Polyharmonic maps: conservation laws and approximations

Zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften

von der KIT-Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

Dissertation

von

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Tag der mündlichen Prüfung: 31. März 2021

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ABSTRACT

In the first part of the thesis we consider elliptic systems in the critical dimension $2m$ that contain a term with antisymmetric structure. An example for such a system is the m-polyharmonic map equation which we investigate throughout the thesis. Following the work of Rivière in the two-dimensional case, we aim to write the system in divergence-free form and establish a conservation law by using a small perturbation of Uhlenbeck's gauge fixing matrix.

In the second part we focus on the m-polyenergy and consider the higher order approximation $E_\varepsilon(u) = \frac{1}{2}\int_\Omega(|D^m u|^2 + \varepsilon|D^{m+1}u|^2)$, which was first introduced by Lamm in the case $m = 1$. We show that critical points $u_\varepsilon : \Omega \to N^n$, $\Omega \subset \mathbb{R}^{2m}$ compact without boundary, of E_ε are smooth. Further we prove that a sequence (u_ε) of critical points converges strongly to an m-polyharmonic map away from finitely many points as $\varepsilon \to 0$. At the points of energy concentration bubbling occurs and we perform a blow-up to show convergence to quasi-m-polyharmonic spheres. To establish the energy identity in the limit we show that no energy is lost in the neck region between bubble and m-polyharmonic map. For mappings into the sphere this is always true. For arbitrary target manifolds N^n we need to impose an additional entropy condition.

Finally, we consider the ε-approximation of the Dirichlet energy and investigate whether every harmonic map occurs as a limit. The answer to this question is no and we derive a gap theorem for ε-harmonic maps $u_\varepsilon : S^2 \to S^2$ of degree zero and ± 1. We show that ε-harmonic maps of degree zero with energy below 8π are constant and maps of degree ± 1 with energy below 12π are of the form Rx with $R \in O(3)$. This stands in contrast to the fact that all rational maps between two-spheres are harmonic. Moreover, we construct non-trivial ε-harmonic maps of degree zero with energy $\geq 8\pi$.

Acknowledgements

First of all I would like to thank my supervisor Prof. Dr. Tobias Lamm without whom none of this would have been possible. Thank you for introducing me to the world of Geometric Analysis and for encouraging me to embark on this journey. Thank you for suggesting such interesting problems and guiding the way with insightful discussions. And thank you for supporting me during difficult parts and never giving up on me. Additionally, I would like to thank all of my colleagues in the Applied Analysis group at KIT and the best office mates I could hope for: Dr. Leonid Chaichenets and Marvin Schulz. Thank you for the great camaraderie, the countless lunches and always cheering me up.

Chapter 4 of this thesis is part of a collaboration with Prof. Dr. Mario Micallef and I would like to thank him for his helpful comments and suggestions. Moreover, I would like to thank my referees Dr. Ben Sharp and apl. Prof. Dr. Peer Kunstmann.

During my time as a PhD student I was fortunate enough to spend six months at Queen Mary University London. I want to thank my host Prof. Dr. Reto Buzano and the Geometric Analysis group for the great hospitality and KHYS for funding this trip.

Thanks also to everyone at RTG 2229 Asymptotic Invariants and Limits of Groups and Spaces and DFG for expanding my horizon in geometry and topology and for funding the first three years of my PhD.

Above all I would like to thank my family and friends. My parents and my sisters Annabell and Fabienne for always believing in me and supporting me in everything I am doing. This is as much your achievement as it is mine. And my friends Co and Jackie for always encouraging me.

Contents

Chapter 1

Introduction

What is the shortest path from one point in space to another? What type of house has the least possible surface area for a given interior volume? And how should a plane be shaped to minimize its resistance to air? For centuries mathematicians all around the world have been pondering questions like these with the goal of finding the optimal object for any given situation. This hunt for maxima and minima is called Calculus of Variations and dates back to Bernoulli and Euler in the eighteenth century. Since then it has become one of the most important analytical techniques with wide ranging applications in mathematics and physics.

All of the above questions can be translated into an abstract mathematical setting. To construct the optimal house, think of a slice made of rubber that bends in every direction. How do you bend it to enclose the most volume? The Inuit got that one right.

To answer the first question, think of many different paths that connect two points A and B. They differ in length and speed given the terrain. Thus, the problem is to minimize one given quantity (the way from A to B) inside another one (the terrain). While a path is a one-dimensional quantity with given start and end point, one can also think of a two-dimensional elastic surface $\mathcal{S}$ and look for the optimal way to place $\mathcal{S}$ in a given surrounding. Mathematically this translates to a variational problem. There are many different ways to place the surface, we can bend or stretch it. The optimal placement is the one that minimizes the elastic deformation energy. To see if there exists an optimal placement and how to find it, let us transfer this problem into mathematical terms.

Let (M, g) and (N, h) be smooth, compact Riemannian manifolds without boundary and let N be isometrically embedded into $\mathbb{R}^d$. For $u \in W^{1,2}(M, N)$ we define the Dirichlet energy

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 dA_M,$$

where dA_M is the volume Element of (M, g). Critical points of this energy are called *(weakly) harmonic maps* and they satisfy

$$\Delta u \perp T_u N,$$

where Δ is the Laplace-Beltrami operator on M and $T_u N$ is the tangent space of N at u. The Euler-Lagrange equation of E, which is called *harmonic map equation*, is given by

$$\Delta u + A(u)(\nabla u, \nabla u) = 0. \tag{1.0.1}$$

Here A denotes the second fundamental form of the embedding $N^n \hookrightarrow \mathbb{R}^d$. More precisely, A is a symmetric bilinear form with $A(u)(X, Y) = \sum_{j=n+1}^{d} \langle X, D_Y e_j \rangle e_j$, where $(e_{n+1}, ..., e_d)$ is an orthonormal basis of $(T_u N)^{\perp}$ and $X, Y \in T_u N$.

If M is a two-dimensional surface and u is not just harmonic but also conformal, which means that it preserves angles, then $u(M)$ is a minimal immersion in N. This is the optimal placement that we have been looking for at the beginning. But is it always possible to find an optimum?

Existence and regularity of critical points of E depend on the choice of M and N. If $M = \mathbb{R}^m$, $N = \mathbb{R}^n$ with the euclidean metric on both the domain and the target, then critical points are the well known harmonic functions and the harmonic map equation simplifies to the linear equation $\Delta u = 0$. In this setting, existence, uniqueness and smoothness of solutions u are fairly easy to prove, since we have powerful tools such as the maximum principle at our disposal. This is no longer the case in the general setting because of the nonlinearity in (1.0.1), and especially domain manifolds of dimension ≥ 2 cause problems.

If M has dimension $m \geq 3$, Schoen and Uhlenbeck [65] showed that minimizers of E are smooth on M except for a closed subset of Hausdorff dimension $\leq m-3$, and Bethuel [7] showed that stationary harmonic maps are smooth up to a closed subset whose $(m-2)$-dimensional Hausdorff measure is zero. However, Rivière [59] discovered everywhere discontinuous weakly harmonic maps for $m = 3$.

Let us restrict ourselves to the two-dimensional case, where M is a surface. In 1948, Morrey [53] showed that if u is a minimizer of E, then $u \in C^\infty(M, N)$. Later Grüter [28] extended this result to conformal weakly harmonic maps. In 1984, Schoen [66] proved regularity of stationary harmonic maps and in 1991, Hélein [32] showed regularity for weakly harmonic maps by applying his famous moving frame method. With a completely different Ansatz Rivière [58] established a conservation law for the harmonic map equation and used this to show continuity of solutions. We will come back to this technique in a short while.

Of course there are many other aspects of harmonic maps that have been studied in the literature (e.g. [14],[17],[18],[31],[49],[51],[56],[71],[79]). Harmonic maps between surfaces and Lie groups (σ-models) play an important role in physics as well. They have strong connections to the Skyme-, Higgs- and Ginzburg-Landau models and are used in the context of (anti)self-dual Yang-Mills connections on 4-manifolds.

For the moment let us ask ourselves if there is a higher order equivalent to harmonic maps that is worth examining. We consider

$$E_m(u) = \frac{1}{2} \int_{M^{2m}} |D^m u|^2 dA_{M^{2m}} \tag{1.0.2}$$

for some $m \in \mathbb{N}$, where D^m is the m^{th} total derivative of $u : M^{2m} \to \mathbb{R}^d$ and M^{2m} is a $2m$-dimensional smooth closed manifold. E_m is called the *m-polyenergy* and critical points are called *(extrinsic-) m-polyharmonic maps*, i.e. $u \in W^{m,2}(M^{2m}, N)$ such that

$$\Delta^m u \perp T_u N. \tag{1.0.3}$$

Here E_m depends on the choice of the embedding $N \hookrightarrow \mathbb{R}^d$. The more natural energy from a geometric point of view is the intrinsic energy which uses covariant derivatives of maps $M^{2m} \to N$. However, the extrinsic energy has analytic advantages such as the bound of u in the full $W^{m,2}$-norm

$$||u||^2_{W^{m,2}(M^{2m},N^n)} \leq cE_m(u),$$

which we will use throughout this thesis. For $m = 2$, critical points are called biharmonic maps and just like harmonic maps they have been studied extensively. In 1999, Chang, Wang and Yang [11] showed regularity for weak biharmonic maps into the sphere (see also [75],[80]), later Wang [78] extended the result to Riemannian manifolds. In 2008, Lamm and Rivière [48] established a conservation law for the biharmonic map equation (see also [72]).

Most of these results carry over to the general case $m \geq 2$. In 2009, Gastel and Scheven [21] used moving frames to show regularity of critical points of (1.0.2). In the same year, Goldstein, Strzelecki and Zatorska [26] showed regularity for m-polyharmonic maps into the sphere. A conservation law was first established by de Longueville and Gastel [20].

Conservation laws

We want to take a closer look at the aforementioned conservation laws. To begin, let us focus on harmonic maps $u \in W^{1,2}(B^2, S^{d-1})$ into the sphere. In this case, the harmonic map equation has the form

$$-\Delta u = u|\nabla u|^2. \tag{1.0.4}$$

Note that the Laplacian of u is only in L^1. This prevents us from applying Calderon-Zygmund estimates (see [25]) to get a control on the full second derivative and use the embedding $W^{2,1} \hookrightarrow C^0(B^2)$ ([1] 4.12). Hence we need to find other ways to improve the regularity of u. One idea is to write the equation in divergence-free form and apply some version of the Wente lemma. Shatah [67] noticed that divergence-free form can be achieved by adding a zero term to (1.0.4). Since u itself is a normal vector in the normal space $N_u S^{d-1}$, we have $\sum_{j=1}^d u_j \nabla u_j = 0$. Thus (1.0.4) is equivalent to

$$-\Delta u_i = \sum_{j=1}^{d} [u_i \nabla u_j - u_j \nabla u_i] \cdot \nabla u_j, \qquad i = 1, ..., d. \tag{1.0.5}$$

Moreover, any solution of (1.0.4) satisfies

$$div\, [u_i \nabla u_j - u_j \nabla u_i] = 0.$$

Hélein [31] later used this to establish a conservation law: By the Poincaré lemma (see [24] Theorem 10.68) there exists $B \in W^{1,2}(B^2)$ such that

$$\nabla^\perp B_{ij} = u_i \nabla u_j - u_j \nabla u_i$$

and (1.0.5) is equivalent to

$$-\Delta u = \nabla^\perp B \cdot \nabla u \quad \Leftrightarrow \quad div\left(\nabla u + B\nabla^\perp u\right) = 0,$$

where $\nabla^\perp$ is ∇ rotated by 90°. (Note that $div \nabla^\perp = 0$) This new structure of (1.0.5) allows us to apply Wente's lemma (see [10]) and thus $u \in C^0(B^2, S^{d-1})$.

Now the question is: Can divergence-free form or "Wente structure" be achieved for arbitrary targets as well and does this method carry over to the m-polyharmonic map equation? This will be the focus of chapter 2.

Let us consider $u \in W^{1,2}(B^2, N)$, where N is an oriented codimension one submanifold of $\mathbb{R}^n$. In this case, the harmonic map equation is given by

$$-\Delta u_i = \sum_{j=1}^{n} n_i \nabla n_j \cdot \nabla u_j, \qquad i = 1, ..., n, \tag{1.0.6}$$

where $n(y)$ is the Gauss map of N at y. Since n is orthogonal to ∇u, i.e. $\sum_{j=1}^n n_j \nabla u_j = 0$, we can add a term to (1.0.6)

$$-\Delta u_i = \sum_{j=1}^{n} [n_i \nabla n_j - n_j \nabla n_i] \cdot \nabla u_j. \tag{1.0.7}$$

However, we cannot assume that $[n_i \nabla n_j - n_j \nabla n_i]$ is divergence free and thus a conservation law is not immediately obvious. What we can derive is that [...] is antisymmetric and we will see in the following that this is enough to establish a conservation law. Consider the equation

$$-\Delta u = \Omega \cdot \nabla u, \tag{1.0.8}$$

with $\Omega \in L^2(B^2, so(n) \otimes \wedge^1\mathbb{R}^2)$. Note that this is equivalent to (1.0.7) if we set

$$\Omega_{ij} = n_i \nabla n_j - n_j \nabla n_i.$$

A crucial step on our way to divergence-free form is the so-called Uhlenbeck gauge, a nonlinear Hodge decomposition for antisymmetric matrix-valued one-forms (see Theorem 2.1.3). Due to the antisymmetry of Ω we can write

$$\Omega = P^{-1}\nabla^{\perp}\xi P + P^{-1}\nabla P,$$

where $P \in W^{1,2}(B^2, SO(n))$ and $\xi \in W^{1,2}(B^2, so(n))$.

In section 2.1 we use a small perturbation of the Uhlenbeck matrix P to establish a conservation law. We proceed as follows: We multiply (1.0.8) with $(id + \varepsilon)P$, $\varepsilon \in W^{1,2} \cap L^{\infty}(B^2, M(n))$ and use the chain rule to write this new equation in the form

$$div((id + \varepsilon)P\nabla u) = [(id + \varepsilon)P\Omega] \cdot \nabla u.$$

We want to choose ε such that [...] is divergence free. At this point we use Uhlenbeck's decomposition of Ω to improve the regularity of this new equation. While the improvement is very small, it is just enough to allow for a version of the Wente lemma by Bethuel and Ghidaglia [6]. Using this, we solve a fixed point problem and find the correct ε to construct the conservation law.

The idea for this approach stems from Rivière's lecture on Conformally Invariant Variational Problems ([62], chapter VI) and we will develop this in more detail in section 2.1. Then we transfer Rivière's Ansatz to the higher order case and examine elliptic systems of the form

$$\Delta^m u = \sum_{k=0}^{m-1} \Delta^k \langle V_k, du\rangle + \sum_{k=0}^{m-2} \Delta^k \delta(w_k du), \tag{1.0.9}$$

with $m \geq 2$, where $u \in W^{m,2}(B^{2m}, \mathbb{R}^n)$ and

$$\begin{aligned}
&w_k \in W^{2k+2-m,2}(B^{2m}, \mathbb{R}^{n\times n}) && \text{for } k \in \{0, ..., m-2\},\\
&V_k \in W^{2k+1-m,2}(B^{2m}, \mathbb{R}^{n\times n} \otimes \wedge^1\mathbb{R}^{2m}) && \text{for } k \in \{0, ..., m-1\}, \text{ where}\\
&V_0 = d\eta + F, \quad \eta \in W^{2-m,2}(B^{2m}, so(n)), && F \in W^{2-m,\frac{2m}{m+1},1}(B^{2m}, \mathbb{R}^{n\times n} \otimes \wedge^1\mathbb{R}^{2m}).
\end{aligned}$$

De Longueville and Gastel [20] showed that the m-polyharmonic map equation is of this form. As in the second order case, the crucial assumption is that η takes values in $so(n)$. A priori $d\eta \in W^{1-m,2}(B^m)$ is not bounded and we need a suitable version of Uhlenbeck's gauge theorem (Theorem 2.3.2) to remove the terms we cannot control. To establish the conservation law we proceed as before and multiply (1.0.9) with an arbitrary perturbation of P, remove problematic terms via the Uhlenbeck decomposition and solve a fixed point argument to determine the suitable perturbation. An important ingredient is a Wente type lemma (Lemma 2.4.1) adapted to our situation.

What makes the computations in the higher order case more challenging is that some components of (1.0.9) are distributions (elements of negative (Lorentz-)Sobolev spaces). To handle these we use a representation in terms of derivatives of Lorentz functions (see Lemma A.2.6) and shift derivatives evenly. For more details on (negative) Lorentz-Sobolev spaces see Appendix A.

Once we have established the conservation law, it is easy to show continuity for solutions u. This provides an alternative to Hélein's moving frame method ([21],[31]).

All in all we show the following

Theorem 1.0.1. *Assume $m \geq 2$, $n \in \mathbb{N}$. We consider the equation*

$$\Delta^m u = \sum_{k=0}^{m-1} \Delta^k \langle V_k, du \rangle + \sum_{k=0}^{m-2} \Delta^k \delta(w_k du) \tag{1.0.10}$$

with coefficient functions

$$\begin{aligned} & w_k \in W^{2k+2-m,2}(B^{2m}, \mathbb{R}^{n\times n}) && \text{for } k \in \{0, ..., m-2\}, \\ & V_k \in W^{2k+1-m,2}(B^{2m}, \mathbb{R}^{n\times n} \otimes \wedge^1 \mathbb{R}^{2m}) && \text{for } k \in \{0, ..., m-1\}, \text{ where} \\ & V_0 = d\eta + F, \quad \eta \in W^{2-m,2}(B^{2m}, so(n)), && F \in W^{2-m, \frac{2m}{m+1},1}(B^{2m}, \mathbb{R}^{n\times n} \otimes \wedge^1 \mathbb{R}^{2m}). \end{aligned} \tag{1.0.11}$$

Then the following statements hold.

(i) Let

$$\begin{aligned} \sigma := & \sum_{k=0}^{m-2} ||w_k||_{W^{2k+2-m,2}(B^{2m})} + \sum_{k=1}^{m-1} ||V_k||_{W^{2k+1-m,2}(B^{2m})} \\ & + ||\eta||_{W^{2-m,2}(B^{2m})} + ||F||_{W^{2-m,\frac{2m}{m+1},1}(B^{2m})}. \end{aligned} \tag{1.0.12}$$

There is $\sigma_0 > 0$, such that whenever $\sigma < \sigma_0$, there exist $\varepsilon \in W^{m,2} \cap L^\infty(B^{2m}_{1/2}; M(n))$ with

$$||\varepsilon||_{W^{m,2}(B^{2m}_{1/2})} + ||\varepsilon||_{L^\infty(B^{2m}_{1/2})} \leq c\sigma,$$

$P \in W^{m,2}(B_{1/2}, SO(n))$, and a distribution $B \in W^{2-m,2}_{loc}(B^{2m}_{1/2}, \mathbb{R}^{n\times n} \otimes \wedge^2 \mathbb{R}^{2m})$ which solve

$$\delta B = \sum_{k=0}^{m-1} \Delta^k((id+\varepsilon)P)V_k - \sum_{k=0}^{m-2} d\Delta^k((id+\varepsilon)P)w_k + d\Delta^{m-1}((id+\varepsilon)P).$$

(ii) A function $u \in W^{m,2}(B^{2m}_{1/2}, \mathbb{R}^n)$ solves (1.0.10) weakly if and only if it is a distributional solution of the conservation law

$$\begin{aligned} \delta\Bigg[& \sum_{l=0}^{m-1} \Delta^l((id+\varepsilon)P)\Delta^{m-l-1} du - \sum_{l=0}^{m-2} d\Delta^l((id+\varepsilon)P)\Delta^{m-l-1} u \\ & - \sum_{k=0}^{m-1}\sum_{l=0}^{k-1} \Delta^l((id+\varepsilon)P)\Delta^{k-l-1} d\langle V_k, du\rangle \\ & + \sum_{k=0}^{m-1}\sum_{l=0}^{k-1} d\Delta^l((id+\varepsilon)P)\Delta^{k-l-1}\langle V_k, du\rangle \\ & - \sum_{k=0}^{m-2}\sum_{l=0}^{k} \Delta^l((id+\varepsilon)P) d\Delta^{k-l-1}\delta(w_k du) \\ & + \sum_{k=0}^{m-2}\sum_{l=0}^{k-1} d\Delta^l((id+\varepsilon)P)\Delta^{k-l-1}\delta(w_k du) - \langle B, du\rangle \Bigg] = 0. \end{aligned} \tag{1.0.13}$$

Corollary 1.0.2. *Weak solutions $u \in W^{m,2}(B^{2m})$ of (1.0.10) are continuous.*

A different variant of this result has been obtained earlier by Lamm and Rivière [48] in the case $m = 2$ and by De Longueville and Gastel [20] for general $m \geq 3$. Here we use a small perturbation

$(id+\varepsilon)P$ of the Uhlenbeck gauge matrix P to establish the conservation law. This Ansatz highlights the strong connection between the conservation law and the matrix P more explicitly than the previous papers. For the sake of completeness and to highlight the differences we include these results in section 2.1. Another new ingredient in our approach is Lemma 2.4.1, a generalization of an estimate by Bethuel and Ghidaglia [6], which we use instead of a Wente type result for the poly-Laplace operator. This allows for more general elliptic operators in divergence form and simplifies the argument.
We also remark that in a recent paper by Guo and Xiang [29] it was shown that weak solutions of (1.0.10) are not only continuous but even Hölder continuous for some positive exponent.

Approximate energy functionals

Now let us consider m-polyharmonic maps as critical points of the functional (1.0.2). In chapter 2 we have assumed that solutions of the m-polyharmonic map equation exist. However, existence of critical points is a priori not clear because E lacks compactness and does not satisfy the Palais-Smale condition. In chapter 3 we will focus on this problem.

In 1981, Sacks and Uhlenbeck [63] introduced their famous α-energy

$$E_\alpha(u) = \frac{1}{2}\int_{M^2}(1+|\nabla u|^2)^\alpha dA_{M^2},$$

with $\alpha > 0$. This perturbation of the Dirichlet functional satisfies the Palais-Smale condition and the existence of critical points of E_α follows. Further, Sacks and Uhlenbeck studied the behavior of critical points u_α as $\alpha \to 1$ and they were able to show strong convergence to a smooth harmonic map on all of M^2 except at finitely many points where bubbles form. More precisely, u_α splits into a harmonic map $u^* : M^2 \to N$ and finitely many non-trivial minimal two-spheres $u^i : S^2 \to N$ in the limit. Additionally, they showed the energy inequality

$$E(u^*) + \frac{vol(M^2)}{2} + \sum_{i=1}^{I} E(u^i) \le \limsup_{\alpha\to 1} E_\alpha(u_\alpha)$$

(see [63] Theorem 4.7). What remained open was the question if equality holds in the limit. To answer this, one needs to take a closer look at the neck region between bubble and harmonic map and determine if energy is lost as $\alpha \to 1$. In 2010, Lamm [43] (see also Jost [39]) solved this problem and established the energy identity

$$\lim_{\alpha\to 1} E_\alpha(u_\alpha) = E(u^*) + \frac{vol(M^2)}{2} + \sum_{i=0}^{I} E(u^i),$$

assuming the entropy condition

$$\liminf_{\alpha\to 1}(\alpha-1)\int_{M^2}\log(1+|\nabla u_\alpha|^2)(1+|\nabla u_\alpha|^2)^\alpha dA_{M^2} = 0.$$

This condition was first introduced by Struwe in [73]. Later, Li and Zhu [50] showed the energy identity if the target manifold is a sphere without this additional assumption. However, the entropy condition cannot be omitted in general (see [51] for a counterexample).

Sacks and Uhlenbeck's idea of approximating the Dirichlet energy to improve compactness was also used in other settings. One example is the Willmore energy

$$\mathcal{W}(u) = \frac{1}{4}\int_{M^2}|H|^2 dA_{M^2},$$

where H denotes the mean curvature vector. $\mathcal{W}$ is invariant under conformal transformations and therefore does not satisfy the Palais-Smale condition. Kuwert, Lamm and Li [40] defined the p-Willmore energy in the spirit of Sacks and Uhlenbeck

$$\mathcal{W}^p(u) = \frac{1}{4}\int_{M^2} \left(1+|H|^2\right)^{\frac{p}{2}} dA_{M^2}.$$

For $p > 2$ this functional is just coercive enough to satisfy the Palais-Smale condition for sequences (u_k) with Willmore energy smaller than 8π. Using this, Kuwert, Lamm and Li proved existence and regularity of critical points of $\mathcal{W}^p$.

Of course one can imagine other types of approximations, for example by adding a regularizing term to the functional. Rivière [60] introduced this approximation of the Willmore energy

$$\mathcal{W}_\sigma(u) = \mathcal{W}(u) + \sigma \int_{M^2} (1+|H|^2)^2 dA_{M^2} + \frac{1}{\log(\frac{1}{\sigma})}\mathcal{O}(u)$$

with parameter $\sigma > 0$ and there are many other examples throughout the literature (e.g. [43],[44],[57],[73]). We want to focus on the ε-approximation of Lamm ([43],[44])

$$E_\varepsilon(u) = \frac{1}{2}\int_{M^2} \left(|\nabla u|^2 + \varepsilon|\Delta u|^2\right) dA_{M^2}.$$

For $\varepsilon > 0$, E_ε satisfies the Palais-Smale condition. Lamm studied sequences of critical points (u_ε) of E_ε as $\varepsilon \to 0$ and discovered the same bubbling phenomenon as Sacks and Uhlenbeck. Moreover, he showed the energy identity in the limit

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = E_0(u^*) + \sum_{i=1}^{I} E_0(u^i).$$

To prevent energy loss in the neck region, he had to assume an entropy condition for general target manifolds similar to the α-energy case

$$\liminf_{\varepsilon\to 0} \varepsilon \log\left(\frac{1}{\varepsilon}\right)\int_{M^2} |\Delta u_\varepsilon|^2 dA_{M^2} = 0.$$

We use his approach and extend it to higher order energies in the critical dimension. To simplify the calculations and for the sake of legibility we choose $\Omega \subset \mathbb{R}^{2m}$ smooth, open and bounded as the domain. However, we expect the results to hold for general domain manifolds M^{2m} as well, since we only pick up lower order curvature terms. For $u \in W^{m+1,2}(\Omega, N)$ we define

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left(|D^m u|^2 + \varepsilon|D^{m+1}u|^2\right) dx, \qquad \varepsilon > 0. \tag{1.0.14}$$

We show that E_ε satisfies the Palais-Smale condition and that critical points are smooth. Following the work of Lamm we establish the energy identity and prove the following two theorems.

Theorem 1.0.3. *Let $\Omega \subset \mathbb{R}^{2m}$ be an open and bounded domain and let $N = S^{d-1} \hookrightarrow \mathbb{R}^d$ be the standard sphere. Further let $(u_\varepsilon)_\varepsilon \in C^\infty(\Omega)$ be a sequence of critical points of E_ε with uniformly bounded energy E_ε. Then there exists a sequence $\varepsilon_k \to 0$ and at most finitely many points $x_1, ..., x_p \in \Omega$ such that $u_{\varepsilon_k} \to u_0$ weakly in $W^{m,2}(\Omega, N)$ and strongly in $C^\infty_{loc}(\Omega \setminus \{x_1, ..., x_p\}, N)$ for all $m \in \mathbb{N}$ and $u_0\colon \Omega \to N$ is a smooth m-polyharmonic map. Moreover, there exist at most finitely many non-trivial smooth quasi-m-polyharmonic maps $\omega^{i,j}\colon S^{2m} \to N$, $1 \le j \le j_i$, sequences of points $x^{i,j}_k \in \Omega$, $x^{i,j}_k \to x_i$ and sequences of radii $t^{i,j}_k \in \mathbb{R}^+$, $t^{i,j}_k \to 0$, such that for $k \to \infty$*

$$\max\left\{\frac{t^{i,j}_k}{t^{i,j'}_k}, \frac{t^{i,j'}_k}{t^{i,j}_k}, \frac{\operatorname{dist}(x^{i,j}_k, x^{i,j'}_k)}{t^{i,j}_k + t^{i,j'}_k}\right\} \to \infty \qquad \forall\ 1 \le i \le p,\ 1 \le j, j' \le j_i,\ j \ne j' \tag{1.0.15}$$

$$\frac{\varepsilon_k}{(t_k^{i,j})^2} \to 0 \qquad \forall\, 1 \le i \le p,\ 1 \le j, \le j_i \tag{1.0.16}$$

and

$$\lim_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k}) = E_0(u_0) + \sum_{i=1}^{p} E_0(\omega^i). \tag{1.0.17}$$

Theorem 1.0.4. *Theorem 1.0.3 holds true for any smooth closed n-dimensional Riemannian target manifold $N \hookrightarrow \mathbb{R}^d$ if we assume additionally that*

$$\varepsilon_k \log\left(\frac{1}{\varepsilon_k}\right) \int_\Omega |D^{m+1} u_{\varepsilon_k}|^2 dx \to 0 \qquad \text{as } k \to \infty. \tag{1.0.18}$$

Remark 1.0.5. *A quasi-m-polyharmonic map is a map $\omega : S^{2m} \to N$ which satisfies (3.5.8). (see [8], [12])*

Chapter 3 is structured in the following way. In sections 3.1 and 3.2 we derive the Euler-Lagrange equation and verify the Palais-Smale condition. Once we know that critical points exist, we employ a hole filling argument and show higher regularity of critical points (see section 3.3).

In section 3.4 we establish a small energy regularity result for critical points u_ε in the spirit of Lamm's work in [44]. Then we show strong convergence of $(u_\varepsilon)_\varepsilon$ away from finitely many points and perform a blow-up analysis to show convergence to a quasi-m-polyharmonic sphere at the energy concentration points.

In section 3.6 we prove our main theorem for mappings into the sphere by adapting work of Wang [80] and Wang/Zheng [81]. We reformulate the Euler-Lagrange equation and use a Hodge decomposition to get a good control on the highest order term. With the results from section 3.4 and Lorentz space theory we show that there is no energy lost in the neck region as ε tends to zero.

In section 3.7 we prove the energy identity for a general target manifold N. Here we have to assume the entropy condition (1.0.18) and we estimate the radial and tangential parts of the derivatives separately. This is influenced by work of Ding and Tian [14] in the approximate harmonic case, Lamm [43] in the ε-harmonic case and Wang and Zheng [82] in the biharmonic case. To estimate the tangential part we approximate u_{ε_k} on annuli around a concentration point by radial m-polyharmonic maps. To estimate the radial derivatives we apply a Pohozaev type argument and introduce cylindrical coordinates (see Ai and Yin [2]) to separate the purely radial derivatives. Together with the entropy condition we show that E_ε tends to zero in the neck region.

Finally, we show existence of critical points u_ε that satisfy the entropy condition. This part is based on the work of Struwe [73] (see also Lamm [43]).

In the last part of this thesis we return to harmonic maps and focus on approximations of the Dirichlet energy. As mentioned before, sequences of critical points (u_α) and (u_ε) of E_α and E_ε respectively converge to a harmonic map and finitely many bubbles as $\alpha \to 1$, $\varepsilon \to 0$. Now one can ask whether every harmonic map is captured by this procedure.

To answer this question, we restrict ourselves to the case $M = N = S^2$. It was shown by Wood and Lemaire ((11.5) in [17]) that all harmonic maps between 2-spheres are precisely the rational maps and their complex conjugates (i.e. rational in z or $\bar{z}$). Further note that a rational map $u : S^2 \to S^2$ has Dirichlet energy $E(u) = 4\pi|\deg(u)|$, which is the least energy that a map of this degree can have.

However, Lamm, Malchiodi and Micallef [47] showed that the only α-harmonic maps $u_\alpha : S^2 \to S^2$ with energy E_α below a certain threshold are the constant maps and rotations. In chapter 4 we establish a similar result for critical points of the ε-approximation of degree zero and ± 1.

The degree of a map $u : S^2 \to S^2$ is defined by

$$\deg(u) = \frac{1}{4\pi} \int_{S^2} J(u) dA_{S^2} \qquad \text{with} \qquad J(u) = u \cdot e_1(u) \wedge e_2(u),$$

where (e_1, e_2) is a local oriented orthonormal frame of TS^2. For every $u \in W^{2,2}(S^2, S^2)$ with $\deg(u) = 1$ we have

$$\begin{aligned} 4\pi(1 + 2\varepsilon) &= \int_{S^2} J(u)\, dA_{S^2} + \frac{\varepsilon}{2\pi} \left(\int_{S^2} J(u)\, dA_{S^2} \right)^2 \\ &\le E(u) + \frac{\varepsilon}{8\pi} \left(\int_{S^2} |\nabla u|^2\, dA_{S^2} \right)^2 \\ &\le E(u) + \frac{\varepsilon}{2} \int_{S^2} |\nabla u|^4\, dA_{S^2} \\ &\le E_\varepsilon(u), \end{aligned} \tag{1.0.19}$$

where we used that $\Delta u = (\Delta u)^T - u|\nabla u|^2$ and

$$\int_{S^2} |(\Delta u)^T|^2\, dA_{S^2} + \int_{S^2} |\nabla u|^4\, dA_{S^2} = \int_{S^2} |\Delta u|^2\, dA_{S^2}$$

in the last step. If u is harmonic, then $(\Delta u)^T = 0$. Thus equality holds in (1.0.19) if and only if u is a conformal map (first inequality) with constant energy density (second inequality)

$$e(u) := \frac{|\nabla u|^2}{2} \equiv 1.$$

For every $R \in SO(3)$ and map $u^R(x) = Rx$ we have

$$E_\varepsilon(u^R) = 4\pi + 8\pi\varepsilon. \tag{1.0.20}$$

Hence rotations are the only minimizers of E_ε among all maps of degree 1. If we compare this to the statement of Wood and Lemaire above, we see that not every harmonic map is approximated by E_ε. For example, a dilation is a rational map of degree one that does not minimize E_ε for $\varepsilon > 0$.

We show the following two results.

Theorem 1.0.6. *For any $\delta > 0$ there exists $\tilde{\varepsilon} > 0$ such that the only critical points u_ε of E_ε of degree zero which satisfy $E_\varepsilon(u_\varepsilon) \le 8\pi - \delta$ and $\varepsilon \le \tilde{\varepsilon}$ are the constant maps.*

Theorem 1.0.7. *For any $\mu > 0$ there exists $\bar{\varepsilon} > 0$ such that the only critical points u_ε of E_ε of degree ± 1 which satisfy $E_\varepsilon(u_\varepsilon) \le 12\pi - \mu$ and $\varepsilon \le \bar{\varepsilon}$ are maps of the form $u^R(x) = Rx$ with $R \in O(3)$.*

Note that we have to include reflections if $\deg u = -1$. The proof of Theorem 1.0.6 follows analogously to [46],[47]. We use the energy identity (1.0.17) and a result by Duzaar and Kuwert [16], which shows that the degree of a sequence (u_ε) is preserved in the limit. The gap theorem for ε-harmonic maps with small energy (Lemma 4.5.1) concludes the proof.

To show Theorem 1.0.7, we use a group of conformal transformations of the sphere called the Möbius group. In section 4.1 we will see that these transformations correspond to $M \in PSL(2, \mathbb{C})$ via stereographic projection to the complex plane. We follow Lamm, Malchiodi and Micallef's idea [47] and apply a Möbius transformation to a critical point u_ε. Then we show that there exists

$M \in PSL(2,\mathbb{C})$ such that $(u_\varepsilon)_M$ is equal to the identity. Moreover, we will show that this M defines a rotation on the sphere.

In a first step we investigate how E_ε changes once we apply u_M. We will see that the transformation relation depends only on the eigenvalue λ of MM^* and it is therefore enough to demonstrate that $\lambda = 1$. To do this we show that critical points u_ε are close to a Möbius transformation in the $\sqrt{\varepsilon}W^{3,2}$-norm and simultaneously establish a bound on λ.

Moreover, we construct rotationally symmetric ε-harmonic maps of degree zero whose ε-energy is bigger or equal to 8π. This shows that the bound in Theorem 1.0.6 is optimal.

Theorem 1.0.8. *For every $\delta > 0$ there exists $\varepsilon_0 > 0$ depending only on δ such that, if $0 < \varepsilon < \varepsilon_0$, there exists an ε-harmonic map $u_\varepsilon : S^2 \to S^2$ with $\deg(u_\varepsilon) = 0$ and*

$$8\pi \leq E_\varepsilon(u_\varepsilon) < 8\pi + \delta.$$

Chapter 2

Conservation laws for even order elliptic systems

In this chapter we establish a conservation law for elliptic systems of the form (1.0.9). Before we get to the higher order case, we take a closer look at second order elliptic equations in dimension two. This will give us a better understanding of the motivation behind our result. Throughout this chapter we denote by B^d the d-dimensional unit ball.

The following is based on joint work with Tobias Lamm [36].

2.1 Second order case

In the introduction we saw how divergence-free form is achieved for mappings to the sphere. In this section we examine the general case and consider $u \in W^{1,2}(B^2, N)$, where N is an oriented submanifold of $\mathbb{R}^n$. As seen in the introduction, the harmonic map equation is equivalent to an equation of the form (2.1.1). In his celebrated paper [58], Rivière used the antisymmetry of Ω to show the following results.

Theorem 2.1.1 (Rivière [58]). *Let $n \in \mathbb{N}$ and let $\Omega \in L^2(B^2, so(n) \otimes \wedge^1\mathbb{R}^2)$, $A \in W^{1,2} \cap L^\infty(B^2, M(n))$ and $B \in W^{1,2}(B^2, M(n) \otimes \wedge^2\mathbb{R}^2)$ satisfying*

$$\nabla_\Omega A := \nabla A - A\Omega = \nabla^\perp B.$$

Then every solution to (2.1.1) satisfies the following conservation law

$$div(A\nabla u + B\nabla^\perp u) = 0.$$

Theorem 2.1.2 (Rivière [58]). *Let $n \in \mathbb{N}$. For every $\Omega \in L^2(B^2, so(n) \otimes \wedge^1\mathbb{R}^2)$ every $u \in W^{1,2}(B^2, \mathbb{R}^n)$ solving*

$$-\Delta u = \Omega \cdot \nabla u \tag{2.1.1}$$

is continuous.

The proof relies heavily on Uhlenbeck's gauge theorem (see [58], [77] or [64]).

Theorem 2.1.3 (Uhlenbeck gauge). *There exists $\sigma > 0$ and $c > 0$ such that for every $\Omega \in L^2(B^2, so(n) \otimes \wedge^1\mathbb{R}^2)$ satisfying $||\Omega||_{L^2(B^2)} < \sigma$, there exist $P \in W^{1,2}(B^2, SO(n))$ and $\xi \in W^{1,2}(B^2, so(n))$ such that*

$$\Omega = P^{-1}\nabla^\perp \xi P + P^{-1}\nabla P$$

and

$$||\xi||_{W^{1,2}(B^2)} + ||\nabla P||_{L^2(B^2)} \le c||\Omega||_{L^2(B^2)}.$$

We will not discuss Rivière's original proof but focus on his subsequent idea to establish a conservation law by using a small perturbation of the Uhlenbeck gauge matrix P. (see [62], chapter VI)

Assume $||\Omega||_{L^2(B^2)} < \sigma$ as in Theorem 2.1.3. Then there exist $P \in W^{1,2}(B^2, SO(n))$, $\xi \in W^{1,2}(B^2, so(n))$ such that

$$\begin{aligned}\Omega =& P^{-1}\nabla^{\perp}\xi P + P^{-1}\nabla P,\\ ||\xi||_{W^{1,2}(B^2)} + ||\nabla P||_{L^2(B^2)} \le& c||\Omega||_{L^2(B^2)}.\end{aligned}$$

We multiply (1.0.8) with $(id+\varepsilon)P$, where $\varepsilon \in W^{1,2} \cap L^\infty(B^2, M(n))$ and id is the identity matrix in $\mathbb{R}^n$.

$$\begin{aligned}&-(id+\varepsilon)P\Delta u = (id+\varepsilon)P\Omega\cdot\nabla u\\ \Leftrightarrow\ &-div\,[(id+\varepsilon)P\nabla u] = [-\nabla\varepsilon P + (id+\varepsilon)(-\nabla P + P\Omega)]\cdot\nabla u\\ \Leftrightarrow\ &-div\,[(id+\varepsilon)P\nabla u] = \left[-\nabla\varepsilon P + (id+\varepsilon)\nabla^{\perp}\xi P\right]\cdot\nabla u.\end{aligned} \tag{2.1.2}$$

As in the case $N = S^{n-1}$, we want to apply a Wente-type estimate to get continuity for a solution u. If [...] on the right-hand side of (2.1.2) were divergence free, we could apply Poincaré's lemma to get the desired Wente structure. Thus we have to choose $\varepsilon \in W^{1,2} \cap L^\infty(B^2, M(n))$ such that

$$div\left[-\nabla\varepsilon P + (id+\varepsilon)\nabla^{\perp}\xi P\right] = 0. \tag{2.1.3}$$

To do this we apply a fixed point argument. Note that $div[(id+\varepsilon)\nabla^{\perp}\xi P]$ is a matrix whose ij-term is a sum of div-curl terms $\nabla((id+\varepsilon)^i_k P^h_j))\cdot\nabla^{\perp}\xi^k_h$. In the below we will abuse notation and write this as $\nabla((id+\varepsilon)P)\cdot\nabla^{\perp}\xi$. Let

$$\begin{aligned}\psi : W^{1,2}\cap L^\infty(B^2) &\to W^{1,2}\cap L^\infty(B^2)\\ \varepsilon &\mapsto \text{solution } \lambda \text{ of } (2.1.4)\end{aligned}$$

$$\begin{cases} div[\nabla\lambda P] = \nabla((id+\varepsilon)P)\cdot\nabla^{\perp}\xi & \text{in } B^2,\\ \lambda = 0 & \text{on } \partial B^2.\end{cases} \tag{2.1.4}$$

To show that ψ is a self-mapping and contraction, we use the following Wente-type estimate by Bethuel and Ghidaglia.

Theorem 2.1.4 (Bethuel/Ghidaglia [6]). *Let φ be a solution of*

$$\begin{cases} -\frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial\varphi}{\partial x_j}\right) = \nabla a\cdot\nabla^{\perp} b & \text{in } B^2,\\ \varphi = 0 & \text{on } \partial B^2,\end{cases}$$

where A_{ij} are bounded functions on B^2 such that

$$\sum_{i,j=1}^{2} A_{ij}(x)\xi_i\xi_j \ge \alpha(\xi_1^2+\xi_2^2) \qquad \text{for all } x\in B^2, \xi\in\mathbb{R}^2$$

and some $\alpha > 0$. *Then there exists a universal constant* $C > 0$ *such that*

$$||\varphi||_{L^\infty(B^2)} + ||\nabla\varphi||_{L^2(B^2)} \leq \frac{C}{\alpha}||\nabla a||_{L^2(B^2)}||\nabla b||_{L^2(B^2)}.$$

Let $\varepsilon_1, \varepsilon_2 \in W^{1,2} \cap L^\infty(B^2)$ and $\psi(\varepsilon_1) = \lambda_1,\ \psi(\varepsilon_2) = \lambda_2$ the corresponding solutions of (2.1.4). Then $\Lambda := \lambda_1 - \lambda_2$ solves

$$\begin{cases} div[\nabla\Lambda P] = \nabla((\varepsilon_1 - \varepsilon_2)P) \cdot \nabla^\perp\xi & \text{in } B^2, \\ \Lambda = 0 & \text{on } \partial B^2. \end{cases}$$

Since P takes values in $SO(n)$ it satisfies the assumptions of Theorem 2.1.4 and we have

$$\begin{aligned} ||\Lambda||_{L^\infty(B^2)} + ||\nabla\Lambda||_{L^2(B^2)} &\leq c\left(||\nabla\varepsilon_1 - \nabla\varepsilon_2||_{L^2(B^2)}||P||_{L^\infty(B^2)} + ||\varepsilon_1 - \varepsilon_2||_{L^\infty(B^2)}||\nabla P||_{L^2(B^2)}\right) \\ &\quad \cdot ||\nabla\xi||_{L^2(B^2)} \\ &\leq c\sigma\left(||\nabla\varepsilon_1 - \nabla\varepsilon_2||_{L^2(B^2)} + ||\varepsilon_1 - \varepsilon_2||_{L^\infty(B^2)}\right). \end{aligned}$$

For σ small enough, we conclude that ψ is a contraction. (To show that ψ is also a self-map we proceed analogously.) The Banach fixed point theorem yields a unique $\varepsilon^* \in W^{1,2} \cap L^\infty(B^2, M(n))$ solving (2.1.3) and with Theorem 2.1.4

$$||\varepsilon^*||_{L^\infty(B^2)} + ||\nabla\varepsilon^*||_{L^2(B^2)} \leq c\sigma.$$

By the Poincaré lemma there exists $B \in W^{1,2}(B^2)$ such that

$$\nabla^\perp B = -\nabla\varepsilon^* P + (id + \varepsilon^*)\nabla^\perp\xi P$$

and (1.0.8) is equivalent to

$$-div((id + \varepsilon^*)P\nabla u) = \nabla^\perp B \cdot \nabla u.$$

Now that we have our equation in the desired divergence-free form, we can show continuity of a solution u. We follow [58] and apply a Hodge decomposition (Corollary 10.70 in [24])

$$(id + \varepsilon^*)P\nabla u = \nabla V + \nabla^\perp W$$

with $V \in W_0^{1,2}(B^2, \mathbb{R}^n)$ and $W \in W^{1,2}(B^2, \mathbb{R}^n)$. Taking the Laplacian of both V and W yields

$$\begin{aligned} \Delta V &= div\nabla V = div((id + \varepsilon^*)P\nabla u) = -\nabla^\perp B \cdot \nabla u \\ \Delta W &= -\nabla^\perp\nabla^\perp W = -\nabla^\perp((id + \varepsilon^*)P) \cdot \nabla u. \end{aligned}$$

Both ΔV and ΔW are of the form $E \cdot F$ with $div E = \nabla^\perp F = 0$ and we can apply results by Coifman, Lions, Meyer and Semmes [13] which yield $V, W \in W^{2,1}_{loc}(B^2)$. Since $(id + \varepsilon^*)P$ is invertible we have $u \in W^{2,1}_{loc} \hookrightarrow C^0(B^2, N)$ (see [1] 4.12).

Remark 2.1.5. *The harmonic map equation is not the only second order elliptic equation with antisymmetric structure. Rivière [58] showed that all critical points of the Lagrangian*

$$F(u) = \int_{B^2} \left(|\nabla u|^2 + \omega(u)(\partial_x u, \partial_y u)\right) dx \wedge dy,$$

where ω *is a* C^1 *two-from on* N *with* $||d\omega||_{L^\infty} \leq c$, *satisfy an equation of the form (1.0.8) with* $\Omega \in L^2(B^2, so(n) \otimes \wedge^1\mathbb{R}^2)$ *and are therefore continuous. This proves a conjecture by Hildebrandt ([33],[34]).*

Let's summarize the results so far:

Theorem 2.1.6. *Let $n \in \mathbb{N}$ and N be an oriented submanifold of $\mathbb{R}^n$. Let $u \in W^{1,2}(B^2, N)$ be a solution of*

$$-\Delta u = \Omega \cdot \nabla u, \tag{2.1.5}$$

where $\Omega \in L^2(B^2, so(n) \otimes \wedge^1\mathbb{R}^2)$ and $\sigma := ||\Omega||_{L^2(B^2)}$. Then the following holds.

(i) There exists $\sigma_0 > 0$ such that whenever $\sigma < \sigma_0$ there exist $\varepsilon \in W^{1,2} \cap L^\infty(B^2, M(n))$, $P \in W^{1,2}(B^2, SO(n))$ and $\xi \in W^{1,2}(B^2, so(n))$ with

$$||\varepsilon||_{L^\infty} + ||\nabla\varepsilon||_{L^2} + ||\xi||_{W^{1,2}(B^2)} + ||P||_{W^{1,2}(B^2)} \leq c\sigma,$$

and $B \in W^{1,2}(B^2)$ that solve

$$\nabla^\perp B = \nabla\varepsilon P - (id + \varepsilon)\nabla^\perp \xi P.$$

(ii) u solves (2.1.5) if and only if it is a solution of

$$-div((id + \varepsilon)P\nabla u) = \nabla^\perp B \cdot \nabla u.$$

(iii) u is continuous.

Remark 2.1.7. *Note that the antisymmetric structure of Ω is crucial and there are counterexamples for $\Omega \in L^2(B^2, M(n) \otimes \wedge^1\mathbb{R}^2)$, where the solution u is in L^∞ but not continuous (see Frehse [19]). One can even construct an example where the solution is not bounded. (see [58])*

In the following we want to apply Rivière's Ansatz to derive conservation laws for higher order elliptic systems in critical dimension.

2.2 Notation

Before we get to the higher order case let us introduce some notation. Let $\wedge^k\mathbb{R}^{2m}$, $k \in \mathbb{N}_0$ be the space of k-forms on $\mathbb{R}^{2m}$. Further let

$$d : W^{1,p}(\mathbb{R}^{2m}, \wedge^k\mathbb{R}^{2m}) \to L^p(\mathbb{R}^{2m}, \wedge^{k+1}\mathbb{R}^{2m})$$

be the exterior derivative and

$$\delta : W^{1,p}(\mathbb{R}^{2m}, \wedge^k\mathbb{R}^{2m}) \to L^p(\mathbb{R}^{2m}, \wedge^{k-1}\mathbb{R}^{2m})$$

the codifferential. We have $dd = \delta\delta = 0$ and the Laplacian is given by

$$\Delta = d\delta + \delta d.$$

If f is a function, the exterior derivative of f is just the gradient ∇f. Let $0 \leq k \leq 2m$, $k \in \mathbb{N}$. Then let

$$* : \wedge^k\mathbb{R}^{2m} \to \wedge^{2m-k}\mathbb{R}^{2m}$$

be the Hodge-Star operator. For a k-form ω we have

$$\delta\omega = (-1)^{2m(k+1)+1} * d * \omega \tag{2.2.1}$$

and

$$** : (-1)^{k(2m-k)} : \wedge^k\mathbb{R}^{2m} \to \wedge^k\mathbb{R}^{2m} \tag{2.2.2}$$

(see e.g. [38]).

2.3 Higher order systems

So far we have introduced a conservation law for the harmonic map equation in two dimensions. In [61], Rivière and Struwe explored this example further and gave a new proof of partial regularity for harmonic maps in higher dimensions. In 2008, Lamm and Rivière [48] applied the same procedure to a fourth order problem such as the biharmonic map equation. Biharmonic maps are critical points $u \in W^{2,2}(B^4, \mathbb{R}^n)$ of

$$\int_{B^4} |\Delta u|^2 dA_{B^4}$$

or equivalently solutions of the biharmonic map equation

$$-\Delta^2 u = \sum_{i=m+1}^{n} \Big(\Delta\langle \nabla u, (d\nu_i \circ u)\nabla u\rangle + \delta\langle \Delta u, (d\nu_i \circ u)\nabla u\rangle + \langle \nabla\Delta u, (d\nu_i \circ u)\nabla u\rangle \Big) \nu_i \circ u,$$

where $\{\nu_i\}_{i=m+1}^n$ is an orthonormal frame of the normal space of $N^m \hookrightarrow \mathbb{R}^n$ near $u(x)$. Lamm and Rivière showed that the biharmonic map equation is of the form

$$\Delta^2 u = \langle V_0, du\rangle + \delta(w du) + \Delta\langle V_1, du\rangle \tag{2.3.1}$$

with

$$w \in L^2(B^4, \mathbb{R}^{n\times n}), \qquad V_1 \in W^{1,2}(B^4, \mathbb{R}^{n\times n} \otimes \wedge^1\mathbb{R}^4), \qquad V_0 \in W^{-1,2}(B^4, \mathbb{R}^{n\times n} \otimes \wedge^1\mathbb{R}^4)$$
$$\text{with } V_0 = d\eta + F, \qquad \eta \in L^2(B^4, so(n)), \qquad F \in L^{\frac{4}{3},1}(B^4, \mathbb{R}^{n\times n} \otimes \wedge^1\mathbb{R}^4).$$

Here V_0 decomposes into the gradient of an antisymmetric term η and a term F, which is of higher integrability. Using this antisymmetry they were able to write (2.3.1) in divergence form and show continuity of solutions u (see Theorem 2.3.1 for $m = 2$). Struwe [72] established partial regularity results for biharmonic maps in dimension greater or equal to four with the same method.

De Longueville and Gastel [20] recently extended the results of Lamm and Rivière to systems of the form (1.0.9). The motivating example behind this equation are the m-polyharmonic maps $u \in W^{m,2}(B^{2m}, N)$, which are critical points of

$$\int_{B^{2m}} |\nabla^m u|^2 dA_{B^{2m}}.$$

They satisfy a $2m^{th}$-order elliptic system called the m-polyharmonic map equation (see [5] or [21]) which is of the form (1.0.9) with antisymmetric component η (see [20]).

Theorem 2.3.1 (Lamm/Rivière [48] and de Longueville/Gastel [20])**.** *Assume $m \geq 2$, $n \in \mathbb{N}$. Let coefficient functions be given as in (1.0.11). For equations of the form (1.0.10) the following statements hold.*

(i) Let

$$\theta := \sum_{k=0}^{m-2} ||w_k||_{W^{2k+2-m,2}(B^{2m})} + \sum_{k=1}^{m-1} ||V_k||_{W^{2k+1-m,2}(B^{2m})} + ||\eta||_{W^{2-m,2}(B^{2m})} + ||F||_{W^{2-m,\frac{2m}{m+1},1}(B^{2m})}.$$

There is $\theta_0 > 0$ such that whenever $\theta < \theta_0$, there are a function $A \in W^{m,2} \cap L^\infty(B^{2m}_{1/4}; GL(n))$ and a distribution $B \in W^{2-m,2}(B^{2m}_{1/4}, \mathbb{R}^{n\times n} \otimes \wedge^2\mathbb{R}^{2m})$ that solve

$$\Delta^{m-1} dA + \sum_{k=0}^{m-1} (\Delta^k A) V_k - \sum_{k=0}^{m-2} (\Delta^k dA) w_k = \delta B.$$

(ii) A function $u \in W^{m,2}(B^{2m}_{1/2}, \mathbb{R}^n)$ solves (1.0.10) weakly in $B^{2m}_{1/4}$ if and only if it is a distributional solution of the conservation law

$$\begin{aligned} 0 = \delta\Bigg[&\sum_{l=0}^{m-1} (\Delta^l A)\Delta^{m-l-1} du - \sum_{l=0}^{m-2} (d\Delta^l A)\Delta^{m-l-1} u \\ &- \sum_{k=0}^{m-1}\sum_{l=0}^{k-1} (\Delta^l A)\Delta^{k-l-1} d\langle V_k, du\rangle + \sum_{k=0}^{m-1}\sum_{l=0}^{k-1} (d\Delta^l A)\Delta^{k-l-1}\langle V_k, du\rangle \\ &- \sum_{k=0}^{m-2}\sum_{l=0}^{k} (\Delta^l A) d\Delta^{k-l-1}\delta(w_k du) + \sum_{k=0}^{m-2}\sum_{l=0}^{k-1} (d\Delta^l A)\Delta^{k-l-1}\delta(w_k du) \\ &- \langle B, du\rangle \Bigg] \end{aligned}$$

(here $d\Delta^{-1}\delta$ means the identity map)

(iii) Every weak solution of (1.0.10) on B^{2m} is continuous on $B^{2m}_{1/16}$ if the smallness condition $\theta < \theta_0$ holds.

The proof of Theorem 2.3.1 shows that the matrix A is a small perturbation of the Uhlenbeck matrix P. However, the methods applied are rather technical and involve solving a dual fixed point problem by using Wente-type estimates for the poly-Laplace operator. Our version is more straight forward and reduces to a simple fixed point problem which we can solve thanks to Lemma 2.4.1. This Lemma is a higher order version of Theorem 2.1.4 and holds for more general elliptic operators in divergence form.

We follow the same method that we used in the harmonic case at the beginning of this chapter. To do this, we need the following version of Uhlenbeck's gauge theorem

Theorem 2.3.2 (de Longueville/Gastel [20])**.** *Assume that $m, n \in \mathbb{N}$ and $B_r \subset \mathbb{R}^{2m}$ is a ball of radius r. Then there exists $\sigma > 0$ such that for all $\Omega \in W^{m-1,2}(B_r, so(n) \otimes \wedge^1\mathbb{R}^{2m})$ satisfying*

$$||\Omega||_{W^{m-1,2}(B_r)} < \sigma,$$

there are functions $P \in W^{m,2}(B_{r/2}, SO(n))$ and $\xi \in W^{m,2}(B_{r/2}, so(n) \otimes \wedge^2\mathbb{R}^{2m})$, such that

$$\Omega = PdP^{-1} + P\delta\xi P^{-1} \tag{2.3.2}$$

holds on $B_{r/2}$. Moreover, we have the estimate

$$||dP||_{W^{m-1,2}(B_{r/2})} + ||\delta\xi||_{W^{m-1,2}(B_{r/2})} \leq c||\Omega||_{W^{m-1,2}(B_r)}. \tag{2.3.3}$$

Before we get to the proof of Theorem 1.0.1 let us note that (1.0.10) contains elements of negative Sobolev and Lorentz-Sobolev spaces. For a definition of these distribution spaces and a brief introduction into Lorentz space theory see Appendix A.

2.4 Main Theorem

An important ingredient for the proof of Theorem 1.0.1 is the following lemma in the spirit of Bethuel and Ghidaglia [6].

Lemma 2.4.1. *Let $\sigma > 0$, $f \in L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}, \mathbb{R}^n)$ for $|\gamma| \leq m-2$ and $P \in W^{m,2}(B^{2m}, SO(n))$ with $||dP||_{W^{m-1,2}(B^{2m})} \leq \sigma$. There exists $\sigma_0 > 0$ such that if $\sigma < \sigma_0$ there exists a unique solution $u \in W^{2m-1,\frac{2m}{2m-1-|\gamma|},1}(B^{2m}, M(n))$ of*

$$\begin{cases} \Delta(\Delta^{m-1}u \cdot P) &= \delta f \qquad \text{in } B^{2m}, \\ \qquad \Delta^j u &= 0 \qquad \text{on } \partial B^{2m} \quad \text{for } j = 0, ..., m-1, \end{cases} \tag{2.4.1}$$

with

$$||D^{2m-1}u||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m})} + ||u||_{L^\infty(B^{2m})} \leq c||f||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m})}.$$

Proof. The boundary conditions determine a solution u of (2.4.1) uniquely. To see this we assume there exist solutions u_1, u_2 and let $v := u_1 - u_2$. Then $\Delta(\Delta^{m-1}v \cdot P) = 0$. Testing this equation with $\Delta^{m-1}v \cdot P$ and integrating by parts gives

$$0 = \int_{B^{2m}} \Delta(\Delta^{m-1}v \cdot P)(\Delta^{m-1}v \cdot P) = -\int_{B^{2m}} |D(\Delta^{m-1}v \cdot P)|^2.$$

Thus we have $D(\Delta^{m-1}v \cdot P) = 0$ and therefore $\Delta^{m-1}v \cdot P = const.$ Because P is invertible and $\Delta^{m-1}v = 0$ on ∂B^{2m} we get $\Delta^{m-1}v = 0$. Iteratively we get $v = 0$ and thus $u_1 = u_2$.
Now we approximate f by $\bar{f} \in C_c^\infty(\mathbb{R}^{2m})$ so that $\bar{f} = 0$ on $\mathbb{R}^{2m} \setminus B^{2m}$ and

$$||\bar{f}||_{L^{\frac{2m}{2m-1-|\gamma|},1}(\mathbb{R}^{2m})} \leq c||f||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m})}.$$

Standard L^p-theory and interpolation results (see [31] Theorem 3.3.3) yield

$$||D(\Delta^{m-1}uP)||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m})} \leq c||f||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m})}.$$

With Hölder's inequality for Lorentz spaces (Lemma A.1.3) and the embedding theorem for Lorentz spaces (Lemma 3.6.3) we estimate

$$\begin{aligned} &||D\Delta^{m-1}u||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m})} \\ &\leq c\left(||f||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m})} + ||D^{2m-2}u||_{L^{\frac{2m}{2m-2-|\gamma|},2}(B^{2m})}||dP||_{L^{2m,2}(B^{2m})}\right) \\ &\leq c\left(||f||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m})} + ||u||_{W^{2m-1,\frac{2m}{2m-1-|\gamma|},1}(B^{2m})}||dP||_{W^{m-1,2}(B^{2m})}\right). \end{aligned}$$

We interchange derivatives and apply the Calderon-Zygmund inequality

$$\begin{aligned} &||D^{2m-1}u||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m})} \\ &\leq c\left(||f||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m})} + ||u||_{W^{2m-1,\frac{2m}{2m-1-|\gamma|},1}(B^{2m})}||dP||_{W^{m-1,2}(B^{2m})}\right). \end{aligned}$$

Since $||dP||_{W^{m-1,2}(B^{2m})} < \sigma$, we choose $\sigma > 0$ small enough, so that we can absorb the second term to the left-hand side. The density of $C_c^\infty(B^{2m})$ in $L^{p,q}(B^{2m})$ finishes the proof.

□

Proof of Theorem 1.0.1. We split the proof in several steps:

Step 1: Gauge fixing

Following the work of de Longueville and Gastel in the proof of Theorem 4.1 (i) in [20] (see also [52]) we find $\Omega \in W^{m-1,2}(B^{2m}, so(m) \otimes \wedge^1\mathbb{R}^{2m})$ by repeatedly solving Neumann problems such that

$$\begin{cases} \Delta^{m-2}\delta(\Omega) = -\eta & \text{in } B^{2m}, \\ ||\Omega||_{W^{m-1,2}(B^{2m})} \leq c||\eta||_{W^{2-m,2}(B^{2m})} \leq c\sigma. \end{cases} \tag{2.4.2}$$

For $\sigma > 0$ sufficiently small we apply Theorem 2.3.2, a higher order version of Uhlenbeck's gauge theorem, and get $\xi \in W^{m,2}(B^{2m}_{1/2}, so(n) \otimes \wedge^2\mathbb{R}^{2m})$ and $P \in W^{m,2}(B^{2m}_{1/2}, SO(n))$ such that

$$\begin{aligned} dP = P\Omega - \delta\xi P, \\ ||dP||_{W^{m-1,2}(B^{2m}_{1/2})} + ||\delta\xi||_{W^{m-1,2}(B^{2m}_{1/2})} \leq c||\Omega||_{L^{m-1,2}(B^{2m})}. \end{aligned} \tag{2.4.3}$$

Step 2: Rewriting the system

Let $\varepsilon \in W^{m,2} \cap L^\infty(B^{2m}_{1/2}, M(n))$. We multiply (1.0.10) with $(id+\varepsilon)P$ and calculate

$$\begin{aligned} (id+\varepsilon)P\Delta^m u &= (id+\varepsilon)P\left[\sum_{k=0}^{m-1}\Delta^k\langle V_k, du\rangle + \sum_{k=0}^{m-2}\Delta^k\delta(w_k du)\right] \\ \Leftrightarrow \Bigg[\sum_{k=0}^{m-1}\Delta^k((id+\varepsilon)P)V_k &- \sum_{k=0}^{m-2} d\Delta^k((id+\varepsilon)P)w_k + d\Delta^{m-1}((id+\varepsilon)P)\Bigg]\cdot du \\ &= \delta\Bigg[\sum_{l=0}^{m-1}\Delta^l((id+\varepsilon)P)\Delta^{m-l-1}du - \sum_{l=0}^{m-2} d\Delta^l((id+\varepsilon)P)\Delta^{m-l-1}u \\ &\quad - \sum_{k=0}^{m-1}\sum_{l=0}^{k-1}\Delta^l((id+\varepsilon)P)\Delta^{k-l-1}d\langle V_k, du\rangle \\ &\quad + \sum_{k=0}^{m-1}\sum_{l=0}^{k-1} d\Delta^l((id+\varepsilon)P)\Delta^{k-l-1}\langle V_k, du\rangle \\ &\quad - \sum_{k=0}^{m-2}\sum_{l=0}^{k}\Delta^l((id+\varepsilon)P)d\Delta^{k-l-1}\delta(w_k du) \\ &\quad + \sum_{k=0}^{m-2}\sum_{l=0}^{k-1} d\Delta^l((id+\varepsilon)P)\Delta^{k-l-1}\delta(w_k du)\Bigg]. \end{aligned} \tag{2.4.4}$$

The right-hand side is already in divergence form, thus we need to find $\varepsilon \in W^{m,2} \cap L^\infty(B^{2m}_{1/2}, M(n))$ such that

$$\delta\left[\sum_{k=0}^{m-1}\Delta^k((id+\varepsilon)P)V_k - \sum_{k=0}^{m-2} d\Delta^k((id+\varepsilon)P)w_k + d\Delta^{m-1}((id+\varepsilon)P)\right] = 0 \qquad \text{on } B^{2m}_{1/2}. \tag{2.4.5}$$

As in section 2.1 we want to apply a fixed point argument to solve this problem. However, to do this, we need to have a certain control on the terms in (2.4.5) and the terms involving V_0 are problematic. We know that $V_0 = d\eta + F$ and we control $F \in W^{2-m,\frac{2m}{m+1},1}(B^{2m})$ by (1.0.12) but $d\eta \in W^{1-m,2}(B^{2m})$ is a priori not bounded. Thus our goal is to remove $d\eta$.
To do this, we take a closer look at $d\Delta^{m-1}((id+\varepsilon)P)$ and note that we can rewrite the highest order term $(id+\varepsilon)d\Delta^{m-1}P$ so that it cancels $(id+\varepsilon)Pd\eta$ in (2.4.5). To see this we use (2.2.1),

(2.2.2), (2.4.2) and (2.4.3).

$$\begin{aligned} d\Delta^{m-1}P &= d\Delta^{m-2}\delta\left(P\Omega - \delta\xi P\right) \\ &= d\Delta^{m-2}(dP\Omega) + d\Delta^{m-2}(P\delta\Omega) - d\Delta^{m-2}(*d*(*d*\xi P)) \\ &= \sum_{i=1}^{2m-2} c_i\nabla^i P\nabla^{2m-2-i}\Omega - d(P\eta) + d\Delta^{m-2}(*(d*\xi\wedge dP)) \\ &= \sum_{i=1}^{2m-2} c_i\nabla^i P\nabla^{2m-2-i}\Omega - dP\eta - P(V_0 - F) + d\Delta^{m-2}(*(d*\xi\wedge dP)), \end{aligned}$$

with constants $c_i \in \mathbb{N}_0$, $1 \leq i \leq 2m-2$ and

$$\nabla^k = \begin{cases} \Delta^{\frac{k}{2}}, & \text{if } k \text{ even}, \\ d\Delta^{\frac{k-1}{2}}, & \text{if } k \text{ odd}. \end{cases}$$

Plugging this back into (2.4.5) and rearranging yields

$$\begin{aligned} \Delta(\Delta^{m-1}\varepsilon\cdot P) = \delta\Bigg[&-\sum_{j=1}^{2m-2}\tilde{c}_j\nabla^j\varepsilon\nabla^{2m-1-j}P - (id+\varepsilon)\Bigg(\sum_{i=1}^{2m-2} c_i\nabla^i P\nabla^{2m-2-i}\Omega \\ &\qquad - dP\eta + PF + d\Delta^{m-2}(*(d*\xi\wedge dP))\Bigg) \\ &- \sum_{k=1}^{m-1}\Delta^k((id+\varepsilon)P)V_k + \sum_{k=0}^{m-2} d\Delta^k((id+\varepsilon)P)w_k\Bigg] \quad \text{in } B^{2m}_{1/2}, \end{aligned} \tag{2.4.6}$$

where $\tilde{c}_j$ are constants in $\mathbb{N}_0$. Now that we have removed the "worst" terms, we want to examine this equation further and take a closer look at the function spaces of the summands. We separate the ε component from the rest and use the embedding results for Lorentz-Sobolev spaces in Lemma A.2.8 and Lemma A.2.9 repeatedly. We use the notation $D^kA \star D^lB$ for any linear combination of D^kA and D^lB, where D denotes the full derivative. For the first term we have

$$\sum_{j=1}^{2m-2} D^j\varepsilon\star D^{2m-1-j}P = \sum_{j=1}^{2m-2} W^{m-j,2}\cdot W^{-m+1+j,2},$$

For the third and fourth term we get

$$\begin{aligned} (id+\varepsilon)dP\eta &= L^\infty\cdot W^{m-1,2}\cdot W^{2-m,2} \hookrightarrow L^\infty\cdot W^{2-m,\frac{2m}{m+1},1}, \\ (id+\varepsilon)PF &= L^\infty\cdot L^\infty\cdot W^{2-m,\frac{2m}{m+1},1}. \end{aligned}$$

The second term is of the from

$$\begin{aligned} &(id+\varepsilon)\Bigg(\sum_{j=1}^{2m-3} D^j\Omega\star D^{2m-2-j}P + \Omega\star D^{2m-2}P\Bigg) \\ &= \sum_{j=1}^{2m-3} L^\infty\cdot W^{m-1-j,2}\cdot W^{-m+2+j,2} + L^\infty\cdot W^{m-1,2}\cdot W^{2-m,2} \\ &\hookrightarrow \sum_{j=1}^{m-2} L^\infty\cdot W^{-m+2+j,\frac{2m}{m+1+j},1} + \sum_{j=m-1}^{2m-3} L^\infty\cdot W^{m-1-j,\frac{2m}{3m-2-j},1} \\ &\quad + L^\infty\cdot W^{2-m,\frac{2m}{m+1},1} \end{aligned}$$

$$\hookrightarrow L^\infty \cdot W^{2-m,\frac{2m}{m+1},1},$$

where we used Lemma A.2.9 in the first step and Lemma A.2.8 with $s = m-2-j,\ p = \frac{2m}{m+1+j},\ t = j$ for $j = 1, ..., m-2$ and $s = -m+1+j,\ p = \frac{2m}{3m-2-j},\ t = 2m-3-j$ for $j = m-1, ..., 2m-3$ in the second step. The fifth term follows in the same way

$$\begin{aligned}(id+\varepsilon)d\Delta^{m-2}\left(*(dP \wedge d*\xi)\right) &= (id+\varepsilon)\sum_{j=1}^{2m-2} D^j\xi \star D^{2m-1-j}P \\ &= \sum_{j=1}^{2m-2} L^\infty \cdot W^{m-j,2} \cdot W^{-m+1+j,2} \\ &\hookrightarrow L^\infty \cdot W^{2-m,\frac{2m}{m+1},1}.\end{aligned}$$

For the last two terms we apply again Lemma A.2.9 and Lemma A.2.8 with $s = m-2k-1,\ p = \frac{2m}{m+2k-j},\ t = 2k-j$ for $2k+1-m < m-2k+j$ and $s = 2k-j-m,\ p = \frac{2m}{3m-2k-1},\ t = 2m-2k-1$ for $m-2k+j \leq 2k+1-m$.

$$\begin{aligned}&\sum_{k=1}^{m-1}\Delta^k((id+\varepsilon)P)V_k = \sum_{k=1}^{m-1}\left(\sum_{j=1}^{2k-1} D^j\varepsilon \star D^{2k-j}P + (id+\varepsilon)\Delta^k P + \Delta^k\varepsilon P\right)V_k \\ &= \sum_{k=1}^{m-1}\sum_{j=1}^{2k-1} W^{m-j,2} \cdot W^{m-2k+j,2} \cdot W^{2k+1-m,2} + \sum_{k=1}^{m-1} L^\infty \cdot W^{m-2k,2} \cdot W^{2k+1-m,2} \\ &\hookrightarrow \sum_{\substack{j,k\in\mathbb{N},\ j\leq 2k-1,\ k\leq m-1 \\ 2k+1-m<m-2k+j}} W^{m-j,2} \cdot W^{2k+1-m,\frac{2m}{m+2k-j}} \\ &\quad + \sum_{\substack{j,k\in\mathbb{N},\ j\leq 2k-1,\ k\leq m-1 \\ m-2k+j\leq 2k+1-m}} W^{m-j,2} \cdot W^{m-2k+j,\frac{2m}{3m-2k-1}} \\ &\quad + \sum_{\substack{k\in\mathbb{N},\ k\leq m-1 \\ 2k+1-m<m-2k}} L^\infty \cdot W^{2k+1-m,\frac{2m}{m+2k},1} + \sum_{\substack{k\in\mathbb{N},\ k\leq m-1 \\ m-2k\leq 2k+1-m}} L^\infty \cdot W^{m-2k,\frac{2m}{3m-2k-1},1} \\ &\hookrightarrow \sum_{j=1}^{2m-3} W^{m-j,2} \cdot W^{-m+1+j,2} + L^\infty \cdot W^{2-m,\frac{2m}{m+1},1}\end{aligned}$$

and analogously

$$\begin{aligned}&\sum_{k=0}^{m-2}\nabla\Delta^k((id+\varepsilon)P)w_k \\ &= \sum_{k=0}^{m-2}\left(\sum_{j=1}^{2k} D^j\varepsilon \star D^{2k+1-j}P + (id+\varepsilon)\delta\Delta^k P + \delta\Delta^k\varepsilon P\right)w_k \\ &= \sum_{k=0}^{m-2}\sum_{j=1}^{2k} W^{m-j,2} \cdot W^{m-2k-1+j,2} \cdot W^{2k+2-m,2} + \sum_{k=0}^{m-2} L^\infty \cdot W^{m-2k+1} \cdot W^{2k+2-m,2} \\ &\hookrightarrow \sum_{j=1}^{2m-3} W^{m-j,2} \cdot W^{-m+1+j,2} + L^\infty \cdot W^{2-m,\frac{2m}{m+1},1}.\end{aligned}$$

Observe that all terms on the right-hand side of (2.4.6) consist of products $W^{m-j,2} \cdot W^{j+1-m,2}$, $j = 1, ..., 2m-2$ and $L^\infty \cdot W^{2-m,\frac{2m}{m+1},1}$. Thus we can simplify (2.4.6) further and write

$$\Delta(\Delta^{m-1}\varepsilon \cdot P) = \delta\Big(\sum_{j=1}^{2m-2} D^j \varepsilon \star K_j + (id+\varepsilon)\star K_0\Big) \tag{2.4.7}$$

with $K_j \in W^{j+1-m,2}(B^{2m}_{1/2})$, $K_0 \in W^{2-m,\frac{2m}{m+1},1}(B^{2m}_{1/2})$. Moreover, with (2.3.3) and (1.0.12) we estimate

$$||K_0||_{W^{2-m,\frac{2m}{m+1},1}(B^{2m}_{1/2})} + \sum_{j=1}^{2m-2} ||K_j||_{W^{j+1-m,2}(B^{2m}_{1/2})} \leq c\sigma. \tag{2.4.8}$$

However, the equation still contains distributions. To take care of these we apply the same technique as de Longueville and Gastel and use a representation of negative Lorentz-Sobolev spaces (see Lemma A.2.6).

$$\begin{aligned}
\varepsilon &= \sum_{|\alpha|\leq m-2} \partial^\alpha \varepsilon_\alpha, & \varepsilon_\alpha &\in W^{2m-1,\frac{2m}{2m-1-|\alpha|},1}(B^{2m}_{1/2}),\\
K_0 &= \sum_{|\alpha|\leq m-2} \partial^\alpha K_0^\alpha, & K_0^\alpha &\in L^{\frac{2m}{m+1},1}(B^{2m}_{1/2}),\\
K_j &= \sum_{|\alpha|\leq m-1-j} \partial^\alpha K_j^\alpha, & K_j^\alpha &\in L^2(B^{2m}_{1/2}).
\end{aligned} \tag{2.4.9}$$

Together with (2.4.8) we get

$$\begin{aligned}
\sum_{|\alpha|\leq m-1-j} ||K_j^\alpha||_{L^2(B^{2m}_{1/2})} &\leq c||K_j||_{W^{j+1-m,2}(B^{2m}_{1/2})} \leq c\sigma,\\
\sum_{|\alpha|\leq m-2} ||K_0^\alpha||_{L^{\frac{2m}{m+1},1}(B^{2m}_{1/2})} &\leq c||K_0||_{W^{2-m,\frac{2m}{m+1},1}(B^{2m}_{1/2})} \leq c\sigma.
\end{aligned} \tag{2.4.10}$$

Note that we assume $\varepsilon \in W^{m+1,\frac{2m}{m+1},1}$ for this representation, which is slightly better than the original assumption $\varepsilon \in W^{m,2}\cap L^\infty$. We will see that we can solve (2.4.6) in this better space and since $W^{m+1,\frac{2m}{m+1},1}(B^{2m}) \hookrightarrow W^{m,2}\cap L^\infty(B^{2m})$ we get the desired result.

This new representation allows us to shift derivatives away from the distributional part. Let $c_{\alpha\gamma},\ c_{\beta\gamma} \in \mathbb{Z}$. With the product rule we get for $j = 1, ..., m-2$

$$D^j\varepsilon \star K_j = \sum_{\substack{|\alpha|\leq m-2\\|\beta|\leq m-1-j}} D^j\partial^\alpha\varepsilon_\alpha \star \partial^\beta K_j^\beta = \sum_{\substack{|\alpha|\leq m-2\\|\beta|\leq m-1-j}} \sum_{\gamma\leq\beta} \partial^\gamma\big(c_{\beta\gamma}\partial^{\beta-\gamma}\partial^\alpha D^j\varepsilon_\alpha \star K_j^\beta\big)$$

The case $j = 0$ follows analogously

$$\begin{aligned}
(id+\varepsilon)\star K_0 &= \sum_{|\gamma|\leq m-2} \partial^\gamma K_0^\gamma + \sum_{\substack{|\alpha|\leq m-2\\|\beta|\leq m-2}} \partial^\alpha\varepsilon_\alpha \star \partial^\beta K_0^\beta\\
&= \sum_{|\gamma|\leq m-2} \partial^\gamma K_0^\gamma + \sum_{\substack{|\alpha|\leq m-2\\|\beta|\leq m-2}} \sum_{\gamma\leq\beta} \partial^\gamma\big(c_{\beta\gamma}\partial^{\beta-\gamma}\partial^\alpha\varepsilon_\alpha \star K_0^\beta\big).
\end{aligned}$$

For $j = m-1, ..., 2m-2$ with $|\alpha| \le j+1-m$ we get

$$D^j\varepsilon \star K_j = \sum_{|\alpha|\le m-2} D^j\partial^\alpha\varepsilon_\alpha \star K_j = \sum_{|\alpha|\le m-2}\sum_{\gamma\le\alpha}\partial^\gamma\big(c_{\alpha\gamma}D^j\varepsilon_\alpha \star \partial^{\alpha-\gamma}K_j\big).$$

If $|\alpha| > j+1-m$ we choose $\beta \le \alpha$ with $|\beta| = j+1-m$ and

$$\begin{aligned} D^j\varepsilon \star K_j &= \sum_{|\alpha|\le m-2} D^j\partial^\alpha\varepsilon_\alpha \star K_j \\ &= \sum_{|\alpha|\le m-2}\sum_{\substack{\gamma\le\beta\\ |\beta|=j+1-m}} \partial^\gamma\big(c_{\beta\gamma}\partial^{\alpha-\beta}D^j\varepsilon_\alpha \star \partial^{\beta-\gamma}K_j\big). \end{aligned}$$

We rewrite the left-hand side of (2.4.7) in the same way.

$$\begin{aligned} \Delta(\Delta^{m-1}\varepsilon\cdot P) &= \sum_{|\alpha|\le m-2}\Delta(\Delta^{m-1}\partial^\alpha\varepsilon_\alpha\cdot P) \\ &= \sum_{|\alpha|\le m-2}\sum_{\gamma\le\alpha}\partial^\gamma\Delta(c_{\alpha\gamma}\Delta^{m-1}\varepsilon_\alpha\partial^{\alpha-\gamma}P) \\ &= \sum_{|\gamma|\le m-2}\partial^\gamma\Delta(\Delta^{m-1}\varepsilon_\gamma\cdot P) + \sum_{|\alpha|\le m-2}\sum_{\gamma<\alpha}\partial^\gamma\Delta(c_{\alpha\gamma}\Delta^{m-1}\varepsilon_\alpha\partial^{\alpha-\gamma}P). \end{aligned}$$

For the last term note that $P \in W^{m,2}(B^{2m}_{1/2}, SO(n))$. Thus we identify P with K_{2m-1} and write

$$\begin{aligned} &\sum_{|\alpha|\le m-2}\sum_{\gamma<\alpha}\partial^\gamma\Delta(c_{\alpha\gamma}\Delta^{m-1}\varepsilon_\alpha\partial^{\alpha-\gamma}P) \\ &= \delta\Bigg[\sum_{|\alpha|\le m-2}\sum_{\gamma<\alpha}\sum_{i=0}^{1}\partial^\gamma\left(c_{\alpha\gamma}D^{2m-2-i}\varepsilon_\alpha \star \partial^{\alpha-\gamma}D^{1-i}K_{2m-1}\right)\Bigg]. \end{aligned}$$

Putting all of this together we get an equation equivalent to (2.4.5)

$$\begin{aligned} &\sum_{|\gamma|\le m-2}\partial^\gamma\Delta(\Delta^{m-1}\varepsilon_\gamma\cdot P) \\ &\quad= \delta\Bigg[\sum_{|\gamma|\le m-2}\partial^\gamma K_0^\gamma + \sum_{\substack{|\alpha|\le m-2\\ |\beta|\le m-2}}\sum_{\gamma\le\beta}\partial^\gamma\big(c_{\beta\gamma}\partial^{\beta-\gamma}\partial^\alpha\varepsilon_\alpha \star K_0^\beta\big) \\ &\qquad + \sum_{j=1}^{m-2}\sum_{\substack{|\alpha|\le m-2\\ |\beta|\le m-1-j}}\sum_{\gamma\le\beta}\partial^\gamma\big(c_{\beta\gamma}\partial^{\beta-\gamma}\partial^\alpha D^j\varepsilon_\alpha \star K_j^\beta\big) \\ &\qquad + \sum_{\substack{j=m-1\\ |\alpha|\le j+1-m}}^{2m-2}\sum_{|\alpha|\le m-2}\sum_{\gamma\le\alpha}\partial^\gamma\big(c_{\alpha\gamma}D^j\varepsilon_\alpha \star \partial^{\alpha-\gamma}K_j\big) \\ &\qquad + \sum_{\substack{j=m-1\\ |\alpha|>j+1-m}}^{2m-2}\sum_{|\alpha|\le m-2}\sum_{\substack{\gamma\le\beta\\ |\beta|=j+1-m}}\partial^\gamma\big(c_{\beta\gamma}\partial^{\alpha-\beta}D^j\varepsilon_\alpha \star \partial^{\beta-\gamma}K_j\big) \\ &\qquad + \sum_{i=0}^{1}\sum_{|\alpha|\le m-2}\sum_{\gamma<\alpha}\partial^\gamma\left(c_{\alpha\gamma}D^{2m-2-i}\varepsilon_\alpha \star \partial^{\alpha-\gamma}D^{1-i}K_{2m-1}\right)\Bigg]. \end{aligned}$$

We simplify this further by setting

$$=: \delta\Big[\sum_{|\gamma|\le m-2} \partial^\gamma\left(\langle\varepsilon, K\rangle_\gamma + K_0^\gamma\right)\Big] \tag{2.4.11}$$

with

$$\begin{aligned} &||K_0^\gamma||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})} + ||\langle\varepsilon, K\rangle_\gamma||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})} \\ &\qquad\le c\sigma\left(\sum_{|\alpha|\le m-2} ||\varepsilon_\alpha||_{W^{2m-1,\frac{2m}{2m-1-|\alpha|},1}(B^{2m}_{1/2})} + 1\right) \end{aligned} \tag{2.4.12}$$

for every γ with $|\gamma| \le m-2$. To see this last inequality we use (2.4.10) and estimate each term separately. This has been done in great detail by de Longueville [52]. We include a short explanation for the sake of completeness.

$$||K_0^\gamma||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})} \le c||K_0^\gamma||_{L^{\frac{2m}{m+1},1}(B^{2m}_{1/2})} \le c\sigma;$$

since $K_0^\gamma \in L^{\frac{2m}{m+1},1}$ and $L^{\frac{2m}{m+1},1} \hookrightarrow L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})$ by Lemma A.1.4. Further we have

$$W^{2m-1-|\beta|+|\gamma|-|\alpha|-j,\frac{2m}{2m-1-|\alpha|},1} \hookrightarrow L^{\frac{2m}{j+|\beta|-|\gamma|},1} \hookrightarrow L^{\frac{2m}{m-1-|\gamma|},1}(B^{2m}_{1/2})$$

by Lemma A.2.2 and Lemma A.1.4 since $|\beta| \le m-j-1$. With Lemma A.1.3 and A.1.4 we have $L^{\frac{2m}{m-1-|\gamma|},1}\cdot L^2 \hookrightarrow L^{\frac{2m}{2m-1-|\gamma|},1}$ and since $\gamma \le \beta$

$$\begin{aligned} &||\partial^{\beta-\gamma}\partial^\alpha D^j\varepsilon_\alpha \star K_j^\beta||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})} \\ &\le c||\varepsilon_\alpha||_{W^{2m-1-j-|\alpha|-|\beta|+|\gamma|,\frac{2m}{2m-1-|\alpha|},1}(B^{2m}_{1/2})}||K_j^\beta||_{L^2(B^{2m}_{1/2})} \le c\sigma||\varepsilon_\alpha||_{W^{2m-1,\frac{2m}{2m-1-|\alpha|},1}(B^{2m}_{1/2})}. \end{aligned}$$

The remaining terms follow in a similar way. With Lemma A.2.2

$$W^{2m-1-|\beta|+|\gamma|-|\alpha|,\frac{2m}{2m-1-|\alpha|},1} \hookrightarrow L^{\frac{2m}{|\beta|-|\gamma|},1}(B^{2m}_{1/2})$$

and by Lemma A.1.3 and Lemma A.1.4 with $|\beta| \le m-2$

$$L^{\frac{2m}{|\beta|-|\gamma|},1}\cdot L^{\frac{2m}{m+1},1} \hookrightarrow L^{\frac{2m}{m+|\beta|-|\gamma|+1},1} \hookrightarrow L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2}).$$

With this and $\gamma \le \beta$

$$\begin{aligned} &||\partial^{\beta-\gamma}\partial^\alpha\varepsilon_\alpha \star K_0^\beta||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})} \\ &\le c||\varepsilon_\alpha||_{W^{2m-1-|\alpha|-|\beta|+|\gamma|,\frac{2m}{2m-1-|\alpha|},1}(B^{2m}_{1/2})}||K_0^\beta||_{L^2(B^{2m}_{1/2})} \le c\sigma||\varepsilon_\alpha||_{W^{2m-1,\frac{2m}{2m-1-|\alpha|},1}(B^{2m}_{1/2})}. \end{aligned}$$

For the next term we have with Lemma A.2.2 and Lemma A.1.3

$$W^{2m-1-j,\frac{2m}{2m-1-|\alpha|},1}\cdot W^{j+1-m-|\alpha|+|\gamma|,2} \hookrightarrow L^{\frac{2m}{j-|\alpha|},1}\cdot L^{\frac{2m}{2m+|\alpha|-|\gamma|-j-1},2} \hookrightarrow L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})$$

so that with $\gamma \le \alpha$

$$\begin{aligned} &||D^j\varepsilon_\alpha \star \partial^{\alpha-\gamma}K_j||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})} \\ &\le c||\varepsilon_\alpha||_{W^{2m-1-j,\frac{2m}{2m-1-|\alpha|},1}(B^{2m}_{1/2})}||K_j||_{W^{j+1-m+|\gamma|-|\alpha|,2}(B^{2m}_{1/2})} \le c\sigma||\varepsilon_\alpha||_{W^{2m-1,\frac{2m}{2m-1-|\alpha|},1}(B^{2m}_{1/2})}. \end{aligned}$$

In the fifth term we use $|\beta| = j+1-m$, Lemma A.2.2 and Lemma A.1.3 to get

$$W^{2m-1-|\alpha|+|\beta|-j,\frac{2m}{2m-1-|\alpha|},1} \cdot W^{j+1-m-|\beta|+|\gamma|,2} \hookrightarrow L^{\frac{2m}{m-1},1} \cdot L^{\frac{2m}{m-|\gamma|},2} \hookrightarrow L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})$$

and

$$\begin{aligned}&||\partial^{\alpha-\beta} D^j \varepsilon_\alpha \star \partial^{\beta-\gamma} K_j||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})} \\ &\le c||\varepsilon_\alpha||_{W^{2m-1-j-|\alpha|+|\beta|,\frac{2m}{2m-1-|\alpha|},1}(B^{2m}_{1/2})} ||K_j||_{W^{j+1-m+|\gamma|-|\beta|,2}(B^{2m}_{1/2})} \le c\sigma ||\varepsilon_\alpha||_{W^{2m-1,\frac{2m}{2m-1-|\alpha|},1}(B^{2m}_{1/2})}.\end{aligned}$$

Finally we estimate for $i = 0,1$ with (2.3.3) and $\gamma \le \alpha$

$$\begin{aligned}&||D^{2m-2-i}\varepsilon_\alpha \star \partial^{\alpha-\gamma} D^{1-i} K_{2m-1}||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})} \\ &\quad \le ||\varepsilon_\alpha||_{W^{1+i,\frac{2m}{2m-1-|\alpha|},1}(B^{2m}_{1/2})} ||P||_{W^{m-|\alpha|+|\gamma|-1+i,2}(B^{2m}_{1/2})} \le c\sigma ||\varepsilon_\alpha||_{W^{2m-1,\frac{2m}{2m-1-|\alpha|},1}(B^{2m}_{1/2})}\end{aligned}$$

and this proves (2.4.12).

Step 3: The fixed point argument

Instead of solving (2.4.11) we solve the system

$$\Delta(\Delta^{m-1}\varepsilon_\gamma \cdot P) = \delta\left(\langle \varepsilon, K\rangle_\gamma + K_0^\gamma\right) \qquad \text{for every } \gamma \text{ with } |\gamma| \le m-2. \tag{2.4.13}$$

To do this we apply a fixed point argument: Let $X_\gamma := \{u \in M(n) : ||u||_{W^{2m-1,\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})} < \infty\}$ and $X = \oplus_{|\gamma|\le m-2} X_\gamma$. We define maps $\psi_\gamma : X_\gamma \to X_\gamma$ by

$$\psi_\gamma : \varepsilon_\gamma \mapsto \text{ solution } \lambda_\gamma \text{ of (2.4.14)}$$

with

$$\begin{cases} \Delta(\Delta^{m-1}\lambda_\gamma \cdot P) &= \delta\left(\langle \varepsilon, K\rangle_\gamma + K_0^\gamma\right) \qquad \text{in } B^{2m}_{1/2}, \\ \qquad\quad \Delta^j \lambda_\gamma &= 0 \qquad\qquad\qquad\qquad \text{on } \partial B^{2m}_{1/2} \text{ for } j = 0,...,m-1. \end{cases} \tag{2.4.14}$$

Let $\hat\lambda = \sum_{|\gamma|\le m-2} \lambda_\gamma$ and $\hat\varepsilon = \sum_{|\gamma|\le m-2} \varepsilon_\gamma$, where λ_γ is a solution of (2.4.14) for every γ with corresponding ε_γ. Let $\Psi = \oplus_{|\gamma|\le m-2}\psi_\gamma$ and

$$\mu := ||\hat\varepsilon||_X := \sum_{|\gamma|\le m-2} ||D^{2m-1}\varepsilon_\gamma||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})}.$$

We apply Lemma 2.4.1 and (2.4.12) to estimate

$$\begin{aligned}||D^{2m-1}\lambda_\gamma||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})} &\le c||\langle\varepsilon, K\rangle_\gamma + K_0^\gamma||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})} \\ &\le c\sigma\left(\sum_{|\gamma|\le m-2} ||\varepsilon_\gamma||_{W^{2m-1,\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})} + 1\right) \\ &\le c_1\sigma(\mu+1).\end{aligned}$$

We choose $\sigma < \frac{\mu}{2c_1(\mu+1)}$ to get

$$||\hat\lambda||_X \le \frac{\mu}{2}.$$

Next we show that ψ_γ is a contraction. Let $\lambda^1_\gamma,\ \lambda^2_\gamma$ be solutions of (2.4.14) with $\varepsilon^1_\gamma,\ \varepsilon^2_\gamma$ respectively. Then $\Lambda_\gamma := \lambda^1_\gamma - \lambda^2_\gamma$ is a solution of

$$\begin{cases} \Delta(\Delta^{m-1}\Lambda_\gamma \cdot P) &= \delta\left(\langle \varepsilon^1 - \varepsilon^2, K\rangle_\gamma\right) \qquad \text{in } B^{2m}_{1/2}, \\ \Delta^j \Lambda_\gamma &= 0 \qquad \text{on } \partial B^{2m}_{1/2} \text{ for } j = 0, ..., m-1. \end{cases}$$

Applying Lemma 2.4.1 and (2.4.12) yields

$$\begin{aligned} &||D^{2m-1}\lambda^1_\gamma - D^{2m-1}\lambda^2_\gamma||_{L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})} \\ &\qquad \le c\sigma \sum_{|\gamma|\le m-2} ||\varepsilon^1_\gamma - \varepsilon^2_\gamma||_{W^{2m-1,\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})}. \end{aligned}$$

With this we have

$$||\hat{\lambda}^1 - \hat{\lambda}^2||_X \le c_2\sigma||\hat{\varepsilon}^1 - \hat{\varepsilon}^2||_X.$$

Choosing $\sigma < \min\{\frac{\mu}{2c_1(\mu+1)}, \frac{1}{2c_2}\}$ shows that Ψ is a contraction. Now we can apply the Banach fixed point theorem which yields a unique $\hat{\varepsilon}^* \in X$ solving (2.4.13) and by Lemma 2.4.1 and (2.4.12)

$$\sum_{|\gamma|\le m-2} ||\varepsilon^*_\gamma||_{W^{2m-1,\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2})} \le c\sigma.$$

Thus we have

$$0 = \delta\left(d\Delta^{m-1}\varepsilon^*_\gamma \cdot P - \langle \varepsilon^*, K\rangle_\gamma + K^\gamma_0\right) \tag{2.4.15}$$

for every γ with $|\gamma| \le m-2$. What is left to show is that these ε^*_γ are the Sobolev functions in the representation (2.4.9) of ε and this ε solves (2.4.5).

Step 4: Back to the original system

We reverse the abbreviations we made at the beginning to get a detailed look at (2.4.15). To do this we go back to (2.4.6). As we have seen before, each term of this equation is a product of a distribution and a Sobolev function. More precisely, the terms are of the form $L^\infty \cdot W^{2-m,\frac{2m}{m-1}}$ and $W^{m-k,2} \cdot W^{-m+1+k,2},\ k = 1, ..., 2m-2$. We use the following representations for the distributions according to Lemma A.2.6

$$\begin{aligned} &FP - d\Delta^{m-2}\delta(\Omega P) + d\Delta^{m-2}\delta\Omega P - d\Delta^{m-2}(*(dP \wedge d*\xi)) \\ &\quad = \sum_{|\alpha|\le m-2} \left(FP - d\Delta^{m-2}\delta(\Omega P) + d\Delta^{m-2}\delta\Omega P - d\Delta^{m-2}(*(dP\wedge d*\xi))\right)^\alpha, \\ &\left(FP - d\Delta^{m-2}\delta(\Omega P) + d\Delta^{m-2}\delta\Omega P - d\Delta^{m-2}(*(dP\wedge d*\xi))\right)^\alpha \in L^{\frac{2m}{m+1},1}(B^{2m}_{1/2}) \end{aligned}$$

$$\begin{aligned} \Delta^k P \cdot V_k &= \sum_{|\alpha|\le m-2} \partial^\alpha(\Delta^k P V_k)^\alpha, & (\Delta^k P V_k)^\alpha &\in L^{\frac{2m}{m+1},1}(B^{2m}_{1/2}),\ k \ne 0 \\ d\Delta^k P w_k &= \sum_{|\alpha|\le m-2} \partial^\alpha(d\Delta^k P w_k)^\alpha, & (d\Delta^k P w_k)^\alpha &\in L^{\frac{2m}{m+1},1}(B^{2m}_{1/2}) \\ \nabla^{2k-l} P \cdot V_k &= \sum_{|\alpha|\le m-1-l} \partial^\alpha(\nabla^{2k-l} P V_k)^\alpha, & (\nabla^{2k-l} P V_k)^\alpha &\in L^2(B^{2m}_{1/2}),\ k \ne 0 \\ \nabla^{2k+1-l} P \cdot w_k &= \sum_{|\alpha|\le m-1-l} \partial^\alpha(\nabla^{2k+1-l} P w_k)^\alpha, & (\nabla^{2k+1-l} P w_k)^\alpha &\in L^2(B^{2m}_{1/2}), \end{aligned}$$

$$\nabla^{2m-1-k}P = \sum_{|\alpha|\le m-1-k} \partial^\alpha(\nabla^{2m-1-k}P)^\alpha, \qquad (\nabla^{2m-1-k}P)^\alpha \in L^2(B^{2m}_{1/2}).$$

Then we shift derivatives to get an equation of the form $\sum_{|\gamma|\le m-2}\partial^\gamma(...)_\gamma = 0$ as in (2.4.11). Using this, we see that (2.4.15) is equivalent to

$$\begin{aligned}
0 = \delta\Bigg[&\sum_{\substack{1\le k\le m-2\\ |\alpha|\le m-1-k}} c_{k,\alpha\gamma}\partial^{\alpha-\gamma}\nabla^k\partial^\beta\varepsilon^*_\beta(\nabla^{2m-1-k}P)^\alpha \\
&+ \sum_{\substack{m-1\le k\le 2m-1\\ |\alpha|\le k+1-m}} c_{k,\alpha\gamma}\nabla^k\varepsilon^*_\alpha\partial^{\alpha-\gamma}\nabla^{2m-1-k}P \\
&+ \sum_{\substack{m-1\le k\le 2m-1\\ |\alpha|> m-1-k\\ |\beta|=m-1-k}} c_{k,\beta\gamma}\partial^{\alpha-\beta}\nabla^k\varepsilon^*_\alpha\partial^{\beta-\gamma}\nabla^{2m-1-k}P \\
&+ \Big(FP - d\Delta^{m-2}\delta(\Omega P) + d\Delta^{m-2}\delta\Omega P - d\Delta^{m-2}(*(dP\wedge d*\xi))\Big)^\gamma \\
&+ \sum_{|\alpha|,|\beta|\le m-2} c_{\beta\gamma}\partial^{\beta-\gamma}\partial^\alpha\varepsilon^*_\alpha\Big(FP - d\Delta^{m-2}\delta(\Omega P) + d\Delta^{m-2}\delta\Omega P \\
&\qquad\qquad - d\Delta^{m-2}(*(dP\wedge d*\xi))\Big)^\beta \\
&+ \sum_{k=1}^{m-1}(\Delta^k PV_k)^\gamma + \sum_{k=0}^{m-1}\sum_{|\alpha|,|\beta|\le m-2} c_{\beta\gamma}\partial^{\beta-\gamma}\partial^\alpha\varepsilon^*_\alpha(\Delta^k PV_k)^\beta \\
&+ \sum_{k=1}^{m-1}\sum_{\substack{1\le l\le m-2\\ l\le 2k\\ |\alpha|\le l+1-m}} c_{l,\alpha\gamma}\partial^{\alpha-\gamma}\nabla^l\partial^\beta\varepsilon^*_\beta(\nabla^{2k-l}PV_k)^\alpha \\
&+ \sum_{k=1}^{m-1}\sum_{\substack{m-1\le l\le 2m-2\\ l\le 2k\\ |\alpha|\le l+1-m}} c_{l,\alpha\gamma}\nabla^l\varepsilon^*_\alpha\partial^{\alpha-\gamma}\nabla^{2k-l}PV_k \\
&+ \sum_{k=1}^{m-1}\sum_{\substack{m-1\le l\le 2m-2\\ l\le 2k\\ |\alpha|> l+1-m\\ |\beta|=l+1-m}} c_{l,\beta\gamma}\partial^{\alpha-\beta}\nabla^l\varepsilon^*_\alpha\partial^{\beta-\gamma}\nabla^{2k-l}PV_k \\
&- \sum_{k=0}^{m-2}(d\Delta^k Pw_k)^\gamma - \sum_{k=0}^{m-2}\sum_{|\alpha|,|\beta|\le m-2} c_{\beta\gamma}\partial^{\beta-\gamma}\partial^\alpha\varepsilon^*_\alpha(d\Delta^k Pw_k)^\beta \\
&- \sum_{k=0}^{m-2}\sum_{\substack{1\le l\le m-2\\ l\le 2k+1\\ |\alpha|\le l+1-m}} c_{l,\alpha\gamma}\partial^{\alpha-\gamma}\nabla^l\partial^\beta\varepsilon^*_\beta(\nabla^{2k+1-l}Pw_k)^\alpha \\
&- \sum_{k=0}^{m-2}\sum_{\substack{m-1\le l\le 2m-3\\ l\le 2k+1\\ |\alpha|\le l+1-m}} c_{l,\alpha\gamma}\nabla^l\varepsilon^*_\alpha\partial^{\alpha-\gamma}\nabla^{2k+1-l}Pw_k
\end{aligned}$$

$$- \sum_{k=0}^{m-2} \sum_{\substack{m-1 \le l \le 2m-3 \\ l \le 2k+1 \\ |\alpha| > l+1-m \\ |\beta| = l+1-m}} c_{l,\beta\gamma} \partial^{\alpha-\beta} \nabla^l \varepsilon^*_\alpha \partial^{\beta-\gamma} \nabla^{2k+1-l} P w_k \Big]$$

$$=: \delta[...]_\gamma.$$

By the Poincaré Lemma (see Lemma 10.68 in [24]) there exist $B_\gamma \in W^{1,\frac{2m}{2m-2-|\gamma|}}_{\text{loc}}(B^{2m}_{1/2}, \mathbb{R}^{n\times n} \otimes \wedge^2\mathbb{R}^{2m})$ for $|\gamma| \le m-2$ such that

$$\delta B_\gamma = [...]_\gamma$$

Now we transform $\hat{\varepsilon}^* = \sum_{|\gamma|\le m-2} \varepsilon^*_\gamma$ and $\hat{B} = \sum_{|\gamma|\le m-2} B_\gamma$ back. Then we have $\varepsilon \in W^{m+1,\frac{2m}{m-1},1}(B^{2m}_{1/2}, M(n))$ with

$$||\varepsilon||_{W^{m+1,\frac{2m}{m-1},1}(B^{2m}_{1/2})} + ||\varepsilon||_{L^\infty(B^{2m}_{1/2})} \le c\sigma$$

and

$$\varepsilon = \sum_{|\gamma|\le m-2} \partial^\gamma \varepsilon^*_\gamma \qquad \text{solves (2.4.5)}.$$

Further $B = \sum_{|\gamma|\le m-2} \partial^\gamma B_\gamma \in W^{2-m,2}_{\text{loc}}(B^{2m}_{1/2}, \mathbb{R}^{n\times n} \otimes \wedge^2\mathbb{R}^{2m})$ with

$$\delta B = \sum_{k=0}^{m-1} \Delta^k((id+\varepsilon)P)V_k - \sum_{k=0}^{m-2} d\Delta^k((id+\varepsilon)P)w_k + d\Delta^{m-1}((id+\varepsilon)P)$$

and

$$\begin{aligned}\delta\Big[&\sum_{l=0}^{m-1} \Delta^l((id+\varepsilon)P)\Delta^{m-l-1}du - \sum_{l=0}^{m-2} d\Delta^l((id+\varepsilon)P)\Delta^{m-l-1}u \\ &- \sum_{k=0}^{m-1}\sum_{l=0}^{k-1} \Delta^l((id+\varepsilon)P)\Delta^{k-l-1}d\langle V_k, du\rangle \\ &+ \sum_{k=0}^{m-1}\sum_{l=0}^{k-1} d\Delta^l((id+\varepsilon)P)\Delta^{k-l-1}\langle V_k, du\rangle \\ &- \sum_{k=0}^{m-2}\sum_{l=0}^{k} \Delta^l((id+\varepsilon)P)d\Delta^{k-l-1}\delta(w_k du) \\ &+ \sum_{k=0}^{m-2}\sum_{l=0}^{k-1} d\Delta^l((id+\varepsilon)P)\Delta^{k-l-1}\delta(w_k du) - \langle B, du\rangle \Big] = 0.\end{aligned}$$

□

Proof of Corollary 1.0.2. We abbreviate the conservation law (1.0.13)

$$\Delta\left((id+\varepsilon)P\Delta^{m-1}u\right) + \delta C = 0 \qquad \text{on } B^{2m}_{1/2}, \tag{2.4.16}$$

where $C \in W^{2-m,\frac{2m}{m+1},1}(B^{2m}_{1/2})$. Since $\varepsilon \in W^{m+1,\frac{2m}{m-1},1} \cap L^\infty(B^{2m}_{1/2})$, $P \in W^{m,2} \cap L^\infty(B^{2m}_{1/2})$ and $\Delta^{m-1}u \in W^{2-m,2}(B^{2m}_{1/2})$, we have

$$(id + \varepsilon)P\Delta^{m-1}u \in W^{2-m,2}(B^{2m}_{1/2}). \tag{2.4.17}$$

Set $f = (id + \varepsilon)P\Delta^{m-1}u$. Then

$$-\Delta f = \delta C \qquad \text{on } B^{2m}_{1/2}.$$

By Theorem 6.2 in [52] we get $f \in W^{3-m,\frac{2m}{m+1},1}(B_\lambda)$ on a smaller ball with radius $0 < \lambda < 1/2$. Since $(id + \varepsilon)P$ is invertible we rewrite (2.4.17)

$$\Delta^{m-1}u = [(id + \varepsilon)P]^{-1} f$$

and $\Delta^{m-1}u \in W^{3-m,\frac{2m}{m+1},1}(B^{2m}_\lambda)$. But this means $u \in W^{m+1,\frac{2m}{m+1},1}(B^{2m}_\lambda)$ and $W^{m+1,\frac{2m}{m+1},1}(B^{2m}_\lambda) \hookrightarrow C^0(B^{2m}_\lambda)$ (see Theorem 2.3 in [20]).

Up until now we have assumed that σ is arbitrarily small so that it satisfies the assumptions of Theorem 2.3.2 and the fixed point argument. A priori this is not true for components V_k, w_k of a system of the form (1.0.10). However, any solution u is continuous. To see this we rescale u (see [52] for a detailed proof). Let $x_0 \in B^{2m}$ and $r > 0$ small enough so that $u_r : B^{2m} \to \mathbb{R}^n$, $u_r(x) := u(x_0 + rx)$ is a solution of (1.0.10) on B^{2m} with corresponding rescaled components $V_{k,r}$ and $w_{k,r}$,

$$\begin{aligned}\sigma_r := &\sum_{k=0}^{m-2} ||w_{k,r}||_{W^{2k+2-m,2}(B^{2m})} + \sum_{k=1}^{m-1} ||V_{k,r}||_{W^{2k+1-m,2}(B^{2m})} \\ &+ ||\eta_r||_{W^{2-m,2}(B^{2m})} + ||F_r||_{W^{2-m,\frac{2m}{m+1},1}(B^{2m})},\end{aligned}$$

$\sigma_r < \sigma_0$ and $B^{2m}_r(x_0) \subset B^{2m}$. By the above we have $u_r \in C^0(B^{2m}_\lambda)$ which is the same as $u \in C^0(B^{2m}_{r\lambda}(x_0))$. A simple covering argument yields $u \in C^0(B^{2m})$.

□

Chapter 3

Energy identity for an approximation of polyharmonic maps

In this chapter we focus on the ε-approximation of the m-polyenergy. We show existence and regularity of critical points u_ε and derive an energy identity for $\varepsilon \to 0$. Many results in this chapter focus on local properties and calculations are often conducted on a ball of small radius R. When estimating certain quantities it is important to keep track of R especially once negative exponents are involved. Thus all constants throughout this chapter are independent of R.

3.1 Euler-Lagrange equation

Let $\Omega \subset \mathbb{R}^{2m}$ be an open bounded domain, N a smooth, closed n-dimensional Riemannian manifold isometrically embedded into $\mathbb{R}^d$ and $u \in W^{m+1,2}(\Omega, N)$. Throughout this chapter we consider the extrinsic energy functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left(|D^m u|^2 + \varepsilon |D^{m+1}u|^2\right) dx,$$

where D^m denotes the m^{th} total derivative of $u : \Omega \to \mathbb{R}^d$. In a first step we derive the Euler-Lagrange equation. Let $\varphi \in C_c^\infty(\Omega, \mathbb{R}^d)$ and define the variation $u_t = \Pi(u + t\varphi)$ of u, where $\Pi : N_\delta \to N$ is the nearest point projection map with $\Pi \in C^\infty(\{x : \operatorname{dist}(x, N) < \delta\}, \mathbb{R}^d)$ (see [69], chapter 2.12.3). Further let $P_{\Pi(y)}[\cdot] : \mathbb{R}^d \to T_{\Pi(y)}N$ be the orthogonal projection onto the tangent space $T_{\Pi(y)}N$ defined by $P_{\Pi(y)}[v] := D_v\Pi(y)$, $y \in N_\delta$. The first variation of E_ε is

$$\begin{aligned}
&\frac{d}{dt}\bigg|_{t=0} \frac{1}{2}\int_\Omega \left(|D^m u_t|^2 + \varepsilon|D^{m+1}u_t|^2\right) dx \\
&= \int_\Omega \left(\langle D^m u, D^m(P_u[\varphi])\rangle + \varepsilon\langle D^{m+1}u, D^{m+1}(P_u[\varphi])\rangle\right) dx \qquad \text{with } \varphi \in C_c^\infty(\Omega, \mathbb{R}^d)
\end{aligned}$$

and thus we have

Lemma 3.1.1. *Let $\varepsilon > 0$. A map $u \colon \Omega \to N$ is a critical point of E_ε if and only if*

$$(\Delta^m u - \varepsilon \Delta^{m+1} u) \perp T_u N. \tag{3.1.1}$$

Angelsberg and Pumberger derived the Euler-Lagrange equation for m-polyharmonic maps in [5], Lemma 2.2. Their results yield the following

Lemma 3.1.2. *A map u is a critical point of E_ε if and only if*

$$\begin{aligned}\Delta^m u - \varepsilon\Delta^{m+1}u &= a + \sum_{\substack{j,l\geq 0\\ 1\leq 2j+l\leq m}} \operatorname{div}^l \cdot \Delta^j b_{jl} + \varepsilon\left(c + \sum_{\substack{j,l\geq 0\\ 1\leq 2j+l\leq m+1}} \operatorname{div}^l \cdot\Delta^j d_{jl}\right)\\ &=: \tilde f + \varepsilon \tilde g,\end{aligned} \tag{3.1.2}$$

with

$$\begin{aligned}
|a| &\leq C\sum_{\lambda\in\Lambda}\prod_{\mu=1}^{m}|D^\mu u|^{\gamma_{\lambda,\mu}}, &&\text{with } \sum_\mu \mu\gamma_{\lambda,\mu} = 2m \text{ for every } \lambda\in\Lambda,\\
|b| &\leq C\sum_{\lambda\in\Lambda_{jl}}\prod_{\mu=1}^{m}|D^\mu u|^{\gamma_{\lambda,\mu}}, &&\text{with } \sum_\mu \mu\gamma_{\lambda,\mu} = 2m-(2j+l) \text{ for every } \lambda\in\Lambda_{jl},\\
|c| &\leq C\sum_{\kappa\in K}\prod_{\mu=1}^{m+1}|D^\mu u|^{\gamma_{\kappa,\mu}}, &&\text{with } \sum_\mu \mu\gamma_{\kappa,\mu} = 2(m+1) \text{ for every } \kappa\in K,\\
|d| &\leq C\sum_{\kappa\in K_{jl}}\prod_{\mu=1}^{m+1}|D^\mu u|^{\gamma_{\kappa,\mu}}, &&\text{with } \sum_\mu \mu\gamma_{\kappa,\mu} = 2(m+1)-(2j+l) \text{ for every } \kappa\in K_{jl}.
\end{aligned}$$

Λ, Λ_{jl}, K, K_{jl} are sets of finitely many indices and $\gamma_{\lambda,\mu}\geq 0$ for every $\lambda\in\Lambda$ or Λ_{jl}, $\gamma_{\kappa,\mu}\geq 0$ for every $\kappa\in K$ or K_{jl} and $0\leq\mu\leq m$.

We want to rewrite (3.1.2) in a form more suitable to our computations. To simplify the notation let $D^{k_1}u * D^{k_2}u$, $k_1,k_2\in\mathbb{N}$ denote any bilinear combination of $D^{k_1}u$ and $D^{k_2}u$. Then

$$\begin{aligned}|D^{k_1}u * D^{k_2}u| &\leq c|D^{k_1}u||D^{k_2}u|,\\ D(D^{k_1}u * D^{k_2}u) &= D^{k_1+1}u * D^{k_2}u + D^{k_1}u * D^{k_2+1}u.\end{aligned}$$

Using the results of Angelsberg and Pumberger we simplify (3.1.2) to

$$\begin{aligned}\Delta^m u - \varepsilon\Delta^{m+1}u = &\sum_{\substack{s,k_1,\dots,k_{2m-1}\in\mathbb{N}_0\\ k_1+2k_2+\dots+(2m-1)k_{2m-1}=2m}} (D^sP_u)[u] * (Du)^{k_1} * \dots * (D^{2m-1}u)^{k_{2m-1}}\\ &+\varepsilon \sum_{\substack{s,k_1,\dots,k_{2m+1}\in\mathbb{N}_0\\ k_1+2k_2+\dots+(2m+1)k_{2m+1}=2m+2}} (D^sP_u)[u] * (Du)^{k_1} * \dots * (D^{2m+1}u)^{k_{2m+1}}.\end{aligned} \tag{3.1.3}$$

3.2 Palais-Smale condition

In this section we show that E_ε satisfies the Palais-Smale condition for every $\varepsilon > 0$. Let

$$W^{m+1,2}(u_j^*TN) = \{w\in W^{m+1,2}(\Omega,\mathbb{R}^d) \mid w(x)\in T_{u_j(x)}N\}$$

and

$$dE_\varepsilon(u_j)(w) = \int_\Omega \left(\langle D^m u_j, D^m w\rangle + \varepsilon\langle D^{m+1}u_j, D^{m+1}w\rangle\right)$$

for $u_j\in W^{m+1,2}(\Omega,N)$ and $w\in W^{m+1,2}(u_j^*TN)$. Then the following holds.

Theorem 3.2.1. *Let $\varepsilon > 0$, $c > 0$, $m \in \mathbb{N}$ and let $(u_j)_{j\in\mathbb{N}} \in W^{m+1,2}(\Omega, N)$ be a sequence such that*

$$E_\varepsilon(u_j) \le c \qquad \text{and} \qquad \limsup_{j\to\infty}\{dE_\varepsilon(u_j)(w) \mid ||w||_{W^{m+1,2}(u_j^*TN)} \le c\} = 0. \tag{3.2.1}$$

Then there exists $u^0 \in W^{m+1,2}(\Omega, N)$ and a subsequence again called $(u_j)_{j\in\mathbb{N}}$ such that

$$u_j \to u^0 \qquad \text{strongly in } W^{m+1,2}(\Omega, \mathbb{R}^d) \quad \text{as} \quad j \to \infty.$$

Proof. Let $\varepsilon > 0$ be fixed and let $(u_j)_{j\in\mathbb{N}}$ be a sequence in $W^{m+1,2}(\Omega, N)$ satisfying (3.2.1). In the following we apply results that only hold up to subsequences. To simplify the notation we will denote these subsequences again by $(u_j)_{j\in\mathbb{N}}$.

With Banach-Alaoglu's theorem, Rellich's theorem and the Sobolev embedding theorem we have the following convergence

$$\begin{aligned} u_j &\rightharpoonup u^0 && \text{weakly in } W^{m+1,2}(\Omega, \mathbb{R}^d) \\ u_j &\to u^0 && \text{strongly in } W^{i,p_i}(\Omega, \mathbb{R}^d)\ \forall\, 1 \le p_i < \frac{2m}{i-1}, \quad 1 \le i \le m \\ u_j &\to u^0 && \text{strongly in } C^{0,\alpha}(\Omega, \mathbb{R}^d)\ \forall\, \alpha \in (0,1), \end{aligned} \tag{3.2.2}$$

and further

$$\begin{aligned} ||D^i u_j||_{L^{\frac{2m}{i-1}}(\Omega)} &\le c \quad \text{for } 2 \le i \le m+1, \\ ||Du_j||_{L^{p_1}(\Omega)} &\le c \quad \text{for all } 1 \le p_1 < \infty. \end{aligned} \tag{3.2.3}$$

The strong $C^{0,\alpha}$-convergence in (3.2.2) implies that the limit u^0 maps into N. Let $P_{\Pi(y)}[\cdot]$ be the projection onto the tangent space $T_{\Pi(y)}N$ as described in section 3.1. Because $u_j \rightharpoonup u^0$ weakly in $W^{m+1,2}(\Omega, N)$ and (3.2.1) we have for $j \to \infty$

$$\begin{aligned} 0 \longleftarrow\ & dE_\varepsilon(u_j)(P_{u_j}[u_j - u^0]) - dE_\varepsilon(u^0)(P_{u^0}[u_j - u^0]) \\ =& \int_\Omega \langle D^m u_j, D^m(P_{u_j}[u_j - u^0])\rangle - \int_\Omega \langle D^m u^0, D^m(P_{u^0}[u_j - u^0])\rangle \\ &+ \varepsilon\int_\Omega \langle D^{m+1} u_j, D^{m+1}(P_{u_j}[u_j - u^0])\rangle - \varepsilon\int_\Omega \langle D^{m+1} u^0, D^{m+1}(P_{u^0}[u_j - u^0])\rangle \\ =& \int_\Omega \langle D^m (u_j - u^0), D^m(P_{u_j}[u_j - u^0])\rangle + \varepsilon\int_\Omega \langle D^{m+1}(u_j - u^0), D^{m+1}(P_{u_j}[u_j - u^0])\rangle \\ &- \int_\Omega \langle D^m u^0, D^m((P_{u^0} - P_{u_j})[u_j - u^0])\rangle - \varepsilon\int_\Omega \langle D^{m+1} u^0, D^{m+1}((P_{u^0} - P_{u_j})[u_j - u^0])\rangle. \end{aligned} \tag{3.2.4}$$

Note that we can bound the derivatives of the projection $P_{\Pi(w)}[v]$ by the derivatives of $v \in W^{m+1,2}(\Omega, \mathbb{R}^d)$ itself using Faà di Bruno's formula

$$\begin{aligned} &|D^m(P_{\Pi(w)}[v])| \\ &= \left| \sum_{\substack{k_i\in\mathbb{N}_0,\ 1\le i\le m \\ k_1+2k_2+\ldots+mk_m=m}} \frac{m!}{k_1!\cdot\ldots\cdot k_m!}(D^{k_1+\ldots+k_m}P_{\Pi(w)})[v] \prod_{i=1}^m \left(\frac{1}{i!}D^i v\right)^{k_i} \right| \\ &\le c \sum_{\substack{k_i\in\mathbb{N}_0,\ 1\le i\le m \\ k_1+2k_2+\ldots+mk_m=m}} |(D^{k_1+\ldots+k_m}P_{\Pi(w)})[v]| \cdot |Dv|^{k_1} \cdot \ldots \cdot |D^m v|^{k_m} \end{aligned} \tag{3.2.5}$$

and $||(D^{k_1+\ldots+k_m}P_{\Pi(w)})[v]||_{L^\infty} \le c$ (see [69]). With this we show that each term in (3.2.4) converges to zero as $j \to \infty$. For the first term we use (3.2.5), Hölder's and Young's inequality ($p_i = \frac{m}{ik_i}$, $i = 1, ..., m$), (3.2.2) and (3.2.3) to estimate

$$\begin{aligned}
&\left|\int_\Omega \langle D^m(u_j - u^0), D^m(P_{u_j}[u_j - u^0])\rangle\right| \\
&\leq \left|\int_\Omega \Bigg\langle D^m(u_j - u^0), \sum_{\substack{k_i \in \mathbb{N}_0,\ 1 \leq i \leq m \\ k_1+2k_2+...+mk_m=m}} |(D^{k_1+...+k_m} P_{u_j})[u_j - u^0]| \right. \\
&\qquad \left. \cdot |D(u_j - u^0)|^{k_1} \cdot ... \cdot |D^m(u_j - u^0)|^{k_m} \Bigg\rangle \right| \\
&\leq c||D^m(u_j - u^0)||_{L^2(\Omega)} \sum_{\substack{k_i \in \mathbb{N}_0,\ 1 \leq i \leq m \\ k_1+2k_2+...+mk_m=m}} ||(D^{k_1+...+k_m} P_{u_j})[u_j - u^0]||_{L^\infty(\Omega)} \\
&\qquad \cdot \sum_{i=1}^m ||D^i(u_j - u^0)||^{\frac{m}{i}}_{L^{\frac{2m}{i}}(\Omega)} \\
&\to 0 \qquad \text{as } j \to \infty.
\end{aligned}$$

Analogously we estimate the third and fourth part in (3.2.4)

$$\begin{aligned}
&\left|\int_\Omega \langle D^m u^0, D^m((P_{u^0} - P_{u_j})[u_j - u^0])\rangle\right| \\
&\leq c\left|\int_\Omega \Bigg\langle D^m u^0, \sum_{\substack{k_i \in \mathbb{N}_0,\ 1 \leq i \leq m \\ k_1+2k_2+...+mk_m=m}} |(D^{k_1+...+k_m}(P_{u^0} - P_{u_j}))[u_j - u^0]| \cdot \right. \\
&\qquad \left. \cdot |D(u_j - u^0)|^{k_1} \cdot ... \cdot |D^m(u_j - u^0)|^{k_m} \Bigg\rangle \right| \\
&\leq c||D^m u^0||_{L^2(\Omega)} \sum_{\substack{k_i \in \mathbb{N}_0,\ 1 \leq i \leq m \\ k_1+2k_2+...+mk_m=m}} ||(D^{k_1+...+k_m}(P_{u^0} - P_{u_j}))[u_j - u^0]||_{L^\infty(\Omega)} \\
&\qquad \cdot \sum_{i=1}^m ||D^i(u_j - u^0)||^{\frac{m}{i}}_{L^{\frac{2m}{i}}(\Omega)} \\
&\to 0 \qquad \text{as } j \to \infty
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_\Omega \langle D^{m+1} u^0, D^{m+1}((P_{u^0} - P_{u_j})[u_j - u^0])\rangle\right| \\
&\leq c\left|\int_\Omega \Bigg\langle D^{m+1} u^0, \sum_{\substack{k_i \in \mathbb{N}_0,\ 1 \leq i \leq m+1 \\ k_1+2k_2+... \\ +(m+1)k_{m+1}=m+1}} |(D^{k_1+...+k_{m+1}}(P_{u^0} - P_{u_j}))[u_j - u^0]| \cdot \right. \\
&\qquad \left. \cdot |D(u_j - u^0)|^{k_1} \cdot ... \cdot |D^{m+1}(u_j - u^0)|^{k_{m+1}} \Bigg\rangle \right| \\
&\leq c||D^{m+1} u^0||_{L^2(\Omega)} ||D^{m+1}(u_j - u^0)||_{L^2(\Omega)} ||(DP_{u^0} - DP_{u_j})[u_j - u^0]||_{L^\infty(\Omega)} \\
&\quad + c||D^{m+1} u^0||_{L^2(\Omega)} \sum_{\substack{k_i \in \mathbb{N}_0,\ 1 \leq i \leq m \\ k_1+2k_2+... \\ +mk_m=m+1}} \Big(||(D^{k_1+...+k_m}(P_{u^0} - P_{u_j}))[u_j - u^0]||_{L^\infty(\Omega)} \\
&\qquad \cdot ||\ |D(u_j - u^0)|^{k_1} \cdot ... \cdot |D^m(u_j - u^0)|^{k_m}||_{L^2(\Omega)} \Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq c||D^{m+1}u^0||_{L^2(\Omega)}||D^{m+1}(u_j-u^0)||_{L^2(\Omega)}||u^0-u_j||_{L^\infty(\Omega)} \\
&\quad \cdot ||\int_0^1 (D^2 P_{\Pi(u_j+t(u^0-u_j))})[u_j-u^0]\cdot D\Pi(u_j+t(u^0-u_j))dt||_{L^\infty(\Omega)} \\
&\quad + c||D^{m+1}u^0||_{L^2(\Omega)} \sum_{\substack{k_i\in\mathbb{N}_0,\ 1\leq i\leq m \\ k_1+2k_2+\ldots+mk_m=m+1}} \Bigg(||(D^{k_1+\ldots+k_m}(P_{u^0}-P_{u_j}))[u_j-u^0]||_{L^\infty(\Omega)} \\
&\quad \cdot \Bigg(\sum_{i=2}^m ||D^i(u_j-u^0)||^{\frac{m}{i-1/2}}_{L^{\frac{2m}{i-1/2}}(\Omega)} + ||D(u_j-u^0)||^{\frac{m}{k_1+1/2(k_2+\ldots+k_m)-1}}_{L^{\frac{2m}{k_1+1/2(k_2+..+k_m)-1}}}\Bigg)\Bigg) \\
&\to 0 \qquad \text{as } j\to\infty,
\end{aligned}
$$

where we used the fundamental theorem of calculus and Hölder's and Young's inequality ($p_1 = \frac{m}{k_1+1/2(k_2+\ldots+k_m)-1}$, $p_i = \frac{m}{k_i(i-1/2)}$, $i = 2,...,m$) in the last estimate. Note that the last term $||D(u_j-u^0)||_{L^{\frac{2m}{k_1+1/2(k_2+..+k_m)-1}}(\Omega)}$ arises only if $k_1\geq 1$ and is therefore well-defined. Thus we are left with

$$
\int_\Omega \langle D^{m+1}(u_j-u^0), D^{m+1}(P_{u_j}[u_j-u^0])\rangle \to 0 \qquad \text{as } j\to\infty.
$$

Let $(\nu^l(\cdot))_{l=1}^{d-n}$ be an orthonormal frame on the normal bundle NN. Then

$$
P_{u_j}[u_j-u^0] = (u_j-u^0) - \sum_{l=1}^{d-n}\nu^l(u_j)\langle\nu^l(u_j), u_j-u^0\rangle
$$

and for the first derivative

$$
\begin{aligned}
&D(P_{u_j}[u_j-u^0]) \\
&= D(u_j-u^0) - \sum_{l=1}^{d-n}\nu^l(u_j)\langle\nu^l(u_j), D(u_j-u^0)\rangle \\
&\quad - \sum_{l=1}^{d-n} D\nu^l(u_j)Du_j\langle\nu^l(u_j), u_j-u^0\rangle - \sum_{l=1}^{d-n}\nu^l(u_j)\langle D\nu^l(u_j)Du_j, u_j-u^0\rangle \\
&= D(u_j-u^0) - \sum_{l=1}^{d-n}\nu^l(u_j)\langle\nu^l(u_j)-\nu^l(u^0), Du^0\rangle \\
&\quad - \sum_{l=1}^{d-n} D\nu^l(u_j)Du_j\langle\nu^l(u_j), u_j-u^0\rangle - \sum_{l=1}^{d-n}\nu^l(u_j)\langle D\nu^l(u_j)Du_j, u_j-u^0\rangle,
\end{aligned}
$$

where we used that $\langle\nu^l(u_j), Du_j\rangle = 0$ and $\langle\nu^l(u^0), Du^0\rangle = 0$ $\forall l = 1,..,d-n$. Using this we have

$$
\begin{aligned}
&\int_\Omega \langle D^{m+1}(u_j-u^0), D^{m+1}(P_{u_j}[u_j-u^0])\rangle \\
&= \int_\Omega \Bigg\langle D^{m+1}(u_j-u^0), D^m\Bigg(D(u_j-u^0) - \sum_{l=1}^{d-n}\nu^l(u_j)\langle\nu^l(u_j)-\nu^l(u^0), Du^0\rangle \\
&\quad - \sum_{l=1}^{d-n} D\nu^l(u_j)Du_j\langle\nu^l(u_j), u_j-u^0\rangle - \sum_{l=1}^{d-n}\nu^l(u_j)\langle D\nu^l(u_j)Du_j, u_j-u^0\rangle\Bigg)\Bigg\rangle \\
&= \int_\Omega \langle D^{m+1}(u_j-u^0), D^{m+1}(u_j-u^0)\rangle
\end{aligned}
$$

$$- \int_\Omega \Big\langle D^{m+1}(u_j - u^0), D^m \Big(\sum_{l=1}^{d-n} \nu^l(u_j)\langle \nu^l(u_j) - \nu^l(u^0), Du^0\rangle$$
$$+ \sum_{l=1}^{d-n} D\nu^l(u_j)Du_j\langle \nu^l(u_j), u_j - u^0\rangle + \sum_{l=1}^{d-n} \nu^l(u_j)\langle D\nu^l(u_j)Du_j, u_j - u^0\rangle \Big)\Big\rangle. \tag{3.2.6}$$

As in (3.2.5) we can bound the m^{th} derivative of $\nu^l(u_j)$ by derivatives of $u_j,\ j \in \mathbb{N}_0$:

$$\begin{aligned} |D^m(\nu_j^l(u_j))| &= \Bigg| \sum_{\substack{k_i\in\mathbb{N}_0,\ 1\le i\le m \\ k_1+2k_2+\ldots+mk_m=m}} \frac{m!}{k_1!\cdot\ldots\cdot k_m!} D^{k_1+\ldots+k_m}\nu^l(u_j)\prod_{i=1}^m \left(\frac{1}{i!}D^i u_j\right)^{k_i} \Bigg| \\ &\le c \sum_{\substack{k_i\in\mathbb{N}_0,\ 1\le i\le m \\ k_1+2k_2+\ldots+mk_m=m}} |D^{k_1+\ldots+k_m}\nu^l(u_j)|\cdot|Du_j|^{k_1}\cdot\ldots\cdot|D^m u_j|^{k_m}, \end{aligned}$$

with $||D^{k_1+\ldots+k_m}\nu^l(u_j)||_{L^\infty(\Omega)} \le c$. Using this we estimate the second term in (3.2.6) with Hölder's and Young's inequality, (3.2.2), (3.2.3) and the fundamental theorem of calculus

$$\begin{aligned} &\Bigg| \int_\Omega \Big\langle D^{m+1}(u_j - u^0), D^m \Big(\sum_{l=1}^{d-n} \nu^l(u_j)\langle \nu^l(u_j) - \nu^l(u_0), Du^0\rangle \Big) \Big\rangle \Bigg| \\ &\le c \sum_{l=1}^{d-n} \sum_{\substack{r,s,t\in\mathbb{N}_0 \\ r+s+t=m}} \int_\Omega \Big(|D^{m+1}(u_j-u^0)|\cdot|D^r\left(\nu^l(u_j)\right)|\cdot|D^s\left(\nu^l(u_j)-\nu^l(u^0)\right)|\cdot|D^{t+1}u^0| \Big) \\ &\le c \sum_{l=1}^{d-n} ||D^{m+1}(u_j-u^0)||_{L^2(\Omega)} \\ &\quad\cdot \Bigg[\sum_{\substack{r,s,t\ge1 \\ r+s+t=m}} ||D^s\left(\nu^l(u_j)-\nu^l(u^0)\right)||_{L^{\frac{2m}{s}}(\Omega)} ||D^r\left(\nu^l(u_j)\right)||_{L^{\frac{2m}{r}}(\Omega)} ||D^{t+1}u^0||_{L^{\frac{2m}{t}}(\Omega)} \\ &\quad + \sum_{\substack{r,s\ge1,\ t=0 \\ r+s=m}} ||D^s\left(\nu^l(u_j)-\nu^l(u^0)\right)||_{L^{\frac{2m}{s}}(\Omega)} ||D^r\left(\nu^l(u_j)\right)||_{L^{\frac{2m}{r-1/2}}(\Omega)} ||Du^0||_{L^{4m}(\Omega)} \\ &\quad + \sum_{\substack{s,t\ge1,\ r=0 \\ s+t=m}} ||D^s\left(\nu^l(u_j)-\nu^l(u^0)\right)||_{L^{\frac{2m}{s}}(\Omega)} ||\nu^l(u_j)||_{L^\infty(\Omega)} ||D^{t+1}u^0||_{L^{\frac{2m}{t}}(\Omega)} \\ &\quad + \sum_{\substack{r,t\ge1,\ s=0 \\ r+t=m}} ||\nu^l(u_j)-\nu^l(u^0)||_{L^\infty(\Omega)} ||D^r\left(\nu^l(u_j)\right)||_{L^{\frac{2m}{r}}(\Omega)} ||D^{t+1}u^0||_{L^{\frac{2m}{t}}(\Omega)} \\ &\quad + \sum_{\substack{r=s=0 \\ t=m}} ||\nu^l(u_j)-\nu^l(u^0)||_{L^\infty(\Omega)} ||\nu^l(u_j)||_{L^\infty(\Omega)} ||D^{m+1}u^0||_{L^2(\Omega)} \\ &\quad + \sum_{\substack{r=t=0 \\ s=m}} ||D^m\left(\nu^l(u_j)-\nu^l(u^0)\right)||_{L^{\frac{2m}{m-1/2}}(\Omega)} ||\nu^l(u_j)||_{L^\infty(\Omega)} ||Du^0||_{L^{4m}(\Omega)} \\ &\quad + \sum_{\substack{s=t=0 \\ s=m}} ||\nu^l(u_j)-\nu^l(u^0)||_{L^\infty(\Omega)} ||D^m\left(\nu^l(u_j)\right)||_{L^{\frac{2m}{m-1/2}}(\Omega)} ||Du^0||_{L^{4m}(\Omega)} \Bigg] \\ &\le c \sum_{l=1}^{d-n} ||D^{m+1}(u_j-u^0)||_{L^2(\Omega)}. \end{aligned}$$

$$
\begin{aligned}
&\cdot\Bigg[\sum_{\substack{r,s,t\in\mathbb{N}\\ r+s+t=m}}\left(\sum_{i=1}^{s}||D^i\left(u_j-u^0\right)||^{\frac{s}{i}}_{L^{\frac{2m}{i}}(\Omega)}\right)\left(\sum_{i=1}^{r}||D^iu_j||^{\frac{r}{i}}_{L^{\frac{2m}{i}}(\Omega)}\right)||D^{t+1}u^0||_{L^{\frac{2m}{t}}(\Omega)}\\
&+\sum_{\substack{r,s\in\mathbb{N}\\ r+s=m}}\left(\sum_{i=1}^{s}||D^i\left(u_j-u^0\right)||^{\frac{s}{i}}_{L^{\frac{2m}{i}}(\Omega)}\right)\left(\sum_{i=1}^{r}||D^iu_j||^{\frac{r-1/2}{i-1/2}}_{L^{\frac{2m}{i-1/2}}(\Omega)}\right)||Du^0||_{L^4(\Omega)}\\
&+\sum_{\substack{s,t\in\mathbb{N}\\ s+t=m}}\left(\sum_{i=1}^{s}||D^i\left(u_j-u^0\right)||^{\frac{s}{i}}_{L^{\frac{2m}{i}}(\Omega)}\right)||\nu^l(u_j)||_{L^\infty(\Omega)}||D^{t+1}u^0||_{L^{\frac{2m}{t}}(\Omega)}\\
&+\sum_{\substack{r,t\in\mathbb{N}\\ r+t=m}}\left(||\int_0^1\frac{d}{dt}\nu^l(u_j-t(u_j-u^0))dt||_{L^\infty(\Omega)}||u_j-u^0||_{L^\infty(\Omega)}\right)\\
&\qquad\cdot\left(\sum_{i=1}^{r}||D^iu_j||^{\frac{r}{i}}_{L^{\frac{2m}{i}}(\Omega)}\right)||D^{t+1}u^0||_{L^{\frac{2m}{t}}(\Omega)}\\
&+\left\|\int_0^1\frac{d}{dt}\nu^l(u_j-t(u_j-u^0))dt\right\|_{L^\infty(\Omega)}||u_j-u^0||_{L^\infty(\Omega)}||\nu^l_j(u_j)||_{L^\infty(\Omega)}||D^{m+1}u^0||_{L^2(\Omega)}\\
&+\left(\sum_{i=1}^{m}||D^i\left(u_j-u^0\right)||^{\frac{m-1/2}{i-1/2}}_{L^{\frac{2m}{i-1/2}}(\Omega)}\right)||\nu^l_j(u_j)||_{L^\infty(\Omega)}||Du^0||_{L^{4m}(\Omega)}\\
&+\Bigg(\left\|\int_0^1\frac{d}{dt}\nu^l(u_j-t(u_j-u^0))dt\right\|_{L^\infty(\Omega)}||u_j-u^0||_{L^\infty(\Omega)}\\
&\qquad\cdot\left(\sum_{i=1}^{m}||D^iu_j||^{\frac{m-1/2}{i-1/2}}_{L^{\frac{2m}{i-1/2}}(\Omega)}\right)||Du^0||_{L^{4m}(\Omega)}\Bigg)\Bigg]\\
\to 0&\qquad\text{as } j\to\infty.
\end{aligned}
$$

We estimate the third and fourth term in (3.2.6) with the same technique

$$
\begin{aligned}
&\left|\int_\Omega\left\langle D^{m+1}(u_j-u^0),D^m\left(\sum_{l=1}^{d-n}D\nu^l(u_j)Du_j\langle\nu^l(u_j),u_j-u^0\rangle+\nu^l(u_j)\langle D\nu^l(u_j)Du_j,u_j-u^0\rangle\right)\right\rangle\right|\\
&\le c\sum_{l=1}^{d-n}\sum_{\substack{r,s,t\in\mathbb{N}_0\\ r+s+t=m}}\int_\Omega|D^{m+1}(u_j-u^0)|\cdot|D^{r+1}\left(\nu^l(u_j)\right)|\cdot|D^s\left(\nu^l(u_j)\right)|\cdot|D^t\left(u_j-u^0\right)|\\
&\le c\sum_{l=1}^{d-n}\Bigg(||D^{m+1}(u_j-u^0)||_{L^2(\Omega)}\\
&\cdot\Bigg[\sum_{\substack{r,s,t\in\mathbb{N}_0\\ s\ge1,t\ge1\\ r+s+t=m}}||D^{r+1}\left(\nu^l(u_j)\right)||_{L^{\frac{2m}{r+1/2}}(\Omega)}||D^s\left(\nu^l(u_j)\right)||_{L^{\frac{2m}{s-1/2}}(\Omega)}||D^t(u_j-u^0)||_{L^{\frac{2m}{t}}(\Omega)}\\
&+\sum_{\substack{r,s,t\in\mathbb{N}_0\\ s\ge1,\ t=0\\ r+s=m}}||D^{r+1}\left(\nu^l(u_j)\right)||_{L^{\frac{2m}{r+1/2}}(\Omega)}||D^s\left(\nu^l(u_j)\right)||_{L^{\frac{2m}{s-1/2}}(\Omega)}||u_j-u^0||_{L^\infty(\Omega)}\\
&+\sum_{\substack{r,s,t\in\mathbb{N}_0\\ s=0,\ t\ge1\\ r+t=m}}||D^{r+1}\left(\nu^l(u_j)\right)||_{L^{\frac{2m}{r+1/2}}(\Omega)}||\nu^l(u_j)||_{L^\infty(\Omega)}||D^t(u_j-u^0)||_{L^{\frac{2m}{t-1/2}}(\Omega)}\\
&+||D^{m+1}\left(\nu^l(u_j)\right)||_{L^2(\Omega)}||\nu^l(u_j)||_{L^\infty(\Omega)}||u_j-u^0||_{L^\infty(\Omega)}\Bigg]\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\leq c \sum_{l=1}^{d-n} \Bigg(||D^{m+1}(u_j - u^0)||_{L^2(\Omega)} \cdot \\
&\cdot \Bigg[\sum_{\substack{r,s,t \in \mathbb{N}_0 \\ s \geq 1, t \geq 1 \\ r+s+t=m}} \sum_{i=1}^{r+1} ||D^i u_j||_{L^{\frac{2m}{i-1/2}}(\Omega)}^{\frac{r+1/2}{i-1/2}} \sum_{i=1}^{s} ||D^i u_j||_{L^{\frac{2m}{i-1/2}}(\Omega)}^{\frac{s-1/2}{i-1/2}} ||D^t(u_j - u^0)||_{L^{\frac{2m}{t}}(\Omega)} \\
&\quad + \sum_{\substack{r \in \mathbb{N}_0, s \in \mathbb{N} \\ r+s=m}} \sum_{i=1}^{r+1} ||D^i u_j||_{L^{\frac{2m}{i-1/2}}(\Omega)}^{\frac{r+1/2}{i-1/2}} \sum_{i=1}^{s} ||D^i u_j||_{L^{\frac{2m}{i-1/2}}(\Omega)}^{\frac{s-1/2}{i-1/2}} ||u_j - u^0||_{L^\infty(\Omega)} \\
&\quad + \sum_{\substack{r \in \mathbb{N}_0, t \in \mathbb{N} \\ r+t=m}} \sum_{i=1}^{r+1} ||D^i u_j||_{L^{\frac{2m}{i-1/2}}(\Omega)}^{\frac{r+1/2}{i-1/2}} ||\nu^l(u_j)||_{L^\infty(\Omega)} ||D^t(u_j - u^0)||_{L^{\frac{2m}{t-1/2}}(\Omega)} \\
&\quad + \sum_{i=1}^{m+1} ||D^i u_j||_{L^{\frac{2(m+1)}{i}}(\Omega)}^{\frac{m+1}{i}} ||\nu^l(u_j)||_{L^\infty(\Omega)} ||u_j - u^0||_{L^\infty(\Omega)} \Bigg] \Bigg) \\
&\to 0 \qquad \text{as } j \to \infty
\end{aligned}
$$

Thus we are left with

$$\int_\Omega \langle D^{m+1}(u_j - u^0), D^{m+1}(u_j - u^0) \rangle \to 0 \qquad \text{as } j \to \infty.$$

□

3.3 Regularity

Theorem 3.3.1. *Let $\varepsilon > 0$ and let $u \in W^{m+1,2}(\Omega, N)$ be a critical point of E_ε. Then u is smooth.*

Proof. Let $\varepsilon > 0$ fixed and let $u \in W^{m+1,2}(\Omega, N)$ be a critical point of E_ε with $c > 0$ such that $E_\varepsilon(u) \leq c$. Then $u \in C^{0,\alpha}$ for any $0 < \alpha < 1$ by the Sobolev embedding $W^{m+1,2} \hookrightarrow C^{0,\alpha}(\Omega, N)$ and thus

$$|u(x) - u(y)| \leq C|x - y|^\alpha, \qquad \forall x, y \in \Omega \text{ and some } C > 0.$$

We use the hole-filling technique to show higher regularity. Without loss of generality we can assume $D^i u(0) = 0,\ i = 0, ..., m-1$. Let $0 < R << 1$ such that $B_R(0) \subset \Omega$ and let $\eta \in C_c^\infty(\Omega)$ with

$$0 \leq \eta \leq 1, \qquad \eta = 1 \text{ on } B_{\frac{R}{2}} \qquad \text{and} \qquad \eta = 0 \text{ on } \Omega \setminus B_R.$$

Further let $A_R = B_R \setminus B_{\frac{R}{2}}$ and define $\overline{u} = ⨍_{A_R} u$ to be the mean value over the annulus A_R. We construct a polynomial U of degree at most m such that

$$\int_{A_R} D^i(u - U) = 0, \qquad \text{for every } i = 0, ..., m.$$

U has the form

$$U(x) = \sum_{\gamma=0}^{m} \sum_{\beta=0}^{m-\gamma} \frac{b_\beta}{\gamma!} \overline{D^{\gamma+\beta} u} x^\gamma$$

with recursively defined constants b_β

$$b_0 = 1, \qquad b_1 = 0, \qquad \text{and} \qquad b_\beta = -\sum_{\sigma=1}^{\beta} \frac{b_{\beta-\sigma}}{\sigma!} \fint_{A_R} x^\sigma dx \qquad \text{if } \beta \geq 2.$$

(see also [15]). Note that $D^m U(x) = \fint_{A_R} D^m u$. Further $b_\beta \sim R^\beta$ and for $x \in B_R$ with (3.2.3)

$$\begin{aligned} |U(x)| &\leq c \sum_{0 \leq \gamma+\beta \leq m} R^{\gamma+\beta} | \fint_{A_R} D^{\gamma+\beta} u| \\ &\leq c \sum_{0 \leq \gamma+\beta \leq m} R^{\gamma+\beta-2m} \left(\int_{B_R} |D^{\gamma+\beta} u|^{\frac{2m}{\gamma+\beta-\alpha}} \right)^{\frac{\gamma+\beta-\alpha}{2m}} R^{2m-\gamma-\beta+\alpha} \leq cR^\alpha. \end{aligned}$$

Therefore

$$||u||_{L^\infty(B_R)} + ||(u-U)||_{L^\infty(B_R)} \leq cR^\alpha. \tag{3.3.1}$$

We multiply (3.1.3) with $\eta^{2(m+1)}(u-U)$ and integrate over Ω. Then

$$\begin{aligned} &(-1)^m \int_\Omega \left\langle \Delta^m u - \varepsilon \Delta^{m+1} u, \eta^{2(m+1)}(u-U) \right\rangle \\ &= \int_\Omega \left\langle D^m u, D^m \left(\eta^{2(m+1)}(u-U)\right) \right\rangle + \varepsilon \int_\Omega \left\langle D^{m+1} u, D^{m+1} \left(\eta^{2(m+1)}(u-U)\right) \right\rangle \\ &\geq \frac{1}{2} \int_\Omega \eta^{2(m+1)} \left(|D^m u|^2 + \varepsilon |D^{m+1} u|^2\right) - c \sum_{i=0}^{m-1} \int_\Omega |D^{m-i}\eta|^2 |D^i(u-U)|^2 \\ &\quad - \varepsilon c \sum_{i=0}^{m} \int_\Omega |D^{m+1-i}\eta|^2 |D^i(u-U)|^2 - \int_\Omega D^{m-1} u D \left(D^m U \eta^{2(m+1)}\right) \\ &\geq \frac{1}{2} \int_\Omega \eta^{2(m+1)} \left(|D^m u|^2 + \varepsilon |D^{m+1} u|^2\right) - c \sum_{i=0}^{m-1} R^{-2(m-i)} \int_{A_R} |D^i(u-U)|^2 \\ &\quad - c\varepsilon \sum_{i=0}^{m} R^{-2(m+1-i)} \int_{A_R} |D^i(u-U)|^2 - cR^{-2} \int_{A_R} |D^{m-1} u|^2 - cR^{2m} \left(\fint_{A_R} D^m u \right)^2 \\ &\geq \frac{1}{2} \int_\Omega \eta^{2(m+1)} \left(|D^m u|^2 + \varepsilon |D^{m+1} u|^2\right) - c\left(\varepsilon + R^2\right) \int_{A_R} |D^{m+1} u|^2 - c \int_{A_R} |D^m u|^2 \\ &\quad - c \left(\int_{A_R} |D^{m-1} u|^{\frac{2m}{m-1}} \right)^{\frac{m-1}{m}} \\ &\geq \frac{1}{2} \int_\Omega \eta^{2(m+1)} \left(|D^m u|^2 + \varepsilon |D^{m+1} u|^2\right) - c \int_{A_R} \left(|D^{m+1} u|^2 + |D^m u|^2\right) - cR^{2\alpha}, \end{aligned} \tag{3.3.2}$$

where we applied the Poincaré inequality for the annulus (Theorem A.10 in [74]) repeatedly as well as the Gagliardo-Nirenberg inequality

$$\begin{aligned} \left(\int_{A_R} |D^{m-1} u|^{\frac{2m}{m-1}} \right)^{\frac{m-1}{m}} &\leq \left(\int_{A_R} |D^m u|^2 \right)^{\frac{m-1}{m}} ||u||_{L^\infty(A_R)}^{\frac{2}{m}} + c||u||_{L^\infty(A_R)}^2 \\ &\leq \int_{A_R} |D^m u|^2 + cR^{2\alpha}. \end{aligned} \tag{3.3.3}$$

For the right-hand side of (3.1.3) we use integration by parts until the highest order terms are $D^m u$ and $D^{m+1} u$ respectively. With Young's inequality ($p_i = \frac{2m}{ik_i}$, $i = 1, ..., m$; $\tilde{p}_i = \frac{2(m+1)}{ik_i}$, $i =$

$1,...,m+1)$ and Hölder's inequality ($p_i = \frac{m}{ik_i},\ i=1,...,m-1,\ p_r = \frac{m}{r},\ p_s = \frac{m}{s};\ \tilde{p}_i = \frac{m}{ik_i},\ i = 1,...,m,\ \tilde{p}_r = \frac{m}{r-1},\ \tilde{p}_s = \frac{m}{s}$) we estimate

$$
\begin{aligned}
&(-1)^m \int_\Omega \Bigg\langle \sum_{\substack{b,k_i\in\mathbb{N}_0\\ k_1+2k_2+...\\ +(2m-1)k_{2m-1}=2m}} (D^b P_u)[u] * (Du)^{k_1} * ... * (D^{2m-1}u)^{k_{2m-1}} \\
&\quad + \varepsilon \sum_{\substack{b,k_i\in\mathbb{N}_0\\ k_1+2k_2+...\\ +(2m+1)k_{2m+1}=2(m+1)}} (D^b P_u)[u] * (Du)^{k_1} * ... * (D^{2m+1}u)^{k_{2m+1}}, \eta^{2(m+1)}(u-U) \Bigg\rangle \\
&\le c\int_\Omega \sum_{\substack{k_i\in\mathbb{N}_0\\ k_1+2k_2+...+mk_m=2m}} |Du|^{k_1}\cdot ...\cdot |D^m u|^{k_m}\eta^{2(m+1)}|(u-U)| \\
&\quad + c\varepsilon\int_\Omega \sum_{\substack{k_i\in\mathbb{N}_0\\ k_1+2k_2+...\\ +(m+1)k_{m+1}=2(m+1)}} |Du|^{k_1}\cdot ...\cdot |D^{m+1} u|^{k_{m+1}}\eta^{2(m+1)}|(u-U)| \\
&\quad + c\int_\Omega \sum_{\substack{r\in\mathbb{N},\ s,k_i\in\mathbb{N}_0\\ 1\le s+r\le m-1\\ k_1+2k_2+...+mk_m+r+s=2m}} |Du|^{k_1}\cdot ...\cdot |D^m u|^{k_m}|D^r(\eta^{2(m+1)})||D^s(u-U)| \\
&\quad + c\varepsilon\int_\Omega \sum_{\substack{r\in\mathbb{N},\ s,k_i\in\mathbb{N}_0\\ 1\le s+r\le m\\ k_1+2k_2+...+(m+1)k_{m+1}+\\ +r+s=2(m+1)}} |Du|^{k_1}\cdot ...\cdot |D^{m+1} u|^{k_{m+1}}|D^r(\eta^{2(m+1)})||D^s(u-U)| \\
&\le (cR^\alpha+\delta)\int_\Omega \eta^{2(m+1)}\left(|D^m u|^2+\varepsilon|D^{m+1}u|^2\right) + cR^\alpha\int_{B_R}\sum_{i=1}^{m-1}|D^i u|^{\frac{2m}{i}} + cR^\alpha\varepsilon\int_{B_R}\sum_{i=1}^{m}|D^i u|^{\frac{2(m+1)}{i}} \\
&\quad + c \sum_{\substack{r\in\mathbb{N},\ s,k_i\in\mathbb{N}_0\\ 1\le s+r\le m-1\\ k_1+2k_2+...+(m-1)k_{m-1}\\ +r+s=m}} \prod_{i=1}^{m-1}\left(\int_{A_R}|D^i u|^{\frac{2m}{i}}\right)^{\frac{ik_i}{m}}\left(\int_{A_R}|D^r\eta|^{\frac{2m}{r}}\right)^{\frac{r}{m}}\left(\int_{A_R}|D^s(u-U)|^{\frac{2m}{s}}\right)^{\frac{s}{m}} \\
&\quad + c_\delta\varepsilon \sum_{\substack{r\ge 2,\ s,k_i\in\mathbb{N}_0\\ 1\le s+r\le m\\ k_1+2k_2+...+(m)k_m+\\ +r+s=m+1}} \prod_{i=1}^{m}\left(\int_{A_R}|D^i u|^{\frac{2m}{i}}\right)^{\frac{ik_i}{m}}\left(\int_{A_R}|D^r\eta|^{\frac{2m}{r-1}}\right)^{\frac{r-1}{m}}\left(\int_{A_R}|D^s(u-U)|^{\frac{2m}{s}}\right)^{\frac{s}{m}} \\
&\quad + c_\delta\varepsilon \sum_{\substack{r=1,\ s,k_i\in\mathbb{N}_0\\ 1\le s+1\le m\\ k_1+2k_2+...+mk_m+\\ +r+s=m+1}} R^{-2}\prod_{i=1}^{m}\left(\int_{A_R}|D^i u|^{\frac{2m}{i}}\right)^{\frac{ik_i}{m}}\left(\int_{A_R}|D^s(u-U)|^{\frac{2m}{s}}\right)^{\frac{s}{m}}. \qquad (3.3.4)
\end{aligned}
$$

To estimate the last three terms we apply the Poincaré-Sobolev inequality $(m-s)$-times to the $D^s(u-U)$ term and in the last step we use the Poincaré inequality

$$
\begin{aligned}
\left(\int_{A_R}|D^s(u-U)|^{\frac{2m}{s}}\right)^{\frac{s}{m}} &\le c\left(\int_{A_R}|D^{s+1}(u-U)|^{\frac{2m}{s+1}}\right)^{\frac{s+1}{m}} \\
&\le ... \\
&\le c\int_{A_R}|D^m(u-U)|^2
\end{aligned}
$$

$$\leq cR^2 \int_{A_R} |D^{m+1}u|^2. \tag{3.3.5}$$

The Gagliardo-Nirenberg inequality and (3.3.1) yield

$$\begin{aligned}\int_{B_R} |D^i u|^{\frac{2m}{i}} &\leq c\left(\int_{B_R} |D^m u|^2\right) ||u||_{L^\infty(B_R)}^{\frac{2(m-i)}{i}} + ||u||_{L^\infty(B_R)}^{\frac{2m}{i}} \\ &\leq cR^{\frac{2\alpha(m-i)}{i}} \left(\int_{B_R} |D^m u|^2\right) + cR^{\frac{2\alpha m}{i}} \qquad \forall i = 1, ..., m-1,\end{aligned} \tag{3.3.6}$$

$$\begin{aligned}\varepsilon \int_{B_R} |D^i u|^{\frac{2(m+1)}{i}} &\leq c\varepsilon\left(\int_{B_R} |D^{m+1} u|^2\right) ||u||_{L^\infty(B_R)}^{\frac{2(m+1-i)}{i}} + c\varepsilon||u||_{L^\infty(B_R)}^{\frac{2(m+1)}{i}} \\ &\leq c\varepsilon R^{\frac{2\alpha(m+1-i)}{i}} \left(\int_{B_R} |D^{m+1} u|^2\right) + c\varepsilon R^{\frac{2\alpha(m+1)}{i}} \qquad \forall i = 1, ..., m.\end{aligned} \tag{3.3.7}$$

With (3.3.1), (3.3.5), (3.3.6), (3.3.7) and the bound $E_\varepsilon(u) \leq c$ we have for (3.3.4)

$$\begin{aligned}(-1)^m \int_\Omega \left\langle \Delta^m u - \varepsilon\Delta^{m+1}u, \eta^{2(m+1)}(u-U)\right\rangle &\leq (cR^\alpha + \delta) \int_\Omega \eta^{2(m+1)} \left(|D^m u|^2 + \varepsilon|D^{m+1}u|^2\right) \\ &\quad + cR^{\frac{\alpha(m+2)}{m}} \int_{B_R} \left(|D^m u|^2 + |D^{m+1}u|^2\right) \\ &\quad + cR^{2\alpha} + c\int_{A_R} |D^{m+1}u|^2.\end{aligned}$$

Together with (3.3.2) and $R, \delta > 0$ small enough we arrive at

$$\begin{aligned}\int_{B_{\frac{R}{2}}} |D^m u|^2 + \varepsilon|D^{m+1}u|^2 &\leq cR^{\frac{\alpha(m+2)}{m}} \int_{B_R} \left(|D^m u|^2 + |D^{m+1}u|^2\right) \\ &\quad + cR^{2\alpha} + c_1\int_{A_R} \left(|D^{m+1}u|^2 + |D^m u|^2\right),\end{aligned}$$

where c_1 does not contain any factor R^{-a}, $a > 0$. We choose R small such that $cR^{\frac{\alpha(m+2)}{m}} < \delta << 1$. Adding $c_1 \int_{B_{R/2}} |D^m u|^2 + |D^{m+1}u|^2$ on both sides

$$(c_1+1)\left(\int_{B_{\frac{R}{2}}} |D^m u|^2 + |D^{m+1}u|^2\right) \leq (c_1+\delta)\left(\int_{B_R} |D^m u|^2 + |D^{m+1}u|^2\right) + cR^{2\alpha}$$

and rearranging we get

$$\left(\int_{B_{\frac{R}{2}}} |D^m u|^2 + |D^{m+1}u|^2\right) \leq \frac{c_1+\delta}{c_1+1}\left(\int_{B_R} |D^m u|^2 + |D^{m+1}u|^2\right) + cR^{2\alpha},$$

where $\frac{c_1+\delta}{c_1+1} < 1$. With Lemma 2.1 in [23] (see p. 86) we have

$$\int_{B_\rho} \left(|D^m u|^2 + |D^{m+1}u|^2\right) \leq c\rho^{2\alpha} \qquad \forall\ 0 < \rho < \frac{R}{2} \tag{3.3.8}$$

and with Morrey's Theorem 5.5 in [24] $Du \in C^{0,\alpha}(B_\rho)$.

To show that $u \in C^{2,\alpha}$ we use difference quotients and repeat this procedure. Let $R < \rho$, $h < \rho - R$ and

$$u_h(x) = \frac{1}{h}\left(u(x + he_\nu) - u(x)\right), \qquad \nu \in \{1, ..., 2m\}.$$

As before there exists a polynomial U_h of degree m such that

$$\int_{A_R} D^i\left(u_h - U_h\right) = 0 \qquad \forall i = 0, ..., m.$$

Since $u \in C^{1,\alpha} \cap W^{m+1,2}(B_\rho)$ we have $||Du||_{L^\infty(B_R)} \le cR^\alpha$ and, if $D^{i+1} \in L^p(B_R)$, then

$$||D^i u_h||_{L^p(B_R)} \le c||D^{i+1}u||_{L^p(B_R)}, \qquad i = 1, ..., m \tag{3.3.9}$$

(see [25] Lemma 7.23). For $x \in B_R(0)$ we have with (3.2.3)

$$\begin{aligned}
|U_h(x)| &\le c \sum_{0\le\gamma+\beta\le m} R^{\gamma+\beta} \Big| ⨍_{A_R} D^{\gamma+\beta} u_h \Big| \\
&\le c \sum_{0\le\gamma+\beta\le m} R^{\gamma+\beta-2m} \left(\int_{B_R} |D^{\gamma+\beta+1}u|^{\frac{2m}{\gamma+\beta}}\right)^{\frac{\gamma+\beta}{2m}} R^{2m-\gamma-\beta} \le c
\end{aligned}$$

and $||(u_h - U_h)||_{L^\infty(B_R)} \le c$, with $c > 0$ independent of R. Applying the difference quotient to (3.1.3) yields

$$\begin{aligned}
\Delta^m u_h - \varepsilon\Delta^{m+1}u_h = &\sum_{\substack{b,k_1,...,k_{2m-1}\in\mathbb{N}_0 \\ k_1+2k_2+... \\ ...+(2m-1)k_{2m-1}=2m}} \left((D^bP_u)[u] * (Du)^{k_1} * ... * (D^{2m-1}u)^{k_{2m-1}}\right)_h \\
&+ \varepsilon \sum_{\substack{b,k_1,...,k_{2m+1}\in\mathbb{N}_0 \\ k_1+2k_2+... \\ ...+(2m+1)k_{2m+1}=2m+2}} \left((D^bP_u)[u] * (Du)^{k_1} * ... * (D^{2m+1}u)^{k_{2m+1}}\right)_h .
\end{aligned} \tag{3.3.10}$$

With the same calculation as before we have

$$\begin{aligned}
&(-1)^m \int_\Omega \left\langle \Delta^m u_h - \varepsilon\Delta^{m+1}u_h, \eta^{2(m+1)}(u_h - U_h)\right\rangle \\
&\ge \frac{1}{2}\int_\Omega \eta^{2(m+1)}\left(|D^m u_h|^2 + \varepsilon|D^{m+1}u_h|^2\right) - \int_\Omega D^{m-1}u_h D\left(\eta^{2(m+1)}D^m U_h\right) \\
&\quad - c\sum_{i=2}^{m-1} R^{2(m-i)}\int_{A_R} |D^i(u_h - U_h)|^2 - c\varepsilon\sum_{i=2}^{m} R^{2(m+1-i)}\int_{A_R} |D^i(u_h - U_h)|^2 \\
&\ge \frac{1}{2}\int_\Omega \eta^{2(m+1)}\left(|D^m u_h|^2 + \varepsilon|D^{m+1}u_h|^2\right) - c\int_{A_R} |D^{m+1}u_h|^2 - cR^{2\alpha},
\end{aligned} \tag{3.3.11}$$

where we used (3.3.3), (3.3.5), (3.3.8), (3.3.9) and

$$\begin{aligned}
\int_\Omega D^{m-1}u_h D\left(\eta^{2(m+1)}D^m U_h\right) &\le cR^{-2}\int_{A_R} |D^{m-1}u_h|^2 + cR^{2m}\left(⨍_{A_R} D^m u_h\right)^2 \\
&\le c\left(\int_{A_R} |D^{m-1}u_h|^{\frac{2m}{m-1}}\right)^{\frac{m-1}{m}} + c\int_{A_R} |D^m u_h|^2 \\
&\le cR^{2\alpha}.
\end{aligned}$$

We have to be a bit more careful with the right-hand side of (3.3.10). Note that

$$\begin{aligned}(D^i u(x)D^j u(x))_h &= \frac{1}{h}\Big[D^i u(x+he_\nu)D^j u(x+he_\nu) - D^i u(x)D^j u(x)\Big]\\ &= \frac{1}{h}\Big[\left(D^i u(x+he_\nu) - D^i(x)\right)D^j u(x+he_\nu) + D^i u(x)\left(D^j u(x+he_\nu) - D^j u(x)\right)\Big]\\ &= D^i u_h(x) D^j u(x+he_\nu) + D^i u(x) D^j u_h(x).\end{aligned}$$

Iteratively the same follows for a product of arbitrarily many factors

$$\begin{aligned}\left((Du)^{k_1} * \dots * (D^{2m+1}u)^{k_{2m+1}}\right)_h = {}& Du_h * (Du)^{k_1-1} * \dots * (D^{2m+1}u)^{k_{2m+1}}\\ &+ (Du)^{k_1} * D^2 u_h * (D^2u)^{k_2-1} * \dots * (D^{2m+1}u)^{k_{2m+1}}\\ &+ \dots\\ &+ (Du)^{k_1} * \dots * D^{2m+1}u_h * (D^{2m+1}u)^{k_{2m+1}-1}.\end{aligned}$$

As before we integrate by parts until the highest order terms are $D^m u$ and $D^{m+1}u$ respectively and estimate

$$\begin{aligned}&(-1)^m \int_\Omega \Big\langle \sum_{\substack{b,k_1,\dots,k_{2m-1}\in\mathbb{N}_0\\ k_1+2k_2+\dots\\ \dots+(2m-1)k_{2m-1}=2m}} \left((D^b P_u)[u] * (Du)^{k_1} * \dots * (D^{2m-1}u)^{k_{2m-1}}\right)_h\\
&\quad + \varepsilon \sum_{\substack{b,k_1,\dots,k_{2m+1}\in\mathbb{N}_0\\ k_1+2k_2+\dots\\ \dots+(2m+1)k_{2m+1}=2m+2}} \left((D^b P_u)[u] * (Du)^{k_1} * \dots * (D^{2m+1}u)^{k_{2m+1}}\right)_h, \eta^{2(m+1)}(u_h - U_h)\Big\rangle\\
&\le c\int_\Omega \sum_{\substack{k_i\in\mathbb{N}_0,\ 1\le t\le m\\ k_1+2k_2+\dots+mk_m=2m}} |Du|^{k_1}\cdot\dots\cdot|D^t u_h||D^t u|^{k_t-1}\cdot\dots\cdot|D^m u|^{k_m}\eta^{2(m+1)}|(u_h-U_h)|\\
&\quad + c\varepsilon\int_\Omega \sum_{\substack{k_i\in\mathbb{N}_0,\ 1\le t\le m+1\\ k_1+2k_2+\dots\\ +(m+1)k_{m+1}=2(m+1)}} |Du|^{k_1}\cdot\dots\cdot|D^t u_h||D^t u|^{k_t-1}\cdot\dots\cdot|D^{m+1} u|^{k_{m+1}}\eta^{2(m+1)}|(u_h-U_h)|\\
&\quad + c\int_\Omega \sum_{\substack{r\in\mathbb{N},\ s,k_i\in\mathbb{N}_0\\ k_m=1\\ 1\le r+s\le m-1\\ 1\le t\le m\\ k_1+2k_2+\dots\\ +mk_m+r+s=2m}} |Du|^{k_1}\cdot\dots\cdot|D^t u_h||D^t u|^{k_t-1}\cdot\dots\cdot|D^m u|^{k_m}|D^r(\eta^{2(m+1)})||D^s(u_h-U_h)|\\
&\quad + c\varepsilon\int_\Omega \sum_{\substack{r\in\mathbb{N},\ s,k_i\in\mathbb{N}_0\\ k_{m+1}=1\\ 1\le r+s\le m\\ 1\le t\le m+1\\ k_1+2k_2+\dots\\ +(m+1)k_{m+1}\\ +r+s=2(m+1)}} |Du|^{k_1}\cdot\dots\cdot|D^t u_h||D^t u|^{k_t-1}\cdot\dots\cdot|D^{m+1} u|^{k_{m+1}}|D^r(\eta^{2(m+1)})||D^s(u_h-U_h)|\\
&= I + II + III + IV.\end{aligned}$$

We estimate each term separately. With Young's inequality ($p_i = \frac{2m}{ik_i},\ i = 1,\dots,m$), (3.3.6), (3.3.8) and (3.3.9) we have

$$I = \int_\Omega \sum_{\substack{k_i\in\mathbb{N}_0,\ 1\le t\le m\\ k_1+2k_2+\dots+mk_m=2m}} |Du|^{k_1}\cdot\dots\cdot|D^t u_h||D^t u|^{k_t-1}\cdot\dots\cdot|D^m u|^{k_m}\eta^{2(m+1)}|(u_h-U_h)|$$

$$
\begin{aligned}
&\leq c||u_h - U_h||_{L^\infty(B_R)} \sum_{i=1}^{m} \int_{B_R} |D^i u|^{\frac{2m}{i}} + c||u_h - U_h||_{L^\infty(B_R)} \sum_{t=1}^{m} \int_{B_R} \left(|D^t u_h||D^t u|^{k_t-1}\right)^{\frac{2m}{tk_t}} \\
&\leq c \sum_{i=1}^{m} \int_{B_R} |D^i u|^{\frac{2m}{i}} + c \sum_{t=1}^{m} \int_{B_R} |D^t u_h|^{\frac{2m}{t}} \\
&\leq cR^{2\alpha}.
\end{aligned}
$$

The second term follows analogously with Young's inequality ($p_i = \frac{2(m+1)}{ik_i}$, $i = 1, ..., m+1$), (3.3.7), (3.3.8) and (3.3.9).

$$
\begin{aligned}
II &= \varepsilon \int_\Omega \sum_{\substack{k_i \in \mathbb{N}_0,\ 1 \leq t \leq m+1 \\ k_1 + 2k_2 + ... \\ +(m+1)k_{m+1} = 2(m+1)}} |Du|^{k_1} \cdot ... \cdot |D^t u_h||D^t u|^{k_t - 1} \cdot ... \cdot |D^{m+1} u|^{k_{m+1}} \eta^{2(m+1)} |(u_h - U_h)| \\
&\leq \delta \int_\Omega \eta^{2(m+1)} |D^{m+1} u_h|^2 + c_\delta \sum_{i=1}^{m+1} \int_{B_R} |D^i u|^{\frac{2(m+1)}{i}} + c \sum_{t=1}^{m} \int_{B_R} \left(|D^t u_h||D^t u|^{k_t - 1}\right)^{\frac{2(m+1)}{tk_t}} \\
&\leq \delta \int_\Omega \eta^{2(m+1)} |D^{m+1} u_h|^2 + c_\delta \sum_{i=1}^{m+1} \int_{B_R} |D^i u|^{\frac{2(m+1)}{i}} + c \sum_{t=1}^{m} \int_{B_R} |D^t u_h|^{\frac{2(m+1)}{t}} \\
&\leq \delta \left(\int_\Omega \eta^{2(m+1)} |D^{m+1} u_h|^2 \right) + cR^{\frac{2\alpha}{m}} \left(\int_{B_R} |D^{m+1} u_h|^2 \right) + c_\delta R^{2\alpha}.
\end{aligned}
$$

For the third term we use (3.3.5), (3.3.6), (3.3.8) and (3.3.9). With Hölder's inequality ($p_i = \frac{2m}{ik_i}$, $i = 1, ..., m$, $p_r = \frac{2m}{r}$, $p_s = \frac{2m}{s}$) we get

$$
\begin{aligned}
III &= \int_\Omega \sum_{\substack{r \in \mathbb{N},\ s, k_i \in \mathbb{N}_0 \\ k_m = 1 \\ 1 \leq r+s \leq m-1 \\ 1 \leq t \leq m \\ k_1 + 2k_2 + ... \\ +mk_m + r + s = 2m}} |Du|^{k_1} \cdot ... \cdot |D^t u_h||D^t u|^{k_t - 1} \cdot ... \cdot |D^m u|^{k_m} |D^r(\eta^{2(m+1)})||D^s(u_h - U_h)| \\
&\leq c \sum_{\substack{r \in \mathbb{N},\ s, k_i \in \mathbb{N}_0 \\ 1 \leq r+s \leq m-1 \\ 1 \leq t \leq m \\ k_1 + 2k_2 + ... \\ +mk_m + r + s = m}} \left[\prod_{\substack{i=1 \\ i \neq t}}^{m} \left(\int_{A_R} |D^i u|^{\frac{2m}{i}} \right)^{\frac{ik_i}{2m}} \left(\int_{A_R} |D^t u_h|^{\frac{2m}{tk_t}} |D^t u|^{\frac{2m(k_t - 1)}{tk_t}} \right)^{\frac{tk_t}{2m}} \right. \\
&\qquad \left. \cdot \left(\int_{A_R} |D^r(\eta^{2(m+1)})|^{\frac{2m}{r}} \right)^{\frac{r}{2m}} \left(\int_{A_R} |D^s(u_h - U_h)|^{\frac{2m}{s}} \right)^{\frac{s}{2m}} \right] \\
&\leq cR \sum_{\substack{r \in \mathbb{N},\ s, k_i \in \mathbb{N}_0 \\ 1 \leq r+s \leq m-1 \\ 1 \leq t \leq m \\ k_1 + 2k_2 + ... \\ +mk_m + r + s = m}} \left[\prod_{\substack{i=1 \\ i \neq t}}^{m} \left(\int_{A_R} |D^i u|^{\frac{2m}{i}} \right)^{\frac{ik_i}{2m}} \left(\int_{A_R} |D^t u_h|^{\frac{2m}{t}} \right)^{\frac{t}{2m}} \right. \\
&\qquad \left. \cdot \left(\int_{A_R} |D^t u|^{\frac{2m}{t}} \right)^{\frac{t(k_t - 1)}{2m}} \left(\int_{A_R} |D^{m+1} u_h|^2 \right)^{\frac{1}{2}} \right] \\
&\leq c \left(\int_{A_R} |D^{m+1} u_h|^2 \right) + cR^{2\alpha}.
\end{aligned}
$$

To estimate IV we use Cauchy-Schwarz and Hölder's inequality (if $r \geq 2$: $p_i = \frac{m}{ik_i}$, $i = 1, ..., m$, $p_r = \frac{m}{r-1}$, $p_s = \frac{m}{s}$; if $r = 1$: $p_i = \frac{m}{ik_i}$, $i = 1, ..., m$, $p_s = \frac{m}{s}$) as well as the Poincaré-Sobolev inequality in

(3.3.5), (3.3.6), (3.3.8) and (3.3.9).

$$IV = \varepsilon \int_\Omega \sum_{\substack{r\in\mathbb{N},\ s,k_i\in\mathbb{N}_0 \\ k_{m+1}=1 \\ 1\le r+s\le m \\ 1\le t\le m+1 \\ k_1+2k_2\ldots \\ +(m+1)k_{m+1}+ \\ +r+s=2(m+1)}} |Du|^{k_1}\cdot\ldots\cdot|D^t u_h||D^t u|^{k_t-1}\cdot\ldots\cdot|D^{m+1}u|^{k_{m+1}}|D^r(\eta^{2(m+1)})||D^s(u_h-U_h)|$$

$$\le c\int_{A_R}\left(|D^{m+1}u|^2+|D^{m+1}u_h|^2\right)$$

$$+ c\varepsilon \sum_{\substack{r\in\mathbb{N},\ s,k_i\in\mathbb{N}_0 \\ 1\le r+s\le m \\ 1\le t\le m \\ k_1+2k_2+\ldots+mk_m+ \\ +r+s=m+1}} \left[\prod_{\substack{i=1 \\ i\ne t}}^m \left(\int_{A_R}|D^i u|^{\frac{2m}{i}}\right)^{\frac{ik_i}{m}} \left(\int_{A_R}|D^t u_h|^{\frac{2m}{t}}\right)^{\frac{t}{m}} \left(\int_{A_R}|D^t u|^{\frac{2m}{t}}\right)^{\frac{t(k_t-1)}{m}}\right.$$

$$\left.\cdot\left(\int_{A_R}|D^r(\eta^{2(m+1)})|^{\frac{2m}{r-1}}\right)^{\frac{r-1}{m}}\left(\int_{A_R}|D^s(u_h-U_h)|^{\frac{2m}{s}}\right)^{\frac{s}{m}}\right]$$

$$+ c\varepsilon \sum_{\substack{r\in\mathbb{N},\ s,k_i\in\mathbb{N}_0 \\ 1\le r+s\le m \\ k_1+2k_2+\ldots+mk_m+ \\ +r+s=m+1}} \left[\prod_{i=1}^m \left(\int_{A_R}|D^i u|^{\frac{2m}{i}}\right)^{\frac{ik_i}{m}} \left(\int_{A_R}|D^r(\eta^{2(m+1)})|^{\frac{2m}{r-1}}\right)^{\frac{r-1}{m}}\right.$$

$$\left.\cdot\left(\int_{A_R}|D^s(u_h-U_h)|^{\frac{2m}{s}}\right)^{\frac{s}{m}}\right]$$

$$+ c\varepsilon R^{-2} \sum_{\substack{r=1,\ s,k_i\in\mathbb{N}_0 \\ 1\le 1+s\le m \\ 1\le t\le m \\ k_1+2k_2+\ldots+mk_m+ \\ +r+s=m+1}} \left[\prod_{\substack{i=1 \\ i\ne t}}^m \left(\int_{A_R}|D^i u|^{\frac{2m}{i}}\right)^{\frac{ik_i}{m}} \left(\int_{A_R}|D^t u_h|^{\frac{2m}{t}}\right)^{\frac{t}{m}} \left(\int_{A_R}|D^t u|^{\frac{2m}{t}}\right)^{\frac{t(k_t-1)}{m}}\right.$$

$$\left.\cdot\left(\int_{A_R}|D^s(u_h-U_h)|^{\frac{2m}{s}}\right)^{\frac{s}{m}}\right]$$

$$+ c\varepsilon R^{-2} \sum_{\substack{r=1,\ s,k_i\in\mathbb{N}_0 \\ 1\le 1+s\le m \\ k_1+2k_2+\ldots+mk_m+ \\ +r+s=m+1}} \prod_{i=1}^m \left(\int_{A_R}|D^i u|^{\frac{2m}{i}}\right)^{\frac{ik_i}{m}} \left(\int_{A_R}|D^s(u_h-U_h)|^{\frac{2m}{s}}\right)^{\frac{s}{m}}$$

$$\le c\left(\int_{A_R}|D^{m+1}u_h|^2\right)+cR^{2\alpha}$$

$$+ c\varepsilon \sum_{\substack{r\in\mathbb{N},\ s,k_i\in\mathbb{N}_0 \\ 1\le r+s\le m \\ 1\le t\le m \\ k_1+2k_2\ldots+mk_m+ \\ +r+s=m+1}} \left[\prod_{\substack{i=1 \\ i\ne t}}^m \left(\int_{A_R}|D^i u|^{\frac{2m}{i}}\right)^{\frac{ik_i}{m}} \left(\int_{A_R}|D^t u_h|^{\frac{2m}{t}}\right)^{\frac{t}{m}} \left(\int_{A_R}|D^t u|^{\frac{2m}{t}}\right)^{\frac{t(k_t-1)}{m}}\right.$$

$$\left.\cdot\left(\int_{A_R}|D^{m+1}u_h|^2\right)\right]$$

$$+ c\varepsilon \sum_{\substack{r\in\mathbb{N},\ s,k_i\in\mathbb{N}_0 \\ 1\le r+s\le m \\ 1\le i\le m \\ k_1+2k_2+...+mk_m+ \\ +r+s=m+1}} \prod_{i=1}^{m} \left(\int_{A_R} |D^i u|^{\frac{2m}{i}}\right)^{\frac{ik_i}{m}} \left(\int_{A_R} |D^{m+1}u_h|^2\right)$$
$$\le c\left(\int_{A_R} |D^{m+1}u_h|^2\right) + cR^{2\alpha}$$

$I - IV$ together yields

$$(-1)^m \int_\Omega \left\langle \Delta^m u_h - \varepsilon\Delta^{m+1}u_h, \eta^{2(m+1)}(u_h - U_h)\right\rangle$$
$$\le \delta\left(\int_\Omega \eta^{2(m+1)}|D^{m+1}u_h|^2\right) + c\left(\int_{A_R} |D^{m+1}u_h|^2\right) + c_\delta R^{2\alpha} + cR^{\frac{2\alpha}{m}} \int_{B_R} |D^{m+1}u_h|^2$$

and with (3.3.11) and $\delta > 0$ small enough

$$\int_{B_{\frac{R}{2}}} |D^m u_h|^2 + |D^{m+1}u_h|^2 \le c_1 \int_{A_R} |D^{m+1}u_h|^2 + cR^{\frac{2\alpha}{m}} \int_{B_R} |D^{m+1}u_h|^2 + cR^{2\alpha}.$$

Applying Lemma 2.1 in [23] as before yields

$$\int_{B_\rho} |D^m u_h|^2 + |D^{m+1}u_h|^2 \le c\rho^{2\alpha} \qquad \forall\ 0<\rho<\frac{R}{2}.$$

Letting $h \to 0$

$$\int_{B_\rho} |D^{m+1}u|^2 + |D^{m+2}u|^2 \le c\rho^{2\alpha} \qquad \forall\ 0<\rho<\frac{R}{2}$$

and thus $D^2 u \in C^{0,\alpha}(B_\rho)$.

We iterate further. Let $R < \rho$ and assume that for some $2 \le i \le m$

$$\int_{B_R} |D^{m+i}u|^2 \le cR^{2\alpha}.$$

Then

$$D^j u \in C^{0,\alpha}(B_R) \quad \text{for} \quad j = 0,...,i \qquad \text{and} \qquad D^j u \in L^{\frac{2m}{j-i}}(B_R) \quad \text{for} \quad j = i+1,...,m+i.$$

Further we have by the Gagliardo-Nirenberg inequality

$$\left(\int_{A_R} |D^j u|^{\frac{2m}{j-i}}\right)^{\frac{j-i}{m}} \le c\left(\int_{A_R} |D^{m+i}u|^2\right) + ||u||^2_{L^\infty(A_R)} \le cR^{2\alpha} \tag{3.3.12}$$

for $j = i+1,...,m+i-1$. Let U_h be a polynomial of degree at most $m+i-1$ so that

$$\int_{A_R} D^j\left(u_h - U_h\right) = 0 \qquad \forall\ j = 1,...,m+i-1,$$

where U_h is defined as earlier and $||D^{i-1}(u_h - U_h)||_{L^\infty(B_R)} \leq c$, since

$$|D^{i-1}U_h(x)| \leq c \sum_{i-1\leq\gamma+\beta\leq m} R^{\gamma+\beta-i+1} | \fint_{A_R} D^{\gamma+\beta} u_h |$$
$$\leq c \sum_{i-1\leq\gamma+\beta\leq m} R^{\gamma+\beta-i+1-2m} \left(\int_{B_R} |D^{\gamma+\beta+1} u|^{\frac{2m}{\gamma+\beta+1-i}} \right)^{\frac{\gamma+\beta+1-i}{2m}} R^{2m-\gamma-\beta-1+i} \leq c$$

for $x \in B_R$. Then

$$\begin{aligned}
(-1)^m & \int_\Omega \left\langle \Delta^m D^{i-1} u_h - \varepsilon \Delta^{m+1} D^{i-1} u_h, \eta^{2(m+1)} D^{i-1}\left(u_h - U_h\right) \right\rangle \\
&= \int_\Omega \left\langle D^{m+i-1} u_h, D^m \left(\eta^{2(m+1)} D^{i-1}(u_h - U_h) \right) \right\rangle \\
&\quad + \varepsilon \int_\Omega \left\langle D^{m+i} u_h, D^{m+1} \left(\eta^{2(m+1)} D^{i-1}(u_h - U_h) \right) \right\rangle \\
&\geq \frac{1}{2} \int_\Omega \eta^{2(m+1)} \left(|D^{m+i-1} u_h|^2 + \varepsilon |D^{m+i} u_h|^2 \right) - \int_\Omega D^{m+i-2} u_h D \left(\eta^{2(m+1)} D^{m+i-1} U_h \right) \\
&\quad - c \sum_{j=0}^{m-1} \int_\Omega |D^{m-j}\eta|^2 |D^{i-1+j}(u_h - U_h)|^2 - c \sum_{j=0}^{m} \int_\Omega |D^{m-j+1}\eta|^2 |D^{i-1+j}(u_h - U_h)|^2 \\
&\geq \frac{1}{2} \int_\Omega \eta^{2(m+1)} \left(|D^{m+i-1} u_h|^2 + \varepsilon |D^{m+i} u_h|^2 \right) - c \int_{A_R} |D^{m+i-1} u_h|^2 \\
&\quad - c \left(\int_{A_R} |D^{m+i-2} u_h|^{\frac{2m}{m-1}} \right)^{\frac{m-1}{m}} - c \sum_{j=0}^{m-1} R^{-2(m-j)} \int_{A_R} |D^{i-1+j}(u_h - U_h)|^2 \\
&\quad - c \sum_{j=0}^{m} R^{-2(m-j+1)} \int_{A_R} |D^{i-1+j}(u_h - U_h)|^2 .
\end{aligned}$$

With the Poincaré inequality we get for the last two terms

$$\int_{A_R} |D^{i-1+j}(u_h - U_h)|^2 \leq R^2 \int_{A_R} |D^{i+j}(u_h - U_h)|^2 \leq ... \leq R^{2(m-j+1)} \int_{A_R} |D^{m+i} u_h|^2$$

and with the Gagliardo-Nirenberg inequality

$$\left(\int_{A_R} |D^{m+i-2} u_h|^{\frac{2m}{m-1}} \right)^{\frac{m-1}{m}} \leq \int_{A_R} |D^{m+i} u|^2 + ||u||^2_{L^\infty(A_R)} \leq c R^{2\alpha}.$$

Thus

$$\begin{aligned}
\int_\Omega & \eta^{2(m+1)} \left(|D^{m+i-1} u_h|^2 + \varepsilon |D^{m+i} u_h|^2 \right) \\
&\leq (-1)^m \int_\Omega \eta^{2(m+1)} \left\langle \Delta^m D^{i-1} u_h - \varepsilon \Delta^{m+1} D^{i-1} u_h, D^{i-1}\left(u_h - U_h\right) \right\rangle \\
&\quad + c \int_{A_R} |D^{m+i} u_h|^2 + c R^{2\alpha}.
\end{aligned}$$

On the other hand we have (see Lemma 3.4.1)

$$(-1)^m \int_\Omega \eta^{2(m+1)} \left\langle \Delta^m D^{i-1} u_h - \varepsilon \Delta^{m+1} D^{i-1} u_h, D^{i-1}\left(u_h - U_h\right) \right\rangle$$

$$
\begin{aligned}
&\leq c \sum_{\substack{1\leq t\leq m+i\\ k_j\in\mathbb{N}_0\\ k_1+\ldots+(m+i)k_{m+i}\\ =2m+i-1}} \int_\Omega \eta^{2(m+1)}|Du|^{k_1}\cdot\ldots\cdot|D^t u_h||D^t u|^{k_t-1}\cdot\ldots\cdot|D^{m+i}u|^{k_{m+i}}|D^{i-1}(u_h-U_h)|\\
&+ c\varepsilon \sum_{\substack{1\leq t\leq m+i\\ k_j\in\mathbb{N}_0\\ k_1+\ldots+(m+i)k_{m+i}\\ =2(m+1)+i-1}} \int_\Omega \eta^{2(m+i)}|Du|^{k_1}\cdot\ldots\cdot|D^t u_h||D^t u|^{k_t-1}\cdot\ldots\cdot|D^{m+i}u|^{k_{m+i}}|D^{i-1}(u_h-U_h)|\\
&+ c \sum_{\substack{1\leq t\leq m+i\\ k_j,s\in\mathbb{N}_0,\ r\in\mathbb{N}\\ k_{m+i}=1\\ 1\leq r+s\leq m-1\\ k_1+\ldots\\ +(m+i)k_{m+i}\\ +r+s\\ =2m+i-1}} \int_\Omega |Du|^{k_1}\cdot\ldots\cdot|D^t u_h||D^t u|^{k_t-1}\cdot\ldots\cdot|D^{m+i}u|^{k_{m+i}}|D^r(\eta^{2(m+1)})||D^{s+i-1}(u_h-U_h)|\\
&+ c\varepsilon \sum_{\substack{1\leq t\leq m+i\\ k_j,s\in\mathbb{N}_0,\ r\in\mathbb{N}\\ k_{m+i}=1\\ 1\leq r+s\leq m\\ k_1+\ldots\\ +(m+i)k_{m+i}\\ +r+s\\ =2(m+1)+i-1}} \int_\Omega |Du|^{k_1}\cdot\ldots\cdot|D^t u_h||D^t u|^{k_t-1}\cdot\ldots\cdot|D^{m+i}u|^{k_{m+i}}|D^r(\eta^{2(m+1)})||D^{s+i-1}(u_h-U_h)|\\
&=: I + II + III + IV.
\end{aligned}
$$

If $k_j = 0\ \forall j = i, ..., m+i-1$, then II consists only of terms $D^j u \in C^{1,\alpha}$ and we estimate these terms by their L^∞-norm.[1] If $k_j \neq 0$ for some $j = i, ..., m+i$, we use Hölder's inequality with $p_i = \frac{2m}{k_i}$, $p_j = \frac{2m}{k_j(j-i)}$, $j = i+1, ..., m+i$. Note that

$$
0 \leq \sum_{j=i}^{m+i} \frac{1}{p_j} = \frac{2m+1+i-\sum_{j=1}^{i-1} jk_j - (i-1)k_i - i\sum_{j=i+1}^{m+i} k_j}{2m} \leq 1,
$$

since $\sum_{j=1}^{m+i} k_j \geq 2$ and $\sum_{j=1}^{m+i} jk_j = 2m+i+1$. Together with (3.3.12) we have

$$
\begin{aligned}
II &= \varepsilon \sum_{\substack{k_j\in\mathbb{N}_0\\ 1\leq t\leq m+i\\ k_1+\ldots+(m+i)k_{m+i}\\ =2(m+1)+i-1}} \int_\Omega \eta^{2(m+1)}|Du|^{k_1}\cdot\ldots\cdot|D^t u_h||D^t u|^{k_t-1}\cdot\ldots\cdot|D^{m+i}u|^{k_{m+i}}|D^{i-1}(u_h-U_h)|\\
&\leq cR^{2m} \sum_{\substack{k_j\in\mathbb{N}_0\\ 1\leq t\leq i-1\\ k_1+\ldots+(i-1)k_{i-1}\\ =2(m+1)+i-1}} \prod_{j=1}^{i-1} R^{\alpha k_j}\\
&\quad + c \sum_{\substack{k_j\in\mathbb{N}_0\\ 1\leq t\leq i-1\\ k_1+\ldots+(m+i)k_{m+i}\\ =2(m+1)+i+1}} \prod_{j=1}^{i-1} R^{\alpha k_j} \left(\int_{B_R} |D^i u|^{2m}\right)^{\frac{k_i}{2m}} \prod_{j=i+1}^{m+i} \left(\int_{B_R} |D^j u|^{\frac{2m}{j-i}}\right)^{\frac{k_j(j-i)}{2m}}
\end{aligned}
$$

[1] Note that we can estimate the L^∞-norm of $D^j u$, $j = 1, ..., i$ on B_R by cR^α, since we can assume wlog that $D^j u(0) = 0$, $j = 1, ..., m-1$. This holds until the last induction step, where $i = m$. Here $D^m u \in C^{0,\alpha}$ but $D^m u(0) \neq 0$ and $||D^m u||_{L^\infty(B_R)} \leq c$ for some constant c. However our estimates still hold in this case because all remaining factors can be estimated by at least $R^{2\alpha}$. For the sake of legibility we omit the case $i = m$ and only write the case $i < m$.

$$
\begin{aligned}
&+ c \sum_{\substack{k_j\in\mathbb{N}_0 \\ t=i \\ k_1+\ldots+(m+i)k_{m+i} \\ =2(m+1)+i+1}} \prod_{j=1}^{i-1} R^{\alpha k_j} \prod_{j=i+1}^{m+i} \left(\int_{B_R} |D^j u|^{\frac{2m}{j-i}}\right)^{\frac{k_j(j-i)}{2m}} \left(\int_{B_R} (|D^i u_h||D^i u|^{k_i-1})^{\frac{2m}{k_i}}\right)^{\frac{k_i}{2m}} \\
&+ c \sum_{\substack{k_j\in\mathbb{N}_0 \\ i+1\le t\le m+i \\ k_1+\ldots+(m+i)k_{m+i} \\ =2(m+1)+i+1}} \Bigg[\prod_{j=1}^{i-1} R^{\alpha k_j} \prod_{\substack{j=i+1 \\ j\neq t}}^{m+i} \left(\int_{B_R} |D^j u|^{\frac{2m}{j-i}}\right)^{\frac{k_j(j-i)}{2m}} \\
&\qquad \cdot \left(\int_{B_R} (|D^t u_h||D^t u|^{k_t-1})^{\frac{2m}{k_t(t-i)}}\right)^{\frac{k_t(t-i)}{2m}} \left(\int_{B_R} |D^i u|^{2m}\right)^{\frac{k_i}{2m}}\Bigg] \\
&\le cR^{2\alpha} + c \sum_{\substack{k_j\in\mathbb{N}_0 \\ t=i \\ k_1+\ldots+(m+i)k_{m+i} \\ =2(m+1)+i+1}} \Bigg[\prod_{j=1}^{i-1} R^{\alpha k_j} \prod_{j=i+1}^{m+i} \left(\int_{B_R} |D^j u|^{\frac{2m}{j-i}}\right)^{\frac{k_j(j-i)}{2m}} \\
&\qquad \cdot \left(\int_{B_R} |D^i u_h|^{2m}\right)^{\frac{1}{2m}} \left(\int_{A_R} |D^i u|^{2m}\right)^{\frac{k_i-1}{2m}}\Bigg] \\
&+ c \sum_{\substack{k_j\in\mathbb{N}_0, \\ i+1\le t\le m+i \\ k_1+\ldots+(m+i)k_{m+i} \\ =2(m+1)+i+1}} \Bigg[\prod_{j=1}^{i-1} R^{\alpha k_j} \prod_{\substack{j=i+1 \\ j\neq t}}^{m+i} \left(\int_{B_R} |D^j u|^{\frac{2m}{j-i}}\right)^{\frac{k_j(j-i)}{2m}} \left(\int_{B_R} |D^t u_h|^{\frac{2m}{t-i}}\right)^{\frac{t-i}{2m}} \\
&\qquad \cdot \left(\int_{B_R} |D^t u|^{\frac{2m}{t-i}}\right)^{\frac{(k_t-1)(t-i)}{2m}} \left(\int_{B_R} |D^i u|^{2m}\right)^{\frac{k_i}{2m}}\Bigg] \\
&\le cR^{\alpha} \int_{B_R} |D^{m+i} u_h|^2 + cR^{2\alpha}.
\end{aligned}
$$

I can be estimated in the same way with Hölder's inequality ($p_j = \frac{2m}{k_j(j+1-i)}$, $j = i, ..., m+i$), since

$$
0 \le \sum_{j=i}^{m+i} \frac{1}{p_j} = \frac{2m + i - 1 - \sum_{j=1}^{i-1} jk_j + (1-i)\sum_{j=i}^{m+i} k_j}{2m} \le 1.
$$

if $k_j \neq 0$ for some $j = i, ..., m+i$. With (3.3.12)

$$
\begin{aligned}
I &= \sum_{\substack{k_j\in\mathbb{N}_0, \\ 1\le t\le m+i, \\ k_1+\ldots+(m+i)k_{m+i} \\ =2m+i-1}} \int_\Omega \eta^{2(m+1)} |Du|^{k_1} \cdot \ldots \cdot |D^t u_h||D^t u|^{k_t-1} \cdot \ldots \cdot |D^{m+i} u|^{k_{m+i}} |D^{i-1}(u_h - U_h)| \\
&\le cR^{2m} \sum_{\substack{1\le t\le i-1, \\ k_j\in\mathbb{N}_0, \\ k_1+\ldots+(i-1)k_{i-1} \\ =2m+i-1}} \prod_{j=1}^{i-1} R^{\alpha k_j} \\
&\quad + c \sum_{\substack{k_j\in\mathbb{N}_0, \\ 1\le t\le i-1 \\ k_1+\ldots \\ +(m+i)k_m=2m+i-1}} \prod_{j=1}^{i-1} R^{\alpha k_j} \prod_{j=i}^{m+i} \left(\int_{B_R} |D^j u|^{\frac{2m}{j+1-i}}\right)^{\frac{k_j(j+1-i)}{2m}}
\end{aligned}
$$

$$+ c \sum_{\substack{k_j\in\mathbb{N}_0,\\ i\le t\le m+i\\ k_1+\ldots\\ +(m+i)k_m=2m+i-1}} \left[\prod_{j=1}^{i-1} R^{\alpha k_j} \prod_{\substack{j=i\\ j\neq t}}^{m+i} \left(\int_{B_R} |D^j u|^{\frac{2m}{j+1-i}} \right)^{\frac{k_j(j+1-i)}{2m}} \left(\int_{B_R} |D^t u_h|^{\frac{2m}{t+1-i}} \right)^{\frac{t+1-i}{2m}} \right.$$

$$\left. \cdot \left(\int_{B_R} |D^t u|^{\frac{2m}{t+1-i}} \right)^{\frac{(k_t-1)(t+1-i)}{2m}} \right]$$

$$\le cR^\alpha \int_{B_R} |D^{m+i}u_h|^2 + cR^{2\alpha}.$$

To estimate IV we assume first that the difference quotient falls onto $D^{m+i}u$. As before we apply Cauchy-Schwarz and Hölder's inequality with $p_j = \frac{m}{k_j(j-i)}$, $j = i+1, ..., m+i-1$, $p_r = \frac{m}{r}$, $p_s = \frac{m}{s}$. Note that

$$0 \le \sum_{j=i+1}^{m+i-1} \frac{1}{p_j} + \frac{1}{p_r} + \frac{1}{p_s} = \frac{m+1-\sum_{j=1}^{i} jk_j - i\sum_{j=i+1}^{m+i-1} k_j}{m} \le 1.$$

With (3.3.12) we have

$$IV_a = \sum_{\substack{k_j,s\in\mathbb{N}_0,\ r\in\mathbb{N}\\ k_{m+i}=1\\ 1\le r+s\le m\\ k_1+\ldots\\ +(m+i)k_{m+i}+r+s\\ =2(m+1)+i-1}} \int_\Omega |Du|^{k_1}\cdot\ldots\cdot|D^{m+i}u_h|^{k_{m+i}}|D^r(\eta^{2(m+1)})||D^{s+i-1}(u_h-U_h)|$$

$$\le \int_{A_R} |D^{m+i}u_h|^2$$

$$+ c \sum_{\substack{k_j,s\in\mathbb{N}_0,\ r\in\mathbb{N}\\ 1\le r+s\le m\\ k_1+\ldots+(m+i-1)k_{m+i-1}\\ +r+s=m+1}} \int_{A_R} |Du|^{2k_1}\cdot\ldots\cdot|D^{m+i-1}u|^{2k_{m+i-1}}|D^r\eta|^2|D^{s+i-1}(u_h-U_h)|^2$$

$$\le \int_{A_R} |D^{m+i}u_h|^2$$

$$+ c \sum_{\substack{k_j,s\in\mathbb{N}_0,\ r\in\mathbb{N}\\ 1\le r+s\le m\\ k_1+\ldots+(m+i-1)k_{m+i-1}\\ +r+s=m+1}} \left[\prod_{j=1}^{i} R^{\alpha k_j} \prod_{j=i+1}^{m+i-1} \left(\int_{A_R} |D^j u|^{\frac{2m}{j-i}} \right)^{\frac{k_j(j-i)}{m}} \left(\int_{A_R} |D^r\eta|^{\frac{2m}{r}} \right)^{\frac{r}{m}} \right.$$

$$\left. \cdot \left(\int_{A_R} |D^{s+i-1}(u_h-U_h)|^{\frac{2m}{s}} \right)^{\frac{s}{m}} \right]$$

$$\le c\int_{A_R} |D^{m+i}u_h|^2$$

$$+ cR^2 \sum_{\substack{k_j,s\in\mathbb{N}_0,\ r\in\mathbb{N}\\ 1\le r+s\le m\\ k_1+\ldots+(m+i-1)k_{m+i-1}\\ +r+s=m+1}} \prod_{j=1}^{i} R^{\alpha k_j} \prod_{j=i+1}^{m+i-1} \left(\int_{A_R} |D^j u|^{\frac{2m}{j-i}} \right)^{\frac{k_j(j-i)}{m}} \left(\int_{A_R} |D^{m+i}u_h|^2 \right)$$

$$\le c\int_{A_R} |D^{m+i}u_h|^2,$$

where we used the Poincaré-Sobolev inequality in the second to last step (see (3.3.5))

$$\left(\int_{A_R} |D^{s+i-1}(u_h - U_h)|^{\frac{2m}{s}}\right)^{\frac{s}{m}} \leq c\int_{A_R} |D^{m+i-1}(u_h - U_h)|^2 \leq cR^2 \int_{A_R} |D^{m+i}u_h|^2.$$

If the difference quotient falls onto a term $D^j u$ with $1 \leq j < m+i$, we apply Cauchy-Schwarz and Hölder's inequality with $p_j = \frac{m}{(j+1-i)k_j}$, $j = i, ..., m+i-1$, $p_r = \frac{m}{r}$, $p_s = \frac{m}{s}$. Note that

$$0 \leq \sum_{j=i}^{m+i-1} \frac{1}{p_j} + \frac{1}{p_r} + \frac{1}{p_s} = \frac{m+1-\sum_{j=1}^{i-1} jk_j - \sum_{j=i}^{m+i-1}(i-1)k_j}{m} \leq 1.$$

Then

$$IV_b = \sum_{\substack{k_j,s\in\mathbb{N}_0,\ r\in\mathbb{N} \\ k_{m+i}=1 \\ 1\leq r+s\leq m+1 \\ 1\leq t\leq m+i-1 \\ k_1+\ldots \\ +(m+i)k_{m+i} \\ +r+s \\ =2(m+1)+i-1}} \int_\Omega |Du|^{k_1} \cdot \ldots \cdot |D^t u_h||D^t u|^{k_t-1} \cdot \ldots \cdot |D^{m+i}u|^{k_{m+i}} |D^r(\eta^{2(m+1)})||D^{s+i-1}(u_h - U_h)|$$

$$\leq c\int_{A_R} |D^{m+i}u|^2$$

$$+ c \sum_{\substack{k_j,s\in\mathbb{N}_0,r\in\mathbb{N} \\ 1\leq r+s\leq m+1 \\ 1\leq t\leq i-1 \\ k_1+\ldots+(m+i-1)k_{m+i-1} \\ +r+s=m+1}} \left[\prod_{j=1}^{i-1} R^{2\alpha k_j} \prod_{j=i}^{m+i-1} \left(\int_{A_R} |D^j u|^{\frac{2m}{j+1-i}}\right)^{\frac{m}{k_j(j+1-i)}} \left(\int_{A_R} |D^r\eta|^{\frac{2m}{r}}\right)^{\frac{r}{m}} \right.$$

$$\left. \cdot \left(\int_{A_R} |D^{s+i-1}(u_h - U_h)|^{\frac{2m}{s}}\right)^{\frac{s}{m}}\right]$$

$$+ c \sum_{\substack{k_j,s\in\mathbb{N}_0,r\in\mathbb{N} \\ 1\leq r+s\leq m+1 \\ i\leq t\leq m+i-1 \\ k_1+\ldots \\ +(m+i-1)k_{m+i-1} \\ +r+s=m+1}} \left[\prod_{j=1}^{i-1} R^{2\alpha k_j} \prod_{\substack{j=i \\ j\neq t}}^{m+i-1} \left(\int_{A_R} |D^j u|^{\frac{2m}{j+1-i}}\right)^{\frac{m}{k_j(j+1-i)}} \left(\int_{A_R} |D^r\eta|^{\frac{2m}{r}}\right)^{\frac{r}{m}} \right.$$

$$\left. \cdot \left(\int_{A_R} (|D^t u_h||D^t u|^{k_t-1})^{\frac{2m}{k_t(t+1-i)}}\right)^{\frac{k_t(t+1-i)}{m}} \left(\int_{A_R} |D^{s+i-1}(u_h - U_h)|^{\frac{2m}{s}}\right)^{\frac{s}{m}}\right]$$

$$\leq cR^{2\alpha}$$

$$+ cR^2 \sum_{\substack{k_j,s\in\mathbb{N}_0,r\in\mathbb{N} \\ 1\leq r+s\leq m+1 \\ 1\leq t\leq i-1 \\ k_1+\ldots+(m+i-1)k_{m+i-1} \\ +r+s=m+1}} \prod_{j=1}^{i-1} R^{2\alpha k_j} \prod_{j=i}^{m+i-1} \left(\int_{A_R} |D^j u|^{\frac{2m}{j+1-i}}\right)^{\frac{m}{k_j(j+1-i)}} \left(\int_{A_R} |D^{m+i}u_h|^2\right)$$

$$+ cR^2 \sum_{\substack{k_j,s\in\mathbb{N}_0,\ r\in\mathbb{N}\\ 1\le r+s\le m\\ i\le t\le m+i-1\\ k_1+\ldots+(m+i-1)k_{m+i-1}\\ +r+s=m+1}} \left[\prod_{j=1}^{i-1} R^{2\alpha k_j} \prod_{\substack{j=i\\ j\neq t}}^{m+i-1} \left(\int_{A_R} |D^j u|^{\frac{2m}{j+1-i}}\right)^{\frac{m}{k_j(j+1-i)}} \left(\int_{A_R} |D^t u_h|^{\frac{2m}{t+1-i}}\right)^{\frac{t+1-i}{m}}\right.$$
$$\left.\cdot \left(\int_{A_R} |D^t u|^{\frac{2m}{t+1-i}}\right)^{\frac{(k_t-1)(t+1-i)}{m}} \left(\int_{A_R} |D^{m+i}u_h|^2\right)\right]$$
$$\le c\int_{A_R} |D^{m+i}u_h|^2 + cR^{2\alpha}$$

Analogously we estimate III.

$$III = \sum_{\substack{r\in\mathbb{N},\ s,k_j\in\mathbb{N}_0\\ 1\le r+s\le m\\ 1\le t\le m+i\\ k_1+\ldots\\ +(m+i)k_{m+i}\\ +r+s\\ =2m+i-1}} \int_\Omega |Du|^{k_1}\cdot\ldots\cdot|D^t u_h||D^t u|^{k_t-1}\cdot\ldots\cdot|D^{m+i}u|^{k_{m+i}}|D^r(\eta^{2(m+1)})||D^{s+i-1}(u_h-U_h)|$$
$$\le c\int_{A_R} |D^{m+i}u_h|^2 + cR^{2\alpha}.$$

All in all we arrive at

$$\int_{B_{\frac{R}{2}}} |D^{m+i}u_h|^2 \le c\int_{A_R} |D^{m+i}u_h|^2 + cR^\alpha \int_{B_R} |D^{m+i}u_h|^2 + cR^{2\alpha}$$

With the same argument as before we get

$$\int_{B_\rho} |D^{m+i}u_h|^2 \le c\rho^{2\alpha} \qquad \forall\ 0<\rho<\frac{R}{2}.$$

Letting $h\to 0$ we have

$$\int_{B_\rho} |D^{m+i+1}u|^2 \le c\rho^{2\alpha} \qquad \text{and} \qquad D^{i+1}u\in C^{0,\alpha}(B_\rho).$$

We iterate until $D^{m+1}u\in C^{0,\alpha}_{loc}$, then classical Schauder estimates show that u is smooth.

□

3.4 Small energy regularity

In this section we want to bound higher derivatives of u on a small ball B_R in terms of the radius R. These estimates will be useful for the analysis of the neck region in section 3.6 and 3.7.

In the following we let $\varepsilon > 0$ be fixed and consider critical points u_ε of E_ε with uniformly bounded energy

$$E_\varepsilon(u_\varepsilon) \le c. \tag{3.4.1}$$

We define a smooth cut-off function $\eta \in C_c^\infty(\Omega)$ satisfying

$$\begin{gathered} 0 \le \eta \le 1, \quad \eta = 1 \text{ on } B_R(x_0), \quad \eta = 0 \text{ on } \Omega \setminus B_{2R}(x_0), \\ \|D^k \eta\|_{L^\infty(\Omega)} \le \frac{c}{R^k} \quad k \in \mathbb{N}, \end{gathered} \tag{3.4.2}$$

where $x_0 \in \Omega$ and $0 < R < \frac{1}{4}\mathrm{inj}_\Omega(x_0)$. Further we define the local energy

$$\tilde{E}_\varepsilon(u, B_R(x_0)) = \int_{B_R(x_0)} \left(|D^m u|^2 + \varepsilon |D^{m+1} u|^2\right) + \int_{B_R(x_0)} \sum_{i=1}^{m-1} |D^i u|^{\frac{2m}{i}}. \tag{3.4.3}$$

Globally we have

$$E_\varepsilon(u, \Omega) \le \tilde{E}_\varepsilon(u, \Omega) \le c E_\varepsilon(u, \Omega) \tag{3.4.4}$$

due to Sobolev embeddings. In a first step we want to calculate $(\Delta^m - \varepsilon \Delta^{m+1}) D^k u_\varepsilon$ for $k \in \mathbb{N}$. We denote by $G_j^i(Du)$ all terms of the form

$$G_j^i(Du) = \sum_{\substack{s, k_1 \dots k_j \in \mathbb{N}_0 \\ k_1 + .. + k_j = i,\, s \le i}} (D^s P_u)[u] * D^{k_1}(Du) * \dots * D^{k_j}(Du).$$

Differentiating gives

$$D(G_j^i(Du)) = G_j^{i+1}(Du) + G_{j+1}^i(Du).$$

Further let

$$\kappa_p(u, A) = \sum_{k=1}^p \int_A |D^k u|^{\frac{2p}{k}}.$$

By induction we get

Lemma 3.4.1. *Let $u_\varepsilon \in C^\infty(\Omega, N)$ be a solution of (3.1.3). Then*

$$(\Delta^m - \varepsilon \Delta^{m+1}) D^k u_\varepsilon = \sum_{\substack{i+j = k+2m \\ j \ge 2}} G_j^i(Du_\varepsilon) + \varepsilon \sum_{\substack{i+j = k+2(m+1) \\ j \ge 2}} G_j^i(Du_\varepsilon) \tag{3.4.5}$$

for every $k \in \mathbb{N}_0$.

The following Lemma is a $2m$-dimensional version of Struwe ([74], Lemma 6.7).

Lemma 3.4.2. *Let $x_0 \in \Omega$ and $R > 0$ such that $B_{2R}(x_0) \subset \Omega$. Further let $\eta \in C_c^\infty(\Omega)$ as above and let $f \in C^\infty(B_{2R}(x_0), N)$. Then*

$$\int_{B_{2R}(x_0)} \eta^2 |f|^{\frac{2(m+1)}{m}} \le c \left(\int_{\mathrm{spt}\,\eta} |f|^2 \right)^{\frac{1}{m}} \left(\int_{B_{2R}(x_0)} \eta^2 |Df|^2 + \frac{c}{R^2} \int_{\mathrm{spt}\,\eta} |f|^2 \right)$$

for some $c > 0$ independent of R.

Proof. For $\Omega \subset \mathbb{R}^{2m}$ the Sobolev embedding yields $W^{1,\frac{2m}{m+1}}(\Omega) \hookrightarrow L^2(\Omega)$. With $g = \eta f^{\frac{m+1}{m}}$ we have

$$\begin{aligned} \int_\Omega \eta^2 |f|^{\frac{2(m+1)}{m}} &= \int_\Omega |g|^2 \le c \left(\int_\Omega |Dg|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{m}} \\ &\le c \left(\int_\Omega \eta^{\frac{2m}{m+1}} |Df|^{\frac{2m}{m+1}} |f|^{\frac{2}{m+1}} \right)^{\frac{m+1}{m}} + c \left(\int_{\mathrm{spt}\,\eta} |D\eta|^{\frac{2m}{m+1}} |f|^2 \right)^{\frac{m+1}{m}} \end{aligned}$$

$$\leq c\Big(\int_\Omega \eta^2|Df|^2\Big)\Big(\int_{\operatorname{spt}\eta}|f|^2\Big)^{\frac{1}{m}}+\frac{c}{R^2}\Big(\int_{\operatorname{spt}\eta}|f|^2\Big)^{\frac{m+1}{m}},$$

where we used Hölder's inequality with $p_1=\frac{m+1}{m}$ and $p_2=m+1$.

□

Note that in the following we always assume $\varepsilon<R^2$.

Corollary 3.4.3. *Let $u\in C_c^\infty(\Omega,N)$. For all $R>0$ and $x_0\in\Omega$ such that $B_{2R}(x_0)\subset\Omega$, $\tilde{E}(u,B_{2R}(x_0))<1$ and $\varepsilon<R^2$ we have*

$$\varepsilon\int_{B_R(x_0)}\sum_{i=1}^{m+1}|D^iu|^{\frac{2(m+1)}{i}}\leq c\tilde{E}_\varepsilon(u,B_{2R}(x_0)). \tag{3.4.6}$$

Proof. Without loss of generality we set $x_0=0$. The case $i=m+1$ follows from (3.4.3). For $i=m$ we use Lemma 3.4.2 and (3.4.3) to estimate

$$\begin{aligned}\int_\Omega \eta^2|D^mu|^{\frac{2(m+1)}{m}}&\leq c\Big(\int_{\operatorname{spt}\eta}|D^mu|^2\Big)^{\frac{1}{m}}\Big(\int_\Omega\eta^2|D^{m+1}u|^2+\frac{c}{R^2}\int_{\operatorname{spt}\eta}|D^mu|^2\Big)\\&\leq\frac{c}{\varepsilon}\tilde{E}_\varepsilon(u,B_{2R}).\end{aligned}$$

For all other cases we use an induction argument. Assume (3.4.6) holds for $i+1$, $1\leq i<m$. Then Lemma 3.4.2 and Young's inequality ($p_1=\frac{m+1}{i+1}$, $p_2=\frac{m+1}{m-i}$) yield

$$\begin{aligned}\int_\Omega\eta^2|D^iu|^{\frac{2(m+1)}{i}}&\leq c\Big(\int_{\operatorname{spt}\eta}|D^iu|^{\frac{2m}{i}}\Big)^{\frac{1}{m}}\Big(\int_\Omega\eta^2|D^{i+1}u|^2|D^iu|^{\frac{2(m-i)}{i}}+\frac{c}{R^2}\int_{\operatorname{spt}\eta}|D^iu|^{\frac{2m}{i}}\Big)\\&\leq c\left(\tilde{E}_\varepsilon(u,B_{2R})\right)^{\frac{1}{m}}\left(\int_\Omega\eta^2|D^{i+1}u|^{\frac{2(m+1)}{i+1}}+\frac{c}{R^2}\tilde{E}_\varepsilon(u,B_{2R})\right)\\&\quad+\delta\int_\Omega\eta^2|D^iu|^{\frac{2(m+1)}{i}}\\&\leq\frac{c}{\varepsilon}\tilde{E}_\varepsilon(u,B_{2R})+\delta\int_\Omega\eta^2|D^iu|^{\frac{2(m+1)}{i}}.\end{aligned}$$

Choosing $\delta>0$ small enough and absorbing this term to the left hand side finishes the proof.

□

Proposition 3.4.4. *For all $\varepsilon,R>0$ such that $\varepsilon<R^2$ there exists $0<\delta_0<1$ and $c>0$ such that if $u_\varepsilon(\Omega,N)$ is a solution of (3.1.3) and $\tilde{E}_\varepsilon(u_\varepsilon,B_{4R}(x_0))<\delta_0$, then*

$$\varepsilon\kappa_{m+2}(u_\varepsilon,B_R(x_0))+\kappa_{m+1}(u_\varepsilon,B_R(x_0))\leq\frac{c}{R^2}\left(\tilde{E}_\varepsilon(u_\varepsilon,B_{4R}(x_0))\right)^{\frac{1}{m}}.$$

Proof. Wlog $x_0=0$. Note that $\kappa_{m+1}(u_\varepsilon)\leq\frac{c}{R^2}\tilde{E}_\varepsilon(u_\varepsilon,B_{4R})$ does not follow directly from Corollary 3.4.3, since $\varepsilon<R^2$. Let u_ε be a solution of (3.1.3). By Lemma 3.4.1 we have

$$\begin{aligned}\int_\Omega\eta^{2m+2}\langle(\Delta^m-\varepsilon\Delta^{m+1})Du_\varepsilon,Du_\varepsilon\rangle&=\int_\Omega\eta^{2m+2}\Big\langle\sum_{\substack{i+j=1+2m\\ j\geq2}}G_j^i(Du_\varepsilon),Du_\varepsilon\Big\rangle\\&\quad+\varepsilon\int_\Omega\eta^{2m+2}\Big\langle\sum_{\substack{i+j=1+2(m+1)\\ j\geq2}}G_j^i(Du_\varepsilon),Du_\varepsilon\Big\rangle.\end{aligned}\tag{3.4.7}$$

First we estimate the left-hand side from below using integration by parts at most $(m+1)$-times, the Leibniz rule as well as Young's and Hölder's inequality ($p_i = \frac{m}{i+1}$, $q_i = \frac{m}{m-i-1}$, $i = 1, ..., m-1$)

$$\begin{aligned}
&(-1)^m \int_\Omega \eta^{2m+2}\langle(\Delta^m - \varepsilon\Delta^{m+1})Du_\varepsilon, Du_\varepsilon\rangle \\
&\geq \int_\Omega \eta^{2m+2}\big(|D^{m+1}u_\varepsilon|^2 + \varepsilon|D^{m+2}u_\varepsilon|^2\big) \\
&\quad - \sum_{i=0}^{m-1}\binom{m}{i}\int_\Omega |D^{m+1}u_\varepsilon|\cdot|D^{i+1}u_\varepsilon|\cdot|D^{m-i}(\eta^{2m+2})| \\
&\quad - \varepsilon\sum_{i=0}^{m}\binom{m+1}{i}\int_\Omega |D^{m+2}u_\varepsilon|\cdot|D^{i+1}u_\varepsilon|\cdot|D^{m-i+1}(\eta^{2m+2})| \\
&\geq \frac{3}{4}\int_\Omega \eta^{2m+2}\big(|D^{m+1}u_\varepsilon|^2 + \varepsilon|D^{m+2}u_\varepsilon|^2\big) - \sum_{i=0}^{m-1}\frac{c}{R^2}\left(\int_{\operatorname{spt}\eta}|D^{i+1}u_\varepsilon|^{\frac{2m}{i+1}}\right)^{\frac{i+1}{m}} \\
&\quad - \sum_{i=0}^{m-1}\frac{c\varepsilon}{R^4}\left(\int_{\operatorname{spt}\eta}|D^{i+1}u_\varepsilon|^{\frac{2m}{i+1}}\right)^{\frac{i+1}{m}} - \frac{c\varepsilon}{R^2}\int_{\operatorname{spt}\eta}|D^{m+1}u_\varepsilon|^2 \\
&\geq \frac{3}{4}\int_\Omega \eta^{2m+2}\big(|D^{m+1}u_\varepsilon|^2 + \varepsilon|D^{m+2}u_\varepsilon|^2\big) - \frac{c}{R^2}\left(\tilde{E}_\varepsilon(u_\varepsilon, B_{2R})\right)^{\frac{1}{m}}.
\end{aligned}$$

We used $\tilde{E}_\varepsilon(u_\varepsilon, B_{4R}(x_0)) < \delta_0$ and $\varepsilon < R^2$ in the last line. Together with (3.4.7) this yields

$$\begin{aligned}
&\int_\Omega \eta^{2m+2}\big(|D^{m+1}u_\varepsilon|^2 + \varepsilon|D^{m+2}u_\varepsilon|^2\big) \\
&\leq (-1)^m\int_\Omega \eta^{2m+2}\left\langle \sum_{\substack{i+j=1+2m\\ j\geq 2}} G^i_j(Du_\varepsilon), Du_\varepsilon\right\rangle \\
&\quad + (-1)^m\varepsilon\int_\Omega \eta^{2m+2}\left\langle \sum_{\substack{i+j=1+2(m+1)\\ j\geq 2}} G^i_j(Du_\varepsilon), Du_\varepsilon\right\rangle + \frac{c}{R^2}\left(\tilde{E}_\varepsilon(u_\varepsilon, B_{2R})\right)^{\frac{1}{m}}.
\end{aligned} \tag{3.4.8}$$

Next we show

$$\sum_{k=1}^{m}\int_\Omega \eta^{2m+2}|D^k u_\varepsilon|^{\frac{2(m+1)}{k}} \leq \frac{c}{R^2}\tilde{E}_\varepsilon(u_\varepsilon, B_{2R}) + c\left(\tilde{E}_\varepsilon(u_\varepsilon, B_{2R})\right)^{\frac{1}{m}}\int_\Omega \eta^{2m+2}|D^{m+1}u_\varepsilon|^2. \tag{3.4.9}$$

We start with the case $k = m$. Lemma 3.4.2 yields

$$\begin{aligned}
\int_\Omega \eta^{2m+2}|D^m u_\varepsilon|^{\frac{2(m+1)}{m}} &\leq \left(\int_{\operatorname{spt}\eta}|D^m u_\varepsilon|^2\right)^{\frac{1}{m}}\left(\int_\Omega \eta^{2m+2}|D^{m+1}u_\varepsilon|^2 + \frac{c}{R^2}\int_{\operatorname{spt}\eta}|D^m u_\varepsilon|^2\right) \\
&\leq \frac{c}{R^2}\tilde{E}_\varepsilon(u_\varepsilon, B_{2R}) + c\left(\tilde{E}_\varepsilon(u_\varepsilon, B_{2R})\right)^{\frac{1}{m}}\int_\Omega \eta^{2m+2}|D^{m+1}u_\varepsilon|^2.
\end{aligned}$$

Analogously we get with Lemma 3.4.2 and Young's inequality ($p_k = \frac{m+1}{k+1}$, $q_k = \frac{m+1}{m-k}$, $k = 1, ..., m-1$)

$$\sum_{k=1}^{m-1}\int_\Omega \eta^{2m+2}|D^k u_\varepsilon|^{\frac{2(m+1)}{k}}$$

$$
\begin{aligned}
&\leq c \sum_{k=1}^{m-1} \left(\int_{\operatorname{spt} \eta} |D^k u_\varepsilon|^{\frac{2m}{k}} \right)^{\frac{1}{m}} \left(\int_\Omega \eta^{2m+2} |D^{k+1} u_\varepsilon|^2 |D^k u_\varepsilon|^{\frac{2(m-k)}{k}} + \frac{c}{R^2} \int_{\operatorname{spt} \eta} |D^k u_\varepsilon|^{\frac{2m}{k}} \right) \\
&\leq c \sum_{k=1}^{m-1} \left(\tilde{E}_\varepsilon(u_\varepsilon, B_{2R}) \right)^{\frac{1}{m}} \left(\int_\Omega \eta^{2m+2} |D^{k+1} u_\varepsilon|^{\frac{2(m+1)}{k+1}} + \int_\Omega \eta^{2m+2} |D^k u_\varepsilon|^{\frac{2(m+1)}{k}} \right. \\
&\qquad \left. + \frac{c}{R^2} \tilde{E}_\varepsilon(u_\varepsilon, B_{2R}) \right) \\
&\leq c \sum_{k=1}^{m} \left(\tilde{E}_\varepsilon(u_\varepsilon, B_{2R}) \right)^{\frac{1}{m}} \int_\Omega \eta^{2m+2} |D^k u_\varepsilon|^{\frac{2(m+1)}{k}} + \frac{c}{R^2} \tilde{E}_\varepsilon(u_\varepsilon, B_{2R}).
\end{aligned}
$$

Putting both inequalities together and choosing $\delta_0 > 0$ small enough so that we can absorb the $D^k u_\varepsilon$-terms to the left-hand side yields (3.4.9).
We use this to estimate the first term of the right-hand side of (3.4.8) further. Note that $\sum_{i+j=2m+1,\ j\geq 2} G_j^i(Du_\varepsilon)$ contains combinations where the degree of every derivative is less than or equal to $m+1$. We will estimate these terms with Young's inequality ($p_i = \frac{2(m+1)}{i l_i}$ $i = 1,...,m+1,\ p_{m+2} = 2(m+1)$) and (3.4.9). The remaining combinations contain derivatives of degree bigger or equal $m+2$. We apply integration by parts until the highest order derivative of u_ε is $m+1$, derivatives $D^r(\eta^{2m+2}),\ 2 \leq r \leq m$ arise only in combinations with derivatives $D^k u_\varepsilon$ of degree $k \leq m$ and no combination contains a single derivative $D(\eta^{2m+2})$. Then we use Young's inequality for $r = 2$ with $p_i = \frac{2m}{i l_i}$, $i = 1, ..., m$ and for $r \geq 3$ we use Hölder's inequality with $p = \frac{2m}{r-2}$, $q = \frac{2m}{2m-r+2}$ as well as Young's inequality with $p_i = \frac{2m-r+2}{i l_i}$, $i = 1, ..., m,\ p_{m+1} = \frac{2m-r+2}{s+1}$ and the definition of the local energy $\tilde{E}$ to estimate these new terms.

$$
\begin{aligned}
&(-1)^m \int_\Omega \eta^{2m+2} \langle \sum_{\substack{i+j=2m+1 \\ j\geq 2}} G_j^i(Du_\varepsilon), Du_\varepsilon \rangle \\
&\leq c \sum_{\substack{l_i \in \mathbb{N}_0,\ i=1,...,m+1 \\ l_1+2l_2+...+(m+1)l_{m+1}=2m+1}} \int_\Omega \left(\eta^{2m+2} |Du_\varepsilon|^{l_1} \cdot ... \cdot |D^{m+1} u_\varepsilon|^{l_{m+1}} |Du_\varepsilon| \right) \\
&\quad + \sum_{\substack{r\in\mathbb{N},\ r\geq 2;\ s, l_i \in \mathbb{N}_0,\ i=1,...,m \\ l_1+2l_2+...+m l_m + r + (s+1) = 2m+2 \\ \sum_{i=1}^m l_i \geq 2,\ r+s\leq m}} \int_{\operatorname{spt}\eta} \left(|Du_\varepsilon|^{l_1} \cdot ... \cdot |D^m u_\varepsilon|^{l_m} |D^r(\eta^{2m+2})| |D^{s+1} u_\varepsilon| \right) \\
&\leq \delta \int_\Omega \eta^{2m+2} |D^{m+1} u_\varepsilon|^2 + c \sum_{i=1}^m \int_\Omega \eta^{2m+2} |D^i u_\varepsilon|^{\frac{2(m+1)}{i}} \\
&\quad + \frac{c}{R^2} \sum_{\substack{s, l_i \in \mathbb{N}_0,\ i=1,...,m \\ l_1+2l_2+...+m l_m + (s+1) = 2m \\ \sum_{i=1}^m l_i \geq 2,\ 2+s\leq m}} \int_{\operatorname{spt}\eta} \left(|Du_\varepsilon|^{l_1} \cdot ... \cdot |D^m u_\varepsilon|^{l_m} |D^{s+1} u_\varepsilon| \right) \\
&\quad + c \sum_{\substack{r\in\mathbb{N},\ r\geq 3;\ s, l_i \in \mathbb{N}_0,\ i=1,...,m \\ l_1+2l_2+...+m l_m + r + (s+1) = 2m+2 \\ \sum_{i=1}^m l_i \geq 2,\ r+s\leq m}} \left(\int_\Omega |D^r(\eta^{2m+2})|^{\frac{2m}{r-2}} \right)^{\frac{r-2}{2m}} \cdot \\
&\qquad \cdot \left(\int_{\operatorname{spt}\eta} |Du_\varepsilon|^{\frac{2m l_1}{2m-r+2}} \cdot ... \cdot |D^m u_\varepsilon|^{\frac{2m l_m}{2m-r+2}} |D^{s+1} u_\varepsilon|^{\frac{2m}{2m-r+2}} \right)^{\frac{2m-r-s}{2m}} \\
&\leq \delta \int_\Omega \eta^{2m+2} |D^{m+1} u_\varepsilon|^2 + c \sum_{i=1}^m \int_\Omega \eta^{2m+2} |D^i u_\varepsilon|^{\frac{2(m+1)}{i}} + \frac{c}{R^2} \sum_{i=1}^m \left(\int_{\operatorname{spt}\eta} |D^i u_\varepsilon|^{\frac{2m}{i}} \right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq \frac{c}{R^2}\left(\tilde{E}(u_\varepsilon, B_{2R})\right)^{\frac{1}{m}} + \left(\delta + c\left(\tilde{E}(u_\varepsilon, B_{2R})\right)^{\frac{1}{m}}\right)\int_\Omega \eta^{2m+2}|D^{m+1}u_\varepsilon|^2.$$

Going back to (3.4.8) we can absorb the $D^{m+1}u_\varepsilon$-term since $\tilde{E}(u_\varepsilon, B_{2R}) < \delta_0$, so that

$$\int_\Omega \eta^{2m+2}(|D^{m+1}u_\varepsilon|^2 + \varepsilon|D^{m+2}u_\varepsilon|^2) \leq (-1)^m\varepsilon\int_\Omega \eta^{2m+2}\left\langle \sum_{\substack{i+j=2(m+1)+1\\ j\geq 2}} G^i_j(Du_\varepsilon), Du_\varepsilon\right\rangle + \frac{c}{R^2}\left(\tilde{E}_\varepsilon(u_\varepsilon, B_{2R})\right)^{\frac{1}{m}}$$

and in particular

$$\kappa_{m+1}(u_\varepsilon, B_R) \leq \frac{c}{R^2}\left(\tilde{E}(u_\varepsilon, B_{2R})\right)^{\frac{1}{m}} + (-1)^m\varepsilon\int_\Omega \eta^{2m+2}\left\langle \sum_{\substack{i+j=2(m+1)+1\\ j\geq 2}} G^i_j(Du_\varepsilon), Du_\varepsilon\right\rangle. \tag{3.4.10}$$

Analogously to (3.4.9) we have

$$\begin{aligned}
&\varepsilon\sum_{k=1}^{m+1}\int_\Omega \eta^{2m+2}|D^k u_\varepsilon|^{\frac{2(m+2)}{k}} \\
&\leq \frac{c}{R^2}\tilde{E}_\varepsilon(u_\varepsilon, B_{4R}) + c\left(\left(\tilde{E}_\varepsilon(u_\varepsilon, B_{2R})\right)^{\frac{1}{m}} + \delta\right)\int_\Omega \eta^{2m+2}\left(|D^{m+1}u_\varepsilon|^2 + \varepsilon|D^{m+2}u_\varepsilon|^2\right).
\end{aligned} \tag{3.4.11}$$

For $k = m+1$ we have with Young's inequality ($p = m+1,\ q = \frac{m+1}{m}$) and Lemma 3.4.2

$$\begin{aligned}
&\varepsilon\int_\Omega \eta^{2m+2}|D^{m+1}u_\varepsilon|^{\frac{2(m+2)}{(m+1)}} = \varepsilon\int_\Omega \eta^{2m+2}|D^{m+1}u_\varepsilon|^{\frac{2}{m+1}}|D^{m+1}u_\varepsilon|^2 \\
&\leq \delta\int_\Omega \eta^{2m+2}|D^{m+1}u_\varepsilon|^2 + c\varepsilon^{\frac{m+1}{m}}\int_\Omega \eta^{2m+2}|D^{m+1}u_\varepsilon|^{\frac{2(m+1)}{m}} \\
&\leq \delta\int_\Omega \eta^{2m+2}|D^{m+1}u_\varepsilon|^2 \\
&\quad + \varepsilon^{\frac{m+1}{m}}\left(\int_{spt\ \eta}|D^{m+1}u_\varepsilon|^2\right)^{\frac{1}{m}}\left(\int_\Omega \eta^{2m+2}|D^{m+2}u_\varepsilon|^2 + \frac{c}{R^2}\int_{spt\ \eta}|D^{m+1}u_\varepsilon|^2\right) \\
&\leq \delta\int_\Omega \eta^{2m+2}|D^{m+1}u_\varepsilon|^2 + c\varepsilon\left(\tilde{E}_\varepsilon(u_\varepsilon, B_{2R})\right)^{\frac{1}{m}}\int_\Omega \eta^{2m+2}|D^{m+2}u_\varepsilon|^2 + \frac{c}{R^2}\tilde{E}_\varepsilon(u_\varepsilon, B_{2R}).
\end{aligned}$$

The remaining terms can also be estimated using Young's inequality ($p = m+1,\ q = \frac{m+1}{m}$), Lemma 3.4.2, Corollary 3.4.3 and Young's inequality ($p = \frac{m+2}{k+1},\ q = \frac{m+2}{m+1-k}$)

$$\begin{aligned}
&\sum_{k=1}^{m}\varepsilon\int_\Omega \eta^{2m+2}|D^k u_\varepsilon|^{\frac{2(m+2)}{k}} = \sum_{k=1}^{m}\varepsilon\int_\Omega \eta^{2m+2}|D^k u_\varepsilon|^{\frac{2}{k}}|D^k u_\varepsilon|^{\frac{2(m+1)}{k}} \\
&\leq c\sum_{k=1}^{m}\int_\Omega \eta^{2m+2}|D^k u_\varepsilon|^{\frac{2(m+1)}{k}} + \delta\varepsilon^{\frac{m+1}{m}}\sum_{k=1}^{m}\int_\Omega \eta^{2m+2}|D^k u_\varepsilon|^{\frac{2(m+1)^2}{mk}} \\
&\leq c\sum_{k=1}^{m}\int_\Omega \eta^{2m+2}|D^k u_\varepsilon|^{\frac{2(m+1)}{k}} + \frac{c\delta}{R^2}\sum_{k=1}^{m}\left(\varepsilon\int_{spt\ \eta}|D^k u|^{\frac{2(m+1)}{k}}\right)^{\frac{m+1}{m}} \\
&\quad + c\delta\varepsilon^{\frac{m+1}{m}}\sum_{k=1}^{m}\left(\int_{spt\ \eta}|D^k u_\varepsilon|^{\frac{2(m+1)}{k}}\right)^{\frac{1}{m}}\left(\int_\Omega \eta^{2m+2}|D^{k+1}u_\varepsilon|^2|D^k u_\varepsilon|^{\frac{2(m+1-k)}{k}}\right)
\end{aligned}$$

$$\leq c\sum_{k=1}^{m}\int_\Omega \eta^{2m+2}|D^k u_\varepsilon|^{\frac{2(m+1)}{k}} + \frac{c}{R^2}\tilde{E}_\varepsilon(u_\varepsilon, B_{4R})$$
$$+ c\delta\varepsilon\sum_{k=1}^{m}\left(\varepsilon\int_{spt\ \eta}|D^k u_\varepsilon|^{\frac{2(m+1)}{k}}\right)^{\frac{1}{m}}\left(\int_\Omega \eta^{2m+2}\left(|D^{k+1}u_\varepsilon|^{\frac{2(m+2)}{k+1}} + |D^k u_\varepsilon|^{\frac{2(m+2)}{k}}\right)\right)$$
$$\leq c\sum_{k=1}^{m}\int_\Omega \eta^{2m+2}|D^k u_\varepsilon|^{\frac{2(m+1)}{k}} + \frac{c}{R^2}\tilde{E}_\varepsilon(u_\varepsilon, B_{4R})$$
$$+ c\delta\varepsilon\left(\tilde{E}_\varepsilon(u_\varepsilon, B_{2R})\right)^{\frac{1}{m}}\sum_{k=1}^{m+1}\int_\Omega \eta^{2m+2}|D^k u_\varepsilon|^{\frac{2(m+2)}{k}}.$$

We add the case $k = m+1$ and choose δ small enough so that we can absorb the $\delta\varepsilon(\tilde{E}_\varepsilon)^{1/m}$-terms to the left hand side. Then (3.4.9) yields (3.4.11).
Using this we turn to the second part of (3.4.8). As before we use integration by part $(m+1)$-times until the highest order term $D^{m+1}u_\varepsilon$ appears only once and at least two derivatives fall onto the cut-off function η. Then we estimate the first term with Young's inequality ($p_i = \frac{2(m+2)}{il_i}$, $i = 1,...,m+2,\ p_{m+3} = 2(m+2)$). The remaining terms, where derivatives fall on the cut-off function, are estimated using Young's inequality:
For $r = 2:\ p_i = \frac{2(m+1)}{il_i},\ i = 1,...,m+1,\ p_{m+2} = \frac{2(m+1)}{s+1}$;
For $r = 3:$ first $q_1 = 2,\ q_2 = 2$, then $p_i = \frac{m}{il_i},\ i = 1,...,m,\ p_{m+1} = \frac{m}{s+1}$;
For $r = 4:\ p_i = \frac{2m}{il_i},\ i = 1,...,m,\ p_{m+1} = \frac{2m}{s+1}$.
For $r \geq 5$ we use Hölder's inequality ($p = \frac{2m}{r-4}, q = \frac{2m}{2m-r+4}$) and Young's inequality for ($p_i = \frac{2m-r+4}{il_i},\ i = 1,...,m+1,\ p_{m+2} = \frac{2m-r+4}{s+1}$).
Further we use (3.4.9), (3.4.11) and Corollary 3.4.3

$$(-1)^m\varepsilon\int_\Omega \eta^{2m+2}\left\langle \sum_{\substack{i+j=2(m+1)+1\\ j\geq 2}} G^i_j(Du_\varepsilon), Du_\varepsilon\right\rangle$$
$$\leq c\varepsilon \sum_{\substack{l_i\in\mathbb{N}_0,\ i=1,\dots,m+2\\ l_1+2l_2+\dots+(m+2)l_{m+2}=2m+3}} \int_\Omega \left(\eta^{2m+2}|Du_\varepsilon|^{l_1}\cdot\ldots\cdot|D^{m+2}u_\varepsilon|^{l_{m+2}}|Du_\varepsilon|\right)$$
$$+ c\varepsilon \sum_{\substack{r\in\mathbb{N},\ r\geq 2;\ s,l_i\in\mathbb{N}_0,\ i=1,\dots,m+1\\ l_1+2l_2+\dots+(m+1)l_{m+1}+r+(s+1)=2m+4\\ \sum_{i=1}^{m+1} l_i\geq 2,\ l_{m+1}=1,\ r+s\leq m+1}} \int_{\mathrm{spt}\,\eta}\Big(|Du_\varepsilon|^{l_1}\cdot\ldots\cdot|D^{m+1}u_\varepsilon|^{l_{m+1}}\cdot$$
$$\cdot|D^r(\eta^{2m+2})|\cdot|D^{s+1}u_\varepsilon|\Big)$$
$$\leq \delta\varepsilon\int_\Omega \eta^{2m+2}|D^{m+2}u_\varepsilon|^2 + c\varepsilon\sum_{i=1}^{m+1}\int_\Omega \eta^{2m+2}|D^i u_\varepsilon|^{\frac{2(m+2)}{i}}$$
$$+ \frac{c\varepsilon}{R^2}\sum_{i=1}^{m+1}\int_{\mathrm{spt}\,\eta}|D^i u_\varepsilon|^{\frac{2(m+1)}{i}} + \frac{c\varepsilon}{R^4}\sum_{i=1}^{m+1}\int_{\mathrm{spt}\,\eta}|D^i u_\varepsilon|^{\frac{2m}{i}}$$
$$+ \frac{c\varepsilon}{R^4}\sum_{\substack{r\in\mathbb{N},\ r=3;\ s,l_i\in\mathbb{N}_0,\ i=1,\dots,m\\ l_1+2l_2+\dots+ml_m+3+(s+1)=m+3\\ \sum_{i=1}^{m+1} l_i\geq 2,\ 3+s\leq m+1}} \int_{\mathrm{spt}\,\eta}|Du_\varepsilon|^{2l_1}\cdot\ldots\cdot|D^m u_\varepsilon|^{2l_m}|D^{s+1}u_\varepsilon|^2$$

$$+ c\varepsilon \sum_{\substack{r\in\mathbb{N},\ r\geq 5;\ s,l_i\in\mathbb{N}_0,\ i=1,\dots,m+1\\ l_1+2l_2+\dots+(m+1)l_{m+1}+r+(s+1)=2m+4\\ \sum_{i=1}^{m+1} l_i\geq 2,\ l_{m+1}=1,\ r+s\leq m+1}} \left(\int_{\operatorname{spt}\eta} |D^r(\eta^{2m+2})|^{\frac{2m}{r-4}}\right)^{\frac{r-4}{2m}}$$

$$\cdot\left(\int_{\operatorname{spt}\eta} |Du_\varepsilon|^{\frac{2ml_1}{2m-r+4}}\cdot\ldots\cdot|D^{m+1}u_\varepsilon|^{\frac{2ml_{m+1}}{2m-r+4}}|D^{s+1}u_\varepsilon|^{\frac{2m}{2m-r+4}}\right)^{\frac{2m-r+4}{2m}}$$

$$\leq \varepsilon\left(\delta + c\left(\tilde{E}_\varepsilon(u_\varepsilon,B_{2R})\right)^{\frac{1}{m}}\right)\int_\Omega \eta^{2m+2}|D^{m+2}u_\varepsilon|^2$$

$$+ c\left(\tilde{E}_\varepsilon(u_\varepsilon,B_{2R})\right)^{\frac{1}{m}}\int_\Omega \eta^{2m+2}|D^{m+1}u_\varepsilon|^2 + \frac{c}{R^2}\tilde{E}_\varepsilon(u_\varepsilon,B_{4R}) + \frac{c\varepsilon}{R^4}\sum_{i=1}^{m+1}\left(\int_{\operatorname{spt}\eta}|D^iu_\varepsilon|^{\frac{2m}{i}}\right)^{\frac{1}{2}}$$

$$\leq \varepsilon\left(\delta + c\left(\tilde{E}_\varepsilon(u_\varepsilon,B_{2R})\right)^{\frac{1}{m}}\right)\int_\Omega \eta^{2m+2}|D^{m+2}u_\varepsilon|^2 + c\left(\tilde{E}_\varepsilon(u_\varepsilon,B_{2R})\right)^{\frac{1}{m}}\int_\Omega \eta^{2m+2}|D^{m+1}u_\varepsilon|^2$$

$$+ \frac{c}{R^2}\tilde{E}_\varepsilon(u_\varepsilon,B_{4R}) + \frac{c\varepsilon}{R^4}\sum_{i=1}^{m}\left(\int_{\operatorname{spt}\eta}|D^iu_\varepsilon|^{\frac{2m}{i}}\right)^{\frac{1}{2}} + \frac{c\varepsilon^{\frac{1}{2}}}{R^3}\left(\varepsilon\int_{\operatorname{spt}\eta}|D^{m+1}u_\varepsilon|^2\right)^{\frac{1}{2}}$$

$$\leq \varepsilon\left(\delta + c\left(\tilde{E}_\varepsilon(u_\varepsilon,B_{2R})\right)^{\frac{1}{m}}\right)\int_\Omega \eta^{2m+2}|D^{m+2}u_\varepsilon|^2$$

$$+ c\left(\tilde{E}_\varepsilon(u_\varepsilon,B_{2R})\right)^{\frac{1}{m}}\int_\Omega \eta^{2m+2}|D^{m+1}u_\varepsilon|^2 + \frac{c}{R^2}\left(\tilde{E}_\varepsilon(u_\varepsilon,B_{4R})\right)^{\frac{1}{m}}.$$

Going back to (3.4.8) and (3.4.10) and absorbing the first two terms yields

$$\varepsilon\kappa_{m+2}(u_\varepsilon,B_R) + \kappa_{m+1}(u_\varepsilon,B_R) \leq \frac{c}{R^2}\left(\tilde{E}(u_\varepsilon,B_{4R})\right)^{\frac{1}{m}}$$

and the Proposition follows.

□

Lemma 3.4.5. *There exist $0<\delta_0<1$, $c>0$ such that for all cut-off functions $\eta\in C^\infty(\Omega)$, $\operatorname{spt}\eta\subset B_{2R}(x_0)$, $R<<1$ and all $u\in C^\infty(\Omega,N)$ there holds: If*

$$\kappa_{q-1}(u,B_{2R}(x_0)) \leq \delta_0 R^{2(m+1-q)}$$

with $q\geq m+1$, then we have for all $1\leq k\leq q-1$

$$\int_\Omega \eta^{2k}|D^ku|^{\frac{2q}{k}} \leq c\big(R^{2(q-m-1)}\kappa_{q-1}(u,B_{2R}(x_0))\big)^{\frac{1}{q-1}}\int_\Omega \eta^{2q}|D^qu|^2 + cR^{-2}\kappa_{q-1}(u,B_{2R}(x_0)).$$

Proof. Wlog $x_0=0$. The case $q=m+1$ follows from Proposition 3.4.4 thus we can assume $q\geq m+2$. For the case $k=q-1$ we use the Sobolev embedding $W^{1,\frac{2m}{m+1}}(\Omega,N)\hookrightarrow L^2(\Omega,N)$ and Hölder's inequality with $p_1=\frac{m+1}{m}$, $p_2=\frac{(m+1)(q-1)}{m}$, $p_3=\frac{(m+1)(q-1)}{q-m-1}$ and $\tilde{p}_1=\frac{(m+1)(q-1)}{mq}$, $\tilde{p}_2=\frac{(m+1)(q-1)}{q-m-1}$

$$\int_\Omega \eta^{2q}|D^{q-1}u|^{\frac{2q}{q-1}} \leq c\bigg(\int_\Omega \eta^{\frac{2qm}{m+1}}|D^qu|^{\frac{2m}{m+1}}|D^{q-1}u|^{\frac{2m}{(q-1)(m+1)}}\bigg)^{\frac{m+1}{m}}$$

$$+ c\bigg(\int_\Omega |D\eta|^{\frac{2m}{m+1}}\eta^{\frac{2m(q-1)}{m+1}}|D^{q-1}u|^{\frac{2mq}{(q-1)(m+1)}}\bigg)^{\frac{m+1}{m}}$$

$$\leq c\bigg(\int_\Omega \eta^{2q}|D^qu|^2\bigg)\bigg(\int_{B_{2R}}|D^{q-1}u|^2\bigg)^{\frac{1}{q-1}}R^{\frac{2(q-m-1)}{q-1}}$$

$$
\begin{aligned}
&+ c\Big(\int_{B_{2R}} |D^{q-1}u|^2\Big)^{\frac{q}{q-1}} R^{-\frac{2m}{q-1}}\\
&\le c\Big(R^{2(q-m-1)}\int_{B_{2R}} |D^{q-1}u|^2\Big)^{\frac{1}{q-1}} \int_\Omega \eta^{2q}|D^q u|^2\\
&+ cR^{-2}\int_{B_{2R}} |D^{q-1}u|^2. \qquad (3.4.12)
\end{aligned}
$$

For $1 \le k \le q-2$ we use Lemma 3.4.2, Hölder's inequality with $p_1 = \frac{(q-1)(m+1)}{qm}$, $p_2 = \frac{(q-1)(m+1)}{q-m-1}$, $s_1 = \frac{q}{k+1}$, $s_2 = \frac{q(m+1)}{qm-km-k}$, $s_3 = \frac{q(m+1)}{q-m-1}$ and Young's inequality with $p_1 = \frac{q(m+1)}{q-m-1}$, $p_2 = \frac{q}{k+1}$, $p_3 = \frac{q(m+1)}{qm-km-k}$.

$$
\begin{aligned}
&\int_\Omega \eta^{2k}|D^k u|^{\frac{2q}{k}}\\
&\le c\left(\int_{\operatorname{spt}\eta} |D^k u|^{\frac{2qm}{k(m+1)}}\right)^{\frac{1}{m}} \left(\int_\Omega \eta^{2k}|D^{k+1}u|^2|D^k u|^{\frac{2(qm-km-k)}{k(m+1)}} + \frac{c}{R^2}\int_{\operatorname{spt}\eta}|D^k u|^{\frac{2qm}{k(m+1)}}\right)\\
&\le cR^{\frac{2(q-m-1)}{(q-1)(m+1)}}\left(\int_{\operatorname{spt}\eta}|D^k u|^{\frac{2(q-1)}{k}}\right)^{\frac{q}{(q-1)(m+1)}}\\
&\quad\cdot\left(\int_\Omega \eta^{2(k+1)}|D^{k+1}u|^{\frac{2q}{k+1}}\right)^{\frac{k+1}{q}}\left(\int_\Omega \eta^{2k}|D^k u|^{\frac{2q}{k}}\right)^{\frac{qm-km-k}{q(m+1)}} R^{\frac{2m(q-m-1)}{q(m+1)}}\\
&\quad + cR^{\frac{-2m}{(q-1)}}\left(\int_{\operatorname{spt}\eta}|D^k u|^{\frac{2(q-1)}{k}}\right)^{\frac{q}{q-1}}\\
&\le cR^{\frac{2m(q-1)+2q}{q-1}}\left(\int_{\operatorname{spt}\eta}|D^k u|^{\frac{2(q-1)}{k}}\right)^{\frac{q^2}{(q-1)(q-m-1)}} + \delta\int_\Omega \eta^{2(k+1)}|D^{k+1}u|^{\frac{2q}{k+1}} + \delta\int_\Omega \eta^{2k}|D^k u|^{\frac{2q}{k}}\\
&\quad + cR^{\frac{-2m}{(q-1)}}\left(\int_{\operatorname{spt}\eta}|D^k u|^{\frac{2(q-1)}{k}}\right) R^{\frac{2(m+1-q)}{(q-1)}}\\
&\le cR^{-2}\int_{\operatorname{spt}\eta}|D^k u|^{\frac{2(q-1)}{k}} + \delta\int_\Omega \eta^{2(k+1)}|D^{k+1}u|^{\frac{2q}{k+1}} + \delta\int_\Omega \eta^{2k}|D^k u|^{\frac{2q}{k}}.
\end{aligned}
$$

Summing over $k = 1, ..., q-1$ and (3.4.12) yields

$$
\begin{aligned}
\sum_{k=1}^{q-1}\int_\Omega \eta^{2k}|D^k u|^{\frac{2q}{k}} &\le \sum_{k=1}^{q-1}\left[\frac{c}{R^2}\int_{\operatorname{spt}\eta}|D^k u|^{\frac{2(q-1)}{k}} + \delta\int_\Omega \eta^{2k}|D^k u|^{\frac{2q}{k}}\right]\\
&+ c\Big(R^{2(q-m-1)}\int_{B_{2R}}|D^{q-1}u|^2\Big)^{\frac{1}{q-1}}\int_\Omega \eta^{2q}|D^q u|^2.
\end{aligned}
$$

Choosing $\delta > 0$ small enough and absorbing the δ-term to the left-hand side finishes the proof. $\square$

In the next theorem we extend Proposition 3.4.4 to higher order.

Theorem 3.4.6. *For all $\varepsilon, R > 0$ with $\varepsilon < R^2$ there exist $0 < \delta_0 < 1$, $c > 0$ such that if $u_\varepsilon \in C^\infty(\Omega, N)$ is a solution of (3.1.3) satisfying $\tilde{E}_\varepsilon(u_\varepsilon, B_R(x_0)) < \delta_0$ for some $x_0 \in \Omega$, then we have for all $p \in \mathbb{N}$*

$$
\sum_{i=1}^{p} R^{2(i-p)}\left[\kappa_{m+i}(u_\varepsilon, B_{2^{-3i+1}R}(x_0)) + \varepsilon\kappa_{m+1+i}(u_\varepsilon, B_{2^{-3i+1}R}(x_0))\right] \le \frac{c}{R^{2p}}\left(\tilde{E}_\varepsilon(u_\varepsilon, B_R(x_0))\right)^{\frac{1}{m}}. \qquad (3.4.13)
$$

Proof. We proof the theorem by induction on p. The case $p=1$ follows from Proposition 3.4.4.
$\underline{p \to p+1:}$ We assume that for some $p \in \mathbb{N}$ and for $\varepsilon > 0$ small enough we have

$$\begin{aligned}&\sum_{i=1}^{p} R^{2(i-p)}\left[\kappa_{m+i}(u_\varepsilon, B_{2^{-3i+1}R}(x_0)) + \varepsilon\kappa_{m+1+i}(u_\varepsilon, B_{2^{-3i+1}R}(x_0))\right]\\ &\leq c\left(\tilde{E}_\varepsilon(u_\varepsilon, B_R(x_0))\right)^{\frac{1}{m}} R^{-2p}. \end{aligned} \tag{3.4.14}$$

From Lemma 3.4.1 we know

$$(\Delta^m - \varepsilon\Delta^{m+1})D^{p+1}u_\varepsilon = \sum_{\substack{i+j=2m+p+1\\ j\geq 2}} G^i_j(Du_\varepsilon) + \varepsilon \sum_{\substack{i+j=2(m+1)+p+1\\ j\geq 2}} G^i_j(Du_\varepsilon).$$

Wlog let $x_0 = 0$. Let $\eta \in C^\infty_c(\Omega)$ be a smooth cut-off function with $\eta = 1$ on $B_{2^{-3p}R}$ and $\eta = 0$ on $\Omega \setminus B_{2^{-3p+1}R}$. Multiplying this equation by $\eta^{2m+2}D^{p+1}u_\varepsilon$ and integrating over Ω yields

$$\begin{aligned}\int_\Omega \eta^{2m+2}\langle(\Delta^m - \varepsilon\Delta^{m+1})D^{p+1}u_\varepsilon, D^{p+1}u_\varepsilon\rangle &= \int_\Omega \eta^{2m+2}\left\langle \sum_{\substack{i+j=2m+p+1\\ j\geq 2}} G^i_j(Du_\varepsilon), D^{p+1}u_\varepsilon\right\rangle\\ &\quad + \varepsilon\int_\Omega \eta^{2m+2}\left\langle \sum_{\substack{i+j=2(m+1)+p+1\\ j\geq 2}} G^i_j(Du_\varepsilon), D^{p+1}u_\varepsilon\right\rangle\\ &=: I + II.\end{aligned}$$

As before we estimate the left-hand side using integration by parts and Young's inequality

$$\begin{aligned}&(-1)^m\int_\Omega \eta^{2m+2}\langle(\Delta^m - \varepsilon\Delta^{m+1})D^{p+1}u_\varepsilon, D^{p+1}u_\varepsilon\rangle\\ &\geq \int_\Omega \eta^{2m+2}|D^{m+p+1}u_\varepsilon|^2 + \varepsilon\int_\Omega \eta^{2m+2}|D^{m+p+2}u_\varepsilon|^2\\ &\quad - \sum_{k=1}^{m}\binom{m}{k}\int_\Omega |D^{m+p+1}u_\varepsilon|\cdot|D^{m+p+1-k}u_\varepsilon|\cdot|D^k(\eta^{2m+2})|\\ &\quad - \varepsilon\sum_{k=1}^{m+1}\binom{m+1}{k}\int_\Omega |D^{m+p+2}u_\varepsilon|\cdot|D^{m+p+2-k}u_\varepsilon|\cdot|D^k(\eta^{2(m+2)})|\\ &\geq \frac{3}{4}\int_\Omega \eta^{2m+2}\left(|D^{m+p+1}u_\varepsilon|^2 + \varepsilon|D^{m+p+2}u_\varepsilon|^2\right)\\ &\quad - \sum_{k=1}^{m}\frac{c}{R^{2k}}\int_{\operatorname{spt}\eta}|D^{m+p+1-k}u_\varepsilon|^2 - \sum_{k=1}^{m+1}\frac{c\varepsilon}{R^{2k}}\int_{\operatorname{spt}\eta}|D^{m+p+2-k}u_\varepsilon|^2\\ &\geq \frac{3}{4}\int_\Omega \eta^{2m+2}\left(|D^{m+p+1}u_\varepsilon|^2 + \varepsilon|D^{m+p+2}u_\varepsilon|^2\right)\\ &\quad - \sum_{k=1}^{m}\frac{c}{R^{2k}}\kappa_{m+p+1-k}(u_\varepsilon, B_{2^{-3p+1}R}) - \sum_{k=1}^{m+1}\frac{c\varepsilon}{R^{2k}}\kappa_{m+p+2-k}(u_\varepsilon, B_{2^{-3p+1}R})\\ &\geq \frac{3}{4}\int_\Omega \eta^{2m+2}\left(|D^{m+p+1}u_\varepsilon|^2 + \varepsilon|D^{m+p+2}u_\varepsilon|^2\right) - \frac{c}{R^{2p+2}}\left(\tilde{E}_\varepsilon(u_\varepsilon, B_R)\right)^{\frac{1}{m}},\end{aligned}$$

where we used the induction assumption (3.4.14) in the last step. Next we want to estimate I further. We use integration by parts $(m-1)$-times as in the proof of Proposition 3.4.4. Note that after integrating by parts $l_{m+p+1} = 1$. With Young's inequality ($q_i = \frac{2(m+p+1)}{il_i}$, $i = 1, ..., m+p+$

$1,\ q_{m+p+2} = \frac{2(m+p+1)}{p+1}$ and $t_i = \frac{m+p+1-r}{il_i},\ i = 1, ..., m+p,\ t_{m+p+1} = \frac{m+p+1-r}{s+p+1}$), Lemma 3.4.5 (for $q = m+p+1$) and the induction assumption (3.4.14) we have

$$\begin{aligned}
I &= \int_\Omega \eta^{2m+2} \left\langle \sum_{\substack{i+j=2m+p+1\\ j\geq 2}} G^i_j(Du_\varepsilon), D^{p+1}u_\varepsilon \right\rangle \\
&\leq c \sum_{\substack{l_i\in\mathbb{N}_0,\ i=1,...,m+p+1,\\ l_1+2l_2+...+(m+p+1)l_{m+p+1}\\ =2m+p+1}} \int_\Omega \eta^{2m+2}|Du_\varepsilon|^{l_1}\cdot ... \cdot |D^{m+p+1}u_\varepsilon|^{l_{m+p+1}}|D^{p+1}u_\varepsilon| \\
&\quad + c \sum_{\substack{l_i\in\mathbb{N}_0,\ i=1,...,m+p+1\\ l_1+2l_2+...+(m+p)l_{m+p}+(m+p+1)\\ +r+s+p+1=2(m+p+1)\\ r,s\in\mathbb{N},\ s+r\leq m-1}} \int_{\operatorname{spt}\eta} \Big(|Du_\varepsilon|^{l_1}\cdot ... \cdot |D^{m+p}u_\varepsilon|^{l_{m+p}}|D^{m+p+1}u_\varepsilon| \\
&\qquad\qquad \cdot |D^r(\eta^{2m+2})||D^{s+p+1}u_\varepsilon| \Big) \\
&\leq \delta \int_\Omega \eta^{2m+2}|D^{m+p+1}u_\varepsilon|^2 + c\sum_{k=1}^{m+p}\int_\Omega \eta^{2m+2}|D^k u_\varepsilon|^{\frac{2(m+p+1)}{k}} \\
&\quad + \sum_{\substack{l_i\in\mathbb{N}_0,\ i=1,...,m+p\\ l_1+2l_2+...+(m+p)l_{m+p}+r+s+p+1\\ =m+p+1\\ r,s\in\mathbb{N},\ s+r\leq m-1}} \frac{c}{R^{2r}}\int_{\operatorname{spt}\eta} \Big(|Du_\varepsilon|^{2l_1}\cdot ... \cdot |D^{m+p}u_\varepsilon|^{2l_{m+p}} \cdot |D^{s+p+1}u_\varepsilon|^2 \Big) \\
&\leq \delta \int_\Omega \eta^{2m+2}|D^{m+p+1}u_\varepsilon|^2 + \frac{c}{R^2}\kappa_{m+p}(u_\varepsilon, B_{2^{-3p+1}R}) + \sum_{r=1}^{m-1}\sum_{i=1}^{m+p}\frac{c}{R^{2r}}\int_{\operatorname{spt}\eta} |D^i u_\varepsilon|^{\frac{2(m+p+1-r)}{i}} \\
&\leq \delta \int_\Omega \eta^{2m+2}|D^{m+p+1}u_\varepsilon|^2 + \sum_{r=1}^{m-1}\frac{c}{R^{2r}}\kappa_{m+p+1-r}(u_\varepsilon, B_{2^{-3p+1}R}) \\
&\leq \delta \int_\Omega \eta^{2m+2}|D^{m+p+1}u_\varepsilon|^2 + \frac{c}{R^{2p+2}}\Big(\tilde{E}_\varepsilon(u_\varepsilon, B_R)\Big)^{\frac{1}{m}}.
\end{aligned}$$

Now we turn to II. Analogously to the above we integrate by parts at most m-times until $l_{m+p+2} = 1$. Using Young's inequality ($q_i = \frac{2(m+p+2)}{il_i},\ i = 1, ..., m+p+2,\ p_{m+p+3} = \frac{2(m+p+2)}{p+1}$ and $t_i = \frac{m+p+2-r}{il_i},\ i = 1, ..., m+p+1,\ t_{m+p+2} = \frac{m+p+2-r}{s+p+1}$), Lemma 3.4.5 (for $q = m+p+2$) and the induction assumption (3.4.14) yield

$$\begin{aligned}
II &= \varepsilon \int_\Omega \eta^{2m+2} \left\langle \sum_{\substack{i+j=2(m+1)+p+1\\ j\geq 2}} G^i_j(Du_\varepsilon), D^{p+1}u_\varepsilon \right\rangle \\
&\leq c\varepsilon \sum_{\substack{l_i\in\mathbb{N}_0,\ i=1,...,m+p+2\\ l_1+2l_2+...+\\ ...+(m+p+2)l_{m+p+2}=2m+p+3}} \int_\Omega \eta^{2m+2}|Du_\varepsilon|^{l_1}\cdot ... \cdot |D^{m+p+2}u_\varepsilon|^{l_{m+p+2}}|D^{p+1}u_\varepsilon| \\
&\quad + c\varepsilon \sum_{\substack{l_i\in\mathbb{N}_0,\ i=1,...,m+p+2\\ l_1+2l_2+...+(m+p+1)l_{m+p+1}+\\ +(m+p+2)+r+s+p+1=2(m+p+2)\\ r,s\in\mathbb{N},\ s+r\leq m}} \int_{\operatorname{spt}\eta} \Big(|Du_\varepsilon|^{l_1}\cdot ... \cdot |D^{m+p+1}u_\varepsilon|^{l_{m+p+1}} \\
&\qquad\qquad \cdot |D^{m+p+2}u_\varepsilon|\cdot|D^r(\eta^{2m+2})|\cdot|D^{s+p+1}u_\varepsilon| \Big)
\end{aligned}$$

$$
\begin{aligned}
&\le \delta\varepsilon \int_\Omega \eta^{2m+2}|D^{m+p+2}u_\varepsilon|^2 + c\varepsilon \sum_{k=1}^{m+p+1} \int_\Omega \eta^{2m+2}|D^k u_\varepsilon|^{\frac{2(m+p+2)}{k}} \\
&\quad + c\varepsilon \sum_{\substack{l_i\in\mathbb{N}_0,\ i=1,\dots,m+p+1 \\ l_1+2l_2+\dots+(m+p+1)l_{m+p+1}+ \\ +r+s+p+1=m+p+2 \\ r,s\in\mathbb{N},\ s+r\le m}} \frac{c}{R^{2r}} \int_{\operatorname{spt}\eta} \left(|Du_\varepsilon|^{2l_1}\cdot\ldots\cdot|D^{m+p+1}u_\varepsilon|^{2l_{m+p+1}}|D^{s+p+1}u_\varepsilon|^2\right) \\
&\le \delta\varepsilon \int_\Omega \eta^{2m+2}|D^{m+p+2}u_\varepsilon|^2 + \frac{c\varepsilon}{R^2}\kappa_{m+p+1}(u_\varepsilon, B_{2^{-3p+1}R}) + \sum_{r=1}^{m}\sum_{i=1}^{m+p+1} \frac{c\varepsilon}{R^{2r}} \int_{\operatorname{spt}\eta} |D^i u_\varepsilon|^{\frac{2(m+p+2-r)}{i}} \\
&\le \delta\varepsilon \int_\Omega \eta^{2m+2}|D^{m+p+2}u_\varepsilon|^2 + \sum_{r=1}^{m} \frac{c\varepsilon}{R^{2r}}\kappa_{m+p+2-r}(u_\varepsilon, B_{2^{-3p+1}R}) \\
&\le \delta\varepsilon \int_\Omega \eta^{2m+2}|D^{m+p+2}u_\varepsilon|^2 + \frac{c}{R^{2p+2}}\left(\tilde{E}_\varepsilon(u_\varepsilon, B_R)\right)^{\frac{1}{m}}.
\end{aligned}
$$

Combining the above estimates and choosing δ and $E_\varepsilon(u_\varepsilon, B_R)$ small enough we have

$$
\int_\Omega \eta^{2m+2}|D^{m+p+1}u_\varepsilon|^2 + \varepsilon \int_\Omega \eta^{2m+2}|D^{m+p+2}u_\varepsilon|^2 \le c\left(\tilde{E}_\varepsilon(u_\varepsilon, B_R)\right)^{\frac{1}{m}} R^{-2p-2},
$$

which finishes the proof.

□

With Theorem 3.4.6 and Sobolev embeddings we have

Corollary 3.4.7. *Let $\varepsilon, R > 0$, $\varepsilon < R^2$ and let u_ε be a smooth solution of (3.1.3). There exist $\delta_0, C, c > 0$ such that if $\tilde{E}_\varepsilon(u_\varepsilon, B_{32R}(x_0)) < \delta_0$ for some $x_0 \in \Omega$, then we have*

$$
||u_\varepsilon||_{C^{k,\alpha}(B_R(x_0))} \le C(R, \delta_0)
$$

for any $\alpha \in (0,1)$ and any $k \in \mathbb{N}$. Further we have

$$
\sum_{i=1}^{k} R^i ||D^i u_\varepsilon||_{L^\infty(B_R(x_0))} \le c\left(\tilde{E}_\varepsilon(u_\varepsilon, B_{32R}(x_0))\right)^{\frac{1}{2m}}.
$$

3.5 Convergence and blow-up

In this section we investigate the limiting process as ε tends to zero. Let $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ be a sequence of critical points of (1.0.14). We show that there exists a subsequence $\varepsilon_k \to 0$ such that $u_{\varepsilon_k}\colon \Omega \to N$ converges weakly in $W^{m,2}(\Omega, N)$ to a smooth m-polyharmonic map u_0 and that away from finitely many points $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ converges in C^s for all $s \in \mathbb{N}$. Afterwards we will perform a blow-up around these singular points .

Remark 3.5.1. *Since $E_{\varepsilon_k}(u_{\varepsilon_k})$ is uniformly bounded by (3.4.1) there exists a subsequence which we will again call $\varepsilon_k \to 0$ such that $u_{\varepsilon_k} \rightharpoonup u_0$ weakly in $W^{m,2}(\Omega, N)$.*

Lemma 3.5.2. *Choose $\delta_0 > 0$ and let*

$$
\Sigma = \bigcap_{R>0} \{x \in \Omega : \limsup_{k\to\infty} \tilde{E}_{\varepsilon_k}(u_{\varepsilon_k}, B_R(x)) > \delta_0\} \tag{3.5.1}
$$

be the set of energy concentration points. Then $|\Sigma| < \infty$.

Proof. To show that there are at most finitely many points where energy concentrates, we take points $x_1, \dots, x_L \in \Sigma$ and choose a radius $R > 0$ small enough so that $B_R(x_i) \cap B_R(x_j) = \emptyset$ for $i \ne j$.

Using (3.4.1) and (3.4.4) we have

$$C \geq \tilde{E}_{\varepsilon_k}(u_{\varepsilon_k}, \Omega) \geq \sum_{i=1}^{L} \tilde{E}_{\varepsilon_k}(u_{\varepsilon_k}, B_R(x_i)) \geq L\delta_0.$$

Since $\delta_0 > 0$ it follows that $L < \infty$.

□

Corollary 3.4.7 implies strong convergence in a neighborhood of $x \in \Omega \setminus \Sigma$. More precisely, there exists a radius $R > 0$ such that $u_{\varepsilon_k} \to u_0$ in $C^s(B_R(x), N)$ for all $s \in \mathbb{N}$. After covering $\Omega \setminus \Sigma$ with such balls we arrive at $u_{\varepsilon_k} \to u_0$ in $C^s_{loc}(\Omega \setminus \Sigma, N)$ for all $s \in \mathbb{N}$. Since $\varepsilon_k \to 0$ it follows that u_0 is a weakly m-polyharmonic map on $\Omega \setminus \Sigma$ which is smooth away from finitely many points.

Next we remove the singular points and extend u_0 to a smooth m-polyharmonic map on all of Ω. To do this we assume without loss of generality $\Sigma = \{x_0\}$. (see [63] for the harmonic and [4] for the biharmonic case)
Let $\psi \in C_c^\infty(\Omega, \mathbb{R}^d)$, $\sigma > 0$ and $\eta_\sigma \in C_c^\infty(\Omega, \mathbb{R}^d)$ with

$$0 \leq \eta_\sigma \leq 1, \qquad \eta_\sigma(x) = 0 \text{ for } |x - x_0| \leq \sigma, \qquad \eta_\sigma(x) = 1 \text{ for } |x - x_0| \geq 2\sigma, \qquad |D^j \eta_\sigma| \leq \frac{c}{\sigma^j}$$

and let $A_\sigma = B_{2\sigma} \setminus B_\sigma$ be the annulus around x_0. Then $\varphi = \eta_\sigma \psi \in C_c^\infty(\Omega \setminus \{x_0\}, \mathbb{R}^d)$. Using this as a test function in the m-polyharmonic map equation and applying Faà di Bruno's formula yields

$$\begin{aligned}
0 &= \int_\Omega \langle D^m u_0, D^m(P_{u_0}[\varphi])\rangle = \int_\Omega \langle D^m u_0, D^m(P_{u_0}[\eta_\sigma\psi])\rangle \\
&= \int_\Omega \left\langle D^m u_0, \sum_{\substack{k_i \in \mathbb{N}_0,\ 1 \leq i \leq m \\ k_1 + 2k_2 + \ldots + mk_m = m}} \frac{m!}{k_1! \cdot \ldots \cdot k_m!} (D^{k_1+\ldots+k_m} P_{u_0})[\eta_\sigma\psi] \prod_{i=1}^m \left(\frac{1}{i!} D^i(\eta_\sigma\psi)\right)^{k_i} \right\rangle \\
&= \int_\Omega \left\langle D^m u_0, \sum_{\substack{k_i \in \mathbb{N}_0,\ 1 \leq i \leq m \\ k_1 + 2k_2 + \\ \ldots + mk_m = m}} \frac{m!}{k_1! \cdot \ldots \cdot k_m!} (D^{k_1+\ldots+k_m} P_{u_0})[\eta_\sigma\psi] \prod_{i=1}^m \left(\frac{1}{i!} \sum_{n=0}^{i} D^n \eta_\sigma D^{i-n}\psi\right)^{k_i} \right\rangle \\
&= \int_\Omega \left\langle D^m u_0, \sum_{\substack{k_i \in \mathbb{N}_0,\ 1 \leq i \leq m \\ k_1 + 2k_2 + \ldots + mk_m = m}} \frac{m!}{k_1! \cdot \ldots \cdot k_m!} (D^{k_1+\ldots+k_m} P_{u_0})[\eta_\sigma\psi] \prod_{i=1}^m \left(\frac{1}{i!} \eta_\sigma D^i\psi\right)^{k_i} \right\rangle \\
&\quad + \int_\Omega \left\langle D^m u_0, \sum_{\substack{k_i \in \mathbb{N}_0, \\ 1 \leq i \leq m \\ k_1 + 2k_2 + \ldots \\ \ldots + mk_m = m}} \frac{m!}{k_1! \cdot \ldots \cdot k_m!} (D^{k_1+\ldots+k_m} P_{u_0})[\eta_\sigma\psi] \prod_{i=1}^m \left(\frac{1}{i!} \sum_{n=1}^{i} D^n \eta_\sigma D^{i-n}\psi\right)^{k_i} \right\rangle
\end{aligned}$$

The first part tends to

$$\int_\Omega \langle D^m u_0, D^m(P_{u_0}[\psi])\rangle$$

as $\sigma \to 0$. We estimate the second part further

$$\left| \int_\Omega \left\langle D^m u_0, \sum_{\substack{k_i \in \mathbb{N}_0, \\ 1 \leq i \leq m \\ k_1 + 2k_2 + \ldots \\ \ldots + mk_m = m}} \frac{m!}{k_1! \cdot \ldots \cdot k_m!} (D^{k_1+\ldots+k_m} P_{u_0})[\eta_\sigma\psi] \prod_{i=1}^m \left(\frac{1}{i!} \sum_{n=1}^{i} D^n \eta_\sigma D^{i-n}\psi\right)^{k_i} \right\rangle \right|$$

$$\leq c \int_{A_\sigma} |D^m u_0| \sum_{\substack{k_i\in\mathbb{N}_0,\ 1\leq i\leq m\\ k_1+2k_2+\ldots+mk_m=m}} \prod_{i=1}^{m}\left(\sum_{n=1}^{i}|D^n\eta_\sigma||D^{i-n}\psi|\right)^{k_i}$$

$$\leq c \int_{A_\sigma} |D^m u_0| \sum_{\substack{k_i\in\mathbb{N}_0,\ 1\leq i\leq m\\ k_1+2k_2+\ldots+mk_m=m}} \prod_{i=1}^{m}\left(\sum_{n=1}^{i}|D^n\eta_\sigma|\right)^{k_i}$$

$$\leq c \int_{A_\sigma} |D^m u_0| \sum_{\substack{k_i\in\mathbb{N}_0,\ 1\leq i\leq m\\ k_1+2k_2+\ldots+mk_m=m}} \left(\sigma^{-k_1}\cdot(\sigma^{-k_2}+\sigma^{-2k_2})\cdot\ldots\cdot(\sigma^{-k_m}+\ldots+\sigma^{-mk_m})\right)$$

$$\leq c \int_{A_\sigma} |D^m u_0| \sum_{\substack{k_i\in\mathbb{N}_0,\ 1\leq i\leq m\\ k_1+2k_2+\ldots+mk_m=m}} \left(\sigma^{-k_1-2k_2-\ldots-mk_m}\right)$$

$$\leq c \int_{A_\sigma} |D^m u_0|\sigma^{-m}$$

$$\leq c \left(\int_{A_\sigma} |D^m u_0|^2\right)^{\frac{1}{2}} \left(\int_{A_\sigma} \sigma^{-2m}\right)^{\frac{1}{2}}$$

$$\leq c \left(\int_{A_\sigma} |D^m u_0|^2\right)^{\frac{1}{2}} \to 0 \qquad \text{for } \sigma\to 0.$$

Thus we have

$$0 = \int_\Omega \langle D^m u_0, D^m(P_{u_0}[\psi])\rangle \qquad \forall\psi\in C_c^\infty(\Omega,\mathbb{R}^d)$$

and u_0 is a weakly m-polyharmonic map on all of Ω. Gastel and Scheven [21] showed that such a map is smooth.

Next we study the behavior of $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ at the energy concentration points as $k\to\infty$ more closely. Since this is a local problem and the set Σ is finite we will assume in the following that $\Sigma=\{x_0\}$.

Lemma 3.5.3. *Let $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ be a sequence of critical points of (1.0.14) with $\varepsilon_k\to 0$ and let $x_0\in\Sigma$. Then there exist sequences $(t_k)_{k\in\mathbb{N}}\in\mathbb{R}^+$, $t_k\to 0$ and $(x_k)_{k\in\mathbb{N}}\in\Omega$, $x_k\to x_0$ and a nontrivial, smooth quasi-m-polyharmonic map $\omega^1: S^{2m}\to N$ such that*

$$\tilde{\varepsilon}_k := \frac{\varepsilon_k}{t_k^2}\to 0 \qquad \textit{and} \tag{3.5.2}$$

$$u_{\varepsilon_k}(x_k+t_k\cdot)\to\omega^1 \qquad \textit{in } C^s_{loc}(\mathbb{R}^{2m},N) \textit{ for all } s\in\mathbb{N}. \tag{3.5.3}$$

Proof. We apply a technique introduced by Brezis and Coron in [9]. Let $\delta_0>0$ and $R_0>0$ such that $B_{2R_0}(x_0)\subset\Omega$ and $\tilde{E}(u_{\varepsilon_k},B_{R_0}(x_0))>\delta_0$. For $t\leq R_0$ we define the maximal concentration function

$$M_{\varepsilon_k}(t) = \max_{y\in B_{R_0}(x_0)}\left[\int_{B_t(y)}(\varepsilon_k|D^{m+1}u_{\varepsilon_k}|^2+|D^m u_{\varepsilon_k}|^2)+\int_{B_t(y)}\sum_{i=1}^{m-1}|D^i u_{\varepsilon_k}|^{\frac{2m}{i}}\right].$$

For $\varepsilon_k\to 0$ there exist sequences $x_k\to x_0$ and $t_k\to 0$ such that

$$M_{\varepsilon_k}(t_k)=\frac{\delta_0}{2}=\tilde{E}_{\varepsilon_k}(u_{\varepsilon_k},B_{t_k}(x_k)) \qquad \forall k\in\mathbb{N}.$$

Without loss of generality we can assume that $x_k \in B_{R_0}(x_0)$, otherwise we choose a subsequence $(x_k)_{k\in\mathbb{N}}$ that is contained in $B_{R_0}(x_0)$. Now we perform a blow-up around the point x_k by setting

$$v_k : B_{\frac{R_0}{t_k}}(0) \subset \mathbb{R}^{2m} \to N, \qquad v_k(x) := u_{\varepsilon_k}(x_k + t_k x).$$

Then v_k solves (3.1.2) if we replace ε_k by $\frac{\varepsilon_k}{t_k^2} =: \tilde{\varepsilon}_k$. Further we have

$$\tilde{E}_{\tilde{\varepsilon}_k}(v_k, B_1(z)) \leq \frac{\delta_0}{2} \qquad \text{for all } z \in B_{\frac{R_0}{2t_k}}(0) \tag{3.5.4}$$

and equality if $z = 0$. Because we are working with the local energy $\tilde{E}_\varepsilon$ we can apply Corollary 3.4.7 for all $s \in \mathbb{N}$

$$\sum_{i=1}^{s} ||D^i v_k||_{L^\infty(B_{\frac{1}{32}}(z))} = \sum_{i=1}^{s} t_k^i ||D^i u_{\varepsilon_k}||_{L^\infty(B_{\frac{t_k}{32}}(x_k+t_k z))} \leq c \sqrt[2m]{\delta_0}.$$

Hence $v_k \to v$ in $C^\infty_{loc}(\mathbb{R}^{2m}, N)$ (up to subsequence) and v is not constant because of (3.5.4).

Next we want to show that $\tilde{\varepsilon}_k \to 0$ and that v is a non-trivial m-polyharmonic map. First we assume that $\tilde{\varepsilon}_k \to c_1$, with $0 < c_1 < \infty$. We rewrite (3.1.2) in terms of v_k and $\tilde{\varepsilon}_k$

$$-\tilde{\varepsilon}_k \Delta^{m+1} v_k + \Delta^m v_k = F[v_k],$$

where $F[\cdot]$ is defined as the right hand side of (3.1.2). The uniform energy bound for critical points in (3.4.1) and (3.4.4) implies

$$0 < \tilde{\varepsilon}_k \int_{B_{\frac{R_0}{t_k}}(0)} |D^{m+1} v_k|^2 + \int_{B_{\frac{R_0}{t_k}}(0)} |D^m v_k|^2 + \sum_{i=1}^{m-1} \int_{B_{\frac{R_0}{t_k}}(0)} |D^i v_k|^{\frac{2m}{i}} \leq c.$$

From the above it follows that v_k converges to a smooth bounded map $v : \mathbb{R}^{2m} \to N \hookrightarrow \mathbb{R}^d$ in C^s_{loc} for all $s \in \mathbb{N}$. Further v satisfies

$$-c_1 \Delta^{m+1} v + \Delta^m v = F[v] \quad \text{on } \mathbb{R}^{2m} \tag{3.5.5}$$

and

$$0 < c_1 \int_{\mathbb{R}^{2m}} |D^{m+1} v|^2 + \int_{\mathbb{R}^{2m}} |D^m v|^2 + \sum_{i=1}^{m-1} \int_{\mathbb{R}^{2m}} |D^i v|^{\frac{2m}{i}} \leq c \tag{3.5.6}$$

in the limit. We define a cut-off function $\phi^l \in C^\infty_c(\mathbb{R}^{2m})$ with

$$0 \leq \phi^l \leq 1, \quad \phi^l = 1 \text{ on } B_l(0), \quad \phi^l = 0 \text{ on } \mathbb{R}^{2m} \setminus B_{2l}(0) \quad \text{and} \quad ||D^s \phi^l||_{L^\infty} \leq \frac{c}{l^s}.$$

We multiply (3.5.5) with $\phi^l(x) x \cdot Dv(x)$ and integrate over $\mathbb{R}^{2m}$. Observe that the right hand side of (3.5.5) lies in the normal space of N and vanishes when multiplied by $Dv \in T_v N$. Thus

$$0 = \int_{\mathbb{R}^{2m}} \langle \Delta^m v(x) - c_1 \Delta^{m+1} v(x), \phi^l(x) x \cdot Dv(x) \rangle dx. \tag{3.5.7}$$

m even: In the following we assume that m is even. Integration by parts yields

$$\int_{\mathbb{R}^{2m}} \Delta^m v(x) \phi^l(x) x \cdot Dv(x) dx = \int_{\mathbb{R}^{2m}} \Delta^{\frac{m}{2}} v(x) \Delta^{\frac{m}{2}} \Big(\phi^l(x) x \cdot Dv(x) \Big) dx$$

$$
\begin{aligned}
&\leq \frac{1}{2}\int_{\mathbb{R}^{2m}} \partial_j(|\Delta^{\frac{m}{2}} v|^2)\phi^l(x)x^j dx + m\int_{\mathbb{R}^{2m}} |\Delta^{\frac{m}{2}} v|^2\phi^l(x)dx\\
&\quad + c\sum_{k=0}^{m-1}\frac{1}{l^k}\int_{B_{2l}\setminus B_l} |\Delta^{\frac{m}{2}} v||D^{m-k}v|\\
&\leq c\left(1+\sum_{k=1}^{m-1}\frac{1}{l^k}\right)\int_{B_{2l}\setminus B_l}\sum_{i=0}^{m-1}|D^{m-i}v|^2
\end{aligned}
$$

and

$$
\begin{aligned}
&c_1\int_{\mathbb{R}^{2m}} \Delta^{m+1}v(x)\phi^l(x)x\cdot Dv(x)dx\\
&= -\frac{c_1}{2}\int_{\mathbb{R}^{2m}} \partial_j\left(|D\Delta^{\frac{m}{2}} v|^2\right)\phi^l(x)x^j dx - c_1(m+1)\int_{\mathbb{R}^{2m}} |D\Delta^{\frac{m}{2}} v|^2\phi^l(x)dx\\
&\quad - c_1c\int_{\mathbb{R}^{2m}} D\Delta^{\frac{m}{2}} v(x)\Delta^{\frac{m-2}{2}}D\cdot(D\phi^l(x)Dv(x))dx\\
&\quad - c_1c\int_{\mathbb{R}^{2m}} D\Delta^{\frac{m}{2}} v(x)D\Delta^{\frac{m-2}{2}}D\cdot(D\phi^l(x)Dv(x))xdx\\
&= -c_1\int_{\mathbb{R}^{2m}} |D\Delta^{\frac{m}{2}} v|^2\phi^l(x)dx + \frac{c_1}{2}\int_{\mathbb{R}^{2m}} |D\Delta^{\frac{m}{2}} v|^2\partial_j\phi^l(x)x^j dx\\
&\quad - c_1c\int_{\mathbb{R}^{2m}} D\Delta^{\frac{m}{2}} v(x)\Delta^{\frac{m-2}{2}}D\cdot(D\phi^l(x)Dv(x))dx\\
&\quad - c_1c\int_{\mathbb{R}^{2m}} D\Delta^{\frac{m}{2}} v(x)D\Delta^{\frac{m-2}{2}}D\cdot(D\phi^l(x)Dv(x))xdx.
\end{aligned}
$$

Inserting both of these terms into (3.5.7) yields

$$
\begin{aligned}
c_1\int_{B_l} |D\Delta^{\frac{m}{2}} v|^2 &\leq c_1\int_{\mathbb{R}^{2m}} |D\Delta^{\frac{m}{2}} v|^2\phi^l(x)dx\\
&= \frac{c_1}{2}\int_{\mathbb{R}^{2m}} |D\Delta^{\frac{m}{2}} v|^2\partial_j\phi^l(x)x^j dx\\
&\quad - \int_{\mathbb{R}^{2m}} \Delta^m v(x)\phi^l(x)x\cdot Dv(x)dx\\
&\quad - c_1c\int_{\mathbb{R}^{2m}} D\Delta^{\frac{m}{2}} v(x)\Delta^{\frac{m-2}{2}}D\cdot(D\phi^l(x)Dv(x))dx\\
&\quad - c_1c\int_{\mathbb{R}^{2m}} D\Delta^{\frac{m}{2}} v(x)D\Delta^{\frac{m-2}{2}}D\cdot(D\phi^l(x)Dv(x))xdx\\
&\leq c\left(1+\sum_{k=1}^{m}\frac{1}{l^k}\right)\int_{B_{2l}\setminus B_l}\sum_{i=0}^{m}|D^{m+1-i}v|^2.
\end{aligned}
$$

The right-hand side tends to zero as $l\to\infty$ because the $W^{m+1,2}$-norm of v is bounded on $\mathbb{R}^{2m}$ by assumption (3.5.6). Thus we have $D\Delta^{\frac{m}{2}} v\equiv 0$ on $\mathbb{R}^{2m}$ and $\Delta^{\frac{m}{2}} v\equiv C$. This constant C must be zero because $\tilde{E}_{\tilde{\varepsilon}}(v)$ is bounded on all of $\mathbb{R}^{2m}$. Subsequently we have $D\Delta^{\frac{m-2}{2}} v\equiv C$ and again C must be zero by (3.5.6). Iteratively we get that $Dv\equiv 0$ and thus $v\equiv const.$ which is a contradiction to (3.5.6).

Next we assume that $\tilde{\varepsilon}_k\to\infty$. Then, for all $R>0$,

$$
0\leq\int_{B_R(0)} |D\Delta^{\frac{m}{2}} v_k|^2 \leq \frac{c}{\tilde{\varepsilon}_k}E_{\tilde{\varepsilon}_k}(v_k,B_R)\leq\frac{c}{\tilde{\varepsilon}_k}\to 0 \qquad \text{as } k\to\infty.
$$

This implies that $D\Delta^{\frac{m}{2}} v\equiv 0$ on $\mathbb{R}^{2m}$ and $\Delta^{\frac{m}{2}} v\equiv C$. With the same argument as before we get the contradiction. Therefore $\tilde{\varepsilon}_k$ must converge to zero as $k \to \infty$ and with (3.5.6) v is in fact a smooth non-trivial m-polyharmonic map from $\mathbb{R}^{2m}$ to N with finite energy.

We can lift this map to a so-called quasi-m-polyharmonic map from S^{2m} to N by using stereographic projection (for $m = 2$ see [41] Theorem 2.2 or [80]).
First we need to extend v to a map on $\mathbb{R}^{2m} \cup \{\infty\}$. We define the inversion on the unit sphere

$$\sigma\colon \mathbb{R}^{2m} \cup \{\infty\} \to \mathbb{R}^{2m} \cup \{\infty\}, \qquad \sigma(x) := \begin{cases} \frac{x}{||x||^2}, & x \notin \{0, \infty\}, \\ \infty, & x = 0, \\ 0, & x = \infty. \end{cases}$$

Then $v \circ \sigma \in C^\infty((\mathbb{R}^{2m} \cup \{\infty\}) \setminus \{0\}, N)$. Now we apply the extension result from earlier in the section to remove the singularity at the origin. Then $v \circ \sigma \in C^\infty(\mathbb{R}^{2m} \cup \{\infty\}, N)$. Applying the inversion again yields $v \in C^\infty(\mathbb{R}^{2m} \cup \{\infty\}, N)$.
Now let $\Pi\colon S^{2m} \to \mathbb{R}^{2m} \cup \{\infty\}$ be the stereographic projection and set $\omega^1 = v \circ \Pi\colon S^{2m} \to N$. Then $\omega^1 \in C^\infty(S^{2m}, N)$. ω^1 does not satisfy (1.0.3) but

$$\left(\prod_{k=0}^{m-1} (-\Delta_{S^{2m}} + k(2m - k - 1)) \right) \omega^1 \perp T_{\omega^1} N, \tag{3.5.8}$$

where $\prod_{k=0}^{m-1}(-\Delta_{S^{2m}} + k(2m-k-1))$ is the $2m$-dimensional Paneitz operator on S^{2m} (see [12] or [8]). Maps $\omega\colon S^{2m} \to \mathbb{R}^d$ that satisfy (3.5.8) are called quasi-m-polyharmonic maps on S^{2m}.

<u>m odd:</u> We repeat the calculations from before.

$$\begin{aligned}
&\int_{\mathbb{R}^{2m}} \Delta^m v(x)\phi^l(x) x \cdot Dv(x) dx \\
&\quad \le -\int_{\mathbb{R}^{2m}} D\Delta^{\frac{m-1}{2}} v(x) D\Delta^{\frac{m-1}{2}} \partial_j v(x) \phi^l(x) x^j dx \\
&\qquad - m \int_{\mathbb{R}^{2m}} D\Delta^{\frac{m-1}{2}} v(x) D\Delta^{\frac{m-3}{2}} \partial_i \partial_j v(x) \phi^l(x) \partial_i x^j dx \\
&\qquad + c \sum_{k=1}^{m} \int_{\mathbb{R}^{2m}} |D\Delta^{\frac{m-1}{2}} v||D^k \phi^l||D^{m-k+1} v||x| dx \\
&\qquad + c \sum_{k=1}^{m-1} \int_{\mathbb{R}^{2m}} |D\Delta^{\frac{m-1}{2}} v||D^k \phi^l||D^{m-k} v| dx \\
&\quad \le -\frac{1}{2} \int_{\mathbb{R}^{2m}} \partial_j (|D\Delta^{\frac{m-1}{2}} v|^2) \phi^l(x) x^j dx - m \int_{\mathbb{R}^{2m}} |D\Delta^{\frac{m-1}{2}} v|^2 \phi^l(x) dx \\
&\qquad + c \sum_{k=0}^{m-1} \frac{1}{l^k} \int_{B_{2l} \setminus B_l} |D\Delta^{\frac{m-1}{2}} v||D^{m-k} v| \\
&\quad \le c \left(1 + \sum_{k=1}^{m-1} \frac{1}{l^k} \right) \int_{B_{2l} \setminus B_l} \sum_{i=0}^{m-1} |D^{m-i} v|^2
\end{aligned}$$

and

$$\begin{aligned}
&c_1 \int_{\mathbb{R}^{2m}} \Delta^{m+1} v \phi^l(x) x \cdot Dv(x) dx \\
&= \frac{c_1}{2} \int_{\mathbb{R}^{2m}} \partial_j (|\Delta^{\frac{m+1}{2}} v|^2) \phi^l(x) x^j dx + c_1(m+1) \int_{R^{2m}} |\Delta^{\frac{m+1}{2}} v|^2 \phi^l(x) dx
\end{aligned}$$

$$
\begin{aligned}
&+ c_1 c \int_{\mathbb{R}^{2m}} \Delta^{\frac{m+1}{2}} v(x) \Delta^{\frac{m-1}{2}} (D\phi^l(x) Dv(x)) dx \\
&+ c_1 c \int_{\mathbb{R}^{2m}} \Delta^{\frac{m+1}{2}} v(x) \Delta^{\frac{m-1}{2}} D \cdot (D\phi^l(x) Dv(x)) x dx \\
= c_1 & \int_{\mathbb{R}^{2m}} |\Delta^{\frac{m+1}{2}} v|^2 \phi^l - \frac{c_1}{2} \int_{\mathbb{R}^{2m}} |\Delta^{\frac{m+1}{2}} v|^2 \partial_j \phi^l(x) x^j dx \\
&+ c_1 c \int_{\mathbb{R}^{2m}} \Delta^{\frac{m+1}{2}} v(x) \Delta^{\frac{m-1}{2}} (D\phi^l(x) Dv(x)) dx \\
&+ c_1 c \int_{\mathbb{R}^{2m}} \Delta^{\frac{m+1}{2}} v(x) \Delta^{\frac{m-1}{2}} D \cdot (D\phi^l(x) Dv(x)) x dx.
\end{aligned}
$$

Inserting this into (3.5.7)

$$
\begin{aligned}
c_1 \int_{B_l(0)} |\Delta^{\frac{m+1}{2}} v|^2 \leq c_1 & \int_{\mathbb{R}^{2m}} |\Delta^{\frac{m+1}{2}} v|^2 \phi^l(x) dx \\
&= \int_{\mathbb{R}^{2m}} \Delta^m v(x) \phi^l(x) x \cdot Dv(x) + \frac{c_1}{2} \int_{\mathbb{R}^{2m}} |\Delta^{\frac{m+1}{2}} v|^2 \partial_j \phi^l(x) x^j dx \\
&\quad - c_1 c \int_{\mathbb{R}^{2m}} \Delta^{\frac{m+1}{2}} v(x) \Delta^{\frac{m-1}{2}} (D\phi^l(x) Dv(x)) dx \\
&\quad - c_1 c \int_{\mathbb{R}^{2m}} \Delta^{\frac{m+1}{2}} v(x) \Delta^{\frac{m-1}{2}} D \cdot (D\phi^l(x) Dv(x)) x dx \\
&\leq c \left(1 + \sum_{k=1}^{m} \frac{1}{l^k} \right) \int_{B_{2l} \setminus B_l} \sum_{i=0}^{m} |D^{m+1-i} v|^2
\end{aligned}
$$

and the Lemma follows with the same argument as in the even case.

□

3.6 Energy identity for $N = S^{d-1}$

Now we want to prove the energy identity in Theorem 1.0.3 and Theorem 1.0.4. Note that if $N \hookrightarrow \mathbb{R}^d$ contains no non-trivial quasi-m-polyharmonic $2m$-sphere, the energy identity is trivial.
In this section we show the energy identity for the target manifold S^{d-1}. In section 3.7 we will prove the general case assuming the entropy condition (1.0.18). The first results of this section, Lemma 3.6.2 and (3.6.12), hold in both cases.
By an argument of Ding and Tian [14] (p.552) it is enough to prove the identity under the assumption that only one bubble forms along the sequence. Theorem 1.0.3 follows with an induction argument.

Theorem 3.6.1. *Let $(u_\varepsilon)_\varepsilon$ satisfy all assumptions of Theorem 1.0.3, assume that $\Sigma = \{x_0\}$ and assume that only one smooth, non-trivial quasi-m-polyharmonic map $\omega^1 : S^{2m} \to N$ forms. Then there exists a sequence $\varepsilon_k \to 0$ such that*

$$
\lim_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k}) = E_0(u_0) + E_0(\omega^1). \tag{3.6.1}
$$

By our previous results we know that there exist $R_0 > 0$, sequences $\varepsilon_k \to 0$ and $x_k \to x_0$ such that $u_{\varepsilon_k} \to u_0$ in $C^s_{loc}(\Omega \setminus B_{R_0}(x_0), N)$ for all $s \in \mathbb{N}$. After performing a blow-up around x_0 we saw in Lemma 3.5.3 that $u_{\varepsilon_k}(x_k + t_k \cdot) \to \omega^1$ in $C^s_{loc}(\mathbb{R}^{2m}, N)$ for all $s \in \mathbb{N}$. Thus, for every constant $M > 1$ we have

$$
E_{\varepsilon_k}(u_{\varepsilon_k}, B_{R_0}(x_k) \setminus B_{R_0/M}(x_k)) + E_{\varepsilon_k}(u_{\varepsilon_k}, B_{MRt_k}(x_k) \setminus B_{Rt_k}(x_k)) \to 0 \tag{3.6.2}
$$

as $k \to \infty$, $R \to \infty$ and $R_0 \to 0$. Note that $R_0 >> Rt_k$ and the quotients of the radii are constant. We have strong convergence on the first annulus and convergence to a bubble on the second annulus.

With this (3.6.1) is equivalent to

$$\lim_{R_0\to 0}\lim_{R\to\infty}\lim_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k}, B_{R_0}(x_k)\setminus B_{Rt_k}(x_k)) = 0. \tag{3.6.3}$$

Lemma 3.6.2. *Let $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ be a sequence of critical points, $\Sigma = \{x_0\}$ and there exists only one smooth, non-trivial quasi-m-polyharmonic map $\omega^1\colon S^{2m}\to N$. Then there exists a subsequence $\varepsilon_k \to 0$ such that*

$$\varepsilon_k \int_{B_{\frac{R_0}{4}}\setminus B_{2Rt_k}(x_0)} |D^{m+1}u_{\varepsilon_k}|^2 \longrightarrow 0 \tag{3.6.4}$$

as $k, R\to\infty$ and $t_k, R_0 \to 0$.

Proof. Wlog $x_0 = 0$. We fix $\delta > 0$. By our assumption there exists only one non-trivial quasi-m-polyharmonic sphere. Hence we claim that there exists $k_1 \in \mathbb{N}$ such that for all $k > k_1$

$$\tilde{E}_{\varepsilon_k}(u_{\varepsilon_k}, B_{2r}\setminus B_r) < \delta \qquad \text{with} \quad Rt_k \le r \le \frac{R_0}{2}. \tag{3.6.5}$$

To prove this claim we argue by contradiction and assume that for $k\to\infty$ there exists $r_k \in [Rt_k, \frac{R_0}{2}]$ such that

$$\tilde{E}_{\varepsilon_k}(u_{\varepsilon_k}, B_{2r_k}\setminus B_{r_k}) = \max_{r\in[Rt_k,\frac{R_0}{2}]} \tilde{E}_{\varepsilon_k}(u_{\varepsilon_k}, B_{2r}\setminus B_r) \ge \delta. \tag{3.6.6}$$

By the definition of the different radii and (3.6.2) we have

$$\frac{R_0}{r_k}\to\infty \qquad \text{and} \qquad \frac{Rt_k}{r_k}\to 0. \tag{3.6.7}$$

As in the previous chapter we perform a blow up by defining

$$w_k\colon B_{\frac{R_0}{r_k}}\setminus B_{\frac{Rt_k}{r_k}} \to N, \qquad w_k(x) = u_{\varepsilon_k}(r_k x)$$

and w_k solves (3.1.3) with ε_k replaced by $\bar{\varepsilon}_k = \frac{\varepsilon_k}{r_k^2}$. By (3.6.6)

$$\tilde{E}_{\bar{\varepsilon}_k}(w_k, B_2\setminus B_1) \ge \delta \tag{3.6.8}$$

and with (3.4.1) and (3.4.4)

$$\tilde{E}_{\bar{\varepsilon}_k}(w_k, B_{\frac{R_0}{r_k}}\setminus B_{\frac{Rt_k}{r_k}}) \le c. \tag{3.6.9}$$

By Lemma 3.5.3 and (3.6.7) we have for large enough k

$$\bar{\varepsilon}_k < \tilde{\varepsilon}_k \to 0. \tag{3.6.10}$$

With (3.6.7), (3.6.9) and (3.6.10) we can argue as in the previous section and assume that $w_k \rightharpoonup w_0$ weakly in $W^{m,2}_{loc}(\mathbb{R}^{2m}\setminus\{0\}, N)$, where w_0 is a weak m-polyharmonic map from $\mathbb{R}^{2m}$ to N with finite energy. There are two possibilities now: Either there exists no point of energy concentration. Then there exists a radius $\tilde{r} > 0$ such that

$$\sup_{k\in\mathbb{N}}\sup_{x\in B_{64}\setminus B_{1/64}} \tilde{E}_{\bar{\varepsilon}_k}(w_k, B_{\tilde{r}}(x)) < \delta_0$$

for $\delta_0 > 0$ small. With Corollary 3.4.7 and a covering argument we have

$$w_k \to w_0 \qquad \text{in } \ C^s(B_2 \setminus B_1, N) \ \ \forall s \in \mathbb{N}.$$

Note that w_0 is non-trivial because of the definition of $\tilde{E}_\varepsilon$ and (3.6.8). Since $\mathbb{R}^{2m} \setminus \{0\}$ is conformally equivalent to $S^{2m} \setminus \{N, S\}$ we can argue as at the end of the last section, remove the singularities and lift w_0 to a smooth non-trivial quasi-m-polyharmonic map from S^{2m} to N. However, this is a contradiction to our assumption that there exists only one bubble ω^1.

On the other hand, if there exists a point $y \in B_{64} \setminus B_{1/64}$ with

$$\tilde{E}_{\bar{\varepsilon}_k}(w_k, B_{\tilde{r}}(y)) > \delta_0 \qquad \forall\ k \in \mathbb{N},\ \forall\ \tilde{r} > 0,$$

we proceed as in the previous chapter by performing a blow-up around y and conclude that there exists a non-trivial quasi-m-polyharmonic map from S^{2m} into N. This is again a contradiction to our assumption of a single bubble. Therefore (3.6.5) must hold.

Next we choose $y \in \Omega$ with $|y| \in [2Rt_k, \frac{R_0}{4}]$. Then $\frac{2|y|}{3}$, $\frac{4|y|}{3} \in (Rt_k, \frac{R_0}{2})$ and $B_{\frac{|y|}{3}}(y) \subset B_{\frac{4|y|}{3}} \setminus B_{\frac{2|y|}{3}}$. With (3.6.5) we have

$$\tilde{E}_{\varepsilon_k}(u_{\varepsilon_k}, B_{\frac{|y|}{3}}(y)) \leq \delta$$

and Corollary 3.4.7 yields

$$\sum_{i=1}^{l} |x|^i |D^i u_{\varepsilon_k}|(x) \leq c \sqrt[2m]{\delta} \qquad\qquad \forall\ x \in B_{\frac{|y|}{3}}(y),\ \forall l \in \mathbb{N}.$$

Covering the annulus $B_{\frac{R_0}{4}} \setminus B_{2Rt_k}$ with such balls we get

$$\sum_{i=1}^{l} |x|^i |D^i u_{\varepsilon_k}|(x) \leq c \sqrt[2m]{\delta} \qquad\qquad \forall\ 2Rt_k \leq |x| \leq \frac{R_0}{4},\ \forall l \in \mathbb{N}. \tag{3.6.11}$$

Together with (3.5.2) we have

$$\begin{aligned} \varepsilon_k \int_{B_{\frac{R_0}{4}} \setminus B_{2Rt_k}} |D^{m+1} u_{\varepsilon_k}|^2 &\leq c \sqrt[m]{\delta} \varepsilon_k \int_{B_{\frac{R_0}{4}} \setminus B_{2Rt_k}} \frac{1}{|x|^{2(m+1)}} \\ &\leq c \sqrt[m]{\delta} \frac{\tilde{\varepsilon}_k}{R^2} \longrightarrow 0 \end{aligned}$$

as $k \to \infty$ and $R \to \infty$.

□

It remains to show that

$$\int_{B_{\frac{R_0}{4}} \setminus B_{2Rt_k}} |D^m u_{\varepsilon_k}|^2 \to 0 \qquad \text{as } k \to \infty.$$

To show this note that we can bound $|D^m u_{\varepsilon_k}|$ uniformly in the Lorentz space $L^{2,\infty}$ (see Definition A.1.1) by using (3.6.11)

$$||D^m u_{\varepsilon_k}||_{L^{2,\infty}(B_{\frac{R_0}{4}}\setminus B_{2Rt_k})} \le c\sqrt[2m]{\delta}\left\|\frac{1}{|x|^m}\right\|_{L^{2,\infty}(\mathbb{R}^{2m})} \le c\sqrt[2m]{\delta} \qquad \forall k\in\mathbb{N} \tag{3.6.12}$$

(see Lemma 5.1.10 in [45]). Since $L^{2,\infty}$ is the dual space to $L^{2,1}$ with respect to the L^2-scalar product (see Lemma A.1.5) it suffices to show that $||D^m u_{\varepsilon_k}||_{L^{2,1}(B_{\frac{R_0}{4}}\setminus B_{2Rt_k})}$ is uniformly bounded.

As stated in chapter 2, Lorentz spaces $L^{p,q}$ are interpolations spaces of L^p-spaces. In short, we have the inclusions

$$L^{p,1}\subset L^{p,q_1}\subset L^{p,p}=L^p\subset L^{p,q_2}\subset L^{p,\infty}$$

with $1<p<\infty$, $1<q_1<p<q_2<\infty$. For a brief introduction into Lorentz space theory, definitions and properties we use, and references in the literature see Appendix A. One of the main tools in this section is the following Sobolev embedding (see [55] Theorem 8.1 and [31] Theorem 3.3.10).

Lemma 3.6.3. *Let $f\colon\mathbb{R}^{2m}\to\mathbb{R}$ and $Df\in L^{p,q}(\mathbb{R}^{2m})$ for some $1<p<2m$ and $1\le q\le\infty$. Then $f\in L^{\frac{2mp}{2m-p},q}(\mathbb{R}^{2m})$ and*

$$||f||_{L^{\frac{2mp}{2m-p},q}(\mathbb{R}^{2m})}\le c||Df||_{L^{p,q}(\mathbb{R}^{2m})}.$$

In particular, if $Df\in L^1(\mathbb{R}^{2m})$ we have $f\in L^{\frac{2m}{2m-1},1}(\mathbb{R}^{2m})$ and

$$||f||_{L^{\frac{2m}{2m-1},1}(\mathbb{R}^{2m})}\le c||Df||_{L^1(\mathbb{R}^{2m})}.$$

Applying the first estimate repeatedly and using $||f||_{L^{p,p}(\mathbb{R}^{2m})}\le c||f||_{L^p(\mathbb{R}^{2m})}$ yields

Corollary 3.6.4. *Let $f:\mathbb{R}^{2m}\to\mathbb{R}$ as above. For $k=1,...,m$*

$$||D^k f||_{L^{\frac{2m}{k},2}(\mathbb{R}^{2m})}\le c||D^m f||_{L^2(\mathbb{R}^{2m})}.$$

Up until now our results hold for arbitrary compact target manifolds N. From now on we assume $N=S^{d-1}$. The following Lemma introduces an alternative formulation of the Euler-Lagrange equation (see [80] for $m=2$).

Lemma 3.6.5. *Let $N=S^{d-1}$. Then (3.1.3) is equivalent to*

$$\begin{aligned}
&\Delta^{m-1}D\cdot(Du\wedge u)\\
&=\sum_{i=1}^{\frac{m}{2}-1}\Delta^{\frac{m}{2}+i-1}D\cdot\left(\Delta^{\frac{m}{2}-i}u\wedge Du\right)+\sum_{i=1}^{\frac{m}{2}}\Delta^{\frac{m}{2}+i-1}\left(D\Delta^{\frac{m}{2}-i}u\wedge Du\right)\\
&\quad-\sum_{i=1}^{\frac{m}{2}-1}\binom{m}{2i}\Delta^{i}\left(\Delta^{\frac{m}{2}}u\wedge\Delta^{\frac{m}{2}-i}u\right)+\sum_{i=1}^{\frac{m}{2}}\binom{m}{2i-1}\Delta^{i-1}D\cdot\left(\Delta^{\frac{m}{2}}u\wedge D\Delta^{\frac{m}{2}-i}u\right)\\
&\quad+\varepsilon\left[\sum_{j=1}^{\frac{m}{2}+1}\binom{m+1}{2j-1}\Delta^{j-1}D\cdot\left(D\Delta^{\frac{m}{2}}u\wedge\Delta^{\frac{m}{2}+1-j}u\right)-\sum_{j=1}^{\frac{m}{2}}\binom{m+1}{2j}\Delta^{j}\left(D\Delta^{\frac{m}{2}}u\wedge D\Delta^{\frac{m}{2}-j}u\right)\right],
\end{aligned}\tag{3.6.13}$$

if m is even and

$$
\begin{aligned}
&\Delta^{m-1} D\cdot(Du\wedge u)\\
&=\sum_{i=1}^{\frac{m-1}{2}}\Delta^{\frac{m-1}{2}+i}\left(D\Delta^{\frac{m-1}{2}-i}u\wedge Du\right)+\sum_{i=1}^{\frac{m-1}{2}}\Delta^{\frac{m-1}{2}+i-1}D\cdot\left(\Delta^{\frac{m+1}{2}-i}u\wedge Du\right)\\
&\quad-\sum_{i=1}^{\frac{m-1}{2}}\binom{m}{2i-1}\Delta^{i-1}D\cdot\left(D\Delta^{\frac{m-1}{2}}u\wedge\Delta^{\frac{m+1}{2}-i}u\right)\\
&\quad+\sum_{i=1}^{\frac{m-1}{2}}\binom{m}{2i}\Delta^{i}\left(D\Delta^{\frac{m-1}{2}}u\wedge D\Delta^{\frac{m-1}{2}-i}u\right)\\
&\quad+\varepsilon\Bigg[\sum_{j=1}^{\frac{m+1}{2}}\binom{m+1}{2j}\Delta^{j}\left(\Delta^{\frac{m+1}{2}}u\wedge\Delta^{\frac{m+1}{2}-j}u\right)-\sum_{j=1}^{\frac{m+1}{2}}\binom{m+1}{2j-1}\Delta^{j-1}D\cdot\left(\Delta^{\frac{m+1}{2}}u\wedge D\Delta^{\frac{m+1}{2}-j}u\right)\Bigg]
\end{aligned}
\tag{3.6.14}
$$

if m is odd.

Proof. Let $u\in C^\infty(\Omega,S^{d-1})$ be a solution of (3.1.2). We know from (3.1.1) that $(\Delta^m u-\varepsilon\Delta^{m+1}u)\perp T_uS^{d-1}$, thus

$$(\Delta^m u-\varepsilon\Delta^{m+1}u)\wedge u=0.$$

Using the product rule we have for even m:

$$\Delta^{m+1}u\wedge u=\sum_{j=1}^{\frac{m}{2}+1}\binom{m+1}{2j-1}\Delta^{j-1}D\cdot\left(D\Delta^{\frac{m}{2}}\wedge\Delta^{\frac{m}{2}+1-j}u\right)-\sum_{j=1}^{\frac{m}{2}}\binom{m+1}{2j}\Delta^{j}\left(D\Delta^{\frac{m}{2}}\wedge D\Delta^{\frac{m}{2}-j}u\right)$$

and

$$
\begin{aligned}
\Delta^m u\wedge u&=\sum_{i=1}^{\frac{m}{2}}\binom{m}{2i}\Delta^{i}\left(\Delta^{\frac{m}{2}}u\wedge\Delta^{\frac{m}{2}-i}u\right)-\sum_{i=1}^{\frac{m}{2}}\binom{m}{2i-1}\Delta^{i-1}D\cdot\left(\Delta^{\frac{m}{2}}u\wedge D\Delta^{\frac{m}{2}-i}u\right)\\
&=\Delta^{m-1}D\cdot(Du\wedge u)-\sum_{i=1}^{\frac{m}{2}-1}\Delta^{\frac{m}{2}+i-1}D\cdot\left(\Delta^{\frac{m}{2}-i}u\wedge Du\right)\\
&\quad-\sum_{i=1}^{\frac{m}{2}}\Delta^{\frac{m}{2}+i-1}\left(D\Delta^{\frac{m}{2}-i}u\wedge Du\right)+\sum_{i=1}^{\frac{m}{2}-1}\binom{m}{2i}\Delta^{i}\left(\Delta^{\frac{m}{2}}u\wedge\Delta^{\frac{m}{2}-i}u\right)\\
&\quad-\sum_{i=1}^{\frac{m}{2}}\binom{m}{2i-1}\Delta^{i-1}D\cdot\left(\Delta^{\frac{m}{2}}u\wedge D\Delta^{\frac{m}{2}-i}u\right).
\end{aligned}
$$

For odd m we have

$$
\begin{aligned}
\Delta^{m+1}u\wedge u&=\sum_{j=1}^{\frac{m+1}{2}}\binom{m+1}{2j}\Delta^{j}\left(\Delta^{\frac{m+1}{2}}u\wedge\Delta^{\frac{m+1}{2}-j}u\right)\\
&\quad-\sum_{j=1}^{\frac{m+1}{2}}\binom{m+1}{2j-1}\Delta^{j-1}D\cdot\left(\Delta^{\frac{m+1}{2}}u\wedge D\Delta^{\frac{m+1}{2}-j}u\right)
\end{aligned}
$$

and

$$\begin{aligned}
\Delta^m u \wedge u &= \sum_{i=1}^{\frac{m-1}{2}} \binom{m}{2i} \Delta^i \left(D\Delta^{\frac{m-1}{2}} u \wedge D\Delta^{\frac{m-1}{2}-i} u \right) \\
&\quad - \sum_{i=1}^{\frac{m+1}{2}} \binom{m}{2i-1} \Delta^{i-1} D \cdot \left(D\Delta^{\frac{m-1}{2}} u \wedge \Delta^{\frac{m+1}{2}-i} u \right) \\
&= -\Delta^{\frac{m-1}{2}} D \cdot \left(D\Delta^{\frac{m-1}{2}} u \wedge u \right) - \sum_{i=1}^{\frac{m-1}{2}} \binom{m}{2i-1} \Delta^{i-1} D \cdot \left(D\Delta^{\frac{m-1}{2}} u \wedge \Delta^{\frac{m+1}{2}-i} u \right) \\
&\quad + \sum_{i=1}^{\frac{m-1}{2}} \binom{m}{2i} \Delta^i \left(D\Delta^{\frac{m-1}{2}} u \wedge D\Delta^{\frac{m-1}{2}-i} u \right) \\
&= -\Delta^{m-1} D \cdot (Du \wedge u) + \sum_{i=1}^{\frac{m-1}{2}} \Delta^{\frac{m-1}{2}+i} \left(D\Delta^{\frac{m-1}{2}-i} u \wedge Du \right) \\
&\quad + \sum_{i=1}^{\frac{m-1}{2}} \Delta^{\frac{m-1}{2}+i-1} D \cdot \left(\Delta^{\frac{m+1}{2}-i} u \wedge Du \right) \\
&\quad - \sum_{i=1}^{\frac{m-1}{2}} \binom{m}{2i-1} \Delta^{i-1} D \cdot \left(D\Delta^{\frac{m-1}{2}} u \wedge \Delta^{\frac{m+1}{2}-i} u \right) \\
&\quad + \sum_{i=1}^{\frac{m-1}{2}} \binom{m}{2i} \Delta^i \left(D\Delta^{\frac{m-1}{2}} u \wedge D\Delta^{\frac{m-1}{2}-i} u \right).
\end{aligned}$$

□

Proof of Theorem 3.6.1. To show the uniform $L^{2,1}$-bound of $D^m u_{\varepsilon_k}$ in the neck region, we adapt the methods of Wang [80] and Wang/Zheng [81]. Since $u_{\varepsilon_k}\colon \Omega \to S^{d-1}$ we have $|u_{\varepsilon_k}| = 1$ and the general Leibniz rule yields

$$0 = D^{m-1}(u_{\varepsilon_k} \cdot Du_{\varepsilon_k}) = D^m u_{\varepsilon_k} \cdot u_{\varepsilon_k} + \sum_{i=1}^{m-1} \binom{m-1}{i} D^i u_{\varepsilon_k} \cdot D^{m-i} u_{\varepsilon_k}.$$

Thus

$$\begin{aligned}
|D^m u_{\varepsilon_k}| &\le |D^m u_{\varepsilon_k} \cdot u_{\varepsilon_k}| + |D^m u_{\varepsilon_k} \wedge u_{\varepsilon_k}| \\
&\le c \sum_{i=1}^{m-1} |D^i u_{\varepsilon_k}||D^{m-i} u_{\varepsilon_k}| + \left| D^{m-1}(Du_{\varepsilon_k} \wedge u_{\varepsilon_k}) \right| + c \sum_{i=1}^{m-2} |D^i (D^{m-1-i} u_{\varepsilon_k} \wedge Du_{\varepsilon_k})| \\
&\le c \sum_{i=1}^{m-1} |D^i u_{\varepsilon_k}||D^{m-i} u_{\varepsilon_k}| + \left| D^{m-1}(Du_{\varepsilon_k} \wedge u_{\varepsilon_k}) \right|.
\end{aligned}$$

With this we estimate

$$\begin{aligned}
||D^m u_{\varepsilon_k}||_{L^{2,1}(B_{\frac{R_0}{8}} \setminus B_{2Rt_k})} &\le c \sum_{i=1}^{m-1} \left\| |D^i u_{\varepsilon_k}| \cdot |D^{m-i} u_{\varepsilon_k}| \right\|_{L^{2,1}(B_{\frac{R_0}{8}} \setminus B_{2Rt_k})} \\
&\quad + c||D^{m-1}(Du_{\varepsilon_k} \wedge u_{\varepsilon_k})||_{L^{2,1}(B_{\frac{R_0}{8}} \setminus B_{2Rt_k})}. \qquad (3.6.15)
\end{aligned}$$

Next we define a smooth cut-off function $\eta \in C_c^\infty(\mathbb{R}^{2m})$ so that

$$\eta = 1 \quad \text{on } B_{\frac{R_0}{8}}, \qquad 0 \le \eta \le 1 \quad \text{on } B_{\frac{R_0}{4}} \setminus B_{\frac{R_0}{8}}, \qquad \eta = 0 \quad \text{on } \mathbb{R}^{2m} \setminus B_{\frac{R_0}{4}}$$

and $|D^l\eta| \le \frac{c}{R_0^l}$, $\forall l \in \mathbb{N}$. We define $\tilde{u}_{\varepsilon_k} \in C_0^\infty(\mathbb{R}^{2m}, S^{d-1})$ to be the continuous extension with compact support $\tilde{u}_{\varepsilon_k} = \eta u_{\varepsilon_k}$. Then $\tilde{u}_{\varepsilon_k}$ satisfies

$$\tilde{u}_{\varepsilon_k} = u_{\varepsilon_k} \qquad \text{on } B_{\frac{R_0}{8}},$$

$$||D^i\tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{i}}(\mathbb{R}^{2m})} \le c\sum_{l=1}^{i} ||D^l u_{\varepsilon_k}||_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})} + c||u_{\varepsilon_k}||_{L^\infty(B_{\frac{R_0}{4}})} \tag{3.6.16}$$

for $i = 1, ..., m$. To see this, we apply Hölder's inequality ($p_l = \frac{i}{l}$, $q_l = \frac{i}{i-l}$ for $i = 1, ..., m$, $l = 1, ..., i$)

$$\begin{aligned}
||D^i\tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{i}}(\mathbb{R}^{2m})} &\le c\sum_{l=0}^{i} \big|\big||D^l u_{\varepsilon_k}| \cdot |D^{i-l}\eta|\big|\big|_{L^{\frac{2m}{i}}(B_{\frac{R_0}{4}})} \\
&\le c\sum_{l=1}^{i} ||D^l u_{\varepsilon_k}||_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})} ||D^{i-l}\eta||_{L^{\frac{2m}{i-l}}(B_{\frac{R_0}{4}})} + c||u_{\varepsilon_k}||_{L^\infty(B_{\frac{R_0}{4}})} \\
&\le c\sum_{l=1}^{i} ||D^l u_{\varepsilon_k}||_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})} + c||u_{\varepsilon_k}||_{L^\infty(B_{\frac{R_0}{4}})}.
\end{aligned}$$

Similarly we have with $p_l = \frac{m}{l}$, $q_l = \frac{m}{m-l}$, $l = 1, ..., m$

$$\begin{aligned}
||D^{m+1}\tilde{u}_{\varepsilon_k}||_{L^2(\mathbb{R}^{2m})} &\le ||D^{m+1}u_{\varepsilon_k}||_{L^2(B_{\frac{R_0}{4}})} + \sum_{l=0}^{m} ||D^l u_{\varepsilon_k} D^{m+1-l}\eta||_{L^2(B_{\frac{R_0}{4}})} \\
&\le ||D^{m+1}u_{\varepsilon_k}||_{L^2(B_{\frac{R_0}{4}})} + \frac{c}{R_0}||u_{\varepsilon_k}||_{L^\infty(B_{\frac{R_0}{4}})} \\
&\quad + \sum_{l=1}^{m} ||D^l u_{\varepsilon_k}||_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})} ||D^{m+1-l}\eta||_{L^{\frac{2m}{m-l}}(B_{\frac{R_0}{4}})} \\
&\le ||D^{m+1}u_{\varepsilon_k}||_{L^2(B_{\frac{R_0}{4}})} + cR_0^{-1}\left(\sum_{l=1}^{m} ||D^l u_{\varepsilon_k}||_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||_{L^\infty(B_{\frac{R_0}{4}})}\right).
\end{aligned} \tag{3.6.17}$$

Note that the constants in (3.6.16) and (3.6.17) are independent of R_0. Now we can estimate the first term in (3.6.15) using Hölder's inequality for Lorentz spaces (see Lemma A.1.3) with $p_i = \frac{2m}{i}$, $q_i = \frac{2m}{m-i}$, $i = 1, ..., m-1$, Corollary 3.6.4 and (3.6.16)

$$\begin{aligned}
\sum_{i=1}^{m-1} \big|\big||D^i u_{\varepsilon_k}| \cdot |D^{m-i}u_{\varepsilon_k}|\big|\big|_{L^{2,1}(B_{\frac{R_0}{8}} \setminus B_{2Rt_k})} &\le c\sum_{i=1}^{m-1} ||D^i\tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{i},2}(\mathbb{R}^{2m})} ||D^{m-i}\tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{m-i},2}(\mathbb{R}^{2m})} \\
&\le c||D^m\tilde{u}_{\varepsilon_k}||^2_{L^2(\mathbb{R}^{2m})} \\
&\le c\left(\sum_{i=1}^{m} ||D^i u_{\varepsilon_k}||^2_{L^{\frac{2m}{i}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right).
\end{aligned} \tag{3.6.18}$$

To estimate the second term in (3.6.15), we first assume that m is even. The case where m is odd follows in the same way with minor modifications.

<u>*m even:*</u> With (3.6.13) we have

$$
\begin{aligned}
&\Delta^{m-1}\Big(D\cdot\big(D\tilde{u}_{\varepsilon_k}\wedge\tilde{u}_{\varepsilon_k}\big)\Big)\\
&=\sum_{i=1}^{\frac{m}{2}-1}\Delta^{\frac{m}{2}+i-1}D\cdot\big(\Delta^{\frac{m}{2}-i}\tilde{u}_{\varepsilon_k}\wedge D\tilde{u}_{\varepsilon_k}\big)+\sum_{i=1}^{\frac{m}{2}}\Delta^{\frac{m}{2}+i-1}\big(D\Delta^{\frac{m}{2}-i}\tilde{u}_{\varepsilon_k}\wedge D\tilde{u}_{\varepsilon_k}\big)\\
&\quad-\sum_{i=1}^{\frac{m}{2}-1}\binom{m}{2i}\Delta^{i}\big(\Delta^{\frac{m}{2}}\tilde{u}_{\varepsilon_k}\wedge\Delta^{\frac{m}{2}-i}\tilde{u}_{\varepsilon_k}\big)\\
&\quad+\sum_{i=1}^{\frac{m}{2}}\binom{m}{2i-1}\Delta^{i-1}D\cdot\big(\Delta^{\frac{m}{2}}\tilde{u}_{\varepsilon_k}\wedge D\Delta^{\frac{m}{2}-i}\tilde{u}_{\varepsilon_k}\big)\\
&\quad+\varepsilon\Bigg[\sum_{j=1}^{\frac{m}{2}+1}\binom{m+1}{2j-1}\Delta^{j-1}D\cdot\big(D\Delta^{\frac{m}{2}}\tilde{u}_{\varepsilon_k}\wedge\Delta^{\frac{m}{2}+1-j}\tilde{u}_{\varepsilon_k}\big)\\
&\qquad-\sum_{j=1}^{\frac{m}{2}}\binom{m+1}{2j}\Delta^{j}\big(D\Delta^{\frac{m}{2}}\tilde{u}_{\varepsilon_k}\wedge D\Delta^{\frac{m}{2}-j}\tilde{u}_{\varepsilon_k}\big)\Bigg]\qquad\text{on } B_{\frac{R_0}{8}}.
\end{aligned}\tag{3.6.19}
$$

Let d be the exterior derivative or differential and d^* be the codifferential. We define the Hodge decomposition of the one-form

$$D\tilde{u}_{\varepsilon_k}\wedge\tilde{u}_{\varepsilon_k}(=d\tilde{u}_{\varepsilon_k}\wedge\tilde{u}_{\varepsilon_k})\in W^{m-1,2}(\mathbb{R}^{2m},\Lambda^1(\mathbb{R}^{2m})).$$

(For an adaption to Lorentz spaces see Gastel/Scheven [21] Lemma 3.1.) Given $D\tilde{u}_{\varepsilon_k}\wedge\tilde{u}_{\varepsilon_k}$ there exist $\Phi_k\in W^{m,2}(\mathbb{R}^{2m})$ and a two-form $\Psi_k\in W^{m,2}(\mathbb{R}^{2m},\Lambda^2(\mathbb{R}^{2m}))$ such that

$$d\tilde{u}_{\varepsilon_k}\wedge\tilde{u}_{\varepsilon_k}=d\Phi_k+d^*\Psi_k\qquad\text{in } \mathbb{R}^{2m},\tag{3.6.20}$$

with $d\Psi_k=0,\ d^*\Phi_k=0$. To get a uniform bound on $||D^{m-1}(Du_{\varepsilon_k}\wedge u_{\varepsilon_k})||_{L^{2,1}(B_{\frac{R_0}{8}}\setminus B_{2Rt_k})}$ it suffices to bound $||D^m\Phi_k||_{L^{2,1}(\mathbb{R}^{2m})}$ and $||D^m\Psi_k||_{L^{2,1}(\mathbb{R}^{2m})}$. For Ψ_k we have

$$\Delta^{\frac{m}{2}}\Psi_k=(d^*d+dd^*)^{\frac{m}{2}}\Psi_k=(dd^*)^{\frac{m-2}{2}}(d\tilde{u}_{\varepsilon_k}\wedge d\tilde{u}_{\varepsilon_k})=\Delta^{\frac{m-2}{2}}(d\tilde{u}_{\varepsilon_k}\wedge d\tilde{u}_{\varepsilon_k}).$$

The Calderon-Zygmund inequality (see [31]), Hölder's inequality for Lorenz spaces ($p_i=\frac{2m}{i},\ q_i=\frac{2m}{m-i},\ i=1,...,m-1$), Corollary 3.6.4 and (3.6.16) yield

$$
\begin{aligned}
||D^m\Psi_k||_{L^{2,1}(\mathbb{R}^{2m})}&\le c\sum_{i=1}^{m-1}\Big|\Big||D^i\tilde{u}_{\varepsilon_k}|\cdot|D^{m-i}\tilde{u}_{\varepsilon_k}|\Big|\Big|_{L^{2,1}(\mathbb{R}^{2m})}\\
&\le c\sum_{i=1}^{m-1}||D^i\tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{i},2}(\mathbb{R}^{2m})}||D^{m-i}\tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{m-i},2}(\mathbb{R}^{2m})}\\
&\le c\left(\sum_{i=1}^{m}||D^iu_{\varepsilon_k}||^2_{L^{\frac{2m}{i}}(B_{\frac{R_0}{4}})}+||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right).
\end{aligned}\tag{3.6.21}
$$

For Φ_k we first note that

$$\Delta^{m-1}D\cdot(D\tilde{u}_{\varepsilon_k}\wedge\tilde{u}_{\varepsilon_k})=\Delta^{m-1}d^*(d\Phi_k+d^*\Psi_k)=\Delta^m\Phi_k.$$

Together with (3.6.19) we have

$$\begin{aligned}\Delta^m \Phi_k = &\sum_{i=1}^{\frac{m}{2}-1} \Delta^{\frac{m}{2}+i-1} D \cdot \left(\Delta^{\frac{m}{2}-i} \tilde{u}_{\varepsilon_k} \wedge D\tilde{u}_{\varepsilon_k}\right) + \sum_{i=1}^{\frac{m}{2}} \Delta^{\frac{m}{2}+i-1} \left(D\Delta^{\frac{m}{2}-i} \tilde{u}_{\varepsilon_k} \wedge D\tilde{u}_{\varepsilon_k}\right) \\ &- \sum_{i=0}^{\frac{m}{2}-1} \binom{m}{2i} \Delta^i \left(\Delta^{\frac{m}{2}} \tilde{u}_{\varepsilon_k} \wedge \Delta^{\frac{m}{2}-i} \tilde{u}_{\varepsilon_k}\right) \\ &+ \sum_{i=1}^{\frac{m}{2}} \binom{m}{2i-1} \Delta^{i-1} D \cdot \left(\Delta^{\frac{m}{2}} \tilde{u}_{\varepsilon_k} \wedge D\Delta^{\frac{m}{2}-i} \tilde{u}_{\varepsilon_k}\right) \\ &+ \varepsilon \Bigg[\sum_{j=1}^{\frac{m}{2}+1} \binom{m+1}{2j-1} \Delta^{j-1} D \cdot \left(D\Delta^{\frac{m}{2}} \tilde{u}_{\varepsilon_k} \wedge \Delta^{\frac{m}{2}+1-j} \tilde{u}_{\varepsilon_k}\right) \\ &\quad - \sum_{j=1}^{\frac{m}{2}} \binom{m+1}{2j} \Delta^j \left(D\Delta^{\frac{m}{2}} \tilde{u}_{\varepsilon_k} \wedge D\Delta^{\frac{m}{2}-j} \tilde{u}_{\varepsilon_k}\right) \Bigg] \qquad \text{in } B_{\frac{R_0}{8}}.\end{aligned}$$

For $i = 1, ..., \frac{m}{2} - 1$ we define

$$\begin{aligned}\xi_k^i(x) &= c_{2m} \int_{\mathbb{R}^{2m}} \ln|x-y| \Delta^{\frac{m}{2}+i-1} D \cdot \left(\Delta^{\frac{m}{2}-i} \tilde{u}_{\varepsilon_k} \wedge D\tilde{u}_{\varepsilon_k}\right)(y) dy, \\ \bar{\xi}_k^i(x) &= -c_{2m} \binom{m}{2i} \int_{\mathbb{R}^{2m}} \ln|x-y| \Delta^i \left(\Delta^{\frac{m}{2}} \tilde{u}_{\varepsilon_k} \wedge \Delta^{\frac{m}{2}-i} \tilde{u}_{\varepsilon_k}\right)(y) dy\end{aligned}$$

and for $i = 1, ..., \frac{m}{2}$

$$\begin{aligned}\hat{\xi}_k^i(x) &= c_{2m} \int_{\mathbb{R}^{2m}} \ln|x-y| \Delta^{\frac{m}{2}+i-1} \left(D\Delta^{\frac{m}{2}-i} \tilde{u}_{\varepsilon_k} \wedge D\tilde{u}_{\varepsilon_k}\right)(y) dy, \\ \tilde{\xi}_k^i(x) &= c_{2m} \binom{m}{2i-1} \int_{\mathbb{R}^{2m}} \ln|x-y| \Delta^{i-1} D \cdot \left(\Delta^{\frac{m}{2}} \tilde{u}_{\varepsilon_k} \wedge D\Delta^{\frac{m}{2}-i} \tilde{u}_{\varepsilon_k}\right)(y) dy\end{aligned}$$

in $\mathbb{R}^{2m}$. For $j = 1, ..., \frac{m}{2}$

$$\vartheta_k^j(x) = -\varepsilon_k c_{2m} \binom{m+1}{2j} \int_{\mathbb{R}^{2m}} \ln|x-y| \Delta^j \left(D\Delta^{\frac{m}{2}} \tilde{u}_{\varepsilon_k} \wedge D\Delta^{\frac{m}{2}-j} \tilde{u}_{\varepsilon_k}\right)(y) dy,$$

and for $j = 1, ..., \frac{m}{2} + 1$

$$\tilde{\vartheta}_k^j(x) = \varepsilon_k c_{2m} \binom{m+1}{2j-1} \int_{\mathbb{R}^{2m}} \ln|x-y| \Delta^{j-1} D \cdot \left(D\Delta^{\frac{m}{2}} \tilde{u}_{\varepsilon_k} \wedge \Delta^{\frac{m}{2}+1-j} \tilde{u}_{\varepsilon_k}\right)(y) dy$$

in $\mathbb{R}^{2m}$. We set

$$\gamma_k = \Phi_k - \sum_{i=1}^{\frac{m}{2}-1} (\xi_k^i + \bar{\xi}_k^i) - \sum_{i=1}^{\frac{m}{2}} (\tilde{\xi}_k^i + \hat{\xi}_k^i) - \sum_{j=1}^{\frac{m}{2}} \vartheta_k^j - \sum_{j=1}^{\frac{m}{2}+1} \tilde{\vartheta}_k^j. \tag{3.6.22}$$

Since $c_{2m} \ln|x-y|$ is the fundamental solution of Δ^m in $\mathbb{R}^{2m}$ we get

$$\Delta^m \gamma_k = 0 \qquad \text{in } B_{\frac{R_0}{8}}.$$

For $\xi_k^i, \bar{\xi}_k^i,\ i = 1, ..., \frac{m}{2} - 1$, we have with the Calderon-Zygmund inequality, Hölder's inequality for Lorentz spaces ($p_i = \frac{2m}{m-2i-b},\ q_i = \frac{2m}{2i-b}$ and $s_i = 2,\ t_i = \frac{2m}{m-2i}$) and Corollary 3.6.4

$$\begin{aligned}||D^m \xi_k^i||_{L^{2,1}(\mathbb{R}^{2m})} &\leq c \left\| \Delta^{i-1} D \cdot (\Delta^{\frac{m}{2}-i} \tilde{u}_{\varepsilon_k} \wedge D\tilde{u}_{\varepsilon_k}) \right\|_{L^{2,1}(\mathbb{R}^{2m})} \\ &\leq c \sum_{b=0}^{2i-1} \left\| |D^{m-2i+b} \tilde{u}_{\varepsilon_k}| \cdot |D^{2i-b} \tilde{u}_{\varepsilon_k}| \right\|_{L^{2,1}(\mathbb{R}^{2m})} \\ &\leq c \sum_{b=0}^{2i-1} \left\| D^{m-2i+b} \tilde{u}_{\varepsilon_k} \right\|_{L^{\frac{2m}{m-2i+b},2}(\mathbb{R}^{2m})} \left\| D^{2i-b} \tilde{u}_{\varepsilon_k} \right\|_{L^{\frac{2m}{2i-b},2}(\mathbb{R}^{2m})} \\ &\leq c ||D^m \tilde{u}_{\varepsilon_k}||^2_{L^2(\mathbb{R}^{2m})}\end{aligned}$$

and

$$\begin{aligned}||D^{2m-2i} \bar{\xi}_k^i||_{L^{\frac{2m}{2m-2i},1}(\mathbb{R}^{2m})} &\leq c \left\| |\Delta^{\frac{m}{2}} \tilde{u}_{\varepsilon_k}| \cdot |\Delta^{\frac{m-2i}{2}} \tilde{u}_{\varepsilon_k}| \right\|_{L^{\frac{2m}{2m-2i},1}(\mathbb{R}^{2m})} \\ &\leq c ||D^m \tilde{u}_{\varepsilon_k}||_{L^{2,2}(\mathbb{R}^{2m})} ||D^{m-2i} \tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{m-2i},2}(\mathbb{R}^{2m})} \\ &\leq c ||D^m \tilde{u}_{\varepsilon_k}||^2_{L^2(\mathbb{R}^{2m})}.\end{aligned}$$

Analogously we estimate

$$\begin{aligned}||D^m \hat{\xi}_k^i||_{L^{2,1}(\mathbb{R}^{2m})} &\leq c ||D^m \tilde{u}_{\varepsilon_k}||^2_{L^2(\mathbb{R}^{2m})} &&\text{and} \\ ||D^{2m-2i+1} \tilde{\xi}_k^i||_{L^{\frac{2m}{2m-2i+1},1}(\mathbb{R}^{2m})} &\leq c ||D^m \tilde{u}_{\varepsilon_k}||^2_{L^2(\mathbb{R}^{2m})} &&\text{for } i = 1, ..., \frac{m}{2}.\end{aligned}$$

With the embedding theorem for Lorentz spaces in Lemma 3.6.3 and (3.6.16) we have

$$\begin{aligned}&||D^m \xi_k^i||_{L^{2,1}(\mathbb{R}^{2m})} + ||D^m \bar{\xi}_k^i||_{L^{2,1}(\mathbb{R}^{2m})} + ||D^m \hat{\xi}_k^i||_{L^{2,1}(\mathbb{R}^{2m})} + ||D^m \tilde{\xi}_k^i||_{L^{2,1}(\mathbb{R}^{2m})} \\ &\leq c \left(\sum_{l=1}^{m} ||D^l u_{\varepsilon_k}||^2_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})} \right)\end{aligned} \tag{3.6.23}$$

for $i = 1, ..., \frac{m}{2} - 1$ or $i = 1, ..., \frac{m}{2}$ respectively.
For $\vartheta_k^j,\ j = 1, ..., \frac{m}{2}$, we use Hölder's inequality ($p_j = 2,\ q_j = \frac{2m}{m-2j}$), Corollary 3.6.4 and the extension property (3.6.17)

$$\begin{aligned}||D^{2m-2j} \vartheta_k^j||_{L^{\frac{2m}{2m-2j},1}(\mathbb{R}^{2m})} &\leq c\varepsilon_k \left\| |D\Delta^{\frac{m}{2}} \tilde{u}_{\varepsilon_k}| \cdot |D\Delta^{\frac{m}{2}-j} \tilde{u}_{\varepsilon_k}| \right\|_{L^{\frac{2m}{2m-2j},1}(\mathbb{R}^{2m})} \\ &\leq c\varepsilon_k ||D^{m+1} \tilde{u}_{\varepsilon_k}||_{L^{2,2}(\mathbb{R}^{2m})} ||D^{m-2j+1} \tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{m-2j},2}(\mathbb{R}^{2m})} \\ &\leq c\varepsilon_k ||D^{m+1} u_{\varepsilon_k}||^2_{L^2(B_{\frac{R_0}{4}})} \\ &\quad + \frac{c\varepsilon_k}{R_0^2} \left(\sum_{l=1}^{m} ||D^l u_{\varepsilon_k}||^2_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})} \right).\end{aligned}$$

Similarly we estimate $||D^{2m-2j+1} \tilde{\vartheta}_k^j||_{L^{\frac{2m}{2m-2j+1}}(\mathbb{R}^{2m})},\ j = 1, ..., \frac{m}{2} + 1$ and again with the embedding theorem for Lorentz spaces

$$\begin{aligned}||D^m \vartheta_k^j||_{L^{2,1}(\mathbb{R}^{2m})} + ||D^m \tilde{\vartheta}_k^j||_{L^{2,1}(\mathbb{R}^{2m})} &\leq c\varepsilon_k ||D^{m+1} u_{\varepsilon_k}||^2_{L^2(B_{\frac{R_0}{4}})} + \frac{c\varepsilon_k}{R_0^2} ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})} \\ &\quad + \frac{c\varepsilon_k}{R_0^2} \sum_{l=1}^{m} ||D^l u_{\varepsilon_k}||^2_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})}\end{aligned} \tag{3.6.24}$$

for $j = 1, ..., \frac{m}{2}$ or $\frac{m}{2} + 1$. Since $\Delta^m \gamma_k = 0$ on $B_{\frac{R_0}{8}}$, we can apply a standard estimate for m-polyharmonic maps (see Lemma A.1.6)

$$\begin{aligned}
||D^m \gamma_k||_{L^{2,1}(B_{\frac{R_0}{8}})} &\leq c||D^m \gamma_k||_{L^2(B_{\frac{R_0}{4}})} \\
&\leq c||D^m \Phi_k||_{L^2(B_{\frac{R_0}{4}})} \\
&+ c \sum_{i=1}^{\frac{m}{2}-1} \left(||D^m \xi_k^i||_{L^2(B_{\frac{R_0}{4}})} + ||D^m \bar{\xi}_k^i||_{L^2(B_{\frac{R_0}{4}})} \right) \\
&+ c \sum_{i=1}^{\frac{m}{2}} \left(||D^m \hat{\xi}_k^i||_{L^2(B_{\frac{R_0}{4}})} + ||D^m \tilde{\xi}_k^i||_{L^2(B_{\frac{R_0}{4}})} \right) \\
&+ c \sum_{j=1}^{\frac{m}{2}} ||D^m \vartheta_k^j||_{L^2(B_{\frac{R_0}{4}})} + c \sum_{j=1}^{\frac{m}{2}+1} ||D^m \tilde{\vartheta}_k^j||_{L^2(B_{\frac{R_0}{4}})}.
\end{aligned}$$

Since $L^{2,1} \hookrightarrow L^2$ we have estimated everything except for the first term. With (3.6.20)

$$\Delta^{\frac{m}{2}} \Phi_k = (dd^* + d^*d)^{\frac{m-2}{2}} d^*(D\tilde{u}_{\varepsilon_k} \wedge \tilde{u}_{\varepsilon_k} - d^* \Psi_k) = \Delta^{\frac{m-2}{2}} D \cdot (D\tilde{u}_{\varepsilon_k} \wedge \tilde{u}_{\varepsilon_k})$$

in $\mathbb{R}^{2m}$. With the Calderon-Zygmund inequality and (3.6.16)

$$\begin{aligned}
||D^m \Phi_k||_{L^2(\mathbb{R}^{2m})} &\leq c \sum_{i=0}^{m} \left|\left| |D^i \tilde{u}_{\varepsilon_k}| \cdot |D^{m-i} \tilde{u}_{\varepsilon_k}| \right|\right|_{L^2(\mathbb{R}^{2m})} \\
&\leq c \sum_{i=0}^{m} ||D^i \tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{i}}(\mathbb{R}^{2m})} ||D^{m-i} \tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{m-i}}(\mathbb{R}^{2m})} \\
&\leq c \left(\sum_{i=1}^{m} ||D^i u_{\varepsilon_k}||^2_{L^{\frac{2m}{i}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})} \right).
\end{aligned}$$

Using this together with (3.6.23) and (3.6.24) yields

$$\begin{aligned}
||D^m \gamma_k||_{L^{2,1}(B_{\frac{R_0}{8}})} \leq c &\left(\sum_{i=1}^{m} ||D^i u_{\varepsilon_k}||^2_{L^{\frac{2m}{i}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})} \right) + c\varepsilon_k ||D^{m+1} u_{\varepsilon_k}||^2_{L^2(B_{\frac{R_0}{4}})} \\
&+ \frac{c\varepsilon_k}{R_0^2} \left(\sum_{l=1}^{m} ||D^l u_{\varepsilon_k}||^2_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})} \right). \qquad (3.6.25)
\end{aligned}$$

Our goal was to get a uniform bound on $D^m \Phi_k$ in the $L^{2,1}$-norm. Using the decomposition of Φ_k in (3.6.22) and the estimates (3.6.23), (3.6.24) and (3.6.25) we arrive at

$$\begin{aligned}
||D^m \Phi_k||_{L^{2,1}(B_{\frac{R_0}{8}})} &\leq c||D^m \gamma_k||_{L^{2,1}(B_{\frac{R_0}{8}})} \\
&+ c \sum_{i=1}^{\frac{m}{2}-1} \left(||D^m \xi_k^i||_{L^{2,1}(B_{\frac{R_0}{8}})} + ||D^m \bar{\xi}_k^i||_{L^{2,1}(B_{\frac{R_0}{8}})} \right) \\
&+ c \sum_{i=1}^{\frac{m}{2}} \left(||D^m \hat{\xi}_k^i||_{L^{2,1}(B_{\frac{R_0}{8}})} + ||D^m \tilde{\xi}_k^i||_{L^{2,1}(B_{\frac{R_0}{8}})} \right) \\
&+ c \sum_{j=1}^{\frac{m}{2}} ||D^m \vartheta_k^j||_{L^{2,1}(B_{\frac{R_0}{8}})} + c \sum_{j=1}^{\frac{m}{2}+1} ||D^m \tilde{\vartheta}_k^j||_{L^{2,1}(B_{\frac{R_0}{8}})}
\end{aligned}$$

$$\leq c\left(\sum_{i=1}^{m}||D^i u_{\varepsilon_k}||^2_{L^{\frac{2m}{i}}(B_{\frac{R_0}{4}})}+||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right)$$
$$+\, c\varepsilon_k||D^{m+1}u_{\varepsilon_k}||^2_{L^2(B_{\frac{R_0}{4}})}$$
$$+\,\frac{c\varepsilon_k}{R_0^2}\left(\sum_{l=1}^{m}||D^l u_{\varepsilon_k}||^2_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})}+||u_{\varepsilon_k}||_{L^\infty(B_{\frac{R_0}{4}})}\right).$$

Going back to (3.6.15) we conclude with (3.6.18) and (3.6.21)

$$||D^m u_{\varepsilon_k}||_{L^{2,1}(B_{\frac{R_0}{8}}\setminus B_{2Rt_k})}\leq c\left(\sum_{i=1}^{m}||D^i u_{\varepsilon_k}||^2_{L^{\frac{2m}{i}}(B_{\frac{R_0}{4}})}+||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right)+c\varepsilon_k||D^{m+1}u_{\varepsilon_k}||^2_{L^2(B_{\frac{R_0}{4}})}$$
$$+\,\frac{c\varepsilon_k}{R_0^2}\left(\sum_{l=1}^{m}||D^l u_{\varepsilon_k}||^2_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})}+||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right).$$

This is uniformly bounded because of (3.4.1), (3.4.4) and (3.5.2). Since we can choose $\delta > 0$ in (3.6.12) arbitrarily small we get

$$||D^m u_{\varepsilon_k}||_{L^2(B_{\frac{R_0}{8}}\setminus B_{4Rt_k})}\leq c||D^m u_{\varepsilon_k}||_{L^{2,\infty}(B_{\frac{R_0}{8}}\setminus B_{4Rt_k})}||D^m u_{\varepsilon_k}||_{L^{2,1}(B_{\frac{R_0}{8}}\setminus B_{4Rt_k})}\leq c\sqrt[2m]{\delta}.$$

Together with (3.6.4) this shows

$$E_{\varepsilon_k}(u_{\varepsilon_k},B_{\frac{R_0}{8}}\setminus B_{4Rt_k})\to 0\quad\text{as } R,k\to\infty,\ R_0\to 0,$$

and Theorem 3.6.1 follows.

<u>*m odd:*</u>
If m is odd the proof needs some minor modifications. With (3.6.14) and (3.6.16) we have

$$\begin{aligned}
&\Delta^{m-1}D\cdot(D\tilde{u}_{\varepsilon_k}\wedge\tilde{u}_{\varepsilon_k})\\
&=\sum_{i=1}^{\frac{m-1}{2}}\Delta^{\frac{m-1}{2}+i}\left(D\Delta^{\frac{m-1}{2}-i}\tilde{u}_{\varepsilon_k}\wedge D\tilde{u}_{\varepsilon_k}\right)\\
&\quad+\sum_{i=1}^{\frac{m-1}{2}}\Delta^{\frac{m-1}{2}+i-1}D\cdot\left(\Delta^{\frac{m+1}{2}-i}\tilde{u}_{\varepsilon_k}\wedge D\tilde{u}_{\varepsilon_k}\right)\\
&\quad-\sum_{i=1}^{\frac{m-1}{2}}\binom{m}{2i-1}\Delta^{i-1}D\cdot\left(D\Delta^{\frac{m-1}{2}}\tilde{u}_{\varepsilon_k}\wedge\Delta^{\frac{m+1}{2}-i}\tilde{u}_{\varepsilon_k}\right)\\
&\quad+\sum_{i=1}^{\frac{m-1}{2}}\binom{m}{2i}\Delta^{i}\left(D\Delta^{\frac{m-1}{2}}\tilde{u}_{\varepsilon_k}\wedge D\Delta^{\frac{m-1}{2}-i}\tilde{u}_{\varepsilon_k}\right)\\
&\quad+\varepsilon\Bigg[\sum_{j=1}^{\frac{m+1}{2}}\binom{m+1}{2j}\Delta^{j}\left(\Delta^{\frac{m+1}{2}}\tilde{u}_{\varepsilon_k}\wedge\Delta^{\frac{m+1}{2}-j}\tilde{u}_{\varepsilon_k}\right)\\
&\qquad-\sum_{j=1}^{\frac{m+1}{2}}\binom{m+1}{2j-1}\Delta^{j-1}D\cdot\left(\Delta^{\frac{m+1}{2}}\tilde{u}_{\varepsilon_k}\wedge D\Delta^{\frac{m+1}{2}-j}\tilde{u}_{\varepsilon_k}\right)\Bigg]\quad\text{in } B_{\frac{R_0}{8}}.
\end{aligned}\tag{3.6.26}$$

As in the even case we introduce the Hodge decomposition

$$d\tilde{u}_{\varepsilon_k} \wedge \tilde{u}_{\varepsilon_k} = d\Phi_k + d^*\Psi_k \qquad \text{in } \mathbb{R}^{2m},$$

where $\Phi_k \in W^{m,2}(\mathbb{R}^{2m})$ and $\Psi_k \in W^{m,2}(\mathbb{R}^{2m}, \Lambda^2(\mathbb{R}^{2m}))$ is a two-form with

$$d\Psi_k = 0, \qquad d^*\Phi_k = 0.$$

To get a uniform bound on $||D^{m-1}(Du_{\varepsilon_k} \wedge u_{\varepsilon_k})||_{L^{2,1}(B_{\frac{R_0}{8}} \setminus B_{2Rt_k})}$ it suffices to bound $||D^m\Phi_k||_{L^{2,1}}$ and $||D^m\Psi_k||_{L^{2,1}}$ uniformly. For Ψ_k we have

$$\Delta^{\frac{m+1}{2}}\Psi_k = (dd^*)^{\frac{m+1}{2}}\Psi_k = (dd^*)^{\frac{m-1}{2}}(d\tilde{u}_{\varepsilon_k} \wedge d\tilde{u}_{\varepsilon_k}) = \Delta^{\frac{m-1}{2}}(d\tilde{u}_{\varepsilon_k} \wedge d\tilde{u}_{\varepsilon_k}).$$

If $m = 1$ we follow Lamm in [44] and apply the results of Coifman, Lions, Meyer and Semmes [13] to estimate

$$\begin{aligned} ||d\tilde{u}_{\varepsilon_k} \wedge d\tilde{u}_{\varepsilon_k}||_{\mathcal{H}^1(\mathbb{R}^2)} &\leq c||D\tilde{u}_{\varepsilon_k}||^2_{L^2(\mathbb{R}^2)} \\ &\leq c\left(||Du_{\varepsilon_k}||^2_{L^2(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right). \end{aligned}$$

(see Appendix A.3 for a definition of the Hardy space $\mathcal{H}^1$) With the work of Fefferman and Stein [70] we have

$$||\Psi_k||_{W^{2,1}(\mathbb{R}^2)} \leq c\left(||Du_{\varepsilon_k}||^2_{L^2(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right)$$

and thus

$$||D\Psi_k||_{L^{2,1}(\mathbb{R}^2)} \leq c\left(||Du_{\varepsilon_k}||^2_{L^2(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right).$$

If $m \geq 3$ we proceed as in (3.6.21) and use the Calderon-Zygmund inequality, Corollary 3.6.4 and (3.6.16) to estimate

$$\begin{aligned} ||D^{m+1}\Psi_k||_{L^{\frac{2m}{m+1},1}(\mathbb{R}^{2m})} &\leq \sum_{i=1}^{m-1} \left\| |D^{m-i}\tilde{u}_{\varepsilon_k}| \cdot |D^{i+1}\tilde{u}_{\varepsilon_k}| \right\|_{L^{\frac{2m}{m+1},1}(\mathbb{R}^{2m})} \\ &\leq c\sum_{i=1}^{m-1} ||D^{m-i}\tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{m-i},2}(\mathbb{R}^{2m})} ||D^{i+1}\tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{i+1},2}(\mathbb{R}^{2m})} \\ &\leq c||D^m\tilde{u}_{\varepsilon_k}||^2_{L^2(\mathbb{R}^{2m})} \\ &\leq c\left(\sum_{i=1}^{m} ||D^i u_{\varepsilon_k}||^2_{L^{\frac{2m}{i}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right). \end{aligned}$$

With the embedding theorem for Lorentz spaces in Lemma 3.6.3 we have

$$\begin{aligned} ||D^m\Psi_k||_{L^{2,1}(\mathbb{R}^{2m})} &\leq c||D^{m+1}\Psi_k||_{L^{\frac{2m}{m+1},1}(\mathbb{R}^{2m})} \\ &\leq c\left(\sum_{i=1}^{m} ||D^i u_{\varepsilon_k}||^2_{L^{\frac{2m}{i}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right). \end{aligned} \tag{3.6.27}$$

For Φ_k we first note that

$$\begin{aligned}\Delta^{m-1}D\cdot(D\tilde{u}_{\varepsilon_k}\wedge\tilde{u}_{\varepsilon_k}) &= \Delta^{m-1}d^*(d\Phi_k+d^*\Psi_k) = (d^*d+dd^*)^{m-1}(d^*d\Phi_k)\\ &= \Delta^m\Phi_k,\end{aligned}$$

since $d^*\Phi_k=0$. Together with (3.6.26) we have

$$\begin{aligned}\Delta^m\Phi_k = &\sum_{i=1}^{\frac{m-1}{2}}\Delta^{\frac{m-1}{2}+i}\left(D\Delta^{\frac{m-1}{2}-i}\tilde{u}_{\varepsilon_k}\wedge D\tilde{u}_{\varepsilon_k}\right)\\
&+\sum_{i=1}^{\frac{m-1}{2}}\Delta^{\frac{m-1}{2}+i-1}D\cdot\left(\Delta^{\frac{m+1}{2}-i}\tilde{u}_{\varepsilon_k}\wedge D\tilde{u}_{\varepsilon_k}\right)\\
&-\sum_{i=1}^{\frac{m-1}{2}}\binom{m}{2i-1}\Delta^{i-1}D\cdot\left(D\Delta^{\frac{m-1}{2}}\tilde{u}_{\varepsilon_k}\wedge\Delta^{\frac{m+1}{2}-i}\tilde{u}_{\varepsilon_k}\right)\\
&+\sum_{i=1}^{\frac{m-1}{2}}\binom{m}{2i}\Delta^{i}\left(D\Delta^{\frac{m-1}{2}}\tilde{u}_{\varepsilon_k}\wedge D\Delta^{\frac{m-1}{2}-i}\tilde{u}_{\varepsilon_k}\right)\\
&+\varepsilon\Bigg[\sum_{j=1}^{\frac{m+1}{2}}\binom{m+1}{2j}\Delta^{j}\left(\Delta^{\frac{m+1}{2}}\tilde{u}_{\varepsilon_k}\wedge\Delta^{\frac{m+1}{2}-j}\tilde{u}_{\varepsilon_k}\right)\\
&\quad-\sum_{j=1}^{\frac{m+1}{2}}\binom{m+1}{2j-1}\Delta^{j-1}D\cdot\left(\Delta^{\frac{m+1}{2}}\tilde{u}_{\varepsilon_k}\wedge D\Delta^{\frac{m+1}{2}-j}\tilde{u}_{\varepsilon_k}\right)\Bigg] \qquad \text{in } B_{\frac{R_0}{8}}.\end{aligned}$$

Similar to the even case we define for $i=1,...,\frac{m-1}{2}$

$$\begin{aligned}
\xi_k^i(x) &= c_{2m}\int_{\mathbb{R}^{2m}}\ln|x-y|\Delta^{\frac{m-1}{2}+i}\left(D\Delta^{\frac{m-1}{2}-i}\tilde{u}_{\varepsilon_k}\wedge D\tilde{u}_{\varepsilon_k}\right)(y)dy,\\
\bar{\xi}_k^i(x) &= c_{2m}\int_{\mathbb{R}^{2m}}\ln|x-y|\Delta^{\frac{m-1}{2}+i-1}D\cdot\left(\Delta^{\frac{m+1}{2}-i}\tilde{u}_{\varepsilon_k}\wedge D\tilde{u}_{\varepsilon_k}\right)(y)dy,\\
\hat{\xi}_k^i(x) &= -c_{2m}\binom{m}{2i-1}\int_{\mathbb{R}^{2m}}\ln|x-y|\Delta^{i-1}D\cdot\left(D\Delta^{\frac{m-1}{2}}\tilde{u}_{\varepsilon_k}\wedge\Delta^{\frac{m+1}{2}-i}\tilde{u}_{\varepsilon_k}\right)(y)dy,\\
\tilde{\xi}_k^i(x) &= c_{2m}\binom{m}{2i}\int_{\mathbb{R}^{2m}}\ln|x-y|\Delta^{i}\left(D\Delta^{\frac{m-1}{2}}\tilde{u}_{\varepsilon_k}\wedge D\Delta^{\frac{m-1}{2}-i}\tilde{u}_{\varepsilon_k}\right)(y)dy
\end{aligned}$$

and for $j=1,...,\frac{m+1}{2}$

$$\begin{aligned}
\vartheta_k^j(x) &= \varepsilon_k c_{2m}\binom{m+1}{2j}\int_{\mathbb{R}^{2m}}\ln|x-y|\Delta^{j}\left(\Delta^{\frac{m+1}{2}}\tilde{u}_{\varepsilon_k}\wedge\Delta^{\frac{m+1}{2}-j}\tilde{u}_{\varepsilon_k}\right)(y)dy,\\
\tilde{\vartheta}_k^j(x) &= -\varepsilon_k c_{2m}\binom{m+1}{2j-1}\int_{\mathbb{R}^{2m}}\ln|x-y|\Delta^{j-1}D\cdot\left(\Delta^{\frac{m+1}{2}}\tilde{u}_{\varepsilon_k}\wedge D\Delta^{\frac{m+1}{2}-j}\tilde{u}_{\varepsilon_k}\right)(y)dy
\end{aligned}$$

in $\mathbb{R}^{2m}$. We set

$$\gamma_k = \Phi_k - \sum_{i=1}^{\frac{m-1}{2}}(\xi_k^i+\bar{\xi}_k^i+\tilde{\xi}_k^i+\hat{\xi}_k^i) - \sum_{j=1}^{\frac{m+1}{2}}(\vartheta_k^j+\tilde{\vartheta}_k^j). \tag{3.6.28}$$

$c_{2m} \ln |x-y|$ is the fundamental solution of Δ^m in $\mathbb{R}^{2m}$ and thus

$$\Delta^m \gamma_k = 0 \qquad \text{in} \ \ B_{\frac{R_0}{8}}.$$

With the Calderon-Zygmund inequality, Hölder's inequality for Lorentz spaces ($p_1 = \frac{m}{m+b-2i}$, $q_1 = \frac{m}{2i-b}$, $p_2 = \frac{m}{m+b-2i+1}$, $q_2 = \frac{m}{2i-b-1}$, $p_3 = \frac{2m-2i+1}{m}$, $q_3 = \frac{2m-2i+1}{m-2i+1}$, $p_4 = \frac{2m-2i}{m}$, $q_4 = \frac{2m-2i}{m-2i}$) and Corollary 3.6.4 we have for $i = 1, ..., \frac{m-1}{2}$

$$\begin{aligned} ||D^m \xi_k^i||_{L^{2,1}(\mathbb{R}^{2m})} &\le c \sum_{b=0}^{2i-1} \Big|\Big| |D^{m+b-2i} \tilde{u}_{\varepsilon_k}| \cdot |D^{2i-b} \tilde{u}_{\varepsilon_k}| \Big|\Big|_{L^{2,1}(\mathbb{R}^{2m})} \\ &\le c \sum_{b=0}^{2i-1} ||D^{m+b-2i} \tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{m+b-2i},2}(\mathbb{R}^{2m})} ||D^{2i-b} \tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{2i-b},2}(\mathbb{R}^{2m})} \\ &\le c ||D^m \tilde{u}_{\varepsilon_k}||^2_{L^2(\mathbb{R}^{2m})}, \end{aligned}$$

$$\begin{aligned} ||D^m \bar{\xi}_k^i||_{L^{2,1}(\mathbb{R}^{2m})} &\le c \sum_{b=0}^{2i-2} \Big|\Big| |D^{m+b-2i+1} \tilde{u}_{\varepsilon_k}| \cdot |D^{2i-b-1} \tilde{u}_{\varepsilon_k}| \Big|\Big|_{L^{2,1}(\mathbb{R}^{2m})} \\ &\le c \sum_{b=0}^{2i-2} ||D^{m+b-2i+1} \tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{m+b-2i+1},2}(\mathbb{R}^{2m})} ||D^{2i-b-1} \tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{2i-b-1},2}(\mathbb{R}^{2m})} \\ &\le c ||D^m \tilde{u}_{\varepsilon_k}||^2_{L^2(\mathbb{R}^{2m})}, \end{aligned}$$

$$\begin{aligned} ||D^{2m-2i+1} \hat{\xi}_k^i||_{L^{\frac{2m}{2m-2i+1},1}(\mathbb{R}^{2m})} &\le c \Big|\Big| |D\Delta^{\frac{m-1}{2}} \tilde{u}_{\varepsilon_k}| \cdot |\Delta^{\frac{m+1}{2}-i} \tilde{u}_{\varepsilon_k}| \Big|\Big|_{L^{\frac{2m}{2m-2i+1},1}(\mathbb{R}^{2m})} \\ &\le c ||D^m \tilde{u}_{\varepsilon_k}||_{L^{2,2}(\mathbb{R}^{2m})} ||D^{m-2i+1} \tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{m-2i+1},2}(\mathbb{R}^{2m})} \\ &\le c ||D^m \tilde{u}_{\varepsilon_k}||^2_{L^2(\mathbb{R}^{2m})}, \end{aligned}$$

$$\begin{aligned} ||D^{2m-2i} \tilde{\xi}_k^i||_{L^{\frac{2m}{2m-2i},1}(\mathbb{R}^{2m})} &\le c \Big|\Big| |D\Delta^{\frac{m-1}{2}} \tilde{u}_{\varepsilon_k}| \cdot |D\Delta^{\frac{m-1}{2}-i} \tilde{u}_{\varepsilon_k}| \Big|\Big|_{L^{\frac{2m}{2m-2i},1}(\mathbb{R}^{2m})} \\ &\le c ||D^m \tilde{u}_{\varepsilon_k}||_{L^{2,2}(\mathbb{R}^{2m})} ||D^{m-2i} \tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{m-2i},2}(\mathbb{R}^{2m})} \\ &\le c ||D^m \tilde{u}_{\varepsilon_k}||^2_{L^2(\mathbb{R}^{2m})}. \end{aligned}$$

With the embedding theorem for Lorentz spaces in Lemma 3.6.3 and (3.6.16) this yields

$$\begin{aligned} &||D^m \xi_k^i||_{L^2(\mathbb{R}^{2m})} + ||D^m \bar{\xi}_k^i||_{L^2(\mathbb{R}^{2m})} + ||D^m \hat{\xi}_k^i||_{L^2(\mathbb{R}^{2m})} + ||D^m \tilde{\xi}_k^i||_{L^2(\mathbb{R}^{2m})} \\ &\le c \Big(||D^m \xi_k^i||_{L^{2,1}(\mathbb{R}^{2m})} + ||D^m \bar{\xi}_k^i||_{L^{2,1}(\mathbb{R}^{2m})} + ||D^m \hat{\xi}_k^i||_{L^{2,1}(\mathbb{R}^{2m})} + ||D^m \tilde{\xi}_k^i||_{L^{2,1}(\mathbb{R}^{2m})} \Big) \\ &\le c \sum_{l=1}^{m} ||D^l u_{\varepsilon_k}||^2_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})} + c ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})} \qquad \text{for } i = 1, ..., \frac{m-1}{2}. \end{aligned} \tag{3.6.29}$$

Analogously we estimate ϑ_k^j, $j = 1, ..., \frac{m+1}{2}$, using Hölder's inequality for Lorentz spaces ($p_1 = \frac{2m-2j}{m}$, $q_1 = \frac{2m-2j}{m-2j}$, $p_2 = \frac{2m-2j+1}{m}$, $q_2 = \frac{2m-2j+1}{m-2j+1}$) and Lemma 3.6.3

$$\begin{aligned} ||D^{2m-2j} \vartheta_k^j||_{L^{\frac{2m}{2m-2j},1}(\mathbb{R}^{2m})} &\le c\varepsilon_k \Big|\Big| |\Delta^{\frac{m+1}{2}} \tilde{u}_{\varepsilon_k}| \cdot |\Delta^{\frac{m+1}{2}-j} \tilde{u}_{\varepsilon_k}| \Big|\Big|_{L^{\frac{2m}{2m-2j},1}(\mathbb{R}^{2m})} \\ &\le c\varepsilon_k ||D^{m+1} \tilde{u}_{\varepsilon_k}||_{L^{2,2}(\mathbb{R}^{2m})} ||D^{m-2j+1} \tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{m-2j},2}(\mathbb{R}^{2m})} \end{aligned}$$

$$\leq c\varepsilon_k ||D^{m+1}\tilde{u}_{\varepsilon_k}||^2_{L^2(\mathbb{R}^{2m})},$$

$$\begin{aligned}||D^{2m-2j+1}\tilde{\vartheta}^j_k||_{L^{\frac{2m}{2m-2j+1}}(\mathbb{R}^{2m})} &\leq c\varepsilon_k \left\| |\Delta^{\frac{m+1}{2}}\tilde{u}_{\varepsilon_k}| \cdot |D\Delta^{\frac{m+1}{2}-j}\tilde{u}_{\varepsilon_k}| \right\|_{L^{\frac{2m}{2m-2j+1},1}(\mathbb{R}^{2m})} \\ &\leq c\varepsilon_k ||D^{m+1}\tilde{u}_{\varepsilon_k}||_{L^{2,2}(\mathbb{R}^{2m})} ||D^{m+2-2j}\tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{m-2j+1},2}(\mathbb{R}^{2m})} \\ &\leq c\varepsilon_k ||D^{m+1}\tilde{u}_{\varepsilon_k}||^2_{L^2(\mathbb{R}^{2m})}.\end{aligned}$$

And again with the embedding theorem for Lorentz spaces in Lemma 3.6.3 and the extension property (3.6.17)

$$\begin{aligned}||D^m\vartheta^j_k||_{L^2(\mathbb{R}^{2m})} + ||D^m\tilde{\vartheta}^j_k||_{L^2(\mathbb{R}^{2m})} &\leq ||D^m\vartheta^j_k||_{L^{2,1}(\mathbb{R}^{2m})} + ||D^m\tilde{\vartheta}^j_k||_{L^{2,1}(\mathbb{R}^{2m})} \\ &\leq c\varepsilon_k ||D^{m+1}u_{\varepsilon_k}||^2_{L^2(B_{\frac{R_0}{4}})} + \frac{c\varepsilon_k}{R_0^2}||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})} \\ &\quad + \frac{c\varepsilon_k}{R_0^2}\sum_{l=1}^m ||D^l u_{\varepsilon_k}||^2_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})}\end{aligned} \tag{3.6.30}$$

for $j = 1, ..., \frac{m+1}{2}$. Since γ_k is a m-polyharmonic function on $B_{\frac{R_0}{8}}$, we can use the standard estimate for m-polyharmonic maps (Lemma A.1.6)

$$\begin{aligned}||D^m\gamma_k||_{L^{2,1}(B_{\frac{R_0}{8}})} &\leq c||D^m\gamma_k||_{L^2(B_{\frac{R_0}{4}})} \\ &\leq c||D^m\Phi_k||_{L^2(B_{\frac{R_0}{4}})} \\ &\quad + c\sum_{i=1}^{\frac{m-1}{2}} \Big(||D^m\xi^i_k||_{L^2(B_{\frac{R_0}{4}})} + ||D^m\bar{\xi}^i_k||_{L^2(B_{\frac{R_0}{4}})} \\ &\qquad + ||D^m\hat{\xi}^i_k||_{L^2(B_{\frac{R_0}{4}})} + ||D^m\tilde{\xi}^i_k||_{L^2(B_{\frac{R_0}{4}})} \Big) \\ &\quad + c\sum_{j=1}^{\frac{m+1}{2}} \Big(||D^m\vartheta^j_k||_{L^2(B_{\frac{R_0}{4}})} + ||D^m\tilde{\vartheta}^j_k||_{L^2(B_{\frac{R_0}{4}})} \Big).\end{aligned}$$

We already estimated everything except for the first term. For this we note that

$$\begin{aligned}\Delta^{\frac{m-1}{2}}D\Phi_k = (dd^*)^{\frac{m-1}{2}}d\Phi_k &= (dd^*)^{\frac{m-1}{2}}(D\tilde{u}_{\varepsilon_k} \wedge \tilde{u}_{\varepsilon_k} - d^*\Psi_k) \\ &= (dd^*)^{\frac{m-1}{2}}(D\tilde{u}_{\varepsilon_k} \wedge \tilde{u}_{\varepsilon_k})\end{aligned}$$

in $\mathbb{R}^{2m}$. With Calderon-Zygmund and (3.6.16)

$$\begin{aligned}||D^m\Phi_k||_{L^2(\mathbb{R}^{2m})} &\leq c\sum_{i=0}^m \left\| |D^i\tilde{u}_{\varepsilon_k}| \cdot |D^{m-i}\tilde{u}_{\varepsilon_k}| \right\|_{L^2(\mathbb{R}^{2m})} \\ &\leq c\sum_{i=0}^m ||D^i\tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{i}}(\mathbb{R}^{2m})} ||D^{m-i}\tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{m-i}}(\mathbb{R}^{2m})} \\ &\leq c\left(\sum_{i=1}^m ||D^i u_{\varepsilon_k}||^2_{L^{\frac{2m}{i}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})} \right).\end{aligned}$$

Thus we have together with (3.6.29) and (3.6.30)

$$\begin{aligned}||D^m\gamma_k||_{L^{2,1}(B_{\frac{R_0}{8}})} \leq c\left(\sum_{i=1}^{m}||D^i u_{\varepsilon_k}||^2_{L^{\frac{2m}{i}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right) + \varepsilon_k c||D^{m+1}u_{\varepsilon_k}||^2_{L^2(B_{\frac{R_0}{4}})}\\ + \frac{c\varepsilon_k}{R_0^2}\left(\sum_{l=1}^{m}||D^l u_{\varepsilon_k}||^2_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right).\end{aligned} \tag{3.6.31}$$

With (3.6.22), (3.6.29), (3.6.30) and (3.6.31) we can now estimate $D^m\Phi_k$.

$$\begin{aligned}||D^m\Phi_k||_{L^{2,1}(B_{\frac{R_0}{8}})} &\leq c\sum_{i=1}^{\frac{m-1}{2}}\Big(||D^m\xi_k^i||_{L^{2,1}(B_{\frac{R_0}{8}})} + ||D^m\bar{\xi}_k^i||_{L^{2,1}(B_{\frac{R_0}{8}})}\\ &\quad + ||D^m\hat{\xi}_k^i||_{L^{2,1}(B_{\frac{R_0}{8}})} + ||D^m\tilde{\xi}_k^i||_{L^{2,1}(B_{\frac{R_0}{8}})}\Big)\\ &\quad + c\sum_{j=1}^{\frac{m+1}{2}}\Big(||D^m\vartheta_k^j||_{L^{2,1}(B_{\frac{R_0}{8}})} + ||D^m\tilde{\vartheta}_k^j||_{L^{2,1}(B_{\frac{R_0}{8}})}\Big) + c||D^m\gamma_k||_{L^{2,1}(B_{\frac{R_0}{8}})}\\ &\leq c\left(\sum_{i=1}^{m}||D^i u_{\varepsilon_k}||^2_{L^{\frac{2m}{i}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right) + c\varepsilon_k||D^{m+1}u_{\varepsilon_k}||^2_{L^2(B_{\frac{R_0}{4}})}\\ &\quad + \frac{c\varepsilon_k}{R_0^2}\left(\sum_{l=1}^{m}||D^l u_{\varepsilon_k}||^2_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right).\end{aligned}$$

Going back to (3.6.15) we conclude

$$\begin{aligned}&||D^m u_{\varepsilon_k}||_{L^{2,1}(B_{\frac{R_0}{8}}\setminus B_{2Rt_k})}\\ &\leq \sum_{i=1}^{m-1}\big|\big||D^i u_{\varepsilon_k}|\cdot|D^{m-i}u_{\varepsilon_k}|\big|\big|_{L^{2,1}(B_{\frac{R_0}{8}})} + ||D^{m-1}(Du_{\varepsilon_k}\wedge u_{\varepsilon_k})||_{L^{2,1}(B_{\frac{R_0}{8}})}\\ &\leq c\sum_{i=1}^{m-1}||D^{i+1}\tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{i+1}}(\mathbb{R}^{2m})}||D^{m-i+1}\tilde{u}_{\varepsilon_k}||_{L^{\frac{2m}{m-i+1}}(\mathbb{R}^{2m})}\\ &\quad + c||D^m\Phi_k||_{L^{2,1}(B_{\frac{R_0}{8}})} + c||D^m\Psi_k||_{L^{2,1}(B_{\frac{R_0}{8}})}\\ &\leq c\left(\sum_{i=1}^{m}||D^i u_{\varepsilon_k}||^2_{L^{\frac{2m}{i}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right) + c\varepsilon_k||D^{m+1}u_{\varepsilon_k}||^2_{L^2(B_{\frac{R_0}{4}})}\\ &\quad + \frac{c\varepsilon_k}{R_0^2}\left(\sum_{l=1}^{m}||D^l u_{\varepsilon_k}||^2_{L^{\frac{2m}{l}}(B_{\frac{R_0}{4}})} + ||u_{\varepsilon_k}||^2_{L^\infty(B_{\frac{R_0}{4}})}\right).\end{aligned}$$

The same argument as in the even case finishes the proof of Theorem 3.6.1.

□

3.7 Energy identity for arbitrary target manifolds N

In this section we want to ease the restrictions on the target manifold and allow arbitrary compact Riemannian manifolds N. Lemma 3.6.2 at the beginning of section 3.6 holds for arbitrary target manifolds. Thus it remains to show that

$$\int_{B_{\frac{R_0}{4}}\backslash B_{2Rt_k}} |D^m u_{\varepsilon_k}|^2 \to 0 \qquad \text{as } k \to \infty$$

and Theorem 1.0.4 follows. In exchange for lowering the assumptions on the target manifold we have to assume a so-called entropy condition introduced by Struwe in [73] (see also Lamm [43]), namely

$$\varepsilon_k \log\left(\frac{1}{\varepsilon_k}\right) \int_\Omega |D^{m+1} u_{\varepsilon_k}|^2 \to 0 \qquad \text{as } k \to \infty \text{ and } \varepsilon_k \to 0. \tag{3.7.1}$$

Using this we show the following result.

Theorem 3.7.1. *Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a sequence such that $\varepsilon_k \to 0$ as $k \to \infty$ and let $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ be a sequence of critical points of E_{ε_k} satisfying all assumptions of Theorem 1.0.4. In particular let $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ satisfy the entropy condition (3.7.1). Further, we assume that $\Sigma = \{x_0\}$ and only one smooth, non-trivial quasi-m-polyharmonic map $\omega^1 \colon S^{2m} \to N$ forms along the sequence. Then*

$$\lim_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k}) = E_0(u_0) + E_0(\omega^1). \tag{3.7.2}$$

Remark 3.7.2. *The existence of a sequence of critical points $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ satisfying the entropy condition (3.7.1) is shown in section 3.8.*

We follow the work of Wang and Zheng [82] and consider the tangential and radial components of $D^m u_{\varepsilon_k}$ separately.

Lemma 3.7.3. *Let u_{ε_k} be a solution of (3.1.2), $\delta > 0$ and $\tilde{E}_{\varepsilon_k}(u_{\varepsilon_k}, B_{2s}\backslash B_s) < \delta$ for all $s \in [Rt_k, \frac{R_0}{4}]$. Then*

$$\int_{B_{\frac{R_0}{4}}\backslash B_{2Rt_k}} |D_T D^{m-1} u_{\varepsilon_k}|^2 \leq c\, \sqrt[m]{\delta}\left(1 + \frac{\varepsilon_k}{(Rt_k)^2}\right),$$

where D_T denotes the tangential component of the derivative.

Proof. We define the radial function $q_k = q_k(|x|)$ on $B_{\frac{R_0}{4}} \setminus B_{2Rt_k}$, so that for $i = 1, ..., l_k$ with $2^{l_k+1}Rt_k = \frac{R_0}{4}$

$$\begin{cases} \Delta^m q_k = 0, & \text{on} \quad B_{2^{i+1}Rt_k} \setminus B_{2^i Rt_k}, \\ D^s q_k(r) = ⨏_{\partial B_{2^{i+1}Rt_k}} D_r^s u_{\varepsilon_k}, & \text{if } r = 2^{i+1}Rt_k, \quad s = 0, ..., m-1, \\ D^s q_k(r) = ⨏_{\partial B_{2^i Rt_k}} D_r^s u_{\varepsilon_k}, & \text{if } r = 2^i Rt_k, \quad s = 0, ..., m-1, \end{cases}$$

where D_r denotes the derivative in radial direction. By (3.1.2) we have

$$\Delta^m(u_{\varepsilon_k} - q_k) = \tilde{f}(u_{\varepsilon_k}) + \varepsilon_k\big(\Delta^{m+1}u_{\varepsilon_k} + \tilde{g}(u_{\varepsilon_k})\big) \quad \text{on } B_{2^{i+1}Rt_k} \setminus B_{2^i Rt_k}.$$

Multiplying with $(u_{\varepsilon_k} - q_k)$, integrating over the annulus $B_{2^{i+1}Rt_k} \setminus B_{2^i Rt_k}$ and using integration by parts yields

$$\begin{aligned} &\int_{B_{2^{i+1}Rt_k}\backslash B_{2^i Rt_k}} |D^m(u_{\varepsilon_k} - q_k)|^2 \\ &= \sum_{s=1}^{m}\left(\int_{\partial B_{2^{i+1}Rt_k}} - \int_{\partial B_{2^i Rt_k}}\right)(-1)^{s+m} D^s \Delta^{m-s}(u_{\varepsilon_k} - q_k) D^{s-1}(u_{\varepsilon_k} - q_k) \end{aligned}$$

$$+(-1)^m \int_{B_{2^{i+1}Rt_k}\setminus B_{2^iRt_k}} \left(\tilde f(u_{\varepsilon_k}) + \varepsilon_k\left(\Delta^m u_{\varepsilon_k} + \tilde g(u_{\varepsilon_k})\right)\right)(u_{\varepsilon_k} - q_k). \tag{3.7.3}$$

With the mean value theorem, Lemma A.4.1 and Corollary 3.4.7 we have for $x \in B_{2^{i+1}Rt_k} \setminus B_{2^iRt_k}$

$$\begin{aligned}
u(x) - q(x) &= u(x) - \left(\fint_{\partial B_{2^iRt_k}} u\right) + \left(\fint_{\partial B_{2^iRt_k}} u\right) - q(x)\\
&\le cRt_k||Du||_{L^\infty(B_{2^{i+1}Rt_k}\setminus B_{2^iRt_k})} + cRt_k||Dq||_{L^\infty(B_{2^{i+1}Rt_k}\setminus B_{2^iRt_k})}\\
&\le c\sqrt[2m]{\delta} + cRt_k\Bigg(\sum_{j=1}^m (Rt_k)^{j-1}\left(|D^jq(2^iRt_k)| + |D^jq(2^{i+1}Rt_k)|\right)\\
&\qquad + (Rt_k)^{-1}|q(2^{i+1}Rt_k) - q(2^iRt_k)|\Bigg)\\
&\le c\sqrt[2m]{\delta}
\end{aligned}$$

and thus

$$\operatorname{osc}_{B_{2^{i+1}Rt_k}\setminus B_{2^iRt_k}}(u_{\varepsilon_k} - q_k) \le c\sqrt[2m]{\delta} \qquad \forall\ 1 \le i \le l_k. \tag{3.7.4}$$

Summing (3.7.3) over $1 \le i \le l_k$ yields

$$\begin{aligned}
&\int_{B_{\frac{R_0}{4}}\setminus B_{2Rt_k}} |D^m(u_{\varepsilon_k} - q_k)|^2\\
&= \sum_{s=1}^m \left(\int_{\partial B_{\frac{R_0}{4}}} - \int_{\partial B_{2Rt_k}}\right)(-1)^{s+m}D^s\Delta^{m-s}(u_{\varepsilon_k} - q_k)D^{s-1}(u_{\varepsilon_k} - q_k)\\
&\quad + (-1)^m\sum_{i=1}^{l_k}\int_{B_{2^{i+1}Rt_k}\setminus B_{2^iRt_k}} \left(\tilde f(u_{\varepsilon_k}) + \varepsilon\left(\Delta^{m+1}u_{\varepsilon_k} + \tilde g(u_{\varepsilon_k})\right)\right)(u_{\varepsilon_k} - q_k)\\
&= \sum_{s=1}^m \left(\int_{\partial B_{\frac{R_0}{4}}} - \int_{\partial B_{2Rt_k}}\right)(-1)^{s+m}D^s\Delta^{m-s}u_{\varepsilon_k}D^{s-1}(u_{\varepsilon_k} - q_k)\\
&\quad + (-1)^m\sum_{i=1}^{l_k}\int_{B_{2^{i+1}Rt_k}\setminus B_{2^iRt_k}} \left(\tilde f(u_{\varepsilon_k}) + \varepsilon\left(\Delta^{m+1}u_{\varepsilon_k} + \tilde g(u_{\varepsilon_k})\right)\right)(u_{\varepsilon_k} - q_k)\\
&=: I + II + III.
\end{aligned}$$

The last equality holds because D^sq_k $(s \in \mathbb{N}_0)$ is constant on $\partial B_{2^iRt_k}$, $i = 1, ..., l_k + 1$. To estimate I we use the assumption $\tilde E_{\varepsilon_k}(u_{\varepsilon_k}, B_{\frac{R_0}{2}} \setminus B_{\frac{R_0}{8}}) < 2\delta$ to apply Corollary 3.4.7 and estimate

$$\begin{aligned}
I &= \sum_{s=1}^m \int_{\partial B_{\frac{R_0}{4}}} (-1)^{s+m}D^s\Delta^{m-s}u_{\varepsilon_k}\left(D^{s-1}u_{\varepsilon_k} - \fint_{\partial B_{\frac{R_0}{4}}} D^{s-1}u_{\varepsilon_k}\right)\\
&\le cR_0^{\frac{2m-1}{2m}}||u_{\varepsilon_k} - q_k||_{L^\infty(\partial B_{\frac{R_0}{4}})}\left(\int_{\partial B_{\frac{R_0}{4}}} |D\Delta^{m-1}u_{\varepsilon_k}|^{\frac{2m}{2m-1}}\right)^{\frac{2m-1}{2m}}\\
&\quad + \sum_{s=2}^m cR_0^{\frac{2m-1}{2m}}\left(\int_{\partial B_{\frac{R_0}{4}}} |D^s\Delta^{m-s}u_{\varepsilon_k}|^{\frac{2m}{2m-s}}\right)^{\frac{2m-s}{2m}}\left(\int_{\partial B_{\frac{R_0}{4}}} |D^{s-1}u_{\varepsilon_k}|^{\frac{2m}{s-1}}\right)^{\frac{s-1}{2m}}
\end{aligned}$$

$$\leq c\sqrt[m]{\delta}$$

and analogously

$$II = \sum_{s=1}^{m} \int_{\partial B_{2Rt_k}} (-1)^s D^s \Delta^{m-s} u_{\varepsilon_k} D^{s-1}(u_{\varepsilon_k} - q_k) \leq c\sqrt[m]{\delta},$$

where c is independent of R, t_k, R_0. For the first part of III we use (3.7.4) and Corollary 3.4.7

$$\begin{aligned}
&\sum_{i=1}^{l_k} \int_{B_{2^{i+1}Rt_k} \setminus B_{2^i Rt_k}} \tilde{f}(u_{\varepsilon_k})(u_{\varepsilon_k} - q_k) \\
&\leq c\sqrt[2m]{\delta} \sum_{i=1}^{l_k} \sum_{\substack{k_j \in \mathbb{N}_0 \\ k_1+2k_2+\ldots+(2m-1)k_{2m-1}=2m}} \int_{B_{2^{i+1}Rt_k} \setminus B_{2^i Rt_k}} |Du_{\varepsilon_k}|^{k_1} \cdot \ldots \cdot |D^{2m-1}u_{\varepsilon_k}|^{k_{2m-1}} \\
&\leq c\sqrt[m]{\delta} \sum_{i=1}^{l_k} \int_{B_{2^{i+1}Rt_k} \setminus B_{2^i Rt_k}} \frac{1}{|x|^{2m}} \\
&\leq c\sqrt[m]{\delta}.
\end{aligned}$$

For the second part of III we use (3.7.4) and (3.6.11) to estimate

$$\begin{aligned}
&\sum_{i=1}^{l_k} \int_{B_{2^{i+1}Rt_k} \setminus B_{2^i Rt_k}} \varepsilon_k \big(\Delta^{m+1} u_{\varepsilon_k} + \tilde{g}(u_{\varepsilon_k})\big)(u_{\varepsilon_k} - q_k) \\
&\qquad\qquad \leq c\sqrt[m]{\delta}\varepsilon_k \int_{2Rt_k}^{\frac{R_0}{4}} \int_{\partial B_\varrho} \frac{1}{|x|^{2(m+1)}} d\mu d\varrho \leq c\sqrt[m]{\delta}\frac{\varepsilon_k}{(Rt_k)^2}.
\end{aligned}$$

Putting all of this together we arrive at

$$\int_{B_{\frac{R_0}{4}} \setminus B_{2Rt_k}} |D^m(u_{\varepsilon_k} - q_k)|^2 \leq c\sqrt[m]{\delta}\left(1 + \frac{\varepsilon_k}{(Rt_k)^2}\right)$$

and because q_k is radial it follows that

$$\int_{B_{\frac{R_0}{4}} \setminus B_{2Rt_k}} |D_T D^{m-1} u_{\varepsilon_k}|^2 \leq c\sqrt[m]{\delta}\left(1 + \frac{\varepsilon_k}{(Rt_k)^2}\right). \tag{3.7.5}$$

□

Next we want to bound the radial component $D_r D^{m-1} u_{\varepsilon_k}$ by the tangential component of $D^m u_{\varepsilon_k}$. Note that $D_T D^{m-1} u_{\varepsilon_k} = D^m u_{\varepsilon_k} - D_r D^{m-1} u_{\varepsilon_k}$.

3.7.1 Radial energy estimate

We use a Pohozaev type argument to show the decay of purely radial derivatives in the neck region. Ai and Yin in [2] developed a procedure for writing $\Delta^m u$ in terms of its radial and tangential components by using cylindrical coordinates (t, θ), where $r = e^t$ and $\theta \in S^{2m-1}$. Then

$$\Delta u(t, \theta) = e^{-2t}\left(\left(\frac{\partial}{\partial t}\right)^2 u + 2(m-1)\frac{\partial}{\partial t} u + \Delta_{S^{2m-1}} u\right)$$

and with Lemma 2.7 in [2]

$$\Delta^m u = e^{-2mt} \sum_{\substack{1\le p+2q\le 2m \\ p,q\in\mathbb{N}\cup\{0\}}} a_{p,q} \left(\frac{\partial}{\partial t}\right)^p \Delta^q_{S^{2m-1}} u \tag{3.7.6}$$

with coefficients $a_{p,q} \in \mathbb{R}$ and

$$\begin{cases} \qquad\qquad a_{p,q} = 0 & \text{for } p = 1, 3, ..., 2m-1, \; q \in \mathbb{N}_0; \\ (-1)^{m-\frac{p}{2}} \quad a_{p,0} > 0 & \text{for } p = 2, 4, ..., 2m. \end{cases} \tag{3.7.7}$$

To simplify notation we assume in the following that m is even. The case where m is odd follows analogously with minor modifications. Let $\tau \in (2Rt_k, \frac{R_0}{4})$. By (3.1.1) we have

$$\int_{B_\tau} \Delta^m u_{\varepsilon_k} (x \cdot Du_{\varepsilon_k}) = \varepsilon_k \int_{B_\tau} \Delta^{m+1} u_{\varepsilon_k} (x \cdot Du_{\varepsilon_k}) . \tag{3.7.8}$$

For the right-hand side we use integration by parts

$$\begin{aligned} \varepsilon_k \int_{B_\tau} \Delta^{m+1} u_{\varepsilon_k} (x \cdot Du_{\varepsilon_k}) = \varepsilon_k & \sum_{k=0}^{\frac{m}{2}} \int_{\partial B_\tau} \partial_l \Delta^{m-k} u_{\varepsilon_k} \Delta^k (x \cdot Du_{\varepsilon_k}) \nu^l \\ & - \varepsilon_k \sum_{k=0}^{\frac{m}{2}-1} \int_{\partial B_\tau} \Delta^{m-k} u_{\varepsilon_k} \partial_l \Delta^k (x \cdot Du_{\varepsilon_k}) \nu^l \\ & - \varepsilon_k \int_{B_\tau} D\Delta^{\frac{m}{2}} u_{\varepsilon_k} D\Delta^{\frac{m}{2}} (x \cdot Du_{\varepsilon_k}) , \end{aligned}$$

where $\nu^l = \frac{x^l}{|x|}$, $l = 1, ..., 2m$ is the outer unit normal. For the last term we have

$$\begin{aligned} \varepsilon_k \int_{B_\tau} D\Delta^{\frac{m}{2}} u_{\varepsilon_k} D\Delta^{\frac{m}{2}} (x \cdot Du_{\varepsilon_k}) &= (m+1)\varepsilon_k \int_{B_\tau} |D\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2 + \varepsilon_k \int_{B_\tau} x \cdot D \left(\frac{|D\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2|}{2} \right) \\ &= \varepsilon_k \int_{B_\tau} Div \left(x \cdot \frac{|D\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2}{2} \right) + \varepsilon_k \int_{B_\tau} |D\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2 \\ &= \varepsilon_k \tau \int_{\partial B_\tau} \frac{|D\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2}{2} + \varepsilon_k \int_{B_\tau} |D\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2 \end{aligned}$$

and thus

$$\begin{aligned} \varepsilon_k \int_{B_\tau} \Delta^{m+1} u_{\varepsilon_k} (x \cdot Du_{\varepsilon_k}) = \varepsilon_k & \sum_{k=0}^{\frac{m}{2}} \int_{\partial B_\tau} \partial_l \Delta^{m-k} u_{\varepsilon_k} \Delta^k (x \cdot Du_{\varepsilon_k}) \nu^l \\ & - \varepsilon_k \sum_{k=0}^{\frac{m}{2}-1} \int_{\partial B_\tau} \Delta^{m-k} u_{\varepsilon_k} \partial_l \Delta^k (x \cdot Du_{\varepsilon_k}) \nu^l \\ & - \varepsilon_k \tau \int_{\partial B_\tau} \frac{|D\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2}{2} - \varepsilon_k \int_{B_\tau} |D\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2 . \end{aligned} \tag{3.7.9}$$

To calculate the left-hand side of (3.7.8) further we switch from spherical coordinates (r, θ) to cylindrical coordinates (t, θ), where $r = e^t$ and $r\frac{\partial}{\partial r} = \frac{\partial}{\partial t}$. Then we use integration by parts with respect to θ, (3.7.6), (3.7.7) and the chain rule (see [2] p.146)

$$\begin{aligned}
&\int_{B_\tau} \Delta^m u_{\varepsilon_k} x \cdot D u_{\varepsilon_k} \\
&= \int_0^\tau \int_{\partial B_r} \Delta^m u_{\varepsilon_k} r \frac{\partial}{\partial r} u_{\varepsilon_k} d\theta dr \\
&= \int_{-\infty}^{\ln(\tau)} \int_{S^{2m-1}} e^{-2mt} \sum_{\substack{1\le p+2q\le 2m \\ p,q\in\mathbb{N}\cup\{0\}}} a_{p,q} \left(\frac{\partial}{\partial t}\right)^p \Delta^q_{S^{2m-1}}(u_{\varepsilon_k}(t,\theta)) \cdot \frac{\partial}{\partial t}(u_{\varepsilon_k}(t,\theta)) e^{2mt} d\theta dt \\
&= \int_{-\infty}^{\ln(\tau)} \int_{S^{2m-1}} \sum_{\substack{1\le p+2q\le 2m \\ p,q\in\mathbb{N}\cup\{0\}}} (-1)^q a_{p,q} \left(\frac{\partial}{\partial t}\right)^p (w_{q,\varepsilon_k}(t,\theta)) \cdot \frac{\partial}{\partial t}(w_{q,\varepsilon_k}(t,\theta)) d\theta dt \\
&= \int_{-\infty}^{\ln(\tau)} \frac{\partial}{\partial t} \int_{S^{2m-1}} \sum_{\substack{1\le 2\tilde{p}+2q\le 2m \\ \tilde{p},q\in\mathbb{N}\cup\{0\}}} (-1)^{q+\tilde{p}+1} \tfrac{2\tilde{p}-1}{2} a_{2\tilde{p},q} \left| \left(\frac{\partial}{\partial t}\right)^{\tilde{p}} (w_{q,\varepsilon_k}(t,\theta)) \right|^2 d\theta dt \\
&\quad + \int_{-\infty}^{\ln(\tau)} \left(\frac{\partial}{\partial t}\right)^2 \int_{S^{2m-1}} \Bigg(\sum_{\substack{1\le 2\tilde{p}+2q\le 2m \\ \tilde{p},q\in\mathbb{N}\cup\{0\}}} (-1)^q a_{2\tilde{p},q} \sum_{k=1}^{\tilde{p}-1} \sum_{l=1}^{\tilde{p}-k} (-1)^{k+l} \left(\frac{\partial}{\partial t}\right)^{k+l-1} (w_{q,\varepsilon_k}(t,\theta)) \\
&\qquad \cdot \left(\frac{\partial}{\partial t}\right)^{2\tilde{p}-k-l} (w_{q,\varepsilon_k}(t,\theta)) \Bigg) d\theta dt,
\end{aligned}$$

where

$$w_{q,\varepsilon_k}(t,\theta) = \begin{cases} \Delta^l_{S^{2m-1}}(u_{\varepsilon_k}(t,\theta)), & q = 2l \\ \nabla_{S^{2m-1}} \Delta^l_{S^{2m-1}}(u_{\varepsilon_k}(t,\theta)), & q = 2l+1, \qquad l \in \mathbb{N}_0. \end{cases}$$

Next we separate the purely radial derivatives, i.e. the terms where $q = 0$. Then

$$\begin{aligned}
&\int_{B_\tau} \Delta^m u_{\varepsilon_k} x \cdot D u_{\varepsilon_k} \\
&= \left[\int_{S^{2m-1}} \sum_{\tilde{p}=1}^{m} (-1)^{\tilde{p}+1} \tfrac{2\tilde{p}-1}{2} a_{2\tilde{p},0} \left| \left(\frac{\partial}{\partial t}\right)^{\tilde{p}} u_{\varepsilon_k} \right|^2 d\theta \right]_{t=-\infty}^{t=\ln(\tau)} \\
&\quad + \left[\frac{\partial}{\partial t} \int_{S^{2m-1}} \sum_{\tilde{p}=1}^{m} a_{2\tilde{p},0} \sum_{k=1}^{\tilde{p}-1} \sum_{l=1}^{\tilde{p}-k} (-1)^{k+l} \left(\frac{\partial}{\partial t}\right)^{k+l-1} u_{\varepsilon_k} \left(\frac{\partial}{\partial t}\right)^{2\tilde{p}-k-l} u_{\varepsilon_k} d\theta \right]_{t=-\infty}^{t=\ln(\tau)} \\
&\quad + \left[\int_{S^{2m-1}} \sum_{\substack{1\le 2\tilde{p}+2q\le 2m \\ \tilde{p},q\in\mathbb{N}\cup\{0\},\ q\neq 0}} (-1)^{q+\tilde{p}+1} \tfrac{2\tilde{p}-1}{2} a_{2\tilde{p},q} \left| \left(\frac{\partial}{\partial t}\right)^{\tilde{p}} (w_{q,\varepsilon_k}(t,\theta)) \right|^2 d\theta \right]_{t=-\infty}^{t=\ln(\tau)} \\
&\quad + \Bigg[\frac{\partial}{\partial t} \int_{S^{2m-1}} \sum_{\substack{1\le 2\tilde{p}+2q\le 2m \\ \tilde{p},q\in\mathbb{N}\cup\{0\},\ q\neq 0}} (-1)^q a_{2\tilde{p},q} \sum_{k=1}^{\tilde{p}-1} \sum_{l=1}^{\tilde{p}-k} (-1)^{k+l} \partial_t^{k+l-1} (w_{q,\varepsilon_k}(t,\theta)) \\
&\qquad \cdot \left(\frac{\partial}{\partial t}\right)^{2\tilde{p}-k-l} (w_{q,\varepsilon_k}(t,\theta)) d\theta \Bigg]_{t=-\infty}^{t=\ln(\tau)}.
\end{aligned}$$

Note that the integrals vanish for $t \to -\infty$ and we are left with the integrals evaluated at $t = \ln(\tau)$. (To see this transform back into spherical coordinates. For every $k \in \mathbb{N}$, u_{ε_k} is smooth and thus bounded. Letting $r \to 0$ we see that the integral converges to zero.)

Now we put everything back into (3.7.8) and transform back into spherical coordinates. Note that by (3.7.7) $(-1)^{\tilde{p}} a_{2\tilde{p},0} > 0$ for m even and $(-1)^{\tilde{p}} a_{2\tilde{p},0} < 0$ for m odd. In the following we assume m to be even. (In case m is odd the calculation follows analogously with switched signs on the right-hand side.) With (3.7.9) we get

$$
\begin{aligned}
&\left[\int_{S^{2m-1}} \sum_{\tilde{p}=1}^{m} (-1)^{\tilde{p}} \tfrac{2\tilde{p}-1}{2} a_{2\tilde{p},0} \left| \left(r\frac{\partial}{\partial r}\right)^{\tilde{p}} (u_{\varepsilon_k}(r,\theta)) \right|^2 d\theta \right]_{r=\tau} \\
&= \tau \left[\frac{\partial}{\partial r} \int_{S^{2m-1}} \sum_{\tilde{p}=1}^{m} \sum_{k=1}^{\tilde{p}-1} \sum_{l=1}^{\tilde{p}-k} \left((-1)^{k+l} a_{2\tilde{p},0} \left(r\frac{\partial}{\partial r}\right)^{k+l-1} (u_{\varepsilon_k}(r,\theta)) \left(r\frac{\partial}{\partial r}\right)^{2\tilde{p}-k-l} (u_{\varepsilon_k}(r,\theta)) \right) d\theta \right]_{r=\tau} \\
&\quad + \left[\int_{S^{2m-1}} \sum_{\substack{1\le 2\tilde{p}+2q\le 2m \\ \tilde{p},q\in\mathbb{N}\cup\{0\},\ q\neq 0}} (-1)^{q+\tilde{p}+1} \tfrac{2\tilde{p}-1}{2} a_{2\tilde{p},q} \left| \left(r\frac{\partial}{\partial r}\right)^{\tilde{p}} (w_{q,\varepsilon_k}(r,\theta)) \right|^2 d\theta \right]_{r=\tau} \\
&\quad + \tau \left[\frac{\partial}{\partial r} \int_{S^{2m-1}} \sum_{\substack{1\le 2\tilde{p}+2q\le 2m \\ \tilde{p},q\in\mathbb{N}\cup\{0\},\ q\neq 0}} \sum_{k=1}^{\tilde{p}-1} \sum_{l=1}^{\tilde{p}-k} (-1)^{k+l} \left((-1)^q a_{2\tilde{p},q} \left(r\frac{\partial}{\partial r}\right)^{k+l-1} (w_{q,\varepsilon_k}(r,\theta)) \right.\right. \\
&\qquad \left.\left. \cdot \left(r\frac{\partial}{\partial r}\right)^{2\tilde{p}-k-l} (w_{q,\varepsilon_k}(r,\theta)) \right) d\theta \right]_{r=\tau} \\
&\quad - \varepsilon_k \sum_{k=0}^{\frac{m}{2}} \int_{\partial B_\tau} \partial_l \Delta^{m-k} u_{\varepsilon_k} \Delta^k (x\cdot Du_{\varepsilon_k}) \nu^l + \varepsilon_k \sum_{k=0}^{\frac{m}{2}-1} \int_{\partial B_\tau} \Delta^{m-k} u_{\varepsilon_k} \partial_l \Delta^k (x\cdot Du_{\varepsilon_k}) \nu^l \\
&\quad + \varepsilon_k \tau \int_{\partial B_\tau} \frac{|D\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2}{2} + \varepsilon_k \int_{B_\tau} |D\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2 .
\end{aligned}
$$

Dividing by $r=\tau$, integrating from $2Rt_k$ to $\frac{R_0}{4}$ and estimating with (3.7.5) and (3.6.11) yields

$$
\begin{aligned}
&\int_{2Rt_k}^{\frac{R_0}{4}} \frac{1}{r} \int_{S^{2m-1}} \sum_{\tilde{p}=1}^{m} (-1)^{\tilde{p}} \tfrac{2\tilde{p}-1}{2} a_{2\tilde{p},0} \left| \left(r\frac{\partial}{\partial r}\right)^{\tilde{p}} (u_{\varepsilon_k}(r,\theta)) \right|^2 d\theta dr \\
&= \int_{\partial B_{2Rt_k} \cup \partial B_{\frac{R_0}{4}}} r^{-(2m-1)} \sum_{\tilde{p}=1}^{m} a_{2\tilde{p},0} \sum_{k=1}^{\tilde{p}-1} \sum_{l=1}^{\tilde{p}-k} (-1)^{k+l} \left(r\frac{\partial}{\partial r}\right)^{k+l-1} u_{\varepsilon_k} \left(r\frac{\partial}{\partial r}\right)^{2\tilde{p}-k-l} u_{\varepsilon_k} \\
&\quad + \int_{B_{\frac{R_0}{4}} \setminus B_{2Rt_k}} r^{-2m} \sum_{\substack{1\le 2\tilde{p}+2q\le 2m \\ \tilde{p},q\in\mathbb{N}\cup\{0\},\ q\neq 0}} (-1)^{q+\tilde{p}+1} \tfrac{2\tilde{p}-1}{2} a_{2\tilde{p},q} \left| \left(r\frac{\partial}{\partial r}\right)^{\tilde{p}} w_{q,\varepsilon_k} \right|^2 \\
&\quad + \int_{\partial B_{2Rt_k} \cup \partial B_{\frac{R_0}{4}}} r^{-(2m-1)} \sum_{\substack{1\le 2\tilde{p}+2q\le 2m \\ \tilde{p},q\in\mathbb{N}\cup\{0\},\ q\neq 0}} \sum_{k=1}^{\tilde{p}-1} \sum_{l=1}^{\tilde{p}-k} \left((-1)^{q+k+l} a_{2\tilde{p},q} \right. \\
&\qquad \left. \cdot \left(r\frac{\partial}{\partial r}\right)^{k+l-1} w_{q,\varepsilon_k} \left(r\frac{\partial}{\partial r}\right)^{2\tilde{p}-k-l} w_{q,\varepsilon_k} \right) \\
&\quad - \varepsilon_k \sum_{k=0}^{\frac{m}{2}} \int_{B_{\frac{R_0}{4}} \setminus B_{2Rt_k}} \frac{1}{|x|} \partial_l \Delta^{m-k} u_{\varepsilon_k} \Delta^k (x\cdot Du_{\varepsilon_k}) \nu^l \\
&\quad + \varepsilon_k \sum_{k=0}^{\frac{m}{2}-1} \int_{B_{\frac{R_0}{4}} \setminus B_{2Rt_k}} \frac{1}{|x|} \Delta^{m-k} u_{\varepsilon_k} \partial_l \Delta^k (x\cdot Du_{\varepsilon_k}) \nu^l
\end{aligned}
$$

$$+\varepsilon_k \int_{B_{\frac{R_0}{4}}\setminus B_{2Rt_k}} \frac{|D\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2}{2} + \int_{2Rt_k}^{\frac{R_0}{4}} \frac{\varepsilon_k}{r} \int_{B_r} |D\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2$$
$$\leq c\sqrt[m]{\delta}\left(1+\frac{\varepsilon_k}{(Rt_k)^2}\right) + \varepsilon_k \log\left(\frac{1}{Rt_k}\right)\int_\Omega |D\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2.$$

With this we can finally show that there is no energy lost in the neck region as $k \to \infty$. Recall (3.6.4), Lemma 3.5.3 and Lemma 3.7.3 to estimate

$$\begin{aligned}E_{\varepsilon_k}(u_{\varepsilon_k}, B_{\frac{R_0}{4}}\setminus B_{2Rt_k}) &\leq \int_{B_{\frac{R_0}{4}}\setminus B_{2Rt_k}} |\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2 + o_k(1)\\
&\leq c\int_{2Rt_k}^{\frac{R_0}{4}} \frac{1}{r}\int_{S^{2m-1}} \sum_{\tilde{p}=1}^{m} \frac{1-2\tilde{p}}{2} a_{2\tilde{p},0}\left|\left(r\frac{\partial}{\partial r}\right)^{\tilde{p}} u_{\varepsilon_k}\right|^2 d\theta + \frac{c\varepsilon_k}{(Rt_k)^2} + o_k(1)\\
&\leq c\varepsilon_k \log\left(\frac{1}{Rt_k}\right)\int_\Omega |D\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2 + o_k(1)\\
&\leq o_k(1).\end{aligned}$$

This finishes the proof of Theorem 3.7.1 and Theorem 1.0.4 follows by applying this procedure to each bubble.

3.8 Entropy condition

To finish this chapter we show that we can always find a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, $\varepsilon_k \to 0$, and critical points $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ so that

$$\varepsilon_k \log\left(\frac{1}{\varepsilon_k}\right)\int_\Omega |D\Delta^{\frac{m}{2}} u_{\varepsilon_k}|^2 \to 0 \qquad \text{as } k\to\infty.$$

More precisely we show the following:

Theorem 3.8.1. *Let $\varepsilon > 0$ and let $\mathcal{F} \subset \mathcal{P}(W^{m+1,2}(\Omega, N))$ be a collection of sets. Let $\phi\colon [0,\infty)\times W^{m+1,2}(\Omega,N) \to W^{m+1,2}(\Omega,N)$ be a semi-flow such that*

$$\begin{cases}\phi(0,\cdot) = \text{id.},\\ \phi(t,\cdot) \text{ a homeomorphism of } W^{m+1,2}(\Omega,N)\ \forall t\geq 0.\end{cases}$$

$E_\varepsilon(\phi(t,u))$ is non-increasing in t for any $u \in W^{m+1,2}(\Omega,N)$. Further assume that $\phi(t,F)\subset F$ $\forall t \in [0,\infty)$ and $\forall F\in\mathcal{F}$. Let

$$\beta_\varepsilon = \inf_{F\in\mathcal{F}}\sup_{u\in F} E_\varepsilon(u) \tag{3.8.1}$$

and assume that $\beta_\varepsilon < \infty$. Then for almost every $\varepsilon > 0$ there exists a critical point $u_\varepsilon \in C^\infty(\Omega,N)$ of E_ε with $E_\varepsilon(u_\varepsilon) = \beta_\varepsilon$ and

$$\lim_{\varepsilon\to 0}\varepsilon\log\left(\frac{1}{\varepsilon}\right)\int_\Omega |D\Delta^{\frac{m}{2}} u_\varepsilon|^2 = 0. \tag{3.8.2}$$

Proof. We follow the work of Struwe in [73] who first introduced the entropy condition for semilinear elliptic equations. With the minmax principle (Theorem 4.2 in [74]) and the results from section 3.2 we obtain existence of a critical point u_ε of E_ε with $E_\varepsilon(u_\varepsilon) = \beta_\varepsilon$ for all $\varepsilon > 0$. What is left to show is the existence of a sequence of critical points u_ε satisfying (3.8.2) as ε tends to zero.

At first we note that

$$\varepsilon \mapsto \beta_\varepsilon$$

is non-decreasing and therefore differentiable almost everywhere with $\frac{d\beta_\varepsilon}{d\varepsilon} \in L^1([0,\sigma_1])$ for $\sigma_1 > 0$ small. Using this we have

$$2\Lambda = \liminf_{\varepsilon \to 0} \varepsilon \log\left(\frac{1}{\varepsilon}\right) \frac{d\beta_\varepsilon}{d\varepsilon} = 0.$$

To show this assume that $\Lambda > 0$. Then we get for $\sigma \in (0,\sigma_1)$ small

$$\int_0^\sigma \frac{d\beta_\varepsilon}{d\varepsilon} d\varepsilon \geq -\Lambda \int_0^\sigma \frac{1}{\varepsilon \log(\varepsilon)} d\varepsilon = \infty,$$

which contradicts $\frac{d\beta_\varepsilon}{d\varepsilon} \in L^1([0,\sigma_1])$. We divide the rest of the proof in three steps.

Step 1: First we want to derive an estimate for $\partial_\varepsilon E_\varepsilon$. We choose $\varepsilon > 0$ fixed such that β_ε is differentiable in ε and we choose a decreasing sequence $\varepsilon_k \downarrow \varepsilon$. For every $k \in \mathbb{N}$ we choose $F_k \in \mathcal{F}$ such that

$$\sup_{u \in F_k} E_{\varepsilon_k}(u) \leq \beta_{\varepsilon_k} + (\varepsilon_k - \varepsilon). \tag{3.8.3}$$

Further we choose $v \in F_k$ such that

$$\beta_\varepsilon - (\varepsilon_k - \varepsilon) \leq E_\varepsilon(v). \tag{3.8.4}$$

Since β_ε is differentiable in ε we have

$$\beta_{\varepsilon_k} \leq \beta_\varepsilon + \left(\frac{d\beta_\varepsilon}{d\varepsilon} + 1\right)(\varepsilon_k - \varepsilon)$$

for k sufficiently large. Putting all these estimates together we arrive at

$$\beta_\varepsilon - (\varepsilon_k - \varepsilon) \leq E_\varepsilon(v) \leq E_{\varepsilon_k}(v) \leq \sup_{u \in F_k} E_{\varepsilon_k}(u) \leq \beta_\varepsilon + \left(\frac{d\beta_\varepsilon}{d\varepsilon} + 2\right)(\varepsilon_k - \varepsilon). \tag{3.8.5}$$

(3.8.4) and (3.8.5) together yield

$$\frac{E_{\varepsilon_k}(v) - E_\varepsilon(v)}{\varepsilon_k - \varepsilon} \leq \frac{d\beta_\varepsilon}{d\varepsilon} + 3$$

and by the mean value theorem there exists $\bar{\varepsilon} \in [\varepsilon, \varepsilon_k]$ such that

$$\partial_\varepsilon E_{\bar{\varepsilon}}(v) \leq \frac{d\beta_\varepsilon}{d\varepsilon} + 3.$$

Because

$$\partial_\varepsilon E_\varepsilon(u) = \int_\Omega |D\Delta^{\frac{m}{2}} u|^2$$

is independent of ε we conclude

$$\partial_\varepsilon E_\varepsilon(v) \leq \frac{d\beta_\varepsilon}{d\varepsilon} + 3 \tag{3.8.6}$$

for all v satisfying (3.8.5).

Step 2: We show the existence of a sequence $(u_{\varepsilon_k})_{k\in\mathbb{N}} \in W^{m+1,2}(\Omega, N)$ satisfying (3.8.5) such that

$$||DE_{\varepsilon_k}(u_{\varepsilon_k})||_{W^{m+1,2}(\Omega,N)^*} \to 0 \qquad \text{as} \quad k \to \infty. \tag{3.8.7}$$

To prove this claim we first note that

$$\sup\{|\,(DE_{\varepsilon_k}(u)(w)) - (DE_\varepsilon(u)(w))\,| \;:\; ||w||_{W^{m+1,2}(u^*TN)} \le 1\} \to 0 \tag{3.8.8}$$

as $\varepsilon_k \to \varepsilon$, where

$$W^{m+1,2}(u^*TN) = \{w \in W^{m+1,2}(\Omega, \mathbb{R}^d) \mid w(x) \in T_{u(x)}N \;\forall x \in \Omega\}.$$

To see this we pick $w \in W^{m+1,2}(u^*TN),\; u \in W^{m+1,2}(\Omega, N)$ and use the Cauchy-Schwarz inequality

$$\begin{aligned}
|\,(DE_{\varepsilon_k}(u)(w)) - (DE_\varepsilon(u)(w))\,| &\le (\varepsilon_k - \varepsilon)\int_\Omega |D\Delta^{\frac{m}{2}}u||D\Delta^{\frac{m}{2}}w| \\
&\le (\varepsilon_k - \varepsilon)\left(\int_\Omega |D\Delta^{\frac{m}{2}}u|^2\right)^{1/2}\left(\int_\Omega |D\Delta^{\frac{m}{2}}w|^2\right)^{1/2} \\
&\le c(\varepsilon_k - \varepsilon) \to 0 \qquad \text{as} \quad k \to \infty.
\end{aligned}$$

Now we assume that (3.8.7) is false, i.e. that there exists a $\delta > 0$ such that

$$||DE_{\varepsilon_k}(u)||_{W^{m+1,2}(\Omega,N)^*} \ge 4\delta$$

for all u satisfying (3.8.5) and k large enough. In the following let u satisfy (3.8.5). We define $X_k\colon W^{m+1,2}(\Omega, N) \to W^{m+1,2}(u^*TN)$ to be a locally Lipschitz continuous pseudo-gradient vector field for E_{ε_k} with

$$||X_k(u)||_{W^{m+1,2}(u^*TN)} \le 1$$

and

$$\langle DE_{\varepsilon_k}(u), X_k(u)\rangle \le -\frac{1}{2}||DE_{\varepsilon_k}(u)||_{W^{m+1,2}(\Omega,N)^*} \le -2\delta. \tag{3.8.9}$$

Further we define a smooth cut-off function $\psi \in C^\infty(\mathbb{R})$ with

$$\begin{cases} 0 \le \psi \le 1, & \\ \psi(s) = 0 & \text{for } s \le 0, \\ \psi(s) = 1 & \text{for } s \ge 1. \end{cases}$$

For k large enough let

$$\psi_k(u) = \psi\left(\frac{E_\varepsilon(u) - (\beta_\varepsilon - (\varepsilon_k - \varepsilon))}{\varepsilon_k - \varepsilon}\right).$$

Then

$$\tilde{X}_k(u) = \psi_k(u)X_k(u)$$

also defines a Lipschitz continuous tangent vector field. We define the flow generated by $\tilde{X}_k$ as $\phi_k\colon \mathbb{R}_0^+ \times W^{m+1,2}(\Omega, N) \to W^{m+1,2}(\Omega, N)$ with

$$\begin{aligned} \frac{d}{dt}\phi_k(t,u) &= \tilde{X}_k(\phi_k(t,u)) \qquad \text{for } t > 0, \\ \phi_k(0,u) &= u. \end{aligned}$$

Let F_k as in (3.8.3). For $v \in F_k$ let $v_t := \phi_k(t,v)$. By the assumtions of the theorem we have $v_t \in F_k$ for all $t \geq 0$ and together with (3.8.3)

$$\sup_{v \in F_k} E_{\varepsilon_k}(v_t) \leq \sup_{v \in F_k} E_{\varepsilon_k}(v) \leq \beta_{\varepsilon_k} + (\varepsilon_k - \varepsilon).$$

Therefore

$$M(t) := \sup_{v \in F_k} E_\varepsilon(v_t) \geq \beta_\varepsilon$$

is only attained at points v_0 for which $(v_0)_t$ satisfies (3.8.5). Note also that $\psi_k((v_0)_t) = 1$ at these points. Now we can estimate

$$\begin{aligned} \frac{d}{dt}E_\varepsilon((v_0)_t) &= \langle dE_\varepsilon((v_0)_t), \frac{d}{dt}(v_0)_t\rangle \\ &= \psi_k((v_0)_t)\langle dE_\varepsilon((v_0)_t), X_k((v_0)_t)\rangle \\ &\leq \langle dE_{\varepsilon_k}((v_0)_t), X_k((v_0)_t)\rangle + |\langle dE_\varepsilon((v_0)_t) - dE_{\varepsilon_k}((v_0)_t), X_k((v_0)_t)\rangle| \\ &\leq -2\delta + o_k(1), \end{aligned}$$

where we used (3.8.8) and (3.8.9) in the last line. With this we have

$$\frac{d}{dt}M(t) \leq -\delta < 0$$

for large enough k and subsequently $M(t) < \beta_\varepsilon$ for large t. However this is a contradiction to the definition of β_ε and therefore (3.8.7) must hold.

Step 3: In the last step we construct a Palais-Smale sequence to obtain strong convergence. Let $(u_k)_{k\in\mathbb{N}} \in W^{m+1,2}(\Omega, N)$ be a sequence satisfying (3.8.5) and (3.8.7). Note that (3.8.7) is not the correct Palais-Smale condition since $DE_{\varepsilon_k}(\cdot)$ depends on ε_k and not on a fixed ε. However, we will use this to show that $\lim_{k\to\infty} ||DE_\varepsilon(u_k)|| = 0$ is satisfied for $(u_k)_{k\in\mathbb{N}}$.
By (3.8.5) the sequence $(u_k)_{k\in\mathbb{N}}$ is uniformly bounded in $W^{m+1,2}(\Omega, N)$ and together with (3.8.8) we have

$$\sup_{k\in\mathbb{N}} E_\varepsilon(u_k) < \infty \qquad \text{and} \qquad ||DE_\varepsilon(u_k)|| \to 0 \quad \text{as } k \to \infty.$$

Therefore $(u_k)_{k\in\mathbb{N}}$ is in fact a Palais-Smale sequence and we have

$$u_k \to u_\varepsilon \qquad \text{strongly in } W^{m+1,2}(\Omega, N)$$

and $u_\varepsilon \in W^{m+1,2}(\Omega, N)$ is a critical point of E_ε. Using (3.8.5) in the limit we get

$$\beta_\varepsilon = \lim_{k\to\infty} E_{\varepsilon_k}(u_k) = E_\varepsilon(u_\varepsilon).$$

$\partial_\varepsilon E_\varepsilon$ is convex and thus lower semi-continuous on $W^{m+1,2}(\Omega, N)$. With (3.8.6) this yields

$$\partial_\varepsilon E_\varepsilon(u_\varepsilon) \leq \liminf_{k\to\infty} \partial_\varepsilon E_\varepsilon(u_{\varepsilon_k}) \leq \frac{d\beta}{d\varepsilon}(\varepsilon) + 3,$$

which finishes the proof.

□

Chapter 4

Limits of ε-harmonic maps

In this chapter we turn our attention to the fourth order approximation of the Dirichlet energy

$$E(u) = \frac{1}{2}\int_{S^2} |\nabla u|^2 dA_{S^2}$$

introduced by Lamm [44] and classify ε-harmonic maps between two-spheres of degree zero and ± 1. The following is based on joint work with Tobias Lamm and Mario Micallef.

To recap, critical points $u_\varepsilon \in W^{2,2}(S^2, S^2)$ of

$$E_\varepsilon(u) = \frac{1}{2}\int_{S^2} |\nabla u|^2 + \varepsilon|\Delta u|^2 dA_{S^2}$$

exist for every $\varepsilon > 0$, are smooth and satisfy

$$\Delta u - \varepsilon\Delta^2 u = -u|\nabla u|^2 + \varepsilon u\Big(\Delta|\nabla u|^2 + \operatorname{div}\langle \Delta u, \nabla u\rangle + \langle \nabla\Delta u, \nabla u\rangle\Big). \tag{4.0.1}$$

Sequences of critical points $(u_{\varepsilon_k})_{k\in\mathbb{N}},\ \varepsilon_k \to 0$ converge to a smooth harmonic map $u^* : S^2 \to S^2$ and finitely many non-trivial harmonic maps $u^i : S^2 \to S^2$, and the energy identity

$$\lim_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k}) = E(u^*) + \sum_{i=1}^{N} E(u^i) \tag{4.0.2}$$

holds (see [44]). While this approximation produces harmonic maps in the limit, it does not detect every harmonic map as we will see in the following.

First we turn our attention to maps of degree one. As mentioned in the introduction, all harmonic maps between two-spheres projected to the complex plane are rational with Dirichlet energy $E(u) = 4\pi|\deg(u)|$. Moreover, all rational maps of degree one are elements of the Möbius group $PSL(2,\mathbb{C})$. In the following we examine this group more closely.

4.1 The Möbius group

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. A holomorphic function

$$m\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \qquad \xi \mapsto \frac{a\xi + b}{c\xi + d}, \qquad \text{with} \qquad ad - bc = 1;\ a,b,c,d \in \mathbb{C}$$

is called Möbius transformation. It is a rational function of degree one. We write the coefficients a, b, c, d in matrix form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{with} \qquad \det M = 1.$$

M is an element of $PSL(2, \mathbb{C})$, the projective special linear group or Möbius group. If we identify $\hat{\mathbb{C}}$ with S^2 via stereographic projection, then m describes conformal transformations of S^2 such as translations, dilations or rotations.

There exist $U, V \in SU(2)$ such that

$$M = UDV^*$$

is the singular value decomposition of M, where D is a diagonal matrix with eigenvalues $D_{11}, D_{22} \in \mathbb{R}$. Since $\det M = 1$ it follows that $\det D = 1$ and thus $D_{11} = \frac{1}{D_{22}}$. Let M^* be the adjoint of M. Then MM^* has eigenvalues $\lambda = D_{11}^2$ and $\lambda^{-1} = D_{22}^2$ ($\lambda > 0$) and D has the form

$$D = \pm \begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix}. \tag{4.1.1}$$

If $M = D$, the corresponding Möbius transformation is a dilation

$$m_\lambda(\xi) := \lambda\xi.$$

Note that the sign of D plays no role here, since $\frac{\lambda^{1/2}\xi+0}{0\cdot\xi+\lambda^{-1/2}} = \lambda\xi = \frac{-\lambda^{1/2}\xi+0}{0\cdot\xi-\lambda^{-1/2}}$.

We want to take a closer look at the group of rotations $SO(3)$ and its interaction with the Möbius group on the complex plane (see [22]). We project $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ onto the complex plane $\hat{\mathbb{C}}$. Let

$$\begin{aligned} \Pi \colon & S^2 \to \hat{\mathbb{C}} \\ & (x, y, z) \mapsto \xi = \xi_1 + i\xi_2, \qquad \text{with} \quad \xi_1 = \frac{x}{1-z}, \quad \xi_2 = \frac{y}{1-z}, \end{aligned}$$

be the stereographic projection from the north pole with inverse

$$\begin{aligned} \Pi^{-1} \colon & \hat{\mathbb{C}} \to S^2 \\ & \xi \mapsto (x, y, z), \qquad \text{with} \quad x = \frac{2\xi_1}{1+|\xi|^2}, \quad y = \frac{2\xi_2}{1+|\xi|^2}, \quad z = \frac{|\xi|^2-1}{|\xi|^2+1}. \end{aligned}$$

Let $R \in SO(3)$ and the corresponding map

$$S^2 \to S^2, \qquad (x, y, z) \mapsto R \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Then

$$\begin{aligned} \Pi \circ R \circ \Pi^{-1} \colon & \hat{\mathbb{C}} \to \hat{\mathbb{C}} \\ & \xi \mapsto \Pi\left(R\Pi^{-1}(\xi)\right) \end{aligned}$$

defines a transformation on $\hat{\mathbb{C}}$. We want to investigate the structure of this transformation further. Let $\phi \in [0, 2\pi]$, then

$$R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

describes a rotation around the z-axis. On the complex plane this transformation has the form

$$\Pi\left(R_\phi \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \Pi \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \\ z \end{pmatrix} = \frac{x\cos\phi - y\sin\phi + i(x\sin\phi + y\cos\phi)}{1-z}$$
$$= \frac{e^{i\phi}x + ie^{i\phi}y}{1-z} = e^{i\phi}\xi = \frac{e^{i\frac{\phi}{2}}\xi + 0}{0\xi + e^{-i\frac{\phi}{2}}},$$

which is a Möbius transfomation with

$$M_\phi = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix}.$$

Let $\psi \in [0, 2\pi]$ and

$$R_\psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}$$

be a rotation around the x-axis. On the complex plane this is

$$\Pi\left(R_\psi \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \Pi \begin{pmatrix} x \\ y\cos\psi - z\sin\psi \\ y\sin\psi + z\cos\psi \end{pmatrix} = \frac{\cos\frac{\psi}{2}\xi + i\sin\frac{\psi}{2}}{i\sin\frac{\psi}{2}\xi + \cos\frac{\psi}{2}},$$

which is again a Möbius transformation with

$$M_\psi = \begin{pmatrix} \cos\frac{\psi}{2} & i\sin\frac{\psi}{2} \\ i\sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{pmatrix}.$$

Every element of $SO(3)$ is generated by the product of three elementary rotations $R_{\phi_1}R_\psi R_{\phi_2}$ (see [22]). An easy calculation shows that a combination of rotations $R_{(\phi_1\psi\phi_2)} = R_{\phi_1}R_\psi R_{\phi_2}$ corresponds to

$$M_{(\phi_1\psi\phi_2)} = M_{\phi_2}M_\psi M_{\phi_1} = \begin{pmatrix} e^{i\frac{\phi_1+\phi_2}{2}}\cos\frac{\psi}{2} & ie^{-i\frac{\phi_2-\phi_1}{2}}\sin\frac{\psi}{2} \\ ie^{i\frac{\phi_2-\phi_1}{2}}\sin\frac{\psi}{2} & e^{-i\frac{\phi_1+\phi_2}{2}}\cos\frac{\psi}{2} \end{pmatrix}, \tag{4.1.2}$$

which is a Möbius transformation. (Note that we have to multiply the complex matrices in reverse order.) Since $M_{\phi_1\psi\phi_2}$ and $-M_{\phi_1\psi\phi_2}$ give rise to the same Möbius transformation $m_{\phi_1\psi\phi_2}$, every rotation $R \in SO(3)$ corresponds (up to sign) to a complex matrix of the form (4.1.2) . Clearly (4.1.2) is not just an element of the Möbius group $PSL(2,\mathbb{C})$ but of its subgroup

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

On the other hand, each element of $SU(2)$ corresponds to a rotation in $SO(3)$. To see this, choose a matrix $U \in SU(2)$ and define ψ, ϕ_1 and ϕ_2 so that

$$\cos\frac{\psi}{2} = |\alpha|, \qquad \sin\frac{\psi}{2} = |\beta|, \qquad \frac{\phi_1+\phi_2}{2} = \arg\alpha, \qquad -\frac{\phi_2-\phi_1}{2} + \frac{\pi}{2} = \arg\beta.$$

Then $U = M_{(\phi_1\psi\phi_2)}$ and $\Pi^{-1}(U) = R_{(\phi_1\psi\phi_2)}$. Hence $SU(2)$ is the double cover of $SO(3)$.

Let us go back to our original problem and use the Möbius transformations in the following way: Let $u_\varepsilon \in W^{1,2}(S^2, S^2)$ be a degree one critical point of E_ε. The idea is to compose u_ε with a Möbius transformation M and show that $(u_\varepsilon)_M$ is close to the identity map $\mathrm{Id} : S^2 \to S^2$. If we are able to show that there exists $M \in PSL(2,\mathbb{C})$ such that $(u_\varepsilon)_M$ is actually equal to Id, then u_ε itself has to be a Möbius transformation. Moreover, if $M \in SU(2)$ then u_ε is a rotation.

In a first step we investigate how E_ε transforms if we apply u_M. To do this we work in stereographic coordinates and consider $u : \hat{\mathbb{C}} \to S^2$. The Riemannian metric on S^2 in stereographic coordinates is given by $g_{ij} = \frac{4}{(1+|\xi|^2)^2}\delta_{ij}$. For $\xi \in \hat{\mathbb{C}}$ we have

$$|\nabla_{S^2} u|^2(\xi) = \frac{(1+|\xi|^2)^2}{4}|\nabla_{\mathbb{C}} u|^2(\xi) \qquad \text{and} \qquad |\Delta_{S^2} u|^2(\xi) = \frac{(1+|\xi|^2)^4}{16}|\Delta_{\mathbb{C}} u|^2(\xi),$$

where $\nabla_{\mathbb{C}}$ is the gradient on $\mathbb{C}$ and $\Delta_{\mathbb{C}}$ the Laplacian on $\mathbb{C}$ with the flat metric on both the domain and the target. The area element is given by

$$dA_{S^2} = \frac{4}{(1+|\xi|^2)^2} dA_{\mathbb{C}},$$

with $dA_{\mathbb{C}} = \frac{\sqrt{-1}}{2} d\xi \wedge d\bar{\xi}$ the Euclidean area element on $\mathbb{C}$. We define u_M by

$$u_M(\xi) = u(M\xi) = u\left(\frac{a\xi+b}{c\xi+d}\right).$$

Using the above and the fact that $M\xi$ is harmonic we have

$$\begin{aligned}
|\Delta_{S^2} u_M|^2(\xi) &= \frac{(1+|\xi|^2)^4}{16}|\Delta_{\mathbb{C}} u_M|^2(\xi) \\
&= \frac{(1+|\xi|^2)^4}{16}\left|\frac{d}{d\xi}\left(\frac{a\xi+b}{c\xi+d}\right)\right|^4 |\Delta_{\mathbb{C}} u|^2(M\xi) \\
&= \frac{(1+|\xi|^2)^4}{16}\frac{1}{|c\xi+d|^8}|\Delta_{\mathbb{C}} u|^2(M\xi) \\
&= \frac{(1+|\xi|^2)^4}{|c\xi+d|^8(1+|M\xi|^2)^4}|\Delta_{S^2} u|^2(M\xi).
\end{aligned}$$

With the singular value decomposition in (4.1.1) we have

$$|c\xi+d|^2(1+|M\xi|^2) = |a\xi+b|^2 + |c\xi+d|^2 = \left|\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\xi \\ 1\end{pmatrix}\right|^2 = \left|\begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix}\begin{pmatrix}\xi \\ 1\end{pmatrix}\right|^2 = \frac{\lambda^2|\xi|^2+1}{\lambda}$$

and therefore

$$|\Delta_{S^2} u_M|^2(\xi) = \frac{\lambda^4(1+|\xi|^2)^4}{(1+\lambda^2|\xi|^2)^4}|\Delta_{S^2} u|^2(M\xi). \tag{4.1.3}$$

Analogously we get

$$|\nabla_{S^2} u_M|^2(\xi) = \frac{\lambda^2(1+|\xi|^2)^2}{(1+\lambda^2|\xi|^2)^2}|\nabla_{S^2} u|^2(M\xi). \tag{4.1.4}$$

Note that the transformation relation depends only on the eigenvalue λ. Hence it is enough to restrict our attention in the following to dilations m_λ. To show that $M \in SU(2)$ it suffices to show that $\lambda = 1$. We set

$$u(m_\lambda(\xi)) = u(\lambda\xi) =: u_\lambda(\xi)$$

and

$$\chi_\lambda(\xi) = \frac{(1+\lambda^2|\xi|^2)^2}{\lambda^2(1+|\xi|^2)^2}.$$

With (4.1.3) and (4.1.4) we have

$$|\nabla_{S^2} u|^2(\lambda\xi) = \chi_\lambda(\xi)|\nabla_{S^2} u_\lambda|^2(\xi) \qquad \text{and} \qquad |\Delta_{S^2} u|^2(\lambda\xi) = \chi_\lambda^2(\xi)|\Delta_{S^2} u_\lambda|^2(\xi).$$

Applying all of this to E_ε we get

$$\begin{aligned}
E_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{C}} \left(|\nabla_{S^2} u|^2(\xi) + \varepsilon|\Delta_{S^2} u|^2(\xi)\right) \frac{4}{(1+|\xi|^2)^2} dA_{\mathbb{C}}(\xi)\\
&= \frac{1}{2}\int_{\mathbb{C}} \left(|\nabla_{S^2} u|^2(\lambda\xi) + \varepsilon|\Delta_{S^2} u|^2(\lambda\xi)\right) \frac{4\lambda^2}{(1+|\lambda\xi|^2)^2} dA_{\mathbb{C}}(\xi)\\
&= \frac{1}{2}\int_{\mathbb{C}} \left(\chi_\lambda(\xi)|\nabla_{S^2} u_\lambda|^2(\xi) + \varepsilon\chi_\lambda^2(\xi)|\Delta_{S^2} u_\lambda|^2(\xi)\right) \frac{4\lambda^2}{(1+|\lambda\xi|^2)^2} dA_{\mathbb{C}}(\xi)\\
&= \frac{1}{2}\int_{\mathbb{C}} \left(|\nabla_{S^2} u_\lambda|^2(\xi) + \varepsilon\chi_\lambda(\xi)|\Delta_{S^2} u_\lambda|^2(\xi)\right) \frac{4}{(1+|\xi|^2)^2} dA_{\mathbb{C}}(\xi)\\
&= \frac{1}{2}\int_{S^2} \left(|\nabla_{S^2} u_\lambda|^2 + \varepsilon\chi_\lambda|\Delta_{S^2} u_\lambda|^2\right) dA_{S^2}\\
&=: E_{\varepsilon,\lambda}(u_\lambda).
\end{aligned} \tag{4.1.5}$$

Hence u is a critical point of E_ε if and only if u_λ is a critical point of $E_{\varepsilon,\lambda}$. Since $E_{\varepsilon,\lambda}(u_\lambda) = E_{\varepsilon,\lambda^{-1}}(u_{\lambda^{-1}})$ we will assume from now on that $\lambda \geq 1$. In the following we omit the subscript and write $\nabla = \nabla_{S^2},\ \Delta = \Delta_{S^2}$. An easy calculation (see [11] Proposition 1.1) shows

Proposition 4.1.1. *Let $\varepsilon > 0$. Every critical point $v \in W^{2,2}(S^2, S^2)$ of $E_{\varepsilon,\lambda}$ satisfies the following Euler-Lagrange equation*

$$-\Delta v + \varepsilon\Delta(\chi_\lambda \Delta v) = v\Big(|\nabla v|^2 - \varepsilon\Delta(\chi_\lambda|\nabla v|^2) - 2\varepsilon \operatorname{div}\langle \chi_\lambda \Delta v, \nabla v\rangle + \varepsilon\chi_\lambda|\Delta v|^2\Big). \tag{4.1.6}$$

4.2 Closeness to the Möbius group

In this section we consider critical points of E_ε of degree 1 whose ε-energy lies below $4\pi(1+2\varepsilon)+\mu$, $\mu > 0$ small. These maps are $W^{1,2}$-close to a Möbius transformation as the following result shows.

Proposition 4.2.1. *For any $\delta > 0$ there exists $\mu > 0$ such that, if $0 \leq \varepsilon \leq 1$ and if $E_\varepsilon(u) \leq 4\pi(1+2\varepsilon)+\mu$, where u is a critical point of E_ε of degree 1, then there exists $M \in PSL(2,\mathbb{C})$ such that*

$$\|\nabla(u_M - \mathrm{Id})\|_{L^2(S^2)} + \sqrt{\varepsilon}\,\|\sqrt{\chi_\lambda}\,\Delta(u_M - \mathrm{Id})\|_{L^2(S^2)} \leq \delta. \tag{4.2.1}$$

Furthermore, there exists a fixed constant $C > 0$ such that if $\lambda \geq 1$ is the largest eigenvalue of MM^* (see (4.1.1)), then

$$\varepsilon(\lambda^2 - 1) \leq C\delta. \tag{4.2.2}$$

Proof. We prove (4.2.1) by contradiction using the energy identity (4.0.2). If (4.2.1) were not true, then we could find a sequence $\mu_n \downarrow 0$, a sequence $\varepsilon_n \in [0,1]$, a sequence $u_n \in W^{2,2}(S^2, S^2)$ of critical points of E_{ε_n} of degree one, with $E_{\varepsilon_n}(u_n) \leq 4\pi(1 + 2\varepsilon_n) + \mu_n$ and $\delta > 0$ such that

$$\left\|\nabla\big((u_n)_M - \mathrm{Id}\big)\right\|_{L^2(S^2)} + \sqrt{\varepsilon_n}\left\|\sqrt{\chi_\lambda}\Delta\big((u_n)_M - \mathrm{Id}\big)\right\|_{L^2(S^2)} > \delta \tag{4.2.3}$$

for all $M \in PSL(2,\mathbb{C})$. Now we have to consider two cases:

$\underline{\varepsilon_n \to 0:}$

There exists $n_0 \in \mathbb{N}$ large enough such that $\varepsilon_n < \frac{1}{4}$ and $\mu_n < \frac{1}{2}$ $\forall n \geq n_0$. Then $E_{\varepsilon_n}(u_n)$ is uniformly bounded by $6\pi + \frac{1}{2}$ for all $n \geq n_0$. By Theorem 1.1 in [44] and Theorem 2 in [16], (u_n) converges to a harmonic map u^* and finitely many non-trivial harmonic two-spheres $u^i : S^2 \to S^2$ with $\deg(u^*) + \sum_{i=1}^k \deg(u^i) = 1$. With the result of Lemaire and Wood mentioned in the introduction, u^*, u^i are rational maps with energy $E(u^*) = 4\pi|\deg(u^*)|$, $E(u^i) = 4\pi|\deg(u^i)|$. Since

$$4\pi = \lim_{n\to\infty} E_{\varepsilon_n}(u_n) = E(u^*) + \sum_{i=1}^k E(u^i) = 4\pi\Big(|\deg(u^*)| + \sum_{i=1}^k |\deg(u^i)|\Big)$$

u^* is either a rational map with $\deg(u^*) = 1$ and $k = 0$, or u^* is a constant map, $k = 1$ and $u^1 : S^2 \to S^2$ is a harmonic map of degree one. In the first case, $u^* = m^*$ with some corresponding $M^* \in PSL(2,\mathbb{C})$ which is a contradiction to (4.2.3).

In the second case, energy concentrates in a small neighborhood of some $x_0 \in S^2$ and u_n converges to a constant map away from x_0. Without loss of generality let x_0 be the south pole S. Let $\sigma_n \downarrow 0$ and D_n be a sequence of small disks around S such that the energy on $S^2 \setminus D_n$ is smaller than σ_n. We project D_n onto the complex plane. Then $\Pi(D_n) = B_{r_n}(0)$, the complex ball with radius r_n and $r_n \to 0$. We perform a blow-up around the origin and define

$$v_n \colon \hat{\mathbb{C}} \to S^2, \qquad v_n(\xi) = u_n \circ \Pi^{-1}\left(\frac{\xi}{r_n}\right).$$

Note that this rescaling corresponds to dilations m_{λ_n} on the sphere with $\lambda_n = \frac{1}{r_n}$ and D_n gets mapped to the lower hemisphere. v_n is a critical point of $E_{\tilde{\varepsilon}_n}$ with $\tilde{\varepsilon}_n = \frac{\varepsilon_n}{r_n^2}$. By Lemma 3.1 in [44] we have

$$v_n \to v^* \qquad \text{in } C^m_{loc}(\mathbb{C}, S^2) \qquad \forall m \in \mathbb{N},$$

where $v^* \colon \mathbb{C} \to S^2$ is a non-trivial harmonic map. With the point removability result of Sacks and Uhlenbeck [63] we can lift v^* to a harmonic map from S^2 to S^2 with corresponding $M^* \in PSL(2,\mathbb{C})$ such that

$$\begin{aligned}
0 \leftarrow &||\nabla(v_n - v^*)||_{L^2(S^2)} + \sqrt{\varepsilon_n}\lambda_n||\Delta(v_n - v^*)||_{L^2(S^2)} \\
&\geq ||\nabla(v_n - v^*)||_{L^2(S^2)} + \sqrt{\varepsilon_n}||\sqrt{\chi_{\lambda_n}}\Delta(v_n - v^*)||_{L^2(S^2)} \\
&= ||\nabla(u_n - (v^*)_{M_{\lambda_n}^{-1}})||_{L^2(S^2)} + \sqrt{\varepsilon_n}||\Delta(u_n - (v^*)_{M_{\lambda_n}^{-1}})||_{L^2(S^2)} \\
&= ||\nabla((u_n)_{M_{\lambda_n}(M^*)^{-1}} - \mathrm{Id})||_{L^2(S^2)} + \sqrt{\varepsilon_n}||\sqrt{\chi_{M_{\lambda_n}(M^*)^{-1}}}\Delta((u_n)_{M_{\lambda_n}(M^*)^{-1}} - \mathrm{Id})||_{L^2(S^2)}.
\end{aligned}$$

$\underline{\varepsilon_n \to \varepsilon_\infty \in (0,1]:}$

Here we have, at least for n large enough, a uniform $W^{2,2}$-bound for the sequence u_n. With the regularity results from chapter 3 we conclude that u_n converges strongly in $W^{2,2}$ to a limiting map u_∞ which is a critical point of E_{ε_∞} and which satisfies

$$E_{\varepsilon_\infty}(u_\infty) = 4\pi(1 + 2\varepsilon_\infty).$$

By (1.0.20) this implies that u_∞ is a rotation, contradicting (4.2.3).

To establish (4.2.2) we set $v := u_M$ and calculate as in section 5 of [47]

$$\begin{aligned}
\log(\chi_\lambda(\xi)) &= 2\log(1 + \lambda^2|\xi|^2) - 2\log\lambda - 2\log(1 + |\xi|^2),\\
\frac{d}{d\lambda}\log(\chi_\lambda(\xi)) &= \frac{4\lambda|\xi|^2}{1 + \lambda^2|\xi|^2} - \frac{2}{\lambda},\\
\frac{d}{d\log\lambda}\log(\chi_\lambda(\xi)) &= \frac{2(\lambda^2|\xi|^2 - 1)}{(\lambda^2|\xi|^2 + 1)},
\end{aligned}$$

to obtain

$$\frac{d}{d\ \log\lambda}E_{\varepsilon,\lambda}(v) = \varepsilon\int_{S^2}\chi_\lambda z(\lambda\cdot)|\Delta v|^2\, dA_{S^2},$$

where $z(\xi) = \frac{|\xi|^2-1}{|\xi|^2+1}$. Since v is a critical point of $E_{\varepsilon,\lambda}$ we have $E'_{\varepsilon,\lambda}(v) = 0$. Using $E_{\varepsilon,\tau}(v) = E_{\varepsilon,\lambda}(v_{\lambda\tau^{-1}})$ we get

$$\frac{d}{d\log\tau}E_{\varepsilon,\tau}(v)|_{\tau=\lambda} = \left(\tau\frac{d}{d\tau}E_{\varepsilon,\lambda}(v_{\lambda\tau^{-1}})\right)\Big|_{\tau=\lambda} = E'_{\varepsilon,\lambda}(v)\left(\tau\frac{d}{d\tau}v_{\lambda\tau^{-1}}\right)\Big|_{\tau=\lambda}$$

and thus

$$\frac{d}{d\ \log\lambda}E_{\varepsilon,\lambda}(v) = 0.$$

Further

$$\begin{aligned}
E_\varepsilon(m_\lambda) = E_{\varepsilon,\lambda}(\mathrm{Id}) &= 4\pi + 2\varepsilon\int_{S^2}\chi_\lambda\, dA_{S^2}\\
&= 4\pi + 2\varepsilon\int_{\mathbb{C}}\frac{(1+\lambda^2|\xi|^2)^2}{\lambda^2(1+|\xi|^2)^2}\frac{4}{(1+|\xi|^2)^2}\, dA_{\mathbb{C}}(\xi)\\
&= 4\pi + 16\pi\varepsilon\int_0^\infty\frac{(1+\lambda^2r^2)^2}{\lambda^2(1+r^2)^2}\frac{r}{(1+r^2)^2}\, dr\\
&= 4\pi + 8\pi\varepsilon\frac{\lambda}{\lambda^2-1}\int_{\lambda^{-1}}^{\lambda} w^2\, dw\\
&= 4\pi\big(1 + \frac{2\varepsilon}{3}(\lambda^2 + 1 + \lambda^{-2})\big), \qquad (4.2.4)
\end{aligned}$$

where we used the substitution $w = \frac{1+\lambda^2r^2}{\lambda(1+r^2)}$. Differentiating this explicit expression for $E_\varepsilon(m_\lambda)$ with respect to $\log\lambda$ yields

$$\begin{aligned}
\frac{d}{d\log\lambda}E_\varepsilon(m_\lambda) &= \frac{16\pi\varepsilon}{3}(\lambda^2 - \lambda^{-2})\\
&= \frac{16\pi\varepsilon}{3}(\lambda^2 - 1)\frac{\lambda^2+1}{\lambda^2}\\
&\geq \frac{16\pi\varepsilon}{3}(\lambda^2 - 1). \qquad (4.2.5)
\end{aligned}$$

Since $\|z\|_{L^\infty(S^2)} \leq 1$, we conclude that

$$\begin{aligned} C\varepsilon(\lambda^2-1) &\leq \frac{d}{d\log\lambda}E_{\varepsilon,\lambda}(\mathrm{Id}) - \frac{d}{d\log\lambda}E_{\varepsilon,\lambda}(v) \\ &= \varepsilon\int_{S^2}\chi_\lambda z(\lambda\cdot)(|\Delta\,\mathrm{Id}\,|^2 - |\Delta v|^2)\,dA_{S^2} \\ &\leq \sqrt{\varepsilon}\|\sqrt{\chi_\lambda}\Delta(v-\mathrm{Id})\|_{L^2(S^2)}\sqrt{\varepsilon}\big(\|\sqrt{\chi_\lambda}\Delta v\|_{L^2(S^2)} + \|\sqrt{\chi_\lambda}\Delta\,\mathrm{Id}\,\|_{L^2(S^2)}\big). \end{aligned} \tag{4.2.6}$$

By assumption,

$$4\pi(1+2\varepsilon)+\mu \geq E_\varepsilon(u) = E_{\varepsilon,\lambda}(u_M) = E_{\varepsilon,\lambda}(v) \geq 4\pi + \frac{\varepsilon}{2}\int_{S^2}\chi_\lambda|\Delta v|^2\,dA_{S^2},$$

where we used

$$\frac{1}{2}\int_{S^2}|\nabla v|^2\,dA_{S^2} \geq 4\pi$$

in the second inequality, which holds because $\deg(v)=1$. Thus

$$\varepsilon\|\sqrt{\chi_\lambda}\Delta v\|^2_{L^2(S^2)} \leq 16\pi\varepsilon + 2\mu. \tag{4.2.7}$$

By the triangle inequality

$$\sqrt{\varepsilon}\|\sqrt{\chi_\lambda}\Delta\,\mathrm{Id}\,\|_{L^2(S^2)} \leq \sqrt{\varepsilon}\big(\|\sqrt{\chi_\lambda}\Delta(\mathrm{Id}\,-v)\|_{L^2(S^2)} + \|\sqrt{\chi_\lambda}\Delta v\|_{L^2(S^2)}\big). \tag{4.2.8}$$

Using (4.2.1), (4.2.7) and (4.2.8) in (4.2.6), we get

$$\varepsilon(\lambda^2-1) \leq C\delta.$$

□

4.3 $\sqrt{\varepsilon}W^{3,2}$-closeness

Next we want to improve the $W^{1,2}$-closeness result from the previous section and get a better bound on the eigenvalue λ.

Proposition 4.3.1 (Polarisation Identity). *Suppose v is a critical point of $E_{\varepsilon,\lambda}$. Setting $\psi := v-\mathrm{Id}$ we have*

$$\Delta\psi + \psi|\nabla\psi|^2 + 2\psi\langle\nabla\psi,\nabla\,\mathrm{Id}\rangle + 2\psi + \mathrm{Id}\,|\nabla\psi|^2 + 2\,\mathrm{Id}\langle\nabla\psi,\nabla\,\mathrm{Id}\rangle = \varepsilon\sum_{j=1}^{3}\Psi_j(\psi,\mathrm{Id}) \tag{4.3.1}$$

with

$$\begin{aligned} \Psi_1(\psi,\mathrm{Id}) &= \chi_\lambda\Big[\Delta^2\psi - 4\psi + (\psi+\mathrm{Id})\Big(4\langle\nabla\Delta\psi,\nabla\psi\rangle + 4\langle\nabla\Delta\psi,\nabla\,\mathrm{Id}\rangle + |\Delta\psi|^2 + 2|\nabla^2\psi|^2 \\ &\qquad + 4\langle\nabla^2\psi,\nabla^2\,\mathrm{Id}\rangle - 4\langle\Delta\psi,\mathrm{Id}\rangle - 8\langle\nabla\psi,\nabla\,\mathrm{Id}\rangle\Big)\Big], \\ \Psi_2(\psi,\mathrm{Id}) &= \nabla_i\chi_\lambda\Big[2\nabla_i\Delta\psi - 4\nabla_i\,\mathrm{Id}\,+(\psi+\mathrm{Id})\Big(4\langle\nabla_i\nabla\psi,\nabla\psi\rangle + 2\langle\Delta\psi,\nabla_i\psi\rangle + 4\langle\nabla_i\nabla\psi,\nabla\,\mathrm{Id}\rangle \\ &\qquad + 2\langle\Delta\psi,\nabla_i\,\mathrm{Id}\rangle + 4\langle\nabla\psi,\nabla_i\nabla\,\mathrm{Id}\rangle - 4\langle\nabla_i\psi,\mathrm{Id}\rangle\Big)\Big], \\ \Psi_3(\psi,\mathrm{Id}) &= \Delta\chi_\lambda\Big[\Delta\psi + 2\psi + (\psi+\mathrm{Id})\Big(|\nabla\psi|^2 + 2\langle\nabla\psi,\nabla\,\mathrm{Id}\rangle\Big)\Big]. \end{aligned}$$

Proof. We use (4.1.6) and replace v with $\psi+\mathrm{Id}$. Note that $\Delta\,\mathrm{Id} = -2\,\mathrm{Id}$, $|\,\mathrm{Id}\,|^2 = 1$ and $|\nabla\,\mathrm{Id}\,|^2 = 2$. Then we have

$$\begin{aligned}
-\Delta\psi - \Delta\,\mathrm{Id} - (\psi+\mathrm{Id})|\nabla\psi + \nabla\,\mathrm{Id}\,|^2 = {} & -\varepsilon\Delta\left(\chi_\lambda(\Delta\psi + \Delta\,\mathrm{Id})\right) \\
& + \varepsilon(\psi+\mathrm{Id})\chi_\lambda|\Delta\psi + \Delta\,\mathrm{Id}\,|^2 \\
& - \varepsilon(\psi+\mathrm{Id})\Delta\left(\chi_\lambda|\nabla\psi + \nabla\,\mathrm{Id}\,|^2\right) \\
& - 2\varepsilon(\psi+\mathrm{Id})\,\mathrm{div}\langle\chi_\lambda(\Delta\psi+\Delta\,\mathrm{Id}), \nabla\psi + \nabla\,\mathrm{Id}\rangle \\
\Leftrightarrow \Delta\psi + \psi|\nabla\psi|^2 + 2\psi + 2\psi\langle\nabla\psi, \nabla\,\mathrm{Id}\rangle & + \mathrm{Id}\,|\nabla\psi|^2 + 2\,\mathrm{Id}\langle\nabla\psi,\nabla\,\mathrm{Id}\rangle \\
= {} & \varepsilon\Delta\left(\chi_\lambda(\Delta\psi + \Delta\,\mathrm{Id})\right) \\
& - \varepsilon(\psi+\mathrm{Id})\chi_\lambda|\Delta\psi + \Delta\,\mathrm{Id}\,|^2 \\
& + \varepsilon(\psi+\mathrm{Id})\Delta\left(\chi_\lambda|\nabla\psi + \nabla\,\mathrm{Id}\,|^2\right) \\
& + 2\varepsilon(\psi+\mathrm{Id})\,\mathrm{div}\langle\chi_\lambda(\Delta\psi+\Delta\,\mathrm{Id}), \nabla\psi + \nabla\,\mathrm{Id}\rangle \qquad (4.3.2)
\end{aligned}$$

We compute the right-hand side further.

$$\varepsilon\Delta\left(\chi_\lambda(\Delta\psi + \Delta\,\mathrm{Id})\right) = \varepsilon\left(\Delta\chi_\lambda(\Delta\psi - 2\,\mathrm{Id}) + 2\langle\nabla\chi_\lambda, \nabla\Delta\psi\rangle - 4\langle\nabla\chi_\lambda, \nabla\,\mathrm{Id}\rangle + \chi_\lambda\Delta^2\psi + 4\chi_\lambda\,\mathrm{Id}\right).$$

For the second term we have

$$-\varepsilon(\psi+\mathrm{Id})\chi_\lambda|\Delta\psi + \Delta\,\mathrm{Id}\,|^2 = -\varepsilon(\psi+\mathrm{Id})\chi_\lambda\Big(|\Delta\psi|^2 + 4|\,\mathrm{Id}\,|^2 - 4\langle\Delta\psi, \mathrm{Id}\rangle\Big)$$

and for the third term

$$\begin{aligned}
\varepsilon(\psi+\mathrm{Id})\Delta\Big(\chi_\lambda|\nabla\psi + \nabla\,\mathrm{Id}\,|^2\Big) = {} & \varepsilon(\psi+\mathrm{Id})\Delta\chi_\lambda\Big(|\nabla\psi|^2 + 2 + 2\langle\nabla\psi, \nabla\,\mathrm{Id}\rangle\Big) \\
& + 4\varepsilon(\psi+\mathrm{Id})\nabla_i\chi_\lambda\Big(\langle\nabla\psi, \nabla_i\nabla\psi\rangle + \langle\nabla_i\nabla\psi, \nabla\,\mathrm{Id}\rangle + \langle\nabla\psi, \nabla_i\nabla\,\mathrm{Id}\rangle\Big) \\
& + 2\varepsilon(\psi+\mathrm{Id})\chi_\lambda\Big(\langle\nabla\Delta\psi, \nabla\psi\rangle + |\nabla^2\psi|^2 + \langle\nabla\Delta\psi, \nabla\,\mathrm{Id}\rangle \\
& \qquad - 2\langle\nabla\psi, \nabla\,\mathrm{Id}\rangle + 2\langle\nabla^2\psi, \nabla^2\,\mathrm{Id}\rangle\Big).
\end{aligned}$$

For the fourth term we have

$$\begin{aligned}
& 2\varepsilon(\psi+\mathrm{Id})\,\mathrm{div}\langle\chi_\lambda(\Delta\psi + \Delta\,\mathrm{Id}), \nabla\psi + \nabla\,\mathrm{Id}\rangle \\
& = 2\varepsilon(\psi+\mathrm{Id})\nabla_i\chi_\lambda\Big(\langle\Delta\psi, \nabla_i\psi\rangle + \langle\Delta\psi, \nabla_i\,\mathrm{Id}\rangle - 2\langle\mathrm{Id}, \nabla_i\psi\rangle\Big) \\
& \quad + 2\varepsilon(\psi+\mathrm{Id})\chi_\lambda\Big(\langle\nabla\Delta\psi, \nabla\psi\rangle + \langle\nabla\Delta\psi, \nabla\,\mathrm{Id}\rangle - 2\langle\nabla\,\mathrm{Id}, \nabla\psi\rangle + |\Delta\psi|^2 - 4\langle\mathrm{Id}, \Delta\psi\rangle\Big).
\end{aligned}$$

Adding all these terms together yields

$$\begin{aligned}
& \varepsilon\chi_\lambda\Big[\Delta^2\psi - 4\psi + (\psi+\mathrm{Id})\Big(4\langle\nabla\Delta\psi, \nabla\psi\rangle + 4\langle\nabla\Delta\psi, \nabla\,\mathrm{Id}\rangle + |\Delta\psi|^2 + 2|\nabla^2\psi|^2 + 4\langle\nabla^2\psi, \nabla^2\,\mathrm{Id}\rangle \\
& \qquad - 4\langle\Delta\psi, \mathrm{Id}\rangle - 8\langle\nabla\psi, \nabla\,\mathrm{Id}\rangle\Big)\Big] \\
& + \varepsilon\nabla_i\chi_\lambda\Big[2\nabla_i\Delta\psi - 4\nabla_i\,\mathrm{Id} + (\psi+\mathrm{Id})\Big(4\langle\nabla_i\nabla\psi, \nabla\psi\rangle + 2\langle\Delta\psi, \nabla_i\psi\rangle + 4\langle\nabla_i\nabla\psi, \nabla\,\mathrm{Id}\rangle \\
& \qquad + 2\langle\Delta\psi, \nabla_i\,\mathrm{Id}\rangle + 4\langle\nabla\psi, \nabla_i\nabla\,\mathrm{Id}\rangle - 4\langle\nabla_i\psi, \mathrm{Id}\rangle\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\Delta\chi_\lambda\Big[\Delta\psi+2\psi+(\psi+\mathrm{Id})\Big(|\nabla\psi|^2+2\langle\nabla\psi,\nabla\,\mathrm{Id}\rangle\Big)\Big]\\
&=:\varepsilon\Big(\Psi_1+\Psi_2+\Psi_3\Big).
\end{aligned}
$$

□

Before we get to the next lemma note that we can estimate χ_λ and its derivatives in terms of λ. Note that we assumed $\lambda\geq 1$. It is easy to see that $|\chi_\lambda|\leq\lambda^2$. Additionally

$$
\partial_i\chi_\lambda(\xi)=\frac{4\xi_i(1+\lambda^2|\xi|^2)(\lambda^2-1)}{\lambda^2(1+|\xi|^2)^3}
$$

and with $g_{ij}=\frac{4}{(1+|\xi|^2)^2}\delta_{ij}$

$$
|\nabla_{S^2}\chi_\lambda|=\sqrt{g^{ij}\partial_i\chi_\lambda\partial_j\chi_\lambda}=\frac{2|\xi|(1+\lambda^2|\xi|^2)(\lambda^2-1)}{\lambda^2(1+|\xi|^2)^2}\leq c(\lambda^2-1). \tag{4.3.3}
$$

Further

$$
\begin{aligned}
\frac{|\nabla_{S^2}\chi_\lambda|^2}{\chi_\lambda}&=\frac{4|\xi|^2(1+\lambda^2|\xi|^2)^2(\lambda^2-1)^2}{\lambda^4(1+|\xi|^2)^4}\cdot\frac{\lambda^2(1+|\xi|^2)^2}{(1+\lambda^2|\xi|^2)^2}=\frac{4|\xi|^2(\lambda^2-1)^2}{\lambda^2(1+|\xi|^2)^2}\\
&\leq c(\lambda^2-1).
\end{aligned} \tag{4.3.4}
$$

To estimate the Laplacian of χ_λ we calculate

$$
\sum_{i=1}^2\partial^2_{ii}\chi_\lambda(\xi)=\frac{8(\lambda^2-1)(1+2\lambda^2|\xi|^2)}{\lambda^2(1+|\xi|^2)^3}-\frac{24(\lambda^2-1)|\xi|^2(1+\lambda^2|\xi|^2)}{\lambda^2(1+|\xi|^2)^4}
$$

and

$$
\begin{aligned}
|\Delta_{S^2}\chi_\lambda|&=\left|\frac{1}{\sqrt{\det g}}\partial_i\left(g^{ij}\sqrt{\det g}\partial_j\chi_\lambda\right)\right|\\
&=\left|\frac{2(\lambda^2-1)(1+2\lambda^2|\xi|^2)}{\lambda^2(1+|\xi|^2)}-\frac{6|\xi|^2(1+\lambda^2|\xi|^2)(\lambda^2-1)}{\lambda^2(1+|\xi|^2)^2}\right|\leq c(\lambda^2-1).
\end{aligned} \tag{4.3.5}
$$

With this we show

Lemma 4.3.2. *There exist $0<\varepsilon_0,\delta_0<1$ and a constant $C>0$ depending only on ε_0 and δ_0 such that for every $0<\varepsilon<\varepsilon_0$, every $0<\delta<\delta_0$ and every critical point $v\in W^{2,2}(S^2,S^2)$ of $E_{\varepsilon,\lambda}$ satisfying (4.2.1) and (4.2.2) we have*

$$
||\sqrt{\chi_\lambda}\nabla^2\psi||_{L^2(S^2)}+\sqrt{\varepsilon}||\chi_\lambda\nabla^3\psi||_{L^2(S^2)}\leq C(\delta+\varepsilon)\lambda,
$$

with $\psi=v-\mathrm{Id}$.

Proof. Note that $||\psi||_{L^\infty(S^2)}\leq 2$. We start by estimating the mean value of ψ with (4.2.1), (4.3.2) and integration by parts. Note that $\int_{S^2}\Delta\left(\chi_\lambda(\Delta\psi+\Delta\,\mathrm{Id})\right)=0$.

$$
\begin{aligned}
\left|2\fint_{S^2}\psi dA_{S^2}\right|=\Bigg|&-\fint_{S^2}\Big(\psi|\nabla\psi|^2+2\psi\langle\nabla\psi,\nabla\,\mathrm{Id}\rangle+\mathrm{Id}\,|\nabla\psi|^2+2\,\mathrm{Id}\langle\nabla\psi,\nabla\,\mathrm{Id}\rangle\Big)dA_{S^2}\\
&+\varepsilon\fint_{S^2}\Big(\Delta\left(\chi_\lambda(\Delta\psi+\Delta\,\mathrm{Id})\right)-(\psi+\mathrm{Id})\chi_\lambda|\Delta\psi+\Delta\,\mathrm{Id}\,|^2\\
&+(\psi+\mathrm{Id})\Delta\left(\chi_\lambda|\nabla\psi+\nabla\,\mathrm{Id}\,|^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + 2(\psi + \mathrm{Id}) \operatorname{div}\langle \chi_\lambda(\Delta\psi + \Delta \mathrm{Id}), \nabla\psi + \nabla \mathrm{Id}\rangle\Big) dA_{S^2}\Big| \\
&\leq c \int_{S^2} |\nabla\psi|^2 dA_{S^2} + c\left(\int_{S^2} |\nabla\psi|^2 dA_{S^2}\right)^{\frac{1}{2}} \\
&\quad + \varepsilon\Big| \fint_{S^2} \Big(-(\psi + \mathrm{Id})\chi_\lambda |\Delta\psi + \Delta \mathrm{Id}|^2 + (\psi + \mathrm{Id})\Delta\left(\chi_\lambda |\nabla\psi + \nabla \mathrm{Id}|^2\right) \\
&\qquad + 2(\psi + \mathrm{Id}) \operatorname{div}\langle \chi_\lambda(\Delta\psi + \Delta \mathrm{Id}), \nabla\psi + \nabla \mathrm{Id}\rangle\Big) dA_{S^2}\Big| \\
&\leq c\delta + \varepsilon\Big| \fint_{S^2} \Big(-(\psi + \mathrm{Id})\chi_\lambda |\Delta\psi + \Delta \mathrm{Id}|^2 + (\Delta\psi + \Delta \mathrm{Id})\left(\chi_\lambda |\nabla\psi + \nabla \mathrm{Id}|^2\right) \\
&\qquad - 2(\nabla\psi + \nabla \mathrm{Id})\langle \chi_\lambda(\Delta\psi + \Delta \mathrm{Id}), \nabla\psi + \nabla \mathrm{Id}\rangle\Big) dA_{S^2}\Big|. \qquad (4.3.6)
\end{aligned}
$$

We estimate the remaining terms using Young's inequality, (4.2.1) and (4.2.2)

$$
\begin{aligned}
\varepsilon\Big| \fint_{S^2} (\psi + \mathrm{Id})\chi_\lambda |\Delta\psi + \Delta \mathrm{Id}|^2 dA_{S^2}\Big| &\leq c\varepsilon \int_{S^2} \chi_\lambda(|\Delta\psi|^2 + 1) dA_{S^2} \\
&\leq c(\delta^2 + \varepsilon) + c\varepsilon(\lambda^2 - 1) \\
&\leq c(\delta + \varepsilon)
\end{aligned}
$$

and

$$
\begin{aligned}
&\varepsilon\Big| \fint_{S^2} (\Delta\psi + \Delta \mathrm{Id})\left(\chi_\lambda |\nabla\psi + \nabla \mathrm{Id}|^2\right) dA_{S^2}\Big| \\
&\quad\leq c\varepsilon \int_{S^2} \chi_\lambda(|\Delta\psi| + 1)\left(|\nabla\psi|^2 + 1\right) dA_{S^2} \\
&\quad\leq c\varepsilon \int_{S^2} \chi_\lambda |\Delta\psi|^2 dA_{S^2} + c\varepsilon \int_{S^2} \chi_\lambda |\nabla\psi|^4 dA_{S^2} + c\varepsilon \int_{S^2} \chi_\lambda |\nabla\psi|^2 dA_{S^2} + c\varepsilon\lambda^2 \\
&\quad\leq c(\delta + \varepsilon) + c\varepsilon \int_{S^2} \chi_\lambda |\nabla\psi|^4 dA_{S^2}. \qquad (4.3.7)
\end{aligned}
$$

For the last term we use the Sobolev embedding $W^{1,1} \hookrightarrow L^2(S^2)$, (4.2.1) and (4.3.4)

$$
\begin{aligned}
\int_{S^2} \chi_\lambda |\nabla\psi|^4 dA_{S^2} &\leq c\left(\int_{S^2} \frac{|\nabla\chi_\lambda|}{\sqrt{\chi_\lambda}} |\nabla\psi|^2 dA_{S^2}\right)^2 + c\left(\int_{S^2} \chi_\lambda |\nabla^2\psi|^2 dA_{S^2}\right)\left(\int_{S^2} |\nabla\psi|^2 dA_{S^2}\right) \\
&\quad + \left(\int_{S^2} \sqrt{\chi_\lambda} |\nabla\psi|^2 dA_{S^2}\right)^2 \\
&\leq c\delta^4\lambda^2 + c\delta^2\left(\int_{S^2} \chi_\lambda |\nabla^2\psi|^2 dA_{S^2}\right).
\end{aligned}
$$

To get an estimate on the full second derivative we integrate by parts and exchange derivatives. By Lemma 2.1.2 in [42] we have $|\nabla\Delta\psi - \Delta\nabla\psi| \leq c(|\nabla\psi|^3 + |\nabla\psi|)$ and therefore

$$
\begin{aligned}
\int_{S^2} \chi_\lambda |\nabla^2\psi|^2 dA_{S^2} &\leq c\int_{S^2} |\nabla\chi_\lambda||\nabla\psi||\nabla^2\psi| dA_{S^2} + c\int_{S^2} \chi_\lambda |\Delta\psi|^2 dA_{S^2} \\
&\quad + c\int_{S^2} \chi_\lambda(|\nabla\psi|^4 + |\nabla\psi|^2) dA_{S^2} \\
&\leq (c\delta^2 + \eta)\int_{S^2} \chi_\lambda |\nabla^2\psi|^2 dA_{S^2} + c_\eta \int_{S^2} \frac{|\nabla\chi_\lambda|^2}{\chi_\lambda} |\nabla\psi|^2 dA_{S^2}
\end{aligned}
$$

$$+ c\int_{S^2}\chi_\lambda|\Delta\psi|^2 dA_{S^2} + c\delta^2\lambda^2.$$

For $\delta, \eta > 0$ small we absorb the first term to the left-hand side and with (4.3.4) we have

$$\int_{S^2}\chi_\lambda|\nabla^2\psi|^2 dA_{S^2} \le c\delta^2\lambda^2 + c\int_{S^2}\chi_\lambda|\Delta\psi|^2 dA_{S^2} \tag{4.3.8}$$

and thus

$$\int_{S^2}\chi_\lambda|\nabla\psi|^4 dA_{S^2} \le c\delta^4\lambda^2 + c\delta^2\int_{S^2}\chi_\lambda|\Delta\psi|^2 dA_{S^2}. \tag{4.3.9}$$

Going back to (4.3.7) and using the above estimates we get

$$\varepsilon\left|⨍_{S^2}(\Delta\psi + \Delta\,\mathrm{Id})\left(\chi_\lambda|\nabla\psi + \nabla\,\mathrm{Id}\,|^2\right) dA_{S^2}\right| \le c(\delta+\varepsilon).$$

Analogously we estimate

$$\varepsilon\left|⨍_{S^2} -2(\nabla\psi + \nabla\,\mathrm{Id})\langle\chi_\lambda(\Delta\psi + \Delta\,\mathrm{Id}), \nabla\psi + \nabla\,\mathrm{Id}\rangle dA_{S^2}\right| \le c(\delta+\varepsilon).$$

Combining all these estimates in (4.3.6) we obtain

$$\left|⨍_{S^2}\psi dA_{S^2}\right| \le c(\delta+\varepsilon). \tag{4.3.10}$$

Further we have with $W^{1,1}\hookrightarrow L^2(S^2)$, (4.3.4), (4.3.8), (4.3.9) and Proposition 4.2.1

$$\begin{aligned}
\varepsilon\int_{S^2}\chi_\lambda^2|\nabla\psi|^6 dA_{S^2} &\le c\varepsilon\left(\int_{S^2}|\nabla\chi_\lambda||\nabla\psi|^3 dA_{S^2}\right)^2 + c\varepsilon\left(\int_{S^2}\chi_\lambda|\nabla^2\psi||\nabla\psi|^2 dA_{S^2}\right)^2\\
&\quad + c\varepsilon\left(\int_{S^2}\chi_\lambda|\nabla\psi|^3 dA_{S^2}\right)^2\\
&\le c\varepsilon\left(\int_{S^2}\chi_\lambda|\nabla\psi|^4 dA_{S^2}\right)\left(\int_{S^2}\frac{|\nabla\chi_\lambda|^2}{\chi_\lambda}|\nabla\psi|^2 dA_{S^2}\right)\\
&\quad + c\varepsilon\left(\int_{S^2}\chi_\lambda|\nabla^2\psi|^2 dA_{S^2}\right)\left(\int_{S^2}\chi_\lambda|\nabla\psi|^4 dA_{S^2}\right)\\
&\quad + c\varepsilon\left(\int_{S^2}\chi_\lambda|\nabla\psi|^4 dA_{S^2}\right)\left(\int_{S^2}\chi_\lambda|\nabla\psi|^2 dA_{S^2}\right)\\
&\le c\delta^4\lambda^2 + c\delta^2\int_{S^2}\chi_\lambda|\Delta\psi|^2 dA_{S^2}
\end{aligned}$$

and similarly

$$\begin{aligned}
\varepsilon^2\int_{S^2}\chi_\lambda^3|\nabla^2\psi|^4 dA_{S^2} &\le c\left(\varepsilon\int_{S^2}\sqrt{\chi_\lambda}|\nabla\chi_\lambda||\nabla^2\psi|^2 dA_{S^2}\right)^2 + c\left(\varepsilon\int_{S^2}\chi_\lambda^{\frac{3}{2}}|\nabla^3\psi||\nabla^2\psi| dA_{S^2}\right)^2\\
&\quad + \left(\varepsilon\int_{S^2}\chi_\lambda^{\frac{3}{2}}|\nabla^2\psi|^2 dA_{S^2}\right)^2\\
&\le c\left(\varepsilon\int_{S^2}\frac{|\nabla\chi_\lambda|}{\sqrt{\chi_\lambda}}\chi_\lambda|\nabla^2\psi|^2 dA_{S^2}\right)^2 + c\left(\varepsilon\int_{S^2}\chi_\lambda^{\frac{3}{2}}|\nabla^2\psi|^2 dA_{S^2}\right)^2\\
&\quad + c\left(\varepsilon\int_{S^2}\chi_\lambda^2|\nabla^3\psi|^2 dA_{S^2}\right)\left(\varepsilon\int_{S^2}\chi_\lambda|\nabla^2\psi|^2 dA_{S^2}\right)
\end{aligned}$$

$$\leq c\delta^4\lambda^2 + c\delta^2\varepsilon\int_{S^2}\chi_\lambda^2|\nabla^3\psi|^2 dA_{S^2}. \tag{4.3.11}$$

With this we can estimate the L^2-norm of $\chi_\lambda\nabla^3\psi$. As above we integrate by parts and exchange derivatives. By Lemma 2.1.2 in [42] we have $|\nabla^2\Delta\psi - \Delta\nabla^2\psi| \leq c(|\nabla^2\psi||\nabla\psi| + |\nabla^2\psi| + |\nabla\psi|^4 + |\nabla\psi|)$. With Proposition 4.2.1 and the estimates above we get

$$\begin{aligned}
\varepsilon\int_{S^2}\chi_\lambda^2|\nabla^3\psi|^2 dA_{S^2} &\leq c\varepsilon\int_{S^2}\chi_\lambda|\nabla\chi_\lambda||\nabla^2\psi||\nabla^3\psi| dA_{S^2} + c\varepsilon\int_{S^2}\chi_\lambda^2|\nabla\Delta\psi|^2 dA_{S^2}\\
&\quad + c\varepsilon\int_{S^2}\chi_\lambda^2|\nabla^2\psi|\left(|\nabla^2\psi||\nabla\psi|^2 + |\nabla^2\psi| + |\nabla\psi|^4 + |\nabla\psi|\right) dA_{S^2}\\
&\leq \eta\varepsilon\int_{S^2}\chi_\lambda^2|\nabla^3\psi|^2 dA_{S^2} + c_\eta\varepsilon\int_{S^2}\chi_\lambda\frac{|\nabla\chi_\lambda|^2}{\chi_\lambda}|\nabla^2\psi|^2 dA_{S^2}\\
&\quad + c\varepsilon\int_{S^2}\chi_\lambda^2|\nabla\Delta\psi|^2 dA_{S^2} + c_\eta\varepsilon^2\int_{S^2}\chi_\lambda^3|\nabla^2\psi|^4 dA_{S^2} + c\int_{S^2}\chi_\lambda|\nabla\psi|^4 dA_{S^2}\\
&\quad + c\varepsilon\int_{S^2}\chi_\lambda^2|\nabla^2\psi|^2 dA_{S^2} + c\varepsilon\int_{S^2}\chi_\lambda^2|\nabla\psi|^6 dA_{S^2} + c\varepsilon\int_{S^2}\chi_\lambda^2|\nabla\psi|^2 dA_{S^2}\\
&\quad + \eta\int_{S^2}\chi_\lambda|\nabla^2\psi|^2 dA_{S^2}\\
&\leq c\delta^2\lambda^2 + (c\delta^2 + \eta)\int_{S^2}\chi_\lambda|\nabla^2\psi|^2 dA_{S^2} + (c_\eta\delta^2 + \eta)\varepsilon\int_{S^2}\chi_\lambda^2|\nabla^3\psi|^2 dA_{S^2}\\
&\quad + c\varepsilon\int_{S^2}\chi_\lambda^2|\nabla\Delta\psi|^2 dA_{S^2}.
\end{aligned}$$

and for $\delta, \eta > 0$ small enough

$$\varepsilon\int_{S^2}\chi_\lambda^2|\nabla^3\psi|^2 dA_{S^2} \leq c\delta^2\lambda^2 + (c\delta^2 + \eta)\int_{S^2}\chi_\lambda|\nabla^2\psi|^2 dA_{S^2} + c\varepsilon\int_{S^2}\chi_\lambda^2|\nabla\Delta\psi|^2 dA_{S^2}. \tag{4.3.12}$$

Now we multiply (4.3.1) with $\chi_\lambda\Delta\psi$ and integrate over S^2. After rearranging we get

$$\begin{aligned}
&\int_{S^2}\langle(\Delta - \varepsilon\chi_\lambda\Delta^2)\psi, \chi_\lambda\Delta\psi\rangle dA_{S^2}\\
&= \int_{S^2}\Big\langle\Big(-\psi|\nabla\psi|^2 - 2\psi\langle\nabla\psi, \nabla\,\mathrm{Id}\rangle - 2\psi - \mathrm{Id}\,|\nabla\psi|^2 - 2\,\mathrm{Id}\langle\nabla\psi, \nabla\,\mathrm{Id}\rangle\Big)\\
&\quad + \varepsilon\Big(\Psi_1 - \chi_\lambda\Delta^2\psi + \Psi_2 + \Psi_3\Big), \chi_\lambda\Delta\psi\Big\rangle dA_{S^2}.
\end{aligned} \tag{4.3.13}$$

We estimate the left-hand side further

$$\begin{aligned}
&\int_{S^2}\langle(\Delta - \varepsilon\chi_\lambda\Delta^2)\psi, \chi_\lambda\Delta\psi\rangle dA_{S^2}\\
&= \int_{S^2}\chi_\lambda|\Delta\psi|^2 dA_{S^2} + \varepsilon\int_{S^2}\chi_\lambda^2|\nabla\Delta\psi|^2 dA_{S^2} + 2\varepsilon\int_{S^2}\chi_\lambda\nabla\chi_\lambda\nabla\Delta\psi\Delta\psi dA_{S^2}\\
&\geq \int_{S^2}\chi_\lambda|\Delta\psi|^2 dA_{S^2} + \frac{3}{4}\varepsilon\int_{S^2}\chi_\lambda^2|\nabla\Delta\psi|^2 dA_{S^2} - c\varepsilon\int_{S^2}|\nabla\chi_\lambda|^2|\Delta\psi|^2 dA_{S^2}.
\end{aligned}$$

Together with (4.2.1), (4.3.4), (4.3.8) and (4.3.12) we have

$$\begin{aligned}
&\int_{S^2}\chi_\lambda|\nabla^2\psi|^2 dA_{S^2} + \varepsilon\int_{S^2}\chi_\lambda^2|\nabla^3\psi|^2 dA_{S^2}\\
&\leq c\int_{S^2}\langle(\Delta - \varepsilon\chi_\lambda\Delta^2)\psi, \chi_\lambda\Delta\psi\rangle dA_{S^2} + c\varepsilon\int_{S^2}\chi_\lambda\frac{|\nabla\chi_\lambda|^2}{\chi_\lambda}|\Delta\psi|^2 dA_{S^2} + c\delta^2\lambda^2
\end{aligned}$$

$$\leq c \int_{S^2} \left\langle (\Delta - \varepsilon\chi_\lambda \Delta^2)\psi, \chi_\lambda \Delta\psi \right\rangle dA_{S^2} + c\delta^2\lambda^2. \tag{4.3.14}$$

On the right-hand side of (4.3.13) we have

$$\begin{aligned}
&\int_{S^2} \Big\langle \Big(-\psi|\nabla\psi|^2 - 2\psi\langle\nabla\psi, \nabla \operatorname{Id}\rangle - 2\psi - \operatorname{Id}|\nabla\psi|^2 - 2\operatorname{Id}\langle\nabla\psi, \nabla\operatorname{Id}\rangle \Big) \\
&\qquad + \varepsilon\Big((\Psi_1 - \chi_\lambda\Delta^2\psi) + \Psi_2 + \Psi_3 \Big), \chi_\lambda\Delta\psi \Big\rangle dA_{S^2} \\
&=: I + II + III + IV.
\end{aligned} \tag{4.3.15}$$

We estimate each term separately. With Young's inequality, (4.2.1) and (4.3.9)

$$\begin{aligned}
I &= \int_{S^2} \Big\langle -\psi|\nabla\psi|^2 - 2\psi\langle\nabla\psi, \nabla\operatorname{Id}\rangle - 2\psi - \operatorname{Id}|\nabla\psi|^2 - 2\operatorname{Id}\langle\nabla\psi, \nabla\operatorname{Id}\rangle, \chi_\lambda\Delta\psi \Big\rangle dA_{S^2} \\
&\leq \eta \int_{S^2} \chi_\lambda |\Delta\psi|^2 dA_{S^2} + c_\eta \int_{S^2} \chi_\lambda \left(|\nabla\psi|^4 + |\nabla\psi|^2 \right) dA_{S^2} - 2\int_{S^2} \langle\psi, \chi_\lambda\Delta\psi\rangle dA_{S^2} \\
&\leq (\eta + c_\eta\delta^2) \int_{S^2} \chi_\lambda |\Delta\psi|^2 dA_{S^2} + c_\eta\delta^2\lambda^2 - 2\int_{S^2} \langle\psi, \chi_\lambda\Delta\psi\rangle dA_{S^2}.
\end{aligned}$$

Let $\bar\psi = \fint_{S^2} \psi$ be the mean value of ψ. Integrating by parts, applying the Poincaré inequality as well as (4.2.1), (4.3.3) and (4.3.10) yields

$$\begin{aligned}
-2\int_{S^2} \langle\psi, \chi_\lambda\Delta\psi\rangle dA_{S^2} &= -2\int_{S^2} \chi_\lambda \left[(\psi - \bar\psi) + \bar\psi \right] \Delta\psi dA_{S^2} \\
&\leq \eta \int_{S^2} \chi_\lambda |\Delta\psi|^2 dA_{S^2} + c_\eta \int_{S^2} \chi_\lambda |\psi - \bar\psi|^2 dA_{S^2} + 2\bar\psi \int_{S^2} \nabla\chi_\lambda \nabla\psi dA_{S^2} \\
&\leq \eta \int_{S^2} \chi_\lambda |\Delta\psi|^2 dA_{S^2} + c_\eta\lambda^2 \int_{S^2} |\nabla\psi|^2 dA_{S^2} \\
&\quad + c(\lambda^2 - 1)|\bar\psi| \left(\int_{S^2} |\nabla\psi|^2 dA_{S^2} \right)^{\frac{1}{2}} \\
&\leq \eta \int_{S^2} \chi_\lambda |\Delta\psi|^2 dA_{S^2} + c_\eta\delta(\delta + \varepsilon)\lambda^2.
\end{aligned}$$

All in all we have

$$I \leq c\left(\eta + c_\eta\delta^2\right) \int_{S^2} \chi_\lambda |\nabla^2\psi|^2 dA_{S^2} + c_\eta\delta(\delta + \varepsilon)\lambda^2.$$

To estimate the second term we use (4.2.1), (4.2.2), (4.3.8), (4.3.9), (4.3.11) and (4.3.12)

$$\begin{aligned}
II &= \varepsilon \int_{S^2} \Big\langle \chi_\lambda \Big[-4\psi + (\psi + \operatorname{Id}) \Big(4\langle\nabla\Delta\psi, \nabla\psi\rangle + 4\langle\nabla\Delta\psi, \nabla\operatorname{Id}\rangle + |\Delta\psi|^2 + 2|\nabla^2\psi|^2 \\
&\qquad + 4\langle\nabla^2\psi, \nabla^2\operatorname{Id}\rangle - 4\Delta\psi\operatorname{Id} - 8\langle\nabla\psi, \nabla\operatorname{Id}\rangle \Big) \Big], \chi_\lambda\Delta\psi \Big\rangle dA_{S^2} \\
&\leq c\varepsilon \int_{S^2} \chi_\lambda^2 \Big(|\nabla\Delta\psi||\Delta\psi||\nabla\psi| + |\nabla\Delta\psi||\Delta\psi| + |\nabla^2\psi|^3 + |\nabla^2\psi|^2 + |\nabla\psi||\Delta\psi| + |\Delta\psi| \Big) dA_{S^2} \\
&\leq \eta\varepsilon \int_{S^2} \chi_\lambda^2 |\nabla^3\psi|^2 dA_{S^2} + \eta \int_{S^2} \chi_\lambda |\nabla^2\psi|^2 dA_{S^2} + c_\eta\varepsilon^2 \int_{S^2} \chi_\lambda^3 |\nabla^2\psi|^4 dA_{S^2} \\
&\quad + c_\eta\varepsilon \int_{S^2} \chi_\lambda^2 |\nabla^2\psi|^2 dA_{S^2} + c_\eta \int_{S^2} \chi_\lambda |\nabla\psi|^4 dA_{S^2} + c\varepsilon \int_{S^2} \chi_\lambda^2 |\nabla\psi|^2 dA_{S^2} + c_\eta\varepsilon^2 \int_{S^2} \chi_\lambda^3 dA_{S^2}
\end{aligned}$$

$$\le (\eta + c_\eta\delta^2)\varepsilon \int_{S^2} \chi_\lambda^2 |\nabla^3\psi|^2 dA_{S^2} + (\eta + c_\eta\delta^2) \int_{S^2} \chi_\lambda |\nabla^2\psi|^2 dA_{S^2} + c(\delta+\varepsilon)^2\lambda^2.$$

Similarly we get for the next term with (4.2.1), (4.2.2), (4.3.4), (4.3.8), (4.3.11) and (4.3.12)

$$\begin{aligned} III &= \varepsilon \int_{S^2} \Big\langle \nabla_i\chi_\lambda \Big[2\nabla_i\Delta\psi - 4\nabla_i \operatorname{Id} + (\psi + \operatorname{Id}) \Big(4\langle \nabla_i\nabla\psi, \nabla\psi\rangle + 2\langle \Delta\psi, \nabla_i\psi\rangle \\ &\quad + 4\langle \nabla_i\nabla\psi, \nabla \operatorname{Id}\rangle + 2\langle \Delta\psi, \nabla_i \operatorname{Id}\rangle + 4\langle \nabla\psi, \nabla_i\nabla \operatorname{Id}\rangle - 4\langle \nabla_i\psi, \operatorname{Id}\rangle \Big) \Big], \chi_\lambda\Delta\psi \Big\rangle dA_{S^2} \\ &\le c\varepsilon \int_{S^2} \chi_\lambda |\nabla\chi_\lambda| \Big(|\nabla\Delta\psi||\Delta\psi| + |\nabla^2\psi|^2|\nabla\psi| + |\nabla^2\psi|^2 + |\Delta\psi||\nabla\psi| + |\Delta\psi| \Big) dA_{S^2} \\ &\le \eta\varepsilon \int_{S^2} \chi_\lambda^2 |\nabla\Delta\psi|^2 dA_{S^2} + \eta \int_{S^2} \chi_\lambda |\nabla^2\psi|^2 dA_{S^2} + c_\eta\varepsilon \int_{S^2} \frac{|\nabla\chi_\lambda|^2}{\chi_\lambda} \chi_\lambda |\Delta\psi|^2 dA_{S^2} \\ &\quad + c\varepsilon^2 \int_{S^2} \chi_\lambda^3 |\nabla^2\psi|^4 dA_{S^2} + c \int_{S^2} \frac{|\nabla\chi_\lambda|^2}{\chi_\lambda} |\nabla\psi|^2 dA_{S^2} + c\varepsilon \int_{S^2} \chi_\lambda |\nabla\chi_\lambda| |\nabla^2\psi|^2 dA_{S^2} \\ &\quad + c\varepsilon \int_{S^2} \chi_\lambda |\nabla\chi_\lambda| |\nabla\psi|^2 dA_{S^2} + c_\eta\varepsilon^2 \int_{S^2} \chi_\lambda |\nabla\chi_\lambda|^2 dA_{S^2} \\ &\le (\eta + c\delta^2)\varepsilon \int_{S^2} \chi_\lambda^2 |\nabla^3\psi|^2 dA_{S^2} + \eta \int_{S^2} \chi_\lambda |\nabla^2\psi|^2 dA_{S^2} + c_\eta(\delta+\varepsilon)^2\lambda^2. \end{aligned}$$

and finally we have with (4.3.5)

$$\begin{aligned} IV &= \varepsilon \int_{S^2} \Big\langle \Delta\chi_\lambda \Big[\Delta\psi + 2\psi + (\psi + \operatorname{Id}) \Big(|\nabla\psi|^2 + 2\langle \nabla\psi, \nabla \operatorname{Id}\rangle \Big) \Big], \chi_\lambda\Delta\psi \Big\rangle dA_{S^2} \\ &\le c\varepsilon \int_{S^2} \chi_\lambda |\Delta\chi_\lambda| \Big(|\Delta\psi|^2 + |\Delta\psi||\nabla\psi|^2 + |\Delta\psi||\nabla\psi| + |\Delta\psi| \Big) dA_{S^2} \\ &\le c\varepsilon \int_{S^2} \chi_\lambda |\Delta\chi_\lambda| |\Delta\psi|^2 dA_{S^2} + c\varepsilon \int_{S^2} \chi_\lambda |\Delta\chi_\lambda| |\nabla\psi|^4 dA_{S^2} + c\varepsilon \int_{S^2} \chi_\lambda |\Delta\chi_\lambda| |\nabla\psi|^2 dA_{S^2} \\ &\quad + \eta \int_{S^2} \chi_\lambda |\Delta\psi|^2 dA_{S^2} + c_\eta\varepsilon^2 \int_{S^2} \chi_\lambda |\Delta\chi_\lambda|^2 dA_{S^2} \\ &\le \eta \int_{S^2} \chi_\lambda |\Delta\psi|^2 dA_{S^2} + c(\delta+\varepsilon)^2\lambda^2. \end{aligned}$$

Now we put (4.3.14), (4.3.15) and the above estimates together. Choosing δ and η small enough so that we can absorb these terms to the left-hand side we arrive at

$$\int_{S^2} \chi_\lambda |\nabla^2\psi|^2 dA_{S^2} + \varepsilon \int_{S^2} \chi_\lambda^2 |\nabla^3\psi|^2 dA_{S^2} \le c(\delta+\varepsilon)^2\lambda^2.$$

□

Corollary 4.3.3. *There exist $\varepsilon_0 > 0$ and $\delta_0 > 0$, possibly smaller than those in Lemma 4.3.2, such that if $v \in W^{2,2}(S^2, S^2)$ is a critical point of $E_{\varepsilon,\lambda}$ satisfying (4.2.1) and (4.2.2), then*

$$\lambda^2 - 1 \le c(\delta + \varepsilon)$$

for some $0 < \varepsilon \le \varepsilon_0$, $0 < \delta \le \delta_0$. Moreover, for $\psi = v - \operatorname{Id}$ the following estimate holds

$$||\psi||_{L^\infty(S^2)} + ||\psi||_{W^{2,2}(S^2)} + \sqrt{\varepsilon}||\nabla^3\psi||_{L^2(S^2)} \le c(\delta+\varepsilon). \tag{4.3.16}$$

Proof. With (4.2.6) and Lemma 4.3.2 we have

$$C\varepsilon(\lambda^2 - 1) \le \frac{d}{d\log\lambda} E_{\varepsilon,\lambda}(\operatorname{Id}) - \frac{d}{d\log\lambda} E_{\varepsilon,\lambda}(v)$$

$$
\begin{aligned}
&= \varepsilon \int_{S^2} \chi_\lambda z(\lambda\cdot)(|\Delta \operatorname{Id}|^2 - |\Delta v|^2)\, dA_{S^2} \\
&\le \sqrt{\varepsilon}\|\sqrt{\chi_\lambda}\Delta(v - \operatorname{Id})\|_{L^2(S^2)}\sqrt{\varepsilon}\big(\|\sqrt{\chi_\lambda}\Delta v\|_{L^2(S^2)} + \|\sqrt{\chi_\lambda}\Delta \operatorname{Id}\|_{L^2(S^2)}\big) \\
&\le c\varepsilon(\delta+\varepsilon)\lambda^2 = c\varepsilon(\delta+\varepsilon)(\lambda^2-1) + c\varepsilon(\delta+\varepsilon).
\end{aligned}
$$

For $\varepsilon + \delta$ small enough

$$\lambda^2 - 1 \le c(\delta+\varepsilon)$$

and $\lambda^2 \le 2$. But then

$$\frac{1}{2} \le \frac{1}{\lambda^2} \le |\chi_\lambda| \le \lambda^2 \le 2.$$

With this and Lemma 4.3.2 we get

$$\int_{S^2} |\nabla^2\psi|^2 dA_{S^2} + \varepsilon \int_{S^2} |\nabla^3\psi|^2 dA_{S^2} \le c(\delta+\varepsilon)^2.$$

By the Sobolev embedding $W^{2,2} \hookrightarrow L^\infty(S^2)$, the Poincaré inequality and (4.3.10) it follows that

$$
\begin{aligned}
||\psi||_{L^\infty(S^2)} &\le c||\psi||_{W^{2,2}(S^2)} \le c(\delta+\varepsilon) + c||\psi - \bar{\psi}||_{L^2(S^2)} + c|\bar{\psi}| \\
&\le c(\delta+\varepsilon) + c||\nabla\psi||_{L^2(S^2)} \le c(\delta+\varepsilon).
\end{aligned}
$$

□

Remark 4.3.4. *Note that*

$$|\chi_\lambda - 1| \le \begin{Bmatrix} \lambda^2 - 1, & \text{if } \chi_\lambda \ge 1 \\ 1 - \frac{1}{\lambda^2}, & \text{if } \chi_\lambda < 1 \end{Bmatrix} \le \lambda^2 - 1 \le c(\delta+\varepsilon). \tag{4.3.17}$$

4.4 Optimal Möbius transformation

This section follows in parts chapter 6 in [47]. For a better comprehension of the arguments we repeat some of the calculations here.

So far our results suggest that there exists $M \in PSL(2,\mathbb{C})$ such that u_M is close to the identity in $\sqrt{\varepsilon}W^{3,2}$, however there is still some freedom in the choice of M. To show that $u_M = \operatorname{Id}$ and λ equal to one, we have to choose the optimal Möbius transformation M with corresponding eigenvalue λ which minimizes $\|\nabla(u_M - \operatorname{Id})\|_{L^2(S^2)}$. To see that an optimal $M \in PSL(2,\mathbb{C})$ exists note that

$$
\begin{aligned}
||\nabla(u_M - \operatorname{Id})||^2_{L^2(S^2)} &= ||\nabla(u - M^{-1})||^2_{L^2(S^2)} \\
&= ||\nabla u||^2_{L^2(S^2)} + 2\langle \nabla u, \nabla M^{-1}\rangle_{L^2(S^2)} + ||\nabla M^{-1}||^2_{L^2(S^2)}.
\end{aligned}
$$

Up to rotations, M can only go to infinity if it approaches a dilation from the south pole to the north pole by a huge factor λ. In this scenario, the energy of m_λ is concentrated on a small disk D centered at the south pole. Let $\delta > 0$ and choose $D \subset S^2$ small enough so that the energy of u on D is less than δ and the energy of m_λ outside of D is less than δ. Further note that $||\nabla M^{-1}||_{L^2(S^2)} = ||\nabla \operatorname{Id}||_{L^2(S^2)}$ due to the conformal invariance of the Dirichlet energy (see (4.1.5)). Then

$$
\begin{aligned}
\langle \nabla u, \nabla M^{-1}\rangle_{L^2(S^2)} &= \langle \nabla u, \nabla M^{-1}\rangle_{L^2(D)} + \langle \nabla u, \nabla M^{-1}\rangle_{L^2(S^2\setminus D)} \\
&\leqslant \delta\left(||\nabla M^{-1}||_{L^2(D)} + ||\nabla u||_{L^2(S^2\setminus D)}\right).
\end{aligned}
$$

Thus

$$||\nabla(u - M^{-1})||^2_{L^2(S^2)} \to ||\nabla u||^2_{L^2(S^2)} + ||\nabla \operatorname{Id}||^2_{L^2(S^2)} \geq 16\pi$$

and it is enough to minimize $\|\nabla(u_M - \operatorname{Id})\|_{L^2(S^2)}$ over a compact subset of $PSL(2,\mathbb{C})$.

From now on we choose the optimal $M \in PSL(2,\mathbb{C})$ that minimizes $\|\nabla(u_M - \operatorname{Id})\|_{L^2(S^2)}$. Let $v := u_M$ satisfy the assumptions of Lemma 4.3.2 and Corollary 4.3.3. Our goal is to improve the bound in (4.3.16) to $\sqrt{\varepsilon}(\lambda^2 - 1)$. In (4.3.1), a problematic term to estimate is $\operatorname{Id}\langle\nabla\psi, \nabla \operatorname{Id}\rangle$, because it involves $\nabla\psi$ of order one. To eliminate this term we exploit that it is an element of the normal space at the identity.

By (4.3.16) v converges pointwise to the identity map as δ and ε tend to zero. Thus we can write v in terms of the tangential component of ψ at the identity

$$v = \operatorname{Id} + \psi = \exp_{\operatorname{Id}} \hat{\psi} \quad (= \operatorname{Id} + \hat{\psi} + O(|\hat{\psi}|^2)), \qquad \hat{\psi} \in T_{\operatorname{Id}} W^{3,2}(S^2, S^2).$$

In the following we want to work with $\hat{\psi}$ instead of ψ. To do this we need formulas to express $\hat{\psi}$ in terms of v and ψ. Let $\mathbf{x} = (x, y, z) \in S^2 \subset \mathbb{R}^3$, then

$$\begin{aligned}
&v(\mathbf{x}) = \mathbf{x}\sqrt{1 - |\hat{\psi}(\mathbf{x})|^2} + \hat{\psi}(\mathbf{x}), && \hat{\psi}(\mathbf{x}) \cdot \mathbf{x} \equiv 0, \\
&\hat{\psi}(\mathbf{x}) = \psi(\mathbf{x}) + \tfrac{1}{2}|\psi(\mathbf{x})|^2\mathbf{x}, && \psi(\mathbf{x}) = \hat{\psi}(\mathbf{x}) - \left(1 - \sqrt{1 - |\hat{\psi}(\mathbf{x})|^2}\right)\mathbf{x}, \\
&|\hat{\psi}|^2 = |\psi|^2(1 - \tfrac{1}{4}|\psi|^2) \leq |\psi|^2 = 2(1 - \sqrt{1 - |\hat{\psi}|^2}).
\end{aligned} \tag{4.4.1}$$

With this we get for the error terms of higher order

$$\begin{aligned}
|\nabla\psi - \nabla\hat{\psi}| &= O(|\hat{\psi}|\,|\nabla\hat{\psi}|) + O(|\hat{\psi}|^2) = O(|\psi|\,|\nabla\psi|) + O(|\psi|^2), \\
|\nabla^2\psi - \nabla^2\hat{\psi}| &= O(|\hat{\psi}||\nabla^2\hat{\psi}|) + O(|\nabla\hat{\psi}|^2) + O(|\hat{\psi}|^2) = O(|\psi||\nabla^2\psi|) + O(|\nabla\psi|^2) + O(|\psi|^2), \\
|\nabla^3\psi - \nabla^3\hat{\psi}| &= O(|\hat{\psi}||\nabla^3\hat{\psi}|) + O(|\nabla^2\hat{\psi}|^2) + O(|\nabla\hat{\psi}|^2) + O(|\hat{\psi}|^2) \\
&= O(|\psi||\nabla^3\psi|) + O(|\nabla^2\psi|^2) + O(|\nabla\psi|^2) + O(|\psi|^2).
\end{aligned} \tag{4.4.2}$$

Let $\mathbf{x}^T$ be the orthogonal projection of $\mathbf{x} \in S^2$ onto the tangent space $T_{\mathbf{x}}S^2$. The tangential component of (4.3.1) is given by

$$\begin{aligned}
\Big[\varepsilon\chi_\lambda\Delta^2\psi - \Delta\psi - 2\psi - 4\varepsilon\chi_\lambda\psi\Big]^T &= \Big[(\operatorname{Id} + \psi)|\nabla\psi|^2 + 2(\operatorname{Id} + \psi)\langle\nabla\psi, \nabla \operatorname{Id}\rangle \\
&\qquad - \varepsilon\Big(\Psi_1(\psi, \operatorname{Id}) - \chi_\lambda(\Delta^2\psi - 4\psi) + \Psi_2(\psi, \operatorname{Id}) + \Psi_3(\psi, \operatorname{Id})\Big)\Big]^T \\
\Leftrightarrow -\varepsilon(\Delta(\Delta\hat{\psi})^T)^T + (\Delta\hat{\psi})^T + 2\hat{\psi} + 4\varepsilon\hat{\psi}& \\
&= -2\hat{\psi}\langle\nabla\hat{\psi}, \nabla \operatorname{Id}\rangle + O(|\hat{\psi}||\nabla^2\hat{\psi}|) + O(|\nabla\hat{\psi}|^2) + O(|\hat{\psi}||\nabla\hat{\psi}|) + O(|\hat{\psi}|^2) \\
&\quad + \varepsilon\Big((\chi_\lambda - 1)(\Delta(\Delta\hat{\psi})^T)^T + \chi_\lambda(\Delta(\Delta(\psi - \hat{\psi}))^T)^T + \chi_\lambda(\Delta(\Delta\psi)^N)^T\Big) \\
&\quad - \varepsilon\Big[\Psi_1 - \chi_\lambda\Delta^2\psi + 4\chi_\lambda\psi + \Psi_2 + \Psi_3\Big]^T.
\end{aligned} \tag{4.4.3}$$

We set

$$J_\varepsilon := \left((\Delta\cdot)^T + 2\right) - \varepsilon\left((\Delta\cdot)^T - 2\right)\left((\Delta\cdot)^T + 2\right).$$

J_ε can be written as a positive operator applied to $J := ((\Delta\cdot)^T + 2)$, namely

$$J_\varepsilon = \left(\mathrm{Id} - \varepsilon((\Delta\cdot)^T - 2)\right)((\Delta\cdot)^T + 2).$$

Thus J and J_ε have the same kernel. Let $\hat{\psi} = \hat{\psi}_0 + \hat{\psi}_1$, where $\hat{\psi}_0 \in \ker J_\varepsilon$ and $\hat{\psi}_1 \in (\ker J_\varepsilon)^\perp$ with respect to the inner product in L^2. $\Delta^T \hat{\psi}_0 = -2\hat{\psi}_0$ since J_ε and J have the same kernel. Further note that J_ε is self-adjoint with respect to the inner product in $L^2(S^2, TS^2)$ and

$$\int_{S^2} \langle J_\varepsilon \hat{\psi}_1, (\Delta\hat{\psi}_0)^T \rangle dA_{S^2} = -2\int_{S^2} \left\langle J_\varepsilon \hat{\psi}_1, \hat{\psi}_0 \right\rangle dA_{S^2} = -2\int_{S^2} \left\langle \hat{\psi}_1, J_\varepsilon \hat{\psi}_0 \right\rangle dA_{S^2} = 0.$$

With this we get

$$\begin{aligned}\int_{S^2} \left\langle J_\varepsilon \hat{\psi}, (\Delta\hat{\psi})^T \right\rangle dA_{S^2} &= \int_{S^2} \left\langle J_\varepsilon \hat{\psi}_1, (\Delta\hat{\psi}_1)^T \right\rangle dA_{S^2} \\ &= \varepsilon \int_{S^2} |\nabla(\Delta\hat{\psi}_1)^T|^2 dA_{S^2} + \int_{S^2} |(\Delta\hat{\psi}_1)^T|^2 dA_{S^2} \\ &\quad + (2+4\varepsilon)\int_{S^2} \langle \hat{\psi}_1, (\Delta\hat{\psi}_1)^T \rangle dA_{S^2}. \end{aligned} \tag{4.4.4}$$

We want to control the $\sqrt{\varepsilon}W^{3,2}$-norm of $\hat{\psi}$ by the left-hand side of (4.4.4). The first two terms on the right-hand side are positive, which leaves us with the last term. To get a control on this term we decompose $\hat{\psi}_1 \in TS^2$ into eigenvectorfields of Δ_{TS^2}, the (rough) connection Laplacian on vector fields on S^2.

First we need to relate $(\Delta\cdot)^T$ and Δ_{TS^2}. Let e_1, e_2 be an orthonormal basis for $T_{\mathbf{x}}S^2$ so that $D_{e_i} e_j(\mathbf{x}) = 0$, where D is the covariant derivative on TS^2. Then we have

$$D_{e_i}\hat{\psi}(\mathbf{x}) = e_i(\hat{\psi})(\mathbf{x}) - (e_i(\hat{\psi}) \cdot \mathbf{x})\mathbf{x} = e_i(\hat{\psi})(\mathbf{x}) + (\hat{\psi}(\mathbf{x}) \cdot e_i(\mathbf{x}))\mathbf{x},$$

since $\hat{\psi}(\mathbf{x}) \cdot \mathbf{x} = 0$ and

$$\Delta_{TS^2}\hat{\psi}(\mathbf{x}) = \sum_{i=1}^{2} \left(D_{e_i}(e_i(\hat{\psi}))(\mathbf{x}) + (\hat{\psi}(\mathbf{x}) \cdot e_i(\mathbf{x}))e_i(\mathbf{x}) \right) = (\Delta\hat{\psi})^T(\mathbf{x}) + \hat{\psi}(\mathbf{x}),$$

where we used that $\hat{\psi}(\mathbf{x}) = \sum_{i=1}^{2} (\hat{\psi}(\mathbf{x}) \cdot e_i) e_i$.

We decompose $W^{3,2}(S^2, TS^2)$ into the eigenspaces of Δ_{TS^2} such that $W^{3,2}(S^2, TS^2) = \oplus_{j=1}^{\infty} E_{\lambda_j}$ with corresponding eigenvalues λ_j and eigenvectorfields $\hat{\psi}_{\lambda_j} \in E_{\lambda_j} \subset W^{3,2}(S^2, TS^2)$. By the above, the eigenvalues of Δ_{TS^2} are the spectrum of Δ shifted up by one. Thus $\lambda_j = -j(j+1)+1,\ j \in \mathbb{N}$ (see [68]). The first eigenvalue is $\lambda_1 = -1$ and the eigenvectorfield $\hat{\psi}_{\lambda_1}$ lies in the kernel of J_ε. Therefore $\hat{\psi}_1 = \sum_{j=2}^{\infty} \hat{\psi}_{\lambda_j}$. Note that $\int_{S^2} \langle \hat{\psi}_{\lambda_i}, \hat{\psi}_{\lambda_j} \rangle = 0$ if $i \neq j$. Then

$$\begin{aligned} 2\int_{S^2} \langle \hat{\psi}_1, (\Delta\hat{\psi}_1)^T \rangle dA_{S^2} &= \sum_{j=2}^{\infty} \int_{S^2} \left\langle \frac{2}{\lambda_j} \Delta_{TS^2}\hat{\psi}_{\lambda_j}, (\Delta\hat{\psi}_{\lambda_j})^T \right\rangle dA_{S^2} \\ &= \sum_{j=2}^{\infty} \int_{S^2} \left\langle \frac{2}{\lambda_j} \left((\Delta\hat{\psi}_{\lambda_j})^T + \hat{\psi}_{\lambda_j} \right), (\Delta\hat{\psi}_{\lambda_j})^T \right\rangle dA_{S^2} \\ &= \sum_{j=2}^{\infty} \frac{2}{\lambda_j} \int_{S^2} |(\Delta\hat{\psi}_{\lambda_j})^T|^2 dA_{S^2} + \sum_{j=2}^{\infty} \int_{S^2} \left\langle \frac{2}{\lambda_j} \hat{\psi}_{\lambda_j}, \left(\Delta_{TS^2}\hat{\psi}_{\lambda_j} - \hat{\psi}_{\lambda_j} \right) \right\rangle dA_{S^2} \\ &= \sum_{j=2}^{\infty} \frac{2}{\lambda_j} \int_{S^2} |(\Delta\hat{\psi}_{\lambda_j})^T|^2 dA_{S^2} + \sum_{j=2}^{\infty} \frac{2(\lambda_j - 1)}{\lambda_j} \int_{S^2} |\hat{\psi}_{\lambda_j}|^2 dA_{S^2}. \end{aligned}$$

Analogously we have

$$\begin{aligned}
4\varepsilon\int_{S^2}\langle\hat\psi_1,(\Delta\hat\psi_1)^T\rangle dA_{S^2} &= -4\varepsilon\sum_{j=2}^{\infty}\int_{S^2}\langle\nabla\hat\psi_{\lambda_j},\nabla\hat\psi_{\lambda_j}\rangle dA_{S^2}\\
&= -4\varepsilon\sum_{j=2}^{\infty}\int_{S^2}\frac{1}{\lambda_j^2}\langle\nabla\Delta_{TS^2}\hat\psi_{\lambda_j},\nabla\Delta_{TS^2}\hat\psi_{\lambda_j}\rangle dA_{S^2}\\
&= -4\varepsilon\sum_{j=2}^{\infty}\int_{S^2}\frac{1}{\lambda_j^2}\left\langle\nabla(\Delta\hat\psi_{\lambda_j})^T+\nabla\hat\psi_{\lambda_j},\nabla(\Delta\hat\psi_{\lambda_j})^T+\nabla\hat\psi_{\lambda_j}\right\rangle dA_{S^2}\\
&= -4\varepsilon\sum_{j=2}^{\infty}\int_{S^2}\frac{1}{\lambda_j^2}|\nabla(\Delta\hat\psi_{\lambda_j})^T|^2 dA_{S^2}-4\varepsilon\sum_{j=2}^{\infty}\int_{S^2}\frac{1}{\lambda_j^2}|\nabla\hat\psi_{\lambda_j}|^2 dA_{S^2}\\
&\quad -8\varepsilon\sum_{j=2}^{\infty}\int_{S^2}\frac{1}{\lambda_j^2}\left\langle\nabla\Delta_{TS^2}\hat\psi_{\lambda_j}-\nabla\hat\psi_{\lambda_j},\nabla\hat\psi_{\lambda_j}\right\rangle dA_{S^2}\\
&= -4\varepsilon\sum_{j=2}^{\infty}\int_{S^2}\frac{1}{\lambda_j^2}|\nabla(\Delta\hat\psi_{\lambda_j})^T|^2 dA_{S^2}+4\varepsilon\sum_{j=2}^{\infty}\int_{S^2}\frac{1-2\lambda_j}{\lambda_j^2}|\nabla\hat\psi_{\lambda_j}|^2 dA_{S^2}.
\end{aligned}$$

Inserting this in (4.4.4) yields

$$\begin{aligned}
\int_{S^2}\left\langle J_\varepsilon\hat\psi,(\Delta\hat\psi)^T\right\rangle dA_{S^2} &= \varepsilon\int_{S^2}|\nabla(\Delta\hat\psi_1)^T|^2 dA_{S^2}+\int_{S^2}|(\Delta\hat\psi_1)^T|^2 dA_{S^2}\\
&\quad+\sum_{j=2}^{\infty}\frac{2}{\lambda_j}\int_{S^2}|(\Delta\hat\psi_{\lambda_j})^T|^2 dA_{S^2}+\sum_{j=2}^{\infty}\frac{2(\lambda_j-1)}{\lambda_j}\int_{S^2}|\hat\psi_{\lambda_j}|^2 dA_{S^2}\\
&\quad-4\varepsilon\sum_{j=2}^{\infty}\int_{S^2}\frac{1}{\lambda_j^2}|\nabla(\Delta\hat\psi_{\lambda_j})^T|^2 dA_{S^2}+4\varepsilon\sum_{j=2}^{\infty}\int_{S^2}\frac{1-2\lambda_j}{\lambda_j^2}|\nabla\hat\psi_{\lambda_j}|^2 dA_{S^2}\\
&= \varepsilon\sum_{j=2}^{\infty}\frac{\lambda_j^2-4}{\lambda_j^2}\int_{S^2}|\nabla(\Delta\hat\psi_{\lambda_j})^T|^2 dA_{S^2}+\sum_{j=2}^{\infty}\frac{\lambda_j+2}{\lambda_j}\int_{S^2}|(\Delta\hat\psi_{\lambda_j})^T|^2 dA_{S^2}\\
&\quad+\varepsilon\sum_{j=2}^{\infty}\int_{S^2}\frac{4(1-2\lambda_j)}{\lambda_j^2}|\nabla\hat\psi_{\lambda_j}|^2 dA_{S^2}+\sum_{j=2}^{\infty}\frac{2(\lambda_j-1)}{\lambda_j}\int_{S^2}|\hat\psi_{\lambda_j}|^2 dA_{S^2}.
\end{aligned}$$

Note that $\lambda_2=-5$ and $\ldots\geq\lambda_j\geq\lambda_{j+1}\geq\ldots$. Therefore

$$\frac{\lambda_j^2-4}{\lambda_j^2}\geq\frac{21}{25},\qquad\frac{\lambda_j+2}{\lambda_j}\geq\frac{3}{5},\qquad\frac{4(1-2\lambda_j)}{\lambda_j^2}\geq 0,\qquad\frac{2(\lambda_j-1)}{\lambda_j}\geq 2\qquad\forall j\geq 2$$

and

$$\int_{S^2}\left\langle J_\varepsilon\hat\psi,(\Delta\hat\psi)^T\right\rangle dA_{S^2}\geq\varepsilon\frac{21}{25}||\nabla(\Delta\hat\psi_1)^T||^2_{L^2(S^2)}+\frac{3}{5}||(\Delta\hat\psi_1)^T||^2_{L^2(S^2)}+2||\hat\psi_1||^2_{L^2(S^2)}. \tag{4.4.5}$$

We want to bound the full second and third derivative of $\hat\psi_1$ in terms of $(\Delta\hat\psi_1)^T$ and $\nabla(\Delta\hat\psi_1)^T$. To do this we take a closer look at the normal part $(\Delta\hat\psi_1)^N=(\Delta\hat\psi_1\cdot\mathbf{x})\mathbf{x}$. Note that with e_1,e_2 the orthonormal basis for $T_{\mathbf{x}}S^2$ as before we have

$$\Delta\hat\psi_1\cdot\mathbf{x}=\sum_{i=1}^{2}e_i(e_i(\hat\psi_1))\cdot\mathbf{x}=\sum_{i=1}^{2}e_i\left(e_i(\hat\psi_1)\cdot\mathbf{x}\right)-\left(e_i(\hat\psi_1)\cdot e_i\right)(\mathbf{x})$$

$$
\begin{aligned}
&= \sum_{i=1}^{2} e_i\left(e_i\left(\hat{\psi}_1 \cdot \mathbf{x}\right)\right) - e_i\left(\hat{\psi}_1 \cdot e_i\right)(\mathbf{x}) - \left(e_i(\hat{\psi}_1) \cdot e_i\right)(\mathbf{x}) \\
&= -2\sum_{i=1}^{2}(e_i(\hat{\psi}_1) \cdot e_i)(\mathbf{x}) = -2 \operatorname{div} \hat{\psi}_1, \qquad (4.4.6)
\end{aligned}
$$

where we used that $\hat{\psi}_1$ is tangential in the second line and $\hat{\psi}_1 \cdot e_i(e_i) = \hat{\psi} \cdot D_{e_i} e_i = 0$ in the last line. Then

$$
\varepsilon\nabla(\Delta\hat{\psi}_1)^T = \varepsilon\nabla(\Delta\hat{\psi}_1) - 2\varepsilon\nabla(\operatorname{div}\hat{\psi}_1 \cdot \mathbf{x}) \qquad \text{and} \qquad (\Delta\hat{\psi}_1)^T = \Delta\hat{\psi}_1 - 2\operatorname{div}\hat{\psi}_1 \cdot \mathbf{x}.
$$

and therefore

$$
\begin{aligned}
\sqrt{\varepsilon}||\nabla\Delta\hat{\psi}_1||_{L^2(S^2)} &\leq \sqrt{\varepsilon}||\nabla(\Delta\hat{\psi}_1)^T||_{L^2(S^2)} + c\sqrt{\varepsilon}\left(||\nabla^2\hat{\psi}||_{L^2(S^2)} + ||\nabla\hat{\psi}_1||_{L^2(S^2)}\right), \\
||\Delta\hat{\psi}_1||_{L^2(S^2)} &\leq ||(\Delta\hat{\psi}_1)^T||_{L^2(S^2)} + c||\nabla\hat{\psi}_1||_{L^2(S^2)}.
\end{aligned}
$$

Integrating by parts we get

$$
\int_{S^2} |\nabla\hat{\psi}_1|^2 dA_{S^2} = -\int_{S^2} \hat{\psi}_1\Delta\hat{\psi}_1 dA_{S^2} \leq \eta||\Delta\hat{\psi}_1||^2_{L^2(S^2)} + c_\eta||\hat{\psi}||^2_{L^2(S^2)}
$$

and for $\eta > 0$ small

$$
\begin{aligned}
&\sqrt{\varepsilon}||\nabla\Delta\hat{\psi}_1||_{L^2(S^2)} + ||\Delta\hat{\psi}_1||_{L^2(S^2)} + ||\nabla\hat{\psi}_1||_{L^2(S^2)} \\
&\qquad \leq c\left(\sqrt{\varepsilon}||\nabla(\Delta\hat{\psi}_1)^T||_{L^2(S^2)} + ||(\Delta\hat{\psi}_1)^T||_{L^2(S^2)} + \sqrt{\varepsilon}||\nabla^2\hat{\psi}||_{L^2(S^2)} + ||\hat{\psi}||^2_{L^2(S^2)}\right).
\end{aligned}
$$

Going back to (4.4.5) this yields

$$
\begin{aligned}
&\sqrt{\varepsilon}||\nabla\Delta\hat{\psi}_1||_{L^2(S^2)} + ||\Delta\hat{\psi}_1||_{L^2(S^2)} + ||\nabla\hat{\psi}_1||_{L^2(S^2)} + ||\hat{\psi}_1||_{L^2(S^2)} \\
&\qquad \leq c\left(\int_{S^2} \langle J_\varepsilon\hat{\psi}, (\Delta\hat{\psi}_1)^T\rangle dA_{S^2}\right)^{\frac{1}{2}} + c\sqrt{\varepsilon}||\nabla^2\hat{\psi}||_{L^2(S^2)}.
\end{aligned}
$$

To get an estimate for the full second and third derivative we integrate by parts and exchange derivatives as in the proof of Lemma 4.3.2. All in all we arrive at

$$
\sqrt{\varepsilon}||\nabla^3\hat{\psi}_1||_{L^2(S^2)} + ||\hat{\psi}_1||_{W^{2,2}(S^2)} \leq c\left(\int_{S^2} \langle J_\varepsilon\hat{\psi}, (\Delta\hat{\psi})^T\rangle dA_{S^2}\right)^{\frac{1}{2}}. \qquad (4.4.7)
$$

The kernel of J, and therefore the kernel of J_ε, consists precisely of the span of the gradient of the linear functions on S^2 and their $90°$ rotations. Thus, the kernel of J_ε is finite dimensional and all norms are equivalent. We estimate

$$
\sqrt{\varepsilon}||\nabla^3\hat{\psi}_0||_{L^2(S^2)} + ||\hat{\psi}_0||_{W^{2,2}(S^2)} \leq c||\hat{\psi}_0||_{L^2(S^2)}.
$$

Together with (4.4.7)

$$
\sqrt{\varepsilon}||\nabla^3\hat{\psi}||_{L^2(S^2)} + ||\hat{\psi}||_{W^{2,2}(S^2)} \leq \left(\int_{S^2} \langle J_\varepsilon\hat{\psi}, (\Delta\hat{\psi})^T\rangle dA_{S^2}\right)^{\frac{1}{2}} + ||\hat{\psi}_0||_{L^2(S^2)}. \qquad (4.4.8)
$$

As noted above, the kernel of J_ε is the same as the kernel of J. Hence we can follow chapter 6 in [47] to estimate $\hat{\psi}_0$. For the sake of completeness we include these calculations here. By assumption

v minimizes $||\nabla(v-\mathrm{Id})||_{L^2(S^2)}$. Therefore

$$0=\int_{S^2}\nabla(v-\mathrm{Id})\cdot\nabla\xi dA_{S^2}=-\int_{S^2}v\cdot\Delta\xi dA_{S^2}\qquad\forall\xi\in\ker J,\tag{4.4.9}$$

where we used that $\nabla\,\mathrm{Id}\cdot\nabla\xi=\operatorname{div}\xi$ and $\int_{S^2}(\operatorname{div}\xi)dA_{S^2}=0$. We decompose $\Delta\xi$ into its tangential and normal part.

$$\Delta\xi(\mathbf{x})=(\Delta\xi)^T(\mathbf{x})+(\Delta\xi\cdot\mathbf{x})\mathbf{x}.$$

For the tangential part we have $(\Delta\xi)^T=-2\xi$, because ξ is in the kernel of J and for the normal part we have $(\Delta\xi\cdot\mathbf{x})\mathbf{x}=-2\mathbf{x}\operatorname{div}\xi(\mathbf{x})$ by (4.4.6). Applying this to (4.4.9) yields

$$\int_{S^2}v\cdot\xi dA_{S^2}+\int_{S^2}(v\cdot\mathbf{x})(\operatorname{div}\xi)dA_{S^2}=0.\tag{4.4.10}$$

With $v=\hat{\psi}+\mathbf{x}\sqrt{1-|\hat{\psi}|^2}$ (see (4.4.1)) we get

$$\int_{S^2}\left(\hat{\psi}+\mathbf{x}\sqrt{1-|\hat{\psi}|^2}\right)\cdot\xi dA_{S^2}=\int_{S^2}\hat{\psi}\cdot\xi dA_{S^2},$$

since $\xi\in T_{\mathbf{x}}S^2$. On the other hand

$$\int_{S^2}(v\cdot\mathbf{x})(\operatorname{div}\xi)dA_{S^2}=\int_{S^2}(\hat{\psi}\cdot\mathbf{x})(\operatorname{div}\xi)dA_{S^2}+\int_{S^2}\sqrt{1-|\hat{\psi}|^2}(\operatorname{div}\xi)dA_{S^2}$$

and (4.4.10) is equivalent to

$$\int_{S^2}\hat{\psi}\cdot\xi dA_{S^2}=-\int_{S^2}\sqrt{1-|\hat{\psi}|^2}(\operatorname{div}\xi)dA_{S^2}=\int_{S^2}\left(1-\sqrt{1-|\hat{\psi}|^2}\right)(\operatorname{div}\xi)dA_{S^2},$$

where we used $\int_{S^2}(\operatorname{div}\xi)dA_{S^2}=0$ in the last step. Now we choose $\xi=\hat{\psi}_0$, which we can do because $\hat{\psi}_0\in\ker J$. With (4.4.1) we estimate

$$||\hat{\psi}_0||^2_{L^2(S^2)}\leq||\hat{\psi}||^2_{L^\infty(S^2)}\int_{S^2}|\nabla\hat{\psi}_0|dA_{S^2}.$$

Hölder's inequality, integration by parts and $(\Delta\hat{\psi}_0)^T=-2\hat{\psi}_0$ yield

$$\int_{S^2}|\nabla\hat{\psi}_0|dA_{S^2}\leq c\left(\int_{S^2}|\nabla\hat{\psi}_0|^2dA_{S^2}\right)^{\frac{1}{2}}=c\left(\int_{S^2}-\Delta\hat{\psi}_0\cdot\hat{\psi}_0dA_{S^2}\right)^{\frac{1}{2}}=2c||\hat{\psi}_0||_{L^2(S^2)}$$

and therefore

$$||\hat{\psi}_0||^2_{L^2(S^2)}\leq||\hat{\psi}||^2_{L^\infty(S^2)}\int_{S^2}|\nabla\hat{\psi}_0|dA_{S^2}\leq c||\hat{\psi}||^2_{L^\infty(S^2)}||\hat{\psi}_0||_{L^2(S^2)}.$$

Dividing by $||\hat{\psi}_0||_{L^2(S^2)}$ and using the Sobolev embedding $W^{2,2}\hookrightarrow L^\infty(S^2)$ we receive

$$||\hat{\psi}_0||_{L^2(S^2)}\leq c||\hat{\psi}||^2_{L^\infty(S^2)}\leq c||\hat{\psi}||_{L^\infty(S^2)}||\hat{\psi}||_{W^{2,2}(S^2)}.$$

Going back to (4.4.8) and choosing $\delta+\varepsilon$ in Corollary 4.3.3 small enough gives

$$\sqrt{\varepsilon}||\nabla^3\hat{\psi}||_{L^2(S^2)}+||\hat{\psi}||_{W^{2,2}(S^2)}\leq\left(\int_{S^2}\langle J_\varepsilon\hat{\psi},(\Delta\hat{\psi})^T\rangle dA_{S^2}\right)^{\frac{1}{2}}.\tag{4.4.11}$$

Now we estimate the right hand side further. By (4.4.3)

$$\begin{aligned}
&\int_{S^2} \left\langle J_\varepsilon \hat{\psi}, (\Delta\hat{\psi})^T \right\rangle dA_{S^2} \\
&= \int_{S^2} \Big\langle -2\hat{\psi}\langle \nabla\hat{\psi}, \nabla \operatorname{Id}\rangle + O(|\hat{\psi}||\nabla^2\hat{\psi}|) + O(|\nabla\hat{\psi}|^2) + O(|\hat{\psi}||\nabla\hat{\psi}|) + O(|\hat{\psi}|^2) \\
&\quad + \varepsilon\Big((\chi_\lambda - 1)(\Delta(\Delta\hat{\psi})^T)^T + \chi_\lambda(\Delta(\Delta(\psi-\hat{\psi}))^T)^T + \chi_\lambda(\Delta(\Delta\psi)^N)^T\Big) \\
&\quad - \varepsilon\Big[\Psi_1 - \chi_\lambda\Delta^2\psi + 4\chi_\lambda\psi + \Psi_2 + \Psi_3\Big]^T, (\Delta\hat{\psi})^T \Big\rangle dA_{S^2}. \qquad (4.4.12)
\end{aligned}$$

We estimate each part separately. For the first part we use $W^{1,1} \hookrightarrow L^2(S^2)$, Hölder's inequality and Corollary 4.3.3

$$\begin{aligned}
&\int_{S^2} \left\langle 2\hat{\psi}\langle\nabla\hat{\psi}, \nabla \operatorname{Id}\rangle + O(|\hat{\psi}||\nabla^2\hat{\psi}|) + O(|\nabla\hat{\psi}|^2) + O(|\hat{\psi}||\nabla\hat{\psi}|) + O(|\hat{\psi}|^2), (\Delta\hat{\psi})^T \right\rangle dA_{S^2} \\
&\leq \int_{S^2} |\nabla^2\hat{\psi}|\Big(|\hat{\psi}||\nabla^2\hat{\psi}| + |\nabla\hat{\psi}|^2 + |\hat{\psi}|^2\Big) dA_{S^2} \\
&\leq (\eta + c(\delta+\varepsilon)) \int_{S^2} |\nabla^2\hat{\psi}|^2 dA_{S^2} + c_\eta \int_{S^2} \left(|\nabla\hat{\psi}|^4 + |\hat{\psi}|^4\right) dA_{S^2} \\
&\leq (\eta + c(\delta+\varepsilon)) \int_{S^2} |\nabla^2\hat{\psi}|^2 dA_{S^2} + c_\eta \left(\int_{S^2} |\nabla^2\hat{\psi}|^2 dA_{S^2}\right)\left(\int_{S^2} |\nabla\hat{\psi}|^2 dA_{S^2}\right) \\
&\quad + c_\eta \left(\int_{S^2} |\nabla\hat{\psi}|^2 dA_{S^2}\right)^2 + c_\eta \left(\int_{S^2} |\nabla\hat{\psi}|^2 dA_{S^2}\right)\left(\int_{S^2} |\hat{\psi}|^2 dA_{S^2}\right) + c_\eta\left(\int_{S^2} |\hat{\psi}|^2 dA_{S^2}\right)^2 \\
&\leq \left(\eta + c(\delta+\varepsilon) + c_\eta(\delta+\varepsilon)^2\right) ||\hat{\psi}||^2_{W^{2,2}(S^2)}.
\end{aligned}$$

Note that this is where the estimate fails if we include the term $\operatorname{Id}\langle\nabla\psi, \nabla \operatorname{Id}\rangle$ and this is the reason why we consider only tangential terms.

In the second part we use integration by parts, $W^{1,1} \hookrightarrow L^2(S^2)$, (4.3.3), (4.3.17), (4.4.2) and Corollary 4.3.3

$$\begin{aligned}
&\varepsilon \int_{S^2} \left\langle (\chi_\lambda - 1)(\Delta(\Delta\hat{\psi})^T)^T + \chi_\lambda(\Delta(\Delta(\psi-\hat{\psi}))^T)^T + \chi_\lambda(\Delta(\Delta\psi)^N)^T, (\Delta\hat{\psi})^T \right\rangle dA_{S^2} \\
&= -\varepsilon \int_{S^2} (\chi_\lambda - 1)|\nabla(\Delta\hat{\psi})^T|^2 + \chi_\lambda \left\langle \nabla\left(\Delta(\psi-\hat{\psi})\right)^T, \nabla(\Delta\hat{\psi})^T \right\rangle dA_{S^2} \\
&\quad - \varepsilon \int_{S^2} \chi_\lambda \left\langle \nabla(\psi|\nabla\psi|^2), \nabla(\Delta\hat{\psi})^T \right\rangle dA_{S^2} \\
&\quad - \varepsilon \int_{S^2} \nabla\chi_\lambda \left\langle \nabla(\Delta\hat{\psi})^T, (\Delta\hat{\psi})^T \right\rangle + \nabla\chi_\lambda \left\langle \nabla\left(\Delta(\psi-\hat{\psi})\right)^T, (\Delta\hat{\psi})^T \right\rangle dA_{S^2} \\
&\quad - \varepsilon \int_{S^2} \nabla\chi_\lambda \left\langle \nabla(\psi|\nabla\psi|^2), (\Delta\hat{\psi})^T \right\rangle dA_{S^2} \\
&\leq \varepsilon((\lambda^2 - 1) + \eta + c_\eta(\delta+\varepsilon)) \int_{S^2} |\nabla^3\hat{\psi}|^2 dA_{S^2} \\
&\quad + (c_\eta + (\lambda^2-1))\varepsilon \int_{S^2} |\nabla^2\hat{\psi}|^4 + |\nabla\hat{\psi}|^4 + |\hat{\psi}|^4 dA_{S^2} + c\varepsilon(\lambda^2-1)\int_{S^2} |\nabla^2\hat{\psi}|^2 dA_{S^2} \\
&\quad + (c_\eta + (\lambda^2-1))\varepsilon \int_{S^2} |\nabla\psi|^6 + |\nabla^2\psi|^2|\nabla\psi|^2|\psi|^2 dA_{S^2} \\
&\leq \varepsilon(\eta + c_\eta(\delta+\varepsilon)) \int_{S^2} |\nabla^3\hat{\psi}|^2 dA_{S^2} + c_\eta\varepsilon(\delta+\varepsilon)||\hat{\psi}||^2_{W^{2,2}(S^2)}.
\end{aligned}$$

In the same way we estimate

$$\begin{aligned}
&\varepsilon \int_{S^2} \left\langle (\Psi_1 - \chi_\lambda \Delta^2 \psi + 4\chi_\lambda \psi)^T, (\Delta \hat{\psi})^T \right\rangle dA_{S^2} \\
&= \varepsilon \int_{S^2} \Big\langle \chi_\lambda \Big[4\langle \nabla \Delta \psi, \nabla \psi \rangle + 2|\nabla^2 \psi|^2 + |\Delta \psi|^2 + 6\langle \nabla \Delta \psi, \nabla \operatorname{Id} \rangle \\
&\qquad + 4\langle \nabla_i \nabla \psi, \nabla_i \nabla \operatorname{Id} \rangle - 4\langle \Delta \psi, \Delta \operatorname{Id} \rangle - 8\langle \nabla \psi, \nabla \operatorname{Id} \rangle \Big]^T, (\Delta \hat{\psi})^T \Big\rangle dA_{S^2} \\
&\leq \varepsilon(\eta + c_\eta(\delta + \varepsilon)) \int_{S^2} |\nabla^3 \hat{\psi}|^2 dA_{S^2} + c_\eta \varepsilon ||\hat{\psi}||^2_{W^{2,2}(S^2)}.
\end{aligned}$$

For the third term we use (4.3.3), $W^{1,1} \hookrightarrow L^2(S^2)$ and Corollary 4.3.3

$$\begin{aligned}
\varepsilon \int_{S^2} \left\langle \Psi_2^T, (\Delta \hat{\psi})^T \right\rangle dA_{S^2} &= \varepsilon \int_{S^2} \Big\langle \Big[\nabla_i \chi_\lambda (\operatorname{Id} + \psi) \Big(4\langle \nabla_i \nabla \psi, \nabla \psi \rangle + 4\langle \nabla_i \nabla \psi, \nabla \operatorname{Id} \rangle + 2\langle \Delta \psi, \nabla_i \psi \rangle \\
&\qquad + 2\langle \Delta \psi, \nabla_i \operatorname{Id} \rangle + 4\langle \nabla \psi, \nabla_i \nabla \operatorname{Id} \rangle - 4\langle \nabla_i \psi, \operatorname{Id} \rangle \Big) \\
&\qquad + 2\nabla_i \chi_\lambda \left(\nabla_i \Delta \psi - 2\nabla_i \operatorname{Id} \right) \Big]^T, (\Delta \hat{\psi})^T \Big\rangle dA_{S^2} \\
&\leq c\varepsilon \int_{S^2} |\nabla \chi_\lambda| |\Delta \hat{\psi}| \left(|\nabla^3 \psi| + |\nabla^2 \psi| |\nabla \psi| + |\nabla^2 \psi| + |\nabla \psi| + 1 \right) dA_{S^2} \\
&\leq c_\eta \varepsilon^2 (\lambda^2 - 1)^2 \int_{S^2} \left(|\nabla^3 \psi|^2 + |\nabla^2 \psi|^2 |\nabla \psi|^2 + |\nabla^2 \psi|^2 + |\nabla \psi|^2 \right) dA_{S^2} \\
&\quad + \eta \int_{S^2} |\nabla^2 \hat{\psi}|^2 dA_{S^2} + c\varepsilon(\lambda^2 - 1) \int_{S^2} |\nabla^2 \hat{\psi}| dA_{S^2} \\
&\leq c_\eta (\delta + \varepsilon) \varepsilon (\lambda^2 - 1)^2 + 2\eta \int_{S^2} |\nabla^2 \hat{\psi}|^2 dA_{S^2}
\end{aligned}$$

and with (4.3.5)

$$\begin{aligned}
&\varepsilon \int_{S^2} \left\langle \Psi_3^T, (\Delta \hat{\psi})^T \right\rangle dA_{S^2} \\
&= \varepsilon \int_{S^2} \left\langle \Delta \chi_\lambda \left((\operatorname{Id} + \psi)(|\nabla \psi|^2 + 2\langle \nabla \psi, \nabla \operatorname{Id} \rangle) + \Delta \psi + 2\psi \right)^T, (\Delta \hat{\psi})^T \right\rangle dA_{S^2} \\
&\leq c_\eta \varepsilon^2 (\lambda^2 - 1)^2 \int_{S^2} \left(|\nabla^2 \psi|^2 + |\nabla \psi|^4 + |\nabla \psi|^2 + |\psi|^2 \right) dA_{S^2} \\
&\quad + \eta \int_{S^2} |\nabla^2 \hat{\psi}|^2 dA_{S^2} \\
&\leq c_\eta (\delta + \varepsilon)^2 \varepsilon^2 (\lambda^2 - 1)^2 + \eta \int_{S^2} |\nabla^2 \hat{\psi}|^2 dA_{S^2}.
\end{aligned}$$

Thus we have in (4.4.12)

$$\int_{S^2} \left\langle J_\varepsilon \hat{\psi}, (\Delta \hat{\psi})^T \right\rangle dA_{S^2} \leq c(\eta + \varepsilon + \delta) \left(\varepsilon ||\nabla^3 \hat{\psi}||^2_{L^2(S^2)} + ||\hat{\psi}||^2_{W^{2,2}(S^2)} \right) + c(\delta + \varepsilon) \varepsilon (\lambda^2 - 1)^2.$$

We apply this to (4.4.11) and choose $\eta, \varepsilon, \delta > 0$ small enough so that we can absorb the higher order terms to the left-hand side.

$$\sqrt{\varepsilon} ||\nabla^3 \hat{\psi}||_{L^2(S^2)} + ||\hat{\psi}||_{W^{2,2}(S^2)} \leq \left(\int_{S^2} \langle J_\varepsilon \hat{\psi}, (\Delta \hat{\psi})^T \rangle dA_{S^2} \right)^{\frac{1}{2}} \leq c(\delta + \varepsilon)^{\frac{1}{2}} \sqrt{\varepsilon} (\lambda^2 - 1). \tag{4.4.13}$$

To get the same bound for ψ we estimate with (4.4.2), $W^{1,1} \hookrightarrow L^2(S^2)$ and Corollary 4.3.3

$$\begin{aligned}\varepsilon \int_{S^2} |\nabla^3\psi|^2 dA_{S^2} &\leq \varepsilon \int_{S^2} |\nabla^3\hat{\psi}|^2 dA_{S^2} + c\varepsilon \int_{S^2} \left(|\psi|^2|\nabla^3\psi|^2 + |\nabla^2\psi|^4 + |\nabla\psi|^4 + |\psi|^4\right) dA_{S^2} \\ &\leq \varepsilon \int_{S^2} |\nabla^3\hat{\psi}|^2 dA_{S^2} + c\varepsilon(\delta+\varepsilon)^2 \int_{S^2} \left(|\nabla^3\psi|^2 + |\nabla^2\psi|^2 + |\nabla\psi|^2 + |\psi|^2\right) dA_{S^2}\end{aligned}$$

Analogously we get

$$\begin{aligned}\int_{S^2} |\nabla^2\psi|^2 dA_{S^2} &\leq \int_{S^2} |\nabla^2\hat{\psi}|^2 dA_{S^2} + c \int_{S^2} \left(|\psi|^2|\nabla^2\psi|^2 + |\nabla\psi|^4 + |\psi|^4\right) dA_{S^2} \\ &\leq \int_{S^2} |\nabla^2\hat{\psi}|^2 dA_{S^2} + c(\delta+\varepsilon)^2 \int_{S^2} \left(|\nabla^2\psi|^2 + |\nabla\psi|^2 + |\psi|^2\right) dA_{S^2}\end{aligned}$$

and

$$\begin{aligned}\int_{S^2} |\nabla\psi| dA_{S^2} &\leq \int_{S^2} |\nabla\hat{\psi}|^2 dA_{S^2} + c(\delta+\varepsilon)^2 \int_{S^2} \left(|\nabla\psi|^2 + |\psi|^2\right) dA_{S^2}, \\ \int_{S^2} |\psi|^2 dA_{S^2} &\leq \int_{S^2} |\hat{\psi}|^2 dA_{S^2} + c(\delta+\varepsilon)^2 \int_{S^2} |\psi|^2 dA_{S^2}.\end{aligned}$$

Putting everything together and absorbing terms yields

$$\sqrt{\varepsilon}||\nabla^3\psi||_{L^2(S^2)} + ||\psi||_{W^{2,2}(S^2)} \leq \left(\int_{S^2} \langle J_\varepsilon\hat{\psi}, (\Delta\hat{\psi})^T\rangle dA_{S^2}\right)^{\frac{1}{2}} \leq c(\delta+\varepsilon)^{\frac{1}{2}}\sqrt{\varepsilon}(\lambda^2-1). \tag{4.4.14}$$

With this we show the following

Theorem 4.4.1. *There exist $\delta > 0$ and $\bar{\varepsilon} > 0$ small such that the only critical points u_ε of E_ε of degree ± 1 with $E_\varepsilon(u_\varepsilon) \leq 4\pi(1+2\varepsilon) + \delta$ and $\varepsilon \leq \bar{\varepsilon}$ are maps of the form $u^R(x) = Rx,\ R \in O(3)$.*

Proof. Since a map of degree -1 only differs from a map of degree 1 by a reflection, we can assume without loss of generality that u_ε is a critical point of degree one. Let M be the Möbius transformation that minimizes $||(u_\varepsilon)_M - \operatorname{Id}||_{L^2(S^2)}$ and let $v = (u_\varepsilon)_M$. We use (4.4.14) to estimate (4.2.6) further

$$\begin{aligned}C\varepsilon(\lambda^2-1) &\leq \frac{d}{d\log\lambda}E_{\varepsilon,\lambda}(\mathrm{Id}) - \frac{d}{d\log\lambda}E_{\varepsilon,\lambda}(v) \\ &\leq 2\sqrt{\varepsilon}||\sqrt{\chi_\lambda}\Delta(v-\mathrm{Id})||_{L^2(S^2)}\sqrt{\varepsilon}\left(||\sqrt{\chi_\lambda}\Delta v||_{L^2(S^2)} + ||\sqrt{\chi_\lambda}\Delta\,\mathrm{Id}\,||_{L^2(S^2)}\right) \\ &\leq c\varepsilon^{\frac{3}{2}}(\delta+\varepsilon)^{\frac{1}{2}}(\lambda^2-1).\end{aligned}$$

Choosing $\varepsilon > 0$ small enough yields $\lambda = 1$. By (4.4.14) ψ must vanish and therefore $v = \mathrm{Id}$ and the Möbius transformation M must be a rotation. Hence u is a rotation.

□

Now we prove the Main Theorem 1.0.7

Proof of Theorem 1.0.7. If the statement is not true, there exist sequences $\varepsilon_k \searrow 0$ and critical points u_{ε_k} with $E_{\varepsilon_k}(u_{\varepsilon_k}) \leq 12\pi - \mu$ and $\deg(u_{\varepsilon_k}) = 1$ but u_{ε_k} is not of the form $u(x) = Rx,\ R \in SO(3)$. With Theorem 1.1 in [44] and Theorem 2 in [16] we get

$$12\pi > \lim_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k}) = \sum_{k=1}^m E(\omega^i) = 4\pi\sum_{i=1}^m |\deg(\omega^i)| \qquad \text{and} \qquad \deg(u_{\varepsilon_k}) = \sum_{i=1}^m \deg(\omega^i) = 1,$$

with ω^i non-trivial harmonic maps. Therefore $m = 1$ and $\deg(\omega^1) = 1$. Since ω^1 is harmonic, $E(\omega^1) = 4\pi$. Thus for every $\delta > 0$ there exists a k large enough so that

$$E_{\varepsilon_k}(u_{\varepsilon_k}) \leq 4\pi + \delta \leq 4\pi(1 + 2\varepsilon) + \delta$$

and Theorem 4.4.1 implies that u_{ε_k} is a rotation which, is a contradiction to our assumption. The proof follows analogously for maps of degree -1.

□

4.5 Gap Theorem for ε-harmonic maps of degree zero

Now we turn our attention to ε-harmonic maps of degree zero. Theorem 1.0.6 follows analogously to [46]. Before we get to the proof, we need a ε-version of the α-harmonic gap theorem of Sacks and Uhlenbeck ([63] Theorem 3.3).

Lemma 4.5.1. *There exists $\delta, \varepsilon > 0$ such that if $u_\varepsilon \in W^{2,2}(S^2, S^2)$ is a critical point of E_ε and $E_\varepsilon(u_\varepsilon) < \delta$, then $E_\varepsilon(u_\varepsilon) = 0$ and u_ε is a constant map.*

Proof. Let u_ε be a critical point of E_ε. We multiply the Euler Lagrange equation (4.0.1) with Δu_ε and integrate by parts.

$$\begin{aligned}\int_{S^2} |\Delta u_\varepsilon|^2 + \varepsilon|\nabla\Delta u_\varepsilon|^2 dA_{S^2} &= \int_{S^2} \langle \Delta u_\varepsilon - \varepsilon\Delta^2 u_\varepsilon, \Delta u_\varepsilon\rangle dA_{S^2} \\ &= \int_{S^2} \left\langle -u_\varepsilon|\nabla u_\varepsilon|^2 + \varepsilon u_\varepsilon \Big(\Delta|\nabla u_\varepsilon|^2 + \operatorname{div}\langle\Delta u_\varepsilon, \nabla u_\varepsilon\rangle + \langle\nabla\Delta u_\varepsilon, \nabla u_\varepsilon\rangle, \Delta u_\varepsilon \right\rangle dA_{S^2} \\ &\leq \eta\int_{S^2} |\Delta u_\varepsilon|^2 dA_{S^2} + \varepsilon\eta\int_{S^2} |\nabla\Delta u_\varepsilon|^2 dA_{S^2} + c_\eta\int_{S^2} |\nabla u_\varepsilon|^4 dA_{S^2} \\ &\quad + c_\eta\varepsilon^2\int_{S^2} |\nabla^2 u_\varepsilon|^4 dA_{S^2}. \end{aligned} \tag{4.5.1}$$

Now we integrate the Bochner formula (A.5.1) to estimate the full second derivative. The left-hand side vanishes, the term involving the Ricci curvature is positive on the sphere and the second derivative of the metric is bounded. After integrating the first term on the right-hand side by parts we get

$$\int_{S^2} |\nabla^2 u_\varepsilon|^2 + |\nabla u_\varepsilon|^2 dA_{S^2} \leq c\int_{S^2} |\Delta u_\varepsilon|^2 + |\nabla u_\varepsilon|^4 dA_{S^2}.$$

By the Sobolev embedding $W^{1,1} \hookrightarrow L^2(S^2)$ and $E_\varepsilon(u_\varepsilon) \leq \delta$ we have

$$\begin{aligned}\int_{S^2} |\nabla u_\varepsilon|^4 dA_{S^2} &\leq c\left(\int_{S^2} |\nabla u_\varepsilon|^2 dA_{S^2}\right)\left(\int_{S^2} |\nabla^2 u_\varepsilon|^2 dA_{S^2}\right) + c\left(\int_{S^2} |\nabla u_\varepsilon|^2 dA_{S^2}\right)^2 \\ &\leq c\delta\int_{S^2} |\nabla^2 u_\varepsilon|^2 dA_{S^2} + c\delta\int_{S^2} |\nabla u_\varepsilon|^2 dA_{S^2}\end{aligned}$$

and with $\delta > 0$ small enough

$$\int_{S^2} |\nabla^2 u_\varepsilon|^2 + |\nabla u_\varepsilon|^2 dA_{S^2} \leq c\int_{S^2} |\Delta u_\varepsilon|^2 dA_{S^2}.$$

Applying all of this to (4.5.1) and choosing η small enough yields

$$\int_{S^2} \varepsilon|\nabla\Delta u_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + |\nabla u_\varepsilon|^2 dA_{S^2} \leq c\varepsilon^2\int_{S^2} |\nabla^2 u_\varepsilon|^4 dA_{S^2}.$$

Using the Sobolev embedding $W^{1,1} \hookrightarrow L^2(S^2)$ and $E_\varepsilon(u_\varepsilon) \le \delta$ again we get

$$\begin{aligned}
\varepsilon^2 \int_{S^2} |\nabla^2 u_\varepsilon|^4 dA_{S^2} &\le c\varepsilon^2 \left(\int_{S^2} |\nabla^2 u_\varepsilon|^2 dA_{S^2} \right) \left(\int_{S^2} |\nabla^3 u_\varepsilon|^2 dA_{S^2} \right) + c\varepsilon^2 \left(\int_{S^2} |\nabla^2 u_\varepsilon|^2 dA_{S^2} \right)^2 \\
&\le c\delta\varepsilon \int_{S^2} |\nabla^3 u_\varepsilon|^2 dA_{S^2} + c\delta\varepsilon \int_{S^2} |\nabla^2 u_\varepsilon|^2 dA_{S^2}, \\
\varepsilon \int_{S^2} |\nabla u_\varepsilon|^6 dA_{S^2} &\le c\varepsilon \left(\int_{S^2} |\nabla^2 u_\varepsilon|^2 dA_{S^2} \right) \left(\int_{S^2} |\nabla u_\varepsilon|^4 dA_{S^2} \right) \\
&\quad + c\varepsilon \left(\int_{S^2} |\nabla u_\varepsilon|^2 dA_{S^2} \right) \left(\int_{S^2} |\nabla u_\varepsilon|^4 dA_{S^2} \right) \\
&\le c\delta^2 \int_{S^2} |\nabla^2 u_\varepsilon|^2 dA_{S^2} + c\delta^2 \int_{S^2} |\nabla u_\varepsilon|^2 dA_{S^2}.
\end{aligned}$$

Now we estimate the full third derivative of u_ε. After integrating by parts, exchanging derivatives (see Lemma 2.1.2 in [42]) and using the above estimates we have

$$\begin{aligned}
\varepsilon \int_{S^2} |\nabla^3 u_\varepsilon|^2 dA_{S^2} &\le c\varepsilon \int_{S^2} |\nabla \Delta u_\varepsilon|^2 dA_{S^2} \\
&\quad + c\varepsilon \int_{S^2} \left(|\nabla^2 u_\varepsilon|^2 |\nabla u_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + |\nabla^2 u_\varepsilon| |\nabla u_\varepsilon|^4 + |\nabla^2 u_\varepsilon| |\nabla u_\varepsilon| \right) dA_{S^2} \\
&\le c\varepsilon \int_{S^2} |\nabla \Delta u_\varepsilon|^2 dA_{S^2} + c_\eta \varepsilon^2 \int_{S^2} |\nabla^2 u_\varepsilon|^4 dA_{S^2} + c \int_{S^2} |\nabla u_\varepsilon|^4 dA_{S^2} \\
&\quad + (\eta + c\varepsilon) \int_{S^2} |\nabla^2 u_\varepsilon|^2 dA_{S^2} + c\varepsilon \int_{S^2} |\nabla u_\varepsilon|^6 dA_{S^2} + c\varepsilon \int_{S^2} |\nabla u_\varepsilon|^2 dA_{S^2} \\
&\le c\varepsilon \int_{S^2} |\nabla \Delta u_\varepsilon|^2 dA_{S^2} + c_\eta \delta\varepsilon \int_{S^2} |\nabla^3 u_\varepsilon|^2 dA_{S^2} \\
&\quad + (c_\eta \delta\varepsilon + \eta + c\varepsilon + c\delta) \int_{S^2} |\nabla^2 u_\varepsilon|^2 dA_{S^2} + c(\varepsilon + \delta) \int_{S^2} |\nabla u_\varepsilon|^2 dA_{S^2}.
\end{aligned}$$

For $\delta, \varepsilon, \eta > 0$ small enough we get with the above

$$\begin{aligned}
\int_{S^2} \left(|\nabla u_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + \varepsilon |\nabla^3 u_\varepsilon|^2 \right) dA_{S^2} &\le c\varepsilon^2 \int_{S^2} |\nabla^2 u_\varepsilon|^4 dA_{S^2} \\
&\le \delta\varepsilon \int_{S^2} \left(|\nabla^3 u_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 \right) dA_{S^2}.
\end{aligned}$$

Hence $E_\varepsilon(u_\varepsilon) = 0$ and u_ε is constant.

□

Proof of Theorem 1.0.6. We assume there exists a sequence $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ of non-constant critical points of E_{ε_k} with $E_{\varepsilon_k}(u_{\varepsilon_k}) \le 8\pi - \delta$. The energy identity (4.0.2) yields

$$8\pi > \lim_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k}) = \sum_{i=1}^N E(u^i) = \sum_{i=1}^N 4\pi |\deg(u^i)|,$$

where $u^i : S^2 \to S^2$, $i = 1, ..., N$ are harmonic maps, which are non-trivial for $i \ge 2$. With the results of Duzaar and Kuwert [16] we have

$$0 = \deg(u_{\varepsilon_k}) = \sum_{i=1}^N \deg(u^i).$$

Thus $N = 1$ and u^1 is a constant harmonic map. Then $\lim_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k}) = 0$ and with Lemma 4.5.1 it follows that $u_{\varepsilon_k} = const.$

□

Next we construct explicit examples of ε-harmonic maps of degree zero with $E_\varepsilon(u_\varepsilon) \geq 8\pi$ which are not constant. This shows that the bound in Theorem 1.0.6 is optimal. We follow [46] and start by defining a class of rotationally symmetric maps. Let $n \in \mathbb{N}$ and

$$\begin{aligned} [n\pi, (n+1)\pi] \times [0, 2\pi] &\to \mathbb{R}^3 \\ (r, \theta) &\mapsto (\sin r\, \cos\theta,\ \sin r\, \sin\theta,\ \cos r) \end{aligned}$$

be a parametrization of S^2. For n even, this parametrization is orientation preserving and for n odd orientation reversing. Further let $f \in C([0,\pi], \mathbb{R})$ with

$$f(0) = 0 \qquad \text{and} \qquad f(\pi) = n\pi.$$

Then we define $u_f \colon S^2 \to S^2$ by

$$\begin{aligned} u_f \colon [0, \pi] \times [0, 2\pi] &\to \mathbb{R}^3 \\ (r, \theta) &\mapsto (\sin(f(r))\cos(\theta), \sin(f(r))\sin(\theta), \cos(f(r)))\,. \end{aligned}$$

u_f is rotationally symmetric and wraps n times around S^2, reversing orientation after each round. Hence u_f has degree zero if n is even and degree one if n is odd.

Let $n = 2$,

$$X = \{f \colon [0, \pi] \to \mathbb{R} : u_f \in W^{2,2}(S^2, \mathbb{R}^3),\ f(0) = 0,\ f(\pi) = 2\pi\}$$

and $M^* = \inf_{f\in X} I(f)$, where

$$I(f) := E_\varepsilon(u_f).$$

$E_\varepsilon(u_f)$ is invariant under rotations about the z-axis and reflections in planes containing the line $(0, 0, z)$. Thus, by the principle of symmetric criticality of Palais (see [54] or Remark 11.4(a) in [3]), f is a critical point of I if and only if u_f is a critical point of E_ε. We show that there exists $f^* \in X$ with $I(f^*) = M^*$. Let (f_j) be a sequence in X with corresponding sequence $(u_{f_j}) \in W^{2,2}(S^2, \mathbb{R}^3)$ and $I(f_j) \searrow M^*$. (u_{f_j}) is bounded in $W^{2,2}(S^2, \mathbb{R}^3)$ and thus contains a subsequence (again denoted u_{f_j}) with $u_{f_j} \rightharpoonup u_{f^*}$ weakly in $W^{2,2}(S^2, \mathbb{R}^3)$ and uniformly in $C^0(S^2, \mathbb{R}^3)$, with $f^* \in X$. By the lower semi-continuity of E_ε with respect to weak convergence in $W^{2,2}(S^2, \mathbb{R}^3)$ we have $I(f^*) = E_\varepsilon(u_{f^*}) = M^*$.

Now we want to express $E_\varepsilon(u_f)$ in terms of f and compute

$$\begin{aligned} \frac{\partial u_f}{\partial r} &= f'(r)\Big(\cos(f(r))\cos(\theta), \cos(f(r))\sin(\theta), -\sin(f(r))\Big) \\ \frac{\partial u_f}{\partial \theta} &= \Big(-\sin(f(r))\sin(\theta), \sin(f(r))\cos(\theta), 0\Big) \\ \frac{\partial^2 u_f}{\partial r^2} &= f''(r)\Big(\cos(f(r))\cos(\theta), \cos f(r)\sin(\theta), -\sin f(r)\Big) \\ &\quad - (f'(r))^2\Big(\sin(f(r))\cos(\theta), \sin(f(r))\sin(\theta), \cos(f(r))\Big) \\ &= \frac{f''(r)}{f'(r)}\frac{\partial u_f}{\partial r} - \Big(f'(r)\Big)^2 u_f \end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u_f}{\partial \theta^2} &= -\Big(\sin(f(r))\cos(\theta), \sin(f(r))\sin(\theta), 0\Big) \\ &= -\sin(f(r))\left(\sin(f(r))u_f + \frac{\cos(f(r))}{f'(r)}\frac{\partial u_f}{\partial r}\right).\end{aligned}$$

Thus we have

$$\frac{1}{2}|\nabla u_f|^2 = \frac{1}{2}\left((f')^2 + \frac{(\sin(f(r)))^2}{(\sin(r))^2}\right).$$

The Laplacian on S^2 is given by $\Delta = \frac{\partial^2}{\partial r^2} + \frac{\cos r}{\sin r}\frac{\partial}{\partial r} + \frac{1}{(\sin r)^2}\frac{\partial^2}{\partial \theta^2}$. Therefore

$$\Delta u_f = \frac{\partial u_f}{\partial r}\left(\frac{f''(r)}{f'(r)} + \frac{\cos(r)}{\sin(r)} - \frac{\cos(f(r))\sin(f(r))}{\sin^2(r)f'(r)}\right) - u_f\left((f'(r))^2 + \frac{\sin^2(f(r))}{\sin^2(r)}\right)$$

and

$$\begin{aligned}|\Delta u_f|^2 &= |(\Delta u_f)^T|^2 + |\nabla u_f|^4 \\ &= \left|\frac{\partial u_f}{\partial r}\right|^2 \left[\frac{(f''(r))^2}{(f'(r))^2} + \frac{\cos^2(r)}{\sin^2(r)} + \frac{\cos^2(f(r))\sin^2(f(r))}{\sin^4(r)(f'(r))^2} + 2\frac{f''(r)\cos(r)}{f'(r)\sin(r)}\right. \\ &\qquad \left. - 2\frac{f''(r)\cos(f(r))\sin(f(r))}{\sin^2(r)(f'(r))^2} - 2\frac{\cos(r)\cos(f(r))\sin(f(r))}{\sin^3(r)f'(r)}\right] \\ &\quad + \left((f'(r))^2 + \frac{\sin^2(f(r))}{\sin^2(r)}\right)^2 \\ &= (f''(r))^2 + \frac{\cos^2(r)(f'(r))^2}{\sin^2(r)} + \frac{\cos^2(f(r))\sin^2(f(r))}{\sin^4(r)} + \frac{2f'(r)f''(r)\cos(r)}{\sin(r)} \\ &\quad - \frac{2f''(r)\cos(f(r))\sin(f(r))}{\sin^2(r)} - \frac{2\cos(r)\cos(f(r))\sin(f(r))f'(r)}{\sin^3(r)} \\ &\quad + (f'(r))^4 + \frac{\sin^4(f(r))}{\sin^4(r)} + 2(f'(r))^2\frac{\sin^2(f(r))}{\sin^2(r)}.\end{aligned}$$

We use this to get a lower bound on $E_\varepsilon(u_{f^*})$

$$\begin{aligned}E_\varepsilon(u_{f^*}) &= \pi\int_0^\pi \left((f^{*\prime})^2 + \frac{(\sin f^*)^2}{(\sin r)^2}\right)\sin r\,dr + \frac{\varepsilon}{2}\int_{S^2}|\Delta u_{f^*}|^2 dA_{S^2} \\ &\geq 2\pi\int_0^\pi f^{*\prime}(\sin f^*)dr.\end{aligned}$$

There exist $r_1 \in (0,\pi)$ such that $f^*(r_1) = \pi$ and

$$\begin{aligned}\int_0^\pi |f^{*\prime}(\sin f^*)|\,dr &\geq \int_0^{r_1} f^{*\prime}(\sin f^*)\,dr - \int_{r_1}^\pi f^{*\prime}(\sin f^*)\,dr \\ &= -\cos f^*(r)|_0^{r_1} + \cos f^*(r)|_{r_1}^\pi \\ &= 4.\end{aligned}$$

Hence

$$E_\varepsilon(u_{f^*}) \geq 8\pi$$

and u_{f^*} is a non-constant ε-harmonic map of degree zero. To complete the proof of Theorem 1.0.8 we show the following

Proposition 4.5.2. *There exists a universal constant $c > 0$, such that for any $0 < \varepsilon < \frac{1}{4}$*

$$E_\varepsilon(u_\varepsilon) < 8\pi + c\varepsilon^{\frac{1}{2}}, \tag{4.5.2}$$

where u_ε is the minimizer of E_ε among all u_f.

Proof. Let $\Lambda > 1$ and

$$f(r) = \begin{cases} 2\arctan(\Lambda\tan(r)), & 0 \le r \le \frac{\pi}{2}, \\ 2\arctan(\Lambda\tan(r)) + 2\pi, & \frac{\pi}{2} < r \le \pi. \end{cases}$$

We consider the corresponding map $u_f \in X$. As r increases from 0 to $\frac{\pi}{2}$, $f(r)$ increases from 0 to π, which means that u_f maps the upper hemisphere to the full sphere with the equator being mapped to the south pole $(0,0,-1)$. As r increases from $\frac{\pi}{2}$ to π, $f(r)$ increases from π to 2π, which means that u_f maps the lower hemisphere to the full sphere but with opposite orientation. Thus u_f has degree zero. We want to estimate $E_\varepsilon(u_f)$. To do this we first calculate

$$f'(r) = \frac{2\Lambda}{\cos^2(r) + \Lambda^2\sin^2(r)},$$

$$f''(r) = \frac{-2\Lambda(-2\cos(r)\sin(r) + 2\Lambda^2\sin(r)\cos(r))}{(\cos^2(r) + \Lambda^2\sin^2(r))^2} = \frac{4\sin(r)\cos(r)\Lambda(1-\Lambda^2)}{(\cos^2(r) + \Lambda^2\sin^2(r))^2}$$

and

$$\cos(f(r)) = \frac{1-\Lambda^2\tan^2(r)}{\Lambda^2\tan^2(r)+1} = \frac{\cos^2(r) - \Lambda^2\sin^2(r)}{\cos^2(r) + \Lambda^2\sin^2(r)} = \frac{(\Lambda^2+1)\cos^2(r) - \Lambda^2}{\cos^2(r) + \Lambda^2\sin^2(r)},$$

$$\sin(f(r)) = \frac{2\Lambda\tan(r)}{1+\Lambda^2\tan^2(r)} = \sin(r)\cos(r)f'(r).$$

Using this and the above we have

$$|\nabla u_f|^2 = 4\Lambda^2\frac{1+\cos^2 r}{(\cos^2(r) + \Lambda^2\sin^2(r))^2}.$$

and

$$\begin{aligned}
|\Delta u_f|^2 = {} & \frac{16\Lambda^2(1-\Lambda^2)^2\cos^2(r)\sin^2(r)}{(\cos^2(r) + \Lambda^2\sin^2(r))^4} + \frac{4\Lambda^2\cos^2(r)}{\sin^2(r)(\cos^2(r) + \Lambda^2\sin^2(r))^2} \\
& + \frac{4\Lambda^2\cos^2(r)((\Lambda^2+1)\cos^2(r) - \Lambda^2)^2}{\sin^2(r)(\cos^2(r) + \Lambda^2\sin^2(r))^4} + \frac{16\Lambda^2(1-\Lambda^2)\cos^2(r)}{(\cos^2(r) + \Lambda^2\sin^2(r))^3} \\
& - \frac{16\Lambda^2(1-\Lambda^2)\cos^2(r)((\Lambda^2+1)\cos^2(r) - \Lambda^2)}{(\cos^2(r) + \Lambda^2\sin^2(r))^4} \\
& - \frac{8\Lambda^2\cos^2(r)((\Lambda^2+1)\cos^2(r) - \Lambda^2)}{\sin^2(r)(\cos^2(r) + \Lambda^2\sin^2(r))^3} + \frac{16\Lambda^4}{(\cos^2(r) + \Lambda^2\sin^2(r))^4} \\
& + \frac{16\Lambda^4\cos^4(r)}{(\cos^2(r) + \Lambda^2\sin^2(r))^4} + \frac{32\Lambda^4\cos^2(r)}{(\cos^2(r) + \Lambda^2\sin^2(r))^4}.
\end{aligned}$$

Thus we have for the second part of $E_\varepsilon(u_f)$

$$
\begin{aligned}
\frac{1}{2}\int_{S^2}|\Delta u_f|^2 dA_{S^2} &= \frac{1}{2}\int_{S^2}\left(|(\Delta u_f)^T|^2+|\nabla u_f|^4\right)dA_{S^2}\\
&=\pi\int_0^\pi\Bigg[\frac{16\Lambda^2(1-\Lambda^2)^2\cos^2(r)\sin^2(r)}{(\cos^2(r)+\Lambda^2\sin^2(r))^4}+\frac{4\Lambda^2\cos^2(r)}{\sin^2(r)(\cos^2(r)+\Lambda^2\sin^2(r))^2}\\
&\quad+\frac{4\Lambda^2\cos^2(r)((\Lambda^2+1)\cos^2(r)-\Lambda^2)^2}{\sin^2(r)(\cos^2(r)+\Lambda^2\sin^2(r))^4}+\frac{16\Lambda^2(1-\Lambda^2)\cos^2(r)}{(\cos^2(r)+\Lambda^2\sin^2(r))^3}\\
&\quad-\frac{16\Lambda^2(1-\Lambda^2)\cos^2(r)((\Lambda^2+1)\cos^2(r)-\Lambda^2)}{(\cos^2(r)+\Lambda^2\sin^2(r))^4}\\
&\quad-\frac{8\Lambda^2\cos^2(r)((\Lambda^2+1)\cos^2(r)-\Lambda^2)}{\sin^2(r)(\cos^2(r)+\Lambda^2\sin^2(r))^3}+\frac{16\Lambda^4(1+\cos^2(r))^2}{(\cos^2(r)+\Lambda^2\sin^2(r))^4}\Bigg]\sin(r)dr\\
&=2\pi\int_0^{\frac{\pi}{2}}\Bigg[\frac{16\Lambda^2(1-\Lambda^2)^2\cos^2(r)\sin^2(r)}{(\cos^2(r)+\Lambda^2\sin^2(r))^4}+\frac{4\Lambda^2\cos^2(r)}{\sin^2(r)(\cos^2(r)+\Lambda^2\sin^2(r))^2}\\
&\quad+\frac{4\Lambda^2\cos^2(r)((\Lambda^2+1)\cos^2(r)-\Lambda^2)^2}{\sin^2(r)(\cos^2(r)+\Lambda^2\sin^2(r))^4}+\frac{16\Lambda^2(1-\Lambda^2)\cos^2(r)}{(\cos^2(r)+\Lambda^2\sin^2(r))^3}\\
&\quad-\frac{16\Lambda^2(1-\Lambda^2)\cos^2(r)((\Lambda^2+1)\cos^2(r)-\Lambda^2)}{(\cos^2(r)+\Lambda^2\sin^2(r))^4}\\
&\quad-\frac{8\Lambda^2\cos^2(r)((\Lambda^2+1)\cos^2(r)-\Lambda^2)}{\sin^2(r)(\cos^2(r)+\Lambda^2\sin^2(r))^3}+\frac{16\Lambda^4(1+\cos^2(r))^2}{(\cos^2(r)+\Lambda^2\sin^2(r))^4}\Bigg]\sin(r)dr,
\end{aligned}
$$

where we used the symmetry of $\cos^2$ and sin about $\frac{\pi}{2}$. Substituting $t=\cos(r)$ and setting $a^2 := 1-\Lambda^{-2}$ gives

$$
\begin{aligned}
\frac{1}{2}\int_{S^2}|\Delta u_f|^2 dA_{S^2} &= 32\pi\int_0^1\Lambda^{-2}\frac{a^4t^2(1-t^2)}{(1-a^2t^2)^4}dt\\
&\quad+8\pi\int_0^1\Lambda^{-2}\frac{t^2}{(1-t^2)(1-a^2t^2)^2}dt+8\pi\int_0^1\Lambda^{-2}\frac{t^2((1+\Lambda^{-2})t^2-1)^2}{(1-t^2)(1-a^2t^2)^4}dt\\
&\quad-32\pi\int_0^1\Lambda^{-2}\frac{a^2t^2}{(1-a^2t^2)^3}dt+32\pi\int_0^1\Lambda^{-2}\frac{a^2t^2((1+\Lambda^{-2})t^2-1)}{(1-a^2t^2)^4}dt\\
&\quad-16\pi\int_0^1\Lambda^{-2}\frac{t^2((1+\Lambda^{-2})t^2-1)}{(1-t^2)(1-a^2t^2)^3}dt+32\pi\int_0^1\Lambda^{-4}\frac{(1+t^2)^2}{(1-a^2t^2)^4}dt\\
&=32\pi\int_0^1\Lambda^{-2}\frac{a^4t^2(1-t^2)}{(1-a^2t^2)^4}dt+32\pi\int_0^1\Lambda^{-2}\frac{t^2(1-t^2)}{(1-a^2t^2)^4}dt\\
&\quad-64\pi\int_0^1\Lambda^{-2}\frac{a^2t^2(1-t^2)}{(1-a^2t^2)^4}dt+32\pi\int_0^1\Lambda^{-4}\frac{(1+t^2)^2}{(1-a^2t^2)^4}dt
\end{aligned}
$$

$$= 32\pi \int_0^1 \Lambda^{-2} \frac{t^2(1-t^2)(a^2-1)^2}{(1-a^2t^2)^4} dt + 32\pi \int_0^1 \Lambda^{-4} \frac{(1+t^2)^2}{(1-a^2t^2)^4} dt$$

$$= 32\pi \int_0^1 \Lambda^{-6} \frac{t^2(1-t^2)}{(1-a^2t^2)^4} dt + 32\pi \int_0^1 \Lambda^{-4} \frac{(1+t^2)^2}{(1-a^2t^2)^4} dt.$$

We estimate each term separately. First we note that $t \in [0,1]$ and $0 \leq a \leq 1$

$$\frac{t^2(1-t^2)}{(1-a^2t^2)^4} \leq \frac{(a^2-a^2t^2)}{a^2(1-a^2t^2)^4} \leq \frac{1}{a^2(1-a^2t^2)^3} = \frac{1}{a^2(1+at)^3(1-at)^3} \leq \frac{1}{a^2(1-at)^3}.$$

Then

$$32\pi \int_0^1 \Lambda^{-6} \frac{t^2(1-t^2)}{(1-a^2t^2)^4} dt \leq 32\pi\Lambda^{-6} \int_0^1 \frac{1}{a^2(1-at)^3} dt = 32\pi\Lambda^{-6} \left(\frac{1}{2a^3(1-a)^2} - \frac{1}{2a^3} \right)$$
$$\leq 32\pi\Lambda^{-6} \frac{(1+a)^2}{a^2(1-a^2)^2} \leq 128\pi\Lambda^{-2} \frac{1}{a^2} = 128\pi \frac{\Lambda^{-2}}{1-\Lambda^{-2}} = 128\pi \frac{1}{\Lambda^2-1},$$

where we used that $\Lambda^{-2} = 1 - a^2$. Analogously we get for the second term

$$\frac{(1+t^2)^2}{(1-a^2t^2)^4} \leq \frac{(1+t)^4}{(1+at)^4(1-at)^4} = \frac{(a+at)^4}{a^4(1+at)^4(1-at)^4} \leq \frac{1}{a^4(1-at)^4}$$

and therefore

$$32\pi \int_0^1 \Lambda^{-4} \frac{(1+t^2)^2}{(1-a^2t^2)^4} dt \leq 32\pi\Lambda^{-4} \int_0^1 \frac{1}{a^4(1-at)^4} dt = 32\pi\Lambda^{-4} \left(\frac{1}{3a^5(1-a)^3} - \frac{1}{3a^5} \right)$$
$$\leq 32\pi\Lambda^{-4} \frac{(1+a)^3}{a^4(1-a^2)^3} \leq 256\pi\Lambda^2 \frac{1}{a^4} = 256\pi \frac{\Lambda^2}{(1-\Lambda^{-2})^2} = 256\pi \frac{\Lambda^6}{(\Lambda^2-1)^2}.$$

All in all we get

$$\frac{1}{2} \int_{S^2} |\Delta u_f|^2 dA_{S^2} \leq 128\pi \frac{1}{\Lambda^2-1} + 256\pi \frac{\Lambda^6}{(\Lambda^2-1)^2}.$$

Analogously we estimate the first part of $E_\varepsilon(u_f)$ (see [46] Proposition 3.2)

$$\frac{1}{2} \int_{S^2} |\nabla u_f|^2 dA_{S^2} = 8\pi \int_0^{\frac{\pi}{2}} \Lambda^2 \frac{1+\cos^2 r}{(\cos^2(r) + \Lambda^2 \sin^2(r))^2} \sin r dr$$
$$= 8\pi \int_0^1 \Lambda^{-2} \frac{(1+t^2)}{(1-a^2t^2)^2} dt$$
$$\leq 8\pi \int_0^1 \frac{\Lambda^{-2}}{2a^2} \left(\frac{1}{(1-at)^2} + \frac{1}{(1+at)^2} \right) dt$$
$$= 8\pi\Lambda^{-2} \frac{1}{a^2(1-a^2)} = 8\pi \frac{1}{a^2} = 8\pi \frac{\Lambda^2}{\Lambda^2-1}.$$

Together with the above we get

$$\frac{1}{2} \int_{S^2} \left(|\nabla u_f|^2 + \varepsilon |\Delta u_f|^2 \right) dA_{S^2} \leq 8\pi \frac{\Lambda^2}{\Lambda^2-1} + 128\pi\varepsilon \frac{1}{\Lambda^2-1} + 256\pi\varepsilon \frac{\Lambda^6}{(\Lambda^2-1)^2}.$$

We choose $\Lambda^2 > 2$, then $\frac{\Lambda^2}{\Lambda^2-1} < 1 + 2\Lambda^{-2} < 2$ and thus

$$E_\varepsilon(u_f) < 8\pi\left(1 + \frac{2}{\Lambda^2}\right) + 128\pi\varepsilon + 1024\pi\varepsilon\Lambda^2.$$

We set $\Lambda := \varepsilon^{-\frac{1}{4}}$. Note that for $\varepsilon \in (0, \frac{1}{4})$, $\Lambda^2 > 2$ still holds. Then we get

$$E_\varepsilon(u_f) < 8\pi\left(1 + 2\varepsilon^{\frac{1}{2}}\right) + 1152\pi\varepsilon^{\frac{1}{2}} = 8\pi + 1168\pi\varepsilon^{\frac{1}{2}}.$$

Since u_ε minimizes E_ε among all maps in X, we have

$$E_\varepsilon(u_\varepsilon) < E_\varepsilon(u_f) < 8\pi + c\varepsilon^{\frac{1}{2}},$$

with $c = 1168\pi$.

□

Proof of Theorem 1.0.8. Given $\delta > 0$ we choose $\varepsilon \in (0, \frac{1}{4})$ such that $\varepsilon < (\frac{\delta}{c})^2$, where c is the constant in (4.5.2). With Proposition 4.5.2 and Theorem 1.0.6 we get

$$8\pi \leq E_\varepsilon(u_\varepsilon) < 8\pi + c\varepsilon^{\frac{1}{2}} < 8\pi + \delta.$$

□

Appendix A

A.1 Lorentz spaces

Lorentz spaces are interpolation spaces of the classical L^p-spaces. They have many useful properties including a version of the Hölder inequality and Sobolev embeddings (see Lemma 3.6.3). In the following we give a brief introduction to the theory and list the results we use in chapter 2 and 3. For detailed proofs see for example [27], [31], [37], [45], [76] and [83].

Definition A.1.1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a measurable function, $s > 0$ and let*

$$f_*(s) = \lambda(f(s)) = |\{x \in \mathbb{R}^n \mid |f(x)| > s\}|$$

be the distribution function of f. We define the non-increasing rearrangement f^* of f

$$f^*(t) = \inf\{s > 0 \mid f_*(s) \le t\}.$$

Further let

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds.$$

Let $1 < p < \infty$, $1 \le q \le \infty$. f is an element of the Lorentz space $L^{p,q}(\mathbb{R}^n)$ if

$$\begin{aligned} ||f||_{L^{p,q}(\mathbb{R}^n)} &= \left(\int_0^\infty (t^{1/p} f^{**}(t))^q \frac{1}{t}\frac{d}{dt}\right)^{1/q} && 1 \le q < \infty, \\ ||f||_{L^{p,\infty}(\mathbb{R}^n)} &= ||t^{1/p} f^{**}(t)||_{L^\infty(0,\infty)} && q = \infty, \end{aligned}$$

is finite. $(L^{p,q}(\mathbb{R}^n), ||\cdot||_{L^{p,q}(\mathbb{R}^n)})$ is a Banach space.

Lemma A.1.2. *Let $1 < p < \infty$ and $f : \mathbb{R}^n \to \mathbb{R}$ be measurable. Then we have*

$$c_1||f||_{L^p(\mathbb{R}^n)} \le ||f||_{L^{p,p}(\mathbb{R}^n)} \le c_2||f||_{L^p(\mathbb{R}^n)}$$

(see [83] Lemma 1.8.10 or [45] Lemma 5.1.7)

Lemma A.1.3 (Hölder inequality)**.** *Let $f \in L^{p_1,q_1}(\mathbb{R}^n)$ and $g \in L^{p_2,q_2}(\mathbb{R}^n)$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$ and $p, p_1, p_2 \in (1, \infty)$, $q, q_1, q_2 \in [1, \infty]$. Then*

$$||fg||_{L^{p,q}(\mathbb{R}^n)} \le ||f||_{L^{p_1,q_1}(\mathbb{R}^n)}||g||_{L^{p_2,q_2}(\mathbb{R}^n)}.$$

(see [37] Theorem 4.5)

Lemma A.1.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable.*

1. *Let $1 < p \le \infty$ and $1 \le q < Q \le \infty$. Then we have*

$$||f||_{L^{p,Q}(\mathbb{R}^n)} \le c||f||_{L^{p,q}(\mathbb{R}^n)}.$$

2. Let $1 < p < P \leq \infty$, $1 \leq q_1, q_2 \leq \infty$ and let $\Omega \subset \mathbb{R}^n$ be bounded. Then we have

$$||f||_{L^{p,q_1}(\Omega)} \leq c|\Omega|^{\frac{1}{p}-\frac{1}{P}}||f||_{L^{P,q_2}(\Omega)}.$$

(see [45] Lemma 5.1.9)

Lemma A.1.5. *Let $1 < p_1, q_1, p_2, q_2 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1$.*

(i) $L^{p_2,q_2}(\mathbb{R}^n)$ is the dual space of $L^{p_1,q_1}(\mathbb{R}^n)$.

(ii) The dual space to $L^{p_1,1}(\mathbb{R}^n)$ is $L^{p_2,\infty}(\mathbb{R}^n)$ but $L^{p_1,1}(\mathbb{R}^n)$ is not reflexive.

(see [37] (2.7))

If f is a m-polyharmonic function on $\mathbb{R}^{2m}$, we can estimate the $L^{2,1}$-norm of $D^m f$ locally in terms of its L^2-norm.

Lemma A.1.6. *Let $f \in C^{2m}(\mathbb{R}^{2m}, \mathbb{R}^n)$ be a m-polyharmonic function, i.e. $\Delta^m f = 0$. For any radius $R > 0$ we have*

$$||D^m f||_{L^{2,1}(B_{\frac{R}{4}})} \leq c||D^m f||_{L^2(B_R)},$$

with $c > 0$ independent of R.

Proof. We show this result for the case m even. The case m odd follows analogously with minor modifications. Let $\eta \in C_c^\infty(\mathbb{R}^{2m})$ be a smooth cut-off function such that $\eta = 1$ on $B_{\frac{R}{2}}$, $0 \leq \eta \leq 1$ in B_R and $\eta = 0$ elsewhere. We want to estimate $\int_{\mathbb{R}^{2m}} \eta^{2(k+1)}|D^{m+k}u|^2$ in terms of $\int_{B_R} |D^m f|^2$ for $k = 1, ..., m-1$. We do this iteratively using integration by parts and Young's inequality.

$$\begin{aligned}
\int_{\mathbb{R}^{2m}} \eta^{2m}|D\Delta^{m-1}f|^2 &\leq c\int_{\mathbb{R}^{2m}} \eta^{2m-1}|D\eta| \cdot |\Delta^{m-1}f| \cdot |D\Delta^{m-1}f| \\
&\leq \delta \int_{\mathbb{R}^{2m}} \eta^{2m}|D\Delta^{m-1}f|^2 + \frac{c}{R^2}\int_{\mathbb{R}^{2m}} \eta^{2m-2}|\Delta^{m-1}f|^2
\end{aligned}$$

and for $k = 2, 4, ..., m-2$

$$\begin{aligned}
\int_{\mathbb{R}^{2m}} \eta^{2k+2}|\Delta^{\frac{m+k}{2}}f|^2 \leq &\int_{\mathbb{R}^{2m}} \eta^{2k+1}|D\eta| \cdot |D\Delta^{\frac{m+k-2}{2}}f| \cdot |\Delta^{\frac{m+k}{2}}f| \\
&+ \int_{\mathbb{R}^{2m}} \eta^{2k+2}|D\Delta^{\frac{m+k-2}{2}}f| \cdot |D\Delta^{\frac{m+k}{2}}f| \\
\leq &\frac{c}{R^2}\int_{\mathbb{R}^{2m}} \eta^{2k}|D\Delta^{\frac{m+k-2}{2}}f|^2 + \delta\int_{\mathbb{R}^{2m}} \eta^{2k+2}|\Delta^{\frac{m+k}{2}}f|^2 \\
&+ \delta\int_{\mathbb{R}^{2m}} \eta^{2k+4}|D\Delta^{\frac{m+k}{2}}f|^2.
\end{aligned}$$

For $k = 1, 3, ..., m-3$

$$\begin{aligned}
\int_{\mathbb{R}^{2m}} \eta^{2k+2}|D\Delta^{\frac{m+k-1}{2}}f|^2 \leq &\int_{\mathbb{R}^{2m}} \eta^{2k+1}|D\eta| \cdot |\Delta^{\frac{m+k-1}{2}}f| \cdot |D\Delta^{\frac{m+k-1}{2}}f| \\
&+ \int_{\mathbb{R}^{2m}} \eta^{2k+2}|\Delta^{\frac{m+k-1}{2}}f| \cdot |\Delta^{\frac{m+k+1}{2}}f| \\
\leq &\frac{c}{R^2}\int_{\mathbb{R}^{2m}} \eta^{2k}|\Delta^{\frac{m+k-1}{2}}f|^2 + \delta\int_{\mathbb{R}^{2m}} \eta^{2k+2}|D\Delta^{\frac{m+k-1}{2}}f|^2 \\
&+ \delta\int_{\mathbb{R}^{2m}} \eta^{2k+4}|\Delta^{\frac{m+k+1}{2}}f|^2.
\end{aligned}$$

Putting everything together and choosing $\delta > 0$ small enough we get

$$\sum_{\substack{k=2\\ k \text{ even}}}^{m-2} \frac{c}{R^{2m-2k}} \int_{\mathbb{R}^{2m}} \eta^{2k+2} |\Delta^{\frac{m+k}{2}} f|^2 + \sum_{\substack{k=1,\\ k \text{ odd}}}^{m-1} \frac{c}{R^{2m-2k}} \int_{\mathbb{R}^{2m}} \eta^{2k+2} |D\Delta^{\frac{m+k-1}{2}} f|^2 \le \frac{c}{R^{2m}} \int_{B_R} |D^m f|^2. \tag{A.1.1}$$

Since $\Delta^m f = 0$ we have

$$\Delta^{\frac{m}{2}}(\eta \Delta^{\frac{m}{2}} f) = \sum_{\substack{k=0\\ k \text{ even}}}^{m-2} \binom{m}{k} \Delta^{\frac{m-k}{2}} \eta \Delta^{\frac{m+k}{2}} f + \sum_{\substack{k=1,\\ k \text{ odd}}}^{m-1} \binom{m}{k} D\Delta^{\frac{m-k-1}{2}} \eta D\Delta^{\frac{m+k-1}{2}} f$$

and with the Calderon-Zygmund inequality (see [25]) and (A.1.1) we arrive at

$$\int_{\mathbb{R}^{2m}} \eta^{2m+2} |D^{2m} f|^2 \le c \int_{\mathbb{R}^{2m}} |\Delta^{\frac{m}{2}}(\eta^{m+1} \Delta^{\frac{m}{2}} f)|^2 \le \frac{c}{R^{2m}} \int_{B_R} |D^m f|^2.$$

With Hölder's inequality we have

$$\int_{B_{\frac{R}{2}}} |D^{2m} f| \le c R^m \left(\int_{B_{\frac{R}{2}}} |D^{2m} f|^2 \right)^{1/2}.$$

Applying the embedding Lemma 3.6.3 repeatedly we get

$$||D^m f||_{L^{2,1}(B_{\frac{R}{4}})} \le ||D^{2m} f||_{L^1(B_{\frac{R}{2}})} \le c ||D^m f||_{L^2(B_R)}$$

with c independent of R.

□

A.2 Lorentz-Sobolev spaces

If $f \in L^{p,q}(\mathbb{R}^n)$ has weak derivatives $D^k f \in L^{p,q}(\mathbb{R}^n)$, $k \in \mathbb{N}$, then f is an element of the so-called Lorentz-Sobolev space $W^{k,p,q}(\mathbb{R}^n)$. These spaces play an important role in chapter 2.

Definition A.2.1. *Let $1 < p < \infty$, $1 \le q \le \infty$ and $k \in \mathbb{N}$. Let $f \in L^{p,q}(\mathbb{R}^n)$ be k times weakly differentiable and for all multiindices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$ let $\frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_n} x_n} f \in L^{p,q}(\mathbb{R}^n)$. Then f is an element of the Lorentz-Sobolev space $W^{k,p,q}(\mathbb{R}^n)$ with norm*

$$||f||_{W^{k,p,q}(\mathbb{R}^n)} := \sum_{0 \le |\alpha| \le k} \left\| \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_n} x_n} f \right\|_{L^{p,q}(\mathbb{R}^n)}.$$

Since we work on the unit ball in $\mathbb{R}^{2m}$, we formulate all results in terms of B^n, $n \in \mathbb{N}$. For detailed proofs of the following properties see [52].

Lemma A.2.2. *Let $k, n \in \mathbb{N}$, $1 < p < \frac{n}{k}$ and $1 \le q \le \infty$. Then*

$$W^{k,p,q}(B^n) \hookrightarrow L^{p^*,q}(B^n)$$

for $\frac{1}{p^*} = \frac{1}{p} + \frac{k}{n}$ and

$$||f||_{L^{p^*,q}(B^n)} \le c ||f||_{W^{k,p,q}(B^n)} \qquad \textit{for any } f \in W^{k,p,q}(B^n).$$

Lemma A.2.3. *Let $s, k \in \mathbb{N}$, $p, p', q, q' \in \mathbb{R}$ with $1 < p, p', q, q' < \infty$, $kp < n, sp' < n, s \leq k, t := \frac{npp'}{n(p+p')-kpp'} > 1$ and $\frac{1}{u} := \min\{\frac{1}{q} + \frac{1}{q'}, 1\}$. Further let $B^n \subset \mathbb{R}^n$. If $f \in W^{k,p,q}(B^n)$, $g \in W^{s,p',q'}(B^n)$, then*

$$fg \in W^{s,t,u}(B^n)$$

and

$$||fg||_{W^{s,t,u}(B^n)} \leq c||f||_{W^{k,p,q}(B^n)}||g||_{W^{s,p'q'}(B^n)}$$

with $c = c(B^n)$.

Lemma A.2.4. *Let $B^n \subset \mathbb{R}^n$. If $f \in W^{k,\frac{n}{k},1}(B^n)$, the f is continuous on B^n.*

In chapter 2 we use Lorentz-Sobolev spaces $W^{-k,p,q}$ with negative exponent $-k$, $k \in \mathbb{N}$. These are distribution spaces and for $p, q > 1$ form dual spaces to $W^{k,p,q}$.

Definition A.2.5. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$ and $k \in \mathbb{N}$. Then $W^{-k,p,q}(B^n)$ is the space of distributions $\Phi \in (C_c^\infty(B^n))'$ such that*

$$|\Phi[f]| \leq c||f||_{W^{k,p',q'}(B^n)} \qquad \forall f \in C_c^\infty(B^n).$$

Each element of $W^{-k,p,q}$ has a representation in terms of derivatives of Lorentz functions:

Lemma A.2.6. *Let $1 < p, q < \infty$, $k \in \mathbb{N}$, $B^n \subset \mathbb{R}^n$ and $f \in W^{-k,p,q}(B^n)$. Then there exist $f_\alpha \in L^{p,q}(B^n)$ so that*

$$f = \sum_{|\alpha| \leq k} \partial^\alpha f_\alpha.$$

Note that this representation is not unique. We define the norm on $W^{-k,p,q}(B^n)$ by

$$||f||_{W^{-k,p,q}(B^n)} := \inf \left\{ \sum_{|\alpha| \leq k} ||f_\alpha||_{L^{p,q}(B^n)} : f = \sum_{|\alpha| \leq k} \partial^\alpha f_\alpha \right\}.$$

The definition of negative Lorentz-Sobolev spaces as dual spaces does not hold for $p, q = 1$ since $L^{p,1}, L^{p',\infty}$ are not reflexive (see Lemma (A.1.5)). We define the space $W^{-k,p,1}$ as follows

Definition A.2.7. *Let $1 < p < \infty$, $k \in \mathbb{N}$. Then*

$$W^{-k,p,1}(B^n) := \left\{ f = \sum_{|\alpha| \leq k} \partial^\alpha f_\alpha : f_\alpha \in L^{p,1}(B^n) \right\}$$

with norm

$$||f||_{W^{-k,p,1}(B^n)} := \inf \left\{ \sum_{|\alpha| \leq k} ||f_\alpha||_{L^{p,1}(B^n)} : f = \sum_{|\alpha| \leq k} \partial^\alpha f_\alpha \right\}$$

Finally we have an embedding theorem and a Hölder inequality.

Lemma A.2.8. *Let $B^n \subset \mathbb{R}^n$, $1 < p < n$, $1 \leq q \leq p$, $l, s, t \in \mathbb{N}_0$ with $tp < n$ and $f \in W^{-s,p,q}(B^n, \wedge^l \mathbb{R}^n)$. Then $f \in W^{-(s+t),\frac{np}{n-tp},q}(B^n, \wedge^l \mathbb{R}^n)$ and*

$$||f||_{W^{-(s+t),\frac{np}{n-tp},q}(B^n)} \leq c||f||_{W^{-s,p,q}(B^n)}.$$

Lemma A.2.9. *Let* $s, t \in \mathbb{N},\ t \leq s,\ 1 < p, p' < \infty$ *with* $\frac{1}{p} + \frac{1}{p'} \leq 1$ *and* $tp < n,\ sp' < n,\ 1 \leq q, q' < \infty$. *Let* $f \in W^{-t,p,q}(B^n)$ *and* $g \in W^{s,p',q'}(B^n)$. *Then*

$$fg \in W^{-t,x,y}(B^n)$$

with $x = \frac{dpp'}{n(p+p')-spp'}$ *and* $\frac{1}{y} = \min\{1, \frac{1}{q}, \frac{1}{q'}\}$. *Further*

$$||fg||_{W^{-t,x,y}(B^n)} \leq c||f||_{W^{-t,p,q}(B^n)}||g||_{W^{s,p',q'}(B^n)}.$$

A.3 Hardy space

Definition A.3.1. *Let* $n \in \mathbb{N}$ *and let* $\psi \in C_c^\infty(\mathbb{R}^n)$ *such that*

$$\int_{\mathbb{R}^n} \psi = 1.$$

Further let

$$\psi_t(x) = t^{-n}\psi\left(\frac{x}{t}\right) \qquad \forall t > 0.$$

For $f \in L^1(\mathbb{R}^n)$ *define*

$$f^\star(x) := \sup_{t>0} |(\psi_t * f)(x)|.$$

Then f *is an element of the Hardy space* $\mathcal{H}^1(\mathbb{R}^n)$ *if and only if* $f^* \in L^1(\mathbb{R}^n)$. *The norm is given by*

$$||f||_{\mathcal{H}^1(\mathbb{R}^n)} := ||f||_{L^1(\mathbb{R}^n)} + ||f^\star||_{L^1(\mathbb{R}^n)}.$$

(see [31] Definition 3.2.4)

A.4 Radial m-polyharmonic functions

We adapt Lemma 5.1 in Hornung and Moser [35] for radial m-polyharmonic functions.

Lemma A.4.1. *There exists a universal constant* C_{2m} *such that for all* $R > 0$ *and for all radial solutions* $q \in C^\infty(B_{2R} \setminus B_R, \mathbb{R}^n)$ *of* $\Delta^m q = 0$ *on* $B_{2R} \setminus B_R$, *the following estimate holds:*

$$||q'||_{C^0(B_{2R}\setminus B_R)} \leq C_{2m}\left(\sum_{i=1}^m R^{i-1}(|q^{(i)}(R)| + |q^{(i)}(2R)|) + R^{-1}|q(2R) - q(R)|\right).$$

Proof. Since q is radial we have

$$\Delta q(|x|) = q''(|x|) + \frac{2m-1}{|x|}q'(|x|).$$

We iterate this equation until $\Delta^m q = 0$. Then q' is a solution of the $(2m-1)$-order system

$$f^{(2m-1)}(t) + c_{2m-2}\frac{f^{(2m-2)}(t)}{t} + c_{2m-3}\frac{f^{(2m-3)}(t)}{t^2} + ... + c_1\frac{f(t)}{t^{2m-1}} = 0 \qquad \text{(A.4.1)}$$

with constants $c_i \in \mathbb{R},\ i = 1, ..., 2m-2$. Let $X \subset C^\infty([R, 2R], \mathbb{R}^n)$ be the space of solutions of (A.4.1) and let

$$L\colon X \to \mathbb{R}^n \times \mathbb{R}^{2m-1},$$

$$f \mapsto \left(f(R), f(2R), f'(R), f'(2R), ..., f^{(m-1)}(R), f^{(m-1)}(2R), \int_R^{2R} f \right).$$

If L is bijective then X is a $(2m-1)$-dimensional subspace and all norms on X are equivalent. Thus we have

$$||f||_{C^0([R,2R],\mathbb{R}^n)} \leq C|Lf| \qquad \forall f \in X$$

and subsequently

$$||q'||_{C^0(B_{2R}\setminus B_R)} \leq C \left(\sum_{i=1}^{m} R^{i-1}(|q^{(i)}(R)| + |q^{(i)}(2R)|) + R^{-1}|q(2R) - q(R)| \right).$$

It is left to show that L is bijective. Let $a \in \mathbb{R}^n \times \mathbb{R}^{2m-1}$. The functional $v \mapsto \int_{B_{2R}\setminus B_R} |D^m v|^2$ has a minimizer in the class of all radial $v \in W^{m,2}(B_{2R})$ with $f(R) = a_1, f(2R) = a_2, f'(R) = a_3, f'(2R) = a_4, ..., f^{(m-1)}(R) = a_{2m-3}, f^{(m-1)}(2R) = a_{2m-2}, \int_R^{2R} f = a_{2m-1}$. Thus v' is a solution of (A.4.1) and $Lv' = a$. Moreover as a solution of (A.4.1) v' is uniquely determined by a.

□

A.5 Bochner formula

The following version of the Bochner formula was calculated by Struwe in [30].

Proposition A.5.1 (Bochner formula). *Let (M,g), (N,h) be Riemannian manifolds and let $u \in C^3(M,N)$. In normal coordinates the following equality holds*

$$\begin{aligned}\Delta_{S^2}\left(\frac{1}{2}|\nabla_{S^2}u|^2\right) &= \partial_\mu(\Delta_{S^2}u^i)\partial_\mu u^i + \partial^2_{\alpha\mu}u^i\partial^2_{\alpha\mu}u^i + R_{\alpha\beta}\partial_\alpha u^i\partial_\beta u^i \\ &\quad + \frac{1}{2}\partial^2_{kl}h_{ij}(u)\partial_\mu u^i\partial_\mu u^j\partial_\alpha u^k\partial_\alpha u^l, \qquad \text{(A.5.1)}\end{aligned}$$

where $R_{\alpha\beta}$ is the Ricci tensor on M.

Proof. We use normal coordinates around points $x_0 \in M$ and $u(x_0) \in N$. Let g_{ij} and h_{ij} be the metrics in normal coordinates on M and N respectively such that $g_{ij} = \delta_{ij}$, $\partial_\alpha g_{ij} = 0$ at x_0 and $h_{ij} = \delta_{ij}$, $\partial_\alpha h_{ij} = 0$ at $u(x_0)$. The second derivative of the inverse at x_0 is given by $\partial^2_{\alpha\beta}g^{ij} = -\partial^2_{\alpha\beta}g_{ij}$. Further we have

$$\begin{aligned}\partial_\alpha(\Delta_{S^2}u) - \Delta_{S^2}(\partial_\alpha u) &= \partial_\alpha\left(\frac{1}{\sqrt{\det g}}\partial_i\left(g^{ij}\sqrt{\det g}\,\partial_j u\right)\right) - \left(\frac{1}{\sqrt{\det g}}\partial_i\left(g^{ij}\sqrt{\det g}\,\partial^2_{j\alpha}u\right)\right) \\ &= \partial^2_{\alpha i}\left(g^{ij}\sqrt{\det g}\right)\partial_j u = \partial^2_{\alpha i}g^{ij}\partial_j u + \frac{1}{2}\partial^2_{\alpha i}g_{kk}\partial_i u \\ &= -\partial^2_{\alpha i}g_{ij}\partial_j u + \frac{1}{2}\partial^2_{\alpha i}g_{kk}\partial_i u\end{aligned}$$

and with this

$$\begin{aligned}\Delta_{S^2}\left(\frac{1}{2}|\nabla_{S^2}u|^2\right) &= \Delta_{S^2}\left(\frac{1}{2}g^{\mu\nu}h_{ij}(u)\partial_\mu u^i\partial_\nu u^j\right) \\ &= \Delta_{S^2}(\partial_\mu u^i)\partial_\mu u^i + \frac{1}{2}\partial^2_{\alpha\alpha}g^{\mu\nu}\partial_\mu u^i\partial_\nu u^i + \partial^2_{\alpha\mu}u^i\partial^2_{\alpha\mu}u^i \\ &\quad + \frac{1}{2}\Delta_{S^2}h_{ij}(u)\partial_\mu u^i\partial_\mu u^j\end{aligned}$$

$$
\begin{aligned}
&= \partial_\mu(\Delta_{S^2} u^i)\partial_\mu u^i + \left(\partial^2_{\mu\alpha} g_{\alpha\beta}\partial_\beta u^i - \frac{1}{2}\partial^2_{\mu\alpha} g_{\rho\rho}\partial_\alpha u^i\right)\partial_\mu u^i + \partial^2_{\alpha\mu}u^i\partial^2_{\alpha\mu}u^i \\
&\quad - \frac{1}{2}\partial^2_{\alpha\alpha} g_{\mu\nu}\partial_\mu u^i \partial_\nu u^i + \frac{1}{2}\partial^2_{kl} h_{ij}(u)\partial_\mu u^i \partial_\mu u^j \partial_\alpha u^k \partial_\alpha u^l \\
&= \partial_\mu(\Delta_{S^2} u^i)\partial_\mu u^i + \partial^2_{\alpha\mu}u^i\partial^2_{\alpha\mu}u^i + R_{\alpha\beta}\partial_\alpha u^i \partial_\beta u^i \\
&\quad + \frac{1}{2}\partial^2_{kl} h_{ij}(u)\partial_\mu u^i \partial_\mu u^j \partial_\alpha u^k \partial_\alpha u^l,
\end{aligned}
$$

where we used that the Ricci tensor in normal coordinates is given by

$$
R_{\alpha\beta} = \frac{1}{2}\left(\partial^2_{\beta\mu} g_{\alpha\mu} + \partial^2_{\alpha\mu} g_{\beta\mu}\right) - \frac{1}{2}\left(\partial^2_{\mu\mu} g_{\alpha\beta} + \partial^2_{\alpha\beta} g_{\mu\mu}\right).
$$

□

Bibliography

[1] Robert A. Adams and John J. F. Fournier. *Sobolev spaces*. Second. Vol. 140. Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, 2003, pp. xiv+305. ISBN: 0-12-044143-8.

[2] Wanjun Ai and Hao Yin. "Neck analysis of extrinsic polyharmonic maps". In: *Ann. Global Anal. Geom.* 52.2 (2017), pp. 129–156. ISSN: 0232-704X. DOI: 10.1007/s10455-017-9551-7. URL: https://doi.org/10.1007/s10455-017-9551-7.

[3] Antonio Ambrosetti and Andrea Malchiodi. *Nonlinear analysis and semilinear elliptic problems*. Vol. 104. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2007, pp. xii+316. ISBN: 978-0-521-86320-9; 0-521-86320-1. DOI: 10.1017/CBO9780511618260. URL: https://doi.org/10.1017/CBO9780511618260.

[4] Gilles Angelsberg. "Large solutions for biharmonic maps in four dimensions". In: *Calc. Var. Partial Differential Equations* 30.4 (2007), pp. 417–447. ISSN: 0944-2669. DOI: 10.1007/s00526-007-0095-8. URL: https://doi.org/10.1007/s00526-007-0095-8.

[5] Gilles Angelsberg and David Pumberger. "A regularity result for polyharmonic maps with higher integrability". In: *Ann. Global Anal. Geom.* 35.1 (2009), pp. 63–81. ISSN: 0232-704X. DOI: 10.1007/s10455-008-9122-z. URL: https://doi.org/10.1007/s10455-008-9122-z.

[6] F. Bethuel and J.-M. Ghidaglia. "Improved regularity of solutions to elliptic equations involving Jacobians and applications". In: *J. Math. Pures Appl. (9)* 72.5 (1993), pp. 441–474. ISSN: 0021-7824.

[7] Fabrice Bethuel. "On the singular set of stationary harmonic maps". In: *Manuscripta Math.* 78.4 (1993), pp. 417–443. ISSN: 0025-2611. DOI: 10.1007/BF02599324. URL: https://doi.org/10.1007/BF02599324.

[8] Thomas P. Branson. "Group representations arising from Lorentz conformal geometry". In: *J. Funct. Anal.* 74.2 (1987), pp. 199–291. ISSN: 0022-1236. DOI: 10.1016/0022-1236(87)90025-5. URL: https://doi.org/10.1016/0022-1236(87)90025-5.

[9] H. Brezis and J.-M. Coron. "Convergence of solutions of H-systems or how to blow bubbles". In: *Arch. Rational Mech. Anal.* 89.1 (1985), pp. 21–56. ISSN: 0003-9527. DOI: 10.1007/BF00281744. URL: https://doi.org/10.1007/BF00281744.

[10] Haïm Brezis and Jean-Michel Coron. "Multiple solutions of H-systems and Rellich's conjecture". In: *Comm. Pure Appl. Math.* 37.2 (1984), pp. 149–187. ISSN: 0010-3640. DOI: 10.1002/cpa.3160370202. URL: https://doi.org/10.1002/cpa.3160370202.

[11] Sun-Yung A. Chang, Lihe Wang, and Paul C. Yang. "A regularity theory of biharmonic maps". In: *Comm. Pure Appl. Math.* 52.9 (1999), pp. 1113–1137. ISSN: 0010-3640. DOI: 10.1002/(SICI)1097-0312(199909)52:9<1113::AID-CPA4>3.0.CO;2-7. URL: https://doi.org/10.1002/(SICI)1097-0312(199909)52:9<1113::AID-CPA4>3.0.CO;2-7.

[12] Sun-Yung A. Chang and Paul C. Yang. "Extremal metrics of zeta function determinants on 4-manifolds". In: *Ann. of Math. (2)* 142.1 (1995), pp. 171–212. ISSN: 0003-486X. DOI: 10.2307/2118613. URL: https://doi.org/10.2307/2118613.

[13] R. Coifman et al. "Compensated compactness and Hardy spaces". In: *J. Math. Pures Appl. (9)* 72.3 (1993), pp. 247–286. ISSN: 0021-7824.

[14] Weiyue Ding and Gang Tian. "Energy identity for a class of approximate harmonic maps from surfaces". In: *Comm. Anal. Geom.* 3.3-4 (1995), pp. 543–554. ISSN: 1019-8385. DOI: 10.4310/CAG.1995.v3.n4.a1. URL: https://doi.org/10.4310/CAG.1995.v3.n4.a1.

[15] Frank Duzaar, Andreas Gastel, and Joseph F. Grotowski. "Optimal partial regularity for nonlinear elliptic systems of higher order". In: *J. Math. Sci. Univ. Tokyo* 8.3 (2001), pp. 463–499. ISSN: 1340-5705.

[16] Frank Duzaar and Ernst Kuwert. "Minimization of conformally invariant energies in homotopy classes". In: *Calc. Var. Partial Differential Equations* 6.4 (1998), pp. 285–313. ISSN: 0944-2669. DOI: 10.1007/s005260050092. URL: https://doi.org/10.1007/s005260050092.

[17] J. Eells and L. Lemaire. "A report on harmonic maps". In: *Bull. London Math. Soc.* 10.1 (1978), pp. 1–68. ISSN: 0024-6093. DOI: 10.1112/blms/10.1.1. URL: https://doi.org/10.1112/blms/10.1.1.

[18] James Eells Jr. and J. H. Sampson. "Harmonic mappings of Riemannian manifolds". In: *Amer. J. Math.* 86 (1964), pp. 109–160. ISSN: 0002-9327. DOI: 10.2307/2373037. URL: https://doi.org/10.2307/2373037.

[19] Jens Frehse. "A discontinuous solution of a mildly nonlinear elliptic system". In: *Math. Z.* 134 (1973), pp. 229–230. ISSN: 0025-5874. DOI: 10.1007/BF01214096. URL: https://doi.org/10.1007/BF01214096.

[20] Andreas Gastel and Frédéric Louis de Longueville. *Conservation laws for even order systems of polyharmonic map type.* 2019. arXiv: 1909.05697 [math.AP].

[21] Andreas Gastel and Christoph Scheven. "Regularity of polyharmonic maps in the critical dimension". In: *Comm. Anal. Geom.* 17.2 (2009), pp. 185–226. ISSN: 1019-8385. DOI: 10.4310/CAG.2009.v17.n2.a2. URL: https://doi.org/10.4310/CAG.2009.v17.n2.a2.

[22] I. M. Gel'fand, R. A. Minlos, and Z. Ja. Šapiro. Представления группы вращени и группы Лоренца, их применения *translated by G. Cummins and T. Boddington: Representations of the rotation and Lorentz groups and their applications.* Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958, p. 368.

[23] Mariano Giaquinta. *Multiple integrals in the calculus of variations and nonlinear elliptic systems.* Vol. 105. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1983, pp. vii+297. ISBN: 0-691-08330-4; 0-691-08331-2.

[24] Mariano Giaquinta and Luca Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs.* Vol. 2. Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]. Edizioni della Normale, Pisa, 2005, pp. xii+302. ISBN: 88-7642-168-8.

[25] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Reprint of the 1998 edition. Springer-Verlag, Berlin, 2001, pp. xiv+517. ISBN: 3-540-41160-7.

[26] Paweł Goldstein, Paweł Strzelecki, and Anna Zatorska-Goldstein. "On polyharmonic maps into spheres in the critical dimension". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 26.4 (2009), pp. 1387–1405. ISSN: 0294-1449. DOI: 10.1016/j.anihpc.2008.10.008. URL: https://doi.org/10.1016/j.anihpc.2008.10.008.

[27] Loukas Grafakos. *Classical Fourier analysis.* Second. Vol. 249. Graduate Texts in Mathematics. Springer, New York, 2008, pp. xvi+489. ISBN: 978-0-387-09431-1.

[28] Michael Grüter. "Regularity of weak H-surfaces". In: *J. Reine Angew. Math.* 329 (1981), pp. 1–15. ISSN: 0075-4102. DOI: 10.1515/crll.1981.329.1. URL: https://doi.org/10.1515/crll.1981.329.1.

[29] Chang-Yu Guo and Chang-Lin Xiang. *Regularity of weak solutions to higher order elliptic systems in critical dimensions.* 2020. arXiv: 2010.09149 [math.AP].

[30] Robert Hardt and Michael Wolf, eds. *Nonlinear partial differential equations in differential geometry.* Vol. 2. IAS/Park City Mathematics Series. Lecture notes from the Summer School held in Park City, Utah, June 20–July 12, 1992. American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1996, pp. xii+339. ISBN: 0-8218-0431-6. DOI: 10.1090/pcms/002. URL: https://doi.org/10.1090/pcms/002.

[31] Frédéric Hélein. *Harmonic maps, conservation laws and moving frames.* Second. Vol. 150. Cambridge Tracts in Mathematics. Translated from the 1996 French original, With a foreword by James Eells. Cambridge University Press, Cambridge, 2002, pp. xxvi+264. ISBN: 0-521-81160-0. DOI: 10.1017/CBO9780511543036. URL: https://doi.org/10.1017/CBO9780511543036.

[32] Frédéric Hélein. "Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne". In: *C. R. Acad. Sci. Paris Sér. I Math.* 312.8 (1991), pp. 591–596. ISSN: 0764-4442.

[33] S. Hildebrandt. "Quasilinear elliptic systems in diagonal form". In: *Systems of nonlinear partial differential equations (Oxford, 1982).* Vol. 111. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. Reidel, Dordrecht, 1983, pp. 173–217.

[34] Stefan Hildebrandt. "Nonlinear elliptic systems and harmonic mappings". In: *Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Vol. 1, 2, 3 (Beijing, 1980).* Sci. Press Beijing, Beijing, 1982, pp. 481–615.

[35] Peter Hornung and Roger Moser. "Energy identity for intrinsically biharmonic maps in four dimensions". In: *Anal. PDE* 5.1 (2012), pp. 61–80. ISSN: 2157-5045. DOI: 10.2140/apde.2012.5.61. URL: https://doi.org/10.2140/apde.2012.5.61.

[36] Jasmin Hörter and Tobias Lamm. "Conservation laws for even order elliptic systems in the critical dimension - a new approach". In: *Calc. Var. Partial Differential Equations* 60.4 (2021), Paper No. 125, 23. ISSN: 0944-2669. DOI: 10.1007/s00526-021-01995-7. URL: https://doi.org/10.1007/s00526-021-01995-7.

[37] Richard A. Hunt. "On $L(p, q)$ spaces". In: *Enseign. Math. (2)* 12 (1966), pp. 249–276. ISSN: 0013-8584.

[38] Jürgen Jost. *Riemannian geometry and geometric analysis.* Sixth. Universitext. Springer, Heidelberg, 2011, pp. xiv+611. ISBN: 978-3-642-21297-0. DOI: 10.1007/978-3-642-21298-7. URL: https://doi.org/10.1007/978-3-642-21298-7.

[39] Jürgen Jost. *Two-dimensional geometric variational problems.* Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Ltd., Chichester, 1991, pp. x+236. ISBN: 0-471-92839-9.

[40] Ernst Kuwert, Tobias Lamm, and Yuxiang Li. "Two-dimensional curvature functionals with superquadratic growth". In: *J. Eur. Math. Soc. (JEMS)* 17.12 (2015), pp. 3081–3111. ISSN: 1435-9855. DOI: 10.4171/JEMS/580. URL: https://doi.org/10.4171/JEMS/580.

[41] Tobias Lamm. "Biharmonic map heat flow into manifolds of nonpositive curvature". In: *Calc. Var. Partial Differential Equations* 22.4 (2005), pp. 421–445. ISSN: 0944-2669. DOI: 10.1007/s00526-004-0283-8. URL: https://doi.org/10.1007/s00526-004-0283-8.

[42] Tobias Lamm. "Biharmonic maps". In: *Dissertation Albert-Ludwig-Universität Freiburg* (2005). URL: https://freidok.uni-freiburg.de/data/1755.

[43] Tobias Lamm. "Energy identity for approximations of harmonic maps from surfaces". In: *Trans. Amer. Math. Soc.* 362.8 (2010), pp. 4077–4097. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-10-04912-3. URL: https://doi.org/10.1090/S0002-9947-10-04912-3.

[44] Tobias Lamm. "Fourth order approximation of harmonic maps from surfaces". In: *Calc. Var. Partial Differential Equations* 27.2 (2006), pp. 125–157. ISSN: 0944-2669. DOI: 10.1007/s00526-005-0001-1. URL: https://doi.org/10.1007/s00526-005-0001-1.

[45] Tobias Lamm. “Geometric Variational Problems”. In: *FU Berlin* (2007). URL: https://www.math.kit.edu/iana1/~lamm/seite/lecturenotes/media/geometricvariationalproblems.pdf.

[46] Tobias Lamm, Andrea Malchiodi, and Mario Micallef. *A gap theorem for α-harmonic maps between two-spheres*. 2020. arXiv: 1903.10217 [math.DG].

[47] Tobias Lamm, Andrea Malchiodi, and Mario Micallef. “Limits of α-harmonic maps”. In: *J. Differential Geom.* 116.2 (2020), pp. 321–348. ISSN: 0022-040X. DOI: 10.4310/jdg/1603936814. URL: https://doi.org/10.4310/jdg/1603936814.

[48] Tobias Lamm and Tristan Rivière. “Conservation laws for fourth order systems in four dimensions”. In: *Comm. Partial Differential Equations* 33.1-3 (2008), pp. 245–262. ISSN: 0360-5302. DOI: 10.1080/03605300701382381. URL: https://doi.org/10.1080/03605300701382381.

[49] Tobias Lamm and Ben Sharp. “Global estimates and energy identities for elliptic systems with antisymmetric potentials”. In: *Comm. Partial Differential Equations* 41.4 (2016), pp. 579–608. ISSN: 0360-5302. DOI: 10.1080/03605302.2015.1116559. URL: https://doi.org/10.1080/03605302.2015.1116559.

[50] Jiayu Li and Xiangrong Zhu. “Energy identity and necklessness for a sequence of Sacks-Uhlenbeck maps to a sphere”. In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 36.1 (2019), pp. 103–118. ISSN: 0294-1449. DOI: 10.1016/j.anihpc.2018.04.002. URL: https://doi.org/10.1016/j.anihpc.2018.04.002.

[51] Yuxiang Li and Youde Wang. “A counterexample to the energy identity for sequences of α-harmonic maps”. In: *Pacific J. Math.* 274.1 (2015), pp. 107–123. ISSN: 0030-8730. DOI: 10.2140/pjm.2015.274.107. URL: https://doi.org/10.2140/pjm.2015.274.107.

[52] Frédéric Louis de Longueville. “Regularität der Lösungen von Systemen (2m)-ter Ordnung vom polyharmonischen Typ in kritischer Dimension”. In: *Dissertation Universität Duisburg-Essen* (2018). URL: https://d-nb.info/1191692124/34.

[53] Charles B. Morrey Jr. “The problem of Plateau on a Riemannian manifold”. In: *Ann. of Math. (2)* 49 (1948), pp. 807–851. ISSN: 0003-486X. DOI: 10.2307/1969401. URL: https://doi.org/10.2307/1969401.

[54] Richard S. Palais. “The principle of symmetric criticality”. In: *Comm. Math. Phys.* 69.1 (1979), pp. 19–30. ISSN: 0010-3616. URL: http://projecteuclid.org/euclid.cmp/1103905401.

[55] Jaak Peetre. “Espaces d’interpolation et théorème de Soboleff”. In: *Ann. Inst. Fourier (Grenoble)* 16.fasc. 1 (1966), pp. 279–317. ISSN: 0373-0956. URL: http://www.numdam.org/item?id=AIF_1966__16_1_279_0.

[56] Jie Qing. “On singularities of the heat flow for harmonic maps from surfaces into spheres”. In: *Comm. Anal. Geom.* 3.1-2 (1995), pp. 297–315. ISSN: 1019-8385. DOI: 10.4310/CAG.1995.v3.n2.a4. URL: https://doi.org/10.4310/CAG.1995.v3.n2.a4.

[57] Tristan Rivière. “A viscosity method in the min-max theory of minimal surfaces”. In: *Publ. Math. Inst. Hautes Études Sci.* 126 (2017), pp. 177–246. ISSN: 0073-8301. DOI: 10.1007/s10240-017-0094-z. URL: https://doi.org/10.1007/s10240-017-0094-z.

[58] Tristan Rivière. “Conservation laws for conformally invariant variational problems”. In: *Invent. Math.* 168.1 (2007), pp. 1–22. ISSN: 0020-9910. DOI: 10.1007/s00222-006-0023-0. URL: https://doi.org/10.1007/s00222-006-0023-0.

[59] Tristan Rivière. “Everywhere discontinuous harmonic maps into spheres”. In: *Acta Math.* 175.2 (1995), pp. 197–226. ISSN: 0001-5962. DOI: 10.1007/BF02393305. URL: https://doi.org/10.1007/BF02393305.

[60] Tristan Rivière. “Willmore Minmax Surfaces and the Cost of the Sphere Eversion”. In: *arXiv e-prints*, arXiv:1512.08918 (Dec. 2015), arXiv:1512.08918. arXiv: 1512.08918 [math.AP].

[61] Tristan Rivière and Michael Struwe. "Partial regularity for harmonic maps and related problems". In: *Comm. Pure Appl. Math.* 61.4 (2008), pp. 451–463. ISSN: 0010-3640. DOI: 10.1002/cpa.20205. URL: https://doi.org/10.1002/cpa.20205.

[62] Tristan Rivière. *Conformally Invariant Variational Problems.* 2012. arXiv: 1206.2116 [math.AP].

[63] J. Sacks and K. Uhlenbeck. "The existence of minimal immersions of 2-spheres". In: *Ann. of Math. (2)* 113.1 (1981), pp. 1–24. ISSN: 0003-486X. DOI: 10.2307/1971131. URL: https://doi.org/10.2307/1971131.

[64] Armin Schikorra. "A remark on gauge transformations and the moving frame method". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 27.2 (2010), pp. 503–515. ISSN: 0294-1449. DOI: 10.1016/j.anihpc.2009.09.004. URL: https://doi.org/10.1016/j.anihpc.2009.09.004.

[65] Richard Schoen and Karen Uhlenbeck. "A regularity theory for harmonic maps". In: *J. Differential Geometry* 17.2 (1982), pp. 307–335. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214436923.

[66] Richard M. Schoen. "Analytic aspects of the harmonic map problem". In: *Seminar on nonlinear partial differential equations (Berkeley, Calif., 1983).* Vol. 2. Math. Sci. Res. Inst. Publ. Springer, New York, 1984, pp. 321–358. DOI: 10.1007/978-1-4612-1110-5_17. URL: https://doi.org/10.1007/978-1-4612-1110-5_17.

[67] Jalal Shatah. "Weak solutions and development of singularities of the SU(2) σ-model". In: *Comm. Pure Appl. Math.* 41.4 (1988), pp. 459–469. ISSN: 0010-3640. DOI: 10.1002/cpa.3160410405. URL: https://doi.org/10.1002/cpa.3160410405.

[68] M. A. Shubin. *Pseudodifferential operators and spectral theory.* Second. Translated from the 1978 Russian original by Stig I. Andersson. Springer-Verlag, Berlin, 2001, pp. xii+288. ISBN: 3-540-41195-X. DOI: 10.1007/978-3-642-56579-3. URL: https://doi.org/10.1007/978-3-642-56579-3.

[69] Leon Simon. *Theorems on regularity and singularity of energy minimizing maps.* Lectures in Mathematics ETH Zürich. Based on lecture notes by Norbert Hungerbühler. Birkhäuser Verlag, Basel, 1996, pp. viii+152. ISBN: 3-7643-5397-X. DOI: 10.1007/978-3-0348-9193-6. URL: https://doi.org/10.1007/978-3-0348-9193-6.

[70] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* Vol. 43. Princeton Mathematical Series. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993, pp. xiv+695. ISBN: 0-691-03216-5.

[71] Michael Struwe. "On the evolution of harmonic mappings of Riemannian surfaces". In: *Comment. Math. Helv.* 60.4 (1985), pp. 558–581. ISSN: 0010-2571. DOI: 10.1007/BF02567432. URL: https://doi.org/10.1007/BF02567432.

[72] Michael Struwe. "Partial regularity for biharmonic maps, revisited". In: *Calc. Var. Partial Differential Equations* 33.2 (2008), pp. 249–262. ISSN: 0944-2669. DOI: 10.1007/s00526-008-0175-4. URL: https://doi.org/10.1007/s00526-008-0175-4.

[73] Michael Struwe. "Positive solutions of critical semilinear elliptic equations on non-contractible planar domains". In: *J. Eur. Math. Soc. (JEMS)* 2.4 (2000), pp. 329–388. ISSN: 1435-9855. DOI: 10.1007/s100970000023. URL: https://doi.org/10.1007/s100970000023.

[74] Michael Struwe. *Variational methods.* Fourth. Vol. 34. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Applications to nonlinear partial differential equations and Hamiltonian systems. Springer-Verlag, Berlin, 2008, pp. xx+302. ISBN: 978-3-540-74012-4.

[75] Paweł Strzelecki. “On biharmonic maps and their generalizations”. In: *Calc. Var. Partial Differential Equations* 18.4 (2003), pp. 401–432. ISSN: 0944-2669. DOI: 10.1007/s00526-003-0210-4. URL: https://doi.org/10.1007/s00526-003-0210-4.

[76] Luc Tartar. “Imbedding theorems of Sobolev spaces into Lorentz spaces”. In: *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)* 1.3 (1998), pp. 479–500. ISSN: 0392-4041.

[77] Karen K. Uhlenbeck. “Connections with L^p bounds on curvature”. In: *Comm. Math. Phys.* 83.1 (1982), pp. 31–42. ISSN: 0010-3616. URL: http://projecteuclid.org/euclid.cmp/1103920743.

[78] Changyou Wang. “Biharmonic maps from $\mathbf{R}^4$ into a Riemannian manifold”. In: *Math. Z.* 247.1 (2004), pp. 65–87. ISSN: 0025-5874. DOI: 10.1007/s00209-003-0620-1. URL: https://doi.org/10.1007/s00209-003-0620-1.

[79] Changyou Wang. “Bubble phenomena of certain Palais-Smale sequences from surfaces to general targets”. In: *Houston J. Math.* 22.3 (1996), pp. 559–590. ISSN: 0362-1588.

[80] Changyou Wang. “Remarks on biharmonic maps into spheres”. In: *Calc. Var. Partial Differential Equations* 21.3 (2004), pp. 221–242. ISSN: 0944-2669. DOI: 10.1007/s00526-003-0252-7. URL: https://doi.org/10.1007/s00526-003-0252-7.

[81] Changyou Wang and Shenzhou Zheng. “Energy identity for a class of approximate biharmonic maps into sphere in dimension four”. In: *Discrete Contin. Dyn. Syst.* 33.2 (2013), pp. 861–878. ISSN: 1078-0947. DOI: 10.3934/dcds.2013.33.861. URL: https://doi.org/10.3934/dcds.2013.33.861.

[82] Changyou Wang and Shenzhou Zheng. “Energy identity of approximate biharmonic maps to Riemannian manifolds and its application”. In: *J. Funct. Anal.* 263.4 (2012), pp. 960–987. ISSN: 0022-1236. DOI: 10.1016/j.jfa.2012.05.008. URL: https://doi.org/10.1016/j.jfa.2012.05.008.

[83] William P. Ziemer. *Weakly differentiable functions*. Vol. 120. Graduate Texts in Mathematics. Sobolev spaces and functions of bounded variation. Springer-Verlag, New York, 1989, pp. xvi+308. ISBN: 0-387-97017-7. DOI: 10.1007/978-1-4612-1015-3. URL: https://doi.org/10.1007/978-1-4612-1015-3.