



# An implicit–explicit time discretization scheme for second-order semilinear wave equations with application to dynamic boundary conditions

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## Abstract

We construct and analyze a second-order implicit–explicit (IMEX) scheme for the time integration of semilinear second-order wave equations. The scheme treats the stiff linear part of the problem implicitly and the nonlinear part explicitly. This makes the scheme unconditionally stable and at the same time very efficient, since it only requires the solution of one linear system of equations per time step. For the combination of the IMEX scheme with a general, abstract, nonconforming space discretization we prove a full discretization error bound. We then apply the method to a nonconforming finite element discretization of an acoustic wave equation with a kinetic boundary condition. This yields a fully discrete scheme and a corresponding a-priori error estimate.

**Keywords** Implicit–explicit time integration · IMEX · Dynamic boundary conditions · Semilinear wave equation · Nonconforming space discretization · Error analysis · A-priori error bounds · Semilinear evolution equations · Operator semigroups

**Mathematics Subject Classification** Primary 65M12 · 65M15 · Secondary 65M60 · 65J08

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## 1 Introduction

In this paper we construct and analyze an implicit–explicit (IMEX) time integration scheme for second-order semilinear wave equations of the form

$$u''(t) + Bu'(t) + Au(t) = f(t, u(t))$$

in a suitable Hilbert space. Here,  $A$  and  $B$  are unbounded operators and  $f$  is a locally Lipschitz continuous nonlinearity. The IMEX scheme is constructed as a combination of the explicit leapfrog method and the implicit Crank–Nicolson scheme. It treats the unbounded linear part of the differential equation implicitly and the nonlinear part explicitly. We show that the scheme is unconditionally stable in the sense that the time-step size is only restricted by the Lipschitz constant of  $f$  but not by the linear operators  $A$  and  $B$ .

We combine this IMEX scheme with an abstract, nonconforming space discretization within the framework of [11–13]. These papers provide a unified error analysis (UEA) which allows one to analyze nonconforming space discretizations of wave equations in a systematic way. Our main result is an error bound that is second order in time and contains abstract space-discretization errors. The error result can then be used to prove convergence rates for specific problems and discretizations by plugging in geometric and interpolation error results. The fully discrete scheme is very efficient. In fact, we show that one time step only requires the solution of one linear system and one application of discretizations of  $A$ ,  $B$ , and  $f$ , respectively. Since the construction of the scheme is based on two second-order methods and our analysis makes use of the specific form of the method, the generalization to higher order is not straightforward and out of the scope of this paper. Higher-order IMEX schemes for second-order equations will be part of future work.

There is a rich literature on IMEX schemes for first-order equations, in particular, there is a well-developed theory for IMEX Runge–Kutta schemes [3,6] or IMEX multistep schemes [1,4,8,15], for instance. In [6,15] an error analysis for ODEs is presented, while [1] contains discretization errors for IMEX schemes applied to conformal space discretizations of quasilinear parabolic evolution equations. IMEX schemes are used in applications, e.g., in structural dynamics and fluid–structure interaction [22], hydrodynamics [16], sea-ice dynamics [20], or atmospheric dynamics, see, e.g., [9], to mention just a few examples. There exists also a so-called Crank–Nicolson-leapfrog IMEX scheme which is obtained from a combination of the Crank–Nicolson and the leapfrog scheme for first-order equations, cf. [17,18], and references therein. However, this scheme is not equivalent to the scheme we construct here, since the leapfrog schemes for first- and second-order equations are not equivalent and indeed have completely different stability properties. More precisely, the Crank–Nicolson-leapfrog scheme is only stable, if the explicitly treated part is skew symmetric.

To solve a second-order equation one can either reformulate it equivalently into a first-order equation and apply an IMEX scheme to it or one can design a scheme for the original second-order form. An example for the first option is the Crank–Nicolson-leapfrog IMEX scheme presented in [17,18]. In contrast, we decided to take the second option and present a new scheme which to the best of our knowledge was not

considered in literature so far. In fact, we are not aware of any IMEX scheme exploiting the special structure of second-order equations. The advantage of this approach is the efficiency of the scheme, as will be discussed in detail in Sect. 2.2.

The main contribution of this paper is to provide a full discretization error analysis of semilinear wave equations in a quite general framework. So far, such an error analysis does not even exist for the Crank–Nicolson scheme, which is also covered as a byproduct of our analysis of the IMEX scheme. The challenge of such a rigorous analysis is that it applies to abstract, non-conforming space discretizations of semilinear wave-type equations.

As an application of our abstract theory, we consider an acoustic wave equation with kinetic boundary conditions that fits into the abstract setting, cf., [13]. Kinetic boundary conditions are a special case of dynamic boundary conditions that contain tangential derivatives and are intrinsically posed on domains with (piecewise) smooth and therefore possibly curved boundaries. Hence, the spatial discretization has to be done on an approximated domain rendering the discretization nonconforming.

The paper is organized as follows: in Sect. 2 we present the problem setting, introduce the IMEX scheme for second-order wave equations, and state a second-order error bound for the time discretization error. In Sect. 3 we briefly recall the UEA and present the fully discrete scheme as a combination of the IMEX scheme with a general space discretization. Afterwards we state and prove the main result, namely the abstract full discretization error bound. Finally, in Sect. 4, we consider a semilinear acoustic wave equation with a kinetic boundary conditions as an example fitting into the abstract setting. We present a finite element space discretization and the full discretization error bound. We finish the paper with numerical experiments underlining the theoretical error bounds and the efficiency of the IMEX scheme.

## 2 The implicit–explicit (IMEX) scheme

In this section we first introduce the problem setting and then present the IMEX scheme and its properties.

### 2.1 Continuous problem

Let  $V, H$  be Hilbert spaces with  $V \subset H$ . We consider the following second-order variational equation as a prototype of second-order wave equations in weak formulation: find  $u: [0, T] \rightarrow V$  s.t.

$$\begin{aligned} m(u''(t), v) + b(u'(t), v) + a(u(t), v) &= m(f(t, u(t)), v) \quad \text{for all } v \in V, t \in (0, T], \\ u(0) &= u^0, \quad u'(0) = v^0, \end{aligned} \quad (1)$$

with bilinear forms

$$\begin{aligned} m: H \times H &\rightarrow \mathbb{R}, \\ a: V \times V &\rightarrow \mathbb{R}, \end{aligned}$$

$$b: V \times H \rightarrow \mathbb{R},$$

and  $f: [0, T] \times V \rightarrow H$ .

For the rest of the paper we require the following assumptions without further mentioning it everywhere:

**Assumption 2.1**

- (a) The bilinear form  $m$  is a scalar product on  $H$ . In the following, we equip  $H$  with  $m$  and denote the corresponding norm by  $\|\cdot\|_m$ .
- (b) The bilinear form  $a$  is symmetric and there exists a constant  $\alpha_G > 0$  s.t.

$$\tilde{a} = a + \alpha_G m$$

is a scalar product on  $V$ . In the following we equip  $V$  with  $\tilde{a}$  and denote the corresponding norm by  $\|\cdot\|_{\tilde{a}}$ .

- (c) The space  $V$  is densely embedded in  $H$ , i.e., there exists an embedding constant  $C_{\text{emb}}$ , s.t.
 
$$\|v\|_m \leq C_{\text{emb}} \|v\|_{\tilde{a}} \quad \text{for all } v \in V. \tag{2}$$
- (d) The bilinear form  $b$  is bounded and quasi-monotone, i.e., there exist constants  $C_B > 0$  and  $\beta_{\text{qm}}$  s.t.

$$b(v, w) \leq C_B \|v\|_{\tilde{a}} \|w\|_m \quad \text{for all } v \in V, w \in H, \tag{3a}$$

$$b(v, v) + \beta_{\text{qm}} m(v, v) \geq 0 \quad \text{for all } v \in V. \tag{3b}$$

- (e) The nonlinearity  $f$  satisfies  $f \in C^1([0, T] \times V; H)$  and is locally Lipschitz-continuous on  $V$  with constant  $L_{T,\rho}$ , i.e., for all  $t \leq T$  and  $v, w \in V$  with  $\|v\|_{\tilde{a}}, \|w\|_{\tilde{a}} \leq \rho$  we have

$$\|f(t, v) - f(t, w)\|_m \leq L_{T,\rho} \|v - w\|_{\tilde{a}}. \tag{4}$$

**Example 2.2** We consider the semilinear damped wave equation

$$u_{tt} - \nabla u_t - \Delta u = |u|u \quad \text{in } (0, T) \times \Omega \tag{5}$$

in a domain  $\Omega \subset \mathbb{R}^d, d = 2, 3$ .

- (a) It is well known, that the weak formulation of (5) equipped with Dirichlet boundary condition  $u = 0$  on  $\partial\Omega$  fits in the setting presented above with

$$V = H_0^1(\Omega), \quad H = L^2(\Omega).$$

- (b) If  $\Omega$  has a  $C^2$  boundary, we can equip (5) with semilinear acoustic boundary conditions that have the form

$$u_{tt} + \partial_{\mathbf{n}} u - \Delta_{\partial\Omega} u = |u|^2 u \quad \text{in } (0, T) \times \partial\Omega.$$

We discuss a more general form of this example in Sect. 4, and show that it also fits in the setting of Sect. 2.1.

In the following we consider (1) as evolution equation on  $H$ :

$$u''(t) + Bu'(t) + Au(t) = f(t, u(t)), \quad u(0) = u^0, \quad u'(0) = v^0, \quad (6)$$

with operators  $A:D(A) \rightarrow H$  and  $B:V \rightarrow H$  induced by  $a$  and  $b$ , i.e.,

$$\begin{aligned} m(Av, w) &= a(v, w), & \text{for all } v \in D(A), w \in V, \\ m(Bv, w) &= b(v, w), & \text{for all } v \in V, w \in H, \end{aligned}$$

with

$$D(A) = \{v \in V \mid \exists C = C(v) > 0 \ \forall w \in V : |a(v, w)| \leq C \|w\|_m\}.$$

The wellposedness of (6) can be obtained by classical semigroup theory applied to the first-order formulation (7) of the equation, as detailed in [13].

**Theorem 2.3** *The problem (6) is locally wellposed, i.e., for all  $u^0 \in D(A), v^0 \in V$  there exists a time  $t^* = t^*(u^0, v^0)$  s.t. for all  $T < t^*$ , (6) has a unique solution*

$$u \in C^2([0, T]; H) \cap C^1([0, T]; V) \cap C([0, T]; D(A)).$$

### 2.2 Construction of the IMEX scheme

We modify the Crank–Nicolson scheme such that it treats the nonlinear part of (6) explicitly and thus avoids the solution of nonlinear equations.

To derive and analyze the scheme we state (6) in a first-order formulation. Set  $u' = v$  and

$$x = \begin{bmatrix} u \\ v \end{bmatrix}, \quad S = \begin{bmatrix} 0 & -I \\ A & B \end{bmatrix}, \quad g(t, x) = \begin{bmatrix} 0 \\ f(t, u) \end{bmatrix}, \quad x_0 = \begin{bmatrix} u^0 \\ v^0 \end{bmatrix}.$$

Then (6) can be written as

$$\begin{aligned} x'(t) + Sx(t) &= g(t, x(t)), & t \in [0, T], \\ x(0) &= x_0. \end{aligned} \quad (7)$$

We consider this equation in the Hilbert space  $(X, p):=(V, \tilde{a}) \times (H, m)$ , where  $p$  is the natural inner product, and  $D(S) = D(A) \times V$ .

Let  $\tau > 0$  be the time step and  $t_n := \tau n$  for  $n \in \mathbb{N}$ . By

$$\begin{bmatrix} u^n \\ v^n \end{bmatrix} = x^n \approx x(t_n)$$

we denote the numerical approximation of the exact solution of (7) at time  $t_n$  and we further use the short notations

$$g^n = \begin{bmatrix} 0 \\ f^n \end{bmatrix} := \begin{bmatrix} 0 \\ f(t_n, u^n) \end{bmatrix} = g(t_n, x^n).$$

The Crank–Nicolson scheme applied to (7) has the form

$$x^{n+1} = x^n + \frac{\tau}{2} \left( -S(x^n + x^{n+1}) + g^n + g^{n+1} \right) \tag{8a}$$

and can be written as

$$R_+ x^{n+1} = R_- x^n + \frac{\tau}{2} \left( g^n + g^{n+1} \right), \quad R_{\pm} := I \pm \frac{\tau}{2} S. \tag{8b}$$

By [11, Lemma 2.14 and Lemma 4.2] we have the following properties of  $R_{\pm}$ :

**Lemma 2.4** *Let  $c_{qm} = \frac{1}{2}\alpha_G C_{emb} + \beta_{qm}$  with  $C_{emb}$  defined in (2) and  $\alpha_G, \beta_{qm}$  from Assumption 2.1. Then, for  $\tau c_{qm} < 2$ , the following assertions hold true:*

- (a)  $R_+$  is invertible with  $\|R_+^{-1}\|_{X \leftarrow X} \leq 1$  and  $R_+^{-1}x \in D(S)$  for all  $x \in X$ .
- (b)  $R := R_+^{-1}R_-$  has a continuous extension on  $X$  satisfying  $\|R\|_{X \leftarrow X} \leq e^{\tau c_{qm}}$ .

By applying  $R_+^{-1}$  to (8b), the Crank–Nicolson scheme reads

$$x^{n+1} = R x^n + \frac{\tau}{2} R_+^{-1} \left( g^n + g^{n+1} \right). \tag{8c}$$

**Lemma 2.5** *The Crank–Nicolson scheme (8a) can equivalently be rewritten in a half-full-half step formulation*

$$v^{n+\frac{1}{2}} = v^n - \frac{\tau}{2} A u^n - \frac{\tau^2}{4} A v^{n+\frac{1}{2}} - \frac{\tau}{2} B v^{n+\frac{1}{2}} + \frac{\tau}{4} (f^n + f^{n+1}), \tag{9a}$$

$$u^{n+1} = u^n + \tau v^{n+\frac{1}{2}}, \tag{9b}$$

$$v^{n+1} = v^{n+\frac{1}{2}} - \frac{\tau}{2} A u^n - \frac{\tau^2}{4} A v^{n+\frac{1}{2}} - \frac{\tau}{2} B v^{n+\frac{1}{2}} + \frac{\tau}{4} (f^n + f^{n+1}). \tag{9c}$$

**Proof** Using

$$v^{n+\frac{1}{2}} := \frac{1}{2} \left( v^n + v^{n+1} \right), \tag{10}$$

we write (8a) component wise:

$$u^{n+1} = u^n + \tau v^{n+\frac{1}{2}},$$

$$v^{n+1} = v^n - \frac{\tau}{2} A (u^n + u^{n+1}) - \tau B v^{n+\frac{1}{2}} + \frac{\tau}{2} (f^n + f^{n+1}).$$

The first equation gives (9b). By eliminating  $u^{n+1}$  in the second equation we obtain

$$v^{n+1} = v^n - \tau Au^n - \frac{\tau^2}{2}Av^{n+\frac{1}{2}} - \tau Bv^{n+\frac{1}{2}} + \frac{\tau}{2}(f^n + f^{n+1}),$$

which is equivalent to the two half steps (9a) and (9c). □

By replacing in (9) the trapezoidal rule for the nonlinear part by the left/right rectangular rule, respectively, we obtain the following IMEX scheme:

$$v^{n+\frac{1}{2}} = v^n - \frac{\tau}{2}Au^n - \frac{\tau^2}{4}Av^{n+\frac{1}{2}} - \frac{\tau}{2}Bv^{n+\frac{1}{2}} + \frac{\tau}{2}f^n, \tag{11a}$$

$$u^{n+1} = u^n + \tau v^{n+\frac{1}{2}}, \tag{11b}$$

$$v^{n+1} = v^{n+\frac{1}{2}} - \frac{\tau}{2}Au^n - \frac{\tau^2}{4}Av^{n+\frac{1}{2}} - \frac{\tau}{2}Bv^{n+\frac{1}{2}} + \frac{\tau}{2}f^{n+1}. \tag{11c}$$

It can be interpreted as a combination of the Crank–Nicolson scheme for the linear and the leapfrog scheme for the nonlinear part, respectively. Obviously, it is time reversible.

**Remark 2.6** An equivalent representation of  $v^{n+1}$  is obtained by subtracting (11a) from (11c), namely

$$v^{n+1} = -v^n + 2v^{n+\frac{1}{2}} + \frac{\tau}{2}(f^{n+1} - f^n). \tag{11d}$$

It is computationally more efficient because of the elimination of the operators  $A$  and  $B$ .

The implementation is comprised by solving the linear system in (11a), and then computing (11b), and (11d). Altogether, each time step requires the solution of one linear system, one application of  $A$  and one evaluation of the nonlinearity (note that  $f^{n+1}$  can be reused in the next time step).

### 2.3 Wellposedness of the IMEX scheme

By (11a) we have

$$Q_+v^{n+\frac{1}{2}} = v^n - \frac{\tau}{2}Au^n + \frac{\tau}{2}f^n, \tag{12}$$

with  $Q_{\pm}:D(A) \rightarrow H$  given by

$$Q_{\pm} = I \pm \frac{\tau}{2}B \pm \frac{\tau^2}{4}A.$$

Since these operators play an important role in the analysis of the method, we collect some of their properties.

**Lemma 2.7** (Properties of  $Q_{\pm}$ ) *Let*

$$\frac{\tau^2}{2}\alpha_G + \tau\beta_{qm} \leq 1. \tag{13}$$

*Then, the operator  $Q_+$  is invertible and its inverse  $Q_+^{-1}:H \rightarrow D(A)$  satisfies*

$$\left\| \left( \frac{\tau}{2}B + \frac{\tau^2}{4}A \right) Q_+^{-1} \right\|_{H \leftarrow H} \leq 1, \tag{14a}$$

$$\left\| Q_+^{-1} \right\|_{H \leftarrow H} \leq \frac{\sqrt{2}}{\tau}, \tag{14b}$$

$$\left\| Q_- Q_+^{-1} \right\|_{H \leftarrow H} \leq e^{\frac{\tau^2}{2}\alpha_G + \tau\beta_{qm}}. \tag{14c}$$

**Proof** (a) We first prove that  $Q_+$  is invertible. By (13) and Assumption 2.1, the bilinear form

$$m + \frac{\tau}{2}b + \frac{\tau^2}{4}a = \underbrace{\left( 1 - \frac{\tau}{2}\beta_{qm} - \frac{\tau^2}{4}\alpha_G \right)}_{\geq 0} m + \frac{\tau}{2}(b + \beta_{qm}m) + \frac{\tau^2}{4}\tilde{a}: V \times V \rightarrow \mathbb{R}$$

is  $V$ -elliptic. Hence, by the Lax–Milgram lemma, for a given  $v \in H \subset V'$  there exists a unique  $z \in V$  such that

$$m(z, w) + \frac{\tau}{2}b(z, w) + \frac{\tau^2}{4}a(z, w) = m(v, w) \quad \text{for all } w \in V,$$

or equivalently

$$\frac{\tau^2}{4}a(z, w) = m\left(v - z - \frac{\tau}{2}Bz, w\right) \quad \text{for all } w \in V.$$

This yields  $z \in D(A)$  and  $Q_+z = v$ , hence  $Q_+$  is invertible.

(b) Proof of the bounds (14): Let  $v \in H$  and set  $z = Q_+^{-1}v \in D(A)$ . Then we have

$$\begin{aligned} \|v\|_m^2 &= \left\| \left( I + \frac{\tau}{2}B + \frac{\tau^2}{4}A \right) z \right\|_m^2 \\ &= \|z\|_m^2 + \left\| \left( \frac{\tau}{2}B + \frac{\tau^2}{4}A \right) z \right\|_m^2 + 2m\left(\left(\frac{\tau}{2}B + \frac{\tau^2}{4}A\right)z, z\right) \\ &= \left( 1 - \frac{\tau^2}{2}\alpha_G - \tau\beta_{qm} \right) \|z\|_m^2 + \left\| \left( \frac{\tau}{2}B + \frac{\tau^2}{4}A \right) z \right\|_m^2 \\ &\quad + 2\frac{\tau}{2}m\left(\left(B + \beta_{qm}I\right)z, z\right) + 2\frac{\tau^2}{4}m\left(\left(A + \alpha_G I\right)z, z\right) \\ &\geq \left\| \left( \frac{\tau}{2}B + \frac{\tau^2}{4}A \right) Q_+^{-1}v \right\|_m^2 + \frac{\tau^2}{2}\|Q_+^{-1}v\|_{\tilde{a}}^2 \end{aligned}$$

due to the quasi-monotonicity of  $b$  (3b) and (13). This immediately yields (14a) and (14b). The bound (14c) can be shown similar to the bound for  $R_+^{-1}R_-$  in the proof of [11, Lemma 2.14].  $\square$

**Corollary 2.8** *The IMEX scheme is wellposed in  $D(A) \times H$ , i.e., for  $u^0 \in D(A)$  and  $v^0 \in H$  the numerical approximations satisfy*

$$u^n \in D(A), \quad v^n \in H, \quad v^{n+\frac{1}{2}} \in D(A), \quad n \geq 0.$$

**Proof** We use induction. The statement holds for  $n = 0$  by assumption, hence we assume that  $u^n \in D(A), v^n \in H$  for some  $n \geq 0$ . By Lemma 2.7,  $Q_+$  is invertible and (12) implies  $v^{n+\frac{1}{2}} \in D(A)$ . From (11b) and (11c) we then get  $u^{n+1} \in D(A), v^{n+1} \in H$ .  $\square$

### 2.4 Error bound for the IMEX scheme

We now state a second-order error bound for the IMEX scheme.

**Theorem 2.9** *Let the assumptions of Theorem 2.3 be satisfied,  $T < t^*(u^0, v^0)$ , and let the exact solution  $u$  of (6) satisfy  $u \in C^4([0, T], H) \cap C^3([0, T], V)$  and  $f(u) \in C^2([0, T], H)$ . Then for all  $\tau$  sufficiently small and all  $t_n < T$ , the approximations  $u^n$  from the IMEX scheme (11) are bounded by*

$$\|u^n\|_{\tilde{\alpha}} \leq \rho := 2\|u\|_{L^\infty([0, T]; V)}.$$

Moreover,  $u^n, v^n$  satisfy the error bound

$$\|u^n - u(t_n)\|_{\tilde{\alpha}} + \|v^n - u'(t_n)\|_m \leq C e^{Mt_n} E(u) \tau^2$$

with  $M = c_{\text{qm}} + \frac{(1+(3/2)^{1/2})L_{T,\rho}}{1-(1+(3/2)^{1/2})L_{T,\rho}\tau}$ ,  $c_{\text{qm}} = \frac{1}{2}\alpha_G C_{\text{emb}} + \beta_{\text{qm}}$ ,

$$E = E(u) = \|u^{(4)}\|_{L^\infty([0, T]; H)} + \|u^{(3)}\|_{L^\infty([0, T]; V)} + \left\| \frac{d}{dt}(f(u)) \right\|_{L^\infty([0, t_n]; H)} + \left\| \frac{d^2}{dt^2}(f(u)) \right\|_{L^\infty([0, t_n]; H)},$$

and a constant  $C$  that only depends on  $T$  but is independent of  $\tau, L$ , and  $u$ .

Since the proof works with the same arguments as the more complicated proof of Theorem 3.3 for the full discretization error of the IMEX scheme, we do not present it here, cf., also Remark 3.4 a).

### 2.5 The IMEX scheme in first-order formulation

For the error analysis we rewrite the IMEX scheme (11) as a perturbation of the one-step formulation of the CN scheme (8b). A similar idea was used in [14] for analyzing the leapfrog scheme and locally implicit schemes for Maxwell equations.

The formulation (8c) of the Crank–Nicolson scheme can be used to prove stability. For the IMEX scheme we now derive a similar formulation.

**Lemma 2.10** (a) *We have the following representations of  $R_+^{-1}$  and  $R$ :*

$$R_+^{-1} = \begin{bmatrix} Q_+^{-1} (I + \frac{\tau}{2} B) & \frac{\tau}{2} Q_+^{-1} \\ -\frac{2}{\tau} + \frac{2}{\tau} Q_+^{-1} (I + \frac{\tau}{2} B) & Q_+^{-1} \end{bmatrix}, \tag{15a}$$

$$R = \begin{bmatrix} -I + Q_+^{-1} (2I + \tau B) & \tau Q_+^{-1} \\ -\frac{4}{\tau} I + \frac{4}{\tau} Q_+^{-1} (4I + 2\tau B) & Q_- Q_+^{-1} \end{bmatrix}. \tag{15b}$$

(b) *For all  $w \in V$  we have*

$$R_+^{-1} \begin{bmatrix} w \\ -Bw \end{bmatrix} = \begin{bmatrix} Q_+^{-1} w \\ -(B + \frac{\tau}{2} A) Q_+^{-1} w \end{bmatrix}. \tag{15c}$$

(c) *The IMEX scheme (11) is equivalent to*

$$x^{n+1} = Rx^n + \frac{\tau}{2} R_+^{-1} (g^n + g^{n+1}) + \frac{\tau^2}{4} \left[ - (B + \frac{\tau}{2} A) Q_+^{-1} (f^n - f^{n+1}) \right]. \tag{16}$$

**Proof** First note that the right-hand side of (15a) is a well-defined mapping from  $X$  to  $D(S)$  by Lemma 2.7. The identities (15) can be verified by straightforward calculations.

(c) To make the following calculations well defined, we assume for the moment that  $v^n, v^{n+1} \in V$ . We eliminate  $v^{n+\frac{1}{2}}$  from the scheme (11), by subtracting (11c) from (11a). This yields

$$v^{n+\frac{1}{2}} = \frac{1}{2} (v^n + v^{n+1}) + \frac{\tau}{4} (f^n - f^{n+1}), \tag{17}$$

which differs from the Crank–Nicolson scheme by the contributions of the nonlinearity  $f$ , cf. (10). Note that we have  $f^n - f^{n+1} \in V$  since, by Corollary 2.8,  $v^n, v^{n+1} \in V$  and  $v^{n+\frac{1}{2}} \in D(A)$ . Inserting (17) into (11b) gives

$$u^{n+1} = u^n + \frac{\tau}{2} (v^n + v^{n+1}) + \frac{\tau^2}{4} (f^n - f^{n+1}). \tag{18}$$

On the other hand, by adding (11a) and (11c) and inserting (11b) and (17) we get

$$v^{n+1} = v^n - \frac{\tau}{2}A(u^n + u^{n+1}) - \frac{\tau}{2}B(v^n + v^{n+1}) + \frac{\tau}{2}(f^n + f^{n+1}) - \frac{\tau^2}{4}B(f^n - f^{n+1}). \tag{19}$$

With the definition (8b) of  $R_{\pm}$  we can express (18) and (19) in first-order formulation as

$$R_+x^{n+1} = R_-x^n + \frac{\tau}{2}(g^n + g^{n+1}) + \frac{\tau^2}{4}\begin{bmatrix} f^n - f^{n+1} \\ -B(f^n - f^{n+1}) \end{bmatrix}.$$

Multiplying by  $R_+^{-1}$  and using (15c) shows that the IMEX scheme is equivalent to (16) under the additional assumption  $v^n, v^{n+1} \in V$ . Since both formulations are also well defined for  $v^n, v^{n+1} \in H$  and since  $V$  is dense in  $H$ , we also get their equivalence for  $v^n, v^{n+1} \in H$ . □

### 3 Full discretization

In this section we combine the IMEX scheme with an abstract space discretization to obtain a fully discrete scheme. We use the framework introduced in [12] for linear equations and extended in [13] to the semilinear case. It is rather general and allows one to cover conforming as well as nonconforming space discretizations, the latter being relevant for the discretization of equations with dynamic boundary conditions.

#### 3.1 Framework

Let  $(V_h)_h$  be a family of finite dimensional vector spaces for the spatial approximation related to a discretization parameter  $h$ , e.g., the mesh width of a spatial grid.

For all  $V_h \in (V_h)_h$  we consider the following discretization of (1): find  $u_h : [0, T] \rightarrow V_h$  s.t.

$$m_h(u_h'', \varphi_h) + b_h(u_h', \varphi_h) + a_h(u_h, \varphi_h) = m_h(f_h(t, u_h), \varphi_h) \quad \forall t \in (0, T], \varphi_h \in V_h, \tag{20}$$

$$u_h(0) = u_h^0, \quad u_h'(0) = v_h^0.$$

Here,  $m_h, a_h, b_h, f_h, u_h^0$  and  $v_h^0$  are approximations of their corresponding continuous counterparts and satisfy similar properties as in Assumption 2.1.

**Assumption 3.1** In the following statements, all constants are independent of  $h$ .

- (a) The bilinear form  $m_h$  is a scalar product on  $V_h$  and we denote  $V_h$  equipped with this scalar product by  $H_h$ .
- (b) The bilinear form  $a_h$  is symmetric and there exists a constant  $\widehat{\alpha}_G \geq 0$  s.t.

$$\widetilde{a}_h = a_h + \widehat{\alpha}_G m_h$$

is a scalar product on  $V_h$ . In the following we equip  $V_h$  with  $\tilde{a}_h$ .

- (c) There exists a constant  $\widehat{C}_{\text{emb}} > 0$  s.t.  $\|v_h\|_{m_h} \leq \widehat{C}_{\text{emb}}\|v_h\|_{\tilde{a}_h}$  for all  $v_h \in V_h$ .
- (d) The bilinear form  $b_h: V_h \times H_h \rightarrow \mathbb{R}$  is continuous and there exists a  $\widehat{\beta}_{\text{qm}} \geq 0$  s.t.

$$b_h(v, v) + \widehat{\beta}_{\text{qm}}\|v\|_{m_h}^2 \geq 0 \quad \text{for all } v_h \in V_h.$$

- (e) The discrete nonlinearity  $f_h: [0, T] \times V_h \rightarrow H_h$  is locally Lipschitz continuous on  $V_h$  with constant  $\widehat{L}_M$ , analogously to (4).

To reformulate (20) as an evolution equation on  $V_h$ , we define  $A_h, B_h \in \mathcal{L}(V_h; V_h)$  via

$$m_h(A_h v_h, \varphi_h) = a_h(v_h, \varphi_h), \quad m_h(B_h v_h, \varphi_h) = b_h(v_h, \varphi_h) \quad \text{for all } v_h, \varphi_h \in V_h.$$

Then (20) is equivalent to

$$\begin{aligned} u_h''(t) + B_h u_h'(t) + A_h u_h(t) &= f_h(t, u_h(t)), \\ u_h(0) = u_h^0, \quad u_h'(0) &= v_h^0. \end{aligned} \tag{21}$$

Analogously to the continuous case we can rewrite this in a first-order formulation. With the Hilbert space  $X_h = V_h \times H_h$  and

$$x_h = \begin{bmatrix} u_h \\ v_h \end{bmatrix}, \quad S_h = \begin{bmatrix} 0 & -I \\ A_h & B_h \end{bmatrix}, \quad g_h(t, x_h(t)) = \begin{bmatrix} 0 \\ f_h(t, u_h(t)) \end{bmatrix},$$

(21) is equivalent to

$$\begin{aligned} x_h'(t) + S_h x_h(t) &= g_h(t, x_h(t)), \quad t \in (0, T], \\ x_h(0) &= x_h^0. \end{aligned} \tag{22}$$

The IMEX scheme (11) applied to (21), reads

$$v_h^{n+\frac{1}{2}} = v_h^n - \frac{\tau}{2} A_h u_h^n - \frac{\tau^2}{4} A_h v_h^{n+\frac{1}{2}} - \frac{\tau}{2} B_h v_h^{n+\frac{1}{2}} + \frac{\tau}{2} f_h^n, \tag{23a}$$

$$u_h^{n+1} = u_h^n + \tau v_h^{n+\frac{1}{2}}, \tag{23b}$$

$$v_h^{n+1} = v_h^{n+\frac{1}{2}} - \frac{\tau}{2} A_h u_h^n - \frac{\tau^2}{4} A_h v_h^{n+\frac{1}{2}} - \frac{\tau}{2} B_h v_h^{n+\frac{1}{2}} + \frac{\tau}{2} f_h^{n+1}, \tag{23c}$$

where we used the short notation  $f_h^n := f_h(t_n, u_h^n)$ . As in the continuous case, (23c) can be replaced by the more efficient update

$$v_h^{n+1} = -v_h^n + 2v_h^{n+\frac{1}{2}} + \frac{\tau}{2} (f_h^{n+1} - f_h^n). \tag{23d}$$

### 3.2 Error analysis

We first introduce some notation that is required for the unified error analysis presented in [12,13].

To relate the discrete and the continuous solution we assume that there exists a lift operator  $\mathcal{L}_h^V \in \mathcal{L}(V_h; V)$  which satisfies

$$\|\mathcal{L}_h^V v_h\|_m \leq C_H \|v_h\|_{m_h}, \quad \|\mathcal{L}_h^V v_h\|_{\tilde{a}} \leq C_V \|v_h\|_{\tilde{a}_h}, \tag{24}$$

for all  $v_h \in V_h$  with constants  $C_H, C_V > 0$  which are independent of  $h$ .

The adjoints

$$\mathcal{L}_h^{H*}: H \rightarrow V_h \quad \text{and} \quad \mathcal{L}_h^{V*}: V \rightarrow V_h$$

of  $\mathcal{L}_h^V$  play an important role in the error analysis. They are defined via

$$\begin{aligned} m_h(\mathcal{L}_h^{H*} v, w_h) &= m(v, \mathcal{L}_h^V w_h) && \text{for all } v \in H, w_h \in H_h, \\ \tilde{a}_h(\mathcal{L}_h^{V*} v, w_h) &= \tilde{a}(v, \mathcal{L}_h^V w_h) && \text{for all } v \in V, w_h \in V_h. \end{aligned}$$

The corresponding first-order operators  $\mathcal{L}_h: X_h \rightarrow X$  and  $\mathcal{L}_h^*: X \rightarrow X_h$  are defined as

$$\mathcal{L}_h \begin{bmatrix} v_h \\ w_h \end{bmatrix} = \begin{bmatrix} \mathcal{L}_h^V v_h \\ \mathcal{L}_h^V w_h \end{bmatrix}, \quad \mathcal{L}_h^* \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} \mathcal{L}_h^{V*} v \\ \mathcal{L}_h^{H*} w \end{bmatrix}.$$

Let  $Z^V \xrightarrow{d} V$  be a dense subspace and  $I_h \in \mathcal{L}(Z^V; V_h)$  be an interpolation operator satisfying

$$\|I_h\|_{H_h \leftarrow Z^V} \leq \widehat{C}_I$$

with  $\widehat{C}_I > 0$  independent of  $h$ . We define  $Z = V \times Z^V \xrightarrow{d} X$  and the first-order reference operator  $J_h: Z \rightarrow X_h$  by

$$J_h \begin{bmatrix} v \\ w \end{bmatrix} := \begin{bmatrix} \mathcal{L}_h^{V*} v \\ I_h w \end{bmatrix}.$$

Furthermore, for  $v_h, w_h \in V_h$ , the errors in the scalar products are defined via.

$$\begin{aligned} \Delta m(v_h, w_h) &:= m(\mathcal{L}_h^V v_h, \mathcal{L}_h^V w_h) - m_h(v_h, w_h), \\ \Delta \tilde{a}(v_h, w_h) &:= \tilde{a}(\mathcal{L}_h^V v_h, \mathcal{L}_h^V w_h) - \tilde{a}_h(v_h, w_h). \end{aligned}$$

For  $z = \begin{bmatrix} v \\ w \end{bmatrix} \in Z$ , the discretization errors in the linear operator and the nonlinearity are given by the following remainder terms:

$$R_h z = (\mathcal{L}_h^* S - S_h J_h) z = \begin{bmatrix} -(\mathcal{L}_h^{V*} - I_h)w \\ \mathcal{L}_h^{H*}(Av + Bw) - (A_h \mathcal{L}_h^{V*} v + B_h I_h w) \end{bmatrix}, \tag{R1}$$

$$r_h(t, z) = \mathcal{L}_h^* g(t, z) - g_h(t, J_h z) = \begin{bmatrix} 0 \\ \mathcal{L}_h^{H*} f(t, v) - f_h(t, \mathcal{L}_h^{V*} v) \end{bmatrix}. \tag{R2}$$

Analogously to the continuous case we define the following operators:

$$\widehat{Q}_\pm := I \pm \frac{\tau}{2} B_h \pm \frac{\tau^2}{4} A_h : V_h \rightarrow V_h, \tag{25a}$$

$$\widehat{R}_\pm := I \pm \frac{\tau}{2} S_h : X_h \rightarrow X_h, \tag{25b}$$

$$\widehat{R} := \widehat{R}_+^{-1} \widehat{R}_-. \tag{25c}$$

Since the setting is the same as in the continuous case with constants independent of  $h$ , Lemmas 2.4 and 2.7 transfer directly to the discrete case with the continuous constants replaced by the discrete ones.

Our analysis relies on the following regularity assumptions.

**Assumption 3.2** Let the assumptions of Theorem 2.3 be satisfied and let  $u$  be the classical solution of (6) that satisfies additionally

$$u \in C^4([0, T], H) \cap C^3([0, T], V) \cap C^2([0, T], Z^V) \quad \text{and} \quad f(u) \in C^2([0, T], H)$$

for a  $T < t^*(u^0, v^0)$ .

We now state our main results. First we present an abstract error bound depending on the radius of a ball containing the exact and the numerical solution. Here, this radius depends on  $\tau$  and  $h$ . Afterwards, in Corollary 3.5, we show that under additional consistency assumptions for the space discretization and for sufficiently small  $\tau$  and  $h$ , the fully discrete approximations  $u_h^n$  are bounded in terms of the exact solution only.

**Theorem 3.3** *Let Assumptions 3.1 and 3.2 be satisfied. For  $T$  given in Assumption 3.2 we set*

$$\rho = \max \left\{ C_V \|u\|_{L^\infty([0, T]; V)}, \max_{t_n \leq T} \|u_h^n\|_{\widetilde{\alpha}_h} \right\}$$

and define  $\widehat{c}_{\text{qm}} = \widehat{\alpha}_G \widehat{C}_{\text{emb}}/2 + \widehat{\beta}_{\text{qm}}$ . If  $\tau$  satisfies the step-size restriction

$$\max \left\{ \tau(1 + (3/2)^{1/2}) \widehat{L}_{T, \rho}, \tau \frac{\widehat{c}_{\text{qm}}}{2}, \frac{\tau^2}{2} \widehat{\alpha}_G + \tau \widehat{\beta}_{\text{qm}} \right\} < 1,$$

then for all  $n > 0$  with  $t_n < T$ , the fully discrete approximations  $u_h^n, v_h^n$  given by the scheme (23) satisfy the error bound

$$\|\mathcal{L}_h^V u_h^n - u(t_n)\|_{\widetilde{\alpha}} + \|\mathcal{L}_h^V v_h^n - u'(t_n)\|_m \leq C e^{\widehat{M}t_n} \left( \sum_{i=0}^4 E_{h,i} + \tau^2 E \right) \tag{26}$$

with

$$\widehat{M} = \widehat{c}_{qm} + \frac{(1 + (3/2)^{1/2})\widehat{L}_{T,\rho}}{1 - (1 + (3/2)^{1/2})\widehat{L}_{T,\rho}\tau}$$

and a constant  $C$  that only depends on  $T$ , but which is independent of  $\tau, h, \widehat{L}$ , and  $u$ . The constants  $E_{h,i} = E_{h,i}(u)$  contain the abstract space discretization errors and are given by

$$\begin{aligned} E_{h,0} &= \|u_h^0 - \mathcal{L}_h^{V*} u^0\|_{\widehat{a}_h} + \|v_h^0 - I_h v^0\|_{m_h}, \\ E_{h,1} &= \|\mathcal{L}_h^{H*} f(\cdot, u) - f_h(\cdot, \mathcal{L}_h^{V*} u)\|_{L^\infty([0,T];H_h)}, \\ E_{h,2} &= \|(\mathbf{I} - \mathcal{L}_h^V I_h)u\|_{L^\infty([0,T];V)} + \|(\mathbf{I} - \mathcal{L}_h^V I_h)u'\|_{L^\infty([0,T];V)} + \|(\mathbf{I} - \mathcal{L}_h^V I_h)u''\|_{L^\infty([0,T];H)} \\ &\quad + \|(\mathbf{I} - \mathcal{L}_h^V I_h)f(\cdot, u)\|_{L^\infty([0,T];V)}, \\ E_{h,3} &= \left\| \max_{\|\varphi_h\|_{\widehat{a}_h}=1} \Delta\widetilde{a}(I_h u, \varphi_h) \right\|_{L^\infty(0,t)} + \left\| \max_{\|\psi_h\|_{m_h}=1} \Delta m(I_h u, \psi_h) \right\|_{L^\infty(0,t)} \\ &\quad + \left\| \max_{\|\varphi_h\|_{\widehat{a}_h}=1} \Delta\widetilde{a}(I_h u', \varphi_h) \right\|_{L^\infty(0,t)} + \left\| \max_{\|\psi_h\|_{m_h}=1} \Delta m(I_h u'', \psi_h) \right\|_{L^\infty(0,t)}, \\ E_{h,4} &= \left\| \max_{\|\psi_h\|_{m_h}=1} |b(u', \mathcal{L}_h^V \psi_h) - b_h(I_h u', \psi_h)| \right\|_{L^\infty(0,t)}, \end{aligned}$$

and  $E = E(u)$  is given in Theorem 2.9.

**Proof** All error terms arising from the space discretization can be expressed within the unified error analysis and were bounded against  $E_{h,i}, i = 0, \dots, 4$  in [12, Theorem 4.8], and [13, Theorem 3.9], respectively.

For the proof of the error bound (26), we use the first-order formulation of the IMEX scheme. We use the notation

$$x_h^n = \begin{bmatrix} u_h^n \\ v_h^n \end{bmatrix}, \quad \widetilde{x}^n = \begin{bmatrix} \widetilde{u}^n \\ \widetilde{v}^n \end{bmatrix} = \begin{bmatrix} u(t_n) \\ u'(t_n) \end{bmatrix}, \quad \widetilde{g}_h^n = g_h(t_n, J_h \widetilde{x}^n) = \begin{bmatrix} 0 \\ f_h(t_n, \mathcal{L}_h^{V*} \widetilde{u}^n) \end{bmatrix} = \begin{bmatrix} 0 \\ \widetilde{f}_h^n \end{bmatrix}.$$

The proof consists of four main steps.

(a) *Splitting of the error.* The error can be split via

$$\mathcal{L}_h x_h^n - \widetilde{x}^n = \mathcal{L}_h e_h^n + (\mathcal{L}_h J_h - \mathbf{I})\widetilde{x}^n, \quad \text{where } e_h^n = x_h^n - J_h \widetilde{x}^n.$$

Due to the continuity of the lift operator and [12, Theorem 4.8] we have

$$\|\mathcal{L}_h x_h^n - \widetilde{x}^n\|_X \leq C \|e_h^n\|_{X_h} + \|(\mathcal{L}_h J_h - \mathbf{I})\widetilde{x}^n\|_X \leq C (\|e_h^n\|_{X_h} + E_{h,3} + E_{h,2}). \tag{27}$$

Hence, it remains to bound the discrete error  $\|e_h^n\|_{X_h}$ .

(b) *Derivation of an error recursion for  $e_h^n$ .* Since the discrete operators share the properties of their continuous counterparts, we can rewrite the fully discrete scheme

(23) analogously to Lemma 2.10 as

$$x_h^{n+1} = \widehat{R}x_h^n + \frac{\tau}{2}\widehat{R}_+^{-1}(g_h^n + g_h^{n+1}) + \frac{\tau^2}{4} \left[ - (B_h + \frac{\tau}{2}A_h) \widehat{Q}_+^{-1}(f_h^n - f_h^{n+1}) \right]. \tag{28}$$

To derive a recursion for the discrete error, we insert  $J_h x$  into the fully discrete scheme (28) and obtain

$$J_h \tilde{x}^{n+1} = \widehat{R}J_h \tilde{x}^n + \frac{\tau}{2}\widehat{R}_+^{-1}(\tilde{g}_h^{n+1} + \tilde{g}_h^n) + \frac{\tau^2}{4} \left[ - (B_h + \frac{\tau}{2}A_h) \widehat{Q}_+^{-1}(\tilde{f}_h^n - \tilde{f}_h^{n+1}) \right] - \Delta_h^{n+1} \tag{29}$$

with a defect  $\Delta_h^{n+1}$  which is yet to be determined. We can interpret  $\Delta_h^{n+1}$  as a perturbation of the defect  $\Delta_{h,CN}^{n+1}$  of the fully discrete Crank–Nicolson scheme given by

$$J_h \tilde{x}^{n+1} = \widehat{R}J_h \tilde{x}^n + \frac{\tau}{2}\widehat{R}_+^{-1}(\tilde{g}_h^{n+1} + \tilde{g}_h^n) - \Delta_{h,CN}^{n+1}. \tag{30}$$

In fact we have

$$\Delta_{h,CN}^{n+1} = \Delta_h^{n+1} - \tilde{\delta}_h^{n+1}, \quad \tilde{\delta}_h^{n+1} = \frac{\tau^2}{4} \left[ - (B_h + \frac{\tau}{2}A_h) \widehat{Q}_+^{-1}(\tilde{f}_h^n - \tilde{f}_h^{n+1}) \right]. \tag{31}$$

A simple calculation shows that  $J_h x$  inserted in the Crank–Nicolson scheme satisfies

$$J_h \tilde{x}^{n+1} = J_h \tilde{x}^n + \frac{\tau}{2} \left( -S_h J_h (\tilde{x}^n + \tilde{x}^{n+1}) + \tilde{g}_h^n + \tilde{g}_h^{n+1} \right) - \delta_h^{n+1} - \mathcal{L}_h^* \delta_{CN}^{n+1}, \tag{32}$$

where

$$\delta_h^{n+1} = - (J_h - \mathcal{L}_h^*) (\tilde{x}^{n+1} - \tilde{x}^n) + \frac{\tau}{2} R_h (\tilde{x}^{n+1} + \tilde{x}^n) - \frac{\tau}{2} (r_h(t_{n+1}, \tilde{x}^{n+1}) + r_h(t_n, \tilde{x}^n)) \tag{33}$$

contains the abstract space discretization errors and

$$\delta_{CN}^{n+1} = \frac{\tau}{2} (x'(t_{n+1}) + x'(t_n)) - (\tilde{x}^{n+1} - \tilde{x}^n)$$

is the defect of the Crank–Nicolson scheme applied to the continuous equation (7). By applying  $\widehat{R}_+^{-1}$  to (32) we obtain with (30) and (31)

$$\Delta_h^{n+1} = \widehat{R}_+^{-1} \delta_h^{n+1} + \widehat{R}_+^{-1} \mathcal{L}_h^* \delta_{CN}^{n+1} + \tilde{\delta}_h^{n+1}. \tag{34}$$

Subtracting (29) from (28) yields the error recursion

$$\begin{aligned}
 e_h^{n+1} &= \widehat{R}e_h^n + \frac{\tau}{2}\widehat{R}_+^{-1}\left(g_h^{n+1} - \widetilde{g}_h^{n+1} + g_h^n - \widetilde{g}_h^n\right) \\
 &\quad + \frac{\tau^2}{4}\left[-(B_h + \frac{\tau}{2}A_h)\widehat{Q}_+^{-1}(f_h^n - \widetilde{f}_h^n - f_h^{n+1} + \widetilde{f}_h^{n+1})\right. \\
 &\quad \left. - (B_h + \frac{\tau}{2}A_h)\widehat{Q}_+^{-1}(f_h^n - \widetilde{f}_h^n - f_h^{n+1} + \widetilde{f}_h^{n+1})\right] + \Delta_h^{n+1}. \tag{35}
 \end{aligned}$$

(c) *Stability.* Solving (35) gives

$$\begin{aligned}
 e_h^n &= \widehat{R}^n e_h^0 + \sum_{m=1}^n \widehat{R}^{n-m} \left( \frac{\tau}{2} \widehat{R}_+^{-1} (g_h^m - \widetilde{g}_h^m + g_h^{m-1} - \widetilde{g}_h^{m-1}) \right. \\
 &\quad \left. + \frac{\tau^2}{4} \left[ - (B_h + \frac{\tau}{2} A_h) \widehat{Q}_+^{-1} (f_h^{m-1} - \widetilde{f}_h^{m-1} - f_h^m + \widetilde{f}_h^m) \right. \right. \\
 &\quad \left. \left. - (B_h + \frac{\tau}{2} A_h) \widehat{Q}_+^{-1} (f_h^{m-1} - \widetilde{f}_h^{m-1} - f_h^m + \widetilde{f}_h^m) \right] + \Delta_h^m \right).
 \end{aligned}$$

Taking the norm, using the triangle inequality, and  $\|\widehat{R}\|_{X_h \leftarrow X_h} \leq e^{\tau\widehat{c}_{qm}}$  yields

$$\begin{aligned}
 \|e_h^n\|_{X_h} &\leq \tau \sum_{m=1}^n e^{(n-m)\tau\widehat{c}_{qm}} \left( \frac{1}{2} \|\widehat{R}_+^{-1}(g_h^m - \widetilde{g}_h^m)\|_{X_h} + \frac{1}{2} \|\widehat{R}_+^{-1}(g_h^{m-1} - \widetilde{g}_h^{m-1})\|_{X_h} \right. \\
 &\quad + \frac{\tau}{4} \left\| \left[ - (B_h + \frac{\tau}{2} A_h) \widehat{Q}_+^{-1} (f_h^{m-1} - \widetilde{f}_h^{m-1}) \right. \right. \\
 &\quad \left. \left. - (B_h + \frac{\tau}{2} A_h) \widehat{Q}_+^{-1} (f_h^{m-1} - \widetilde{f}_h^{m-1}) \right] \right\|_{X_h} \\
 &\quad + \frac{\tau}{4} \left\| \left[ - (B_h + \frac{\tau}{2} A_h) \widehat{Q}_+^{-1} (f_h^m - \widetilde{f}_h^m) \right. \right. \\
 &\quad \left. \left. - (B_h + \frac{\tau}{2} A_h) \widehat{Q}_+^{-1} (f_h^m - \widetilde{f}_h^m) \right] \right\|_{X_h} \Bigg) \\
 &\quad + \left\| \widehat{R}^n e_h^0 + \sum_{m=1}^n \widehat{R}^{n-m} \Delta_h^m \right\|_{X_h}.
 \end{aligned}$$

We investigate the different terms in the first sum separately. For this we use the Lipschitz-continuity of the discrete nonlinearity and the bounds from Lemmas 2.7 and 2.4 for the discrete case. Note that by (38) we have  $\|\mathcal{L}_h^{V,*}u(t)\|_{\widetilde{a}_h} \leq \rho$  for all  $t \leq T$ . With  $\tau\widehat{c}_{qm} < 2$  and  $\|\widehat{R}_+^{-1}\|_{X_h \leftarrow X_h} \leq 1$ , we obtain:

$$\|\widehat{R}_+^{-1}(g_h^m - \widetilde{g}_h^m)\|_{X_h} \leq \widehat{L}_{T,\rho} \|e_h^m\|_{X_h}.$$

Using

$$\|\widehat{Q}_+^{-1}\|_{V_h \leftarrow H_h} \leq \frac{\sqrt{2}}{\tau} \quad \text{and} \quad \left\| (B_h + \frac{\tau}{2} A_h) \widehat{Q}_+^{-1} \right\|_{H_h \leftarrow H_h} \leq \frac{2}{\tau},$$

we have for  $\frac{\tau^2}{2}\widehat{\alpha}_G + \tau\widehat{\beta}_{qm} \leq 1$

$$\left\| \left[ - (B_h + \frac{\tau}{2} A_h) \widehat{Q}_+^{-1} (f_h^m - \widetilde{f}_h^m) \right. \right. \\
 \left. \left. - (B_h + \frac{\tau}{2} A_h) \widehat{Q}_+^{-1} (f_h^m - \widetilde{f}_h^m) \right] \right\|_{X_h} \leq \frac{\widehat{L}_{T,\rho}}{\tau} \sqrt{6} \|e_h^m\|_{X_h}.$$

With  $C_{3/2} = 1 + (3/2)^{1/2}$ , this yields

$$e^{-n\tau\widehat{c}_{\text{qm}}}\|e_h^n\|_{X_h} \leq C_{3/2}\widehat{L}_{T,\rho}\tau \sum_{m=1}^n e^{-m\tau\widehat{c}_{\text{qm}}}\|e_h^m\|_{X_h} + e^{-n\tau\widehat{c}_{\text{qm}}}\left\|\widehat{R}^n e_h^0 + \sum_{m=1}^n \widehat{R}^{n-m} \Delta_h^m\right\|_{X_h}.$$

By applying a discrete Gronwall Lemma, multiplying by  $e^{n\tau\widehat{c}_{\text{qm}}}$ , and inserting (34), we obtain for

$$\tau \leq \frac{1}{C_{3/2}L_{T,\rho}}$$

$$\begin{aligned} \|e_h^n\|_{X_h} &\leq e^{\frac{C_{3/2}\widehat{L}_{T,\rho}n\tau}{1-C_{3/2}\widehat{L}_{T,\rho}\tau}} \left( \left\| \widehat{R}^n e_h^0 + \sum_{m=1}^n \widehat{R}^{n-m} \left( \widehat{R}_+^{-1} \delta_h^m + \widehat{R}_+^{-1} \mathcal{L}_h^* \delta_{\text{CN}}^m + \widetilde{\delta}_h^m \right) \right\|_{X_h} \right) \\ &\leq e^{\frac{C_{3/2}\widehat{L}_{T,\rho}tn}{1-C_{3/2}\widehat{L}_{T,\rho}\tau}} \left( e^{n\tau\widehat{c}_{\text{qm}}} \left( \|e_h^0\|_{X_h} + \sum_{m=1}^n (\|\delta_h^m\|_{X_h} + \|\mathcal{L}_h^* \delta_{\text{CN}}^m\|_{X_h}) \right) \right. \\ &\quad \left. + \left\| \sum_{m=1}^n \widehat{R}^{n-m} \widetilde{\delta}_h^m \right\|_{X_h} \right). \end{aligned} \tag{36}$$

(d) *Defects.* The initial error  $e_h^0$  is given by the discretization errors of the initial values and bounded by

$$\|e_h^0\|_{X_h} \leq CE_{h,0}.$$

From (33) we obtain for the defect containing the space discretization errors

$$\|\delta_h^m\|_{X_h} = \tau \left\| \frac{1}{\tau} \int_{t_{m-1}}^{t_m} (J_h - \mathcal{L}_h^*) x'(s) \, ds + \frac{1}{2} R_h (\widetilde{x}^{n+1} + \widetilde{x}^n) - \frac{1}{2} (r_h(t_{n+1}, \widetilde{x}^{n+1}) + r_h(t_n, \widetilde{x}^n)) \right\|_{X_h}.$$

By [12, Theorem 4.8] and [13, Theorem 3.9] we have

$$\|\delta_h^m\|_{X_h} \leq C\tau \sum_{i=1}^4 E_{h,i}.$$

The Crank–Nicolson defect was bounded in [11, Lemma 2.15] by

$$\begin{aligned} \|\mathcal{L}_h^* \delta_{\text{CN}}^m\|_{X_h} &\leq C \|\delta_{\text{CN}}^m\|_X \leq C\tau^3 \|x^{(3)}\|_{L^\infty([t_m, t_{m-1}]; X)} \\ &\leq C\tau^3 \left( \|u^{(3)}\|_{L^\infty([t_m, t_{m-1}]; V)} + \|u^{(4)}\|_{L^\infty([t_m, t_{m-1}]; H)} \right) \\ &\leq C\tau^3 E. \end{aligned}$$

To bound the additional IMEX defect we split it into

$$\widetilde{\delta}_h^m = \frac{\tau^2}{4} \left[ - \left( B_h + \frac{\tau}{2} A_h \right) \widehat{Q}_+^{-1} (\widetilde{f}_h^{m-1} - \widetilde{f}_h^m) - \widehat{Q}_+^{-1} (\widetilde{f}_h^{m-1} - \widetilde{f}_h^m) \right] = \widetilde{\delta}_{h,1}^m + \widetilde{\delta}_{h,2}^m + \widetilde{\delta}_{h,3}^m$$

with

$$\begin{aligned} \tilde{\delta}_{h,1}^m &= \frac{\tau^2}{4} \left[ \widehat{Q}_+^{-1} (\tilde{f}_h^{m-1} - \mathcal{L}^{H*} \tilde{f}_h^{m-1} - \tilde{f}_h^m + \mathcal{L}^{H*} \tilde{f}_h^m) \right. \\ &\quad \left. - (B_h + \frac{\tau}{2} A_h) \widehat{Q}_+^{-1} (\tilde{f}_h^{m-1} - \mathcal{L}^{H*} \tilde{f}_h^{m-1} - \tilde{f}_h^m + \mathcal{L}^{H*} \tilde{f}_h^m) \right], \\ \tilde{\delta}_{h,2}^m &= \frac{\tau}{4} \left[ \tau \widehat{Q}_+^{-1} \mathcal{L}^{H*} (\tilde{f}_h^{m-1} - \tilde{f}_h^m) \right. \\ &\quad \left. - \widehat{Q}_- \widehat{Q}_+^{-1} \mathcal{L}^{H*} (\tilde{f}_h^{m-1} - \tilde{f}_h^m) \right], \\ \tilde{\delta}_{h,3}^m &= \frac{\tau}{4} \left[ \begin{matrix} 0 \\ -\mathcal{L}^{H*} (\tilde{f}_h^{m-1} - \tilde{f}_h^m) \end{matrix} \right], \end{aligned}$$

where we used the additional notation  $\tilde{f}^m = f(t_m, \tilde{u}^m)$ . The first term is bounded by  $\|\tilde{\delta}_{h,1}^m\|_{X_h} \leq C\tau E_{h,1}$ .

The terms  $\tilde{\delta}_{h,2}^m$  and  $\tilde{\delta}_{h,3}^m$  are only of order  $\tau^2$ , which is not sufficient to obtain a global error of order two. Moreover, a combination of both terms from two successive time steps allows to gain an additional factor of  $\tau$ . With the explicit representation of  $\widehat{R}$  analogous to that of  $R$  in (15b), we obtain

$$\tilde{\delta}_{h,2}^m + \widehat{R}\tilde{\delta}_{h,3}^{m-1} = \frac{\tau}{2} \left[ \begin{matrix} \frac{\tau}{2} \widehat{Q}_+^{-1} \mathcal{L}^{H*} (-\tilde{f}^{m-2} + 2\tilde{f}^{m-1} - \tilde{f}^m) \\ \frac{1}{2} \widehat{Q}_- \widehat{Q}_+^{-1} \mathcal{L}^{H*} (-\tilde{f}^{m-2} + 2\tilde{f}^{m-1} - \tilde{f}^m) \end{matrix} \right].$$

Using this together with the bound bounds for  $\widehat{Q}_+^{-1}$  and  $\widehat{Q}_- \widehat{Q}_+^{-1}$  from Lemma 2.7 and the continuity of the adjoint lift operator, leads to the bound

$$\begin{aligned} \|\tilde{\delta}_{h,2}^m + \widehat{R}\tilde{\delta}_{h,3}^{m-1}\|_{X_h} &\leq C\tau \|\mathcal{L}^{H*} (-\tilde{f}^{m-2} + 2\tilde{f}^{m-1} - \tilde{f}^m)\|_{m_h} \\ &\leq C\tau^3 \left\| \frac{d^2}{dt^2} (f(t, u(t))) \right\|_{L^\infty([t_{m-2}, t_m]; H)} \\ &\leq C\tau^3 E, \end{aligned}$$

and hence,

$$\begin{aligned} \left\| \sum_{m=1}^n \widehat{R}^{n-m} \tilde{\delta}_h^m \right\|_{X_h} &\leq C e^{n\tau\widehat{c}_{qm}} E_{h,1} + \left\| \widehat{R}^{n-1} \tilde{\delta}_{h,2}^1 + \tilde{\delta}_{h,3}^n + \sum_{m=2}^n \widehat{R}^{n-m} (\tilde{\delta}_{h,2}^m + \widehat{R}\tilde{\delta}_{h,3}^{m-1}) \right\|_{X_h} \\ &\leq e^{n\tau\widehat{c}_{qm}} \left( C E_{h,1} + \|\tilde{\delta}_{h,2}^1\|_{X_h} + \|\tilde{\delta}_{h,3}^n\|_{X_h} + \sum_{m=2}^n \|\tilde{\delta}_{h,2}^m + \widehat{R}\tilde{\delta}_{h,3}^{m-1}\|_{X_h} \right) \\ &\leq C e^{n\tau\widehat{c}_{qm}} \tau^2 (E_{h,1} + E). \end{aligned}$$

Inserting all bounds into (36) yields

$$\|e_h^n\|_{X_h} \leq C e^{\widehat{M}t_n} \left( E_{h,0} + \sum_{i=1}^4 E_{h,i} + \tau^2 E \right). \tag{37}$$

Finally, the error bound (26) follows from

$$\|\mathcal{L}_h^V u_h^n - u(t_n)\|_{\tilde{a}} + \|\mathcal{L}_h^V v_h^n - u'(t_n)\|_m \leq \sqrt{2} \|\mathcal{L}_h x_h^n - x(t_n)\|_X$$

and (27). □

**Remark 3.4**

- (a) Theorem 2.9 can be proven by replacing all discrete quantities by their continuous counterparts in the proof of Theorem 3.3. Since all assumptions in the discrete setting remain valid in the continuous case, the proof applies to the latter as well and all space discretization errors vanish.
- (b) The step-size restriction in Theorem 2.9 is not a CFL condition, since it only depends on constants that are independent of the mesh width  $h$ . Note that in the special case of a linear problem with  $\widehat{\alpha}_G = \widehat{\beta}_{qm} = 0$ , the scheme is unconditionally stable.
- (c) Theorem 3.3 and Corollary 3.5 are also valid for the Crank–Nicolson scheme. In this case, the error recursion (35) simplifies to

$$e_h^{n+1} = \widehat{R}e_h^n + \frac{\tau}{2} \widehat{R}_+^{-1} \left( g_h^{n+1} - \widetilde{g}_h^{n+1} + g_h^n - \widetilde{g}_h^n \right) + \delta_h^{n+1} + \widehat{R}_+^{-1} \mathcal{L}_h^* \delta_{CN}^{n+1}$$

and the error bound (26) holds with  $E_{CN}(u) = \|u^{(4)}\|_{L^\infty([0,T];H)} + \|u^{(3)}\|_{L^\infty([0,T];V)}$  (instead of  $E$ ) and  $1 + (3/2)^{1/2}$  replaced by 1 in the CFL condition and the error bound.

Under additional consistency assumptions for the space discretization, the following corollary states that for sufficiently small  $\tau$  and  $h$ , the fully discrete approximations are bounded in terms of the exact solution and converge.

**Corollary 3.5** *Let Assumptions 3.1 and 3.2 be satisfied.*

a) For  $T$  given in Assumption 3.2 define

$$\rho = 2C_V \|u\|_{L^\infty([0,T];V)}. \tag{38}$$

If  $E_{h,i} \xrightarrow{h \rightarrow 0} 0$  for  $i = 0, \dots, 4$ , then there exist  $h^* > 0$  and  $\tau^* > 0$  with  $\tau^*$  independent of  $h$  s.t. for all  $h < h^*$ ,  $\tau < \tau^*$  we have

$$\max_{t_n \leq T} \|u_h^n\|_{\tilde{a}_h} \leq \rho, \tag{39}$$

and the fully discrete solution converges, i.e.,

$$\max_{t_n \leq T} \{ \|\mathcal{L}_h^V u_h^n - u(t_n)\|_{\tilde{a}} + \|\mathcal{L}_h^V v_h^n - u'(t_n)\|_m \} \rightarrow 0, \quad \tau, h \rightarrow 0.$$

b) If additionally  $E_{h,i} \leq h^k$  for a  $k > 0$  and  $i = 0, \dots, 4$ , we obtain the error bound

$$\max_{t_n \leq T} \{ \|\mathcal{L}_h^V u_h^n - u(t_n)\|_{\tilde{a}} + \|\mathcal{L}_h^V v_h^n - u'(t_n)\|_m \} \leq C e^{\widehat{M}t_n} (\tau^2 + h^k) \tag{40}$$

with  $\widehat{M}$  defined in Theorem 3.3 and a constant  $C$  independent of  $\tau$  and  $h$ .

**Proof** It remains to prove the bound (39) for  $\tau$  and  $h$  sufficiently small, since all other assertions then follow immediately from Theorem 3.3.

Let  $f_h^\rho: [0, T] \times V_h \rightarrow H_h$  be a function that is globally Lipschitz-continuous on  $V_h$  with Lipschitz constant  $\widehat{L}_{T,\rho}$  and satisfies

$$f_h^\rho(t, v_h) = f_h(t, v_h) \text{ for all } t \in [0, T] \text{ and all } v_h \in V_h \text{ with } \|v_h\|_{\widetilde{a}_h} \leq \rho.$$

Further let  $u_h^{\rho,n}$  be the iterates of the IMEX scheme (23) with  $f_h$  replaced by  $f_h^\rho$ . Note that due to (38) we have

$$f_h(t, \mathcal{L}_h^{V*} u(t)) = f_h^\rho(t, \mathcal{L}_h^{V*} u(t)) \text{ for all } t \in [0, T].$$

Hence, as in the proof of Theorem 3.3, we obtain similar to the bound of the first component in (37)

$$\|u_h^{\rho,n} - \mathcal{L}_h^{V*} \widetilde{u}_h^n\|_{\widetilde{a}_h} \leq C e^{\widehat{M}t_n} \left( \sum_{i=0}^4 E_{h,i} + \tau^2 E \right)$$

for all  $t_n \leq T$ . Since  $E_{h,i} \xrightarrow{h \rightarrow 0} 0$ , and  $E$  is independent of  $h$  and  $\tau$ , we can choose  $h^*, \tau^* > 0$  s.t. for all  $h < h^*, \tau < \tau^*$  we have

$$\|u_h^{\rho,n} - \mathcal{L}_h^{V*} \widetilde{u}_h^n\|_{\widetilde{a}_h} \leq \frac{\rho}{2}$$

and hence together with (38)

$$\|u_h^{\rho,n}\|_{\widetilde{a}_h} \leq \|u_h^{\rho,n} - \mathcal{L}_h^{V*} \widetilde{u}_h^n\|_{\widetilde{a}_h} + \|\mathcal{L}_h^{V*} \widetilde{u}_h^n\|_{\widetilde{a}_h} \leq \frac{\rho}{2} + C_V \|\widetilde{u}_h^n\|_{\widetilde{a}} \leq \rho.$$

This implies, that for all  $t_n \leq T$  the iterates  $u_h^{\rho,n}$  coincide with the original iterates  $u_h^n$  and thereby  $\|u_h^n\|_{\widetilde{a}_h} = \|u_h^{\rho,n}\|_{\widetilde{a}_h} \leq \rho$ . □

#### 4 Application: semilinear wave equation with kinetic boundary conditions

In this section we consider the IMEX scheme applied to a finite element discretization of a semilinear acoustic wave equation with a kinetic boundary condition. Kinetic boundary conditions serve as an effective model for the interaction of waves with a boundary covered by a thin layer. A derivation can be found in, e.g., [10], and the wellposedness was proven in [21]. The space discretization we present in this section was analyzed in [11,12,19].

We show that this example fits into the abstract theory presented in the previous sections.

### 4.1 Formulation of the equations

Let  $\Omega \subset \mathbb{R}^d, d = 2, 3$ , be a bounded domain with smooth boundary  $\Gamma = \partial\Omega$ . With  $\Delta_\Gamma$  we denote the Laplace-Beltrami operator on  $\Gamma$  and with  $\mathbf{n}$  the outer normal vector.

We consider the following semilinear acoustic wave equations with kinetic boundary conditions. Seek  $u: [0, T] \times \overline{\Omega} \rightarrow \mathbb{R}$  satisfying

$$\begin{cases} u_{tt} + (\alpha_\Omega + \beta_\Omega \cdot \nabla) u_t - \Delta u = f_\Omega(t, \mathbf{x}, u), & \text{in } (0, T) \times \Omega, \\ u_{tt} + \partial_{\mathbf{n}} u + (\alpha_\Gamma + \beta_\Gamma \cdot \nabla_\Gamma) u_t - \Delta_\Gamma u = f_\Gamma(t, \mathbf{x}, u), & \text{in } (0, T) \times \partial\Omega, \\ u(0, \mathbf{x}) = u^0(\mathbf{x}), \quad u_t(0, \mathbf{x}) = v^0(\mathbf{x}), & \text{in } \overline{\Omega}, \end{cases} \quad (41)$$

where the nonlinearities and the coefficients satisfy the following conditions.

**Assumption 4.1** (a) The nonlinearities satisfy

$$f_\Omega \in C^1([0, T] \times \overline{\Omega} \times \mathbb{R}; \mathbb{R}), \quad f_\Gamma \in C^1([0, T] \times \Gamma \times \mathbb{R}; \mathbb{R}). \quad (42)$$

Moreover, they satisfy the following growth condition, that there exist

$$\zeta_\Omega \begin{cases} < \infty, & d = 2, \\ \leq \frac{d}{d-2}, & d \geq 3, \end{cases} \quad \text{and} \quad \zeta_\Gamma \begin{cases} < \infty, & d = 2, 3, \\ \leq \frac{d-1}{d-3}, & d \geq 4 \end{cases} \quad (43)$$

such that for all  $(t, \mathbf{x}, u) \in [0, T] \times \Omega \times \mathbb{R}$

$$|f_\Omega(t, \mathbf{x}, u)| \leq C(1 + |u|^{\zeta_\Omega}), \quad |\nabla f_\Omega(t, \mathbf{x}, u)| \leq C(1 + |u|^{\zeta_\Omega-1}), \quad (44)$$

and for all  $(t, \mathbf{x}, u) \in [0, T] \times \Gamma \times \mathbb{R}$

$$|f_\Gamma(t, \mathbf{x}, u)| \leq C(1 + |u|^{\zeta_\Gamma}), \quad |\nabla f_\Gamma(t, \mathbf{x}, u)| \leq C(1 + |u|^{\zeta_\Gamma-1})$$

hold true.

(b) The coefficients  $\alpha_\Omega \in C(\overline{\Omega}), \beta_\Omega \in C^1(\overline{\Omega})^d, \alpha_\Gamma \in C(\Gamma)$  and  $\beta_\Gamma \in C^1(\Gamma)^d$  are non-negative and satisfy

$$\alpha_\Omega - \frac{1}{2} \operatorname{div} \beta_\Omega \geq 0 \quad \text{in } \Omega, \quad \alpha_\Gamma + \frac{1}{2} (\beta_\Omega \cdot \mathbf{n} - \operatorname{div}_\Gamma \beta_\Gamma) \geq 0 \quad \text{on } \Gamma.$$

In [13] was shown that the weak formulation of (41) is of the form (2) with

$$\begin{aligned}
 H &= L^2(\Omega) \times L^2(\Gamma) \\
 V &= H^1(\Omega; \Gamma) := \{v \in H^1(\Omega) \mid \gamma(v) \in H^1(\Gamma)\} \subset H^1(\Omega) \times H^1(\Gamma), \\
 m(v, \varphi) &= \int_{\Omega} v\varphi \, dx + \int_{\Gamma} v\varphi \, ds, \\
 b(v, \varphi) &= \int_{\Omega} (\alpha_{\Omega}v + \beta_{\Omega} \cdot \nabla v) \varphi \, dx + \int_{\Gamma} (\alpha_{\Gamma}v + \beta_{\Gamma} \cdot \nabla_{\Gamma}v) \varphi \, ds, \\
 a(v, \varphi) &= \int_{\Omega} \nabla v \nabla \varphi \, dx + \int_{\Gamma} \nabla_{\Gamma}v \nabla_{\Gamma}\varphi \, ds,
 \end{aligned} \tag{45}$$

and  $f: [0, T] \times V \rightarrow H$  defined via

$$m(f(t, v), \varphi) = \int_{\Omega} (f_{\Omega}(t, \cdot, v(\cdot))) \varphi \, dx + \int_{\Gamma} (f_{\Gamma}(t, \cdot, v(\cdot))) \varphi \, ds.$$

Note that for the subset relation  $V \subset H$ , each element  $v \in V$  is identified with  $(v, \gamma(v)) \in H$ . Furthermore, Assumption 2.1 is satisfied and we have  $D(A) = H^2(\Omega; \Gamma)$ . Thus, Theorem 2.3 yields the existence of a solution  $u$  of (41).

### 4.2 Space discretization

As in [13], we use the bulk-surface finite element method presented in [7] to discretize (41) in space. This discretization was also considered in [11,12] for linear problems.

We start by giving a short summary of this method and refer to [7,11] for more details.

#### Bulk-surface finite element method

Let  $(\mathcal{T}_h)_h$  be a quasi-uniform family of consistent meshes of order  $p$  isoparametric elements with maximal mesh width  $h$ . For each  $\mathcal{T}_h \in (\mathcal{T}_h)_h$  the discretized domain and its boundary are denoted by

$$\Omega_h := \bigcup_{K \in \mathcal{T}_h} K \approx \Omega \quad \text{and} \quad \Gamma_h := \partial \Omega_h.$$

We define the bulk and the surface finite element space of order  $p \geq 1$  via

$$\begin{aligned}
 V_{h,p}^{\Omega} &:= \{v_h \in C(\Omega_h) \mid v_h|_K = \widehat{v}_h \circ (F_K)^{-1} \text{ with } \widehat{v}_h \in \mathbb{P}_p(\widehat{K}) \text{ for all } K \in \mathcal{T}_h\}, \\
 V_{h,p}^{\Gamma} &:= \{\vartheta_h \in C(\Gamma_h) \mid \vartheta_h = v_h|_{\Gamma_h} \text{ with } v_h \in V_{h,p}^{\Omega}\}.
 \end{aligned}$$

Here  $\mathbb{P}_p(\widehat{K})$  denotes the space of polynomials of degree  $p$  on a reference triangle  $\widehat{K}$  and  $F_K$  is a transformation from  $\widehat{K}$  to  $K$ . This discretization is nonconforming because

$\Omega_h \neq \Omega$ . In [7], an elementwise smooth homeomorphism  $G_h: \Omega_h \rightarrow \Omega$  with

$$G_h|_K \in C^{p+1}(K), \quad \text{for all } p \leq k \text{ and } K \in \mathcal{T}_h$$

is constructed. This allows us to define lifted versions of  $v_h \in V_{h,p}^\Omega$  and  $\vartheta_h \in V_{h,p}^\Gamma$  as

$$v_h^\ell := v_h \circ G_h^{-1} \quad \text{and} \quad \vartheta_h^\ell := \vartheta_h \circ G_h^{-1}. \tag{46}$$

The mapping  $G_h$  is constructed in such a way, that  $G_h(a_i) = a_i, i = 1, \dots, N = \dim V_h$ , where  $a_1, \dots, a_N \in \Omega_h$  are the nodes corresponding to the finite element discretization. This implies  $v_h^\ell(a_i) = v_h(a_i)$  for  $i = 1, \dots, N$  and for all  $v_h \in V_{h,p}^\Omega$ .

By  $I_{h,\Omega}: C(\overline{\Omega}) \rightarrow V_{h,p}^\Omega$  and  $I_{h,\Gamma}: C(\Gamma) \rightarrow V_{h,p}^\Gamma$  we denote the nodal interpolation operator in  $\Omega$  and on  $\Gamma$ , respectively. By construction, the nodes on the surface coincide with the bulk nodes and therefore we have

$$\gamma(I_{h,\Omega} v) = I_{h,\Gamma} \gamma(v) \quad \text{for all } v \in C(\overline{\Omega}).$$

*The semidiscrete equation*

We now present the space discretization in the framework of Sect. 3. As finite element space we choose  $V_h = V_{h,p}^\Omega$ . Furthermore we set  $I_h := I_{h,\Omega}|_{Z^V}: Z^V \rightarrow V_h$ , where

$$Z^V := D(A) = H^2(\Omega; \Gamma) \xrightarrow{d} V = H^1(\Omega; \Gamma), \tag{47}$$

and define the lift operator via

$$\mathcal{L}_h^V v := v^\ell$$

with  $v^\ell$  given in (46). The spatial discretization of (41) can then be written as (20) where the discretizations  $m_h, b_h, a_h: V_h \times V_h \rightarrow \mathbb{R}$  of  $m, b$ , and  $a$  are defined via

$$\begin{aligned} m_h(v_h, \varphi_h) &:= \int_{\Omega_h} v_h \varphi_h \, dx + \int_{\Gamma_h} v_h \varphi_h \, ds, \\ b_h(v_h, \varphi_h) &:= \int_{\Omega_h} (I_{h,\Omega} \alpha_\Omega v_h + I_{h,\Omega} \beta_\Omega \cdot \nabla v_h) \varphi_h \, dx \\ &\quad + \int_{\Gamma_h} (I_{h,\Gamma} \alpha_\Gamma v_h + I_{h,\Gamma} \beta_\Gamma \cdot \nabla_\Gamma v_h) \varphi_h \, ds, \\ a_h(v_h, \varphi_h) &:= \int_{\Omega_h} \nabla v_h \nabla \varphi_h \, dx + \int_{\Gamma_h} \nabla_{\Gamma_h} u_h \nabla_{\Gamma_h} \varphi_h \, ds. \end{aligned}$$

The discretized nonlinearity  $f_h: [0, T] \times V_h \rightarrow H_h$  is given by

$$\begin{aligned} m_h(f_h(t, v_h), \varphi_h) &:= \int_{\Omega_h} I_{h,\Omega} f_\Omega(t, \cdot, v_h^\ell(\cdot))(\mathbf{x}) \varphi_h(\mathbf{x}) \, dx \\ &\quad + \int_{\Gamma_h} I_{h,\Gamma} f_\Gamma(t, \cdot, v_h^\ell(\cdot))(\mathbf{x}) \varphi_h(\mathbf{x}) \, ds \end{aligned} \tag{48}$$

for all  $\varphi_h \in V_h$ .

In [11,13] it was shown, that Assumption 3.1 is satisfied.

**Remark 4.2** The nodal interpolation only requires function evaluations in the nodes  $a_1, \dots, a_N$ . Since they are invariant under the lift operator, the computation of  $v_h^\ell$  is not necessary. It is only needed for the definition of  $f_h$  since the interpolation operator acts on functions defined on  $\Omega$ .

### 4.3 Full discretization error bound

We now state an error bound for the full discretization of (41) with the bulk-surface finite element method and the IMEX scheme (23).

**Corollary 4.3** *Let  $1 \leq p \leq k$  and  $\Gamma \in C^{k+1}$ . Furthermore let Assumption 4.1 be satisfied and let  $u$  be a solution of (41) on  $[0, T]$  satisfying*

$$\begin{aligned} u &\in C^4([0, T]; L^2(\Omega) \times L^2(\Gamma)) \cap C^3([0, T]; H^1(\Omega; \Gamma)) \cap C^2([0, T]; H^2(\Omega; \Gamma)), \\ u, u' &\in L^\infty([0, T]; H^{p+1}(\Omega; \Gamma)), \\ u'' &\in L^\infty([0, T]; H^p(\Omega; \Gamma)), \\ f_\Omega(t, \cdot, u(t, \cdot)) &\in L^\infty([0, T]; H^{\max\{2, p\}}(\Omega)), \\ f_\Gamma(t, \cdot, u(t, \cdot)) &\in L^\infty([0, T]; H^{\max\{2, p\}}(\Gamma)). \end{aligned}$$

Then there exist  $\tau^*, h^*, \rho > 0$  s.t. for all  $0 < h < h^*, 0 < \tau < \tau^*$ , and  $t_n \leq T$ , the approximations  $u_h^n$  and  $v_h^n$  given by (23) with bulk-surface elements of order  $p$ , satisfy

$$\|(u_h^n)^\ell - u(t_n)\|_{H^1(\Omega; \Gamma)} + \|(v_h^n)^\ell - u'(t_n)\|_{L^2(\Omega) \times L^2(\Gamma)} \leq C e^{\left(\frac{1}{2} + \frac{(1+(3/2)^{1/2})\widehat{L}_{T,\rho}}{1-(1+(3/2)^{1/2})\widehat{L}_{T,\rho}\tau}\right)t_n} (h^p + \tau^2) \tag{49}$$

with a constant  $C$  independent of  $\tau$  and  $h$ . The Lipschitz constant of the discretized nonlinearity is given by

$$\widehat{L}_{T,\rho} = C \left( \sigma(\Omega)^{\frac{\zeta_\Omega - 1}{2\zeta_\Omega}} + \sigma(\Gamma)^{\frac{\zeta_\Gamma - 1}{2\zeta_\Gamma}} + 2\rho^{\zeta_\Omega - 1} + 2\rho^{\zeta_\Gamma - 1} \right), \tag{50}$$

where  $\sigma(\Omega)$  and  $\sigma(\Gamma)$  denote the measure of  $\Omega$  and  $\Gamma$ , respectively, and  $\zeta_\Gamma$  and  $\zeta_\Omega$  are given in Assumption 4.1.

**Proof** In [13] it was shown, that Assumptions 2.1 and 3.1 are satisfied with  $\widehat{C}_{\text{emb}} = \widehat{\alpha}_G = 1, \widehat{\beta}_{\text{qm}} = 0$ , and  $\widehat{L}_{T,\rho}$  given in (50). The regularity assumptions on  $u$  are such that  $u \in C^4([0, T], H) \cap C^3([0, T], V) \cap C^2([0, T], Z^V)$ , cf. (45) and (47). Since additionally  $Z^V = D(A) = H^2(\Omega; \Gamma)$ , we have that  $Au \in C^2([0, T]; H)$  and hence  $f(u) = u'' + Bu' + Au \in C^2([0, T]; H)$ .

Thus, also Assumption 3.2 is satisfied, and we can apply Corollary 3.5.

Under the above assumptions, in [12,13] it was shown that the space discretization error terms are bounded by  $E_{h,i} \leq Ch^p$ . So the bound (49) follows then directly by (40). □

### 4.4 Implementation

In the following numerical experiments we compare the IMEX scheme with the Crank–Nicolson and the explicit classical Runge–Kutta scheme of order 4 applied to the space discretized wave equation with kinetic boundary conditions. For the implementation we used the C++ finite element-library `deal.II` (version 9.2) [2,5]. The codes to reproduce the experiments are available on <https://doi.org/10.5445/IR/1000127003>. We ran the experiments on a computer with an i5 processor (3.5 GHz) and 16 GB RAM.

To comment on the implementation we first introduce some additional notation. For a finite element function  $u_h \in V_h$  we denote by  $\mathbf{u} \in \mathbb{R}^N$ , the corresponding coefficient vector in the finite element basis. Furthermore,  $\mathbf{M} \in \mathbb{R}^{N \times N}$  is the mass matrix,  $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{N \times N}$  are the stiffness matrices related to  $A_h$  and  $B_h$ , respectively, and  $\mathbf{f}^n \in \mathbb{R}^N$  denotes the load vector corresponding to  $f_h^n = f_h(t_n, u^n)$ ,  $n \in \mathbb{N}$ .

*IMEX scheme*

The fully discrete IMEX scheme (23) reads

$$\mathbf{M}\mathbf{v}^{n+\frac{1}{2}} = \mathbf{M}\mathbf{v}^n - \frac{\tau}{2}\mathbf{A}\mathbf{u}^n - \frac{\tau^2}{4}\mathbf{A}\mathbf{v}^{n+\frac{1}{2}} - \frac{\tau}{2}\mathbf{B}\mathbf{v}^{n+\frac{1}{2}} + \frac{\tau}{2}\mathbf{f}^n, \tag{51a}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \tau\mathbf{v}^{n+\frac{1}{2}}, \tag{51b}$$

$$\mathbf{M}\mathbf{v}^{n+1} = -\mathbf{M}\mathbf{v}^n + 2\mathbf{M}\mathbf{v}^{n+\frac{1}{2}} + \frac{\tau}{2}(\mathbf{f}^{n+1} - \mathbf{f}^n). \tag{51c}$$

The linear system in (51a) has the form

$$\mathbf{Q}_+\mathbf{v}^{n+\frac{1}{2}} = \mathbf{M}\mathbf{v}^n - \frac{\tau}{2}\mathbf{A}\mathbf{u}^n + \frac{\tau}{2}\mathbf{f}^n, \quad \mathbf{Q}_+ = \mathbf{M} + \frac{\tau}{2}\mathbf{B} + \frac{\tau^2}{4}\mathbf{A}.$$

We solve this system with the GMRES solver provided by `deal.II` and either a sparse incomplete LU or a geometric multigrid preconditioner. For the measure of the error in the GMRES iterations, the residual  $r$  with corresponding coefficient vector  $\mathbf{r}$  is used. A suitable stopping criteria would be

$$\|r\|_{\tilde{a}_h} \leq \tau^2 \text{tol},$$

where  $\text{tol}$  is a given tolerance, since then in (51b) the error in  $\mathbf{u}^{n+1}$  caused by the solution of the linear system in the  $\|\cdot\|_{\tilde{a}_h}$  norm is of order  $\tau^3$ , which corresponds to the local error of the IMEX scheme. In practice, the computation of  $\|r\|_{\tilde{a}_h}$  is quite expensive and we thus used the stopping criterion

$$\|r\|_{h,2} = \|\mathbf{r}\|_{h,2} \leq \tau^2 \text{tol},$$

in a grid dependent scaled Euclidean norm  $\|\cdot\|_{h,2} = h^{d/2}\|\cdot\|_2$ . This is much more efficient, since this norm is available within the GMRES code at no additional cost. The criterion worked well in our numerical experiments as we will show in Sect. 4.5.

We always use  $\text{tol} = 0.01$  in our numerical examples, which is chosen s.t. the errors in solving the linear systems do not destroy the overall order of convergence.

Note that in the IMEX scheme (23), only  $\mathbf{M}\mathbf{v}^{n+1}$  is required so that we neither compute nor store  $\mathbf{v}^{n+1}$  itself.

*Crank–Nicolson scheme*

The fully discrete Crank–Nicolson scheme can be written in the form

$$\mathbf{Q}_+\mathbf{u}^{n+1} - \frac{\tau^2}{4}\mathbf{f}^{n+1} = (\mathbf{M} + \frac{\tau}{2}\mathbf{B} - \frac{\tau^2}{4}\mathbf{A})\mathbf{u}^n + \tau\mathbf{M}\mathbf{v}^n + \frac{\tau^2}{4}\mathbf{f}^n, \tag{52a}$$

$$\mathbf{M}\mathbf{v}^{n+1} = \mathbf{M}\mathbf{v}^n - \frac{\tau}{2}\mathbf{A}(\mathbf{u}^n + \mathbf{u}^{n+1}) + \mathbf{B}(\mathbf{u}^n - \mathbf{u}^{n+1}) + \frac{\tau}{2}(\mathbf{f}^n + \mathbf{f}^{n+1}). \tag{52b}$$

We solve the nonlinear equation (52a) with a simplified Newton method where we use  $\mathbf{Q}_+$  as an approximation to the Jacobian. The linear equations are solved as in the IMEX scheme. We stop the Newton scheme when the update  $\Delta u$  satisfies  $\|\Delta u\|_{h,2} \leq \tau^3 \text{tol}$  with a given tolerance  $\text{tol}$ . In the numerical examples we use  $\text{tol} = 0.1$ , which is again chosen s.t. the Newton errors do not destroy the overall order of convergence. All matrix vector products appearing in (52a) and (52b) are computed only ones and saved in temporary vectors, as well as all terms that can be reused in the next time step. As in the IMEX scheme we do not compute  $\mathbf{v}^{n+1}$  but only  $\mathbf{M}\mathbf{v}^{n+1}$ .

*Classical Runge–Kutta scheme*

The classical Runge–Kutta scheme is an explicit scheme of order four that is suited for hyperbolic problems because its stability region contains an interval on the imaginary axis. This is in contrast to the second-order schemes by Heun and Runge, which intersect with the imaginary axis in the origin only. We implemented it using mass lumping to obtain a fully explicit scheme. Note that the space discretization with mass lumping also fits into the setting of Sect. 3, as it was shown in [12] for a linear acoustic wave equation.

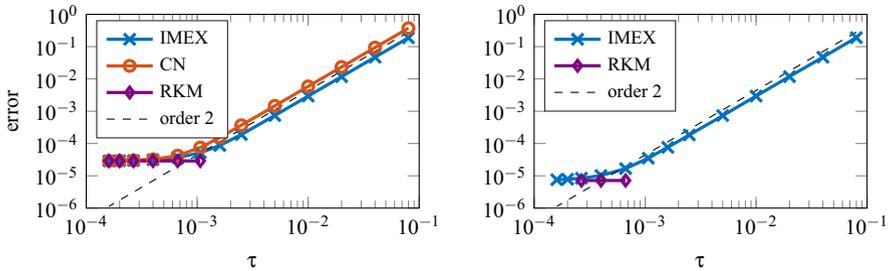
**4.5 Numerical examples**

We consider the semilinear wave equation with kinetic boundary conditions (41) with  $\alpha_\Omega = 1, \beta_\Omega(\mathbf{x}) = \mathbf{x}$  and  $\alpha_\Gamma = \beta_\Gamma = 0$  on the unit disc  $\Omega = B(0, 1) \subset \mathbb{R}^2$ . As nonlinearities, we choose

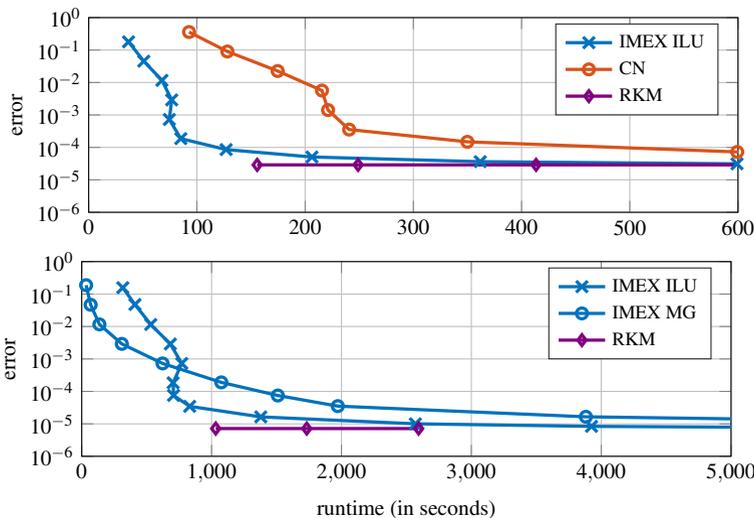
$$\begin{aligned} f_\Omega(t, \mathbf{x}, u) &= |u|u + \eta_\Omega(t, \mathbf{x}), \\ f_\Gamma(t, \mathbf{x}, u) &= |u|^2u + \eta_\Gamma(t, \mathbf{x}) \end{aligned}$$

with

$$\begin{aligned} \eta_\Omega(t, \mathbf{x}) &= -\left(4\pi^2 + |\sin(2\pi t)\mathbf{x}_1\mathbf{x}_2|\right) \sin(2\pi t)\mathbf{x}_1\mathbf{x}_2 + 6\pi \cos(2\pi t)\mathbf{x}_1\mathbf{x}_2, \\ \eta_\Gamma(t, \mathbf{x}) &= -4\pi^2 \sin(2\pi t)\mathbf{x}_1\mathbf{x}_2 + 6 \sin(2\pi t)\mathbf{x}_1\mathbf{x}_2 - (\sin(2\pi t)\mathbf{x}_1\mathbf{x}_2)^3, \end{aligned}$$



**Fig. 1** Error  $E_h(0.8)$  of the IMEX scheme (solved with GMRES and ILU preconditioner), the Crank–Nicolson scheme, and the classical Runge–Kutta method plotted against step size  $\tau$  for coarse space discretization (328,193 degrees of freedom, left) and fine space discretization (18,3118,745 degrees of freedom, right)



**Fig. 2** Error  $E_h(0.8)$  of the IMEX scheme, solved with GMRES and ILU/Multigrid(MG, F-cycle with 8 levels) preconditioner, the Crank–Nicolson scheme, and the classical Runge–Kutta method plotted against runtime for coarse space discretization (3288,193 degrees of freedom, top) and fine space discretization (18,3118,745 degrees of freedom, bottom)

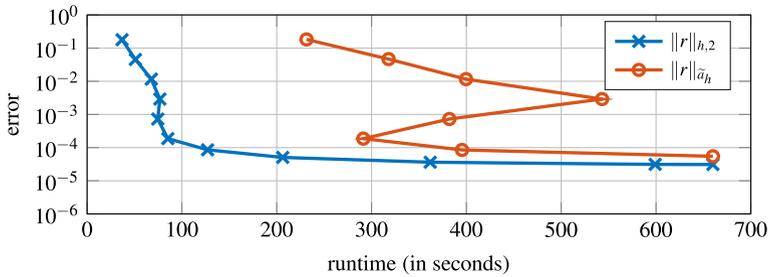
and as initial values

$$u(0, \mathbf{x}) = 0, \quad u_t(0, \mathbf{x}) = 2\pi \mathbf{x}_1 \mathbf{x}_2.$$

The example is chosen such that the exact solution is given by

$$u(t, \mathbf{x}) = \sin(2\pi t) \mathbf{x}_1 \mathbf{x}_2$$

which allows us to compute reliable errors. For the space discretization we use isoparametric elements of order  $p = 2$ , and choose  $u_h^0 = I_{h,\Omega} u^0$  and  $v_h^0 = I_{h,\Omega} v^0$  as discrete initial values.



**Fig. 3** Error  $E_h(0.8)$  of the IMEX scheme plotted against the runtime when using the two different error estimates as stopping criteria for the GMRES scheme as discussed in Sect. 4.4 for a coarse space discretization (3288,193 degrees of freedom)

As we cannot compute the lift of a finite element function exactly, we consider the error

$$E_h(t) := \|u_h(t) - u(t)\|_{\Omega_h} \|_{H^1(\Omega_h; \Gamma_h)} + \|u'_h(t) - u'(t)\|_{\Omega_h} \|_{L^2(\Omega_h) \times L^2(\Gamma_h)}.$$

We evaluate the integrals with a quadrature rule of order 4 such that the quadrature error is negligible. The restriction of  $u$  to  $\Omega_h$  is possible since for convex domains we have  $\Omega_h \subset \Omega$ .

In Fig. 1 the errors of the IMEX, the Crank–Nicolson, and the classical Runge–Kutta scheme are plotted against the time-step size  $\tau$  for a coarse ( $h \approx 0.014$ ) and a fine ( $h \approx 0.007$ ) space discretization, respectively. As predicted by Corollary 1, the IMEX and the Crank–Nicolson scheme converge with order two until the error of the space discretization is reached. The explicit Runge–Kutta scheme is only stable under a strong CFL condition and then the error reaches immediately the space discretization error plateau.

Figure 2 shows the errors of the different schemes plotted against the runtime for the same coarse and fine space discretization as in Fig. 1. It can be observed, that the IMEX scheme is significantly faster than the Crank–Nicolson scheme. For errors of the magnitude of the space discretization error plateau, the classical Runge–Kutta scheme is more efficient than the IMEX scheme, but the IMEX scheme outperforms the Runge–Kutta scheme if less accuracy is sufficient. The Runge–Kutta method has the disadvantage that the stability limit in applications is not exactly known, and therefore there is a risk that it will not be stable if a too large time-step size is chosen, or the effort is unnecessarily high if the time-step size is too small. For the large system obtained by the fine space discretization and large time-step sizes, it can be observed, that the use of the multigrid preconditioner is quite efficient.

Finally, Fig. 3 shows a comparison of the runtimes of the IMEX scheme when using the different stopping criteria for the GMRES solver discussed in Sect. 4.4, namely using  $\|r\|_{\tilde{a}_h}$  or  $\|r\|_{h,2}$  as estimate for the error, respectively. It can be seen that the afford of computing the (better suited)  $\|r\|_{\tilde{a}_h}$  is too high and does not pay off.

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