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IIB matrix model: Emergent spacetime from the master field

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We argue that the large-N master field of the Lorentzian IIB matrix model can give the points and metric of a classical spacetime.

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1. Introduction

The IIB matrix model [1,2] has been suggested as a nonperturbative formulation of type-IIB superstring theory. First results on the partition function of the Euclidean IIB matrix model were reported in Refs. [3,4]. Later, numerical simulations [5–7] of the Lorentzian IIB matrix model suggested the appearance of a $3 + 6$ split of the nine spatial dimensions (matching Euclidean results were presented in Ref. [8]). Still, the physical interpretation of the emergence of a classical spacetime in Refs. [1,2,5–8] is not really satisfactory, because there is no manifest small dimensionless parameter to motivate a saddle-point approximation.

Recently, we have revived an old idea, the large-N master field of Witten [9], for a possible origin of classical spacetime in the context of the IIB matrix model; see Appendix B in the earlier preprint version [10] of Ref. [11]. But we did not give any details about where precisely in the master field the classical spacetime is encoded. In the present paper, we try to be more explicit.

Before we set out on our search for classical spacetime in the IIB matrix model, we have five preliminary remarks. First, we take the Lorentzian signature in the IIB matrix model, because it is not clear how to interpret an emerging Euclidean "spacetime" from the Euclidean IIB matrix model. Second, our discussion of the Lorentzian path integrals will be strictly formal, omitting all convergence issues. Third, we introduce a length scale "ℓ" into the IIB matrix model, in order to give the dimension of length to the bosonic matrix variable. Fourth, such a length scale "ℓ" may enter the effective metric of the regularized big bang singularity [12–15]. Fifth, the focus of the present paper is solely on the IIB matrix model, but it is possible that some of our results could carry over to other matrix models [16–18].

We will now start by recalling the IIB matrix model and the concept of the master field, and will then turn to the emergence of the spacetime points and the spacetime metric.

2. Model

The action of the Lorentzian IIB matrix model is given by [1,2]

$$S[A,\Psi] = S_b[A] + S_f[A,\Psi]$$

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$$= \mathrm{Tr}\left(\frac{1}{4}\left[A^{\mu},\, A^{\nu}\right]\left[A^{\kappa},\, A^{\lambda}\right]\widetilde{\eta}_{\mu\kappa}\,\widetilde{\eta}_{\nu\lambda} + \frac{1}{2}\,\overline{\Psi}_{\beta}\,\widetilde{\Gamma}^{\mu}_{\beta\alpha}\,\widetilde{\eta}_{\mu\nu}[A^{\nu},\, \Psi_{\alpha}]\right), \tag{1a}$$

$$\widetilde{\eta}_{\mu\nu} = \Big[\mathrm{diag}\,(-1,\, 1, \ldots, 1)\Big]_{\mu\nu}, \tag{1b}$$

with vector indices $\mu, \nu, \kappa, \lambda \in \{0,\, 1, \ldots, 9\}$ and spinor indices $\alpha, \beta \in \{1,\, 2, \ldots, 32\}$. The vector A^{μ} and the Majorana–Weyl spinor Ψ_{α} are both $N \times N$ traceless Hermitian matrices. They live in a 10D spacetime consisting of a single point, a special case of the Eguchi–Kawai reduction [19] operative in the large-N limit of certain field theories; see Ref. [20] for a review.

The action (1) is invariant under the following global gauge transformation:

$$A^{\mu} \to \Omega\, A^{\mu}\, \Omega^{\dagger}, \tag{2a}$$

$$\Psi_{\alpha} \to \Omega\, \Psi_{\alpha}\, \Omega^{\dagger}, \tag{2b}$$

$$\Omega \in SU(N). \tag{2c}$$

In addition, there is $SO(1,\, 9)$ Lorentz invariance and an $\mathcal{N} = 2$ supersymmetry [2].

The partition function Z is defined by the following Lorentzian "path" integral [5]:

$$Z = \int dA\, d\Psi\, \exp\left(i\, S[A, \Psi]/\ell^{4}\right). \tag{3}$$

Here, we have introduced a length scale "ℓ", so that A^{μ} from Eq. (1) must have the dimension of length and Ψ_{α} the dimension of (length)$^{3/2}$.

The length scale "ℓ" is solely introduced to simplify the physics discussion later on and can be removed by considering dimensionless variables A' and Ψ'. The IIB-matrix-model path integral (3) in terms of dimensionless variables A' and Ψ' has, as emphasized in Appendix B of Ref. [10], no obvious small dimensionless parameter and, therefore, no obvious saddle-point approximation.

As the fermions appear quadratically in the action, they can be integrated out [3,4] and the partition function becomes

$$Z = \int dA\, \exp\left(i\, S_{\mathrm{eff}}[A]/\ell^{4}\right), \tag{4a}$$

with an effective action

$$S_{\mathrm{eff}}[A] = S_{b}[A] + S_{\mathrm{induced}}[A]. \tag{4b}$$

For completeness, we mention that the integration measure dA in Eqs. (3) and (4a) is standard [21], except for the restriction to tracelessness.

3. Master field

A particular gauge-invariant bosonic observable is given by

$$w^{\mu_1 \cdots \mu_m} = \mathrm{Tr}\left(A^{\mu_1} \cdots A^{\mu_m}\right). \tag{5}$$

Its expectation values are given by the following Lorentzian path integrals:

$$\langle w^{\mu_1\cdots\mu_m}\; w^{\nu_1\cdots\nu_n}\cdots\rangle = Z^{-1}\int dA\,\left(w^{\mu_1\cdots\mu_m}\; w^{\nu_1\cdots\nu_n}\cdots\right)\exp\left[i\,S_{\text{eff}}/\ell^4\right], \tag{6}$$

with normalization factor Z from Eq. (4).

The expectation values (6) have the following factorization property:

$$\langle w^{\mu_1\cdots\mu_m}\; w^{\nu_1\cdots\nu_n}\cdots w^{\omega_1\cdots\omega_z}\rangle \stackrel{N}{=} \langle w^{\mu_1\cdots\mu_m}\rangle\;\langle w^{\nu_1\cdots\nu_n}\rangle\cdots\langle w^{\omega_1\cdots\omega_z}\rangle, \tag{7}$$

which holds to leading order in N (see Sect. III A of Ref. [20] for further discussion). From Eq. (7) follows the result that, to leading order in N, the expectation value of the square of w equals the square of the expectation value of w,

$$\langle\,\left(w^{\mu_1\cdots\mu_m}\right)^2\rangle \stackrel{N}{=} \left(\,\langle w^{\mu_1\cdots\mu_m}\rangle\,\right)^2, \tag{8}$$

which is a truly remarkable result for a statistical (quantum) theory.

According to Witten [9], the factorization results (7) and (8) imply that the path integrals (6) are saturated by a *single* configuration, the master field $\widehat{A}^{\mu}$. For just one observable w from Eq. (5) and its expectation value ("Wilson loop"), we then have

$$\langle w^{\mu_1\cdots\mu_m}\rangle \stackrel{N}{=} \text{Tr}\left(\widehat{A}^{\mu_1}\cdots\widehat{A}^{\mu_m}\right). \tag{9}$$

In principle, it is possible that there is more than one master field, as long as these master fields give, in the large-N limit, exactly the same results for all possible observables of the type (5). For simplicity, we will talk, in the following, about a single master field.

The explicit expression for the IIB-matrix-model master field $\widehat{A}^{\mu}$ is not known, but it is possible to give an *algebraic equation* for it. Based on previous work by Greensite and Halpern [22], the IIB-matrix-model master field takes the following form [10]:

$$\widehat{A}^{\mu}_{ab}(\tau_{\text{eq}}) = \exp\left[i\,(\widehat{p}_a-\widehat{p}_b)\,\tau_{\text{eq}}\right]\,\widehat{a}^{\mu}_{ab}, \tag{10a}$$

where τ_{eq} must have a sufficiently large value (it traces back to the fictitious Langevin time τ of stochastic quantization) and where the τ-independent matrix $\widehat{a}^{\mu}$ on the right-hand side solves the following algebraic equation:

$$i\left(\widehat{p}_a-\widehat{p}_b\right)\widehat{a}^{\mu}_{ab} = -\left.\frac{\delta S_{\text{eff}}}{\delta A_{\mu\;ba}}\right|_{A=\widehat{a}} + \widehat{\eta}^{\mu}_{ab}, \tag{10b}$$

in terms of the master momenta $\widehat{p}_a$ (uniform random numbers) and the master noise matrices $\widehat{\eta}^{\mu}_{ab}$ (Gaussian random numbers); see Ref. [22] for further details and Refs. [23,24] for some interesting results.

Further remarks on the IIB-matrix-model master field also appear in Appendix B of Ref. [10], but, here, we just assume that the master field has been obtained, in the form as given by Eq. (10) or otherwise.

4. Emergent spacetime points

As argued in Appendix B of Ref. [10], the only place where "classical spacetime" can reside in the IIB matrix model is the master field $\widehat{A}^{\mu}$ of the model. But precisely where? In the following, we present a few rather naive ideas (hopefully, not too naive).

Following Refs. [5–7], we begin by making a particular global gauge transformation (2a),

$$\underline{\widehat{A}}^{\mu} = \underline{\Omega}\,\widehat{A}^{\mu}\,\underline{\Omega}^{\dagger}, \tag{11a}$$

$$\underline{\Omega} \in SU(N), \tag{11b}$$

so that the transformed 0-component [singled out by the Minkowski "metric" (1b)] is diagonal and has ordered eigenvalues $\widehat{\alpha}_i \in \mathbb{R}$,

$$\underline{\widehat{A}}^{0} = \text{diag}\Big(\widehat{\alpha}_1, \widehat{\alpha}_2, \ldots, \widehat{\alpha}_{N-1}, \widehat{\alpha}_N\Big), \tag{12a}$$

$$\widehat{\alpha}_1 \leq \widehat{\alpha}_2 \leq \cdots \leq \widehat{\alpha}_{N-1} \leq \widehat{\alpha}_N, \tag{12b}$$

$$\sum_{i=1}^{N} \widehat{\alpha}_i = 0, \tag{12c}$$

where the last equality from tracelessness implies that some $\widehat{\alpha}_i$ are negative and some positive. The ordering (12b) will turn out to be crucial for the time coordinates $\widetilde{t}$ and $\widehat{t}$ obtained below.

Indeed, we can introduce a continuous function $\widetilde{x}^{0}(\widetilde{\zeta}) \equiv \widetilde{c}\,\widetilde{t}(\widetilde{\zeta})$ for $\widetilde{\zeta} \in (0,\,1]$ by identifying (cf. Ref. [21])

$$\widetilde{x}^{0}(i/N) \equiv \widetilde{c}\,\widetilde{t}(i/N) = \widehat{\alpha}_i, \tag{13}$$

with $i \in \{1, \ldots, N\}$ and a velocity $\widetilde{c}$ that is expected to be related to the vacuum velocity of light in the low-energy theory. From Eq. (12b), we immediately have

$$\widetilde{t}\big(1/N\big) \leq \widetilde{t}\big(2/N\big) \leq \cdots \leq \widetilde{t}\big(1-1/N\big) \leq \widetilde{t}\big(1\big), \tag{14}$$

where the ordering is the defining property of what makes physical time.

The problem now is how to extract the *corresponding* space coordinates $\widetilde{x}^{m}(\widetilde{\zeta})$ from the Hermitian $\underline{\widehat{A}}^{m}$ matrices. The simplest idea (following Ref. [2]) is to calculate the eigenvalues of the nine matrices $\underline{\widehat{A}}^{m}$, but then it is unclear how to order them with respect to the eigenvalues from Eq. (12). We will use a relatively simple procedure, which approximates the $\underline{\widehat{A}}^{m}$ eigenvalues but still manages to order them along the diagonal. Our procedure corresponds, in fact, to a type of coarse graining of some of the information contained in the IIB-matrix-model master field. There is, however, more information in the master field that we will not consider, and even information not in the master field, as there are also non-factorizing observables [20] in the IIB matrix model.

We start from the following trivial observation: if M is an $N \times N$ Hermitian matrix, then *any* $n \times n$ block centered on the diagonal of M is *also* Hermitian, which holds for $n \geq 1$ and $n \leq N$. With $N \gg 1$, we take n so that $1 \ll n \ll N$. Specifically, we proceed by the following six steps.

The first step is to let K be an odd divisor of N, so that

$$N = K\,n, \tag{15a}$$

$$K = 2\,L + 1, \tag{15b}$$

where both L and n are positive integers (we have chosen an odd value of K for later convenience). In the limit $N \to \infty$, we also take $K \to \infty$ but are not sure exactly how fast (with n staying finite or not).

The second step is to consider, in each of the 10 matrices $\underline{\widehat{A}}^{\mu}$ from Eqs. (11) and (12), the K blocks of size $n \times n$ centered on the diagonals.

The third step is to realize that we already know the diagonalized blocks of $\underline{\widehat{A}}^{0}$ from Eq. (12a). This allows us to define the following time coordinate $\widehat{t}(\zeta)$, for $\zeta \in (0,\, 1]$, as the average of the $\widehat{\alpha}_i$ eigenvalues of each $n \times n$ block:

$$\widehat{x}^{0}\left(k/K\right) \equiv \widetilde{c}\,\widehat{t}\left(k/K\right) \equiv \left(\frac{1}{n}\sum_{j=1}^{n}\widehat{\alpha}_{(k-1)\,n+j}\right) + \widetilde{c}\,\widehat{t}_{\text{shift}}, \tag{16}$$

with $k \in \{1, \ldots, K\}$, an arbitrary real constant $\widehat{t}_{\text{shift}}$, and the velocity $\widetilde{c}$ mentioned below Eq. (13). The time coordinates from Eq. (16) are ordered,

$$\widehat{t}\left(1/K\right) \le \widehat{t}\left(2/K\right) \le \cdots \le \widehat{t}\left(1-1/K\right) \le \widehat{t}\left(1\right), \tag{17}$$

because the $\widehat{\alpha}_i$ are, according to Eq. (12b). With an appropriate value of $\widehat{t}_{\text{shift}}$ in Eq. (16), we can set $\widehat{t} = 0$ for the halfway block at $k = L + 1$. The blocks with $k < L + 1$ will generically have negative time coordinates $\widehat{t}$ and those with $k > L + 1$ generically positive time coordinates $\widehat{t}$.

The fourth step is to obtain the eigenvalues of the $n \times n$ blocks of the nine spatial matrices $\underline{\widehat{A}}^{m}$ and to denote these real eigenvalues $\left(\widehat{\beta}^{\,m}\right)_i$, with $i \in \{1, \ldots, N\}$. How the n eigenvalues are ordered in each block is irrelevant, as they will be averaged over in the next step.

The fifth step is to define, just as in step three, the following nine spatial coordinates $\widehat{x}^{m}(\zeta)$, for $\zeta \in (0,\, 1]$, as the averages of the $\left(\widehat{\beta}^{\,m}\right)_i$ eigenvalues of the $n \times n$ blocks:

$$\widehat{x}^{m}\left(k/K\right) \equiv \frac{1}{n}\sum_{j=1}^{n}\left[\,\widehat{\beta}^{\,m}\,\right]_{(k-1)\,n+j}, \tag{18}$$

with $k \in \{1, \ldots, K\}$. The averaging is done independently for each value of m.

The sixth and last step is, first, to observe that $\widehat{t}(\zeta)$ from Eqs. (16) and (17) is a nondecreasing function of $\zeta \equiv k/K$ and, then, to eliminate ζ between $\widehat{t}(\zeta)$ from Eq. (16) and $\widehat{x}^{m}(\zeta)$ from Eq. (18), in order to obtain

$$\widehat{x}^{m} = \widehat{x}^{m}\left(\widehat{t}\,\right), \tag{19}$$

which corresponds to a particular foliation of what will become the classical spacetime.

If the master-field matrices $\underline{\widehat{A}}^{\mu}$ are more or less block-diagonal (with a width $\Delta N \ll N$, as suggested by the numerical results from Refs. [5–7]) and if an appropriate value of n can be chosen (perhaps $n \sim \Delta N$, for sufficiently large values of N), then the expressions (16) and (18) may provide suitable spacetime points. In a somewhat different notation, these spacetime points are denoted

$$\widehat{x}^{\mu}_{k} = \left(\widehat{x}^{0}_{k}, \widehat{x}^{m}_{k}\right) \equiv \left(\widehat{x}^{0}\left(k/K\right), \widehat{x}^{m}\left(k/K\right)\right), \tag{20}$$

where k runs over $\{1, \ldots, K\}$ with K given by Eq. (15). Each of these 10 coordinates has the dimension of length, which traces back to the dimension of the bosonic matrix variable A^{μ}, as discussed in Sect. 2. The points (20), and those obtained from different choices of block size n and block position along the diagonals of the master-field matrices, effectively build a spacetime manifold with continuous (interpolating) coordinates x^{μ} if there is also an emerging metric $g_{\mu\nu}(x)$.

5. Emergent spacetime metric

In Sect. 4, we have obtained K points $\widehat{x}_k^{\mu}$ as given by Eq. (20), which sample a 10D classical spacetime. (We have put a hat on our coordinates in order to remind us of their master-field origin.) The idea now is that low-energy fields propagate over a spacetime manifold which interpolates between these discrete spacetime points $\widehat{x}_k^{\mu}$. The low-energy fields include the matter fields (scalar, vector, spinor) and the metric field (tensor). In fact, Aoki et al. [2] have argued that the propagation of a matter field (e.g., the propagation of a scalar field σ) determines the effective inverse metric, which is found to depend on the density function of the spacetime points $\widehat{x}_k^{\mu}$ and the correlations of these density functions.

The crucial result in Ref. [2] is Eq. (4.16), which we rewrite as follows:

$$g^{\mu\nu}(x) \sim \int_{\mathbb{R}^D} d^D y\, \langle\langle\, \rho(y)\,\rangle\rangle\, (x-y)^{\mu}\, (x-y)^{\nu}\, f(x-y)\, r(x,y), \tag{21}$$

where $D = 10$ is the spacetime dimension and the average $\langle\langle\, \rho(y)\,\rangle\rangle$ corresponds, for the procedure used in Sect. 4, to averaging over different block sizes and block positions along the diagonals of the master-field matrices (details will be presented elsewhere).

The quantities that enter the multiple integral (21) are the density function

$$\rho(x) \;\equiv\; \sum_{k=1}^{K} \delta^{(D)}\big(x - \widehat{x}_k\big), \tag{22}$$

the dimensionless density correlation function $r(x, y)$ defined by

$$\langle\langle\, \rho(x)\, \rho(y)\,\rangle\rangle \;\equiv\; \langle\langle\, \rho(x)\,\rangle\rangle\, \langle\langle\, \rho(y)\,\rangle\rangle\, r(x,y), \tag{23}$$

and a strongly localized function $f(x)$, which appears in the effective action of a low-energy scalar degree of freedom σ “propagating” over the discrete spacetime points $\widehat{x}_k^{\mu}$,

$$S_{\text{eff}}[\sigma] \propto \sum_{k,l} \frac{1}{2} f\big(\widehat{x}_k - \widehat{x}_l\big)\, \big(\sigma_k - \sigma_l\big)^2 + \sum_{k} \frac{1}{2}\, \mu^2\, \ell^{-2}\, \big(\sigma_k\big)^2, \tag{24}$$

where $f(x) = f\left(x^0, x^1, \ldots, x^{D-1}\right)$ has dimension $1/(\text{length})^2$, μ is dimensionless, and ℓ is the model length scale introduced in Eq. (3). Here, σ_k is the field value at the point $\widehat{x}_k$ and the continuous field $\sigma(x)$ has $\sigma(\widehat{x}_k) = \sigma_k$. After averaging over different block structures in the master-field matrices (see above) and making a Taylor expansion, the continuous field $\sigma(x)$ is found to have a standard kinetic term $g^{\mu\nu}\, \partial_\mu\sigma\, \partial_\nu\sigma$ in the action, with the inverse metric given by Eq. (21). See Sect. 4.2 of Ref. [2] for further details, Appendix A for a sample calculation, and Ref. [25] for earlier work on random-lattice field theories.

The inverse metric $g^{\mu\nu}(x)$ from Eq. (21) is manifestly dimensionless and the metric $g_{\mu\nu}$ is simply obtained as the matrix inverse of $g^{\mu\nu}$. In fact, general covariance is also expected to emerge dynamically [2] and the quantity determined by the integral (21) will, for a strongly localized function f, transform approximately like $dx^{\mu}\, dx^{\nu}$, that is, approximately like a rank-2 contravariant tensor. Taking the matrix inverse of this quantity gives an object that transforms approximately like a rank-2 covariant tensor, so that this object can indeed be interpreted as the emergent metric $g_{\mu\nu}(x)$.

The outstanding tasks are to obtain the master-field matrices $\widehat{A}^{\mu}$, to identify an effective scalar σ from it (cf. Sect. 4.1 of Ref. [2]), and to recover the effective action (24). The explicit results for

$\rho(x), f(x)$, and $r(x, y)$ must also explain how the inverse metric from Eq. (21) acquires a Lorentzian signature.

Using appropriate units to set $\ell = 1$, we have performed a toy-model calculation with the function $f_{\text{test},2}(x) = \alpha + x^0 x^1$ inserted into the multiple integral (21) for $D = 2$, where we also assume $\rho(x) = r(x, y) = 1$ and cut the integration ranges off symmetrically at ± 1. The resulting inverse metric at $x^\mu = 0$ is found to change continuously from a Euclidean to a Lorentzian signature as the parameter α changes continuously from $\alpha = 1$ to $\alpha = 0$ (see Appendix B for further details and a trivial extension to $D = 4$). The conclusion is that, in principle, it is possible to obtain a Lorentzian inverse metric from the expression (21). But it will be a challenge to establish, if at all relevant, the effective Lorentzian metric of the regularized big bang singularity with $b \sim \ell$ as the length parameter [12–15].

For the record, we give a further result, based on Eq. (4.17) of Ref. [2], which concerns the background value of the dilaton field Φ,

$$\sqrt{-g(x)}\,\exp\big[-\Phi(x)\big] \propto \langle\langle\, \rho(x)\, \rangle\rangle, \tag{25}$$

with $g \equiv \det g_{\mu\nu}$ and the meaning of the average on the right-hand side explained in the text below Eq. (21).

Returning to the expression (21) for the emergent inverse metric, we observe that it depends not only on the density distribution ρ of emerged spacetime points and their correlation function r, but also on the localization function f from the scalar effective action (24). In this way, the metric only exists if matter is present, which reminds us of Dicke's interpretation of spacetime (see Appendix 4, p. 50 and Appendix 5, p. 60 in Ref. [26]). The new insight from the IIB matrix model is that matter and spacetime are expected to emerge simultaneously.

Note added

Two subsequent papers [27,28] give details on the extraction of the spacetime points and the spacetime metric, assuming that the IIB-matrix-model master field is known. A further paper [29] shows that the IIB-matrix-model master field can, in principle, give rise to the regularized big bang metric [12] of general relativity.

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Appendix A. Effective action of a scalar degree of freedom

The expression (21) for the emergent inverse metric in Sect. 5 was obtained from an *assumed* effective action (24) of a scalar degree of freedom σ. Even though the particular form of this effective action is entirely reasonable (cf. the discussion of random-lattice scalars in Sect. 6 of Ref. [25]), it is desirable to understand in some detail how this effective action could arise in the IIB matrix model. This is done in the present appendix, where we show that the IIB matrix model can, in principle, produce the effective action (24).

We start by noting that we should not be led astray by the notation A^{μ} resembling 10 gauge fields and that the IIB-matrix-model master field $\widehat{A}^{\mu}$ is really a *single* $10 \times N \times N$ matrix with entries having the dimension of length. The last observation suggests that, in order to get an effective *field* $\phi(x^0, \ldots, x^9)$ in the continuum, the perturbation ϕ_k of the master-field matrix must be taken *equal* on all 10 "slices" of the matrix (an explicit example will be given below).

For simplicity, we focus on the four "large" spacetime dimensions [5,6],

$$D = 4, \tag{A.1}$$

and let the indices $\mu, \nu, \ldots$ run over $\{0,\ 1,\ 2,\ 3\}$. We now present an explicit construction of a perturbation of the master field for the case

$$N = K\,n = 6, \quad n = 3, \tag{A.2}$$

where n corresponds to the averaging block used in Sect. 4 for the extraction of the spacetime points (here, there are only two spacetime points, $\widehat{x}_1^{\mu}$ and $\widehat{x}_2^{\mu}$). For the sake of argument, we simply assume that $N = 6$ is large enough so that there exists a master field (later, we will extend the explicit construction to $N \gg 1$).

The 6×6 master-field matrices are assumed to have a band-diagonal structure [5–7] and are given by

$$\underline{\widehat{A}}^{\mu} = \begin{pmatrix} \mathcal{B}_{11}^{\mu} & \mathcal{B}_{12}^{\mu} \\ \mathcal{B}_{21}^{\mu} & \mathcal{B}_{22}^{\mu} \end{pmatrix}, \tag{A.3a}$$

in terms of 3×3 blocks $\mathcal{B}_{kl}^{\mu}$, where

$$\mathcal{B}_{12}^{\mu} \sim 0, \quad \mathcal{B}_{21}^{\mu} \sim 0, \tag{A.3b}$$

and the block $\mathcal{B}_{11}^{\mu}$ has real eigenvalues $\{\widehat{x}_{1,a}^{\mu}, \widehat{x}_{1,b}^{\mu}, \widehat{x}_{1,c}^{\mu}\}$ with an average value

$$\widehat{x}_1^{\mu} = \frac{1}{3}\left(\widehat{x}_{1,a}^{\mu} + \widehat{x}_{1,b}^{\mu} + \widehat{x}_{1,c}^{\mu}\right), \tag{A.3c}$$

and similarly for the block $\mathcal{B}_{22}^{\mu}$, with real eigenvalues $\{\widehat{x}_{2,a}^{\mu}, \widehat{x}_{2,b}^{\mu}, \widehat{x}_{2,c}^{\mu}\}$ and an average value

$$\widehat{x}_2^{\mu} = \frac{1}{3}\left(\widehat{x}_{2,a}^{\mu} + \widehat{x}_{2,b}^{\mu} + \widehat{x}_{2,c}^{\mu}\right). \tag{A.3d}$$

Now consider the following 6×6 matrices A^{μ} involving the perturbations $\phi_1, \phi_2 \in \mathbb{R}$:

$$A^{\mu} = \text{diag}\left(B_{<11>}^{\mu}, B_{<12>}^{\mu}, B_{<22>}^{\mu}\right), \tag{A.4a}$$

in terms of 2×2 blocks

$$B_{<11>}^{\mu} = \begin{pmatrix} \widehat{x}_1^{\mu} & c^{\mu}\,\phi_1\,\left(1 - \phi_1^2/\ell^2\right) \\ c^{\mu}\,\phi_1\,\left(1 - \phi_1^2/\ell^2\right) & \widehat{x}_1^{\mu} + \phi_1 \end{pmatrix}, \tag{A.4b}$$

$$B_{<12>}^{\mu} = \begin{pmatrix} \widehat{x}_1^{\mu} & k_{12}\,(\phi_1 - \phi_2) \\ k_{12}\,(\phi_1 - \phi_2) & \widehat{x}_2^{\mu} \end{pmatrix}, \tag{A.4c}$$

$$B^{\mu}_{<22>} = \begin{pmatrix} \widehat{x}^{\mu}_{2} & d^{\mu}\,\phi_2\,\left(1-\phi_2^2/\ell^2\right) \\ d^{\mu}\,\phi_2\,\left(1-\phi_2^2/\ell^2\right) & \widehat{x}^{\mu}_{2}+\phi_2 \end{pmatrix}, \tag{A.4d}$$

for a dimensionless coupling k_{12} and dimensionless constants c^{μ} and d^{μ},

$$k_{12} = k_{12}\big(\Delta x\big) \in \mathbb{R}, \tag{A.4e}$$

$$c^{\mu} = \big(c^0,\, c,\, c,\, c\big) \in \mathbb{R}^4, \tag{A.4f}$$

$$d^{\mu} = \big(d^0,\, d,\, d,\, d\big) \in \mathbb{R}^4, \tag{A.4g}$$

with definition

$$\Delta x^{\mu} \equiv \widehat{x}^{\mu}_{2} - \widehat{x}^{\mu}_{1}. \tag{A.5}$$

Three remarks are in order. First, the *same* perturbation ϕ_1 appears in *all* four matrices A^{μ}, and similarly for ϕ_2. Second, the parameter k_{12} depends on the coordinate distance Δx^{μ} and is assumed to drop rapidly as this distance increases (otherwise, the emerging scalar theory does not make sense [25]). Third, the matrices A^{μ} reduce, for $\phi_1 = \phi_2 = 0$, to diagonal matrices with approximately the same eigenvalues as the master-field matrices (A.3), which were assumed to be band-diagonal.

Next, insert the perturbation matrices A^{μ} from Eq. (A.4) in the bosonic action (1) and find

$$\begin{aligned} S_b\Big|^{\text{(pert)}} = \frac{1}{2}\Big[&3\,\big(\Delta x^0\big)^2 - \big(\Delta x^1\big)^2 - \big(\Delta x^3\big)^2 - \big(\Delta x^1\big)^2 - 2\,\Delta x^0\,\big(\Delta x^1 + \Delta x^2 + \Delta x^3\big) \\ &+ 2\,\Delta x^1\,\Delta x^2 + 2\,\Delta x^2\,\Delta x^3 + 2\,\Delta x^3\,\Delta x^1\Big]\,\Big(k_{12}\big(\Delta x\big)\Big)^2\,(\phi_1-\phi_2)^2 \\ &+ \frac{2}{3}\,\ell^{-4}\,(c^0-c)^2\,{\phi_1}^4\,\big(\ell^2-{\phi_1}^2\big)^2 + \frac{2}{3}\,\ell^{-4}\,(d^0-d)^2\,{\phi_2}^4\,\big(\ell^2-{\phi_2}^2\big)^2. \end{aligned} \tag{A.6}$$

Apparently, we have already recovered the "kinetic" term $(\sigma_1-\sigma_2)^2$ of Eq. (24), which gives rise to the emergent inverse metric (21). The mass-squared terms ${\sigma_1}^2$ and ${\sigma_2}^2$ of Eq. (24) result from spontaneous symmetry breaking, at least for the simple model considered. Indeed, with shifted scalar variables,

$$\phi_1 = \ell + \chi_1, \quad \phi_2 = \ell + \chi_2, \tag{A.7}$$

the effective action (A.6) becomes, in a shorthand notation,

$$\begin{aligned} S_b\Big|^{\text{(pert)}} &= \frac{1}{2}\Big[\cdots\Big]\,\Big(k_{12}\big(\Delta x\big)\Big)^2\,(\chi_1-\chi_2)^2 \\ &\quad + 6\,\big(c^0-c\big)^2\,\ell^2\,{\chi_1}^2 + 6\,\big(d^0-d\big)^2\,\ell^2\,{\chi_2}^2 + \cdots, \end{aligned} \tag{A.8}$$

where the ellipsis at the end stands for cubic and higher-order self-interaction terms of the scalars χ_1 and χ_2. Note that the square bracket in Eq. (A.8), which is explicitly shown in Eq. (A.6), can be positive, zero, or negative, whereas the mass-square terms in Eq. (A.8) are strictly nonnegative. The indefinite sign of the square bracket in Eqs. (A.6) and (A.8) traces back to the Lorentzian "signature" of the coupling constants (1b) in the IIB matrix model.

By adding appropriate (generalized) blocks to Eq. (A.4a) we can easily obtain matrices with larger values of N. In this way, we keep essentially the same properties as discussed for the $(N,\, n) = (6,\, 3)$

case and obtain, in particular, an effective action with kinetic terms as shown in Eq. (24), but now in terms of scalars χ_k.

Appendix B. Emergent Lorentzian signature

In this appendix, we present some details of the 2D toy-model calculation for the emergent inverse metric mentioned in Sect. 5. The aim of this 2D toy-model calculation is to present a possible mechanism for obtaining, in the emergent inverse metric, two eigenvalues with opposite signs. For completeness, we will also discuss an extended 4D toy-model calculation, which is slightly more realistic as it allows for a direct interpolation between the standard 4D Euclidean inverse metric and the standard 4D Minkowski inverse metric. Throughout this appendix, we use length units that set the IIB-matrix-model length scale to unity, $\ell = 1$.

Both calculations start from the multiple integral (21) for spacetime dimension $D = 2$ or 4 at the spacetime point

$$x^\mu = 0, \tag{B.1a}$$

with a simplified integrand having

$$\langle\langle\, \rho(y)\, \rangle\rangle = 1, \tag{B.1b}$$

$$r(x, y) = 1, \tag{B.1c}$$

and symmetric cutoffs on the integrals,

$$\int_{-1}^{1} dy^0 \cdots \int_{-1}^{1} dy^{D-1}. \tag{B.1d}$$

The only nontrivial contribution to the integrand of Eq. (21) then comes from the correlation function $f(x-y)$, for which we will make two Ansätze.

B.1. 2D calculation

For the first toy-model calculation, we take

$$D = 2, \tag{B.2a}$$

$$f_{\text{test},2}(y) = \alpha + y^0\, y^1, \tag{B.2b}$$

where the Ansatz function (B.2b) combines a term that is even in both y^0 and y^1 with a term that is odd in both y^0 and y^1. From Eq. (21) with simplifications (B.1), we then get the following multiple integral for the emerging inverse metric:

$$g^{\mu\nu}_{\text{test},2}(0) = \int_{-1}^{1} dy^0 \int_{-1}^{1} dy^1\, y^\mu\, y^\nu\, f_{\text{test},2}(y). \tag{B.3}$$

The integrals are trivial and we obtain the inverse metric

$$g^{\mu\nu}_{\alpha}(0) = \begin{pmatrix} 4\,\alpha/3 & 4/9 \\ 4/9 & 4\,\alpha/3 \end{pmatrix}, \tag{B.4a}$$

which has the following set of eigenvalues:

$$\mathcal{E}_\alpha = \frac{4}{9}\left\{ \left(3\,\alpha - 1\right),\, \left(3\,\alpha + 1\right) \right\}. \tag{B.4b}$$

We now introduce an interpolation parameter ρ,

$$\alpha(\rho) = 1 - \rho, \tag{B.5a}$$

$$\rho \in [0,\, 1], \tag{B.5b}$$

so that the inverse metric (B.4a) and its eigenvalues (B.4b) are given by

$$g_\rho^{\mu\nu}(0) = \begin{pmatrix} 4\,(1-\rho)/3 & 4/9 \\ 4/9 & 4\,(1-\rho)/3 \end{pmatrix}, \tag{B.6a}$$

$$\mathcal{E}_\rho = \frac{4}{9}\Big\{\big(2 - 3\,\rho\big),\, \big(4 - 3\,\rho\big)\Big\}. \tag{B.6b}$$

We see that we have obtained an inverse metric that interpolates between a Euclidean signature for $\rho = 0$ and a Lorentzian signature for $\rho = 1$:

$$\mathcal{E}_{\rho=0} = \Big\{8/9,\, 16/9\Big\}, \tag{B.7a}$$

$$\mathcal{E}_{\rho=1} = \Big\{-4/9,\, 4/9\Big\}. \tag{B.7b}$$

At $\rho = 2/3$, the inverse metric (B.6) is degenerate, with a vanishing determinant.

The origin of the Lorentzian signature (B.7b) is easy to understand. For $\rho = 1$, the Ansatz parameter $\alpha = 1 - \rho$ in Eq. (B.2b) equals zero, so that the integrand of Eq. (B.3) becomes simply $y^\mu\, y^\nu\, y^0\, y^1$. The symmetric integrals (B.3) then vanish unless $\{\mu,\, \nu\} = \{0,\, 1\}$ or $\{\mu,\, \nu\} = \{1,\, 0\}$. In other words, the matrix for the emergent inverse metric (B.3) is off-diagonal with entries $(2/3)^2 = 4/9$, so that the eigenvalues are $\pm 4/9$. The off-diagonal matrix structure traces back to the assumption that the correlation function $f(y)$, for $\rho = 1$ or $\alpha = 0$, is given by a single monomial $y^0\, y^1$, which is odd in both y^0 and y^1.

A final remark on this 2D calculation of a Lorentzian signature is in order. From the $\rho = 1$ inverse metric (B.6a), we obtain, after a suitable coordinate transformation (with $g^{\mu\nu} \to g^{\prime\,\mu\nu}$) and a rescaling of x^0 and x^1 by an identical factor (here, a factor $2/3$), the standard Minkowski form, $g^{\prime\,\mu\nu} = \mathrm{diag}(-1,\, 1)$. Instead of rescaling the coordinates, it is also possible, for this simple case, to multiply the Ansatz function (B.2b) by an appropriate overall factor (here, a factor 9/4).

B.2. 4D calculation

For the second toy-model calculation, we take

$$D = 4, \tag{B.8a}$$

$$f_{\text{test},4}(y) = \alpha + \beta\left[\big(y^2\big)^2 + \big(y^3\big)^2\right] + \gamma\, y^0\, y^1, \tag{B.8b}$$

where the Ansatz function (B.8b) combines two terms that are even in both y^0 and y^1 with one term that is odd in both y^0 and y^1. From Eq. (21) with simplifications (B.1), we then get the emergent inverse metric

$$g_{\text{test},4}^{\mu\nu}(0) = \int_{-1}^{1} dy^0 \int_{-1}^{1} dy^1 \int_{-1}^{1} dy^2 \int_{-1}^{1} dy^3\; y^\mu\, y^\nu\, f_{\text{test},4}(y). \tag{B.9}$$

Again, the integrals are trivial and we obtain

$$g^{\mu\nu}_{\alpha\beta\gamma}(0) = \frac{16}{9}\begin{pmatrix} 3\,\alpha + 2\,\beta & \gamma & 0 & 0 \\ \gamma & 3\,\alpha + 2\,\beta & 0 & 0 \\ 0 & 0 & 3\,\alpha + (14/5)\,\beta & 0 \\ 0 & 0 & 0 & 3\,\alpha + (14/5)\,\beta \end{pmatrix}, \tag{B.10a}$$

which has the following set of eigenvalues:

$$\mathcal{E}_{\alpha\beta\gamma} = \frac{16}{9}\left\{\Big(3\,\alpha + 2\,\beta + \gamma\Big), \Big(3\,\alpha + 2\,\beta - \gamma\Big), \Big(3\,\alpha + \frac{14}{5}\,\beta\Big), \Big(3\,\alpha + \frac{14}{5}\,\beta\Big)\right\}. \tag{B.10b}$$

Let us now introduce an interpolation parameter σ,

$$\alpha(\sigma) = \frac{3}{32}\,\big(2 - 7\,\sigma\big), \tag{B.11a}$$

$$\beta(\sigma) = \frac{45}{64}\,\sigma, \tag{B.11b}$$

$$\gamma(\sigma) = -\frac{9}{16}\,\sigma, \tag{B.11c}$$

$$\sigma \in [0,\ 1], \tag{B.11d}$$

so that the inverse metric (B.10a) and its eigenvalues (B.10b) are given by

$$g^{\mu\nu}_{\sigma}(0) = \begin{pmatrix} 1-\sigma & -\sigma & 0 & 0 \\ -\sigma & 1-\sigma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{B.12a}$$

$$\mathcal{E}_{\sigma} = \Big\{1 - 2\,\sigma,\ 1,\ 1,\ 1\Big\}. \tag{B.12b}$$

From Eq. (B.12a) for $\sigma = 0$, we immediately have the standard Euclidean inverse metric,

$$g^{\mu\nu}_{\sigma=0}(0) = \text{diag}\Big(1,\ 1,\ 1,\ 1\Big), \tag{B.13a}$$

while, from Eq. (B.12a) for $\sigma = 1$, we obtain, after a suitable coordinate transformation (with $g^{\mu\nu} \to g^{\prime\,\mu\nu}$), the standard Minkowski inverse metric,

$$g^{\prime\,\mu\nu}_{\sigma=1}(0) = \text{diag}\Big(-1,\ 1,\ 1,\ 1\Big). \tag{B.13b}$$

Again, we interpolate smoothly between a Euclidean signature ($\sigma = 0$) and a Lorentzian signature ($\sigma = 1$). At $\sigma = 1/2$, the inverse metric (B.12) is degenerate, with a vanishing determinant.

The expression (21) for the emergent inverse metric, first proposed in Ref. [2] and reinterpreted in the present paper, has the potential to give either a Euclidean or a Lorentzian inverse metric, depending on the functional behavior of the correlation functions $r(x, y)$ and $f(x - y)$, which result from the detailed structure of the emerging spacetime points. In principle, it is even possible to get a Lorentzian emergent inverse metric from a Euclidean IIB matrix model, provided that the correlation functions have the appropriate structure [a Euclidean toy-model calculation for $D = 4$ may give the inverse metric (B.9) with y^0 replaced by y^4 and $f_{\text{test},4}(y)$ by $f_{\text{test,E4}}(y) = 1 - \gamma\,(y^1y^2 + y^1y^3 + y^1y^4 + y^2y^3 + y^2y^4 + y^3y^4)$, and then finds the Lorentzian signature $(-+++)$ for parameter values $\gamma > 1$].

This last observation, if applicable, would remove the need for working with the (possibly more difficult) Lorentzian IIB matrix model and the first two of the five preliminary remarks in Sect. 1 would no longer apply.

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