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The moving particles method for reliability estimation: recent advances

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In the following, recent advances of the moving particles method are highlighted. A multi level version of the estimator is introduced that balances the statistical error and the numerical approximation error by computing a telescoping sum of estimates for the number of moves. An extension to general model classes in the sense of a multi fidelity method is obtained based on the estimation of bivariate Poisson distributions for censored data. Finally, the local sensitivity of the reliability estimate with respect to the parameters of the probability characteristics of the model input is considered. It is shown how to obtain estimates of the local sensitivity without any additional function evaluations.

1 Introduction

Structural reliability analysis is concerned with the computation of the failure probability

$$P_F = \int I_{g(x)<0}(x) p(x,\theta) \mathrm{d}x, \tag{1}$$

where the real valued function $g(x)$, $x \in \mathbb{R}^n$, denotes the performance function of the structure, $p(x,\theta)$ is the probability density function of those structural parameters that are assumed to be random variables and θ contains the parameters of the probability distributions. The indicator function $I_{g(x)<0}(x)$ is given by

$$I_{g(x)<0}(x) = \begin{cases} 1, & \text{if } g(x) < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

The limit state function $g(x) = 0$ separates the safe domain $\{x \in \mathbb{R}^n : g(x) > 0\}$ from the failure domain $F = \{x \in \mathbb{R}^n : g(x) < 0\}$. In engineering applications, the failure probability is small and direct Monte Carlo simulation for its estimation requires a huge number of evaluations of the performance function and thus a large computational effort.

The moving particles algorithm [1] yields an estimate of P_F. It starts with an initial Monte Carlo simulation (MCS) with N_m samples. These initial samples are then moved to the failure region by the following procedure: The values $g(\boldsymbol{\theta}_j)$, $j = 1, ..., N$, of the N samples are ranked. The sample with the maximum value of the performance function is moved: a Markov chain Monte Carlo simulation (MCMC) is carried out starting from one of the remaining samples and the final state of the Markov chain is accepted, if the value of the performance function could be reduced. Otherwise, the sample is simply replaced by the initial value of the Markov chain.

Each move of a particle gives rise to a subset $F_i = \{x \in \mathbb{R}^n : g(x) < g_i\}$, $i = 1, ..., M_T$, where $g_i = \max_{1 \le j \le N} g(x_j^{(i)})$, $i = 1, ..., M_T$, denotes the maximum value of the performance function of the N particles $x_j^{(i)}$, $j = 1, ..., N$, when carrying out the ith move. For the conditional probability $P_i = P(G < g_i | G < g_{i-1})$, where for simplicity, the random variable $g(X)$ is abbreviated by G, one finds the estimator $\hat{P}_i = \frac{N-1}{N}$, because only one particle is moved per each iteration step of the algorithm. With $F_0 = \mathbb{R}^n$ the failure probability is obtained as

$$P_F = \prod_{i=1}^{M_T} P(F_i | F_{i-1}) = \prod_{i=1}^{M_T} P_i. \tag{3}$$

The number of moves to get an initial sample into the failure region follows a Poisson distribution with parameter $\lambda = -\log P_F$. The estimator for the parameter of the Poisson distribution is obtained from $\lambda = E[M]$ as

$$\hat{\lambda} = \frac{\sum_{j=1}^{N} M_j}{N} = \frac{M_T}{N}, \tag{4}$$

where M_j, $j = 1..., N$ denotes the number of moves until the initial sample j reaches the failure state.

The coefficient of variation for the failure probability estimated with the moving particles algorithm is given by

$$\delta_{mp} = \sqrt{\frac{-\log P_F}{N}}, \tag{5}$$

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and the average number of function evaluations is

$$N_{mp} = N(1 - T \log P_F), \tag{6}$$

where the first term accounts for the initial Monte Carlo simulation and the second term for the Markov chain samples (with burn-in period T).

2 Multi level extension

For the moving particles algorithm, denote by M_l the number of moves when an approximation $g_l(\boldsymbol{\theta})$ of the performance function is applied. The estimator for the Poisson distribution parameter at level L and thus for $-\log P_{F_L}$, where P_{F_L} denotes the failure probability computed with the approximated performance function $g_L(\boldsymbol{\theta})$ at approximation order L is obtained from

$$E[M_L] = E[M_0] + \sum_{l=1}^{L} E[M_l - M_{l-1}] \tag{7}$$

and reads

$$\hat{M}_L^M = \frac{1}{N_0} \sum_{i=1}^{N_0} M_0^{(i)} + \sum_{l=1}^{L} \frac{1}{N_l} \sum_{i=1}^{N_l} (M_l^{(i)} - M_{l-1}^{(i)}). \tag{8}$$

Making use of the fact that M_l are Poisson distributed random variables, it can be shown that the multi level estimator for the failure probability is more efficient in terms of computational cost than the single level version [2].

Example. Stochastic heat equation with random heat source. Consider the linear stochastic partial differential equation

$$\mathrm{d}u = \frac{\partial^2 u}{\partial x^2}\mathrm{d}t + \theta u \mathrm{d}t + \sigma \mathrm{d}W_t \tag{9}$$

with parameters θ and σ, where $\mathrm{d}W$ are the increments of a standard cylindrical Wiener process

$$W_t = \sum_{k=1}^{\infty} W_t^k \sin(k\pi x), \tag{10}$$

and W_t^k are standard independent Brownian motions.

The boundary conditions are $u(x = 0, t) = u(x = 1, t) = 0$ and the initial condition reads

$$u(x, 0) = \sqrt{2} \sum_{k=1}^{\infty} \sin(k\pi x), \tag{11}$$

so that the solution of the stochastic partial differential equation decouples and can be written as

$$u(x, t) = \sqrt{2} \sum_{k=1}^{\infty} u_k(t) \sin(k\pi x), \tag{12}$$

where the coefficients $u_k(t)$ are described by the linear stochastic ordinary differential equations

$$\mathrm{d}u_k = (-\pi^2 k^2 + \theta) u_k \mathrm{d}t + \sigma \mathrm{d}W_t^k. \tag{13}$$

Their solutions at time t are

$$u_k(t) = \exp((-\pi^2 k^2 + \theta)t) + \xi_t^k, \tag{14}$$

where ξ_t^k is a normal random variable with mean zero and variance

$$\frac{\sigma^2(1 - \exp(2(-\pi^2 k^2 + \theta)t))}{2(\pi^2 k^2 - \theta)}. \tag{15}$$

Failure is assumed to occur if $u(x = 0.5, t = T)$ is larger than a given threshold. In the following, $\theta = 1$, $\sigma = 1$, $T = 1/50$ were set.

The solution of the stochastic partial differential equation (9) is approximated by truncating the solution at $k = k_{max}$. In addition, time integration is applied in order to solve the stochastic ordinary differential equations (13). The approximation parameters for this problem are the truncation parameter k_{max} and the time step.

Figure 1 displays the variance of the single level and the multi level estimator for the number of moves in dependence of a single approximation parameter h, where h is computed by comparing the exact and the approximate failure probability. The threshold value for failure has been set to 1.5. It can be seen that the variance of the multilevel estimator is smaller than that of the single level estimator and that it decreases with decreasing discretization parameter h, while the variance of the single level estimator is nearly independent of h. As the number of moves is a Poisson distributed random variable, the same result holds for the mean values.

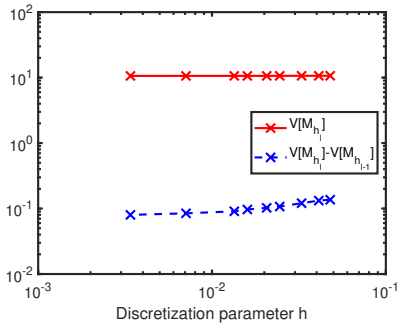
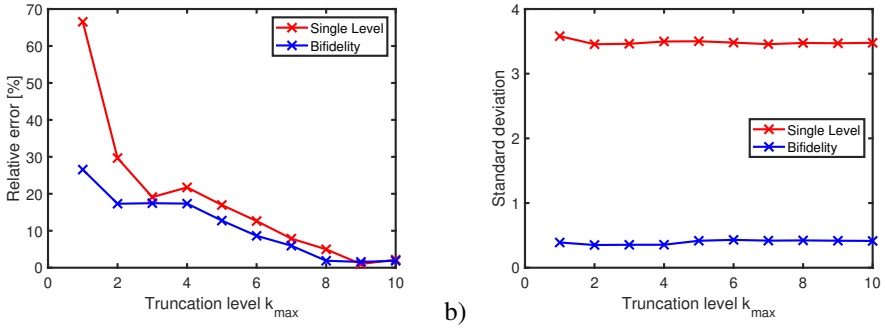


Fig. 1: Variance of M_{h_l} and $M_{h_l} - M_{h_{l-1}}$

3 Multi fidelity extension

Consider without loss of generality the bilevel situation with a high fidelity and a low fidelity model. M_h and M_l denote the random variables for the moves of the particles related to the high-fidelity and the low-fidelity model, respectively. As the low fidelity model is assumed to be computationally more efficient, suppose that samples M_{l_i}, $i = 1, ..., N$, are generated for particles with the low-fidelity models and pairs of samples (M_{l_i}, M_{h_i}), $i = 1, ..., M$ that were obtained with both the low and the high fidelity model , where $M \ll N$. The objective is to estimate the parameters λ_l and λ_h of the two dependent Poisson distributed random variables M_h and M_l. To this end, three independent Poisson distributed random variables are introduced and the parameters of the three Poisson distributions are estimated by maximum likelihood estimation via the expectation maximization algorithm [3].

Example. Consider again the stochastic heat equation with random heat source. In this example only the truncation order k_{max} has been varied. The moving particles method has been applied with 1000 initial samples (particles). For the bifidelity method described above, only ten particles were considered, for which the number of moves to reach the failure region were computed with both the low- and high-fidelity model. Figure 2 summarizes the results for the single level and the bifidelity method. For low truncation order, the multiplicative bifidelity method reduces the relative error compared to the single level method. Moreover, due to the maximum likelihood estimation via expectation maximization, also the standard deviation of the estimator is considerably reduced. This demonstrates the advantages of the bifidelity method over the single level moving particles algorithm.



a) b)

Fig. 2: Relative error and standard deviation for the single level estimator and the bifidelity method with multiplicative information fusion.

4 Extension to local sensitivity analysis

Applying the product rule to equation (3), one finds for the partial derivative of the failure probability with respect to θ:

$$\frac{\partial P_F}{\partial \theta} = P_F \sum_{i=1}^{M_T} \frac{1}{P_i} \frac{\partial P_i}{\partial \theta}, \tag{16}$$

where

$$P_i = P(G < g_i | G < g_{i-1}) = \int I_{g<g_i}(x) \frac{I_{g<g_{i-1}}(x) p(x)}{P(G < g_{i-1})} \mathrm{d}x \tag{17}$$

and

$$P(G < g_{i-1}) = \int I_{g<g_{i-1}}(x) p(x) \mathrm{d}x. \tag{18}$$

The partial derivative of the conditional probability P_i is given by

$$\begin{aligned} \frac{\partial P_i}{\partial \theta} &= \frac{\partial}{\partial \theta} \int I_{g<g_i}(x) \frac{I_{g<g_{i-1}}(x) p(x)}{P(G < g_{i-1})} \mathrm{d}x \\ &= \int I_{g<g_i}(x) \frac{I_{g<g_{i-1}}(x)}{P(G < g_{i-1})} \frac{\partial p(x)}{\partial \theta} \mathrm{d}x - \int I_{g<g_i}(x) \frac{I_{g<g_{i-1}}(x) p(x)}{P(G < g_{i-1})^2} \frac{\partial P(G < g_{i-1})}{\partial \theta} \mathrm{d}x \\ &= \int I_{g<g_i}(x) \frac{1}{P(G < g_{i-1})} \frac{\partial p(x)}{\partial \theta} \mathrm{d}x - \int I_{g<g_i}(x) \frac{p(x)}{P(G < g_{i-1})^2} \frac{\partial P(G < g_{i-1})}{\partial \theta} \mathrm{d}x, \end{aligned} \tag{19}$$

as $g_i < g_{i-1}$ and thus $I_{g<g_i}(x) I_{g<g_{i-1}}(x) = I_{g<g_i}(x)$. The last expression can also be written as

$$\frac{\partial P_i}{\partial \theta} = E_{G<g_{i-1}}[I_{g<g_i}(x) \frac{\partial \ln p(x)}{\partial \theta}] - \frac{P_i}{P(G < g_{i-1})} \frac{\partial P(G < g_{i-1})}{\partial \theta}. \tag{20}$$

The expression in the last term involves the partial derivative of the probability related to a set with larger bound, namely F_{i-1}. Analogously to equation (16), we find the expression

$$\frac{\partial P(G < g_{i-1})}{\partial \theta} = P(G < g_{i-1}) \sum_{j=1}^{i-1} \frac{1}{P_j} \frac{\partial P_j}{\partial \theta} \tag{21}$$

for this partial derivative. Thus, the expression for the partial derivative of the conditional probability P_i with respect to θ reads

$$\frac{\partial P_i}{\partial \theta} = E_{G<g_{i-1}}[I_{g<g_i}(x) \frac{\partial \ln p(x)}{\partial \theta}] - \sum_{j=1}^{i-1} \frac{1}{P_j} \frac{\partial P_j}{\partial \theta}. \tag{22}$$

Inserting this expression into equation (16) and observing that $\hat{P}_1 = \hat{P}_2 = ... = \hat{P}_{M_T}$ yields a telescoping sum, such that the estimator for the sensitivity of the failure probability is obtained as

$$\frac{\partial \hat{P}_F}{\partial \theta} = \left(\frac{N-1}{N} \right)^{M_T - 1} \left(\frac{1}{N} \sum_{j=1}^{N} I_{g<g_{M_T}}(x_j^{(M_T)}) \left. \frac{\partial \ln p(x)}{\partial \theta} \right|_{x=x_j^{(M_T)}} \right). \tag{23}$$

Thus, the score function $\frac{\partial \ln p(x)}{\partial \theta}$ is evaluated and averaged for the remaining $N-1$ particles before the move of the last particle into the failure domain, which does not involve any additional evlauations of the performance function $g(x)$.

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