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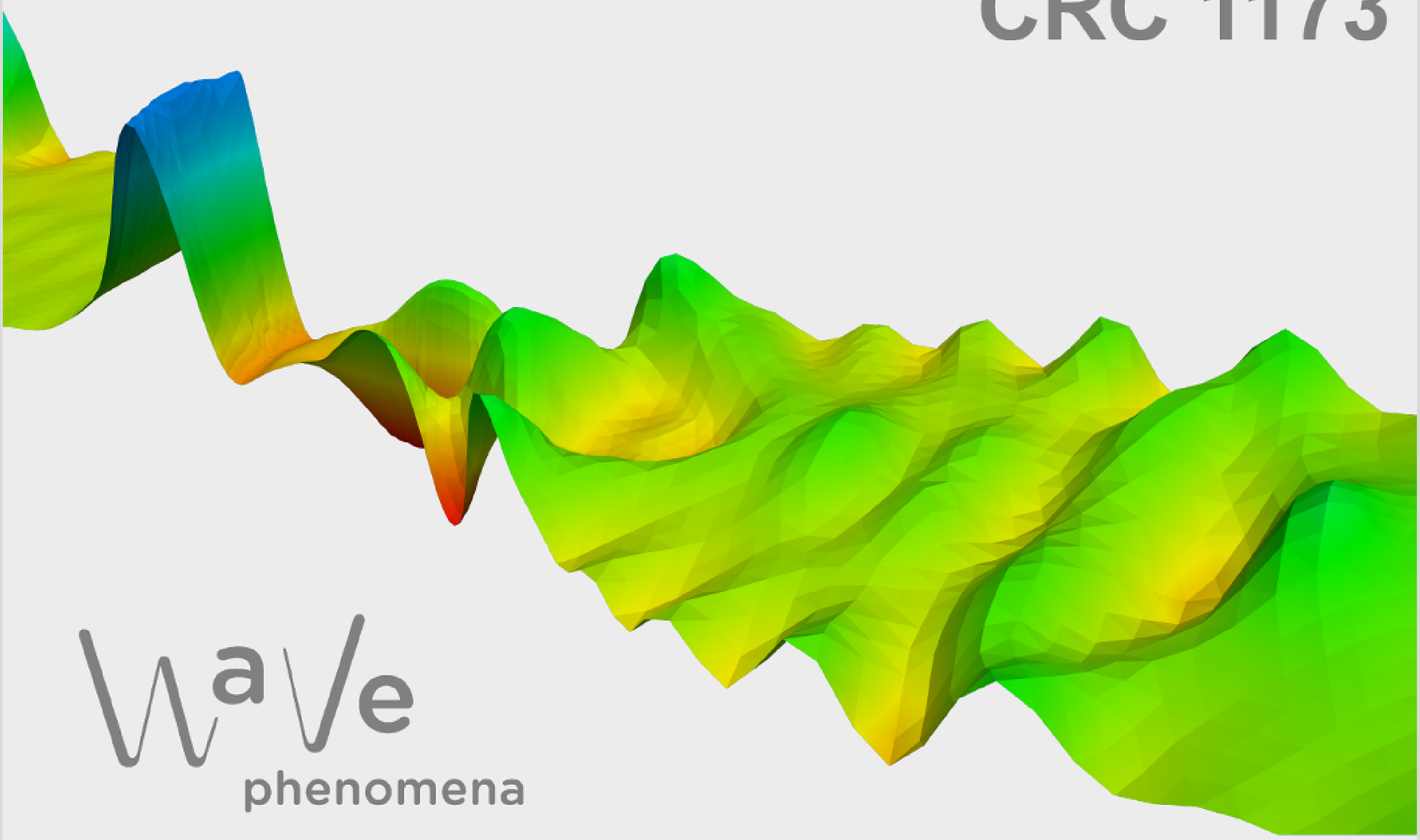

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NONLINEAR HELMHOLTZ EQUATIONS WITH SIGN-CHANGING DIFFUSION COEFFICIENT

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ABSTRACT. In this paper we study nonlinear Helmholtz equations with sign-changing diffusion coefficients on bounded domains. The existence of an orthonormal basis of eigenfunctions is established making use of weak T-coercivity theory. All eigenvalues are proved to be bifurcation points and the bifurcating branches are investigated both theoretically and numerically. In a one-dimensional model example we obtain the existence of infinitely many bifurcating branches that are mutually disjoint, unbounded, and consist of solutions with a fixed nodal pattern.

1. Introduction and main results

In this paper we are interested in nonlinear sign-changing transmission problems of the form

$$-\operatorname{div}(\sigma(x)\,\nabla u) - \lambda\, c(x)\, u = g(x,u) \quad \text{in } \Omega, \qquad u \in \mathrm{H}_0^1(\Omega) \tag{1}$$

for a bounded open domain $\Omega \subset \mathbb{R}^N$. The diffusion coefficient σ is sign-changing in the sense that there are two mutually disjoint subdomains Ω_+ and Ω_-, where σ is assumed to be positive and negative, respectively. Up to now, the linear theory dealing with the well-posedness of such problems for right-hand sides $g(x,z) = f(x)$ has been studied to some extent both analytically and numerically [5, 4, 8, 2, 7, 3]. Here, the main difficulty is that the differential operator $u \mapsto -\operatorname{div}(\sigma(x)\,\nabla u)$ is not elliptic on the whole domain Ω, but only on the two subdomains Ω_-, Ω_+. This implies that the associated bilinear form $(u,v) \mapsto \int_\Omega \sigma(x)\,\nabla u \cdot \nabla v \,\mathrm{d}x$ is not semi-bounded. Accordingly, the standard theory for elliptic boundary value problems does not apply. In the papers [5, 4] the (weak) T-coercivity approach was introduced to develop a solution theory for such linear problems. Our intention is to use and extend this technique in order to study nonlinear equations of the form Eq. (1) both analytically and numerically. We believe this to be interesting for the following reasons:

(i) Nonlinear permittivity and permeability effects in optical (meta-) materials lead to the study of nonlinear Helmholtz-type problems like Eq. (1). Given the importance of the Kerr-nonlinearity $g(x,u) = u^3$ we will mostly focus on this special case to keep the presentation simple.

(ii) The linear theory and especially the spectral theory for linear weakly T-coercive problems is fundamental for our analysis of Eq. (1), which relies on Bifurcation Theory and Critical Point Theory (or Calculus of Variations). Our theoretically predicted and numerically computed solutions of Eq. (1) from Theorem 2 emanate from eigenvalues of the underlying linear problem. As a consequence, our nonlinear analysis may be seen both as an application and as an extension of the linear theory. It is the first contribution in this field.

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(iii) The analysis of classical nonlinear elliptic boundary value problems like

$$-\Delta u - \lambda\, c(x)\, u = g(x,u) \quad \text{in } \Omega, \qquad u \in \mathrm{H}_0^1(\Omega)$$

is very well understood. Typical features of such elliptic boundary value problems are the existence of infinitely many unbounded branches that bifurcate from the trivial solution and carry solutions having a specific property that prevents the branches from intersecting each other. One outcome of our analysis is that similar properties may be expected to hold in the context of Eq. (1) where σ is sign-changing. In a one-dimensional model example we prove this rigorously.

We now fix our assumptions on Ω and the coefficient functions σ, c appearing in Eq. (1).

Assumption (A).

(1) $\Omega \subset \mathbb{R}^N$ for $N \geq 1$ is a bounded domain and there are open subsets $\Omega_+, \Omega_- \subset \Omega$ such that $\overline{\Omega_+ \cup \Omega_-} = \overline{\Omega}$ and $\Omega_+ \cap \Omega_- = \varnothing$.
(2) $\sigma > 0$ on Ω_+, $\sigma < 0$ on Ω_- and $|\sigma| + |\sigma|^{-1} \in \mathrm{L}^\infty(\Omega)$.
(3) $c \in \mathrm{L}^\infty(\Omega)$ with $c(x) \geq \alpha > 0$ for almost all $x \in \Omega$.

Remark 1.

(a) We can equally assume $c(x) \leq -\alpha < 0$ simply by considering $-\lambda$ instead of λ. So this is just a matter of convenience. On the contrary we cannot assume c to be sign-changing since we will need that $\langle u, v\rangle_c := \int_\Omega c(x)\, u(x) v(x)\, \mathrm{d}x$ defines an inner product on $\mathrm{L}^2(\Omega)$. In Remark 9 (b) we will show that one cannot expect our results to hold for general sign-changing c. This motivates why we restrict our considerations to positive coefficient functions.
(b) It is interesting to compare our results for sign-changing σ and $c \equiv 1$ with analogous ones dealing with sign-changing c and $\sigma \equiv 1$. We will see in Remark 9 (c) that the corresponding differential operators have similar qualitative spectral properties. On the other hand, it turns out that there are subtle differences regarding the existence of positive eigenfunctions.
(c) We need not require a priori smoothness properties of Ω or the interface $\Gamma := \overline{\Omega_+} \cap \overline{\Omega_-}$, but imposing those is natural when it comes to verify Assumption (B) below.

We say that λ is an eigenvalue if Eq. (1) has a nontrivial solution for $g \equiv 0$. In this case any such solution is called an eigenfunction. In our context, the geometric multiplicity will coincide with the algebraic multiplicity because the corresponding abstract formulation involves self-adjoint compact operators. As usual, eigenvalues turn out to be the only candidates for bifurcation points on the trivial solution branch $\{(0,\lambda) : \lambda \in \mathbb{R}\}$. We recall that $(0,\lambda)$ is a bifurcation point if there is a sequence $(u_n, \lambda_n)_{n\in\mathbb{N}} \subset \mathrm{H}_0^1(\Omega) \times \mathbb{R}$ of solutions with $u_n \neq 0$ for all $n \in \mathbb{N}$ and $(u_n, \lambda_n) \to (0, \lambda)$ in $\mathrm{H}_0^1(\Omega) \times \mathbb{R}$. To get a good spectral theory for the differential operator $u \mapsto -c(x)^{-1} \operatorname{div}(\sigma(x)\nabla u)$ we impose some weak $\mathtt{T}$-coercivity assumption on the bilinear form

$$a : \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0^1(\Omega) \to \mathbb{R}, \quad (u, v) \mapsto \int_\Omega \sigma(x)\, \nabla u \cdot \nabla v\, \mathrm{d}x.$$

Assumption (B). There is a bounded linear invertible operator $\mathtt{T} : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ such that the bilinear form $(u, v) \mapsto a(u, \mathtt{T}v) + \langle \mathtt{K}u, v\rangle_{\mathrm{H}_0^1(\Omega)}$ is continuous and coercive on $\mathrm{H}_0^1(\Omega) \times \mathrm{H}_0^1(\Omega)$ for some compact operator $\mathtt{K} : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$.

We will see in Section 3 and in particular Corollary 8 that Assumptions (A) and (B) ensure the existence of an orthonormal basis of eigenfunctions $(\phi_j)_{j\in\mathbb{Z}}$ with associated eigenvalues $(\lambda_j)_{j\in\mathbb{Z}}$ satisfying $\pm\lambda_j \nearrow +\infty$ as $j \to \pm\infty$. Explicit criteria for the validity of Assumption (B) can be found in the Theorems 2.1, 3.1, 3.3, 3.7, 3.10 from [4]. On the other hand, it is known that Assumption (B) does not always hold, for example in 2D if $\sigma_-/\sigma_+ \in \left[-3, -\frac{1}{3}\right]$ and the interface Γ has a right angle corner, see [4, 2]. In our main result we denote by $\mathcal{S}$ the closure of the nontrivial solutions in $\mathrm{H}_0^1(\Omega) \times \mathbb{R}$ of Eq. (1).

Theorem 2. *Assume (A) and (B) and $g(x,u) = u^3, N \in \{1,2,3\}$. Let $(\lambda_j)_{j\in\mathbb{Z}}$ denote the unbounded sequence of eigenvalues of Eq. (1) from Corollary 8. Then each $(0,\lambda_j)$ is a bifurcation point for Eq. (1) and for all $j \in \mathbb{Z}$ such that λ_j has odd geometric multiplicity, the connected component $\mathcal{C}_j$ in $\mathcal{S}$ containing $(0,\lambda_j)$ satisfies Rabinowitz' alternative:*
- *(i) $\mathcal{C}_j$ is unbounded or*
- *(ii) $\mathcal{C}_j$ contains another trivial solution $(0,\lambda_k)$ with $k \neq j$.*

This bifurcation result admits improvements in many directions.

Remark 3.

(a) We will prove much more than the unboundedness of the sequence of eigenvalues. This includes a min-max-characterization, $\pm\lambda_j \nearrow +\infty$ as $j \to \pm\infty$ as well as some eigenvalue asymptotics of the form $c\,(1+|j|)^{\frac{2}{N}} \le 1+|\lambda_j| \le C\,(1+|j|)^{\frac{2}{N}}$ for all $j \in \mathbb{Z}$ and some $c, C > 0$. This implies the following weak Weyl law
$$c'\Lambda^{\frac{N}{2}} \le \mathrm{Card}\left(\{\lambda_j : j \in \mathbb{Z}\} \cap [-\Lambda,\Lambda]\right) \le C'\Lambda^{\frac{N}{2}}, \quad \text{for } \Lambda \ge \Lambda_0$$
and some $c', C', \Lambda_0 > 0$. We refer to Corollary 8 for the details.

(b) The existence of solution continua $\mathcal{C}_j$ can be proved under much weaker assumptions on the nonlinearity since the underlying abstract bifurcation theorems allow for this generality. As an example, one may as well consider nonlinearities $g(x,u) = \Gamma(x)\,|u|^{p-1}u$ with $\Gamma \in \mathrm{L}^\infty(\Omega)$ and $1 < p < \infty$ for $N \in \{1,2\}$ or $1 < p < \frac{N+2}{N-2}$ where now all $N \in \mathbb{N}, N \ge 3$ are possible. Truncating the nonlinearity, *i.e.*, considering $\widetilde{g}(x,u) := g(x,\chi(u))$ for some bounded function $\chi \in C^\infty(\mathbb{R})$ such that $\chi(z) = z$ for $|z| \le 1$, one may even extend the local bifurcation analysis near the trivial solution branch to supercritical $p \ge \frac{N+2}{N-2}$.

(c) We shall see in Corollary 4 that it is possible to rule out alternative (ii) at least in some special one-dimensional case. So the continua $\mathcal{C}_j$ are unbounded in this case, which is in line with our illustrations, see Section 2. We believe this behavior to be typical also in the general case. The standard way to prove the unboundedness of branches is to identify some characteristic j-dependent property that all nontrivial solutions on $\mathcal{C}_j$ satisfy and that makes alternative (ii) impossible. In the context of elliptic problems, nodal patterns ("number of interior zeros") are typically considered. We show that at least in the one-dimensional case similar characterizations of the branches are possible.

(d) In the case of a simple eigenvalue the Crandall-Rabinowitz Theorem (cf. Theorem 12) allows to say more about $\mathcal{C}_j$ in a small neighborhood of the bifurcation point. This theorem shows that for each such j there is a smooth curve $s \mapsto (\widehat{u}_j(s), \widehat{\lambda}_j(s)) \in \mathrm{H}_0^1(\Omega) \times \mathbb{R}$ such that the continuum $\mathcal{C}_j$ coincides with this curve near $(\widehat{u}_j(0), \widehat{\lambda}_j(0)) = (0, \lambda_j)$. Moreover, if $\phi_j \in \mathrm{H}_0^1(\Omega)$ denotes the eigenfunction associated with the eigenvalue λ_j with $\int_\Omega c(x)\phi_j(x)^2\,\mathrm{d}x = 1$, then
$$\widehat{u}_j'(0) = \phi_j, \qquad \widehat{\lambda}_j'(0) = 0, \qquad \widehat{\lambda}_j''(0) = -2\int_\Omega \phi_j(x)^4\,\mathrm{d}x. \tag{2}$$
A proof of these formulas will be given in Remark 3 (a). We emphasize that the predicted bending to the left is reproduced in our numerically produced pictures

from Section 2. Given that the cubic nonlinearity is a real analytic function, further statements about the global shape of $\mathcal{C}_j$ can be deduced from [6, Theorem 9.1.1].

To get a better idea of Theorem 2 we discuss some one-dimensional model example. In this case we can provide more information about the following points:

(i) Assumptions (A) and (B) are satisfied.
(ii) The eigenvalues (λ_j) are simple and Remark 3 (d) applies.
(iii) The eigenpairs (ϕ_j, λ_j) are almost explicitly known.
(iv) Rabinowitz' Alternative (ii) is ruled out by identifying a characteristic property for the nontrivial solutions on $\mathcal{C}_j$, see Remark 3 (c). So $\mathcal{C}_j$ is unbounded for all $j \in \mathbb{Z}$ and $\mathcal{C}_j \cap \mathcal{C}_k = \emptyset$ for $j \neq k$.

We consider the following situation: Assume that $\overline{\Omega} = \overline{\Omega_-} \cup \overline{\Omega_+}$ is an interval with precisely two non-void sub-intervals $\Omega_- = (a_-, 0)$ and $\Omega_+ = (0, a_+)$ with $a_- < 0 < a_+$. (In Section 2 our numerical plots deal with the special case $a_- = -5, a_+ = 5$.) The coefficient function c and σ satisfy $c(x) = c_\pm$ resp. $\sigma(x) = \sigma_\pm$ on $\Omega_\pm$ where $c_\pm > 0$ and $\sigma_+ > 0 > \sigma_-$ are constants. For such domains and coefficients we consider the nonlinear problem

$$-\frac{\mathrm{d}}{\mathrm{d}x}\big(\sigma(x)u'(x)\big) - \lambda c(x)u = u^3 \quad \text{in } \Omega, \qquad u \in \mathrm{H}^1_0(\Omega). \tag{3}$$

Corollary 4. *Assume that $\Omega \subset \mathbb{R}$ and c, σ are as above. Let $(\lambda_j)_{j\in\mathbb{Z}}$ denote the unbounded sequence of eigenvalues of Eq. (3) from Corollary 8. Then the connected component $\mathcal{C}_j \subset \mathrm{H}^1_0(\Omega) \times \mathbb{R}$ in $\mathcal{S}$ containing $(0, \lambda_j)$ is unbounded and we have $\mathcal{C}_j \cap \mathcal{C}_k = \emptyset$ for $j \neq k$. More precisely, for $(u, \lambda) \in \mathcal{C}_j$ with $u \neq 0$ the following holds:*

- *If $j = 0$ then u has no interior zeros in Ω and satisfies $|u'| > 0$ on $\overline{\Omega_\pm}$.*
- *If $j \geq 1$ then u has j interior zeros in Ω_+ and satisfies $|u'| > 0$ on $\overline{\Omega_-}$.*
- *If $j \leq -1$ then u has $|j|$ interior zeros in Ω_- and satisfies $|u'| > 0$ on $\overline{\Omega_+}$.*

Here, $|u'| > 0$ on $\overline{\Omega_\pm}$ means that the continuous extension of $|u'| : \Omega_\pm \to \mathbb{R}$ to $\overline{\Omega_\pm}$ is positive. We stress that nontrivial solutions u are smooth away from the interface $x = 0$ and continuous at $x = 0$, but they are not continuously differentiable at this point. In fact, $\sigma u'$ is continuous on $\overline{\Omega}$ so that $u'(0)$ does not exist in the classical sense. The eigenvalues (λ_j) are characterized by Eqs. (17) and (19) from Lemma 15, which allows to deduce more precise asymptotic as $|j| \to \infty$ beyond the weak Weyl law mentioned in Remark 3 (a). Furthermore, the associated eigenfunctions are given explicitly in terms of λ_j, see Eqs. (16), (18) and (20). They are responsible for the nodal characterization along the bifurcating branches.

Numerical illustrations related to Theorem 2 and Corollary 4 are given in the next section. They illustrate the global behavior of the $\mathcal{C}_j$ as well as the evolution of solutions along these branches. Next, in Section 3, we develop the spectral theory for $\mathtt{T}$-coercive problems. We stress that this theory is essentially known, see for instance [7, Section 1], but we could not find a self-contained and complete treatise in the literature that covers our setting. In Section 4, we use the linear theory and well-known bifurcation theorems to prove Theorem 2 as well as Corollary 4. The proof of the former is abstract while the proof of the latter relies on explicit computations. In Section 5, we will present an alternative variational approach to nonlinear $\mathtt{T}$-coercive problems. It gives the existence of infinitely many nontrivial solutions of Eq. (3) for any given $\lambda \in \mathbb{R}$. The proof is based on the Critical Point Theory from [23, Chapter 4].

2. Visualization of bifurcation results via PDE2path

In this section, we illustrate our theoretical results of Theorem 2 and Corollary 4 with numerical bifurcation diagrams. The results were obtained with the package pde2path [24, 11], version 2.9b and using Matlab 2018b. The code to reproduce the numerical results is available on Zenodo with doi 10.5281/zenodo.5140021.

2.1. **One-dimensional example.** We consider $\Omega = (-5,5)$ with $\Omega_- = (-5,0), \Omega_+ = (0,5)$ and $c \equiv 1$. The diffusion coefficient σ is chosen piecewise constant, set $\sigma_+ = 1$ and compare two different values for σ_-, namely $\sigma_- \in \{-2, -1.001\}$. We consider Eq. (3) in this special case, *i.e.*,

$$-\frac{\mathrm{d}}{\mathrm{d}x}\big(\sigma(x)\,u'\big) - \lambda u = u^3 \quad \text{in } \Omega, \qquad u \in H_0^1(\Omega).$$

We choose a tailored finite element mesh which is refined close to $\Gamma = \{0\}$ in the following way. We start with an equidistant mesh with $h = 2^{-9}$, *i.e.*, Ω is divided into 5120 equal subintervals. Then, we refine all intervals which are closer than 0.1 to Γ five times by halving them. This finally means that intervals close to Γ are only 2^{-14} long. We point out that this finely resolved mesh is required to faithfully represent the sharp interface behavior at $\Gamma = \{0\}$, especially for $\sigma_- = -1.001$. An insufficient mesh resolution does not only influence the numerical quality of the eigenfunctions or solutions along the branches, but also the (qualitative picture) of the bifurcation diagram. We validated our results by assuring that a further refinement of the mesh (halving all intervals) leads to the same results and conclusions.

2.1.1. **Bifurcation diagrams and eigenfunctions for different contrasts.** We first investigate whether $\frac{\sigma_+}{\sigma_-} \approx -1$ influences the bifurcation diagrams. For this, we allow λ to vary in the interval $[-10, 15]$. The bifurcation diagrams are depicted in Fig. 1 for $\sigma_- = -2$ and $\sigma_- = -1.001$. Qualitatively, they are quite similar with clearly separated, apparently unbounded branches without secondary bifurcations. Note that the bending direction of the branches to the left is determined by the sign of the nonlinear term and can be predicted by the bifurcation formulae coming from the Crandall-Rabinowitz Theorem, *cf.* Eq. (2). The first striking phenomenon due to the sign-changing coefficient is the occurrence of eigenvalues and, hence, bifurcation points, with negative value. In fact, for sign-changing σ, there are two families of eigenvalues diverging to $\pm\infty$, see Theorem 2. We use the following labeling of branches (*cf.* Fig. 1): The branch starting closest to zero is labeled as $\mathcal{C}_0$ and the branches for negative and positive bifurcation points are labeled as $\mathcal{C}_{-i}$ and $\mathcal{C}_i$ with $i \in \mathbb{N}$, respectively. The value of i increases as $|\lambda| \to \infty$.

Besides the eigenvalues, we also study the eigenfunctions by considering the solutions at the first point of each branch in Fig. 2. We display the branch name according to Fig. 1 as well as the value of λ at the bifurcation point. As (partly) expected from [7], we make the following observations. Firstly, the solutions are concentrated (w.r.t. the L^2-norm) on the "oscillatory part", which is Ω_- for negative eigenvalues (left column of Fig. 2) and Ω_+ for positive eigenvalues (right column of Fig. 2). The eigenvalue closest to zero (from which $\mathcal{C}_0$ emanates) plays a special role (middle column of Fig. 2). Secondly, with increasing $|\lambda|$, the number of maxima and minima increases as one observes also for the eigenfunctions of the Laplacian. Thirdly, the transmission condition at Γ requires the (normal) derivative of u to change sign, such that the solutions show a sharp interface behavior. As the jump of the gradient depends on the contrast, this effect becomes more and more pronounced the closer the contrast gets to -1. Taking a closer look at the bifurcation values and the corresponding solutions in Fig. 2, we note that $\mathcal{C}_0$ starts much closer to zero for $\sigma_- = -1.001$ than for $\sigma_- = -2$. This illustrates the theoretical

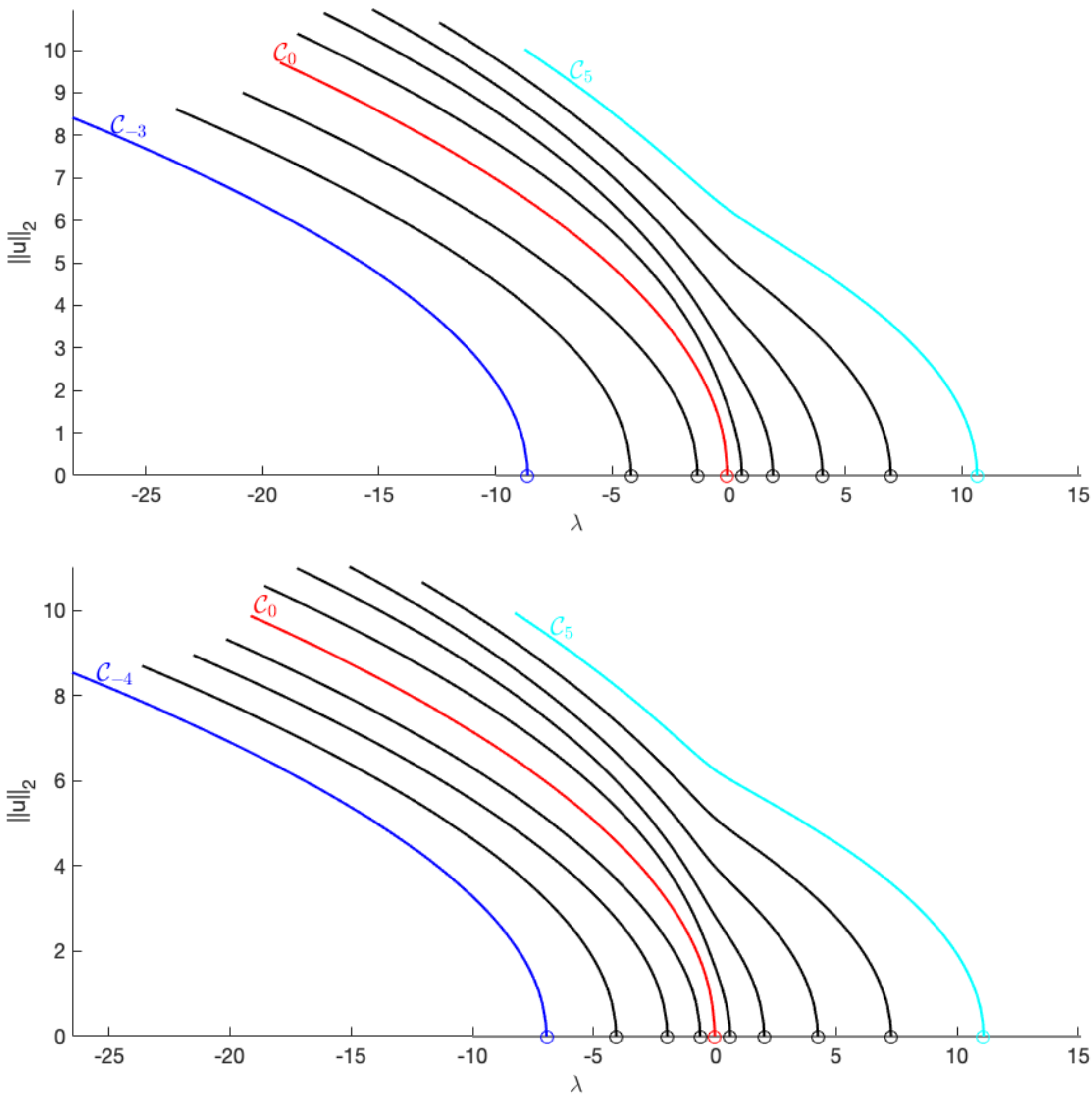

FIGURE 1. Bifurcation diagrams for $\sigma_- = -2$ (top) and $\sigma_- = -1.001$ (bottom).

expectation that for a contrast close to -1, we have an eigenvalue which approaches zero. Moreover, we observe a certain shrinking of the negative bifurcation values towards zero when the contrast approaches -1.

2.1.2. **Patterns of solutions along branches.** We now take a closer look at how solutions evolve along branches — depending on whether the corresponding bifurcation value is negative, close to zero or positive. According to the previous discussion, we focus on $\sigma_- = -1.001$ in the following because it shows the phenomena in a particularly pronounced form and is close to the interesting "critical" contrast of -1. In general, we observe that a certain limit pattern or profile of the solution evolves on each branch which remains qualitatively stable (values of maxima, minima and plateaus of course change with λ). As example for a negatively indexed bifurcation branch away from zero, we consider $\mathcal{C}_{-2}$, *cf.* Fig. 1. The first, 50th, and 100th solution on the branch are depicted in Fig. 3. As described above, the solution concentrates in Ω_- where it oscillates, while it decays exponentially in Ω_+. This profile remains stable over the branch, but we note that the maxima and minima become wider along the branch. This widening of the extrema in Ω_- is also

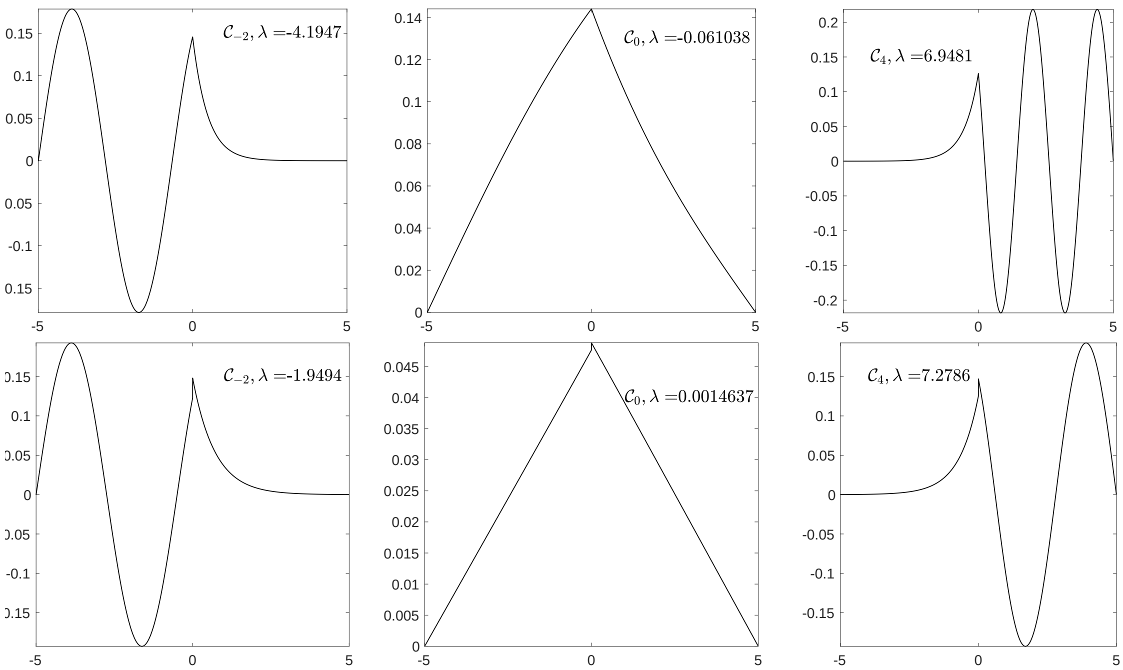

FIGURE 2. First solution on branches $\mathcal{C}_{-2}$, $\mathcal{C}_0$ and $\mathcal{C}_4$ (from left to right) for $\sigma_- = -2$ (top row) as well as $\sigma_- = -1.001$ (bottom row).

noted for the other branches emanating from a negative bifurcation point. Yet, the more oscillations occur for the branches as $\lambda \to -\infty$, the less pronounced the effect becomes because we have more extrema over the same interval. We emphasize that this effect of widening extrema is specific to the sign-changing case and especially to bifurcations starting at negative λ.

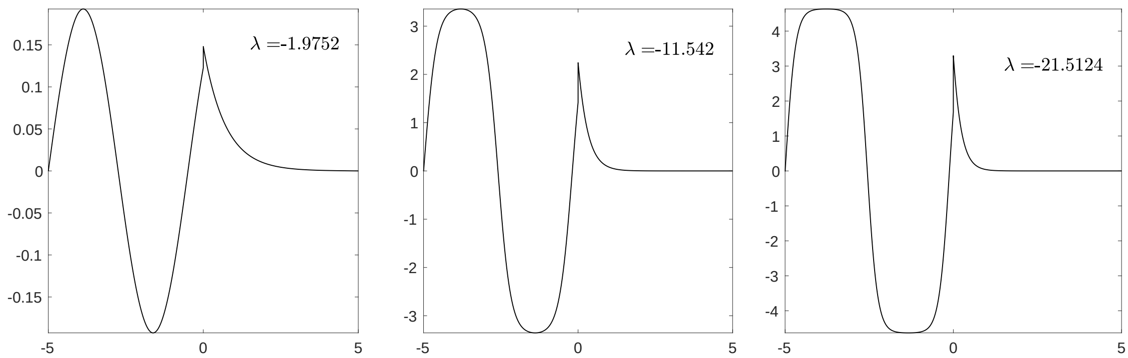

FIGURE 3. Solution at first, 50th and 100th point of branch $\mathcal{C}_{-2}$ for $\sigma_- = -1.001$.

As an example for a positively indexed bifurcation branch away from zero, we study the branch $\mathcal{C}_5$, cf. Fig. 1. As expected, we observe in Fig. 4 that the first solution concentrates on Ω_+, where it oscillates as typical for an eigenfunction of the Laplacian, and shows an exponential decay in Ω_-. The oscillatory pattern in Ω_+ is preserved along the branch. The behavior in Ω_- changes when λ gets negative: Instead of an exponential decay to zero, we now see an exponential decay to (almost) a plateau (with value $\pm\sqrt{-\lambda}$) and a sharp transition to the zero boundary value. Once this pattern is established, it remains stable as well. This appearance of a plateau different from zero is also a specific phenomenon of the sign-changing case.

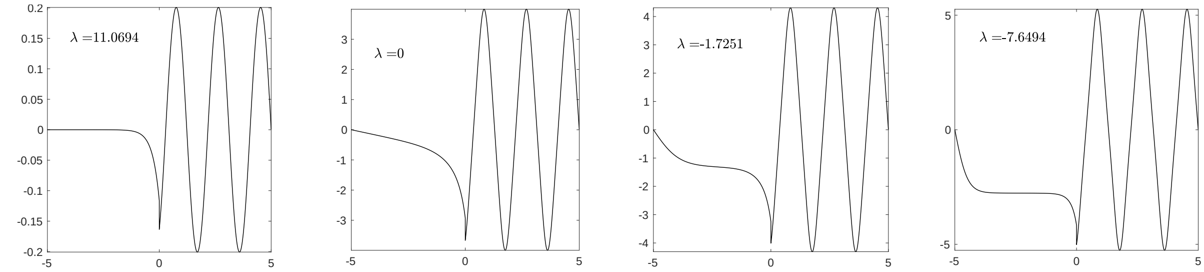

FIGURE 4. Solution at first, 61st, 70th, and 100th point of branch $\mathcal{C}_5$ for $\sigma_- = -1.001$.

The occurrence of a plateau in Ω_- is also observed in Fig. 5 for the branch $\mathcal{C}_0$ closest to zero, cf. Fig. 1. While the first solution has a similar shape in Ω_+ and Ω_- with a linear decay in each subdomain, the ensuing solutions on the branch quickly evolve a plateau in Ω_- and an exponential decay in Ω_+. This pattern then remains stable along the branch. For the case of the standard Laplacian, in contrast, the solutions on the branch of the first, smallest eigenvalue decays exponentially on both sides of $x = 0$ and thereby establishes a spike-like form without any plateaus.

All in all, we observe a certain stability of profiles along branches. The form of the profiles depends on where the bifurcation starts. Moreover, we always recognize a concentration to the oscillatory part and further the establishment of plateaus different from zero in Ω_-. As already emphasized, both effects are specific to the sign-changing case. This qualitative description of solutions seems to transfer to other contrasts, but the bifurcation points closest to zero and the (quantitative) decay in $\Omega_\pm$ significantly depend on the contrast as already discussed above.

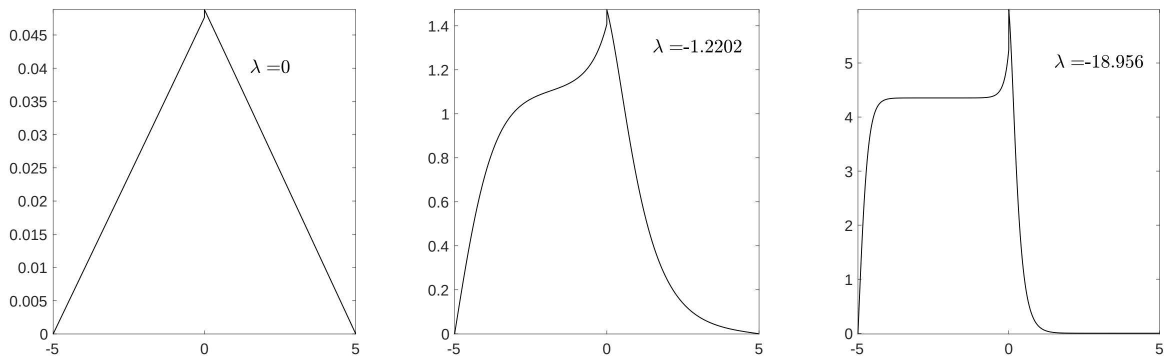

FIGURE 5. Solution at first, 10th, and 100th point of branch $\mathcal{C}_0$ for $\sigma_- = -1.001$.

2.2. **Two-dimensional example.** We consider $\Omega = (-2,2)^2$ with $\Omega_- = (-2,0)\times(-2,2)$, as well as $\sigma_+ = 1$ and $\sigma_- = -2$. The finite element mesh is tailored similar to the one-dimensional experiment: We start with a symmetric uniform mesh with $h = 2^{-4}$ and refine three times all elements in the strip of width 0.1 around the interface $\Gamma = \{0\} \times (-2, 2)$.

We focus on the behavior of solutions in this numerical experiment and let λ vary in $[-12, 15]$. There are three different types of eigenfunctions either concentrated on Ω_-, on Γ, or on Ω_+. In contrast to the one-dimensional case, there are several different eigenfunctions concentrated on Γ. As before, the eigenfunctions concentrate on Ω_- or Γ are associated with negative values of λ. In Figs. 6 to 8, we show the evolution of solutions along a branch for each of the three types described above.

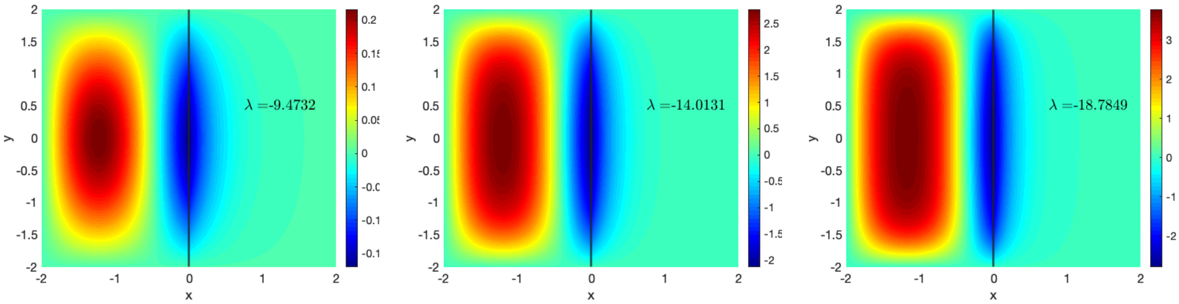

FIGURE 6. Solution at first, 25th, and 50th point of branch associated with an eigenfunction concentrated on Ω_-.

Similar to the one-dimensional case, we observe a widening of the extrema along the branch with concentration in Ω_- in Fig. 6, in particular in the y-direction. Furthermore,

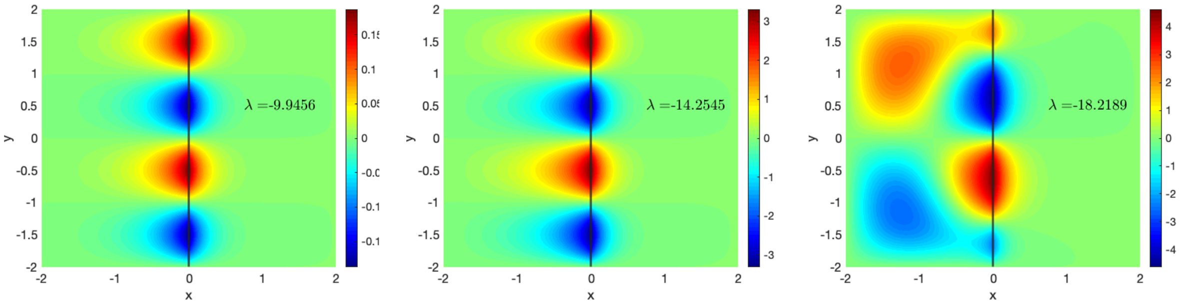

FIGURE 7. Solution at first, 25th, and 50th point of branch associated with an eigenfunction concentrated on Γ.

plateaus in Ω_- evolve for negative λ in Figs. 7 and 8. Due to the second space dimension in the problem, we can have two (or more) different plateaus evolving in Ω_-. In Fig. 7 for a branch with concentration on Γ, we note that the plateaus and the transition between them seems to slightly change the oscillatory pattern on Γ as well. While the two maxima have almost the same height for the first and 25th point (Fig. 7 left and middle), one maximum becomes predominant for the 50th point on the branch, see Fig. 7 right. Finally,

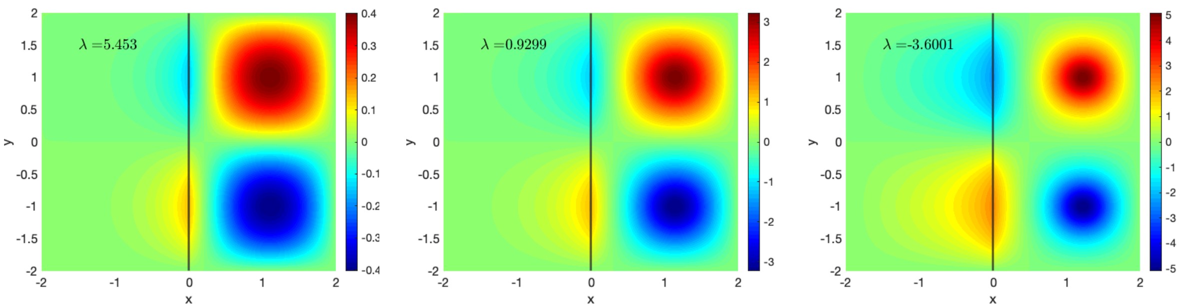

FIGURE 8. Solution at first, 25th, and 50th point of branch associated with an eigenfunction concentrated on Ω_+.

for Fig. 8 and a branch with concentration on Ω_+, we emphasize that the solution in Ω_+ evolves like a solution of the standard Laplacian along a branch. In particular, the extrema become thinner, *i.e.*, more spatially localized, which should be contrasted with solutions concentrated in Ω_- in Fig. 6.

3. Linear Theory

In this section we want to describe the linear theory for weakly $\mathtt{T}$-coercive problems. As pointed out earlier, this theory is essentially well-known [5, 4, 7]. Since it is short and rather self-contained, we provide the details here, which will moreover allow us to fix the required notation. Furthermore, we prove some Weyl law asymptotics that have not appeared in the literature yet. We want to deal with linear problems of the form

$$\int_\Omega \sigma(x)\,\nabla u\cdot\nabla v\,\mathrm{d}x-\lambda\int_\Omega c(x)\,u\,v\,\mathrm{d}x=F(v),\qquad \forall v\in \mathrm{H}_0^1(\Omega). \tag{4}$$

The a priori unknown solution u is to be found in the Sobolev space $\mathrm{H}_0^1(\Omega)$ and the coefficient functions σ and c are assumed to satisfy the conditions (A) and (B) from Section 1. To develop a solution theory for the variational problem Eq. (4) both in $\mathrm{H}_0^1(\Omega)$ and $\mathrm{L}^2(\Omega)$ we assume $F\in \mathrm{H}^{-1}(\Omega)=\mathrm{H}_0^1(\Omega)'$. We introduce the inner products

$$\langle u,v\rangle_{\mathrm{H}_0^1(\Omega)}=\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x,\qquad \langle u,v\rangle_c=\int_\Omega c(x)\,u\,v\,\mathrm{d}x.$$

In the case of diffusion coefficients σ with a fixed sign, the bilinear form

$$a:\mathrm{H}_0^1(\Omega)\times\mathrm{H}_0^1(\Omega)\to\mathbb{R},\quad (u,v)\mapsto\int_\Omega \sigma(x)\nabla u\cdot\nabla v\,\mathrm{d}x$$

is coercive so that the Lax-Milgram Lemma allows to transform Eq. (4) into a linear equation of Fredholm type in the Hilbert space $(\mathrm{L}^2(\Omega),\langle\cdot,\cdot\rangle_c)$, see [13, Section 6.2.3]. The solution theory for such equations is perfectly understood and the natural question is to which extent this theory carries over to the case of sign-changing coefficients σ where the bilinear form a is no longer coercive. In that case a is not even weakly coercive in the sense that there is no compact operator $K:\mathrm{H}_0^1(\Omega)\to\mathrm{H}_0^1(\Omega)$ such that $(u,v)\mapsto a(u,v)+\langle Ku,v\rangle_{\mathrm{H}_0^1(\Omega)}$ is coercive. In [5, 4] the $\mathtt{T}$-coercivity approach was developed to get a solution theory for such strongly indefinite linear problems. The idea is to require the existence of an invertible linear operator $\mathtt{T}:\mathrm{H}_0^1(\Omega)\to\mathrm{H}_0^1(\Omega)$ such that the bilinear form $(u,v)\mapsto a(u,\mathtt{T}v)$ is coercive. In the case of weak $\mathtt{T}$-coercivity one requires that $(u,v)\mapsto a(u,\mathtt{T}v)$ is weakly coercive in the sense explained above. In Assumption (B) we require that the bilinear form a is weakly $\mathtt{T}$-coercive, which is equivalent to assuming that the self-adjoint operator generating the bilinear form is Fredholm. This will allow us to deduce a Fredholm Alternative for Eq. (4) involving self-adjoint operators as well as the existence of an orthonormal basis of eigenfunctions. In the following we provide the counterpart of the classical Fredholm Theory and Spectral Theory for Eq. (4) with Assumptions (A) and (B).

We introduce the bounded linear operator $A:\mathrm{H}_0^1(\Omega)\to\mathrm{H}_0^1(\Omega)$ defined by the relation $\langle Au,v\rangle_{\mathrm{H}_0^1(\Omega)}:=a(u,v)$. This is possible by Riesz' Representation Theorem and $\sigma\in\mathrm{L}^\infty(\Omega)$, see Assumption (A). In the following we denote by $\iota:\mathrm{H}_0^1(\Omega)\to\mathrm{L}^2(\Omega)$ the compact embedding operator, $(-\Delta)^{-1}:\mathrm{L}^2(\Omega)\to\mathrm{H}_0^1(\Omega)$ is given by

$$\left\langle(-\Delta)^{-1}\iota u,v\right\rangle_{\mathrm{H}_0^1(\Omega)}=\left\langle u,(-\Delta)^{-1}\iota v\right\rangle_{\mathrm{H}_0^1(\Omega)}=\int_\Omega uv\,\mathrm{d}x,\qquad \text{for } u,v\in\mathrm{H}_0^1(\Omega). \tag{5}$$

The compact operator $C=(-\Delta)^{-1}(c\,\cdot)\iota:\mathrm{H}_0^1(\Omega)\to\mathrm{H}_0^1(\Omega)$ is given by

$$\langle Cu,v\rangle_{\mathrm{H}_0^1(\Omega)}=\langle u,v\rangle_c\qquad \text{for } u,v\in\mathrm{H}_0^1(\Omega).$$

Proposition 5. *Under Assumptions (A) and (B), there exists $\ell\in\mathbb{R}$ such that the bounded linear operator $A_\ell:=A+\ell C:\mathrm{H}_0^1(\Omega)\to\mathrm{H}_0^1(\Omega)$ is self-adjoint and invertible.*

Proof. The self-adjointness follows from $a(u,v) = a(v,u)$ and

$$\langle Cu, v\rangle_{\mathrm{H}_0^1(\Omega)} = \langle u, v\rangle_c = \langle v, u\rangle_c = \langle u, Cv\rangle_{\mathrm{H}_0^1(\Omega)},$$

for all $u, v \in \mathrm{H}_0^1(\Omega)$. To prove the invertibility of A_ℓ define the family of operators $z \mapsto A_z := A + zC$ for $z \in \mathbb{C}$ on the complex Hilbert space $\mathrm{H}_0^1(\Omega;\mathbb{C})$. The bilinear form associated with A_z is given by $(u,v) \mapsto a(u,v) + z\langle u, v\rangle_c$. From Assumption (B) and the Lax-Milgram Lemma we infer that $\mathtt{T}^*A + \mathtt{K}$ is invertible. Moreover, we have the relation

$$A_z = (\mathtt{T}^*)^{-1}[\mathtt{T}^*A + \mathtt{K}] - (\mathtt{T}^*)^{-1}\mathtt{K} + zC.$$

Here, the first summand is invertible while the other two summands are compact. Therefore $\{A_z : z \in \mathbb{C}\}$ is a holomorphic family of Fredholm operators. For $z \in \mathbb{C}\setminus\mathbb{R}$, the operator A_z is injective. Indeed, if $A_z u = 0$ then $\langle A_z u, \overline{u}\rangle_{\mathrm{H}_0^1(\Omega)} = 0$ and

$$0 = \Im\left(\langle A_z u, \overline{u}\rangle_{\mathrm{H}_0^1(\Omega)}\right) = \Im\left(a(u,\overline{u}) + z\,\|u\|_c^2\right) = \Im(z)\,\|u\|_c^2.$$

So, we have $\ker(A_z) = \{0\}$, which implies that A_z has a bounded inverse as an injective Fredholm operator. Using the analytic Fredholm theorem on $\mathbb{C}$, see [12, Theorem C.8], the set $\{A_z^{-1} : z \in \mathbb{C}\}$ is a meromorphic family of operators with poles of finite rank. Therefore, the operator $(A + zC)^{-1}$ exists for all $z \in \mathbb{C}\setminus\Lambda$ for a discrete set $\Lambda \subset \mathbb{R}$. In particular, there exists $\ell \in \mathbb{R}$ such that A_ℓ is an invertible Fredholm operator. □

Using this result we may rewrite Eq. (4) as follows. We start with the equivalent version

$$\langle A_\ell u, v\rangle_{\mathrm{H}_0^1(\Omega)} - (\lambda + \ell)\langle Cu, v\rangle_{\mathrm{H}_0^1(\Omega)} = \left\langle (-\Delta)^{-1}F, v\right\rangle_{\mathrm{H}_0^1(\Omega)}, \qquad \forall v \in \mathrm{H}_0^1(\Omega).$$

Here, $(-\Delta)^{-1}F \in \mathrm{H}_0^1(\Omega)$ denotes the uniquely defined function $w \in \mathrm{H}_0^1(\Omega)$ satisfying $\int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}x = F(v)$ for all $v \in \mathrm{H}_0^1(\Omega)$. The proposition shows that this can be recast as

$$u - (\lambda + \ell)K_c u = KF, \qquad u \in \mathrm{L}^2(\Omega)$$

where

$$K_c := \iota A_\ell^{-1}(-\Delta)^{-1}(c\cdot) \quad \text{and} \quad K := \iota A_\ell^{-1}(-\Delta)^{-1}. \tag{6}$$

We stress that $K_c : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ is self-adjoint with respect to $\langle\cdot,\cdot\rangle_c$ because of

$$\begin{aligned}\langle K_c u, v\rangle_c &\overset{(5)}{=} \left\langle A_\ell^{-1}(-\Delta)^{-1}(cu), (-\Delta)^{-1}(cv)\right\rangle_{\mathrm{H}_0^1(\Omega)}\\ &= \left\langle (-\Delta)^{-1}(cu), A_\ell^{-1}(-\Delta)^{-1}(cv)\right\rangle_{\mathrm{H}_0^1(\Omega)}\\ &= \langle u, K_c v\rangle_c\end{aligned}$$

for all $u, v \in \mathrm{L}^2(\Omega)$. We have thus proved the following.

Proposition 6. *Let Assumptions (A) and (B) hold as well as $F \in H^{-1}(\Omega)$. Then Eq. (4) is equivalent to*

$$u - (\lambda + \ell)K_c u = KF, \qquad u \in \mathrm{L}^2(\Omega)$$

where $K_c, K : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ are the compact operators given by Eq. (6). Moreover, K_c is self-adjoint with respect to $\langle\cdot,\cdot\rangle_c$. In particular, the linear problem Eq. (4) satisfies the Fredholm Alternative in $\mathrm{L}^2(\Omega)$ in the sense of [13, Appendix D, Theorem 5].

The Spectral Theorem for compact self-adjoint operators [13, Appendix D, Theorem 7] provides an orthonormal basis of eigenfunctions as pointed out in [7]. For notational simplicity we introduce $\mathbb{Z}^* := \mathbb{Z}\setminus\{0\}$.

Proposition 7. *Let Assumptions (A) and (B) hold. Then there is an* $(\mathrm{L}^2(\Omega), \langle\cdot,\cdot\rangle_c)$*-orthonormal basis consisting of eigenfunctions* $(\phi_j)_{j\in\mathbb{Z}^*}$ *with associated eigenvalue sequence* $(\mu_j)_{j\in\mathbb{Z}^*}$ *of the operator* K_c *such that*

$$\mu_{-1} \le \mu_{-2} \le \cdots \le \mu_{-n} \nearrow 0 \swarrow \mu_n \le \cdots \le \mu_2 \le \mu_1$$

as well as

$$\langle\phi_i, \phi_j\rangle_c = \delta_{i,j} \quad \text{and} \quad a(\phi_i, \phi_j) = \left(\mu_j^{-1} - \ell\right)\delta_{i,j} \qquad (i, j \in \mathbb{Z}^*). \tag{7}$$

In particular, for $j \ge 1$*, we have*

$$\mu_{-j} = \min_{\substack{X\subset \mathrm{L}^2(\Omega)\\ \dim(X)=j}} \max_{\phi\in X\setminus\{0\}} \frac{\|\phi\|_c^2}{a(\phi,\phi) + \ell\|\phi\|_c^2}, \qquad \mu_j = \max_{\substack{X\subset \mathrm{L}^2(\Omega)\\ \dim(X)=j}} \min_{\phi\in X\setminus\{0\}} \frac{\|\phi\|_c^2}{a(\phi,\phi) + \ell\|\phi\|_c^2}. \tag{8}$$

Moreover, there are constants $d_1, d_2 > 0$ *such that*

$$d_1(1+|j|)^{-\frac{2}{N}} \le |\mu_j| \le d_2(1+|j|)^{-\frac{2}{N}}, \qquad \text{for all } j \in \mathbb{Z}^*. \tag{9}$$

Proof. By Proposition 6 the compact operator K_c is self-adjoint on $(\mathrm{L}^2(\Omega), \langle\cdot,\cdot\rangle_c)$. Therefore, using the spectral theorem for self-adjoint compact operators [13, Appendix D, Theorem 7], there exists an orthonormal basis $(\phi_j)_{j\in\mathbb{Z}^*}$ of $(\mathrm{L}^2(\Omega), \langle\cdot,\cdot\rangle_c)$ consisting of eigenfunctions of K_c where the corresponding eigenvalue sequence $(\mu_j)_{j\in\mathbb{Z}^*}$ is a null sequence. Notice that $K_c\phi_j = \mu_j\phi_j$ implies $C\phi_j = \mu_j A_\ell \phi_j$ and hence, for all $i, j \in \mathbb{Z}^*$, we compute

$$\delta_{i,j} = \langle\phi_i, \phi_j\rangle_c = \langle\phi_i, C\phi_j\rangle_{\mathrm{H}_0^1(\Omega)} = \mu_j \langle\phi_i, A_\ell\phi_j\rangle_{\mathrm{H}_0^1(\Omega)} = \mu_j\left(a(\phi_i, \phi_j) + \ell\delta_{i,j}\right)$$

which gives $\mu_j a(\phi_i, \phi_j) = (1 - \mu_j\ell)\delta_{i,j}$. In particular, all μ_j are non-zero. Moreover,

$$\frac{\|\phi\|_c^2}{a(\phi,\phi) + \ell\|\phi\|_c^2} = \frac{\sum_{j\in\mathbb{Z}^*} c_j^2}{\sum_{j\in\mathbb{Z}^*} \mu_j^{-1} c_j^2} \quad \text{for } \phi = \sum_{j\in\mathbb{Z}^*} c_j\phi_j \text{ and } (c_j)_{j\in\mathbb{Z}^*} \in \ell^2(\mathbb{Z}^*). \tag{10}$$

Next we show that infinitely many μ_j are positive and infinitely many of them are negative. Indeed, choose $x_0 \in \Omega_+$, a test function $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^N)$ and $\chi_n(x) := \chi(n(x - x_0))$. Then, for large enough n we have $\chi_n \in \mathrm{H}_0^1(\Omega)$ with $\left[a(\chi_n, \chi_n) + \ell\|\chi_n\|_c^2\right] / \|\chi_n\|_c^2 \to +\infty$ as $n \to +\infty$. So we must have $\mu_j > 0$ for infinitely many $j \in \mathbb{Z}^*$. Similarly, for $x_0 \in \Omega_-$ and $\chi_n(x) := \chi(n(x - x_0))$ we obtain $\left[a(\chi_n, \chi_n) + \ell\|\chi_n\|_c^2\right] / \|\chi_n\|_c^2 \to -\infty$, hence $\mu_j < 0$ for infinitely many $j \in \mathbb{Z}^*$. This and Eq. (10) implies the min-max characterization of the eigenvalues from Eq. (8) after a suitable reordering of the eigenpairs. Finally, to prove Eq. (9), for $\phi \in \mathrm{L}^2(\Omega)$, we compute

$$\begin{aligned}
|\langle K_c\phi, \phi\rangle_c| &= \left|\left\langle K_c\phi, (-\Delta)^{-1}(c\phi)\right\rangle_{\mathrm{H}_0^1(\Omega)}\right| \\
&= \left|\left\langle A_\ell^{-1}(-\Delta)^{-1}(c\phi), (-\Delta)^{-1}(c\phi)\right\rangle_{\mathrm{H}_0^1(\Omega)}\right| \\
&\le \left|\left|\left|A_\ell^{-1}\right|\right|\right| \left\|(-\Delta)^{-1}(c\phi)\right\|_{\mathrm{H}_0^1(\Omega)}^2 \\
&= \left|\left|\left|A_\ell^{-1}\right|\right|\right| \left\langle(-\Delta)^{-1}(c\phi), \phi\right\rangle_c.
\end{aligned}$$

So Eq. (8) implies that the modulus of the j-th largest and j-th smallest eigenvalue is bounded from above by $|||A_\ell^{-1}|||\kappa_j(\Omega)^{-1}$ where $\kappa_j(\Omega)$ is the j-th smallest Dirichlet eigenvalue of $-c(x)^{-1}\Delta$. The typical monotonicity properties of Dirichlet eigenvalues with respect to the underlying domain and coefficient functions (which in turn follow from a min-max characterization) imply that $\kappa_j(\Omega)$ is bounded from below and from above by a multiple of the j-th Dirichlet eigenvalue on a ball of suitable radius. This implies $\widetilde{d_1}\,(1+|j|)^{\frac{2}{N}} \le \kappa_j(\Omega) \le \widetilde{d_2}(1+|j|)^{\frac{2}{N}}$ for some $\widetilde{d_1}, \widetilde{d_2} > 0$. We thus obtain

$$\max\left(\mu_j, |\mu_{-j}|\right) \le \left|\!\left|\!\left| A_\ell^{-1} \right|\!\right|\!\right| \widetilde{d_1}^{-1}\,(1+|j|)^{-\frac{2}{N}}.$$

The lower bounds result from Eq. (8) and $|a(\phi,\phi)| \le \|\sigma\|_\infty \int_\Omega |\nabla\phi|^2 \,\mathrm{d}x$ because

$$\min\left(\mu_j, |\mu_{-j}|\right) \ge \left(\|\sigma\|_\infty \kappa_j(\Omega) + |\ell|\right)^{-1} \ge \left(\|\sigma\|_\infty \widetilde{d_2} + |\ell|\right)^{-1} (1+|j|)^{-\frac{2}{N}}. \qquad \square$$

To facilitate the application of this result we add some corollary.

Corollary 8. *Let Assumptions (A) and (B) hold. Then there is a sequence* $(\lambda_j)_{j\in\mathbb{Z}}$ *containing all eigenvalues of the differential operator* $\phi \mapsto -c(x)^{-1}\operatorname{div}(\sigma(x)\nabla\phi)$ *on* $\mathrm{H}_0^1(\Omega)$ *that satisfies*

$$-\infty \swarrow \cdots \le \lambda_{-j} \le \cdots \le \lambda_{-1} \le \lambda_0 \le \lambda_1 \le \cdots \le \lambda_j \le \cdots \nearrow +\infty$$

as well as $c\,(1+|j|)^{\frac{2}{N}} \le 1 + |\lambda_j| \le C\,(1+|j|)^{\frac{2}{N}}$ *for all* $j \in \mathbb{Z}$ *and some* $c, C > 0$.

Proof. It suffices to choose the sequence $(\lambda_j)_{j\in\mathbb{Z}}$ in such a way that $\{\lambda_j : j \in \mathbb{Z}\} = \left\{\mu_j^{-1} - \ell : j \in \mathbb{Z}^*\right\}$, $j \mapsto \lambda_j$ is nondecreasing. Then the estimates for μ_j from Eq. (9) imply $c\,(1+|j|)^{\frac{2}{N}} \le 1+|\lambda_j| \le C\,(1+|j|)^{\frac{2}{N}}$ for all $j \in \mathbb{Z}$ and some $c, C > 0$ as claimed. $\square$

Remark 9.

(a) We may consider Eq. (4) also as an equation in $\mathrm{H}_0^1(\Omega)$, namely

$$A_\ell u - (\lambda + \ell)Cu = (-\Delta)^{-1}F, \qquad u \in \mathrm{H}_0^1(\Omega). \tag{11}$$

For frequently used bounded linear functionals of the form $F(v) = \int_\Omega f(x)v(x)\,\mathrm{d}x$ we have $F \in \mathrm{H}^{-1}(\Omega)$ provided that

$$f \in \mathrm{L}^{\frac{2N}{N+2}}(\Omega) \text{ if } N \ge 3, \quad f \in \mathrm{L}^p(\Omega),\ p > 1 \text{ if } N = 2, \quad f \in \mathrm{L}^1(\Omega) \text{ if } N = 1. \tag{12}$$

This is a consequence of Sobolev's Embedding Theorem. We will write $F = f$ in this case.

(b) In [7, Section 1], the authors provide some explicit one-dimensional example showing that all statements in this section may be false when $c \in \mathrm{L}^\infty(\Omega)$ is sign-changing. In fact they showed that for some tailor-made σ as in Assumption (A) and $c := \sigma$ the operator $u \mapsto -c(x)^{-1}\operatorname{div}(\sigma(x)\,\nabla u)$ may have the whole complex plane as spectrum. In particular, the spectral theory of (compact) self-adjoint operators does not apply in this context.

(c) We mention some similarities and differences concerning the spectral properties of the differential operator $u \mapsto -c(x)^{-1}\operatorname{div}(\sigma(x)\nabla u)$ for

(I) sign-changing σ and $c = 1$, (II) $\sigma = 1$ and sign-changing c.

In the case (I), Proposition 7 and Corollary 8 yield two sequences of eigenvalues going to $-\infty$ or $+\infty$, respectively, as well as the corresponding min-max-characterization. This is also true for (II), see the Propositions 1.10 and 1.11 in [10]. On the other hand, there are subtle differences. As we will see in Lemma 15, in our one-dimensional model example for case (I) there is precisely one positive eigenfunction ϕ_0 with associated eigenvalue λ_0 that need not be of smallest absolute value among all eigenvalues. In fact, $|\lambda_0|$ can be much larger than $|\lambda_{-1}|$ for large contrasts $\frac{\sigma_+|a_-|}{\sigma_- a_+}$. This can be read off from the formulas Eq. (17), Eq. (19). In particular, there is little hope to prove the existence of positive eigenvalues via some straightforward application of the Krein-Rutman theorem. This is different for the case (II) where Manes-Micheletti [17] (see also [10, Theorem 1.13]) proved the existence of one positive and one negative principal eigenvaluse *i.e.*, algebraically simple eigenvalues coming with positive eigenfunctions that have smallest absolute value among the positive and negative eigenvalues, respectively. So here the two models exhibit different phenomena. As demonstrated by Hess-Kato [15, Theorem 2] in a partially more general context, such piece of information can be used to prove global bifurcation results, so it would be interesting to find some replacement for the Krein-Rutman Theorem in our setting.

4. Proof of Theorem 2

We now prove the theoretical bifurcation results by rather direct applications of well-known bifurcation results for equations of the form $F(u,\lambda)=0$ for mappings $F\in\mathscr{C}^2(H\times\mathbb{R},H)$ defined on a Hilbert space H. We will consider bifurcation from the trivial solution branch, so F is supposed to satisfy $F(0,\lambda)=0$ for all $\lambda\in\mathbb{R}$. In the context of our problem the Hilbert space is given by $(\mathrm{H}_0^1(\Omega),\langle\cdot,\cdot\rangle_{\mathrm{H}_0^1(\Omega)})$ and

$$F(u,\lambda)=A_\ell u-(\lambda+\ell)Cu-\Gamma(u), \tag{13}$$

where $A_\ell:\mathrm{H}_0^1(\Omega)\to\mathrm{H}_0^1(\Omega)$ is a bounded linear self-adjoint and invertible operator and $C:\mathrm{H}_0^1(\Omega)\to\mathrm{H}_0^1(\Omega)$ is a linear compact self-adjoint operator and $\Gamma:\mathrm{H}_0^1(\Omega)\to\mathrm{H}_0^1(\Omega), u\mapsto(-\Delta)^{-1}(u^3)$, see Remark 9 (a). Sobolev's Embedding Theorem and the Rellich-Kondrachov Theorem imply that H is compact provided that $N\in\{1,2,3\}$. We prove Theorem 2 in two steps: First we use variational bifurcation theory to prove that each $(0,\lambda_j)$ is a bifurcation point where λ_j is taken from Corollary 8. Then we apply Rabinowitz' Global Bifurcation Theorem to prove our statements about those λ_j with odd geometric multiplicity.

Local Variational Bifurcation. Our first claim is that bifurcations occur at any eigenvalue associated with Eq. (1) or equivalently Eq. (13). Here, no assumption on the multiplicity of the eigenvalue is needed. This is proved by exploiting the variational structure of Eq. (13) that we shall now explain. We define the energy functional $\Psi(\cdot,\lambda):=\Psi_\lambda:\mathrm{H}_0^1(\Omega)\to\mathbb{R}$ via

$$\Psi_\lambda(u):=\frac{1}{2}\int_\Omega\sigma(x)|\nabla u|^2\,\mathrm{d}x-\frac{\lambda}{2}\int_\Omega c(x)u^2\,\mathrm{d}x-\frac{1}{4}\int_\Omega u^4\,\mathrm{d}x.$$

Then $\Psi\in\mathscr{C}^2(\mathrm{H}_0^1(\Omega)\times\mathbb{R},\mathbb{R})$ and the (partial) Fréchet derivative $\Psi_\lambda'(u):\mathrm{H}_0^1(\Omega)\to\mathbb{R}$ at $u\in\mathrm{H}_0^1(\Omega)$ is the bounded linear operator given by the following formula:

$$\begin{aligned}\Psi_\lambda'(u)[\phi]&=\int_\Omega\sigma(x)\nabla u\cdot\nabla\phi\,\mathrm{d}x-\lambda\int_\Omega c(x)u\phi\,\mathrm{d}x-\int_\Omega u^3\phi\,\mathrm{d}x\\&=\langle Au-\lambda Cu-\Gamma(u),\phi\rangle_{\mathrm{H}_0^1(\Omega)}\\&=\langle A_\ell u-(\lambda+\ell)Cu-\Gamma(u),\phi\rangle_{\mathrm{H}_0^1(\Omega)}.\end{aligned}\tag{14}$$

This shows that $\nabla\Psi_\lambda(u)=F(u,\lambda)=0$ is the Euler-Lagrange equation of the functional Ψ_λ. The nonstandard feature about this problem is that the Hessian of the functional $A-\lambda C$ is self-adjoint and Fredholm, but A is strongly indefinite. So it is not possible to use the more well-known variational bifurcation theory going back to Marino [18], Böhme [1, Satz II.1] and Rabinowitz [21, Theorem 11.4]. We recall that these results apply if the self-adjoint operator A or, more generally, $A+\tilde{\ell}C$ for some $\tilde{\ell}\in\mathbb{R}$, generates a norm. This assumption typically holds in the context of classical nonlinear elliptic boundary value problems involving the Laplacian or, more generally, divergence-form operators with diffusion coefficients σ having a fixed sign. In our case, however, this is not true. We need to resort to a much more advanced tool called spectral flow that has been developed for such purposes [14]. Its definition is rather cumbersome, but we will not need this concept in its full generality. The following simplified version is based on Theorem 2.1(i) and the following Remark (3) in [19], which is a slightly improved version of [14, Corollary 3].

Theorem 10. *Suppose H is a separable real Hilbert space and $\Psi \in \mathscr{C}^2(H\times\mathbb{R},\mathbb{R})$ satisfies $\nabla\Psi_\lambda(0)=0$ for all $\lambda\in\mathbb{R}$. Moreover suppose $\nabla\Psi_\lambda(u)=Lu-\lambda Ku-\Gamma(u)$ where*

(i) $L : H \to H$ is a linear invertible self-adjoint Fredholm operator,
(ii) $K : H \to H$ is a linear compact and positive self-adjoint operator,
(iii) $\Gamma : H \to H$ satisfies $\Gamma'(0) = 0$.

Then each $\lambda_\star\in\mathbb{R}$ such that $\ker(\Psi'_{\lambda_\star}(0))\neq\{0\}$ is a bifurcation point for $\nabla\Psi'_\lambda(u)=0$.

Proof. Our assumptions (i), (ii), (iii) imply that $(\Psi_\lambda)_{\lambda\in\mathbb{R}}$ is a continuous family of $\mathscr{C}^2$-functionals as in [19]. If $\lambda_\star\in\mathbb{R}$ is as required, then Theorem 2.1(i) in [19] proves that the interval $[\lambda_\star-\varepsilon,\lambda_\star+\varepsilon]$ contains a bifurcation point provided that the Hessians $\Psi''_{\lambda_\star\pm\varepsilon}(0)$ are invertible and the spectral flow of this family over the interval $I:=[\lambda_\star-\varepsilon,\lambda_\star+\varepsilon]$ is nonzero. In fact, since L is invertible and K is compact, the linear operator $\Psi''_\lambda(0)=L-\lambda K$ has a nontrivial kernel only for λ belonging to a discrete subset of $\mathbb{R}$. So we may choose $\varepsilon>0$ so small that $\Psi''_{\lambda_\star+\varepsilon}(0),\Psi''_{\lambda_\star-\varepsilon}(0)$ are invertible and $\lambda_\star$ is the only candidate for bifurcation in I by the Implicit Function Theorem. Using then the positivity of K we get from Remark (3) in [19] that the spectral flow over I is the dimension of $\ker(\Psi'_{\lambda_\star}(0))$ which is positive by assumption. So $\lambda_\star$ is a bifurcation point. □

As in the well-known special case of positive operators L it cannot be expected that bifurcation always comes in the form of a continuous curve, see [1, Section 6]. We also mention the open problem to prove stronger results for families of even functionals as in [21, Corollary 11.30]. We now show how to apply this theorem in our context. We choose $L=A_\ell$, $K=C$ and the bifurcation parameter $\lambda+\ell$ becomes λ, *i.e.*, we assume $\ell=0$ without loss of generality. Choosing $\lambda_\star=\lambda_j$ for $j\in\mathbb{Z}$ for λ_j as in Corollary 8, one obtains from Theorem 10 that each λ_j is a bifurcation point for Eq. (1). This finishes the variational part of our bifurcation theoretical result.

Global Bifurcation. We continue with the proof of global bifurcation for our problem with the aid of Rabinowitz' Global Bifurcation Theorem [20]. This result states that the bifurcating solutions lie on solution continua that are unbounded or return to the trivial solution branch $\{(0,\lambda):\lambda\in\mathbb{R}\}$ at some other bifurcation point. Here, a solution continuum is defined as a closed and connected sets consisting of solutions. Given that the proof of this bifurcation theorem uses Leray-Schauder degree theory, more restrictive compactness assumptions are required compared to Theorem 10. In order to avoid technicalities, we state a simplified variant of this result from Theorem II.3.3 in [16]. The set $\mathcal{S}\subset H\times\mathbb{R}$ denotes the closure of nontrivial solutions of $F(u,\lambda)=0$ in $H\times\mathbb{R}$.

Theorem 11 (Rabinowitz)**.** *Suppose H is a separable real Hilbert space and that $F\in\mathscr{C}^1(H\times\mathbb{R},H)$ is given by $F(u,\lambda)=Lu-\lambda Ku-\Gamma(u)$ where*

(i) $L : H \to H$ is a linear invertible self-adjoint Fredholm operator,
(ii) $K : H \to H$ is a linear compact and positive self-adjoint operator,
(iii) $\Gamma : H \to H$ is compact with $\Gamma'(0) = 0$.

Suppose that $\lambda_\star\in\mathbb{R}$ is such that the dimension of $\ker(\Psi'_{\lambda_\star}(0))$ is odd. Then $(0,\lambda_\star)\in\mathcal{S}$. Moreover, if $\mathcal{C}$ denotes the connected component of $(0,\lambda_\star)$ in $\mathcal{S}$, then

(A) $\mathcal{C}$ is unbounded or
(B) $\mathcal{C}$ contains a point $(0,\lambda^\star)$ with $\lambda^\star\neq\lambda_\star$.

We mention that the general version of this result is formulated in Banach spaces and does not involve any self-adjointness assumption. It then claims the above-mentioned properties of $\mathcal{C}$ assuming that $\lambda_\star$ is an eigenvalue of odd algebraic multiplicity. Under our more restrictive assumptions including self-adjointness the algebraic multiplicity of $\lambda_\star$ is equal to its geometric multiplicity and hence to the dimension of the corresponding eigenspace. As before, our Eq. (13) fits in this abstract framework so that our claim from Theorem 2 about global bifurcation follows from choosing $\lambda_\star = \lambda_j$ and denoting by $\mathcal{C}_j \subset \mathcal{S}$ the associated connected component given by Theorem 11. This proves the theorem. □

Bifurcation from Simple Eigenvalues — Proof of Remark 3 (d). For completeness, we state the Crandall-Rabinowitz Theorem [9, Theorem 1.7]. It shows that in the case of simple eigenvalues local bifurcation occurs in the form of differentiable curves the bending direction of which can be computed. Our version is a simplified variant of Theorem I.5.1 in [16]. The bifurcation formulae Eq. (15) are taken from Eq. (I.6.11) in [16].

Theorem 12 (Crandall-Rabinowitz). *Let H be a Hilbert space and let $F \in \mathscr{C}^3(H \times \mathbb{R}, H)$ satisfy $F(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}$. Assume that $\lambda^* \in \mathbb{R}, \phi \in H, \|\phi\| = 1$ is such that $F_x(0,\lambda^*) : H \to H$ is a self-adjoint linear operator with*

$$\ker(F_x(0,\lambda^*)) = \operatorname{span}\{\phi\} \quad \textit{and} \quad \langle F_{x\lambda}(0,\lambda^*)[\phi], \phi\rangle \neq 0.$$

Then there is an $\varepsilon > 0$ and a continuously differentiable curve $(\widehat{x}, \widehat{\lambda}) : (-\varepsilon, \varepsilon) \to H \times \mathbb{R}$ such that $\widehat{\lambda}(0) = \lambda^*$, $\widehat{x}(0) = 0$, $\widehat{x}'(0) = \phi$, and

$$F(\widehat{x}(s), \widehat{\lambda}(s)) = 0, \qquad \textit{for } |s| < \varepsilon.$$

Furthermore, in a small neighborhood of $(0,\lambda^*) \in H \times \mathbb{R}$ there are no other solutions and in case $F_{xx}(0,\lambda^*) \equiv 0$ we have $\widehat{\lambda}'(0) = 0$ and

$$\widehat{\lambda}''(0) = -\frac{1}{3}\frac{\langle F_{xxx}(0,\lambda^*)[\phi,\phi,\phi], \phi\rangle}{\langle F_{x\lambda}(0,\lambda^*)[\phi], \phi\rangle}. \tag{15}$$

Remark 13.

(a) Consider a simple eigenvalue λ_j. Theorem 12 shows that bifurcation occurs in the form of smooth curves $(\widehat{u}_j, \widehat{\lambda}_j)$. We now compute the bifurcation direction at the bifurcation point using Eq. (15). Theorem 12 immediately gives $\widehat{\lambda}_j(0) = \lambda_j, \widehat{u}_j(0) = 0$ and $\widehat{u}_j'(0) = \phi_j$. Given that Eq. (1) does not have quadratic terms, we moreover find $F_{uu}(0,\lambda_0) \equiv 0$. So we obtain $\widehat{\lambda}_j'(0) = 0$ and Eq. (15) gives

$$\widehat{\lambda}_j''(0) = -\frac{1}{3}\frac{\left\langle -6K(\phi_j^3), \phi_j\right\rangle_c}{\left\langle -K_c\phi_j, \phi_j\right\rangle_c} = -2\frac{\left\langle \phi_j^3, K_c\phi_j\right\rangle_{\mathrm{L}^2(\Omega)}}{\left\langle K_c\phi_j, \phi_j\right\rangle_c} = -2\int_\Omega \phi_j(x)^4 \,\mathrm{d}x.$$

Here we used $K_c\phi_j = \mu_j\phi_j$ and $\|\phi_j\|_c^2 = 1$ (by convention).

(b) The numeric suggest that the branches do not become unbounded for finite λ. Related a priori bounds are, however, missing. In the context of elliptic problems, such bounds are typically available for positive solutions or solutions with uniformly bounded Morse-Index. It is an open problem how to adapt these methods to strongly indefinite problems.

4.1. **Proof of Corollary 4.** We now sharpen our results from Theorem 2 for the one-dimensional boundary value problem

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\sigma(x)\, u'(x)\right) - \lambda\, c(x)\, u = u^3 \quad \text{in } \Omega, \qquad u \in \mathrm{H}_0^1(\Omega)$$

from Eq. (3). The assumptions on σ, c, and $\Omega = (a_-, a_+) \subset \mathbb{R}$ were specified in the Introduction. We want to verify that Assumptions (A) and (B) are satisfied in this context. While Assumption (A) is trivial, the verification Assumption (B) dealing with the weak $\mathtt{T}$-coercivity of $(u,v) \mapsto a(u,v) := \int_\Omega \sigma(x)\, u'v'\, \mathrm{d}x$ requires some work. The following result seems to be well-known to experts, but a reference appears to be missing in the literature.

Lemma 14. *Let Ω, σ, and c be given as in Corollary 4. Then the bilinear form a is weakly $\mathtt{T}$-coercive. In particular, Assumption (B) holds.*

Proof. Let $\chi \in \mathscr{C}_0^\infty(\Omega)$ with $\chi(x) = 1$ for x close to 0. Then define

$$\mathtt{T}u(x) = u(x) \quad \text{if } 0 < x < a_+, \qquad \mathtt{T}u(x) = 2\chi(x)u(-mx) - u(x) \quad \text{if } a_- < x < 0,$$

where $m \in \left(0, \frac{a_+}{|a_-|}\right)$ will be chosen sufficiently small. Then $\mathtt{T}$ is a well-defined bijective operator on $\mathrm{H}_0^1(\Omega)$ because of $\mathtt{T} \circ \mathtt{T} = \mathrm{id}$. Moreover, it satisfies

$$\begin{aligned} a(u, \mathtt{T}u) = \int_{a_-}^{a_+} |\sigma(x)|\, u'(x)^2\, \mathrm{d}x - 2m\sigma_- \int_{a_-}^{0} \chi(x)u'(-mx)u'(x)\, \mathrm{d}x& \\ + 2\sigma_- \int_{a_-}^{0} \chi'(x)u(-mx)u'(x)\, \mathrm{d}x& \\ \geq \min(\sigma_+, |\sigma_-|)\, \|u\|_{\mathrm{H}_0^1(\Omega)}^2 - 2m\, |\sigma_-|\, \|\chi\|_\infty\, \|u'(-m\,\cdot)\|_{\mathrm{L}^2(a_-,0)}\, \|u'\|_{\mathrm{L}^2(a_-,0)}& \\ - 2\, |\sigma_-|\, \|\chi'\|_\infty\, \|u(-m\,\cdot)\|_{\mathrm{L}^2(a_-,0)}\, \|u'\|_{\mathrm{L}^2(a_-,0)}& \\ \geq \min(\sigma_+, |\sigma_-|)\, \|u\|_{\mathrm{H}_0^1(\Omega)}^2 - 2\sqrt{m}\, |\sigma_-|\, \|\chi\|_\infty\, \|u'\|_{\mathrm{L}^2(0,m|a_-|)}\, \|u'\|_{\mathrm{L}^2(a_-,0)}& \\ - \frac{2}{\sqrt{m}}\, |\sigma_-|\, \|\chi'\|_\infty\, \|u\|_{\mathrm{L}^2(0,m|a_-|)}\, \|u'\|_{\mathrm{L}^2(a_-,0)}&. \end{aligned}$$

Then, using the definition of m, we have the estimations $\|u'\|_{\mathrm{L}^2(0,m|a_-|)} \leq \|u\|_{\mathrm{H}_0^1(\Omega)}$ and $\|u\|_{\mathrm{L}^2(0,m|a_-|)} \leq \|u\|_{\mathrm{L}^2(\Omega)}$. We obtain

$$a(u, \mathtt{T}u) + \left\langle \widetilde{C}u, u \right\rangle_{\mathrm{H}_0^1(\Omega)} \geq \left[\min(\sigma_+, |\sigma_-|) - \sqrt{m}\,(2\,|\sigma_-|\,\|\chi\|_\infty + 1)\right] \|u\|_{\mathrm{H}_0^1(\Omega)}^2$$

where $\widetilde{C} = m^{-\frac{3}{2}}\, |\sigma_-|^2\, \|\chi'\|_\infty^2\, (-\Delta)^{-1}\iota$ is a compact operator. Choosing $m > 0$ so small that $\min(\sigma_+, |\sigma_-|) - \sqrt{m}\,(2\,|\sigma_-|\,\|\chi\|_\infty + 1)$ is positive, we find that a is weakly $\mathtt{T}$-coercive. □

We conclude that the assumptions of Theorem 2 are verified and hence the existence of infinitely many bifurcating branches $\mathcal{C}_j$ is ensured. So the claim is proved once we have shown $\mathcal{C}_j \cap \mathcal{C}_k = \varnothing$ for $j \neq k$. This will be achieved by proving the following property for nontrivial solutions $(u, \lambda) \in \mathcal{C}_j$:

- If $j = 0$ then u has no interior zeros in Ω and satisfies $|u'| > 0$ on $\overline{\Omega_\pm}$.
- If $j \geq 1$ then u has j interior zeros in Ω_+ and satisfies $|u'| > 0$ on $\overline{\Omega_-}$.
- If $j \leq -1$ then u has $|j|$ interior zeros in Ω_- and satisfies $|u'| > 0$ on $\overline{\Omega_+}$.

Here, $|u'| > 0$ on $\overline{\Omega_\pm}$ means that the continuous extension of $|u'| : \Omega_\pm \to \mathbb{R}$ to $\overline{\Omega_\pm}$ is positive. We first prove the corresponding property for the eigenfunction ϕ_j at the bifurcation point $(0, \lambda_j)$. Recall from Eq. (2) that the solutions "look like" this eigenfunction close to the bifurcation point. The first step is to compute the eigenpairs of the linear problem.

Step 1: Nodal characterization of the eigenfunctions.

Lemma 15. *Let $(\phi_j, \lambda_j)_{j\in\mathbb{Z}}$ denote the sequence of eigenpairs for the one-dimensional boundary value problem (3) as in Corollary 8. Then each λ_j is simple and in particular*

$$-\infty \swarrow \cdots < \lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < \cdots \nearrow +\infty.$$

This sequence can be ordered in the following way:

- *λ_0 is the only eigenvalue in the interval $\left(-\frac{\pi^2}{4k_-^2a_-^2}, \frac{\pi^2}{4k_+^2a_+^2}\right)$ and ϕ_0 has no interior zeros in Ω with $|\phi_0'| > 0$ on $\overline{\Omega_\pm}$. Moreover,*
$$\lambda_0 > 0 \;\Leftrightarrow\; \frac{\sigma_+a_-}{a_+\sigma_-} > 1, \qquad \lambda_0 < 0 \;\Leftrightarrow\; \frac{\sigma_+a_-}{a_+\sigma_-} < 1, \qquad \lambda_0 = 0 \;\Leftrightarrow\; \frac{\sigma_+a_-}{a_+\sigma_-} = 1.$$
- *For $j \geq 1$, $\lambda_j > 0$ is the only eigenvalue in the interval $\left(\frac{j^2\pi^2}{k_+^2a_+^2}, \frac{(2j+1)^2\pi^2}{4k_+^2a_+^2}\right)$ and ϕ_j has j interior zeros in Ω_+ with $\left|\phi_j'\right| > 0$ on $\overline{\Omega_-}$.*
- *For $j \leq -1$, $\lambda_j < 0$ is the only eigenvalue in the interval $\left(-\frac{(2j-1)^2\pi^2}{4k_-^2a_-^2}, -\frac{j^2\pi^2}{k_-^2a_-^2}\right)$ and ϕ_j has $|j|$ interior zeros in Ω_- with $\left|\phi_j'\right| > 0$ on $\overline{\Omega_+}$.*

Proof. Any eigenpair $(\phi, \lambda) \in \mathrm{H}_0^1(\Omega) \times \mathbb{R}$ satisfies

$$\begin{aligned} -\phi''(x) &= -\lambda k_-^2 \phi(x), \quad \text{on } (a_-, 0), \qquad k_- := \sqrt{c_-/|\sigma_-|}, \\ -\phi''(x) &= +\lambda k_+^2 \phi(x), \quad \text{on } (0, a_+), \qquad k_+ := \sqrt{c_+/\sigma_+}, \\ \phi, \sigma\phi' &\in \mathscr{C}([a_-, a_+]), \qquad \phi(a_-) = \phi(a_+) = 0. \end{aligned}$$

Positive eigenvalues. Solving the ODE and exploiting the continuity of eigenfunctions as well as the homogeneous Dirichlet boundary conditions, we find the following formula for eigenfunctions ϕ_j associated with positive eigenvalues

$$\phi_j(x) := \alpha_j \begin{cases} \sin(\sqrt{\lambda_j}k_+a_+)\sinh(\sqrt{\lambda_j}k_-(x-a_-)) & \text{if } a_- < x < 0, \\ \sinh(\sqrt{\lambda_j}k_-a_-)\sin(\sqrt{\lambda_j}k_+(x-a_+)) & \text{if } 0 < x < a_+, \end{cases} \qquad (\lambda_j > 0). \tag{16}$$

The parameter $\alpha_j \in \mathbb{R}\setminus\{0\}$ is chosen such that $\|\phi_j\|_c = 1$. The equation for λ_j now results from the condition that $\sigma\phi_j'$ has to be continuous. This means

$$\sigma_-k_- \sin(\sqrt{\lambda_j}k_+a_+)\cosh(\sqrt{\lambda_j}k_-a_-) = \sigma_+k_+\sinh(\sqrt{\lambda_j}k_-a_-)\cos(\sqrt{\lambda_j}k_+a_+)$$

or equivalently

$$\frac{\tan\left(\sqrt{\lambda_j}k_+a_+\right)}{\tanh\left(\sqrt{\lambda_j}k_-a_-\right)} \cdot \frac{\sigma_-k_-}{\sigma_+k_+} = 1 \qquad (\lambda_j > 0). \tag{17}$$

By elementary monotonicity considerations one finds that this equation has a unique solution such that $\sqrt{\lambda_j}k_+a_+ \in \left(j\pi, (j+\frac{1}{2})\pi\right)$ for $j \geq 1$. Moreover, it has a unique solution such that $\sqrt{\lambda_0}k_+a_+ \in \left(0, \frac{1}{2}\pi\right)$ if and only if $\frac{\sigma_+a_-}{\sigma_-a_+} > 1$. No further solutions exist. We thus obtain:

- For $j \geq 1$, there is a unique solution λ_j in the interval $\left(\frac{j^2\pi^2}{k_+^2a_+^2}, \frac{(2j+1)^2\pi^2}{4k_+^2a_+^2}\right)$ and ϕ_j has j interior zeros in Ω_+ with $\left|\phi_j'\right| > 0$ on $\overline{\Omega_-}$.
- If $\frac{\sigma_+a_-}{\sigma_-a_+} > 1$, then there is a unique solution λ_0 in the interval $\left(0, \frac{\pi^2}{4k_+^2a_+^2}\right)$ and ϕ_0 has no interior zeros in Ω with $|\phi_0'| > 0$ on $\overline{\Omega_\pm}$.

Negative eigenvalues. Similarly, we obtain for the negative eigenvalues

$$\phi_j(x) := \alpha_j \begin{cases} \sinh(\sqrt{|\lambda_j|}k_+a_+)\sin(\sqrt{|\lambda_j|}k_-(x-a_-)) & \text{if } a_- < x < 0, \\ \sin(\sqrt{|\lambda_j|}k_-a_-)\sinh(\sqrt{|\lambda_j|}k_+(x-a_+)) & \text{if } 0 < x < a_+, \end{cases} \qquad (\lambda_j < 0) \tag{18}$$

and

$$\frac{\tan\left(\sqrt{|\lambda_j|}k_-a_-\right)}{\tanh\left(\sqrt{|\lambda_j|}k_+a_+\right)} \cdot \frac{\sigma_+k_+}{\sigma_-k_-} = 1 \qquad (\lambda_j < 0). \tag{19}$$

As for the positive eigenvalues one finds:

- For $j \geq 1$, there is a unique solution λ_{-j} in the interval $\left(-\frac{(2j+1)^2\pi^2}{4k_-^2a_-^2}, -\frac{j^2\pi^2}{k_-^2a_-^2}\right)$ and ϕ_j has $|j|$ interior zeros in Ω_- with $|\phi_j'| > 0$ on $\overline{\Omega_+}$.
- If $\frac{\sigma_+a_-}{\sigma_-a_+} < 1$, then there is a unique solution λ_0 in the interval $\left(-\frac{\pi^2}{4k_-^2a_-^2}, 0\right)$ and ϕ_0 has no interior zeros in Ω with $|\phi_0'| > 0$ on $\overline{\Omega_\pm}$.

Zero eigenvalue. The eigenvalue zero only occurs if $\frac{\sigma_-}{\sigma_+} = \frac{a_-}{a_+}$. Here the associated eigenfunction is given by

$$\phi_j(x) := \alpha_j \begin{cases} 1 - \frac{x}{a_-} & \text{if } a_- < x < 0, \\ 1 - \frac{x}{a_+} & \text{if } 0 < x < a_+, \end{cases} \qquad (\lambda_j = 0). \tag{20}$$

We find:

- If $\frac{\sigma_+a_-}{\sigma_-a_+} = 1$, then $\lambda_0 = 0$ and ϕ_0 has no interior zeros in Ω with $|\phi_0'| > 0$ on $\overline{\Omega_\pm}$.

□

Step 2: Nodal characterization close to the bifurcation points. Next we deduce that the nontrivial solutions $(u,\lambda) \in \mathcal{C}_j$ sufficiently close to $(0,\lambda_j)$ have this nodal pattern. Indeed, if $(u^n,\lambda^n) \in \mathcal{C}_j$ converges to $(0,\lambda_j)$ in $\mathrm{H}_0^1(\Omega)$, then $u^n/\|u^n\|_{\mathrm{H}_0^1(\Omega)}$ converges to a multiple of ϕ_j in the $\mathrm{H}_0^1(\Omega)$-topology. This follows from the fact that each (u^n,λ^n) solves Eq. (3) and that a suitable subsequence of $(u^n/\|u^n\|_{\mathrm{H}_0^1(\Omega)})$ converges uniformly by compact embeddings of Sobolev spaces. Integrating Eq. (3) once, one finds that the convergence even holds in $\mathscr{C}^1(\overline{\Omega_+})$ and $\mathscr{C}^1(\overline{\Omega_-})$. So if infinitely many u^n had more than j interior zeros in $\Omega_\pm$, then the collapse of zeros would cause at least one double zero of ϕ_j, but this is false in view of our formulas for these eigenfunctions from above. So almost all u^n have at most j interior zeros in $\Omega_\pm$. Similarly, almost all u^n have at least j zeros. So we conclude that the solutions close to the bifurcation point have exactly j interior zeros in $\Omega_\pm$ and are strictly monotone in $\Omega_\mp$.

Step 3: Nodal characterization along the whole branch. We finally claim that this nodal property is preserved on connected subsets of $\mathcal{S}$ that do not contain the trivial solution. Indeed, the set of solutions on $\mathcal{C}_j \setminus \{(0,\lambda_j)\}$ with this property is open in $\mathcal{S}$ with respect to the topology of $\mathrm{H}_0^1(\Omega) \times \mathbb{R}$. It is also closed in $\mathcal{S}$ since double zeros cannot occur (by the same arguments as above) and zeros cannot converge to the interface at $x = 0$ as the solutions evolve along the branch. Indeed, in the latter case the equation on the monotone part would imply that the solution has to vanish identically there, whence $u \equiv 0$ on Ω, which is impossible. So we conclude that all elements on $\mathcal{C}_j \setminus \{(0,\lambda_j)\}$ have the claimed property and the proof is finished. □

5. Variational methods

We want to show that variational methods can be used to prove further existence and multiplicity results for Eq. (1). To this end we follow the generalized Nehari manifold (or Nehari-Pankov manifold) approach presented in [23, Chapter 4]. It turns out that the results from this paper apply almost verbatim to problems of the form

$$-\operatorname{div}(\sigma(x)\,\nabla u) - \lambda\, c(x)\, u = g(x,u), \quad \text{in } \mathrm{H}_0^1(\Omega) \tag{21}$$

for σ, c, and Ω satisfying Assumptions (A) and (B) from Section 1. We first prove some existence result in general bounded domains $\Omega \subset \mathbb{R}^N, N \in \mathbb{N}$ and for a rather large class of nonlinearities. Unfortunately, we have to impose some technical assumption that appears to be difficult to check in general, see Assumption (C) below. Afterwards, we check all hypotheses in the one-dimensional model example presented earlier and deduce the existence of infinitely many solutions for any given $\lambda \in \mathbb{R}$.

5.1. The general case.

The variational approach aims at proving the existence of critical points of energy functionals associated with the given problem. In our case such a functional is given by $\Phi : \mathrm{H}_0^1(\Omega) \to \mathbb{R}$ where

$$\Phi(u) \coloneqq \frac{1}{2}\int_\Omega \sigma(x)\,|\nabla u(x)|^2\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega c(x)\,u(x)^2\,\mathrm{d}x - \int_\Omega \left(\int_0^{u(x)} g(x,s)\,\mathrm{d}s\right)\mathrm{d}x \tag{22}$$

where $\lambda \in \mathbb{R}$ is fixed. Our Assumptions (A) and (B) and on the nonlinearity will ensure that $u \in \mathrm{H}_0^1(\Omega)$ satisfies $\Phi'(u) = 0$ if and only if u is a weak solution to Eq. (1). In view of the sign change of σ, this functional is strongly indefinite, which makes it much harder to prove the existence of nontrivial critical points. Strong indefiniteness means that the quadratic part of the functional is positive definite on an infinite-dimensional subspace of $\mathrm{H}_0^1(\Omega)$ and it is negative definite on another infinite-dimensional subspace. In such a situation, global minima or global maxima cannot exist and nowadays classical Critical Point Theorems like the Mountain Pass Theorem or the Linking Theorem [25] do not apply either. For this reason we resort to the much more recent Critical Point Theory for strongly indefinite functional by Szulkin and Weth [23, Chapter 4]. To verify their assumptions we need Assumptions (A) and (B) from before, but also the following rather delicate one:

Assumption (C). There is a constant $D > 0$ such that

$$\left|\sum_{i,j\in\mathbb{Z}} c_i c_j \,\langle\phi_i,\phi_j\rangle_{\mathrm{H}_0^1(\Omega)}\right| \le D\sum_{j\in\mathbb{Z}}(1+|\lambda_j|)\,|c_j|^2$$

for all finite sequences $(c_j)_{j\in\mathbb{Z}}$ where $(\lambda_j,\phi_j)_{j\in\mathbb{Z}}$ are the eigenpairs of $-c(x)^{-1}\operatorname{div}(\sigma(x)\,\nabla)$ from Corollary 8.

This technical assumption is used to show that the norm $\|\cdot\|_{\mathrm{H}_0^1(\Omega)}$ is equivalent to the norm that we will define in Eq. (24). Roughly speaking, it says that the orthonormal family $(\phi_i)_{i\in\mathbb{Z}}$ with respect to the L^2 inner product $\langle\cdot,\cdot\rangle_c$ is close to being an orthogonal family with respect to $\langle\cdot,\cdot\rangle_{\mathrm{H}_0^1(\Omega)}$ and $\lambda_j = a(\phi_j,\phi_j) \sim \langle\phi_j,\phi_j\rangle_{\mathrm{H}_0^1(\Omega)}$ as $|j| \to \infty$. Unfortunately, it turns out to be hard to verify this assumption even in a one-dimensional setting where explicit formulas are available, see Lemma 19. In the higher-dimensional setting, we only know of some related estimates [7, Corollary 3.1] to bound the eigenfunctions away from the interface $\Gamma = \overline{\Omega_+} \cap \overline{\Omega_-}$, but the more delicate contributions close to the interface are not understood sufficiently well. Our variational existence result reads as follows.

Theorem 16. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let Assumptions (A) and (B) hold. Moreover suppose that the continuous function $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies*

(i) $|g(x,u)| \le C(1+|u|)^{q-1}$ *for some* $2 < q < 2^*$
(ii) $g(x,u) = o(|u|)$ *as* $u \to 0$ *uniformly with respect to* $x \in \Omega$
(iii) $z \mapsto |z|^{-1} g(x,z)$ *strictly increasing on* $(0,\infty)$ *and on* $(-\infty,0)$
(iv) $z^{-2} \int_0^z g(x,s)\,\mathrm{d}s \to +\infty$ *uniformly in* x *as* $z \to \infty$.

Then Eq. (21) has a least energy solution. If $g(x,\cdot)$ is odd for all $x \in \Omega$, then it has infinitely many other solutions in $\mathrm{H}^1_0(\Omega)$.

We recall that a least energy solution $u^* \in \mathrm{H}^1_0(\Omega) \setminus \{0\}$ satisfies

$$\Phi(u^*) = \min\{\Phi(u) : u \in \mathrm{H}^1_0(\Omega) \setminus \{0\} \text{ solves } (1)\}.$$

Theorem 16 can be visualized as follows: Fix $\lambda \in \mathbb{R}$ and draw a vertical line at the position λ in the bifurcation diagram for Eq. (1). Then the theorem says that this vertical line hits infinitely many solutions. From our bifurcation analysis it seems plausible to conjecture that these solutions lie on infinitely many distinct branches that bifurcate from the trivial solution branch at some bifurcation point $\lambda_j > \lambda$, see Theorem 2. Moreover, we expect these solutions to form an unbounded sequence in $\mathrm{H}^1_0(\Omega)$.

We will see that the Theorem 16 is a direct consequence of some abstract Critical Point Theorem applied to the corresponding energy functional Eq. (22). To see this we introduce the terminology that is based on the linear theory developed earlier. We recall from Corollary 8 that due to the validity of Assumptions (A) and (B) there is a basis $(\phi_j)_{j\in\mathbb{Z}}$ on $\mathrm{L}^2(\Omega)$ such that

$$\langle \phi_i, \phi_j \rangle_c = \delta_{i,j}, \qquad a(\phi_i,\phi_j) = \lambda_j \delta_{i,j}, \qquad \pm\lambda_j \nearrow +\infty \text{ as } j \to \pm\infty. \tag{23}$$

In particular, $(\phi_j, \lambda_j - \lambda)$ are eigenpairs of the operator $u \mapsto -c(x)^{-1}\operatorname{div}(\sigma(x)\nabla u) - \lambda u$. The corresponding bilinear form is $a^*(u,v) := a(u,v) - \lambda \langle u,v\rangle_c$. We use the orthogonal decomposition $\mathrm{H}^1_0(\Omega) = E^+ \oplus_\perp E^0 \oplus_\perp E^-$ where

$$\begin{aligned} E^+ &:= \operatorname{span}\{\phi_j : \lambda_j - \lambda > 0\}, \\ E^0 &:= \operatorname{span}\{\phi_j : \lambda_j - \lambda = 0\}, \\ E^- &:= \operatorname{span}\{\phi_j : \lambda_j - \lambda < 0\}. \end{aligned}$$

The subspaces E^+, E^- are infinite-dimensional whereas E^0 is finite-dimensional, which is a consequence of Eq. (23). Here, $E^0 = \{0\}$ may be possible. Let $\Pi^\pm : \mathrm{H}^1_0(\Omega) \to E^\pm$ denote the corresponding orthogonal projectors and we will write $u^\pm := \Pi^\pm u$ in the following. Then define the inner product

$$\langle u,v\rangle := a^*(u^+,v^+) - a^*(u^-,v^-) + \langle u^0,v^0\rangle_c, \qquad \|u\| := \sqrt{\langle u,u\rangle}. \tag{24}$$

Proposition 17. *Let the Assumptions (A) to (C) hold and fix $\lambda \in \mathbb{R}$. Then the map $\langle\cdot,\cdot\rangle$ from Eq. (24) is well-defined and defines an inner product that induces a norm which is equivalent to the standard norm on $\mathrm{H}^1_0(\Omega)$. Moreover, $(\psi_j)_{j\in\mathbb{Z}}$ given by $\psi_j := \phi_j / \|\phi_j\|$ is an orthonormal basis of $\mathrm{H}^1_0(\Omega)$ equipped with this inner product.*

Proof. We have for each finite linear combination $u = \sum_{i\in\mathbb{Z}} c_i \phi_i$

$$\|u\|^2 = \left\langle \sum_{i\in\mathbb{Z}} c_i\phi_i, \sum_{j\in\mathbb{Z}} c_j\phi_j \right\rangle = \sum_{i,j\in\mathbb{Z}} c_i c_j \langle \phi_i,\phi_j\rangle$$

$$
\begin{aligned}
&\overset{(23)}{=} \sum_{i,j\in\mathbb{Z}} c_i c_j \cdot \delta_{i,j} \begin{cases} \lambda_j - \lambda & \text{if } \phi_j \in E^+ \\ 1 & \text{if } \phi_j \in E^0 \\ -(\lambda_j-\lambda) & \text{if } \phi_j \in E^- \end{cases} \\
&= \sum_{j\in\mathbb{Z}} c_j^2 \begin{cases} |\lambda_j - \lambda| & \text{if } \phi_j \in E^+ \oplus E^- \\ 1 & \text{if } \phi_j \in E^0 \end{cases} \\
&\geq c \sum_{j\in\mathbb{Z}} c_j^2 \,(1+|\lambda_j|) \\
&\overset{(C)}{\geq} cD^{-1} \sum_{i,j\in\mathbb{Z}} c_i c_j \left\langle \phi_i, \phi_j \right\rangle_{\mathrm{H}_0^1(\Omega)} \\
&= cD^{-1} \left\| u \right\|_{\mathrm{H}_0^1(\Omega)}^2
\end{aligned}
$$

The converse estimate is trivial, namely for all $w \in \mathrm{H}_0^1(\Omega)$

$$
\begin{aligned}
|a^*(w,w)| &\leq \left\|\sigma\right\|_\infty \int_\Omega |\nabla w|^2 + |\lambda| \left\|c\right\|_\infty \int_\Omega |w|^2 \,\mathrm{d}x \\
&\leq \left(\left\|\sigma\right\|_\infty + C_P(\Omega)^2 |\lambda| \left\|c\right\|_\infty\right) \int_\Omega |\nabla w|^2 \,\mathrm{d}x \\
&= \left(\left\|\sigma\right\|_\infty + C_P(\Omega)^2 |\lambda| \left\|c\right\|_\infty\right) \left\|w\right\|_{\mathrm{H}_0^1(\Omega)}^2 .
\end{aligned}
$$

Here, $C_P(\Omega)$ denotes the best constant of Poincaré's Inequality on $\mathrm{H}_0^1(\Omega)$. This gives $\left\|u\right\| \leq C \left\|u\right\|_{\mathrm{H}_0^1(\Omega)}$ for some $C > 0$. So we conclude that $\left\|\cdot\right\|$ is equivalent to the standard norm on $\mathrm{H}_0^1(\Omega)$. ☐

So we see that the energy functional Φ takes the form

$$
\Phi(u) = \frac{1}{2}\left(\left\|u^+\right\|^2 - \left\|u^-\right\|^2\right) - I(u) \qquad \text{where } I(u) := \int_\Omega \left(\int_0^{u(x)} g(x,s)\,\mathrm{d}s\right) \mathrm{d}x.
$$

We recall from [23, p.31] two assumptions of the abstract critical point theorem that we will have to check:

Assumption (D).

(1) $\Phi(u) = \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\left\|u^-\right\|^2 - I(u)$ where $I(0) = 0$, $\frac{1}{2}I'(u)[u] > I(u) > 0$ for all $u \neq 0$ and I is weakly lower semicontinuous.
(2) For each $w \in E \setminus (E^0 \oplus E^-)$ there exists a unique nontrivial critical point of $\Phi|_{\mathbb{R}^+ w \oplus E^0 \oplus E^-}$, which is the unique global maximizer.

Given that Eq. (21) is equivalent to $\Phi'(u) = 0$ under our assumptions by [23, Theorem 3(i)], Theorem 16 is proved once the hypotheses of the following Critical Point Theorem [23, Theorem 35] are satisfied.

Theorem 18 (Szulkin, Weth)**.** *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and suppose that $\Phi : H \to \mathbb{R}$ satisfies Assumption (D) and*

(i) $I'(u) = o(\left\|u\right\|)$ as $u \to 0$,
(ii) $I(su)/s^2 \to \infty$ uniformly for u on weakly compact subsets of $H \setminus \{0\}$ as $s \to \infty$.
(iii) I' is completely continuous.

Then equation $\Phi'(u) = 0$ has a least energy solution. Moreover, if I is even, then this equation has infinitely many pairs of solutions.

All of these assumptions have been verified in [23, Theorem 37] for the very similar equation

$$-\Delta u - \lambda u = g(x,u) \quad \text{in } \Omega, \qquad u \in \mathrm{H}^1_0(\Omega) \tag{25}$$

where λ is bigger than or equal to the lowest (positive) Dirichlet eigenvalue of the negative Laplacian. (The only reason for the latter assumption is that it ensures $E^+ \neq \{0\}$ as well as $E^0 \oplus E^- \neq \{0\}$. In our case, this assumption is satisfied for all $\lambda \in \mathbb{R}$ because of Eq. (23).) The only significant difference between the situations is that the subspace E^- for Eq. (25) is finite-dimensional, whereas it is infinite-dimensional in our setting. However, it was emphasized explicitly on p. 32, l. 15–17 of [23] that the corresponding computations do not rely on $\dim(E^-) < \infty$.

Proof of Theorem 16: So we verify Assumption (D) and (i), (ii), (iii) using the results from [23]. We start with the properties (i), (ii), (iii) that only depend on the nonlinear part and are therefore entirely identical to the discussion related to Eq. (25). In fact, under our assumptions from Theorem 16 the assumptions (i), (ii) were verified in the proof of [23, Theorem 16] and [23, Theorem 3(iii)] gives (iii). So the assumptions (i), (ii), (iii) of Theorem 18 are satisfied. Assumption (D) (1) is a consequence of [23, Theorem 3(ii)] and [23, Lemma 21]. The most difficult assumption Assumption (D) (2) is verified on [23, pp. 31–32], and the arguments carry over to our setting simply by replacing the bilinear form B by our a^*. □

One may even prove this result for nonlinearities of the opposite sign, so for g replaced by $-g$. This is possible because the roles of E^+, E^- are exchangeable. In the context of strongly indefinite semilinear Schrödinger equation with periodic potentials, this fact was exploited in [22, Theorem 4.1].

5.2. **An example in 1D.** We now show that the general result from above applies in the one-dimensional setting that we already discussed in our bifurcation analysis from Corollary 4. So we consider the problem Eq. (3), namely

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\sigma(x)\,u'(x)\right) - \lambda\, c(x)\, u = u^3 \quad \text{in } \Omega, \qquad u \in \mathrm{H}^1_0(\Omega).$$

The assumptions on σ, c, Ω are as before. It is trivial to check that the cubic nonlinearity $g(x,z) = z^3$ satisfies the assumptions (i) to (iv) from Theorem 16 and Assumptions (A) and (B) have been verified earlier, see Lemma 14.

Lemma 19. *Let Ω, σ, c be given as in Corollary 4. Then Assumption (C) holds.*

Proof. We recall from the proof of Corollary 4 and Lemma 15 that the orthonormal basis of eigenfunctions $(\phi_j)_{j\in\mathbb{Z}}$ is given by

$$\phi_j(x) := \alpha_j \begin{cases} \sin\left(\sqrt{\lambda_j}k_+a_+\right)\sinh\left(\sqrt{\lambda_j}k_-(x-a_-)\right) & \text{if } a_- < x < 0, \\ \sinh\left(\sqrt{\lambda_j}k_-a_-\right)\sin\left(\sqrt{\lambda_j}k_+(x-a_+)\right) & \text{if } 0 < x < a_+, \end{cases} \qquad \text{if } \lambda_j > 0$$

$$\phi_j(x) := \alpha_j \begin{cases} \sinh\left(\sqrt{|\lambda_j|}k_+a_+\right)\sin\left(\sqrt{|\lambda_j|}k_-(x-a_-)\right) & \text{if } a_- < x < 0, \\ \sin\left(\sqrt{|\lambda_j|}k_-a_-\right)\sinh\left(\sqrt{|\lambda_j|}k_+(x-a_+)\right) & \text{if } 0 < x < a_+, \end{cases} \qquad \text{if } \lambda_j < 0$$

and $\alpha_j \in \mathbb{R}$ ensures $\|\phi_j\|_c = 1$. We then have for $\lambda_j > 0$

$$\alpha_j^{-2} = \sin\left(\sqrt{\lambda_j}k_+a_+\right)^2 \int_{a_-}^0 \sinh\left(\sqrt{\lambda_j}k_-(x-a_-)\right)^2 \mathrm{d}x + \sinh\left(\sqrt{\lambda_j}k_-a_-\right)^2 \int_0^{a_+} \sin\left(\sqrt{\lambda_j}k_+(x-a_+)\right)^2 \mathrm{d}x$$

$$= \frac{\sin\left(\sqrt{\lambda_j}k_+a_+\right)^2}{k_-\sqrt{\lambda_j}} \int_{\sqrt{\lambda_j}k_-a_-}^{0} \sinh(y)^2 \,\mathrm{d}y + \frac{\sinh\left(\sqrt{\lambda_j}k_-a_-\right)^2}{k_+\sqrt{\lambda_j}} \int_0^{\sqrt{\lambda_j}k_+a_+} \sin(y)^2 \,\mathrm{d}y$$
$$= \frac{\sin\left(\sqrt{\lambda_j}k_+a_+\right)^2}{4k_-\sqrt{\lambda_j}} \mathsf{e}^{2\sqrt{\lambda_j}k_-|a_-|}(1+\mathcal{O}(1)) + \frac{\mathsf{e}^{2\sqrt{\lambda_j}k_-|a_-|}(1+\mathcal{O}(1))}{4k_+\sqrt{\lambda_j}} \left[\frac{\sqrt{\lambda_j}k_+a_+}{2} + \mathcal{O}(1)\right]$$
$$= \frac{a_+}{8}\mathsf{e}^{2\sqrt{\lambda_j}k_-|a_-|}(1+\mathcal{O}(1)) \qquad \text{as } j\to\infty.$$

Similarly we get for $\lambda_j < 0$

$$\alpha_j^{-2} = \frac{|a_-|}{8}\mathsf{e}^{2\sqrt{|\lambda_j|}k_+a_+}(1+\mathcal{O}(1)) \qquad \text{as } j\to-\infty.$$

With these preparations we may now bound the inner products of our basis functions. We first do this for $\lambda_i, \lambda_j > 0$.

$$\begin{aligned}
\left|\langle\phi_j,\phi_i\rangle_{\mathrm{H}_0^1(\Omega)}\right| &= \left|\int_{a_-}^{a_+} \phi_j'(x)\phi_i'(x)\,\mathrm{d}x\right| \\
&= \left|\sigma_-^{-1}\int_{a_-}^{0} \sigma_-\phi_j'(x)\phi_i'(x)\,\mathrm{d}x + \sigma_+^{-1}\int_0^{a_+} \sigma_+\phi_j'(x)\phi_i'(x)\,\mathrm{d}x\right| \\
&\le \left(|\sigma_-|^{-1}+\sigma_+^{-1}\right)\left|\int_{a_-}^{0} \sigma_-\phi_j'(x)\phi_i'(x)\,\mathrm{d}x\right| + \sigma_+^{-1}\left|\int_{a_-}^{a_+} \sigma(x)\phi_j'(x)\phi_i'(x)\,\mathrm{d}x\right| \\
&\overset{(7)}{\lesssim} \left|\int_{a_-}^{0} \phi_j'(x)\phi_i'(x)\,\mathrm{d}x\right| + |\lambda_j|\,\delta_{i,j} \\
&\lesssim |\lambda_j|\,\delta_{i,j} + |\alpha_i|\,|\alpha_j|\left|\sin\left(\sqrt{\lambda_i}k_+a_+\right)\sin\left(\sqrt{\lambda_j}k_+a_+\right)\right|\sqrt{\lambda_i\lambda_j}k_-\cdot \\
&\qquad\left|\int_{a_-}^{0}\cosh\left(\sqrt{\lambda_i}k_-(x-a_-)\right)\cosh\left(\sqrt{\lambda_j}k_-(x-a_-)\right)\mathrm{d}x\right| \\
&\lesssim |\lambda_j|\,\delta_{i,j} + |\alpha_i|\,|\alpha_j|\sqrt{\lambda_i\lambda_j}\left|\int_{a_-}^{0}\cosh\left(\sqrt{\lambda_i}k_-y\right)\cosh\left(\sqrt{\lambda_j}k_-y\right)\mathrm{d}y\right| \\
&\lesssim |\lambda_j|\,\delta_{i,j} + |\alpha_i|\,|\alpha_j|\sqrt{\lambda_i\lambda_j}\int_{a_-}^{0}\mathsf{e}^{\left(\sqrt{\lambda_i}+\sqrt{\lambda_j}\right)k_-|y|}\,\mathrm{d}y \\
&\lesssim |\lambda_j|\,\delta_{i,j} + |\alpha_i|\,|\alpha_j|\sqrt{\lambda_i\lambda_j}\frac{e^{\left(\sqrt{\lambda_i}+\sqrt{\lambda_j}\right)k_-|a_-|}}{\left(\sqrt{\lambda_i}+\sqrt{\lambda_j}\right)k_-} \\
&\lesssim |\lambda_j|\,\delta_{i,j} + \frac{\sqrt{\lambda_i\lambda_j}}{\sqrt{\lambda_i}+\sqrt{\lambda_j}} \\
&\lesssim \sqrt{(1+\lambda_i)(1+\lambda_j)}\left(\delta_{i,j}+\frac{1}{1+|i|+|j|}\right).
\end{aligned}$$

By symmetry, for $\lambda_i, \lambda_j < 0$, we have the same result

$$\left|\langle\phi_j,\phi_i\rangle_{\mathrm{H}_0^1(\Omega)}\right| \lesssim \sqrt{(1+|\lambda_i|)(1+|\lambda_j|)}\left(\delta_{i,j}+\frac{1}{1+|i|+|j|}\right).$$

For $\lambda_i < 0 < \lambda_j$ we obtain in a similar way

$$\left|\langle\phi_j,\phi_i\rangle_{\mathrm{H}_0^1(\Omega)}\right| = \left|\int_{a_-}^{a_+} \phi_j'(x)\phi_i'(x)\,\mathrm{d}x\right|$$

$$
\begin{aligned}
&\lesssim |\alpha_i\alpha_j| \sqrt{|\lambda_i|\,\lambda_j} \\
&\qquad \left[\sinh\left(\sqrt{|\lambda_i|}k_+a_+\right)\left|\int_{a_-}^{0} \cos\left(\sqrt{|\lambda_i|}k_-y\right)\cosh\left(\sqrt{\lambda_j}k_-y\right)\mathrm{d}y\right|\right. \\
&\qquad\qquad \left. + \sinh\left(\sqrt{\lambda_j}k_-|a_-|\right)\left|\int_{0}^{a_+} \cos\left(\sqrt{\lambda_j}k_+y\right)\cosh\left(\sqrt{|\lambda_i|}k_+y\right)\mathrm{d}y\right|\right] \\
&\lesssim |\alpha_i\alpha_j| \sqrt{|\lambda_i|\,\lambda_j} \sum_{\tau_i,\tau_j\in\{-1,+1\}} \left[\mathsf{e}^{\sqrt{|\lambda_i|}k_+a_+}\left|\int_{a_-}^{0} \mathsf{e}^{\left(\tau_i\mathsf{i}\sqrt{|\lambda_i|}+\tau_j\sqrt{\lambda_j}\right)k_-y}\,\mathrm{d}y\right|\right. \\
&\qquad\qquad \left. + \mathsf{e}^{\sqrt{\lambda_j}k_-|a_-|}\left|\int_{0}^{a_+} \mathsf{e}^{\left(\tau_i\sqrt{|\lambda_i|}+\tau_j\mathsf{i}\sqrt{\lambda_j}\right)k_+y}\,\mathrm{d}y\right|\right] \\
&\lesssim |\alpha_i\alpha_j| \sqrt{|\lambda_i|\,\lambda_j} \sum_{\tau_i,\tau_j\in\{-1,+1\}} \left[\mathsf{e}^{\sqrt{|\lambda_i|}k_+a_+}\left|\frac{\mathsf{e}^{\left(\tau_i\mathsf{i}\sqrt{|\lambda_i|}+\tau_j\sqrt{\lambda_j}\right)k_-a_-}}{\tau_i\mathsf{i}\sqrt{|\lambda_i|}+\tau_j\sqrt{\lambda_j}}\right|\right. \\
&\qquad\qquad \left. + \mathsf{e}^{\sqrt{\lambda_j}k_-|a_-|}\left|\frac{\mathsf{e}^{\left(\tau_i\sqrt{|\lambda_i|}+\tau_j\mathsf{i}\sqrt{\lambda_j}\right)k_+a_+}}{\tau_i\sqrt{|\lambda_i|}+\tau_j\mathsf{i}\sqrt{\lambda_j}}\right|\right] \\
&\lesssim \sqrt{|\lambda_i|\,\lambda_j}\,|\alpha_i\alpha_j|\,\mathsf{e}^{\sqrt{|\lambda_i|}k_+a_++\sqrt{\lambda_j}k_-|a_-|}\frac{1}{\sqrt{|\lambda_i|+\lambda_j}} \\
&\lesssim \frac{\sqrt{(1+|\lambda_i|)(1+\lambda_j)}}{1+|i|+|j|}.
\end{aligned}
$$

For the special case where 0 is an eigenvalue, for $0 = \lambda_0 < \lambda_i$, we compute

$$
\begin{aligned}
\left|\langle\phi_0,\phi_i\rangle_{\mathrm{H}_0^1(\Omega)}\right| &= \left|\int_{a_-}^{a_+} \phi_0'(x)\phi_i'(x)\,\mathrm{d}x\right| \\
&\lesssim |\alpha_i|\sqrt{\lambda_i}\left[\left|\int_{a_-}^{0}\cosh\left(\sqrt{\lambda_i}k_-y\right)\mathrm{d}y\right|\right. \\
&\qquad\qquad \left. + \sinh\left(\sqrt{\lambda_i}k_-|a_-|\right)\left|\int_{0}^{a_+}\cos\left(\sqrt{\lambda_i}k_+y\right)\mathrm{d}y\right|\right] \\
&\lesssim \sqrt{\lambda_i}\,|\alpha_i|\,\frac{\sinh\left(\sqrt{\lambda_i}k_-|a_-|\right)}{\sqrt{\lambda_i}} \\
&\lesssim \frac{\sqrt{1+\lambda_i}}{1+|i|}.
\end{aligned}
$$

Again, by symmetry, for $\lambda_i < 0 = \lambda_0$, we have

$$
\left|\langle\phi_0,\phi_i\rangle_{\mathrm{H}_0^1(\Omega)}\right| \lesssim \frac{\sqrt{1+|\lambda_i|}}{1+|i|}.
$$

Therefore, using $\widetilde{c}_i \coloneqq |c_i|\sqrt{1+|\lambda_i|} + |c_{-i}|\sqrt{1+|\lambda_{-i}|}$, we find

$$
\left|\sum_{i,j\in\mathbb{Z}} c_ic_j\,\langle\phi_j,\phi_i\rangle_{\mathrm{H}_0^1(\Omega)}\right| \lesssim \sum_{i,j\in\mathbb{Z}} |c_i||c_j|\sqrt{(1+|\lambda_i|)(1+|\lambda_j|)}\,\left(\delta_{i,j}+\frac{1}{1+|i|+|j|}\right)
$$

$$
\begin{aligned}
&\lesssim \sum_{j\in\mathbb{Z}}(1+|\lambda_j|)c_j^2 + \sum_{i,j=0}^{+\infty}\frac{|\widetilde{c}_i||\widetilde{c}_j|}{1+i+j} \\
&\lesssim \sum_{j\in\mathbb{Z}}(1+|\lambda_j|)c_j^2 + \sum_{i=0}^{+\infty}|\widetilde{c}_i|\,\frac{1}{1+i}\sum_{j=0}^{i}|\widetilde{c}_j| \\
&\lesssim \sum_{j\in\mathbb{Z}}(1+|\lambda_j|)c_j^2 + \|(\widetilde{c}_i)\|_{\ell^2(\mathbb{N})}\left\|\left(\frac{1}{1+i}\sum_{j=0}^{i}|\widetilde{c}_j|\right)\right\|_{\ell^2(\mathbb{N})} \\
&\lesssim \sum_{j\in\mathbb{Z}}(1+|\lambda_j|)c_j^2 + \|(\widetilde{c}_j)\|_{\ell^2(\mathbb{N})}^2 \\
&\lesssim \sum_{j\in\mathbb{Z}}(1+|\lambda_j|)c_j^2
\end{aligned}
$$

In the second last estimate we used Hardy's inequality. This proves Assumption (C). ☐

Combining Lemma 14, Lemma 19 and Theorem 16 we thus obtain:

Corollary 20. *Let Ω, σ, c be given as in Corollary 4 and $\lambda \in \mathbb{R}$. Then equation Eq. (3) has infinitely many nontrivial solutions in $\mathrm{H}_0^1(\Omega)$, among which a least energy solution.*

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