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Biharmonic nonlinear scalar field equations

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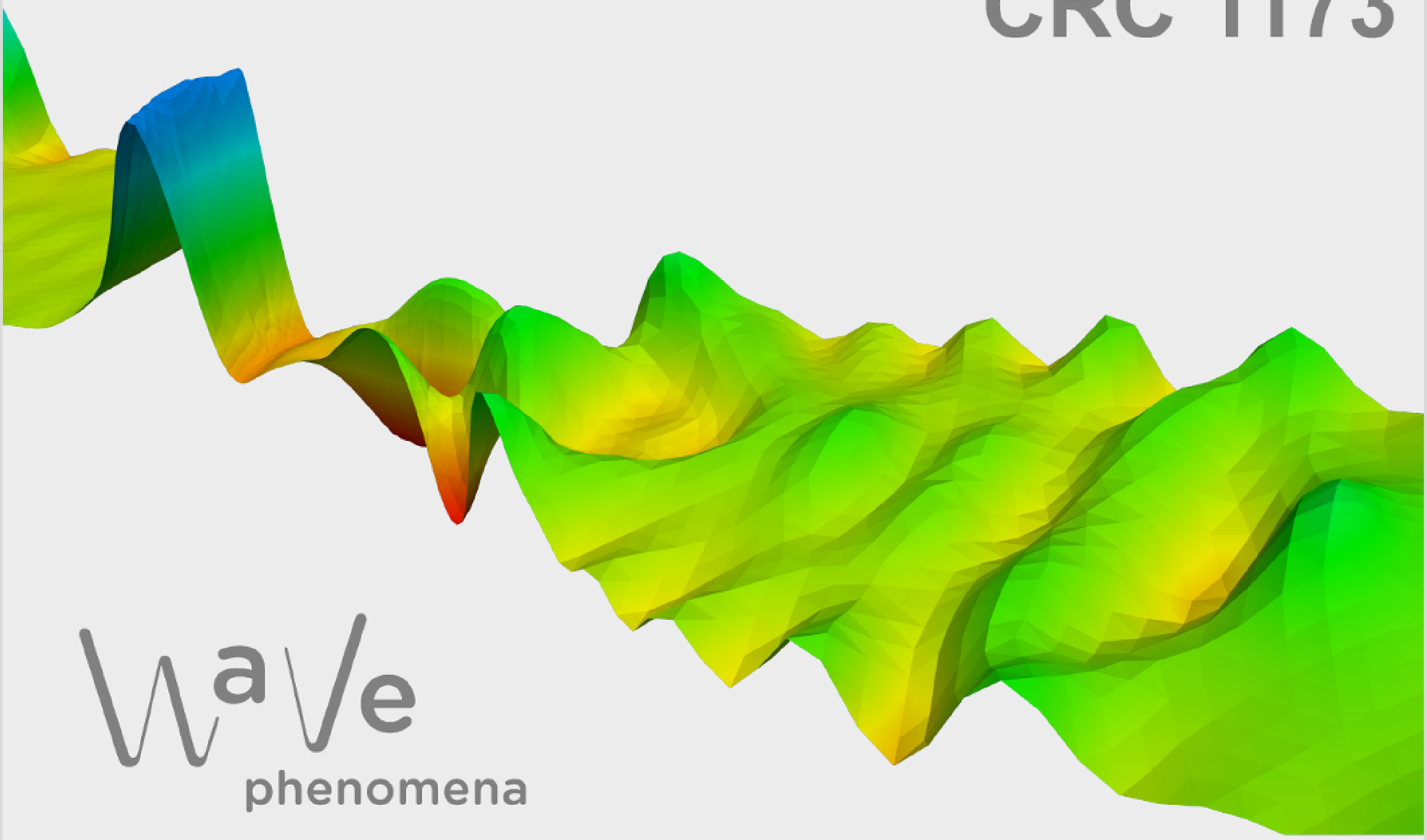

Wave phenomena

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Biharmonic nonlinear scalar field equations

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ABSTRACT. We prove a Brezis-Kato-type regularity result for weak solutions to the biharmonic nonlinear equation

$$\Delta^2 u = g(x,u) \qquad \text{in } \mathbb{R}^N$$

with a Carathéodory function $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $N \geq 5$. The regularity results give rise to the existence of ground state solutions provided that g has a general subcritical growth at infinity. We also conceive a new biharmonic logarithmic Sobolev inequality

$$\int_{\mathbb{R}^N} |u|^2 \log |u| \, dx \leq \frac{N}{8} \log \left(C \int_{\mathbb{R}^N} |\Delta u|^2 \, dx \right), \quad \text{for } u \in H^2(\mathbb{R}^N),\ \int_{\mathbb{R}^N} u^2 \, dx = 1,$$

for a constant $0 < C < \left(\frac{2}{\pi e N}\right)^2$ and we characterize its minimizers.

1. Introduction

The study of higher-order differential elliptic operators is important, e.g. in nonlinear elasticity [3], low Reynolds number hydrodynamics, in structural engineering [21, 24] as well as in nonlinear optics [11], and has attracted attention from the mathematical point of view [12]. The methods developed for the second order problem, e.g. involving the Laplacian $-\Delta$, may no longer be available. For instance, it is the well-known that the bi-Laplacian $(-\Delta)^2 = \Delta^2$ cannot be studied by means of some classical methods such as maximum principles, Polya-Szegő inequalities, or even if $(\Delta u)^2 \in L^1(\mathbb{R}^N)$, then it is possible that $\Delta|u| \notin L^1_{loc}(\mathbb{R}^N)$.

The first aim is of this work is to establish a regularity result in the spirit of Brezis-Kato [6] of weak solutions to

$$\Delta^2 u = g(x,u), \quad x \in \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a domain, $N \geq 2$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. If we suppose that Ω is bounded, then there is an extensive literature devoted to this problem. Namely, recall that if $g(x,u) = f(x)$, then Agmon, Douglis, Nirenberg [2] showed that for $1 < q < \infty$, $f \in L^q(\Omega)$, there exists a unique strong solution $u \in W^{2,2}_0(\Omega) \cap W^{4,q}(\Omega)$ to (1.1) provided that $\partial\Omega \in C^4$ see also [12, Corollary 2.21] and references therein. Recently Mayboroda and Maz'ya [17] showed L^∞-estimates of u (resp. ∇u), where $f \in C^\infty_0(\Omega)$, Ω is an arbitrary bounded domain and $N = 4, 5$ (resp. $N = 2, 3$). To the best of our knowledge, a variant of Brezis-Kato result [6] for (1.1) is known only on a bounded domain in a particular case. Namely, Van der Vorst [25] showed that, if $N \geq 5$, $g(x,u) = a(x)u$ and $a(x) \in L^{N/4}(\Omega)$, then any weak solution $u \in W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)$ to (1.1) satisfies $u \in L^q(\Omega)$ for all $1 \leq q < \infty$. This result is suitable to show the regularity for the biharmonic equation with the nonlinearities of the special form $g(x,u) = f(u)u$ cf. [25, Lemma B3]. In this paper we give a full answer to the problem on an arbitrary domain and for general g with the adequate Brezis-Kato growth as we shall see below.

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From now on we assume that $\Omega \subset \mathbb{R}^N$ possibly unbounded domain and $N \geq 5$. Inspired by [6], we impose on g the following growth assumption:

$$|g(x,s)| \leq a(x)\big(1+|s|\big), \quad \text{for } s \in \mathbb{R} \text{ and a.e. } x \in \Omega, \text{ where } 0 \leq a \in L^{N/4}_{\text{loc}}(\Omega). \tag{1.2}$$

The first main result reads as follows.

THEOREM 1.1. *Let $u \in W^{2,2}_{loc}(\Omega)$ be a weak solution to (1.1), where g satisfies (1.2). Then $u \in C^{3,\alpha}_{loc}(\Omega) \cap W^{4,q}_{loc}(\Omega)$, for any $0 < \alpha < 1$ and $1 \leq q < \infty$.*

It is worth mentioning that in proof of Theorem 1.1 we can no longer apply classical techniques for Laplacian, e.g. due to Brezis and Kato [6], or Brezis and Lieb [7, Theorem 2.3], since $\Delta|u|$ may not be well-defined for $u \in W^{2,2}_{\text{loc}}(\Omega)$. Moreover, the Moser iteration technique does not seem to be applicable straightforwardly for g.

We shall present some consequences of Theorem 1.1 in $\Omega = \mathbb{R}^N$. Let us define $\mathcal{D}^{2,2}(\mathbb{R}^N)$ as a completion of the space $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{\mathcal{D}^{2,2}} := \Big(\sum_{|\alpha|=2} \|\partial^\alpha u\|^2_{L^2(\mathbb{R}^N)}\Big)^{\frac{1}{2}}$. By the use of the Fourier transform and the Plancharel theorem we find a constant $c > 0$ such that, for $u \in C_0^\infty(\mathbb{R}^N)$,

$$\frac{1}{c}\|u\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \leq \|\Delta u\|_{L^2(\mathbb{R}^N)} \leq c\|u\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)}.$$

Therefore, the norms $\|u\| := \|\Delta u\|_{L^2(\mathbb{R}^N)}$ and $\|u\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)}$ are equivalent on $\mathcal{D}^{2,2}(\mathbb{R}^N)$. Moreover, $\mathcal{D}^{2,2}(\mathbb{R}^N)$ is a Hilbert space with the inner product

$$\langle u, v\rangle := \int_{\mathbb{R}^N} \Delta u \Delta v \, dx \qquad \text{for } u,\ v \in \mathcal{D}^{2,2}(\mathbb{R}^N)$$

and $u \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ is a *weak solution* to (1.1) provided that

$$\langle u, v\rangle = \int_{\mathbb{R}^N} g(x,u)v \qquad \text{for any } v \in C_0^\infty(\mathbb{R}^N).$$

As usually expected, the following general Pohožaev-type result holds, cf. [23].

THEOREM 1.2. *Let $u \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ be a weak solution to (1.1), where g satisfies (1.2). Then*

$$\int_{\mathbb{R}^N} |\Delta u|^2 \, dx = \frac{2N}{N-4}\int_{\mathbb{R}^N} G(x,u)\, dx + \frac{2}{N-4}\int_{\mathbb{R}^N} x \cdot \partial_x G(x,u)\, dx. \tag{1.3}$$

provided that $G(x,u)$, $x \cdot \partial_x G(x,u) \in L^1(\mathbb{R}^N)$, where $G(x,s) := \int_0^s g(x,t)\, dt$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$.

We demonstrate that the Brezis-Kato result for biharmonic Laplacean as well as Theorem 1.2 open the way to study the existence of solutions and their regularity for (1.1). Indeed, let us assume that g is independent of x and the following condition holds:

$(g0)$ there is a constant $c > 0$ such that $|g(s)| \leq c\big(1+|s|^{2^{**}-1}\big)$ for $s \in \mathbb{R}$,

where $2^{**} := \frac{2N}{N-4}$. Then $a(x) := g(u(x))/(1+|u(x)|) \in L^{N/4}_{loc}(\mathbb{R}^N)$ for $u \in L^{2^{**}}(\mathbb{R}^N)$ and in view of Theorem 1.1, weak solutions to the semilinear problem (1.1) belong to $C^{3,\alpha}_{\text{loc}}(\mathbb{R}^N) \cap W^{4,q}_{\text{loc}}(\mathbb{R}^N)$. We introduce the energy functional

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2 - \int_{\mathbb{R}^N} G(u)\, dx, \tag{1.4}$$

where $G(s) = \int_0^s g(t)\, dt$. Next, we show the existence of weak solutions to (1.1) under growth assumption at 0 and at infinity inspired by a seminal paper due to Berestycki and Lions [5] (cf. [19, 20]). We assume that g is continuous, $g(0) = 0$ and $(g0)$ holds. Let

$$G_+(s) := \begin{cases} \int_0^s \max\{g(t), 0\}\, dt & \text{for } s \geq 0, \\ \int_s^0 \max\{-g(t), 0\}\, dt & \text{for } s < 0, \end{cases}$$

and $g_+(s) = G'_+(s)$. Suppose in addition, that and the following conditions are satisfied:

$(g1)$ $\lim_{s\to 0} G_+(s)/|s|^{2^{**}} = 0$,
$(g2)$ there exists $\xi_0 > 0$ such that $G(\xi_0) > 0$,
$(g3)$ $\lim_{|s|\to\infty} G_+(s)/|s|^{2^{**}} = 0$.

We introduce the Pohožaev manifold

$$\mathcal{M} := \Big\{u \in \mathcal{D}^{2,2}(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} |\Delta u|^2 = 2^{**} \int_{\mathbb{R}^N} G(u)\, dx\Big\}, \tag{1.5}$$

and in view of Theorem 1.2, $\mathcal{M}$ contains all nontrivial solutions. The existence result reads as follows.

THEOREM 1.3. *Let $(g0)$–$(g3)$ be satisfied. Then $\inf_{\mathcal{M}} J > 0$ and there is a ground state solution $u_0 \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ to (1.1), i.e. $u_0 \in \mathcal{M}$ solves (1.1) and $J(u_0) = \inf_{\mathcal{M}} J$. Moreover $u_0 \in C^{3,\alpha}_{loc}(\mathbb{R}^N) \cap W^{4,q}_{loc}(\mathbb{R}^N)$, for any $0 < \alpha < 1$ and $1 \le q < \infty$.*

Theorem 1.3 enables us to consider the following nonlinearity

$$G(s) = s^2 \log|s| \quad \text{for } s \neq 0, \text{ and } G(0) = 0 \tag{1.6}$$

which satisfies $(g0)$–$(g3)$. In view of Theorem 1.3 there is a ground state solution to (1.1) and

$$C_{N,log} := 2^{**}\Big(\frac{1}{2} - \frac{1}{2^{**}}\Big)^{-\frac{4}{N-4}} (\inf_{\mathcal{M}} J)^{\frac{4}{N-4}}.$$

We gain the following new *biharmonic logarithmic Sobolev inequality*.

THEOREM 1.4. *For any $u \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} |u|^2\, dx = 1$, there holds*

$$\frac{N}{8} \log\left(\left(\frac{8e}{C_{N,log}(N-4)}\right)^{(N-4)/N} \int_{\mathbb{R}^N} |\Delta u|^2\, dx\right) \ge \int_{\mathbb{R}^N} |u|^2 \log|u|\, dx \tag{1.7}$$

and

$$\left(\frac{8e}{C_{N,log}(N-4)}\right)^{(N-4)/N} < \Big(\frac{2}{\pi e N}\Big)^2.$$

Moreover the equality in (1.7) holds provided that $u = u_0/\|u_0\|_{L^2(\mathbb{R}^N)}$ and u_0 is a ground state solution to (1.1). If the equality in (1.7) holds for u, then there are uniquely determined $\lambda > 0$ and $r > 0$ such that $u_0 := \lambda u(r\cdot) \in \mathcal{M}$ and u_0 is a ground state solution to (1.1).

Recall that the classical logarithmic Sobolev inequality given in [26]:

$$\frac{N}{4} \log\Big(\frac{2}{\pi e N} \int_{\mathbb{R}^N} |\nabla u|^2\, dx\Big) \ge \int_{\mathbb{R}^N} |u|^2 \log(|u|)\, dx, \quad \text{for } u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^2\, dx = 1, \tag{1.8}$$

which is equivalent to the Gross inequality [13], cf. [14]. Recall that the optimality of (1.8) and the characterization of minimizers have been already proved by Carlen [8] in the context of the Gross inequality as well as by del Pino and Dolbeault [9, 10] for the interpolated Gagliardo–Nirenberg inequalities and the L^p-Sobolev logarithmic inequality. A generalization of the optimal Gross inequality in Orlicz spaces is given by Adams [1]. However, to the best of our knowledge, the logarithmic Sobolev inequality for higher order operators have not been obtained in the literature so far and (1.8) seems to be the first one for the biharmonic Laplacian. Note that, in contrast to (1.8) and the Laplacian problem involving (1.6), we do not know ground state solutions to (1.1) explicitly. Hence the exact computation of $C_{N,log}$ remains an open question.

The paper is organized as follows. In Section 2 we prove Theorem 1.1 and in Section 3 we obtain the Pohožaev-type result. The main result of Section 4 is a general variant of Lion's lemma (Lemma 4.1) in $\mathcal{D}^{2,2}(\mathbb{R}^N)$, which is crucial for the proof of Theroem 1.3 given in Section 5. The last Section 6 is devoted to the biharmonic logarithmic Sobolev inequality.

2. Regularity theory and proof of Theorem 1.1

Let N, $k \in \mathbb{N}$ and $1 \leq p < \infty$ with $N > kp$. We define $\mathcal{D}^{k,p}(\mathbb{R}^N)$ as a completion of the space $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{k,p}} := \left(\sum_{|\alpha|=k} \|D^\alpha u\|^p_{L^p(\mathbb{R}^N)} \right)^{\frac{1}{p}}, \qquad u \in C_0^\infty(\mathbb{R}^N).$$

Hence

$$\mathcal{D}^{k,p}(\mathbb{R}^N) \subset \mathcal{D}^{k-l,\frac{Np}{N-lp}}(\mathbb{R}^N), \qquad 0 \leq l \leq k, \tag{2.1}$$

and

$$\sum_{j=0}^{k} \sum_{|\alpha|=k-j} \|D^\alpha u\|_{L^{\frac{Np}{N-jp}}(\mathbb{R}^N)} \leq c\|u\|_{\mathcal{D}^{k,p}}, \qquad u \in \mathcal{D}^{k,p}(\mathbb{R}^N). \tag{2.2}$$

We fix an open set $\Omega \subset \mathbb{R}^N$. We recall that by the standard approach based on mollifiers and the Calderon–Zygmund L^p–estimates for higher order elliptic operators [**22**, (2.6)] we have the following lemma.

Lemma 2.1. *Let $1 < p < \infty$ and k be a positive integer. If $w \in L^p_{loc}(\Omega)$ and $\Delta^k w \in L^p_{loc}(\Omega)$, then $w \in W^{2k,p}_{loc}(\Omega)$.*

Suppose that $u \in W^{2,2}_{\text{loc}}(\Omega)$ is a weak solution to (1.1), where g satisfies (1.2). Clearly $u \in L^{2^{**}}_{\text{loc}}(\Omega)$. Fix $U \subset\subset \Omega$. Since $\frac{2N}{N+4} < \frac{N}{4}$ and $\frac{2N}{N+4} = 2^{**}\frac{N-4}{N+4}$, by the Hölder inequality

$$\int_U |g(x,u)|^{\frac{2N}{N+4}}\, dx \leq c \int_U |a(x)|^{\frac{2N}{N+4}} + |a(x)|^{\frac{N}{4}\frac{8}{N+4}} |u|^{2^{**}\frac{N-4}{N+4}}\, dx < \infty,$$

for some constant $c > 0$. Then, by the distributional equality

$$\Delta^2 u = g(x,u) \in L^{\frac{2N}{N+4}}_{\text{loc}}(\Omega),$$

and Lemma 2.1, we infer that $u \in W^{4,\frac{2N}{N+4}}_{\text{loc}}(\Omega)$.

Now the crucial step is the following lemma.

Lemma 2.2. *Let $p \geq \frac{2N}{N+4}$ and $u \in W^{4,p}_{loc}(\Omega)$ be a weak solution to (1.1), where g satisfies (1.2). Then*

$$u \in \begin{cases} L^{Np/N-5p}_{loc}(\Omega), & \text{if } 5p < N,\\ L^q_{loc}(\Omega) \ \text{ for every } 1 \leq q < \infty, & \text{if } 5p \geq N.\end{cases}$$

Proof. If $4p \geq N$, then the conclusion follows immediately by the Sobolev embedding $W^{4,p}_{\text{loc}}(\Omega) \subset L^q_{\text{loc}}(\Omega)$, $q \geq 1$. Thus, we can clearly assume that $4p < N$. Let us define

$$\tilde{a}(x) := \begin{cases} \frac{g(x,u(x))}{u(x)} \chi_{\{x\in\Omega \,|\, |u(x)|>1\}}(x), & \text{for } u(x) \neq 0,\\ 0 & \text{for } u(x) = 0,\end{cases}$$
$$b(x) := g(x,u(x))\chi_{\{x\in\Omega \,|\, |u(x)|\leq 1\}}(x),$$

and observe that $g(x,u) = \tilde{a}(x)u + b(x)$ and $\tilde{a}, b \in L^{N/4}_{\text{loc}}(\Omega)$.

Let U be an arbitrary open bounded subset of Ω such that $U \subset \overline{U} \subset \Omega$. We find an open bounded V with C^∞-smooth boundary such that $\overline{U} \subset V \subset \overline{V} \subset \Omega$. Indeed, let $\xi \in C_0^\infty(\Omega)$ be a smooth cut-off function such that $\xi \equiv 1$ on U and $0 \leq \xi \leq 1$. By Sard's theorem, there is a regular value $c \in (0,1)$. Then $V = \xi^{-1}((c,1])$ is an open bounded subset with the smooth boundary $\partial V = \xi^{-1}(\{c\})$ satisfying $\overline{U} \subset V \subset \overline{V} \subset \Omega$.

Now take $\eta \in C_0^\infty(V)$ such that $\eta = 1$ on U and $0 \le \eta \le 1$. We restrict our problem to V. By the assumption $u \in W^{4,p}(V)$ is a distributional solution of

$$\Delta^2 u = \tilde{a}(x)u + b(x) \qquad \text{in } V \tag{2.3}$$

and $\tilde{a}, b \in L^{N/4}(V)$. We define

$$v := u\eta.$$

Certainly, we have $v \in W^{4,p}(V) \subset H^2(V)$ and $v \in H_0^1(V)$, since $\operatorname{supp}\eta \subset\subset V$. Standard calculations yield

$$\begin{aligned}\Delta^2 v &= (\Delta^2 u)\eta + \underbrace{4\nabla\Delta u\cdot\nabla\eta + 4\sum_{i=1^N}\nabla u_{x_i}\cdot\nabla\eta_{x_i} + 2\Delta u\Delta\eta + 4\nabla u\cdot\nabla\Delta\eta + u\Delta^2\eta}_{=:K(u)} \\ &=: (\Delta^2 u)\eta + K(u).\end{aligned} \tag{2.4}$$

Observe that $u \in W^{4,p}(V) \subset W^{3,p^*}(V)$, $p^* = \frac{Np}{N-p}$ and $\eta \in C_0^\infty(V)$ imply that

$$\|K(u)\|_{L^{p^*}(V)} \le c\|u\|_{W^{3,p^*}(V)}\|\eta\|_{W^{4,\infty}(V)} \le c(\eta)\|u\|_{W^{4,p}(V)}, \tag{2.5}$$

for some constant $c(\eta) > 0$.

In view of [**25**, Lemma B.2], for every $\varepsilon > 0$ there are $q_\varepsilon \in L^{N/4}(V)$ and $\widehat{f_\varepsilon} \in L^\infty(V)$ such that

$$\tilde{a}(x)v = q_\varepsilon(x)v + \widehat{f_\varepsilon}, \tag{2.6}$$

and

$$\|q_\varepsilon\|_{L^{N/4}(V)} \le \varepsilon. \tag{2.7}$$

By (2.4), (2.3) and (2.6) we get

$$\begin{aligned}\Delta^2 v &= (\Delta^2 u)\eta + K(u) \\ &= \tilde{a}(x)v + b(x)\eta + K(u) \\ &= q_\varepsilon(x)v + f_\varepsilon + K(u),\end{aligned} \tag{2.8}$$

where

$$f_\varepsilon := \widehat{f_\varepsilon} + b(x)\eta \in L^{\frac{N}{4}}(V). \tag{2.9}$$

We recall some needed regularity results from [**2**] (see also [**12**, Thm 2.20]), for all $1 < q < \infty$, $\bar{g} \in L^q(V)$, there exists a unique strong solution $u \in W^{4,q}(V)$ to the problem

$$\begin{cases}(-\Delta)^2 u = \bar{g} & \text{in } V,\\ u = \Delta u = 0 & \text{on } \partial V.\end{cases}$$

satisfying

$$\|u\|_{W^{4,q}(V)} \le c_q\|\bar{g}\|_{L^q(V)},$$

where $c_q > 0$ depends only on N, q and V. Denote by T_q the linear operator $g \mapsto u$ considered as an operator from $L^q(V)$ to $W^{4,q}(V)$ and rewrite the above inequality as

$$\|T_q\bar{g}\|_{W^{4,q}(V)} \le c_q\|\bar{g}\|_{L^q(V)}. \tag{2.10}$$

Obviously, T_q is the L^q-inverse of the bilaplacian $(-\Delta)^2$ considered with the Navier boundary conditions $u = \Delta u = 0$ on ∂V.

Now we can rephrase (2.8) in the language of operators

$$v - A_{\varepsilon,q}v = h_{\varepsilon,q}, \tag{2.11}$$

where $A_{\varepsilon,q}v := T_q(q_\varepsilon v)$ and $h_{\varepsilon,q} := T_q(f_\varepsilon + K(u))$.

We consider two cases separately.

Case I: $5p < N$**.**

In what follows we take $q = p^*$. By the Sobolev embedding $W^{4,p^*}(V) \subset L^{\frac{Np}{N-5p}}(V)$, (2.10), (2.9) and (2.5), we have

$$\begin{aligned}\|h_{\varepsilon,p^*}\|_{L^{\frac{Np}{N-5p}}(V)} &\le c_{\text{Sobolev}}\|T_{p^*}(f_\varepsilon + K(u))\|_{W^{4,p^*}(V)}\\ &\le c_{\text{Sobolev}}c_{p^*}\|f_\varepsilon + K(u)\|_{L^{p^*}(V)}\\ &\le c\left(\|f_\varepsilon\|_{L^{\frac{N}{4}}(V)} + \|K(u)\|_{L^{p^*}(V)}\right)\\ &\le c\left(\|f_\varepsilon\|_{L^{\frac{N}{4}}(V)} + c(\eta)\|u\|_{W^{4,p}(V)}\right),\end{aligned} \tag{2.12}$$

where $c > 0$ is some constant. We estimate the norm of the linear operator $A_{\varepsilon,p^*} : L^{\frac{Np}{N-5p}}(V) \to L^{\frac{Np}{N-5p}}(V)$ applying the Sobolev embedding $W^{4,p^*}(V) \subset L^{\frac{Np}{N-5p}}(V)$ and (2.10)

$$\|A_{\varepsilon,p^*}v\|_{L^{\frac{Np}{N-5p}}(V)} \le c_{\text{Sobolev}}\|T_{p^*}(q_\varepsilon v)\|_{W^{4,p^*}(V)} \le c_{\text{Sobolev}}c_{p^*}\|q_\varepsilon v\|_{L^{p^*}(V)}. \tag{2.13}$$

We use the Hölder inequality with the exponents

$$\frac{1}{\frac{N}{4}} + \frac{1}{\frac{Np}{N-5p}} = \frac{1}{p^*}$$

to obtain

$$\|q_\varepsilon v\|_{L^{p^*}(V)} \le \|q_\varepsilon\|_{L^{N/4}(V)}\|v\|_{L^{\frac{Np}{N-5p}}(V)}. \tag{2.14}$$

In view of (2.13), (2.14) and (2.7) we gain

$$\|A_{\varepsilon,p^*}v\|_{L^{\frac{Np}{N-5p}}(V)} \le c_{\text{Sobolev}}c_{p^*}\varepsilon\|v\|_{L^{\frac{Np}{N-5p}}(V)}.$$

We choose $\varepsilon := (2c_{\text{Sobolev}}c_{p^*})^{-1}$ to deduce

$$\|A_{\varepsilon,p^*}\|_{L^{\frac{Np}{N-5p}} \to L^{\frac{Np}{N-5p}}} \le \frac{1}{2}. \tag{2.15}$$

Then $(I - A_{\varepsilon,p^*})$ is invertible on the space $L^{\frac{Np}{N-5p}}(V)$ with the norm bounded by 2 and by (2.11)

$$v = (I - A_{\varepsilon,p^*})^{-1}h_{\varepsilon,p^*}, \tag{2.16}$$

so by the above and by (2.12)

$$\begin{aligned}\|v\|_{L^{\frac{Np}{N-5p}}(V)} &\le \left\|(I - A_{\varepsilon,p^*})^{-1}\right\|_{L^{\frac{Np}{N-5p}} \to L^{\frac{Np}{N-5p}}}\|h_{\varepsilon,p^*}\|_{L^{\frac{Np}{N-5p}}(V)}\\ &\le 2c\left(\|f_\varepsilon\|_{L^\infty(V)} + c(\eta)\|u\|_{W^{4,p}(V)}\right) < \infty.\end{aligned}$$

Hence $v \in L^{\frac{Np}{N-5p}}(V)$ and, since $u = v$ on $U \subset \Omega$ and U is arbitrary, we finally get $u \in L^{\frac{Np}{N-5p}}_{\text{loc}}(\Omega)$ as claimed. This finishes the proof of Case I.

Case II: $5p \ge N$**.**

We proceed similarly as in Case I. Fix any $\frac{Np}{N-4p} \le q < \infty$ and define $r := \frac{Nq}{N+4q}$. Then we have $1 < r < \frac{N}{4} \le \frac{Np}{N-p}$. We employ the Sobolev embedding $W^{4,r}(V) \subset L^q(V)$, (2.10), (2.9) and (2.5) to estimate

$$
\begin{aligned}
\|h_{\varepsilon,r}\|_{L^q(V)} &\le c_{\text{Sobolev}}\|T_r(f_\varepsilon+K(u))\|_{W^{4,r}(V)}\\
&\le c_{\text{Sobolev}}c_r\|f_\varepsilon+K(u)\|_{L^r(V)}\\
&\le c\left(\|f_\varepsilon\|_{L^{\frac{N}{4}}(V)}+\|K(u)\|_{L^{p^*}}\right)\\
&\le c\left(\|f_\varepsilon\|_{L^{\frac{N}{4}}(V)}+c(\eta)\|u\|_{W^{4,p}(V)}\right),
\end{aligned}
\tag{2.17}
$$

for some constant $c>0$. We bound the norm of $A_{\varepsilon,r}: L^q(V)\to L^q(V)$ by exploiting the Sobolev embedding $W^{4,r}(V)\subset L^q(V)$ and (2.10)

$$
\|A_{\varepsilon,r}\|_{L^q(V)} \le c_{\text{Sobolev}}\|T_r(q_\varepsilon v)\|_{W^{4,r}(V)} \le c_{\text{Sobolev}}c_r\|q_\varepsilon v\|_{L^r(V)}. \tag{2.18}
$$

We use Hölder's inequality with exponents

$$
\frac{1}{\frac{N}{4}}+\underbrace{\frac{1}{\frac{Nr}{N-4r}}}_{=\frac{1}{q}}=\frac{1}{r}
$$

and (2.7) to obtain

$$
\|q_\varepsilon v\|_{L^r(V)} \le \|q_\varepsilon\|_{L^{\frac{N}{4}}(V)}\|v\|_{L^q(V)} \le \varepsilon\|v\|_{L^q(V)}. \tag{2.19}
$$

We choose $\varepsilon=(2c_{\text{Sobolev}}c_r)^{-1}$ and from (2.18), (2.19) deduce that

$$
\|A_{\varepsilon,r}\|_{L^q\to L^q}\le\frac{1}{2}.
$$

As in the last part of Case I, we then show that $v\in L^q(V)$. This implies that $u\in L^q(U)$ and, since $U\subset\Omega$ and $q\ge\frac{Np}{N-4p}$ were arbitrary, the proof of Case II is completed. $\square$

Proof of Theorem 1.1. Let $u\in W^{2,2}_{\text{loc}}(\Omega)$ be a weak solution to (1.1). Then $u\in W^{4,\frac{2N}{N+4}}_{\text{loc}}(\Omega)$. We show that $u\in L^q_{\text{loc}}(\Omega)$, for every $q\ge1$. If $N=5$ or $N=6$, then, by Lemma 2.2, $u\in L^q_{\text{loc}}(\Omega)$, for every $q\ge1$, and we are done. If $N>6$, then we define $p_1:=\frac{2N}{N+4}$, $5p_1<N$, and we use Lemma 2.2 to obtain $u\in L^{\frac{Np_1}{N-5p_1}}_{\text{loc}}(\Omega)$. Since $\frac{Np_1}{N-5p_1}=\frac{2N}{N-6}$,

$$
p_1<p_2:=\frac{Np_1}{N-5p_1}\frac{N-6}{N+2}=\frac{2N}{N+2}<\frac{N}{4}.
$$

Fix $U\subset\subset\Omega$. Observe that $p_2\frac{N+2}{8}=\frac{N}{4}$ and by the Hölder inequality

$$
\int_U|g(x,u)|^{p_2}\,dx\le c\int_U|a(x)|^{p_2}\,dx+c\Big(\int_U|a(x)|^{p_2\frac{N+2}{8}}\,dx\Big)^{\frac{8}{N+2}}\Big(\int_U|u|^{\frac{Np_1}{N-5p_1}}\,dx\Big)^{\frac{N-6}{N+2}}<\infty
$$

for some constant $c>0$. Therefore we get $\Delta^2u=g(x,u)\in L^{p_2}_{\text{loc}}(\Omega)$. Since $u\in W^{4,p_1}_{\text{loc}}(\Omega)\subset L^{p_2}_{\text{loc}}(\Omega)$, we use Lemma 2.1 to get $u\in W^{4,p_2}_{\text{loc}}(\mathbb{R}^N)$. Let K be the largest natural number less than $\frac{N-4}{2}$. We continue applying Lemma 2.2 in this fashion and get a finite sequence $(p_k)_{k=1}^K$ such that for $k=1,\dots,K$

$$
\begin{aligned}
&p_k:=\frac{2N}{N+6-2k},\\
&p_k\frac{N+6-2k}{8}=\frac{N}{4},\\
&p_{k+1}=\frac{Np_k}{N-5p_k}\frac{N-4-2k}{N+4-2k},\quad\text{if } k\ge1.
\end{aligned}
$$

By the definition of K, we get $5p_K < N$, $\frac{Np_K}{N-5p_K} \geq N$ and $u \in L_{\mathrm{loc}}^{\frac{Np_K}{N-5p_K}}(\Omega)$. Finally, by Lemma 2.2 we obtain that $u \in L^q_{\mathrm{loc}}(\Omega)$, for every $q \geq 1$. Since $\Delta^2 u = g(x,u) \in L^q_{\mathrm{loc}}(\Omega)$, for every $1 \leq q < \infty$, by Lemma 2.1, $u \in W^{4,q}_{\mathrm{loc}}(\Omega)$, $q \geq 1$, so by the Sobolev embedding $u \in C^{3,\alpha}_{\mathrm{loc}}(\Omega)$, for every $0 < \alpha < 1$. □

3. Pohožaev identity

Proof of Theorem 1.2. One can find $\varphi \in C^\infty(\mathbb{R})$ satisfying $\varphi|_{(-\infty,1]} \equiv 1$, $\varphi|_{[2,\infty)} \equiv 0$ and $0 \leq \varphi \leq 1$. For every $n \geq 1$, we define $\varphi_n \in C_0^\infty(\mathbb{R}^N)$ by $\varphi_n(x) := \varphi\left(\frac{|x|^2}{n^2}\right)$.

By Theorem 1.1, we may assume that $u \in C^{3,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap W^{4,q}_{\mathrm{loc}}(\mathbb{R}^N)$, $0 < \alpha < 1$, $1 \leq q < \infty$, so

$$0 = \Delta^2 u - g(x,u) \quad \text{a.e. in } \mathbb{R}^N.$$

Thus, for a.e. $x \in \mathbb{R}^N$ and for every n, we obtain

$$0 = (\Delta^2 u - g(x,u))\varphi_n x \cdot \nabla u. \tag{3.1}$$

The following identities hold

$$g(x,u)\varphi_n x \cdot \nabla u = \operatorname{div}(\varphi_n G(x,u)x) - G(x,u)x \cdot \nabla\varphi_n - N\varphi_n G(x,u) - \varphi_n x \cdot \partial_x G(x,u)$$

and

$$\Delta^2 u \varphi_n x \cdot \nabla u = \operatorname{div}(\varphi_n (x\cdot\nabla u)\nabla\Delta u) - (x \cdot \nabla u)(\nabla\varphi_n \cdot \nabla\Delta u) - \varphi_n \nabla(x\cdot\nabla u)\cdot\nabla(\Delta u).$$

We transform the rightmost term of the above equation

$$\begin{aligned}\varphi_n \nabla(x\cdot\nabla u)\cdot\nabla(\Delta u) &= -\varphi_n \Delta u \Delta(x\cdot\nabla u) + \varphi_n \operatorname{div}(\Delta u \nabla(x\cdot\nabla u))\\ &= -\varphi_n\Delta u(2\Delta u + x\cdot\nabla\Delta u) + \operatorname{div}(\varphi_n\Delta u\nabla(x\cdot\nabla u)) - \Delta u\nabla\varphi_n\cdot\nabla(x\cdot\nabla u)\\ &= -2\varphi_n(\Delta u)^2 - \varphi_n\Delta u x\cdot\nabla\Delta u + \operatorname{div}(\varphi_n\Delta u\nabla(x\cdot\nabla u)) - \Delta u\nabla\varphi_n\cdot\nabla(x\cdot\nabla u).\end{aligned}$$

Finally, we rewrite the second term of the above line as follows

$$\varphi_n\Delta u x\cdot\nabla\Delta u = \operatorname{div}\left(\varphi_n\frac{(\Delta u)^2}{2}x\right) - \frac{1}{2}(\Delta u)^2\nabla\varphi_n\cdot x - \frac{N}{2}\varphi_n(\Delta u)^2.$$

Putting the above identities into (3.1) we get

$$\begin{aligned}0 = &-\operatorname{div}(\varphi_n G(x,u)x) + G(x,u)x\cdot\nabla\varphi_n + N\varphi_n G(x,u) + \varphi_n x\cdot\partial_x G(x,u)\\ &+ \operatorname{div}(\varphi_n(x\cdot\nabla u)\nabla\Delta u) - (x\cdot\nabla u)(\nabla\varphi_n\cdot\nabla\Delta u) - \operatorname{div}\left(\varphi_n\left(\Delta u\nabla(x\cdot\nabla u) - \frac{(\Delta u)^2}{2}x\right)\right)\\ &- \frac{N-4}{2}\varphi_n(\Delta u)^2 - \frac{1}{2}(\Delta u)^2 x\cdot\nabla\varphi_n + \Delta u\nabla\varphi_n\cdot\nabla(x\cdot\nabla u)\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\operatorname{div}\left(\varphi_n\left(G(x,u)x + \Delta u\nabla(x\cdot\nabla u) - x\cdot\nabla u\nabla\Delta u - \frac{(\Delta u)^2}{2}x\right)\right)\\ &\quad= G(x,u)x\cdot\nabla\varphi_n + N\varphi_n G(x,u) + \varphi_n x\cdot\partial_x G(x,u) - (x\cdot\nabla u)(\nabla\varphi_n\cdot\nabla\Delta u)\\ &\qquad- \frac{N-4}{2}\varphi_n(\Delta u)^2 - \frac{1}{2}(\Delta u)^2 x\cdot\nabla\varphi_n + \Delta u\nabla\varphi_n\cdot\nabla(x\cdot\nabla u).\end{aligned} \tag{3.2}$$

Fix $n \geq 1$ and take $R > 0$ such that $\operatorname{supp}\varphi_n \subset B_R$. By the divergence theorem, we obtain

$$\begin{aligned}0 = \int_{B_R} &G(x,u)x\cdot\nabla\varphi_n + N\varphi_n G(x,u) + \varphi_n x\cdot\partial_x G(x,u) - (x\cdot\nabla u)(\nabla\varphi_n\cdot\nabla\Delta u)\\ &- \frac{N-4}{2}(\Delta u)^2\varphi_n - \frac{1}{2}(\Delta u)^2 x\cdot\nabla\varphi_n + \Delta u\nabla\varphi_n\cdot\nabla(x\cdot\nabla u)\,dx.\end{aligned}$$

Note that

$$-\int_{B_R}(x\cdot\nabla u)(\nabla\varphi_n\cdot\nabla\Delta u)\,dx=\int_{B_R}\Delta u\nabla\varphi_n\cdot\nabla(x\cdot\nabla u)+\Delta u\Delta\varphi_n x\cdot\nabla u\,dx-\underbrace{\int_{B_R}\mathrm{div}\ (x\cdot\nabla u\Delta u\nabla\varphi_n)\,dx}_{=0}.$$

Summing up, we have

(3.3)

$$\begin{aligned}0&=\int_{B_R}G(u)x\cdot\nabla\varphi_n+N\varphi_n G(x,u)+\varphi_n x\cdot\partial_x G(x,u)+2\Delta u\nabla\varphi_n\cdot\nabla(x\cdot\nabla u)+\Delta u\Delta\varphi_n x\cdot\nabla u\\&\qquad-\frac{N-4}{2}(\Delta u)^2\varphi_n-\frac{1}{2}(\Delta u)^2x\cdot\nabla\varphi_n\,dx\\&=\int_{\mathbb{R}^N}G(u)x\cdot\nabla\varphi_n+N\varphi_n G(x,u)+\varphi_n x\cdot\partial_x G(x,u)+2\Delta u\nabla\varphi_n\cdot\nabla(x\cdot\nabla u)+\Delta u\Delta\varphi_n x\cdot\nabla u\\&\qquad-\frac{N-4}{2}(\Delta u)^2\varphi_n-\frac{1}{2}(\Delta u)^2x\cdot\nabla\varphi_n\,dx.\end{aligned}$$

We return to (3.3) and pass to the limit as $n\to\infty$ to obtain

$$0=N\int_{\mathbb{R}^N}G(x,u)\,dx+\int_{\mathbb{R}^N}x\cdot\partial_x G(x,u)\,dx-\frac{N-4}{2}\int_{\mathbb{R}^N}|\Delta u|^2\,dx,$$

where we used Lebesgue's dominated convergence theorem and the properties of φ_n. The proof is completed. □

4. Lions lemma

We prove a biharmonic variant of Lion's lemma, cf. [15, 16], [20, Section 2].

LEMMA 4.1. *Suppose that (u_n) is bounded in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ and for some $r>0$*

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B(y,r)}|u_n|^2\,dx=0. \tag{4.1}$$

Then

$$\int_{\mathbb{R}^N}\Psi(u_n)\,dx\to 0\quad \text{as } n\to\infty$$

for every continuous $\Psi:\mathbb{R}\to\mathbb{R}$ satisfying

$$\lim_{s\to 0}\frac{\Psi(s)}{|s|^{2^{**}}}=\lim_{|s|\to\infty}\frac{\Psi(s)}{|s|^{2^{**}}}=0. \tag{4.2}$$

We prove the following result, which implies the variant of Lions's lemma in $\mathcal{D}^{2,2}(\mathbb{R}^N)$.

LEMMA 4.2. *Suppose that $(u_n)\subset\mathcal{D}^{2,2}(\mathbb{R}^N)$ is bounded. Then $u_n(\cdot+y_n)\rightharpoonup 0$ in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ for any $(y_n)\subset\mathbb{Z}^N$ if and only if*

$$\int_{\mathbb{R}^N}\Psi(u_n)\,dx\to 0\quad \text{as } n\to\infty$$

for any continuous $\Psi:\mathbb{R}\to\mathbb{R}$ satisfying (4.2).

PROOF. Let (u_n) be a sequence in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ be such that $u_n(\cdot+y_n)\rightharpoonup 0$ in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ for every $(y_n)\subset\mathbb{Z}^N$. Take any $\varepsilon>0$ and $2^*<p<2^{**}$ and suppose that Ψ satisfies (4.2). Then we find $0<\delta<M$ and $c(\varepsilon)>0$ such that

$$\begin{aligned}\Psi(s)&\le\varepsilon|s|^{2^{**}}&&\text{for } |s|\le\delta,\\\Psi(s)&\le\varepsilon|s|^{2^{**}}&&\text{for } |s|>M,\\\Psi(s)&\le c(\varepsilon)|s|^p&&\text{for } |s|\in(\delta,M].\end{aligned}$$

Let us define (w_n) by

$$w_n(x) := \begin{cases} |u_n(x)| & \text{for } |u_n(x)| > \delta, \\ |u_n(x)|^{2^{**}/2^*}\delta^{1-2^{**}/2^*} & \text{for } |u_n(x)| \le \delta. \end{cases}$$

We are about to show that (w_n) is bounded in $W^{1,2^*}(\mathbb{R}^N)$. First of all, we have

$$\begin{aligned} \int_{\mathbb{R}^N} |w_n(x)|^{2^*}\,dx &= \int_{\{|u_n|\le\delta\}} \delta^{2^*-2^{**}}|u_n|^{2^{**}}\,dx + \int_{\{|u_n|\ge\delta\}} |u_n|^{2^*}\,dx \\ &= \delta^{2^*-2^{**}}\int_{\{|u_n|\le\delta\}} |u_n|^{2^{**}}\,dx + \int_{\{|u_n|>\delta\}} \frac{|u_n|^{2^{**}}}{|u_n|^{2^{**}-2^*}}\,dx \\ &\le \delta^{2^*-2^{**}}\int_{\{|u_n|\le\delta\}} |u_n|^{2^{**}}\,dx + \int_{\{|u_n|>\delta\}} \frac{|u_n|^{2^{**}}}{\delta^{2^{**}-2^*}}\,dx \\ &= \delta^{2^*-2^{**}}\int_{\mathbb{R}^N} |u_n|^{2^{**}}\,dx. \end{aligned} \tag{4.3}$$

By the absolute continuous characterization (see §1.1.3 in [18]), we infer that each u_n is absolutely continuous on almost every line parallel to the $0x_i$-axis, for $i = 1, \dots, N$. Thus the same holds for each w_n, since $w_n = F(u_n)$, where $F(t) = \min\{\delta^{1-2^{**}/2^*}|t|^{2^{**}/2^*}, |t|\}$ is a globally Lipschitz function. Moreover, for every $i = 1, \dots, N$, we have

$$\frac{\partial w_n}{\partial x_i} = \begin{cases} \frac{2^{**}}{2^*}\delta^{1-2^{**}/2^*}\operatorname{sign}(u_n)|u_n|^{2^{**}/2^*-1}\frac{\partial u_n}{\partial x_i}, & \text{for } |u_n(x)| \le \delta, \\ \operatorname{sign}(u_n)\frac{\partial u_n}{\partial x_i}, & \text{for } |u_n(x)| > \delta. \end{cases}$$

Thus

$$\begin{aligned} \int_{\mathbb{R}^N} \left|\frac{\partial w_n}{\partial x_i}\right|^{2^*} dx &= \left(\frac{2^{**}}{2^*}\right)^{2^*} \delta^{2^*-2^{**}} \int_{\{|u_n|\le\delta\}} |u_n|^{2^{**}-2^*}\left|\frac{\partial u_n}{\partial x_i}\right|^{2^*} dx + \int_{\{|u_n|>\delta\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{2^*} dx \\ &\le \left(\frac{2^{**}}{2^*}\right)^{2^*} \int_{\{|u_n|\le\delta\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{2^*} dx + \int_{\{|u_n|>\delta\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{2^*} dx \\ &\le \left(\frac{2^{**}}{2^*}\right)^{2^*} \int_{\mathbb{R}^N} \left|\frac{\partial u_n}{\partial x_i}\right|^{2^*} dx. \end{aligned} \tag{4.4}$$

By (4.3), (4.4) (again using an absolute continuous characterization on lines from §1.1.3 [18]) and the fact that (u_n) is bounded in $\mathcal{D}^{2,2}(\mathbb{R}^N)$, we conclude that (w_n) is bounded in $W^{1,2^*}(\mathbb{R}^N)$.

Let $\Omega = (0,1)^N$ and $y \in \mathbb{R}^N$ be arbitrary. Then, by the Sobolev inequality one has

$$\begin{aligned} \int_{\Omega+y} \Psi(u_n)\,dx &= \int_{(\Omega+y)\cap\{\delta<|u_n|\le M\}} \Psi(u_n)\,dx + \int_{(\Omega+y)\cap(\{|u_n|>M\}\cup\{|u_n|\le\delta\})} \Psi(u_n)\,dx \\ &\le c(\varepsilon)\int_{(\Omega+y)\cap\{\delta<|u_n|\le M\}} |w_n|^p\,dx + \varepsilon\int_{(\Omega+y)\cap(\{|u_n|>M\}\cup\{|u_n|\le\delta\})} |u_n|^{2^{**}}\,dx \\ &\le c(\varepsilon)C\Big(\int_{\Omega+y} |w_n|^{2^*} + |\nabla w_n|^{2^*}\,dx\Big)\Big(\int_{\Omega+y} |w_n|^p\,dx\Big)^{1-2^*/p} + \varepsilon\int_{\Omega+y} |u_n|^{2^{**}}\,dx, \end{aligned}$$

where $C > 0$ is a constant from the Sobolev inequality. Then we sum the inequalities over $y \in \mathbb{Z}^N$ and get

$$\int_{\mathbb{R}^N} \Psi(u_n)\,dx \le c(\varepsilon)C\left(\int_{\mathbb{R}^N} |w_n|^{2^*} + |\nabla w_n|^{2^*}\,dx\right)\left(\sup_{y\in\mathbb{Z}^N}\int_\Omega |w_n(\cdot+y)|^p\,dx\right)^{1-2^*/p} + \varepsilon\int_{\mathbb{R}^N} |u_n|^{2^{**}}\,dx.$$

Let us take $(y_n) \subset \mathbb{Z}^N$ such that

$$\sup_{y\in\mathbb{Z}^N}\int_\Omega |w_n(\cdot+y)|^p\,dx \le 2\int_\Omega |w_n(\cdot+y_n)|^p\,dx$$

for any $n \geq 1$. By the assumption $u_n(\cdot + y_n) \rightharpoonup 0$ in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ and passing to a subsequence we obtain $u_n(\cdot + y_n) \to 0$ in $L^p(\Omega)$.

Since $|w_n(x)| \leq |u_n(x)|$, we infer that $w_n(\cdot + y_n) \to 0$ in $L^p(\Omega)$. Therefore

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \Psi(u_n)\, dx \leq \varepsilon \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^{**}}\, dx,$$

and since $\varepsilon > 0$ is arbitrary, the assertion follows.

On the other hand, suppose that $u_n(\cdot + y_n)$ does not converge to 0 in $\mathcal{D}^{2,2}(\mathbb{R}^N)$, for some (y_n) in $\mathbb{Z}^N$, and $\Psi(u_n) \to 0$ in $L^1(\mathbb{R}^N)$. We may assume that $u_n(\cdot + y_n) \to u_0 \neq 0$ in $L^p(\Omega)$ for some bounded domain $\Omega \subset \mathbb{R}^N$ and $1 < p < 2^{**}$. Take any $\varepsilon > 0$, $q > 2^{**}$ and let us define $\Psi(s) := \min\{|s|^p, \varepsilon^{p-q}|s|^q\}$ for $s \in \mathbb{R}$. Then

$$\begin{aligned}
\int_{\mathbb{R}^N} \Psi(u_n)\, dx &\geq \int_{\Omega+y_n\cap\{|u_n|\geq\varepsilon\}} |u_n|^p\, dx + \int_{\Omega+y_n\cap\{|u_n|\leq\varepsilon\}} \varepsilon^{q-p}|u_n|^q\, dx \\
&= \int_{\Omega+y_n} |u_n|^p\, dx + \int_{\Omega+y_n\cap\{|u_n|\leq\varepsilon\}} \varepsilon^{p-q}|u_n|^q - |u_n|^p\, dx \\
&\geq \int_{\Omega+y_n} |u_n|^p\, dx - \varepsilon^p|\Omega|.
\end{aligned}$$

Thus we get $u_n(\cdot + y_n) \to 0$ in $L^p(\Omega)$ and this contradicts $u_0 \neq 0$. □

Proof of Lemma 4.1. Suppose that there is $(y_n) \subset \mathbb{Z}^N$ such that $u_n(\cdot + y_n)$ does not converge weakly to 0 in $\mathcal{D}^{2,2}(\mathbb{R}^N)$. Since $u_n(\cdot + y_n)$ is bounded, there is $u_0 \neq 0$ such that, up to a subsequence,

$$u_n(\cdot + y_n) \rightharpoonup u_0 \quad \text{in } \mathcal{D}^{2,2}(\mathbb{R}^N),$$

as $n \to \infty$. We find $y \in \mathbb{R}^N$ such that $u_0\chi_{B(y,r)} \neq 0$ in $L^2(B(y,r))$. Observe that, passing to a subsequence, we may assume that $u_n(\cdot + y_n) \to u_0$ in $L^2(B(y,r))$. Then, in view of (4.1)

$$\int_{B(y,r)} |u_n(\cdot + y_n)|^2\, dx = \int_{B(y_n+y,r)} |u_n|^2\, dx \to 0$$

as $n \to \infty$, which contradicts the fact $u_n(\cdot + y_n) \to u_0 \neq 0$ in $L^2(B(y,r))$. Therefore $u_n(\cdot + y_n) \rightharpoonup 0$ in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ for any $(y_n) \subset \mathbb{Z}^N$ and by Lemma 4.2 we conclude. □

5. Proof of Theorem 1.3

In this section we adapt a variational approach from [**20**, Section 3] for the bi-Laplacian. Let

$$G_-(s) := \begin{cases} \int_0^s \max\{-g(t), 0\}\, dt & \text{for } s \geq 0, \\ \int_s^0 \max\{g(t), 0\}\, dt & \text{for } s < 0. \end{cases}$$

Notice that $G_+, G_- \geq 0$ and $G = G_+ - G_-$.

First, we sketch our approach with an approximation J_ε of J and present some auxiliary lemmas. The proof of Theorem 1.3 is postponed to the end of the section. Let

$$g_+(s) := G_+'(s) \quad \text{and} \quad g_-(s) := g_+(s) - g(s), \qquad s \in \mathbb{R}.$$

Notice that $G_-(s) = \int_0^s g_-(t)\, dt \geq 0$, for $s \in \mathbb{R}$. In view of (g1) and (g3), there is some $c > 0$ such that for every $s \in \mathbb{R}$

$$|G_+(s)| \leq c|s|^{2^{**}}, \tag{5.1}$$

so $G_+(u) \in L^1(\mathbb{R}^N)$ whenever $u \in \mathcal{D}^{2,2}(\mathbb{R}^N) \subset L^{2^{**}}(\mathbb{R}^N)$. On the other hand, $G_-(u)$ may not be integrable, for $u \in \mathcal{D}^{2,2}(\mathbb{R}^N)$, unless $G_-(u) \leq c|u|^{2^{**}}$ for some $c > 0$. To overcome this problem, for

$\varepsilon \in (0,1)$, we define $\varphi_\varepsilon : \mathbb{R} \to [0,1]$ by

$$\varphi_\varepsilon(s) := \begin{cases} \frac{1}{\varepsilon^{2^{**}-1}}|s|^{2^{**}-1} & \text{for } |s| \le \varepsilon, \\ 1 & \text{for } |s| \ge \varepsilon. \end{cases}$$

We introduce a new functional

$$J_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2 + \int_{\mathbb{R}^N} G_-^\varepsilon(u)\,dx - \int_{\mathbb{R}^N} G_+(u)\,dx, \tag{5.2}$$

where $G_-^\varepsilon(s) := \int_0^s \varphi_\varepsilon(t)g_-(t)\,dt$, $s \in \mathbb{R}$. By (g0), there is $c(\varepsilon) > 0$ such that

$$|\varphi_\varepsilon(s)g_-(s)| \le c(\varepsilon)|s|^{2^{**}-1}, \quad s \in \mathbb{R}. \tag{5.3}$$

This implies that $G_-^\varepsilon(s) \le c(\varepsilon)|s|^{2^{**}}$ for any $s \in \mathbb{R}$ and some constant $c(\varepsilon) > 0$ depending on $\varepsilon > 0$. Hence, for $\varepsilon \in (0,1)$, J_ε is well-defined on $\mathcal{D}^{2,2}(\mathbb{R}^N)$, continuous and $J_\varepsilon'(u)(v)$ exists for any $u \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ and $v \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. Therefore, we say that u is a *critical point* of J_ε provided that $J_\varepsilon'(u)(v) = 0$ for any $v \in \mathcal{C}_0^\infty(\mathbb{R}^N)$.

We define, for $\varepsilon \in (0,1)$,

$$\begin{aligned} G_\varepsilon &:= G_+ - G_-^\varepsilon, \\ \mathcal{M}_\varepsilon &:= \Big\{u \in \mathcal{D}^{2,2}(\mathbb{R}^N)\setminus\{0\} : \int_{\mathbb{R}^N} |\Delta u|^2 - 2^{**}\int_{\mathbb{R}^N} G_\varepsilon(u)\,dx = 0\Big\}, \\ \mathcal{P}_\varepsilon &:= \Big\{u \in \mathcal{D}^{2,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} G_\varepsilon(u)\,dx > 0\Big\} \neq \emptyset, \\ c_\varepsilon &:= \inf_{u\in\mathcal{M}_\varepsilon} J_\varepsilon(u). \end{aligned}$$

and introduce the map $m_{\mathcal{P}_\varepsilon} : \mathcal{P}_\varepsilon \to \mathcal{M}_\varepsilon$ given by

$$m_{\mathcal{P}_\varepsilon}(u) = u(r_\varepsilon\cdot),$$

where

$$r_\varepsilon = r_\varepsilon(u) := \left(\frac{2^{**}\int_{\mathbb{R}^N} G_\varepsilon(u)\,dx}{\int_{\mathbb{R}^N}|\Delta u|^2}\right)^{1/4} = \frac{\left(2^{**}\int_{\mathbb{R}^N} G_\varepsilon(u)\,dx\right)^{\frac{1}{4}}}{\|u\|^{1/2}}.$$

We check that $m_{\mathcal{P}_\varepsilon}$ is well-defined. If $u \in \mathcal{P}_\varepsilon$, then

$$\begin{aligned} \int_{\mathbb{R}^N} |\Delta(m_{\mathcal{P}_\varepsilon}(u)(x)|^2\,dx &= r_\varepsilon^{4-N}\int_{\mathbb{R}^N}|\Delta u|^2\,dx \\ &= \left(2^{**}\int_{\mathbb{R}^N} G_\varepsilon(u)\,dx\right)^{\frac{4-N}{4}} \|u\|^{\frac{N-4}{2}}\|u\|^2 \\ &= \left(2^{**}\int_{\mathbb{R}^N} G_\varepsilon(u)\,dx\right)\frac{\|u\|^{\frac{N}{2}}}{\left(2^{**}\int_{\mathbb{R}^N} G_\varepsilon(u)\,dx\right)^{\frac{N}{4}}} \\ &= \left(2^{**}\int_{\mathbb{R}^N} G_\varepsilon(u)\,dx\right) r_\varepsilon^{-N} \\ &= 2^{**}\int_{\mathbb{R}^N} G_\varepsilon(m_{\mathcal{P}_\varepsilon}(u)(x))\,dx. \end{aligned}$$

□

LEMMA 5.1. *For every $\delta > 0$ there is $c_\delta > 0$ such that*

$$G_\varepsilon(u+v) - G_\varepsilon(u) - \delta|u|^{2^{**}} \le c_\delta|v|^{2^{**}}$$

for all $u, v \in \mathbb{R}$.

PROOF. First, we show that for every $\delta > 0$ there is $c(\delta) > 0$ such that

$$|G^{\varepsilon}_{-}(u+v) - G^{\varepsilon}_{-}(u)| \leq \delta |u|^{2^{**}} + c(\delta)|v|^{2^{**}}, \quad u,\ v \in \mathbb{R}. \tag{5.4}$$

Fix $\delta > 0$ and u, $v \in \mathbb{R}$. By the mean value theorem, there is $\theta \in (0,1)$ such that

$$\begin{aligned} |G^{\varepsilon}_{-}(u+v) - G^{\varepsilon}_{-}(u)| &\leq |\varphi_{\varepsilon}(u+\theta v) g_{-}(u+\theta v)||v| \\ &\leq c(\varepsilon)|u+\theta v|^{2^{**}-1}|v| \\ &\leq c_1(\varepsilon)|u|^{2^{**}-1}|v| + c_1(\varepsilon)|v|^{2^{**}}, \end{aligned}$$

where we used (5.3). We exploit the Young inequality with $\delta / c_1(\varepsilon)$

$$|u|^{2^{**}-1}|v| \leq \frac{\delta}{c_1(\varepsilon)}|u|^{(2^{**}-1)p} + c_2(\delta,\varepsilon)|v|^{q}, \quad \text{where } p = \frac{2^{**}}{2^{**}-1},\ q = 2^{**},$$

to obtain

$$|G^{\varepsilon}_{-}(u+v) - G^{\varepsilon}_{-}(u)| \leq \delta |u|^{2^{**}} + c_3(\delta,\varepsilon)|v|^{2^{**}},$$

what proves the assertion.

Now, we show that for every $\delta > 0$ there is $c(\delta) > 0$ such that

$$G_{+}(u+v) - G_{+}(u) - \delta |u|^{2^{**}} \leq c(\delta)|v|^{2^{**}}, \qquad u,\ v \in \mathbb{R}.$$

Fix $\delta > 0$ and u, $v \in \mathbb{R}$. By (g1) and (g3), there are $0 < \eta < M$ such that

$$G_{+}(s) \leq \frac{2}{2^{2^{**}}}\delta |s|^{2^{**}},$$

if $|s| < \eta$ or $|s| > M$. We consider four cases.

Case I: $|u+v| < \eta$ or $|u+v| > M$.

We use the fact that $G_{+} \geq 0$ and obtain

$$G_{+}(u+v) - G_{+}(u) \leq G_{+}(u+v) \leq \frac{2}{2^{2^{**}}}\delta |u+v|^{2^{**}} \leq \delta \left(|u|^{2^{**}} + |v|^{2^{**}}\right),$$

what proves the assertion.

Case II: $\eta \leq |u+v| \leq M$ and $|v| > M$.

There is $c > 0$ such that $G_{+}(s) \leq c|s|^{2^{**}}$, for every $s \in \mathbb{R}$, so

$$G_{+}(u+v) - G_{+}(u) \leq G_{+}(u+v) \leq c|u+v|^{2^{**}} \leq cM^{2^{**}} \leq c|v|^{2^{**}}$$

and we are done.

Case III: $\eta \leq |u+v| \leq M$ and $\eta/2 \leq |v| \leq M$.

The set $C := \left\{(u,v) \in \mathbb{R}^2 \mid \eta \leq |u+v| \leq M \text{ and } \eta/2 \leq |v| \leq M\right\}$ is compact and the function $h : C \to \mathbb{R}$, given by $h(u,v) := \frac{G_{+}(u+v) - G_{+}(u) - \delta|u|^{2^{**}}}{|v|^{2^{**}}}$, is continuous. Thus, there is $c(\delta) > 0$ such that $\max_{(u,v)\in C} h(u,v) \leq c(\delta)$ and we are done.

Case IV: $\eta \leq |u+v| \leq M$ and $|v| < \eta/2$.

By the continuity of g_{+} and by (g0), there is $c(\eta)$ such that

$$|g_{+}(s)| \leq c(\eta)|s|^{2^{**}-1}, \qquad |s| \geq \frac{\eta}{2}.$$

By the mean value theorem, there is $\theta \in (0,1)$ such that

$$G_{+}(u+v) - G_{+}(u) = g_{+}(u+\theta v)v.$$

Notice that $|u+\theta v| \geq |u+v| - (1-\theta)|v| > \eta - \eta/2 = \eta/2$, so combining the above we obtain

$$G_{+}(u+v) - G_{+}(u) \leq c(\eta)|u+\theta v|^{2^{**}-1}|v|.$$

We then proceed as in the first part of the proof.

Finally, we use the above results to deduce

$$\begin{aligned} G_\varepsilon(u+v)-G_\varepsilon(u) &= G_+(u+v)-G_+(u)-\left(G_-^\varepsilon(u+v)-G_-^\varepsilon(u)\right) \\ &\le \delta|u|^{2^{**}}+c(\delta)|v|^{2^{**}}+|G_-^\varepsilon(u+v)-G_-^\varepsilon(u)| \\ &\le 2\left(\delta|u|^{2^{**}}+c(\delta)|v|^{2^{**}}\right). \end{aligned}$$

□

LEMMA 5.2. *Suppose that* $(u_n)\subset\mathcal{M}_\varepsilon$, $J_\varepsilon(u_n)\to c_\varepsilon$ *and*

$$u_n\rightharpoonup\widetilde{u}\neq 0 \text{ in } \mathcal{D}^{2,2}(\mathbb{R}^N),\ u_n(x)\to\widetilde{u}(x) \quad \text{for a.e. } x\in\mathbb{R}^N$$

for some $\widetilde{u}\in\mathcal{D}^{2,2}(\mathbb{R}^N)$. *Then* $u_n\to\widetilde{u}$, $\widetilde{u}$ *is a critical point of* J_ε *and* $J_\varepsilon(\widetilde{u})=c_\varepsilon$.

PROOF. It follows, by Lemma 5.1, that for every $\delta>0$ theres is $c(\delta)>0$ such that

$$|G_\varepsilon(u+v)-G_\varepsilon(u)|\le\delta|u|^{2^{**}}+c(\delta)|v|^{2^{**}},\quad u,\ v\in\mathbb{R}.$$

Thus taking any $v\in\mathcal{C}_0^\infty(\mathbb{R}^N)$ and $t\in\mathbb{R}$ we observe that $(G_\varepsilon(u_n+tv)-G_\varepsilon(u_n))$ is uniformly integrable and tight. In view of Vitali's convergence theorem we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}G_\varepsilon(u_n+tv)-G_\varepsilon(u_n)\,dx=\int_{\mathbb{R}^N}G_\varepsilon(\widetilde{u}+tv)-G_\varepsilon(\widetilde{u})\,dx.$$

Since each $u_n\in\mathcal{M}_\varepsilon$, we get

$$c_\varepsilon\leftarrow J_\varepsilon(u_n)=\frac{1}{2}\int_{\mathbb{R}^N}|\Delta u_n|^2\,dx-\int_{\mathbb{R}^N}G_\varepsilon(u_n)\,dx=\left(\frac{2^{**}}{2}-1\right)\int_{\mathbb{R}^N}G_\varepsilon(u_n)\,dx,$$

so

$$A:=\lim_{n\to\infty}\int_{\mathbb{R}^N}G_\varepsilon(u_n)\,dx=\frac{1}{2^{**}}\left(\frac{1}{2}-\frac{1}{2^{**}}\right)^{-1}c_\varepsilon>0. \tag{5.5}$$

Combining the above we have

$$\begin{aligned} \lim_{n\to\infty}\int_{\mathbb{R}^N}G_\varepsilon(u_n+tv)\,dx &= \lim_{n\to\infty}\int_{\mathbb{R}^N}G_\varepsilon(u_n)\,dx+\int_{\mathbb{R}^N}G_\varepsilon(\widetilde{u}+tv)\,dx-\int_{\mathbb{R}^N}G_\varepsilon(\widetilde{u})\,dx \\ &= A+\int_{\mathbb{R}^N}G_\varepsilon(\widetilde{u}+tv)\,dx-\int_{\mathbb{R}^N}G_\varepsilon(\widetilde{u})\,dx. \end{aligned} \tag{5.6}$$

By (5.5) and Lemma 5.1, $u_n+tv\in\mathcal{P}_\varepsilon$ for sufficiently large n and sufficiently small $|t|$. Thus and by (5.6), for sufficiently small $|t|$, we have

$$\begin{aligned} &\lim_{n\to\infty}\frac{1}{t}\left(\left(\int_{\mathbb{R}^N}G_\varepsilon(u_n+tv)\,dx\right)^{\frac{N-4}{N}}-\left(\int_{\mathbb{R}^N}G_\varepsilon(u_n)\,dx\right)^{\frac{N-4}{N}}\right) \\ &=\frac{1}{t}\left(\left(A+\int_{\mathbb{R}^N}G_\varepsilon(\widetilde{u}+tv)\,dx-\int_{\mathbb{R}^N}G_\varepsilon(\widetilde{u})\,dx\right)^{\frac{N-4}{N}}-A^{\frac{N-4}{N}}\right). \end{aligned}$$

and, consequently, by the Lebesgue dominated convergence theorem

$$\lim_{t\to 0}\frac{1}{t}\left(\left(A+\int_{\mathbb{R}^N}G_\varepsilon(\widetilde{u}+tv)\,dx-\int_{\mathbb{R}^N}G_\varepsilon(\widetilde{u})\,dx\right)^{\frac{N-4}{N}}-A^{\frac{N-4}{N}}\right)=\frac{N-4}{N}A^{\frac{-4}{N}}\int_{\mathbb{R}^N}g_\varepsilon(\widetilde{u})v\,dx, \tag{5.7}$$

where $g_\varepsilon:=G_\varepsilon'=g_+-\varphi_\varepsilon g_-$.

If $u_n+tv\in\mathcal{P}_\varepsilon$, then $J_\varepsilon(m_{\mathcal{P}_\varepsilon}(u_n+tv))\ge c_\varepsilon$, so

$$r_\varepsilon(u_n+tv)^{4-N}\left(\frac{1}{2}-\frac{1}{2^{**}}\right)\int_{\mathbb{R}^N}|\Delta(u_n+tv)|^2\,dx\ge c_\varepsilon.$$

Raising both sides to the $4/N$-power yields

$$\left(\frac{1}{2}-\frac{1}{2^{**}}\right)^{\frac{4}{N}}\int_{\mathbb{R}^N}|\Delta(u_n+tv)|^2\,dx \geq c_\varepsilon^{\frac{4}{N}}\left(2^{**}\int_{\mathbb{R}^N}G_\varepsilon(u_n+tv)\,dx\right)^{\frac{N-4}{N}}. \tag{5.8}$$

Assumptions $u_n \in \mathcal{M}_\varepsilon$ and $J_\varepsilon(u_n) \to c_\varepsilon$ imply that

$$\int_{\mathbb{R}^N}|\Delta u_n|^2\,dx \to c_\varepsilon\left(\frac{1}{2}-\frac{1}{2^{**}}\right)^{-1}. \tag{5.9}$$

For all n and t, we have

$$\int_{\mathbb{R}^N}\Delta u_n\Delta v\,dx+\frac{t}{2}\int_{\mathbb{R}^N}|\Delta v|^2\,dx=\frac{1}{2t}\left(\int_{\mathbb{R}^N}|\Delta(u_n+tv)|^2\,dx-\int_{\mathbb{R}^N}|\Delta u_n|^2\,dx\right).$$

Hence, by (5.8) and since $u_n \in \mathcal{M}_\varepsilon$, for $t>0$,

$$\begin{aligned}&\int_{\mathbb{R}^N}\Delta u_n\Delta v\,dx+\frac{t}{2}\int_{\mathbb{R}^N}|\Delta v|^2\,dx \geq\\ &\frac{1}{2t}\left(c_\varepsilon^{\frac{4}{N}}\left(\frac{1}{2}-\frac{1}{2^{**}}\right)^{\frac{-4}{N}}\left(2^{**}\int_{\mathbb{R}^N}G_\varepsilon(u_n+tv)\,dx\right)^{\frac{N-4}{N}}-\left(2^{**}\int_{\mathbb{R}^N}G_\varepsilon(u_n)\,dx\right)^{\frac{N-4}{N}}\left(\int_{\mathbb{R}^N}|\Delta u_n|^2\,dx\right)^{\frac{4}{N}}\right).\end{aligned}$$

Letting $n\to\infty$, by (5.6), (5.5) and (5.9), we deduce that, for sufficiently small $t>0$,

$$\begin{aligned}&\int_{\mathbb{R}^N}\Delta\widetilde{u}\Delta v\,dx+\frac{t}{2}\int_{\mathbb{R}^N}|\Delta v|^2\,dx\\ &\quad\geq\frac{1}{2t}c_\varepsilon^{\frac{4}{N}}\left(\frac{1}{2}-\frac{1}{2^{**}}\right)^{\frac{-4}{N}}\left(\left(2^{**}\left(A+\int_{\mathbb{R}^N}G_\varepsilon(\widetilde{u}+tv)\,dx-\int_{\mathbb{R}^N}G_\varepsilon(\widetilde{u})\,dx\right)\right)^{\frac{N-4}{N}}-(2^{**}A)^{\frac{N-4}{N}}\right)\\ &\quad=\frac{2^{**}}{2}A^{\frac{4}{N}}\frac{1}{t}\left(\left(A+\int_{\mathbb{R}^N}G_\varepsilon(\widetilde{u}+tv)\,dx-\int_{\mathbb{R}^N}G_\varepsilon(\widetilde{u})\,dx\right)^{\frac{N-4}{N}}-A^{\frac{N-4}{N}}\right).\end{aligned}$$

We pass to the limit as $t\to 0^+$ and use (5.7) to get

$$\int_{\mathbb{R}^N}\Delta\widetilde{u}\Delta v\,dx \geq \frac{2^{**}}{2}\frac{N-4}{N}\int_{\mathbb{R}^N}g_\varepsilon(\widetilde{u})v\,dx=\int_{\mathbb{R}^N}g_\varepsilon(\widetilde{u})v\,dx.$$

Since $v\in C_0^\infty(\mathbb{R}^N)$ was arbitrary we infer that $\widetilde{u}$ is a critical point of J_ε. We use the Pohožaev identity Theorem 1.2 to the equation $\Delta^2 u = g_\varepsilon(u)$ with $G_\varepsilon\in L^1(\mathbb{R}^N)$, to deduce that $\widetilde{u}\in\mathcal{M}_\varepsilon$, what leads to

$$c_\varepsilon \leq J_\varepsilon(\widetilde{u})=\left(\frac{1}{2}-\frac{1}{2^{**}}\right)\int_{\mathbb{R}^N}|\Delta\widetilde{u}|^2\,dx \leq \liminf_{n\to\infty}\left(\frac{1}{2}-\frac{1}{2^{**}}\right)\int_{\mathbb{R}^N}|\Delta u_n|^2\,dx=\lim_{n\to\infty}J_\varepsilon(u_n)=c_\varepsilon,$$

where the weak l.s.c of the norm was used. Thus, $J_\varepsilon(\widetilde{u})=c_\varepsilon$ and $\|u_n\|\to\|\widetilde{u}\|$, so $u_n\to\widetilde{u}$ in $\mathcal{D}^{2,2}(\mathbb{R}^N)$. □

PROOF OF THEOREM 1.3. Take a minimizing sequence (u_n) in $\mathcal{M}_\varepsilon$ of J_ε, i.e., $J_\varepsilon(u_n)\to c_\varepsilon$. Since $u_n\in\mathcal{M}_\varepsilon$, $n\geq 1$, we have

$$J_\varepsilon(u_n)=\left(\frac{1}{2}-\frac{1}{2^{**}}\right)\int_{\mathbb{R}^N}|\Delta u_n|^2\,dx\to c_\varepsilon,$$

and so (u_n) is bounded in $\mathcal{D}^{2,2}(\mathbb{R}^N)$. Moreover, we have

$$2^{**}\int_{\mathbb{R}^N}G_+(u_n)\,dx \geq \int_{\mathbb{R}^N}|\Delta u_n|^2\,dx\to\left(\frac{1}{2}-\frac{1}{2^{**}}\right)c_\varepsilon.$$

By the assumption G_+ satisfies (4.2), so (4.1) is not satisfied. Passing to a subsequence, we may choose (y_n) in $\mathbb{R}^N$ and $0\neq u_\varepsilon\in\mathcal{D}^{2,2}(\mathbb{R}^N)$ such that

$$u_n(\cdot+y_n)\rightharpoonup u_\varepsilon\quad\text{in }\mathcal{D}^{2,2}(\mathbb{R}^N),\quad u_n(x+y_n)\to u_\varepsilon(x)\quad\text{for a.e. }x\in\mathbb{R}^N,$$

as $n \to \infty$. In view of Lemma 5.2, $u_\varepsilon \in \mathcal{M}_\varepsilon$ is a critical point of J_ε at the level c_ε.

Choose $\varepsilon_n \to 0^+$. Fix an arbitrary $u \in \mathcal{M}$. Since $G_\varepsilon(s) \geq G(s)$, for all $s \in \mathbb{R}$ and $\varepsilon \in (0,1)$, we deduce that

$$\int_{\mathbb{R}^N} G_{\varepsilon_n}(u)\,dx \geq \int_{\mathbb{R}^N} G(u)\,dx = \frac{1}{2^{**}} \int_{\mathbb{R}^N} |\Delta u|^2\,dx > 0,$$

so $m_{\mathcal{P}_{\varepsilon_n}}(u) \in \mathcal{M}_{\varepsilon_n}$ is well-defined. We have

$$\begin{aligned}
J_{\varepsilon_n}(u_{\varepsilon_n}) \leq J_{\varepsilon_n}(m_{\mathcal{P}_{\varepsilon_n}}(u)) &= \left(\frac{1}{2} - \frac{1}{2^{**}}\right) \underbrace{\left(\frac{2^{**}\int_{\mathbb{R}^N} G_{\varepsilon_n}(u)\,dx}{\int_{\mathbb{R}^N} |\Delta u|^2\,dx}\right)^{\frac{4-N}{4}}}_{r_{\varepsilon_n}(u)^{4-N}} \int_{\mathbb{R}^N} |\Delta u|^2\,dx \\
&= \left(\frac{1}{2} - \frac{1}{2^{**}}\right)\left(\int_{\mathbb{R}^N} |\Delta u|^2\,dx\right)^{\frac{N}{4}} \left(2^{**}\int_{\mathbb{R}^N} G_{\varepsilon_n}(u)\,dx\right)^{-\frac{N-4}{4}} \\
&\leq \left(\frac{1}{2} - \frac{1}{2^{**}}\right)\left(\int_{\mathbb{R}^N} |\Delta u|^2\,dx\right)^{\frac{N}{4}} \underbrace{\left(2^{**}\int_{\mathbb{R}^N} G(u)\,dx\right)^{-\frac{N-4}{4}}}_{=\left(\int_{\mathbb{R}^N}|\Delta u|^2\,dx\right)^{-\frac{N-4}{4}}} \\
&= \left(\frac{1}{2} - \frac{1}{2^{**}}\right)\int_{\mathbb{R}^N} |\Delta u|^2\,dx = J(u).
\end{aligned}$$

Thus $J_{\varepsilon_n}(u_{\varepsilon_n}) \leq \inf_{\mathcal{M}} J$ and

$$\int_{\mathbb{R}^N} |\Delta u_{\varepsilon_n}|^2\,dx \leq \left(\frac{1}{2} - \frac{1}{2^{**}}\right)^{-1} \inf_{\mathcal{M}} J, \qquad \text{for every } n. \tag{5.10}$$

We have $G_\varepsilon(s) \leq G_{1/2}(s)$, for all $s \in \mathbb{R}$ and $\varepsilon \in (0,1/2)$, so

$$\int_{\mathbb{R}^N} G_{1/2}(u_\varepsilon)\,dx \geq \int_{\mathbb{R}^N} G_\varepsilon(u_\varepsilon)\,dx = \frac{1}{2^{**}}\int_{\mathbb{R}^N} |\Delta u_\varepsilon|^2\,dx > 0 \implies u_\varepsilon \in \mathcal{P}_{1/2},$$

and some calculations yield

$$J_\varepsilon(u_\varepsilon) \geq J_{1/2}(m_{\mathcal{P}_{1/2}}(u_\varepsilon)) \geq J_{1/2}(u_{1/2}).$$

Therefore, we get

$$2^{**}\int_{\mathbb{R}^N} G_+(u_{\varepsilon_n})\,dx \geq \int_{\mathbb{R}^N} |\Delta u_{\varepsilon_n}|^2\,dx = \left(\frac{1}{2} - \frac{1}{2^{**}}\right)^{-1} J_{\varepsilon_n}(u_{\varepsilon_n}) \geq \left(\frac{1}{2} - \frac{1}{2^{**}}\right)^{-1} J_{1/2}(u_{1/2}) > 0.$$

By (5.10), (u_{ε_n}) is bounded in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} G_+(u_{\varepsilon_n})\,dx > c > 0$, for some constant c. In view of Lemma 4.1, (4.1) is not satisfied. Passing to a subsequence, there is (y_n) in $\mathbb{R}^N$ such that $u_{\varepsilon_n}(\cdot + y_n) \rightharpoonup u_0 \neq 0$ and $u_{\varepsilon_n}(x + y_n) \to u_0(x)$ a.e. in $\mathbb{R}^N$. We write $\widetilde{u}_n := u_{\varepsilon_n}(\cdot + y_n)$ for short. Since g_- is continuous and $g_-(0) = 0$, one may check that, for every $v \in C_0^\infty(\mathbb{R}^N)$,

$$\left|\frac{1}{\varepsilon_n^{2^{**}-1}}|\widetilde{u}_n|^{2^{**}-1}\chi_{\{|\widetilde{u}_n|\leq\varepsilon_n\}}g_-(\widetilde{u}_n)v\right| \leq \left|\chi_{\{|\widetilde{u}_n|\leq\varepsilon_n\}}g_-(\widetilde{u}_n)v\right| \to 0 \quad \text{a.e. in } \mathbb{R}^N$$

and

$$\left|\chi_{\{|\widetilde{u}_n|>\varepsilon_n\}}g_-(\widetilde{u}_n)v - g_-(u_0)v\right| \to 0 \quad \text{a.e. in } \mathbb{R}^N.$$

Due to the estimate $|g_-(\widetilde{u}_n)v| \leq c\left(1 + |\widetilde{u}_n|^{2^{**}-1}\right)|v|$, the family $\{g_-(\widetilde{u}_n)v\}$ is uniformly integrable (and tight because of the compact support). In view of Vitali's convergence theorem

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\varphi_{\varepsilon_n}(\widetilde{u}_n)g_-(\widetilde{u}_n)v - g_-(u_0)v|\,dx \\
&\quad \leq \int_{\mathbb{R}^N} \left|\frac{1}{\varepsilon_n^{2^{**}-1}}|\widetilde{u}_n|^{2^{**}-1}\chi_{\{|\widetilde{u}_n|\leq\varepsilon_n\}}g_-(\widetilde{u}_n)v\right|\,dx + \int_{\mathbb{R}^N} \left|\chi_{\{|\widetilde{u}_n|>\varepsilon_n\}}g_-(\widetilde{u}_n)v - g_-(u_0)v\right|\,dx \to 0,
\end{aligned}$$

as $n \to \infty$. Similarly, we obtain

$$\int_{\mathbb{R}^N} g_+(\widetilde{u}_n)v\,dx \to \int_{\mathbb{R}^N} g_+(u_0)v\,dx.$$

Gathering the above we deduce that

$$\begin{aligned} J'_{\varepsilon_n}(\widetilde{u}_n)(v) &= \int_{\mathbb{R}^N} \Delta\widetilde{u}_n \Delta v\,dx - \int_{\mathbb{R}^N} g_+(\widetilde{u}_n)v\,dx + \int_{\mathbb{R}^N} \varphi_{\varepsilon_n}(\widetilde{u}_n)g_-(\widetilde{u}_n)v\,dx \\ &\to \int_{\mathbb{R}^N} \Delta u_0 \Delta v\,dx - \int_{\mathbb{R}^N} g_+(u_0)v\,dx + \int_{\mathbb{R}^N} g_-(u_0)v\,dx. \end{aligned}$$

Each $\widetilde{u}_n$ is a critical point of J_{ε_n}, since so is u_{ε_n} (translation invariance), hence

$$\int_{\mathbb{R}^N} \Delta u_0 \Delta v\,dx = \int_{\mathbb{R}^N} g(u_0)v\,dx,$$

i.e., u_0 is a weak solution to (1.1). By Lebesgue's dominated convergence theorem one may show that

$$G_-^{\varepsilon_n}(\widetilde{u}_n) \to G_-(u_0) \quad \text{a.e. in } \mathbb{R}^N,$$

as $n \to \infty$, and, on the other hand,

$$2^{**}\int_{\mathbb{R}^N} G_-^{\varepsilon_n}(\widetilde{u}_n)\,dx = 2^{**}\int_{\mathbb{R}^N} G_+(\widetilde{u}_n)\,dx - \int_{\mathbb{R}^N} |\Delta\widetilde{u}_n|^2\,dx \le c\left(\sup_{n\ge 1}\|\widetilde{u}_n\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)}\right) < \infty,$$

where we used the fact that $\widetilde{u}_n \in \mathcal{M}_{\varepsilon_n}$, (5.1) and (5.10). By Fatou's lemma and by the above

$$\int_{\mathbb{R}^N} G_-(u_0)\,dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} G_-^{\varepsilon_n}(\widetilde{u}_n)\,dx < \infty,$$

namely, we have shown that $G_-(u_0) \in L^1(\mathbb{R}^N)$. By the Pohožaev identity, we infer that $u_0 \in \mathcal{M}$. Lastly, we show that $J(u_0) = \inf_{\mathcal{M}} J$. We use the weak l.s.c. of the norm and (5.10) to find that

$$\begin{aligned} J(u_0) &= \left(\frac{1}{2} - \frac{1}{2^{**}}\right)\int_{\mathbb{R}^N} |\Delta u_0|^2\,dx \le \liminf_{n\to\infty}\left(\frac{1}{2} - \frac{1}{2^{**}}\right)\int_{\mathbb{R}^N} |\Delta\widetilde{u}_n|^2\,dx \\ &= \liminf_{n\to\infty}\left(\frac{1}{2} - \frac{1}{2^{**}}\right)\int_{\mathbb{R}^N} |\Delta u_{\varepsilon_n}|^2\,dx \le \inf_{\mathcal{M}} J. \end{aligned}$$

□

6. Biharmonic logarithmic inequality

LEMMA 6.1. *If $u \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} |u|^2\,dx = 1$, then*

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx < \left(\int_{\mathbb{R}^N} |\Delta u|^2\,dx\right)^{1/2}.$$

PROOF. We rely on ideas from [4]. Let us define the Fourier transform $\widehat{u}$ of u (whenever possible) as

$$\widehat{u}(\xi) = \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} e^{-ix\cdot\xi}u(x)\,dx, \quad \xi \in \mathbb{R}^N.$$

If $u \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} |u|^2\,dx = 1$, then $u \in H^2(\mathbb{R}^N)$ and by the Plancharel theorem

$$\begin{aligned} \|u\|_{L^2(\mathbb{R}^N)} &= \|\widehat{u}\|_{L^2(\mathbb{R}^N)}, \\ \|\nabla u\|_{L^2(\mathbb{R}^N)} &= \|\widehat{\nabla u}\|_{L^2(\mathbb{R}^N)} = \|\xi\widehat{u}\|_{L^2(\mathbb{R}^N)}, \\ \|\Delta u\|_{L^2(\mathbb{R}^N)} &= \|\widehat{\Delta u}\|_{L^2(\mathbb{R}^N)} = \||\xi|^2\widehat{u}\|_{L^2(\mathbb{R}^N)}. \end{aligned}$$

By the Cauchy–Schwartz inequality we get

$$\Big(\int_{\mathbb{R}^N} |\xi\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2} \le \Big(\int_{\mathbb{R}^N} ||\xi|^2\widehat{u}(\xi)|^2\,d\xi\Big)^{1/4}\Big(\int_{\mathbb{R}^N} |\widehat{u}(\xi)|^2\,d\xi\Big)^{1/4}.$$

and the assertion follows with the non-strict inequality. Recall that the equality in the Cauchy–Schwartz inequality holds if and only if $|\xi|^2\widehat{u}(\xi) = \lambda\widehat{u}(\xi)$ for some λ, what implies $\widehat{u} = 0$. Hence the inequality in the statement is in fact strict. □

Proof of Theorem 1.4. Observe that the following inequality holds

$$\Big(\int_{\mathbb{R}^N} |\Delta u|^2\,dx\Big)^{\frac{N}{N-4}} \ge C_{N,log}\int_{\mathbb{R}^N} |u|^2\log|u|\,dx, \quad \text{for any } u \in \mathcal{D}^{2,2}(\mathbb{R}^N), \tag{6.1}$$

where

$$C_{N,log} = 2^{**}\Big(\frac{1}{2} - \frac{1}{2^{**}}\Big)^{-\frac{4}{N-4}}(\inf_{\mathcal{M}} J)^{\frac{4}{N-4}}.$$

Indeed, it is enough to consider $u \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} |u|^2\log|u|\,dx > 0$. We then obtain $u(r\cdot) \in \mathcal{M}$, where

$$r := \Big(\frac{2^{**}\int_{\mathbb{R}^N} |u|^2\log|u|\,dx}{\int_{\mathbb{R}^N}|\Delta u|^2}\Big)^{1/4}.$$

Hence $J(u(r\cdot)) \ge \inf_{\mathcal{M}} J$ and we get (6.1).

Now note that (6.1) is equivalent to

$$\Big(\int_{\mathbb{R}^N} |\Delta u|^2\,dx\Big)^{\frac{N}{N-4}} \ge C_{N,log}\max_{\alpha\in\mathbb{R}}\Big\{e^{-\alpha 2^{**}}\int_{\mathbb{R}^N} |e^{\alpha}u|^2\log|e^{\alpha}u|\,dx\Big\}, \quad \text{for } u \in \mathcal{D}^{2,2}(\mathbb{R}^N). \tag{6.2}$$

Assuming that $\int_{\mathbb{R}^N}|u|^2\,dx = 1$, the maximum of the right hand side of (6.2) is attained at $\alpha = \frac{N-4}{8} - \int_{\mathbb{R}^N}|u|^2\log|u|\,dx$. Hence we get

$$\frac{N}{N-4}\log\Big(\int_{\mathbb{R}^N}|\Delta u|^2\,dx\Big) \ge \log(C_{N,log}) - \alpha 2^{**} + 2\alpha + \log\Big(\frac{N-4}{8}\Big)$$

that is

$$\frac{N}{N-4}\log\Big(\int_{\mathbb{R}^N}|\Delta u|^2\,dx\Big) \ge \log\Big(C_{N,log}\frac{N-4}{8}e^{-1}\Big) + \frac{8}{N-4}\int_{\mathbb{R}^N}|u|^2\log|u|\,dx$$

and

$$\frac{N}{8}\log\Big(\int_{\mathbb{R}^N}|\Delta u|^2\,dx\Big) \ge \frac{N-4}{8}\log\Big(C_{N,log}\frac{N-4}{8}e^{-1}\Big) + \int_{\mathbb{R}^N}|u|^2\log|u|\,dx$$

thus (1.7) holds.

We show that the constant in (1.7) is optimal, i.e., there is $u \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ such that the equality holds. First of all, notice that if u_0 is a minimizer given by Theorem 1.3, then for u_0 we have the equality in (6.1):

$$\Big(\int_{\mathbb{R}^N} |\Delta u_0|^2\,dx\Big)^{\frac{N}{N-4}} = C_{N,log}\int_{\mathbb{R}^N} |u_0|^2\log|u_0|\,dx. \tag{6.3}$$

We use (6.1) for the family of functions $\frac{e^{\alpha}}{\|u_0\|_{L^2}}u_0 \in \mathcal{D}^{2,2}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, to get

$$\Big(\int_{\mathbb{R}^N} |\Delta u_0|^2\,dx\Big)^{\frac{N}{N-4}} \ge C_{N,log}\|u_0\|_{L^2}^{2^{**}-2}e^{(2-2^{**})\alpha}\int_{\mathbb{R}^N}|u_0|^2\log\Big|\frac{e^{\alpha}}{\|u_0\|_{L^2}}u_0\Big|\,dx, \quad \alpha \in \mathbb{R}. \tag{6.4}$$

Now let us consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(\alpha) := C_{N,log}\|u_0\|_{L^2}^{2^{**}-2}e^{(2-2^{**})\alpha}\int_{\mathbb{R}^N}|u_0|^2\log\Big|\frac{e^{\alpha}}{\|u_0\|_{L^2}}u_0\Big|\,dx - \Big(\int_{\mathbb{R}^N}|\Delta u_0|^2\,dx\Big)^{\frac{N}{N-4}}$$

Note that

$$f'(\alpha) = 0 \iff \alpha = \frac{N-4}{8} - \int_{\mathbb{R}^N} \left| \frac{u_0}{\|u_0\|_{L^2}} \right|^2 \log \left| \frac{u_0}{\|u_0\|_{L^2}} \right| dx.$$

On the other hand, f attains maximum at $\alpha = \log(\|u_0\|_{L^2})$ in view of (6.4) and (6.3), thus

$$\int_{\mathbb{R}^N} \left| \frac{u_0}{\|u_0\|_{L^2}} \right|^2 \log \left| \frac{u_0}{\|u_0\|_{L^2}} \right| dx = \frac{N-4}{8} - \log(\|u_0\|_{L^2})$$

or, equivalently,

$$\frac{1}{\|u_0\|_{L^2}^2} \int_{\mathbb{R}^N} |\Delta u_0|^2 \, dx = \frac{N}{4},$$

where we used the fact that $u_0 \in \mathcal{M}$. Therefore we obtain the equality in (1.7) for the function $\frac{u_0}{\|u_0\|_{L^2}}$.

Let us now suppose that

$$\frac{N}{8} \log \left(\left(\frac{8e}{C_{N,log}(N-4)} \right)^{(N-4)/N} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx \right) = \int_{\mathbb{R}^N} |u|^2 \log |u| \, dx$$

for some $u \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ such that $\|u\|_{L^2(\mathbb{R}^N)} = 1$. Then

$$\left(\int_{\mathbb{R}^N} |\Delta u|^2 \, dx \right)^{\frac{N}{N-4}} = C_{N,log} e^{-\alpha 2^{**}} \int_{\mathbb{R}^N} |e^\alpha u|^2 \log |e^\alpha u| \, dx$$

for $\alpha = \frac{N-4}{8} - \int_{\mathbb{R}^N} |u|^2 \log |u| \, dx$ and the equality in (6.1) holds for $u_1 := e^\alpha u$. Hence $J(u_0) = \inf_{\mathcal{M}} J$ for

$$u_0 := u_1(r\cdot) \in \mathcal{M}, \quad \text{where } r = \left(\frac{2^{**} \int_{\mathbb{R}^N} |u_1|^2 \log |u_1| \, dx}{\int_{\mathbb{R}^N} |\Delta u_1|^2 \, dx} \right)^{1/4}.$$

Let us sketch the proof that u_0 is a critical point of J. Firstly, note that, for every $v \in C_0^\infty(\mathbb{R}^N)$, $G(u_0+v) \in L^1(\mathbb{R}^N)$, for $G(s) := s^2 \log |s|$. Fix an arbitrary $v \in C_0^\infty(\mathbb{R}^N)$. We use the fact that G is C^1-smooth and the Lebesgue dominated convergence theorem to get

$$\lim_{t \to 0} \frac{1}{t} \left(\int_{\mathbb{R}^N} G(u_0 + tv) \, dx - \int_{\mathbb{R}^N} G(u_0) \, dx \right) = \int_{\mathbb{R}^N} g(u_0) v \, dx.$$

By the continuity, $\int_{\mathbb{R}^N} G(u_0 + tv) \, dx > 0$, for sufficiently small $|t| > 0$, so $(u_0 + tv)(r\cdot) \in \mathcal{M}$, where

$$r = \left(\frac{2^{**} \int_{\mathbb{R}^N} G(u_0 + tv) \, dx}{\int_{\mathbb{R}^N} |\Delta(u_0 + tv)|^2 \, dx} \right)^{1/4}.$$

Hence

$$J((u_0 + tv)(r\cdot)) \geq \inf_{\mathcal{M}} J = J(u_0)$$

or, equivalently,

$$\left(\frac{1}{2} - \frac{1}{2^{**}} \right)^{4/N} \int_{\mathbb{R}^N} |\Delta(u_0 + tv)|^2 \, dx \geq J(u_0)^{4/N} \left(2^{**} \int_{\mathbb{R}^N} G(u_0 + tv) \, dx \right)^{(N-4)/N}.$$

We then proceed similarly as in the last part of the proof of Lemma 5.2 to conclude that

$$\int_{\mathbb{R}^N} \Delta u_0 \Delta v \, dx \geq \int_{\mathbb{R}^N} g(u_0) v \, dx,$$

which yields that u_0 is a critical point of J.

Finally, we show the estimate of the constant $C_{N,log}$ from Theorem 1.7. Observe that if $u \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} |u|^2\,dx = 1$, then $u \in H^2(\mathbb{R}^N)$. In view of Lemma 6.1 and the logarithmic Sobolev inequality (1.8) we obtain

$$\int_{\mathbb{R}^N} |u|^2 \log(|u|)\,dx < \frac{N}{4}\log\left(\frac{2}{\pi e N}\left(\int_{\mathbb{R}^N} |\Delta u|^2\,dx\right)^{1/2}\right) = \frac{N}{8}\log\left(\left(\frac{2}{\pi e N}\right)^2 \int_{\mathbb{R}^N} |\Delta u|^2\,dx\right),$$

and so

$$\left(\frac{8e}{C_{N,log}(N-4)}\right)^{(N-4)/N} < \left(\frac{2}{\pi e N}\right)^2.$$

□

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