The QED initial state corrections to the forward-backward asymmetry of $e^+e^- \to \gamma^*/Z^{0*}$ to higher orders

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A B S T R A C T

The QED initial state corrections are calculated to the forward–backward asymmetry for $e^+e^- \to \gamma^*/Z^{0*}$ in the leading logarithmic approximation to $O(\alpha^6 L^6)$, with $L = \ln(s/m_e^2)$, extending the known corrections up to $O(\alpha^2 L^2)$ in analytic form. We use the method of massive on-shell operator matrix elements and present the radiators both in Mellin-N and momentum fraction z-space. Numerical results are presented for various energies around the Z-peak by also including energy cuts. These corrections are of relevance for the precision measurements at the FCC_ee.

1. Introduction

The measurement of the forward–backward asymmetry for the process $e^+e^- \to \gamma^*/Z^*$ provides an excellent possibility to determine the running fine structure constant $\alpha_{\text{QED}}(s) = 4\pi a(s)$ near $s = M_Z^2$ at high precision. At the planned future e^+e^- facilities which operate at high energy and at large luminosity, like the ILC, CLIC [1–4], the FCC_ee [5], and also muon colliders [6], if operating in the vicinity of the Z-peak, one will obtain highly precise data. Any measurement based on these data needs a theoretical description of even higher precision [7]. Recently, higher order inclusive corrections were calculated for $e^+e^- \to \gamma^*/Z^*$ to $O(\alpha^6 L^5)$ up to the first three orders in L at the respective order in α in Refs. [8–11] confirming the results to $O(\alpha^2)$ in Ref. [12] and correcting Refs. [13,14], where $L = \ln(s/m_e^2)$. This will lead to a change of the analysis codes TOPAZ [15,16] and ZFITTER [17] and may require a re-analysis of the data taken at LEP [18].

The first order QED initial state radiative (ISR) corrections to the forward–backward asymmetry have been mutually calculated, see [16,19–24]. Furthermore, the initial-final state interference and final state corrections are known at this order, cf. [16] for a survey. Furthermore, electroweak [25,26] and QCD corrections [27] have also been calculated. Starting at $O(\alpha^2)$ also the contributions to the leading order series of $O(\alpha^k L^k)$, $k \geq 1$ receive besides the inclusive contribution another one, related to the angular structure, being present in all sub-leading terms as well.[1] Yet the $O(\alpha^k L^k)$ terms are universal, since they do not depend on the process-dependent Wilson coefficients, cf. [11]. The inclusive terms were computed in [11] and include besides the anomalous dimensions and massive OMEs also the inclusive massless Wilson coefficients up to $O(\alpha^2)$ [29,30].

In the present paper we will calculate the angular dependent leading logarithmic contributions to the radiators to $O(\alpha^6 L^6)$ as a first specific contribution which emerges for the forward–backward asymmetry. To 2nd order, these corrections were obtained in [23]. The corresponding radiators can be represented by iterated integrals over the alphabet of the harmonic polylogarithms [31] and cyclotomic harmonic polylogarithms [32] for cyclotomy c = 4. We also determine efficient representations for these quantities allowing a fast numerical analysis and present the corresponding corrections for the forward–backward asymmetry in the vicinity of the Z-resonance. These corrections, unlike the inclusive ones, do not lead to distribution–valued radiators.

The paper is organized as follows. In Section 2 we calculate the higher order QED initial state corrections to the forward–backward asymmetry. Here we use the packages Sigma [33,34] and HarmonicSums [31,32,35–38]. The leading logarithmic radiators of $H_{FB}(z)$ to $O(\alpha^6 L^6)$ of the angular-dependent terms are presented in Section 3. We also derive the expansions of the radiators in the regions

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$z \to 0$ and $z \to 1$. Numerical and phenomenological results are presented in Section 4. In Appendix A we give a brief account of the aforementioned radiators in Mellin N-space. The radiators are given in computer-readable form in an attachment to this paper.

2. The forward-backward asymmetry around the Z-peak

The forward-backward asymmetry is formed out of the partial cross sections integrating over the angle θ for the forward and backward hemispheres separately,

$$\sigma_F = 2\pi \int\limits_0^1 d\cos(\theta)\frac{d\sigma}{d\Omega}, \quad \sigma_B = 2\pi \int\limits_{-1}^0 d\cos(\theta)\frac{d\sigma}{d\Omega}. \tag{1}$$

The angle θ is defined between the incoming electron e^- and the outgoing muon μ^- from γ^*/Z^* decay. The forward-backward asymmetry is defined by

$$A_{\text{FB}}(s) = \frac{\sigma_F(s) - \sigma_B(s)}{\sigma_T(s)}, \tag{2}$$

with $\sigma_T(s) = \sigma_F(s) + \sigma_B(s)$. At Born level this reduces to [39]

$$\sigma_{FB}^{(0)}(s) = \sigma_F^{(0)}(s) - \sigma_B^{(0)}(s) = \frac{\pi\alpha^2}{s}N_{C,f}\left(1 - \frac{4m_f^2}{s}\right)G_3(s), \tag{3}$$

$$\sigma_T^{(0)}(s) = \sigma_F^{(0)}(s) + \sigma_B^{(0)}(s) = \frac{4\pi\alpha^2}{3s}N_{C,f}\sqrt{1 - \frac{4m_f^2}{s}}\left[\left(1 + \frac{2m_f^2}{s}\right)G_1(s) - 6\frac{m_f^2}{s}G_2(s)\right], \tag{4}$$

with m_f the final state fermion mass, $m_f \equiv m_\mu$. $N_{C,f}$ is the number of colors of the final state fermion, with $N_{C,f} = 1$ in the present case. s is the squared cms energy, and the effective couplings $G_i(s)|_{i=1\ldots3}$ read

$$G_1(s) = G_{1,1} + G_{1,2} + G_{1,3} = Q_e^2 Q_f^2 + 2Q_e Q_f v_e v_f \text{Re}[\chi_Z(s)] + (v_e^2 + a_e^2)(v_f^2 + a_f^2)|\chi_Z(s)|^2, \tag{5}$$

$$G_2(s) = (v_e^2 + a_e^2)a_f^2|\chi_Z(s)|^2, \tag{6}$$

$$G_3(s) = G_{3,1} + G_{3,2} = 2Q_e Q_f a_e a_f \text{Re}[\chi_Z(s)] + 4v_e v_f a_e a_f|\chi_Z(s)|^2. \tag{7}$$

For later use we define

$$\sigma_T^{\gamma\gamma} = \qquad F_1 G_{1,1}, \quad \sigma_T^{\gamma Z} = \sigma_T^{Z\gamma} = \frac{1}{2}F_1 G_{1,2}, \quad \sigma_T^{ZZ} = F_1 G_{1,3} - 6F_3 G_2 \tag{8}$$

$$\sigma_{FB}^{\gamma Z} = \sigma_{FB}^{Z\gamma} = \frac{1}{2}F_2 G_{3,1}, \quad \sigma_{FB}^{ZZ} = F_2 G_{3,2}. \tag{9}$$

with

$$F_1 = \frac{4\pi\alpha^2}{3s}N_{C,f}\sqrt{1 - \frac{4m_f^2}{s}}\left(1 + \frac{2m_f^2}{s}\right), \quad F_2 = \frac{\pi\alpha^2}{s}N_{C,f}\left(1 - \frac{4m_f^2}{s}\right),$$

$$F_3 = \frac{4\pi\alpha^2}{3s}N_{C,f}\left(1 - \frac{4m_f^2}{s}\right)\frac{m_f^2}{s}. \tag{10}$$

The reduced Z–propagator is given by

$$\chi_Z(s) = \frac{s}{s - M_Z^2 + iM_Z\Gamma_Z}, \tag{11}$$

where M_Z and Γ_Z are the mass and the width of the Z boson. $Q_{e,f}$ are the electromagnetic charges of the electron ($Q_e = -1$) and the final state fermion, respectively, and the electro–weak couplings v_i and a_i read

$$v_e = \frac{1}{\sin\theta_w \cos\theta_w}\left[I_{w,e}^3 - 2Q_e \sin^2\theta_w\right], \qquad a_e = \frac{1}{\sin\theta_w \cos\theta_w}I_{w,e}^3, \tag{12}$$

$$v_f = \frac{1}{\sin\theta_w \cos\theta_w}\left[I_{w,f}^3 - 2Q_f \sin^2\theta_w\right], \qquad a_f = \frac{1}{\sin\theta_w \cos\theta_w}I_{w,f}^3. \tag{13}$$

θ_w denotes the weak mixing angle, and $I_{w,i}^3 = \pm1/2$ the third component of the weak isospin for up and down particles, respectively.

When accounting for initial-state-radiation, one obtains the following representation of the A_{FB} introducing the radiators H_e^{LL} and H_{FB}^{LL} [23], using the notation in [19],

$$A_{\text{FB}}(s) = \frac{1}{\sigma_T(s)} \int\limits_{z_0}^1 dz \frac{4z}{(1+z)^2} \tilde{H}_e^{LL}(z) \sigma_{FB}^{(0)}(zs), \qquad \sigma_T(s) = \int\limits_{z_0}^1 dz\, H_e(z) \sigma_T^{(0)}(zs),$$

(14)

with

$$\tilde{H}_e^{LL}(z) = \left[H_e^{LL}(z) + H_{FB}^{LL}(z) \right].$$

(15)

The normalization factor $\sigma_T(s)$ will be calculated considering all corrections derived in Ref. [11]. We will consider different values for the threshold z_0. Like in Refs. [8–10] z_0 is chosen as $z_0 = 4m_\tau^2/s$, with m_τ the mass of the τ lepton. Another choice will be $z_0 = 0.99$ or 0.999 in accordance with [40]. The variable z is given by $z = s'/s$, where s' denotes the virtuality of the gauge boson γ^* or Z^*. Furthermore, $H_e(z)$ is the radiator in the inclusive case, given in Ref. [11] and $H_{FB}^{LL}(z)$ denotes the leading-log radiator in the angular dependent case up to $O(\alpha^6 L^6)$,

$$H_{FB}^{LL}(z) = \int\limits_0^1 dx_1 \int\limits_0^1 dx_2 \left(\frac{(1+z)^2}{(x_1+x_2)^2} - 1 \right) \Gamma_{ee}^{LL}(x_2) \Gamma_{ee}^{LL}(x_1) \delta(x_1 x_2 - z).$$

(16)

Here $\Gamma_{ee}^{LL}(x)$ is the leading-log operator matrix element. The product $\Gamma_{ee}^{LL}(x_1)\Gamma_{ee}^{LL}(x_2)$ is consistently expanded up to $O(\alpha^6 L^6)$. The leading-log radiators do only contain universal contributions. The radiators obey the expansion

$$H_i(z) = \delta_{ie}\delta(1-z) + \sum_{k=1}^\infty a^k \sum_{l=0}^k L^l H_i^{(k,l)}(z), \quad i = e, FB,$$

(17)

$$H_i^{LL}(z) = \delta_{ie}\delta(1-z) + \sum_{k=1}^\infty (aL)^k H_i^{(k),LL}(z),$$

(18)

where $a = \alpha/(4\pi)$ and α denotes the fine structure constant.

We proceed in the following way in order to evaluate Eq. (16) analytically. First, we calculate the Mellin transform

$$\mathcal{M}[H_{FB}^{LL}(z)](n) = \int\limits_0^1 dz\, z^n H_{FB}^{LL}(z) = \int\limits_0^1 dx_1 \int\limits_0^1 dx_2 x_1^n x_2^n \left(\frac{(1+x_1 x_2)^2}{(x_1+x_2)^2} - 1 \right) \Gamma_{ee}^{LL}(x_2) \Gamma_{ee}^{LL}(x_1).$$

(19)

Since this integral is not suited to be integrated with the package `HarmonicSums` directly, we compute the generating function

$$\mathcal{G}[H_{FB}^{LL}(z)](t) = \sum_{n=0}^\infty t^n \mathcal{M}[H_{FB}^{LL}(z)](n)$$

$$= \int\limits_0^1 dx_1 \int\limits_0^1 dx_2 \frac{1}{1-tx_1 x_2} \left(\frac{(1+x_1 x_2)^2}{(x_1+x_2)^2} - 1 \right) \Gamma_{ee}^{LL}(x_2) \Gamma_{ee}^{LL}(x_1),$$

(20)

which resums the Mellin-kernel into a denominator which can be easily integrated over. After the integration over x_1 and x_2 we are left with generalized iterated integrals evaluated at argument $z = 1$ which contain the parameter t in their letters. We use differential equations to pull this parameter into the argument. This is straightforward since the limit $t \to 0$ always exists and is easily expressed in terms of known constants. Afterwards we can use the `HarmonicSums` command `GetMoment` to get the n-space expression and `GeneralInvMellin` to arrive at the final result. The radiators are consistently expanded in (aL) to $O((aL)^6)$ and are expressed using the variable $\sqrt{z}$ rather than z to obtain a unique representation concerning the contributing iterated integrals.

At $O(\alpha)$ the complete QED initial state corrections [16,21,22] are known and are given by

$$A_{\text{FB}}^{(1)}(s) = \frac{1}{\sigma_T(s)} \int\limits_{z_0}^1 dz \frac{4z}{(1+z)^2} a(s) [\tilde{H}_e^{(1),LL}(z) L + \tilde{H}_e^{(1,0)}(z)] \sigma_T^{(0)}(zs),$$

(21)

and

$$H_e^{(1),LL}(z) = 4 \left[\frac{1+z^2}{1-z} \right]_+$$

(22)

$$\tilde{H}_e^{(1,0)}(z) = 4 \left[-\left[\frac{1+z^2}{1-z} \right]_+ + \left(2\zeta_2 - \frac{1}{2} \right) \delta(1-z) + \frac{1+z^2}{1-z} [2\ln(1+z) - 2\ln(2) - \ln(z)] \right].$$

(23)

3. The leading-log radiators H_{FB}^{LL}

We obtain the following leading-log radiators $H_{FB}^{(k),LL}$:

$$H_{FB}^{(1),LL}(z) = 0 \tag{24}$$

$$H_{FB}^{(2),LL}(z) = \frac{2(1-z)(1+z)^2}{z} + 2\pi \frac{(1-z)^2}{\sqrt{z}} - 8(1+z)H_0 - 8(1-z)^2 \frac{H_{\{4,0\}}}{\sqrt{z}} \tag{25}$$

$$\begin{aligned}
H_{FB}^{(3),LL}(z) = &-\frac{16(1-z)(4+11z+4z^2)}{3z} - \pi\left[\frac{4(2-3z-2z^2-3z^3+2z^4)}{3z^{3/2}}\right.\\
&+\frac{4(1-z)(1+5z)}{\sqrt{z}}H_0 + \left.\frac{16(1-z)^2}{\sqrt{z}}H_{\{4,1\}}\right] + \left[\frac{4(1+z)(5-18z-19z^2)}{3z}\right.\\
&-\frac{16(1-z)(1-7z)}{\sqrt{z}}H_{\{4,0\}} - \left.96(1+z)H_{\{4,1\}}\right]H_0 - 8(1+z)H_0^2\\
&+\left[\frac{16(1-z)(1+z)^2}{z} - \frac{64(1-z)^2}{\sqrt{z}}H_{\{4,0\}}\right]H_1 + \left[\frac{16(2-3z-2z^2-3z^3+2z^4)}{3z^{3/2}}\right.\\
&+\left.\frac{64(1-z)^2}{\sqrt{z}}H_{\{4,1\}}\right]H_{\{4,0\}} + \left[-\frac{16(1-z)(1+z)^2}{z} + 64(1+z)H_0\right.\\
&+\left.\frac{64(1-z)^2}{\sqrt{z}}H_{\{4,0\}}\right]H_{-1} - 64(1+z)H_{0,1} + \frac{32(1-z)^2}{\sqrt{z}}H_{0,\{4,0\}} + 96(1+z)H_{0,\{4,1\}}\\
&+\frac{64(1-z)^2}{\sqrt{z}}H_{1,\{4,0\}} - \frac{64(1-z)^2}{\sqrt{z}}H_{\{4,0\},\{4,1\}} - 64(1+z)H_{-1,0}\\
&-\frac{64(1-z)^2}{\sqrt{z}}H_{-1,\{4,0\}} + 20(1+z)\zeta_2,
\end{aligned} \tag{26}$$

with $H_{\vec{w}} \equiv H_{\vec{w}}(\sqrt{z})$ and $\zeta_k, k \in \mathbb{N}, k \geq 2$ are the values of Riemann's ζ function at integer argument. The iterated integrals are given by

$$H_{b,\vec{a}}(z) = \int_0^z dy f_b(y) H_{\vec{a}}(y), \quad H_\emptyset = 1. \tag{27}$$

The indices b, a_i refer to the letters in the following alphabet

$$\mathfrak{A} = \left\{f_0 = \frac{1}{z}, f_1 = \frac{1}{1-z}, f_{-1} = \frac{1}{1+z}, f_{\{4,0\}} = \frac{1}{1+z^2}, f_{\{4,1\}} = \frac{z}{1+z^2}\right\}, \tag{28}$$

where the last two letters in (28) are cyclotomic. The function $H_{FB}^{(2),LL}(z)$ has been calculated in [23] in an equivalent representation. The radiators $H_{FB}^{(4-6),LL}(z)$ are too voluminous to be presented here and are given in computer-readable form in the attachment.

We finally illustrate the structure of the radiators expanding them for their leading small and a few large z terms. One obtains

$$z H_{FB}^{(2),LL}(z) \simeq 2 + O(\sqrt{z}) \tag{29}$$

$$z H_{FB}^{(3),LL}(z) \simeq -\frac{8\pi}{3\sqrt{z}} + \frac{10}{3}\ln(z) - \frac{32}{3} + O(\sqrt{z}\ln(z)) \tag{30}$$

$$\begin{aligned}
z H_{FB}^{(4),LL}(z) \simeq &\frac{32}{9z} - \frac{11}{6}\ln^2(z) + \frac{10}{27}\ln(z) - 38 - 43\zeta_2\\
&+\frac{\pi}{\sqrt{z}}\left(\frac{4}{3}\ln(z) + \frac{328}{27}\right) + O(\sqrt{z}\ln^2(z))
\end{aligned} \tag{31}$$

$$\begin{aligned}
z H_{FB}^{(5),LL}(z) \simeq &-\frac{512}{81z} + \frac{5}{36}\ln^3(z) + \frac{97}{18}\ln^2(z) + \left[-\frac{299}{27} + \frac{215\zeta_2}{18}\right]\ln(z) + \frac{72283}{243}\\
&+\frac{2551}{27}\zeta_2 + \frac{320}{9}\zeta_3 + \frac{\pi}{\sqrt{z}}\left(\frac{2}{3}\ln^2(z) + \frac{92}{27}\ln(z) + \frac{460}{81}\right.\\
&+\left.\frac{4}{9}\zeta_2\right) + O(\sqrt{z}\ln^3(z))
\end{aligned} \tag{32}$$

$$\begin{aligned}
z H_{FB}^{(6),LL}(z) \simeq &-\frac{10240}{729z} - \frac{256}{81z}\ln(z) + \frac{97}{720}\ln^4(z) + \frac{79}{162}\ln^3(z) + \left(\frac{17059}{2430} - \frac{19}{180}\zeta_2\right)\ln^2(z)\\
&+\left(-\frac{159109}{7290} - \frac{2825}{81}\zeta_2 + \frac{136}{9}\zeta_3\right)\ln(z) - \frac{2459744}{10935} + \frac{87173}{1215}\zeta_2 - \frac{53216}{405}\zeta_3
\end{aligned}$$

$$+ \frac{5659}{300} \zeta_2^2 + \frac{\pi}{\sqrt{z}} \left(-\frac{1}{18} \ln^3(z) - \frac{32}{9} \ln^2(z) + \left[-\frac{39212}{1215} - \frac{73}{9} \zeta_2 \right] \ln(z) \right.$$

$$\left. - \frac{401806}{3645} - \frac{15904}{405} \zeta_2 - \frac{128}{9} \zeta_3 \right) + O\left(\sqrt{z} \ln^4(z)\right). \tag{33}$$

For the large z representation we set $u = 1 - \sqrt{z}$ and obtain

$$H_{FB}^{(2),LL}(z) \simeq 32u - 16u^2 + \frac{112}{3} u^3 + \frac{56}{3} u^4 + O\left(u^5\right) \tag{34}$$

$$H_{FB}^{(3),LL}(z) \simeq \frac{64}{3} u - \frac{416}{3} u^2 + \frac{1408}{9} u^3 - \frac{736}{9} u^4 + \left(-256u + 128u^2 - \frac{896}{3} u^3 - \frac{448}{3} u^4 \right)$$

$$\times \ln(2u) + O\left(u^5\right) \tag{35}$$

$$H_{FB}^{(4),LL}(z) \simeq \frac{27328}{27} u - \frac{13664}{27} u^2 + \frac{162080}{81} u^3 + \frac{36112}{81} u^4 + \left(1024u - 512u^2 \right.$$

$$\left. + \frac{3584}{3} u^3 + \frac{1792}{3} u^4 \right) \left(\ln^2(2u) - \zeta_2 \right) + \left(1024 - \frac{3328}{3} u + \frac{2176}{3} u^2 \right)$$

$$\times u^2 \ln(2u) + O\left(u^5\right) \tag{36}$$

$$H_{FB}^{(5),LL}(z) \simeq \frac{387712}{81} u - \frac{523328}{81} u^2 + \frac{2954144}{243} u^3 - \frac{328688}{243} u^4 + \left(\frac{2048}{3} + \frac{11264}{3} u \right.$$

$$\left. - \frac{34304}{9} u^2 + \frac{28928}{9} u^3 \right) u \left(\zeta_2 - \ln^2(2u) \right) + \left(-\frac{8192}{3} + \frac{4096}{3} u - \frac{28672}{9} u^2 \right.$$

$$\left. - \frac{14336}{9} u^3 \right) u \left(2\zeta_3 - 3\zeta_2 \ln(2u) + \ln^3(2u) \right) + \left(-\frac{73216}{9} + \frac{30464}{9} u \right.$$

$$\left. - \frac{419840}{27} u^2 - \frac{108544}{27} u^3 \right) u \ln(2u) + O\left(u^5\right) \tag{37}$$

$$H_{FB}^{(6),LL}(z) \simeq \frac{153977536}{3645} u - \frac{136846688}{3645} u^2 + \frac{1140995264}{10935} u^3 + \frac{60397216}{10935} u^4 + \left(-\frac{894976}{27} \right.$$

$$\left. + \frac{300032}{27} u - \frac{24477184}{405} u^2 - \frac{7326464}{405} u^3 \right) u \left(\zeta_2 - \ln^2(2u) \right) + \left(\frac{16384}{5} - \frac{8192}{5} u \right.$$

$$\left. + \frac{57344}{15} u^2 + \frac{28672}{15} u^3 \right) u \left(\zeta_2^2 + \frac{5}{3} \ln^4(2u) \right) + \left(\frac{32768}{9} + \frac{81920}{9} u - \frac{1132544}{135} u^2 \right.$$

$$\left. + \frac{1276928}{135} u^3 \right) u \left(2\zeta_3 - 3\zeta_2 \ln(2u) + \ln^3(2u) \right) + \left(-\frac{886784}{27} + \frac{1338368}{27} u \right.$$

$$\left. - \frac{35939456}{405} u^2 + \frac{5414336}{405} u^3 - \left[-\frac{131072}{3} + \frac{65536}{3} u - \frac{458752}{9} u^2 \right. \right.$$

$$\left. \left. - \frac{229376}{9} u^3 \right] \zeta_3 \right) u \ln(2u) + \left(-32768 + 16384u - \frac{114688}{3} u^2 - \frac{57344}{3} u^3 \right)$$

$$\times u \zeta_2 \ln^2(2u) + O\left(u^5\right). \tag{38}$$

These terms are the beginning of the respective series used in the numerical representation to which we turn now.

4. Numerical results

For the numerical evaluation of the new radiators we calculated 60 terms of the expansions around $\sqrt{z} = 0$ and $\sqrt{z} = 1$ using `HarmonicSums`. The constants associated with cyclotomic harmonic polylogarithms for cyclotomy c = 4, which are needed for the expansion around $\sqrt{z} = 1$ are known at least up to weight w = 5 [32]. However, all cyclotomic constants except of π cancel in the final results. We switch between these two expansions at $z = 1/4$, where the absolute difference of the respective approximation is always smaller than 10^{-12}. The usual harmonic polylogarithms are calculated using the implementations in [42,43].[2] In the numerical illustration we refer to the electroweak parameters given in [41].

In Fig. 1 we illustrate the effect of the initial–state QED corrections on the forward–backward asymmetry for a wider range of cms energies around the Z-peak with $\sqrt{s} \in [85, 96]$ GeV to provide an overall impression on the size of the radiative corrections to the born cross-section (black, dashed line), which will be detailed below. One sees that the bulk of the corrections are due to the $O(\alpha)$ corrections (included in the blue, dotted line), which are themselves dominated by the leading-log term, making up $\sim 98\%$ of the complete $O(\alpha)$ correction. However, the additional leading-log corrections up to $O(\alpha^6 L^6)$ (included in red, full line) show visible effects which are important for high precision analysis.

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[2] Here we also give the sub-leading values for $O(\alpha L^0)$. For similar sub-leading corrections see [44].
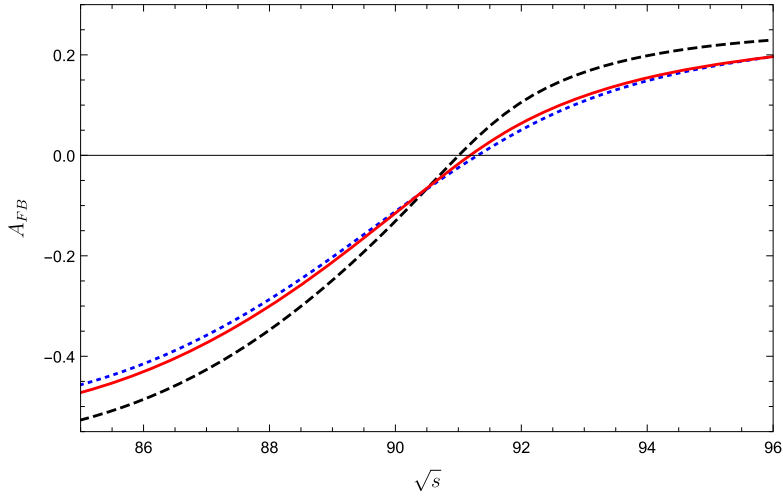
Fig. 1. A_{FB} and its initial state QED corrections as a function of $\sqrt{s}$. Black (dashed) the Born approximation, blue (dotted) the $O(\alpha)$ improved approximation, red (full) also including the leading-log improvement up to $O(\alpha^6)$ for $s'/s \geq 4m_\tau^2/s$.
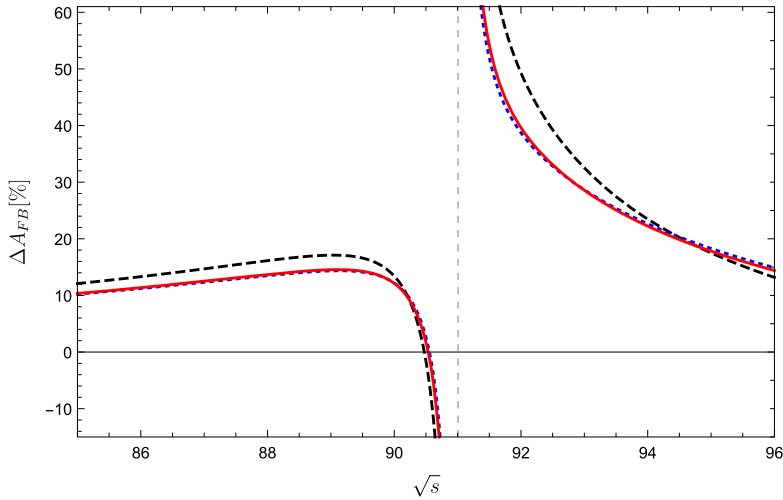

Fig. 2. $\Delta A_{FB}^{(l)}$, for $l = 1, 2, 6$, in % as defined in Eq. (39) as a function of $\sqrt{s}$. Black (dashed) the $O(\alpha)$ improved approximation, blue (dotted) the $O(\alpha^2 L^2)$ improved approximation, red (full) also including the leading-log improvement up to $O(\alpha^6)$ for $s'/s \geq 4m_\tau^2/s$.

Let us define

$$\Delta A_{FB}^{(l)} = 1 - \frac{A_{FB}^{(l)}}{A_{FB}^{(0)}}, \tag{39}$$

where l denotes the order to which the initial state radiation is taken into account. $l = 0$ corresponds to the Born approximation. In Fig. 2 we illustrate the relative size of the ISR corrections to the Born cross section. Close to the Z-boson mass $\Delta A_{FB}^{(l)}$ is not a good metric to study the size of the corrections since it diverges because of the zero-crossing of A_{FB}. Away from the Z-boson mass, one sees that the $O(\alpha^2 L^2)$ contributions (blue, dotted line) correct the $O(\alpha)$ prediction (black, dashed line) by several percent. The additional leading-log corrections up to $O(\alpha^6 L^6)$ (red, full line) can lead to further corrections up to the percent level.

In Table 1 we illustrate the effect of the different orders of the QED initial state corrections of the forward–backward asymmetry at the Z-peak and two more values of $s = s_\pm = (91.1 \pm 3.2\,\text{GeV})^2$ [40]. The ISR corrections are not monotonic order by order, even leading to a sign change in the corrected values of A_{FB} at $s = M_Z^2$. Here the ISR corrections are largest.

If we compare the known one-loop corrected values of A_{FB} with the highest radiative correction calculated in this paper we obtain further corrections of -3% for s_- and -1% for s_+. These values supersede previous estimates, based on assumptions, in [40].

If cuts of s' are applied the ISR corrections turn out to be smaller, as illustrated in Tables 2 and 3. As noted in [19] also an approximative treatment of the forward–backward asymmetry

$$A_{\text{FB}}(s) \approx \frac{1}{\sigma_T(s)} \int_{z_0}^{1} dz \, \frac{4z}{(1+z)^2} H_e(z) \sigma_{FB}^{(0)}(zs), \tag{40}$$

has been used in the literature [24] close to the Z-peak, where $H_e(z)$ denotes the inclusive radiator appearing in case of $\sigma_T(s)$ [11]. In the

Table 1
A_{FB} evaluated at $s_- = (87.9\,\text{GeV})^2$, M_Z^2 and $s_+ = (94.3\,\text{GeV})^2$ for the cut $z > 4m_\tau^2/s$.

	$A_{FB}(s_-)$	$A_{FB}(M_Z^2)$	$A_{FB}(s_+)$
$\mathcal{O}(\alpha^0)$	−0.3564803	0.0225199	0.2052045
$+\mathcal{O}(\alpha L^1)$	−0.2945381	−0.0094232	0.1579347
$+\mathcal{O}(\alpha L^0)$	−0.2994478	−0.0079610	0.1611962
$+\mathcal{O}(\alpha^2 L^2)$	−0.3088363	0.0014514	0.1616887
$+\mathcal{O}(\alpha^3 L^3)$	−0.3080578	0.0000198	0.1627252
$+\mathcal{O}(\alpha^4 L^4)$	−0.3080976	0.0001587	0.1625835
$+\mathcal{O}(\alpha^5 L^5)$	−0.3080960	0.0001495	0.1625911
$+\mathcal{O}(\alpha^6 L^6)$	−0.3080960	0.0001499	0.1625911

Table 2
A_{FB} evaluated at $s_- = (87.9\,\text{GeV})^2$, M_Z^2 and $s_+ = (94.3\,\text{GeV})^2$ for the cut $z > 0.99$.

	$A_{FB}(s_-)$	$A_{FB}(M_Z^2)$	$A_{FB}(s_+)$
$\mathcal{O}(\alpha^0)$	−0.2714847	0.01545540	0.2539839
$+\mathcal{O}(\alpha L^1)$	−0.2913715	0.01244012	0.2787800
$+\mathcal{O}(\alpha L^0)$	−0.2923065	0.01266498	0.2794039
$+\mathcal{O}(\alpha^2 L^2)$	−0.2922610	0.01489549	0.2760952
$+\mathcal{O}(\alpha^3 L^3)$	−0.2917702	0.01430419	0.2764488
$+\mathcal{O}(\alpha^4 L^4)$	−0.2918380	0.01439578	0.2763865
$+\mathcal{O}(\alpha^5 L^5)$	−0.2918313	0.01438608	0.2763934
$+\mathcal{O}(\alpha^6 L^6)$	−0.2918318	0.01438681	0.2763929

Table 3
A_{FB} evaluated at $s_- = (87.9\,\text{GeV})^2$, M_Z^2 and $s_+ = (94.3\,\text{GeV})^2$ for the cut $z > 0.999$.

	$A_{FB}(s_-)$	$A_{FB}(M_Z^2)$	$A_{FB}(s_+)$
$\mathcal{O}(\alpha^0)$	−0.2683019	0.01538044	0.2576633
$+\mathcal{O}(\alpha L^1)$	−0.2905494	0.01623701	0.2797643
$+\mathcal{O}(\alpha L^0)$	−0.2913658	0.01630129	0.2805232
$+\mathcal{O}(\alpha^2 L^2)$	−0.2899912	0.01655798	0.2787005
$+\mathcal{O}(\alpha^3 L^3)$	−0.2898133	0.01641903	0.2787137
$+\mathcal{O}(\alpha^4 L^4)$	−0.2898401	0.01645212	0.2786941
$+\mathcal{O}(\alpha^5 L^5)$	−0.2898362	0.01644636	0.2786984
$+\mathcal{O}(\alpha^6 L^6)$	−0.2898368	0.01644713	0.2786978

Table 4
Contribution to A_{FB} at different orders of αL evaluated at $s_- = (87.9\,\text{GeV})^2$, M_Z^2 and $s_+ = (94.3\,\text{GeV})^2$ for the cut $z > 4m_\tau^2/s$. All: full contributions, ang.: only the angular dependent contributions.

	all (s_-)	ang. (s_-)	all (M_Z^2)	ang. (M_Z^2)	all (s_+)	ang. (s_+)
$\mathcal{O}(\alpha^2 L^2)$	$-9 \cdot 10^{-3}$	$-9 \cdot 10^{-5}$	$9 \cdot 10^{-3}$	$-1 \cdot 10^{-5}$	$5 \cdot 10^{-4}$	$-6 \cdot 10^{-5}$
$\mathcal{O}(\alpha^3 L^3)$	$8 \cdot 10^{-4}$	$-4 \cdot 10^{-6}$	$-1 \cdot 10^{-3}$	$-8 \cdot 10^{-7}$	$1 \cdot 10^{-3}$	$-3 \cdot 10^{-6}$
$\mathcal{O}(\alpha^4 L^4)$	$-4 \cdot 10^{-5}$	$-3 \cdot 10^{-7}$	$1 \cdot 10^{-4}$	$-7 \cdot 10^{-8}$	$-1 \cdot 10^{-4}$	$-7 \cdot 10^{-8}$
$\mathcal{O}(\alpha^5 L^5)$	$2 \cdot 10^{-6}$	$-1 \cdot 10^{-8}$	$-9 \cdot 10^{-6}$	$-4 \cdot 10^{-9}$	$8 \cdot 10^{-6}$	$1 \cdot 10^{-8}$
$\mathcal{O}(\alpha^6 L^6)$	$-6 \cdot 10^{-8}$	$-6 \cdot 10^{-10}$	$4 \cdot 10^{-7}$	$-1 \cdot 10^{-10}$	$-3 \cdot 10^{-8}$	$2 \cdot 10^{-9}$

Table 5
Contribution to A_{FB} at different orders of αL evaluated at $s_- = (87.9\,\text{GeV})^2$, M_Z^2 and $s_+ = (94.3\,\text{GeV})^2$ for the cut $z > 0.99$. All: full contributions, ang.: only the angular dependent contributions.

	all (s_-)	ang. (s_-)	all (M_Z^2)	ang. (M_Z^2)	all (s_+)	ang. (s_+)
$\mathcal{O}(\alpha^2 L^2)$	$5 \cdot 10^{-5}$	$-5 \cdot 10^{-8}$	$2 \cdot 10^{-3}$	$-2 \cdot 10^{-9}$	$-3 \cdot 10^{-3}$	$5 \cdot 10^{-8}$
$\mathcal{O}(\alpha^3 L^3)$	$5 \cdot 10^{-4}$	$-3 \cdot 10^{-8}$	$-6 \cdot 10^{-4}$	$-1 \cdot 10^{-9}$	$4 \cdot 10^{-4}$	$3 \cdot 10^{-8}$
$\mathcal{O}(\alpha^4 L^4)$	$-7 \cdot 10^{-5}$	$-9 \cdot 10^{-9}$	$9 \cdot 10^{-5}$	$-3 \cdot 10^{-10}$	$-6 \cdot 10^{-5}$	$9 \cdot 10^{-9}$
$\mathcal{O}(\alpha^5 L^5)$	$7 \cdot 10^{-6}$	$-2 \cdot 10^{-9}$	$-1 \cdot 10^{-5}$	$-4 \cdot 10^{-11}$	$7 \cdot 10^{-6}$	$2 \cdot 10^{-9}$
$\mathcal{O}(\alpha^6 L^6)$	$-5 \cdot 10^{-7}$	$-2 \cdot 10^{-10}$	$7 \cdot 10^{-7}$	$-4 \cdot 10^{-12}$	$-5 \cdot 10^{-7}$	$2 \cdot 10^{-10}$

sub–leading corrections to the radiators [11], Wilson coefficients appear, which are different for σ_T and σ_{FB}. Therefore the radiator for σ_T cannot be used in its sub–leading terms for this reason. The numerical results in Table 4–6 show, that contributions due to the angular radiators are indeed suppressed if measuring at the Z-peak w.r.t. the inclusive ones also in higher orders. This holds quite irrespectively of the cuts in s'. For completeness we also quantify the final state and initial–final interference contributions at $O(\alpha)$, cf. e.g. [16,19,21,22] for the respective numerator, σ_{FB}, and denominator, σ_T contributions to $A_{FB} = \sigma_{FB}/\sigma_T$. The inclusive final state correction to σ_{FB} and σ_T imply the factors

$$1 + \delta_{FB}^F, \quad \text{with } \delta_{FB}^F = 0 \quad \text{and} \quad 1 + \delta_T^F, \quad \text{with } \delta_T^F = 3a \tag{41}$$

Table 6

Contribution to A_{FB} at different orders of αL evaluated at $s_- = (87.9\,\text{GeV})^2$, M_Z^2 and $s_+ = (94.3\,\text{GeV})^2$ for the cut $z > 0.999$. All: full contributions, ang.: only the angular dependent contributions.

	all (s_-)	ang. (s_-)	all (M_Z^2)	ang. (M_Z^2)	all (s_+)	ang. (s_+)
$\mathcal{O}(\alpha^2 L^2)$	$1 \cdot 10^{-3}$	$-5 \cdot 10^{-10}$	$3 \cdot 10^{-4}$	$2 \cdot 10^{-11}$	$-2 \cdot 10^{-3}$	$5 \cdot 10^{-10}$
$\mathcal{O}(\alpha^3 L^3)$	$2 \cdot 10^{-4}$	$-4 \cdot 10^{-10}$	$-1 \cdot 10^{-4}$	$2 \cdot 10^{-11}$	$1 \cdot 10^{-5}$	$4 \cdot 10^{-10}$
$\mathcal{O}(\alpha^4 L^4)$	$-3 \cdot 10^{-5}$	$-2 \cdot 10^{-10}$	$3 \cdot 10^{-5}$	$9 \cdot 10^{-12}$	$-2 \cdot 10^{-5}$	$2 \cdot 10^{-10}$
$\mathcal{O}(\alpha^5 L^5)$	$4 \cdot 10^{-6}$	$-5 \cdot 10^{-11}$	$-6 \cdot 10^{-6}$	$3 \cdot 10^{-12}$	$4 \cdot 10^{-6}$	$5 \cdot 10^{-11}$
$\mathcal{O}(\alpha^6 L^6)$	$-5 \cdot 10^{-7}$	$-1 \cdot 10^{-11}$	$8 \cdot 10^{-7}$	$5 \cdot 10^{-13}$	$-6 \cdot 10^{-7}$	$1 \cdot 10^{-11}$

for the $\mu^+\mu^-$ final state. The multiplicative numerical correction to A_{FB} is 0.998149.

The initial–final interference term [16,22] has a more involved representation compared to the final state corrections, since there are different correction terms to the different pieces of the angular averaged Born cross section. In particular these terms are important because of logarithmically large terms of $\ln(1 - s'_{\min}/s)$ in the region $s'_{\min}$ close to s as chosen in [40]. The correction term $1 + \delta_{FB}^{FI}$ modifying A_{FB} to $O(\alpha)$ is given by

$$\delta_{FB}^{FI} = -\frac{1}{\sigma_T^{\text{Born}}} \left\{ \sigma_{FB}^{\gamma Z} \text{Re}[R_T^{\gamma\gamma} + R_T^{ZZ}] + \sigma_{FB}^{ZZ} \text{Re}[R_T^{ZZ}] \right\}$$
$$+ \frac{1}{A_{FB}} \left\{ \text{Re}[R_{FB}^{\gamma\gamma}] \frac{\sigma_T^{\gamma\gamma}}{\sigma_T^{\text{Born}}} + \text{Re}[R_{FB}^{\gamma\gamma} + R_{FB}^{ZZ}] \frac{\sigma_T^{\gamma Z}}{\sigma_T^{\text{Born}}} + \text{Re}[R_{FB}^{ZZ}] \frac{\sigma_T^{ZZ}}{\sigma_T^{\text{Born}}} \right\}, \tag{42}$$

with R_A^{ij}, $A = T, FB$, $i, j = \gamma, Z$, the corresponding radiators [16,22,26] and

$$\delta_{FB}^{FI} = -0.0821 \qquad \text{for } \sqrt{s} = 87.9\,\text{GeV} \quad s'/s > 0.99 \tag{43}$$

$$\delta_{FB}^{FI} = +0.0928 \qquad \text{for } \sqrt{s} = 94.3\,\text{GeV} \quad s'/s > 0.99 \tag{44}$$

$$\delta_{FB}^{FI} = -0.1954 \qquad \text{for } \sqrt{s} = 87.9\,\text{GeV} \quad s'/s > 0.999 \tag{45}$$

$$\delta_{FB}^{FI} = +0.2223 \qquad \text{for } \sqrt{s} = 94.3\,\text{GeV} \quad s'/s > 0.999 \tag{46}$$

The tight cuts in (43)–(46) imply a correction term of up to $\pm 9\%$ for $z > 0.99$ and up to $\pm 22\%$ for $z > 0.999$.

We have calculated the QED initial state corrections to the forward-backward asymmetry for $e^+e^- \to \gamma^*/Z^*$ in the leading logarithmic approximation up to $O(\alpha^6 L^6)$ in analytic form, which extends the known corrections up to $O(\alpha^2 L^2)$. The radiators turn out to be expressible in terms of cyclotomic harmonic sums in Mellin-space and cyclotomic harmonic polylogarithms in z-space. With the new radiators we find corrections of a few percent for the cut $z > 4m_\tau^2$, which decrease if tighter cuts are applied. These corrections become important at planned future e^+e^- facilities which operate with high luminosities at energies at or close to the Z-boson mass.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

Appendix A. *N* space results

In the following we are presenting the first radiators in Mellin N-space.

$$H_{FB}^{(1),LL}(N) = 0 \tag{47}$$

$$H_{FB}^{(2),LL}(N) = \frac{8(3N^2 + 3N - 1)P_1}{(N-1)N^2(N+1)^2(N+2)(2N-1)(2N+3)}$$
$$- \frac{32(4N^2 + 4N - 1)(-1)^N}{(2N-1)(2N+1)(2N+3)}[S_{-1} + \ln(2)] \tag{48}$$

$$H_{FB}^{(3),LL}(N) = \frac{64(3N^2 + 3N - 1)P_2}{(N-1)N^2(N+1)^2(N+2)(2N-1)(2N+3)} S_1$$
$$- \frac{4P_7}{3(N-1)^2 N^3(N+1)^3(N+2)^2(2N-3)(2N-1)^2(2N+3)^2(2N+5)}$$
$$+ (-1)^N \left\{ \frac{8P_4}{N(N+1)(2N-1)(2N+1)(2N+3)} S_{-2} \right.$$

$$-\frac{256(4N^2+4N-1)}{(2N-1)(2N+1)(2N+3)}\left[S_{-1,1}-\frac{1}{2}\ln^2(2)+\sum_{i=1}^{N}\frac{\ln(2)+S_{-1}(i)}{1+2i}\right]$$

$$-\frac{128P_5}{3(2N-3)(2N-1)^2(2N+1)^2(2N+3)^2(5+2N)}S_{-1}$$

$$+\frac{4P_3}{N(N+1)(2N-1)(2N+1)(2N+3)}\zeta_2$$

$$\left.-\frac{128P_6}{3(2N-3)(2N-1)^2(2N+1)^2(2N+3)^2(2N+5)}\ln(2)\right\}, \tag{49}$$

with the polynomials

$$P_1 = 4N^4+8N^3-N^2-5N-3 \tag{50}$$

$$P_2 = 4N^4+8N^3-N^2-5N-3, \tag{51}$$

$$P_3 = 32N^4+16N^3-48N^2-14N+9, \tag{52}$$

$$P_4 = 64N^4+80N^3-24N^2-22N+9, \tag{53}$$

$$P_5 = 128N^7+256N^6-320N^5-144N^4+1128N^3+104N^2-936N+27, \tag{54}$$

$$P_6 = 896N^7+2944N^6-512N^5-7344N^4-2760N^3+2816N^2+36N-243, \tag{55}$$

$$P_7 = 5120N^{14}+35840N^{13}+73856N^{12}-22784N^{11}-252848N^{10}-201200N^9$$
$$+193136N^8+264800N^7-43255N^6-113445N^5+7512N^4+13587N^3$$
$$+1539N^2+8262N-1620. \tag{56}$$

Since the expressions for the higher order radiators are voluminous, they are given in a computer-readable file attached to this paper. Here the functions $S_{\vec{a}} \equiv S_{\vec{a}}(N)$ denote the harmonic sums [36,37]

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N}\frac{(\text{sign}(b))^k}{k^{|b|}}S_{\vec{a}}(k), \quad S_\emptyset = 1, \quad a,b_i \in \mathbb{Z}\backslash\{0\}. \tag{57}$$

Despite the fact that already at 2nd order in z–space cyclotomic harmonic polylogarithms occur, cyclotomic harmonic sums are only contributing from 3rd order onward. As usual, the N–space expressions turn out to be structurally simpler than the z–space expressions. However, there are evanescent poles at half integer arguments $N = 1/2, 3/2, \ldots$. As has been seen in the small z expansion of the radiators, they are all tractable for $N > 1$. The pure singlet kernels are introducing a rightmost pole at $N = 1$ and poles left to this are allowed.

One may numerically represent the higher order corrections also starting from Mellin N space, as described in Refs. [38,45]. For this asymptotic representations and the recurrences of the respective expressions need to be known, which can be easily obtained using the package HarmonicSums. The recurrences are implied by the quantities itself in terms of the hierarchic sum–structures.

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