

The averaging bias - A standard miscalculation, which extensively underestimates real CO₂ emissions

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Funding information

DFG, Grant/Award Numbers: Project 237267381 (TRR 150), Project 426888090 (SFB 1441)

The substitution of energy based on fossil fuels in different sectors like household or traffic by electric energy saves CO₂ of this specific sector due to decreased fossil fuel consumption. An important quantity is the additional CO₂ emission $\Delta F(\bar{D}, \Delta D)$ due to an increased electric power demand ΔD for the average electricity power demand \bar{D} . Commonly, the formula $\Delta F(\bar{D}, \Delta D) \approx M(\bar{D})\Delta D$ is used (called simplified formula), where $M(\bar{D})$ represents mean average CO₂ footprint. It is shown in the present manuscript, that the simplified formula may underestimate the CO₂ footprint significantly if the average CO₂ footprint depends on the average electricity power demand, which is the case for most of mixed partly renewable and partly non-renewable electric energy systems. Therefore, the real CO₂ emissions would outmatch those according to simplified easily by factor 2 in reality depending on the status of the electricity system. In order to establish a more precise calculation of the CO₂ footprint, the general formula $\Delta F(\bar{D}, \Delta D) = \bar{D}\Delta M(\bar{D}, \Delta D) + \Delta D M(\bar{D} + \Delta D)$ which is exact and contains the simplified formula as a special case, is derived in this article. The simplified formula requires an additional term that takes into account the change of the mean average CO₂ footprint ΔM depending on the electricity power demand.

KEYWORDS

CO₂ emissions, electricity, fossil-based energy, Leibniz's fundamental theorem of calculus, non-fossil-based energy

1 | GENERAL INTRODUCTION

The rapid reduction of global CO₂ emissions is the key recommendation of the Intergovernmental Panel of Climate Change IPCC [1]. Policymakers around the world are responding to enable this ambitious target [2, 3]. A total global remaining CO₂ budget of 420 Gt for all humanity was analyzed by the IPCC to limit global warming to 1.5 °C. Detailed probabilities for the achievement of the warming limit have been determined but are unimportant for the focus of this publication.

A policy approach to manage and analyze the reduction of CO₂ emissions is to define different sectors such as electric power, transport, industry, and households. Each sector is typically regulated with a tighter limit on CO₂ emissions, that is, a 50% reduction. However, looking at each sector in isolation can lead to inaccurate estimates of CO₂ emissions because the sectors interact.

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TABLE 1 CO₂ equivalents CO_{2i} of different technologies, according to [18]; oil is of minor importance and neglected in this publication

<i>i</i>	Technology	$\bar{E}_i^{CO_2}$ in g _{CO₂} /kWh	Classification
1	Wind Power ^a	9	Regenerative
2	Hydropower	23	Regenerative
3	Photovoltaics	50	Regenerative
4	Biomass	70	Regenerative
5	Nuclear	24	Non-regenerative
6	Gas	499	Non-regenerative
7	Hard Coal	830	Non-regenerative
8	Brown Coal	1075	Non-regenerative

^aWind Power technology includes the respective contributions of onshore and offshore.

An example is the heat supply of a building. The advisable substitution of an oil-burner by a modern heat-pump eliminates the CO₂ emissions of the sector “households”, as the oil consumption is eliminated. As a consequence the power demand of the sector “electric energy” is increased in order to operate the new electric heat-pump. The CO₂ reduction of the sector “household” can be easily determined. For instance, a decrease of oil consumption of $\Delta V_{Oil}/\Delta t = -1000$ l/year leads to a decrease of CO₂ emissions of $\Delta m_{CO_2}/\Delta t = -3200$ kg/year.

But how does the additional demand for electrical energy increase CO₂ emissions from the “electrical energy” sector? For the sector “electric energy” a constant average CO₂ footprint M (unit: g_{CO₂}/kWh) is available. The standard calculation of the CO₂ impact of increased electric power demand ΔD (unit: GW) for a exemplary time period Δt is typically calculated as [4–14]:

$$\frac{\Delta m_{CO_2 \text{ sector electric energy}}}{\Delta t} = M \Delta D. \quad (1)$$

Please note that the unit of M respectively ΔD needs to be adapted in order to calculate the correct dimension of the result. The outline of the article is as follows: Section 2 gives an overview with respect to the different energy sectors and the corresponding CO₂ footprint. In Section 3, the fundamental theorem of calculus is used to relate the CO₂ impact due to an increased electric power demand to the average CO₂ footprint M . It is shown that the dependence of the average CO₂ on the power demand ΔD has to be taken into account in order not to underestimate the CO₂ footprint. Several examples are discussed. Section 4 summarizes the results.

2 | ANALYSIS

In order to derive the CO₂ emissions of the sector “electric energy”, the characteristics of electric power generation must be analyzed. An hourly resolved matrix P_{ij} electric power generation for Germany in the year 2017 [15, 16] is the basis of the analysis with i specifying the electricity source and j the hour in one representative year. For $j \in [1, 8760]$ hours, the electric power P_{ij} of eight electricity sources i is known. The year 2017 has been chosen, as the Matrix P_{ij} of 2017 has been the latest available hourly resolved complete dataset. However, the chosen year does not influence the general analysis of the averaging bias at all. The CO₂ impact for all technologies i is depicted in Table 1.

As an example; P_{1j} denotes Wind Power, P_{8j} denotes Brown Coal Power for each hour j . “Regenerative Power” P_j^{reg} , “Non-Regenerative Power” P_j^{nreg} and “Supply” S_j are defined as hourly averaged values as follows

$$P_j^{\text{reg}} = \sum_{i=1}^4 P_{ij}, \quad (2)$$

$$P_j^{\text{nreg}} = \sum_{i=5}^8 P_{ij}, \quad (3)$$

$$S_j = \sum_{i=1}^8 P_{ij} = \sum_{i=1}^4 P_{ij} + \sum_{i=5}^8 P_{ij} = P_j^{\text{reg}} + P_j^{\text{nreg}}. \quad (4)$$

Note that the expression “regenerative energy” is not correct from the thermodynamic perspective. Nevertheless, it is used in this article as it represents a common definition for photovoltaics, wind, water, and biomass based electric power.

The indices depict an average value over 1 h, that is, $P_{3\ 1849}$, represents the average “Photovoltaics Power” for hour 1849. Please note, that the consideration of import and export energy transport via the system boundary requires an extra balance factor B_j . Also an additional energy storage capacity based power storage and supply contribution P_j^{st} will be necessary. Losses due to electric resistance and the electric power transformation are denoted by H_j . Therefore, the electric energy demand D of hour j is defined as a function of the relevant energy contributors as

$$D_j = P_j^{\text{reg}} + P_j^{\text{nreg}} + B_j + P_j^{\text{st}} + H_j = S_j + B_j + P_j^{\text{st}} + H_j. \quad (5)$$

The yearly average of B_j amounts to roughly 10% of the yearly average total energy demand D_j . H_j typically scales between 4% and 10% of D_j [11, 17]. Within the next decades the energy storage capacity based power request and support P_j^{st} becomes more important as the following situation will occur more and more frequently, especially after the year 2030

$$P_j^{\text{reg}} > D_j. \quad (6)$$

However, for the derivation of the averaging bias, the import/export balance B_j , the electric resistance H_j as well as the energy storage P_j^{st} are set equal to zero, which implies $S_j = D_j$. In general, the supply is a function of the energy demand, since the supply must satisfy the demand. Following Equation (5), the supply of energy S_j is a function of energy demand D_j , with priority of energy contributor P_j^{reg} . Index j again denotes the average value of a certain hour j . This can be expressed as

$$S_j = f_1(D_j), \quad (7)$$

$$P_j^{\text{reg}} = f_2(\text{weather, status electricity grid, } \dots), \quad (8)$$

$$P_j^{\text{nreg}} = f_3(D_j, P_j^{\text{reg}}). \quad (9)$$

The functions f_1, f_2, f_3 symbolize the general and complex dependency of electric demand and supply as well as the interaction between weather and boundary conditions on P_j^{reg} and the resulting dependency of P_j^{nreg} .

Figure 1 depicts the contribution of different energy sources to power generation in the year 2017. The transient behavior of P_{1j} (Wind), P_{8j} (Brown Coal) as well as P_j^{reg} (Sum Regenerative) and P_j^{nreg} (Sum Non Regenerative) can be seen. Already in the year 2017 an impressive contribution of renewable electric energy P_j^{reg} was established. Of course a rising importance of P_j^{reg} is anticipated according to [19, 20].

The electricity demand D is not plotted in Figure 1, but fluctuates between 40 and 80 GW within a year. The renewable energy supply P_j^{reg} varies between 7.7 and 61 GW, wherein biomass enables a regenerative baseload. Non-regenerative energy is typically needed to close the gap with a P_j^{nreg} peak of 62 GW. The following analysis considers an unlimited energy transport within the system boundaries.

In order to determine the CO₂ impact of the complete sector “electric energy”, the detailed contribution of different electric energy sources must be considered as defined in Table 1. The combination of a detailed knowledge of each electric energy source (i.e., Hydropower, Wind, Gas, etc.) and the specific CO₂ equivalent impact of each technology $\bar{E}_i^{\text{CO}_2}$ enables the calculation of CO₂ emissions of $P_j^{\text{reg}}, P_j^{\text{nreg}}$ and S_j , according to Equations (10)-(12) for every hour j . The result is depicted in Figure 2. The necessary equations are

$$E_j^{\text{reg CO}_2} = \frac{1}{\sum_{i=1}^4 P_{ij}} \sum_{i=1}^4 P_{ij} \cdot \bar{E}_i^{\text{CO}_2} = \frac{1}{P_j^{\text{reg}}} \sum_{i=1}^4 P_{ij} \cdot \bar{E}_i^{\text{CO}_2}, \quad (10)$$

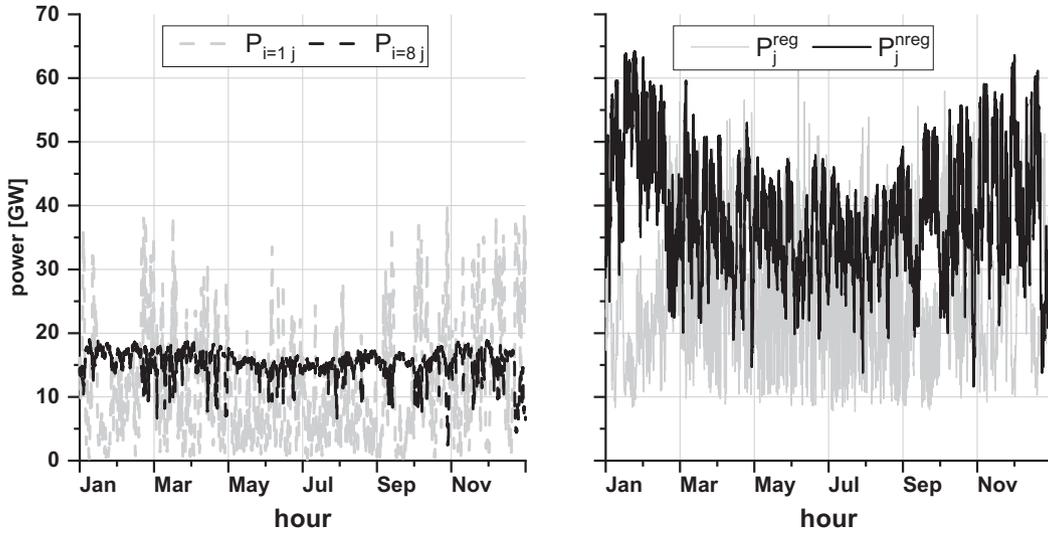


FIGURE 1 Evolution of electric power generation over 8760 h: Evolution of P_{1j} (Wind), P_{8j} (Brown Coal), P_j^{reg} (Sum Regenerative) and P_j^{nreg} (Sum Non Regenerative). Source: Germany 2017 data according to [15]

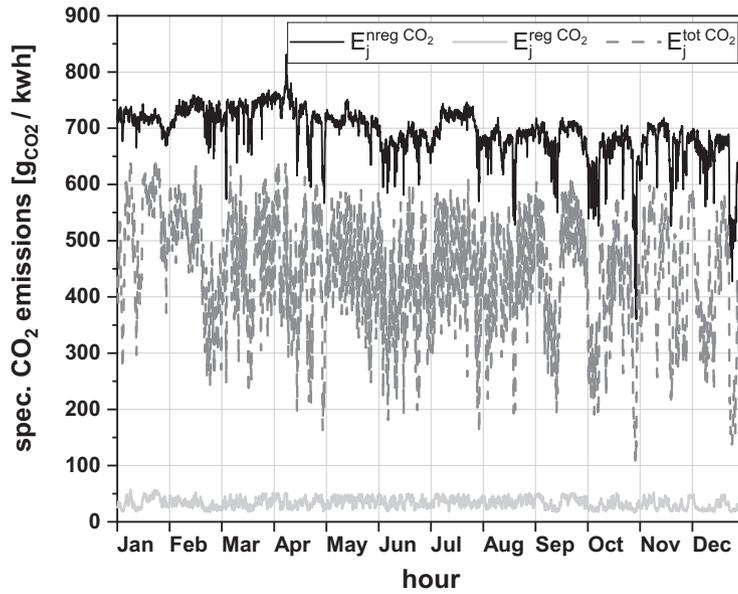


FIGURE 2 CO₂ emissions of electric power production of 8760 h, evolution of E_j^{reg} , E_j^{nreg} and E_j^{tot} , year 2017

$$E_j^{\text{nreg CO}_2} = \frac{1}{\sum_{i=5}^8 P_{ij}} \sum_{i=5}^8 P_{ij} \cdot \bar{E}_i^{\text{CO}_2} = \frac{1}{P_j^{\text{nreg}}} \sum_{i=5}^8 P_{ij} \cdot \bar{E}_i^{\text{CO}_2} \quad \text{and} \quad (11)$$

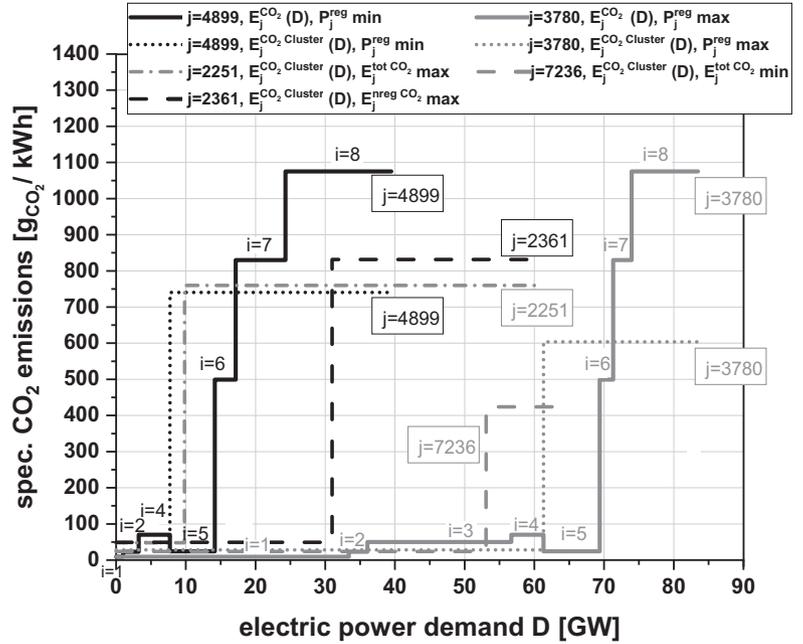
$$E_j^{\text{tot CO}_2} = \frac{1}{\sum_{i=1}^8 P_{ij}} \sum_{i=1}^8 P_{ij} \cdot \bar{E}_i^{\text{CO}_2}. \quad (12)$$

Note that P_j^{reg} is mainly depending on the weather and the status of the electric grid. But especially P_j^{nreg} is a function of energy demand D_j . Therefore, $E_j^{\text{nreg CO}_2}$ as well as $E_j^{\text{tot CO}_2}$ are also a function of energy demand D_j .

In order to define the CO₂ emissions as a function of the hourly averaged electric power demand D_j , two different possibilities are presented in Equations (13)-(15). In the first approach one assumes

$$E_j^{\text{CO}_2} = \bar{E}_{k_j^{\text{min}}}^{\text{CO}_2} \quad (13)$$

FIGURE 3 Year 2017: CO₂ emissions of selected hours as a function of electric energy demand
 $j = 4899$, 24.7.2017; 03.00-04.00: minimal regenerative power P_j^{reg} ; $P_j^{reg} = 7.7$ GW, note: $P_3 = 0$
 $j = 3780$, 7.6.2017; 12.00-13.00: maximal regenerative power P_j^{reg} ; $P_j^{reg} = 61.3$ GW
 $j = 2251$, 4.4.2017; 20.00-21.00: worst total energy mix $E_j^{tot CO_2} max$; $E_j^{tot CO_2} = 643.6$ gCO₂/kWh
 $j = 7236$, 29.10.2017; 13.00-14.00: best total energy mix $E_j^{tot CO_2} min$; $E_j^{tot CO_2} = 96.7$ gCO₂/kWh
 $j = 2361$, 9.4.2017; 09.00-10.00: worst non-regenerative energy mix $E_j^{nreg CO_2} max$; $E_j^{nreg CO_2} = 831.5$ gCO₂/kWh



with $1 \leq k_j^{\min} \leq 8$ defined for each hour j by as the minimum k satisfying

$$\sum_{i=1}^k P_{ij} \geq D_j. \quad (14)$$

Equation (13) and the condition (14) imply, that regenerative energy has priority and within the electricity system and for a given electricity demand D_j the electricity contribution of technologies i with lowest CO₂ impact is supplied with priority.

Figure 3 illustrates, that for hour 4899 and 3780 the theoretical behavior of the CO₂ impact $E_j^{CO_2}$ is illustrated. As the renewable energy is typically not sufficient as $D_j > P_j^{reg}$, additional energy P_j^{nreg} must be provided. Therefore, a decrease of CO₂ impact is depicted because of nuclear power ($i = 5$) while afterwards the CO₂ impact increases again up to a specific brown coal energy value of 1075 gCO₂/kWh.

Indeed Equation (13) is only the consequence of the aforementioned theoretical assumption, that only the technology i with the lowest CO₂ impact is applied step by step. More technologies i would be added consecutively to satisfy the demand in theory. However, most electricity contributors co-contribute simultaneously due to electricity net constraints and long distance electricity transport challenges. Therefore the CO₂ impacts of $E_j^{reg CO_2}$ and $E_j^{nreg CO_2}$ are determined in a second approach as a function of electric power generation P_j , which is equivalent to the power demand D for the given assumptions

$$E_j^{CO_2 \text{ cluster}}(D_j) = \begin{cases} E_j^{reg CO_2}, & D_j \leq P_j^{reg}, \\ E_j^{nreg CO_2}, & D_j > P_j^{reg}. \end{cases} \quad (15)$$

Equation (15) represents a further adaption to realistic boundary conditions as all regenerative contributors are bundled for low electric power demand and all non-regenerative contributors are bundled for high electric power demand. The result of Equation (13)-(15) is depicted in Figure 3. It illustrates selected representative elements of the 8760 h matrix.

The results for both, Equation (13) and (15), respectively, are shown for hour 4899 and 3780. The minimal regenerative power ($j = 4899$) of 7.7 GW at night with a contribution of hydropower ($i = 2$) and biomass ($i = 4$) is quite limited. The result $E_j^{CO_2 \text{ Cluster}}$ of the alternative Equation (15) is plotted additionally and depicts the low regenerative CO₂ footprint and a non-regenerative average specific emission of 741 gCO₂/kWh. Both results are also plotted for hour $j = 3780$ with a maximal regenerative power of 61.3 GW, wherein wind ($i = 1$) and photovoltaics ($i = 3$) dominate the regenerative contribution. Note that the non-regenerative footprint of 604 gCO₂/kWh is smaller compared to hour $j = 4899$. The relative contribution of nuclear power ($i = 5$) to the total non-regenerative power supply in hour $j = 3780$ causes the difference.

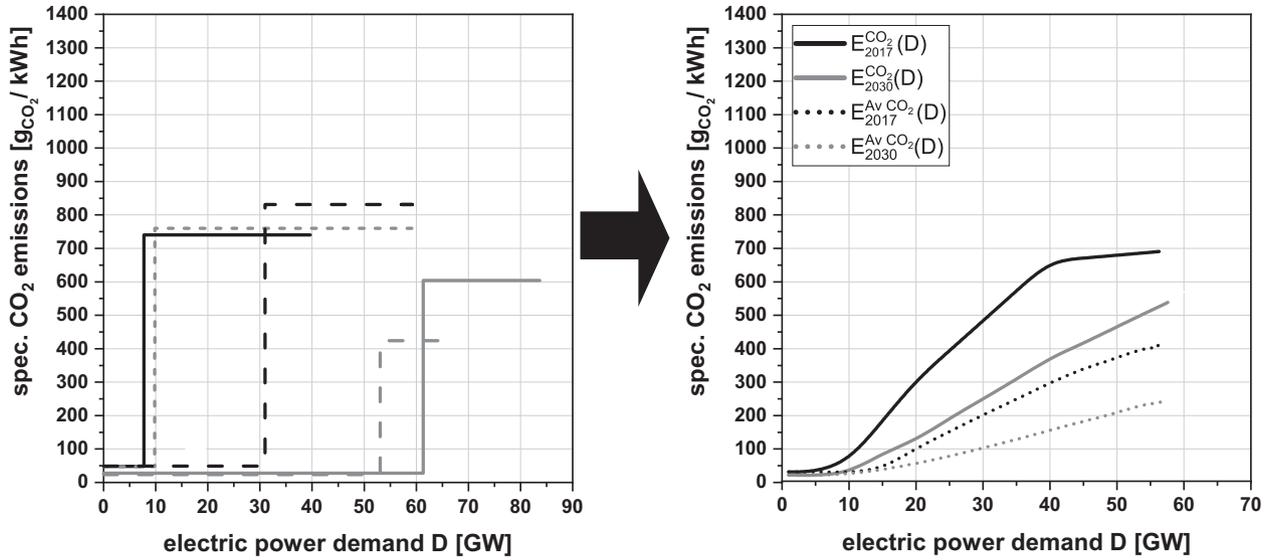


FIGURE 4 left: examples of $E_j^{CO_2 \text{ cluster}}$ of the year 2017 as in Figure 3; right: evolution of $E_{2017}^{CO_2}$ and $E_{2017}^{Av CO_2}$, 2017: $\bar{D}=56.3$ GW, $E_{2017}^{Av CO_2}$ (56.3 GW)=412 g_{CO_2}/kWh ; evolution of $E_{2030}^{CO_2}$ and $E_{2030}^{Av CO_2}$, 2030: $\bar{D} = 57.6$ GW; $E_{2030}^{Av CO_2}$ (57.6 GW) = 244 g_{CO_2}/kWh

Hour $j = 2251$ represents the highest total energy specific footprint $E_j^{\text{tot } CO_2}$ of 644 g_{CO_2}/kWh as the regenerative output contributes only 9.8 GW but the total demand was 60.1 GW. On the other hand, hour $j = 7236$ illustrates the lowest total energy specific footprint $E_j^{\text{tot } CO_2}$ of 96.7 g_{CO_2}/kWh with a dominant regenerative contribution of 53.1 GW. Finally, hour 2361 illustrates the highest non-regenerative specific footprint $E_j^{\text{nreg } CO_2}$ of 831.5 g_{CO_2}/kWh due to a dominant coal energy contribution.

The yearly averaged CO_2 impact $E_{2017}^{CO_2}$ as a function of the energy demand D can be calculated by the following equation

$$E_{2017}^{CO_2}(D) = \frac{1}{8760} \sum_{j=1}^{8760} E_j^{CO_2 \text{ cluster}}(D_j). \quad (16)$$

Note that D is identical to the total electric power generation P according to the simplifying assumptions and Equation (5). Furthermore, the moving average value $E_{2017}^{Av CO_2}$ is defined as

$$E_{2017}^{Av CO_2}(D) = \frac{1}{D} \int_0^D E_{2017}^{CO_2}(\bar{D}) d\bar{D} \quad (17)$$

with the yearly averaged electricity demand \bar{D}

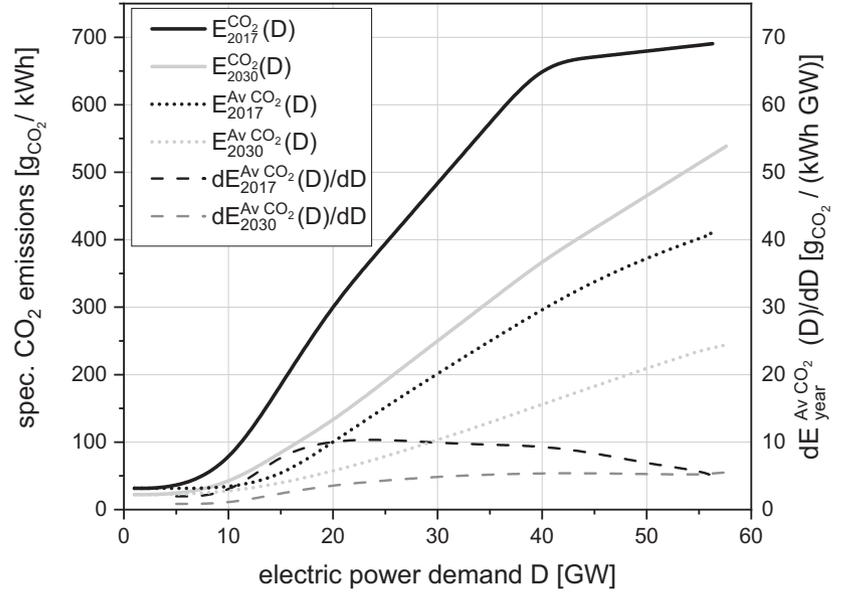
$$\bar{D} = \frac{1}{8760} \sum_{j=1}^{8760} D_j. \quad (18)$$

Please note that $E^{CO_2}(\bar{D})$ depends on the year via weather conditions, technology change and the adapted demand, indicated by an index denoting the year, for example, by $E_{2017}^{CO_2}(\bar{D})$ and later by $E_{2030}^{CO_2}(\bar{D})$. Note that $E_{2017}^{Av CO_2}(\bar{D})$ corresponds to M in Equation (1).

Figure 4 illustrates the results of Equations (16) and (17) for the data of 2017. The evolution $E_j^{CO_2 \text{ cluster}}$ of selected hours were discussed in Figure 3 but are replotted on the left hand side of Figure 4 in order to demonstrate the derivation of $E_{2017}^{CO_2}$. Also the moving average value $E_{2017}^{Av CO_2}$ is plotted according to Equation (17) on the right hand side.

In addition a simulation of the year 2030 has been accomplished [16] with detailed information about the scale up of regenerative power installation according to [19, 20]. Furthermore, the increase of energy storage capacities is considered

FIGURE 5 $E_{2017}^{CO_2}(D)$, $E_{2030}^{CO_2}(D)$ and $E_{2017}^{Av CO_2}(D)$, $E_{2030}^{Av CO_2}(D)$ as in Figure 4 (right diagram), $dE_{2017}^{Av CO_2}(D)/dD$ and $dE_{2030}^{Av CO_2}(D)/dD$ on the right axis. Please note that $dE_j^{Av CO_2}(D)/dD$ is equivalent to the simplified derivative dM/dD of equation (1) respectively dM/dx of the general formula (see, Equation (31))



as well as the increase of electricity demand due to ambitious heat pump or battery electric vehicle penetration scenarios with increased \bar{D} as a consequence. These results are also depicted in Figure 4. However, detailed explanations of the 2030 calculation are not in the focus of this publication, as the general analysis is of major interest.

The main question remains the analysis of the CO₂ impact of an increased electricity demand ΔD . Substituting $\Delta m_{CO_2, sector\ electric\ energy}$ in the following by the simplified notation Δm_{CO_2} leads to the commonly used equation (see also Equation (1)): $\Delta m_{CO_2} = M \Delta D \Delta t$. Indeed the correct calculus, that is, for the year 2030 is defined as shown in Equation (19)

$$\Delta m_{CO_2} = \Delta t \int_{\bar{D}}^{\bar{D}+\Delta D} E_{2030}^{CO_2}(\bar{D}) d\bar{D}. \tag{19}$$

Besides $E_{2017}^{CO_2}(D)$ and $E_{2017}^{Av CO_2}(D)$ the derivative of $dE_{2017}^{Av CO_2}(D)/dD$ becomes of major importance, which is explained in the next section. Note that $dE_{2017}^{Av CO_2}(D)/dD$ is equivalent to the derivative $dM(D)/dD$ according to the nomenclature of Equation (1), which is equivalent to $dM(x)/dx$ of the general formula (see, (31) in Section 3). Also note that this derivative remains important over the years, especially for an electric power demand in the range of 60 GW. Although $E_{2030}^{CO_2}(D)$ and $E_{2030}^{Av CO_2}(D)$ are significantly smaller than in the year 2017, the derivative $dE_{2030}^{Av CO_2}(D)/dD$ becomes more important, as $E_{2030}^{Av CO_2}(D)$ has even a slightly steeper gradient for D approximately equal to 60 GW, which is depicted in Figure 5.

3 | MATHEMATICAL FORMULATION OF THE PROBLEM

3.1 | Fundamental theorem of calculus

The fundamental theorem of calculus (Erster Hauptsatz der Differential- und Integralrechnung) relates the two fundamental concepts of calculus, that of integration and that of differentiation. It states that derivation and integration are mutual inverses (up to a constant) [21, 22]. The theorem is stated in the following: Let $I = [a, b]$ be a closed interval on the real line \mathbb{R} and $c \in I$. Furthermore, let $f : I \rightarrow \mathbb{R}$ be a real-valued continuous function defined on I . Then, the function

$$F(x) = \int_c^x f(s) ds \tag{20}$$

is continuous on I and continuously differentiable on the open interval (a, b) with

$$\frac{d}{dx}F(x) = F'(x) = f(x). \quad (21)$$

The total differential of $F(x)$ is

$$dF(x) = F' dx = f(x) dx. \quad (22)$$

If $f(x)$ can be assumed to be non-negative, then the integral $F(x)$ can be interpreted based on the area under the curve $f(x)$ in the interval $I = [c, x]$. This implies that an increment $dF(x) = F'(x) dx$ represents an infinitesimal small area in the range x and $x + dx$.

Example 3.1a: One may consider the linear function $f(x) = \alpha x$ with the real constant $\alpha > 0$ and the interval $I = [c, x]$. Then

$$F(x) = \frac{\alpha}{2}(x^2 - c^2) \quad (23)$$

holds. The derivative of $F(x)$ is

$$F'(x) = \alpha x, \quad (24)$$

and the total differential of $F(x)$ takes the form

$$dF(x) = \alpha x dx. \quad (25)$$

Example 3.1b: In the context of estimating the specific CO₂ footprint due to a power demand, one may specifically consider x as electrical power demand D in kW, $f(x)$ as the specific CO₂ emission in g_{CO₂}/kWh and $F(x)$ as the CO₂ footprint (CO₂ impact) within the range of energy demand $c = D_1$, $x = D_2$ with $D_1 \leq D_2$.

3.2 | Implication of the fundamental theorem for moving averages

The mean value M of the function $f(x)$ on the interval $I = [a, b]$ computes as

$$M(a, b) = \frac{1}{b-a} \int_a^b f(s) ds. \quad (26)$$

For the special case that $a = 0$ and $b = x$, one obtains for variable x a moving average

$$M(0, x) = \frac{1}{x} \int_0^x f(s) ds, \quad (27)$$

or, equivalently, with $M(x) = M(0, x)$,

$$x M(x) = \int_0^x f(s) ds. \quad (28)$$

Determining the total differential of both sides of (28) gives, taking into account the relations (21) and (22) (note that $dF(x) = f(x) dx$),

$$d(xM(x)) = f(x) dx = dF(x). \quad (29)$$

Applying the product rule

$$d(xM(x)) = M(x) dx + x dM(x) \quad (30)$$

gives (*general formula*)

$$M(x) dx + x dM(x) = f(x) dx = dF(x), \quad (31)$$

which states that the increment of $F(x)$ is equal to the sum of the increment of x multiplied by M and the increment of M multiplied by x . The special case (*simplified formula*)

$$M(x) dx = f(x) dx = dF(x) \quad (32)$$

is only valid, if $|dM(x)| \ll 1$ holds exactly or approximately.

Equation (31) can be reformulated using finite increments. With

$$F(x) = \int_0^x f(s) ds, \quad \Delta F(x, \Delta x) = \int_x^{x+\Delta x} f(s) ds \quad (33)$$

one obtains

$$M(x) = \frac{F(x)}{x}, \quad M(x + \Delta x) = \frac{F(x) + \Delta F(x, \Delta x)}{x + \Delta x} \quad (34)$$

or, equivalently,

$$xM(x) = F(x), \quad (x + \Delta x)M(x + \Delta x) = F(x) + \Delta F(x, \Delta x). \quad (35)$$

Taking the difference of the last two equations results in

$$\boxed{x\Delta M(x, \Delta x) + \Delta x M(x + \Delta x) = \Delta F(x, \Delta x)} \quad (36)$$

with

$$\Delta M(x, \Delta x) = M(x + \Delta x) - M(x). \quad (37)$$

In simplified notation Equation (36) may be recast as

$$\boxed{x\Delta M + \Delta x M = \Delta F} \quad (38)$$

which will be called *general formula for estimating the CO₂ impact* in the following¹. The Equation (38) is valid for increments of arbitrary size, but the arguments of the functions entering in Equation (36) have to be taken into account carefully.

It should be noted that, in the context of estimating the CO₂ footprint, the *simplified formula*

$$\boxed{\Delta x M \approx \Delta F} \quad (39)$$

is commonly used. In particular, by (39), the increase in $\Delta F(x, \Delta x)$ may be severely underestimated. As demonstrated in the previous section, such positive values ΔM are not uncommon.

Example 3.2a: Assume the function $f(x)$ to be constant on the interval $I = [0, \infty)$: $f(x) = f_0$. Then, the mean value of the function is constant ($dM(x) = 0$) and equal to the constant value of the function: $M(x) = f_0 = M_0$. As a result, the equations

$$M(x) dx = f(x) dx \quad (40)$$

¹ Note added in proof: The general formula (38) with finite increments is derived in Equations (33)-(36) without reference to the fundamental theorem.

and

$$\Delta x M = \Delta F \quad (41)$$

hold exactly. Therefore, for the special case of constant functions $f(x)$, the simplified formula (39) for the CO₂ impact is exact.

Example 3.2b: Assume that the function $f(x)$ is piecewise constant² except for a jump at $x = x_0$ from 0 to $f_0 > 0$, that is,

$$f(x) = \begin{cases} 0, & x \leq x_0, \\ f_0, & x > x_0. \end{cases} \quad (42)$$

Then it follows for $F(x)$ and for the moving average $M(x)$

$$F(x) = \begin{cases} 0, & x \leq x_0, \\ f_0(x - x_0), & x > x_0 \end{cases} \quad (43)$$

and

$$M(x) = \begin{cases} 0, & x \leq x_0, \\ f_0 \frac{x - x_0}{x}, & x > x_0. \end{cases} \quad (44)$$

At $x = x_0$ the moving average changes from zero to positive values with slope $M'(x = x_0) = f_0/x_0$. This implies that, for a large jump f_0 , there is a rapid change of the mean value close to $x = x_0$. Additionally, the approximation $\Delta F(x) \approx M(x)\Delta x$ is clearly inaccurate.

Example 3.2c: Again consider the linear function $f(x) = \alpha x$ or $df(x) = \alpha dx$ involving a positive constant $\alpha > 0$. This implies $M(x) = M(0, x) = \alpha x/2$ and $dM(x) = \alpha dx/2$.

It follows that the general formula (see Equations (30), (31), (38))

$$d(xM(x)) = f(x) dx \quad (45)$$

is naturally satisfied, whereas the simplified formula (see equations (32), (39))

$$M(x) dx = f(x) dx \quad (46)$$

is not fulfilled. Indeed, for general α , the term $x dM(x)$, that is, the change of $M(x)$ with x , is not taken into account.

For the integral one obtains

$$\Delta F(x, \Delta x) = \int_{x_0}^{x_0 + \Delta x} \alpha x dx = \alpha x_0 \Delta x + \frac{\alpha \Delta x^2}{2} = f_0 \Delta x + \frac{\Delta f}{2} \Delta x. \quad (47)$$

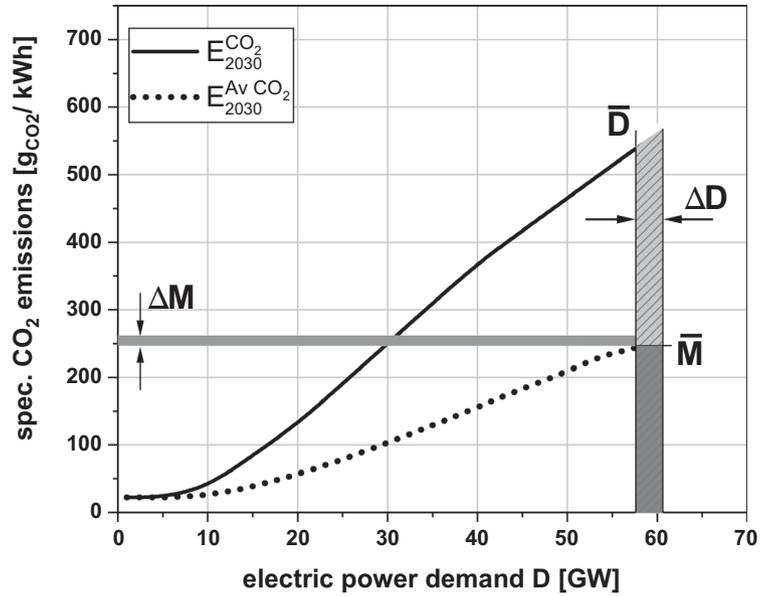
Based on $f_0 = \alpha x_0$, $\Delta f = \alpha \Delta x$ and $M(x_0) = f_0/2$, this result may be decomposed into

$$\Delta F(x, \Delta x) = M(x_0) \Delta x + \frac{f_0}{2} \Delta x + \frac{\Delta f}{2} \Delta x. \quad (48)$$

With this example in mind it becomes clear, that the average $M(x_0)$ multiplied by Δx as an estimator for $\Delta F(x, \Delta x)$, that is, $\Delta F(x, \Delta x) \approx M(x_0)\Delta x$, produces an erroneous result, because the terms $f_0\Delta x/2$ and $\Delta f\Delta x/2$ have been neglected.

²Note added in proof: In the submitted version of the article, a piecewise continuous function was assumed in Equation (20). If one considers piecewise continuous integrands f , as only done in this example, only the one-sided derivatives of the running integral exist at the discontinuity points of f (which moreover coincide with the left- or right-sided limits of the integrand in the discontinuity points).

FIGURE 6 Graphical illustration of Equations (50) and (51) Please note that the depicted areas represent $\bar{M}(\bar{D})\Delta\bar{D}$ and $\bar{D}\Delta M(\bar{D}, \Delta\bar{D})$



4 | DISCUSSION AND CONCLUSION

For the calculation of CO₂ emissions of additional electric energy demand, insufficient simplified mathematic models are typically used, which might be motivated by the complexity of the electricity supply sources and the grid situation. An example for such a simplified formula to analyze the additional CO₂ emissions per time interval $\Delta F(\bar{D}, \Delta D)$ caused by additional electric power ΔD (unit: Watt) is the direct utilization of the average CO₂ emission footprint $M(\bar{D})$ (unit g_{CO₂}/kWh) for a given average electricity demand \bar{D} of the electricity sector by the equation

$$\Delta F(\bar{D}, \Delta D) \approx M(\bar{D})\Delta D, \quad (49)$$

which corresponds to the simplified formula introduced in Section 4 (see equation (39)). As shown in Section 3, the following integral would be the exact formulation

$$\Delta F(\bar{D}, \Delta D) = \int_{\bar{D}}^{\bar{D}+\Delta D} f(D) dD. \quad (50)$$

Here, $f(D)$ represents the specific CO₂ emissions as a function of electric power demand D .

The mathematical analysis showed that Equation (49) is only valid, when the CO₂ emissions are completely independent from the energy supply situation, that is, if the complete electric energy would be either supplied constantly only by one technology, that is, wind power, or would be supplied by a constant mix of several technologies, that is, a combination of wind power and photovoltaics power, which is both by far not the case.

The examples discussed in Section 3 show for the specific assumption of a discontinuous, piecewise constant function and a linear function that the simplified formula is generally invalid and leads to erroneous results. The simplified formula is only valid for a constant function. Indeed, there is a clear interaction between electric power demand D and CO₂ emissions of the electricity sector, as additional electric energy supply typically requires the support of additional fossil power plants also in the future. It is clear that Equation (49) cannot be generally utilized as it may significantly underestimate real CO₂ emissions.

By applying the fundamental theorem of differential and integral calculation of Leibniz of the 17th century, the general and exact formula can be written as follows (see Equations (36) and (38))

$$\Delta F(\bar{D}, \Delta D) = \bar{D}\Delta M(\bar{D}, \Delta D) + \Delta D M(\bar{D} + \Delta D). \quad (51)$$

TABLE 2 Definition of variables

Identifier 1	Identifier 2	Variable	Unit	Dimension
D	x, s	Electric energy demand	W	$M \cdot L^2 \cdot T^3$
$E_{2017}^{CO_2}, E_{2030}^{CO_2}$	$f(x), f(s)$	Yearly averaged CO ₂ impact as function of the energy demand D	g_{CO_2}/kWh	$M_{CO_2} / (M \cdot L^2 / T^2)$
$\Delta m_{CO_2} / \Delta t$	$F(x)$	CO ₂ impact per time period	g_{CO_2}/h	M_{CO_2} / T
$E_{2017}^{Av CO_2}$	$M(x) = M(0, x)$	Average value of CO ₂ impact as Function of energy demand D	g_{CO_2}/kWh	$M_{CO_2} / (M \cdot L^2 / T^2)$

TABLE 3 Notation

Symbol	Physical quantity	Unit	Dimension
M	average CO ₂ footprint	g_{CO_2}/kWh	$M_{CO_2} \cdot M^{-1} \cdot L^{-2} \cdot T^2$
$\Delta m_{CO_2, sector}^{electric energy}$	CO ₂ impact of the sector electric energy	g_{CO_2}	M_{CO_2}
B_j	extra balance power considering, that is, import/export	GW	$M \cdot L^2 \cdot T^{-3}$
$\bar{E}_i^{CO_2}$	average specific CO ₂ equivalent impact of each technology i	g_{CO_2}/kWh	$M_{CO_2} \cdot M^{-1} \cdot L^{-2} \cdot T^2$
D_j	electric energy demand	GW	$M \cdot L^2 \cdot T^{-3}$
\bar{D}	yearly averaged electricity demand	GW	$M \cdot L^2 \cdot T^{-3}$
$E_i^{reg CO_2}$	specific CO ₂ impact of regenerative electric power	g_{CO_2}/kWh	$M_{CO_2} \cdot M^{-1} \cdot L^{-2} \cdot T^2$
$E_i^{nreg CO_2}$	specific CO ₂ impact of non-regenerative electric power	g_{CO_2}/kWh	$M_{CO_2} \cdot M^{-1} \cdot L^{-2} \cdot T^2$
$E_i^{tot CO_2}$	specific CO ₂ impact of total electric power	g_{CO_2}/kWh	$M_{CO_2} \cdot M^{-1} \cdot L^{-2} \cdot T^2$
$E_j^{CO_2 cluster}$	specific CO ₂ impact of energy demand		
	with average reg / nreg contribution	g_{CO_2}/kWh	$M_{CO_2} \cdot M^{-1} \cdot L^{-2} \cdot T^2$
P_j^{st}	electric power based on additional energy storage capacities	GW	$M \cdot L^2 \cdot T^{-3}$
H_j	electric losses	GW	$M \cdot L^2 \cdot T^{-3}$
P_{ij}	electric power	GW	$M \cdot L^2 \cdot T^{-3}$
P_j^{reg}	regenerative power	GW	$M \cdot L^2 \cdot T^{-3}$
P_j^{nreg}	non-regenerative power	GW	$M \cdot L^2 \cdot T^{-3}$
S_j	total electric power supply	GW	$M \cdot L^2 \cdot T^{-3}$

The term $\Delta M(\bar{D}, \Delta D)\bar{D}$ is missing in the simplified formula (49) and is important for most of mixed partly renewable and partly non-renewable electric energy systems. It can be even significantly larger than the term $\Delta DM(\bar{D} + \Delta D)$. Figure 6 illustrates the contribution of both terms in order to define the increase of CO₂ emissions, according to Equations (50) and (51). Note that the light grey area is equivalent to the left grey area, which represents the summand $\bar{D}\Delta M$ of Equation (51) and the error of the simplified Equation (49).

The real CO₂ emissions of the electricity system may be significantly underestimated if only the simplified formula (49) is utilized. The real CO₂ emission would outmatch those according to the simplified equation (49) easily by factor 2 in reality depending on the status of the electricity system.

5 | NOTATION

Table 3 explains the definition of major variables.

ACKNOWLEDGEMENTS

The author Thomas Koch acknowledges the financial support by German Research Foundation (Deutsche Forschungsgemeinschaft, DFG) through project 237267381 (TRR 150) as well as 426888090 (SFB 1441). Furthermore, the support by Alexander Heinz, Kai Scheiber, Philipp Weber and Christian Böhmeke is highly appreciated. The authors thank the reviewers for their valuable comments, which led to improvements in the manuscript.

Open access funding enabled and organized by Projekt DEAL.

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How to cite this article: Koch, T., Böhlke, T. The averaging bias - A standard miscalculation, which extensively underestimates real CO₂ emissions. *Z Angew Math Mech.* 2021;101:e202100205.
<https://doi.org/10.1002/zamm.202100205>