{E.37}$$

where N_ϵ is defined in Eq. (6.128) and $P_{ij}(z,\epsilon) \equiv P_{ij}^{(0)}(z) + P_{ij}^{\perp}(z)/2$.

F. Phase space parametrization

In this appendix the parametrization that is used in this calculation is presented. To calculate the collinear subtraction term analytically it is crucial that the d- dimensional phase space is parametrized in such a way that it has a simple factorized behavior in the double- and triple-collinear limits.

We separate energy and angular phase of real emissions with momenta p_5 and p_6 by writing

$$[\mathrm{d}p_5][\mathrm{d}p_6] = \mathrm{d}E_5\, E_5^{1-2\epsilon}\, \mathrm{d}E_6\, E_6^{1-2\epsilon} \times [\mathrm{d}\Omega_5][\mathrm{d}\Omega_6]\,, \tag{F.1}$$

where

$$[\mathrm{d}\Omega_i] = \frac{\mathrm{d}\Omega_i^{(d-1)}}{2(2\pi)^{d-1}}\,. \tag{F.2}$$

is the element of a $(d-1)$-dimensional solid angle of parton i.

We discuss the parametrization of the angular phase space $[\mathrm{d}\Omega_5][\mathrm{d}\Omega_6]$ in double-collinear partitions $5i, 6j$, with $i, j \in \{1, 4\}$ and $i \neq j$, in Section F.1 and in triple-collinear partitions $5i, 6i$, with $i \in \{1, 4\}$, in Section F.2.

F.1. Double-collinear partitions $5i, 6j$, $i, j \in \{1, 4\}$, $i \neq j$

We consider double-collinear partitions $5i, 6j$, with $i, j \in \{1, 4\}$ and $i \neq j$. Directions of momenta p_5 and p_6 are parametrized independently and in the same way. For the sake of definiteness we show the parametrization of p_5 for which the collinear singularity ($\vec{p}_5 \parallel \vec{p}_i$) is present. The direction is written as

$$n_5^\mu = t^\mu + \cos\theta_5\, e_i^\mu + \sin\theta_5\, b^\mu\,, \tag{F.3}$$

with $e_i = (0, \vec{n}_i)$ is the direction of hard momentum p_i and $t = (1, \vec{0})$. The vector b^μ is chosen in such a way that

$$t \cdot b = e_i \cdot b = 0\,. \tag{F.4}$$

Given this choice, the angular phase space is written as

$$[\mathrm{d}\Omega_5] = \frac{\mathrm{d}\Omega_b^{(d-2)}}{2(2\pi)^{d-1}}\, \mathrm{d}\eta_5\, [\eta_5(1-\eta_5)]^{-\epsilon}\,, \tag{F.5}$$

where $\eta_5 = \eta_{i5} = (1-\cos\theta_5)/2$.

We require the phase space in the double-collinear limit C_{5i}. It reads

$$\left[C_{5i}\,\mathrm{d}\Omega_5\right] = \frac{\mathrm{d}\Omega_b^{(d-2)}}{2(2\pi)^{d-1}}\,\mathrm{d}\eta_5\,\eta_5^{-\epsilon}\,. \tag{F.6}$$

F.2. Triple-collinear partitions $5i, 6i$, $i \in \{1,4\}$

In this section we present the parametrization of $[\mathrm{d}\Omega_5][\mathrm{d}\Omega_6]$ in triple-collinear partitions $5i, 6i$, with $i \in \{1,4\}$. The following parametrization is taken from Ref. [6]. We use the notation defined in Appendix A.2 for four-momenta.

Directions of the momenta are written as

$$\begin{aligned} n_5^\mu &= t^\mu + \cos\theta_5\, e_i^\mu + \sin\theta_5\, b^\mu\,, \\ n_6^\mu &= t^\mu + \cos\theta_6\, e_i^\mu + \sin\theta_6\left(\cos\varphi\, b^\mu + \sin\varphi\, a^\mu\right)\,, \end{aligned} \tag{F.7}$$

with $e_i = (0, \vec{n}_i)$, $t = (1, \vec{0})$ and $i \in \{1,4\}$, depending on the considered partition $w^{5i,6i}$. The vectors a^μ and b^μ are chosen in such a way that

$$t\cdot b = e_i \cdot b = 0\,, \quad t\cdot a = e_i\cdot a = b\cdot a = 0\,. \tag{F.8}$$

Given this choice, the angular phase space volume element is written as

$$\begin{aligned} &[\mathrm{d}\Omega_{56}] \equiv [\mathrm{d}\Omega_5][\mathrm{d}\Omega_6] \\ &= \frac{\mathrm{d}\Omega_b^{(d-2)}\mathrm{d}\Omega_a^{(d-3)}}{2^{6\epsilon}(2\pi)^{2d-2}}[\eta_5(1-\eta_5)]^{-\epsilon}[\eta_6(1-\eta_6)]^{-\epsilon}\frac{|\eta_5-\eta_6|^{1-2\epsilon}}{D^{1-2\epsilon}}\frac{\mathrm{d}\eta_5\,\mathrm{d}\eta_6\,\mathrm{d}\lambda}{[\lambda(1-\lambda)]^{\frac{1}{2}+\epsilon}}\,, \end{aligned} \tag{F.9}$$

where $\eta_5 = \eta_{i5} = (1-\cos\theta_5)/2$, $\eta_6 = \eta_{i6} = (1-\cos\theta_6)/2$ and λ is related to η_{56} through

$$\sin^2\varphi = 2\lambda(1-\lambda)\eta_{56}\,, \quad \eta_{56} = \frac{|\eta_5-\eta_6|^2}{D}\,. \tag{F.10}$$

In Eqs. (F.9, F.10) we also used

$$D = \eta_5 + \eta_6 - 2\eta_5\eta_6 + 2(2\lambda-1)\sqrt{\eta_5\eta_6(1-\eta_5)(1-\eta_6)}\,. \tag{F.11}$$

In the triple-collinear partitions $5i, 6i$ the angular phase space is split into four sectors, see Fig. 6.3; they are labeled with (a) - (d). In the different sectors we use

$$\begin{aligned} &(a) \quad \eta_5 = x_3\,, && \eta_6 = \frac{x_3x_4}{2}\,, \\ &(b) \quad \eta_5 = x_3\,, && \eta_6 = x_3\left(1-\frac{x_4}{2}\right)\,, \\ &(c) \quad \eta_5 = \frac{x_3x_4}{2}\,, && \eta_6 = x_3\,, \\ &(d) \quad \eta_5 = x_3\left(1-\frac{x_4}{2}\right)\,, && \eta_6 = x_3\,. \end{aligned} \tag{F.12}$$

The parametrization Eq. (F.12) is chosen to ensure consistency with the angular ordering in the different sectors. Since η_5 and η_6 enter Eq. (F.9) in a symmetric way we obtain the same result for the phase space in sectors (a) and (c) as well as in sectors (b) and (d).

For sectors (a,c) we obtain

$$\begin{aligned}\left[\mathrm{d}\Omega_{56}\,\theta^{(a,c)}\right] = &\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right] \times \frac{\mathrm{d}\Omega_b^{(d-2)}}{\Omega^{d-2}}\frac{\mathrm{d}\Omega_a^{(d-3)}}{\Omega^{d-3}} \\ &\times \frac{\mathrm{d}x_3}{x_3^{1+2\epsilon}}\frac{\mathrm{d}x_4}{x_4^{1+2\epsilon}}\frac{\mathrm{d}\lambda}{\pi[\lambda(1-\lambda)]^{\frac{1}{2}+\epsilon}}\left(256F_\epsilon^{(a,c)}\right)^{-\epsilon} 4F_0^{(a,c)}x_3^2x_4\,.\end{aligned} \tag{F.13}$$

In Eq. (F.13) we introduced

$$F_\epsilon^{(a,c)} \equiv \frac{(1-x_3)(1-x_3x_4/2)(1-x_4/2)^2}{2N(x_3,x_4,\lambda)^2}\,, \quad F_0^{(a,c)} \equiv \frac{1-x_4/2}{2N(x_3,x_4/2,\lambda)}\,, \tag{F.14}$$

with

$$N(x_3,x_4,\lambda) = 1 + x_4(1-2x_3) - 2(1-2\lambda)\sqrt{x_4(1-x_3)(1-x_3x_4)}\,. \tag{F.15}$$

We require the phase space element in double-collinear limits that are present in sectors (a) and (c). In parametrization Eq. (F.12) they correspond to the limit $x_4 \to 0$. The relevant limits read

$$\lim_{x_4\to 0} F_\epsilon^{(a,c)} = \frac{1-x_3}{2}\,, \quad \lim_{x_4\to 0} F_0^{(a,c)} = \frac{1}{2}\,. \tag{F.16}$$

We continue with sectors (b,d). Using parametrizations Eq. (F.12) in the phase space Eq. (F.9) we obtain

$$\begin{aligned}\left[\mathrm{d}\Omega_{56}\,\theta^{(b,d)}\right] = &\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]^2\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right] \times \frac{\mathrm{d}\Omega_b^{(d-2)}}{[\Omega^{(d-2)}]}\frac{\mathrm{d}\Omega_a^{(d-3)}}{[\Omega^{(d-3)}]} \\ &\times \frac{\mathrm{d}x_3}{x_3^{1+2\epsilon}}\frac{\mathrm{d}x_4}{x_4^{1+2\epsilon}}\frac{\mathrm{d}\lambda}{\pi(\lambda(1-\lambda))^{\frac{1}{2}+\epsilon}}\left(256F_\epsilon^{(b,d)}\right)^{-\epsilon} 4F_0^{(b,d)}x_3^2x_4^2\,,\end{aligned} \tag{F.17}$$

In Eq. (F.17) functions $F_\epsilon^{(b,d)}$ and $F_0^{(b,d)}$ are defined as

$$F_\epsilon^{(b,d)} \equiv \frac{(1-x_3)(1-x_4/2)(1-x_3(1-x_4/2))}{4N\left(x_3,1-x_4/2,\lambda\right)^2}\,, \quad F_0^{(b,d)} \equiv \frac{1}{4N\left(x_3,1-x_4/2,\lambda\right)}\,, \tag{F.18}$$

where N is defined in Eq. (F.15). In sectors (b,d) the double-collinear $(p_5 \parallel p_6)$ singularity is present. In this parametrization η_{56} reads

$$\eta_{56} = \frac{x_3x_4^2}{4N(x_3,1-x_4,\lambda)} \underset{x_4\to 0}{\longrightarrow} \frac{x_3x_4^2}{16\lambda(1-x_3)} \equiv \bar{\eta}_{56}\,. \tag{F.19}$$

Hence, the double collinear limit is, similarly to sectors (a,c), given by $x_4 \to 0$. With the limits

$$\lim_{x_4\to 0} F_\epsilon^{(b,d)} = \frac{1}{2^6\lambda^2}\,, \quad \lim_{x_4\to 0} F_0^{(b,d)} = \frac{1}{16(1-x_3)\lambda}\,, \tag{F.20}$$

we can write Eq. (F.17) in the $x_4 \to 0$ limit as

$$\begin{aligned}
&\lim_{x_4\to 0}\left[\mathrm{d}\Omega_{56}\theta^{(b,d)}\right] \\
&= \left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]\left[\frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)}\right] \times [\mathrm{d}\Omega_{56}]\;\eta_{i,56}^{-\epsilon}\left(1-\eta_{i,56}\right)^{\epsilon} \\
&\times \mathrm{d}\Lambda\,\frac{\mathrm{d}\Omega_a^{(d-3)}}{[\Omega^{(d-3)}]} \times 2\bar{\eta}_{56}\frac{\mathrm{d}x_4}{x_4^{1+2\epsilon}}\,.
\end{aligned} \tag{F.21}$$

In Eq. (F.21) we already rewrote integration over b and x_3 to obtain $[\mathrm{d}\Omega_{56}]$ that corresponds to the angular phase space element of the combined direction $p_5 + p_6$ after taking the double collinear C_{56} limit. We also defined the normalized volume element

$$\mathrm{d}\Lambda \equiv \left[\frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}{\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)}\right] \times \frac{1}{\pi(\lambda(1-\lambda))^{\frac{1}{2}+\epsilon}} \times \mathrm{d}\lambda\,, \tag{F.22}$$

with

$$\int \mathrm{d}\Lambda = 1\,, \quad \int \mathrm{d}\Lambda\,\lambda = \frac{1+2\epsilon}{2}\,, \quad \int \mathrm{d}\Lambda\,(1-\lambda) = \frac{1-2\epsilon}{2}\,. \tag{F.23}$$

F.3. Unresolved phase space integral in the C_{56} limit

In this section we discuss the phase space parametrization dependent integration over unresolved phase space in the double-collinear subtraction term Eq. (6.114). We consider sector (b); the relevant limit is written in Eq. (6.115).

We begin with the integration over the second term on the right hand side of Eq. (6.115). The relevant parametrization dependent integral is given in Eq. (6.121). The integral in question reads

$$g_{s,b}^2 \int \left(C_{56}[\mathrm{d}\Omega_5][\mathrm{d}\Omega_6]\theta^{(b)}\right)\frac{1}{p_5\cdot p_6}\,\kappa_{\perp\mu}\kappa_{\perp\nu}F_{\mathrm{LM}}^{\mu\nu}\left(1,4\,|\,\frac{1}{z}\cdot 5\right)\,. \tag{F.24}$$

Vector $\kappa_\perp$ is defined around Eq. (6.120) and the properties of function $F_{\mathrm{LM}}^{\mu\nu}$ are discussed around Eq. (6.118).

In the phase space parametrization Eq. (F.21) integral Eq. (F.24) reads

$$
\begin{aligned}
&g_{s,b}^2 \int (C_{56}[\mathrm{d}\Omega_5][\mathrm{d}\Omega_6]\theta^{(b)})\ \frac{1}{p_5\cdot p_6}\ \kappa_{\perp\mu}\kappa_{\perp\nu}F_{\mathrm{LM}}^{\mu\nu}\left(1,4\,|\,\frac{1}{z}\cdot 5\right) \\
&= [\alpha_{s,b}]\left[\frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)}\right]\frac{1}{E_5E_6}\int [\mathrm{d}\Omega_{56}]\ \eta_{1,56}^{-\epsilon}(1-\eta_{1,56})^{\epsilon}\times \underbrace{\int_0^1 \frac{\mathrm{d}x_4}{x_4^{1+2\epsilon}}}_{=-\frac{1}{2\epsilon}} \\
&\times \int \mathrm{d}\Lambda\ \frac{\mathrm{d}\Omega_a^{(d-3)}}{[\Omega^{(d-3)}]}\ \kappa_{\perp\mu}\kappa_{\perp\nu}\ F_{\mathrm{LM}}^{\mu\nu}(1,4\,|\,56)\ ,
\end{aligned}
\tag{F.25}
$$

where the last argument of function $F_{\mathrm{LM}}^{\mu\nu}$ has to be understood as gluon that carries momentum $p_{56} = (E_5+E_6)\cdot n_{56}$. In this parametrization, the vector $\kappa_\perp$ reads[1]

$$
\kappa_\perp^\mu = a^\mu\sqrt{1-\lambda} + r^\mu\sqrt{\lambda} \quad \text{with} \quad r^\mu = \sin\theta_{15}\ e_1^\mu - \cos\theta_{15}\ b^\mu\,. \tag{F.26}
$$

We can further average over directions of $\kappa_\perp$. We find

$$
\langle \kappa_\perp^\mu \kappa_\perp^\nu \rangle \equiv \int \mathrm{d}\Lambda \frac{\mathrm{d}\Omega_a^{(d-3)}}{[\Omega^{(d-3)}]}\kappa_\perp^\mu\kappa_\perp^\nu = -\frac{1}{2}g_\perp^{\mu\nu} + \epsilon r^\mu r^\nu\,. \tag{F.27}
$$

where $g_\perp^{\mu\nu}$ is the metric tensor of the space tangential to p_{56}. Contracting Eq. (F.27) with $F_{\mathrm{LM}}^{\mu\nu}(1,4\,|\,56)$ we obtain

$$
\left[-\frac{1}{2}g_{\mu\nu}^{\perp} + \epsilon r_\mu r_\nu\right]F_{\mathrm{LM}}^{\mu\nu}(1,4\,|\,56) = \frac{1}{2}F_{\mathrm{LM}}(1,4\,|\,56) + \epsilon r_\mu r_\nu\ F_{\mathrm{LM}}^{\mu\nu}(1,4\,|\,56)\,. \tag{F.28}
$$

Using Eqs. (F.27, F.28) to rewrite Eq. (F.25) we obtain

$$
\begin{aligned}
&g_{s,b}^2 \int (C_{56}[\mathrm{d}\Omega_5][\mathrm{d}\Omega_6]\theta^{(b)})\ \frac{1}{p_5\cdot p_6}\ \kappa_{\perp\mu}\kappa_{\perp\nu}F_{\mathrm{LM}}^{\mu\nu}\left(1,4\,|\,\frac{1}{z}\cdot 5\right) \\
&= -\frac{[\alpha_{s,b}]}{2\epsilon}\left[\frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)}\right]\frac{1}{E_5E_6}\int [\mathrm{d}\Omega_{56}]\ \eta_{1,56}^{-\epsilon}(1-\eta_{1,56})^{\epsilon} \\
&\times \left[\frac{1}{2}F_{\mathrm{LM}}(1,4\,|\,56) + \epsilon r_\mu r_\nu\ F_{\mathrm{LM}}^{\mu\nu}(1,4\,|\,56)\right]\,.
\end{aligned}
\tag{F.29}
$$

We now consider the integration of the first term on the right-hand side of Eq. (6.115) over the angular phase space of the unresolved momentum. In this case the direction of the gluon $g(p_6)$ completely factorizes from the function F_{LM}. Hence, after employing parametrization Eq. (F.21) we use

$$
\int \mathrm{d}\Lambda = 1\,, \quad \int \frac{\mathrm{d}\Omega_a^{(d-3)}}{[\Omega^{(d-3)}]} = 1\,, \tag{F.30}
$$

[1] For completeness, in sector (d) we find $r^\mu = -\sin\theta_{16}\ e_1^\mu + \cos\theta_{16}\ b^\mu$.

we immediately obtain

$$
\begin{aligned}
& g_{s,b}^2 \int (C_{56}[\mathrm{d}\Omega_5][\mathrm{d}\Omega_6]\theta^{(b)})\ \frac{1}{p_5 \cdot p_6}\ F_{\mathrm{LM}}\left(1,4\,|\,\frac{1}{z}\cdot 5\right) \\
& = -\frac{[\alpha_{s,b}]}{2\epsilon}\left[\frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)}\right]\frac{1}{E_5 E_6}\int [\mathrm{d}\Omega_{56}]\eta_{1,56}^{-\epsilon}(1-\eta_{1,56})^{\epsilon} F_{\mathrm{LM}}(1,4\,|\,56)\,.
\end{aligned}
\tag{F.31}
$$

Using the results in Eqs. (F.29, F.31) we write the contribution of sector (b) to the subtraction term Eq. (6.114) as

$$
\begin{aligned}
& \Big\langle [1-\mathbb{S}]\,[1-S_6]\,\theta^{(b)} C_{56}[\mathrm{d}p_6][\mathrm{d}p_7] w^{51,61} F_{\mathrm{LM}}(1,4\,|\,5,6)\Big\rangle \\
& = -\frac{[\alpha_{s,b}]}{2\epsilon}\left[\frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)}\right] \\
& \quad \times \int [\mathrm{d}p_5] w_{\mathrm{tc}}^{1} x_3^{-\epsilon}(1-x_3)^{\epsilon} \int_0^{E_5} \mathrm{d}E_6\ E_6^{-1-2\epsilon}\,[I-\mathbb{S}]\,[I-S_6]\,\mathcal{P}_{56}(1,4,5,6)\,,
\end{aligned}
\tag{F.32}
$$

where we defined

$$
\mathcal{P}_{56}(1,4,5,6) \equiv \frac{E_6}{E_5}\left[P_{gg}(z,\epsilon)F_{LM}(1,4|5+6) + \epsilon P_{gg}^{\perp}(z,k_{\perp}) r_{\mu} r_{\nu} F_{LM}^{\mu\nu}(1,4|5+6)\right]\,, \tag{F.33}
$$

and

$$
P_{gg}(z,\epsilon) \equiv P_{gg}^{(0)}(z) + \frac{1}{2}P_{gg}^{\perp}(z) = 2C_A\left(\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)(1-\epsilon)\right)\,. \tag{F.34}
$$

G. Some phase space integrals

In this appendix we collect phase space integrals that appear regularly throughout the calculation.

G.1. Energy integrals

Required energy integrals read

$$\int_0^{E_{\rm max}} \mathrm{d}E_i \, E_i^{1-n\epsilon} \frac{1}{E_i^2} = -\frac{E_{\rm max}^{-n\epsilon}}{n\epsilon} \,, \tag{G.1}$$

$$\int_{E_{\rm min}}^{E_{\rm max}} \mathrm{d}E_i \, E_i^{1-n\epsilon} \frac{1}{E_i^2} = -\frac{E_{\rm min}^{-n\epsilon} - E_{\rm max}^{-n\epsilon}}{n\epsilon} \,. \tag{G.2}$$

G.2. Angular integrals

For the d-dimensional solid-angle element we use the notations

$$[\mathrm{d}\Omega_q] \equiv \frac{\mathrm{d}\Omega_q^{(d-1)}}{2(2\pi)^{d-1}} \,. \tag{G.3}$$

We define the solid angle volume in $(d-1)$-dimensions as

$$[\Omega^{(d-2)}] \equiv \int \frac{\mathrm{d}\Omega^{(d-2)}}{2(2\pi)^{d-1}} = \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right] . \tag{G.4}$$

Basic integrals read

$$\int [\mathrm{d}\Omega_k] \frac{1}{\rho_{ki}} = -\frac{2^{-2\epsilon}}{\epsilon} \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right] \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right] , \tag{G.5}$$

$$\int [\mathrm{d}\Omega_k] \frac{\rho_{ij}}{\rho_{ki}\rho_{kj}} = -\frac{2^{1-2\epsilon}}{\epsilon} \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right] \eta_{ij}^{-\epsilon} K_{ij} \,, \tag{G.6}$$

where ρ_{ij} and η_{ij} are defined in Appendix A.2 and

$$K_{ij} = \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right] \eta_{ij}^{1+\epsilon} {}_2\mathrm{F}_1\big(1,1;1-\epsilon;1-\eta_{ij}\big) \,. \tag{G.7}$$

An expansion of K_{ij} in the dimensional regularization parameter ϵ is given in Eq. (A.22).

In addition we need several variations of integrals Eqs. (G.5, G.6). A generic variation reads

$$\int [\mathrm{d}\Omega_k] \left(\frac{\rho_{ki}}{4}\right)^{-\epsilon} \frac{1}{\rho_{ki}} = \int \frac{\mathrm{d}\Omega_6^{(d-2)}}{2(2\pi)^{(d-1)}} \times \underbrace{\frac{1}{2^{1-\epsilon}} \int_0^1 \frac{2\mathrm{d}x}{[4x(1-x)]^\epsilon} \frac{1}{x^{1+\epsilon}}}_{= -\frac{2^{-\epsilon}}{2\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}}$$
$$= -\frac{2^{-2\epsilon}}{\epsilon} \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right] \left[\frac{2^\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right], \tag{G.8}$$

where in the first step we parametrized $\vec{n}_k$ relative to $\vec{n}_i$ and used the substitution $x = (1 - \cos\theta_{ki})/2$.

Other required variations depend on the phase space parametrization, see Appendix F, and sector decomposition, see Appendix A.3.3. We use

$$\int \left[C_{ki}\, \mathrm{d}\Omega_k\right] \frac{1}{\rho_{ki}} = -\frac{2^{-2\epsilon}}{\epsilon} \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right], \tag{G.9}$$

and

$$\int \theta\left(\eta_{ki} < \frac{\eta_{ji}}{2}\right) \left[C_{ki}\, \mathrm{d}\Omega_k\right] \frac{1}{\rho_{ki}} = -\frac{1}{\epsilon} \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right] \rho_{ji}^{-\epsilon}. \tag{G.10}$$

G.3. Generic solid angle integrals

In this appendix we collect various generic angular integrals that do not appear directly but are required indirectly, for instance in the computation of partitioning dependent integrals in Appendix H. These integrals read

$$\int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]} \frac{1}{\rho_{1q}^\alpha} = 2^{1+\alpha-2\epsilon} \times \frac{\Gamma(1-\epsilon)\Gamma(1+\alpha-\epsilon)}{\Gamma(2+\alpha-2\epsilon)}, \quad \text{with} \quad \mathrm{Re}(\alpha) \geq -1, \tag{G.11}$$

$$\int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]} \frac{1}{\rho_{1q}^\alpha \rho_{2q}^\beta} = 2^{1-(\alpha+\beta)-2\epsilon} \times \frac{\Gamma(1-\epsilon-\alpha)\Gamma(1-\epsilon-\beta)}{\Gamma(2-(\alpha+\beta)-2\epsilon)} \times {}_2\mathrm{F}_1(\alpha,\beta;1-\epsilon;1-\eta_{12}), \tag{G.12}$$

$$\int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]} \frac{1}{(\rho_{1q}+\rho_{2q})^\alpha} = 2^{1-\alpha-2\epsilon} \times \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \times {}_2\mathrm{F}_1\left(\frac{\alpha}{2}, \frac{1}{2}+\frac{\alpha}{2}; \frac{3}{2}-\epsilon; 1-\eta_{12}\right). \tag{G.13}$$

Moreover, we require

$$\int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]} \frac{1}{\rho_{1q}^\alpha (\rho_{1q}+\rho_{2q})^\beta}, \quad \text{for} \quad \begin{cases} \alpha = 1+\epsilon, & \beta = 1, \\ \alpha = \epsilon, & \beta = 2, \\ \alpha = -1+\epsilon, & \beta = 3. \end{cases} \tag{G.14}$$

Result for these integrals read

$$\int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]} \, \frac{1}{\rho_{1q}^{1+\epsilon}(\rho_{1q}+\rho_{2q})} = -\frac{2^{-2\epsilon}}{\epsilon} 2^{-1-2\epsilon}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right] \times {}_2\mathrm{F}_1\left(1,1+\frac{\epsilon}{2};1-\frac{3}{2}\epsilon;1-\eta_{12}\right), \tag{G.15}$$

$$\int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]} \, \frac{1}{\rho_{1q}^{\epsilon}(\rho_{1q}+\rho_{2q})^2} = 2^{-2-4\epsilon}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-2\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-3\epsilon)}\right] \times \left[{}_2\mathrm{F}_1\left(1,1+\frac{\epsilon}{2};1-\frac{3}{2}\epsilon;1-\eta_{12}\right) + \frac{3+3\epsilon}{1-3\epsilon}\,{}_2\mathrm{F}_1\left(1,\frac{3}{2}+\frac{3\epsilon}{2};\frac{3}{2}-\frac{3}{2}\epsilon;1-\eta_{12}\right)\right], \tag{G.16}$$

$$\begin{aligned}\int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]} \, \frac{1}{\rho_{1q}^{-1+\epsilon}(\rho_{1q}+\rho_{2q})^3} &= 2^{-2-4\epsilon}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right] \\ &\times \left[\left(\frac{1}{2}-\frac{\epsilon}{4}\right) \times {}_2\mathrm{F}_1\left(1,1+\frac{\epsilon}{2};1-\frac{3}{2}\epsilon;1-\eta_{12}\right)\right. \\ &+ \frac{3}{2}\times\frac{1-\epsilon^2}{1-3\epsilon}\times {}_2\mathrm{F}_1\left(1,\frac{3}{2}+\frac{\epsilon}{2};\frac{3}{2}-\frac{3}{2}\epsilon;1-\eta_{12}\right) \\ &\left.+ \frac{3}{4}\times\frac{\epsilon(2+\epsilon)}{2-3\epsilon}\times {}_2\mathrm{F}_1\left(1,2+\frac{\epsilon}{2};2-\frac{3}{2}\epsilon;1-\eta_{12}\right)\right].\end{aligned} \tag{G.17}$$

All integrals in Eqs. (G.11-G.17) can be computed along similar lines and we show some of the computations below. As a first example we consider the integral Eq. (G.13). We parametrize the direction $\vec{n}_q$ with respect to $\vec{n}_1+\vec{n}_2$.To this end, we introduce a normalized vector

$$\vec{n}_{12} \equiv \frac{\vec{n}_1+\vec{n}_2}{L_{12}}, \quad \text{with} \quad L_{12}^2 = 2(2-\rho_{12}), \tag{G.18}$$

and write the denominator of the integrand in Eq. (G.13) as

$$\rho_{1q}+\rho_{2q} = 1-(\vec{n}_q\cdot\vec{n}_1)+1-(\vec{n}_q\cdot\vec{n}_2) = 2-L_{12}(\vec{n}_q\cdot\vec{n}_{12}). \tag{G.19}$$

We use $\cos\theta = \vec{n}_q\cdot\vec{n}_{12}$ and integrate over the remaining $d-2$ directions. We obtain

$$\begin{aligned}&\int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]} \, \frac{1}{(\rho_{1q}+\rho_{2q})^\alpha} \\ &= \underbrace{\frac{1}{[\Omega^{(d-2)}]}\times\int\frac{\mathrm{d}\Omega^{(d-2)}}{2(2\pi)^{(d-1)}}}_{=1} \int_{-1}^{1} \mathrm{d}\cos\theta\,(\sin\theta)^{d-4}\frac{1}{[2-L_{12}\cos\theta]^\alpha} \\ &= \int_{-1}^{1}\mathrm{d}\cos\theta\,[1-\cos^2\theta]^{-\epsilon}\frac{1}{[2-L_{12}\cos\theta]^\alpha},\end{aligned} \tag{G.20}$$

where $d = 4 - 2\epsilon$ was used. We substitute

$$\xi = \frac{1+\cos\theta}{2} \quad \Rightarrow \quad \int_{-1}^{1} \mathrm{d}\cos\theta\,[1-\cos^2\theta]^{-\epsilon} = 2\times 2^{-2\epsilon}\int_0^1 \mathrm{d}\xi\,\xi^{-\epsilon}(1-\xi)^{-\epsilon}\,, \tag{G.21}$$

in the remaining integral on the right-hand side in Eq. (G.20) and obtain

$$\begin{aligned}
&\int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]}\,\frac{1}{(\rho_{1q}+\rho_{2q})^{\alpha}} \\
&= 2\times 2^{-2\epsilon}\int_0^1 \mathrm{d}\xi\,\xi^{-\epsilon}(1-\xi)^{-\epsilon}\frac{1}{[2-L_{12}\cos\theta]^{2}} \\
&= (2+L_{12})^{-\alpha}\times 2\times 2^{-2\epsilon}\int_0^1 \mathrm{d}\xi\,\xi^{-\epsilon}(1-\xi)^{-\epsilon}\left[1-\left(\frac{2L_{12}}{2+L_{12}}\right)\xi\right]^{-\alpha}.
\end{aligned} \tag{G.22}$$

We use the integral representation of the hypergeometric function [81]

$${}_2\mathrm{F}_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \mathrm{d}t\,t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,, \quad \mathrm{Re}(c) > \mathrm{Re}(b) > 0\,, \tag{G.23}$$

with

$$z = \frac{2L_{12}}{2+L_{12}}\,, \quad a = \alpha\,, \quad b = 1-\epsilon\,, \quad c = 2-2\epsilon\,, \tag{G.24}$$

and re-write the integral on the right-hand side of Eq. (G.22) as

$$\int_0^1 \mathrm{d}\xi\,\xi^{-\epsilon}(1-\xi)^{-\epsilon}\left[1-\left(\frac{2L_{12}}{2+L_{12}}\right)\xi\right]^{-\alpha} = \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}\times{}_2\mathrm{F}_1\left(\alpha, 1-\epsilon; 2-2\epsilon; \frac{2L_{12}}{2+L_{12}}\right). \tag{G.25}$$

The above result can be further simplified. With the help of the identity [81]

$${}_2\mathrm{F}_1(a,b;2b;z) = \left(1-\frac{z}{2}\right)^{-a}\times{}_2\mathrm{F}_1\left(\frac{a}{2},\frac{1}{2}+\frac{a}{2};b+\frac{1}{2};z^2(2-z)^{-2}\right), \tag{G.26}$$

we can write the hypergeometric function in Eq. (G.25) in the following form

$$\begin{aligned}
&{}_2\mathrm{F}_1\left(2, 1-\epsilon; 2-2\epsilon; \frac{2L_{14}}{2+L_{14}}\right) \\
&= (2+L_{14})^{\alpha}\times 2^{-\alpha}\times{}_2\mathrm{F}_1\left(\frac{\alpha}{2},\frac{1}{2}+\frac{\alpha}{2};\frac{3}{2}-\epsilon;\frac{L_{14}^2}{4}\right) \\
&= (2+L_{14})^{\alpha}\times 2^{-\alpha}\times{}_2\mathrm{F}_1\left(\frac{\alpha}{2},\frac{1}{2}+\frac{\alpha}{2};\frac{3}{2}-\epsilon;1-\eta_{14}\right).
\end{aligned} \tag{G.27}$$

We combine this result with Eqs. (G.22, G.25) we find for the final result

$$\int \frac{[\mathrm{d}\Omega_6]}{[\Omega^{(d-2)}]} \, \frac{1}{(\rho_{61}+\rho_{64})^{\alpha}} = 2^{1-\alpha-2\epsilon} \times \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \times {}_2\mathrm{F}_1\left(\frac{\alpha}{2}, \frac{1}{2}+\frac{\alpha}{2}; \frac{3}{2}-\epsilon; 1-\eta_{14}\right). \tag{G.28}$$

As second and last example, we consider the integral Eq. (G.16). It reads

$$\int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]} \, \frac{1}{\rho_{1q}^{\alpha}(\rho_{1q}+\rho_{2q})^{\beta}}, \quad \text{with} \quad \alpha = \epsilon, \quad \beta = 2. \tag{G.29}$$

Parts of the calculation can be done for generic α and β. We begin with introducing Feynman parameters [82] to combine the denominators[1]

$$\begin{aligned} &\int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]} \frac{1}{\rho_{1q}^{\alpha}(\rho_{1q}+\rho_{2q})^{\beta}} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \mathrm{d}x \, x^{\alpha-1}(1-x)^{\beta-1} \int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]} \, \frac{1}{[x\rho_{1q}+(1-x)(\rho_{1q}+\rho_{2q})]^{\alpha+\beta}} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \mathrm{d}x \, x^{\alpha-1}(1-x)^{\beta-1} \int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]} \, \frac{1}{[\rho_{1q}+(1-x)\rho_{2q}]^{\alpha+\beta}}. \end{aligned} \tag{G.30}$$

Similar to our previous discussion we combine vectors in the denominator to obtain a single scalar product. We write

$$\begin{aligned} \rho_{1q}+(1-x)\rho_{2q} &= [1-\vec{n}_q\cdot\vec{n}_1] + (1-x)[1-\vec{n}_q\cdot\vec{n}_2] \\ &= 1+(1-x) - \vec{n}_q\cdot[\vec{n}_1+(1-x)\vec{n}_2] = (2-x) - L_x(\vec{n}_q\cdot\vec{n}_x), \end{aligned} \tag{G.31}$$

where we introduced a vector $\vec{n}_x$ and the function L_x, that possess the following properties

$$\begin{aligned} \vec{n}_x^2 = 1 \quad \text{and} \quad L_x^2 &= [\vec{n}_1+(1-x)\vec{n}_2]^2 \\ &= 1+(1-x)^2+2(1-x)(\vec{n}_1\cdot\vec{n}_2) \\ &= (2-x)^2 - 2(1-x)\rho_{12}. \end{aligned} \tag{G.32}$$

We first consider the angular integral on the right-hand side in Eq. (G.30). We parametrize $\vec{n}_q$ with respect to the direction $\vec{n}_x$ and write $\cos\theta = \vec{n}_q\cdot\vec{n}_x$. The integral then becomes

$$\begin{aligned} &\int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]} \, \frac{1}{[\rho_{1q}+(1-x)\rho_{2q}]^{\alpha+\beta}} \\ &= \underbrace{\frac{1}{[\Omega^{(d-2)}]} \times \int \frac{\mathrm{d}\Omega^{(d-2)}}{2(2\pi)^{(d-1)}}}_{=1} \int_{-1}^{1} \mathrm{d}\cos\theta \frac{[1-\cos^2\theta]^{-\epsilon}}{[(2-x)-L_x\cos\theta]^{\alpha+\beta}}. \end{aligned} \tag{G.33}$$

[1]Note that for the application of Feynman parameters in the first step we need $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$.

This integral can be rewritten into the integral representation of the hypergeometric function. Therefore we substitute

$$\xi = \frac{1+\cos\theta}{2} \quad \Rightarrow \quad \int_{-1}^{1} \mathrm{d}\cos\theta\,[1-\cos^2\theta]^{-\epsilon} \to 2\times 2^{-2\epsilon}\int_0^1 \mathrm{d}\xi\,\xi^{-\epsilon}(1-\xi)^{-\epsilon}, \tag{G.34}$$

in the integral in Eq. (G.33). We obtain

$$\begin{aligned}&\int_{-1}^{1} \mathrm{d}\cos\theta \frac{[1-\cos^2\theta]^{-\epsilon}}{[(2-x)-L_x\cos\theta]^{\alpha+\beta}}\\ &= 2\times 2^{-2\epsilon}(2-x+L_x)^{-\alpha-\beta}\int_0^1 \mathrm{d}\xi\,\xi^{-\epsilon}(1-\xi)^{-\epsilon}\left[1-\left(\frac{2L_x}{2-x+L_x}\right)\xi\right]^{-\alpha-\beta}.\end{aligned} \tag{G.35}$$

We use Eq. (G.23) with $a=\alpha+\beta$, $b=1-\epsilon$ and $c=2(1-\epsilon)$ to write Eq. (G.35) in terms of a hypergeometric function. We find

$$\begin{aligned}&\int_0^1 \mathrm{d}\xi\,\xi^{-\epsilon}(1-\xi)^{-\epsilon}\left[1-\left(\frac{2L_x}{2-x+L_x}\right)\xi\right]^{-\alpha-\beta}\\ &= \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}\times {}_2\mathrm{F}_1\left(\alpha+\beta, 1-\epsilon; 2(1-\epsilon); \frac{2L_x}{2-x+L_x}\right).\end{aligned} \tag{G.36}$$

We re-write this result, using the identity Eq. (G.26), as

$$\begin{aligned}&{}_2\mathrm{F}_1\left(\alpha+\beta, 1-\epsilon; 2(1-\epsilon); \frac{2L_x}{2-x+L_x}\right)\\ &= (2-x)^{-\alpha-\beta}(2-x+L_x)^{\alpha+\beta}\times {}_2\mathrm{F}_1\left(\frac{\alpha+\beta}{2}, \frac{1}{2}+\frac{\alpha+\beta}{2}; \frac{3}{2}-\epsilon; \frac{L_x^2}{(2-x)^2}\right).\end{aligned} \tag{G.37}$$

Together with Eqs. (G.30, G.33, G.35) we obtain for the full integral, before integration over Feynman parameter x, the following result

$$\begin{aligned}&\int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]}\,\frac{1}{\rho_{1q}^{\alpha}(\rho_{1q}+\rho_{2q})^{\beta}}\\ &= \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} 2\times 2^{-2\epsilon}\\ &\times \int_0^1 \mathrm{d}x\, x^{\alpha-1}(1-x)^{\beta-1}(2-x)^{-\alpha-\beta}{}_2\mathrm{F}_1\left(\frac{\alpha+\beta}{2}, \frac{1}{2}+\frac{\alpha+\beta}{2}; \frac{3}{2}-\epsilon; 1-\frac{2(1-x)}{(2-x)^2}\rho_{12}\right).\end{aligned} \tag{G.38}$$

To continue, from this point on we consider the particular values $\alpha=\epsilon$ and $\beta=2$ for the powers of the denominators in the integrand in Eq. (G.38). To perform the Feynman parameter integration we use the Mellin-Barns integral representation of the hypergeometric function [83]

$$
\begin{aligned}
{}_2\mathrm{F}_1(a,b;c;z) = &\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \\
&\times \frac{1}{2\pi i}\int\limits_{-i\infty}^{+i\infty} \mathrm{d}t\, \Gamma(a+t)\Gamma(b+t)\Gamma(c-a-b-t)\Gamma(-t)(1-z)^t\,,
\end{aligned}
\tag{G.39}
$$

which applied to the hypergeometric function on the right-hand side of Eq. (G.38) reads

$$
\begin{aligned}
&{}_2\mathrm{F}_1\left(\frac{2+\epsilon}{2},\frac{3+\epsilon}{2};\frac{3}{2}-\epsilon;1-\frac{1-x}{(2-x)^2}2\rho_{12}\right) = -\frac{3\epsilon\times 2^{-2\epsilon}}{\pi}\frac{\Gamma\left(\frac{3}{2}-\epsilon\right)}{\Gamma(1-3\epsilon)\Gamma(2+\epsilon)} \\
&\times \frac{1}{2\pi i}\int\limits_{-i\infty}^{+i\infty} \mathrm{d}t\, \Gamma\left(\frac{2+\epsilon}{2}+t\right)\Gamma\left(\frac{3+\epsilon}{2}+t\right)\Gamma(-1-2\epsilon-t)\Gamma(-t)\left(\frac{1-x}{(2-x)^2}\right)^t (2\rho_{12})^t\,.
\end{aligned}
\tag{G.40}
$$

Inserting this representation into Eq. (G.38) we obtain

$$
\begin{aligned}
\int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]}\,\frac{1}{\rho_{1q}^{\epsilon}(\rho_{1q}+\rho_{2q})^2} &= \frac{\Gamma(1-\epsilon)}{\Gamma(1-3\epsilon)\Gamma(1+\epsilon)}\left(-\frac{3\epsilon^2\times 2^{-2\epsilon}}{\sqrt{\pi}}\right) \\
&\times \int\limits_0^1 \mathrm{d}x\, x^{\epsilon-1}(1-x)(2-x)^{-2-\epsilon} \\
&\times \frac{1}{2\pi i}\int\limits_{-i\infty}^{+i\infty} \mathrm{d}t\, \Gamma\left(\frac{2+\epsilon}{2}+t\right)\Gamma\left(\frac{3+\epsilon}{2}+t\right)\Gamma(-1-2\epsilon-t)\Gamma(-t)\left(\frac{1-x}{(2-x)^2}\right)^t (2\rho_{12})^t\,.
\end{aligned}
\tag{G.41}
$$

In this form integration over Feynman parameter x can be done with any computer algebra system. We find

$$
\begin{aligned}
\int\limits_0^1 \mathrm{d}x\, x^{\epsilon-1}(1-x)(2-x)^{-2-\epsilon}\left(\frac{1-x}{(2-x)^2}\right)^t &= \int\limits_0^1 \mathrm{d}x\, x^{\epsilon-1}(1-x)^{1+t}(2-x)^{-2-\epsilon-2t} \\
&= \frac{2^{-2-\epsilon-2t}\Gamma(1+\epsilon)\Gamma(2+t)}{\epsilon\Gamma(2+\epsilon+t)}\,{}_2\mathrm{F}_1\left(\epsilon,2+\epsilon+2t;2+\epsilon+t;\frac{1}{2}\right).
\end{aligned}
\tag{G.42}
$$

The hypergeometric function on the right-hand side in Eq. (G.42) can be expressed in terms of standard Γ-functions with the help of the identity [81]

$$
{}_2\mathrm{F}_1\left(a,b;\frac{a}{2}+\frac{b}{2}+1;\frac{1}{2}\right) = \frac{2\pi^{\frac{1}{2}}\Gamma\left(\frac{a}{2}+\frac{b}{2}+1\right)}{a-b}\left(\frac{1}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{1}{2}+\frac{b}{2}\right)}-\frac{1}{\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}\right). \tag{G.43}
$$

Applied to the hypergeometric function in Eq. (G.42) we obtain

$$
\begin{aligned}
&{}_2F_1\left(\epsilon, 2+\epsilon+2t; 2+\epsilon+t; \frac{1}{2}\right) \\
&\quad = -\frac{\pi^{\frac{1}{2}}\Gamma(2+\epsilon+t)}{1+t}\left(\frac{1}{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(\frac{3}{2}+\frac{\epsilon}{2}+t\right)} - \frac{1}{\Gamma\left(\frac{1+\epsilon}{2}\right)\Gamma\left(1+\frac{\epsilon}{2}+t\right)}\right).
\end{aligned}
\tag{G.44}
$$

We collect everything that depends on the variable t , and use again Eq. (G.39) to write the remaining integral over t in terms of hypergeometric functions. We use

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}t\, \Gamma(-1-2\epsilon-t)\Gamma(-t)\Gamma(1+t)\left(\frac{\Gamma\left(1+\frac{\epsilon}{2}+t\right)}{\Gamma\left(\frac{\epsilon}{2}\right)} - \frac{\Gamma\left(\frac{3}{2}+\frac{\epsilon}{2}+t\right)}{\Gamma\left(\frac{1+\epsilon}{2}\right)}\right)(1-(1-\eta_{12}))^t \\
&\quad = \frac{\Gamma\left(1+\frac{\epsilon}{2}\right)\Gamma\left(-\frac{3}{2}\epsilon\right)\Gamma(-2\epsilon)}{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(1-\frac{3}{2}\epsilon\right)} \times {}_2\mathrm{F}_1\left(1, 1+\frac{\epsilon}{2}; 1-\frac{3}{2}\epsilon; 1-\eta_{12}\right) \\
&\quad\quad - \frac{\Gamma\left(\frac{3}{2}+\frac{\epsilon}{2}\right)\Gamma\left(\frac{1}{2}-\frac{3}{2}\epsilon\right)\Gamma(-2\epsilon)}{\Gamma\left(\frac{3}{2}-\frac{3}{2}\epsilon\right)\Gamma\left(\frac{1+\epsilon}{2}\right)} \times {}_2\mathrm{F}_1\left(1, \frac{3}{2}+\frac{\epsilon}{2}; \frac{3}{2}-\frac{3}{2}\epsilon; 1-\eta_{12}\right) \\
&\quad = \frac{\Gamma(1-2\epsilon)}{2\epsilon}\left[\frac{1}{3} \times {}_2\mathrm{F}_1\left(1, 1+\frac{\epsilon}{2}; 1-\frac{3}{2}\epsilon; 1-\eta_{12}\right)\right. \\
&\quad\quad \left. + \frac{1+\epsilon}{1-3\epsilon} \times {}_2\mathrm{F}_1\left(1, \frac{3}{2}+\frac{\epsilon}{2}; \frac{3}{2}-\frac{3}{2}\epsilon; 1-\eta_{12}\right)\right].
\end{aligned}
\tag{G.45}
$$

We combine this formula with Eqs. (G.41, G.42, G.44) and obtain the following final result

$$
\begin{aligned}
&\int \frac{[\mathrm{d}\Omega_q]}{[\Omega^{(d-2)}]}\frac{1}{\rho_{1q}^{\epsilon}(\rho_{1q}+\rho_{2q})^2} = 2^{-2-4\epsilon}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-2\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-3\epsilon)}\right] \\
&\quad \times \left[{}_2\mathrm{F}_1\left(1, 1+\frac{\epsilon}{2}; 1-\frac{3}{2}\epsilon; 1-\eta_{12}\right) + \frac{3+3\epsilon}{1-3\epsilon} \times {}_2\mathrm{F}_1\left(1, \frac{3}{2}+3\times\frac{\epsilon}{2}; \frac{3}{2}-\frac{3}{2}\epsilon; 1-\eta_{12}\right)\right].
\end{aligned}
\tag{G.46}
$$

G.4. Generic solid angle integral identities

The following integral identities have proven to simplify computations. We used

$$
\int [\mathrm{d}\Omega_q]\left[\frac{\rho_{q1}}{\rho_{q_1}+\rho_{q2}}\right]^2\frac{1}{\rho_{q1}\rho_{q2}} = \frac{1}{2}\int [\mathrm{d}\Omega_q]\frac{1}{\rho_{q1}\rho_{q2}} - \int [\mathrm{d}\Omega_q]\frac{1}{(\rho_{q1}+\rho_{q2})^2}, \tag{G.47}
$$

$$
\int [\mathrm{d}\Omega_q]\frac{1}{\rho_{q1}(\rho_{q1}+\rho_{q2})}\rho_{q2}^{-\epsilon} = \int [\mathrm{d}\Omega_q]\frac{1}{\rho_{q1}\rho_{q2}^{1+\epsilon}} - \int [\mathrm{d}\Omega_q]\frac{1}{\rho_{q2}^{1+\epsilon}(\rho_{q1}+\rho_{q2})}. \tag{G.48}
$$

To prove Eq. (G.47) consider the integral

$$\begin{aligned}\int[\mathrm{d}\Omega_q]\,\frac{1}{\rho_{q1}\rho_{q2}} &= \int[\mathrm{d}\Omega_q]\,\left(\frac{\rho_{q1}+\rho_{q2}}{\rho_{q1}+\rho_{q2}}\right)^2\frac{1}{\rho_{q1}\rho_{q2}}\\
&= \int[\mathrm{d}\Omega_q]\,\left(\frac{\rho_{q1}}{\rho_{q1}+\rho_{q2}}\right)^2\frac{1}{\rho_{q1}\rho_{q2}} + \int[\mathrm{d}\Omega_q]\,\left(\frac{\rho_{q2}}{\rho_{q1}+\rho_{q2}}\right)^2\frac{1}{\rho_{q1}\rho_{q2}}\\
&\quad + \int[\mathrm{d}\Omega_q]\,\frac{2\rho_{q1}\rho_{q2}}{(\rho_{q1}+\rho_{q2})^2}\frac{1}{\rho_{q1}\rho_{q2}}\\
&= 2\int[\mathrm{d}\Omega_q]\,\left(\frac{\rho_{q1}}{\rho_{q1}+\rho_{q2}}\right)^2\frac{1}{\rho_{q1}\rho_{q2}} + 2\int[\mathrm{d}\Omega_q]\,\frac{1}{(\rho_{q1}+\rho_{q2})^2}\,.\end{aligned} \tag{G.49}$$

In the last step we have used that because we are integrating over all directions of $\vec{n}_q$ we can exchange the two directions $\vec{n}_{i=1,2}$ without changing the result of the integral. Identity Eq. (G.48) can be obtained upon straightforward manipulations. We write

$$\begin{aligned}&\int[\mathrm{d}\Omega_q]\,\frac{1}{\rho_{q1}(\rho_{q1}+\rho_{q2})}\rho_{q2}^{-\epsilon}\\
&= \int[\mathrm{d}\Omega_q]\,\left[\frac{1}{\rho_{q1}(\rho_{q1}+\rho_{q2})}\rho_{q2}^{-\epsilon} + \frac{1}{\rho_{q2}(\rho_{q1}+\rho_{q2})}\rho_{q2}^{-\epsilon}\right] - \int[\mathrm{d}\Omega_q]\,\frac{1}{\rho_{q2}(\rho_{q1}+\rho_{q2})}\rho_{q2}^{-\epsilon}\\
&= \int[\mathrm{d}\Omega_q]\,\frac{1}{(\rho_{q1}+\rho_{q2})}\rho_{q2}^{-\epsilon}\underbrace{\left[\frac{1}{\rho_{q1}}+\frac{1}{\rho_{q2}}\right]}_{=\frac{(\rho_{q1}+\rho_{q2})}{\rho_{q1}\rho_{q2}}} - \int[\mathrm{d}\Omega_q]\,\frac{1}{\rho_{q2}^{1+\epsilon}(\rho_{q1}+\rho_{q2})}\\
&= \int[\mathrm{d}\Omega_q]\,\frac{1}{\rho_{q1}\rho_{q2}^{1+\epsilon}} - \int[\mathrm{d}\Omega_q]\,\frac{1}{\rho_{q2}^{1+\epsilon}(\rho_{q1}+\rho_{q2})}\,.\end{aligned} \tag{G.50}$$

H. Partitioning-dependent integrals

In this appendix we collect integrals that depend on the choice of partition functions. We defined them as

$$\langle \mathcal{O} \rangle_{S_5} \equiv \left(-\frac{2^{-2\epsilon}}{\epsilon} \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] \right)^{-1} \int \frac{\mathrm{d}\Omega_5^{(d-1)}}{2(2\pi)^{(d-1)}} \frac{\rho_{14}}{\rho_{15}\rho_{45}} \, \mathcal{O}(\Omega_5) \,, \tag{H.1}$$

where $\mathcal{O}$ has a residual dependence on the partitioning and contains no further singularities for $\epsilon \to 0$. Explicit formulas for the partition functions are given in Appendix A.3.2. For these, the required integrals read

$$\begin{aligned}
\langle \Delta_{61} \rangle_{S_5} = \langle \Delta_{64} \rangle_{S_5} = \frac{3}{2} + \epsilon \left(\frac{\ln 2}{2} - 2 \ln \eta_{14} \right) + \epsilon^2 \bigg(- \frac{\pi^2}{2} - \ln 2 + \frac{\ln^2 2}{4} - \frac{1}{2\sqrt{1-\eta_{14}}} \\
\times \ln \left(\frac{1+\sqrt{1-\eta_{14}}}{1-\sqrt{1-\eta_{14}}} \right) + \frac{\ln \eta_{14}}{2} - \ln 2 \ln \eta_{14} + \frac{3 \ln^2 \eta_{14}}{2} \\
+ \frac{5}{2} \mathrm{Li}_2(1-\eta_{14}) \bigg) + \mathcal{O}(\epsilon^3) \,,
\end{aligned} \tag{H.2}$$

$$\langle \Delta_{56} \rangle_{S_5} = 1 - 2\epsilon \ln \eta_{14} + \epsilon^2 \left(\mathrm{Li}_2\big((1-\eta_{14})^2\big) + 2 \ln^2 \eta_{14} - \frac{2}{2-\eta_{14}} \right) + \mathcal{O}(\epsilon^3) \,, \tag{H.3}$$

where

$$\begin{aligned}
\Delta_{61} = w_{\mathrm{dc}}^{54} + \left(\frac{\rho_{51}}{4} \right)^{-\epsilon} w_{\mathrm{tc}}^{51} \,, \quad \Delta_{64} = w_{\mathrm{dc}}^{51} + \left(\frac{\rho_{54}}{4} \right)^{-\epsilon} w_{\mathrm{tc}}^{54} \,, \\
\Delta_{56} = \sum_{i \in \{1,4\}} w_{\mathrm{tc}}^{i} \left(\frac{\eta_{i5}}{1-\eta_{i5}} \right)^{-\epsilon} \,.
\end{aligned} \tag{H.4}$$

We also use the $\mathcal{O}(\epsilon^2)$ coefficient of $\langle \Delta_{61} \rangle_{S_5}$ and $\langle \Delta_{56} \rangle_{S_5}$ defined as $1/2 \langle \Delta_{61} \rangle''_{S_5}$ and $1/2 \langle \Delta_{56} \rangle''_{S_5}$, respectively. These functions read

$$\begin{aligned}
\langle \Delta_{61} \rangle''_{S_5} = -\pi^2 - 2 \ln 2 + \frac{\ln^2 2}{2} - \frac{1}{\sqrt{1-\eta_{14}}} \ln \left(\frac{1+\sqrt{1-\eta_{14}}}{1-\sqrt{1-\eta_{14}}} \right) \\
+ \ln \eta_{14} - 2 \ln 2 \ln \eta_{14} + 3 \ln^2 \eta_{14} + 5 \mathrm{Li}_2(1-\eta_{14}) \,,
\end{aligned} \tag{H.5}$$

$$\langle \Delta_{56} \rangle''_{S_5} = 2 \mathrm{Li}_2\big((1-\eta_{14})^2\big) + 4 \ln^2 \eta_{14} - \frac{4}{2-\eta_{14}} \,. \tag{H.6}$$

Finally, we defined the following finite integral

$$\langle r^\mu r^\nu \rangle_{\rho_5} \equiv \sum_{i\in\{1,4\}} \int \frac{d^3\Omega_5}{2\pi} \left[\left(\frac{n_1 \cdot r^{(i)}}{n_1 \cdot n_5} - \frac{n_4 \cdot r^{(i)}}{n_4 \cdot n_5} \right)^2 - 2 \frac{n_1 \cdot n_4}{(n_1 \cdot n_5)(n_4 \cdot n_5)} \right] w^i_{\rm dc} \,, \tag{H.7}$$

where $n_i^\mu = p_i^\mu / E_i$ and vectors $r_\mu^{(i)}$ depend on the phase space parametrization and are defined in Appendix F. The result for this integral reads

$$\langle r^\mu r^\nu \rangle_{\rho_5} = 2 \left[\frac{1}{2-\eta_{14}} - 1 - \ln(2-\eta_{14}) \right] . \tag{H.8}$$

Only the order $\mathcal{O}(\epsilon^2)$ coefficient of the integrals in Eqs. (H.2, H.3) depends on the explicit form of the partition functions. In the following chapter we show how to expand the integrals without using explicit formulas for the partition functions. In Appendix H.2 we demonstrate how these integrals can be computed.

H.1. Generic expansions of the $\langle \cdot \rangle_{S_5}$ integrals

We discuss how to expand integrals in Eqs. (H.2, H.3) without using explicit formulas for the partition functions. As an example we consider[1]

$$\langle \Delta_{61} \rangle_{S_5} = -\epsilon 2^{2\epsilon} \int \frac{[d\Omega_5]}{[\Omega^{(d-2)}]} \frac{\rho_{14}}{\rho_{15}\rho_{45}} \left(w^{54}_{\rm dc} + w^{51}_{\rm tc} \left(\frac{\rho_{15}}{4} \right)^{-\epsilon} \right) . \tag{H.9}$$

We need this integral up to $\mathcal{O}(\epsilon^2)$ where only terms of $\mathcal{O}(\epsilon^2)$ contribute to the finite part of the subtractions. Since the integral on the right-hand side in Eq. (H.9) is multiplied with ϵ we require the integral itself only to order $\mathcal{O}(\epsilon)$.

The integral in Eq. (H.9) has two contributions. We first consider the second term, which is proportional to the partition function $w^{51}_{\rm tc}$. The partition function regulates the singularity in the $\vec{p}_5 \parallel \vec{p}_4$ limit. We subtract the $\vec{p}_5 \parallel \vec{p}_1$ singularity and write

$$\begin{aligned} &\int \frac{[d\Omega_5]}{[\Omega^{(d-2)}]} \frac{\rho_{14}}{\rho_{15}\rho_{45}} w^{51}_{\rm tc} \left(\frac{\rho_{51}}{4} \right)^{-\epsilon} \\ &= \int \frac{[d\Omega_5]}{[\Omega^{(d-2)}]} \left(\frac{\rho_{15}}{4} \right)^{-\epsilon} \frac{1}{\rho_{15}} \left[\frac{\rho_{14}}{\rho_{45}} w^{51}_{\rm tc} - 1 \right] + \int \frac{[d\Omega_5]}{[\Omega^{(d-2)}]} \left(\frac{\rho_{15}}{4} \right)^{-\epsilon} \frac{1}{\rho_{15}} . \end{aligned} \tag{H.10}$$

The first term on the right-hand side of Eq. (H.10) depends on the partition function. It is finite and, therefore, contributes to order $\mathcal{O}(\epsilon^0)$ of the final result. However, dependence on the partition function should appear first at order $\mathcal{O}(\epsilon)$. To make this manifest, we use

$$\left(\frac{\rho_{15}}{4} \right)^{-\epsilon} = \left[\left(\frac{\rho_{15}}{4} \right)^{-\epsilon} - 1 \right] + 1 . \tag{H.11}$$

[1]Note that this integral is calculated exact in Appendix H.2 for the chosen partitioning given in Appendix A.3.2.

in Eq. (H.10) and obtain

$$\begin{aligned}
&\int \frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]} \frac{\rho_{14}}{\rho_{15}\rho_{45}} w_{\mathrm{tc}}^{51} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \\
&= \int \frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]} \frac{1}{\rho_{15}} \left[\left(\frac{\rho_{15}}{4}\right)^{-\epsilon} - 1 \right] \left[\frac{\rho_{14}}{\rho_{45}} w_{\mathrm{tc}}^{51} - 1 \right] \\
&\quad + \int \frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]} \frac{\rho_{14}}{\rho_{15}\rho_{45}} w_{\mathrm{tc}}^{51} - \int \frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]} \frac{1}{\rho_{15}} + \int \frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \frac{1}{\rho_{15}} .
\end{aligned} \tag{H.12}$$

The second term on the right-hand side of Eq. (H.12) is the only one that depend on the partition function and contributes to lower orders in the ϵ expansion. However, we can combine this term with the first term in Eq. (H.9) and use the following relation

$$w_{\mathrm{dc}}^{54} + w_{\mathrm{tc}}^{51} = 1 , \tag{H.13}$$

which follows independently of the chosen partitioning from the C_{61} limit of Eq. (A.11). We obtain

$$\begin{aligned}
&\int \frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]} \frac{\rho_{14}}{\rho_{16}\rho_{46}} \left[w_{\mathrm{dc}}^{61} + w_{\mathrm{tc}}^{64} \left(\frac{\rho_{64}}{4}\right)^{-\epsilon} \right] \\
&= \int \frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]} \frac{1}{\rho_{46}} \left[\left(\frac{\rho_{64}}{4}\right)^{-\epsilon} - 1 \right] \left[\frac{\rho_{14} w_{\mathrm{tc}}^{64}}{\rho_{16}} - 1 \right] + \int \frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]} \frac{\rho_{14}}{\rho_{16}\rho_{46}} \\
&\quad - \int \frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]} \frac{1}{\rho_{46}} + \int \frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]} \left(\frac{\rho_{64}}{4}\right)^{-\epsilon} \frac{1}{\rho_{46}} .
\end{aligned} \tag{H.14}$$

The first term on the right-hand side of Eq. (H.14) still depends on the partitioning. However, it is regulated and by constructions contributes first at $\mathcal{O}(\epsilon)$.

All arising integrals in Eq. (H.14) that do not depend on the partition function are given in Eqs. (G.5, G.6, G.8). The result reads

$$\begin{aligned}
&\int \frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]} \frac{\rho_{14}}{\rho_{16}\rho_{46}} - \int \frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]} \frac{1}{\rho_{46}} + \int \frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]} \left(\frac{\rho_{64}}{4}\right)^{-\epsilon} \frac{1}{\rho_{46}} \\
&= -\frac{2^{-2\epsilon}}{\epsilon} \left[2\eta_{14}^{-\epsilon} K_{14} - \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} + \frac{2^{\epsilon}}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} \right] .
\end{aligned} \tag{H.15}$$

Combining Eqs. (H.9, H.14, H.15) we obtain the final result

$$\begin{aligned}
\langle \Delta_{61} \rangle_{S_5} &= 2\eta_{14}^{-\epsilon} K_{14} - \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right] + \left[\frac{2^{\epsilon}}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} \right] \\
&\quad - \epsilon 2^{2\epsilon} \int \frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]} \frac{1}{\rho_{15}} \left[\left(\frac{\rho_{15}}{4}\right)^{-\epsilon} - 1 \right] \left[\frac{\rho_{14}}{\rho_{45}} w_{\mathrm{tc}}^{51} - 1 \right] .
\end{aligned} \tag{H.16}$$

We perform similar manipulations to the integral $\langle \Delta_{56} \rangle_{S_5}$ and obtain

$$\langle \Delta_{56} \rangle_{S_5} = \eta_{14}^{-\epsilon} K_{14} - \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right] + 2^{-\epsilon} \left[\frac{2^\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} \right]$$
$$- \epsilon 2^{2\epsilon} \int \frac{[d\Omega_5]}{[\Omega^{(d-2)}]} \left\{ \frac{\rho_{14}}{\rho_{15}\rho_{45}} w_{\rm tc}^{1} \left[\left(1 - \frac{\rho_{15}}{2}\right)^\epsilon - 1 \right] + \frac{1}{\rho_{15}} \left[\left(\frac{\rho_{15}}{2}\right)^{-\epsilon} - 1 \right] \right.$$
$$\left. \times \left[\frac{\rho_{14}}{\rho_{45}} w_{\rm tc}^{1} \left(1 - \frac{\rho_{15}}{2}\right)^\epsilon - 1 \right] \right\}. \tag{H.17}$$

We note that, since dependence in the partition function appears first at $\mathcal{O}(\epsilon^2)$, Eqs. (H.16, H.17) prove the independence of IR $1/\epsilon$ poles in the subtraction terms on the chosen partition functions.

H.2. Computation of the $\langle \cdot \rangle_{S_5}$ integrals

We now demonstrate how to compute the $\langle \cdot \rangle_{S_5}$ integrals. We begin with the integral $\langle \Delta_{61} \rangle_{S_5}$ in Eq. (H.9) that can be computed *exact*. We consider contributions that are proportional to double-collinear and triple-collinear partition functions separately, and write

$$\langle \Delta_{61} \rangle_{S_5} = \left\langle w_{\rm dc}^{54} \right\rangle_{S_5} + \left\langle w_{\rm tc}^{51} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \right\rangle_{S_5}. \tag{H.18}$$

We start with the first term on the right-hand side of Eq. (H.18) and use the explicit form of the partition function in Eq. (A.14). We find

$$\left\langle w_{\rm dc}^{54} \right\rangle_{S_5} = -\epsilon 2^{2\epsilon} \int \frac{[d\Omega_5]}{[\Omega^{(d-2)}]} \left(\frac{\rho_{15}}{\rho_{15}+\rho_{45}} \right)^2 \frac{\rho_{14}}{\rho_{15}\rho_{45}}. \tag{H.19}$$

With the help of the identity in Eq. (G.47) we obtain

$$\left\langle w_{\rm dc}^{54} \right\rangle_{S_5} = -\epsilon 2^{2\epsilon} \left[\frac{1}{2} \int \frac{[d\Omega_5]}{[\Omega^{(d-2)}]} \frac{\rho_{14}}{\rho_{15}\rho_{45}} - \int \frac{[d\Omega_5]}{[\Omega^{(d-2)}]} \frac{\rho_{14}}{(\rho_{15}+\rho_{45})^2} \right]. \tag{H.20}$$

The first integral on the right-hand side of Eq. (H.20) can be found in Eq. (G.6) and the second in Eq. (G.13). The final result reads

$$\left\langle w_{\rm dc}^{54} \right\rangle_{S_5} = \eta_{14}^{-\epsilon} K_{14} + \epsilon\, \eta_{14}^{-\epsilon} \bar{K}_{14}, \tag{H.21}$$

where we have defined

$$\bar{K}_{14} \equiv \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \eta_{14}^{1+\epsilon} {}_2F_1\left(1, \frac{3}{2}; \frac{3}{2} - \epsilon; 1 - \eta_{14}\right). \tag{H.22}$$

The triple-collinear contribution to Eq. (H.18) can be computed in the same way, using identities and integrals presented in Appendix G. We obtain

$$\left\langle w_{\text{tc}}^{51} \left(\frac{\rho_{15}}{4}\right)^{-\epsilon} \right\rangle_{S_5} = \left[\frac{2^{\epsilon}}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right] \eta_{14} \Bigg\{ \left(1 - \frac{\epsilon}{2}\right) \\ \times {}_2\text{F}_1\left(1, 1+\frac{\epsilon}{2}; 1-\frac{3}{2}\epsilon; 1-\eta_{12}\right) - \left(\frac{3}{2}\epsilon \frac{1+\epsilon}{1-3\epsilon}\right) {}_2\text{F}_1\left(1, \frac{3}{2}+\frac{\epsilon}{2}; \frac{3}{2}-\frac{3}{2}\epsilon; 1-\eta_{12}\right) \Bigg\}. \tag{H.23}$$

We combine Eqs. (H.21, H.23) and obtain the following result

$$\boxed{\langle \Delta_{61} \rangle_{S_5} = \eta_{14}^{-\epsilon} K_{14} + \epsilon\, \eta_{14}^{-\epsilon} \bar{K}_{14} + \left[\frac{2^{\epsilon}}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right] \eta_{14} \Bigg\{ \left(1 - \frac{\epsilon}{2}\right) \\ \times {}_2\text{F}_1\left(1, 1+\frac{\epsilon}{2}; 1-\frac{3}{2}\epsilon; 1-\eta_{12}\right) - \left(\frac{3}{2}\epsilon \frac{1+\epsilon}{1-3\epsilon}\right) {}_2\text{F}_1\left(1, \frac{3}{2}+\frac{\epsilon}{2}; \frac{3}{2}-\frac{3}{2}\epsilon; 1-\eta_{12}\right) \Bigg\}.} \tag{H.24}$$

We discuss the computation of the function $\langle \Delta_{56} \rangle_{S_5}$ in Eq. (H.3). It reads

$$\langle \Delta_{56} \rangle_{S_5} = -\epsilon 2^{2\epsilon} \int \frac{[\text{d}\Omega_5]}{[\Omega^{(d-2)}]} \frac{\rho_{14}}{\rho_{15}\rho_{45}} w_{\text{tc}}^{1} \left(\frac{\rho_{15}}{2}\right)^{-\epsilon} \left(1 - \frac{\rho_{15}}{2}\right)^{\epsilon}, \tag{H.25}$$

This integral is more complicated then the one discussed above and we compute it as an expansion in ϵ. We require Eq. (H.25) up to order $\mathcal{O}(\epsilon^2)$. Hence, we have to compute the following integral to order $\mathcal{O}(\epsilon)$

$$\int \frac{[\text{d}\Omega_5]}{[\Omega^{(d-2)}]} \frac{\rho_{14}}{\rho_{15}\rho_{45}} w_{\text{tc}}^{1} \left(\frac{\rho_{15}}{2}\right)^{-\epsilon} \left(1 - \frac{\rho_{15}}{2}\right)^{\epsilon}. \tag{H.26}$$

To this end, we follow the discussion in the previous section and use

$$\left(\frac{\rho_{15}}{2}\right)^{-\epsilon} \left(1 - \frac{\rho_{15}}{2}\right)^{\epsilon} \\ = \left(\frac{\rho_{15}}{2}\right)^{-\epsilon} \left[\left(1 - \frac{\rho_{15}}{2}\right)^{\epsilon} - 1\right] + \left(\frac{\rho_{15}}{2}\right)^{-\epsilon} \\ = \left[\left(\frac{\rho_{15}}{2}\right)^{-\epsilon} - 1\right] \left[\left(1 - \frac{\rho_{15}}{2}\right)^{\epsilon} - 1\right] + \left[\left(1 - \frac{\rho_{15}}{2}\right)^{\epsilon} - 1\right] + \left(\frac{\rho_{15}}{2}\right)^{-\epsilon}, \tag{H.27}$$

to re-write the integral Eq. (H.26) as

$$\int \frac{[\text{d}\Omega_5]}{[\Omega^{(d-2)}]} \frac{\rho_{14}}{\rho_{15}\rho_{45}} w_{\text{tc}}^{1} \left(\frac{\rho_{15}}{2}\right)^{-\epsilon} \left(1 - \frac{\rho_{15}}{2}\right)^{\epsilon} \\ = \int \frac{[\text{d}\Omega_5]}{[\Omega^{(d-2)}]} \frac{\rho_{14}}{\rho_{15}\rho_{45}} w_{\text{tc}}^{1} \\ \times \left(\left[\left(\frac{\rho_{15}}{2}\right)^{-\epsilon} - 1\right] \left[\left(1 - \frac{\rho_{15}}{2}\right)^{\epsilon} - 1\right] + \left[\left(1 - \frac{\rho_{15}}{2}\right)^{\epsilon} - 1\right] + \left(\frac{\rho_{15}}{2}\right)^{-\epsilon} \right). \tag{H.28}$$

Since

$$\left[\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon}-1\right]\xrightarrow[5\|1]{}0\,, \tag{H.29}$$

the first and the second term on the right-hand side of Eq. (H.28) are regulated and therefore finite. By construction, the first term contributes first at $\mathcal{O}(\epsilon^2)$. Since we need the integral only up to $\mathcal{O}(\epsilon)$ there is no need to calculate this part of the integral. The second term contributes first at $\mathcal{O}(\epsilon)$ hence it only contributes to the finite part of the calculation while the third term contains the singularity in ϵ and the finite part of the integral Eq. (H.28).

We begin with the following contribution to Eq. (H.28)

$$\begin{aligned}
&\int\frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]}\,\frac{\rho_{14}}{\rho_{15}\rho_{45}}w_{\mathrm{tc}}^{1}\left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\\
&=\int\frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]}\,\frac{\rho_{14}}{\rho_{15}\rho_{45}}\left(\frac{\rho_{45}}{\rho_{15}+\rho_{45}}\right)^{2}\left[1+\frac{2\rho_{15}}{\rho_{15}+\rho_{45}}\right]\left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\\
&=\int\frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]}\,\frac{\rho_{14}}{\rho_{15}}\frac{(\rho_{15}+\rho_{45})-\rho_{15}}{(\rho_{15}+\rho_{45})^{2}}\left[1+\frac{2\rho_{15}}{\rho_{15}+\rho_{45}}\right]\left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\\
&=\int\frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]}\left[\frac{\rho_{14}}{\rho_{15}\,(\rho_{15}+\rho_{45})}+\frac{\rho_{14}}{(\rho_{15}+\rho_{45})^{2}}-\frac{2\rho_{14}\rho_{15}}{(\rho_{15}+\rho_{45})^{3}}\right]\left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\,.
\end{aligned}\tag{H.30}$$

These integrals are calculated in Sec. G.3, see Eqs. (G.15, G.16, G.17). Inserting these results into Eq. (H.30) we obtain for the integral

$$\begin{aligned}
&\int\frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]}\,\frac{\rho_{14}}{\rho_{15}\rho_{45}}w_{\mathrm{tc}}^{1}\left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\\
&=-2^{-2-3\epsilon}\left[\frac{2^{\epsilon}}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]\rho_{14}\Bigg[-\left(\frac{2}{\epsilon}+\frac{\epsilon}{2}\right)\times{}_2\mathrm{F}_1\left(1,1+\frac{\epsilon}{2};1-\frac{3}{2}\epsilon;1-\eta_{14}\right)\\
&+3\epsilon\times\frac{1+\epsilon}{1-3\epsilon}\times{}_2\mathrm{F}_1\left(1,\frac{3}{2}+\frac{\epsilon}{2};\frac{3}{2}-\frac{3}{2}\epsilon;1-\eta_{14}\right)\\
&+\frac{3}{2}\epsilon\times\frac{2+\epsilon}{2-3\epsilon}\times{}_2\mathrm{F}_1\left(1,2+\frac{\epsilon}{2};2-\frac{3}{2}\epsilon;1-\eta_{14}\right)\Bigg]\,.
\end{aligned}\tag{H.31}$$

Expanding in ϵ we obtain

$$\begin{aligned}
&-\epsilon 2^{2\epsilon}\int\frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]}\,\frac{\rho_{14}}{\rho_{15}\rho_{45}}w_{\mathrm{tc}}^{1}\left(\frac{\rho_{15}}{2}\right)^{-\epsilon}\\
&=\frac{1}{2}-\epsilon\ln(\eta_{14})-\epsilon^{2}\left[\frac{1}{2}+\frac{\pi^{2}}{6}-\ln(\eta_{14})^{2}-\frac{3}{2}\mathrm{Li}_2(1-\eta_{14})\right]+\mathcal{O}(\epsilon^{3})\,.
\end{aligned}\tag{H.32}$$

Next, we need to compute

$$\int\frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]}\,\frac{\rho_{14}}{\rho_{15}\rho_{45}}w_{\mathrm{tc}}^{1}\left[\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon}-1\right]. \tag{H.33}$$

This integral is more complicated. However, by construction, this integral is regulated and we only need the leading contribution in ϵ. We expand it in ϵ and find

$$\begin{aligned}&\int \frac{[\mathrm{d}\Omega_5]}{[\Omega^{(d-2)}]}\,\frac{\rho_{14}}{\rho_{15}\rho_{45}} w_{\mathrm{tc}}^{1}\left[\left(1-\frac{\rho_{15}}{2}\right)^{\epsilon}-1\right]\\&=\epsilon\times\frac{1}{2\pi}\times\int \mathrm{d}\Omega_5^{(3)}\,\frac{\rho_{14}}{\rho_{15}\rho_{45}}\left(\frac{\rho_{45}}{\rho_{15}+\rho_{45}}\right)^2\left(1+\frac{2\rho_{15}}{\rho_{15}+\rho_{45}}\right)\ln\left(1-\frac{\rho_{15}}{2}\right)+\mathcal{O}(\epsilon^2)\,,\end{aligned} \tag{H.34}$$

where we already used the explicit form of the partition function w_{tc}^{1} in Eq. (A.15). The rational part of the integrand in Eq. (H.34) can be further simplified. We write

$$\begin{aligned}&\frac{\rho_{14}}{\rho_{15}\rho_{45}}\left(\frac{\rho_{45}}{\rho_{15}+\rho_{45}}\right)^2\left(1+\frac{2\rho_{15}}{\rho_{15}+\rho_{45}}\right)\\&=\frac{\rho_{14}}{\rho_{15}(\rho_{15}+\rho_{45})}\left(1-\frac{\rho_{15}}{\rho_{15}+\rho_{45}}\right)\left(1+\frac{2\rho_{15}}{\rho_{15}+\rho_{45}}\right).\end{aligned} \tag{H.35}$$

Note that after using Eq. (H.35) in Eq. (H.34) the dependence on ρ_{45} is given by powers of $1/(\rho_{51}+\rho_{45})$. To compute this integral, we parametrize $\vec{n}_5$ with respect to the direction $\vec{n}_1$. Given this choice, the logarithm in Eq. (H.34) becomes independent of the azimuthal angle φ_5 and we can integrate over it using well known formulas

$$\int \mathrm{d}\varphi_5\,\frac{1}{(a-b\cos\varphi_5)^n}=2\pi\times\begin{cases}1, & n=0\,,\\ (a^2-b^2)^{-\frac{1}{2}}, & n=1\,,\\ a(a^2-b^2)^{-\frac{3}{2}}, & n=2\,.\end{cases} \tag{H.36}$$

After integration over φ_5, only squares of $\sin\theta_5$ appear. As a result, the dependence on integration variable θ_5 is given through square roots of polynomials of $\cos\theta_5$ that can be rationalized and integrated. We obtain

$$\begin{aligned}&\frac{1}{2\pi}\times\int \mathrm{d}\Omega_5^{(3)}\,\frac{\rho_{14}}{\rho_{15}\rho_{45}} w_{\mathrm{tc}}^{1}\ln\left(1-\frac{\rho_{15}}{2}\right)\\&=-\frac{\pi^2}{6}+\frac{\eta_{14}}{4-2\eta_{14}}+\frac{\mathrm{Li}_2(1-\eta_{14})}{2}-\mathrm{Li}_2(-1+\eta_{14})\,.\end{aligned} \tag{H.37}$$

Together with the result in Eqs. (H.25, H.32) an Eq. (H.28) we write the final result as

$$\langle\Delta_{56}\rangle_{S_5}=\frac{1}{2}-\epsilon\ln(\eta_{14})-\epsilon^2\left[\frac{1}{2-\eta_{14}}-\ln(\eta_{14})^2-\frac{\mathrm{Li}_2\left((1-\eta_{14})^2\right)}{2}\right]+\mathcal{O}(\epsilon^3)\,. \tag{H.38}$$

Finally, we note that the finite integral $\langle r^\mu r^\nu\rangle_{\rho_5}$ in Eq. (H.7) can be computed following the previous discussion of Eq. (H.34).

I. Subtraction terms

In this appendix we collect required double-soft and triple-collinear subtraction terms whose computation in not discussed in this thesis. They are computed in Refs. [67,68] in a general case and we report required formulas for the case of DIS.

I.1. Double-soft subtraction terms

The UV-renormalized double-soft subtraction term for two soft gluons reads

$$\left\langle \mathbb{S}F_{\rm LM}(1_q,4_q\,|\,5_g,6_g)\right\rangle = [\alpha_s]^2\left(\frac{4E_{\rm max}^2}{\mu^2}\right)^{-2\epsilon}\left\langle\left[C_F^2\,2\eta_{14}^{-2\epsilon}K_{14}^2 + C_A C_F\,S_{gg}^{\rm (nab)}\right]F_{\rm LM}(1_q,4_q)\right\rangle_\delta, \quad \text{(I.1)}$$

where K_{14} is given in Eq. (A.20) and the non-abelian coefficient $S_{gg}^{\rm (nab)}$ reads [67]

$$\begin{aligned}
S_{gg}^{\rm (nab)} = \Bigg\langle\Bigg\{ &\frac{1}{2\epsilon^4} + \frac{1}{\epsilon^3}\left[\frac{11}{12} - \ln(\eta_{14})\right] \\
&+ \frac{1}{\epsilon^2}\left[2{\rm Li}_2(1-\eta_{14}) + \ln^2(\eta_{14}) - \frac{11}{6}\ln(\eta_{14}) + \frac{11}{3}\ln 2 - \frac{\pi^2}{4} - \frac{16}{9}\right] \\
&+ \frac{1}{\epsilon}\Bigg[6{\rm Li}_3(\eta_{14}) + 2{\rm Li}_3(1-\eta_{14}) + \left(2\ln(\eta_{14}) + \frac{11}{3}\right){\rm Li}_2(1-\eta_{14}) - \frac{2}{3}\ln^3(\eta_{14}) \\
&\quad + \left(3\ln(1-\eta_{14}) + \frac{11}{6}\right)\ln^2(\eta_{14}) - \left(\frac{22}{3}\ln 2 + \frac{\pi^2}{2} - \frac{32}{9}\right)\ln(\eta_{14}) \\
&\quad - \frac{45}{4}\zeta_3 - \frac{11}{3}\ln^2 2 - \frac{11}{36}\pi^2 - \frac{137}{18}\ln 2 + \frac{217}{54}\Bigg] \\
&- 4\,{\rm HPL}(\{-1,0,0,1\},\eta_{14}) - 7\,{\rm HPL}(\{0,1,0,1\},\eta_{14}) + \frac{22}{3}{\rm Ci}_3\left(2\arcsin\left(\sqrt{\eta_{14}}\right)\right) \\
&+ \frac{\sqrt{1-\eta_{14}}{\rm Si}_2\left(2\arcsin\left(\sqrt{\eta_{14}}\right)\right)}{3\sqrt{\eta_{14}}} + 2{\rm Li}_4(1-\eta_{14}) - 14{\rm Li}_4(\eta_{14}) + 4{\rm Li}_4\left(\frac{1}{1+\eta_{14}}\right) \\
&- 2{\rm Li}_4\left(\frac{1-\eta_{14}}{1+\eta_{14}}\right) + 2{\rm Li}_4\left(\frac{\eta_{14}-1}{1+\eta_{14}}\right) + {\rm Li}_4(1-\eta_{14}^2) + \left[10\ln(\eta_{14}) - 4\ln(1+\eta_{14}) + \frac{11}{3}\right] \\
&\quad \times {\rm Li}_3(1-\eta_{14}) + \left[14\ln(1-\eta_{14}) + 2\ln(\eta_{14}) + 4\ln(1+\eta_{14}) + \frac{22}{3}\right]{\rm Li}_3(\eta_{14}) \\
&+ 4\ln(1-\eta_{14}){\rm Li}_3(-\eta_{14}) + \frac{9}{2}{\rm Li}_2^2(1-\eta_{14}) - 4{\rm Li}_2(1-\eta_{14}){\rm Li}_2(-\eta_{14}) \\
&+ \left[7\ln(1-\eta_{14})\ln(\eta_{14}) - \ln^2(\eta_{14}) - \frac{5}{2}\pi^2 + \frac{22}{3}\ln 2 - \frac{131}{18}\right]{\rm Li}_2(1-\eta_{14})
\end{aligned} \quad \text{(I.2)}$$

$$
\begin{aligned}
&+\left[\frac{2}{3}\pi^2 - 4\ln(1-\eta_{14})\ln(\eta_{14})\right]\mathrm{Li}_2(-\eta_{14}) + \frac{\ln^4(\eta_{14})}{3} + \frac{\ln^4(1+\eta_{14})}{6} \\
&-\ln^3(\eta_{14})\left[\frac{4}{3}\ln(1-\eta_{14}) + \frac{11}{9}\right] + \ln^2(\eta_{14})\left[7\ln^2(1-\eta_{14}) + \frac{11}{3}\ln(1-\eta_{14})\right. \\
&\quad\left. + \frac{\pi^2}{3} + \frac{22}{3}\ln 2 - \frac{32}{9}\right] - \frac{\pi^2}{6}\ln^2(1+\eta_{14}) \\
&+\zeta_3\left[\frac{17}{2}\ln(\eta_{14}) - 11\ln(1-\eta_{14}) + \frac{7}{2}\ln(1+\eta_{14}) - \frac{21}{2}\ln 2 - \frac{99}{4}\right] + \ln(\eta_{14})\times \\
&\left[-\frac{7\pi^2}{2}\ln(1-\eta_{14}) + \frac{22}{3}\ln^2 2 - \frac{11}{18}\pi^2 + \frac{137}{9}\ln 2 - \frac{208}{27}\right] - 12\mathrm{Li}_4\left(\frac{1}{2}\right) \\
&+\frac{143}{720}\pi^4 - \frac{\ln^4 2}{2} + \frac{\pi^2}{2}\ln^2 2 - \frac{11}{6}\pi^2\ln 2 + \frac{125}{216}\pi^2 + \frac{22}{9}\ln^3 2 \\
&\left.\left.+\frac{137}{18}\ln^2 2 + \frac{434}{27}\ln 2 - \frac{649}{81} + \mathcal{O}(\epsilon)\right\} \times F_{\mathrm{LM}}(1_q, 4_q)\right\rangle_\delta .
\end{aligned}
$$

The counter term for a soft quark-anti-quark pair reads [67]

$$
\begin{aligned}
\left\langle \mathbb{S}F_{\mathrm{LM,ns}}(1_q, 4_q \,|\, 5'_q, 6'_q)\right\rangle &= C_F N_f T_R\,[\alpha_s]^2\left(\frac{4E_{\max}^2}{\mu^2}\right)^{-2\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]^2\left\langle\left\{-\frac{1}{3\epsilon^3}\right.\right. \\
&+\frac{1}{\epsilon^2}\left[\frac{2}{3}\ln(\eta_{14}) - \frac{4}{3}\ln 2 + \frac{13}{18}\right] + \frac{1}{\epsilon}\left[-\frac{4}{3}\mathrm{Li}_2(1-\eta_{14}) - \frac{2}{3}\ln^2(\eta_{14}) + \ln(\eta_{14})\right. \\
&\quad\left.\times\left(\frac{8}{3}\ln 2 - \frac{13}{9}\right) + \frac{\pi^2}{9} + \frac{4}{3}\ln^2 2 + \frac{35}{9}\ln 2 - \frac{125}{54}\right] - \frac{8}{3}\mathrm{Ci}_3\left(2\arcsin\left(\sqrt{\eta_{14}}\right)\right) \\
&-\frac{2\sqrt{1-\eta_{14}}\,\mathrm{Si}_2\left(2\arcsin\left(\sqrt{\eta_{14}}\right)\right)}{3\sqrt{\eta_{14}}} - \frac{4}{3}\mathrm{Li}_3(1-\eta_{14}) - \frac{8}{3}\mathrm{Li}_3(\eta_{14}) \\
&+\mathrm{Li}_2(1-\eta_{14})\left[\frac{29}{9} - \frac{8}{3}\ln 2\right] + \frac{4}{9}\ln^3(\eta_{14}) + \ln^2(\eta_{14})\left[-\frac{4}{3}\ln(1-\eta_{14})\right. \\
&\quad\left.-\frac{8}{3}\ln 2 + \frac{13}{9}\right] + \ln(\eta_{14})\left[-\frac{8}{3}\ln^2 2 - \frac{70}{9}\ln 2 + \frac{2}{9}\pi^2 + \frac{107}{27}\right] + 9\zeta_3 + \frac{2\pi^2}{3}\ln 2 \\
&\left.\left.-\frac{8}{9}\ln^3 2 - \frac{23}{108}\pi^2 - \frac{35}{9}\ln^2 2 - \frac{223}{27}\ln 2 + \frac{601}{162} + \mathcal{O}(\epsilon)\right\} \times F_{\mathrm{LM,ns}}(1_q, 4_q)\right\rangle_\delta .
\end{aligned}
\tag{I.3}
$$

In Eqs. (I.2, I.3) $\mathrm{HPL}(\{a_1, \ldots, a_n\}, z)$ are harmonic polylogarithms [84] and the Clausen functions are defined as

$$
\mathrm{Ci}_n(z) = \frac{\mathrm{Li}_n(e^{iz}) + \mathrm{Li}_n(e^{-iz})}{2}, \quad \mathrm{Si}_n(z) = \frac{\mathrm{Li}_n(e^{iz}) - \mathrm{Li}_n(e^{-iz})}{2i}. \tag{I.4}
$$

I.2. Triple-collinear subtraction terms

Quark-initiated processes

The UV-renormalized triple-collinear counter terms for the process $q + e^- \to e^- + q + gg$ read [68]

$$\begin{aligned}
&\Big\langle [I - \mathbb{S}]\,[I - S_6]\Big[\theta^{(a)}\mathbb{C}_1[1 - C_{51}] + \theta^{(b)}\mathbb{C}_1[1 - C_{56}] + \theta^{(c)}\mathbb{C}_1[1 - C_{61}] \\
&\quad + \theta^{(d)}\mathbb{C}_1[1 - C_{56}]\Big][\mathrm{d}p_5][\mathrm{d}p_6] w^{51,61} F_{\mathrm{LM}}(1_q, 4_q \,|\, 5_g, 6_g)\Big\rangle = [\alpha_s]^2 \left(\frac{E_1^2}{\mu^2}\right)^{-2\epsilon} \int_0^1 \mathrm{d}z \\
&\quad \times \Big\langle \Big[R^{(1)}_{\mathrm{reg}} + R^{(1)}_{+}\left(\mathcal{D}_0(z) - 4\epsilon\mathcal{D}_1(z) + \mathcal{O}(\epsilon^2)\right) + R^{(1)}_{\delta}\delta(1-z)\Big] \times \frac{F_{\mathrm{LM}}(z\cdot 1_q, 4_q)}{z}\Big\rangle_\delta ,
\end{aligned} \tag{I.5}$$

and

$$\begin{aligned}
&\Big\langle [I - \mathbb{S}]\,[I - S_6]\Big[\theta^{(a)}\mathbb{C}_4[1 - C_{54}] + \theta^{(b)}\mathbb{C}_4[1 - C_{56}] + \theta^{(c)}\mathbb{C}_4[1 - C_{64}] \\
&\quad + \theta^{(d)}\mathbb{C}_4[1 - C_{56}]\Big][\mathrm{d}p_5][\mathrm{d}p_6] w^{54,64} F_{\mathrm{LM}}(1_q, 4_q \,|\, 5_g, 6_g)\Big\rangle \\
&= [\alpha_s]^2 \Big\langle \left(\frac{E_4^2}{\mu^2}\right)^{-2\epsilon} R^{(4)}_{\mathrm{reg}} \times F_{\mathrm{LM}}(1_q, 4_q)\Big\rangle_\delta .
\end{aligned} \tag{I.6}$$

Functions $R^{(i)}$ are decomposed into different colour factors

$$R^{(i)}_{\{\delta,+,\mathrm{reg}\}} = C_F^2 R^{(i),\mathrm{a}}_{\{\delta,+,\mathrm{reg}\}} + C_F C_A R^{(i),\mathrm{na}}_{\{\delta,+,\mathrm{reg}\}} , \tag{I.7}$$

where $R^{(4),\mathrm{a}}_{\{\delta,+\}} = R^{(4),\mathrm{na}}_{\{\delta,+\}} = 0$. The coefficients read [68]

$$R^{(1),\mathrm{a}}_{\delta} = \frac{1}{\epsilon}\left[\frac{\pi^2}{3}\ln 2\right] - \frac{7\pi^2}{6}\ln^2(2) + 8\zeta_3 \ln 2 + \mathcal{O}(\epsilon) , \tag{I.8}$$

$$\begin{aligned}
R^{(1),\mathrm{na}}_{\delta} &= \frac{1}{\epsilon}\Big[-\frac{1571}{216} + \frac{11\pi^2}{36} + \frac{3}{8}\zeta_3 + \frac{\pi^2}{3}\ln 2 + \frac{11}{2}\ln^2(2) + \left(-\frac{32}{9} + \frac{\pi^2}{6} - \frac{11\ln 2}{3}\right) \\
&\quad \times \ln\left(\frac{E_{\max}}{E_1}\right)\Big] - \frac{1}{12}\ln^4 2 - \frac{176}{9}\ln^3 2 - \left(\frac{79}{9} + \frac{11\pi^2}{12}\right)\ln^2 2 \\
&\quad + \frac{513\zeta_3 + 913 + 165\pi^2}{108}\ln 2 + \left(\frac{64}{9} - \frac{\pi^2}{3} + \frac{22\ln 2}{3}\right)\ln^2\left(\frac{E_{\max}}{E_1}\right) \\
&\quad + \left(\frac{11\zeta_3}{2} + \frac{383}{54} - \frac{22\pi^2}{9} - 11\ln^2(2) + \frac{\ln 2}{3} - \frac{2}{3}\pi^2\ln 2\right)\ln\left(\frac{E_{\max}}{E_1}\right) + \mathcal{O}(\epsilon) ,
\end{aligned} \tag{I.9}$$

$$R^{(1),\mathrm{a}}_{+} = -\frac{4\pi^2}{3}\ln 2 + \mathcal{O}(\epsilon) , \tag{I.10}$$

$$R^{(1),\mathrm{na}}_{+} = \frac{1}{\epsilon}\left[\frac{11}{3}\ln 2 - \frac{\pi^2}{6} + \frac{32}{9}\right] - 11\ln^2 2 - \frac{1+2\pi^2}{3}\ln 2 - 7\zeta_3 + \frac{11\pi^2}{9} + 22 + \mathcal{O}(\epsilon) , \tag{I.11}$$

$$
\begin{aligned}
R_{\text{reg}}^{(1),\text{a}} &= \frac{1}{\epsilon}\left(-\frac{z+1}{2}\ln 2\ln z+(1-z)\ln 2+\frac{(z^2+3)}{4(z-1)}\ln^2 z-z\ln z+\frac{3(z-1)}{2}\right)\\
&+\frac{z^2\left(-36\zeta_3+33+4\pi^2\right)-2\left(33+2\pi^2\right)z-60\zeta_3+33}{6(z-1)}+\frac{7(z-1)}{2}\ln^2 2\\
&+\left(-6z+\pi^2(z+1)+6\right)\ln 2+\frac{\left(3(z-1)z-\pi^2\left(3z^2+5\right)\right)}{3(z-1)}\ln z\\
&+\frac{z}{2}\ln^2 z+\frac{(9z^2+19)}{12(1-z)}\ln^3 z+\frac{7(z+1)}{4}\ln^2 2\ln z+\frac{(z^2+7)}{2(1-z)}\ln 2\ln^2 z\\
&+(3z-1)\ln 2\ln z+6(1-z)\ln(1-z)-4(1-z)\ln(1-z)\ln 2\\
&+\left(-2(z+1)\ln 2-\frac{2\left(z^2+1\right)}{z-1}\ln z-4z\right)\text{Li}_2(z)+\left(\frac{2\left(3z^2+5\right)}{z-1}\right)\text{Li}_3(z)+\mathcal{O}(\epsilon)\,,
\end{aligned}
\tag{I.12}
$$

$$
\begin{aligned}
R_{\text{reg}}^{(1),\text{na}} &= \frac{1}{\epsilon}\left[\frac{\left(6\pi^2-61\right)z^2-15z+76}{36(z-1)}-\frac{11(z+1)}{6}\ln 2+\frac{(11z^2+2)}{12(z-1)}\ln z\right.\\
&\quad\left.+\frac{(z^2+1)}{2(1-z)}\ln(1-z)\ln z+\left(\frac{1+z^2}{2(1-z)}\right)\text{Li}_2(z)\right]\\
&+\frac{3\left(z^2(48\zeta_3-119)-46z-36\zeta_3+165\right)+\pi^2\left(-50z^2+12z+12\right)}{36(z-1)}\\
&+\frac{\left(\left(61-6\pi^2\right)z^2+15z-76\right)}{9(z-1)}\ln(1-z)+\frac{(49z^2+57z-20)}{36(z-1)}\ln z\\
&+\frac{2\left(z^2+1\right)}{z-1}\ln^2 1-z\ln z+\frac{(z-1)}{2}\ln(1-z)\ln z+\frac{(11z^2+2)}{8(1-z)}\ln^2 z\\
&+\frac{2\left(z^2+1\right)}{z-1}\ln(1-z)\ln z\ln 2+\frac{22(z+1)}{3}\ln(1-z)\ln 2+\frac{(z^2+1)}{4(z-1)}\ln(1-z)\ln^2 z\\
&+\frac{11(z+1)}{2}\ln^2 2+\frac{(11z^2+2)}{3(1-z)}\ln 2\ln z+\frac{(-7z^2+6z+4\pi^2+1)}{6(1-z)}\ln 2\\
&+\left(\frac{2\left(z^2+1\right)}{z-1}\ln(1-z)+\frac{2\left(z^2+1\right)}{z-1}\ln 2+\frac{(z^2+1)}{2(z-1)}\ln z+\frac{25z^2-6z+7}{6(z-1)}\right)\text{Li}_2(z)\\
&+\left(\frac{2\left(z^2+1\right)}{z-1}\right)\text{Li}_3(1-z)+\left(\frac{(z^2+1)}{2(1-z)}\right)\text{Li}_3(z)+\mathcal{O}(\epsilon)\,,
\end{aligned}
\tag{I.13}
$$

$$
\begin{aligned}
R_{\text{reg}}^{(4),\text{a}} &= \frac{1}{\epsilon}\left[\frac{31}{16}+\frac{9\ln 2}{8}+\frac{\pi^2\ln 2}{3}-2\zeta_3\right]+\frac{715}{32}-\frac{7\pi^2}{30}+\frac{17\ln 2}{8}+\pi^2\ln 2-\frac{63\ln^2 2}{16}\\
&-\frac{7\pi^2\ln^2 2}{6}+16\ln 2\zeta_3+\mathcal{O}(\epsilon)\,,
\end{aligned}
\tag{I.14}
$$

$$
\begin{aligned}
R_{\text{reg}}^{(4),\text{na}} &= \frac{1}{\epsilon}\left[-\frac{1015}{108}+\frac{\pi^2}{8}-\frac{11\ln 2}{4}+\frac{\pi^2\ln 2}{3}+\frac{11\ln^2 2}{2}+\ln\left(\frac{E_{\text{max}}}{E_4}\right)\right.\\
&\quad\left.\times\left(-\frac{32}{9}+\frac{\pi^2}{6}-\frac{11\ln 2}{3}\right)+\frac{19}{8}\zeta_3\right]-\frac{2281}{48}-\frac{119\pi^2}{144}+\frac{173\pi^4}{480}-\frac{1247\ln 2}{108}\\
&+\frac{161}{36}\pi^2\ln 2-\frac{19\ln^2 2}{36}-\frac{11}{12}\pi^2\ln^2 2-\frac{176\ln^3 2}{9}-\frac{\ln^4 2}{12}+\ln\left(\frac{E_{\text{max}}}{E_4}\right)
\end{aligned}
\tag{I.15}
$$

$$\times\left(\frac{383}{54}-\frac{22}{9}\pi^2+\frac{1}{3}\ln 2-\frac{2}{3}\pi^2\ln 2-11\ln^2 2+\frac{11}{2}\zeta_3\right)+\ln^2\left(\frac{E_{\max}}{E_4}\right)$$
$$\times\left(\frac{64}{9}-\frac{1}{3}\pi^2+\frac{22}{3}\ln 2\right)-2\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{25\zeta_3}{24}-\frac{13}{4}\zeta_3\ln 2+\mathcal{O}(\epsilon)\,.$$

The non-singlet contributions read

$$\begin{aligned}&\sum_{i\in\{1,4\}}\Big\langle[I-\mathbb{S}]\mathbb{C}_i\Big[\theta_i^{(a)}+\Big(\theta_i^{(b)}+\theta_i^{(d)}\Big)[I-C_{56}]+\theta_i^{(d)}\Big][\mathrm{d}p_5][\mathrm{d}p_6]\\
&\qquad\qquad\times w^{5i,6i}F_{\mathrm{LM,ns}}(1,4\,|\,5,6)\Big\rangle\\
&=[\alpha_s]^2\left(\frac{E_1^2}{\mu^2}\right)^{-2\epsilon}\int_0^1\mathrm{d}z\,\Big\langle\Big[R^{(1)}_{\mathrm{reg,ns}}+R^{(1)}_{+,\mathrm{ns}}\left(\mathcal{D}_0(z)-4\epsilon\mathcal{D}_1(z)+\mathcal{O}(\epsilon^2)\right)+R^{(1)}_{\delta,\mathrm{ns}}\delta(1-z)\Big]\\
&\quad\times\frac{F_{\mathrm{LM}}(z\cdot 1_q,4_q)}{z}\Big\rangle_\delta+[\alpha_s]^2\Big\langle\left(\frac{E_4^2}{\mu^2}\right)^{-2\epsilon}R^{(4)}_{\mathrm{reg,ns}}F_{\mathrm{LM}}(1_q,4_q)\Big\rangle_\delta\,.\end{aligned}\tag{I.16}$$

Split into different colour factors the functions $R^{(i)}$ read

$$R^{(i)}_{\{\delta,+,\mathrm{reg}\},\mathrm{ns}}=\frac{C_F(C_A-2C_F)}{2}R^{(i),1}_{\{\delta,+,\mathrm{reg}\},\mathrm{ns}}+N_fC_FT_RR^{(i),2}_{\{\delta,+,\mathrm{reg}\},\mathrm{ns}}\,,\tag{I.17}$$

where $R^{(4),i}_{\{\delta,+\},\mathrm{ns}}=0$, with $i\in\{1,2\}$, and $R^{(1),1}_{\{\delta,+\},\mathrm{ns}}=0$. The coefficients read [68]

$$\begin{aligned}R^{(1),1}_{\mathrm{reg,ns}}&=\frac{1}{\epsilon}\Big[-\frac{\pi^2(1+z^2)+3(8-15z+7z^2)}{12(1-z)}+\frac{(-5+2z^2)\ln z}{4(1-z)}\\
&\quad+\frac{(1+z^2)\ln(1-z)\ln z}{2(1-z)}-\frac{(1+z^2)\ln^2 z}{4(1-z)}+\frac{(1+z^2)\mathrm{Li}_2(z)}{2(1-z)}\Big]\\
&-\frac{\left(\pi^2\left(1+z^2\right)+3\left(8-15z+7z^2\right)\right)\ln 2}{3(-1+z)}-\frac{\left(\pi^2\left(1+z^2\right)+3\left(8-15z+7z^2\right)\right)}{3(-1+z)}\\
&\quad\times\ln(1-z)-\frac{\left(66-57z-39z^2+4\pi^2\left(1+z^2\right)\right)\ln z}{12(-1+z)}+\frac{\left(-5+2z^2\right)\ln 2\ln z}{-1+z}\\
&-\frac{3}{2}(1-z)\ln(1-z)\ln z+\frac{2\left(1+z^2\right)\ln 2\ln(1-z)\ln z}{-1+z}+\frac{2\left(1+z^2\right)\ln(1-z)^2\ln z}{-1+z}\\
&+\frac{\left(-25+12z+4z^2\right)\ln z^2}{8(-1+z)}-\frac{\left(1+z^2\right)\ln 2\ln z^2}{-1+z}-\frac{\left(1+z^2\right)\ln(1-z)\ln z^2}{4(-1+z)}\\
&-\frac{7\left(1+z^2\right)\ln z^3}{12(-1+z)}-2(1+z)\ln z\ln(1+z)-2(1+z)\mathrm{Li}_2(-z)\\
&-\frac{2\left(1+z^2\right)\ln z\mathrm{Li}_2(-z)}{-1+z}-\frac{\left(-13+6z+z^2\right)\mathrm{Li}_2(z)}{2(-1+z)}+\frac{2\left(1+z^2\right)\ln 2\,\mathrm{Li}_2(z)}{-1+z}\\
&+\frac{2\left(1+z^2\right)\ln(1-z)\mathrm{Li}_2(z)}{-1+z}-\frac{3\left(1+z^2\right)\ln z\mathrm{Li}_2(z)}{2(-1+z)}+\frac{3\left(1+z^2\right)\mathrm{PolyLog}[3,1-z]}{-1+z}\\
&+\frac{4\left(1+z^2\right)\mathrm{Li}_3(-z)}{-1+z}+\frac{9\left(1+z^2\right)\mathrm{Li}_3(z)}{2(-1+z)}\end{aligned}\tag{I.18}$$

$$- \frac{\pi^2\left(11-6z+z^2\right)+3\left(-1-6z+6\zeta_3+z^2(7+6\zeta_3)\right)}{12(-1+z)} + \mathcal{O}(\epsilon)\,,$$

$$\begin{aligned} R^{(1),2}_{\delta,\mathrm{ns}} &= \frac{1}{\epsilon}\left[\frac{275}{108} - \frac{\pi^2}{9} - 2\ln^2 2 + \ln\left(\frac{E_{\max}}{E_1}\right)\left(\frac{10}{9} + \frac{4\ln 2}{3}\right)\right] + \frac{509}{108} + \frac{97\pi^2}{216} \\ &- \frac{265\ln 2}{54} - \frac{5\pi^2\ln 2}{9} + \frac{59\ln^2 2}{9} + \frac{64\ln^3 2}{9} + \ln\left(\frac{E_{\max}}{E_1}\right)\left(-\frac{83}{27} + \frac{8\pi^2}{9} - \frac{4\ln 2}{3}\right. \\ &\left. + 4\ln^2 2\right) - \ln^2\left(\frac{E_{\max}}{E_1}\right)\left(\frac{20}{9} + \frac{8\ln 2}{3}\right) - \frac{11}{2}\zeta_3 + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{I.19}$$

$$R^{(1),2}_{+,\mathrm{ns}} = \frac{1}{\epsilon}\left[-\frac{10}{9} - \frac{4\ln 2}{3}\right] - \frac{4}{9}(16+\pi^2) + 4\ln^2 2 + \mathcal{O}(\epsilon)\,, \tag{I.20}$$

$$\begin{aligned} R^{(1),2}_{\mathrm{reg,ns}} &= \frac{1}{\epsilon}\left[\frac{13}{18} + \frac{7z}{18} + \frac{2\ln 2}{3} + \frac{2z\ln 2}{3} + \frac{(1+z^2)\ln z}{3(1-z)}\right] + \frac{\ln 2}{3}\left(-5+z\right. \\ &\left. - 8(1+z)\ln(1-z) - \frac{4(1+z^2)\ln z}{(1-z)}\right) - \frac{(7-6z-5z^2)\ln z}{9(1-z)} - \frac{(1+z^2)\ln^2 z}{2(1-z)} \\ &+ \frac{4(3(1+z^2)\mathrm{Li}_2(z) - \pi^2 z^2)}{9(1-z)} + \frac{29(1+z)}{9} - \frac{2}{9}(13+7z)\ln(1-z) + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{I.21}$$

$$\begin{aligned} R^{(4),1}_{\mathrm{reg,ns}} &= \frac{1}{\epsilon}\left[\frac{13}{8} - \frac{\pi^2}{4} + \zeta_3\right] + \frac{335}{16} - \frac{5\pi^2}{6} + \frac{7\pi^4}{45} - \frac{13\ln 2}{2} + \pi^2\ln 2 \\ &- \frac{39}{2}\zeta_3 - 4\zeta_3\ln 2 + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{I.22}$$

$$\begin{aligned} R^{(4),2}_{\mathrm{reg,ns}} &= \frac{1}{\epsilon}\left[\frac{329}{108} + \ln 2 - 2\ln^2 2 + \ln\left(\frac{E_{\max}}{E_4}\right)\left(\frac{10}{9} + 2\ln 2\right)\right] + \frac{2773}{216} + \frac{35\pi^2}{72} + \frac{43\ln 2}{27} \\ &- \frac{13\pi^2\ln 2}{9} + \frac{32\ln^2 2}{9} + \frac{64\ln^3 2}{9} + \ln\left(\frac{E_{\max}}{E_4}\right)\left(-\frac{83}{27} + \frac{8\pi^2}{9} - \frac{4\ln 2}{3} + 4\ln^2 2\right) \\ &+ \ln^2\left(\frac{E_{\max}}{E_4}\right)\left(-\frac{20}{9} - \frac{8\ln 2}{3}\right) + \frac{19}{6}\zeta_3 + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{I.23}$$

The triple-collinear counterterm for the singlet contribution reads [68]

$$\begin{aligned} &\left\langle \mathbb{C}_1\left([I-C_{51}]\theta_1^{(a)} + \theta_1^{(b)} + \theta_1^{(c)} + \theta_1^{(d)}\right)[\mathrm{d}p_5][\mathrm{d}p_6]w^{51,61}F_{\mathrm{LM,s}}(1_q,4_{q'}\,|\,5_q,6_{q'})\right\rangle \\ &= C_F T_R\,[\alpha_s]^2\left(\frac{E_1^2}{\mu^2}\right)^{-2\epsilon}\int_0^1 \mathrm{d}z\,\left\{\frac{1}{\epsilon}\left[\frac{5}{2} - \frac{13}{9z} - \frac{5z}{2} + \frac{13z^2}{9} - \frac{\ln 2}{2} - \frac{2\ln 2}{3z} + \frac{1}{2}z\ln 2\right.\right. \\ &\quad\left. + \frac{2}{3}z^2\ln 2 - \frac{\ln z}{2} - \frac{4\ln z}{3z} - \frac{1}{2}z\ln z - \ln 2\ln z - z\ln 2\ln z - \frac{\ln^2 z}{2} - \frac{1}{2}z\ln^2 z\right] \\ &\quad + \mathrm{Li}_2(z)\left(-\frac{20}{3z} - 4z + \frac{4z^2}{3} - 4\ln 2 - 4z\ln 2\right) - \frac{4\mathrm{Li}_2(-z)\left(2+3z+3z^2+2z^3\right)}{3z} \\ &\quad - \frac{31}{18} - \frac{\pi^2}{3} + \frac{197}{27z} + \frac{8\pi^2}{9z} - \frac{41z}{18} + \frac{\pi^2 z}{3} - \frac{89z^2}{27} - \frac{4\pi^2 z^2}{9} - \frac{71\ln 2}{6} + \frac{2}{3}\pi^2\ln 2 \\ &\quad + \frac{61\ln 2}{9z} + \frac{71}{6}z\ln 2 + \frac{2}{3}\pi^2 z\ln 2 - \frac{61}{9}z^2\ln 2 + \frac{7\ln^2 2}{4} + \frac{7\ln^2 2}{3z} - \frac{7}{4}z\ln^2 2 \end{aligned}$$

$$
\begin{aligned}
&- \frac{7}{3} z^2 \ln^2 2 - 10 \ln(1-z) + \frac{52 \ln(1-z)}{9z} + 10z \ln(1-z) - \frac{52}{9} z^2 \ln(1-z) \\
&+ 2 \ln 2 \ln(1-z) + \frac{8 \ln 2 \ln(1-z)}{3z} - 2z \ln 2 \ln(1-z) - \frac{8}{3} z^2 \ln 2 \ln(1-z) + \frac{19 \ln z}{3} \\
&+ \frac{2}{3} \pi^2 \ln z + \frac{70 \ln z}{9z} + 6z \ln z + \frac{2}{3} \pi^2 z \ln z + \frac{26}{9} z^2 \ln z + 6 \ln 2 \ln z + \frac{20 \ln 2 \ln z}{3z} \\
&+ 6z \ln 2 \ln z + \frac{4}{3} z^2 \ln 2 \ln z + \frac{7}{2} \ln^2 2 \ln z + \frac{7}{2} z \ln^2 2 \ln z + 2 \ln(1-z) \ln z \\
&- \frac{4 \ln(1-z) \ln z}{3z} - 2z \ln(1-z) \ln z + \frac{4}{3} z^2 \ln(1-z) \ln z + \frac{11 \ln^2 z}{4} + \frac{4 \ln^2 z}{z} \\
&+ \frac{19}{4} z \ln^2 z + \frac{2}{3} z^2 \ln^2 z + 3 \ln 2 \ln^2 z + 3z \ln 2 \ln^2 z + \frac{5 \ln^3 z}{6} + \frac{5}{6} z \ln^3 z \\
&- 4 \ln z \ln(1+z) - \frac{8 \ln z \ln(1+z)}{3z} - 4z \ln z \ln(1+z) - \frac{8}{3} z^2 \ln z \ln(1+z) \\
&- 4 \mathrm{Li}_3(z) - 4z \mathrm{Li}_3(z) + 4\zeta_3 + 4z\zeta_3 + \mathcal{O}(\epsilon) \Big\} \\
&\times \sum_{f \in \{q, \bar{q}\}} \left\langle \frac{F_{\mathrm{LM}}(z \cdot 1_f, 4_f)}{z} \right\rangle_\delta .
\end{aligned}
\tag{I.24}
$$

Gluon-initiated process

There is only the initial state triple-collinear counterterm for the gluon-initiated process. It reads

$$
\begin{aligned}
&\Big\langle [I - S_6] \mathbb{C}_1 \Big[\theta_1^{(a)} [I - C_{61}] + \theta_1^{(b)} [I - C_{56}] + \theta_1^{(c)} [I - C_{51}] \\
&\quad + \theta_1^{(d)} [I - C_{56}] \Big] [\mathrm{d}p_5][\mathrm{d}p_6] w^{51,61} w_g^{51} F_{\mathrm{LM},g}(1, 4 \,|\, 5, 6) \Big\rangle \\
&= [\alpha_s]^2 \left(\frac{E_1^2}{\mu^2} \right)^{-2\epsilon} \int_0^1 \mathrm{d}z \, \left[C_F^2 R_{\mathrm{reg}}^{(1),\mathrm{a}} + C_F C_A R_{\mathrm{reg}}^{(1),\mathrm{na}} \right] \sum_{f \in \{q, \bar{q}\}} \left\langle \frac{F_{\mathrm{LM}}(z \cdot 1_f, 4_f)}{z} \right\rangle_\delta .
\end{aligned}
\tag{I.25}
$$

where [68]

$$
\begin{aligned}
R_{\mathrm{reg}}^{(1),\mathrm{a}} = \frac{1}{\epsilon} \Bigg[&\frac{8\pi^2 z^2 - 8\pi^2 z - 15z + 4\pi^2 - 3}{12} + 3 \left(2z^2 - 2z + 1 \right) \ln(1-z) \ln 2 \\
&+ \left(-2z^2 + 2z - 1 \right) \ln(1-z) \ln z + \frac{1-2z}{2} \ln z \ln 2 + \frac{-9z^2 + 11z - 5}{2} \ln 2 \\
&+ \frac{4z^2 - 6z + 3}{4} \ln^2 z - \frac{3}{4} \ln z - \left(2z^2 - 2z + 1 \right) \mathrm{Li}_2(z) \\
&- 3(1 - 2z + 2z^2) \ln 2 \ln \left(\frac{E_{\mathrm{max}}}{E_1} \right) \Bigg] \\
&+ \frac{-3\pi^2 z^2 + 12z\zeta_3 + 3\pi^2 z - 24z - 6\zeta_3 - \pi^2}{3} - 9 \left(2z^2 - 2z + 1 \right) \ln^2 1 - z \ln 2 \\
&+ 4 \left(2z^2 - 2z + 1 \right) \ln^2 1 - z \ln z - \frac{19 \left(2z^2 - 2z + 1 \right)}{2} \ln(1-z) \ln^2 2 \\
&+ 4 \left(2z^2 - 2z + 1 \right) \ln(1-z) \ln 2 \ln 2 + \left(18z^2 - 22z + 7 \right) \ln(1-z) \ln 2
\end{aligned}
\tag{I.26}
$$

$$
\begin{aligned}
&+\frac{\left(2z^2-2z+1\right)}{2}\ln(1-z)\ln^2 z+\ln(1-z)\ln z+\frac{7(2z-1)}{4}\ln z\ln^2 2\\
&+\frac{3-4\pi^2z^2+4\pi^2z+15z-2\pi^2}{3}\ln(1-z)+\frac{57z^2-71z+32}{4}\ln^2 2\\
&+\frac{-8z^2+14z-7}{2}\ln^2 z\ln 2+2(z+2)\ln z\ln 2\\
&+\frac{-4\pi^2z^2-117z^2+8\pi^2z+150z-4\pi^2-27}{6}\ln 2+\frac{-28z^2+38z-19}{12}\ln^3 z\\
&+\frac{(8z+9)}{8}\ln^2 z+\frac{-32\pi^2z^2+40\pi^2z-21z-20\pi^2+9}{12}\ln z\\
&+\left(\ln 2\left(8z^2-12z+6\right)+\left(-2z^2+2z-1\right)\left(\ln z-4\ln(1-z)\right)-2\right)\mathrm{Li}_2(z)\\
&+\left(8z^2-8z+4\right)\mathrm{Li}_3(1-z)+\left(14z^2-18z+9\right)\mathrm{Li}_3(z)\\
&+3(1-2z+2z^2)\ln 2\ln^2(E_{\max}/E_1)\\
&+\ln\left(\frac{E_{\max}}{E_1}\right)\left(\frac{19(1-2z+2z^2)}{2}\ln^2(2)+6(1-2z+2z^2)\ln(1-z)\ln 2+3\ln 2\right.\\
&\qquad\left.-\frac{2\pi^2(1-2z+2z^2)}{3}\right)+\mathcal{O}(\epsilon)\,,
\end{aligned}
$$

$$
\begin{aligned}
R^{(1),\mathrm{na}}_{\mathrm{reg}}=&\ \frac{1}{\epsilon}\left[\frac{-6\pi^2z^3-67z^3+3\pi^2z^2+81z^2-3\pi^2z-27z+13}{9z}\right.\\
&+\left(2z^2-2z+1\right)\ln(1-z)\ln 2+\left(2z^2-2z+1\right)\ln(1-z)\ln z\\
&-\left(2z^2+2z+1\right)\ln(1+z)\ln z+(4z+1)\ln z\ln 2+\frac{4-31z^3+24z^2+3z}{6z}\ln 2\\
&+\frac{6z+1}{2}\ln^2 z+\frac{12z+1}{2}\ln z-\left(2z^2+2z+1\right)\mathrm{Li}_2(-z)+\left(2z^2-2z+1\right)\mathrm{Li}_2(z)\\
&\left.-(1-2z+2z^2)\ln 2\ln\left(\frac{E_{\max}}{E_1}\right)\right]\\
&+\left(\left(8z^2+8z+4\right)\left(\ln(1-z)+\ln 2\right)+\left(2z^2-6z+1\right)\ln z\right)\mathrm{Li}_2(-z)\\
&+\left(\left(-8z^2+8z-4\right)\ln(1-z)-8(z-3)z\ln 2-4z\ln z\right)\mathrm{Li}_2(z)\\
&+\frac{44z^3+48z^2+15z+8}{3z}\mathrm{Li}_2(-z)+\frac{-22z^3+96z^2-3z+20}{3z}\mathrm{Li}_2(z)\\
&-\left(18z^2-2z+9\right)\mathrm{Li}_3(1-z)+\left(10z^2+26z+5\right)\mathrm{Li}_3(-z)\\
&+\left(4z^2+4z+2\right)\left(3\mathrm{Li}_3\left(\frac{z}{1+z}\right)+\mathrm{Li}_3(1-z^2)\right)+(32z+4)\mathrm{Li}_3(z)\\
&+(1-2z+2z^2)\ln 2\ln^2\left(\frac{E_{\max}}{E_1}\right)\\
&+\left(\frac{7(1-2z+2z^2)}{2}\ln 2+2(1-2z+2z^2)\ln(1-z)+1\right)\ln 2\ln\left(\frac{E_{\max}}{E_1}\right)+\mathcal{O}(\epsilon)\,.
\end{aligned}
\tag{I.27}
$$

J. Finite contributions of subtractions

In this appendix we collect finite contributions of the subtraction terms.

J.1. Quark non-singlet contributions

Regular matrix element: $F_{\mathrm{LM}}(1,4)$

For Δ_{ns} we define

$$\Delta_{\mathrm{ns}}(E_1, E_4, E_{\max}, \eta_{14}) = C_F^2\, \Delta_{\mathrm{ns}}^1 + C_F C_A\, \Delta_{\mathrm{ns}}^2 + C_F N_f\, \Delta_{\mathrm{ns}}^3\,, \tag{J.1}$$

with

$$\begin{aligned}
\Delta_{\mathrm{ns}}^1 = {} & \mathrm{Li}_2(1-\eta_{14})\Bigg\{6\ln^2\left(\frac{E_1}{E_4}\right) + 8\ln^2\left(\frac{E_1}{E_{\max}}\right) + 3\ln\left(\frac{4E_1^2}{\mu^2}\right) \\
& + \ln\left(\frac{E_1}{E_4}\right)\left(-8\ln\left(\frac{E_1}{E_{\max}}\right) + 4\ln\eta_{14} + 6\right) + (-8\ln\eta_{14} - 6)\ln\left(\frac{E_1}{E_{\max}}\right) \\
& + 6\ln\eta_{14} - 2\pi^2 + 20\Bigg\} + \ln\left(\frac{E_1}{E_4}\right)\Bigg\{\ln\left(\frac{E_1}{E_{\max}}\right)\left(-12\ln^2\eta_{14} - \ln^2(2) + \frac{4\pi^2}{3}\right) \\
& + 9\ln^2\eta_{14} + \frac{3}{4}(\ln^2(2)+1) + \ln\eta_{14}\left(3\ln\left(\frac{4E_1^2}{\mu^2}\right) - 2\pi^2 + 13\right) + 12\zeta_3 - 2\pi^2\Bigg\} \\
& + \ln^2\left(\frac{E_1}{E_4}\right)\Bigg\{4\ln^2\eta_{14} + \frac{1}{2}(\ln^2(2)+13) + \frac{3}{2}\ln\left(\frac{4E_1^2}{\mu^2}\right) - 4\ln\eta_{14}\ln\left(\frac{E_1}{E_{\max}}\right) \\
& + 3\ln\eta_{14} - \frac{5\pi^2}{3}\Bigg\} + \ln\left(\frac{E_1}{E_{\max}}\right)\Bigg\{ - 15\ln^2\eta_{14} - \frac{3}{4}\ln^2(2) \\
& + \ln\eta_{14}\left(-6\ln\left(\frac{4E_1^2}{\mu^2}\right) + 4\pi^2 - 26\right) + \pi^2\Bigg\} + \left(\frac{9}{8} - \frac{\pi^2}{3}\right)\ln^2\left(\frac{4E_1^2}{\mu^2}\right) \\
& + 2\ln\eta_{14}\ln^3\left(\frac{E_1}{E_4}\right) + \frac{1}{2}\ln^4\left(\frac{E_1}{E_4}\right) + \left(12\ln^2\eta_{14} + \ln^2(2) - \frac{4\pi^2}{3}\right)\ln^2\left(\frac{E_1}{E_{\max}}\right) \\
& + 8\ln^2\eta_{14} + \frac{7}{8}\ln^2(2) + \ln\eta_{14}\left(\frac{9}{2}\ln\left(\frac{4E_1^2}{\mu^2}\right) - 3\pi^2 + \frac{39}{2}\right) \\
& + \left(2\zeta_3 + \frac{75}{8} - \pi^2\right)\ln\left(\frac{4E_1^2}{\mu^2}\right) + 2\mathrm{Li}_2(1-\eta_{14})^2 + \frac{9\zeta_3}{2} + \frac{7\pi^4}{10} - \frac{107\pi^2}{12} + \frac{405}{32}\,,
\end{aligned}$$

$$\begin{aligned}
\Delta_{\mathrm{ns}}^2 = {} & -4\,\mathrm{HPL}(\{-1,0,0,1\},\eta_{14}) - 7\,\mathrm{HPL}(\{0,1,0,1\},\eta_{14}) + \frac{22}{3}\mathrm{Ci}_3\left(2\arcsin\left(\sqrt{\eta_{14}}\right)\right) \\
& + \ln\left(\frac{E_1}{E_4}\right)\Bigg\{ - \frac{11}{6}\ln^2\eta_{14} + \ln\eta_{14}\left(-\frac{11}{3}\ln\left(\frac{4E_1^2}{\mu^2}\right) - \frac{\pi^2}{3} + \frac{233}{18}\right) - 15\zeta_3 - \frac{65\pi^2}{24}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{2353}{108} + \frac{2\ln 2}{3}\bigg\} + \ln^2\left(\frac{E_1}{E_4}\right)\left(-\frac{11}{6}\ln\left(\frac{4E_1^2}{\mu^2}\right) + \frac{11\ln\eta_{14}}{6} - \frac{\pi^2}{6} + \frac{233}{36}\right) \\
&+ \mathrm{Li}_2(1-\eta_{14})\bigg\{ -2\ln^2\eta_{14} - \frac{11}{3}\ln\left(\frac{4E_1^2}{\mu^2}\right) + \frac{22}{3}\ln\left(\frac{E_1}{E_{\max}}\right) - 4\mathrm{Li}_2(-\eta_{14}) \\
&\quad + 7\ln(1-\eta_{14})\ln\eta_{14} - \frac{\pi^2}{2} - \frac{131}{18} + \frac{22\ln 2}{3}\bigg\} + \ln\left(\frac{E_1}{E_{\max}}\right)\bigg\{\frac{11}{3}\ln^2\eta_{14} \\
&\quad + \ln\eta_{14}\left(\frac{22}{3}\ln\left(\frac{4E_1^2}{\mu^2}\right) + \frac{2\pi^2}{3} - \frac{134}{9}\right) + 4\zeta_3 + \frac{11\pi^2}{9} - \frac{4}{9}(1+3\ln 2)\bigg\} \\
&+ \ln\eta_{14}\bigg\{22\ln^2(2) - \frac{11}{2}\ln\left(\frac{4E_1^2}{\mu^2}\right) - \frac{7}{2}\pi^2\ln(1-\eta_{14}) + \frac{9\zeta_3}{2} - \frac{677}{27} \\
&\quad + \pi^2\left(\frac{107}{72} + \frac{4\ln 2}{3}\right)\bigg\} - \frac{11}{8}\ln^2\left(\frac{4E_1^2}{\mu^2}\right) + \frac{11}{6}\ln^3\left(\frac{E_1}{E_4}\right) \\
&- \frac{22}{3}\ln\eta_{14}\ln^2\left(\frac{E_1}{E_{\max}}\right) - \frac{1}{6}\pi^2\ln^2(\eta_{14}+1) + \frac{5}{12}\ln^4(\eta_{14}) + \frac{1}{6}\ln^4(\eta_{14}+1) \\
&+ \ln^2\eta_{14}\left(7\ln^2(1-\eta_{14}) + \frac{11}{6}\ln(1-\eta_{14}) + \frac{11\pi^2}{6} - \frac{361}{36} + \frac{22\ln 2}{3}\right) \\
&- 3\ln(1-\eta_{14})\ln^3\eta_{14} + \frac{\sqrt{1-\eta_{14}}\mathrm{Si}_2\left(2\arcsin\left(\sqrt{\eta_{14}}\right)\right)}{3\sqrt{\eta_{14}}} + \pi^2\left(\frac{2}{3}\ln^2(2) - \frac{607}{108}\right) \\
&- 2\ln^2(2) - \frac{2}{3}\ln^4(2) + \left(10\zeta_3 - \frac{1931}{216} + \frac{107\pi^2}{72}\right)\ln\left(\frac{4E_1^2}{\mu^2}\right) + \mathrm{Li}_4\left(1-\eta_{14}^2\right) \\
&+ \frac{7\mathrm{Li}_2(1-\eta_{14})^2}{2} + 10\mathrm{Li}_4(1-\eta_{14}) + 10\mathrm{Li}_4\left(-\frac{1-\eta_{14}}{\eta_{14}}\right) - 6\mathrm{Li}_4(\eta_{14}) \\
&+ 4\mathrm{Li}_4\left(\frac{1}{\eta_{14}+1}\right) - 2\mathrm{Li}_4\left(\frac{1-\eta_{14}}{\eta_{14}+1}\right) + 2\mathrm{Li}_4\left(\frac{\eta_{14}-1}{\eta_{14}+1}\right) \\
&+ \mathrm{Li}_2(-\eta_{14})\left(\frac{2\pi^2}{3} - 4\ln(1-\eta_{14})\ln\eta_{14}\right) + \mathrm{Li}_3(1-\eta_{14})(4\ln\eta_{14} - 4\ln(\eta_{14}+1)) \\
&+ 4\mathrm{Li}_3(-\eta_{14})\ln(1-\eta_{14}) + \mathrm{Li}_3(\eta_{14})\left(14\ln(1-\eta_{14}) - 4\ln\eta_{14} + 4\ln(\eta_{14}+1) + \frac{11}{3}\right) \\
&- 11\zeta_3\ln(1-\eta_{14}) + \frac{7}{2}\zeta_3\ln(\eta_{14}+1) - 16\mathrm{Li}_4\left(\frac{1}{2}\right) + \zeta_3\left(-\frac{1205}{36} - 14\ln 2\right) \\
&+ \frac{181\pi^4}{720} + \frac{127265}{2592} + 2\ln 2,
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{\mathrm{ns}}^3 = &-\frac{4}{3}\mathrm{Ci}_3\left(2\arcsin\left(\sqrt{\eta_{14}}\right)\right) + \ln^2\left(\frac{E_1}{E_4}\right)\left(\frac{1}{3}\ln\left(\frac{4E_1^2}{\mu^2}\right) - \frac{\ln\eta_{14}}{3} - \frac{19}{18}\right) \\
&+ \ln\left(\frac{E_1}{E_4}\right)\left(\frac{1}{3}\ln^2\eta_{14} + \ln\eta_{14}\left(\frac{2}{3}\ln\left(\frac{4E_1^2}{\mu^2}\right) - \frac{19}{9}\right) + \frac{7\pi^2}{12} - \frac{221}{54} - \frac{2\ln 2}{3}\right) \\
&+ \ln\left(\frac{E_1}{E_{\max}}\right)\left(-\frac{2}{3}\ln^2\eta_{14} + \ln\eta_{14}\left(\frac{20}{9} - \frac{4}{3}\ln\left(\frac{4E_1^2}{\mu^2}\right)\right) - \frac{2\pi^2}{9} + \frac{4}{9}(1+3\ln 2)\right) \\
&+ \ln\eta_{14}\left(-4\ln^2(2) + \ln\left(\frac{4E_1^2}{\mu^2}\right) - \frac{13\pi^2}{36} + \frac{223}{54}\right) + \frac{1}{4}\ln^2\left(\frac{4E_1^2}{\mu^2}\right) - \frac{1}{3}\ln^3\left(\frac{E_1}{E_4}\right) \\
&+ \frac{4}{3}\ln\eta_{14}\ln^2\left(\frac{E_1}{E_{\max}}\right) + \left(\frac{4}{9}(4-3\ln 2) - \frac{1}{3}\ln(1-\eta_{14})\right)\ln^2\eta_{14}
\end{aligned}
$$

$$
-\frac{\sqrt{1-\eta_{14}}}{3\sqrt{\eta_{14}}}\,\mathrm{Si}_2(2\arcsin(\sqrt{\eta_{14}}))+2\ln^2(2)+\mathrm{Li}_2(1-\eta_{14})\bigg\{\frac{2}{3}\ln\left(\frac{4E_1^2}{\mu^2}\right)
-\frac{4}{3}\ln\left(\frac{E_1}{E_{\max}}\right)+\frac{29}{18}-\frac{4\ln 2}{3}\bigg\}+\ln\left(\frac{4E_1^2}{\mu^2}\right)\left(\frac{175}{108}-\frac{13\pi^2}{36}\right)-\frac{2\mathrm{Li}_3(\eta_{14})}{3}
+\frac{85\zeta_3}{18}+\frac{25\pi^2}{27}-\frac{11641}{1296}-2\ln 2\,.
$$

Here, $\mathrm{HPL}(\{a_1,\ldots,a_n\},z)$ are harmonic polylogarithms [84] and the Clausen functions $\mathrm{Ci}_n(z)$ and $\mathrm{Si}_n(z)$ are given in Eq. (I.4).

Boosted matrix element: $F_{\mathrm{LM}}(z\cdot 1,4)$

For $\mathcal{T}_{\mathrm{ns}}$ we define

$$
\mathcal{T}_{\mathrm{ns}}(E_1,E_4,E_{\max},\eta_{14},z)=C_F^2\,\mathcal{T}_{\mathrm{ns}}^1+C_AC_F\,\mathcal{T}_{\mathrm{ns}}^2+C_FN_f\,\mathcal{T}_{\mathrm{ns}}^3+(C_A-2C_F)C_F\,\mathcal{T}_{\mathrm{ns}}^4\,, \tag{J.2}
$$

with

$$
\begin{aligned}
\mathcal{T}_{\mathrm{ns}}^1 = {}& \frac{\pi^2(5-2z)}{3(z-1)}+5z+8\mathcal{D}_3(z)+12\mathcal{D}_2(z)\ln\left(\frac{4E_1^2}{\mu^2}\right) \\
&+\ln\left(\frac{E_1}{E_4}\right)\Bigg(\ln\eta_{14}\left(-2z-2(z+1)\ln\left(\frac{4E_1^2}{\mu^2}\right)-4(z+1)\ln(1-z)+2\right) \\
&+(2-2z)\ln z-\frac{2(z^2+1)\ln(4E_1^2/\mu^2)\ln z}{z-1}-\frac{4(z^2+1)\ln(1-z)\ln z}{z-1}\Bigg) \\
&+\ln\eta_{14}\Bigg(-3z+2(z-1)\ln z+\ln\left(\frac{4E_1^2}{\mu^2}\right)\left(\frac{2(z^2+1)\ln z}{z-1}-3(z+1)\right) \\
&\quad+\ln(1-z)\left(\frac{4(z^2+1)\ln z}{z-1}-6(z+1)\right)+3\Bigg)+\ln z\Bigg(\frac{-22z^2-5z+17}{2(z-1)} \\
&\quad+\frac{\pi^2\left(2z^2+2\right)}{3(1-z)}+4(z+1)\ln(1+z)\Bigg)+\ln^2\left(\frac{E_1}{E_4}\right)\Bigg(-z+(-z-1)\ln\left(\frac{4E_1^2}{\mu^2}\right) \\
&\quad-2(z+1)\ln(1-z)+1\Bigg)-2(z+1)\ln(1-z)\ln^2\eta_{14} \\
&+\ln\left(\frac{E_1}{E_{\max}}\right)\Bigg(-\frac{1}{2}\ln^2(2)(z+1)-2\ln^2\eta_{14}(z+1)+\frac{2}{3}\pi^2(z+1) \\
&\quad+\ln\eta_{14}\left(4(z-1)+4(z+1)\ln\left(\frac{4E_1^2}{\mu^2}\right)+8(z+1)\ln(1-z)\right)\Bigg) \\
&+\ln^2\left(\frac{4E_1^2}{\mu^2}\right)\left(\frac{1}{2}(-z-5)-2(z+1)\ln(1-z)+\frac{(3z^2+1)\ln z}{2(z-1)}\right) \\
&+\left(2(z-1)+\frac{(1-7z^2)\ln z}{2(z-1)}\right)\ln^2(1-z)-\frac{3\left(2z^2+2z-7\right)\ln^2 z}{4(z-1)} \\
&+\ln(1-z)\Bigg(\frac{\pi^2\left(27-29z^2\right)}{6(1-z)}+\frac{\left(-7z^2+2z-7\right)\ln z}{z-1}+\frac{-7z-(z+1)\ln^2(2)-46}{2} \\
&\quad+\frac{5(z^2+1)\ln^2 z}{2(z-1)}\Bigg)+\ln\left(\frac{4E_1^2}{\mu^2}\right)\Bigg(-3z+2\pi^2(z+1)+\frac{(-4z^2-2z+3)\ln z}{z-1}
\end{aligned}
$$

$$
\begin{aligned}
&+\ln(1-z)\left(-z+\frac{2(z^2+1)\ln z}{z-1}-5\right)-6(z+1)\ln^2(1-z)\\
&+\frac{(3z^2+1)\ln^2 z}{2(z-1)}-10\Bigg)-4(z+1)\ln^3(1-z)+\frac{1}{12}(z+1)\ln^3(z)\\
&+\mathrm{Li}_2(1-\eta_{14})\Bigg(-2z-4(z+1)\ln\left(\frac{E_1}{E_{\max}}\right)-2(z+1)\ln\left(\frac{4E_1^2}{\mu^2}\right)\\
&-8(z+1)\ln(1-z)+2\Bigg)+\mathcal{D}_1(z)\Bigg(8\ln\left(\frac{E_1}{E_4}\right)\ln\eta_{14}-16\ln\left(\frac{E_1}{E_{\max}}\right)\ln\eta_{14}\\
&+12\ln\eta_{14}+6\ln\left(\frac{4E_1^2}{\mu^2}\right)+\ln^2(2)+4\ln^2\left(\frac{E_1}{E_4}\right)+4\ln^2\eta_{14}+4\ln^2\left(\frac{4E_1^2}{\mu^2}\right)\\
&+16\mathrm{Li}_2(1-\eta_{14})-8\pi^2+26\Bigg)+\left(4(z+1)+\frac{4(z^2+1)\ln z}{z-1}\right)\mathrm{Li}_2(-z)\\
&+\mathrm{Li}_2(z)\left(\frac{2\left(2z^2-5\right)}{z-1}-2(z+1)\ln\left(\frac{4E_1^2}{\mu^2}\right)+\frac{\left(3-5z^2\right)\ln(1-z)}{z-1}+\frac{\left(z^2+1\right)\ln z}{z-1}\right)\\
&+\frac{\left(9z^2+1\right)\mathrm{Li}_3(1-z)}{1-z}-\frac{8(z^2+1)\mathrm{Li}_3(-z)}{z-1}+\frac{\left(1-3z^2\right)\mathrm{Li}_3(z)}{z-1}\\
&+\mathcal{D}_0(z)\Bigg\{4\ln\left(\frac{E_1}{E_4}\right)\ln\eta_{14}\ln\left(\frac{4E_1^2}{\mu^2}\right)+6\ln\eta_{14}\ln\left(\frac{4E_1^2}{\mu^2}\right)+2\ln^2\left(\frac{E_1}{E_4}\right)\ln\left(\frac{4E_1^2}{\mu^2}\right)\\
&+\left(13-\frac{10\pi^2}{3}\right)\ln\left(\frac{4E_1^2}{\mu^2}\right)+\ln\left(\frac{E_1}{E_{\max}}\right)\Bigg(-8\ln\eta_{14}\ln\left(\frac{4E_1^2}{\mu^2}\right)+\ln^2(2)\\
&\quad+4\ln^2\eta_{14}-\frac{4\pi^2}{3}\Bigg)+3\ln^2\left(\frac{4E_1^2}{\mu^2}\right)+\left(8\ln\left(\frac{E_1}{E_{\max}}\right)+4\ln\left(\frac{4E_1^2}{\mu^2}\right)\right)\mathrm{Li}_2(1-\eta_{14})\\
&+16\zeta_3\Bigg\}+\frac{\left(1-11z^2\right)\zeta_3}{z-1}-2\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_{\mathrm{ns}}^2=\;&\mathcal{D}_0(z)\Bigg(-\frac{11}{6}\ln^2\left(\frac{4E_1^2}{\mu^2}\right)+\left(\frac{67}{9}-\frac{\pi^2}{3}\right)\ln\left(\frac{4E_1^2}{\mu^2}\right)+9\zeta_3+\frac{11\pi^2}{6}\\
&-\frac{2}{27}(104+9\ln 2)\Bigg)+\mathcal{D}_1(z)\left(-\frac{22}{3}\ln\left(\frac{4E_1^2}{\mu^2}\right)-\frac{2\pi^2}{3}+\frac{134}{9}\right)-\frac{22}{3}\mathcal{D}_2(z)\\
&+\ln\left(\frac{4E_1^2}{\mu^2}\right)\Bigg(\frac{\left(z^2+1\right)\ln^2(z)}{2-2z}+\frac{\left(5z^2+17\right)\ln(z)}{6-6z}+\frac{1}{9}(10-77z)+\frac{1}{6}\pi^2(z+1)\\
&+\frac{11}{3}(z+1)\ln(1-z)\Bigg)+\frac{11}{12}(z+1)\ln^2\left(\frac{4E_1^2}{\mu^2}\right)+\frac{\left(7z^2-12z+27\right)\ln^2(z)}{8-8z}\\
&-\frac{7\left(z^2+1\right)\ln^3(z)}{12(z-1)}+\left(\frac{\left(z^2+1\right)\ln(z)}{z-1}+\frac{11(z+1)}{3}\right)\ln^2(1-z)\\
&+\frac{5\left(z^2+1\right)\mathrm{Li}_3(1-z)}{z-1}+\frac{4\left(z^2+1\right)\mathrm{Li}_3(-z)}{z-1}+\frac{4\left(z^2+1\right)\mathrm{Li}_3(z)}{z-1}\\
&+\mathrm{Li}_2(-z)\left(-\frac{2\left(z^2+1\right)\ln(z)}{z-1}-2(z+1)\right)+\mathrm{Li}_2(z)\Bigg(\frac{2\left(4z^2-3z+10\right)}{3(z-1)}\\
&+\frac{\left(z^2+1\right)\ln(1-z)}{z-1}+\frac{\left(z^2+1\right)\ln(z)}{1-z}\Bigg)+\frac{\left(7-11z^2\right)\zeta_3}{2(z-1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\pi^2\left(-26z^2+3z+1\right)}{18(z-1)}+\ln(1-z)\bigg(\frac{\pi^2\left(z^2-3\right)}{6(z-1)}+\frac{1}{18}(40-299z)\\
&\quad+(z-1)\ln(z)\bigg)+\ln(z)\bigg(\frac{\pi^2\left(z^2+1\right)}{3-3z}+\frac{83z^2+114z-109}{18(z-1)}\\
&\quad-2(z+1)\ln(z+1)\bigg)+\frac{2}{27}(z+67+9\ln 2)\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_{\rm ns}^3=&\,\mathcal{D}_0(z)\left(\frac{1}{3}\ln^2\left(\frac{4E_1^2}{\mu^2}\right)-\frac{10}{9}\ln\left(\frac{4E_1^2}{\mu^2}\right)-\frac{\pi^2}{3}+\frac{2}{27}(17+9\ln 2)\right)\\
&+\mathcal{D}_1(z)\left(\frac{4}{3}\ln\left(\frac{4E_1^2}{\mu^2}\right)-\frac{20}{9}\right)+\frac{4}{3}\mathcal{D}_2(z)+\frac{1}{6}(-z-1)\ln^2\left(\frac{4E_1^2}{\mu^2}\right)\\
&+\frac{\left(z^2+1\right)\ln^2(z)}{4(z-1)}-\frac{2}{3}(z+1)\ln^2(1-z)+\left(\frac{\left(z^2+1\right)\ln(z)}{3(z-1)}+\frac{2}{9}(4z+1)\right.\\
&\quad\left.-\frac{2}{3}(z+1)\ln(1-z)\right)\ln\left(\frac{4E_1^2}{\mu^2}\right)+\frac{2\left(z^2+1\right)\mathrm{Li}_2(z)}{3(1-z)}+\frac{\pi^2\left(1-5z^2\right)}{18(1-z)}\\
&+\frac{\left(-5z^2-6z+7\right)\ln(z)}{18(z-1)}+\frac{1}{54}(-19z-67-36\ln 2)+\frac{4}{9}(4z+1)\ln(1-z)\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_{\rm ns}^4=&\ln\left(\frac{4E_1^2}{\mu^2}\right)\bigg(-\frac{(z^2+1)\ln^2 z}{2(1+z)}+\frac{\pi^2(z^2+1)}{6(1+z)}+\ln z\left(\frac{2(z^2+1)\ln(1+z)}{z+1}-z-1\right)\\
&\quad+2(z-1)\bigg)-\frac{7(z^2+1)\ln^3(z)}{12(z+1)}-\frac{(z^2+1)\ln^3(1+z)}{z+1}\\
&+\left(\frac{(z^2+1)\ln(1+z)}{2(1+z)}-z-2\right)\ln^2 z+\ln z\bigg(\frac{3(z^2+1)\ln^2(1+z)}{z+1}-\frac{\pi^2(z^2+1)}{3(1+z)}\\
&\quad+\frac{1}{4}(-11z-19)+3(z+1)\ln(1+z)\bigg)+\mathrm{Li}_2(-z)\bigg(\frac{2(z^2+1)}{z+1}\\
&\quad+\frac{4(z^2+1)\ln(1-z)}{z+1}\ln\left(\frac{4E_1^2}{\mu^2}\right)-\frac{(z^2+1)\ln z}{z+1}+3(z+1)\bigg)\\
&-\frac{4(z^2+1)\mathrm{Li}_3(1-z)}{z+1}+\frac{9(z^2+1)\mathrm{Li}_3(-z)}{z+1}+\frac{4(z^2+1)\mathrm{Li}_3(z)}{z+1}\\
&+\frac{6(z^2+1)}{z+1}\mathrm{Li}_3\left(\frac{z}{z+1}\right)+\frac{2(z^2+1)\mathrm{Li}_3\left(1-z^2\right)}{z+1}\\
&+\mathrm{Li}_2(z)\left(-\frac{(z^2+1)\ln z}{z+1}+z+3\right)-\frac{5\left(z^2(2\zeta_3-3)+2\zeta_3+3\right)}{4(z+1)}\\
&+\frac{\pi^2(z^2+1)\ln(1+z)}{2(1+z)}+\ln(1-z)\left(\frac{\pi^2(z^2+1)}{3(1+z)}+4(z-1)\right)\\
&+\ln(1-z)\ln z\left(\frac{4(z^2+1)\ln(1+z)}{z+1}-z+1\right)+\frac{1}{12}\pi^2(z-3)\,.
\end{aligned}
$$

J.2. Quark singlet contributions

For quark singlet contributions we defined

$$
\begin{aligned}
\mathcal{T}_{\mathrm{s}}(E_1, z) = C_F T_R \Big\{ & \frac{340 + 177z - 357z^2 - 160z^3}{54z} + \frac{2\pi^2 \left(2 + 3z + 12z^2 + 4z^3\right)}{9z} \\
& - \frac{2(-1+z)^3 \ln 2}{3z} + \left(\frac{11}{4} + \frac{8}{3z} + \frac{27z}{4} + 2z^2\right) \ln^2 z + \frac{1}{6}(1+z) \ln^3 z \\
& + \ln^2(1-z) \left(2 + \frac{8}{3z} - 2z - \frac{8z^2}{3} - 4(1+z)\ln z\right) + \ln^2\left(\frac{4E_1^2}{\mu^2}\right) \\
& \quad \times \left(\frac{4 + 3z - 3z^2 - 4z^3}{6z} + (1+z)\ln z\right) + \frac{2\ln(1-z)}{9z}\Big(26 + 6\left(-5+\pi^2\right) z \\
& \quad + 6\left(8+\pi^2\right) z^2 - 44z^3 + \left(-6 + 9z - 9z^2 + 6z^3\right)\ln z\Big) \\
& + \frac{1}{9z} \ln\left(\frac{4E_1^2}{\mu^2}\right)\Big(26 - 30z + 6\pi^2 z + 48z^2 + 6\pi^2 z^2 - 44z^3 \\
& \quad - 6\left(-4 - 3z + 3z^2 + 4z^3\right)\ln(1-z) + 3\left(8 + 15z + 21z^2 + 8z^3\right)\ln z \\
& \quad + 9z(1+z)\ln^2 z\Big) + \frac{1}{9}\ln z\Big(75 + \frac{64}{z} + 126z + 32z^2 \\
& \qquad - \frac{12(1+z)\left(2 + z + 2z^2\right)\ln(1+z)}{z}\Big) - \frac{4\left(2 + 3z + 3z^2 + 2z^3\right)\mathrm{Li}_2(-z)}{3z} \\
& + \left(-8 - \frac{20}{3z} - 16z - 4z^2 - 4(1+z)\ln\left(\frac{4E_1^2}{\mu^2}\right) - 8(1+z)\ln(1-z)\right)\mathrm{Li}_2(z) \\
& - 8(1+z)\mathrm{Li}_3(1-z) - 4(1+z)\mathrm{Li}_3(z) + 4(1+z)\zeta_3 \Big\}.
\end{aligned} \tag{J.3}
$$

J.3. Gluon contributions

For $\mathcal{T}_g$ in gluon-initiated contributions we define

$$
\mathcal{T}_g(E_1, E_4, E_{\max}, \eta_{14}, z) = C_F T_R\, \mathcal{T}_g^1 + C_A T_R\, \mathcal{T}_g^2, \tag{J.4}
$$

with

$$
\begin{aligned}
\mathcal{T}_g^1 = & -62z^2 + \frac{307z}{4} + \frac{1}{12}\pi^2\left(58z^2 - 66z + 11\right) + \ln\left(\frac{E_1}{E_{\max}}\right)\Big(\frac{1}{3}\pi^2\left(-2z^2 + 2z - 1\right) \\
& + \ln\eta_{14}\Big(8(z-1)z + \left(-8z^2 + 8z - 4\right)\ln\left(\frac{4E_1^2}{\mu^2}\right) - 8\left(2z^2 - 2z + 1\right)\ln(1-z)\Big)\Big) \\
& + \left(-8z^2 + \frac{35z}{4} + \frac{4}{3}\pi^2\left(2z^2 - 2z + 1\right) + 1\right)\ln z + \ln\left(\frac{E_1}{E_4}\right)\Big(\ln\eta_{14}\Big(-4(z-1)z \\
& \quad + \left(4z^2 - 4z + 2\right)\ln\left(\frac{4E_1^2}{\mu^2}\right) + \left(8z^2 - 8z + 4\right)\ln(1-z)\Big) - 4(z-1)z\ln z \\
& + \left(4z^2 - 4z + 2\right)\ln\left(\frac{4E_1^2}{\mu^2}\right)\ln z + \left(8z^2 - 8z + 4\right)\ln(1-z)\ln z\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\ln\eta_{14}\bigg(-6(z-1)z+4(z-1)\ln zz+\ln(1-z)\Big[6\left(2z^2-2z+1\right)\\
&\qquad+\left(-8z^2+8z-4\right)\ln z\Big]+\ln\left(\frac{4E_1^2}{\mu^2}\right)\left(6z^2-6z+\left(-4z^2+4z-2\right)\ln z+3\right)\bigg)\\
&+\left(-2(z-1)z+\left(2z^2-2z+1\right)\ln\left(\frac{4E_1^2}{\mu^2}\right)+\left(4z^2-4z+2\right)\ln(1-z)\right)\ln^2\left(\frac{E_1}{E_4}\right)\\
&+\left(z+\left(2z^2-2z+1\right)\ln(1-z)+\left(-2z^2+z-\frac{1}{2}\right)\ln z-\frac{1}{4}\right)\ln^2\left(\frac{4E_1^2}{\mu^2}\right)\\
&+\left(-17z^2+21z+\left(-z^2+5z-\frac{5}{2}\right)\ln z-7\right)\ln^2(1-z)+\left(z^2-\frac{z}{2}-\frac{3}{8}\right)\ln^2 z\\
&+\ln\left(\frac{4E_1^2}{\mu^2}\right)\bigg\{\pi^2\left(-\frac{8z^2}{3}+2z-1\right)+\frac{1}{2}\left(62z^2-73z+29\right)+\left(10z^2-10z+\frac{3}{2}\right)\ln z\\
&\quad+\ln(1-z)\left(\left(-4z^2+4z-2\right)\ln z-2\left(5z^2-7z+2\right)\right)+\left(6z^2-6z+3\right)\ln^2(1-z)\\
&\quad+\left(\frac{1}{2}-z\right)\ln^2 z\bigg\}+\ln(1-z)\bigg\{70z^2-\frac{161z}{2}+\frac{1}{6}\pi^2\left(-34z^2+26z-13\right)\\
&\quad+2\left(7z^2-7z+3\right)\ln z+\left(-5z^2+5z-\frac{5}{2}\right)\ln^2 z+29\bigg\}\\
&+\frac{11}{6}\left(2z^2-2z+1\right)\ln^3(1-z)+\left(\frac{7z^2}{3}-\frac{5z}{2}+\frac{5}{4}\right)\ln^3(z)\\
&+\mathrm{Li}_2(1-\eta_{14})\bigg\{-4(z-1)z+\left(4z^2-4z+2\right)\ln\left(\frac{4E_1^2}{\mu^2}\right)+\left(8z^2-8z+4\right)\ln(1-z)\bigg\}\\
&+\mathrm{Li}_2(z)\bigg\{(4z-2)\ln\left(\frac{4E_1^2}{\mu^2}\right)+\left(-2z^2+10z-5\right)\ln(1-z)+\left(2z^2-2z+1\right)\ln z+3\bigg\}\\
&+\left(-10z^2+18z-9\right)\mathrm{Li}_3(1-z)+\left(-14z^2+18z-9\right)\mathrm{Li}_3(z)\\
&+\left(32z^2-36z+18\right)\zeta_3-\frac{69}{4},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_g^2=&\left(\frac{1}{6}\left(-31z^2+24z+\frac{4}{z}+3\right)+\left(2z^2-2z+1\right)\ln(1-z)+(4z+1)\ln z\right)\ln^2\left(\frac{4E_1^2}{\mu^2}\right)\\
&+\ln\left(\frac{4E_1^2}{\mu^2}\right)\bigg\{\left(6z^2-6z+3\right)\ln^2(1-z)+(6z+1)\ln^2 z\\
&\quad+\left(-\frac{62z^2}{3}+16z+\frac{8}{3z}+2\right)\ln(1-z)+\ln z\bigg(\frac{44z^2}{3}-2\left(2z^2+2z+1\right)\ln(1+z)\\
&\qquad+16z+\frac{8}{3z}+5\bigg)+\frac{-71z^3+93z^2-39z+26}{9z}-\frac{2}{3}\pi^2 z(2z-5)\bigg\}\\
&+\frac{13}{6}\left(2z^2-2z+1\right)\ln^3(1-z)+\left(2z^2+2z+1\right)\ln^3(1+z)+\ln^2(1-z)\bigg(-\frac{59z^2}{3}\\
&\quad+\left(2z^2-18z-3\right)\ln z+15z+\frac{8}{3z}+2\bigg)+\ln^2 z\bigg(11z^2+\left(-z^2-z-\frac{1}{2}\right)\ln(1+z)\\
&\quad+17z+\frac{8}{3z}+\frac{11}{4}\bigg)+\ln z\bigg(-3\left(2z^2+2z+1\right)\ln^2(1+z)\\
&\quad+\frac{1}{9}\left(68z^2+315z+\frac{64}{z}+75\right)-\frac{2\left(19z^3+21z^2+6z+4\right)\ln(1+z)}{3z}+\frac{4\pi^2 z}{3}\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(3z+\frac{1}{6}\right)\ln^3(z)+\mathrm{Li}_2(z)\Bigg(-4(4z+1)\ln\left(\frac{4E_1^2}{\mu^2}\right)-22z^2+\left(2z^2-34z-7\right)\ln(1-z)\\
&\quad-40z-\frac{20}{3z}+4z\ln z-8\Bigg)+\mathrm{Li}_2(-z)\Bigg(-2\left(2z^2+2z+1\right)\ln\left(\frac{4E_1^2}{\mu^2}\right)\\
&\quad-4\left(2z^2+2z+1\right)\ln(1-z)+\left(-2z^2+6z-1\right)\ln z-\frac{2\left(19z^3+21z^2+6z+4\right)}{3z}\Bigg)\\
&+\left(18z^2-34z+1\right)\mathrm{Li}_3(1-z)+\left(-10z^2-26z-5\right)\mathrm{Li}_3(-z)\\
&-6\left(2z^2+2z+1\right)\mathrm{Li}_3\left(\frac{z}{z+1}\right)-2\left(2z^2+2z+1\right)\mathrm{Li}_3\left(1-z^2\right)-4(8z+1)\mathrm{Li}_3(z)\\
&+\left(10z^2+16z+9\right)\zeta_3+\frac{1}{9}\pi^2\left(53z^2+27z+\frac{4}{z}+6\right)+\pi^2\left(-z^2-z-\frac{1}{2}\right)\ln(1+z)\\
&+\ln(1-z)\Bigg(\frac{1}{18}\left(-320z^2+399z+\frac{104}{z}-156\right)+\pi^2\left(-3z^2+7z-\frac{1}{6}\right)\\
&\quad+\ln z\left(\frac{22z^2}{3}-4\left(2z^2+2z+1\right)\ln(1+z)-8z-\frac{4}{3z}+2\right)\Bigg)\\
&+\frac{-970z^3+129z^2+447z-36(z-1)^3\ln 2+340}{54z}.
\end{aligned}
$$

Abbreviations

LO	Leading order
NLO	Next-to-leading order
NNLO	Next-to-next-to-leading order
QCD	(perturbative) Quantum Chromodynamics
$\overline{\text{MS}}$	Modified minimal subtraction scheme
SM	Standard Model of particle physics
IR	Infrared
UV	Ultraviolet
DIS	Deep-inelastic scattering
LHC	Large Hadron Collider
CERN	European Organization for Nuclear Research
FKS	Frixione-Kunszt-Signer
COM	Center-of-mass

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