2D and 3D proximity maps for major and minor keys and chords

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[1] This paper updates and extends the last section of manuscript "Constructing rhythmic fugues" from a selection of my papers *Eine kleine Mathmusik* presented at the IRCAM's MaMuX seminar, Paris, January 25, 2003, but by thematic reasons included in the materials of the seminar of February 9, 2002; http://repmus.ircam.fr/_media/mamux/saisons/saison01-2001-2002/tangian_2002-2003_einekleinemathmusik_1-2_with-articles.pdf .

Abstract

Minor and major keys/chords are arranged along the joint subdominant-dominant axis. For this purpose, the relative minor for a major tonic (e.g. Am regarding C major) is put between the tonic and its subdominant (F), being interpreted as the 'semi-subdominant'. Correspondingly, the relative major for a minor key (C for Am) is called its 'semi-dominant'. Thereby, two axes of fifths for major and minor keys are merged into one. To reflect the proximity of other types of key/chord relations (parallel major-minor keys, major dominants in minor keys and minor subdominants in major keys), this axis is closed by analogy with the circle of fifths and twisted, as if wrapping a torus. The torus unfolded results in a key/chord proximity map. Due to using the subdominant-dominant axis, it is free from inconsistencies inherent in some known maps.

Keywords: Music theory, diatonic functions, key proximity map, tonic, dominant, subdominant, mediant

Contents

1 Introduction

Musical harmony is commonly considered from the viewpoint of either psychoacoustics or music theory/composition. The musical acoustics approach deals mainly with such chords' properties like consonance-dissonance, belonging to particular classes, and their perceptual proximity explained in terms of pitches and timbres. The music theory/composition approach focuses on 'correctly' connecting chords, tonal tension/relaxation effects, and their use in building the musical form of a piece. For reviews see [Acotto and Andreatta 2012; Bigo et al. 2014–2015; Chew 2002; Cohn 2012; Krumhansl and Kessler 1982; Tymoczko 2011].

This article suggests a 2D proximity map for major and minor harmonies within the second approach. The idea was however inspired by the first-approach talk at the 7th International Conference of Music Perception and Cognition [Krumhansl 2002], which referred to the key proximity map [Krumhansl and Kessler 1982, p. 346]. Following [Shepard 1982], the major keys were ordered according the circle of fifths (Figure 1), and the same was done for the minor keys. To visualize the proximity of relative major-minor (A-Am) and parallel major-minor (C-Cm), the two circles of fifths were helically wounded on a torus, and then the torus was unfolded resulting in a 2D map in Figure 2.

We develop this idea further but instead of ordering keys according to their acoustical proximity, we focus on the chords' diatonic functions. As one can notice, the chord order by fifth in Figure 1 implies subdominant-tonic-dominant triplets: F-C-G, C-G-D, etc. Inserting the relative minor chord before each major chord (like Am before C), we obtain the major-minor circle Figure 3. The subdominant-tonic-dominant triplets are then extended to subdominant-submediant-tonic-mediant-dominant quintuplets (like F-Am-C-Em-G).

The insertion of relative minor keys into the circle of major keys is justified by the (sub)mediant's ambivalent (or even intermediate) functionality. [Weber 1817–1821; Tchaikovski 1872; Rimski-Korsakov 1886] characterized the (sub)mediants as auxiliary to the diatonic functions of tonic, dominant and subdominant. To emphasize the relation to these main functions, [Riemann 1893] called mediant (Em in C major) tonic parallel (in this context, the German 'parallel' is the same as 'relative' in English), and submediant (Am in C major) subdominant parallel. Alternatively, [Catuar 1924; Schenker 1906–1935; Tulin-Privano 1965] proposed their context-dependent functional interpretation. For instance, Am in C major is interpreted either as tonic or subdominant. In minor keys, the relative major is often used instead of the dominant: in weaker cadences, for the second themes of sonatas and symphonies, etc. A close relation of (sub)mediants to three main diatonic functions is also demonstrated by [Andreatta and Baroin 2016], where harmonic paths in Andreatta's song *La sera non è più la tua canzone* are visualized; see Figure 4. There, the zig-zag arrow lines with subdominant-tonic-dominant transitions through submediants and mediants follow the logic of successive arcs in the subdominant-dominant circle in Figure 3.

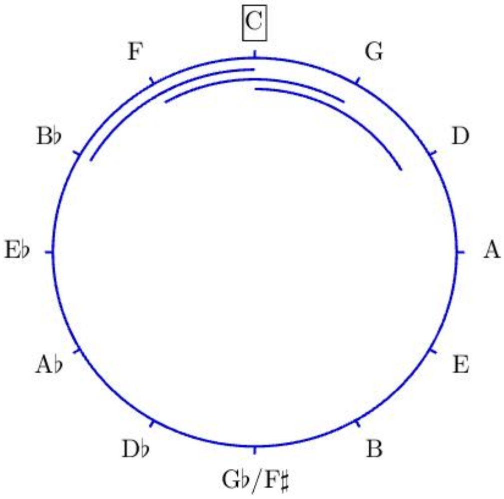
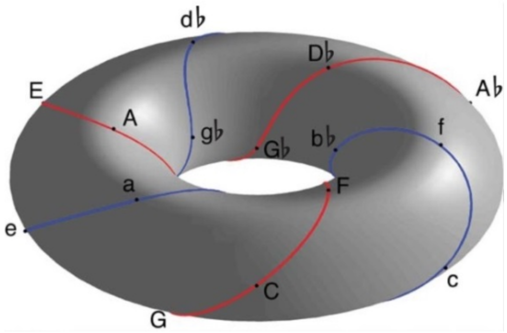


Figure 1: The circle of fifths with three subdominant-tonic-dominant triplets underlined by arcs

(a)



(b)

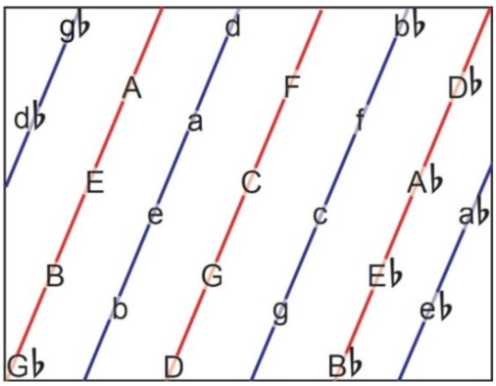


Figure 2: (a) The major-key and minor-key circles of fifths wound on a torus with one azimuth turn in the clockwise direction and three tube turns in the counterclockwise direction. (b) The unfolded torus with both circles of fifths (the horizontal section is along the torus' equator, and the vertical section is at the torus' "12 o'clock"). Source: [Zatorre and Krumhansl 2002].

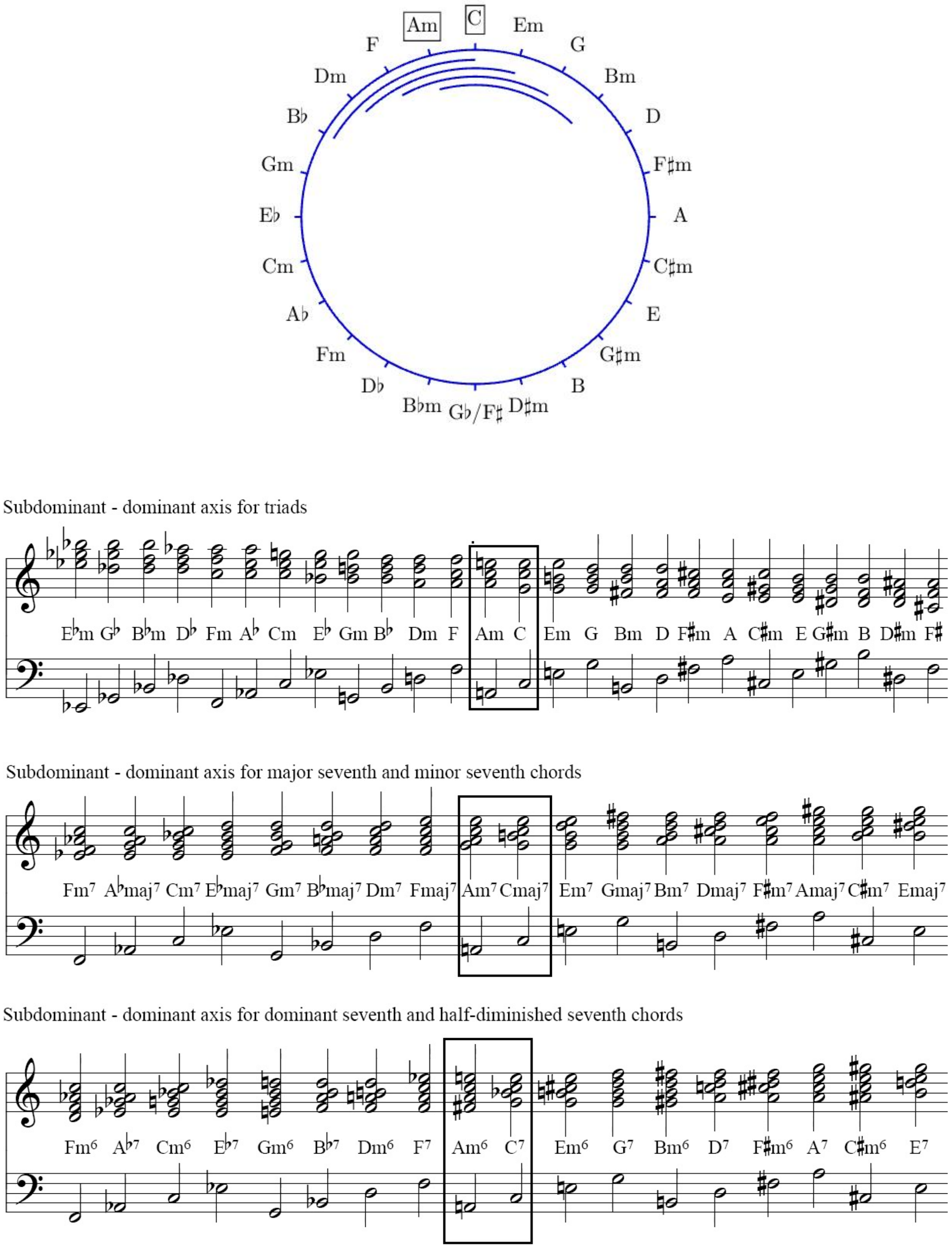


Figure 3: The subdominant-dominant circle with four subdominant-submediant-tonic-subdominant-dominant quintuplets underlined by arcs, and the corresponding chords in symbolic and standard notation

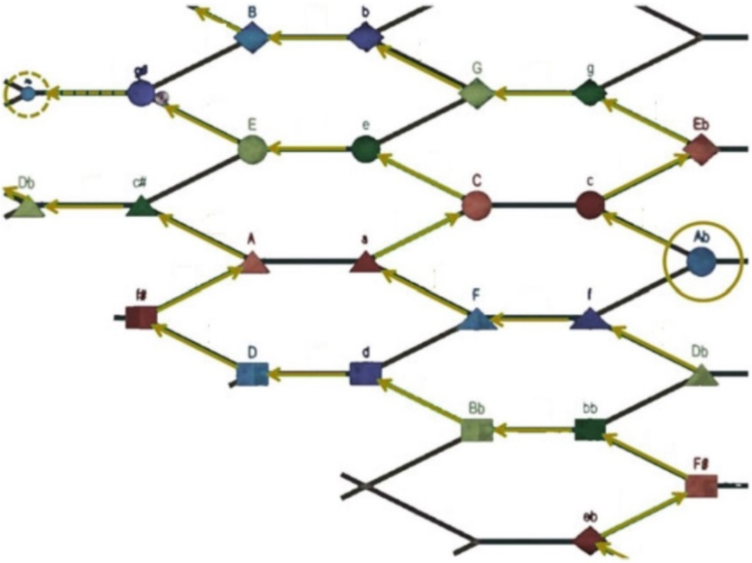


Figure 4: Subdominant-submediant-tonic-mediant-dominant paths in M. Andreatta's song *La sera non è più la tua canzone.* Source: [Andreatta and Baroin 2016, p. 267]

What we are going to do is to use the subdominant-dominant axis for constructing a key/chord proximity map. Among other things, it enables to visualize the proximity of keys at the distance of major third (C-E) — the second most important interval after fifths for tuning systems since Pythagoras — with no additional third dimension required by [Bigo Ghisi Spicher Andreatta 2015; Gollin 1998].

In Section 2, 'Subdominant-dominant axis', the merge of the circles of fifths for major and minor keys is discussed.

In Section 3, 'Subdominant-dominant helix', the subdominant-dominant axis is twisted, bringing closer some chord types.

In Section 4, 'Enhancing the enharmonic equivalence', the subdominant-dominant helix closes in on itself to identify the six-sharp and six-flat keys ($F_\sharp = G_\flat$).

In Section 5, 'Alternative winding the subdominant-dominant axis on a torus', we analyze several 3D toroidal maps of chord proximity.

In Section 6, 'The uniqueness of the 2D chord proximity map and its application', it is shown that wrapping the torus by the subdominant-dominant axis in alternative ways does not change the 2D key proximity map obtained by unfolding the torus. In addition, we illustrate the use of this unique map for finding harmonic paths with an increasing harmonic tension.

Section 7, 'Conclusions', recapitulates the main statements of the paper.

2 Subdominant-dominant axis

In music theory, fifths determine diatonic functions relative to the tonally central chord, called the *tonic*. The chord whose root is one fifths higher than that of the tonic is called the *dominant*, and the chord whose root is one fifths lower is called the *subdominant*. Examples of such chord triplets are underlined by arcs in Figure 1. They are harmonic determinants of every major or minor tonality, and most simple melodies (e.g. classical blues) can be harmonized using these three chords. The keys whose tonics differ in one fifth — like F and C or C and G — have two common basic chords, enabling smooth transitions (modulations) from each other.

In addition to subdominant and dominant, each key (e.g. C) has two intermediate chords — the *submediant* (Am) and the *mediant* (Em), which we call the *semi-subdominant* and the *semi-dominant*, respectively. Then every key is characterized by harmonic quintuplets that include three main and two auxiliary diatonic functions, as underlined by arcs in the circle of Figure 3.

Let us index the main and auxiliary diatonic functions. For each chord, the degree of its 'subdominance' and 'dominance' relative to the 0-indexed tonic is expressed in ½-steps as shown at the bottom of Table 1. For C major, the relative minor Am is the semi-subdominant, D is the second dominant, F♯m is the 2½-dominant, and the parallel Cm is the 3½-subdominant.

Table 1: Positive and negative diatonic function indices

Diatonic function	1½-subdominant	Subdominant	Semi-subdominant (Submediant)	Tonic	Semi-dominant (Mediant)	Dominant	1½-dominant
Scale degree	II	IV	vi	I	iii	V	
Case of C major	Dm	F	Am	C	Em	G	Bm
Diatonic function index for C major	–1½	–1	–½	0	+½	+1	+1½

Such an indexation of diatonic functions focuses on the dominant or subdominant direction, which is perceived as more important than the distance from the tonic. For example, the three chord progressions in Table 2 are perceived as functionally quite similar. All the three can back up the same melody (e.g. *Let's twist again* by K Mann and D Appell recorded by Chubby Checker in 1961). This is explained by the fact that all of them perform two successive subdominant descents with a return to the tonic C via the dominant G. Respecting the subdominant-dominant direction is more critical than the degree values. A similar phenomenon is inherent in melodic variations: a melody remains recognizable if the ascending/descending direction of melodic intervals at metrical accents is preserved, while the interval values being less important [Zaripov 1983].

Table 2: Similarly perceived chord progressions

Chord progression					Diatonic indices				
C	Am	F	G	C	0	–½	–1	+1	0
C	Am	Dm	G	C	0	–½	–1½	+1	0
C	F	Dm	G	C	0	–1	–1½	+1	0

In a similar way, a tonic's 'clockwise neighbors' at the subdominant-dominant circle produce a dominant tension. As mentioned in Introduction, the relative major to a minor chord (C with respect to Am) was often used in Viennese sonatas instead of the major dominant (E) avoiding the alteration of the seventh scale degree (g→g♯).

3 Subdominant-dominant helix

The subdominant-dominant circle visualizes the proximity of diatonic functions for so-called natural (purely diatonic) modes but fails to do this for tonics of parallel keys (C-Cm), major dominants to minor chords (like E-Am whose diatonic indices differ by 4½) and harmonic (minor) subdominants to major chords (like Fm-C whose diatonic indices also differ by 4½). To bring these chords closer, the axes of major and minor keys are helically twisted like in Figure 2. The corresponding subdominant-dominant helix is shown in Figure 5.

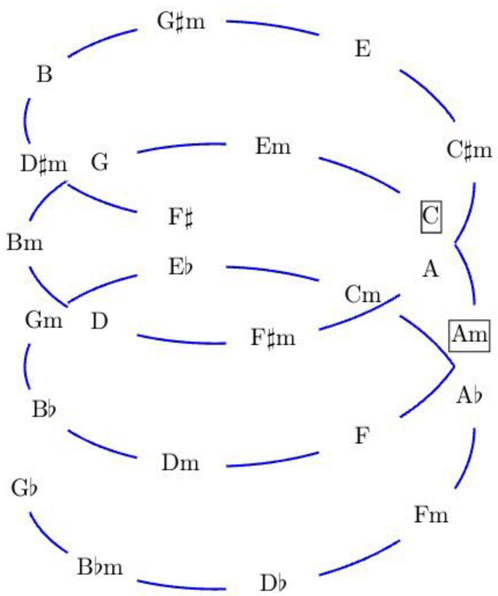


Figure 5: The subdominant-dominant helix

4 Enhancing the enharmonic equivalence

The last step is enhancing the enharmonic equivalence, that is, equalizing the six-sharp and six-flat keys ($F_\sharp$ = $G_\flat$). This is attained by closing the subdominant-dominant helix in a toroidal coil as done in Figure 2a for both helixes of major and minor keys. The result for the subdominant-dominant helix is shown in Figure 6a. Referring to Figure 2a, the red helix for major keys is integrated into the blue helix for minor keys, giving a single helix with a double 'chord density'. Figure 6b depicts the unfolding of the torus obtained by cutting the torus' equator and transecting the tube at the torus' "six o'clock".

Compared with Figure 2, Figure 6 more consistently visualizes the relations between the diatonic functions of chords. For example, the shortest path from C major to its 3rd dominant A major in Figure 2 goes through C's submediant (= semi-subdominant) Am. The second distinction is using the mathematically standard counterclockwise direction of rotation in Figure 6a, both in the horizontal plane and the vertical dimension. Hence, the dominant direction in Figure 2b is ascending, as opposed to the descending dominant direction in Figure 2b. The relations between chords with respect to their reciprocal diatonic functionality are shown in Figure 7 by color links.

5 Alternatively wrapping the torus by the subdominant-dominant axis

The unfolding in Figure 7b can be folded back into a torus in two ways. If it is rolled from bottom to top and the resulting tube is rolled into a ring, then we obtain the torus in Figure 7a. If the unfolding is first rolled from left to right and then the tube is rolled into a ring, then we obtain the torus in Figure 8a. The difference is that the subdominant-dominant axis in Figure 7 makes a single azimuth (horizontal) turn and three tube (vertical) turns, whereas in Figure 8 it makes three azimuth turns and one tube turn. It does not change the chord interrelations because the new unfolding in Figure 8b is in fact the one in Figure 7b turned by 90° in the counterclockwise direction and then reversed from left to right.

The subdominant-dominant axis can be wound on a torus in four other ways differing in the number of azimuth and tube turns. These ways, including the ones already discussed are characterized in Table 3. The table also includes the characteristics of the 'chromatic axis', which consists of the links between the relative major-minor chords (C-Cm) and counterparallel major-minor chords (C-$C_\sharp$m), and the horizontal-vertical positions of eight axes of major thirds.

Similarly to Figures 7-8, Figures 9-10 and 11-12 constitute pairs with exchanging numbers of azimuth and tube turns. Moreover, the characteristics of the subdominant-dominant and chromatic axes are interdependent. Both always have 3 either azimuth or tube turns, and the sum of the turns counted along the table rows is always equal to 14. That is, the average number of turns per axis is equal to seven — the number considered fundamental to music by Pythagoras.

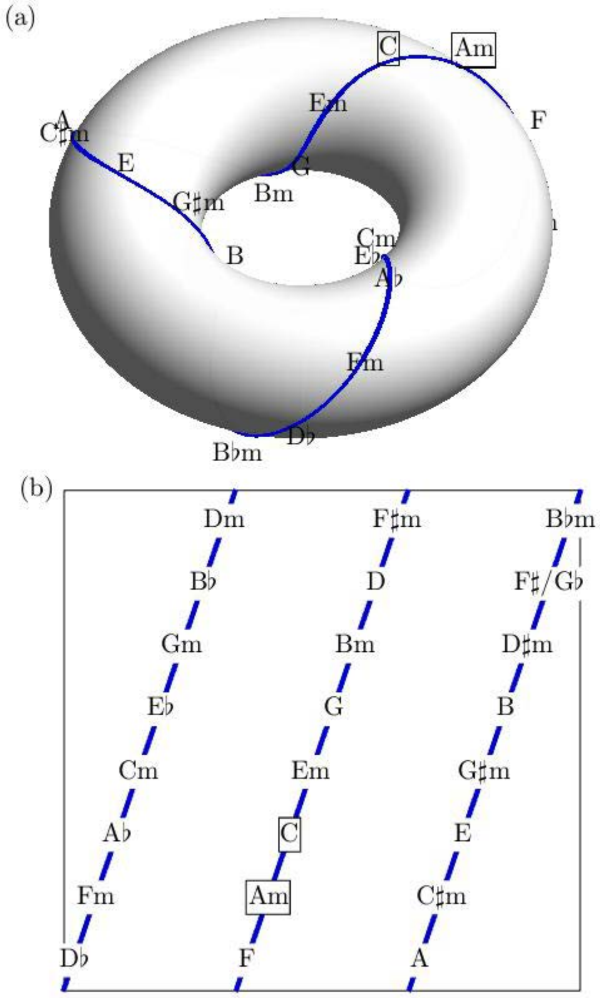


Figure 6: (a) The subdominant-dominant circle wound on a torus with one azimuth turn and three tube turns in the counterclockwise directions. (b) The unfolded torus (the horizontal section is along the torus' equator, and the vertical section is at the torus' "six o'clock").

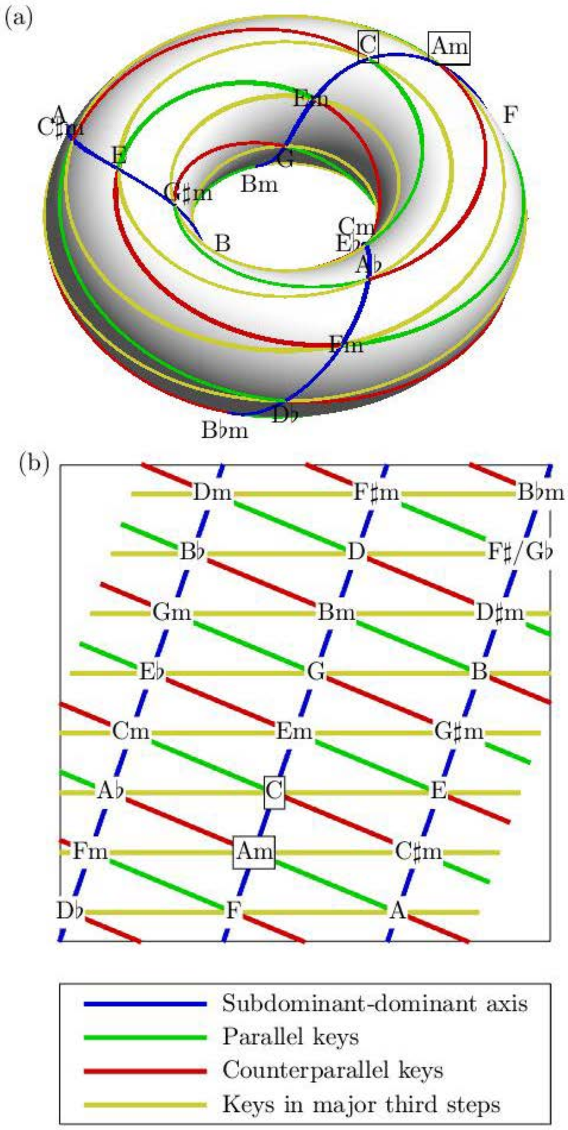


Figure 7: The same as Figure 6 but with links between close keys/chords

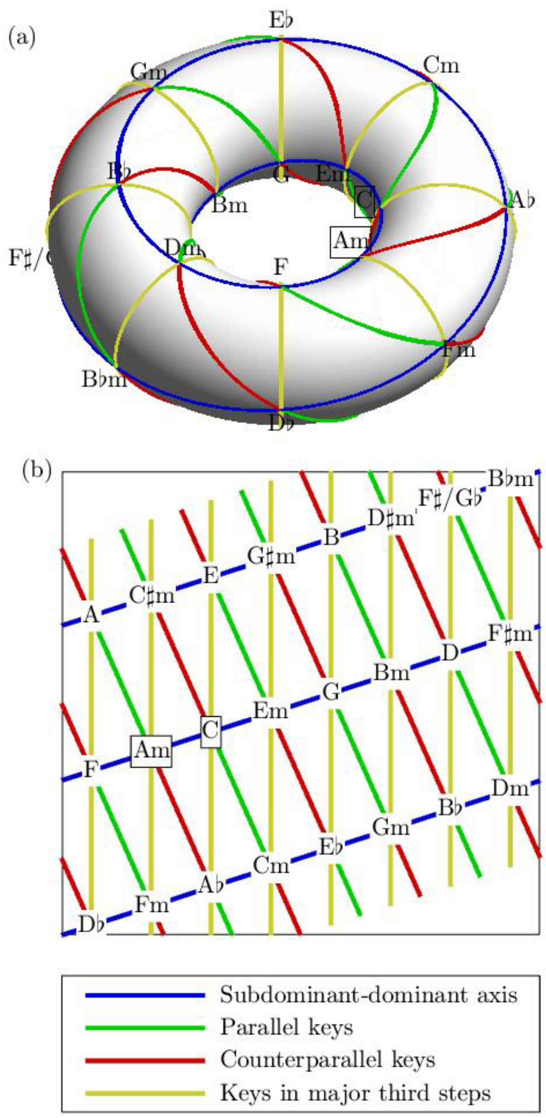


Figure 8: (a) The subdominant-dominant circle wound on a torus with 3 azimuth turn and one tube turns in the counterclockwise directions. (b) The unfolded torus (the horizontal section is along the torus' equator, and the vertical section is at the torus' "six o'clock").

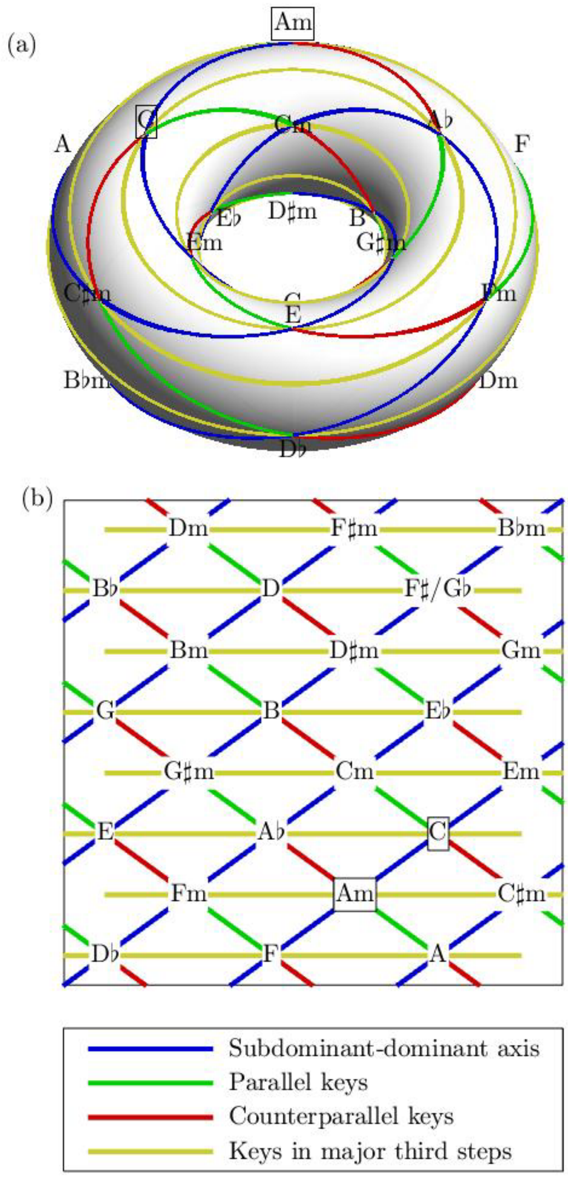


Figure 9: (a) The subdominant-dominant circle wound on a torus with 4 azimuth turns and 3 tube turns in the counterclockwise directions. (b) The unfolded torus (the horizontal section is along the torus' equator, and the vertical section is at the torus' "six o'clock").

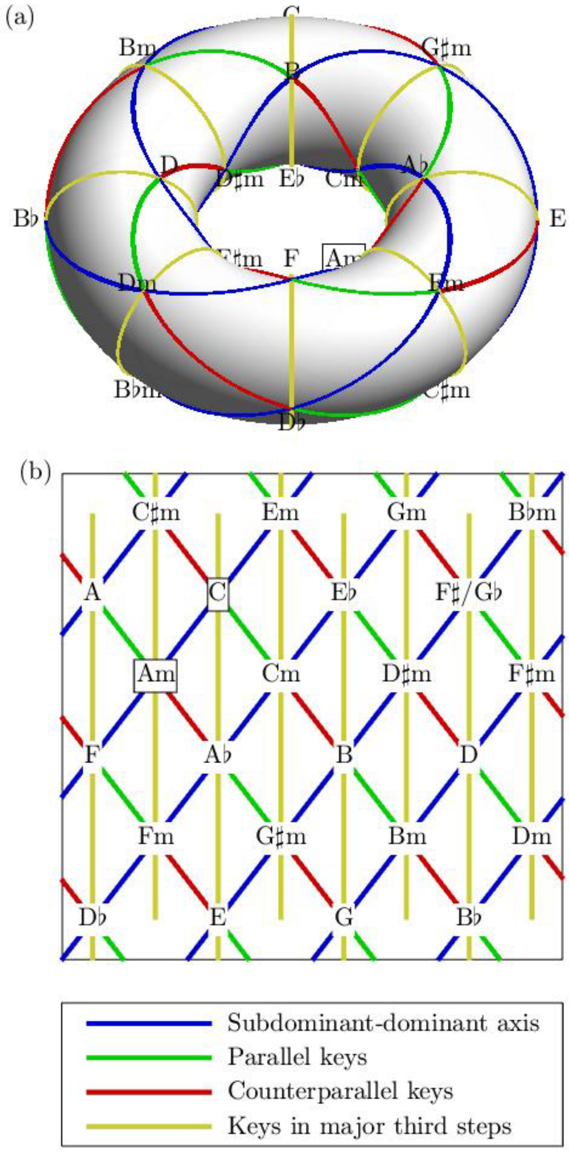


Figure 10: (a) The subdominant-dominant circle wound on a torus with 3 azimuth turns and 4 tube turns in the counterclockwise directions. (b) The unfolded torus (the horizontal section is along the torus' equator, and the vertical section is at the torus' "six o'clock").

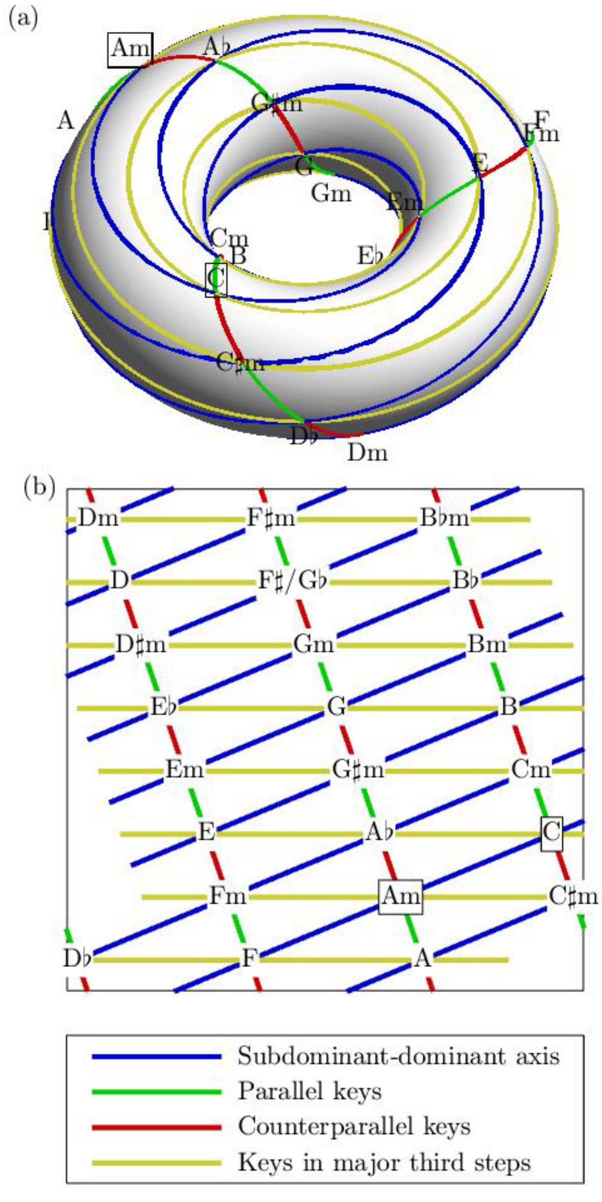


Figure 11: (a) The subdominant-dominant circle wound on a torus with 7 azimuth turns and 3 tube turns in the counterclockwise directions. (b) The unfolded torus (the horizontal section is along the torus' equator, and the vertical section is at the torus' "six o'clock").

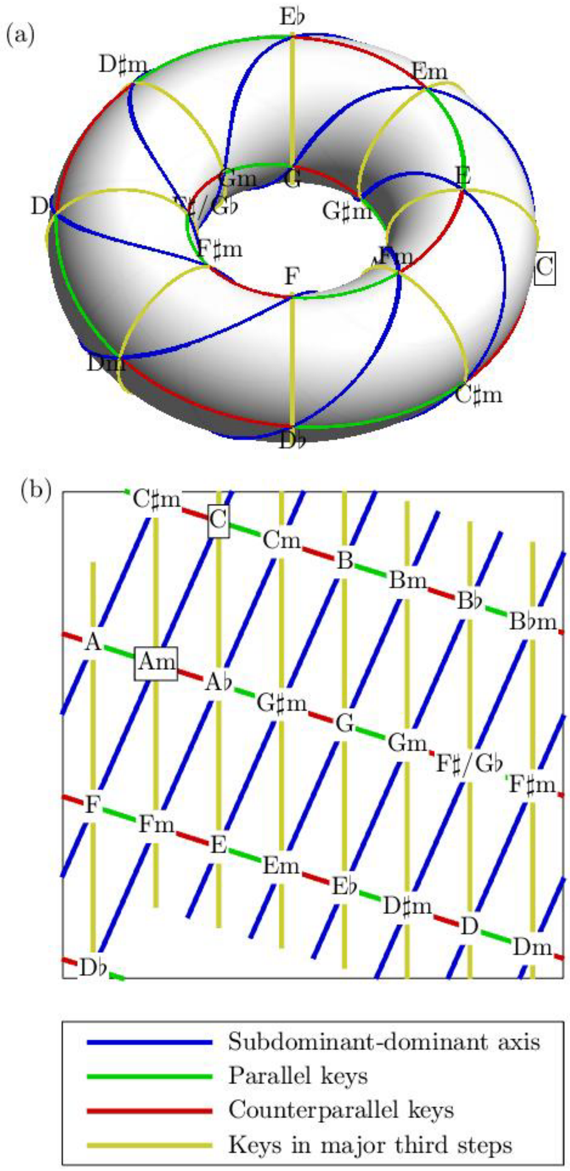


Figure 12: (a) The subdominant-dominant circle wound on a torus with 3 azimuth turns and 7 tube turns in the counterclockwise directions. (b) The unfolded torus (the horizontal section is along the torus' equator, and the vertical section is at the torus' "six o'clock").

Table 3: Subdominant-dominant axis windings on a torus

	Subdominant-dominant axis		Chromatic axis		Position of circles of major thirds
	Number of azimuth turns	Number of tube turns	Number of azimuth turns	Number of tube turns	
Figure 6	1	3			
Figure 7	1	3	7	3	Horizontal
Figure 8	3	1	3	7	Vertical
Figure 9	4	3	4	3	Horizontal
Figure 10	3	4	3	4	Vertical
Figure 11	7	3	1	3	Horizontal
Figure 12	3	7	3	1	Vertical

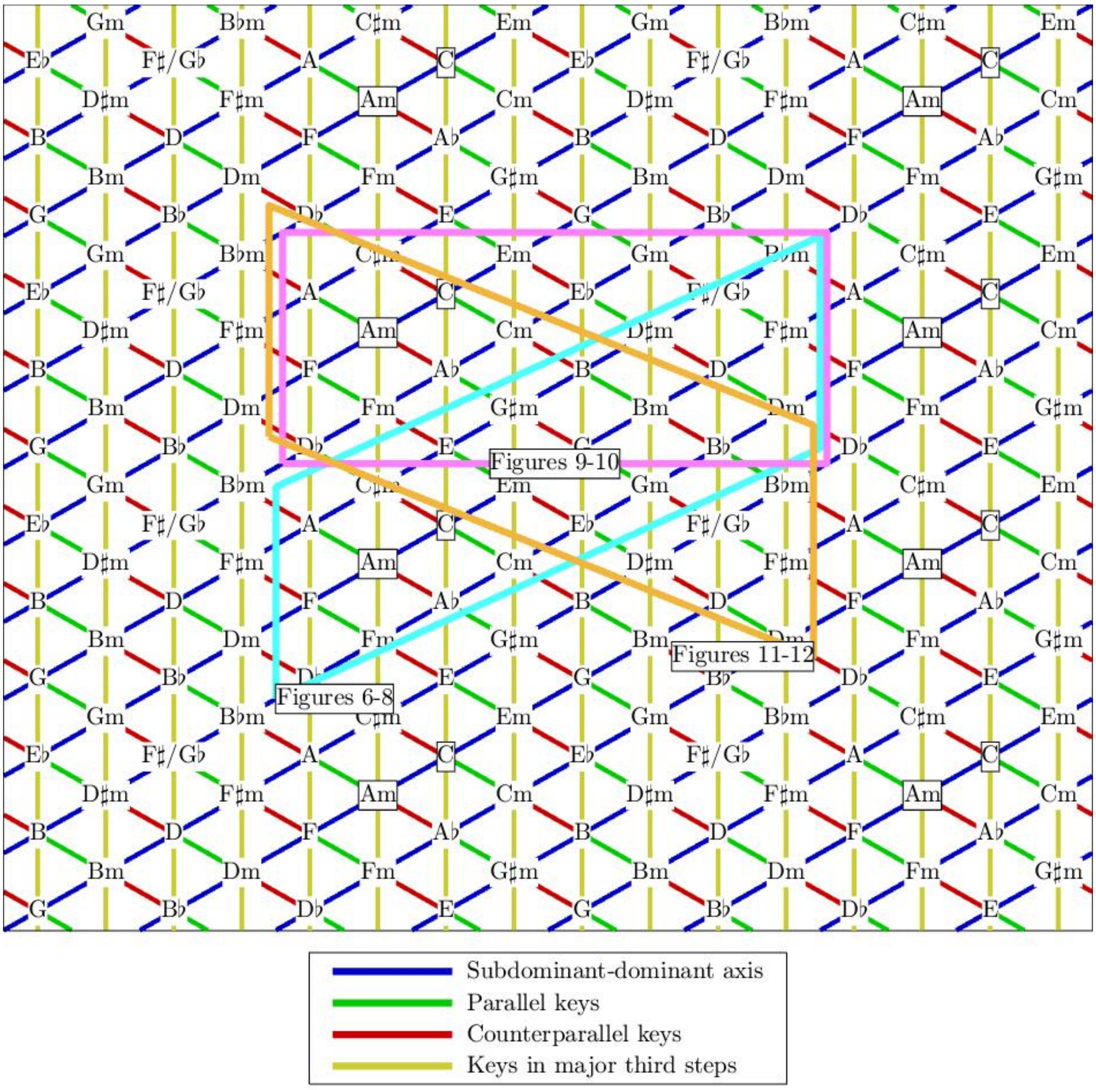


Figure 13: Extended key proximity map with the areas covered by Figures 6--12

6 The uniqueness of the 2D chord proximity map and its application

In spite of different windings of the subdominant-dominant axis, the resulting key/chord proximity maps are the same. This is demonstrated by the extended map in Figure 13, showing the areas covered by each pair of figures.

Among other things, this proximity map suggests the harmonic paths with increasing either dominant or subdominant tension. The simplest two-step paths are collected in Figure 14.

Finally we note that the inharmonic equivalence in the chord proximity map shouldn’t be misinterpreted. The circle of fifths is a tuning rather than a composition device. Most composers, applying distant modulations, return to the tonic not by the enharmonic shortcuts but through successive back-steps. Indeed, the enharmonic tonic has a quite distinct quality than the home tonic, and the seventh dominant should not be therefore considered the fifth subdominant.

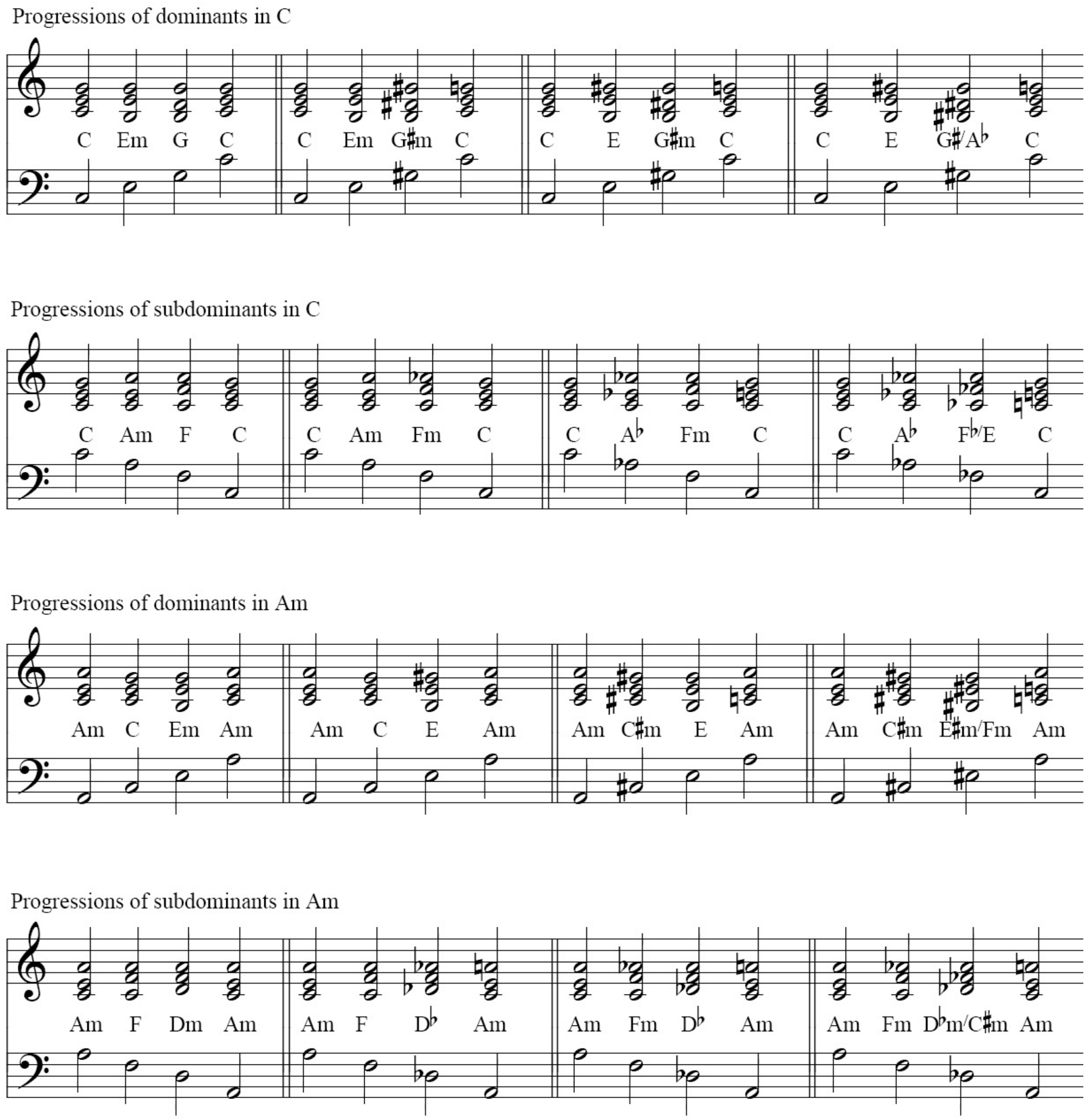


Figure 14: Two-step harmonic paths with progressive dominant or subdominant tension resolving in cadences

7 Conclusions

Let us summarize the main points of the paper.

1. The circles of fifths for major and minor keys are merged in a joint major-minor subdominant-dominant circle, where all the chords are considered subdominants or dominants of different degrees to a given tonic.

2. To reflect the proximity of certain chords, this circle is wound on a torus. To obtain a 2D visualization, the torus is unfolded, resulting in a 2D key/chord proximity map with improved consistency and local symmetry.

3. The key/chord proximity map is shown to be the same when derived from alternative 3D toroidal representations, proving its invariant character. It is noteworthy that all the consistent 3D representations make use of the Pythagorean fundamental number seven.

References

Acotto E, Andreatta M (2012) Between mind and mathematics. Different kinds of computational representations of music. *Math. Sci. hum / Mathematics and Social Sciences*, 199(3), 7–25.

Andreatta M, Baroin G (2016) An introduction on formal and computational models in popular music analysis and generation. In: Kapoula Z, Vernet M (Eds) *Aesthetics and Neuroscience*. Springer, 257–269.

Bigo L, Andreatta M (2014) A geometric model for the analysis of pop music. *Sonus* 35(1), 36–48.

Bigo L, Andreatta M (2015) Topological structures in computer-aided music analysis. In: Meredith D (Ed) *Computational Music Analysis*. Springer, 57–80.

Bigo L, Ghisi D, Spicher A, Andreatta M (2015) Representation of musical structures and processes in simplicial chord spaces. *Computer Music Journal*, 39(3), 9–24.

Catuar GL (1924) *Prakticheskij kurs garmonii* (=Practical Course of Harmony). *Part I*. Moscow (Russian).

Chew E (2002) The spiral array: an algorithm for determining key boundaries. In: Anagnostopoulou C, Ferrand M, Smaill A (Eds) *Music and Artificial Intelligence*. Berlin, Springer, 18–31.

Cohn R (2012) *Audacious Euphony: Chromatic Harmony and the Triad's Second Nature*. Oxford, Oxford University Pres.

Gollin E (1998) Some aspects of three-dimensional Tonnetze. *Journal of Music Theory* 42(2), pp. 195–206

Krumhansl CL (2002) Music as cognition: Mental maps and models. In: Stevens C, Burnham D, McPherson G, Schubert E, Renwick J (Eds) *Proceedings of the 7th International Conference on Music Perception and Cognition, 17–21 July 2002, Sydney, Australia*. Sydney, University of New South Wales, p. 2.

Krumhansl CL, Kessler EJ (1982). Tracing the dynamic changes in perceived tonal organization in a spatial representation of musical keys. *Psychological Review*, 89(4), 334–368.

Riemann H (1893) *Vereinfachte Harmonielehre oder die Lehre von den tonalen Funktionen der Akkorde*. London, Augener – New York, Schirmer.

Rimsky-Korsakov NA (1886) *Практический учебник гармонии.* С.-Петербург, Битнер. Engl. transl. *Practical Manual of Harmony*. New York, Carl Fisher, 1930.

Shenker H (1906–1935) *Neue musikalische Theorien und Phantasien, Bd. 1-3.* Stuttgart-Berlin-Wien.

Tchaikovski PI (1872) *Руководство к практическому изучению гармонии.* Москва, типография П. Юргенсона. Engl. Transl. *Guide to the Practical Study of Harmony*. Leipzig; P. Jurgenson, 1900.

Tymoczko D (2011) *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice*. Oxford, Oxford University Press.

Tjulin JuN, Privano NG (1965) *Теоретические основы гармонии (= Theoretical Foundations of Harmony)*. Muzyka, Moscow (Russian).

Weber JG (1817–1821) *Versuch einer geordneten Theorie der Tonsatzkunst*. *Bd. 1–3.* Mainz.

Zaripov RKh (1983) *Машинный поиск вариантов при моделировании творческого процесса (= Computer search for options while modeling the creative process)*. Moscow: Nauka (Russian).

Zatorre RJ, Krumhansl CL (2002) Mental models and musical minds. *Science's Compass*, 298 (December 13), 2138–2139.

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