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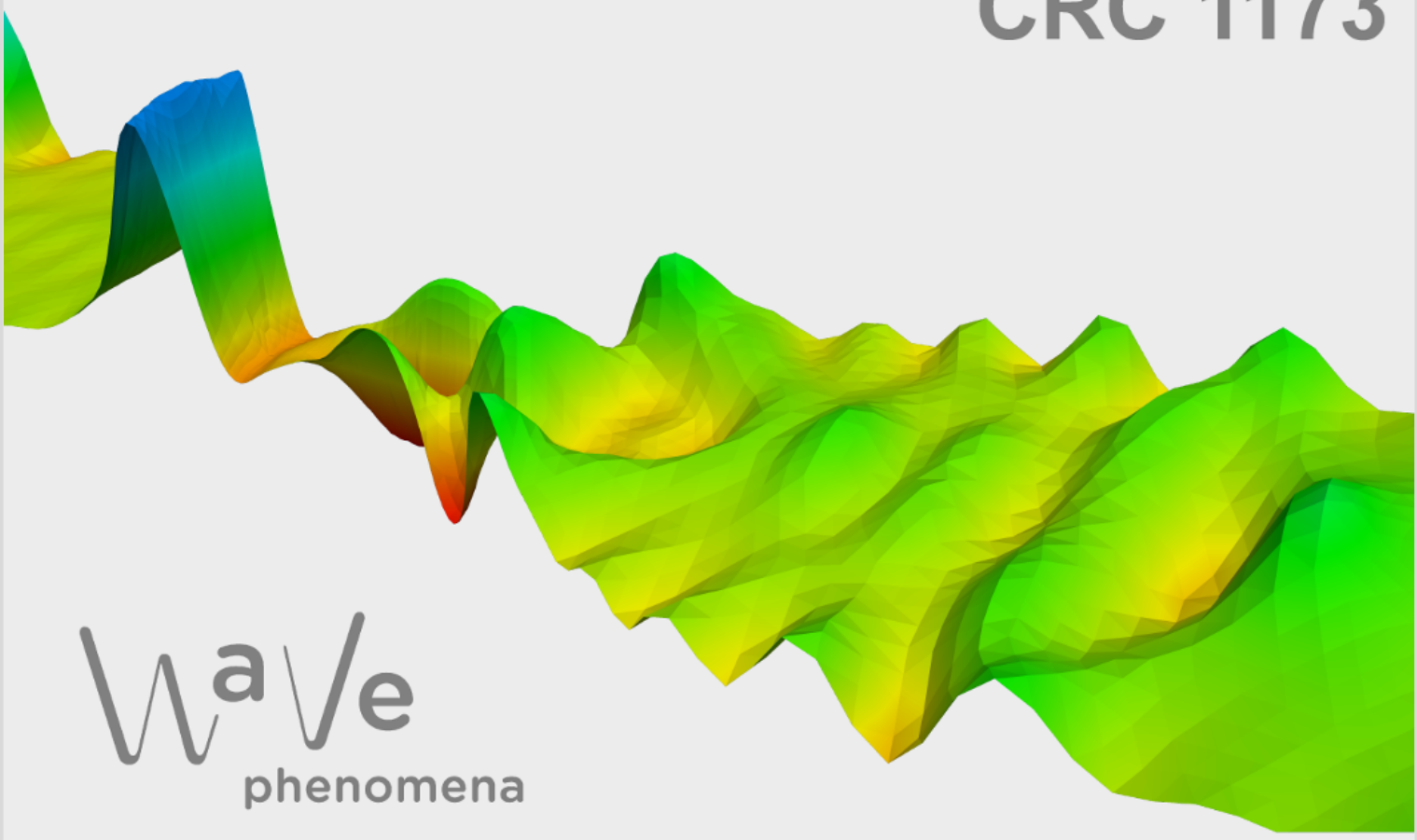

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WELL-POSEDNESS FOR MAXWELL EQUATIONS WITH KERR NONLINEARITY IN THREE DIMENSIONS VIA STRICHARTZ ESTIMATES

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ABSTRACT. We show new local well-posedness results for quasilinear Maxwell equations in three spatial dimensions with an emphasis on the Kerr nonlinearity. For this purpose, new Strichartz estimates are proved for solutions with rough permittivity by conjugation to half-wave equations. We use the Strichartz estimates in a known combination with energy estimates to derive the new well-posedness results.

1. INTRODUCTION

In the following Maxwell equations in three spatial dimensions, the physically most relevant case (cf. [2, 5]), are analyzed. These describe the propagation of *electric and magnetic fields* $(E, B) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$, and *displacement and magnetizing fields* $(D, H) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$. The system of equations is given by

$$\begin{cases} \partial_t D = \nabla \times H - J_e, & \nabla \cdot D = \rho_e, \\ \partial_t B = -\nabla \times E - J_m, & \nabla \cdot B = \rho_m, \\ D(0,\cdot) = D_0, & B(0,\cdot) = B_0. \end{cases} \tag{1}$$

$(\rho_e, \rho_m) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}$ denote *electric and magnetic charges* and $(J_e, J_m) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ *electric and magnetic currents*. There is no physical evidence for the existence of magnetic charges or magnetic currents, but we include them to highlight a key aspect of the analysis.

The notations follow the previous work [11] on Maxwell equations in two spatial dimensions. We denote space-time coordinates $x = (x^0, x^1, \dots, x^n) = (t, x') \in \mathbb{R} \times \mathbb{R}^n$ and the dual variables in Fourier space by $\xi = (\xi^0, \xi^1, \dots, \xi^n) = (\tau, \xi') \in \mathbb{R} \times \mathbb{R}^n$. In this work we supplement Maxwell equations with time-instantaneous material laws, linking E with D and H with B:

$$\begin{aligned} D(x) &= \varepsilon(x) E(x), \quad \varepsilon : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}, \\ B(x) &= \mu(x) H(x), \quad \mu : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}. \end{aligned} \tag{2}$$

ε is referred to as *permittivity*, and μ is referred to as *permeability*. In the following we consider $\mu \equiv 1$. This means that the considered material is magnetically isotropic, which is a common assumption in nonlinear optics (cf. [9]). In fact, in the constant coefficient case this is no additional assumption (cf. [8]). μ below denotes a regularity parameter unrelated with the permeability. As in the preceding work

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[11], we want to describe the propagation in possibly anisotropic and inhomogeneous media. We suppose that ε is a matrix-valued function $\varepsilon : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ with $\lambda, \Lambda > 0$ such that for any $\xi' \in \mathbb{R}^3$ and $x \in \mathbb{R} \times \mathbb{R}^3$

$$\lambda |\xi'|^2 \leq \sum_{i,j=1}^{3} \varepsilon^{ij}(x) \xi_i' \xi_j' \leq \Lambda |\xi'|^2, \quad \varepsilon^{ij}(x) = \varepsilon^{ji}(x). \tag{3}$$

Sum convention is in use, e.g.,

$$\varepsilon^{ij}(x) \xi_i' \xi_j' = \sum_{i,j=1}^{3} \varepsilon^{ij}(x) \xi_i' \xi_j'.$$

Here we focus on $\varepsilon = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ because this covers the physically relevant case

$$\varepsilon(E) = (1 + |E|^2) 1_{3\times 3} \tag{4}$$

of the Kerr nonlinearity.

We denote[1]

$$\mathcal{B}(\partial) = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}, \quad P(x, \partial) = \begin{pmatrix} \partial_t 1_{3\times 3} & -\mathcal{B} \\ \mathcal{B}\varepsilon^{-1} & \partial_t 1_{3\times 3} \end{pmatrix}. \tag{5}$$

(1) becomes

$$P(x, \partial) \begin{pmatrix} D \\ H \end{pmatrix} = - \begin{pmatrix} J_e \\ J_m \end{pmatrix}, \quad \begin{cases} \nabla \cdot D = \rho_e, \\ \nabla \cdot B = \rho_m. \end{cases} \tag{6}$$

As in [11], we make use of the FBI transform and analyze the equation in phase space. $P(x, \partial)$ is conjugated to half-wave equations whose dispersive properties depend on the number of different eigenvalues of ε. This was previously analyzed in the constant-coefficient case by Liess [6] and Lucente–Ziliotti [7]; see also [10, 8]. It was proved that for $\varepsilon(x) \equiv \varepsilon$ satisfying (3) solutions to (6) with ε having less than three different eigenvalues decay like solutions to the three-dimensional wave equation. However, if ε has three different eigenvalues, the decay is weakened to the decay of the two-dimensional wave equation. We prove the first result for variable, rough, possibly anisotropic coefficients; see Dumas–Sueur [1] for smooth scalar coefficients. Below let

$$(|D|^\alpha u)\widehat{\ }(\xi) = |\xi|^\alpha \hat{u}(\xi), \quad (|D'|^\alpha u)\widehat{\ }(\xi) = |\xi'|^\alpha \hat{u}(\xi),$$

and (ρ, p, q, d) is referred to as Strichartz pair if $d \in \mathbb{Z}_{\geq 2}$, $\rho = d\big(\frac{1}{2} - \frac{1}{q}\big) - \frac{1}{p}$, $p, q \geq 2$, $\frac{2}{p} + \frac{d-1}{q} \leq \frac{d-1}{2}$, and $(p, q, d) \neq (2, \infty, 3)$.

Theorem 1.1. *Let $\varepsilon = diag(\varepsilon_1, \varepsilon_2, \varepsilon_3) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ be a matrix-valued function satisfying (3). Let $u = (D, H) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ with $\nabla \cdot D = \rho_e$ and $\nabla \cdot H = \rho_m$, and P as in (5).*
If ε has no more than two different eigenvalues for any $x \in \mathbb{R} \times \mathbb{R}^3$, and $\|\partial^2 \varepsilon_i\|_{L^\infty} \leq \mu^4$, then we find the following estimate to hold:

$$\||D|^{-\rho} u\|_{L^p L^q} \lesssim \mu \|u\|_{L^2} + \mu^{-1} \|Pu\|_{L^2} + \||D|^{-\frac{1}{2}} \rho_e\|_{L^2} + \||D|^{-\frac{1}{2}} \rho_m\|_{L^2} \tag{7}$$

provided that the right-hand side is finite and $(\rho, p, q, 3)$ is a Strichartz pair.

[1] In this work we denote differential operators by ∂ to avoid confusion with the displacement field, deviating from the usual notation for pseudo-differential operators.

The theorem states that in case of small charges the dispersive properties of wave equations are recovered. As in the two-dimensional case, note that on the one hand, if

$$\|\rho_e\|_{\dot{H}^{-\frac{1}{2}}} \sim \|D\|_{\dot{H}^{\frac{1}{2}}}, \quad \|\rho_m\|_{\dot{H}^{-\frac{1}{2}}} \sim \|B\|_{\dot{H}^{\frac{1}{2}}}, \tag{8}$$

(7) follows from Sobolev embedding. Moreover, if one omits the contribution of charges on the right-hand side in (7), we can find stationary solutions $D = \nabla\varphi$ and $H = 0$ for $\varepsilon = 1_{3\times 3}$, which clearly violate the Strichartz estimates.

Corresponding Strichartz estimates with additional derivative loss under weaker regularity assumptions on ε follow (cf. [17, 11]). In the following, for $\lambda \in 2^{\mathbb{Z}}$ we denote Littlewood-Paley projections by

$$(S_\lambda f)\widehat{(}\xi) = \beta(\lambda^{-1}\xi)\hat{f}(\xi), \quad (S'_\lambda f)\widehat{(}\xi) = \beta'(\lambda^{-1}\xi)\hat{f}(\xi),$$

where

$$\operatorname{supp}(\beta) \subseteq \{\xi \in \mathbb{R}^{n+1} \,:\, |\xi| \sim 1\}, \sum_{\lambda\in 2^{\mathbb{N}_0}} \beta(\lambda^{-1}\xi) \equiv 1 \text{ for } |\xi| \geq 1,$$

$$\operatorname{supp}(\beta') \subseteq \{\xi \in \mathbb{R}^{n+1} \,:\, |\xi| \sim 1,\, |\xi_0| \lesssim |\xi'|\}, \sum_{\lambda\in 2^{\mathbb{N}_0}} \beta'(\lambda^{-1}\xi) \equiv 1 \text{ for } |\xi| \geq 1,\, |\xi_0| \lesssim |\xi'|.$$

We have the following for C^s-coefficients:

Theorem 1.2. *Let $\varepsilon : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ be a matrix-valued function with coefficients in C^s, $0 < s < 2$, satisfying (3). Let $u = (D, H) : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3\times\mathbb{R}^3$ with $\nabla\cdot D = \rho_e$ and $\nabla\cdot B = \rho_m$. Then, we obtain the estimate*

$$\||D|^{-\rho-\frac{\sigma}{2}}u\|_{L^pL^q} \lesssim \mu\|u\|_{L^2} + \mu^{-1}\|Pu\|_{\dot{H}^{-\sigma}} + \||D|^{-\frac{1}{2}-\frac{\sigma}{2}}\rho_e\|_{L^2} + \||D|^{-\frac{1}{2}-\frac{\sigma}{2}}\rho_m\|_{L^2} \tag{9}$$

provided that the right hand-side is finite, $(\rho, p, q, 3)$ is a Strichartz pair,

$$\sigma = \frac{2-s}{2+s}, \text{ and } \|\varepsilon^{ij}\|_{\dot{C}^s} \leq \mu^4.$$

Moreover, by the arguments from [11], Strichartz estimates for coefficients $\partial^2\varepsilon \in L^1L^\infty$ (cf. [11, Theorem 1.3]) and also the inhomogeneous equation (cf. [11, Theorem 1.5]) are derived. We have the following theorem, which is important to treat the quasilinear equation.

Theorem 1.3. *Let $\varepsilon : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ be a matrix-valued function with Lipschitz coefficients, satisfying (3) and $\partial_x^2\varepsilon \in L^1L^\infty$. Let u, ρ_e, ρ_m be as in Theorem 1.3, and $(\rho, p, q, 3)$ be a Strichartz pair. Then,*

$$\begin{aligned}\||D'|^{-\rho}u\|_{L^p(0,T;L^q)} &\lesssim \mu^{\frac{1}{p}}\|u\|_{L^\infty L^2} + \mu^{-\frac{1}{p'}}\|P(x,\partial)u\|_{L^1L^2} \\ &\quad + T^{\frac{1}{2}}\||D'|^{-\frac{1}{2}}\rho_e(0)\|_{L^2(\mathbb{R}^3)} + T^{\frac{1}{2}}\||D'|^{-\frac{1}{2}}\partial_t\rho_e\|_{L^1L^2} \\ &\quad + T^{\frac{1}{2}}\||D'|^{-\frac{1}{2}}\rho_m(0)\|_{L^2(\mathbb{R}^3)} + T^{\frac{1}{2}}\||D'|^{-\frac{1}{2}}\partial_t\rho_m\|_{L^1L^2},\end{aligned} \tag{10}$$

whenever the right hand-side is finite, provided that $\mu \geq 1$, and $T\|\partial_x^2\varepsilon\|_{L^1L^\infty} \leq \mu^2$.

The reason for additional terms $\||D'|^{-\frac{1}{2}}\partial_t\rho\|_{L^1L^2}$ showing up, compared to Theorem 1.1, is that we use Duhamel's formula in the reductions. For applying the estimates to solve quasilinear equations, $L^\infty L^2$- and L^1L^2-norms are needful. We

further have to reduce the regularity of ε to control $\|\partial \varepsilon\|_{L^p L^\infty}$ for energy estimates. We denote homogeneous Besov spaces by $\dot{B}^{pqr}_s$ with norm

$$\|u\|^r_{\dot{B}^{pqr}_s} = \sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{rs} \|S_\lambda u\|^r_{L^p L^q}$$

with the obvious modification for $r = \infty$. For the coefficients of ε, we use the microlocalizable scale of space (cf. [18, 11, 19]):

$$\|v\|_{\mathcal{X}^s} = \sup_{\lambda \in 2^{\mathbb{Z}}} \lambda^s \|S_\lambda v\|_{L^1 L^\infty}.$$

Theorem 1.4. *Let $\varepsilon \in \mathcal{X}^s$, $0 < s < 2$, and $u = (D, H)$, $(\rho, p, q, 3)$, and σ as in the assumptions of Theorem 1.2. Then, we find the following estimate to hold:*

$$\begin{aligned} \||D|^{-\rho - \frac{\sigma}{p}} u\|_{\dot{B}^{pq\infty}_0} &\lesssim \mu^{\frac{1}{p}} \|u\|_{L^\infty L^2} + \mu^{-\frac{1}{p'}} \||D|^{-\sigma} P u\|_{L^1 L^2} \\ &+ T^{\frac{1}{2}} \||D|^{-\frac{1}{2} - \frac{\sigma}{p}} \rho_e\|_{L^\infty L^2} + T^{\frac{1}{2}} \||D|^{-\frac{1}{2} - \frac{\sigma}{p}} \partial_t \rho_e\|_{L^1 L^2} \\ &+ T^{\frac{1}{2}} \||D|^{-\frac{1}{2} - \frac{\sigma}{p}} \rho_m\|_{L^\infty L^2} + T^{\frac{1}{2}} \||D|^{-\frac{1}{2} - \frac{\sigma}{p}} \partial_t \rho_m\|_{L^1 L^2} \end{aligned} \tag{11}$$

for all u compactly supported in $[0, T]$, and μ, T verifying

$$T^s \|\varepsilon\|^2_{\mathcal{X}^s} \lesssim \mu^{2+s}.$$

Inhomogeneous Strichartz estimates can be derived by similar means as in [11], which is omitted here. We record the following corollary, which becomes useful when we treat quasilinear equations. The corollary is proved following along the lines of [11, Corollary 1.7].

Corollary 1.5. *Assume that $\|\partial_x \varepsilon\|_{L^2 L^\infty} \lesssim 1$ and for some $\tilde{s} \in [1, 2)$, suppose that $\|\varepsilon\|_{\mathcal{X}^s} \lesssim 1$. Let $(\rho, p, q, 3)$ be a Strichartz pair, and $P(x, \partial)$ as in (6). Then the solution $u = (D, H)$ to*

$$\begin{cases} P(x, \partial) u = f, & \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = \rho_e, \\ u(0) = u_0, & \partial_1 u_4 + \partial_2 u_5 + \partial_6 u_6 = \rho_m \end{cases} \tag{12}$$

satisfies

$$\begin{aligned} \|\langle D' \rangle^{-\alpha} u\|_{L^p(0,T;L^q)} &\lesssim_T \|u_0\|_{L^2} + \|f\|_{L^1(0,T;L^2)} \\ &+ \|\langle D' \rangle^{-\frac{1}{2} - \frac{\sigma}{p}} \rho_e(0)\|_{L^2} + \|\langle D' \rangle^{-\frac{1}{2} - \frac{\sigma}{p}} \partial_t \rho_e\|_{L^1 L^2} \\ &+ \|\langle D' \rangle^{-\frac{1}{2} - \frac{\sigma}{p}} \rho_m(0)\|_{L^2} + \|\langle D' \rangle^{-\frac{1}{2} - \frac{\sigma}{p}} \partial_t \rho_m\|_{L^1 L^2} \end{aligned} \tag{13}$$

for $\alpha > \rho + \frac{\sigma}{p}$ and $\sigma = \sigma(\tilde{s}) = \frac{2 - \tilde{s}}{2 + \tilde{s}}$.

As in [11], after conjugation of $P(x, \partial)$ the key ingredient in the proof of Strichartz estimates are estimates for the half-wave equations. We use the following result, shown in [11]:

Proposition 1.6 ([11, Proposition 1.8])**.** *Let $\lambda \in 2^{\mathbb{N}_0}$, $\lambda \gg 1$, and $d \geq 2$. Assume $\varepsilon = \varepsilon^{ij}(x)$ satisfies $\varepsilon^{ij} \in C^2$, $\|\partial_x^2 \varepsilon\|_{L^\infty} \leq 1$, and (3). Let $Q(x, \partial)$ denote the pseudo-differential operator with symbol*

$$Q(x, \xi) = -\xi_0 + \big(\varepsilon^{ij}_{\lambda^{\frac{1}{2}}}(x) \xi_i \xi_j\big)^{1/2}.$$

Moreever, let $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ decay rapidly outside the unit cube and (ρ, p, q, d) be a Strichartz pair. Then, we find the estimates

$$\lambda^{-\rho} \|S_\lambda u\|_{L^p L^q} \lesssim \|S_\lambda u\|_{L^2} + \|Q(x, \partial) S_\lambda u\|_{L^2} \tag{14}$$

to hold with an implicit constant uniform in λ. *For Lipschitz coefficients* ε^{ij} *with* $\|\partial_x^2 \varepsilon\|_{L^1 L^\infty} \leq 1$, *we obtain*

$$\lambda^{-\rho} \|S_\lambda u\|_{L^p L^q} \lesssim \|S_\lambda u\|_{L^\infty L^2} + \|Q(x,\partial) S_\lambda u\|_{L^2}. \tag{15}$$

The Strichartz estimates yield an improvement of the local well-posedness theory for Maxwell equations

$$\begin{cases} P(x,\partial)(D,H) &= 0, \quad \nabla \cdot D = \nabla \cdot H = 0, \\ (D,H)(0) &\in H^s(\mathbb{R}^3;\mathbb{R})^6, \end{cases} \tag{16}$$

where $\varepsilon^{-1}(D) = \psi(|D|^2) 1_{3\times 3}$, and $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a smooth monotone increasing function with $\psi(0) = 1$. This covers the Kerr nonlinearity $\varepsilon = (1 + |E|^2) 1_{3\times 3}$. The energy method (cf. [3]) yields local well-posedness for $s > 5/2$. We also refer to Spitz's works [13, 14], where Maxwell equations with Kerr nonlinearity were proved to be locally well-posed in $H^3(\Omega)$ on domains. By the same means as in [11], we obtain the following improvement over energy arguments via Strichartz estimates:

Theorem 1.7. (16) *is locally well-posed for* $s > 13/6$.

In the two-dimensional case we have shown that the derivative loss for Strichartz estimates with rough coefficients is sharp (cf. [11, Section 7]). In the three dimensional case we do not have an example showing sharpness. However, the fact that the derivative loss matches the loss for second order hyperbolic operators indicates sharpness of the Strichartz estimates in the present work. We contend that $s > 13/6$ is the limit of showing local well-posedness for (16) via Strichartz estimates. Nonetheless, an improvement of local well-posedness might still be possible by adapting the arguments for the proof of sharp well-posedness for quasilinear wave equations as in [12, 4].

Outline of the paper. The strategy of the proofs follows [11] closely. In Section 2, we point out how standard localization arguments reduce Theorem 1.1 to a dyadic estimate with frequency truncated coefficients. Then, the symbol is symmetrized to two degenerate and four non-degenerate half wave equations. We see that the divergence conditions ameliorate the contribution of the degenerate components as in the two-dimensional case. The estimates for the non-degenerate half-wave equations for ε having less than three eigenvalues are provided by Proposition 1.6. Estimates for less regular coefficients (Theorem 1.2, Theorem 1.4, and Corollary 1.5) and inhomogeneous estimates follow as in the two-dimensional case. Hence, the proofs are omitted. In Section 3 we prove Theorem 1.7.

2. Reduction to half-wave equations

The purpose of this section is to reduce (1) to half-wave equations. The Strichartz estimates then follow from Proposition 1.6. The key point is to diagonalize the principal symbol of

$$P(x,\partial) = \begin{pmatrix} \partial_t 1_{3\times 3} & -\mathcal{B}(\partial) \\ \mathcal{B}(\partial)\varepsilon^{-1} & \partial_t 1_{3\times 3} \end{pmatrix}.$$

The diagonalization argument follows the two-dimensional case, but is more involved. The eigenpairs had been computed in case of constant coefficients in [10]. This suffices on the level of the symbols. Further reductions are standard, i.e., localization to a cube of size 1, reduction to dyadic estimates, truncating frequencies of the coefficients, and we shall be brief. We start with diagonalizing the principal symbol:

2.1. **Diagonalizing the principal symbol.** Corresponding to (5), let

$$\mathcal{B}(\xi) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}. \tag{17}$$

We find

$$\tilde{p}(x,\xi) = i \begin{pmatrix} \xi_0 1_{3\times 3} & -\mathcal{B}(\xi) \\ \mathcal{B}(\xi)\varepsilon^{-1}(x) & \xi_0 1_{3\times 3} \end{pmatrix}. \tag{18}$$

We suppose that $\varepsilon^{-1} = \mathrm{diag}(a,b,b)$. Let

$$\|\xi\|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2, \quad \|\xi\|_\varepsilon^2 = b(x)\xi_1^2 + a(x)\xi_2^2 + a(x)\xi_3^2,$$
$$\xi_i' = \xi_i/\|\xi\|, \qquad \tilde{\xi}_i = \xi_i/\|\xi\|_\varepsilon, \qquad i = 1,2,3.$$

The eigenvalues of $\tilde{p}(x,\xi)$ are

$$\lambda_{1,2} = i\xi_0, \quad \lambda_{3,4} = i\xi_0 \mp i\sqrt{b(x)}\|\xi\|, \quad \lambda_{5,6} = i\xi_0 \mp i\|\xi\|_\varepsilon.$$

Let

$$d(x,\xi) = i\mathrm{diag}(\xi_0, \xi_0, \xi_0 - \sqrt{b(x)}\|\xi\|, \xi_0 + \sqrt{b(x)}\|\xi\|, \xi_0 - \|\xi\|_\varepsilon, \xi_0 + \|\xi\|_\varepsilon).$$

We find the following corresponding eigenvectors, which are normalized to zero-homogeneous entries. Eigenvectors to $i\xi_0$ are

$$v_1^t = \big(0,0,0,\xi_1',\xi_2',\xi_3'\big),$$
$$v_2^t = \big(\frac{\tilde{\xi}_1}{a}, \frac{\tilde{\xi}_2}{b}, \frac{\tilde{\xi}_3}{b}, 0,0,0\big).$$

Eigenvectors to $i\xi_0 \pm i\sqrt{b(x)}\|\xi\|$ are given by

$$v_3^t = \big(0, -\frac{\xi_3'}{\sqrt{b}}, \frac{\xi_2'}{\sqrt{b}}, -(\xi_2'^2 + \xi_3'^2), \xi_1'\xi_2', \xi_1'\xi_3'\big),$$
$$v_4^t = \big(0, \frac{\xi_3'}{\sqrt{b}}, -\frac{\xi_2'}{\sqrt{b}}, -(\xi_2'^2 + \xi_3'^2), \xi_1'\xi_2', \xi_1'\xi_3'\big).$$

Eigenvectors to $i\xi_0 \pm i\|\xi\|_\varepsilon$ are given by

$$v_5^t = \big(\tilde{\xi}_2^2 + \tilde{\xi}_3^2, -\tilde{\xi}_1\tilde{\xi}_2, -\tilde{\xi}_1\tilde{\xi}_3, 0, -\tilde{\xi}_3, \tilde{\xi}_2\big),$$
$$v_6^t = \big(-(\tilde{\xi}_2^2 + \tilde{\xi}_3^2), \tilde{\xi}_1\tilde{\xi}_2, \tilde{\xi}_1\tilde{\xi}_3, 0, -\tilde{\xi}_3, \tilde{\xi}_2\big).$$

Set

$$m(x,\xi) = (v_1, \dots, v_6).$$

We find

$$m^{-1}(x,\xi) = \begin{pmatrix} 0 & 0 & 0 & \xi_1' & \xi_2' & \xi_3' \\ ab\tilde{\xi}_1 & ab\tilde{\xi}_2 & ab\tilde{\xi}_3 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{b}\|\xi\|}{2\|\xi\|_\varepsilon}\frac{\tilde{\xi}_3}{\tilde{\xi}_2^2+\tilde{\xi}_3^2} & \frac{\sqrt{b}\|\xi\|}{2\|\xi\|_\varepsilon}\frac{\tilde{\xi}_2}{\tilde{\xi}_2^2+\tilde{\xi}_3^2} & -\frac{1}{2} & \frac{\xi_1'\xi_2'}{2(\xi_2'^2+\xi_3'^2)} & \frac{\xi_1'\xi_3'}{2(\xi_2'^2+\xi_3'^2)} \\ 0 & \frac{\sqrt{b}\|\xi\|}{2\|\xi\|_\varepsilon}\frac{\tilde{\xi}_3}{\tilde{\xi}_2^2+\tilde{\xi}_3^2} & -\frac{\sqrt{b}\|\xi\|}{2\|\xi\|_\varepsilon}\frac{\tilde{\xi}_2}{\tilde{\xi}_2^2+\tilde{\xi}_3^2} & -\frac{1}{2} & \frac{\xi_1'\xi_2'}{2(\xi_2'^2+\xi_3'^2)} & \frac{\xi_1'\xi_3'}{2(\xi_2'^2+\xi_3'^2)} \\ \frac{a}{2} & -\frac{b\tilde{\xi}_1\tilde{\xi}_2}{2(\tilde{\xi}_2^2+\tilde{\xi}_3^2)} & -\frac{b\tilde{\xi}_1\tilde{\xi}_3}{2(\tilde{\xi}_2^2+\tilde{\xi}_3^2)} & 0 & -\frac{\xi_3'\|\xi\|_\varepsilon}{2\|\xi\|(\xi_2'^2+\xi_3'^2)} & \frac{\|\xi\|_\varepsilon\xi_2'}{2\|\xi\|(\xi_2'^2+\xi_3'^2)} \\ -\frac{a}{2} & \frac{b\xi_1\tilde{\xi}_2}{2(\tilde{\xi}_2^2+\tilde{\xi}_3^2)} & \frac{b\xi_1\tilde{\xi}_3}{2(\tilde{\xi}_2^2+\tilde{\xi}_3^2)} & 0 & -\frac{\|\xi\|_\varepsilon}{2\|\xi\|}\frac{\xi_3'}{(\xi_2'^2+\xi_3'^2)} & \frac{\|\xi\|_\varepsilon\xi_2'}{2\|\xi\|(\xi_2'^2+\xi_3'^2)} \end{pmatrix}.$$

In the constant-coefficient case, Lucente–Ziliotti [7] used a similar argument, but did not give the eigenvectors. It turns out that these have to be normalized carefully

to find uniformly L^p-bounded conjugation operators. More precisely, note that the matrix becomes singular for $|\xi_2|+|\xi_3|\to 0$. The remedy is to renormalize $v_3,\dots,v_6$ with

$$\alpha(x,\xi)=\frac{(\xi_2^2+\xi_3^2)^{\frac12}}{(\|\xi\|\|\xi\|_\varepsilon)^{\frac12}}. \tag{19}$$

In fact, we find by elementary matrix operations, that is adding and subtracting the third and fourth, and fifth and sixth eigenvector, that

$$|\det m(x,\xi)|\sim_\varepsilon \begin{vmatrix} 0&0&0&1&0&0\\ 0&0&0&0&\xi_2'&\xi_3'\\ 0&0&0&0&-\tilde\xi_3&\tilde\xi_2\\ 1&0&0&0&0&0\\ 0&\xi_3'&-\xi_2'&0&0&0\\ 0&\tilde\xi_2&\tilde\xi_3&0&0&0 \end{vmatrix} \sim (\xi_2'\tilde\xi_2+\xi_3'\tilde\xi_3)^2=\alpha^4(x,\xi).$$

This suggests renormalizing the eigenvectors from above with (19), as for the associated eigenvectors of $v_3/\alpha(x,\xi),\dots,v_6/\alpha(x,\xi)$ we can verify L^pL^q-boundedness. We give the details. Let $\delta=\|\xi\|/\|\xi\|_\varepsilon$. Note that

$$\alpha(x,\xi)=\frac{(\xi_2^2+\xi_3^2)^{\frac12}}{(\|\xi\|\|\xi\|_\varepsilon)^{\frac12}}=\frac{(\tilde\xi_2^2+\tilde\xi_3^2)^{\frac12}}{\delta^{\frac12}}=(\delta(\xi_2'^2+\xi_3'^2))^{\frac12}.$$

We find

$$\tilde m(x,\xi)= \tag{20}$$

$$\begin{pmatrix} 0 & \frac{\tilde\xi_1}{a} & 0 & 0 & (\delta(\tilde\xi_2^2+\tilde\xi_3^2))^{\frac12} & -(\delta(\tilde\xi_2^2+\tilde\xi_3^2))^{\frac12}\\ 0 & \frac{\tilde\xi_2}{b} & -\frac{\xi_3'}{\sqrt{b}(\delta(\xi_2'^2+\xi_3'^2))^{\frac12}} & \frac{\xi_3'}{\sqrt{b}(\delta(\xi_2'^2+\xi_3'^2))^{\frac12}} & -\frac{\delta^{\frac12}\tilde\xi_1\tilde\xi_2}{(\tilde\xi_2^2+\tilde\xi_3^2)^{1/2}} & \frac{\delta^{\frac12}\tilde\xi_1\tilde\xi_2}{(\tilde\xi_2^2+\tilde\xi_3^2)^{\frac12}}\\ 0 & \frac{\tilde\xi_3}{b} & \frac{\xi_2'}{\sqrt{b}(\delta(\xi_2'^2+\xi_3'^2))^{\frac12}} & -\frac{\xi_2'}{\sqrt{b}(\delta(\xi_2'^2+\xi_3'^2))^{\frac12}} & -\frac{\delta^{\frac12}\tilde\xi_1\tilde\xi_3}{(\tilde\xi_2^2+\tilde\xi_3^2)^{\frac12}} & \frac{\delta^{\frac12}\tilde\xi_1\tilde\xi_3}{(\tilde\xi_2^2+\tilde\xi_3^2)^{\frac12}}\\ \xi_1' & 0 & -\frac{(\xi_2'^2+\xi_3'^2)^{\frac12}}{\delta^{\frac12}} & -\frac{(\xi_2'^2+\xi_3'^2)^{\frac12}}{\delta^{\frac12}} & 0 & 0\\ \xi_2' & 0 & \frac{\xi_1'\xi_2'}{(\delta(\xi_2'^2+\xi_3'^2))^{\frac12}} & \frac{\xi_1'\xi_2'}{(\delta(\xi_2'^2+\xi_3'^2))^{\frac12}} & -\frac{\delta^{\frac12}\tilde\xi_3}{(\tilde\xi_2^2+\tilde\xi_3^2)^{\frac12}} & -\frac{\delta^{\frac12}\tilde\xi_3}{(\tilde\xi_2^2+\tilde\xi_3^2)^{\frac12}}\\ \xi_3' & 0 & \frac{\xi_1'\xi_3'}{(\delta(\xi_2'^2+\xi_3'^2))^{\frac12}} & \frac{\xi_1'\xi_3'}{(\delta(\xi_2'^2+\xi_3'^2))^{\frac12}} & \frac{\tilde\xi_2\delta^{\frac12}}{(\tilde\xi_2^2+\tilde\xi_3^2)^{\frac12}} & \frac{\tilde\xi_2\delta^{\frac12}}{(\tilde\xi_2^2+\tilde\xi_3^2)^{\frac12}} \end{pmatrix}.$$

By Cramer's rule, we find $\tilde m(x,\xi)^{-1}$ from $m^{-1}(x,\xi)$ by modifying the rows 3-6:

$$\tilde m^{-1}(x,\xi)=$$

$$\begin{pmatrix} 0 & 0 & 0 & \xi_1' & \xi_2' & \xi_3'\\ ab\tilde\xi_1 & ab\tilde\xi_2 & ab\tilde\xi_3 & 0 & 0 & 0\\ 0 & -\frac{\sqrt{b}\delta^{\frac12}\tilde\xi_3}{2(\tilde\xi_2^2+\tilde\xi_3^2)^{\frac12}} & \frac{\sqrt{b}\delta^{\frac12}\tilde\xi_2}{2(\tilde\xi_2^2+\tilde\xi_3^2)^{\frac12}} & -\frac{(\tilde\xi_2^2+\tilde\xi_3^2)^{\frac12}}{2\delta^{\frac12}} & \frac{\xi_1'\xi_2'\delta^{\frac12}}{2(\xi_2'^2+\xi_3'^2)^{\frac12}} & \frac{\xi_1'\xi_3'\delta^{\frac12}}{2(\xi_2'^2+\xi_3'^2)^{\frac12}}\\ 0 & \frac{\sqrt{b}\delta^{\frac12}\tilde\xi_3}{2(\tilde\xi_2^2+\tilde\xi_3^2)^{\frac12}} & -\frac{\sqrt{b}\delta^{\frac12}\tilde\xi_2}{2(\tilde\xi_2^2+\tilde\xi_3^2)^{1/2}} & -\frac{(\tilde\xi_2^2+\tilde\xi_3^2)^{\frac12}}{2\delta^{\frac12}} & \frac{\delta^{\frac12}\xi_1'\xi_2'}{2(\xi_2'^2+\xi_3'^2)^{\frac12}} & \frac{\delta^{\frac12}\xi_1'\xi_3'}{2(\xi_2'^2+\xi_3'^2)^{\frac12}}\\ \frac{a(\tilde\xi_2+\tilde\xi_3^2)^{\frac12}}{2\delta^{\frac12}} & -\frac{b\tilde\xi_1\tilde\xi_2}{2(\delta(\tilde\xi_2^2+\tilde\xi_3^2))^{\frac12}} & -\frac{b\tilde\xi_1\tilde\xi_3}{2(\delta(\tilde\xi_2^2+\tilde\xi_3^2))^{\frac12}} & 0 & -\frac{\xi_3'}{2(\delta(\xi_2'^2+\xi_3'^2))^{\frac12}} & \frac{\xi_2'}{2(\delta(\xi_2'^2+\xi_3'^2))^{\frac12}}\\ -\frac{a(\tilde\xi_2+\tilde\xi_3^2)^{\frac12}}{2\delta^{\frac12}} & \frac{b\tilde\xi_1\tilde\xi_2}{2(\delta(\tilde\xi_2^2+\tilde\xi_3^2))^{\frac12}} & \frac{b\tilde\xi_1\tilde\xi_3}{2(\delta(\tilde\xi_2^2+\tilde\xi_3^2))^{\frac12}} & 0 & -\frac{\xi_3'}{2(\delta(\xi_2'^2+\xi_3'^2))^{\frac12}} & \frac{\xi_2'}{2(\delta(\xi_2'^2+\xi_3'^2))^{\frac12}} \end{pmatrix}.$$

Conclusively, we find

$$\tilde p(x,\xi)=\tilde m(x,\xi)d(x,\xi)\tilde m^{-1}(x,\xi).$$

Next, we associate pseudo-differential operators with the symbols. A little care is required when having symbols $(\xi_2^2+\xi_3^2)^{\frac{1}{2}}$ in the denominator. These must not be separated from ∂_2 or ∂_3 to recover bounded operators in L^2.

Let ∂_{ij}^2 denote the second partial derivative with respect to coordinates i,j and

$$D_{ij} = Op((\xi_i^2+\xi_j^2)^{\frac{1}{2}}), \quad D = Op(\|\xi'\|), \quad D_\varepsilon = Op((\xi_i\xi_j\varepsilon^{ij})^{\frac{1}{2}}).$$

We give the expressions for $\mathcal{M}$:

$$\begin{aligned}
&\mathcal{M}_{11} = 0, \quad \mathcal{M}_{12} = \frac{-i}{D_\varepsilon}\partial_1(a^{-1}\cdot), \quad \mathcal{M}_{13} = 0,\\
&\quad \mathcal{M}_{14} = 0, \quad \mathcal{M}_{15} = \frac{1}{D_\varepsilon^{\frac{3}{2}}}\cdot D^{\frac{1}{2}}D_{23}, \quad \mathcal{M}_{16} = -\frac{1}{D_\varepsilon^{\frac{3}{2}}}\cdot D^{\frac{1}{2}}D_{23},\\
&\mathcal{M}_{21} = 0, \quad \mathcal{M}_{22} = \frac{-i}{D_\varepsilon}\partial_2(b^{-1}\cdot), \quad \mathcal{M}_{23} = \frac{i}{D^{\frac{1}{2}}}\cdot D_\varepsilon^{\frac{1}{2}}\frac{\partial_3}{D_{23}}(\frac{1}{\sqrt{b}}\cdot),\\
&\quad \mathcal{M}_{24} = -\frac{i}{D^{\frac{1}{2}}}\cdot D_\varepsilon^{\frac{1}{2}}\frac{\partial_3}{D_{23}}(\frac{1}{\sqrt{b}}\cdot), \quad \mathcal{M}_{25} = \frac{1}{D_\varepsilon^{\frac{3}{2}}}\cdot D^{\frac{1}{2}}\frac{\partial_{12}^2}{D_{23}}, \quad \mathcal{M}_{26} = -\frac{1}{D_\varepsilon^{\frac{3}{2}}}\cdot D^{\frac{1}{2}}\frac{\partial_{12}^2}{D_{23}},\\
&\mathcal{M}_{31} = 0, \quad \mathcal{M}_{32} = -\frac{i}{D_\varepsilon}\partial_3(b^{-1}\cdot), \quad \mathcal{M}_{33} = -\frac{i}{D^{\frac{1}{2}}}\cdot D_\varepsilon^{\frac{1}{2}}\frac{\partial_2}{D_{23}}(\frac{1}{\sqrt{b}}\cdot),\\
&\quad \mathcal{M}_{34} = \frac{i}{D^{\frac{1}{2}}}\cdot D_\varepsilon^{\frac{1}{2}}\frac{\partial_2}{D_{23}}(\frac{1}{\sqrt{b}}\cdot), \quad \mathcal{M}_{35} = \frac{1}{D_\varepsilon^{\frac{3}{2}}}\cdot D^{\frac{1}{2}}\frac{\partial_{13}^2}{D_{23}}, \quad \mathcal{M}_{36} = -\frac{1}{D_\varepsilon^{\frac{3}{2}}}\cdot D^{\frac{1}{2}}\frac{\partial_{13}^2}{D_{23}},\\
&\mathcal{M}_{41} = -\frac{i}{D}\partial_1, \quad \mathcal{M}_{42} = 0, \quad \mathcal{M}_{43} = -D_\varepsilon^{\frac{1}{2}}\cdot\frac{D_{23}}{D^{\frac{3}{2}}},\\
&\quad \mathcal{M}_{44} = -D_\varepsilon^{\frac{1}{2}}\cdot\frac{D_{23}}{D^{\frac{3}{2}}}, \quad \mathcal{M}_{45} = \mathcal{M}_{46} = 0,\\
&\mathcal{M}_{51} = -\frac{i}{D}\partial_2, \quad \mathcal{M}_{52} = 0, \quad \mathcal{M}_{53} = -D_\varepsilon^{\frac{1}{2}}\cdot\frac{\partial_{12}^2}{D^{\frac{3}{2}}D_{23}},\\
&\quad \mathcal{M}_{54} = -D_\varepsilon^{\frac{1}{2}}\cdot\frac{\partial_{12}^2}{D^{\frac{3}{2}}D_{23}}, \quad \mathcal{M}_{55} = \frac{i}{D_\varepsilon^{\frac{1}{2}}}\cdot\frac{D^{\frac{1}{2}}\partial_3}{D_{23}}, \quad \mathcal{M}_{56} = \frac{i}{D_\varepsilon^{\frac{1}{2}}}\frac{D^{\frac{1}{2}}\partial_3}{D_{23}},\\
&\mathcal{M}_{61} = -\frac{i}{D}\partial_3, \quad \mathcal{M}_{62} = 0, \quad \mathcal{M}_{63} = -D_\varepsilon^{\frac{1}{2}}\frac{\partial_{13}^2}{D^{\frac{3}{2}}D_{23}},\\
&\quad \mathcal{M}_{64} = -D_\varepsilon^{\frac{1}{2}}\cdot\frac{\partial_{13}^2}{D^{\frac{3}{2}}D_{23}}, \quad \mathcal{M}_{65} = -\frac{i}{D_\varepsilon^{\frac{1}{2}}}\cdot\frac{D^{\frac{1}{2}}\partial_2}{D_{23}}, \quad \mathcal{M}_{66} = -\frac{i}{D_\varepsilon^{\frac{1}{2}}}\cdot\frac{D^{\frac{1}{2}}\partial_2}{D_{23}}.
\end{aligned}$$

For the diagonal symbol $d(x,\xi)$ we consider

$$\mathcal{D} = \mathrm{diag}(\partial_t, \partial_t, \partial_t - i\sqrt{b(x)}D, \partial_t + i\sqrt{b(x)}D, \partial_t + iD_\varepsilon, \partial_t - iD_\varepsilon).$$

We associate operators to $\tilde{m}^{-1}$ as follows:

$$
\begin{aligned}
&\mathcal{N}_{11} = \mathcal{N}_{12} = \mathcal{N}_{13} = 0,\\
&\quad \mathcal{N}_{14} = -i\partial_1 \frac{1}{D}, \quad \mathcal{N}_{15} = -i\partial_2 \frac{1}{D}, \quad \mathcal{N}_{16} = -i\partial_3 \frac{1}{D},\\
&\mathcal{N}_{21} = -iab\partial_1 \frac{1}{D_\varepsilon}, \quad \mathcal{N}_{22} = -iab\partial_2 \frac{1}{D_\varepsilon}, \quad \mathcal{N}_{23} = -iab\partial_3 \frac{1}{D_\varepsilon},\\
&\quad \mathcal{N}_{24} = \mathcal{N}_{25} = \mathcal{N}_{26} = 0,\\
&\mathcal{N}_{31} = 0, \quad \mathcal{N}_{32} = i\frac{\sqrt{b}}{2} \cdot \frac{\partial_3}{D_{23}} D^{\frac{1}{2}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}}, \quad \mathcal{N}_{33} = -i\frac{\sqrt{b}}{2} \frac{\partial_2}{D_{23}} D^{\frac{1}{2}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}},\\
&\quad \mathcal{N}_{34} = -\frac{D_{23}}{2D^{\frac{1}{2}}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}}, \quad \mathcal{N}_{35} = -\frac{\partial_{12}^2}{2D_{23}} \frac{1}{D^{\frac{1}{2}}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}}, \quad \mathcal{N}_{36} = -\frac{\partial_{13}^2}{D_{23}} \frac{1}{D^{\frac{1}{2}}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}}.
\end{aligned}
$$

The remaining expressions are given by

$$
\begin{aligned}
&\mathcal{N}_{41} = 0, \quad \mathcal{N}_{42} = -i\frac{\sqrt{b}}{2} \frac{\partial_3}{D_{23}} D^{\frac{1}{2}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}}, \quad \mathcal{N}_{43} = i\frac{\sqrt{b}}{2} \frac{\partial_2}{D_{23}} D^{\frac{1}{2}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}},\\
&\quad \mathcal{N}_{44} = -\frac{D_{23}}{2D^{\frac{1}{2}}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}}, \quad \mathcal{N}_{45} = -\frac{\partial_{12}^2}{2D_{23}} \frac{1}{D^{\frac{1}{2}}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}}, \quad \mathcal{N}_{46} = -\frac{\partial_{13}^2}{2D_{23}} \frac{1}{D^{\frac{1}{2}}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}},\\
&\mathcal{N}_{51} = \frac{a}{2} \cdot \frac{D_{23}}{D^{\frac{1}{2}}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}}, \quad \mathcal{N}_{52} = \frac{b}{2} \cdot \frac{\partial_{12}^2}{D^{\frac{1}{2}} D_{23}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}}, \quad \mathcal{N}_{53} = \frac{b}{2} \cdot \frac{\partial_{13}^2}{D^{\frac{1}{2}} D_{23}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}},\\
&\quad \mathcal{N}_{54} = 0, \quad \mathcal{N}_{55} = i\frac{\partial_3}{2D_{23}D^{\frac{1}{2}}} \cdot D_\varepsilon^{\frac{1}{2}}, \quad \mathcal{N}_{56} = -i\frac{\partial_2}{2D_{23}D^{\frac{1}{2}}} \cdot D_\varepsilon^{\frac{1}{2}},\\
&\mathcal{N}_{61} = -\frac{a}{2} \cdot \frac{D_{23}}{D^{\frac{1}{2}}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}}, \quad \mathcal{N}_{62} = -\frac{b}{2} \cdot \frac{\partial_{12}^2}{D^{\frac{1}{2}} D_{23}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}}, \quad \mathcal{N}_{63} = -\frac{b}{2} \cdot \frac{\partial_{13}^2}{D^{\frac{1}{2}} D_{23}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}},\\
&\quad \mathcal{N}_{64} = 0, \quad \mathcal{N}_{65} = i\frac{\partial_3}{2D_{23}D^{\frac{1}{2}}} \cdot D_\varepsilon^{\frac{1}{2}}, \quad \mathcal{N}_{66} = -i\frac{\partial_2}{2D_{23}D^{\frac{1}{2}}} \cdot D_\varepsilon^{\frac{1}{2}}.
\end{aligned}
$$

After a long, but straight-forward computation we find the composite expressions to be

$$
\begin{aligned}
&(\mathcal{M}\mathcal{D}\mathcal{N})_{11} = -\frac{1}{D_\varepsilon}\partial_1(a^{-1}\cdot)\partial_t ab\partial_1 \frac{1}{D_\varepsilon} + \frac{1}{D_\varepsilon^{\frac{3}{2}}} D^{\frac{1}{2}} D_{23}\partial_t a \frac{D_{23}}{D^{\frac{1}{2}}} \frac{1}{D_\varepsilon^{\frac{1}{2}}},\\
&(\mathcal{M}\mathcal{D}\mathcal{N})_{12} = \frac{-1}{D_\varepsilon}\partial_1(a^{-1}\cdot)(\partial_t ab\partial_2 \frac{1}{D_\varepsilon}) + \frac{1}{D_\varepsilon^{\frac{3}{2}}} D^{\frac{1}{2}} D_{23}\partial_t b \frac{\partial_{12}^2}{D^{\frac{1}{2}} D_{23}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}},\\
&(\mathcal{M}\mathcal{D}\mathcal{N})_{13} = -\frac{1}{D_\varepsilon}\partial_1(a^{-1}\cdot)\partial_t ab\partial_3 \frac{1}{D_\varepsilon} + \frac{1}{D_\varepsilon^{\frac{3}{2}}} D^{\frac{1}{2}} D_{23}\partial_t \big(b \cdot \frac{\partial_{13}^2}{D^{\frac{1}{2}} D_{23}} \cdot \frac{1}{D_\varepsilon^{\frac{1}{2}}}\big), \quad (\mathcal{M}\mathcal{D}\mathcal{N})_{14} = 0,\\
&(\mathcal{M}\mathcal{D}\mathcal{N})_{15} = \frac{1}{D_\varepsilon^{\frac{3}{2}}} D^{\frac{1}{2}} D_{23} D_\varepsilon \frac{\partial_3}{D_{23}D^{\frac{1}{2}}} D_\varepsilon^{\frac{1}{2}}, \quad (\mathcal{M}\mathcal{D}\mathcal{N})_{16} = -\frac{1}{D_\varepsilon^{\frac{3}{2}}} D^{\frac{1}{2}} D_{23} D_\varepsilon \frac{\partial_2}{D_{23}D^{\frac{1}{2}}} D_\varepsilon^{\frac{1}{2}}.
\end{aligned}
$$

$$(\mathcal{MDN})_{21} = -\frac{1}{D_\varepsilon}\partial_2(b^{-1}\cdot)(\partial_t ab\partial_1\frac{1}{D_\varepsilon}) + \frac{1}{D_\varepsilon^{\frac{3}{2}}}D^{\frac{1}{2}}\frac{\partial_{12}^2}{D_{23}}\partial_t a\frac{D_{23}}{D^{\frac{1}{2}}}\cdot\frac{1}{D_\varepsilon^{\frac{1}{2}}},$$

$$(\mathcal{MDN})_{22} = -\frac{1}{D_\varepsilon}\partial_2(b^{-1}\cdot)\partial_t ab\partial_2\frac{1}{D_\varepsilon} - \frac{1}{D^{\frac{1}{2}}}D_\varepsilon^{\frac{1}{2}}\frac{\partial_3}{D_{23}}\big(\frac{1}{\sqrt{b}}\cdot\big)\partial_t\big(\sqrt{b}\frac{\partial_3}{D_{23}}D^{\frac{1}{2}}\cdot\frac{1}{D_\varepsilon^{\frac{1}{2}}}\big)$$
$$+\frac{1}{D_\varepsilon^{\frac{3}{2}}}D^{\frac{1}{2}}\frac{\partial_{12}^2}{D_{23}}\partial_t\big(b\frac{\partial_{12}^2}{D^{\frac{1}{2}}D_{23}}\big)\cdot\frac{1}{D_\varepsilon^{\frac{1}{2}}},$$

$$(\mathcal{MDN})_{23} = -\frac{1}{D_\varepsilon}\partial_2(b^{-1}\cdot)\partial_t ab\partial_3\frac{1}{D_\varepsilon} + \frac{1}{D_\varepsilon^{\frac{1}{2}}}D^{\frac{1}{2}}\frac{\partial_3}{D_{23}}\big(\frac{1}{\sqrt{b}}\cdot\big)\partial_t\frac{\sqrt{b}}{2}\frac{\partial_2}{D_{23}}D^{\frac{1}{2}}\cdot\frac{1}{D_\varepsilon^{\frac{1}{2}}}$$
$$+\frac{1}{D_\varepsilon^{\frac{3}{2}}}D^{\frac{1}{2}}\frac{\partial_{12}^2}{D_{23}}\partial_t b\frac{\partial_{13}^2}{D^{\frac{1}{2}}D_{23}}\frac{1}{D_\varepsilon^{\frac{1}{2}}},$$

$$(\mathcal{MDN})_{24} = -\frac{1}{D^{\frac{1}{2}}}D_\varepsilon^{\frac{1}{2}}\partial_3 D^{\frac{1}{2}}\frac{1}{D_\varepsilon^{\frac{1}{2}}},$$

$$(\mathcal{MDN})_{25} = -\frac{1}{D^{\frac{1}{2}}}D_\varepsilon^{\frac{1}{2}}\frac{\partial_3}{D_{23}}D\frac{\partial_{12}^2}{D_{23}}\frac{1}{D^{\frac{1}{2}}}\frac{1}{D_\varepsilon^{\frac{1}{2}}} + \frac{1}{D_\varepsilon^{\frac{3}{2}}}D^{\frac{1}{2}}\frac{\partial_{12}^2}{D_{23}}D_\varepsilon\frac{\partial_3}{D_{23}D^{\frac{1}{2}}}D_\varepsilon^{\frac{1}{2}},$$

$$(\mathcal{MDN})_{26} = -\frac{1}{D^{\frac{1}{2}}}D_\varepsilon^{\frac{1}{2}}\frac{\partial_3}{D_{23}}D\frac{\partial_{13}^2}{D_{23}}\frac{1}{D^{\frac{1}{2}}}\frac{1}{D_\varepsilon^{\frac{1}{2}}} - \frac{1}{D_\varepsilon^{\frac{3}{2}}}D^{\frac{1}{2}}\frac{\partial_{12}^2}{D_{23}}D_\varepsilon\frac{\partial_2}{D_{23}D^{\frac{1}{2}}}D_\varepsilon^{\frac{1}{2}}.$$

$$(\mathcal{MDN})_{31} = -\frac{1}{D_\varepsilon}\partial_3(b^{-1}\cdot)\partial_t(ab\partial_1\frac{1}{D_\varepsilon}) + \frac{1}{D_\varepsilon^{\frac{3}{2}}}D^{\frac{1}{2}}\frac{\partial_{13}^2}{D_{23}}\partial_t a\frac{D_{23}}{D^{\frac{1}{2}}}\cdot\frac{1}{D_\varepsilon^{\frac{1}{2}}},$$

$$(\mathcal{MDN})_{32} = -\frac{1}{D_\varepsilon}\partial_3(b^{-1}\cdot)\partial_t ab\partial_2\frac{1}{D_\varepsilon} + \frac{1}{D^{\frac{1}{2}}}D_\varepsilon^{\frac{1}{2}}\frac{\partial_2}{D_{23}}\big(\frac{1}{\sqrt{b}}\cdot\big)\partial_t\big(\sqrt{b}\frac{\partial_3}{D_{23}}D^{\frac{1}{2}}\cdot\frac{1}{D_\varepsilon^{\frac{1}{2}}}\big)$$
$$+\frac{1}{D_\varepsilon^{\frac{3}{2}}}D^{\frac{1}{2}}\frac{\partial_{13}^2}{D_{23}}\partial_t\big(b\frac{\partial_{12}^2}{D^{\frac{1}{2}}D_{23}}\cdot\frac{1}{D_\varepsilon^{\frac{1}{2}}}\big),$$

$$(\mathcal{MDN})_{33} = -\frac{1}{D_\varepsilon}\partial_3(b^{-1}\cdot)\partial_t ab\partial_3\frac{1}{D_\varepsilon} - \frac{1}{D^{\frac{1}{2}}}D_\varepsilon^{\frac{1}{2}}\frac{\partial_2}{D_{23}}\big(\frac{1}{\sqrt{b}}\cdot\big)\partial_t\big(\sqrt{b}\frac{\partial_2}{D_{23}}D^{\frac{1}{2}}\cdot\frac{1}{D_\varepsilon^{\frac{1}{2}}}\big)$$
$$+\frac{1}{D_\varepsilon^{\frac{3}{2}}}D^{\frac{1}{2}}\frac{\partial_{13}^2}{D_{23}}\partial_t b\frac{\partial_{13}^2}{D^{\frac{1}{2}}D_{23}}\cdot\frac{1}{D_\varepsilon^{\frac{1}{2}}},$$

$$(\mathcal{MDN})_{34} = \frac{1}{D^{\frac{1}{2}}}D_\varepsilon^{\frac{1}{2}}\partial_2 D^{\frac{1}{2}}\cdot\frac{1}{D_\varepsilon^{\frac{1}{2}}},$$

$$(\mathcal{MDN})_{35} = \frac{1}{D^{\frac{1}{2}}}D_\varepsilon^{\frac{1}{2}}\frac{\partial_2}{D_{23}}D^{\frac{1}{2}}\frac{\partial_{12}^2}{D_{23}}\cdot\frac{1}{D_\varepsilon^{\frac{1}{2}}} + \frac{1}{D_\varepsilon^{\frac{3}{2}}}D^{\frac{1}{2}}\frac{\partial_{13}^2}{D_{23}}D_\varepsilon\frac{\partial_3}{D_{23}D^{\frac{1}{2}}}D_\varepsilon^{\frac{1}{2}},$$

$$(\mathcal{MDN})_{36} = \frac{1}{D^{\frac{1}{2}}}D_\varepsilon^{\frac{1}{2}}\frac{\partial_2}{D_{23}}D\frac{\partial_{13}^2}{D_{23}}\frac{1}{D^{\frac{1}{2}}}\frac{1}{D_\varepsilon^{\frac{1}{2}}} - \frac{1}{D_\varepsilon^{\frac{3}{2}}}D^{\frac{1}{2}}\frac{\partial_{13}^2}{D_{23}}D_\varepsilon\frac{\partial_2}{D_{23}D^{\frac{1}{2}}}D_\varepsilon^{\frac{1}{2}}.$$

$$(\mathcal{MDN})_{41}=0,\qquad (\mathcal{MDN})_{42}=-D_\varepsilon^{\frac12}\frac{D_{23}}{D^{\frac32}}(\sqrt{b}D)\sqrt{b}\frac{\partial_3}{D_{23}}D^{\frac12}\cdot\frac{1}{D_\varepsilon^{\frac12}},$$

$$(\mathcal{MDN})_{43}=D_\varepsilon^{\frac12}\frac{D_{23}}{D^{\frac32}}(\sqrt{b}D)(\sqrt{b}\frac{\partial_2}{D_{23}}D^{\frac12}\cdot\frac{1}{D_\varepsilon^{\frac12}}),\qquad (\mathcal{MDN})_{44}=-\partial_t\frac{\partial_1^2}{D^2}+D_\varepsilon^{\frac12}\frac{D_{23}}{D^{\frac32}}\partial_t\frac{D_{23}}{D^{\frac12}}\cdot\frac{1}{D_\varepsilon^{\frac12}},$$

$$(\mathcal{MDN})_{45}=-\partial_t\frac{\partial_{12}^2}{D^2}+D_\varepsilon^{\frac12}\frac{D_{23}}{D^{\frac32}}\partial_t\frac{\partial_{12}^2}{D_{23}}\frac{1}{D^{\frac12}}\cdot\frac{1}{D_\varepsilon^{\frac12}},\qquad (\mathcal{MDN})_{46}=-\partial_t\frac{\partial_{13}^2}{D^2}+D_\varepsilon^{\frac12}\partial_t\frac{\partial_{13}^2}{D^2}\frac{1}{D_\varepsilon^{\frac12}}.$$

$$(\mathcal{MDN})_{51}=\frac{1}{D_\varepsilon^{\frac12}}\frac{D^{\frac12}\partial_3}{D_{23}}D_\varepsilon a\frac{D_{23}}{D^{\frac12}}\frac{1}{D_\varepsilon^{\frac12}},$$

$$(\mathcal{MDN})_{52}=-D_\varepsilon^{\frac12}\frac{\partial_{12}^2}{D^{\frac32}D_{23}}\sqrt{b}D\sqrt{b}\frac{\partial_3}{D_{23}}D^{\frac12}\frac{1}{D_\varepsilon^{\frac12}}+\frac{1}{D_\varepsilon^{\frac12}}\frac{D^{\frac12}\partial_3}{D_{23}}D_\varepsilon b\frac{\partial_{12}^2}{D^{\frac12}D_{23}}\frac{1}{D_\varepsilon^{\frac12}},$$

$$(\mathcal{MDN})_{53}=D_\varepsilon^{\frac12}\frac{\partial_{12}^2}{D^{\frac32}D_{23}}\sqrt{b}D\sqrt{b}\frac{\partial_2}{D_{23}}D^{\frac12}\frac{1}{D_\varepsilon^{\frac12}}+\frac{1}{D_\varepsilon^{\frac12}}\frac{D^{\frac12}\partial_3}{D_{23}}D_\varepsilon b\frac{\partial_{13}^2}{D^{\frac12}D_{23}}\frac{1}{D_\varepsilon^{\frac12}},$$

$$(\mathcal{MDN})_{54}=-\partial_t\frac{\partial_{12}^2}{D^2}+D_\varepsilon^{\frac12}\frac{\partial_{12}^2}{D^2}\partial_t\frac{1}{D_\varepsilon^{\frac12}},$$

$$(\mathcal{MDN})_{55}=-\partial_t\frac{\partial_2^2}{D^2}+D_\varepsilon^{\frac12}\frac{\partial_{12}^2}{D^{\frac32}D_{23}}\partial_t\frac{\partial_{12}^2}{D_{23}}\frac{1}{D^{\frac12}}\cdot\frac{1}{D_\varepsilon^{\frac12}}-\frac{1}{D_\varepsilon^{\frac12}}\frac{\partial_3}{D_{23}^2}\partial_t\partial_3D_\varepsilon^{\frac12},$$

$$(\mathcal{MDN})_{56}=-\partial_t\frac{\partial_{23}^2}{D^2}+D_\varepsilon^{\frac12}\frac{\partial_{12}^2}{D^{\frac32}D_{23}}\partial_t\frac{\partial_{13}^2}{D_{23}}\frac{1}{D^{\frac12}}\cdot\frac{1}{D_\varepsilon^{\frac12}}+\frac{1}{D_\varepsilon^{\frac12}}\frac{D^{\frac12}\partial_3}{D_{23}}\partial_t\frac{\partial_2}{D_{23}D^{\frac12}}\cdot D_\varepsilon^{\frac12}.$$

$$(\mathcal{MDN})_{61}=-\frac{1}{D_\varepsilon^{\frac12}}\frac{D^{\frac12}\partial_2}{D_{23}}D_\varepsilon a\frac{D_{23}}{D^{\frac12}}\frac{1}{D_\varepsilon^{\frac12}},$$

$$(\mathcal{MDN})_{62}=-D_\varepsilon^{\frac12}\frac{\partial_{13}^2}{D^{\frac32}D_{23}}\sqrt{b}D\sqrt{b}\frac{\partial_3}{D_{23}}D^{\frac12}\frac{1}{D_\varepsilon^{\frac12}}-\frac{1}{D_\varepsilon^{\frac12}}\frac{D^{\frac12}\partial_2}{D_{23}}D_\varepsilon b\frac{\partial_{12}^2}{D^{\frac12}D_{23}}\frac{1}{D_\varepsilon^{\frac12}},$$

$$(\mathcal{MDN})_{63}=D_\varepsilon^{\frac12}\frac{\partial_{13}^2}{D^{\frac32}D_{23}}\sqrt{b}D\sqrt{b}\frac{\partial_2}{D_{23}}D^{\frac12}\frac{1}{D_\varepsilon^{\frac12}}-\frac{1}{D_\varepsilon^{\frac12}}\frac{D^{\frac12}\partial_2}{D_{23}}D_\varepsilon b\frac{\partial_{13}^2}{D^{\frac12}D_{23}}\frac{1}{D_\varepsilon^{\frac12}},$$

$$(\mathcal{MDN})_{64}=-\partial_t\frac{\partial_{31}^2}{D^2}+D_\varepsilon^{\frac12}\frac{\partial_{13}^2}{D^{\frac32}}\partial_t\frac{1}{D^{\frac12}}\frac{1}{D_\varepsilon^{\frac12}},$$

$$(\mathcal{MDN})_{65}=-\partial_t\frac{\partial_{23}^2}{D^2}+D_\varepsilon^{\frac12}\frac{\partial_{13}^2}{D^{\frac32}D_{23}}\partial_t\frac{\partial_{12}^2}{D_{23}}\frac{1}{D^{\frac12}}\frac{1}{D_\varepsilon^{\frac12}}+\frac{1}{D_\varepsilon^{\frac12}}\frac{\partial_2\partial_t}{D_{23}^2}\partial_3D_\varepsilon^{\frac12},$$

$$(\mathcal{MDN})_{66}=-\partial_t\frac{\partial_3^2}{D^2}+D_\varepsilon^{\frac12}\frac{\partial_{13}^2}{D^{\frac32}D_{23}}\partial_t\frac{\partial_{13}^2}{D_{23}}\frac{1}{D^{\frac12}}\frac{1}{D_\varepsilon^{\frac12}}-\frac{1}{D_\varepsilon^{\frac12}}\frac{\partial_2}{D_{23}}\partial_t\frac{\partial_2}{D_{23}}D_\varepsilon^{\frac12}.$$

In Subsection 2.4 we shall prove that the difference with P, after further reductions, is bounded in L^2.

2.2. **Reductions for C^2-coefficients.** Next, we carry out reductions as in [11] for the proof of Theorem 1.1. Precisely, we apply the following:

- Localization to a cube of size 1,
- Reduction to dyadic estimates,
- Truncating the coefficients of P at frequency $\lambda^{\frac12}$,

- Reduction to half-wave equations.

To begin with, by scaling we suppose that $\|\partial^2 \varepsilon\|_{L^\infty} \leq 1$ and $\mu = 1$.

2.2.1. *Localization to a cube of size* 1. Let $s(\xi)$ denote a symbol supported in $B(0,2)\backslash B(0,1/2)$ such that

$$\sum_{j \in \mathbb{Z}} s(2^{-j}\xi) = 1, \qquad \xi \in \mathbb{R}^4 \backslash \{0\}.$$

For $\lambda \in 2^{\mathbb{N}_0}$, let $S_\lambda = S(D/\lambda)$ be the Littlewood-Paley multiplier and $S_0 = 1 - \sum_{j \geq 0} S_{2^j}$. Let $u = S_0 u + (1 - S_0)u$. As in [11, Paragraph 3.2.1], the contribution of $S_0 u$ is treated by Sobolev embedding and the Hardy-Littlewood-Sobolev inequality. It remains to prove the claim for the inhomogeneous norm for the high frequencies:

$$\|\langle D \rangle^{-\rho} u\|_{L^p L^q} \lesssim \|u\|_{L^2} + \|Pu\|_{L^2} + \|\langle D' \rangle^{-\frac{1}{2}} \rho_e\|_{L^2} + \|\langle D' \rangle^{-\frac{1}{2}} \rho_m\|_{L^2}$$

with $\langle D \rangle = Op((1 + |\xi|^2)^{\frac{1}{2}})$ and $\langle D' \rangle = Op((1 + |\xi'|^2)^{\frac{1}{2}})$. We introduce a smooth partition of unity:

$$1 = \sum_{j \in \mathbb{Z}^{n+1}} \chi_j(x), \quad \chi_j(x) = \chi(x - j), \quad \text{supp}\chi \subseteq B(0,2).$$

By commutator estimates, we find (cf. [11, Paragraph 3.2.1])

$$\sum_j \|\chi_j u\|_{L^2}^2 + \|P \chi_j u\|_{L^2}^2 \lesssim \|u\|_{L^2}^2 + \|Pu\|_{L^2}^2,$$

and

$$\|\langle D \rangle^{-\rho} u\|_{L^p L^q}^2 \lesssim \sum_j \|\langle D \rangle^{-\rho} \chi_j u\|_{L^p L^q}^2.$$

2.2.2. *Reduction to dyadic estimates.* By Littlewood-Paley theory and commutator arguments, we find that it is enough to prove

$$\lambda^{-\rho} \|S_\lambda u\|_{L^p L^q} \lesssim \|S_\lambda u\|_{L^2} + \|P S_\lambda u\|_{L^2} + \lambda^{-\frac{1}{2}} \|S_\lambda \rho_e\|_{L^2} + \lambda^{-\frac{1}{2}} \|S_\lambda \rho_m\|_{L^2}. \tag{21}$$

Details are given in [11, Paragraph 3.2.2].

2.2.3. *Truncating the coefficients of P at frequency $\lambda^{\frac{1}{2}}$.* Finally, we reduce (21) to ε_{ii} having Fourier transform supported in $\{|\xi| \leq \lambda^{\frac{1}{2}}\}$. Note that for $\lambda \gg 1$, $\varepsilon^{ij}_{\lambda^{\frac{1}{2}}}$, denoting the Fourier truncated coefficients, is still uniformly elliptic. The error estimate is shown as in [11, Paragraph 3.2.3]. It is enough to show

$$\lambda^{-\rho} \|S_\lambda u\|_{L^p L^q} \lesssim \|S_\lambda u\|_{L^2} + \|P_\lambda S_\lambda u\|_{L^2} + \lambda^{-\frac{1}{2}} \|S_\lambda \rho_e\|_{L^2} + \lambda^{-\frac{1}{2}} \|S_\lambda \rho_m\|_{L^2}, \tag{22}$$

where

$$P_\lambda = \begin{pmatrix} \partial_t 1_{3\times 3} & -\mathcal{B}(\partial) \\ \mathcal{B}(\partial) \varepsilon^{-1}_{\lambda^{\frac{1}{2}}} & \partial_t 1_{3\times 3} \end{pmatrix}. \tag{23}$$

2.2.4. *Reduction to half-wave equations.* We consider the two regions $\{|\xi_0| \gg \|\xi\|\}$ and $\{|\xi_0| \lesssim \|\xi\|\}$. The first region is away from the characteristic surface, hence, P is elliptic in this region. The contribution can be estimated by Sobolev embedding. To make the argument precise, we use the FBI transform (cf. [16]-[18], [11, Section 2]). For $\lambda \in 2^{\mathbb{Z}}$, we define the FBI for $f \in L^1(\mathbb{R}^m;\mathbb{C})$ by

$$T_\lambda f(z) = C_m \lambda^{\frac{3m}{4}} \int_{\mathbb{R}^m} e^{-\frac{\lambda}{2}(z-y)^2} f(y)dy, \quad z = x - i\xi \in T^*\mathbb{R}^m \equiv \mathbb{R}^{2m},$$
$$C_m = 2^{-\frac{m}{2}}\pi^{-\frac{3m}{4}}.$$

$T_\lambda : L^2(\mathbb{R}^m) \to L^2_\Phi(T^*\mathbb{R}^m)$ is an isometric mapping with $\Phi(z) = e^{-\lambda\xi^2}$. We write $z = x - i\xi$ because $T_\lambda f$ is holomorphic. $T_\lambda f$ is related with the Fourier transform by

$$T_\lambda f(z) = C_m \lambda^{\frac{3m}{4}} e^{\frac{\lambda}{2}\xi^2} \int_{\mathbb{R}^m} e^{-\frac{\lambda}{2}(x-y)^2} e^{i\lambda\xi.(x-y)} f(y)dy.$$

We have an inversion formula for the FBI transform by taking the adjoint mapping in L^2_Φ:

$$T^*_\lambda F(y) = C_m \lambda^{\frac{3m}{4}} \int_{\mathbb{R}^{2m}} e^{-\frac{\lambda}{2}(\overline{z}-y)^2} \Phi(z) F(z) dx d\xi.$$

The FBI transform allows to conjugate rough symbols to multiplication in phase space. We consider $a(x,\xi) \in C^s_x C^\infty_c$, $a(x,\xi) = 0$ for $\xi \notin B(0,2)$. Let $a_\lambda(x,\xi) = a(x,\xi/\lambda)$ denote the scaled symbol supported at frequencies $\lesssim \lambda$. Let

$$\tilde{a}^s_\lambda = \sum_{|\alpha|+|\beta|<s} (\partial_\xi - \lambda\xi)^\alpha \frac{\partial^\alpha_x \partial^\beta_\xi a(x,\xi)}{|\alpha|!|\beta|!(-i\lambda)^{|\alpha|}\lambda^{|\beta|}} (\frac{1}{i}\partial_x - \lambda\xi)^\beta.$$

For $s \leq 1$, we have

$$\tilde{a}^s_\lambda = a.$$

Define the remainder

$$R^s_{\lambda,a} = T_\lambda A_\lambda - \tilde{a}^s_\lambda T_\lambda.$$

We need the following approximation result:

Theorem 2.1 ([17, Theorem 5, p. 393])**.** *Suppose that $a \in C^s_x C^\infty_c$. Then,*

$$\|R^s_{\lambda,a}\|_{L^2 \to L^2_\Phi} \lesssim \lambda^{-\frac{s}{2}},$$
$$\|(\partial_\xi - \lambda) R^s_{\lambda,a}\|_{L^2 \to L^2_\Phi} \lesssim \lambda^{\frac{1}{2}-\frac{s}{2}}.$$

We make further use of the following multiplier result:

Proposition 2.2 ([11, Proposition 2.2])**.** *Let $1 \leq p,q \leq \infty$, $a \in C^s_x C^\infty_c(\mathbb{R}^m \times \mathbb{R}^m)$, $a(x,\xi) = 0$ for $\xi \notin B(0,2)$, and*

$$\sup_{x\in\mathbb{R}^m} \Big(\sum_{|\alpha|\leq m+1} \|D^\alpha_\xi a(x,\cdot)\|_{L^1_\xi} \Big) \leq C.$$

Then, we find the following estimate to hold:

$$\|T^*_\lambda a(x,\xi) T_\lambda f\|_{L^pL^q} \lesssim C\|f\|_{L^pL^q}.$$

To estimate pseudo-differential operators, we use the following:

Lemma 2.3 ([11, Lemma 2.3]). *Let $1 \leq p, q \leq \infty$, and $a \in C_x^s C_c^\infty(\mathbb{R}^m \times \mathbb{R}^m)$ with $a(x,\xi) = 0$ for $\xi \notin B(0,2)$. Suppose that*

$$\sup_{x \in \mathbb{R}^m} \sum_{0 \leq |\alpha| \leq m+1} \|D_\xi^\alpha a(x,\cdot)\|_{L^1_\xi} \leq C.$$

Then, we find the following estimate to hold:

$$\|a(x,\partial) f\|_{L^p L^q} \lesssim C \|f\|_{L^p L^q}.$$

We return to the reduction to half-wave estimates: By applying Theorem 2.1, we find

$$\|T_\lambda\big(\frac{P(x,\partial)}{\lambda} S_\lambda u\big) - p(x,\xi) T_\lambda S_\lambda u\|_{L^2_\Phi} \lesssim \lambda^{-\frac{1}{2}} \|S_\lambda u\|_{L^2}.$$

Denote $v_\lambda = T_\lambda S_\lambda u$, and we observe for $|\xi_0| \gg \|\xi'\|$

$$\|p(x,\xi) v_\lambda\|_{L^2_\Phi} \gtrsim \|v_\lambda\|_{L^2_\Phi}$$

by the diagonalization

$$p(x,\xi) = \tilde{m}(x,\xi) d(x,\xi) \tilde{m}^{-1}(x,\xi), \quad |d_{ii}| \gtrsim |\xi_0| \gtrsim 1.$$

The claim now follows as in [11, Paragraph 3.4.3] by the L^2-mapping properties of the FBI transform and Sobolev embedding.

We handle the main contribution coming from $\{|\xi_0| \lesssim \|\xi'\|\}$ following along the lines of [11]. In the following assume that the space-time Fourier transform of u is supported in $\{|\xi_0| \lesssim \|\xi'\|\}$. We start with the proof of

$$\lambda^{-\rho} \|S_\lambda w\|_{L^p L^q} \lesssim \|S_\lambda u\|_{L^2} + \|\mathcal{D} S_\lambda w\|_{L^2} + \lambda^{-\frac{1}{2}} \|S_\lambda \rho_e\|_{L^2} + \lambda^{-\frac{1}{2}} \|S_\lambda \rho_m\|_{L^2},$$

where

$$\mathcal{D} = \mathrm{diag}(\partial_t, \partial_t, \partial_t + i\sqrt{b} D, \partial_t - i\sqrt{b} D, \partial_t + i D_\varepsilon, \partial_t - i D_\varepsilon)$$

and $w = \tilde{S}_\lambda \mathcal{N} S_\lambda u$. $\tilde{S}_\lambda = \sum_{|j| \leq 2} S_{2^j \lambda}$ denotes a mildly enlarged version of S_λ. For the estimates of w_3 to w_6 we invoke Proposition 1.6. For the first and second component, we use Theorem 2.1:

$$\|T_\lambda S_\lambda w_i - [\tilde{m}^{-1}(x,\xi) T_\lambda S_\lambda u]_i\|_{L^2_\Phi} \lesssim \lambda^{-\frac{1}{2}} \|S_\lambda u\|_{L^2}, \quad i = 1, 2.$$

Denote $v_\lambda = T_\lambda S_\lambda w$. By $\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = \rho_e$, $\partial_1 u_4 + \partial_2 u_5 + \partial_3 u_3 = \rho_m$, and Theorem 2.1, we find

$$\begin{aligned}
&\|[\tilde{m}^{-1}(x,\xi) T_\lambda S_\lambda u]_1\|_{L^2_\Phi} \\
&\leq \|\frac{1}{D} S_\lambda \rho_m\|_{L^2} + \|T_\lambda(\partial_1 \frac{1}{D} S_\lambda u_4 + \partial_2 \frac{1}{D} S_\lambda u_5 + \partial_3 \frac{1}{D} S_\lambda u_6) - i(\xi_1' v_{\lambda,4} + \xi_2' v_{\lambda,5} + \xi_3' v_{\lambda,6})\|_{L^2_\Phi} \\
&\lesssim \lambda^{-\frac{1}{2}} \|S_\lambda \rho_m\|_{L^2} + \lambda^{-\frac{1}{2}} \|S_\lambda u\|_{L^2}.
\end{aligned}$$

The ultimate estimate is a consequence of Lemma 2.3 and the frequency localization $|\xi'| \sim |\xi|$. Similarly,

$$\begin{aligned}
&\|[\tilde{m}^{-1}(x,\xi) T_\lambda S_\lambda u]_2\|_{L^2_\Phi} \\
&\lesssim \|\frac{1}{D_\varepsilon} S_\lambda \rho_e\|_{L^2} + \|T_\lambda(\frac{1}{D_\varepsilon} \partial_1 S_\lambda u_1 + \frac{1}{D_\varepsilon} \partial_2 S_\lambda u_2 + \frac{1}{D_\varepsilon} \partial_3 \lambda u_3) - i(\tilde{\xi}_1 v_{\lambda,1} + \tilde{\xi}_2 v_{\lambda,2} + \tilde{\xi}_3 v_{\lambda,3})\|_{L^2_\Phi} \\
&\lesssim \lambda^{-1} \|S_\lambda \rho_e\|_{L^2} + \lambda^{-\frac{1}{2}} \|S_\lambda u\|_{L^2}.
\end{aligned}$$

By the triangle inequality and mapping properties of the FBI transform, we find the estimates

$$\|S_\lambda w_1\|_{L^2} + \|S_\lambda w_2\|_{L^2} = \|v_{\lambda,1}\|_{L^2_\Phi} + \|v_{\lambda,2}\|_{L^2_\Phi} \lesssim \lambda^{-\frac{1}{2}} \|S_\lambda u\|_{L^2} + \lambda^{-1} \|S_\lambda \rho_e\|_{L^2} + \lambda^{-1} \|S_\lambda \rho_m\|_{L^2}.$$

This allows for an estimate of $\|S_\lambda w_1\|_{L^pL^q} + \|S_\lambda w_2\|_{L^pL^q}$ by Sobolev embedding. For details we refer to [11, Paragraph 3.4.3]. Also the conjugation with $\tilde{m}$, $\tilde{m}^{-1}$ in phase space can be estimated as in [11].

2.3. **Reductions for $\partial_x^2\varepsilon \in L^1L^\infty$.** The reduction of Theorem 1.3 to the dyadic estimates

$$\lambda^{-\rho}\|S_\lambda u\|_{L^pL^q} \lesssim \|S_\lambda u\|_{L^\infty L^2} + \|P_\lambda S_\lambda u\|_{L^1L^2} + \lambda^{-\frac{1}{2}}\|S_\lambda \rho_e\|_{L^2} + \lambda^{-\frac{1}{2}}\|S_\lambda \rho_m\|_{L^2} \tag{24}$$

is a variant of the above argument. The steps are:

- reduction to the case $\mu = 1$,
- confining the support of u to the unit cube and the frequency support to large frequencies,
- estimate away from the characteristic surface,
- reduction to dyadic estiamtes,
- truncating the coefficients at frequency $\lambda^{\frac{1}{2}}$.

For details we refer to [11, Subsection 3.4]. Since the error estimates proved in Proposition 2.4 is valid for frequency truncated C^1-coefficients, we can use the diagonalization and Proposition 2.4 to (24) to an application of half-wave estimates recalled in Proposition 1.6.

2.4. **Error estimates.** In this section we prove the following:

Proposition 2.4. *Let $\lambda \in 2^{\mathbb{N}_0}$, $\mathcal{M}$, $\mathcal{D}$, $\mathcal{N}$ with frequency truncated C^1 coefficients $\varepsilon_{ij}^{\lambda^{\frac{1}{2}}}$ as in Subsection 2.1 and P_λ as in (23). Then, we find the following estimate to hold:*

$$\|\mathcal{M}\mathcal{D}\mathcal{N}S'_\lambda - P_\lambda S'_\lambda\|_{L^2\to L^2} \lesssim 1. \tag{25}$$

For the proof commutator estimates for pseudo-differential operators are crucial. We consider symbols $p \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m\backslash 0)$, which are homogeneous in ξ, i.e., $p(x,\nu\xi) = \nu^\alpha p(x,\xi)$ for some $\nu > 0$ and any $\xi \neq 0$, and satisfy

$$\sup_{x\in\mathbb{R}^m,\xi\in B(0,2)\backslash B(0,1/2)} |\partial_x^\beta p(x,\xi)| \lesssim_\beta \lambda^{\frac{|\beta|-1}{2}} \text{ for } |\beta| \geq 1. \tag{26}$$

To estimate $p(x,\partial)$ in L^2, we use Lemma 2.3.

Furthermore, in [11, Section 2] was shown that compositions of these operators admit an expansion as in the classical Kohn–Nirenberg theorem with suitable L^2-bounds. Recall the following:

Proposition 2.5 ([19, Proposition 0.3C])**.** *Given $P(x,\partial) \in OPS^{m_1}_{\rho_1,\delta_1}$, $Q(x,\partial) \in OPS^{m_2}_{\rho_2,\delta_2}$, suppose that*

$$0 \leq \delta_2 < \rho \leq 1 \text{ with } \rho = \min(\rho_1,\rho_2).$$

Then, $(P\circ Q)(x,\partial) \in OPS^{m_1+m_2}_{\rho,\delta}$ with $\delta = \max(\delta_1,\delta_2)$, and $P(x,\partial)\circ Q(x,\partial)$ satisfies the asymptotic expansion

$$(P\circ Q)(x,\partial) = \sum_\alpha \frac{1}{\alpha!}(D_\xi^\alpha P\, \partial_x^\alpha Q)(x,\partial) + R,$$

where $R : \mathcal{S}' \to C^\infty$ is a smoothing operator and $D_\xi^\alpha = (-i)^{|\alpha|}\partial_\xi^\alpha$.

The terms can be estimated by Lemma 2.3. For the proof of Proposition 2.4 the following commutator estimates, which follow from this expansion and Lemma 2.3, suffice:

Proposition 2.6. *Let $\lambda \in 2^{\mathbb{N}_0}$. Suppose that $p_1(x,\xi)$ and $p_2(x,\xi)$ are symbols, which are β_1 and β_2 homogeneous, respectively, and satisfy (26). Then, we find the following estimates to hold:*

$$\|p_1(x,\partial) \circ p_2(x,\partial) S_\lambda\|_{L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^m)} \lesssim \lambda^{\beta_1+\beta_2},$$
$$\|[p(x,\partial) S_\lambda, q(x,\partial) S_\lambda]\|_{L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^m)} \lesssim \lambda^{\beta_1+\beta_2-1}.$$

We record the following useful special case (cf. [11, Lemma 2.5.])

Lemma 2.7. *Let $\lambda, \mu \in 2^{\mathbb{N}_0}$ with $\min(\lambda,\mu) \ll \max(\lambda,\mu)$ and $\alpha \in \mathbb{R}$. Then, we find the following estimate to hold:*

$$\|S'_\lambda D_\varepsilon^\alpha S'_\mu\|_{L^2 \to L^2} \lesssim_\alpha (\lambda \vee \mu)^{-N} \text{ for any } N. \tag{27}$$

We are ready for the proof of Proposition 2.4.

Proof of Proposition 2.4. We verify the estimate componentwise and note that we are free to include frequency localizations S'^{2}_λ between the single operators by Lemma 2.7 and the frequency localization of a and b. To denote operators bounded in L^2, we write $O_{L^2}(1)$.
We start with the estimate for $(\mathcal{MDN})_{11}$: Momentarily assume that a, b are time-independent. By Proposition 2.6, we compute

$$\begin{aligned}
&(\mathcal{MDN})_{11} S'_\lambda \\
&= \partial_t(-\frac{1}{D_\varepsilon} S'_\lambda \partial_1(b\cdot)\partial_1 S'_\lambda \frac{1}{D_\varepsilon} S'_\lambda + \frac{1}{D_\varepsilon^{\frac{3}{2}}} S'_\lambda D^{\frac{1}{2}} D_{23} a S'_\lambda \frac{D_{23}}{D^{\frac{1}{2}}} \frac{1}{D_\varepsilon^{\frac{1}{2}}} S'_\lambda) + O_{L^2}(1) \\
&= \partial_t(-\frac{1}{D_\varepsilon} S'_\lambda \partial_1(b\cdot)\partial_1 S'_\lambda \frac{1}{D_\varepsilon} S'_\lambda + \frac{1}{D_\varepsilon^{\frac{3}{2}}} S'_\lambda a D_{23}^2 S'_\lambda \frac{1}{D_\varepsilon^{\frac{1}{2}}} S'_\lambda) + O_{L^2}(1) \\
&= \partial_t(-\frac{1}{D_\varepsilon} S'_\lambda (b\partial_1^2 + a\partial_2^2 + a\partial_3^2) S'_\lambda \frac{1}{D_\varepsilon} S'_\lambda) + O_{L^2}(1) \\
&= \partial_t S'_\lambda + O_{L^2}(1).
\end{aligned}$$

For time-dependent a, b the estimate is found likewise, with additional commutator estimates. Hence, in the following we suppose for simplicity that coefficients a and b are time-independent. For the estimate of $(\mathcal{MDN})_{12}$ observe

$$\begin{aligned}
(\mathcal{MDN})_{12} S'_\lambda &= \partial_t\big(- \frac{1}{D_\varepsilon} S'_\lambda b \partial_{12}^2 S'_\lambda \frac{1}{D_\varepsilon} S'_\lambda + \frac{1}{D_\varepsilon^{\frac{3}{2}}} S'_\lambda b \partial_{12}^2 S'_\lambda \frac{1}{D_\varepsilon^{\frac{1}{2}}} S'_\lambda\big) + O_{L^2}(1) \\
&= \partial_t\big(- \frac{1}{D_\varepsilon} S'_\lambda b \partial_{12}^2 S'_\lambda \frac{1}{D_\varepsilon} S'_\lambda + \frac{1}{D_\varepsilon} S'_\lambda \partial_{12}^2 S'_\lambda \frac{1}{D_\varepsilon} S'_\lambda\big) + O_{L^2}(1) \\
&= O_{L^2}(1).
\end{aligned}$$

The estimate for $(\mathcal{MDN})_{13}$ follows likewise. Furthermore, the estimates for $(\mathcal{MDN})_{15}$ and $(\mathcal{MDN})_{16}$ follow directly from Proposition 2.4. For $(\mathcal{MDN})_{21}$ we find

$$\begin{aligned}
(\mathcal{MDN})_{21} S'_\lambda &= \partial_t\big(- \frac{1}{D_\varepsilon} S'_\lambda a \partial_{12}^2 S'_\lambda \frac{1}{D_\varepsilon} S'_\lambda + \frac{1}{D_\varepsilon} S'_\lambda a \partial_{12}^2 S'_\lambda \frac{1}{D_\varepsilon} S'_\lambda\big) + O_{L^2}(1) \\
&= O_{L^2}(1).
\end{aligned}$$

[2]Strictly speaking, they have to be slightly enlarged every time we include them, but this will be omitted for the sake of brevity.

For the estimate of $(\mathcal{MDN})_{22}$ note that

$$\begin{aligned}(\mathcal{MDN})_{22}S'_\lambda &= \partial_t\big((-\frac{1}{D_\varepsilon}S'_\lambda a\partial_2^2 S'_\lambda\frac{1}{D_\varepsilon}S'_\lambda) - \frac{1}{D^{\frac12}}S'_\lambda D_\varepsilon^{\frac12}S'_\lambda\frac{\partial_3^2}{D_{23}^2}D^{\frac12}\frac{1}{D_\varepsilon^{\frac12}}S'_\lambda\\ &\quad+\frac{1}{D_\varepsilon^{\frac32}}S'_\lambda D^{\frac12}\frac{\partial_{12}^2}{D_{23}}b\frac{\partial_{12}^2}{D^{\frac12}D_{23}}S'_\lambda\frac{1}{D_\varepsilon^{\frac12}}S'_\lambda\big)+O_{L^2}(1)\\ &= \partial_t\big(\big(-\frac{1}{D_\varepsilon}S'_\lambda a\partial_2^2S'_\lambda\frac{1}{D_\varepsilon}S'_\lambda\big)-\frac{\partial_3^2}{D_{23}^2}S'_\lambda+\frac{1}{D_\varepsilon}S'_\lambda b\partial_1^2\frac{\partial_2^2}{D_{23}^2}S'_\lambda\frac{1}{D_\varepsilon}S'_\lambda\big)+O_{L^2}(1).\end{aligned}$$

We use the identity

$$b\partial_1^2+a\partial_2^2+a\partial_3^2=-D_\varepsilon^2 \text{ in } L^2 \tag{28}$$

for the last term:

$$\begin{aligned}(\mathcal{MDN})_{22}S'_\lambda &= \partial_t\big(\big(-\frac{1}{D_\varepsilon}S'_\lambda a\partial_2^2S'_\lambda\frac{1}{D_\varepsilon}S'_\lambda\big)-\frac{\partial_3^2}{D_{23}^2}S'_\lambda-\frac{1}{D_\varepsilon}S'_\lambda D_\varepsilon^2\frac{\partial_2^2}{D_{23}^2}S'_\lambda\frac{1}{D_\varepsilon}S'_\lambda\\ &\quad-\frac{1}{D_\varepsilon}S'_\lambda a(\partial_2^2+\partial_3^2)\frac{\partial_2^2}{D_{23}^2}S'_\lambda\frac{1}{D_\varepsilon}S'_\lambda\big)+O_{L^2}(1)\\ &= \partial_t\big(-\frac{\partial_2^2+\partial_3^2}{D_{23}^2}S'_\lambda\big)+O_{L^2}(1)\\ &= \partial_tS'_\lambda+O_{L^2}(1).\end{aligned}$$

The estimate for $(\mathcal{MDN})_{23}$ is found by similar means. After straight-forward applications of Proposition 2.6, we find

$$(\mathcal{MDN})_{23}S'_\lambda=-\frac{1}{D_\varepsilon}S'_\lambda a\partial_{23}^2S'_\lambda\frac{1}{D_\varepsilon}S'_\lambda+\frac{\partial_{23}^2}{D_{23}^2}S'_\lambda+\frac{1}{D_\varepsilon}S'_\lambda b\partial_1^2S'_\lambda\frac{\partial_{23}^2}{D_{23}^2}\frac{1}{D_\varepsilon}S'_\lambda+O_{L^2}(1)$$

By plugging (28) into the last term, it follows like above

$$(\mathcal{MDN})_{23}S'_\lambda=O_{L^2}(1).$$

The estimates for $(\mathcal{MDN})_{24}$ and $(\mathcal{MDN})_{25}$ follow directly from Proposition 2.6. For $(\mathcal{MDN})_{26}$ we use Proposition 2.6 to write

$$\begin{aligned}(\mathcal{MDN})_{26}S'_\lambda &= -\frac{\partial_1\partial_2^2}{D_{23}^2}S'_\lambda-\frac{\partial_1\partial_3^2}{D_{23}^2}S'_\lambda+O_{L^2}(1)\\ &= \partial_1S'_\lambda+O_{L^2}(1).\end{aligned}$$

The estimate for $(\mathcal{MDN})_{31}$ follows as for $(\mathcal{MDN})_{12}$. For $(\mathcal{MDN})_{32}$ we can argue as for $(\mathcal{MDN})_{23}$ and for $(\mathcal{MDN})_{33}$ as for $(\mathcal{MDN})_{22}$. The estimate for $(\mathcal{MDN})_{34}$ follows as for $(\mathcal{MDN})_{24}$, for estimating $(\mathcal{MDN})_{35}$ we refer to $(\mathcal{MDN})_{26}$ and for $(\mathcal{MDN})_{36}$ to $(\mathcal{MDN})_{25}$.

Furthermore, the estimates for $(\mathcal{MDN})_{42}$ and $(\mathcal{MDN})_{43}$ follow directly from Proposition 2.6. For $(\mathcal{MDN})_{44}$ we note that

$$\begin{aligned}(\mathcal{MDN})_{44}S'_\lambda &= \partial_t\big(-\frac{\partial_1^2}{D^2}S'_\lambda+D_\varepsilon^{\frac12}S'_\lambda\frac{D_{23}^2}{D^2}\frac{1}{D_\varepsilon^{\frac12}}S'_\lambda\big)+O_{L^2}(1)\\ &= -\partial_t\big(\frac{\partial_1^2+\partial_2^2+\partial_3^2}{D^2}\big)S'_\lambda+O_{L^2}(1),\end{aligned}$$

and the claim follows because

$$-D^2=\partial_1^2+\partial_2^2+\partial_3^2 \text{ in } L^2. \tag{29}$$

The estimates for $(\mathcal{MDN})_{45}$, $(\mathcal{MDN})_{46}$, $(\mathcal{MDN})_{51}$, and $(\mathcal{MDN})_{52}$ are again direct consequences of Proposition 2.6.
Next,

$$\begin{aligned}(\mathcal{MDN})_{53}S_\lambda' &= b\frac{\partial_1\partial_2^2}{D_{23}^2}S_\lambda' + b\frac{\partial_1\partial_2^2}{D_{23}^2}S_\lambda' + O_{L^2}(1)\\ &= -b\partial_1 S_\lambda' + O_{L^2}(1).\end{aligned}$$

For $(\mathcal{MDN})_{55}$ we find

$$(\mathcal{MDN})_{55}S_\lambda' = -\partial_t\frac{\partial^2}{D^2}S_\lambda' + \partial_t\frac{\partial_1^2}{D^2}\frac{\partial_2^2}{D_{23}^2}S_\lambda' - \partial_t\frac{\partial_3^2}{D_{23}^2}S_\lambda' + O_{L^2}(1)$$

By (29), we can rewrite the second term to find

$$\begin{aligned}(\mathcal{MDN})_{55}S_\lambda' &= -\partial_t\frac{\partial_2^2}{D^2}S_\lambda' - \frac{\partial_2^2+\partial_3^2}{D^2}\frac{\partial_2^2}{D_{23}^2}S_\lambda' - \partial_t\frac{\partial^2}{D_{23}^2}S_\lambda' - \partial_t\frac{\partial_3^2}{D_{23}^2}S_\lambda' + O_{L^2}(1)\\ &= \partial_t S_\lambda' + O_{L^2}(1).\end{aligned}$$

We find by Proposition 2.6 and (29),

$$\begin{aligned}(\mathcal{MDN})_{56}S_\lambda' &= -\partial_t\frac{\partial_{23}^2}{D^2}S_\lambda' + \frac{\partial_1^2}{D^2}\frac{\partial_{23}^2}{D_{23}^2}\partial_t S_\lambda' + \partial_t\frac{\partial_{23}^2}{D_{23}^2}S_\lambda' + O_{L^2}(1)\\ &= -\partial_t\frac{\partial_{23}^2}{D^2}S_\lambda' - \frac{\partial_2^2+\partial_3^2}{D_{23}^2}\partial_{23}^2\partial_t S_\lambda' - \frac{\partial_{23}^2}{D_{23}^2}\partial_t S_\lambda' + \frac{\partial_{23}^2}{D_{23}^2}\partial_t S_\lambda' + O_{L^2}(1)\\ &= O_{L^2}(1).\end{aligned}$$

The estimate for $(\mathcal{MDN})_{61}$ can be carried out as for $(\mathcal{MDN})_{51}$. For $(\mathcal{MDN})_{62}$ and $(\mathcal{MDN})_{63}$ we refer to $(\mathcal{MDN})_{53}$. $(\mathcal{MDN})_{64}$ can be estimated as $(\mathcal{MDN})_{54}$, and $(\mathcal{MDN})_{65}$ is handled as $(\mathcal{MDN})_{66}$ and $(\mathcal{MDN})_{66}$ as $(\mathcal{MDN})_{55}$.
The proof is complete. □

3. Improved local well-posedness for quasilinear Maxwell equations

The purpose of this section is to improve the local well-posedness for

$$\begin{cases} P(x,\partial)(D,H) &= 0, \quad \nabla\cdot D = \nabla\cdot H = 0,\\ (D,H)(0) &\in H^s(\mathbb{R}^3;\mathbb{R})^6, \end{cases} \tag{30}$$

where $\varepsilon^{-1}(D) = \psi(|D|^2)$, where $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a smooth, monotone increasing function with $\psi(0) = 1$. Energy arguments give local well-posedness for initial data in $H^s(\mathbb{R}^3)$, $s > 5/2$. We shall use the previously derived Strichartz estimates to lower the regularity to $s > 13/6$. The proof follows the argument from [11] closely. Let $A = \sup_{0\leq t'\leq t}\|u(t')\|_{L^\infty_{x'}}$ and $B(t) = \|\nabla_{x'}u(t)\|_{L^\infty_{x'}}$ and suppose that smooth solutions

$$u = (D,H) : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3\times\mathbb{R}^3$$

exist locally in time.

The argument to prove local well-posedness for $s > 13/6$ consists of three steps:

- Showing energy estimates for solutions u to (30):

$$E^s(u(t)) \lesssim e^{C(A)\int_0^t B(s)ds}E^s(u(0)), \tag{31}$$

  where $E^s(u) \approx_A \|u\|_{H^s}$ for $s \geq 0$.

- Proving L^2-Lipschitz bounds for differences of solutions $v = u^1 - u^2$, where u^i solves (30) for $i = 1, 2$:

$$\|v(t)\|_{L^2}^2 \lesssim e^{C(A)\int_0^t B(s)ds} \|v(0)\|_{L^2}^2, \tag{32}$$

  where $A = \sup_{0\le t'\le t} \|u_1(t')\|_{L^\infty_{x'}} + \sup_{0\le t'\le t} \|u_2(t')\|_{L^\infty_{x'}}$, $B(t) = \|\nabla_{x'} u_1(t)\|_{L^\infty_{x'}} + \|\nabla_{x'} u_2(t)\|_{L^\infty_{x'}}$.
- Inferring continuous dependence using frequency envelopes (cf. [15, 3]).

Up to modifying the definition of the energy norm in three dimensions, the proof follows [11]. Hence, we only give the proof of the energy estimate.

Proposition 3.1. *Let $s \ge 0$. Then, we find (31) to hold. Moreover, for $u_0 \in H^s$, $s > 13/6$, we find that there is $T = T(\|u_0\|_{H^s})$ with T lower semicontinuous, such that*

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s}.$$

Proof. We consider energy norms as in [11]:

$$\|u\|_{E^s}^2 = \langle \langle D' \rangle^s u, C(u) \langle D' \rangle^s u \rangle \approx_A \|u\|_{H^s}^2.$$

To determine $C(u)$, we rewrite $\partial_t u = \mathcal{A}^j(u)\partial_j u$. For $j = 1, 2, 3$, we find

$$\mathcal{A}^j(u) = \begin{pmatrix} 0 & A_1^j(u) \\ A_2^j(u) & 0 \end{pmatrix}, \quad A_{1,2}^j \in \mathbb{R}^{3\times 3}.$$

It holds

$$(A_1^j)_{mn} = -\varepsilon_{imn},$$

where ε denotes the Levi–Civita symbol. Furthermore,

$$A_2^1 = 2\psi'(u) \cdot \begin{pmatrix} 0 & 0 & 0 \\ D_1D_3 & D_2D_3 & D_3^2 \\ -D_1D_2 & -D_2^2 & -D_2D_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \psi(u) \\ 0 & -\psi(u) & 0 \end{pmatrix},$$

$$A_2^2 = 2\psi'(u) \cdot \begin{pmatrix} -D_1D_3 & -D_2D_3 & -D_3^2 \\ 0 & 0 & 0 \\ D_1^2 & D_1D_2 & D_1D_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\psi(u) \\ 0 & 0 & 0 \\ \psi(u) & 0 & 0 \end{pmatrix},$$

$$A_2^3 = 2\psi'(u) \cdot \begin{pmatrix} D_1D_2 & D_2^2 & D_2D_3 \\ -D_1^2 & -D_1D_2 & -D_1D_3 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \psi(u) & 0 \\ -\psi(u) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To cancel the top-order term in the time derivative of $\|u\|_{E^s}^2$, we require

$$\mathcal{A}^j(u)^* C(u) = C(u)\mathcal{A}^j(u). \tag{33}$$

A suitable choice is found with the ansatz

$$C(u) = \begin{pmatrix} C_1 & 0 \\ 0 & 1_{3\times 3} \end{pmatrix}.$$

A straightforward computation then shows that C with

$$(C_1)_{ij} = 2\psi' \cdot D_i D_j + \psi \cdot \delta_{ij}$$

satisfies (33). It remains to verify $E^s(u) \approx_A \|u\|_{H^s}^2$. For this it is enough to show that

$$\langle \xi, C_1 \xi \rangle \approx_A |\xi|^2, \quad |\xi| = 1.$$

The boundedness from above is clear. For the boundedness from below, we compute

$$\begin{aligned}\langle \xi, C_1 \xi \rangle &= \psi \cdot |\xi|^2 + 2\psi' \cdot (D_1^2 |\xi_1|^2 + D_2^2 |\xi_2|^2 + D_3^2 |\xi_3|^2) \\ &\quad + 4\psi' \cdot (D_1 D_2 \xi_1 \xi_2 + D_1 D_3 \xi_1 \xi_3 + D_2 D_3 \xi_2 \xi_3) \\ &= \psi \cdot |\xi|^2 + 2 \cdot \psi' \cdot ((D_1 \xi_1 + D_2 \xi_2 + D_3 \xi_3)^2),\end{aligned}$$

which suffices by the monotonicity of ψ and $\psi(0) = 1$. For any $s > 13/6$, we can choose $\delta > 0$ such that due to Strichartz estimates provided by Corollary 1.5

$$\| \langle D' \rangle^{1-s} u \|_{L^{2+\delta}(0,T;L^\infty)} \lesssim_{\| \partial_x \varepsilon \|_{L^2 L^\infty}, T} \| u_0 \|_{L^2}$$

holds. We use the previous estimate to prove

$$\| \nabla_{x'} u \|_{L^{2+\delta}(0,T;L^\infty)} \lesssim \| u_0 \|_{H^s} \tag{34}$$

for $s > \frac{13}{6}$ and $\delta = \delta(s)$. By smoothness of solution, we require that $\| \nabla_{x'} u \|_{L^{2+\delta}(0,T_0;L^\infty)} \leq K$ for fixed $K > 0$ and maximally defined $T_0 > 0$. Take $T \in (0, T_0)$ with

$$\begin{aligned}&\| \partial_x \varepsilon \|_{L^2(0,T;L^\infty)} \leq T^{\delta'} \| \partial_x \varepsilon \|_{L^{2+\delta}(0,T;L^\infty)} \lesssim_A T^{\delta'} K \leq 1 \\ &\text{and } \| \partial_x \varepsilon \|_{L^1(0,T;L^\infty)} \lesssim_A T^{\frac{1}{2}+\delta'} K \leq 1.\end{aligned}$$

This yields uniform constants in the energy inequality (31)

$$E^s(u(t)) \lesssim e^{c(A) \int_0^t B(t') dt'} E^s(u(0)) \tag{35}$$

and in the Strichartz estimate

$$\| \langle D' \rangle^{-\alpha} w \|_{L^p(0,T;L^q)} \lesssim \| w_0 \|_{L^2} + \| P(x, \partial) w \|_{L^1 L^2} \tag{36}$$

for $\alpha > \rho + \frac{1}{3p}$ from Corollary 1.5 with $\tilde{s} = 1$ for $\partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3 = 0$ and $\partial_1 w_4 + \partial_2 w_5 + \partial_3 w_6 = 0$. For low frequencies, Bernstein's inequality and (35) yield

$$\| S'_{\lesssim 1} \nabla_{x'} u \|_{L^{2+\delta} L^\infty} \lesssim T^{\frac{1}{2}+\delta'} \| u_0 \|_{L^2}.$$

For high frequencies, we consider the auxiliary function $v = \langle D' \rangle^s u$, which still satisfies the divergence condition. Moreover,

$$\| P(x, u, \partial) v(t) \|_{L^2} = \| [P(x, u, \partial), \langle D' \rangle^s] u(t) \|_{L^2} \lesssim_A \| \nabla u(t) \|_{L^\infty_{x'}} \| u(t) \|_{H^s}, \tag{37}$$

where $P(x, u, \partial)$ has coefficients $\varepsilon(u)^{-1}$. For the Strichartz pair $(1 + \varepsilon, 2 + \delta, \infty, 3)$, estimate (34) yields

$$\| S'_{\geq 1} \nabla_{x'} u \|_{L^{2+\delta} L^\infty} \lesssim \| \langle D' \rangle^{1-s} v \|_{L^{2+\delta} L^\infty} \lesssim \| v_0 \|_{L^2} + \| P(x, u, \partial) v \|_{L^1 L^2}$$

since we can choose ε and δ small enough such that $s > 2 + \varepsilon + \frac{1}{3(2+\delta)}$. By (37) and (35), we conclude

$$\begin{aligned}\| S'_{\geq 1} \nabla_{x'} u \|_{L^{2+\delta} L^\infty} &\lesssim_A \| u_0 \|_{H^s} + \| \nabla_{x'} u \|_{L^1 L^\infty_{x'}} \| u \|_{L^\infty H^s} \\ &\lesssim_A \| u_0 \|_{H^s} (1 + T^{\frac{1}{2}} \| \nabla_{x'} u \|_{L^{2+\delta} L^\infty}).\end{aligned}$$

At this point the argument follows the proof of [11, Proposition 6.1]. □

The L^2-Lipschitz bounds for differences of solutions follow with the definition of the energy norm from above as in [11], which allows to conclude the proof of Theorem 1.7 using frequency envelopes. We omit the details to avoid repetition.

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