The Moduli Space of Algebraic Translation Surfaces

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Contents

List of Symbols

Sch	Category of schemes
$C, X, Y, S, B, \ldots$	Schemes
$\mathscr{O}_X$	Structure sheaf of a scheme X
X_y	Fiber over y of a morphism $X \to Y$
$\mathbb{A}^n_X$	Affine n-space over X
$\mathbb{P}^n_X$	Projective n-space over X
ε	Variable satisfying $\varepsilon^2 = 0$
$\mathcal{C}, \mathcal{X}, \ldots$	First-order deformations
$\mathscr{F}, \mathscr{G}, \ldots$	Sheaves
$\mathscr{F}^\vee$	Dual sheaf
$\operatorname{Sym}^\bullet \mathscr{F}$	Symmetric power of $\mathscr{F}$
$\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$	Internal sheaf of homomorphisms
$\mathscr{F}_x$	Stalk of $\mathscr{F}$ in x
$\mathscr{F}\|_X$	Pullback of a sheaf $\mathscr{F}$ to X
$\mathscr{F}\|_x$	Fiber of a sheaf $\mathscr{F}$ in x
$\Omega_{X/Y}$	Cotangent sheaf of $X \to Y$
$\mathcal{T}_{X/Y}$	Tangent sheaf of $X \to Y$
Γ	Global section functor
$\kappa(x)$	Residue field of a point $x \in X$ of a locally ringed space
$\mathscr{S}\!pec$	Relative Spec construction
$\mathscr{P}\!roj$	Relative Proj construction
$\mathbb{V}(\mathscr{F})$	Vector bundle associated with a quasi-coherent sheaf $\mathscr{F}$
$\mathbb{P}(\mathscr{F})$	Projective bundle associated with a quasi-coherent sheaf $\mathscr{F}$
$\mathbf{Mor}_B(X, Y)$	Scheme of morphisms
$\mathbf{Isom}_B(X, Y)$	Scheme of isomorphisms
$(\pi \colon C \to B; p_1, \ldots, p_n)$	Family of smooth curves with n marked points

$\mathcal{M}_{g,n}$	Moduli stack of smooth curves of genus g with n marked points
$\Omega\mathcal{M}_g$	Moduli space of algebraic translation surfaces of genus g
$\pi_g\colon \mathcal{C}_g \to \mathcal{M}_g$	Modular family of genus g
μ	Partition of $2g-2$ into positive integers
$\lvert\mu\rvert$	Length of the partition μ
$\mathcal{H}(\mu)$	Stratum of $\Omega\mathcal{M}_g$ associated with a partition μ
ω	Regular differential on a connected smooth projective curve
L_ω	Lie derivative associated with ω
$\mathcal{L}_\omega$	Twisted Lie derivative associated with ω
$\mathcal{A}^\bullet$	Two term complex associated with a map $\mathcal{A}\colon \mathscr{F} \to \mathscr{G}$
PGL_2	Projective linear group
$\mathcal{S}_g$	Schottky space of Schottky groups of rank g

Introduction

Throughout the history of mathematics, many results were achieved by generalizing the original object. Algebraic geometry in the style of Grothendieck can be regarded as one major example. In this spirit we completely rebuild the moduli space of translation surfaces from the ground up in the category of schemes over $\mathbb{Z}$. In particular this enables considerations in positive characteristic that might solve problems over $\mathbb{C}$.

A *translation surface* is a connected 2-dimensional real manifold M such that all transition maps are translations, that is, maps $\mathbb{R}^2 \to \mathbb{R}^2$ of the form $x \mapsto x + c$. Every translation surface is equipped with a natural flat metric. It is given by the usual metric on $\mathbb{R}^2$, which is compatible with the transition maps and therefore induces a metric on a translation surface. If the metric completion X of M is a compact surface, obtained by adding finitely many points to M, we call X an *algebraic translation surface*.

Using a triangulation of X, it is easy to see that X can be represented as polygons in the plane together with gluing instructions for parallel edges. This allows for hands-on descriptions of these objects, as seen in Figure 1. Such representations were the starting point for the theory of translation surfaces. In [FK36] they were used to study the geodesic flow on a rational polyhedron which is known as the study of rational billiards in modern terms.

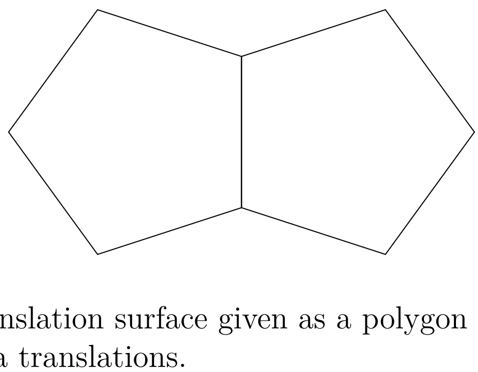


Figure 1: An algebraic translation surface given as a polygon in the plane. Parallel sides are identified via translations.

There is a third description of algebraic translation surfaces, justifying the prefix "algebraic". Given a translation surface M, we can pull back the differential $\mathrm{d}z$ on $\mathbb{R}^2 \cong \mathbb{C}$ along the charts to obtain a nowhere vanishing holomorphic differential on M. If M is further algebraic, the differential extends to a holomorphic differential ω on the metric completion X. Hence, every algebraic translation surface can be identified with a pair (X, ω), where X is a compact Riemann surface (i.e., a connected smooth projective curve over $\mathbb{C}$) and ω is a holomorphic differential on X. This implicitly defined map is bijective. Given a pair (X, ω), we obtain an algebraic translation surface by removing the zeros $\Sigma \subseteq X$ of ω from X and using the atlas given by the charts

$$\varphi\colon M \to \mathbb{C} = \mathbb{R}^2, \qquad z \mapsto \int_{z_0}^{z} \omega \in \mathbb{C} = \mathbb{R}^2$$

around a point $z_0 \in M := X \setminus \Sigma$. Equivalently, those are the charts φ on M, such that $\omega = \mathrm{d}\varphi$. For more information on algebraic translation surfaces we suggest one of the surveys [Wri15], [Zor06], [MT02] or [Mas06].

We instead turn our focus to the space of all algebraic translation surfaces ΩM_g. As a point set, it corresponds to the set of all equivalence classes of pairs (X, ω), where X is a compact Riemann surface of genus g and ω is a holomorphic differential on X. Using Teichmüller space T_g together with its universal curve $\pi\colon C_g \to T_g$, Hubbard and Masur [HM79] constructed the set ΩM_g as a quotient of the total space of the vector bundle $\pi_* \Omega_{C_g/T_g}$. In particular, ΩM_g is an analytic space. Alternatively, [FM14] contains a description of this analytic structure of ΩM_g as an application of the compact-open topology.

The Riemann-Roch Theorem implies that the orders of zeros of every holomorphic differential on a compact Riemann surface of genus g sum up to $2g - 2$, i.e., naturally form a partition of $2g - 2$ into positive integers. The subset $\mathcal{H}(\mu) \subseteq \Omega M_g$, consisting of translation surfaces that induce a fixed partition μ, is called a *stratum* of ΩM_g. It was a great insight of Veech [Vee86] that there is a natural coordinate system on the set $\mathcal{H}(\mu)$, called *period coordinates*, turning $\mathcal{H}(\mu)$ into a smooth orbifold of dimension $2g + |\mu| - 1$. To put it crudely, period coordinates of an algebraic translation surface are given by the edges in a polygon representation seen as complex numbers. One of the biggest achievements at the beginning of this century was the classification of the connected components of a stratum by Kontsevich and Zorich [KZ03], showing that almost all of the strata are connected.

The local study of algebraic translation surfaces and the global study of a stratum is closely connected via the natural action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathcal{H}(\mu)$, illustrated in Figure 2. One such example is the famous Veech dichotomy [Vee89], which asserts certain properties of the geodesic flow on a translation surface if the stabilizer of said surface is $\mathrm{SL}_2(\mathbb{Z})$. The

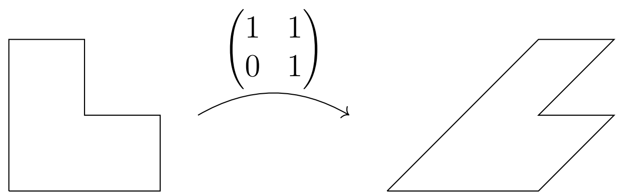


Figure 2: Illustration of the $\mathrm{GL}_2^+(\mathbb{R})$ action, using the polygon representation of algebraic translation surfaces.

description of orbit closures by Eskin, Mirzakhani and Mohammadi [EMM15] as *affine invariant submanifolds*, commonly known as a "magic wand" [Zor14] for the study of translation surfaces, resulted in Mirzakhani being awarded the Fields Medal in 2014.

In this thesis we construct the moduli space of algebraic translation surfaces and its strata in the category of schemes over $\mathbb{Z}$. To this end, we generalize and adapt results that are well-known in the analytic category over $\mathbb{C}$ to schemes over $\mathbb{Z}$. Moreover, we introduce the Lie derivative to algebraic geometry, which is an established tool in differential geometry. This toolbox we built allows us to verify basic properties of the moduli space that are expected from the complex results.

The main advantage of generalizing to schemes over $\mathbb{Z}$ is the potential for future applications. Having done this work once in full generality transfers all our results to any category that is geometric in nature. For example, every analytic category, whether over $\mathbb{C}$ or over some non-Archimedean field, is contained in this class. In particular, we obtain new purely algebraic proofs of some results that were previously constrained by the use of complex analytic tools. Moreover, our results open the door to non-algebraically closed fields and fields of positive characteristic, which in turn can be used for considerations in algebraic number theory or complex geometry.

The structure of this thesis is as follows. In Chapter 1 we introduce all the necessary tools for the construction and study of the spaces that we consider. Most notably, we generalize the Proper Base Change Theorem 1.4 to the non-Noetherian case. This in turn allows us to build a space equivalent to ΩM_g in the category of schemes over $\mathbb{Z}$, and the theory of deformations describes the local properties of said space.

Chapter 2 contains the construction of the moduli space of algebraic translation surfaces $\Omega\mathcal{M}_g$ as a vector bundle over the stack of algebraic curves of genus g. We show that for every partition μ of $2g-2$ into positive integers, the stratum $\mathcal{H}(\mu) \subseteq \Omega\mathcal{M}_g$ is locally closed.

Chapter 3 is concerned with the study of local properties of a stratum $\mathcal{H}(\mu)$. To ease the exposition we no longer work on the stack and instead construct a special family of curves over a smooth variety containing all the local information of $\mathcal{H}(\mu)$.

The main results of this thesis are Theorem 2.17, Theorem 3.16 and Theorem 3.17 which are summarized by the following theorem.

Theorem. *Let $g > 1$ be an integer and let μ be a partition of $2g-2$ into n positive integers. Then the stratum $\mathcal{H}(\mu) \subseteq \Omega\mathcal{M}_g$ consisting of pairs (C,ω), where C is a connected smooth projective curve of genus g and $\omega \in H^0(C,\Omega_C)$ induces the partition μ, is locally closed. Moreover, for every k-valued point of $\mathcal{H}(\mu)$, where k is a field, there is a natural isomorphism*

$$\mathcal{T}_{\mathcal{H}(\mu)}\Big|_{(C,\omega)} \cong \mathbb{H}^1(C,\mathcal{L}_\omega)$$

of the tangent space of $\mathcal{H}(\mu)$ at a point (C,ω) and the first hypercohomology of the twisted Lie derivative associated with ω. In particular, if the characteristic of k is 0 or every part of the partition μ is strictly smaller than $\operatorname{char} k > 0$, the dimension of the tangent space at any k-valued point is $2g+n-1$.

Finally, Chapter 4 illustrates the strength of our methods by transferring our results to Berkovich analytic spaces over $\mathbb{C}_p$. Moreover, we give some ideas how the results could be further extended using special properties of Berkovich analytic spaces.

Chapter 1

Preliminaries

In this chapter we collect all the necessary results that we need in the construction and analysis of the strata of the Hodge bundle. In a nutshell, we aim to construct a space whose points correspond to pairs (C, ω), where C is a connected smooth projective curve over an algebraically closed field and ω is a regular differential on C. Of course, we could just take the scheme theoretic disjoint union, but this is hardly satisfactory.

Instead, we want to use the following observation. Given a separated morphism of schemes $f\colon X \to Y$ of finite type, the fiber X_y over $y \in Y$ is a variety over $\kappa(y)$ and carries a sheaf of differentials $\Omega_{X_y/\kappa(y)}$. Those sheaves glue to a sheaf $\Omega_{X/Y}$ on X and under reasonable hypothesis it is a vector bundle. Subjected to stricter assumptions, the pushforward $f_*\Omega_{X/Y}$ is a vector bundle over Y and the fiber over $y \in Y$ is $H^0(X_y, \Omega_{X_y/\kappa(y)})$. Hence, to construct our desired space, we need to find a morphism of schemes $X \to Y$ such that Y is a variety over some algebraically closed field k and the set of fibers X_y is in bijection with the set of isomorphism classes of curves over k. Then, if the morphism satisfies the required assumptions, the total space of the pushforward of $\Omega_{X/Y}$ has the points we desire. As we will see, there is only one reasonable[2] geometric choice for Y in the category of schemes, the moduli space of curves M.

To make the construction described above precise, we first review the notion of a smooth morphism of schemes and afterwards the Proper Base Change Theorem. This theorem, which we extend to the non-Noetherian case, is essential in the construction of the Hodge bundle and its stratification. As the bundle lives on the moduli stack of curves, we review the notion of moduli functors and give the necessary definitions. Next up, we discuss deformation theory, hypercohomology and the Lie derivative, which we use to study local properties of the moduli space of algebraic translation surfaces.

[2] Unfortunately, in this case there is no morphism $X \to M$ having our desired properties.

1.1 Smooth Morphisms of Schemes

We start our discussion with the property that guarantees that the sheaf $\Omega_{X/Y}$ is a vector bundle on X. As this sheaf is of utmost importance for this thesis, we shortly review its construction and some of its properties.

Definition 1.1. A morphism $f\colon X \to Y$ satisfying the following conditions is called *smooth*.

i) The morphism f is flat and locally of finite presentation.

ii) The fiber $X_y = X \times_Y \operatorname{Spec} \kappa(y)$ is regular for every geometric point $y \to Y$.

The sheaf of differentials for a morphism $f\colon X \to Y$ is a quasi-coherent sheaf $\Omega_{X/Y}$ together with a universal Y-derivation $\mathrm{d}_{X/Y}\colon \mathscr{O}_X \to \Omega_{X/Y}$ such that the map

$$\operatorname{Hom}_{\mathscr{O}_X}(\Omega_{X/Y}, \mathscr{F}) \to \operatorname{Der}_Y(\mathscr{O}_X, \mathscr{F}), \qquad \alpha \mapsto \alpha \circ \mathrm{d}_{X/Y}$$

induces an isomorphism of functors $\mathbf{Mod}_{\mathscr{O}_X} \to \mathbf{Set}$. Hence, the pair $(\Omega_{X/Y}, \mathrm{d}_{X/Y})$ is unique up to unique isomorphism. It can be constructed as the conormal sheaf of the diagonal morphism $\delta\colon X \to X \times_Y X$. In this case the universal derivation is obtained by using the sheaf morphisms associated with the two canonical projections $X \times_Y X \to X$ and forming their difference.

For the reader's convenience we collect some equivalent conditions for a morphism to be smooth.

Proposition 1.2. *Let $f\colon X \to Y$ be locally of finite presentation. Then the following statements are equivalent.*

i) The morphism f is smooth.

ii) The morphism f is flat and the sheaf of relative differentials $\Omega_{X/Y}$ is locally free of rank equal to the relative dimension of X/Y.

iii) Locally on X, f factors into $X \xrightarrow{g} \mathbb{A}^n_Y \to Y$, where g is étale.

iv) For any $x \in X$ there is an affine neighborhood $\operatorname{Spec} S$ *of x and an affine neighborhood* $\operatorname{Spec} R$ *of $f(x)$ such that*

$$S = R[T_1, \dots, T_n] \big/ (g_1, \dots, g_m)$$

and the ideal generated by the $m \times m$ minors of the matrix $\left(\frac{\partial g_i}{\partial T_j}\right)_{ij}$ contains 1.

v) The morphism f is formally[1] *smooth.*

Proof. See [Vak17, Exercise 21.2.Q] or [SP, Lemma 02G1] for the implication $i) \Rightarrow ii)$, which is the only one we use in the remainder of this work. $\square$

Finally, we need a result about how the sheaf of differentials behaves in a Cartesian diagram, which is provided by the following theorem.

Theorem 1.3 (Pullback of Differentials)**.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{\alpha} & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

be a commutative diagram of schemes. Then there is a natural homomorphism

$$\alpha^*\Omega_{X/Y} \to \Omega_{X'/Y'}.$$

If furthermore $X' \cong X \times_Y Y'$, i.e., the above diagram is Cartesian, then the morphism of sheaves is an isomorphism.

Proof. See [Vak17, Theorem 21.2.27]. Alternatively the first statement can be found in [SP, Lemma 01UV] and the second in [SP, Lemma 01V0]. $\square$

1.2 Proper Base Change

We want to study the fibers of the direct image of a quasi-coherent sheaf. In general, not much can be said. However, under suitable assumptions, the fibers are isomorphic to the global sections over the fiber. To make this precise, we need some notation. Let $\pi\colon X \to Y$ and $\alpha\colon Z \to Y$ be morphisms of schemes, $p \in \mathbb{Z}$ and $\mathscr{F}$ be a quasi-coherent sheaf on X. Then

$$\beta^p_\alpha(\mathscr{F})\colon \alpha^*(R^p\pi_*\mathscr{F}) \to R^p\hat{\pi}_*\hat{\alpha}^*\mathscr{F}$$

denotes the natural map arising from the Cartesian diagram

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\hat{\alpha}} & X \\ \downarrow{\scriptstyle \hat{\pi}} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{\alpha} & Y \end{array}$$

[1] See [SP, Section 02GZ] for the definition of a formally smooth morphism.

using the following construction: We start by applying $R^p\pi_*$ to the natural adjunction morphism $\mathrm{id} \to \hat{\alpha}_*\hat{\alpha}^*$, which yields the map $R^p\pi_* \to R^p\pi_*\hat{\alpha}_*\hat{\alpha}^*$. This morphism is extended using the natural map

$$R^p\pi_*\hat{\alpha}_*\hat{\alpha}^* \to R^p(\pi \circ \hat{\alpha})_*\hat{\alpha}^* = R^p(\alpha \circ \hat{\pi})_*\hat{\alpha}^* \to \alpha_* R^p\hat{\pi}_*\hat{\alpha}^*,$$

where the left and right morphism are edge maps in the Grothendieck spectral sequence. Then the morphism $\beta^p_\alpha(\mathscr{F})$ is the image under the adjunction isomorphism

$$\mathrm{Hom}_{\mathscr{O}_Y}(R^p\pi_*\mathscr{F}, \alpha_* R^p\hat{\pi}_*\hat{\alpha}^*\mathscr{F}) \cong \mathrm{Hom}_{\mathscr{O}_Z}(\alpha^* R^p\pi_*\mathscr{F}, R^p\hat{\pi}_*\hat{\alpha}^*\mathscr{F}).$$

We call this morphism *base change map.* In the special case $Z = y \in Y$, we let $\beta^p_y(\mathscr{F})$ denote the map constructed from the inclusion $\alpha\colon \operatorname{Spec}\kappa(y) \hookrightarrow Y$.

The base change map looks quite complicated at a first glance. We therefore describe it once more locally, using the fact that all sheaves appearing in the formulation are quasi-coherent ([SP, Lemma 01XJ]). Let $Y = \operatorname{Spec} R$ and $Z = \operatorname{Spec} A$ be affine. Making the necessary substitutions, we obtain that

$$\beta^p_\alpha(\mathscr{F})\colon H^p(X, \mathscr{F}) \otimes_R A \to H^p(X \times_Y Z, \mathscr{F} \otimes_R A)$$

is the natural map induced by the tensor product using the pullback map on cohomology $H^p(X, \mathscr{F}) \to H^p(X \times_Y Z, \mathscr{F} \otimes_R A)$ and the A-module structure on $H^p(X \times_Y Z, \mathscr{F} \otimes_R A)$. To make the last map even more explicit, the reader is encouraged to use Čech cohomology, which gives the same result as sheaf cohomology in our situation.

We state the central theorem of this section.

Theorem 1.4 (Proper Base Change)**.** *Let $\pi\colon X \to Y$ be a proper finitely presented morphism, $\mathscr{F}$ be a finitely presented quasi-coherent sheaf on X and flat over Y and $y \in Y$ be so that $\beta^p_y(\mathscr{F})$ is surjective for some fixed p. Then the following results hold.*

i) There exists an open neighborhood U of y such that for any $\alpha\colon Z \to U$, $\beta^p_\alpha(\mathscr{F})$ is an isomorphism. In particular, $\beta^p_y(\mathscr{F})$ is an isomorphism.

ii) The map $\beta^{p-1}_y(\mathscr{F})$ is surjective if and only if $R^p\pi_*\mathscr{F}$ is locally free in some open neighborhood of y.

In particular, if $\beta^p_y(\mathscr{F})$ is surjective for all $y \in Y$, then the map $\beta^p_\alpha(\mathscr{F})$ is an isomorphism for any $\alpha\colon Z \to Y$. In this case we say that $\mathscr{F}$ commutes with arbitrary base changes*.*

Before we give the proof, we state a useful proposition that is a direct generalization of [Fan+05, Proposition 4.35] to higher direct images.

Lemma 1.5. *Let p be an integer,*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ U & \xrightarrow{d} & V & \xrightarrow{e} & W \end{array}$$

be a commutative diagram of schemes such that both squares are Cartesian and $\mathscr{F}$ be a quasi-coherent sheaf on C. Then the diagram of $\mathscr{O}_U$-modules

$$\begin{array}{ccc} d^*e^*R^pc_*\mathscr{F} & \xrightarrow{\sim} & (e\circ d)^*R^pc_*\mathscr{F} \\ {\scriptstyle d^*\beta^p_e(\mathscr{F})}\downarrow & & \downarrow{\scriptstyle \beta^p_{e\circ d}(\mathscr{F})} \\ d^*R^pb_*g^*\mathscr{F} & & R^pa_*(g\circ f)^*\mathscr{F} \\ & {\scriptstyle \beta^p_d(g^*\mathscr{F})}\searrow \quad \swarrow{\scriptstyle \sim} & \\ & R^pa_*f^*g^*\mathscr{F} & \end{array}$$

commutes. If additionally the base change homomorphism $\beta^p_e(\mathscr{F})$ is an isomorphism, then $\beta^p_{e\circ d}(\mathscr{F})$ is an isomorphism if and only if $\beta^p_d(g^*\mathscr{F})$ is an isomorphism.

Proof. The first part is immediately verified locally by taking a section $s \in R^pc_*\mathscr{F}(\tilde{W}) = H^p(c^{-1}(\tilde{W}), \mathscr{F})$, where $\tilde{W} \subseteq W$ is an affine open set, and calculating the two paths in the diagram. The second claim is obvious. □

Proof of Theorem 1.4. The result is well known if Y is locally Noetherian, see for example [Vak17, Theorem 28.1.6] or [EGA, III Theorem 7.7.5]. A short complete proof can be found in [Ten13]. Note for the general case that the statement is local on Y. Hence, we may assume that $Y = \operatorname{Spec} R$ is affine. We apply a standard reduction argument by Grothendieck as suggested by [Vak17, Section 28.2.9]. Using [EGA, IV Theorem 8.10.5 and Corollary 11.2.7], we can construct the following Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\hat{\varphi}_0} & X_0 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_0} \\ Y & \xrightarrow{\varphi_0} & Y_0, \end{array}$$

where $Y_0 = \operatorname{Spec} R_0$ is an affine Noetherian scheme and π_0 is proper. Moreover, there exists a coherent sheaf $\mathcal{F}_0$ on X_0 that is flat over Y_0, such that $\mathscr{F}$ is canonically isomorphic to the pullback $\hat{\varphi}_0^*\mathscr{F}$. In fact, R_0 can be chosen as a finite type $\mathbb{Z}$-subalgebra of R and in this case the morphism φ is induced by the canonical inclusion $R_0 \hookrightarrow R$. We argue similarly to [Con00, Lemma 5.1.1]. Let $\{R_i\}$ denote the set of finitely generated R_0-subalgebras of R, partially ordered by inclusion, and let $Y_i = \operatorname{Spec} R_i$ be the corresponding schemes. By pulling back along the canonical morphisms $\psi_i\colon Y_i \to Y_0$, we obtain proper morphisms $\pi_i\colon X_i \to Y_i$ together with coherent sheaves $\mathscr{F}_i$ on X_i, flat over Y_i, compatible with the data on X, i.e., the pair $(\pi, \mathscr{F})$ is the limit of the system $\{(\pi_i, \mathscr{F}_i)\}$. For $y \in Y$, let $y_i \in Y_i$ be the image of y under the canonical map $\varphi_i\colon Y \to Y_i$.

Using Čech cohomology or a modified version of [SP, Lemma 07TB], it is clear that the base change map β^p_y is the colimit of the base change maps

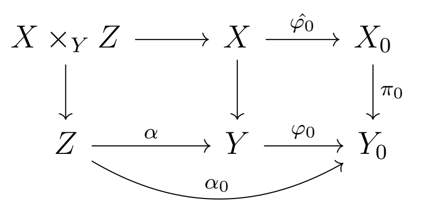
$$\beta^p_{y_i}(\mathscr{F}_i)\colon H^p(X_i, \mathscr{F}_i) \otimes_{R_i} \kappa(y_i) = (R^p\pi_{i*}\mathscr{F}_i)_{y_i} \otimes_{\mathscr{O}_{Y_i,y_i}} \kappa(y_i) \to H^p((X_i)_{y_i}, \mathscr{F}_i|_{y_i}),$$

where we use the fact that higher direct images of quasi-coherent sheaves are again quasi-coherent, see [SP, Lemma 01XJ]. The compatible isomorphisms

$$H^p((X_i)_{y_i}, \mathscr{F}_i|_{y_i}) \otimes_{\kappa(y_i)} \kappa(y_j) \cong H^p((X_j)_{y_j}, \mathscr{F}_j|_{y_j})$$

for $j \geq i$ from flat base change [SP, Lemma 02KH] imply that for any $y \in Y$ the base change $\beta^p_y(\mathscr{F})$ is surjective if and only if $\beta^p_{y_i}(\mathscr{F}_i)$ is surjective for large i.

Let $\beta^p_y(\mathscr{F})$ be surjective at a point y. By the discussion above we may assume that $\beta^p_{y_0}(\mathscr{F}_0)$ is surjective. Applying the theorem in the Noetherian case, we get an open neighborhood U of y_0 such that for any morphism with image in U the base change map is an isomorphism. If we consider the open neighborhood $\varphi_0^{-1}(U)$ of y and a morphism $\alpha\colon Z \to \varphi_0^{-1}(U)$, we get the following two Cartesian squares[1]:



From the Noetherian base change we know that the base change maps $\beta^p_{\alpha_0}(\mathscr{F}_0)$ and $\beta^p_{\varphi_0}(\mathscr{F}_0)$ are isomorphisms. Hence, Lemma 1.5 implies that $\beta^p_\alpha(\hat{\varphi}_0^*\mathscr{F}_0) = \beta^p_\alpha(\mathscr{F})$ is an isomorphism, i.e., property $i)$ holds. It remains to verify the second assertion.

Let $\beta^{p-1}_y(\mathscr{F})$ be surjective. Then for i large enough $\beta^{p-1}_{y_i}(\mathscr{F}_i)$ is surjective and hence

[1] To get a Cartesian square, we technically need to replace X by the open subset $\pi^{-1}(\varphi_0^{-1}(U))$. As this mainly hinders the exposition, we skip this detail. Alternatively it is not hard to see that the base change map exists for arbitrary commutative squares and the equivalent of Lemma 1.5 holds in this more general setting.

$R^p\pi_{i*}\mathscr{F}_i$ is locally free. Using the base change isomorphism $\beta^p_{\varphi_i}(\mathscr{F}_i)$, we get that

$$R^p\pi_*\mathscr{F} \cong R^p\pi_*\hat{\varphi}_i^*\mathscr{F}_i \cong \varphi_i^* R^p\pi_{i*}\mathscr{F}_i$$

is locally free as a pullback of a locally free sheaf.

Conversely, if $R^p\pi_*\mathscr{F}$ is locally free, it follows from [SP, Lemma 0B8W] that $R^p\pi_{i*}\mathscr{F}_i$ is locally free for i large enough[1]. We get that $\beta^{p-1}_{y_i}(\mathscr{F}_i)$ is surjective for large i from the Noetherian case. Hence, $\beta^{p-1}_y(\mathscr{F})$ is surjective. □

We close this section with two easy but useful applications of Proper Base Change.

Lemma 1.6. *Let $\pi\colon X \to Y$ be a proper finitely presented morphism, $\mathscr{F}$ be a finitely presented quasi-coherent sheaf on X and flat over Y, $p \in \mathbb{Z}$ and $H^p(X_y, \mathscr{F}|_y) = 0$ for all $y \in Y$. Then $R^p\pi_*\mathscr{F} = 0$.*

Proof. By Theorem 1.4, the condition on the cohomology groups implies that all the fibers of the sheaf $R^p\pi_*\mathscr{F}$ vanish. If $R^p\pi_*\mathscr{F}$ is of finite type, we can apply Nakayama to obtain the result[2]. In the general case, we reduce to a Noetherian base as in the proof of Theorem 1.4. Since the statement is local, we may assume that $Y = \operatorname{Spec} R$ is affine and $\mathscr{F}$ is obtained by base change from a sheaf $\mathscr{F}_0$ along some Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\hat{\varphi}} & X_0 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_0} \\ Y & \xrightarrow{\varphi} & Y_0, \end{array}$$

where $Y_0 = \operatorname{Spec} R_0$ is Noetherian and the objects with index 0 have the same properties as their counterparts. The Semicontinuity Theorem ([Vak17, Theorem 28.1.1] or [EGA, Theorem 7.6.9]) implies that the set of points y_0 in Y_0 having nontrivial $H^p((X_0)_{y_0}, \mathscr{F}_0|_{y_0})$ is closed. Clearly, the image of φ lies in its complement. Hence, we may assume that $H^p((X_0)_{y_0}, \mathscr{F}_0|_{y_0})$ vanishes for all $y_0 \in Y_0$. Using Theorem 1.4 two more times, we first get that $R^p\pi_{0*}\mathscr{F}_0$ vanishes and then that $R^p\pi_*\mathscr{F} \cong \varphi^* R^p\pi_{0*}\mathscr{F}_0 = 0$. □

Corollary 1.7. *Let $\pi\colon X \to Y$ be a proper finitely presented morphism and $\mathscr{F}$ be a finitely presented quasi-coherent sheaf on X that is flat over Y. Suppose further that $H^p(X_y, \mathscr{F}|_y) = 0$ and $\beta^{p-2}_y(\mathscr{F})$ is surjective for all $y \in Y$ and some $p \in \mathbb{Z}$. Then $R^{p-1}\pi_*\mathscr{F}$ is locally free on Y.*

[1] As the necessary statement is hidden in the proof of the above lemma, we repeat the argument here. By [SP, Lemma 02JO] the sheaves $R^p\pi_{i*}\mathscr{F}_i$ are flat over Y_i for i large enough, which is equivalent to being locally free, see for example [SP, Lemma 00NX].

[2] The higher direct image is of finite type, and indeed of finite presentation, if the dimension of the fibers is bounded by p, see [SP, Lemma 0EX5].

Proof. Applying Theorem 1.4 yields that all the fibers of $R^p\pi_*\mathscr{F}$ vanish. Therefore $R^p\pi_*\mathscr{F} = 0$ by Lemma 1.6 and $\beta_y^{p-1}(\mathscr{F})$ is surjective for all $y \in Y$ by Theorem 1.4. The result follows from another application of Proper Base Change. □

We remark that this corollary in the case $p = 1$ can also be found in [Fan+05, Proposition 4.37]. This is also the most useful instance of the result above as in this case the condition on the base change map $\beta^{p-2} = \beta^{-1}$ is vacuous.

1.3 Moduli Functors

In this section we introduce the notion of representable functors and give examples, which illustrate their geometric significance.

Definition 1.8. Let **C** be a (locally small) category. A contravariant functor $F\colon \mathbf{C} \to \mathbf{Set}$ is called *representable* if it is isomorphic to $\mathrm{Hom}(\,\cdot\,, M)$ for some object $M \in \mathbf{C}$. We say that M *represents* the functor F.

In the special case $\mathbf{C} = \mathbf{Sch}$, covariant functors $\mathbf{Sch} \to \mathbf{Set}$ are called *moduli functors*. If they are representable the representing object is called a *fine moduli space*.

As usual, a representing object is unique up to unique isomorphism. This follows immediately from the Yoneda Lemma.

Given a representable functor F together with a representing object M, there is a distinguished element C, corresponding to $\mathrm{id} \in \mathrm{Hom}(M, M)$, called the *universal object*, or *universal family* in the case $\mathbf{C} = \mathbf{Sch}$. The pair (M, C) has the following universal property that is easy to verify: For objects $X \in \mathbf{C}$ and $Y \in F(X)$, there is a unique map $f\colon X \to M$ such that $F(f)(C) = Y$. In practice this means that every family over X is obtained by pulling back the universal family and is the reason why this notion is so useful.

We illustrate this on some examples.

Definition 1.9. The *Hilbert functor* $\mathrm{hilb}_{\mathbb{P}^n}$ is the contravariant functor that assigns to every locally Noetherian scheme S the set of closed subschemes $X \subseteq \mathbb{P}^n_S = \mathbb{P}^n \times S$ that are flat over S. For a morphism $f\colon T \to S$ between locally Noetherian schemes the corresponding map $\mathrm{hilb}_{\mathbb{P}^n}(f)\colon \mathrm{hilb}_{\mathbb{P}^n}(S) \to \mathrm{hilb}_{\mathbb{P}^n}(T)$ is defined via pullback.

Theorem 1.10. *The Hilbert functor is representable with fine moduli space* $\mathrm{Hilb}_{\mathbb{P}^n}$*, called* Hilbert scheme*. Furthermore, the following properties hold.*

i) The Hilbert scheme is projective.

ii) The connected components of $\mathrm{Hilb}_{\mathbb{P}^n}$ *are given by projective schemes* $\mathrm{Hilb}_{\mathbb{P}^n}^{p(t)}$*, where* $p(t)$ *is a rational polynomial and* $\mathrm{Hilb}_{\mathbb{P}^n}^{p(t)}$ *represents the subfunctor* $\mathrm{hilb}_{\mathbb{P}^n}^{p(t)}$ *of* $\mathrm{hilb}_{\mathbb{P}^n}$ *that assigns to any locally Noetherian scheme* S *the set*

$$\mathrm{hilb}_{\mathbb{P}^n}^{p(t)}(S) = \{\, X \subseteq \mathbb{P}^n_S \text{ closed} \,|\, X \text{ flat over } S \text{ with Hilbert polynomial } p(t)\,\}$$

iii) The tangent space to $\mathrm{Hilb}_{\mathbb{P}^n}$ *at the point corresponding to* X *is given by* $H^0(X, \mathcal{N})$*, where* $\mathcal{N}$ *is the normal sheaf of* X *in* $\mathbb{P}^n$*.*

iv) If X *is a locally complete intersection and* $H^1(X, \mathcal{N}) = 0$*, then the point of* $\mathrm{Hilb}_{\mathbb{P}^n}$ *corresponding to* X *is non-singular and the local dimension is equal to* $h^0(X, \mathcal{N})$*.*

Proof. The original existence proof by Grothendieck for the quotient scheme, which is a direct generalization of the Hilbert scheme, can be found in [FGA]. The usual modern English reference is [Fan+05, Theorem 5.15]. They state and give a proof originally by [AK80]. The statements about the tangent space and local dimension can be found in [Fan+05, Corollary 6.4.11]. □

The Hilbert functor and Hilbert scheme were envisioned by Grothendieck as building blocks to solve more complicated moduli problems. Two such examples are given in this section. We use the Hilbert scheme once more in Chapter 3 to construct a modular family of curves.

In practice it is useful to have a more flexible Hilbert functor available. Let X be a scheme over B. The general Hilbert functor $\mathrm{hilb}_{X/B}$ assigns to any scheme S over B the set

$$\mathrm{hilb}_{X/B}(S) := \{\text{closed subschemes } Y \subseteq X \times_B S,\ \text{flat over } S\}$$

and morphisms are once again defined by pullback.

Proposition 1.11. *Let* B *be a Noetherian scheme and* X *be a closed subscheme of* $\mathbb{P}^n_B$*. Then the functor* $\mathrm{hilb}_{X/B}$ *is representable by a projective scheme* $\mathrm{Hilb}_{X/B}$*. Moreover, there is a natural decomposition*

$$\mathrm{hilb}_{X/B} = \coprod_{p(t)} \mathrm{hilb}_{X/B}^{p(t)}, \qquad \mathrm{Hilb}_{X/B} = \coprod_{p(t)} \mathrm{Hilb}_{X/B}^{p(t)},$$

where the functors and schemes on the right hand side correspond to families with fixed Hilbert polynomial $p(t)$*.*

This was proved by Grothendieck in [FGA] and covered in [Fan+05, Theorem 5.15]. Note that $\mathrm{hilb}_{\mathbb{P}^n} = \mathrm{hilb}_{\mathbb{P}^n/\mathbb{Z}}$ is just a special case of this result. It is also possible to argue the other way around: Using the universal property of fiber products, we get $\mathrm{Hilb}_{\mathbb{P}^n \times B/B} = \mathrm{Hilb}_{\mathbb{P}^n} \times B$. Then it is possible to construct $\mathrm{Hilb}_{X/B}$ as a closed subscheme of $\mathrm{Hilb}_{\mathbb{P}^n \times B/B}$, see [ACG11, p. 43] for details.

We apply this construction to build a fine moduli space for the functor that parameterizes morphisms between schemes.

Lemma 1.12. *Let B be a Noetherian scheme and X and Y be closed subschemes of $\mathbb{P}^n_B$. Assume further that X is flat over B. Then the functor given for any B-scheme S by*

$$\mathrm{mor}_B(X, Y)(S) = \mathrm{Mor}_S(X \times_B S, Y \times_B S)$$

is representable by a scheme $\mathbf{Mor}_B(X, Y)$. If furthermore Y is flat over B, the functor

$$\mathrm{isom}_B(X, Y)(S) = \mathrm{Isom}_S(X \times_B S, Y \times_B S)$$

is representable by a scheme $\mathbf{Isom}_B(X, Y)$.

Proof. The idea is to construct $\mathbf{Mor}_B(X, Y)$ as an open subscheme of $\mathrm{Hilb}_{X \times_B Y/B}$ using the observation that a map can be identified with its graph. More precisely, for separated schemes S and T over B the set of B-morphisms $S \to T$ can be naturally identified with closed subschemes Γ of $S \times_B T$ that project isomorphically onto S. In [ACG11, IX Lemma 7.5] it is shown that the condition of a closed subscheme of a fiber product being a graph is open. Therefore, we can realize $\mathbf{Mor}_B(X, Y)$ as the open subset of $\mathrm{Hilb}_{X \times_B Y/B}$ consisting of pairs (b, Γ), where b is a point of B and Γ is the graph of a morphism $X_b \to Y_b$.

If Y is flat, the scheme $\mathbf{Mor}_B(Y, X)$ exists as an open subscheme of $\mathrm{Hilb}_{X \times_B Y/B}$. Clearly $\mathbf{Isom}_B(X, Y) \coloneqq \mathbf{Mor}_B(X, Y) \cap \mathbf{Mor}_B(Y, X) \subseteq \mathrm{Hilb}_{X \times_B Y/B}$ parameterizes isomorphisms between X and Y over B and represents the functor $\mathrm{isom}_B(X, Y)$. $\square$

For later use, we remark that $\mathbf{Isom}_B(X, Y)$ has the following universal property. It is the smallest scheme over B such that $X \times_B \mathbf{Isom}_B(X, Y)$ and $Y \times_B \mathbf{Isom}_B(X, Y)$ are isomorphic over $\mathbf{Isom}_B(X, Y)$, i.e., for any B-scheme S with the same property there is a unique map $S \to \mathbf{Isom}_B(X, Y)$ such that the isomorphism over S is obtained by pulling back the isomorphism over $\mathbf{Isom}_B(X, Y)$.

1.3.1 Moduli Functor of Smooth Curves

Similar to the Hilbert functor and scheme we want to build a functor that parameterizes families of curves and show that it is representable. Unfortunately, we show that this is impossible. We work with the following definition of a curve.

Definition 1.13. A *curve* over a field k is a reduced and separated scheme of finite type over $\operatorname{Spec} k$ such that all irreducible components have dimension 1.

In practice all of the curves we consider are connected, projective and non-singular. For the sake of clarity, as it can be quite difficult to add the right requirements if needed, we proceed with the utmost precision.

Although it is not strictly needed for our purposes, we introduce the notion of a family of curves with marked points. In many cases our arguments apply to such curves without change and they can be quite handy in certain situations. We see two applications in the construction of the strata and the calculation of a tangent space.

Definition 1.14. Let $g, n \geq 0$. A *family of smooth curves of genus g with n marked points over a scheme B* is a tuple

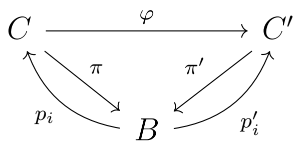
$$(\pi\colon C \to B; p_1, \dots, p_n\colon B \to C)$$

with the following properties:

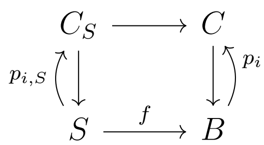
i) The map π is a smooth proper surjective morphism such that the fiber C_b over any geometric point b of B is a connected smooth projective curve of genus g.

ii) The morphisms $p_1, \dots, p_n$ are pairwise disjoint sections of π.

Two families $(\pi\colon C \to B; p_1, \dots, p_n)$ and $(\pi'\colon C' \to B; p'_1, \dots, p'_n)$ over the same scheme B are said to be *isomorphic* if there exists an isomorphism $\varphi\colon C \to C'$ over B such that the diagram



commutes for all i. The *pullback of a family* $(\pi\colon C \to B; p_1, \dots, p_n)$ *under a morphism* $f\colon S \to B$ is the family $(\pi_S\colon C_S \to S; p_{1,S}, \dots, p_{n,S})$ fitting in the following diagram, where $C_S = C \times_B S$ denotes the fiber product and $p_{i,S} = (p_i \circ f) \times \mathrm{id}_S$ are the induced sections.



If the structure morphism and markings are clear from the context, we abbreviate a family by C/B.

Definition 1.15. The *moduli functor of smooth curves of genus g with n marked points* is the functor $\mathcal{M}_{g,n}$ sending a scheme B to the set

$$\mathcal{M}_{g,n}(B) \coloneqq \{(\pi\colon C \to B; p_1, \dots, p_n\colon B \to C) \mid C \text{ is a smooth curve over } B\}\big/\sim$$

of families of smooth curves of genus g with n marked points over B, up to isomorphism. For a morphism $f\colon S \to B$, the induced map

$$\mathcal{M}_{g,n}(B) \to \mathcal{M}_{g,n}(S)$$

is defined by pulling back families of curves over B to S.

Lemma 1.16. *For all $g, n \geq 0$ the functor $\mathcal{M}_{g,n}$ is not representable.*

Proof. We show the claim in the case $g = 1 = n$ only. However, the argument readily transfers to other choices of parameters. In a nutshell, curves with automorphisms (that always exist for every genus and any amount of marked points) prevent the existence of a universal family.

We may work over an algebraically closed field k. Assume for the sake of contradiction that $\mathcal{M}_{1,1}$ is represented by a scheme M. Let $\mathcal{C}$ denote the universal curve over M. Consider as a base for a family a nodal curve B, i.e., $\mathbb{P}^1$ where 0 and 1 are identified. Let (E, O) denote any elliptic curve. Then E is a connected smooth projective curve of genus 1 over k and O is a closed point of E. It is well known that there exists an automorphism[1] σ of order 2 of E fixing the point O. We get a new scheme C over B by gluing a copy of E at every point of B. To get something nontrivial we use the automorphism σ to glue the fibers over 0 and 1, see Figure 3 for an illustration. Clearly there is still a section $B \to C$ mapping every point b to O in the fiber $C_b = E$. Hence, we get a family C/B that is not isomorphic to the trivial family $B \times E$, for which all fibers are isomorphic. In particular C can not be obtained by pulling back the universal family $\mathcal{C}$ along a morphism $B \to M$. This can be seen as follows. Since all the fibers over B are isomorphic such a morphism has to be constant. But then the pullback gives a trivial family over B. Of course, one has to verify that C is not isomorphic to $B \times E$. One way to see this is to consider sections $p\colon B \to C$. By construction a section of C/B is a morphism $\mathbb{P}^1 \to \mathbb{P}^1 \times E$ that respects the gluing, i.e., a morphism $\mathbb{P}^1 \to E$ that is compatible with the automorphism σ. By Riemann-Hurwitz any such morphism is constant and by the second condition maps to a fixed point of σ. In particular there are only finitely many sections $p\colon B \to C$, as there are only finitely many fixed points of σ (once again by Riemann-Hurwitz). □

[1] This automorphism is typically called the elliptic involution.

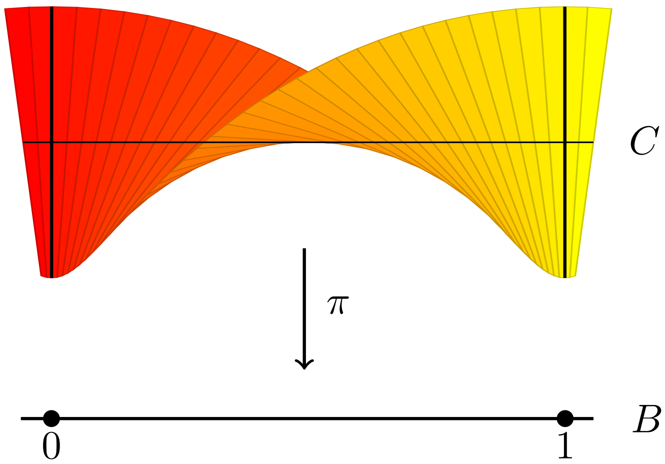


Figure 3: An example of a nontrivial family with isomorphic fibers. Here the points 0 and 1 in the base and the corresponding curves in the family are glued together.

The previous result tells us that there is no fine moduli space for the functor $\mathcal{M}_{g,n}$. We could still ask whether there exists a scheme that best approximates the functor of curves in the following sense. A *coarse moduli scheme* for the functor $\mathcal{M}_{g,n}$ is a scheme M together with a natural transformation $\eta\colon \mathcal{M}_{g,n} \to \mathrm{Mor}(\,\cdot\,, M)$ such that

i) the tuple (M, η) is initial under all such pairs,

ii) the natural transformation η induces a bijection

$$\eta(\operatorname{Spec} k)\colon \mathcal{M}_{g,n}(\operatorname{Spec} k) \xrightarrow{\sim} \mathrm{Mor}(\operatorname{Spec} k, M) = M(k)$$

on k-valued points for all algebraically closed fields k.

A celebrated result by Mumford [MFK93, Theorem 5.11] assures that this is indeed the case.

Theorem 1.17. *For any choice of parameters $g, n \geq 0$ with $2g - 2 + n > 0$, there exists a coarse moduli space $M_{g,n}$ of $\mathcal{M}_{g,n}$. It is an irreducible quasi-projective normal algebraic variety of dimension $3g - 3 + n$.*

The condition $2g-2+n > 0$ is simply a shorthand to exclude the pairs $(0,0)$, $(0,1)$, $(0,2)$ and $(1,0)$ and due to the existence of curves with infinite automorphism groups in these cases.

Remark 1.18. Since $M_{g,n}$ is not a fine moduli space it, is not equipped with a universal family. Still, it is reasonable to ask if there exists something close to it, a family over

$M_{g,n}$ such that every fiber over a point is isomorphic to the curve represented by the point. But even without the universality condition such a curve does not exist, giving us another argument why $\mathcal{M}_{g,n}$ can not be representable. Details in the case $g = 1 = n$ can be found in [Har10, Remark 26.3.1].

In Chapter 2 and 3 we introduce the notion of a stack and a modular family to remedy the non-existence of a fine moduli space for the moduli functor of curves.

1.4 Deformation Theory

In this section we develop the necessary tools to analyze the tangent space of the moduli space of algebraic translation surfaces and its strata. Deformation theory is an interesting branch of algebraic geometry with many applications outside of moduli theory. One such example is the definition of a formally smooth morphism, which is a deformation theoretic description of smoothness. For a more detailed discussion, we refer to the excellent textbooks [Har10] and [Ser06].

Throughout this section, let k be a field and X be a scheme over k with structure morphism π. Let ε be a variable fulfilling $\varepsilon^2 = 0$ and $k[\varepsilon] \cong k[X]/(X^2)$ denote the *dual numbers* over k.

Definition 1.19. A *(first-order) deformation* of X with n marked points $p_i\colon \operatorname{Spec} k \to X$ is a Cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ {\scriptstyle p_i}\uparrow\downarrow{\scriptstyle \pi} & & {\scriptstyle \pi_\varepsilon}\downarrow\uparrow{\scriptstyle p_{\varepsilon,i}} \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[\epsilon], \end{array}$$

where π_ε is a flat morphism and the markings $p_{\varepsilon,i}$ pull back to the markings p_i for all i.

Two deformations $\mathcal{X}$ and $\mathcal{Y}$ are called *isomorphic* if there is an isomorphism $\varphi\colon \mathcal{X} \to \mathcal{Y}$ over $\operatorname{Spec} k[\varepsilon]$ that respects the marked points such that

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & & \searrow & \\ \mathcal{X} & & \xrightarrow{\ \varphi\ } & & \mathcal{Y} \\ & \searrow & & \swarrow & \\ & & \operatorname{Spec} k[\varepsilon] & & \end{array}$$

commutes.

A deformation is called *trivial* if it is isomorphic to the deformation $X \times_k \operatorname{Spec} k[\epsilon]$ obtained by pulling back the data on X along the canonical morphism $\operatorname{Spec} k[\varepsilon] \to \operatorname{Spec} k$.

We denote by $\operatorname{Def}(X) := \operatorname{Def}_k(X; p_1, \ldots, p_n)$ the set of all deformations of X with markings p_i up to isomorphism.

Note that in the definition of a deformation of X, we do not allow arbitrary Cartesian diagrams of the form

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[\varepsilon]. \end{array}$$

Instead, we require that the vertical map on the left is the structure morphism of X. This prevents automorphism of X to come into play. Alternatively, one can fix the upper horizontal map, i.e., a closed embedding $X \hookrightarrow \mathcal{X}$. This is the approach followed by Hartshorne in [Har10].

Remark 1.20. Every deformation $\mathcal{X}$ of a connected smooth projective curve over $\operatorname{Spec} k$ is a family of smooth curves over $\operatorname{Spec} k[\varepsilon]$. This follows from [Ser06, Lemma 1.2.3], which implies that in this case $\mathcal{X}$ is Noetherian and hence proper by the valuation criterion for properness.

We start our investigation of deformations by characterizing the group of automorphisms of the trivial deformation $\operatorname{Aut}_{\operatorname{Def}}(X \times_k \operatorname{Spec} k[\varepsilon])$ of an affine scheme X. The following lemma identifies the group of automorphisms with a subgroup of the global sections of the tangent sheaf $\mathcal{T}_X = \Omega_X^\vee$, the dual of the cotangent sheaf. If $X = \operatorname{Spec} A$, then $\mathcal{T}_X$ is the quasi-coherent sheaf constructed from the k-vector space $\operatorname{Der}_k(A, A)$ of k-derivations. Note that although $\mathcal{T}_X$ is in general not a line bundle, it still makes sense for a section $t \in \mathcal{T}_X(U)$ over an open set $U \subseteq X$ to vanish at a point $x \in U$. This is the case precisely if t maps to 0 under the canonical map

$$\mathcal{T}_X(U) \to \mathcal{T}_X|_x = \mathcal{T}_{X,x} \otimes_{\mathscr{O}_{X,x}} \kappa(x).$$

Therefore, we get for any marked points $p_1, \ldots, p_n \colon \operatorname{Spec} k \to X$ a well-defined subgroup $H^0(X, \mathcal{T}_X(-p_1 - \cdots - p_n))$ of $H^0(X, \mathcal{T}_X)$ consisting of global sections vanishing at the points $p_i(\operatorname{Spec} k) \in X$. In the case of a regular curve X, this notation is consistent with twisting the line bundle $\mathcal{T}_X$ with the divisor $D = -p_1 - \cdots - p_n$.

Lemma 1.21. *Let X be an affine scheme over k with n marked points $p_1, \ldots, p_n$ and* *let* $\mathcal{X} = X \times_k \operatorname{Spec} k[\varepsilon]$ *denote the trivial deformation. Then*

$$\operatorname{Aut}_{\mathrm{Def}}(\mathcal{X}; p_1, \dots, p_n) \cong H^0(X, \mathcal{T}_X(-p_1 - \cdots - p_n)).$$

Proof. We first assume that there are no marked points. Write $X = \operatorname{Spec} A$, where A is a k-algebra. For $A[\varepsilon] := A \oplus \varepsilon A$ we have $\mathcal{X} = \operatorname{Spec} A[\varepsilon]$. An automorphism of the deformation $\mathcal{X}$ is a $k[\varepsilon]$-algebra isomorphism

$$\varphi\colon A[\varepsilon] \to A[\varepsilon]$$

that is the identity modulo ε. Let $\pi_1, \pi_2\colon A[\varepsilon] \to A$ denote the projection on the first and second component, respectively, and write $\varphi = \varphi_1 + \varepsilon\varphi_2$, where $\varphi_i = \pi_i \circ \varphi$ is k-linear. By assumption $\varphi_1(a + \varepsilon b) = a$ and (using $k[\varepsilon]$-linearity) $\varphi_2(a + \varepsilon b) = \varphi_2(a) + b$ for all $a, b \in A$. Therefore, every automorphism of the trivial deformation defines a map $D := \varphi_2|_A\colon A \to A$.

We show that D is a k-derivation. Clearly, D is k-linear. For $a, b \in A$ we have $\varphi(ab) = \varphi(a)\varphi(b)$ which immediately implies the Leibniz formula $D(ab) = aD(b) + bD(a)$.

Conversely, given a derivation D, we can construct the map

$$\varphi\colon A[\varepsilon] \to A[\varepsilon], \qquad \varphi = \pi_1 + \varepsilon(D \circ \pi_1 + \pi_2).$$

It is easy to check that this map is multiplicative and $k[\varepsilon]$-linear. Moreover, only 0 gets mapped to 0 and $a + \varepsilon(b - D(a))$ is contained in the preimage of $a + \varepsilon b$, that is, φ is an isomorphism. Consequently, we have an isomorphism of groups

$$\operatorname{Aut}_{\mathrm{Def}(X)}(\mathcal{X}) \cong \operatorname{Der}_k(A, A) = H^0(X, \mathcal{T}_X).$$

It remains to show that this isomorphism restricts to an isomorphism of the respective subgroups.

To keep the exposition clear we only consider the case of one marked point. The argument easily generalizes to more markings. Let $q\colon A \to k$, $q_\varepsilon\colon A[\varepsilon] \to k[\varepsilon]$ be the marked points, i.e.,

$$\begin{array}{ccc} A & \longleftarrow & A[\varepsilon] \\ {\scriptstyle q}\downarrow\uparrow & & \uparrow\downarrow{\scriptstyle q_\varepsilon} \\ k & \longleftarrow & k[\varepsilon] \end{array}$$

is a commutative diagram, and $\varphi\colon A[\varepsilon] \to A[\varepsilon]$ be an automorphism respecting the marking. As before, φ can be written as $\varphi = \pi_1 + \varepsilon(D \circ \pi_1 + \pi_2)$ and $q_\varepsilon = q \circ \pi_1 + \varepsilon(q \circ \pi_2)$

by definition. Plugging in the compatibility assumption $q_\varepsilon = q_\varepsilon \circ \varphi$, we get

$$q \circ \pi_1 + \varepsilon(q \circ \pi_2) = q \circ \pi_1 + \varepsilon(q \circ D \circ \pi_1 + q \circ \pi_2)$$

which implies $q \circ D \circ \pi_1 = 0$, i.e., $D(A) \subseteq p := \ker q$. Under the canonical map

$$\operatorname{Der}_k(A, A) \to \operatorname{Der}_k(A, A)_p = \operatorname{Der}_k(A_p, A_p), \qquad Q \mapsto \left(\frac{a}{b} \mapsto \frac{aQ(b) - bQ(a)}{b^2}\right),$$

D maps to a derivation D_p with $D_p(A_p) \subseteq p_p$ and hence vanishes in the fiber over p. In particular $D \in H^0(X, \mathcal{T}_X(-p))$. Clearly, this process is completely reversible. □

This construction allows us to characterize deformations of smooth varieties using the fact that deformations of smooth varieties are locally trivial.

Lemma 1.22. *Let X be a non-singular affine scheme over k with n marked points. Then the deformations of X are trivial, that is* $\operatorname{Def}(X) = \{0\}$.

Proof. Write $X = \operatorname{Spec} A$. The result is well known in the case of no marked points and follows immediately from the deformation theoretic description of smoothness, see [Har10, Corollary 4.8] for details. Therefore, it is enough to show that for an affine deformation diagram with one marked point

$$\begin{array}{ccc} A & \longleftarrow & A[\varepsilon] \\ {\scriptstyle q}\downarrow\uparrow{\scriptstyle \pi} & & \uparrow\downarrow{\scriptstyle q'} \\ k & \longleftarrow & k[\varepsilon] \end{array}$$

there is an automorphism φ of $A[\varepsilon] = A \oplus \varepsilon A$ of unmarked deformations mapping q' to the trivial marked point $q_\varepsilon = q \circ \pi_1 + \varepsilon(q \circ \pi_2)$.

Similar to the proof of Lemma 1.21, we can write $q' = (q \circ \pi_1) + \varepsilon(D' \circ \pi_1 + q \circ \pi_2)$, where $D' \in \operatorname{Der}_k(A, k)$ is a k-derivation, k carries an A-module structure via q and $\pi_i \colon A[\varepsilon] \to A$ denotes the projection on the first and second component, respectively. Using Lemma 1.21 with no marked points, the map

$$\varphi \colon A[\varepsilon] \to A[\varepsilon], \qquad \varphi = \pi_1 + \varepsilon(\pi \circ D' \circ \pi_1 + \pi_2),$$

is an automorphism of the given deformation diagram without marked points. Having in mind the identity $q \circ \pi = \mathrm{id}_k$, it is easy to verify that $q' \circ \varphi = q_\varepsilon$. □

Proposition 1.23. *Let $(X; p_1, \dots, p_n)$ be a smooth variety over k with n marked points. Then there is a natural isomorphism*

$$\mathrm{Def}(X; p_1, \dots, p_n) \cong H^1(X, \mathcal{T}_X(-p_1 \cdots - p_n)).$$

Proof. We strengthen the proof of [Har10, Theorem 5.3]. In fact, the only difference is the use of Lemma 1.21 and Lemma 1.22, but since the correspondence is used later, we include the proof for the reader's convenience.

Let $\mathcal{X}$ be a deformation of the marked variety X, i.e., there is a Cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ {\scriptstyle p_i}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle p_{\varepsilon,i}} \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[\varepsilon] \end{array}$$

together with n sections $p_{\varepsilon,i}\colon \operatorname{Spec} k[\varepsilon] \to \mathcal{X}$ such that everything commutes for fixed i. Let $\{U_\alpha\}$ denote an affine open covering of X. For every α we get an induced deformation $\mathcal{U}_\alpha = \mathcal{X} \cap U_\alpha$ of U_α and since deformations of affine smooth varieties are trivial we have isomorphisms

$$\varphi_\alpha\colon U_\alpha \times_k \operatorname{Spec} k[\varepsilon] \xrightarrow{\sim} \mathcal{U}_\alpha.$$

On the intersections $U_{\alpha\beta} := U_\alpha \cap U_\beta$ we have transition maps

$$\varphi_{\alpha\beta} = \varphi_\beta^{-1} \circ \varphi_\alpha\colon U_{\alpha\beta} \times_k \operatorname{Spec} k[\varepsilon] \to U_{\alpha\beta} \times_k \operatorname{Spec} k[\varepsilon]$$

that fix the $p_{\varepsilon,i}$ and therefore correspond to elements $D_{\alpha\beta} \in H^0(U_{\alpha\beta}, \mathcal{T}_X(-p_1 - \cdots - p_n))$ by Lemma 1.21. By construction, those elements satisfy the cocycle condition

$$D_{\alpha\beta} + D_{\beta\gamma} + D_{\gamma\alpha} = 0$$

on triple overlaps since the addition of derivations corresponds to composition of automorphisms. Hence, the tuple $(D_{\alpha\beta})$ is a one-cocycle. This results in a well defined map on cohomology: If φ'_α denote different isomorphisms, the $\varphi'^{-1}_\alpha \circ \varphi_\alpha$ are automorphisms of $U_\alpha \times_k \operatorname{Spec} k[\varepsilon]$ fixing the $p_{\varepsilon,i}$, hence are induced by an element $\psi_\alpha\colon H^0(U_\alpha, \mathcal{T}_X(-p_1 - \ldots - p_n))$. The new derivations $D'_{\alpha\beta}$, corresponding to φ', satisfy the relation

$$D'_{\alpha\beta} = D_{\alpha\beta} + \psi_\alpha|_{U_{\alpha\beta}} - \psi_\alpha|_{U_{\alpha\beta}}.$$

Therefore, the map is well defined on the level of cohomology. Evidently, the construction is reversible and defines the claimed natural bijection. □

Remark 1.24. Using Proposition 1.23, the set of deformations of a smooth variety over k can be fitted with the structure of a vector space. It is also possible to describe this structure intrinsically. We start with scalar multiplication. Given a scalar $\lambda \in k$ and a deformation $\mathcal{X}$ of X, the deformation $\lambda\mathcal{X}$ is defined by the Cartesian diagram

$$\begin{CD} \lambda\mathcal{X} @>>> \mathcal{X} \\ @VVV @VVV \\ \operatorname{Spec} k[\varepsilon] @>{\varphi_\lambda}>> \operatorname{Spec} k[\varepsilon], \end{CD}$$

where φ_λ is the morphism induced by the automorphism $a + \varepsilon b \mapsto a + \varepsilon\lambda b$. Addition is slightly more complicated and uses the fiber product in the category of k-algebras. To give two deformations $\mathcal{X}$ and $\mathcal{Y}$ of X is the same datum as a deformation $\mathcal{X}'$ over $\operatorname{Spec}(k[\varepsilon] \times_k k[\varepsilon])$, the spectrum of the algebra consisting of pairs $(a+\varepsilon b, a+\varepsilon c)$. In one direction the pair $(\mathcal{X}, \mathcal{Y})$ gets mapped to the scheme $\mathcal{X}' = (X, \mathscr{O}_{\mathcal{X}} \times_{\mathscr{O}_X} \mathscr{O}_{\mathcal{Y}})$ and the inverse is given by pulling back along the canonical morphisms $\operatorname{Spec} k[\varepsilon] \to \operatorname{Spec}(k[\varepsilon] \times_k k[\varepsilon])$. The sum $\mathcal{X} + \mathcal{Y}$ is then defined by the Cartesian diagram

$$\begin{CD} \mathcal{X} + \mathcal{Y} @>>> \mathcal{X}' \\ @VVV @VVV \\ \operatorname{Spec} k[\varepsilon] @>{\varphi}>> \operatorname{Spec}(k[\varepsilon] \times_k k[\varepsilon]), \end{CD}$$

where φ is dual to the k-algebra morphism

$$k[\varepsilon] \times_k k[\varepsilon] \to k[\varepsilon], \qquad (a+\varepsilon b, a+\varepsilon c) \mapsto a + \varepsilon(b+c).$$

This linear structure is natural. It is possible to show that the moduli functor $\mathcal{M}_{g,n}$ restricted to Artin rings and a fixed curve is a deformation functor[1] [Ser06, Theorem 2.4.1]. By the criterion of Schlessinger [Ser06, Theorem 2.3.2] the image of $k[\varepsilon]$, i.e., the set of first order deformations of a curve C, carries a linear structure which is precisely the one we described above. Hence, the bijection of Proposition 1.23 is an isomorphism of vector spaces.

[1] Technically, we have to use the moduli stack, which we introduce in the next chapter, as automorphisms are still present when using the moduli functor $\mathcal{M}_{g,n}$. Hence, $\mathcal{M}_{g,n}(\operatorname{Spec} k[\varepsilon])$ corresponds to deformations where the structure morphism is not fixed.

1.5 Hypercohomology

Intuitively a tangent vector of a space parameterizing pairs of objects corresponds to changes in the two components that may depend on each other. Often this dependence is captured using the concept of hypercohomology. Indeed, this is the case for the spaces we construct in the next chapter.

Here we review the important facts about hypercohomology, which can all be found in [Bry08, Chapter I.2 and I.3]. The general idea is to construct a cohomology theory that takes as argument a complex of abelian sheaves. Of course, it should satisfy the usual properties such as a long exact sequence in cohomology arising from a short exact sequence and the ability to calculate it using ideas of Čech. Moreover, if the complex consists of just one term, hypercohomology should (and does) coincide with the usual sheaf cohomology.

Let X denote a topological space and $\mathscr{F}^\bullet$ a complex of sheaves of abelian groups[1] on X which is bounded below[2]. Let $f\colon \mathscr{F}^\bullet \to \mathcal{I}^\bullet$ be an *injective resolution of* $\mathscr{F}^\bullet$, i.e., $\mathcal{I}^\bullet$ is a complex of injective sheaves such that f is injective in every degree and the diagram

$$\begin{array}{ccccc} \mathscr{F}^{i-1} & \longrightarrow & \mathscr{F}^{i} & \longrightarrow & \mathscr{F}^{i+1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{I}^{i-1} & \longrightarrow & \mathcal{I}^{i} & \longrightarrow & \mathcal{I}^{i+1} \end{array}$$

commutes and induces an isomorphism in cohomology. The usual properties hold:

i) Injective resolutions of $\mathscr{F}^\bullet$ exist.

ii) Two different injective resolutions of $\mathscr{F}^\bullet$ are homotopic. In particular, they induce a natural isomorphism in cohomology.

This allows us to define the *hypercohomology of the complex* $\mathscr{F}^\bullet$ as the cohomology of the complex $\Gamma(\mathcal{I})^\bullet$, that is,

$$\mathbb{H}^\bullet(X, \mathscr{F}^\bullet) \coloneqq H^\bullet(\Gamma(\mathcal{I})^\bullet).$$

Similar to the usual cohomology it is functorial and well defined up to canonical isomorphism. The most important property for us is the long exact sequence resulting from a short exact sequence of complexes.

[1] The results extend without change to complexes of sheaves with values in R-modules and indeed in any abelian category with enough injective objects.

[2] In all our applications the complex is finite and all degrees different from 0 and 1 vanish.

Proposition 1.25. *Let $0 \to \mathscr{F}^\bullet \to \mathscr{G}^\bullet \to \mathscr{H}^\bullet \to 0$ be a short exact sequence of bounded below complexes of sheaves of abelian groups on X. Then for every i there are functorial maps $\delta\colon \mathbb{H}^i(X, \mathscr{H}^\bullet) \to \mathbb{H}^{i+1}(X, \mathscr{F}^\bullet)$ such that*

$$\cdots \to \mathbb{H}^i(X, \mathscr{F}^\bullet) \to \mathbb{H}^i(X, \mathscr{G}^\bullet) \to \mathbb{H}^i(X, \mathscr{H}^\bullet) \xrightarrow{\delta} \mathbb{H}^{i+1}(X, \mathscr{F}^\bullet) \to \cdots$$

is a long exact sequence of abelian groups.

The following example relates hypercohomology to the usual cohomology.

Example 1.26. Let $\mathscr{F}$ be a sheaf of abelian groups on X and let $\mathscr{F}[i]$ denote the complex of abelian sheaves on X with value $\mathscr{F}$ in degree i and 0 for all other degrees. In particular, an injective resolution of $\mathscr{F}$ gives rise to an injective resolution of $\mathscr{F}[0]$. By definition we get

$$\mathbb{H}^\bullet(X, \mathscr{F}[0]) = H^\bullet(X, \mathscr{F}).$$

Similarly, an injective resolution of $\mathscr{F}$ gives rise to an injective resolution of $\mathscr{F}[i]$ where the degree is shifted by i. Hence

$$\mathbb{H}^n(X, \mathscr{F}[i]) = H^{n-i}(X, \mathscr{F}).$$

In view of the previous example, hypercohomology is a direct generalization of the usual cohomology. Next we discuss a method to calculate hypercohomology groups.

There is a notion of Čech cohomology that works for hypercohomology. To simplify the exposition, we assume that the complex $\mathscr{F}^\bullet$ is bounded below by 0, i.e., $\mathscr{F}^i = 0$ for all $i < 0$. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X. For every i we have the usual Čech complex $\check{C}^\bullet(\mathcal{U}, \mathscr{F}^i)$. Since this construction is functorial we get the following commutative diagram, where d and δ denote the boundary maps of $\mathscr{F}^\bullet$ and the Čech complex, respectively.

$$\begin{array}{ccccccc}
\check{C}^0(\mathcal{U}, \mathscr{F}^0) & \xrightarrow{d} & \check{C}^0(\mathcal{U}, \mathscr{F}^1) & \xrightarrow{d} & \check{C}^0(\mathcal{U}, \mathscr{F}^2) & \xrightarrow{d} & \cdots \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
\check{C}^1(\mathcal{U}, \mathscr{F}^0) & \xrightarrow{d} & \check{C}^1(\mathcal{U}, \mathscr{F}^1) & \xrightarrow{d} & \check{C}^1(\mathcal{U}, \mathscr{F}^2) & \xrightarrow{d} & \cdots \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
\check{C}^2(\mathcal{U}, \mathscr{F}^0) & \xrightarrow{d} & \check{C}^2(\mathcal{U}, \mathscr{F}^1) & \xrightarrow{d} & \check{C}^2(\mathcal{U}, \mathscr{F}^2) & \xrightarrow{d} & \cdots \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

Using this data, we can build the *total complex* $\mathrm{Tot}^\bullet(\check{C}^\bullet(\mathcal{U}, \mathscr{F}^\bullet))$. The term in degree n is given by the diagonal

$$\mathrm{Tot}^n(\check{C}^\bullet(\mathcal{U}, \mathscr{F}^\bullet)) \coloneqq \bigoplus_{i+j=n} \check{C}^i(\mathcal{U}, \mathscr{F}^j)$$

and the boundary map into the (i, j)th component is given by $\delta + (-1)^j d$. Similar to sheaf cohomology, we get a map

$$H^\bullet(\mathrm{Tot}^\bullet(\check{C}^\bullet(\mathcal{U}, \mathscr{F}^\bullet))) \to \mathbb{H}^\bullet(X, \mathscr{F}^\bullet)$$

and under reasonable assumptions this map is an isomorphism of complexes.

Proposition 1.27. *Let $\mathcal{U}$ be an open cover of X such that the sheaf and Čech cohomology agree. Then the hypercohomology can be calculated using the cohomology of the total complex. In particular, if X is a separated Noetherian scheme and $\mathcal{U}$ is an open cover of X by affine open sets, the hypercohomology of a complex of quasi-coherent sheaves can be calculated using the Čech double complex.*

We close this section with the only instance of hypercohomology we encounter in Chapter 3.

Example 1.28. Let $\mathscr{F}^\bullet = \cdots \to 0 \to \mathscr{F}^0 \to \mathscr{F}^1 \to 0 \to \cdots$ be a complex of sheaves of abelian groups in degree 0 and 1 and let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X that can be used to calculate the hypercohomology of $\mathscr{F}^\bullet$ as described in Proposition 1.27. We want to analyze the steps that are necessary to calculate the first hypercohomology group $\mathbb{H}^1(X, \mathscr{F}^\bullet)$. The diagram of the double complex reads

$$\begin{array}{ccccccc}
\check{C}^0(\mathcal{U}, \mathscr{F}^0) & \xrightarrow{d} & \check{C}^0(\mathcal{U}, \mathscr{F}^1) & \xrightarrow{d} & 0 & \xrightarrow{d} & \cdots \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
\check{C}^1(\mathcal{U}, \mathscr{F}^0) & \xrightarrow{d} & \check{C}^1(\mathcal{U}, \mathscr{F}^1) & \xrightarrow{d} & 0 & \xrightarrow{d} & \cdots \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
\check{C}^2(\mathcal{U}, \mathscr{F}^0) & \xrightarrow{d} & \check{C}^2(\mathcal{U}, \mathscr{F}^1) & \xrightarrow{d} & 0 & \xrightarrow{d} & \cdots \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

and the part of the total complex relevant for the calculation of the degree 1 term of hypercohomology is given by

$$\check{C}^0(\mathcal{U},\mathscr{F}^0) \xrightarrow{(\delta,-d)} \check{C}^1(\mathcal{U},\mathscr{F}^0)\oplus\check{C}^0(\mathcal{U},\mathscr{F}^1) \xrightarrow{(\delta,\delta-d)} \check{C}^2(\mathcal{U},\mathscr{F}^0)\oplus\check{C}^1(\mathcal{U},\mathscr{F}^1).$$

In this case

$$\operatorname{im}(\delta,-d)=\left\{(\delta(f_\alpha),-d(f_\alpha))\,\middle|\, f_\alpha\in\check{C}^0(\mathcal{U},\mathscr{F}^0)\right\}$$

and

$$\ker(\delta,\delta-d)=\left\{(f_{\alpha\beta},g_\alpha)\,\middle|\,\delta(f_{\alpha\beta})=0, d(f_{\alpha\beta})+g_\alpha|_{U_{\alpha\beta}}-g_\beta|_{U_{\alpha\beta}}=0\right\}.$$

This explicit description comes in handy in the calculation of tangent spaces in Chapter 3.

1.6 Lie Derivative

In this section we introduce the complex of sheaves whose first hypercohomology describes the tangent space of a moduli space.

Let C be a connected smooth projective curve over an algebraically closed field k and let $\omega\in H^0(C,\Omega_C)$ be an abelian differential. There is a natural map

$$L_\omega\colon \mathcal{T}_C\to\Omega_C,$$

of quasi-coherent sheaves on C, called the *Lie derivative associated with* ω. Locally, using a trivializing element $\mathrm{d}x$ of Ω_C with dual $\frac{\partial}{\partial x}$, the Lie derivative is given by

$$g\frac{\partial}{\partial x}\mapsto\frac{\partial}{\partial x}(fg)\,\mathrm{d}x,$$

where $\omega=f\,\mathrm{d}x$. We quickly verify that this is a sensible definition.

Lemma 1.29. *The Lie derivative is well-defined.*

Proof. We have to check that the definition is independent of the chosen uniformizer on open subsets where both sheaves trivialize and therefore glues to a map of sheaves. Let $\mathrm{d}y$ be a different local generator of Ω_C and let $\frac{\partial}{\partial y}$ denote the dual local generator of $\mathcal{T}_C$. Since both sheaves are line bundles, there is an invertible regular function h such that

$$h\,\mathrm{d}y=\mathrm{d}x,\qquad h^{-1}\frac{\partial}{\partial y}=\frac{\partial}{\partial x}.$$

The calculation

$$\frac{\partial}{\partial x}(fg)\,\mathrm{d}x=h^{-1}\frac{\partial}{\partial y}(fg)h\,\mathrm{d}y=\frac{\partial}{\partial y}(fg)\,\mathrm{d}y=\frac{\partial}{\partial y}(hfh^{-1}g)\,\mathrm{d}y$$

shows that L_ω is indeed well-defined as a map $\mathcal{T}_C \to \Omega_C$. □

If $Z = \operatorname{div}\omega = \sum_{i=1}^n \mu_i p_i$ denotes the zero divisor of ω and $Z_{\text{red}} = \sum_{i=1}^n p_i$ is the associated reduced divisor, the Lie derivative restricts to a map

$$\mathcal{L}_\omega\colon \mathcal{T}_C(-Z_{\text{red}}) \to \Omega_C(-Z)$$

that we call the *twisted Lie derivative associated with* ω. Using the same notation as above, this follows from the property

$$\operatorname{ord}_{p_i} fg = \operatorname{ord}_{p_i} f + \operatorname{ord}_{p_i} g = \mu_i + \operatorname{ord}_{p_i} g$$

which implies that $\operatorname{ord}_{p_i} L_\omega\left(g\frac{\partial}{\partial x}\right) = \operatorname{ord}_{p_i} \frac{\partial}{\partial x}(fg) \geq (\mu_i + 1) - 1 = \mu_i$ for a local section $g\frac{\partial}{\partial x}$ of $\mathcal{T}_X(-Z_{\text{red}})$.

Notation 1.30. We use the notation $f' \coloneqq D(f)$ if the derivative D is clear from the context. This is mostly used for a uniformizer $\frac{\partial}{\partial x}$ in which case we also write $\frac{\partial f}{\partial x} \coloneqq \frac{\partial}{\partial x} f$.

Remark 1.31. The map defined above has a geometric interpretation. It is closely related to the Lie derivative in differential geometry which can axiomatically be described as follows: Let $D \in \mathcal{T}_C(C)$ be a vector field. The *Lie derivative associated with* D is a collection of maps

$$L_D\colon \Omega_C^l \to \Omega_C^l$$

for every $l \in \mathbb{Z}$ satisfying the following properties.

i) For $l = 0$ the map L_D is the directional derivative D and commutes with the exterior derivative d on functions, i.e., the diagram

$$\begin{array}{ccc} \mathcal{O}_C & \xrightarrow{D} & \mathcal{O}_C \\ \downarrow{\scriptstyle \mathrm{d}} & & \downarrow{\scriptstyle \mathrm{d}} \\ \Omega_C & \xrightarrow{L_D} & \Omega_C \end{array}$$

commutes.

ii) The Leibniz rule and the Leibniz rule for contractions hold for L_D.

Now one can easily check that the equation $L_\omega(D) = L_D(\omega)$ is satisfied using the first property.

Actually, we are quite surprised that we could not find any traces of the Lie derivative in algebraic geometry outside of complex geometry.

Chapter 2

Algebraic Construction

"Of course, here I'm working with the moduli stack rather than with the moduli space. For those of you who aren't familiar with stacks, don't worry: basically, all it means is that I'm allowed to pretend that the moduli space is smooth and that there's a universal family over it."
Who hasn't heard these words, or their equivalent, spoken in a talk? And who hasn't fantasized about grabbing the speaker by the lapels and shaking him until he says what – exactly – he means by them? But perhaps you're now thinking that all that is in the past, and that at long last you're going to learn what a stack is and what they do. **Fat chance**.

— Joe Harris and Ian Morrison [HM98, p. 139]

The goal of this chapter is the construction of the moduli space of algebraic translation surfaces and its strata as Deligne-Mumford stacks. We give a complete self-contained definition of the stacks we are working with. However, we do not verify that they are indeed stacks. For example, the stack condition cannot be found in this thesis. Nonetheless, for readers having the necessary background (we suggest [Ols16]), it should be easy to fill in the gaps. For everyone else, we discuss the idea behind stacks and later shift to a chart that lives in the category of schemes (even varieties), which should make the results accessible to a broader audience. In particular, we shed light on how the moduli stack enables us to use arguments that would otherwise be impossible using the coarse moduli space.

The first section is concerned with the description of the moduli stack of smooth curves $\mathcal{M}_g$ as a fibered category and the definition of vector bundles in this setting. After collecting important results about families of smooth curves, we construct the Hodge bundle over $\mathcal{M}_g$. This is mostly an application of the previous section together with the Proper Base Change Theorem from Chapter 1. The final section of this chapter contains one of the main results of this thesis, namely, the algebraic construction of the strata as locally closed subsets of the Hodge bundle.

2.1 Moduli Stack of Smooth Curves

One of the reasons the moduli functor of smooth curves is not representable, is the existence of automorphisms of curves which are invisible for the functor. The definition of a stack should therefore make it possible to recover automorphisms of families. Starting with this requirement (and a flash of genius) one might arrive at the following definition.

Definition 2.1. The *moduli stack of smooth curves of genus g with n marked points* $\mathcal{M}_{g,n}$ is the category consisting of the following data[1].

i) The objects of $\mathcal{M}_{g,n}$ are families of smooth curves.

ii) The set of morphisms between two families $(\pi\colon C \to B; p_i)$ and $(\pi'\colon C' \to B'; p'_i)$ is given by

$$\mathrm{Mor}(C'/B', C/B) := \left\{ \begin{array}{ccc} C' & \xrightarrow{\hat{\alpha}} & C \\ {\scriptstyle p_i}\uparrow\downarrow{\scriptstyle \pi'} & & {\scriptstyle \pi}\downarrow\uparrow{\scriptstyle p'_i} \\ B' & \xrightarrow{\alpha} & B \end{array} \;\middle|\; (\hat{\alpha}, \alpha) \text{ makes } \pi' \text{ a pullback of } \pi \right\}.$$

Note that automorphisms of a family of smooth curves C/B are precisely the morphisms $(\hat{\alpha}, \alpha) \in \mathrm{Mor}(C/B, C/B)$ with $\alpha = \mathrm{id}_B$. There is a canonical functor $F\colon \mathcal{M}_{g,n} \to \mathbf{Sch}$ that sends a family C/B to the base scheme B and a morphism $(\hat{\alpha}, \alpha)$ to α. This makes $\mathcal{M}_{g,n}$ into a *fibered category.*

Even more properties hold for the category $\mathcal{M}_{g,n}$. For example, it satisfies the stack condition, the diagonal is representable and it has an étale chart. Unfortunately, we do not explain the meaning behind all these properties and they are not needed apart from the étale chart, which we construct explicitly in the following chapter. Instead we convince the reader of the usefulness of stacks with the informal discussion in the following remark.

Remark 2.2. As we have seen in Chapter 1, representable moduli functors carry a lot of geometric meaning in the form of a fine moduli space together with a universal family. Unfortunately, not all natural moduli functors have a fine moduli space, at least in the category of schemes.

[1] We use the same symbol as for the functor. There should be no confusion.

Using the Yoneda Embedding

$$\mathbf{Sch} \to \mathbf{Set}^{\mathbf{Sch}},$$

we can see the category of schemes as a subcategory of the functor category $\mathbf{Sch} \to \mathbf{Set}$. Note that under this embedding every moduli functor is trivially representable. Indeed, the representing object is the moduli functor itself. Unfortunately, the category $\mathbf{Set}^{\mathbf{Sch}}$ is way too big and most of the objects carry no geometric meaning. For example, we have seen in the proof of Lemma 1.16 that even the moduli functor of smooth curves is not a sheaf with respect to the Zariski topology.

The basic idea behind stacks (and algebraic spaces for that matter) is to provide a small extension of the category **Sch** such that most interesting moduli functors are representable without losing all the techniques that were developed for schemes. A good reference for the geometric significance of stacks is [Ols16]. The following discussion describes how $\mathcal{M}_{g,n}$ represents the moduli functor of smooth curves.

Let X be a scheme and let $\mathbf{Sch}_X$ denote the category of schemes over X. This defines an embedding of the category of schemes in the category of stacks

$$\mathbf{Sch} \to \mathbf{Stacks}, \qquad X \mapsto \mathbf{Sch}_X.$$

A *morphism* from a scheme X to $\mathcal{M}_{g,n}$ is a functor $G\colon \mathbf{Sch}_X \to \mathcal{M}_{g,n}$ of fibered categories, i.e., fitting in the diagram

$$\begin{array}{ccc} \mathbf{Sch}_X & \xrightarrow{\;G\;} & \mathcal{M}_{g,n}. \\ & \searrow \quad \swarrow_{F} & \\ & \mathbf{Sch} & \end{array}$$

Unwinding definitions, this is precisely equivalent to choosing a family $\pi\colon C \to X$ of smooth curves over X (together with n marked points).

We close this first discussion on the stack of smooth curves with the definition of a vector bundle.

Definition 2.3. A *quasi-coherent sheaf* $\mathscr{F}$ on $\mathcal{M}_{g,n}$ consists of the following data:

i) For all families $(C/B) \in \mathrm{Ob}(\mathcal{M}_{g,n})$ a quasi-coherent sheaf $\mathscr{F}_{C/B}$ on B.

ii) For all morphisms $A = (\hat{\alpha}, \alpha) \in \mathrm{Mor}(C'/B', C/B)$ an isomorphism

$$\varphi_A \colon \alpha^*(\mathscr{F}_{C/B}) \to \mathscr{F}_{C'/B'}$$

satisfying the cocycle condition, i.e., for each pair

$$A_1 = (\hat{\alpha_1}, \alpha_1) \in \mathrm{Mor}(C_1/B_1, C_2/B_2), \qquad A_2 = (\hat{\alpha_2}, \alpha_2) \in \mathrm{Mor}(C_2/B_2, C_3/B_3)$$

the diagram

$$\begin{array}{ccc} \alpha_1^*\alpha_2^*\mathscr{F}_{C_3/B_3} & \xrightarrow{\sim} & (\alpha_2 \circ \alpha_1)^*\mathscr{F}_{C_3/B_3} \\ \downarrow{\scriptstyle \alpha_1^*(\varphi_{A_2})} & & {\scriptstyle \varphi_{A_2 \circ A_1}}\downarrow \\ \alpha_1^*\mathscr{F}_{C_2/B_2} & \xrightarrow{\varphi_{A_1}} & \mathscr{F}_{C_1/B_1} \end{array}$$

commutes.

A quasi-coherent sheaf $\mathscr{F}$ is called *coherent* if all $\mathscr{F}_{C/B}$ are coherent sheaves on B and called a *vector bundle* if all $\mathscr{F}_{C/B}$ are locally free.

2.2 Properties of Families of Smooth Curves

In this short section we collect two useful properties of families of smooth curves that are well known to experts but for which there seems to be no good reference in the literature.

Proposition 2.4. *The following results hold for a family of smooth curves $\pi\colon C \to B$.*

i) The morphism π is $\mathscr{O}$-connected, i.e., $\pi_*\mathscr{O}_C \cong \mathscr{O}_B$.

ii) The sheaf $\Omega_{C/B}$ is dualizing. In particular there exists a covariant functor

$$\pi^!\colon \mathbf{QCoh}_B \to \mathbf{QCoh}_C$$

with $\Omega_{C/B} = \pi^!\mathscr{O}_B$ and a natural isomorphism

$$\pi_* \mathscr{H}\!om_{\mathscr{O}_C}(\mathscr{F}, \pi^!\mathscr{G}) \cong \mathscr{H}\!om_{\mathscr{O}_B}(R^1\pi_*\mathscr{F}, \mathscr{G})$$

for all quasi-coherent sheaves $\mathscr{F}$ on C and $\mathscr{G}$ on B.

iii) (Grothendieck duality) The sheaf $R^1\pi_*\Omega_{C/B}$ is free of rank one.

Proof. i) We follow ideas from [Vak17, 28.1.H]. Consider the morphism $\mathscr{O}_B \to \pi_*\mathscr{O}_C$ on the level of sheaves on B. For $b \in B$ we get the map

$$\kappa(b) = \mathscr{O}_B|_b \longrightarrow \pi_*\mathscr{O}_C|_b \xrightarrow{\beta_b^0(\mathscr{O}_C)} H^0(C_b, \mathscr{O}_{C_b}) \cong \kappa(b) \ ,$$

where $\beta_b^0(\mathscr{O}_C)$ denotes the base change map. Here we use that the fiber over b is geometrically irreducible and geometrically reduced. Since the composition is surjective, the same holds for $\beta_b^0(\mathscr{O}_C)$. Using Theorem 1.4, we find that $\pi_*\mathscr{O}_C$ is locally free of rank 1, i.e., invertible. We can verify that the map $\mathscr{O}_B \to \pi_*\mathscr{O}_C$ is an isomorphism by arguing locally. Given a ring homomorphism $f\colon R \to S$ that makes S into a free R-module of rank 1 we have to show that f is an isomorphism. By assumption there exists an element $s \in S$ such that the map

$$R \to S, \qquad r \mapsto f(r) \cdot s,$$

is an isomorphism of R-modules. In particular, s is a unit in S and therefore multiplication by s^{-1} is an R-linear isomorphism as well. Hence, f is an isomorphism as a composition of two isomorphisms.

ii) Note that π is locally projective. Indeed, using [EGA, Corollary 9.6.4] we see that the line bundle $\Omega_{C/B}$ is relatively ample since this is true over every fiber. See also the discussion in Chapter 3 on the construction of a modular family. Hence, the first claim follows from [Kle80, Proposition (22)] and the second one from the definition of a dualizing sheaf, see [Kle80, Proposition (9)].

iii) This follows formally from i) and ii). Plugging in $\mathscr{G} = \mathscr{O}_B$ and $\mathscr{F} = \Omega_{C/B}$ in the natural isomorphism, we get

$$\begin{aligned} \pi_*\mathscr{O}_C &\cong \pi_* \mathscr{H}\!om_{\mathscr{O}_C}(\Omega_{C/B}, \Omega_{C/B}) = \pi_* \mathscr{H}\!om_{\mathscr{O}_C}(\Omega_{C/B}, \pi^!\mathscr{O}_B) \\ &\cong \mathscr{H}\!om_{\mathscr{O}_B}(R^1\pi_*\Omega_{C/B}, \mathscr{O}_B) = (R^1\pi_*\Omega_{C/B})^\vee, \end{aligned}$$

where $\cdot^\vee$ denotes the dual sheaf as usual. The result follows from i) using the fact that a line bundle is trivial if and only if its dual is.

□

2.3 Hodge Bundle

We construct the moduli space of algebraic translation surfaces as the total space of a vector bundle on $\mathcal{M}_{g,n}$. The first and most complicated[1] step is to construct the vector bundle on the base of each family. The compatibility condition is then verified quite easily.

Theorem 2.5. *Let $\pi\colon C \to B$ be a family of smooth curves of genus g. Then $\pi_*\Omega_{C/B}$ is a locally free sheaf on B with fiber $H^0(C_b, \Omega_{C_b})$ over $b \in B$.*

[1] Luckily, we have done most of the work already.

Proof. Since π is smooth of relative dimension 1, $\Omega_{C/B}$ is a line bundle by Proposition 1.2. In particular it is finitely presented. Therefore, it is enough to show that the base change map β_b^0 is surjective for every b: Since β_b^{-1} is surjective for trivial reasons, Proper Base Change 1.4 implies in this case that $R^0\pi_*\Omega_{C/B} = \pi_*\Omega_{C/B}$ is locally free and that the fibers are as claimed.

Let b be a point of B. For dimension reasons $H^2(C_b, \Omega_{C_b}) = 0$, i.e., β_b^2 is a surjection. Moreover, this also forces $R^2\pi_*\Omega_{C/B}$ to be 0 using Lemma 1.6. Hence, $R^2\pi_*\Omega_{C/B}$ is locally free. By Proper Base Change the map

$$\beta_b^1 \colon (R^1\pi_*\Omega_{C/B})_b \to H^1(C_b, \Omega_{C_b})$$

is an isomorphism. The claim follows after another application of Proper Base Change since $R^1\pi_*\Omega_{C/B}$ is free by Grothendieck duality Proposition 2.4(iii). □

The construction of the locally free sheaf above extends to a vector bundle on $\mathcal{M}_{g,n}$. We only have to check that the compatibility condition is satisfied.

Let $\pi\colon C \to B$ and $\pi'\colon C' \to B'$ be two families of curves and $(\hat{\alpha}, \alpha)$ be a morphism between them, i.e.,

$$\begin{array}{ccc} C' & \xrightarrow{\hat{\alpha}} & C \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{\alpha} & B \end{array}$$

is a Cartesian square. Using the natural isomorphism $\hat{\alpha}^*\Omega_{C/B} \cong \Omega_{C'/B'}$ from Theorem 1.3 as well as the natural isomorphism $\alpha^*\pi_*\Omega_{C/B} \cong \pi'_*\hat{\alpha}^*\Omega_{C/B}$ from Proper Base Change with $p = 0$ and $Z = B'$, we get

$$\pi'_*\Omega_{C'/B'} \cong \pi'_*\hat{\alpha}^*\Omega_{C/B} \cong \alpha^*\pi_*\Omega_{C/B}.$$

Since this isomorphism is natural, it clearly satisfies the cocycle condition. Hence, the locally free sheaves $\pi_*\Omega_{C/B}$ glue to a vector bundle on $\mathcal{M}_{g,n}$.

Definition 2.6 (Hodge Bundle)**.** The vector bundle on $\mathcal{M}_{g,n}$ given by $\pi_*\Omega_{C/B}$ for a family $\pi\colon C \to B$ of smooth curves is called the *Hodge bundle* and denoted by $\Omega_{\mathcal{M}_{g,n}}$. The total space $\Omega\mathcal{M}_{g,n}$ of this vector bundle, i.e., the stack whose objects are the morphisms $\mathbb{V}(\pi_*\Omega_{C/B}) \to B$, where

$$\mathbb{V}(\pi_*\Omega_{C/B}) \coloneqq \mathscr{S}\!pec(\mathrm{Sym}^\bullet(\pi_*\Omega_{C/B})^\vee)$$

is the total space of the Hodge bundle, is called the *moduli space of algebraic translation surfaces of genus g*.

The existence of the Hodge bundle as a vector bundle on the moduli stack of smooth curves immediately implies the following result.

Corollary 2.7. *Let*

$$\begin{array}{ccc} C' & \xrightarrow{\hat{\alpha}} & C \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{\alpha} & B \end{array}$$

be a Cartesian diagram of families of smooth curves. Then there are natural isomorphisms

$$\pi'_*\Omega_{C'/B'} \cong \alpha^*\pi_*\Omega_{C/B}, \qquad \mathbb{V}(\pi'_*\Omega_{C'/B'}) \cong \alpha^*\mathbb{V}(\pi_*\Omega_{C/B}) \cong B' \times_B \mathbb{V}(\pi_*\Omega_{C/B}),$$

i.e., the Hodge bundle over B' is obtained by pulling back the Hodge bundle over B and this construction is compatible with forming the total space of the vector bundle.

Remark 2.8. The previous corollary could be used to construct the Hodge bundle on $\mathcal{M}_{g,n}$ without the need to generalize Proper Base Change to non-Noetherian schemes as we did in Theorem 1.4. Indeed, let $\pi\colon C \to B$ be any family of smooth curves and assume without loss of generality that B is affine. Then, using the techniques of Grothendieck that we applied in the proof of Theorem 1.4, we can construct a Cartesian diagram

$$\begin{array}{ccc} C & \xrightarrow{\hat{\alpha}} & C_0 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_0} \\ B & \xrightarrow{\alpha} & B_0, \end{array}$$

where $\pi_0\colon C_0 \to B_0$ is a family of smooth curves and B_0 is an affine Noetherian scheme. It follows that $\pi_*\Omega_{C/B} \cong \alpha^*\pi_{0*}\Omega_{C_0/B_0}$ is a vector bundle since we know from the Noetherian case that $\pi_{0*}\Omega_{C_0/B_0}$ is a vector bundle.

2.4 Stratification of the Hodge Bundle

Having constructed the moduli space of algebraic curves we can build the strata.

We start the discussion with the case of a connected smooth projective curve C/k of genus g. For every non-zero differential $\omega \in H^0(C, \Omega_C)$ we have

$$\deg \operatorname{div} \omega = 2g - 2$$

by Riemann-Roch. In particular, every differential defines a partition of $2g - 2$ into positive integers[1]. Let $\mathcal{P}_g$ denote the set of partitions of $2g-2$ into positive integers up to permutation. Then we have a map

$$P\colon H^0(C,\Omega_C)\setminus\{0\}\to\mathcal{P}_g,\qquad \omega\mapsto\operatorname{div}\omega=\sum_{i=1}^n\mu_i p_i\mapsto(\mu_1,\dots,\mu_n),$$

where the $p_i\in C(k)$ are the pairwise different points in the support of $\operatorname{div}\omega$. Define $\mathcal{H}_{C/k}(\mu):=P^{-1}(\mu)$ for $\mu\in\mathcal{P}_g$. This clearly defines a decomposition of $H^0(C,\Omega_C)$, i.e.,

$$H^0(C,\Omega_C)=\bigsqcup_{\mu\in\mathcal{P}}\mathcal{H}_{C/k}(\mu)\sqcup\{0\}\,.$$

The construction generalizes to families of curves using the universal property of a vector bundle which we now recall. If $\mathcal{E}$ denotes a quasi-coherent sheaf on a scheme S, the definition $\mathbb{V}(\mathcal{E}):=\mathscr{S}\!pec(\operatorname{Sym}^\bullet\mathcal{E}^\vee)$ allows us to canonically identify $\mathcal{E}$ with the sheaf of sections of $\mathbb{V}(\mathcal{E})$ (see [EGA, II 1.7.9]). This is also the justification why we used the dual sheaf $(\pi_*\Omega_{C/B})^\vee$ in the construction of the moduli space of algebraic curves. Special thanks to Piotr Achinger [MOA] for pointing out the right universal property to me.

Proposition 2.9. *Let $\mathcal{E}$ be a quasi-coherent sheaf on some scheme S. Then scheme morphisms $X\to\mathbb{V}(\mathcal{E})$ correspond to morphisms $f\colon X\to S$ together with a section e of $f^*\mathcal{E}$.*

Proof. This follows immediately from the previous discussion together with the fact that for every $f\colon X\to S$ the diagram

$$\begin{array}{ccc}\mathbb{V}(f^*\mathcal{E}) & \longrightarrow & \mathbb{V}(\mathcal{E})\\ \downarrow & & \downarrow\\ X & \longrightarrow & S\end{array}$$

is Cartesian. □

Given a family $\pi\colon C\to B$ of smooth curves of genus g, a geometric point of $\mathbb{V}(\pi_*\Omega_{C/B})$ corresponds to a connected smooth projective curve $\tilde{C}$ over some algebraically closed field together with an element $\omega\in H^0(\tilde{C},\Omega_{\tilde{C}})$. Clearly, the partition of $\operatorname{div}\omega$ only depends on the image of the geometric point and not the chosen algebraically closed field. Hence, we obtain a well-defined map[2]

$$P\colon\mathbb{V}(\pi_*\Omega_{C/B})\setminus B\to\mathcal{P}_g$$

[1] In the case $g=1$ we also see the empty partition of 0 as a valid partition into positive integers.
[2] Here we identify B with the zero section of $\mathbb{V}(\pi_*\Omega_{C/B})$.

and we set $\mathcal{H}_{C/B}(\mu) \coloneqq P^{-1}(\mu)$ for any partition μ of $2g-2$ into positive integers.

Definition 2.10. The set $\mathcal{H}_{C/B}(\mu) \subseteq \mathbb{V}(\pi_*\Omega_{C/B})$ is called a *stratum* of $\mathbb{V}(\pi_*\Omega_{C/B})$.

Combining the fact that the definition of the $\mathcal{H}(\mu)$ behaves well with respect to pullback together with Corollary 2.7 we arrive at the following result.

Lemma 2.11. *Let μ be a partition of $2g-2$ into positive integers,*

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ B' & \longrightarrow & B \end{array}$$

be a Cartesian diagram of families of smooth curves and $\alpha\colon \mathbb{V}(\pi'_*\Omega_{C'/B'}) \to \mathbb{V}(\pi_*\Omega_{C/B})$ denote the induced map on bundles. Then the stratum $\mathcal{H}_{C/B}(\mu)$ pulls back to $\mathcal{H}_{C'/B'}(\mu)$, i.e.,

$$\mathcal{H}_{C'/B'}(\mu) = \alpha^{-1}(\mathcal{H}_{C/B}(\mu)).$$

For the sake of completeness we give a definition of the strata as a substack of the moduli space of algebraic translation surfaces.

Definition 2.12. Let $g \geq 2$ be an integer and μ be a partition of $2g-2$ into positive integers. Then $\mathcal{H}(\mu)$ is the substack of $\Omega\mathcal{M}_g$ consisting of objects $\mathcal{H}_{C/B}(\mu)$ for every family of smooth curves C/B and called a *stratum* of $\Omega\mathcal{M}_g$.

Of course the definition above implies that the category $\mathcal{H}(\mu)$ carries a structure of a stack, which we have not yet verified. Even calling the set $\mathcal{H}_{C/B}(\mu)$ a stratum of $\mathbb{V}(\pi_*\Omega_{C/B})$ requires the existence of a scheme structure on $\mathcal{H}_{C/B}(\mu)$. We show that each $\mathcal{H}_{C/B}(\mu)$ is a locally closed subset of $\mathbb{V}(\pi_*\Omega_{C/B})$ and hence carries the natural reduced scheme structure. This justifies the name *stratification* for the decomposition

$$\mathbb{V}(\pi_*\Omega_{C/B}) = \bigsqcup_{\mu \in \mathcal{P}_g} \mathcal{H}_{C/B}(\mu) \sqcup B.$$

In general, nothing more can be said. For example, depending on the curve C, the smallest stratum $\mathcal{H}_{C/k}(2g-2)$ could be either one-dimensional or empty.

Before we can prove that the strata are locally closed, we need to introduce a partial ordering on the set of all partitions $\mathcal{P}_g$. Let $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu = (\nu_1, \ldots, \nu_m)$ be two partitions of $2g-2$. We define

$$\nu \leq \mu$$

if there is a partition of $\{1, \dots, n\}$ into non-empty disjoint sets $I_1, \dots, I_m$, such that $\nu_i = \sum_{j \in I_i} \mu_j$. Intuitively, the partition ν is smaller than μ if it arises by combining parts from μ.

We first consider the case of a connected smooth projective curve over an algebraically closed field. Special thanks to Giulio Bresciani [MOB] for discussing the following proof with me.

Proposition 2.13. *Let C be a connected smooth projective curve of genus $g \geq 2$ over an algebraically closed field k and let $\mu = (\mu_1, \dots, \mu_n)$ be a partition of $2g-2$ into positive integers, i.e., $\mu_i \in \mathbb{N}$ and*

$$\sum_{i=1}^{n} \mu_i = 2g - 2.$$

Then the set $\mathcal{H}_{C/k}(\mu) \subseteq H^0(C, \Omega_C) \cong \mathbb{A}^g_k$ is locally closed and in particular carries a natural structure as a reduced scheme.

Proof. For a point $p = (p_1, \dots, p_n) \in C^n$ let D_p denote the divisor $\sum_{i=1}^n \mu_i p_i$. Consider the following Cartesian diagram:

$$\begin{array}{ccc} C^n \times C & \xrightarrow{\varphi} & C \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\pi} \\ C^n & \longrightarrow & \operatorname{Spec} k \end{array}$$

We construct a line bundle on $C^n \times C$ as follows. Let $\varphi_i \colon C^n \times C \to C \times C$ denote the projection on the i-th and last component for $i = 1, \dots, n$. Since C is separated, the diagonal $\Delta \subseteq C \times C$ is closed and a divisor (see also Lemma 2.16). Let $\Delta_i \coloneqq \varphi_i^{-1}(\Delta)$. These subsets define divisors on $C^n \times C$. Finally, let $\mathscr{L} \coloneqq \varphi^*\Omega_C \otimes \mathscr{O}_{C^n \times C}(-\sum_{i=1}^n \mu_i \Delta_i)$. This is clearly a line bundle on $C^n \times C$ and the fiber over $p \in C^n$ is given by

$$\begin{aligned} \mathscr{L}|_{p \times C} &= \varphi^*\Omega_C|_{p \times C} \otimes_{\mathscr{O}_{C^n \times C}|_{p \times C}} \mathscr{O}_{C^n \times C}\left(-\sum \mu_i \Delta_i\right)\Big|_{p \times C} \cong \Omega_C \otimes_{\mathscr{O}_C} \mathscr{O}_C\left(-\sum \mu_i p_i\right) \\ &= \Omega_C \otimes_{\mathscr{O}_C} \mathscr{O}_C(-D_p) = \Omega_C(-D_p). \end{aligned}$$

Since π is proper, the same is true for ψ. In particular, using [SP, Proposition 02O5], $\psi_*\mathscr{L}$ is a coherent sheaf on C^n. Hence, $S \coloneqq \operatorname{supp} \psi_*\mathscr{L} \subseteq C^n$ is a closed set and proper over k (with any induced structure, see [SP, Lemma 0CYL]). By construction, S coincides with the set of points $p \in C^n$ such that $\Omega_C(-D_p)$ has a non-zero global section. Furthermore, an easy calculation (or application of Proper Base Change in the disguise of Proposition 2.5) shows that

$$\psi_*\varphi^*\Omega_C = H^0(C, \Omega_C) \otimes_k \mathscr{O}_{C^n}$$

is the trivial vector bundle with fiber $H^0(C, \Omega_C)$ on C^n. The pushforward of the natural map $\mathscr{L} \to \varphi^*\Omega_C$ induces the injective map $\psi_*\mathscr{L} \to \psi_*\varphi^*\Omega_C$. Finally, since the restriction of $\psi_*\mathscr{L}$ to S is a line bundle by a theorem of Grauert [Har77, Corollary 12.9], we get a natural map[1]

$$S \cong \mathbb{P}(\psi_*\mathscr{L}|_S) \to \mathbb{P}(H^0(C, \Omega_C) \otimes_k \mathscr{O}_S) \to \mathbb{P}_k(H^0(C, \Omega_C)),$$

whose image is closed (using the Cancellation Theorem [Vak17, Theorem 10.1.19] and the fact that S is proper over k) and consists of the union of all the smaller strata $\bigcup_{\nu \leq \mu} \mathcal{H}_{C/k}(\nu)$. The claim follows using the next lemma. □

Lemma 2.14. *Let X be a topological space and let $A, A_1, \ldots, A_d$ be closed subsets of X with $A_i \subseteq A$. Then $A \setminus (A_1 \cup \cdots \cup A_d)$ is locally closed.*

Proof. Clearly $A \setminus (A_1 \cup \cdots \cup A_d) = A \cap (A_1 \cap \cdots \cap A_d)^{\mathsf{c}}$ is an intersection of a closed and an open set. □

Remark 2.15. The previous proposition and proof are also true if Ω_C is replaced by any other line bundle on C.

The result and proof about connected smooth projective curves generalize to families of smooth curves. The proof uses once again the techniques by Grothendieck to reduce to a Noetherian base. At this point, most of the proof above generalizes without much work. This is due to the following lemma.

Lemma 2.16. *Let $f\colon X \to Y$ be a separated and smooth morphism of relative dimension 1 and $\sigma\colon Y \to X$ be a section. Then $\sigma(Y)$ is a relative effective Cartier divisor on X. In particular, the diagonal $\Delta \subseteq X \times_Y X$ is a relative effective Cartier divisor.*

Proof. Since f is separated, the image $\sigma(Y)$ is closed. By the assumption on the dimension the fibers $\sigma(Y)_y \subseteq X_y$ are effective Cartier divisors for all $y \in Y$. Hence, $\sigma(Y)$ is a relative effective Cartier divisor on X by [SP, Lemma 062Y] or a quick direct argument[2].

---

[1] The map on points can be described as follows. Consider the pullback map $k \cong \psi_*\mathscr{L}|_s \to H^0(C, \Omega_C)$ for $s \in S$ which is clearly injective. Hence, the image is a one-dimensional subspace of $H^0(C, \Omega_C)$ and corresponds to a well defined point of $\mathbb{P}_k(H^0(C, \Omega_C))$.

[2] The fibers are smooth curves. Hence, the different notions of divisors agree. In particular the sheaf of ideals $\mathcal{I}$ defining the closed subscheme $\sigma(Y)$ is invertible over every fiber and hence invertible by Nakayama. Flatness is clear since the morphism f is flat.

For the second claim note that the morphism f' in

$$\begin{array}{ccc} X' = X \times_Y X & \xrightarrow{f'} & X \\ {\scriptstyle \mathrm{id}\times\mathrm{id}} \uparrow \downarrow {\scriptstyle f'} & & \downarrow {\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

is also separated and smooth of relative dimension 1, i.e., every section of f' is a relative effective Cartier divisor. □

Theorem 2.17. *Let $\pi\colon C \to B$ be a family of smooth curves of genus $g \geq 2$ and μ be a partition of $2g-2$ into positive integers. Then the set*

$$\mathcal{H}_{C/B}(\mu) \subseteq \mathbb{V}(\pi_*\Omega_{C/B})$$

is locally closed and in particular carries a natural structure as a reduced scheme.

Proof. We generalize the argument used in Proposition 2.13, starting with a Noetherian base B. Replace $H^0(C, \Omega_C)$ with the Hodge bundle $\pi_*\Omega_{C/B}$ and $\mathbb{P}_k(H^0(C, \Omega_C))$ with the analog relative construction $\mathbb{P}(\pi_*\Omega_{C/B})$. Note that the diagonal $\Delta \subseteq C \times_B C$ is a relative effective Cartier divisor by Lemma 2.16. Hence, the pullback of the divisor is well defined and gives us an effective Cartier divisor on C^{n+1}, see [SP, Lemma 056Q]. The calculation from here is identical. Note that the use of the result of Grauert should be replaced with the more general Theorem 1.4.

Now let $\pi\colon C \to B$ be an arbitrary family of smooth curves. Notice that the same proof used above breaks down at the point where we claim that $\psi_*\mathscr{L}$ is coherent. In fact it is not clear that this sheaf is of finite type, which would be enough to imply that the support is closed. We instead reduce to the Noetherian case using the same ideas applied in the proof of Theorem 1.4.

The statement is local on B. Hence, we may assume that $B = \operatorname{Spec} R$ is affine. Using techniques of Grothendieck, we get a Cartesian diagram

$$\begin{array}{ccc} C & \xrightarrow{\hat{\alpha}} & C_0 \\ \downarrow {\scriptstyle \pi} & & \downarrow {\scriptstyle \pi_0} \\ B & \xrightarrow{\alpha} & B_0, \end{array}$$

where $\pi_0\colon C_0 \to B_0$ is a family of smooth curves and $B_0 = \operatorname{Spec} R_0$ is an affine Noetherian scheme.

From the Noetherian case we get that $\mathcal{H}_{C_0/B_0}(\mu)$ is locally closed and by Lemma 2.11 the construction commutes with arbitrary base changes. Hence, $\mathcal{H}_{C/B}(\mu) = \alpha^{-1}(\mathcal{H}_{C_0/B_0}(\mu))$

is locally closed, where we identify α with the induces map on bundles. □

We close this chapter with an alternative construction of a stratum using the concept of marked points. This point of view is useful in the description of the tangent space of a stratum.

Remark 2.18. Let $(\pi\colon C \to B; p_1, \dots, p_n)$ be a family of smooth curves of genus g with n marked points. By Lemma 2.16 the images $D_i := p_i(B)$ are relative effective Cartier divisors on C. Let $\mu = (\mu_1, \dots, \mu_n)$ be a partition of $2g-2$ into n positive integers and consider the line bundles

$$\mathscr{L} := \mathscr{O}_C\left(\sum_{i=1}^n \mu_i D_i\right) := \bigotimes_{i=1}^n \mathscr{O}_C(D_i)^{\mu_i}, \qquad \mathscr{F} := \Omega_{C/B} \otimes \mathscr{L}.$$

By construction, $\pi_*\mathscr{F}$ is a subsheaf of $\pi_*\Omega_{C/B}$ with geometric fibers

$$\begin{aligned}\pi_*\mathscr{F}|_b &= H^0\left(C_b, \Omega_{C/B}\Big|_{C_b} \otimes \mathscr{L}|_{C_b}\right) = H^0\left(C_b, \Omega_{C_b} \otimes \mathscr{O}_{C_b}\left(\sum_{i=1}^n \mu_i p_i(b)\right)\right)\\ &= H^0\left(C_b, \Omega_{C_b}\left(\sum_{i=1}^n \mu_i p_i(b)\right)\right) = \left\{\omega \in \Omega_{C_b}(C_b) \,\Big|\, \operatorname{ord}_{p_i(b)} \omega = \mu_i\right\}.\end{aligned}$$

If the last set is not zero, it is a one-dimensional vector space by Riemann-Roch[1]. Doing this for all possible markings, i.e., applying this construction to the forgetful functor

$$\mathcal{M}_{g,n} \to \mathcal{M}_g,$$

the resulting image is the stratum $\mathcal{H}(\mu)$.

[1] For any two differentials ω_1, ω_2 in this vector space let $f \in k(C_b)$ be a rational function on C_b with $\omega_1 = f\omega_2$. Then $f = \frac{\omega_1}{\omega_2}$ has no zeroes or poles on C_b and is therefore constant.

Chapter 3

Local Properties

In this chapter we analyze local properties of the spaces we have constructed. More precisely, we give a description of tangent space using hypercohomology and the Lie derivative, which allows us to conclude that the moduli space of algebraic translation surfaces and its strata are smooth in characteristic 0. We arrive at the same conclusion for all partitions μ in almost all positive characteristics.

The first section contains a discussion about the tangent space of a stack and how it can be geometrically interpreted. To that end, we introduce the notion of a modular family of smooth curves that plays a similar role as Teichmüller space in complex geometry. **From this point onward, all schemes are assumed to be defined over a fixed algebraically closed field k.** This is not strictly necessary but it simplifies the exposition tremendously as it allows us to work in the category of varieties over k.

We start our discussion of local properties with the simplest case, the whole moduli space of algebraic translation surfaces. Since it is a vector bundle over the moduli space of smooth curves it is clear that it inherits most of its tangent vectors from the this space. Nonetheless, it is instructive to apply our techniques first to a case, where we know what results to expect.

In the last section we use the same ideas to give a description of the tangent space of a stratum.

Our main source of inspiration for the local description of the strata was [Bai+19] where the authors answer the same question for l-differentials over the field of complex numbers. Essentially, in our arguments we remove any dependence on the field $\mathbb{C}$ to make them applicable to any algebraically closed field. Note that similar calculations can already be found in [Möl08]. Different approaches for calculating the closely related dimension of the strata directly can be found in [Pol06] and [Sch18].

3.1 Tangent Space of a Stack and Modular Families

Fix an algebraically closed field k. Recall the definition of a tangent space in algebraic geometry. The first definition one usually encounters for varieties uses a locally affine embedding and linearization of the defining equations. To see that this description is independent of the chosen embedding one can construct an isomorphism with the Zariski tangent space $(\mathfrak{m}/\mathfrak{m}^2)^\vee$. In the category of schemes, having nilpotent elements and in particular the dual numbers $k[\varepsilon]$, we can give another natural definition of the tangent space, following [Har77, Exercise II.2.8]. Note that this result can not be stated in the category of varieties. This is one of the big advantages of working with schemes even if all the objects we are interested in are varieties.

Proposition 3.1. *Let X be a scheme over k and let $x \in X$ be a k-rational point. Then there is a natural isomorphism of k-vector spaces*

$$\{\, t \in \operatorname{Hom}(\operatorname{Spec} k[\varepsilon], X) \mid \operatorname{im} t = \{\, x \,\}\,\} \cong \left(\mathfrak{m}_x \big/ \mathfrak{m}_x^2\right)^\vee = \mathcal{T}_X|_x,$$

where $\mathcal{T}_X|_x$ denotes the tangent space of X in x.

Proof. The statement is local so without loss of generality let $X = \operatorname{Spec} R$ be affine. In fact, since $\operatorname{Spec} k[\varepsilon]$ contains only one point, we could assume R to be $\mathscr{O}_{X,x}$ but this is not necessary. Since x is a rational point, there is a maximal ideal $\mathfrak{m}_x \subseteq R$ corresponding to x such that $R\big/\mathfrak{m}_x \cong k$, i.e., $R \cong \mathfrak{m}_x \oplus k$. Hence, the left hand side is equal to

$$\left\{ \varphi\colon R \to k[\varepsilon] \,\middle|\, \varphi^{-1}((\varepsilon)) = \mathfrak{m}_x \right\} = \left\{ \varphi\colon R\big/\mathfrak{m}_x^2 \to k[\varepsilon] \,\middle|\, \varphi^{-1}((\varepsilon)) = \mathfrak{m}_x \right\} = \left(\mathfrak{m}_x\big/\mathfrak{m}_x^2\right)^\vee.$$

□

Hence, it is reasonable to define the tangent space of a stack $\mathscr{S}$ at an object O over k as

$$\mathcal{T}_{\mathscr{S}}|_O \coloneqq \{\, t \in \operatorname{Mor}(\operatorname{Spec} k[\varepsilon], \mathscr{S}) \mid \operatorname{im} t = O \,\}.$$

We dwell a little longer on this definition and consider it in more detail for the moduli stack of smooth curves.

By construction, maps $t\colon \operatorname{Spec} k[\varepsilon] \to \mathcal{M}_g$ correspond to Cartesian diagrams

$$\begin{array}{ccc} C & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[\varepsilon]. \end{array}$$

Note that the vertical arrow on the left is precisely the smooth curve in the image of the tangent vector t. Hence, we obtain a canonical identification of the tangent space of $\mathcal{M}_g$ at a curve C and first order deformations of C, i.e., $\mathcal{T}_{\mathcal{M}_g}\big|_C \cong \mathrm{Def}(C)$.

There is a canonical scheme attached to $\mathcal{M}_g$, the coarse moduli space M_g. At a first glance it would seem like a good idea to define the tangent space of $\mathcal{M}_g$ at C as the tangent space of M_g at the point corresponding to C. And indeed, this would give the right answer in all but the singular points. The problem is that M_g has lost a lot of the structure that is present in $\mathcal{M}_g$. In the classical solution of the moduli problem (for example [MFK93]) M_g is constructed as a quotient of a smooth variety B with finite stabilizers. The variety B can be chosen in such a way that $\mathcal{T}_{\mathcal{M}_g}\big|_C = \mathcal{T}_B|_b$, where b is any point in the preimage of the point corresponding to C under the projection $B \to M_g$.

We discuss hereafter the construction of the variety B in more detail. The following notion first appeared in [Mum65] with a proof of existence following in [DM69].

Definition 3.2. A *modular family* of smooth curves of genus g is a family X/B of smooth curves of genus g over k with the following properties:

i) For each connected smooth projective curve C over k, there is at least one and at most finitely many closed points $b \in B$ such that the fiber X_b is isomorphic to C.

ii) For each $b \in B$, the functor of local deformations of the fiber X_b is pro-represented by the complete local ring $\hat{\mathscr{O}}_{B,b}$ together with the formal family induced by X.

iii) For any other family of smooth curves X'/B' over k, there exists a scheme S, a surjective étale morphism $S \to B'$ and a morphism $S \to B$ such that

$$X' \times_{B'} S \cong X \times_B S$$

as families over S.

Remark 3.3. The second property of a modular family X/B implies that for each k-valued point b there is a canonical isomorphism

$$\mathrm{Def}(X_b) \cong \mathcal{T}_B|_b.$$

In particular, the tangent space of a modular family and of $\mathcal{M}_g$ coincide at points corresponding to the same smooth curve.

To be more precise, the second property in the previous definition means that for every local Artin k-algebra A the morphisms $\operatorname{Spec} A$ to B with image b naturally correspond to deformations of C_b over $\operatorname{Spec} A$, see [Fan+05, Section 6.2], [Ser06] or [Har10]. We use this property once to verify that any family satisfying the first and second property of a modular family is already a modular family.

A large portion of the remainder of this section contains the proof of the existence of a modular family. The arguments we use are an extended version of the proof in [Har10, Thereom 27.2]. Another complete proof for stable curves over $\mathbb{C}$ in the analytic category can be found in [ACG11, Chapter XI Theorem 6.5].

We remark that most sources do not construct a modular family explicitly and only show the existence. Such non-constructive arguments are used for example in the proof of [Ols16, Theorem 8.4.5] or in [DM69].

Theorem 3.4. *For every $g \geq 2$ there is a smooth variety $\mathcal{Z}$ over k together with a family $\mathcal{C}_{\mathcal{Z}} \to \mathcal{Z}$ satisfying the first property of a modular curve.*

Proof. The first part of the construction is identical to the construction of the coarse moduli space M_g in [MFK93]. The starting point is the observation that any line bundle $\mathscr{L}$ with $\deg \mathscr{L} \geq 2g+1$ on a smooth connected projective curve C is very ample and therefore induces a closed embedding into a projective space. While the canonical line bundle Ω_C in general does not correspond to an embedding, the tricanonical one $\Omega_C^{\otimes 3}$ clearly has the property $d := \deg \Omega_C^{\otimes 3} = 6g-6 > 2g+1$ for $g \geq 2$ and embeds the curve C as a regular curve of degree d in $\mathbb{P}^n$, where $n = \dim H^0(C, \Omega_C^{\otimes 3}) - 1 = 5g-6$. The next observation is that the Hilbert polynomial of a tricanonically embedded curve is $p(t) = \chi(\Omega_C^{\otimes 3t}) = (6g-6)t - g + 1$, independent of the curve C. In particular, the embeddings correspond to points of $\mathrm{Hilb}_{\mathbb{P}^n}^{p(t)}$.

The scheme $\mathrm{Hilb}_{\mathbb{P}^n}^{p(t)}$ comes equipped with a universal family. However, it contains more points than just the ones corresponding to smooth curves embedded via the tricanonical divisor. The next step is to remove points corresponding to singular curves. Let $\mathcal{W} \subseteq \mathrm{Hilb}_{\mathbb{P}^n}^{p(t)} \times \mathbb{P}^n$ denote the universal closed subscheme with flat morphism f to $\mathrm{Hilb}_{\mathbb{P}^n}^{p(t)}$ and $\mathcal{Q} \subseteq \mathcal{W}$ the subset of points where f is not smooth. Then $\mathcal{Q}$ is closed since smoothness is an open condition. Using properness of f, we obtain that $f(\mathcal{Q}) \subseteq \mathrm{Hilb}_{\mathbb{P}^n}^{p(t)}$ is a closed subset containing the points of $\mathrm{Hilb}_{\mathbb{P}^n}^{p(t)}$ corresponding to singular curves. Let $H' \subseteq \mathrm{Hilb}_{\mathbb{P}^n}^{p(t)}$ denote its open complement. Then it is not hard to see that H' is smooth[1] of dimension $25(g-1)^2 + 4(g-1)$.

[1] The tangent space at a point of $\mathrm{Hilb}_{\mathbb{P}^n}^{p(t)}$ corresponding to a smooth curve C is, by the universal property of the Hilbert scheme, an embedded deformation of C. The latter deformations are parameterized by $H^0(C, \mathcal{N}_{C/\mathbb{P}^n})$, where $\mathcal{N}_{C/\mathbb{P}^n}$ denotes the normal bundle, see [Har10, Corollary 2.5]. We have a short exact sequence

$$0 \to \mathcal{T}_C \to \mathcal{T}_{\mathbb{P}^n}|_C \to \mathcal{N}_{C/\mathbb{P}^n} \to 0,$$

characterizing the normal bundle. The Euler sequence $0 \to \mathscr{O}_C \to \mathscr{O}_C(1)^{\oplus n+1} \to \mathcal{T}_{\mathbb{P}^n}|_C \to 0$ and

The scheme H' still contains smooth curves not embedded via the tricanonical divisor. Up to equivalence, two embeddings of the same curve only differ by an element of the Picard scheme of C, which has dimension $H^1(C, \mathscr{O}_C^\times) = g$. Hence, we expect the dimension of the subscheme $H \subseteq H'$ consisting of tricanonically embedded curves to be $25(g-1)^2 + 3g - 4$. The precise properties of H are collected in the following proposition, whose proof can be found in [MFK93, Proposition 5.1] or [ACG11, Chapter XI Proposition 5.1] for stable curves over $\mathbb{C}$.

Proposition 3.5. *There is a unique locally closed subscheme $H \subseteq \mathrm{Hilb}_{\mathbb{P}^n}^{p(t)}$ with the following property: A morphism $\varphi\colon B \to \mathrm{Hilb}_{\mathbb{P}^n}^{p(t)}$ factors through H if and only if*

i) the induced subscheme $C \subseteq B \times \mathbb{P}^n$ is a smooth curve of genus g over B,

ii) the invertible sheaf on C induced by $\mathscr{O}_{\mathbb{P}^n}(1)$ is isomorphic to

$$\Omega_{C/B}^{\otimes 3} \otimes \pi^*(\mathscr{L})$$

for some invertible sheaf $\mathscr{L}$ on B, where $\pi\colon C \to B$ denotes the structure morphism,

iii) for every geometric point $b \in B$, the fiber C_b is a tricanonical curve in $\mathbb{P}^n_{\kappa(b)}$.

By construction (as the scheme representing the obvious functor given by the conditions in the proposition) the scheme H is equipped with a universal family $\mathcal{C}_H$. This is just the restriction of the universal family $\mathcal{W}$ of $\mathrm{Hilb}_{\mathbb{P}^n}^{p(t)}$ to H. Moreover, H is also a smooth scheme, see [MFK93, Proposition 5.3], and hence a variety.

Certainly, every curve appears as a fiber over some point of H. However, different choices of a basis in $H^0(C, \Omega_C^{\otimes 3})$ result in different points of H corresponding to isomorphic curves[1]. Since any isomorphism has to respect the tricanonical divisor, this is the only degree of freedom. In particular, every curve appears infinitely many times as a fiber and the orbit under the natural operation of $G = \mathrm{PGL}_{n+1}(k)$ on H, which is the restriction of the natural action of G on $\mathrm{Hilb}_{\mathbb{P}^n}^{p(t)}$, consists of all the points corresponding to isomorphic curves.

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the fact that $\mathscr{O}_C(1)$ has degree d and is therefore non-special imply the vanishing of $H^1(C, \mathcal{T}_{\mathbb{P}^n}|_C)$. Additivity of the degree and Riemann-Roch for vector bundles imply $\deg \mathcal{T}_{\mathbb{P}^n}|_C = \deg \mathscr{O}_C(1)^{\oplus n+1} = (n+1)d$ and $H^0(C, \mathcal{T}_{\mathbb{P}^n}|_C) = \chi(\mathcal{T}_{\mathbb{P}^n}|_C) = \deg \mathcal{T}_{\mathbb{P}^n}|_C - \mathrm{rk}\, \mathcal{T}_{\mathbb{P}^n}|_C (g-1) = (n+1)d - n(g-1)$. The exactness of

$$0 \to H^0(C, \mathcal{T}_{\mathbb{P}^n}|_C) \to H^0(C, \mathcal{N}_{C/\mathbb{P}^n}) \to H^1(C, \mathcal{T}_C) \to 0$$

implies $h^0(C, \mathcal{N}_{C/\mathbb{P}^n}) = (n+1)d - n(g-1) + (3g-3) = 25(g-1)^2 + 4(g-1)$ which shows the smoothness and dimension claim.

[1] Unless the choice of a different basis results in the same embedding, i.e., is an automorphism of the curve.

In the construction of the coarse moduli space, the next (and final) step would be to factor by this group action (the hard part is to see that this is possible). As $\dim G = (n+1)^2 - 1 = 25(g-1)^2 - 1$ the resulting quotient space has the expected dimension $3g-3$. See [MFK93, Proposition 5.4] and [MFK93, Theorem 5.11] for details.

We continue with the construction of the modular family. The idea is to first build a local family around a curve C and then take a finite disjoint union of local families to obtain a modular family. Consider the following useful point of view. We constructed H as a locally closed smooth subscheme of $\mathrm{Hilb}^{p(t)}_{\mathbb{P}^n}$, which itself is a closed subscheme of some $\mathbb{P}^m$. This allows us to see H as a locally closed smooth subvariety of $\mathbb{P}^m$ where the action of G on H is induced by the action of $\mathrm{PGL}_m(k)$ via an inclusion $G \hookrightarrow \mathrm{PGL}_m(k)$.

Let $h \in H$ be a point corresponding to a curve C and consider the orbit $G \cdot h$, which is a closed subset of H, together with the reduced structure. This is again a variety and since every point can be mapped to any other point via an automorphism induced by an element of G, it is smooth. Moreover, the stabilizer G_h can be identified with the automorphism group of C. Hence, it is finite. This implies that $\dim(G \cdot h) = \dim G = 25(g-1)^2 - 1$. Consider the linear subspace $T \subseteq \mathbb{P}^m$ tangent to $G \cdot h$ at h. Since T is clearly[1] G_h-invariant, we find another G_h-invariant linear subspace $L \subseteq \mathbb{P}^m$ of complementary dimension such that $T \cap L = \{h\}$. By construction, the subvariety $Z \coloneqq L \cap H$ intersects the orbit $G \cdot h$ transversely and is of dimension $3g-3 = \mathrm{codim}_H(G \cdot h)$. In the following, we replace Z by an open subset containing h to obtain a family that locally has properties *i)* and *ii)* of a modular family.

First note that h is a smooth point of Z as the intersection of L and H is transversal by construction. The set of points $z \in Z$ such that the intersection of $G \cdot z$ and Z is not transversal at z is closed. Since h is not one of those points, they form a proper subset of Z. In particular, we may assume that the intersection of $G \cdot z$ and Z is transversal for every z and that Z is smooth. Note that this already implies that the intersection $G \cdot z \cap Z$ is finite for every $z \in Z$.

We call the smooth variety Z together with the restriction $\mathcal{C}_Z$ of the universal family $\mathcal{C}_H$ to Z a *local modular family* for the curve C. Note that it has the property that every smooth curve C' appears at most finitely many times as a fiber over a closed point.

To obtain a modular family, we observe that the image of $G \times Z$ contains an open subset of H since this is certainly true for $G \times (L \cap H)$. As H is a variety and in particular quasi-compact, we find finitely many local modular families Z_i such that the corresponding orbits cover H. The disjoint union $\mathcal{Z}$ of the Z_i together with the family $\mathcal{C}_{\mathcal{Z}} \coloneqq \mathcal{C}_{Z_1} \sqcup \cdots \sqcup \mathcal{C}_{Z_n}$ has by construction the property that every connected smooth projective curve over k appears at least once and at most finitely many times as a fiber

[1] Any automorphism maps tangent spaces to tangent spaces.

over a closed point $z \in \mathcal{Z}$, i.e., satisfies property i). □

It remains to verify that the smooth variety $\mathcal{Z}$ together with the family $\mathcal{C}_{\mathcal{Z}}$ constructed in Theorem 3.4, is a modular family.

Proposition 3.6. *Let $z \in \mathcal{Z}$ be a closed point. Then there is a natural isomorphism*

$$\mathcal{T}_{\mathcal{Z}}\big|_z \cong H^1(\mathcal{C}_{\mathcal{Z},z}, \mathcal{T}_{\mathcal{C}_{\mathcal{Z},z}}).$$

Proof. Write $C := \mathcal{C}_{\mathcal{Z},z}$. Consider the short exact sequence

$$0 \to \mathcal{T}_C \to \mathcal{T}_{\mathbb{P}^n}\big|_C \to \mathcal{N}_{C/\mathbb{P}^n} \to 0$$

characterizing the normal bundle. The fact that $\mathscr{O}_C(1)$ has degree d and is therefore non-special together with the Euler sequence $0 \to \mathscr{O}_C \to \mathscr{O}_C(1)^{\oplus n+1} \to \mathcal{T}_{\mathbb{P}^n}\big|_C \to 0$ implies the vanishing of $H^1(C, \mathcal{T}_{\mathbb{P}^n}\big|_C)$. Hence, the long exact sequence in cohomology asserts that the map

$$H^0(C, \mathcal{N}_{C/\mathbb{P}^n}) \to H^1(C, \mathcal{T}_C)$$

is surjective with kernel $H^0(C, \mathcal{T}_{\mathbb{P}^n}\big|_C)$. Using [Har10, Corollary 2.5], we get a natural identification of the tangent space of $\mathcal{Z}$ at the point z and a subspace of $H^0(C, \mathcal{N}_{C/\mathbb{P}^n})$. It is easy to verify that the induced map $\mathcal{T}_{\mathcal{Z}}\big|_z \to H^1(C, \mathcal{T}_C)$ is still surjective and hence an isomorphism. Indeed, the tangent vectors of $\mathrm{Hilb}^{p(t)}_{\mathbb{P}^n}$ at z that get mapped to zero span the tangent space of $G \cdot z$ at z since those tangent vectors point to the same curve C and therefore induce the trivial deformation. □

This result together with the smoothness of $\mathcal{Z}$ already implies that the family $\mathcal{C}_{\mathcal{Z}} \to \mathcal{Z}$ has the second property of a modular family. We refer the reader to [Har10, Theorem 27.2] for details and collect the following result without proof.

Proposition 3.7. *Let $z \in \mathcal{Z}$ be a closed point with corresponding curve C. Then the complete local ring $\hat{\mathscr{O}}_{\mathcal{Z},z}$ together with the induced formal family pro-represents the local deformation functor of C.*

Theorem 3.8. *For every $g \geq 2$ a modular family of smooth curves of genus g over k exists and can be chosen to be a smooth variety.*

The proof of this theorem uses the scheme of isomorphisms between two families of curves, C_1 and C_2, over the same base scheme B. Note that we have only shown the existence of $\mathbf{Isom}_B(C_1, C_2)$ if B is Noetherian and C_1 and C_2 are projective over B. Mumford, however, claims in [Mum65, Section 3] that $\mathbf{Isom}_B(C_1, C_2)$ exists in complete generality, referring to general results of Grothendieck [FGA, exposé 221, p. 20].

Indeed, using those results it is possible to remove the projectivity assumption entirely. Unfortunately, it seems that all the results of Grothendieck in the source above only concern (locally) Noetherian bases. The right reference for our needs is [AK80, (2.6) Theorem] or more precisely [AK80, (2.7) Corollary]. They show that the Hilbert scheme of a strongly projective morphism exists and is itself strongly projective. This is done with no assumption on the base scheme. In the case of a family of curves C/B this means that $\mathrm{Hilb}_{C/B}$ exists, as long as we keep track of the tricanonical divisor[1] on C and similar for $\mathrm{Hilb}_{C_1\times_B C_2/B}$ using the pullbacks of the two tricanonical divisors. From here, the construction of the scheme $\mathbf{Isom}_B(C_1, C_2)$ can be carried out as before.

Proof of Theorem 3.8. In light of Theorem 3.4 and Proposition 3.7 it would be enough to show that the family $\mathcal{C}_{\mathcal{Z}} \to \mathcal{Z}$ satisfies the third property of a modular family. Instead, we show the stronger statement that any family X/B having properties *i*) and *ii*) of a modular family is already a modular family.

To see this let X'/B' be any other family of smooth curves and examine the two families

$$\begin{array}{ccc} X' \times B & & X \times B' \\ & \searrow \quad \swarrow & \\ & B' \times B & \end{array}$$

over $B' \times B$. Consider the scheme of isomorphisms $S \coloneqq \mathbf{Isom}_{B'\times B}(X \times B', X' \times B)$. Using the universal property, we get an S-isomorphism

$$X \times_B S \cong X \times B' \times_{B'\times B} S \cong X' \times B \times_{B'\times B} S \cong X' \times_{B'} S$$

and have to show that the canonical morphism $\varphi\colon S \to B'$ is surjective and étale.

Let $b' \in B'$ be a point and $C = X_{b'}$ be the corresponding curve. Let $b_1, \dots, b_n$ denote the finitely many points of B with fiber isomorphic to C. Note that $\mathbf{Isom}_k(X'_{b'}, X_{b_i})$ is finite for every i since the automorphism group of C is finite. Hence, the fiber of $S \to B'$ over b' is finite and not empty, i.e., φ is surjective with finite fibers. It remains to verify that φ is étale.

Let $s \in S$ be a point over $b' \in B$. Then s corresponds to an element of $\mathbf{Isom}_k(X'_{b'}, X_{b_i})$, i.e., fixes the point $b_i \in B$, and an isomorphism $X_{b_i} \cong X'_{b'}$. Consider an Artin ring A that is a quotient of $\mathscr{O}_{B',b'}$. By pulling back the family over B' along the corresponding

[1] Note that the tricanonical divisor $\Omega^{\otimes 3}_{C/B}$ on C enables us to embed C in a projective space over B. Indeed, the canonical map $\pi^*\pi_*\Omega^{\otimes 3}_{C/B} \to \Omega^{\otimes 3}_{C/B}$ is surjective and induces the embedding $C \to \mathbb{P}_B(\pi_*\Omega^{\otimes 3}_{C/B})$ using the construction described in [SP, Section 01O8]. Then $\mathrm{Hilb}_{C/B}$ can be identified with the Hilbert scheme of this projective embedding in the sense of Altman and Kleiman, i.e., one has to keep track of the canonical line bundle.

morphisms $\operatorname{Spec} A \to B'$ we obtain a family of smooth curves over $\operatorname{Spec} A$. Using property *ii*) of a modular family, there is a unique morphism $\operatorname{Spec} A \to B$ with image b_i inducing an isomorphic family. Note that this family has no non-trivial automorphisms since those imply the existence of non-trivial elements[1] in $H^0(X_{b_i}, \mathcal{T}_{X_{b_i}}) = 0$. Hence, there is a unique morphism $\operatorname{Spec} A \to S$ with image s, i.e., there is a unique map $\mathscr{O}_{S,s} \to A$ such that

$$\begin{array}{ccc} \mathscr{O}_{B',b'} & \xrightarrow{\varphi^{\#}} & \mathscr{O}_{S,s} \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

commutes. Using the universal property of the completion, this implies that the complete local rings $\hat{\mathscr{O}}_{B',b'}$ and $\hat{\mathscr{O}}_{S,s}$ are isomorphic, i.e., φ is étale at s and hence étale. □

Remark 3.9. In general it is not true that there is a, in some sense, best modular family, see [Har10, Remark 26.6.2]. However, if X_1/B_1 and X_2/B_2 are two modular families, it follows from the definition that there is a third modular family X_3/B_3 together with two surjective étale morphisms $B_3 \to B_1$ and $B_3 \to B_2$ such that

$$X_1 \times_{B_1} B_3 \cong X_3 \cong X_2 \times_{B_2} B_3$$

as families over B_3. Hence, properties that are stable under étale maps are independent of the choice of a modular family. In particular every modular family is a smooth variety of dimension $3g - 3$.

Remark 3.10. A modular family also exists for $g = 1$ (and $g = 0$). If the characteristic of k is different from 2 and 3, one modular family of elliptic curves is explicitly given by

$$B = \operatorname{Spec} k[\lambda, \lambda^{-1}, (\lambda - 1)^{-1}],$$

together with the family X of plane cubic curves over B given by the equation $y^2 = x(x-1)(x-\lambda)$, see [Mum65, Chapter 4] or [Har10, Theorem 26.4] for more details.

Caution 3.11. For the remainder of this thesis we fix a modular family $\pi\colon \mathcal{C}_{\mathcal{Z}} \to \mathcal{Z}$ of genus $g \geq 2$ and identify it with the moduli stack of smooth curves, i.e.,

$$\mathcal{M}_g = \mathcal{Z}, \qquad \mathcal{C} = \mathcal{C}_{\mathcal{Z}}.$$

[1] Arguing by induction, we can reduce to the case of a small extension, i.e., in essence an extension over the dual numbers $k[\varepsilon]$. Then the result is essentially Lemma 1.21. See also [Har10, Corollary 18.3].

This also means that we identify $\Omega\mathcal{M}_g = \mathbb{V}(\pi_*\Omega_{\mathcal{C}_\mathcal{Z}/\mathcal{Z}})$ and $\mathcal{H}(\mu) = \mathcal{H}_{\mathcal{C}_\mathcal{Z}/\mathcal{Z}}(\mu)$.

Note that since we are only interested in the tangent spaces of the moduli space of algebraic translation surfaces and its strata, this identification makes essentially no difference considering Remark 3.9. Nonetheless, it is important to keep this convention in mind.

3.2 Tangent Space of the Moduli Space of Algebraic Translation Surfaces

Fix an integer $g \geq 2$ and an algebraically closed field k. In this section we give a description of the tangent space of $\Omega\mathcal{M}_g$, reminding ourselves that we are working on a modular family, see Caution 3.11.

It is clear that the moduli space of algebraic translation surfaces is a smooth variety of dimension $4g-3$ since $\mathcal{M}_g$ is a smooth variety of dimension $3g-3$ and $\pi_*\Omega_{\mathcal{C}_g/\mathcal{M}_g}$ is a vector bundle of rank g.

We begin by giving a description of the tangent space at a point (C, ω) in terms of deformations of the curve C and the differential ω.

Let $\operatorname{Spec} k[\varepsilon] \to \Omega\mathcal{M}_g$ be a tangent vector. Applying the universal property of the vector bundle, this is equivalent to a morphism $t\colon \operatorname{Spec} k[\varepsilon] \to \mathcal{M}_g$ together with a section ω_ε of $t^*\pi_*\Omega_{\mathcal{C}_g/\mathcal{M}_g}$. The universal property of $\mathcal{M}_g$ in the guise of Proposition 3.6 translates the map t into the pullback diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{t'} & \mathcal{C}_g \\ \downarrow{\scriptstyle \pi_\varepsilon} & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} k[\varepsilon] & \xrightarrow{t} & \mathcal{M}_g. \end{array}$$

Hence, the calculation

$$t^*\pi_*\Omega_{\mathcal{C}_g/\mathcal{M}_g} = \pi_{\varepsilon*}t'^*\Omega_{\mathcal{C}_g/\mathcal{M}_g} = \pi_{\varepsilon*}\Omega_{\mathcal{C}/\operatorname{Spec} k[\varepsilon]} = H^0(\mathcal{C}, \Omega_{\mathcal{C}/\operatorname{Spec} k[\varepsilon]})$$

gives us the following description of the tangent space at a point (C, ω):

Proposition 3.12. *The tangent vectors of $\Omega\mathcal{M}_g$ at a closed point corresponding to (C, ω) can be canonically identified with the deformations of the pair (C, ω), i.e., the deformation* *diagrams*

$$\begin{array}{ccc} (C,\omega) & \longrightarrow & (\mathcal{C},\omega_\varepsilon) \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[\varepsilon] \end{array}$$

such that the differential $\omega_\varepsilon \in H^0(\mathcal{C}, \Omega_{\mathcal{C}/\operatorname{Spec} k[\varepsilon]})$ *pulls back to* ω*, up to isomorphism of deformations.*

We use this description to write the tangent space at a point as a hypercohomology group. To this end, note that every map of sheaves $A\colon \mathscr{F} \to \mathscr{G}$ on some topological space induces a two term complex in degree zero and one

$$A^\bullet := \cdots \to 0 \to \mathscr{F} \to \mathscr{G} \to 0 \to \cdots$$

whose first hypercohomology can be calculated using Example 1.28.

Theorem 3.13. *For every closed point of* $\Omega\mathcal{M}_g$ *corresponding to a pair* (C,ω) *there is a canonical isomorphism*

$$\mathcal{T}_{\Omega\mathcal{M}_g}\Big|_{(C,\omega)} \cong \mathbb{H}^1(C, L_\omega^\bullet),$$

where the vector space on the right is the first hypercohomology group of the Lie derivative $L_\omega\colon \mathcal{T}_C \to \Omega_C$ *associated with* ω*.*

To the best of our knowledge, this idea first appeared in Chapter IV of [HM79] for the moduli space of quadratic differentials and can also be found in [Möl08] in the proof of Theorem 2.1.

Proof of Theorem 3.13. Using Proposition 3.12, we have to construct an isomorphism between the hypercohomology group and the deformations of the pair (C,ω). Let

$$\begin{array}{ccc} (C,\omega) & \longrightarrow & (\mathcal{C},\omega_\varepsilon) \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[\varepsilon] \end{array}$$

be such a deformation, i.e., the diagram above is Cartesian and $\omega_\varepsilon \in H^0(\mathcal{C}, \Omega_{\mathcal{C}/\operatorname{Spec} k[\varepsilon]})$ pulls back to the differential ω. Let U_α be an affine open cover of C such that $\mathcal{C}$ is given by transition functions $\varphi_{\alpha\beta}$ with

$$\varphi_{\alpha\beta}^{\#} = \pi_1 + \varepsilon(D_{\alpha\beta} \circ \pi_1 + \pi_2)$$

on the intersection $U_{\alpha\beta} \coloneqq U_\alpha \cap U_\beta$. Here $D_{\alpha\beta}$ denotes derivations on $U_{\alpha\beta}$ and π_i is the projection on the ith component, see the proof of Lemma 1.21. Sections of $\Omega_{\mathcal{C}/\operatorname{Spec} k[\varepsilon]}$ can be locally written as $\omega_1 + \varepsilon\omega_2$, where the ω_i are sections of Ω_C over the same open set. The transition function for this sheaf on the intersection $U_{\alpha\beta}$ is therefore given by

$$\mathrm{d}f + \varepsilon\,\mathrm{d}\tilde{f} \mapsto \mathrm{d}(f + \varepsilon D_{\alpha\beta}(f)) + \varepsilon\,\mathrm{d}\Big(\tilde{f} + \varepsilon D_{\alpha\beta}(\tilde{f})\Big) = \mathrm{d}f + \varepsilon\,\mathrm{d}\tilde{f} + \varepsilon\,\mathrm{d}D_{\alpha\beta}(f).$$

Note that the term $\mathrm{d}D_{\alpha\beta}(f)$ is related to the Lie derivative. If $\mathrm{d}x$ denotes a trivializing element for Ω_C on $U_{\alpha\beta}$ and g, h are regular functions such that $\mathrm{d}f = g\,\mathrm{d}x$ and $D_{\alpha\beta} = h\frac{\partial}{\partial x}$, we have

$$\mathrm{d}D_{\alpha\beta}(f) = \mathrm{d}\left(h\frac{\partial}{\partial x}f\right) = \mathrm{d}(hg) = \frac{\partial}{\partial x}(hg)\,\mathrm{d}x = L_{\mathrm{d}f}(D_{\alpha\beta}).$$

Hence, local sections $\omega_{1,\alpha} + \varepsilon\omega_{2,\alpha} \in H^0(U_\alpha, \Omega_{\mathcal{C}/\operatorname{Spec} k[\varepsilon]})$ pull back to $\omega_\alpha \coloneqq \omega|_{U_\alpha}$ if and only if $\omega_{1,\alpha} = \omega_\alpha$, i.e., they can be written as $\omega_\alpha + \omega'_\alpha$ for some $\omega'_\alpha \in H^0(U_\alpha, \Omega_{\mathcal{C}/\operatorname{Spec} k[\varepsilon]})$. Those local descriptions glue to a global section if and only if they agree under the transition functions. This translates to the condition

$$\omega'_\beta\big|_{U_{\alpha\beta}} - \omega'_\alpha|_{U_{\alpha\beta}} = L_{\omega_\alpha}(D_{\alpha\beta}).$$

Given a deformation $(\mathcal{C}, \omega_\varepsilon)$ and writing $\omega_\varepsilon|_{U_\alpha} = \omega_\alpha + \varepsilon\omega'_\alpha$, we obtain the pair

$$(D_{\alpha\beta}, \omega'_\alpha) \in \check{C}^1(\mathcal{U}, \mathcal{T}_C) \oplus \check{C}^0(\mathcal{U}, \Omega_C)$$

which is a one-cochain of the complex associated with the Lie derivative by the arguments above. Using Example 1.28, it is easy to verify that this map induces an isomorphism in cohomology. □

Corollary 3.14. *The moduli space of algebraic translation surfaces $\Omega\mathcal{M}_g$ is a smooth variety of dimension $4g-3$.*

Proof. As remarked before, the statement is clear since $\mathcal{M}_g$ is a smooth variety (remember Caution 3.11) of dimension $3g-3$ and $\Omega\mathcal{M}_g$ is (the total space of) a vector bundle of rank g. We show the same assertion using a completely different argument that generalizes in the case of a stratum. We calculate the dimension of the tangent space of $\Omega\mathcal{M}_g$ at every point using the description of Theorem 3.13. The dimension turns out to be independent of the point (C, ω) of $\Omega\mathcal{M}_g$ implying that $\Omega\mathcal{M}_g$ is smooth.

Let (C, ω) correspond to a closed point of $\Omega\mathcal{M}_g$ and $L_\omega\colon \mathcal{T}_C \to \Omega_C$ denote the Lie derivative. We have a short exact sequence of complexes in degree zero and one

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{T}_C & \longrightarrow & \mathcal{T}_C & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle A} & & \big\downarrow{\scriptstyle L_\omega} & & \big\downarrow{\scriptstyle B} & & \\ 0 & \longrightarrow & \Omega_C & \longrightarrow & \Omega_C & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

which induces the exact sequence

$$\mathbb{H}^0(C, B^\bullet) \to \mathbb{H}^1(C, A^\bullet) \to \mathbb{H}^1(C, L_\omega^\bullet) \to \mathbb{H}^1(C, B^\bullet) \to \mathbb{H}^2(C, A^\bullet)$$

in hypercohomology. Using Example 1.26, this translates to the short exact sequence

$$0 = H^0(C, \mathcal{T}_C) \to H^0(C, \Omega_C) \to \mathbb{H}^1(C, L_\omega^\bullet) \xrightarrow{\psi} H^1(C, \mathcal{T}_C) \xrightarrow{\varphi} H^1(C, \Omega_C) \cong k,$$

where the map ψ is surjective by construction of the first hypercohomology group of the Lie derivative. Hence, the map $\varphi = H^1(C, L_\omega)$ is 0 and

$$\dim \mathbb{H}^1(C, L_\omega^\bullet) = h^1(C, \mathcal{T}_C) + h^0(C, \Omega_C) = (3g-3) + g = 4g - 3$$

since $g > 1$. □

3.3 Tangent Space of a Stratum

Fix an integer $g \geq 2$, an algebraically closed field k and a partition $\mu = (\mu_1, \ldots, \mu_n)$ of $2g-2$ into $n = |\mu|$ positive integers.

In this section we give a description of the tangent space of the stratum $\mathcal{H}(\mu)$ using the first hypercohomology group associated with the twisted Lie derivative. This uses techniques similar to the previous section. In characteristic 0 this representation of the tangent space implies that every irreducible component of $\mathcal{H}(\mu)$ is smooth of the same dimension.

We begin by giving a correspondence between tangent vectors at $(C, \omega) \in \mathcal{H}(\mu)$ and certain deformations of the pair (C, ω).

Proposition 3.15. *Let $p_1, \ldots, p_n \colon \operatorname{Spec} k \to C$ denote the zeros of ω on C of an element $(C, \omega) \in \mathcal{H}(\mu)$. Then tangent vectors of $\mathcal{H}(\mu)$ at a closed point corresponding to (C, ω) can be canonically identified with marked deformations of the tuple $(C; p_1, \ldots, p_n; \omega)$, i.e.,* *the marked deformation diagrams*

$$\begin{array}{ccc} (C,\omega) & \longrightarrow & (\mathcal{C},\omega_\varepsilon) \\ {\scriptstyle p_i}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle p_{\varepsilon,i}} \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[\varepsilon] \end{array}$$

such that the differential $\omega_\varepsilon \in H^0(\mathcal{C}, \Omega_{\mathcal{C}/\operatorname{Spec} k[\varepsilon]})$ *vanishes at the marked points and pulls back to* ω*, up to isomorphism of deformations with marked points.*

Proof. Let $T\colon \operatorname{Spec} k[\varepsilon] \to \Omega\mathcal{M}_g$ be a tangent vector with image (C,ω) and denote by $t\colon \operatorname{Spec} k[\varepsilon] \to \mathcal{M}_g$ the induced tangent vector of $\mathcal{M}_g$. Using the description of a stratum in Remark 2.18, we see that T corresponds to a tangent vector of $\mathcal{H}(\mu)$ if and only if there exist sections $q_i\colon \mathcal{M}_g \to \mathcal{C}_g$ such that $q_1(C),\dots,q_n(C)$ are the zeroes of ω on C and the differential $\omega_\varepsilon \in H^0(\mathcal{C}, \Omega_{\mathcal{C}/\operatorname{Spec} k[\varepsilon]})$ pulling back to ω is a section of

$$t^*\pi_{g*}\left(\Omega_{\mathcal{C}_g/\mathcal{M}_g} \otimes \mathscr{O}_{\mathcal{C}_g}\left(\sum \mu_i q_i(\mathcal{M}_g)\right)\right) \subseteq t^*\pi_{g*}\Omega_{\mathcal{C}_g/\mathcal{M}_g} = H^0(\mathcal{C}, \Omega_{\mathcal{C}/\operatorname{Spec} k[\varepsilon]}).$$

Considering the Cartesian diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{t'} & \mathcal{C}_g \\ \downarrow{\scriptstyle \pi_\varepsilon} & & {\scriptstyle \pi_g}\downarrow\uparrow{\scriptstyle q_i} \\ \operatorname{Spec} k[\varepsilon] & \xrightarrow{t} & \mathcal{M}_g, \end{array}$$

the last assertion is equivalent to $\omega_\varepsilon \in H^0(\mathcal{C}, \Omega_{\mathcal{C}/\operatorname{Spec} k[\varepsilon]})$ pulling back to ω and vanishing at the marked points in the deformation diagram

$$\begin{array}{ccc} (C,\omega) & \longrightarrow & (\mathcal{C},\omega_\varepsilon) \\ {\scriptstyle p_i}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle p_{\varepsilon,i}} \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[\varepsilon], \end{array}$$

where both p_i and $p_{\varepsilon,i}$ are obtained by pulling back q_i. □

Using this description of tangent vectors, we can construct an isomorphism of the tangent space at a point with a hypercohomology group. This argument is a generalization of the case[1] $k = 1$ in [Bai+19, Theorem 2.1] to arbitrary algebraically closed fields.

[1] The paper [Bai+19] is concerned with k-differentials ($k \in \mathbb{N}$) on compact Riemann surfaces.

Theorem 3.16. *For every closed point of $\mathcal{H}(\mu)$ corresponding to a pair (C, ω) there is a canonical isomorphism*

$$\mathcal{T}_{\mathcal{H}(\mu)}\Big|_{(C,\omega)} \cong \mathbb{H}^1(C, \mathcal{L}_\omega^\bullet),$$

where the vector space on the right is the first hypercohomology group of the twisted Lie derivative associated with ω.

Proof. Write $\mu = (\mu_1, \ldots, \mu_n)$ and set $Z = \operatorname{div} \omega$. Using Proposition 3.15, we have to show that the first-order deformations of the datum $(C; p_1, \ldots, p_n; \omega)$, where the p_i are the zeroes of the differential ω on C, are isomorphic to the claimed hypercohomology group. Let

$$\begin{array}{ccc} (C,\omega) & \longrightarrow & (\mathcal{C}, \omega_\varepsilon) \\ {\scriptstyle p_i}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle p_{\varepsilon,i}} \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[\varepsilon] \end{array}$$

be such a deformation. The deformation of the marked curve is by Proposition 1.23 given by an element of $H^1(C, \mathcal{T}_C(-Z_{\mathrm{red}}))$. Consider a one-cocycle representing the deformation. Hence, we are given an affine open cover $\mathcal{U} = \{U_\alpha\}$ of C and compatible elements $D_{\alpha\beta} \in \mathcal{T}_C(-Z_{\mathrm{red}})(U_\alpha \cap U_\beta)$ such that $\mathcal{C}$ is given by transition functions $\varphi_{\alpha\beta}$ with

$$\varphi_{\alpha\beta}^{\#} = \pi_1 + \varepsilon(D_{\alpha\beta} \circ \pi_1 + \pi_2)$$

on the intersection $U_{\alpha\beta} := U_\alpha \cap U_\beta$, where π_i denotes the projection on the ith component. Completely analogously to the calculation for the space $\Omega\mathcal{M}_g$, the transition functions for the sheaf $\Omega_{\mathcal{C}/\operatorname{Spec} k[\varepsilon]}$ are given by

$$\mathrm{d}f + \varepsilon\,\mathrm{d}\tilde{f} \mapsto \mathrm{d}(f + \varepsilon D_{\alpha\beta}(f)) + \varepsilon\,\mathrm{d}\Big(\tilde{f} + \varepsilon D_{\alpha\beta}(\tilde{f})\Big) = \mathrm{d}f + \varepsilon\,\mathrm{d}\tilde{f} + \varepsilon\,\mathrm{d}D_{\alpha\beta}(f)$$

and $\mathrm{d}D_{\alpha\beta}(f) = L_{\mathrm{d}f}(D_{\alpha\beta})$. Note that $L_{\omega_\alpha}(D_{\alpha\beta}) = \mathcal{L}_{\omega_\alpha}(D_{\alpha\beta})$.

Since $\omega_\varepsilon \in H^0(\mathcal{C}, \Omega_{\mathcal{C}/\operatorname{Spec} k[\varepsilon]})$ is a global section pulling back to ω with zeros in the marked points, it can be locally written as $\omega_\alpha + \varepsilon\omega'_\alpha$ for some sections ω'_α of $\Omega_C(-Z)$ that satisfy the equations

$$\omega'_\beta\Big|_{U_{\alpha\beta}} - \omega'_\alpha|_{U_{\alpha\beta}} = \mathcal{L}_{\omega_\alpha}(D_{\alpha\beta}).$$

Hence, the pair $(D_{\alpha\beta}, \omega'_\alpha) \in \check{C}^1(\mathcal{U}, \mathcal{T}_C(-Z_{\mathrm{red}})) \oplus \check{C}^0(\mathcal{U}, \Omega_C(-Z))$ forms a one-cochain of the complex associated with the twisted Lie derivative.

Conversely, given a one-cochain $(D_{\alpha\beta}, \omega'_\alpha) \in \check{C}^1(\mathcal{U}, \mathcal{T}_C(-Z_{\mathrm{red}})) \oplus \check{C}^0(\mathcal{U}, \Omega_C(-Z))$, we can build a deformation $(\mathcal{C}, \omega_\varepsilon)$ of (C, ω) by reversing the steps above. Using Example 1.28, it is easy to check that the two maps induce inverse isomorphisms in cohomology. $\square$

Unlike the case of the whole moduli space of algebraic translation surfaces, it is not at all clear whether or not the strata are smooth varieties and what the dimension of their irreducible components are. Using the description of the tangent space given in Theorem 3.16, we are able to completely answer these questions in characteristic 0 and partially in positive characteristic.

Theorem 3.17. *Let k be an algebraically closed field such that $\operatorname{char} k = 0$ or each part μ_i of the partition μ is strictly smaller than $p = \operatorname{char} k > 0$. Then*

$$\dim \mathcal{T}_{\mathcal{H}(\mu)}\Big|_{(C,\omega)} = 2g + n - 1,$$

where C is a curve over k and $n = |\mu|$ is the size of the partition.

Proof. We have to calculate $\dim \mathbb{H}^1(C, \mathcal{L}_\omega^\bullet)$. Consider the following short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{T}_C(-Z_{\mathrm{red}}) & \longrightarrow & \mathcal{T}_C(-Z_{\mathrm{red}}) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathcal{A}} & & \big\downarrow{\scriptstyle \mathcal{L}_\omega} & & \big\downarrow{\scriptstyle \mathcal{B}} & & \\ 0 & \longrightarrow & \Omega_C(-Z) & \longrightarrow & \Omega_C(-Z) & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

of complexes in degree zero and one. It induces the following long exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathbb{H}^0(C, \mathcal{A}^\bullet) \longrightarrow \mathbb{H}^0(C, \mathcal{L}_\omega^\bullet) \longrightarrow \mathbb{H}^0(C, \mathcal{B}^\bullet) \\ \hookrightarrow \mathbb{H}^1(C, \mathcal{A}^\bullet) \longrightarrow \mathbb{H}^1(C, \mathcal{L}_\omega^\bullet) \longrightarrow \mathbb{H}^1(C, \mathcal{B}^\bullet) \\ \hookrightarrow \mathbb{H}^2(C, \mathcal{A}^\bullet) \longrightarrow \mathbb{H}^2(C, \mathcal{L}_\omega^\bullet) \longrightarrow \mathbb{H}^2(C, \mathcal{B}^\bullet) \longrightarrow \cdots \end{array}$$

of hypercohomology. Using Example 1.26, the hypercohomology groups of the complexes $\mathcal{A}^\bullet$ and $\mathcal{B}^\bullet$ are $\mathbb{H}^i(C, \mathcal{A}^\bullet) = H^{i-1}(C, \Omega_C(-Z))$ and $\mathbb{H}^i(C, \mathcal{B}^\bullet) = H^i(C, \mathcal{T}_C(-Z_{\mathrm{red}}))$. Substituting this into the long exact sequence and noting that $H^0(C, \mathcal{T}_C(-Z_{\mathrm{red}}))$ vanishes, we obtain the exact sequence

$$\begin{array}{l} 0 \xrightarrow{\sim} H^0(C, \mathcal{T}_C(-Z_{\mathrm{red}})) \longrightarrow H^0(C, \Omega_C(-Z)) \longrightarrow \mathbb{H}^1(C, \mathcal{L}_\omega^\bullet) \\ \hookrightarrow H^1(C, \mathcal{T}_C(-Z_{\mathrm{red}})) \xrightarrow{\varphi} H^1(C, \Omega_C(-Z)) \longrightarrow \mathbb{H}^2(C, \mathcal{L}_\omega^\bullet) \longrightarrow 0 \end{array}$$

of finite dimensional k-vector spaces. Since the alternating sum of the dimensions of vector spaces in a finite exact sequence vanishes, Riemann-Roch yields

$$\begin{aligned}
\dim \mathbb{H}^1(C, \mathcal{L}_\omega^\bullet) &= (h^0(C, \Omega_C(-Z)) - h^1(C, \Omega_C(-Z))) \\
&\quad - (h^0(C, \mathcal{T}_C(-Z_{\mathrm{red}})) - h^1(C, \mathcal{T}_C(-Z_{\mathrm{red}}))) + \dim \mathbb{H}^2(C, \mathcal{L}_\omega^\bullet) \\
&= (\deg \Omega_C(-Z) - g + 1) - (\deg \mathcal{T}_C(-Z_{\mathrm{red}}) - g + 1) + \dim \mathbb{H}^2(C, \mathcal{L}_\omega) \\
&= (0 - g + 1) - (-2g + 2 - n - g + 1) + \dim \mathbb{H}^2(C, \mathcal{L}_\omega^\bullet) \\
&= 2g - 2 + n + \dim \mathbb{H}^2(C, \mathcal{L}_\omega^\bullet).
\end{aligned}$$

Therefore, it is enough to show that $\mathbb{H}^2(C, \mathcal{L}_\omega^\bullet)$ is one-dimensional. The exact sequence implies that $\mathbb{H}^2(C, \mathcal{L}_\omega^\bullet)$ is the cokernel of the map

$$\varphi = H^1(C, \mathcal{L}_\omega) \colon H^1(C, \mathcal{T}_C(-Z_{\mathrm{red}})) \to H^1(C, \Omega_C(-Z)).$$

Since we are only interested in the dimension of the cokernel we can consider the dual map $\varphi^\vee$ and calculate the dimension of the kernel $K \coloneqq \ker \varphi^\vee$. To this end, we apply Serre duality. Fix an isomorphism $\int \colon H^1(C, \Omega_C) \to k$. By Serre duality we have natural isomorphisms

$$H^1(C, \mathcal{T}_C(-Z_{\mathrm{red}}))^\vee \cong H^0(C, \Omega_C^2(Z_{\mathrm{red}})), \qquad H^1(C, \Omega_C(-Z))^\vee \cong H^0(C, \mathscr{O}_C(Z))$$

coming from perfect pairings $\langle \cdot, \cdot \rangle$ on

$$H^1(C, \mathcal{T}_C(-Z_{\mathrm{red}})) \times H^0(C, \Omega_C^2(Z_{\mathrm{red}}))$$

and

$$H^1(C, \Omega_C(-Z)) \times H^0(C, \mathscr{O}_C(Z)),$$

respectively. We recall that in both cases $\langle \cdot, \cdot \rangle$ is given by the cup product in cohomology composed with the trace map $\int$. To summarize, we have the following commutative diagram

$$\begin{array}{ccc}
H^1(C, \mathcal{T}_C(-Z_{\mathrm{red}})) \times H^0(C, \Omega_C^2(Z_{\mathrm{red}})) & \longrightarrow & k \\
{\scriptstyle \varphi = H^1(C, \mathcal{L}_\omega)} \downarrow \qquad \uparrow {\scriptstyle \varphi^\vee = H^0(C, \mathcal{L}_\omega^\vee)} & & \| \\
H^1(C, \Omega_C(-Z)) \times H^0(C, \mathscr{O}_C(Z)) & \longrightarrow & k,
\end{array}$$

where $\mathcal{L}_\omega^\vee \colon \mathscr{O}_C(Z) = \Omega_C \otimes \Omega_C^\vee \otimes \mathscr{O}_C(Z) \to \Omega_C \otimes \Omega_C \otimes \mathscr{O}_C(Z_{\mathrm{red}}) = \Omega_C^{\otimes 2}(Z_{\mathrm{red}})$ is the dual map of $\mathcal{L}_\omega$. The pairing allows us to calculate $\mathcal{L}_\omega^\vee$ locally using the explicit description of

$\mathcal{L}_\omega$. Let h and D be sections of $\mathscr{O}_C(Z)$ and $\mathcal{T}_C(-Z_{\mathrm{red}})$, respectively, and write

$$\omega = f\,\mathrm{d}x, \qquad D = g\frac{\partial}{\partial x}$$

on some trivializing neighborhood of Ω_C. Using the equality

$$0 = \mathrm{d}(fgh) = (f'gh + fg'h + fgh')\,\mathrm{d}x$$

in cohomology, we get locally

$$\langle D, \mathcal{L}_\omega^\vee(h)\rangle = \langle \mathcal{L}_\omega(D), h\rangle = \int h\mathcal{L}_\omega(D) = \int h(f'g + fg')\,\mathrm{d}x = \int -h'fg\,\mathrm{d}x.$$

Hence, the dual Lie derivative is locally given by $\mathcal{L}_\omega^\vee(h) = -h'f\,\mathrm{d}x \otimes \mathrm{d}x$. In particular, $K = \ker\varphi^\vee$ consists of the global sections $h \in H^0(C, \mathscr{O}_C(Z))$ that fulfill the differential equation $-h'f = 0$ locally everywhere. Therefore, since C is irreducible, K coincides with the kernel of the map

$$\alpha\colon H^0(C, \mathscr{O}_C(Z)) \to k(C)^l, \qquad h \mapsto \left(h|_{U_i}{}'\right)_i,$$

where $k(C)$ denotes the function field of C and $U_1, \ldots, U_l$ is an open cover of C such that $\mathscr{O}_C(Z)$ trivializes. If $\operatorname{char} k = 0$ the kernel of α only consists of constant functions, i.e., $K \cong k$ is one-dimensional. In positive characteristic p every element that is a pth power has a vanishing derivative. Using the representation of $H^0(C, \mathscr{O}_C(Z))$ as a subsheaf of the constant sheaf $k(C)$, we can identify α with the universal derivative of $\Omega_{k(C)/k}$. Then $\ker\alpha = H^0(C, \mathscr{O}_C(Z)) \cap k(C)^p$, where $k(C)^p = \{\, f^p \mid f \in k(C)\,\}$. Hence, the assumption $\mu_i < p = \operatorname{char} k$ guarantees that every non-constant element $h \in H^0(C, \mathscr{O}_C(Z))$ is not a pth power over some zero of ω, i.e., not a pth power. □

Corollary 3.18. *Let k be an algebraically closed field such that* $\operatorname{char} k = 0$ *or each part μ_i of the partition μ is strictly smaller than $p =$* $\operatorname{char} k > 0$. *Then each connected component of the stratum $\mathcal{H}(\mu)$ of a modular family over k is smooth of dimension* $2g + |\mu| - 1$. *In particular, $\mathcal{H}(\mu)$ is a smooth variety.*

Proof. The dimension of a connected component of a variety equals the dimension of the tangent space in a smooth point of said connected component. Since being smooth is an open condition and the dimension of the tangent space is the same at every point, the claim follows. □

We close out this section with a closer look at the case of positive characteristic by considering hyperelliptic curves. To investigate those, we start with a small observation.

Lemma 3.19. *Let C be a connected smooth projective curve of genus g and D be a divisor on C.*

i) The divisor D is principle if and only if $\deg D = 0$ and $h^0(C, \mathscr{O}_C(D)) > 0$.

ii) Let $\deg D = 2g - 2$ and $h^0(C, \mathscr{O}_C(D)) \geq g$. Then D is a canonical divisor. In particular $h^0(C, \mathscr{O}_C(D)) = g$.

Proof. i) Let $f \in h^0(C, \mathscr{O}_C(D))$. Then $\operatorname{div} f + D \geq 0$ and $\deg(\operatorname{div} f + D) = 0$ implies $\operatorname{div} f + D = 0$. Hence, $D = \operatorname{div} f^{-1}$ is principle.

ii) Let K denote a canonical divisor. Using Riemann-Roch, we obtain $h^0(K - D) - h^0(K - (K - D)) = \deg(K - D) - g + 1$ which implies that $h^0(K - D) \geq 1$.

□

A connected smooth curve C over k of genus $g \geq 2$ is called *hyperelliptic* if there exists a separable morphism $f : C \to \mathbb{P}^1_k$ of degree 2. By σ we denote the induced automorphism of order 2 of C.

Proposition 3.20. *Let C be a hyperelliptic curve of genus g with covering map f and automorphism σ.*

i) For any two points P, Q of C the divisors $P + \sigma(P)$ and $Q + \sigma(Q)$ are equivalent.

ii) Let D be a divisor with $\deg D = 0$. Then $D + \sigma(D)$ is a principle divisor.

iii) Let D be a divisor with $\deg D = g - 1$. Then $D + \sigma(D)$ is a canonical divisor.

iv) Let K be an effective canonical divisor. Then $\sigma(K) = K$.

Proof. i) The divisor $P + \sigma(P) - (Q + \sigma(Q))$ is the image of the principle divisor $f(P) - f(Q)$ on $\mathbb{P}^1_k$ under the pullback map. Hence, it is principle.

ii) This follows immediately from *i*).

iii) Let P be a fixed point of σ. Then

$$h^0(C, \mathscr{O}(2 \cdot P)) \geq 2,\ h^0(C, \mathscr{O}(4 \cdot P)) \geq 3, \dots, h^0(C, \mathscr{O}((2g-2) \cdot P)) \geq g.$$

Lemma 3.19 together with *ii*) imply that $(g - 1) \cdot (P + \sigma(P)) \sim D + \sigma(D)$ is a canonical divisor.

iv) This follows from the fact that σ acts as $-\operatorname{id}$ on $H^0(C, \Omega_C)$, see [Liu02, Remark 4.28].

□

Equipped with those results, we can easily generate examples of pairs (C, ω) in positive characteristic p such that the dimension of $k(C)^p \cap H^0(C, \mathscr{O}_C(\operatorname{div}\omega))$ is greater than 1.

Theorem 3.21. *Let k be an algebraically closed field of positive characteristic. Then there exist integers $g \in \mathbb{N}$ and partitions μ of $2g-2$ into positive integers such that*

$$\dim \mathcal{T}_{\mathcal{H}(\mu)}\Big|_{(C,\omega)} > 2g + |\mu| - 1$$

for some point (C, ω) of $\mathcal{H}(\mu)$.

At the same time, the results on hyperelliptic curves imply that nothing new happens in the smallest example not covered by Theorem 3.17, the stratum $\mathcal{H}(2)$ in characteristic 2.

Theorem 3.22. *Let k be an algebraically closed field of characteristic 2. Then $\mathcal{H}(2)$ is a smooth variety of dimension 4.*

Proof. Every connected smooth projective curve of genus 2 over an algebraically closed field is hyperelliptic, i.e., for every point (C, ω) of $\mathcal{H}(2)$ the curve C is hyperelliptic. Using Proposition 3.20 we can write $\operatorname{div}\omega = 2 \cdot P$ where P is a fixed point of the hyperelliptic involution. We have to show that $k(C)^2 \cap \mathscr{O}_C(2 \cdot P) = k$. Assume for the sake of contradiction that there exists a non-constant element $f \in k(C)$ such that $f^2 \in k(C)^2 \cap \mathscr{O}_C(2 \cdot P)$. Then $\operatorname{ord}_P(f) = -1$ and $h^0(C, \mathscr{O}_C(P)) \geq 2$. This is a contradiction to $h^0(C, \mathscr{O}_C(\operatorname{div}\omega)) = 2 = 2g - 2$. □

Chapter 4

p-adic Analytic Construction

One of the big advantages of working over the complex numbers in the context of translation surfaces is the existence of the Teichmüller space T_g. Since T_g is a Stein manifold, every vector bundle on it is trivial. In particular, this holds for the Hodge bundle, simplifying calculations significantly.

There are analytic theories for different fields other than $\mathbb{C}$ and in this last chapter we want to present one such approach focusing on the algebraically closed complete field $\mathbb{C}_p$ containing the p-adic numbers $\mathbb{Q}_p$. We construct a cover of the moduli space of curves, the Schottky space $\mathcal{S}_g$, over which the Hodge bundle is trivial. The big difference to the complex case is that $\mathcal{S}_g$ only parameterizes curves contained in an (analytically) open set of M_g. We also give some ideas on how the notion of period coordinates might be extended to this space.

4.1 Berkovich Analytic Spaces

We present this exposition using the analytic theory of Berkovich [Ber90]. It is not entirely clear that this is the right point of view. However, it is the most geometric one known to us and has an interesting integration theory that we discuss in a later section.

Berkovich built on the rigid viewpoint in a rather straightforward way, extending the range of allowed radii for affinoid domains. He later expanded his category after recognizing that more exotic gluing was covered by his theory, see [Ber93]. Our interest lies in the first spaces, called *good* by Berkovich. We are working exclusively over $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$ for a fixed prime p. However, all the results are also valid for different complete non-Archimedean fields. And indeed, different fields arise naturally in the context of Berkovich analytic spaces. Only the final section about integration requires $\mathbb{C}_p$ as a ground field.

The easiest examples of analytic spaces are spaces that locally look like an open or closed subset of affine space. In fact, in the complex world all analytic spaces are of this form and also the Schottky space $\mathcal{S}_g$ is contained in this class.

As a point set the n-dimensional analytic affine space $\mathbb{A}^{n,\mathrm{an}}$ over $\mathbb{C}_p$ is the set of all multiplicative seminorms on the polynomial ring $\mathbb{C}_p[X_1,\dots,X_n]$ such that the restriction to $\mathbb{C}_p$ is bounded. Every polynomial $f \in \mathbb{C}_p[X_1,\dots,X_n]$ defines a real-valued function

$$\mathbb{A}^{n,\mathrm{an}} \to \mathbb{R}, \qquad x \mapsto x(f).$$

We equip $\mathbb{A}^{n,\mathrm{an}}$ with the weakest topology such that all functions of this form are continuous. To enable geometric considerations on $\mathbb{A}^{n,\mathrm{an}}$, we need to define a sheaf of analytic functions. Note that every point x determines a valuation field $\mathscr{H}(x)$ via the following construction: The set

$$\wp_x := \{\, f \in \mathbb{C}_p[X_1,\dots X_n] \mid x(f) = 0 \,\}$$

is a prime ideal of $\mathbb{C}_p[X_1,\dots,X_n]$ and the seminorm x on $\mathbb{C}_p[X_1,\dots,X_n]/\wp_x$ is a multiplicative norm. Therefore, x extends to an absolute value on the corresponding field of fractions $\mathscr{K}(x)$. Finally, the field $\mathscr{H}(x)$ is the completion of $\mathscr{K}(x)$ with respect to this norm. The image of a polynomial $f \in \mathbb{C}_p[X_1,\dots,X_n]$ in $\mathscr{H}(x)$ is denoted by $f(x)$. A rational function $f \in \mathbb{C}_p(X_1,\dots,X_n)$ is *defined* on a subset $U \subseteq \mathbb{A}^{n,\mathrm{an}}$ if f can be written as a quotient $\frac{g}{h}$ with $g, h \in \mathbb{C}_p[X_1,\dots,X_n]$ and $h(x) \neq 0$ for all $x \in U$.

Definition 4.1. Let $U \subseteq \mathbb{A}^{n,\mathrm{an}}$ be a subset. An *analytic function* on U is a map

$$f \colon U \to \prod_{x \in U} \mathscr{H}(x), \qquad x \mapsto f(x),$$

satisfying $f(x) \in \mathscr{H}(x)$ and every x has an open neighborhood $U_x \subseteq \mathbb{A}^{n,\mathrm{an}}$ contained in U such that for every $\epsilon > 0$ there is an element $g \in \mathbb{C}_p(X_1,\dots,X_n)$ defined on U_x with

$$|f(y) - g(y)| < \epsilon$$

for all $y \in U_x$.

The sheaf $\mathscr{O}_{\mathbb{A}^{n,\mathrm{an}}}$, which associates to every open subset $U \subseteq \mathbb{A}^{n,\mathrm{an}}$ the ring of analytic functions on U makes the pair $(\mathbb{A}^{n,\mathrm{an}}, \mathscr{O}_{\mathbb{A}^{n,\mathrm{an}}})$ into a locally ringed space, the *Berkovich analytic affine space of dimension* n.

We remark that there is a natural analytification functor $\cdot^{\mathrm{an}}$ from the category of schemes of locally finite type over $\mathbb{C}_p$ to the category of Berkovich analytic spaces which is fully faithful on proper schemes together with a GAGA-like theory[1], see [Ber90, Chapter 3.4]. In particular, all the algebraic tools we developed so far are also available in the analytic context. An illustrating example of the usefulness of an analytic theory is [Ber90, Theorem 3.4.8], which translates the separatedness of a scheme X into the property that the analytification X^{an} is Hausdorff. The analytification functor is most easily described for affine varieties $X \subseteq \mathbb{A}^n$. The very same equations defining X as a closed subset in $\mathbb{A}^n$, also define a closed subset V of $\mathbb{A}^{n,\mathrm{an}}$ together with a sheaf of rings. This pair is the analytification X^{an} of X. In the general case we cover X by affine open sets and glue the resulting analytic spaces back together.

4.2 Mumford Curves

In this section we review the theory of p-adic Schottky groups, which uniformize Mumford curves. This additional datum is similar to a marking in Teichmüller theory and the key idea behind Schottky space. The classical reference for this material is [Gv80]. A modern treatment using Berkovich spaces can be found in [PT21].

Definition 4.2. A *Schottky group* is a subgroup $\Gamma \subseteq \mathrm{PGL}_2(\mathbb{C}_p)$ such that

i) the group Γ is finitely generated,

ii) all non-trivial elements of Γ are hyperbolic[2] and

iii) there exists a non-empty Γ-invariant connected open subset U of $\mathbb{P}^{1,\mathrm{an}}$ on which Γ acts properly and freely.

Theorem 4.3. *Let Γ be a Schottky group. Then Γ is a free group of finite rank g. Moreover, there is a maximal open subset $O \subseteq \mathbb{P}^{1,\mathrm{an}}$, called the* ordinary points of Γ, *on which Γ acts proper and free. The quotient*

$$X := \Gamma \backslash O$$

is a connected smooth projective analytic curve of genus g.

Proof. [PT21, Proposition 6.4.7] shows that Schottky groups are free and finitely generated. Although at this point they use a different definition which is equivalent to the one given here by [PT21, Proposition 6.4.24 and Theorem 6.4.26]. The second claim can be found in [PT21, Theorem 6.4.18] □

[1] Here, GAGA stands for Géometrie Algébrique et Géométrie Analytique and refers to [Ser56].
[2] A matrix $M \in \mathrm{GL}_2(\mathbb{C}_p)$ is called *hyperbolic* if the two eigenvalues of M have different absolute values.

Curves that admit such a uniformization are called *Mumford curves*. Since Mumford curves are in particular proper, they are the analytification of an algebraic curve. We remark that there is a purely algebraic characterization of Mumford curves that can be found as a special case in [Mum72, Theorem 4.20] and is the content of the following lemma:

Lemma 4.4. *The following statements are equivalent for an analytic space X.*

i) X is a Mumford curve.

ii) X is the analytification of a connected smooth projective curve C with totally split reduction[1].

There is a notion of families of Mumford curves, developed by Piwek in [Piw86]. The idea is to extend the definition of Schottky groups to subgroups of $\mathrm{PGL}_2(\mathscr{O}_B(B)) \subseteq \mathrm{Aut}_B\, \mathbb{P}^{1,\mathrm{an}}_B$ for a base space B. In the case $B = (\operatorname{Spec} \mathbb{C}_p)^{\mathrm{an}}$ we obtain the original definition of a Schottky group. In the next section we consider a special family that is in fact a fine moduli space for the corresponding moduli problem.

4.3 Schottky Space and the Universal Curve

We sketch a construction of the fine moduli space of families of Mumford curves of genus g, following [Ger81] and [Piw86]. It parameterizes Schottky groups of rank g and, as such, Mumford curves of genus g together with a choice of uniformization. We want to point out the existence of the paper [PT20], which is currently being worked on, that generalizes the results of Gerritzen and Piwek to Berkovich spaces over $\mathbb{Z}$.

The idea is quite elegant. Rigid points of the fine moduli space should encode a Schottky group, i.e., g elements of $\mathrm{PGL}_2(\mathbb{C}_p)$ that generate a Schottky group. If

$$R_g \coloneqq \{ (\gamma_1, \dots, \gamma_g) \in \mathrm{PGL}_2(\mathbb{C}_p)^g \mid \langle \gamma_1, \dots, \gamma_g \rangle \text{ is Schottky group of rank } g \}$$

denotes the set of all Schottky groups together with a generating ordered set, then, using the fact that two Mumford curves are isomorphic if and only if the corresponding Schottky groups are conjugated, we obtain that as a set

$$M_g^{\mathrm{Mum}}(\mathbb{C}_p) = \mathrm{PGL}_2(\mathbb{C}_p) \Big\backslash R_g \Big/ \mathrm{Aut}(F_g)\,.$$

[1] A curve C over $\mathbb{C}_p$ has totally split reduction if there is a model over the ring of integers $\mathbb{C}_p^\circ$ of $\mathbb{C}_p$ such that the base change C_k to the residue field $k = \mathbb{C}_p^\circ \big/ \mathbb{C}_p^{\circ\circ}$ consists of copies of $\mathbb{P}^1_k$ that intersect transversally.

Here the left hand side denotes the $\mathbb{C}_p$-valued points of the coarse moduli space M_g^{an} consisting of Mumford curves, $\mathrm{PGL}_2(\mathbb{C}_p)$ acts via conjugation on R_g and $\mathrm{Aut}(F_g)$ by identifying $\langle \gamma_1, \ldots, \gamma_g \rangle$ with the free group F_g of rank g. Similar to the construction of Teichmüller space, the key idea is to only factor out one of the groups to obtain a cover of the moduli space, i.e., the rigid points of Schottky space $\mathcal{S}_g$ correspond to $\mathrm{PGL}_2(\mathbb{C}_p)\backslash R_g$. In the following, we make this precise.

Lemma 4.5. *Every hyperbolic element of* $\mathrm{PGL}_2(\mathbb{C}_p)$ *is uniquely determined by*

i) the attracting and repulsing fixed point α *and* α' *in* $\mathbb{P}^{1,\mathrm{an}}(\mathbb{C}_p)$ *and*

ii) the multiplier β*, i.e., the ratio of the eigenvalues such that* $|\beta| < 1$.

The inverse of the corresponding bijection is given by

$$M\colon (\alpha, \alpha', \beta) \mapsto \begin{pmatrix} \beta u'v - uv' & (1-\beta)uu' \\ (\beta - 1)vv' & \beta u'v - uv' \end{pmatrix},$$

where $\alpha = (u : v)$ *and* $\alpha' = (u' : v')$.

This coordinate system is called *Koebe coordinates* by Poineau and Turchetti. Since $\mathrm{PGL}_2(\mathbb{C}_p)$ acts 3-transitively on $\mathbb{P}^{1,\mathrm{an}}$, every conjugacy class of a hyperbolic element contains exactly one representative with coordinates $(0, \infty, \beta)$, where $0 < |\beta| < 1$.

Example 4.6. For $g = 1$ this already tells us everything about the fine moduli space $\mathcal{S}_1$. In this case, every Schottky group is generated by one element conjugated to $M(0, \infty, \beta)$ for some β with $0 < |\beta| < 1$. Hence, if X denotes the coordinate on $\mathbb{A}^{1,\mathrm{an}}$, we have

$$\mathcal{S}_1 = \left\{ x \in \mathbb{A}^{1,\mathrm{an}} \,\middle|\, 0 < |X(x)| < 1 \right\},$$

which is a punctured open disc in the affine line.

For the remainder of this section we fix an integer $g \geq 2$ and consider the affine space $\mathbb{A}^{3g-3,\mathrm{an}}$ with coordinates $X_3, \ldots, X_g, X'_2, \ldots, X'_g, Y_1, \ldots, Y_g$. It is convenient to set values for the clearly missing variables. As it turns out, $X_1 \coloneqq 0, X_2 \coloneqq 1$ and $X'_1 = \infty$ (seen as constant morphisms to $\mathbb{P}^{1,\mathrm{an}}$) are good choices. Let $\mathcal{U}_g$ denote the open subset of $\mathbb{A}^{3g-3,\mathrm{an}}$ defined by the inequalities

$$0 < |Y_i| < 1, \qquad X_i^{\sigma_i} \neq X_j^{\sigma_j},$$

for $i \neq j \in \{1, \ldots, g\}$ and $\sigma_i, \sigma_j \in \{\emptyset, '\}$. Finally, the definition of the bijection M in

Lemma 4.5 readily extends to regular functions on $\mathcal{U}_g$. Hence, we obtain matrices

$$M_i := M(X_i, X_i', Y_i) \in \mathrm{PGL}_2(\mathcal{O}_{\mathcal{U}_g}(\mathcal{U}_g)),$$

such that over every point $x \in \mathcal{U}_g$ the matrix $M_i(x) \in \mathrm{PGL}_2(\mathscr{H}(x))$ is hyperbolic.

Definition 4.7. The *Schottky space of rank g* over $\mathbb{C}_p$, denoted by $\mathcal{S}_g$, is the subspace of $\mathcal{U}_g$ consisting of points $x \in \mathcal{U}_g$ such that the subgroup

$$\Gamma_x := \langle M_1(x), \ldots, M_g(x) \rangle \subseteq \mathrm{PGL}_2(\mathscr{H}(x))$$

is a Schottky group[1] of rank g.

Theorem 4.8. *The Schottky space $\mathcal{S}_g$ is an open path-connected subset of $\mathbb{A}^{3g-3,\mathrm{an}}$.*

The first topological property is given in [PT20, Theorem 3.3.5]. Also the path-connectedness over $\mathbb{Z}$ can be found in the same paper as [PT20, Theorem 4.2.1] but this result can not be immediately transferred to our case. They use, in an essential way, that the Archimedean points form a path-connected subset. But those points vanish after making a base change to a non-Archimedean field. Nonetheless, given some other results of the same paper, it is not too hard to show the claim over a non-Archimedean base field.

One result we need, is that the group $\mathrm{Aut}(F_g)$ acts analytically on $\mathcal{S}_g$ and the corresponding action factors through $\mathrm{Out}(F_g)$, see [PT20, Section 4.1]. The second result is a combination of [PT20, Corollary 3.4.4] and [PT20, Corollary 3.3.4] and summarized in the following proposition.

Proposition 4.9. *Let $\mathcal{SB}_g$ be the subset of $\mathcal{U}_g$ given by the inequalities*[2]

$$|Y_i| \cdot \left| [X_j^{\sigma_j}, X_k^{\sigma_k}; X_i, X_i'] \right| < 1$$

for all $i, j, k \in \{1, \ldots, g\}$ with $j \neq i \neq k$ and $\sigma_j, \sigma_k \in \{\emptyset, '\}$, where $[\cdot, \cdot; \cdot, \cdot]$ denotes the cross-ratio. Then $\mathcal{SB}_g$ is an open path-connected subset of $\mathcal{S}_g$, whose orbit under the action of $\mathrm{Out}(F_g)$ covers $\mathcal{S}_g$.

Proof of Theorem 4.8. Using the property that $\mathcal{SB}_g$ is path-connected and that

$$\mathrm{Out}(F_g) \cdot \mathcal{SB}_g = \mathcal{S}_g,$$

[1] This is one instance where different non-Archimedean fields arise naturally in the framework of Berkovich. The reader should be able to extend the definition of a Schottky group in the obvious way.

[2] The points of $\mathcal{SB}_g$ correspond to *Schottky bases*.

it is enough to show that $\varphi \cdot \mathcal{SB}_g \cap \mathcal{SB}_g \neq \emptyset$ for every φ in a generating set of $\mathrm{Out}(F_g)$. Applying a classical result from Nielsen [Nie24], the group $\mathrm{Out}(F_g)$ is generated by four automorphisms $\varphi_i\colon F_g \to F_g$. If $\{e_1, \ldots, e_g\}$ denotes a basis of F_g, the maps are given by

$$\varphi_1\colon \begin{cases} e_1 \mapsto e_g, \\ e_i \mapsto e_{i-1}, \text{ for all } i > 1, \end{cases} \qquad \varphi_2\colon \begin{cases} e_1 \mapsto e_2, \\ e_2 \mapsto e_1, \\ e_i \mapsto e_i, \text{ for all } i > 2, \end{cases}$$

$$\varphi_3\colon \begin{cases} e_1 \mapsto e_1^{-1}, \\ e_i \mapsto e_i, \text{ for all } i > 1, \end{cases} \qquad \varphi_4\colon \begin{cases} e_1 \mapsto e_2^{-1}e_1, \\ e_i \mapsto e_i, \text{ for all } i > 1. \end{cases}$$

Since $\mathcal{SB}_g$ is obviously invariant under the action of the group generated by φ_1, φ_2 and φ_3, it is enough to show that there is a Schottky basis $(\gamma_1, \ldots, \gamma_g)$ such that $(\varphi_4 \cdot \gamma_1, \ldots, \varphi_4 \cdot \gamma_g)$ is still a Schottky basis. But this is obvious, considering the definition of $\mathcal{SB}_g$, by choosing the multipliers small enough. For example, choose $\gamma_1 = M(0, \infty, \beta_1)$ and $\gamma_2 = M(-1, 1, \beta_2)$. Then it is clear, by either arguing geometrically or numerically, that the absolute value of the multiplier of $\gamma_2^{-1}\gamma_1$ is $|\beta_1\beta_2|$. Hence, given any Schottky basis containing γ_1 and γ_2 as the first two generators, we can modify the multipliers of γ_1 and γ_2 such that the two points corresponding to $(\gamma_1, \ldots, \gamma_g)$ and $\left(\gamma_2^{-1}\gamma_1, \gamma_2, \ldots, \gamma_g\right)$ are both contained in $\mathcal{SB}_g$. ☐

As indicated before, Schottky space is a solution to a moduli problem. As such it should be equipped with a universal Schottky group. As we have not talked about families in the context of Schottky uniformized curves, we do not verify that the following object deserves its name. Fortunately, due to the explicit construction, this is not necessary for understanding the remainder of this thesis.

Definition 4.10. The group

$$\mathcal{G}_g \coloneqq \langle M(0, \infty, Y_1), M(1, X_2', Y_2), \ldots, M(X_g, X_g', Y_g) \rangle \subseteq \mathrm{PGL}_2(\mathcal{O}_{\mathcal{S}_g}(\mathcal{S}_g))$$

is called *universal Schottky group of rank g*.

Theorem 4.11. *Let* $O_g \coloneqq \bigcup_{x \in \mathcal{S}_g} O_x \subseteq \mathbb{P}^{1,\mathrm{an}}_{\mathcal{S}_g}$ *denote the set of ordinary points of* $\mathcal{G}_g$*, where* $O_x \subseteq \mathbb{P}^{1,\mathrm{an}}_{\mathscr{H}(x)}$ *denotes the ordinary points of the Schottky group corresponding to x, see Theorem 4.3. Then* O_g *is open and the action of* $\mathcal{G}_g$ *on* O_g *is proper and free. In particular, the quotient*

$$\mathcal{C}_g \coloneqq \mathcal{G}_g \backslash O_g$$

carries the structure of an analytic space. It is called the universal Mumford curve of genus g. *We obtain a commutative diagram*

$$
\begin{array}{ccc}
O_g & \xrightarrow{\varphi} & \mathcal{C}_g, \\
& \searrow \quad \swarrow_{\pi} & \\
& \mathcal{S}_g &
\end{array}
$$

where φ is a local isomorphism and π is proper and smooth[1] *of relative dimension 1.*

The proof in the classical language can be found in [Piw86, Kapitel II Satz 4] for more general families of Schottky groups. The same statement for Berkovich spaces over $\mathbb{Z}$ is presented in [PT20, Theorem 5.1.2].

The family $\pi\colon \mathcal{C}_g \to \mathcal{S}_g$ takes the role of a modular family. Applying our algebraic tools, we immediately obtain the following result.

Theorem 4.12. *Let $g \geq 2$ and $\pi\colon \mathcal{C}_g \to \mathcal{S}_g$ denote the universal curve over Schottky space. Then $\pi_*\Omega_{\mathcal{C}_g/\mathcal{S}_g}$ is a trivial vector bundle of rank g on $\mathcal{S}_g$. Denote the total space of this bundle by $\Omega\mathcal{S}_g$.*

For every partition μ of $2g-2$ into n positive integers the subset $\mathcal{H}(\mu) \subseteq \Omega\mathcal{S}_g$ consisting of pairs (C_x, ω), where C_x is the Mumford curve corresponding to $x \in \mathcal{S}_g$ and ω is an element of $H^0(C_x, \Omega_{C_x})$ inducing the partition μ, is locally closed. Decorated with the canonical reduced structure, the analytic space $\mathcal{H}(\mu)$ is smooth of dimension $2g+n-1$. Moreover, there is a natural identification

$$\mathcal{T}_{\mathcal{H}(\mu)}\Big|_{(C_x,\omega)} \cong \mathbb{H}^1(C_x, \mathcal{L}_\omega^\bullet)$$

of the tangent space of $\mathcal{H}(\mu)$ at the point (C_x, ω) and the first hypercohomology of the twisted Lie derivative associated with ω.

Proof. Most of the statements are clear and follow without change from the algebraic proof for families of smooth curves. The only new statement is that $\pi_*\Omega_{\mathcal{C}_g/\mathcal{S}_g}$ is not only a vector bundle but that it has global generators. This follows from [Piw86, Kapitel II Satz 5] where the construction of differentials associated with a Schottky group is extended to the case of families. □

[1] We use the definition of smoothness introduced by Berkovich in [Ber90], which we recall for the reader's convenience. A morphism $f\colon X \to Y$ of analytic spaces is called *smooth of relative dimension* n if it is flat and for every y the fiber X_y is either empty or a smooth $\mathscr{H}(x)$-analytic space of dimension n, i.e., all the local rings of X_y are regular. Alternatively, an analytic space X is smooth over a field if it is locally the analytification of a smooth scheme.

4.4 p-adic Integration

Algebraic geometry over the field of complex numbers is significantly better understood than over an arbitrary algebraically closed field. This is in part due to the availability of analytic tools. One such tool is the (complex) integral, which can be used to give a description of the tangent space of a stratum in the guise of *period coordinates*. We recall the construction around a fixed compact Riemann surface X_0 of genus g together with an abelian differential $\omega_0 \in H^0(X_0, \Omega_{X_0})$. Let $\Sigma(\omega_0)$ denote the set of zeroes of ω_0 and choose a basis

$$\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_{n-1} \in H_1(X_0, \Sigma(\omega_0), \mathbb{Z})$$

of the first relative singular homology, see Figure 4 for an illustration. Using the Gauß-Manin connection, it is possible to find an open subset U of the stratum containing (X_0, ω_0) such that the paths on X_0 naturally form a basis of $H_1(X, \Sigma(\omega), \mathbb{Z})$ for all $(X, \omega) \in U$. Then the period map

$$\Theta\colon U \to \mathbb{C}^{2g+n-1}, \qquad (X,\omega) \mapsto \left(\int_{\alpha_1} \omega, \int_{\beta_1} \omega, \dots, \int_{\alpha_g} \omega, \int_{\beta_g} \omega, \int_{\gamma_1} \omega, \dots, \int_{\gamma_{n-1}} \omega \right),$$

integrates the paths against the abelian differentials to obtain charts for the stratum. For more details we refer to [FM14, Section 2.3] for a readable proof sketch and the original paper [Vee86, Theorem 26.2] by Veech.

In order to obtain similar results over $\mathbb{C}_p$ it therefore seems necessary to investigate a notion of integral that is available in the category of Berkovich spaces. Recall that the complex integral is heavily dependent on the logarithm, for example disguised as the Cauchy integral formula. Hence, we start our investigation with the p-adic logarithm.

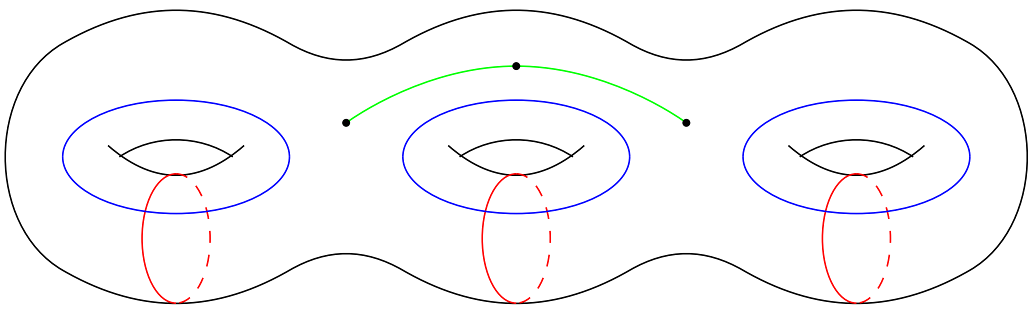

Figure 4: Base of relative singular homology on a surface of genus 3 with 3 marked points.

As usual, it is defined as the power series

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

on its radius of convergence, i.e., for $x \in \mathbb{C}_p^{\circ\circ}$, the maximal ideal of the valuation ring $\mathbb{C}_p^{\circ}$. For formal reasons it satisfies the functional equation $\log(ab) = \log(a) + \log(b)$. By requiring that this property still holds, it is possible to extend the logarithm uniquely to a function $\log\colon (\mathbb{C}_p^{\circ})^{\times} \to \mathbb{C}_p$. Indeed, every element $x \in (\mathbb{C}_p^{\circ})^{\times}$ can be written in the form $x = \zeta \cdot \tilde{x}$, where ζ is a root of unity and $\tilde{x}$ is an element of $1 + \mathbb{C}_p^{\circ\circ}$. Since $\mathbb{C}_p$ has no torsion, the functional equation implies that elements of finite order get mapped to 0 by log. Therefore, the only reasonable choice is to set $\log(x) \coloneqq \log(\tilde{x})$. That this is well defined follows from [Kob84, Section IV. 2]. To extend the logarithm any further we have to make a choice. This is due to the following lemma:

Lemma 4.13. *The groups $\mathbb{C}_p^{\times}$ and $(\mathbb{C}_p^{\circ})^{\times} \oplus \mathbb{Q}$ are isomorphic.*

Proof. There is a short exact sequence

$$0 \to (\mathbb{C}_p^{\circ})^{\times} \to \mathbb{C}_p^{\times} \xrightarrow{v} \mathbb{Q} \to 0$$

of abelian groups, where v denotes the p-adic valuation. Since the morphism

$$\mathbb{Q} \to \mathbb{C}_p^{\times}, \qquad q \mapsto p^q,$$

is a right split, the result follows from the Splitting Lemma. □

Lemma 4.13 implies that a morphism $\mathbb{C}_p^{\times} \to \mathbb{C}_p$ which agrees with log on the subset $(\mathbb{C}_p^{\circ})^{\times} \subseteq \mathbb{C}_p^{\times}$ is uniquely determined by a group homomorphism $\mathbb{Q} \to \mathbb{C}_p$, which itself is uniquely determined by the image of 1. This last choice can be made arbitrarily. Hence, every choice $\lambda \in \mathbb{C}_p$ yields a unique extension of log satisfying the functional equation by defining

$$\log^{\lambda}(p^a \cdot x) \coloneqq a\lambda + \log(x)$$

for $a \in \mathbb{Q}$ and $x \in (\mathbb{C}_p^{\circ})^{\times}$. Unfortunately, there is in general no canonical choice for $\lambda \in \mathbb{C}_p$. Therefore, we arbitrarily choose a value $\lambda \in \mathbb{C}_p$ once and for all and call $\mathrm{Log} \coloneqq \log^{\lambda}$ the *p-adic logarithm.*

Building on old ideas of Coleman, Berkovich [Ber07] constructed a very general theory of integration, which has all the usual properties if restricted to spaces coming from algebraic geometry. Since we only use the integral on such spaces, we do not present the results of Berkovich in full generality. Instead, we follow [KRZ16] and [KRZ18], who applied this integration theory successfully to bound the number of rational points on a certain class of curves.

Let X be a smooth $\mathbb{C}_p$-analytic space. We denote by

$$\mathcal{P}(X) \coloneqq \{\, \gamma \mid \gamma\colon [0,1] \to X \text{ continuous}, \gamma(0), \gamma(1) \in X(\mathbb{C}_p)\,\}$$

the set of all continuous paths between $\mathbb{C}_p$-valued points of X and by $Z^1_{dR}(X)$ the space of closed analytic one-forms on X.

Theorem 4.14 (Berkovich-Coleman Integral)**.** *For every smooth $\mathbb{C}_p$-analytic space X there is a unique pairing*

$$\int\colon \mathcal{P}(X) \times Z^1_{dR}(X) \to \mathbb{C}_p, \qquad (\gamma, \omega) \mapsto \int_\gamma \omega,$$

satisfying the following properties for all $\gamma \in \mathcal{P}(X)$ and $\omega \in Z^1_{dR}(X)$.

i) The map $\omega \mapsto \int_\gamma \omega$ is $\mathbb{C}_p$-linear in ω.

ii) The value $\int_\gamma \omega$ depends only on the fixed end-point homotopy class of γ.

iii) If $\gamma * \gamma'$ denotes the concatenation of two elements $\gamma, \gamma' \in \mathcal{P}(X)$ with $\gamma(0) = \gamma'(1)$, then
$$\int_{\gamma * \gamma'} \omega = \int_\gamma \omega + \int_{\gamma'} \omega.$$

iv) If X' is a smooth $\mathbb{C}_p$-analytic space, $\varphi\colon X \to X'$ a morphism and $\omega' \in Z^1_{dR}(X')$, then
$$\int_\gamma \varphi^* \omega' = \int_{\varphi(\gamma)} \omega'.$$

v) If $\omega = \mathrm{d}f$ is exact, then $\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0))$.

vi) For $X = \mathbb{G}_m^{1,\mathrm{an}} = (\operatorname{Spec} \mathbb{C}_p[t^{\pm 1}])^{\mathrm{an}}$ and the invariant differential $\omega = \mathrm{d}t/t$ the value of the integral is given by

$$\int_\gamma \frac{\mathrm{d}t}{t} = \operatorname{Log}(\gamma(1)) - \operatorname{Log}(\gamma(0)).$$

Remark 4.15. There are (at least) two different notions of integrals used in the literature, the Berkovich-Coleman integral defined above and the abelian integral, which is only dependent on the endpoints of a path. Since we actually want non-trivial values along closed loops the abelian integral does not seem to be the right tool for our problem.

Notation 4.16. If X is simply connected, the homotopy class of a path $\gamma \in \mathcal{P}(X)$ depends only on the end-points $\gamma(0)$ and $\gamma(1)$. In this case we write $\int_a^b := \int_\gamma$, where γ is any path from a to b.

In the following, we consider the integral in the case $X = Y^{\text{an}}$ the analytification of a connected smooth projective variety Y over $\mathbb{C}_p$. In this case, X is path-connected and the integral along a closed path $\gamma \in \mathcal{P}(X)$ depends only[1] on the first singular homology class of γ.

Note that in the case of an algebraic curve C, there is a canonical inclusion $H^0(C, \Omega_C) \subseteq Z^1_{dR}(C)$. Hence, in this case the Berkovich-Coleman integral induces a well defined pairing

$$\int : H_1(C^{\text{an}}, \mathbb{Z}) \times H^0(C, \Omega_C) \to \mathbb{C}_p.$$

The Berkovich-Coleman integral is not just a purely theoretical construction. Indeed, it can be computed very explicitly. Using property *iv*), the functoriality of the integral, the formula for exact differentials, *v*), and a well-built open covering of X, there are algorithms for the integral almost ready to implement[2], see [KK20]. Another possible way to calculate the integral on a curve is by using the Jacobian. This works especially well for Mumford curves and is illustrated in the following example.

Example 4.17. For simplicity assume that p is odd. Let E be an elliptic curve over $\mathbb{C}_p$ given by the affine equation $y^2 = x(x-1)(x-\lambda)$ for some $\lambda \in \mathbb{C}_p \setminus \{0, 1\}$ and let $j(E) = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(1-\lambda)^2}$ denote the j-invariant of E. Since S_3 acts on $\mathbb{C}_p \setminus \{0, 1\}$ via the automorphisms

$$\lambda \mapsto \frac{1}{\lambda}, \qquad \lambda \mapsto \frac{1}{1-\lambda},$$

which identifies isomorphic elliptic curves, there are essentially only two different cases to consider.

For $|\lambda| = |1-\lambda| = 1$ the curve E is an elliptic curve with good reduction, i.e., not a Tate curve, and the topological space E^{an} is simply connected.

[1] Indeed, let $\gamma' \in \mathcal{P}(X)$ be another closed path with $[\gamma] = [\gamma']$ in $\pi_1(Y)$, i.e., if P, P' denotes the start point of γ and γ', respectively, and $\delta \in \mathcal{P}(X)$ is a path from P to P' with inverse path δ^{-1} then γ and $\delta^{-1} * \gamma' * \delta$ are fixed end-point homotopic. Using properties *ii*) and *iii*) of the integral, one computes for $\omega \in Z^1_{dR}(X)$

$$\int_\gamma \omega = \int_{\delta^{-1}*\gamma'*\delta} \omega = \int_{\delta^{-1}} \omega + \int_{\gamma'} \omega + \int_\delta \omega = \int_{\gamma'} \omega + \int_{\delta^{-1}*\delta} \omega = \int_{\gamma'} \omega.$$

Finally, using again properties *ii*) and *iii*), the integral along the commutators of $\pi_1(X)$ is zero. Hence, the integral depends only on the class of γ in $\pi_1(X)^{ab} \cong H_1(X, \mathbb{Z})$.

[2] The issue preventing an implementation comes mostly from the field $\mathbb{C}_p$ itself.

In the other case, we can assume without loss of generality that $|\lambda|$ is smaller than 1. Then E is a Tate curve. Topologically E^{an} is a circle connected to $\mathbb{R}$-trees. We calculate the value of integrating $\omega \in H^0(E, \Omega_E)$ along a generator of $H_1(E^{\mathrm{an}}, \mathbb{Z})$. By the theory of Tate, there exists a non-zero $q \in \mathbb{C}_p^{\circ\circ}$ and an isomorphism $E(\mathbb{C}_p) \cong \mathbb{C}_p^\times \big/ q^{\mathbb{Z}}$ of groups that extends to an analytic isomorphism

$$\mathbb{G}_m^{1,\mathrm{an}} \big/ q^{\mathbb{Z}} \cong E^{\mathrm{an}},$$

which makes $\varphi\colon \mathbb{G}_m^{1,\mathrm{an}} \to \mathbb{G}_m^{1,\mathrm{an}} \big/ q^{\mathbb{Z}} \cong E^{\mathrm{an}}$ a universal covering map. Let γ denote the (unique) path from 1 to q in $\mathbb{G}_m^{1,\mathrm{an}}$. Then $\varphi(\gamma)$ is a closed loop in E^{an} and a generator of $H_1(E^{\mathrm{an}}, \mathbb{Z})$. Finally, the pullback $\varphi^*\omega$ of ω is an invariant differential on $\mathbb{G}_m^{1,\mathrm{an}}$, i.e., it can be written in the form $\varphi^*\omega = \alpha\,\mathrm{d}t/t$ for a unique $\alpha \in \mathbb{C}_p$, where t denotes the coordinate function on $\mathbb{G}_m^{1,\mathrm{an}}$. Using the functoriality of the integral, we see that

$$\int_{\varphi(\gamma)} \omega = \int_1^q \alpha \frac{\mathrm{d}t}{t} = \alpha \int_1^q \frac{\mathrm{d}t}{t} = \alpha \operatorname{Log}(q).$$

In particular, the value α, which fixes ω, appears in the formula above.

The previous example can be generalized. Let C denote an algebraic Mumford curve of genus g and let J be the Jacobian of C. In this case the universal cover of J^{an} is an analytic torus $T \cong \mathbb{G}_m^{g,\mathrm{an}} = (\operatorname{Spec} \mathbb{C}_p[t_1^{\pm 1}, \ldots, t_g^{\pm 1}])^{\mathrm{an}}$ and we have an analytic uniformization $J^{\mathrm{an}} = T \big/ H$, where H denotes a lattice of rank g. Let $\varphi\colon T \to J^{\mathrm{an}}$ denote the corresponding covering map.

Choose a point $P \in C(\mathbb{C}_p)$ and let $\iota = \iota_P\colon C \to J$ be the Abel-Jacobi map with respect to P. Let $\widetilde{C}^{\mathrm{an}}$ denote the topological universal cover of C^{an} with covering map $\widetilde{\varphi}\colon \widetilde{C}^{\mathrm{an}} \to C^{\mathrm{an}}$. Note that $\widetilde{C}^{\mathrm{an}}$ itself is an analytic space and $\widetilde{\varphi}$ is an analytic map. Fix a point $\widetilde{P} \in \varphi^{-1}(P)$. Using the universal property of coverings, the map ι extends uniquely to a map $\widetilde{\iota}\colon \widetilde{C}^{\mathrm{an}} \to T$ such that the diagram

$$\begin{array}{ccc} \widetilde{C}^{\mathrm{an}} & \xrightarrow{\ \widetilde{\iota}\ } & T \\ \Big\downarrow{\scriptstyle \widetilde{\varphi}} & & \Big\downarrow{\scriptstyle \varphi} \\ C^{\mathrm{an}} & \xrightarrow{\ \iota\ } & J^{\mathrm{an}} \end{array} \qquad\qquad \begin{array}{ccc} \widetilde{P} & \longmapsto & 1 \\ \Big\downarrow & & \Big\downarrow \\ P & \longmapsto & 0 \end{array}$$

commutes.

Lemma 4.18. *In the situation above $H \cong H_1(C, \mathbb{Z})$ and the isomorphism is given by mapping closed paths to the endpoints of lifts in T.*

Proof. The Abel-Jacobi map ι induces an isomorphism of the singular homology groups

$H_1(C^{\mathrm{an}}, \mathbb{Z})$ and $H_1(J^{\mathrm{an}}, \mathbb{Z})$ using [BR15, Proposition 4.7]. Covering theory yields

$$H \cong \pi_1(J^{\mathrm{an}}) \cong H_1(J^{\mathrm{an}}, \mathbb{Z}),$$

since H is abelian. Hence, the claim follows by going the top left path in the commutative diagram. □

To arrive at similar results as in Example 4.17, we choose a basis of T, i.e., an isomorphism $T \cong (\operatorname{Spec} \mathbb{C}_p[t_1^{\pm 1}, \ldots, t_g^{\pm 1}])^{\mathrm{an}}$. In this way, we may assume that H is generated by some $q_1, \ldots, q_g \in (\mathbb{C}_p^\times)^g$. Let $\gamma_i \in H_1(C^{\mathrm{an}}, \mathbb{Z})$ denote the image of q_i using the isomorphism of Lemma 4.18. Finally, let $\omega \in H^0(C, \Omega_C)$ be a regular differential.

By [Mil86, Prop 2.2] the Abel-Jacobi map ι induces an isomorphism of differentials via pullback $\iota^*\colon H^0(J, \Omega_J) \to H^0(C, \Omega_C)$. For the unique lift $\widetilde{\gamma}_i$ of γ_i starting at $\widetilde{P}$ and $\omega' \in H^0(J, \Omega_J)$ with $\iota^*(\omega') = \omega$ we get

$$\int_{\gamma_i} \omega = \int_{\iota(\gamma_i)} \omega' = \int_{\widetilde{\iota}(\widetilde{\gamma}_i)} \widetilde{\varphi}^*(\omega').$$

Since $\widetilde{\varphi}^*(\omega')$ is an invariant differential on T, there are $\alpha_1, \ldots, \alpha_g \in \mathbb{C}_p$ such that $\widetilde{\varphi}^*(\omega') = \sum_{i=j}^g \alpha_j \frac{\mathrm{d}t_j}{t_j}$. Because T is simply connected, the integral along $\widetilde{\iota}(\widetilde{\gamma}_i)$ depends only on the endpoints of this path. Hence

$$\int_{\gamma_i} \omega = \int_1^{q_i} \sum_{j=1}^g \alpha_i \frac{\mathrm{d}t_j}{t_j} = \sum_{j=1}^g \alpha_j \operatorname{Log}(q_{ij}),$$

where q_{ij} denotes the j-th component of q_i.

It is an open problem whether this integral can be used to obtain coordinates on the strata of $\Omega\mathcal{S}_g$, maybe using ideas of [Ger86], [De 90] or [Ich19]. The most glaring problem is that $\dim H_1(C, \mathbb{Z}) = g < 2g$ for a Mumford curve C of genus g, i.e., there are not enough paths to obtain $2g + n - 1$ coordinates using a construction similar to period coordinates. However, it is precisely for this reason that we think this problem is worth investigating.

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