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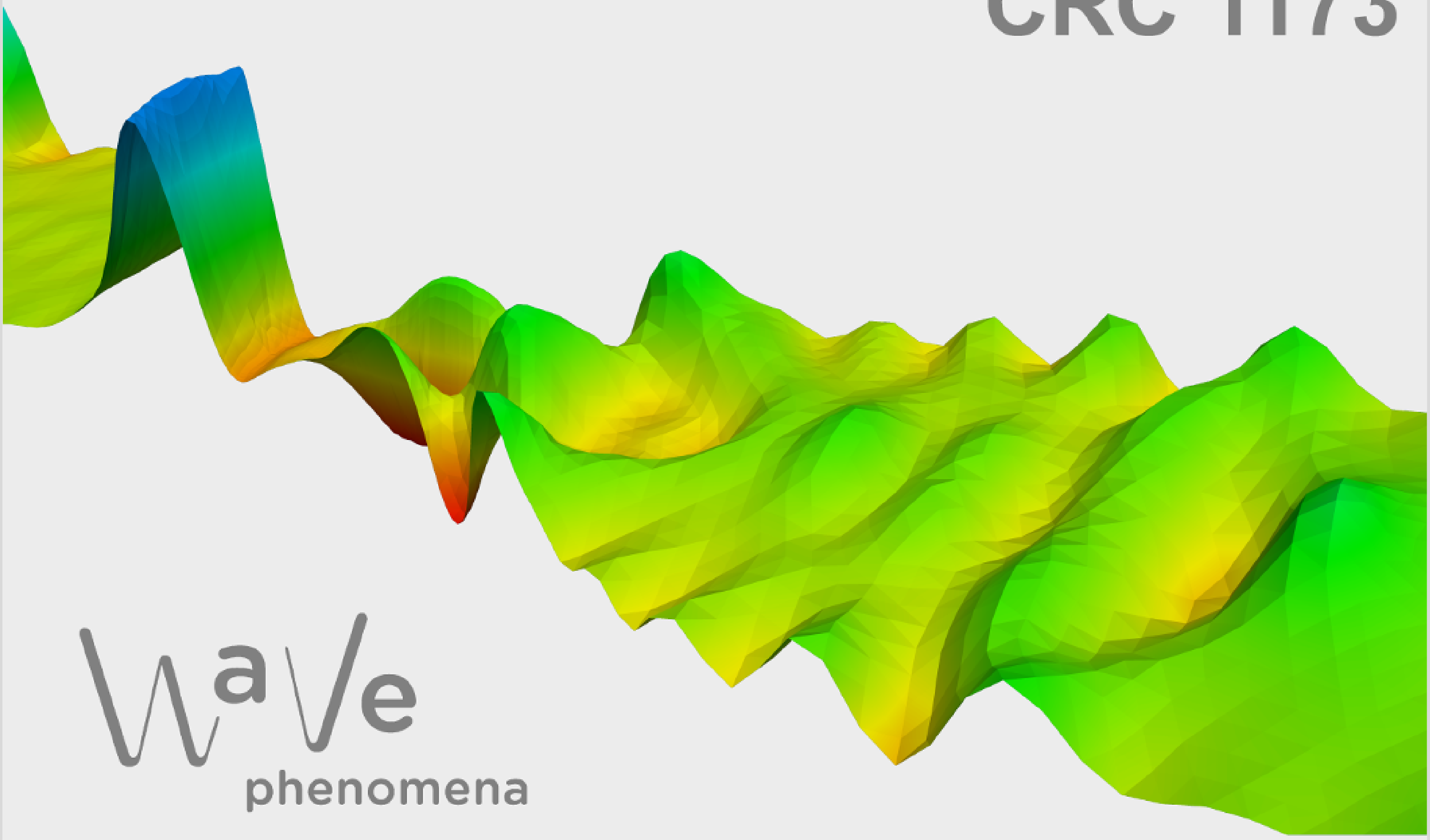

Wave phenomena

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BORN-INFELD PROBLEM WITH GENERAL NONLINEARITY

JAROSŁAW MEDERSKI AND ALESSIO POMPONIO

Abstract. In this paper, using variational methods, we look for non-trivial solutions for the following problem

$$\begin{cases} -\mathrm{div}\left(a(|\nabla u|^2)\nabla u\right) = g(u), & \text{in } \mathbb{R}^N,\ N \geq 3, \\ u(x) \to 0, & \text{as } |x| \to +\infty, \end{cases}$$

under general assumptions on the continuous nonlinearity g. We assume only growth conditions of g at 0, however no growth conditions at infinity are imposed. If $a(s) = (1-s)^{-1/2}$, we obtain the well-known Born-Infeld operator, but we are able to study also a general class of a such that $a(s) \to +\infty$ as $s \to 1^-$. We find a radial solution to the problem with finite energy.

1. Introduction

Almost a century ago, Born and Infeld introduced a new electromagnetic theory in a series of papers (see [16–19]) as a nonlinear alternative to the classical Maxwell theory. This theory was proposed to provide a model presenting a unitarian point of view to describe electrodynamics and had the notable feature to be a fine answer to the well-known *infinity energy problem*. In the Born-Infeld model, indeed, the electromagnetic field generated by a point charge has finite energy. A crucial role is played by the following peculiar differential operator

$$\mathcal{Q}(u) = -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right).$$

Such operator is present also in classical relativity, where it represents the mean curvature operator in Lorentz-Minkowski space, see for instance [6, 20].

In last years many authors focused their attention to problems related to $\mathcal{Q}$ in the whole $\mathbb{R}^N$, with $N \geq 1$. In particular, some results for

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \rho, \qquad \text{in } \mathbb{R}^N,$$

can be found in [10, 12–15, 26, 27], under different assumptions on ρ. Here ρ can be considered as an assigned charges source. See also [5], where the Born-Infeld equation in coupled with the nonlinear Schrödinger one.

Few is still known, at contrary, in presence of a nonlinearity, namely for equations of this type

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = g(u), \qquad \text{in } \mathbb{R}^N. \tag{1.1}$$

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Let us observe that classical variational techniques do not work directly for this problem, due to the particular nature of the operator $\mathcal{Q}$. Indeed, at least formally, solutions of (1.1) are critical points of the functional

$$I(u)=\int_{\mathbb{R}^N}\left(1-\sqrt{1-|\nabla u|^2}\right)-\int_{\mathbb{R}^N}G(u)\,dx,$$

where G is a primitive of g. However, since we have to impose the condition $|\nabla u|\leq 1$, a.e. in $\mathbb{R}^N$, the lack of regularity of the functional on the set $\{x\in\mathbb{R}^N:|\nabla u|=1\}$ requires different and non-standard strategies.

One of the first paper dealing with this kind of problem using variational methods is [11], where $g(s)=|s|^{p-2}s$, for $p>2^*$ and $N\geq 3$. By means of suitable truncation arguments (that will be crucial in our approach, as we will see later), the existence of *finite energy* solutions is proved.

We mention, moreover, [2, 3, 32] where (1.1) has been attached by means of ODE-techniques finding solutions which could have infinite energy. In particular, in [2, 3], the existence of positive or sign-changing radial solutions is considered for a pure power nonlinearity or under suitable sign assumptions on g (a prototype of such nonlinearity is $g(s)=-\lambda s+s^p$, for $\lambda>0$ and $p>1$). In [32], instead, the existence of oscillating solutions of (1.1), namely with an unbounded sequence of zeros, is proved for nonlinearities such that $g'(0)>0$. Finally, in [7], a similar problem is considered in an exterior domain.

Our aim is to show existence of finite energy radial solutions involving a large class of operators and nonlinearities in the spirit of Berestycki and Lions [8, 9] and we will present an adequate variational approach for the problem. More precisely we consider

$$\begin{cases}-\mathrm{div}\left(a(|\nabla u|^2)\nabla u\right)=g(u), & \text{in } \mathbb{R}^N,\ N\geq 3,\\[2ex] u(x)\to 0, & \text{as } |x|\to+\infty,\end{cases}\tag{1.2}$$

under the following assumptions on a:

- (a0) $a:[0,1)\to(0,+\infty)$ is continuous, of class $\mathcal{C}^1$ on $(0,1)$, and $[0,1)\ni s\mapsto a(s)s$ is strictly convex;
- (a1) $\lim_{s\to 1^-}a(s)=+\infty$;

and on the nonlinearity g:

- (g0) $g:\mathbb{R}\to\mathbb{R}$ is continuous and $g(0)=0$;
- (g1) for some $\gamma\geq 2^*/2$, we have
$$-\infty<\liminf_{s\to 0}\frac{g(s)}{|s|^{\gamma-1}}\leq\limsup_{s\to 0}\frac{g(s)}{|s|^{\gamma-1}}=-m\leq 0;$$
- (g2) there exists $\xi_0>0$ such that $G(\xi_0)>0$, where
$$G(s)=\int_0^s g(t)\,dt,\quad \text{for } s\in\mathbb{R}.$$

Clearly, $a(s)=(1-s)^\alpha$ with $\alpha<0$ satisfies (a0), (a1), and we get the operator $\mathcal{Q}$ for $\alpha=-1/2$. Another important example is the following general mean curvature operator arising in the study of hypersurfaces in the Lorentz–Minkowski space $\mathbb{L}^{N+1}$ and in $\mathbb{R}^{N+1}$ given by

$$a(s):=\beta(1-s)^{-1/2}-\gamma(1+s)^{-1/2},\quad \beta>0,\gamma\geq 0,\tag{1.3}$$

see [20, 23, 28] and references therein.

With regard to g, by assumption (g1), the problem is in the so called *positive mass case*. We will consider also the *zero mass case* namely, instead of (g1), we will assume

- (g1′) $\limsup_{s\to 0}g(s)/|s|^{\gamma-1}=0$, for all $\gamma\geq 2^*/2$.

We recall that these kinds of hypotheses on g have been introduced for the first time in [8, 9] for the study of

$$-\Delta u = g(u), \qquad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $\gamma = 2$. However, we want to remark that, in contrast to what happens in these previous papers, in our case there is no assumption on the behaviour at infinity of g. This is a direct consequence of the natural framework associated to (1.2) which has to take in account the condition $|\nabla u| \leq 1$, a.e. in $\mathbb{R}^N$: this assures that each function is, actually, bounded. See Section 2 for more details.

An intermediate step for the study of (1.2), based on an approximation argument, has been widely studied in the literature, e.g. see [33] and references therein. Indeed by the Taylor expansion of $\frac{1}{\sqrt{1-|u|}}$ to the k-th order, we arrive at the approximated problem

$$\mathcal{Q}(u) \approx -\Delta u - \frac{1}{2}\Delta_4 u - \frac{3}{2\cdot 2^2}\Delta_6 u - \cdots - \frac{(2k-3)!!}{(k-1)!\cdot 2^{k-1}}\Delta_{2k} u = g(u) \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

Note that [33] deals precisely with (1.5), where g satisfying more restrictive Berestycki-Lions-type assumptions. In [33] (see also the references therein), it is not clear if one can solve (1.1) passing to the limit, as $k \to +\infty$. We would like to mention that some partial results using this approximation process have been obtained only in case of the fixed charges source ρ on the right hand side instead of the nonlinear term $g(u)$, see e.g. [12, 13, 26, 27]. Therefore (1.1) requires a different variational approach presented in this work.

Our main result reads as follows.

Theorem 1.1. *Suppose that a satisfies (a0), (a1) and g satisfies (g0), (g2) and one between (g1) and (g1′). Then there exists a nontrivial radial solution u of (1.2) such that*

$$\int_{\mathbb{R}^N} A(|\nabla u|^2)\,dx, \int_{\mathbb{R}^N} a(|\nabla u|^2)|\nabla u|^2\,dx, \int_{\mathbb{R}^N} |G(u)|\,dx < +\infty,$$

where $A(s) = \int_0^s a(t)\,dt$.

We use a truncation argument applied to a similarly as in [11] but due to the lack of scaling of the nonlinearity we use a different variational approach for (1.2). Inspired by [24, 25] (see also [1, 4, 21, 22]), we will adapt for our problem the method explored considering an auxiliary functional that allows to construct a suitable Palais-Smale sequence, which almost satisfies a Pohozaev type identity. The compactness properties of the general nonlinear term will be investigated similarly as in [30, 31], see Sections 3 and 4 for more details.

The paper is organized as follows. In Section 2 we introduce our functional framework and some technical tools. Section 3 and Section 4 will be devoted, respectively, to the positive mass case and to the zero mass one and, therein, we will prove our main result.

We conclude this introduction fixing some notations. For any $p \geq 1$, we denote by $L^p(\mathbb{R}^N)$ the usual Lebesgue spaces equipped by the standard norm $|\cdot|_p$. In our estimates, we will frequently denote by $C > 0$, $c > 0$ fixed constants, that may change from line to line, but are always independent of the variable under consideration. We also use the notation $o_n(1)$ to indicate a quantity which goes to zero as $n \to +\infty$. Moreover, for any $R > 0$, we denote by B_R the ball of $\mathbb{R}^N$ centred in the origin with radius R. Finally, if u is a radial function of $\mathbb{R}^N$, with an abuse of notation, for any $x \in \mathbb{R}^N$, we denote $u(x) = u(r)$, with $r = |x|$.

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2. Functional framework

In this section we introduce the functional framework related to (1.2) with some useful continuous and compact embedding properties. Moreover, following [11], we present a truncated problem which will play a crucial role in our arguments.

Take any $q > N$. Let $\mathcal{X}_0^{2,q}$ be the completion of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the following norm

$$\|u\|_0 = \left(|\nabla u|_2^2 + |\nabla u|_q^2\right)^{1/2}.$$

Recall that $\mathcal{X}_0^{2,q}$ is continuously embedded into $L^p(\mathbb{R}^N)$ for $p \in [2^*, +\infty]$ and

$$\mathcal{X}_0 := \mathcal{X}_{0,\mathrm{rad}}^{2,q} = \left\{u \in \mathcal{X}_0^{2,q} : u \text{ radially symmetric } \right\}$$

embeds compactly into $L^p(\mathbb{R}^N)$, for $p \in (2^*, +\infty)$, see e.g. [11,33]. Moreover, as in [11,34], we have the following

Lemma 2.1. *Let $p \in [2, N)$. Then there exists $C > 0$ (depending only on N and p) such that for all $u \in \mathcal{X}_0$, there holds*

$$|u(x)| \leq C|x|^{-\frac{N-p}{p}}|\nabla u|_p,$$

for almost every $x \in \mathbb{R}^N \setminus \{0\}$.

Let $\mathcal{X}^{2,q,\gamma}$ be the completion of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the following norm

$$\|u\| = \left(|\nabla u|_2^2 + |\nabla u|_q^2 + |u|_\gamma^2\right)^{1/2}$$

and clearly if $\gamma \geq 2^*$, then $\mathcal{X}^{2,q,\gamma}$ and $\mathcal{X}^{2,q}$ coincides. Moreover $\mathcal{X}^{2,q,\gamma}$ is continuously embedded into $L^p(\mathbb{R}^N)$ for $p \in [\min\{2^*, \gamma\}, +\infty]$ and

$$\mathcal{X} := \mathcal{X}_{\mathrm{rad}}^{2,q,\gamma} = \left\{u \in \mathcal{X}^{2,q,\gamma} : u \text{ radially symmetric } \right\}$$

embeds compactly into $L^p(\mathbb{R}^N)$, for $p \in (\min\{2^*, \gamma\}, +\infty)$.

Similarly as in [11] for $\mathcal{Q}$ we introduce a truncated problem. Let us fix $\theta_1 \in (0, 1)$. For any $\theta \in (0, \theta_1]$ we fix $q = q(\theta) > N$ such that

$$q \geq 2\frac{a'(1-\theta)(1-\theta) + a(1-\theta)}{a(1-\theta)}. \tag{2.1}$$

Then we define a continuous function $a_\theta : [0, +\infty) \to \mathbb{R}^+$ by

$$a_\theta(s) := \begin{cases} a(s) & \text{if } 0 \leq s \leq 1-\theta, \\ (1-\theta)^{-\frac{q-2}{2}} a(1-\theta) s^{\frac{q-2}{2}} & \text{if } s > 1-\theta. \end{cases}$$

The functions $a_\theta(s)$ and $\varphi(s) := a_\theta(s)s$ are differentiable in $[0, +\infty) \setminus \{1-\theta\}$ and, by (2.1) and (a0), we deduce that $\varphi'(s_1) < \varphi'_-(1-\theta) \leq \varphi'_+(1-\theta) < \varphi'(s_2)$, for any $s_1 < 1-\theta < s_2$.

Lemma 2.2. *The map $\varphi(s)$ is strictly convex.*

Proof. Clearly φ is strictly convex on $[0,1-\theta]$ and on $[1-\theta,+\infty)$. Take $0<s<1-\theta<t$. If $\frac{s+t}{2}\le 1-\theta$, then by the convexity we obtain

$$\varphi(s)-\varphi\Big(\frac{s+t}{2}\Big)>\varphi'\Big(\frac{s+t}{2}\Big)\Big(s-\frac{s+t}{2}\Big),$$
$$\varphi(1-\theta)-\varphi\Big(\frac{s+t}{2}\Big)>\varphi'\Big(\frac{s+t}{2}\Big)\Big(1-\theta-\frac{s+t}{2}\Big),$$
$$\varphi(t)-\varphi(1-\theta)>\varphi'_+(1-\theta)(t-1+\theta).$$

In view of (2.1) we get $\varphi'_+(1-\theta)\ge\varphi'\big(\frac{s+t}{2}\big)$ and we conclude

$$\frac{\varphi(s)+\varphi(t)}{2}>\varphi\Big(\frac{s+t}{2}\Big).$$

Similarly we argue if $\frac{s+t}{2}>1-\theta$ and we conclude. □

For the positive mass case we will consider the following truncated problem

$$\begin{cases} -\operatorname{div}\big(a_\theta(|\nabla u|^2)\nabla u\big)\,u=g(u) & \text{in } \mathbb{R}^N,\\ u\in\mathcal{X}. \end{cases}\tag{2.2}$$

For the zero mass case, instead, we will consider the following truncated problem

$$\begin{cases} -\operatorname{div}\big(a_\theta(|\nabla u|^2)\nabla u\big)\,u=g(u) & \text{in } \mathbb{R}^N,\\ u\in\mathcal{X}_0. \end{cases}\tag{2.3}$$

Clearly, if u_θ is a solution of (2.2) or of (2.3) such that $|\nabla u_\theta|\le 1-\theta$, then u_θ is a solution also of (1.2).

Observe that there exists $\bar c_\theta=\bar c_\theta(\theta)>0$ such that

$$\bar c\left(s^2+|s|^q\right)\le a_\theta(s^2)s^2\le\bar c_\theta\left(s^2+|s|^q\right),\qquad\text{for all } s\in\mathbb{R},\tag{2.4}$$

$$\bar c\left(s^2+|s|^q\right)\le A_\theta(s^2)\le\bar c_\theta\left(s^2+|s|^q\right),\qquad\text{for all } s\in\mathbb{R},\tag{2.5}$$

where $A_\theta(s)=\int_0^s a_\theta(t)\,dt$ and

$$\bar c:=\frac{2}{q}\cdot\frac{(1-\theta_1)^{\frac{q-2}{2}}}{1+(1-\theta_1)^{q-2}}\cdot\min_{s\in[0,1)}a(s)$$

is independent of θ.

We conclude this section with the following lemma, which is also new for $\mathcal{Q}$ and which will play a crucial role in our arguments.

Lemma 2.3. *Suppose that $u_n\rightharpoonup u_0$ in $\mathcal{X}_0$ and*

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}a_\theta(|\nabla u_n|^2)|\nabla u_n|^2\,dx=\int_{\mathbb{R}^N}a_\theta(|\nabla u_0|^2)|\nabla u_0|^2\,dx.\tag{2.6}$$

Then $u_n\to u_0$ strongly in $\mathcal{X}_0$.

Proof. Let $\varphi:\mathbb{R}^N\to\mathbb{R}$ be given by $\varphi(v):=a_\theta(|v|^2)|v|^2$, for $v\in\mathbb{R}^N$. By Lemma 2.2, φ is strictly convex, hence the map $\Phi:\mathcal{X}_0\to\mathbb{R}$, such that

$$\Phi(u):=\int_{\mathbb{R}^N}\varphi(\nabla u)\,dx,\qquad\text{for } u\in\mathcal{X}_0,$$

is well defined and strictly convex as well. So, since $\frac12(\nabla u_n+\nabla u_0)\rightharpoonup\nabla u_0$, we obtain

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^N}\varphi\Big(\frac12(\nabla u_n+\nabla u_0)\Big)\,dx\ge\int_{\mathbb{R}^N}\varphi(\nabla u_0)\,dx.\tag{2.7}$$

Then, taking into account the convexity of φ, we know that, a.e. in $\mathbb{R}^N$,

$$\xi_n := \frac{1}{2}\big(\varphi(\nabla u_n) + \varphi(\nabla u_0)\big) - \varphi\Big(\frac{1}{2}(\nabla u_n + \nabla u_0)\Big) \geq 0,$$

hence, by (2.6) and (2.7),

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} \xi_n \, dx = 0. \tag{2.8}$$

For any $k \geq 1$ we define

$$\mu_k := \inf\left\{\frac{1}{2}\big(\varphi(v_1)+\varphi(v_2)\big) - \varphi\Big(\frac{1}{2}(v_1+v_2)\Big) : v_1, v_2 \in \mathbb{R}^N \text{ s.t. } |v_1|, |v_2| \leq k, |v_1 - v_2| \geq \frac{1}{k}\right\},$$

$$\Omega_{n,k} := \left\{x \in \mathbb{R}^N : |\nabla u_n|, |\nabla u_0| \leq k, |\nabla u_n - \nabla u_0| \geq \frac{1}{k}\right\}.$$

Since $\mu_k > 0$, by the strict convexity of φ, and (2.8) holds, we infer that the Lebesgue measure $|\Omega_{n,k}| \to 0$, as $n \to +\infty$. Take any $\varepsilon > 0$, we find a subsequence $\{n_k\}$ such that $|\bigcup_{k=1}^\infty \Omega_{n_k,k}| < \varepsilon$. Again letting $\varepsilon \to 0$ and passing to a subsequence we obtain that $\nabla u_n \to \nabla u_0$ a.e. on $\mathbb{R}^N$. Note that a_θ is of class $\mathcal{C}^1$ on $(0, 1-\theta)$ and $(1-\theta, +\infty)$, hence φ' exists almost everywhere. Now take $s \in [0,1]$, by (2.4) we observe that the sequence $\{\varphi'(\nabla u_n - s\nabla u_0)\nabla u_0\}$ is uniformly integrable and tight and converges a.e. to $\varphi'\big((1-s)\nabla u_0\big)\nabla u_0$. In view of the Vitali Convergence Theorem we get

$$\begin{aligned}
\int_{\mathbb{R}^N} \varphi(\nabla u_n)\, dx - \int_{\mathbb{R}^N} \varphi(\nabla u_n - \nabla u_0)\, dx &= \int_0^1 \int_{\mathbb{R}^N} \varphi'(\nabla u_n - s\nabla u_0)\nabla u_0 \, dx\, ds \\
&\xrightarrow[n\to+\infty]{} \int_0^1 \int_{\mathbb{R}^N} \varphi'\big((1-s)\nabla u_0\big)\nabla u_0\, dx\, ds \\
&= \int_{\mathbb{R}^N} \varphi(\nabla u_0)\, dx.
\end{aligned}$$

Since (2.6) holds, we get

$$\int_{\mathbb{R}^N} \varphi(\nabla u_n - \nabla u_0)\, dx \to 0,$$

as $n \to +\infty$, and by (2.4) we conclude. □

3. The positive mass case

In this section we deal with the positive mass case, namely, we will assume on g (g0), (g1) and (g2).

Let $g_1(s) := \max\{g(s) + ms^{\gamma-1}, 0\}$, for $s \geq 0$, and $g_1(s) := \min\{g(s) + m|s|^{\gamma-1}, 0\}$, for $s < 0$, and $g_2(s) = g_1(s) - g(s)$, for $s \geq 0$, and $g_i(s) = -g_i(-s)$ for, $s < 0$. Then $g_1(s), g_2(s) \geq 0$, for $s \geq 0$,

$$\lim_{s\to 0} g_1(s)/s^{\gamma-1} = 0, \tag{3.1}$$

$$g_2(s) \geq ms^{\gamma-1}, \qquad \text{for } s \geq 0. \tag{3.2}$$

If we set

$$G_i(s) = \int_0^s g_i(t)\, dt, \qquad \text{for } i = 1, 2,$$

then by (3.2) we have

$$G_2(s) \geq \frac{m}{\gamma}|s|^\gamma, \qquad \text{for } s \in \mathbb{R}. \tag{3.3}$$

By (g1) and (3.1), we have that there exist two fixed positive constants, $\bar{c}_1, \bar{c}_2$ such that

$$|g(s)| \le \bar{c}_1 |s|^{\gamma-1}, \qquad \text{for all } |s| \le \bar{c}_2, \tag{3.4}$$

$$|G(s)| \le \bar{c}_1 |s|^{\gamma}, \qquad \text{for all } |s| \le \bar{c}_2, \tag{3.5}$$

$$|g_1(s)| \le \bar{c}_1 |s|^{\gamma-1}, \qquad \text{for all } |s| \le \bar{c}_2, \tag{3.6}$$

$$|G_1(s)| \le \bar{c}_1 |s|^{\gamma}, \qquad \text{for all } |s| \le \bar{c}_2. \tag{3.7}$$

Lemma 3.1. *For any $u \in \mathcal{X}$, $\int_{\mathbb{R}^N} G(u)\,dx$ and $\int_{\mathbb{R}^N} g(u)u\,dx$ are well defined. The same is true for $\int_{\mathbb{R}^N} G_i(u)\,dx$ and $\int_{\mathbb{R}^N} g_i(u)u\,dx$, for $1 = 1, 2$.*

Proof. Let $u \in \mathcal{X}$. Since $\mathcal{X}$ is embedded into $L^\gamma(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we have that

$$\begin{aligned}
\int_{\mathbb{R}^N} |G(u)|\,dx &= \int_{\{|u| \le \bar{c}_2\}} |G(u)|\,dx + \int_{\{|u| > \bar{c}_2\}} |G(u)|\,dx \\
&\le \bar{c}_1 \int_{\{|u| \le \bar{c}_2\}} |u|^\gamma\,dx + \operatorname{meas}\{|u| > \bar{c}_2\} \cdot \max_{\{s \le \|u\|_\infty\}} |G(s)| \\
&\le \bar{c}_1 |u|_\gamma^\gamma + \operatorname{meas}\{|u| > \bar{c}_2\} \cdot \max_{\{s \le \|u\|_\infty\}} |G(s)| < +\infty.
\end{aligned}$$

The arguments are similar for $\int_{\mathbb{R}^N} g(u)u\,dx$, $\int_{\mathbb{R}^N} G_i(u)\,dx$ and $\int_{\mathbb{R}^N} g_i(u)u\,dx$, $1 = 1, 2$. ☐

Lemma 3.2. *If $u_n \rightharpoonup u_0$ in $\mathcal{X}$, then*

$$\lim_n \int_{\mathbb{R}^N} g_1(u_n)u_n\,dx = \int_{\mathbb{R}^N} g_1(u_0)u_0\,dx \tag{3.8}$$

and

$$\lim_n \int_{\mathbb{R}^N} G_1(u_n)\,dx = \int_{\mathbb{R}^N} G_1(u_0)\,dx. \tag{3.9}$$

Proof. Here we follow some ideas of [30, Corollary 3.6] (cf. [31]) and we divide the proof into two intermediate steps by which the conclusion follows immediately.
Step 1: We claim that

$$\lim_n \int_{\mathbb{R}^N} g_1(u_n)(u_n - u_0)\,dx = 0. \tag{3.10}$$

Since $\{u_n\}$ is bounded in $\mathcal{X}$ then, by the continuous embedding of $\mathcal{X}$ into $L^\infty(\mathbb{R}^N)$, we infer that there exists $M > 0$ such that $|u_n|_\infty \le M$, for any $n \ge 1$. Take any $\varepsilon > 0$ and $\beta > 2^*$. Then, by (3.1), we find $0 < \delta < M$ and $c_\varepsilon > 0$ such that

$$\begin{aligned}
|g_1(s)| &\le \varepsilon |s|^{\gamma-1} \quad \text{if } |s| \in [0, \delta], \\
|g_1(s)| &\le c_\varepsilon |s|^{\beta-1} \quad \text{if } |s| \in (\delta, M].
\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N} |g_1(u_n)(u_n - u_0)|\,dx \le \varepsilon \int_{\mathbb{R}^N} |u_n|^{\gamma-1}|u_n - u_0|\,dx + c_\varepsilon \int_{\mathbb{R}^N} |u_n|^{\beta-1}|u_n - u_0|\,dx,$$

and, by the compact embedding of $\mathcal{X}$ into L^β, the boundedness of the sequence $\{u_n\}$ in $\mathcal{X}$, we infer that

$$\limsup_n \int_{\mathbb{R}^N} |g_1(u_n)(u_n - u_0)|\,dx \le \varepsilon C$$

for some constant $C > 0$ and so (3.10) is proved.
Step 2: We claim that

$$\lim_n \int_{\mathbb{R}^N} g_1(u_n)u_0\,dx = \int_{\mathbb{R}^N} g_1(u_0)u_0\,dx.$$

Since the sequence $\{g_1(u_n)u_0\}$ is uniformly integrable and tight, then the conclusion follows by Vitali Convergence Theorem.

Step 3: We claim that

$$\lim_n \left(\int_{\mathbb{R}^N} g_1(u_n)u_n \, dx - \int_{\mathbb{R}^N} g_1(u_n)(u_n - u_0) \, dx \right) = \int_{\mathbb{R}^N} g_1(u_0)u_0 \, dx.$$

Indeed, if we set $\phi_n(s) = g_1(u_n)(u_n - su_0)$, for any $n \in \mathbb{N}$ and $s \in [0,1]$, taking in account Step 2, we have

$$\begin{aligned}
&\lim_n \left(\int_{\mathbb{R}^N} g_1(u_n)u_n \, dx - \int_{\mathbb{R}^N} g_1(u_n)(u_n - u_0) \, dx \right) \\
&\quad = \lim_n \int_{\mathbb{R}^N} \big(\phi_n(0) - \phi_n(1)\big) dx = - \lim_n \int_{\mathbb{R}^N} \left(\int_0^1 \phi_n'(s) \, ds \right) dx \\
&\quad = \int_0^1 \left(\lim_n \int_{\mathbb{R}^N} g_1(u_n)u_0 \, dx \right) ds = \int_0^1 \left(\int_{\mathbb{R}^N} g_1(u_0)u_0 \, dx \right) ds = - \int_0^1 \left(\int_{\mathbb{R}^N} \phi_0'(s) \, dx \right) ds \\
&\quad = \int_{\mathbb{R}^N} \big(\phi_0(0) - \phi_0(1)\big) dx = \int_{\mathbb{R}^N} g_1(u_0)u_0 \, dx.
\end{aligned}$$

The proof of (3.9) is similar. □

Solutions of (2.2) will be found as critical points of the functional $I_\theta : \mathcal{X} \to \mathbb{R}$ defined as

$$I_\theta(u) = \frac{1}{2} \int_{\mathbb{R}^N} A_\theta(|\nabla u|^2) \, dx + \int_{\mathbb{R}^N} G_2(u) \, dx - \int_{\mathbb{R}^N} G_1(u) \, dx.$$

The functional is well defined in $\mathcal{X}$ by (2.5).

Lemma 3.3. *For any $\theta \in (0, \theta_1]$, the functional $I_\theta : \mathcal{X} \to \mathbb{R}$ verifies the mountain pass geometry. More precisely:*

(i) there are $\alpha, \rho > 0$ such that $I_\theta(u) \geq \alpha$, for $\|u\| = \rho$;

(ii) there is $\bar{u} \in \mathcal{X} \setminus \{0\}$, independent of $\theta \in (0, \theta_1]$, with $\|\bar{u}\| > \rho$ and $|\nabla \bar{u}| < 1 - \theta_1$, almost everywhere in $\mathbb{R}^N$, and such that $I_\theta(\bar{u}) < 0$.

Proof. (i) By the continuous embedding of $\mathcal{X}$ into $L^\infty(\mathbb{R}^N)$, and by (3.1), we can consider $\rho > 0$ sufficiently small such that

$$G_1(u(x)) \leq \frac{m}{2\gamma} |u(x)|^\gamma, \qquad \text{a.e. } x \in \mathbb{R}^N \text{ and for any } u \in \mathcal{X} \text{ with } \|u\| = \rho.$$

Hence, by (3.3) and (2.5), for any $u \in \mathcal{X}$ with $\|u\| = \rho$, we have

$$I_\theta(u) \geq \frac{\bar{c}}{2} \left(|\nabla u|_2^2 + |\nabla u|_q^q \right) + \frac{m}{2\gamma} |u|_\gamma^\gamma \geq c\|u\|^\beta \geq \alpha > 0,$$

where $\beta = \max\{2, q, \gamma\}$.

(ii) Let $u_R \in \mathcal{X}$ such that, for any $x \in \mathbb{R}^N$,

$$u_R(x) := \begin{cases} \xi_0 & \text{in } B_R, \\ -\frac{\xi_0}{\sqrt{R}}|x| + \xi_0(1 + \sqrt{R}) & \text{in } B_{R+\sqrt{R}} \setminus B_R, \\ 0 & \text{in } \mathbb{R}^N \setminus B_{R+\sqrt{R}}. \end{cases}$$

Arguing as in [8], for R sufficiently large, we have $\int_{\mathbb{R}^N} G(u_R) \, dx > 0$ and, clearly, $|\nabla u_R| < 1 - \theta_1$. Moreover, for any $t > 1$, we have also that $|\nabla u_R(\cdot/t)| \leq 1 - \theta_1$ and so, denoting $\bar{u} = u_R(\cdot/t)$, with R and t sufficiently large and independently by $\theta \in (0, \theta_1]$, we have $\|\bar{u}\| > \rho$ and

$$I_\theta(\bar{u}) \leq c_1 \left(t^{N-2} |\nabla u_R|_2^2 + t^{N-q} |\nabla u_R|_q^q \right) - t^N \int_{\mathbb{R}^N} G(u_R) \, dx < 0.$$

□

Let us define the mountain pass level for the functional I_θ

$$m_\theta := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\theta(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in \mathcal{C}([0,1],\mathcal{X}) \mid \gamma(0) = 0, \gamma(1) = \bar{u}\}.$$

By Lemma 3.3, we deduce that $m_\theta \geq \alpha$, for any $\theta \in (0,\theta_1]$.

Observe that, since $|\nabla \bar{u}| < 1-\theta_1$, we have that $I_{\theta_1}(t\bar{u}) = I_\theta(t\bar{u})$, for any $t\in[0,1]$ and for any $\theta \in (0,\theta_1]$. Hence we deduce that

$$m_\theta \leq \max_{t\in[0,1]} I_\theta(t\bar{u}) = \max_{t\in[0,1]} I_{\theta_1}(t\bar{u}),$$

for any $\theta\in(0,\theta_1]$. Hence there exists $c>0$ (independent of $\theta\in(0,\theta_1]$) such that

$$0 < m_\theta \leq c, \qquad \text{for any } \theta\in(0,\theta_1]. \tag{3.11}$$

Following [24, 25], we define the functional $J_\theta : \mathbb{R}\times\mathcal{X}\to\mathbb{R}$ as

$$J_\theta(\sigma,u) = I_\theta(u(e^{-\sigma}\cdot)) = \frac{e^{N\sigma}}{2}\int_{\mathbb{R}^N} A_\theta(e^{-2\sigma}|\nabla u|^2)\,dx + e^{N\sigma}\int_{\mathbb{R}^N} G_2(u)\,dx - e^{N\sigma}\int_{\mathbb{R}^N} G_1(u)\,dx.$$

With similar arguments of Lemma 3.3, also J_θ has a mountain pass geometry and we can define its mountain pass level as

$$\tilde{m}_\theta := \inf_{(\sigma,\gamma)\in\Sigma\times\Gamma}\max_{t\in[0,1]} J_\theta\big(\sigma(t),\gamma(t)\big),$$

where

$$\Sigma := \{\sigma\in\mathcal{C}([0,1],\mathbb{R}) \mid \sigma(0) = \sigma(1) = 0\}.$$

Observe that arguing as in [24, Lemma 3.1], we obtain

Lemma 3.4. *For any $\theta\in(0,\theta_1]$, the mountain pass levels of I_θ and J_θ coincide, namely $m_\theta = \tilde{m}_\theta$.*

Now, as an immediate consequence of Ekeland's variational principle [35, Theorem 2.8] (cf. [25, Lemma 2.3]) we obtain the following results.

Lemma 3.5. *Let $\theta\in(0,\theta_1]$ and $\varepsilon>0$. Suppose that $\tilde{\gamma}\in\Sigma\times\Gamma$ satisfies*

$$\max_{t\in[0,1]} J_\theta(\tilde{\gamma}(t)) \leq m_\theta + \varepsilon,$$

then there exists $(\sigma,u)\in\mathbb{R}\times\mathcal{X}$ such that

(1) $\operatorname{dist}_{\mathbb{R}\times\mathcal{X}}\big((\theta,u),\tilde{\gamma}([0,1])\big) \leq 2\sqrt{\varepsilon}$*;*
(2) $J_\theta(\sigma,u)\in[m_\theta-\varepsilon, m_\theta+\varepsilon]$*;*
(3) $\|DJ_\theta(\sigma,u)\|_{\mathbb{R}\times\mathcal{X}^*} \leq 2\sqrt{\varepsilon}$*.*

Proposition 3.6. *For any $\theta\in(0,\theta_1]$, there exists a sequence $\{(\sigma_n,u_n)\}\subset\mathbb{R}\times\mathcal{X}$ such that, as $n\to+\infty$, we get*

(1) $\sigma_n\to 0$*;*
(2) $J_\theta(\sigma_n,u_n)\to m_\theta$*;*
(3) $\partial_\sigma J_\theta(\sigma_n,u_n)\to 0$*;*
(4) $\partial_u J_\theta(\sigma_n,u_n)\to 0$ *strongly in* $\mathcal{X}^*$*.*

Proof. In view of Lemma 3.5 we conclude by letting $\varepsilon\to 0$. □

Now we find a radial solution of the truncated problem (2.2).

Proposition 3.7. *For any $\theta \in (0,\theta_1]$, there exists $u_\theta \in \mathcal{X}$ a non-trivial solution of (2.2) such $I_\theta(u_\theta) = m_\theta$. Moreover there exists $C>0$ such that*

$$\|u_\theta\|_0 \le C, \quad \text{for any } \theta \in (0,\theta_1]. \tag{3.12}$$

Finally u_θ is a weak solution of

$$-\left(r^{N-1} a_\theta(|u_\theta'(r)|^2) u_\theta'(r)\right)' = r^{N-1} g(u_\theta(r)), \tag{3.13}$$

namely

$$\int_0^{+\infty} r^{N-1} a_\theta(|u_\theta'(r)|^2) u_\theta'(r) v'(r)\, dr = \int_0^{+\infty} r^{N-1} g(u_\theta(r)) v(r)\, dr,$$

for all $v \in \mathcal{X}$.

Proof. Fix $\theta \in (0,\theta_1]$. By Proposition 3.6, there exists a sequence $\{(\sigma_n,u_n)\} \subset \mathbb{R}\times\mathcal{X}$ such that

$$\begin{cases} \dfrac{e^{N\sigma_n}}{2}\displaystyle\int_{\mathbb{R}^N} A_\theta(e^{-2\sigma_n}|\nabla u_n|^2)\,dx + e^{N\sigma_n}\int_{\mathbb{R}^N} G_2(u_n)\,dx - e^{N\sigma_n}\int_{\mathbb{R}^N} G_1(u_n)\,dx = m_\theta + o_n(1), \\ \dfrac{Ne^{N\sigma_n}}{2}\displaystyle\int_{\mathbb{R}^N} A_\theta(e^{-2\sigma_n}|\nabla u_n|^2)\,dx - e^{(N-2)\sigma_n}\int_{\mathbb{R}^N} a_\theta(e^{-2\sigma_n}|\nabla u_n|^2)|\nabla u_n|^2\,dx \\ \qquad\qquad + Ne^{N\sigma_n}\displaystyle\int_{\mathbb{R}^N} G_2(u_n)\,dx - Ne^{N\sigma_n}\int_{\mathbb{R}^N} G_1(u_n)\,dx = o_n(1), \\ e^{(N-2)\sigma_n}\displaystyle\int_{\mathbb{R}^N} a_\theta(e^{-2\sigma_n}|\nabla u_n|^2)|\nabla u_n|^2\,dx + e^{N\sigma_n}\int_{\mathbb{R}^N} g_2(u_n)u_n\,dx - e^{N\sigma_n}\int_{\mathbb{R}^N} g_1(u_n)u_n\,dx = o_n(1)\|u_n\|. \end{cases} \tag{3.14}$$

From the first and the second equation of the previous system we get

$$e^{(N-2)\sigma_n}\int_{\mathbb{R}^N} a_\theta(e^{-2\sigma_n}|\nabla u_n|^2)|\nabla u_n|^2\,dx = Nm_\theta + o_n(1).$$

Therefore, since $\sigma_n \to 0$, as $n\to+\infty$, by (2.4) we deduce that $\{u_n\}$ is a bounded sequence in $\mathcal{X}_0$ and so also in $L^\infty(\mathbb{R}^N)$, namely there exists $\bar C>0$ such that $|u_n|_\infty \le \bar C$, for any $n\ge 1$. This implies that, by (3.1) and Lemma 2.1, there exists $R>1$ such that

$$G_1(u_n(x)) \le \frac{m}{2\gamma}|u_n(x)|^\gamma, \qquad \text{a.e. } x\in\mathbb{R}^N \text{ with } |x|\ge R \text{ and for any } n\ge 1.$$

Hence

$$\int_{\mathbb{R}^N} G_1(u_n)\,dx = \int_{B_R} G_1(u_n)\,dx + \int_{B_R^c} G_1(u_n)\,dx \le C \max_{\{s\le\bar C\}}|G_1(s)| + \frac{m}{2\gamma}\int_{\mathbb{R}^N}|u_n(x)|^\gamma\,dx.$$

By this, by (3.3) and by the first equation of (3.14), we infer that $\{u_n\}$ is a bounded sequence also in $\mathcal{X}$. Then there exists $u_\theta\in\mathcal{X}$ such that $u_n \rightharpoonup u_\theta$ in $\mathcal{X}$. Since $\partial_u J_\theta(\sigma_n,u_n)\to 0$ strongly in $\mathcal{X}^*$ and $\sigma_n\to 0$, we have that u_θ is a weak (possibly trivial) solution of (2.2) and so it satisfies

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2\,dx + \int_{\mathbb{R}^N} g_2(u_\theta)u_\theta\,dx = \int_{\mathbb{R}^N} g_1(u_\theta)u_\theta\,dx.$$

Since $u_n\rightharpoonup u_\theta$ in $\mathcal{X}$, by the weak lower semicontinuity and the Fatou's Lemma we have that

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2\,dx \le \liminf_{n\to+\infty}\int_{\mathbb{R}^N} a_\theta(|\nabla u_n|^2)|\nabla u_n|^2\,dx,$$

$$\int_{\mathbb{R}^N} g_2(u_\theta)u_\theta\,dx \le \liminf_{n\to+\infty}\int_{\mathbb{R}^N} g_2(u_n)u_n\,dx;$$

while, by Lemma 3.2, we have

$$\int_{\mathbb{R}^N} g_1(u_\theta)u_\theta \, dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} g_1(u_n)u_n \, dx.$$

Therefore, by the third equation of (3.14),

$$\begin{aligned}
&\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2 \, dx + \int_{\mathbb{R}^N} g_2(u_\theta)u_\theta \, dx \\
&\qquad \le \liminf_{n\to+\infty} \left[\int_{\mathbb{R}^N} a_\theta(|\nabla u_n|^2)|\nabla u_n|^2 \, dx + \int_{\mathbb{R}^N} g_2(u_n)u_n \, dx \right] \\
&\qquad = \liminf_{n\to+\infty} \left[e^{(N-2)\sigma_n} \int_{\mathbb{R}^N} a_\theta(e^{-2\sigma_n}|\nabla u_n|^2)|\nabla u_n|^2 \, dx + e^{N\sigma_n} \int_{\mathbb{R}^N} g_2(u_n)u_n \, dx \right] \\
&\qquad = \liminf_{n\to+\infty} \left[e^{N\sigma_n} \int_{\mathbb{R}^N} g_1(u_n)u_n \, dx + o_n(1)\|u_n\| \right] \\
&\qquad = \int_{\mathbb{R}^N} g_1(u_\theta)u_\theta \, dx = \int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2 \, dx + \int_{\mathbb{R}^N} g_2(u_\theta)u_\theta \, dx
\end{aligned}$$

and so

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2 \, dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} a_\theta(|\nabla u_n|^2)|\nabla u_n|^2 \, dx, \tag{3.15}$$

$$\int_{\mathbb{R}^N} g_2(u_\theta)u_\theta \, dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} g_2(u_n)u_n \, dx. \tag{3.16}$$

In view of Lemma 2.3 equation (3.15) implies that $u_n \to u_\theta$ strongly in $\mathcal{X}_0$.
Moreover, since, by (3.2), we know that for any $s \in \mathbb{R}$ we can write $g_2(s)s = m|s|^\gamma + h(s)$, where h is a non-negative continuous function, by Fatou's Lemma we deduce that

$$\int_{\mathbb{R}^N} |u_\theta|^\gamma \, dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^\gamma \, dx,$$

$$\int_{\mathbb{R}^N} h(u_\theta) \, dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^N} h(u_n) \, dx.$$

These last two inequalities and (3.16) imply that

$$\int_{\mathbb{R}^N} |u_\theta|^\gamma \, dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^\gamma \, dx$$

and so, actually, $u_n \to u_\theta$ strongly in $\mathcal{X}$ and so $I_\theta(u_\theta) = m_\theta$.
Finally, since

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2 \, dx = N m_\theta,$$

by (3.11) and (2.4), we prove that there exists $C > 0$ such that $\|u_\theta\|_0 \le C$, for any $\theta \in (0, \theta_1]$. □

We are now able to conclude the proof of our main theorem in the positive mass case.

Proof of Theorem 1.1. By Proposition 3.7, for any $\theta \in (0, \theta_1]$, there exists $u_\theta \in \mathcal{X}$ a nontrivial solution of (2.2) such $I_\theta(u_\theta) = m_\theta$. Since $q > N$, by [29], we deduce that $u_\theta \in \mathcal{C}^{1,\alpha}$, for some $\alpha \in (0, 1)$.
Let us prove the following

Claim: there exists $C > 0$ such that

$$|a_\theta(|u_\theta'(r)|^2)u_\theta'(r)| \le C, \qquad \text{for any } r \ge 0 \text{ and } \theta \in (0, \theta_1]. \tag{3.17}$$

By the regularity of u_θ, we infer that $u_\theta'(0)=0$ and so also

$$a_\theta(|u_\theta'(0)|^2)u_\theta'(0)=0.$$

We now consider the case $r>0$. Integrating the equation (3.13), for any $r>0$, we have

$$-a_\theta(|u_\theta'(r)|^2)u_\theta'(r)=\frac{1}{r^{N-1}}\int_0^r s^{N-1}g(u_\theta(s))\,ds.$$

By Lemma 2.1 and by (3.12), we deduce that there exists $R>1$, such that

$$|u_\theta(r)|\le \bar c_2, \quad \text{for any } \theta\in(0,\theta_1] \text{ and for any } r>R, \tag{3.18}$$

where $\bar c_2$ is defined in (3.4).
By the continuous embedding of $\mathcal{X}_0$ in $L^\infty(\mathbb{R}^N)$ and (3.12), there exists $C>0$ such that $|u_\theta|_\infty\le C\|u_\theta\|_0\le C$, for any $\theta\in(0,\theta_1]$, and so we have that, for any $0<r\le R$ and $\theta\in(0,\theta_1]$,

$$|a_\theta(|u_\theta'(r)|^2)u_\theta'(r)|\le\frac{1}{r^{N-1}}\int_0^r s^{N-1}|g(u_\theta(s))|\,ds\le C.$$

While, for any $r>R$,

$$\begin{aligned}|a_\theta(|u_\theta'(r)|^2)u_\theta'(r)| &\le \frac{1}{r^{N-1}}\int_0^r s^{N-1}|g(u_\theta(s))|\,ds\\ &\le \frac{1}{r^{N-1}}\left(\int_0^R s^{N-1}|g(u_\theta(s))|\,ds+\int_R^r s^{N-1}|g(u_\theta(s))|\,ds\right)\\ &\le \frac{C}{r^{N-1}}+\underbrace{\frac{c_1}{r^{N-1}}\int_1^r s^{N-1}|g(u_\theta(s))|\,ds}_{(A)}.\end{aligned}$$

We have to estimate (A). First of all, by Lemma 2.1 and (3.12), for $r>1$, we have that

$$|u_\theta(r)|\le Cr^{-\frac{N-2}{2}}|\nabla u_\theta|_2\le \bar C r^{-\frac{N-2}{2}}.$$

Hence, by (3.18) and (3.4), since $\gamma\ge 2^*/2$,

$$(A)\le\frac{C}{r^{N-1}}\int_1^r s^{N-1}|u_\theta(s)|^{\gamma-1}\,ds\le\frac{C}{r^{N-1}}\int_1^r s^{N-1-\frac{N-2}{2}(\gamma-1)}\,ds\le C\left(r^{1-\frac{N-2}{2}(\gamma-1)}+1\right)\le C.$$

Therefore the claim is proved.
Now we conclude if we show the existence of $\bar\theta\in(0,\theta_1]$ such that

$$|u_{\bar\theta}'(r)|\le 1-\bar\theta, \qquad \text{for any } r\ge 0. \tag{3.19}$$

Suppose by contradiction that (3.19) does not hold, then there exists a sequence $\{\theta_n\}\subset(0,\theta_1]$ which tends to zero and a sequence $\{r_n\}\subset\mathbb{R}_+$ such that

$$\lim_n |u_{\theta_n}'(r_n)|=1,$$

which implies that (by (a1))

$$\lim_n a_{\theta_n}(|u_{\theta_n}'(r_n)|)|u_{\theta_n}'(r_n)|=+\infty.$$

Thus we obtain a contradiction with (3.17).
Finally, taking into account (2.4), (2.5) and Lemma 3.1, we get

$$\int_{\mathbb{R}^N}A(|\nabla u_{\bar\theta}|^2)\,dx,\int_{\mathbb{R}^N}a(|\nabla u_{\bar\theta}|^2)|\nabla u_{\bar\theta}|^2\,dx,\int_{\mathbb{R}^N}|G(u_{\bar\theta})|\,dx<+\infty.$$

□

4. The zero mass case

In this section we deal with the zero mass case, namely, we will assume on g (g0), (g1$'$) and (g2). Since in several steps the arguments are close to those of the previous section, we skip some details.

Let

$$g_1(s) := \begin{cases} \max\{g(s), 0\} & \text{for } s \geq 0, \\ 0 & \text{for } s < 0. \end{cases}$$

and $g_2(s) := g_1(s) - g(s)$ for $s \geq 0$ and then we can extend them as odd functions for $s < 0$. Then $g_1(s) \geq 0$ for $s \geq 0$,

$$\lim_{s \to 0} g_1(s)/|s|^{\gamma-1} = 0, \quad \text{for all } \gamma \geq 2^*/2, \tag{4.1}$$

$$g_2(s) \geq 0, \quad \text{for } s \geq 0. \tag{4.2}$$

If we set

$$G_i(s) = \int_0^s g_i(t)\, dt \quad \text{for } i = 1, 2,$$

then by (4.2) we have

$$G_2(s) \geq 0, \quad \text{for } s \in \mathbb{R}.$$

By (g1$'$), there exist two positive constants, $\bar{c}_1$ and $\bar{c}_2$ such that

$$|g(s)| \leq \bar{c}_1 |s|^{\gamma-1}, \qquad \text{for all } |s| \leq \bar{c}_2, \tag{4.3}$$

$$|G(s)| \leq \bar{c}_1 |s|^{\gamma}, \qquad \text{for all } |s| \leq \bar{c}_2, \tag{4.4}$$

$$|g_1(s)| \leq \bar{c}_1 |s|^{\gamma-1}, \qquad \text{for all } |s| \leq \bar{c}_2, \tag{4.5}$$

$$|G_1(s)| \leq \bar{c}_1 |s|^{\gamma}, \qquad \text{for all } |s| \leq \bar{c}_2. \tag{4.6}$$

Arguing as in the proof of Lemma 3.1 and Lemma 3.2, we have

Lemma 4.1. *For any $u \in \mathcal{X}_0$, $\int_{\mathbb{R}^N} G(u)\, dx$ and $\int_{\mathbb{R}^N} g(u)u\, dx$ are well defined. The same is true for $\int_{\mathbb{R}^N} G_i(u)\, dx$ and $\int_{\mathbb{R}^N} g_i(u)u\, dx$, for $1 = 1, 2$.*

Lemma 4.2. *If $u_n \rightharpoonup u_0$ in $\mathcal{X}_0$, then*

$$\lim_n \int_{\mathbb{R}^N} g_1(u_n)u_n\, dx = \int_{\mathbb{R}^N} g_1(u_0)u_0\, dx$$

and

$$\lim_n \int_{\mathbb{R}^N} G_1(u_n)\, dx = \int_{\mathbb{R}^N} G_1(u_0)\, dx.$$

Solutions of (2.3) will be found as critical points of the functional $I_\theta : \mathcal{X}_0 \to \mathbb{R}$ defined as

$$I_\theta(u) = \frac{1}{2} \int_{\mathbb{R}^N} A_\theta(|\nabla u|^2)\, dx + \int_{\mathbb{R}^N} G_2(u)\, dx - \int_{\mathbb{R}^N} G_1(u)\, dx.$$

which is well defined in $\mathcal{X}_0$. Here and in what follows, with an abuse of notation, we use I_θ, J_θ, m_θ, $\tilde{m}_\theta$, Γ, and Σ in the zero mass setting, as well.

We show that I_θ satisfies the mountain pass geometry.

Lemma 4.3. *For any $\theta \in (0, \theta_1]$, the functional $I_\theta : \mathcal{X}_0 \to \mathbb{R}$ verifies the mountain pass geometry. More precisely:*

(i) there are $\alpha, \rho > 0$ such that $I_\theta(u) \geq \alpha$, for $\|u\| = \rho$;
(ii) there is $\bar{u} \in \mathcal{X}_0 \setminus \{0\}$, independent of $\theta \in (0, \theta_1]$, with $\|\bar{u}\| > \rho$ and $|\nabla \bar{u}| < 1 - \theta_1$, almost everywhere in $\mathbb{R}^N$, and such that $I_\theta(\bar{u}) < 0$.

Proof. (i) Let us fix $\gamma > \max\{q, 2^*\}$. By the continuous embedding of $\mathcal{X}_0$ into $L^\infty(\mathbb{R}^N)$, and by (4.4), we can consider $\rho > 0$ sufficiently small such that

$$G(u(x)) \le \bar{c}_1 |u(x)|^\gamma, \qquad \text{a.e. } x \in \mathbb{R}^N \text{ and for any } u \in \mathcal{X}_0 \text{ with } \|u\| = \rho.$$

Hence, by (2.5) and since $\mathcal{X}_0$ is embedded into $L^\gamma(\mathbb{R}^N)$, for any $u \in \mathcal{X}_0$ with $\|u\| = \rho$, we have

$$I_\theta(u) \ge c\left(|\nabla u|_2^2 + |\nabla u|_q^q - |u|_\gamma^\gamma\right) \ge c\left(|\nabla u|_2^2 + |\nabla u|_q^q - |\nabla u|_2^\gamma - |\nabla u|_q^\gamma\right) \ge \alpha > 0.$$

(ii) As in the proof of Lemma 3.3. □

Let us define the mountain pass level for the functional I_θ

$$m_\theta := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\theta(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], \mathcal{X}_0) \mid \gamma(0) = 0, \gamma(1) = \bar{u}\}.$$

By Lemma 3.3, we deduce that $m_\theta \ge \alpha$, for any $\theta \in (0, \theta_1]$.

Observe that, since $|\nabla \bar{u}| < 1 - \theta_1$, we have that $I_{\theta_1}(t\bar{u}) = I_\theta(t\bar{u})$, for any $t \in [0,1]$ and for any $\theta \in (0, \theta_1]$. Hence we deduce that

$$m_\theta \le \max_{t \in [0,1]} I_\theta(t\bar{u}) = \max_{t \in [0,1]} I_{\theta_1}(t\bar{u}),$$

for any $\theta \in (0, \theta_1]$. Hence there exists $c > 0$ (independent of $\theta \in (0, \theta_1]$) such that

$$0 < m_\theta \le c_2, \qquad \text{for any } \theta \in (0, \theta_1]. \tag{4.7}$$

As done in Section 3, we define the functional $J_\theta : \mathbb{R} \times \mathcal{X}_0 \to \mathbb{R}$ as

$$J_\theta(\sigma, u) = I_\theta(u(e^{-\sigma} \cdot)) = \frac{e^{N\sigma}}{2} \int_{\mathbb{R}^N} A_\theta(e^{-2\sigma} |\nabla u|^2)\, dx + e^{N\sigma} \int_{\mathbb{R}^N} G_2(u)\, dx - e^{N\sigma} \int_{\mathbb{R}^N} G_1(u)\, dx.$$

The functional J_θ has a mountain pass geometry and we can define its mountain pass level as

$$\tilde{m}_\theta := \inf_{(\sigma,\gamma) \in \Sigma \times \Gamma} \max_{t \in [0,1]} J_\theta\big(\sigma(t), \gamma(t)\big),$$

where

$$\Sigma := \{\sigma \in \mathcal{C}([0,1], \mathbb{R}) \mid \sigma(0) = \sigma(1) = 0\}.$$

The following holds

Lemma 4.4. *For any $\theta \in (0, \theta_1]$, the mountain pass levels of I_θ and J_θ coincide, namely $m_\theta = \tilde{m}_\theta$.*

Lemma 4.5. *Let $\theta \in (0, \theta_1]$ and $\varepsilon > 0$. Suppose that $\tilde{\gamma} \in \Sigma \times \Gamma$ satisfies*

$$\max_{t \in [0,1]} J_\theta(\tilde{\gamma}(t)) \le m_\theta + \varepsilon,$$

then there exists $(\sigma, u) \in \mathbb{R} \times \mathcal{X}_0$ such that

(1) $\mathrm{dist}_{\mathbb{R} \times \mathcal{X}_0}\big((\theta, u), \tilde{\gamma}([0,1])\big) \le 2\sqrt{\varepsilon}$;
(2) $J_\theta(\sigma, u) \in [m_\theta - \varepsilon, m_\theta + \varepsilon]$;
(3) $\|DJ_\theta(\sigma, u)\|_{\mathbb{R} \times \mathcal{X}^*} \le 2\sqrt{\varepsilon}$.

Proposition 4.6. *For any $\theta \in (0, \theta_1]$, there exists a sequence $\{(\sigma_n, u_n)\} \subset \mathbb{R} \times \mathcal{X}_0$ such that, as $n \to +\infty$, we get*

(1) $\sigma_n \to 0$;
(2) $J_\theta(\sigma_n, u_n) \to m_\theta$;
(3) $\partial_\sigma J_\theta(\sigma_n, u_n) \to 0$;
(4) $\partial_u J_\theta(\sigma_n, u_n) \to 0$ *strongly in* $\mathcal{X}_0^*$.

Proposition 4.7. *For any $\theta \in (0,\theta_1]$, there exists $u_\theta \in \mathcal{X}_0$ a non-trivial solution of (2.2) such $I_\theta(u_\theta) = m_\theta$. Moreover there exists $C > 0$ such that*

$$\|u_\theta\|_0 \le C, \qquad \textit{for any } \theta \in (0,\theta_1].$$

Finally u_θ is a weak solution of

$$-\big(r^{N-1} a_\theta(|u_\theta'(r)|^2) u_\theta'(r)\big)' = r^{N-1} g(u_\theta(r)),$$

namely

$$\int_0^{+\infty} r^{N-1} a_\theta(|u_\theta'(r)|^2) u_\theta'(r) v'(r)\, dr = \int_0^{+\infty} r^{N-1} g(u_\theta(r)) v(r)\, dr,$$

for all $v \in \mathcal{X}_0$.

Proof. Fix $\theta \in (0,\theta_1]$. By Proposition 4.6, there exists a sequence $\{(\sigma_n, u_n)\} \subset \mathbb{R} \times \mathcal{X}_0$ such that

$$\begin{cases} \dfrac{e^{N\sigma_n}}{2} \displaystyle\int_{\mathbb{R}^N} A_\theta(e^{-2\sigma_n}|\nabla u_n|^2)\, dx + e^{N\sigma_n} \int_{\mathbb{R}^N} G_2(u_n)\, dx - e^{N\sigma_n} \int_{\mathbb{R}^N} G_1(u_n)\, dx = m_\theta + o_n(1), \\ \dfrac{N e^{N\sigma_n}}{2} \displaystyle\int_{\mathbb{R}^N} A_\theta(e^{-2\sigma_n}|\nabla u_n|^2)\, dx - e^{(N-2)\sigma_n} \int_{\mathbb{R}^N} a_\theta(e^{-2\sigma_n}|\nabla u_n|^2)|\nabla u_n|^2\, dx \\ \qquad\qquad + N e^{N\sigma_n} \displaystyle\int_{\mathbb{R}^N} G_2(u_n)\, dx - N e^{N\sigma_n} \int_{\mathbb{R}^N} G_1(u_n)\, dx = o_n(1), \\ e^{(N-2)\sigma_n} \displaystyle\int_{\mathbb{R}^N} a_\theta(e^{-2\sigma_n}|\nabla u_n|^2)|\nabla u_n|^2\, dx + e^{N\sigma_n} \int_{\mathbb{R}^N} g_2(u_n) u_n\, dx - e^{N\sigma_n} \int_{\mathbb{R}^N} g_1(u_n) u_n\, dx = o_n(1)\|u_n\|. \end{cases}$$

From the first and the second equation of the previous system we get

$$e^{(N-2)\sigma_n} \int_{\mathbb{R}^N} a_\theta(e^{-2\sigma_n}|\nabla u_n|^2)|\nabla u_n|^2\, dx = N m_\theta + o_n(1).$$

Therefore, since $\sigma_n \to 0$, as $n \to +\infty$, by (2.4) we deduce that $\{u_n\}$ is a bounded sequence in $\mathcal{X}_0$ and so also in $L^\infty(\mathbb{R}^N)$, namely there exists $\bar{C} > 0$ such that $|u_n|_\infty \le \bar{C}$, for any $n \ge 1$. Then there exists $u_\theta \in \mathcal{X}_0$ such that $u_n \rightharpoonup u_\theta$ in $\mathcal{X}_0$. Since $\partial_u J_\theta(\sigma_n, u_n) \to 0$ strongly in $\mathcal{X}_0^*$ and $\sigma_n \to 0$, we have that u_θ is a weak (possibly trivial) solution of (2.3) and so it satisfies

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2\, dx + \int_{\mathbb{R}^N} g_2(u_\theta) u_\theta\, dx = \int_{\mathbb{R}^N} g_1(u_\theta) u_\theta\, dx.$$

Arguing as in proof of Proposition 3.7 we can show that

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2\, dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} a_\theta(|\nabla u_n|^2)|\nabla u_n|^2\, dx.$$

In view of Lemma 2.3, we have that $u_n \to u_\theta$ strongly in $\mathcal{X}_0$ and so $I_\theta(u_\theta) = m_\theta$.
Finally, since

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2\, dx = N m_\theta,$$

by (4.7) and (2.4), we prove that there exists $C > 0$ such that $\|u_\theta\|_0 \le C$, for any $\theta \in (0,\theta_1]$. □

We are now able to conclude the proof of Theorem 1.1 repeating the arguments of Section 3.

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