The three-loop unpolarized and polarized non-singlet anomalous dimensions from off shell operator matrix elements

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Abstract

We calculate the unpolarized and polarized three–loop anomalous dimensions and splitting functions P_{NS}^+, P_{NS}^- and P_{NS}^{s} in QCD in the $\overline{\text{MS}}$ scheme by using the traditional method of space–like off shell massless operator matrix elements. This is a gauge–dependent framework. For the first time we also calculate the three–loop anomalous dimensions $P_{\text{NS}}^{\pm,\text{tr}}$ for transversity directly. We compare our results to the literature.
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1. Introduction

The anomalous dimensions of local quark and gluon operators determine the scaling violations of the deep–inelastic scattering structure functions [1,2] by the scale evolution of the parton densities and are therefore instrumental in the measurement of the strong coupling constant $a(M_Z^2) = \alpha_s(M_Z^2)/(4\pi)$ [3] for this inclusive precision data. They have been calculated to 3–loop order both in the unpolarized and polarized case [4–7] using the method of on–shell for-

ward Compton amplitudes, in which the scale is set by the virtuality $Q^2 = -q^2$ of the exchanged current. At four–loop order a series of low moments for the non–singlet anomalous dimensions has been calculated in Refs. [8] and at five–loop order in [9]. The $O(T_F)$ contributions at three–loop order have been confirmed by the calculation of massive on-shell operator matrix elements (OMEs) [10–14]. The 3–loop unpolarized anomalous dimensions $\gamma_{ij}^{(2)}$ have been obtained in implicit form also by calculating the inclusive hadronic Higgs-boson production process $gg \to H^0$ and the Drell–Yan process at N^{3}LO in [15].

The traditional way of calculating the anomalous dimensions consists in computing the off shell massless local OMEs, cf. [16–20] in the one–loop case, which in general implies the breaking of gauge invariance to be dealt with. The two–loop anomalous dimensions have been calculated in [10–14,21–36].

In this paper we are calculating the unpolarized and polarized three–loop anomalous dimensions for the first time using the method of massless off shell OMEs in the flavor non–singlet case, which is the first recalculation in the polarized case γ_{NS}^- and for the anomalous dimension γ_{NS}^s. The present calculation requires the knowledge of the corresponding massless off shell OMEs to two–loop order, cf. [33,34,37], up to the terms of $O(\varepsilon^0)$ in the dimensional parameter $\varepsilon = D - 4$. The off shell OMEs are gauge–dependent quantities. We will calculate the anomalous dimensions and splitting functions: P_{NS}^+, P_{NS}^- and P_{NS}^s. For the first time we also calculate the three–loop anomalous dimension $P_{NS}^{\pm,tr}$ for transversity in a direct way.

The paper is organized as follows. In Section 2 we derive the structure of the physical part of the flavor non–singlet unrenormalized off shell OMEs to three–loop order. From their pole terms of $O(1/\varepsilon)$ one can extract the non–singlet anomalous dimensions. Due to a known Ward identity, cf. e.g. [10,14], the polarized anomalous dimension can be calculated by applying anticommuting γ_5. We also calculate the polarized OMEs in the Larin scheme [34,38] from which one can determine the Z–factor $Z_5^{NS}(N)$ of the corresponding finite renormalization to three–loop order. The details of the calculation are described in Section 3. In Section 4 we present the three–loop anomalous dimensions and splitting functions. We compare with results in the literature in Section 5 and Section 6 contains the conclusions. In an appendix we briefly summarize the transition from the Larin to the $\overline{\text{MS}}$ scheme for the polarized anomalous dimension in the vector case.

2. The unrenormalized operator matrix elements

The massless off shell non–singlet OMEs are defined as expectation values of the local operators

$$O_{q,r;\mu_1...\mu_N}^{NS} = i^{N-1} \mathbf{S} \left[\overline{\psi} \gamma_{\mu_1} D_{\mu_2}...D_{\mu_N} \frac{\lambda_r}{2} \psi \right] - \text{ trace terms}, \tag{1}$$

$$O_{q,r;\mu_1...\mu_N}^{NS,5} = i^{N-1} \mathbf{S} \left[\overline{\psi} \gamma_5 \gamma_{\mu_1} D_{\mu_2}...D_{\mu_N} \frac{\lambda_r}{2} \psi \right] - \text{ trace terms} \tag{2}$$

between quark (antiquark) states ψ ($\overline{\psi}$) of space–like momentum p, $p^2 < 0$, and are given by

$$\hat{A}_{qq}^{NS,(5)} = \langle q(p)|O^{NS,(5)}|q(p)\rangle. \tag{3}$$

Here $\mathbf{S}$ is the symmetry operator, λ_r a $SU(N_F)$ flavor matrix and $D_\mu = \partial_\mu + ig_s t_a A_\mu^a$ the covariant derivative, with A_μ^a the gluon field, ψ the quark field, t_a the generators of $SU(N_C)$, and

$g_s = \sqrt{4\pi\alpha_s}$. The Feynman rules of QCD are given in [39] and for the local operators in [14,40]. The local operator in the case of transversity is given by

$$O^{\text{NS,tr}}_{q,r;\mu\mu_1...\mu_N} = i^{N-1}\mathbf{S}\left[\overline{\psi}\sigma_{\mu\mu_1}D_{\mu_2}...D_{\mu_N}\frac{\lambda_r}{2}\psi\right] - \text{trace terms},\tag{4}$$

where $\sigma_{\mu\nu} = (i/2)[\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu]$.

The operator matrix elements have the representation

$$\hat{A}^{\text{NS}}_{qq} = \left[\slashed{\Delta}\hat{A}^{\text{NS,phys}}_{qq} + \slashed{p}\frac{\Delta.p}{p^2}\hat{A}^{\text{NS,EOM}}_{qq}\right](\Delta.p)^{N-1}.\tag{5}$$

Here Δ denotes a light–like vector, $\Delta.\Delta = 0$. The following projectors are applied to separate the physical (phys) contribution and the one vanishing by the equation of motion (EOM), which does not hold in the off shell case,

$$\hat{A}^{\text{NS,phys}}_{qq} = \frac{1}{4(\Delta.p)^N}\text{tr}\left[\left(\slashed{p} - \frac{p^2}{\Delta.p}\slashed{\Delta}\right)\hat{A}^{\text{NS}}_{qq}\right],\tag{6}$$

$$\hat{A}^{\text{NS,EOM}}_{qq} = \frac{1}{4(\Delta.p)^N}\text{tr}\left[\slashed{\Delta}\hat{A}^{\text{NS}}_{qq}\right].\tag{7}$$

In the polarized case the operator (1) is replaced by the operator of Eq. (2) and the following relations hold

$$\hat{A}^{\text{NS,5}}_{qq} = \left[\gamma_5\slashed{\Delta}\hat{A}^{\text{NS,5,phys}}_{qq} + \gamma_5\slashed{p}\frac{\Delta.p}{p^2}\hat{A}^{\text{NS,5,EOM}}_{qq}\right](\Delta.p)^{N-1}\tag{8}$$

with

$$\hat{A}^{\text{NS,5,phys}}_{qq} = \frac{1}{4(\Delta.p)^N}\text{tr}\left[\left(\slashed{p} - \frac{p^2}{\Delta.p}\slashed{\Delta}\right)\gamma_5\hat{A}^{\text{NS,5}}_{qq}\right],\tag{9}$$

$$\hat{A}^{\text{NS,5,EOM}}_{qq} = \frac{1}{4(\Delta.p)^N}\text{tr}\left[\slashed{\Delta}\gamma_5\hat{A}^{\text{NS,5}}_{qq}\right].\tag{10}$$

In the case of transversity we consider the unrenormalized Green's function [41]

$$\hat{G}^{ij,\text{NS,tr}}_{\mu,qQ} = \delta_{ij}(\Delta.p)^{N-1}\left[\Delta_\rho\sigma^{\mu\rho}\Delta_T\hat{A}^{\text{NS,phys}}_{qq}\left(\frac{-p^2}{\mu^2},\varepsilon,N\right) + c_1\Delta^\mu + c_2 p^\mu\right.$$

$$\left. + c_3\gamma_\mu\slashed{p} + c_4\slashed{\Delta}\slashed{p}\Delta^\mu + c_5\slashed{\Delta}\slashed{p}p^\mu\right],\tag{11}$$

where i, j are external color indices and the coefficients $c_k|_{k=1...5}$ denote other OMEs than those we are going to deal with.

Since the non–singlet anomalous dimensions receive only contributions from the unrenormalized OME $\hat{A}^{\text{NS,(5),phys}}_{qq}$ we will consider only this operator matrix element in the following. In Mellin N space it has the representation

$$\hat{A}^{\text{NS,(5)}}_{qq} = 1 + \sum_{k=1}^{\infty}\hat{a}^k S^k_\varepsilon\left(\frac{-p^2}{\mu^2}\right)^{\varepsilon k/2}\hat{A}^{(k),\text{NS,(5)}}_{qq},\tag{12}$$

with the spherical factor

$$S_\varepsilon = \exp\left[\frac{\varepsilon}{2}(\gamma_E - \ln(4\pi))\right], \tag{13}$$

where γ_E is the Euler–Mascheroni number and $\hat{a}$ the bare coupling constant. The free gluon propagator is given by[1]

$$D_{\mu\nu}^{ab}(k) = \frac{\delta^{ab}}{k^2 + i0}\left[-g_{\mu\nu} + (1 - \hat{\xi})\frac{k_\mu k_\nu}{k^2 + i0}\right], \tag{14}$$

which defines the gauge parameter in the $R_{\hat{\xi}}$ gauge. The renormalization of the massive off shell non–singlet OMEs encounters the renormalization of the coupling constant and the gauge parameter, as well as that of the local operator. In the following we will deviate from Refs. [33,34] and perform the renormalization of the coupling constant and the gauge parameter and use the resulting expression, $\tilde{A}$, at $\mu^2 = -p^2$ to extract the anomalous dimensions. In the unrenormalized OME obtained in the diagrammatic calculation the coupling constant and the gauge parameter are renormalized before comparing to $\tilde{A}$ in Eq. (25). The unrenormalized coupling is given by

$$\hat{a} = a\left[1 + \frac{2}{\varepsilon}\beta_0 a + \left(\frac{4}{\varepsilon^2}\beta_0^2 + \frac{1}{\varepsilon}\beta_1\right)a^2\right] + O(a^3), \tag{15}$$

where a denotes the renormalized strong coupling constant. The expansion coefficients of the QCD β–function are given by [42][2]

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F N_F, \tag{16}$$

$$\beta_1 = \frac{34}{3}C_A^2 - \frac{20}{3}C_A T_F N_F - 4C_F T_F N_F. \tag{17}$$

The bare gauge parameter $\hat{\xi}$ is renormalized by

$$\hat{\xi} = \xi\, Z_3(\xi), \tag{18}$$

where Z_3 is the Z–factor of the gluon propagator, cf. [43–46],

$$Z_3(\xi) = 1 + a\frac{z_{11}}{\varepsilon} + a^2\left[\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon}\right] + O(a^3), \tag{19}$$

with

$$z_{11} = C_A\left[-\frac{13}{3} + \xi\right] + \frac{8}{3}T_F N_F, \tag{20}$$

$$z_{22} = C_A^2\left[-\frac{13}{2} - \frac{17}{6}\xi + \xi^2\right] + C_A T_F N_F\left[4 + \frac{8}{3}\xi\right], \tag{21}$$

$$z_{21} = C_A^2\left[-\frac{59}{8} + \frac{11}{8}\xi + \frac{1}{4}\xi^2\right] + 4\, C_F T_F N_F + 5\, C_A T_F N_F. \tag{22}$$

The color factors are $C_F = (N_C^2 - 1)/(2N_C)$, $C_A = N_C$, $T_F = 1/2$ for $SU(N_C)$ and $N_C = 3$ for QCD; N_F denotes the number of massless quark flavors.

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[1] Note a typo in [33], Eq. (2.6).
[2] Note a typographical error in [33], Eq. (2.13) and [34], Eq. (2.14).

In Mellin N space the Z-factor of a local non–singlet operator reads [40]

$$
Z^{\text{NS}} = 1 + a \frac{\gamma_{qq}^{(0),\text{NS}}}{\varepsilon} + a^2 \left[\frac{1}{\varepsilon^2} \left(\frac{1}{2} \gamma_{qq}^{(0),\text{NS}^2} + \beta_0 \gamma_{qq}^{(0),\text{NS}} \right) + \frac{1}{2\varepsilon} \gamma_{qq}^{(0),\text{NS}} \right]
$$

$$
+ a^3 \left[\frac{1}{\varepsilon^3} \left(\frac{1}{6} \gamma_{qq}^{(0),\text{NS}^3} + \beta_0 \gamma_{qq}^{(0),\text{NS}^2} + \frac{4}{3} \beta_0^2 \gamma_{qq}^{(0),\text{NS}} \right) + \frac{1}{\varepsilon^2} \left(\frac{1}{2} \gamma_{qq}^{(0),\text{NS}} \gamma_{qq}^{(1),\text{NS}} \right. \right.
$$

$$
\left. \left. + \frac{2}{3} \beta_0 \gamma_{qq}^{(1),\text{NS}} + \frac{2}{3} \beta_1 \gamma_{qq}^{(0),\text{NS}} \right) + \frac{1}{3\varepsilon} \gamma_{qq}^{(2),\text{NS}} \right]. \tag{23}
$$

In (23) the terms $\gamma_{qq}^{(k),\text{NS}}$, $k = 0, 1, 2, \dots$ denote the expansion coefficients of the anomalous dimension

$$
\gamma_{\text{NS}} = \sum_{k=1}^{\infty} a^k \gamma_{\text{NS}}^{(k-1)}. \tag{24}
$$

The partly renormalized OME, $\tilde{A}_{qq}^{\text{NS,phys}}$, reads

$$
\tilde{A}_{qq}^{\text{NS,phys}} = 1 + a \left[\frac{a_{qq}^{\text{NS},(1,-1)}}{\varepsilon} + a_{qq}^{\text{NS},(1,0)} + a_{qq}^{\text{NS},(1,1)} \varepsilon \right]
$$

$$
+ a^2 \left[\frac{a_{qq}^{\text{NS},(2,-2)}}{\varepsilon^2} + \frac{a_{qq}^{\text{NS},(2,-1)}}{\varepsilon} + a_{qq}^{\text{NS},(2,0)} \right]
$$

$$
+ a^3 \left[\frac{a_{qq}^{\text{NS},(3,-3)}}{\varepsilon^3} + \frac{a_{qq}^{\text{NS},(3,-2)}}{\varepsilon^2} + \frac{a_{qq}^{\text{NS},(3,-1)}}{\varepsilon} \right]. \tag{25}
$$

The expansion coefficients $a_{qq}^{\text{NS},(i,j)}$ are in general gauge dependent. The renormalized OMEs are given by

$$
A_{qq}^{\text{NS,phys}} = \frac{\tilde{A}_{qq}^{\text{NS,phys}}}{Z^{\text{NS}}}, \tag{26}
$$

expanded to $O(a^3)$ and setting $S_\varepsilon = 1$. The anomalous dimensions are iteratively extracted form the $1/\varepsilon$ pole terms and the other expansion coefficients $a_{qq}^{\text{NS},(i,j)}$ are given in Ref. [37].

Eq. (25) is understood to hold both for the unpolarized as well as the polarized case, by relabeling the corresponding quantities to $f \to \Delta f$. Similar expressions hold for transversity. From them we will determine $\gamma_{\text{NS}}^{(2)}$ and $\Delta \gamma_{\text{NS}}^{(2)}$ in both cases. The further three–loop non–singlet anomalous dimensions $\gamma_{\text{NS}}^{(2),\text{s}}$ can be derived from other quarkonic diagrams at three–loop order.[3] Because $\gamma_{\text{NS}}^{(2),\text{s}}$ occurs for the first time at the three–loop loop level, there is no renormalization of the OME

$$
\frac{2}{1+(-1)^N} \hat{A}_{qq}^{\text{PS},(3),\text{phys}}(N) \Big|_{d_{abc}d^{abc}} = \hat{a}_s^3 \frac{1}{3\varepsilon} \gamma_{\text{NS}}^{(2),\text{s}}(N) + O(\varepsilon^0), \quad N \in \mathbb{N}, \text{ odd}, N \geq 1. \tag{27}
$$

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[3] There is a further non–singlet anomalous dimension $\Delta \gamma_{\text{NS}}^{(2),\text{s}}$ [7] occurring in the pole–term of an axialvector–vector current interference contribution in the forward Compton amplitude, related to the polarized structure function g_5, introduced in Ref. [47], which has *even* moments. These aspects are of importance but are not discussed in [48]. We will consider this quantity elsewhere.

The other expansion coefficients occurring in (25) potentially coming from lower orders in the coupling a do all vanish in this case. The anomalous dimension $\gamma_{\mathrm{NS}}^{(2)\mathrm{s}}$ is formally obtained as the $O(1/\varepsilon)$ pole term of the pure–singlet OME by considering in the unpolarized case the analytic continuation from odd values of N. The $d_{abc}d^{abc}$ terms in the non–singlet $+$ contributions vanish. One considers the contributions $\propto d_{abc}d^{abc}$ of this OME for the *odd* moments. In this way $\gamma_{\mathrm{NS}}^{(2),\mathrm{s}}$ corresponds to the non–singlet combination $\gamma_{\mathrm{qq}}^{(2),\mathrm{s}} - \gamma_{\mathrm{q\bar q}}^{(2),\mathrm{s}}$.

In deep–inelastic scattering one may form up to three different combinations of quark distributions in the unpolarized and polarized case

$$q_{\mathrm{NS},ik}^{+} = q_i + \bar q_i - (q_k + \bar q_k), \tag{28}$$

$$q_{\mathrm{NS},ik}^{-} = q_i - \bar q_i - (q_k - \bar q_k), \tag{29}$$

$$q_{\mathrm{NS}}^{\mathrm{v}} = \sum_{k=1}^{N_F} (q_k - \bar q_k) \tag{30}$$

and analogously for $(q_k, \bar q_k) \to (\Delta q_k, \Delta \bar q_k)$. Here i, k denote the different flavors. These combinations can be obtained by combining the scattering cross sections for different neutral and charged current exchanges off proton and neutron targets.[4] The corresponding anomalous dimensions ruling the evolution of these non–singlet distributions are γ_{NS}^{+}, γ_{NS}^{-} and $\gamma_{\mathrm{NS}}^{\mathrm{v}} = \gamma_{\mathrm{NS}}^{-} + \gamma_{\mathrm{NS}}^{\mathrm{s}}$. In the polarized case mostly pure virtual photon exchange has been studied experimentally, which is described by the structure functions $g_{1,2}(x, Q^2)$. Their non–singlet contributions evolve with $\Delta\gamma_{\mathrm{NS}}^{+}$. The following relations hold

$$\Delta\gamma_{\mathrm{NS}}^{+} = \gamma_{\mathrm{NS}}^{-}, \tag{31}$$

$$\Delta\gamma_{\mathrm{NS}}^{-} = \gamma_{\mathrm{NS}}^{+}. \tag{32}$$

Let us finally describe the differences of the present method to calculate the anomalous dimensions to the one used in Refs. [4–7]. In order for massless calculations to lead to non-vanishing results when using dimensional regularization at least one non–vanishing mass scale is required. For massless (forward) OMEs this is the off-shellness of the external partons, which in general leads to gauge–variant expressions, as has been outlined above. The anomalous dimensions also appear in the pole terms of the massless forward Compton amplitude used in [4–7]. It is the sub–system cross section $V^*(q) + p(p) \to V^*(q) + p(p)$ of a virtual electro–weak gauge boson scattering with momentum q and space–like virtuality $q^2 = -Q^2$ off a parton p with momentum p and $p^2 = 0$. This cross section is process–dependent, unlike the off–shell massless OMEs, which are process–independent. The sub–process $V^*(q) + p(p) \to V^*(q) + p(p)$ can be associated to the partonic hadronic tensors and after suitable projections to structure functions. The hadronic tensor $W_{\mu\nu}$ is related to the forward Compton amplitude $T_{\mu\nu}$ by

$$W_{\mu\nu}(p, q) = \frac{1}{2\pi}\mathrm{Im}T_{\mu\nu}(p, q). \tag{33}$$

We follow [50]. The structure functions are given by

$$F_i(x, Q^2) = \sum_{l=q,g} \left(C_{i,l}\left(\alpha_s(\mu_r^2), \frac{Q^2}{\mu_f^2}, \frac{\mu_f^2}{\mu_r^2}\right) \otimes f_l\left(\alpha_s(\mu_r^2), \frac{\mu_f^2}{\mu^2}, \frac{\mu_f^2}{\mu_r^2}\right) \right)(x), \tag{34}$$

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[4] In the case of deuteron or He^3 targets nuclear wave function corrections have to be applied. Heavier nuclear targets have quite a variety of different corrections, known as EMC effect [49].

where $\mu_{f,r}$ are the renormalization and factorization scale and μ denotes a hadronic reference scale at which the non-perturbative partonic distributions, f_l, are defined and $\otimes$ denotes the Mellin convolution

$$[A(z) \otimes B(z)](x) = \int\limits_0^1 dz_1 \int\limits_0^1 dz_2 \delta(x - z_1 z_2) A(z_1) B(z_2). \tag{35}$$

The renormalized parton density is defined as

$$f_l\left(z, \alpha_s(\mu_r^2), \frac{\mu_f^2}{\mu^2}, \frac{\mu_f^2}{\mu_r^2}\right) = \sum_{k=q,g} \left(\Gamma_{lk}\left(\alpha_s(\mu_r^2), \frac{\mu_f^2}{\mu^2}, \frac{\mu_f^2}{\mu_r^2}, \varepsilon\right) \otimes \hat{f}_k\right)(z). \tag{36}$$

The renormalization group equation [51] of the parton densities follow from the one presented for the transition functions Γ_{lk}. The latter takes the following form

$$\left(\left[\left\{\mu_f^2 \frac{\partial}{\partial \mu_f^2} + \beta(a_s(\mu_f^2)) \frac{\partial}{\partial a_s(\mu_f^2)}\right\} \mathbf{1} \delta_{lm} - \frac{1}{2} P_{lm}(a_s(\mu_f^2), \varepsilon)\right]\right.$$
$$\left.\otimes \Gamma_{mk}\left(a_s(\mu_f^2), \frac{\mu_f^2}{\mu^2}, 1, \varepsilon\right)\right)(z) = 0,$$

$$a_s(\mu_f^2) \equiv \frac{\alpha_s(\mu_f^2)}{4\pi} \quad , \quad \mathbf{1} = \delta(1-z), \tag{37}$$

where we have set $\mu_r = \mu_f$ for simplicity. The functions $P_{ij}\left(a_s, \varepsilon, z\right)$ appearing in the above equation are the splitting functions. The same equation as in Eq. (37) also applies to the parton densities because of the definition in Eq. (36). The scale dependence of the coefficient function in Eq. (34) is given by

$$\left(\left[\left\{\mu_f^2 \frac{\partial}{\partial \mu_f^2} + \beta(a_s(\mu_f^2)) \frac{\partial}{\partial a_s(\mu_f^2)}\right\} \mathbf{1} \delta_{lm} + \frac{1}{2} P_{lm}(a_s(\mu_f^2), \varepsilon)\right]\right.$$
$$\left.\otimes C_{i,m}\left(a_s(\mu_f^2), \frac{Q^2}{\mu_f^2}, 1\right)\right)(z) = 0. \tag{38}$$

This renormalization group equation implies, that the pole terms of the unrenormalized functions $\hat{C}_{i,m}$ depend on the anomalous dimensions. At k–loop order the $1/\varepsilon$ term contains the anomalous dimension $\gamma_{qq}^{\text{NS},(k-1)}$ in the flavor non–singlet case. It can be extracted by expanding (33) in powers of $1/x$, where x denotes the Bjorken variable $x = Q^2/(2p.q)$, cf. also [1,47], with

$$\text{Im} T_{\mu\nu}^j \sim \sum_{N=N_0}^{\infty} p_{\mu\nu}^j \left(\frac{1}{2x}\right)^N C^j(N), \tag{39}$$

cf. [35], where j denotes the respective part of the forward Compton amplitude, $p_{\mu\nu}^j$ the associated tensor structure, and N_0 is implied by the crossing relations [1,47]. Depending on the kind of the amplitude, the values of N run over the even or the odd integers. It is now easy to see that the Mellin moments $C^j(N)$ can be obtained by the residue theorem directly and the pole terms of $O(1/\varepsilon)$ allow to extract the anomalous dimensions of the desired loop order. The method has

the advantage that the expansion coefficient of $O(\varepsilon^0)$ leads to the moments of the Wilson coefficients. The advantage of the method based on the off–shell massless OMEs in the singlet case consists in the direct access to all anomalous dimensions, while using the method based on the forward Compton amplitude requires (artificial) Higgs and graviton couplings to compute the gluonic anomalous dimensions $\gamma_{gi}^{(k)}$ in the unpolarized and polarized cases, respectively.

3. Details of the calculation

The Feynman diagrams for the massless off shell OMEs are generated by QGRAF [40,52] and the Dirac and Lorentz algebra is performed by FORM [53]. The color algebra is performed by using Color [54]. The local operators are resummed into propagators by observing the current crossing relations, cf. [1,47], as has been described in Ref. [37], in the corresponding OMEs $\hat{A}_{qq}^{\text{NS},(5)}$ for even or odd moments, which will depend on the resummation variable t quadratically only. To calculate the anomalous dimension $\gamma_{\text{NS}}^{(2),s}$ we resum first, using the variable t itself.

In the flavor non–singlet case 684 irreducible diagrams contribute. The reducible diagrams are accounted for by wave–function renormalization [44–46], decorating the OMEs at lower order in the coupling constant [33,34,37]. The different local operator insertions are resummed using generating functions of the type

$$\sum_{N=0}^{\infty} (\Delta.k)^N \left(t^N \pm (-t)^N \right) \to \left[\frac{1}{1 - \Delta.k\, t} \pm \frac{1}{1 + \Delta.k\, t} \right], \tag{40}$$

where t denotes an auxiliary parameter for the resummation of the formal Taylor series, see [55]. Eq. (40) implements the corresponding current crossing relations in the unpolarized (+) and the polarized case (−) [1,47], which is not just a formality. Only the moments contributing to the respective cases exist.[5]

In the calculation of the one– and two–loop contributions we also used the package EvaluateMultiSums [58] and also applied LiteRed [59] for some checks, cf. [37]. The irreducible three–loop diagrams are reduced to 252 master integrals using the code Crusher [60] by applying the integration–by–parts relations [61,62]. Relations between a small number of t-dependent master integrals are difficult to prove analytically for general values of D. However, they can be proven for the whole finite range of Mellin N and ε used in the present analysis by the method of arbitrary large moments [63]. For the calculation of the necessary initial values for the difference equations we use the results given in [62,64].

The method of arbitrary large moments implemented within the package SolveCoupledSystem [65] is also used to generate a large number of moments for the massless OMEs. By using the method of guessing [66,67] and its implementation in Sage [68,69] we determine the difference equations, which correspond to the different color and multiple zeta value factors [70]. To calculate $\gamma_{\text{NS}}^{(2),\pm}$ we generate 3000 even resp. odd moments and for $\gamma_{\text{NS}}^{(2),s}$ 500 moments. It turns out that the determination of the largest recurrence requires 1537 moments for $\gamma_{\text{NS}}^{(2),+}$, 1568 moments for $\gamma_{\text{NS}}^{(2),-}$, 1104 moments for $\gamma_{\text{NS}}^{(2)+,\text{tr}}$ and for $\gamma_{\text{NS}}^{(2)-,\text{tr}}$, and 348 moments for

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[5] This representation is sometimes misinterpreted. A prominent example is the Burkhardt-Cottingham sumrule. The fact that the 0th moment does not occur in the Mellin moment decomposition of the polarized structure function $g_1(x, Q^2)$ does not mean that the associated integral vanishes as a consequence of the light cone expansion. In fact, the proof of the Burkhardt-Cottingham sumrule needs quite different techniques [56,57].

$\gamma_{\text{NS}}^{(2),s}$. The difference equations are solved by using methods from difference field theory [71] implemented in the package Sigma [72,73] utilizing functions from HarmonicSums [74–81], to obtain the three–loop anomalous dimensions. The largest difference equation contributing has order o = 16 and degree d = 304. Comparing to the reconstruction of the anomalous dimensions out of their moments performed in Ref. [67] the largest difference equation had order o = 16 and degree d = 192, requiring 1079 moments. The overall computation time using the automated chain of codes described amounted to about 20 days of CPU time on Intel(R) Xeon(R) CPU E5-2643 v4 processors. In the present calculation we kept only one power in the gauge parameter $\hat{\xi}$ to check the renormalization, which has been sufficient to compute the non–singlet anomalous dimensions. In calculating the complete OMEs, no gauge–dependent contribution can be neglected.

The anomalous dimensions, γ_{NS}, can be expressed by harmonic sums [74,75]

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_{\emptyset} = 1, \quad b, a_i \in \mathbb{Z}\backslash\{0\}, N \in \mathbb{N}\backslash\{0\}. \tag{41}$$

Their Mellin inversion to the splitting functions $P_{qq}(z)$

$$\gamma_{qq}(N) = -\int_0^1 dz z^{N-1} P_{qq}(z) \tag{42}$$

can be performed using routines of the packages HarmonicSums and is expressed in terms of harmonic polylogarithms [76] given by

$$H_{b,\vec{a}}(z) = \int_0^z dx f_b(x) H_{\vec{a}}(x), \quad H_{\emptyset} = 1, \quad b, a_i \in \{-1, 0, 1\}, \tag{43}$$

with the alphabet of letters

$$\mathfrak{A}_{\text{H}} = \left\{ f_0(z) = \frac{1}{z}, \quad f_{-1}(z) = \frac{1}{1+z}, \quad f_1(z) = \frac{1}{1-z} \right\}. \tag{44}$$

In z–space one usually distinguishes three contributions to the individual splitting functions, because of their different treatment in Mellin convolutions,

$$P(z) = P^{\delta}(z) + P^{\text{plu}}(z) + P^{\text{reg}}(z), \tag{45}$$

where $P^{\delta}(z) = p_0\delta(1-z)$, $P^{\text{reg}}(z)$ is a regular function in $z \in [0,1]$ and $P^{\text{plu}}(z)$ denotes the remaining genuine +-distribution, the Mellin transformation of which is given by

$$\int_0^1 dz(z^{N-1} - 1) P^{\text{plu}}(z). \tag{46}$$

We will use this representation in Section 4.

4. The anomalous dimensions and splitting functions

In the following we use the minimal representations in terms of the contributing harmonic sums and harmonic polylogarithms by applying the algebraic relations between the harmonic sums and the harmonic polylogarithms [82]. 26 harmonic sums up to weight w = 5 contribute. Both the anomalous dimensions in the vector case, $\gamma_{\rm NS}^{(2),\pm}(N)$, and for transversity, $\gamma_{\rm NS}^{(2),\pm,\rm tr}(N)$, are sometimes written in terms of the difference $\gamma_{\rm NS}^{(2),+}(N) - \gamma_{\rm NS}^{(2),-}(N)$. This is somewhat problematic, since $\gamma_{\rm NS}^{(2),+}(N)$ is defined for positive even integers only, while $\gamma_{\rm NS}^{(2),-}(N)$ refers to positive odd integers. Later the respective analytic continuations from $N \in \mathbb{N} \to \mathbb{C}$ proceeds from the even or the odd integers [81]. We will therefore refer to the complete expressions, respectively, as long as they are written in terms of harmonic sums. Considering their Mellin inversion to z space allows then to consider the respective difference term, since the corresponding expression is free of N.

We obtain the following expressions for the non–singlet anomalous dimensions in Mellin N space, using the shorthand notation $S_{\vec{a}}(N) \equiv S_{\vec{a}}$. In the vector case they are given by

$$
\gamma_{\rm NS}^{(2),+} = \frac{1}{2}\left[1 + (-1)^N\right]
$$

$$
\times \Bigg\{ C_F^2 \Bigg\{ C_A \Bigg[\frac{72 P_3}{N^2(1+N)^2}\zeta_3 + \frac{32 P_{15}}{9N^2(1+N)^2}S_{-2,1} - \frac{16 P_{17}}{9N^2(1+N)^2}S_3 + \frac{P_{33}}{18N^4(1+N)^4}
$$

$$
+ \Bigg(-\frac{16 P_{29}}{9N^4(1+N)^4} - \frac{4288}{9}S_2 + \frac{64(-12+31N+31N^2)}{3N(1+N)}S_3 + 320 S_4 - 1024 S_{3,1}
$$

$$
+ \frac{64(-84+31N+31N^2)}{3N(1+N)}S_{-2,1} + 3712 S_{-2,2} + 3840 S_{-3,1} - 7168 S_{-2,1,1}\Bigg)S_1
$$

$$
+ \Bigg(256 S_3 + 1792 S_{-2,1}\Bigg)S_1^2 + \Bigg(\frac{4 P_{19}}{9N^2(1+N)^2} - 832 S_3 - 5248 S_{-2,1}\Bigg)S_2 + \frac{352}{3}S_2^2
$$

$$
+ \frac{16(-30+151N+151N^2)}{3N(1+N)}S_4 + \Bigg(-\frac{16 P_{22}}{9N^2(1+N)^3} + \Bigg(-\frac{64 P_9}{9N^2(1+N)^2} - 256 S_2\Bigg)S_1
$$

$$
+ \frac{32(12+31N+31N^2)S_2}{3N(1+N)} + 64 S_3 + 5376 S_{2,1} - 384 S_{-2,1} + 576\zeta_3\Bigg)S_{-2}
$$

$$
+ \Bigg(-\frac{32(8+3N+3N^2)}{N(1+N)} + 512 S_1\Bigg)S_{-2}^2 + \Bigg(\frac{32(108+31N+31N^2)}{3N(1+N)}S_1 - \frac{16 P_{16}}{9N^2(1+N)^2}
$$

$$
- 1152 S_1^2 + 2624 S_2 + 960 S_{-2}\Bigg)S_{-3} + \Bigg(\frac{16(138+35N+35N^2)}{3N(1+N)} - 1472 S_1\Bigg)S_{-4}
$$

$$
+ 2304 S_{-5} + 768 S_{2,3} + 2688 S_{2,-3} - \frac{64(-24+29N+29N^2)}{3N(1+N)}S_{3,1} - 768 S_{4,1}
$$

$$
+ \frac{32(-174+31N+31N^2)}{3N(1+N)}S_{-2,2} - 3648 S_{-2,3} - \frac{1920}{N(1+N)}S_{-3,1} + 1728 S_{-4,1}
$$

$$-5376S_{2,1,-2} + 1536S_{3,1,1} - \frac{128(-84 + 31N + 31N^2)}{3N(1+N)}S_{-2,1,1} - 1536S_{-2,1,-2}$$

$$-5376S_{-2,2,1} - 5376S_{-3,1,1} + 10752S_{-2,1,1,1}\Bigg] + T_F N_F\Bigg[-\frac{16P_5}{9N^2(1+N)^2}S_2$$

$$+\frac{4P_{34}}{9N^4(1+N)^4} + \left(-\frac{8P_{13}}{9N^2(1+N)^2} + \frac{1280}{9}S_2 - \frac{512}{3}S_3 - \frac{512}{3}S_{-2,1} + 128\zeta_3 \right)S_1$$

$$-\frac{128}{3}S_2^2 + \frac{64(12 + 29N + 29N^2)}{9N(1+N)}S_3 - \frac{512}{3}S_4 + \left(-\frac{128(-3 + 10N + 16N^2)}{9N^2(1+N)^2} \right.$$

$$+\frac{2560}{9}S_1 - \frac{256}{3}S_2 \Bigg)S_{-2} + \left(\frac{128(3 + 10N + 10N^2)}{9N(1+N)} - \frac{256}{3}S_1 \right)S_{-3}$$

$$-\frac{256}{3}S_{-4} - \frac{256(-3 + 10N + 10N^2)}{9N(1+N)}S_{-2,1}$$

$$+\frac{256}{3}S_{3,1} - \frac{256}{3}S_{-2,2} + \frac{1024}{3}S_{-2,1,1} - \frac{32(2 + 3N + 3N^2)}{N(1+N)}\zeta_3 \Bigg]\Bigg\}$$

$$+C_F\Bigg\{ T_F^2 N_F^2\Bigg[\frac{8P_{28}}{27N^3(1+N)^3} - \frac{128}{27}S_1 - \frac{640}{27}S_2 + \frac{128}{9}S_3 \Bigg] + C_A^2\Bigg[-\frac{24P_3}{N^2(1+N)^2}\zeta_3$$

$$-\frac{32P_{11}}{9N^2(1+N)^2}S_{-2,1} + \frac{8P_{18}}{9N^2(1+N)^2}S_3 + \frac{P_{32}}{54N^3(1+N)^3} + \left(\frac{4P_{35}}{3N^4(1+N)^4} \right.$$

$$-\frac{16(-8 + 11N + 11N^2)}{N(1+N)}S_3 - 256S_4 + 512S_{3,1} - \frac{64(-24 + 11N + 11N^2)}{3N(1+N)}S_{-2,1}$$

$$-1024S_{-2,2} - 1024S_{-3,1} + 2048S_{-2,1,1} \Bigg)S_1 + \left(-128S_3 - 512S_{-2,1} \right)S_1^2 + \left(-\frac{8344}{27} \right.$$

$$+384S_3 + 1536S_{-2,1} \Bigg)S_2 - \frac{16(-24 + 55N + 55N^2)}{3N(1+N)}S_4 + 64S_5 + \left(\frac{32P_{10}}{9N^2(1+N)^2}S_1 \right.$$

$$+\frac{16P_{27}}{9N^3(1+N)^3} - \frac{352}{3}S_2 - 64S_3 - 1536S_{2,1} + 128S_{-2,1} - 192\zeta_3 \Bigg)S_{-2}$$

$$+\left(\frac{48(2 + N + N^2)}{N(1+N)} - 192S_1 \right)S_{-2}^2 + \left(\frac{16P_{12}}{9N^2(1+N)^2} - \frac{32(24 + 11N + 11N^2)}{3N(1+N)}S_1 \right.$$

$$+256S_1^2 - 768S_2 - 320S_{-2} \Bigg)S_{-3} + \left(-\frac{16(30 + 13N + 13N^2)}{3N(1+N)} + 320S_1 \right)S_{-4}$$

$$-704S_{-5} - 384S_{2,3} - 768S_{2,-3} + \frac{64(-12 + 11N + 11N^2)}{3N(1+N)}S_{3,1} + 384S_{4,1}$$

$$-\frac{32(-48 + 11N + 11N^2)}{3N(1+N)}S_{-2,2} + 1088S_{-2,3} + \frac{512}{N(1+N)}S_{-3,1} - 448S_{-4,1}$$

$$+1536S_{2,1,-2} - 768S_{3,1,1} + \frac{128\left(-24 + 11N + 11N^2\right)}{3N(1+N)}S_{-2,1,1} + 512S_{-2,1,-2}$$

$$+1536[S_{-2,2,1} + S_{-3,1,1}] - 3072S_{-2,1,1,1}\Bigg] + C_A T_F N_F\Bigg[-\frac{8P_{31}}{27N^3(1+N)^3}$$

$$+\left(-\frac{16P_{30}}{27N^3(1+N)^3} + 64S_3 + \frac{256}{3}S_{-2,1} - 128\zeta_3\right)S_1 + \frac{5344}{27}S_2$$

$$-\frac{32\left(3 + 14N + 14N^2\right)}{3N(1+N)}S_3 + \frac{320}{3}S_4 + \left(-\frac{1280}{9}S_1 + \frac{64\left(-3 + 10N + 16N^2\right)}{9N^2(1+N)^2}\right.$$

$$+\frac{128}{3}S_2\Bigg)S_{-2} + \left(-\frac{64\left(3 + 10N + 10N^2\right)}{9N(1+N)} + \frac{128}{3}S_1\right)S_{-3}$$

$$+\frac{128}{3}S_{-4} - \frac{256}{3}S_{3,1} + \frac{128\left(-3 + 10N + 10N^2\right)}{9N(1+N)}S_{-2,1} + \frac{128}{3}S_{-2,2} - \frac{512}{3}S_{-2,1,1}$$

$$+\frac{32\left(2 + 3N + 3N^2\right)}{N(1+N)}\zeta_3\Bigg]\Bigg\} + C_F^3\Bigg\{-\frac{48P_3}{N^2(1+N)^2}\zeta_3 + \frac{8P_4}{N^2(1+N)^2}S_3 + \frac{P_{36}}{N^5(1+N)^5}$$

$$+\left(\frac{8P_{26}}{N^4(1+N)^4} - \frac{128(1+2N)}{N^2(1+N)^2}S_2 + 128S_2^2 - 384S_3 + 128S_4 + 512S_{3,1} - 3328S_{-2,2}\right.$$

$$-\frac{384\left(-4 + N + N^2\right)}{N(1+N)}S_{-2,1} - 3584S_{-3,1} + 6144S_{-2,1,1}\Bigg)S_1 + \left(-\frac{64\left(1 + 3N + 3N^2\right)}{N^3(1+N)^3}\right.$$

$$-1536S_{-2,1}\Bigg)S_1^2 + \left(\frac{4P_{25}}{N^3(1+N)^3} + 512S_3 + 4352S_{-2,1}\right)S_2 - \frac{32\left(2 + 3N + 3N^2\right)}{N(1+N)}S_2^2$$

$$-\frac{32\left(2 + 15N + 15N^2\right)}{N(1+N)}S_4 + \left(\frac{32P_{24}}{N^3(1+N)^3} + \left(-\frac{128\left(5 + 7N + 3N^2\right)}{N^2(1+N)^2} + 512S_2\right)S_1\right.$$

$$-\frac{64\left(4 + 3N + 3N^2\right)}{N(1+N)}S_2 + 128S_3 - 4608S_{2,1} + 256S_{-2,1} - 384\zeta_3\Bigg)S_{-2} + \left(\frac{128}{N(1+N)}\right.$$

$$-256S_1\Bigg)S_{-2}^2 + \left(\frac{32\left(8 + 5N + 9N^2\right)}{N^2(1+N)^2} - \frac{64\left(20 + 3N + 3N^2\right)}{N(1+N)}S_1 + 1280S_1^2 - 2176S_2\right.$$

$$-640S_{-2}\Bigg)S_{-3} + \left(-\frac{32\left(26 + 3N + 3N^2\right)}{N(1+N)} + 1664S_1\right)S_{-4} - 1792S_{-5} - 384S_{2,3}$$

$$-2304S_{2,-3} + \frac{128\left(-2 + 3N + 3N^2\right)}{N(1+N)}S_{3,1} + 384S_{4,1} - \frac{64\left(4 - N + 3N^2\right)}{N^2(1+N)^2}S_{-2,1}$$

$$-\frac{64\left(-26 + 3N + 3N^2\right)}{N(1+N)}S_{-2,2} + 2944S_{-2,3} + \frac{1792}{N(1+N)}S_{-3,1} - 1664S_{-4,1}$$

$$+4608S_{2,1,-2} - 768S_{3,1,1} + \frac{768\left(-4 + N + N^2\right)}{N(1+N)}S_{-2,1,1} + 1024S_{-2,1,-2}$$

$$+4608[S_{-2,2,1} + S_{-3,1,1}] - 9216 S_{-2,1,1,1}\Biggr\}\Biggr\},\tag{47}$$

$$\gamma_{\mathrm{NS}}^{(2),-} = \frac{1}{2}\Bigl[1 - (-1)^N\Bigr]$$

$$\times \Biggl\{ C_F^2 \Biggl\{ C_A \Biggl[\frac{16\bigl(-126 + 6N + 427N^2 + 770N^3 + 385N^4\bigr)}{9N^2(1+N)^2}[1 - S_3] + \frac{72\zeta_3 P_{37}}{N^2(1+N)^2}$$

$$+ \frac{32 P_{43}}{9N^2(1+N)^2} S_{-2,1} - \frac{16 P_{17}}{9N^2(1+N)^2} + \frac{P_{57}}{18N^5(1+N)^5} + \Biggl(-\frac{16 P_{50}}{9N^4(1+N)^4} - \frac{4288}{9} S_2$$

$$+ \frac{64\bigl(-12 + 31N + 31N^2\bigr)}{3N(1+N)} S_3 + 320 S_4 - 1024 S_{3,1} + \frac{64\bigl(-84 + 31N + 31N^2\bigr)}{3N(1+N)} S_{-2,1}$$

$$+ 3712 S_{-2,2} + 3840 S_{-3,1} - 7168 S_{-2,1,1}\Biggr) S_1 + \Bigl(256 S_3 + 1792 S_{-2,1}\Bigr) S_1^2$$

$$+ \Biggl(\frac{4 P_{53}}{9N^3(1+N)^3} - 832 S_3 - 5248 S_{-2,1}\Biggr) S_2 + \frac{352}{3} S_2^2 + \frac{16\bigl(-30 + 151N + 151N^2\bigr)}{3N(1+N)} S_4$$

$$+ \Biggl(-\frac{16 P_{45}}{9N^3(1+N)^2} + \Biggl(-\frac{64 P_{40}}{9N^2(1+N)^2} - 256 S_2\Biggr) S_1 + \frac{32\bigl(12 + 31N + 31N^2\bigr)}{3N(1+N)} S_2$$

$$+ 64 S_3 + 5376 S_{2,1} - 384 S_{-2,1} + 576\zeta_3\Biggr) S_{-2} + \Biggl(-\frac{32\bigl(8 + 3N + 3N^2\bigr)}{N(1+N)} + 512 S_1\Biggr) S_{-2}^2$$

$$+ \Biggl(\frac{32\bigl(108 + 31N + 31N^2\bigr)}{3N(1+N)} S_1 - \frac{16 P_{44}}{9N^2(1+N)^2} - 1152 S_1^2 + 2624 S_2 + 960 S_{-2}\Biggr) S_{-3}$$

$$+ \Biggl(\frac{16\bigl(138 + 35N + 35N^2\bigr)}{3N(1+N)} - 1472 S_1\Biggr) S_{-4} + 2304 S_{-5} + 768 S_{2,3} + 2688 S_{2,-3}$$

$$- \frac{64\bigl(-24 + 29N + 29N^2\bigr)}{3N(1+N)} S_{3,1} - 768 S_{4,1} + \frac{32\bigl(-174 + 31N + 31N^2\bigr)}{3N(1+N)} S_{-2,2}$$

$$- 3648 S_{-2,3} - \frac{1920 S_{-3,1}}{N(1+N)} + 1728 S_{-4,1} - 5376 S_{2,1,-2} + 1536 S_{3,1,1}$$

$$- \frac{128\bigl(-84 + 31N + 31N^2\bigr)}{3N(1+N)} S_{-2,1,1} - 1536 S_{-2,1,-2} - 5376 S_{-2,2,1} - 5376 S_{-3,1,1}$$

$$+ 10752 S_{-2,1,1,1}\Biggr] + T_F N_F \Biggl[-\frac{16 P_5}{9N^2(1+N)^2} S_2 + \frac{4 P_{54}}{9N^4(1+N)^4} + \Biggl(-\frac{8 P_{51}}{9N^3(1+N)^3}$$

$$+ \frac{1280}{9} S_2 - \frac{512}{3} S_3 - \frac{512}{3} S_{-2,1} + 128\zeta_3\Biggr) S_1 - \frac{128}{3} S_2^2 + \frac{64\bigl(12 + 29N + 29N^2\bigr)}{9N(1+N)} S_3$$

$$- \frac{512}{3} S_4 + \Biggl(-\frac{128\bigl(-3 + 10N + 16N^2\bigr)}{9N^2(1+N)^2} + \frac{2560}{9} S_1 - \frac{256}{3} S_2\Biggr) S_{-2}$$

$$+\left(\frac{128\left(3+10N+10N^2\right)}{9N(1+N)}-\frac{256}{3}S_1\right)S_{-3}-\frac{256}{3}S_{-4}+\frac{256}{3}S_{3,1}$$

$$-\frac{256\left(-3+10N+10N^2\right)}{9N(1+N)}S_{-2,1}-\frac{256}{3}S_{-2,2}$$

$$+\frac{1024}{3}S_{-2,1,1}-\frac{32\left(2+3N+3N^2\right)}{N(1+N)}\zeta_3\bigg]\bigg\}+C_F\bigg\{T_F^2N_F^2\bigg[\frac{8P_{28}}{27N^3(1+N)^3}-\frac{128}{27}S_1$$

$$-\frac{640}{27}S_2+\frac{128}{9}S_3\bigg]+C_A^2\bigg[-\frac{24P_{37}}{N^2(1+N)^2}\zeta_3-\frac{32P_{41}}{9N^2(1+N)^2}S_{-2,1}+\frac{8P_{18}}{9N^2(1+N)^2}S_3$$

$$+\frac{P_{59}}{54N^5(1+N)^5}+\left(\frac{4P_{55}}{3N^4(1+N)^4}-\frac{16\left(-8+11N+11N^2\right)}{N(1+N)}S_3-256S_4+512S_{3,1}\right.$$

$$\left.-\frac{64\left(-24+11N+11N^2\right)}{3N(1+N)}S_{-2,1}-1024S_{-2,2}-1024S_{-3,1}+2048S_{-2,1,1}\right)S_1$$

$$+\left(-128S_3-512S_{-2,1}\right)S_1^2+\left(-\frac{8344}{27}+384S_3+1536S_{-2,1}\right)S_2+64S_5$$

$$-\frac{16\left(-24+55N+55N^2\right)}{3N(1+N)}S_4+\left(\frac{32P_{39}}{9N^2(1+N)^2}S_1+\frac{16P_{49}}{9N^3(1+N)^3}-\frac{352}{3}S_2-64S_3\right.$$

$$\left.-1536S_{2,1}+128S_{-2,1}-192\zeta_3\right)S_{-2}+\left(\frac{48\left(2+N+N^2\right)}{N(1+N)}-192S_1\right)S_{-2}^2$$

$$+\left(\frac{16P_{42}}{9N^2(1+N)^2}-\frac{32\left(24+11N+11N^2\right)}{3N(1+N)}S_1+256S_1^2-768S_2-320S_{-2}\right)S_{-3}$$

$$+\left(-\frac{16\left(30+13N+13N^2\right)}{3N(1+N)}+320S_1\right)S_{-4}-704S_{-5}-384S_{2,3}$$

$$-768S_{2,-3}+\frac{64\left(-12+11N+11N^2\right)}{3N(1+N)}S_{3,1}+384S_{4,1}$$

$$-\frac{32\left(-48+11N+11N^2\right)S_{-2,2}}{3N(1+N)}+1088S_{-2,3}+\frac{512S_{-3,1}}{N(1+N)}-448S_{-4,1}+1536S_{2,1,-2}$$

$$-768S_{3,1,1}+\frac{128\left(-24+11N+11N^2\right)}{3N(1+N)}S_{-2,1,1}+512S_{-2,1,-2}+1536[S_{-2,2,1}$$

$$+S_{-3,1,1}]-3072S_{-2,1,1,1}\bigg]+C_AT_FN_F\bigg[-\frac{8P_{56}}{27N^4(1+N)^4}+\left(-\frac{16P_{52}}{27N^3(1+N)^3}\right.$$

$$\left.+64S_3+\frac{256}{3}S_{-2,1}-128\zeta_3\right)S_1+\frac{5344}{27}S_2-\frac{32\left(3+14N+14N^2\right)}{3N(1+N)}S_3+\frac{320}{3}S_4$$

$$+\left(\frac{64\left(-3+10N+16N^2\right)}{9N^2(1+N)^2}-\frac{1280}{9}S_1+\frac{128}{3}S_2\right)S_{-2}+\left(-\frac{64\left(3+10N+10N^2\right)}{9N(1+N)}\right.$$

$$
+\frac{128}{3}S_1\bigg)S_{-3}+\frac{128}{3}S_{-4}-\frac{256}{3}S_{3,1}+\frac{128\big(-3+10N+10N^2\big)}{9N(1+N)}S_{-2,1}+\frac{128}{3}S_{-2,2}
$$

$$
-\frac{512}{3}S_{-2,1,1}+\frac{32\big(2+3N+3N^2\big)}{N(1+N)}\zeta_3\bigg]\bigg\}+C_F^3\bigg\{-\frac{48\zeta_3 P_{37}}{N^2(1+N)^2}+\frac{8P_{38}}{N^2(1+N)^2}S_3
$$

$$
+\frac{P_{58}}{N^5(1+N)^5}+\bigg(\frac{8P_{48}}{N^4(1+N)^4}-\frac{128(1+2N)}{N^2(1+N)^2}S_2+128S_2^2-384S_3+128S_4
$$

$$
+512S_{3,1}-\frac{384\big(-4+N+N^2\big)}{N(1+N)}S_{-2,1}-3328S_{-2,2}-3584S_{-3,1}+6144S_{-2,1,1}\bigg)S_1
$$

$$
+\bigg(-\frac{64\big(1+3N+3N^2\big)}{N^3(1+N)^3}-1536S_{-2,1}\bigg)S_1^2+\bigg(\frac{4P_{47}}{N^3(1+N)^3}+512S_3+4352S_{-2,1}\bigg)S_2
$$

$$
-\frac{32\big(2+3N+3N^2\big)}{N(1+N)}S_2^2-\frac{32\big(2+15N+15N^2\big)}{N(1+N)}S_4+\bigg(\frac{32P_{46}}{N^3(1+N)^3}+512S_2
$$

$$
+\bigg(\frac{128\big(1-N+3N^2\big)}{N^2(1+N)^2}+512S_2\bigg)S_1-\frac{64\big(4+3N+3N^2\big)}{N(1+N)}S_2-512S_2
$$

$$
+128S_3-4608S_{2,1}+256S_{-2,1}-384\zeta_3\bigg)S_{-2}+\bigg(\frac{128}{N(1+N)}-256S_1\bigg)S_{-2}^2
$$

$$
+\bigg(\frac{32\big(20+17N+21N^2\big)}{N^2(1+N)^2}+1280S_1^2-2176S_2-\frac{64\big(20+3N+3N^2\big)}{N(1+N)}S_1
$$

$$
-640S_{-2}\bigg)S_{-3}+\bigg(-\frac{32\big(26+3N+3N^2\big)}{N(1+N)}+1664S_1\bigg)S_{-4}-1792S_{-5}-384S_{2,3}
$$

$$
-2304S_{2,-3}+\frac{128\big(-2+3N+3N^2\big)}{N(1+N)}S_{3,1}+384S_{4,1}-1664S_{-4,1}+4608S_{2,1,-2}
$$

$$
-\frac{64\big(16+11N+15N^2\big)}{N^2(1+N)^2}S_{-2,1}-\frac{64\big(-26+3N+3N^2\big)}{N(1+N)}S_{-2,2}+2944S_{-2,3}
$$

$$
+\frac{1792}{N(1+N)}S_{-3,1}-768S_{3,1,1}+\frac{768\big(-4+N+N^2\big)}{N(1+N)}S_{-2,1,1}+1024S_{-2,1,-2}
$$

$$
+4608[S_{-2,2,1}+S_{-3,1,1}]-9216S_{-2,1,1,1}\bigg\}\bigg\}. \tag{48}
$$

In the case of transversity we obtain

$$
\gamma_{\mathrm{NS}}^{(2),\mathrm{tr},+}=\frac{1}{2}\Big[1+(-1)^N\Big]
$$

$$
\times\bigg\{C_F\bigg\{T_F^2 N_F^2\bigg[\frac{8\big(-8+17N+17N^2\big)}{9N(1+N)}-\frac{128}{27}S_1-\frac{640}{27}S_2+\frac{128}{9}S_3\bigg]
$$

$$
+C_A T_F N_F\bigg[-\frac{16\big(-22+45N+45N^2\big)}{9N(1+N)}+\bigg(-\frac{16\big(9+209N+209N^2\big)}{27N(1+N)}+64S_3
$$

$$+\frac{256}{3}S_{-2,1}-128\zeta_3\Big)S_1+\frac{5344}{27}S_2-\frac{448}{3}S_3+\frac{320}{3}S_4+\Big(-\frac{1280}{9}S_1+\frac{128}{3}S_2\Big)S_{-2}$$

$$+\Big(-\frac{640}{9}+\frac{128}{3}S_1\Big)S_{-3}+\frac{128}{3}S_{-4}-\frac{256}{3}S_{3,1}+\frac{1280}{9}S_{-2,1}+\frac{128}{3}S_{-2,2}-\frac{512}{3}S_{-2,1,1}$$

$$+96\zeta_3\Big]+C_A^2\Big[\frac{-968+1657N+1657N^2}{18N(1+N)}+\Big(\frac{4P_{14}}{3(-1+N)N(1+N)(2+N)}-176S_3$$

$$-256S_4+512S_{3,1}-\frac{704}{3}S_{-2,1}-1024S_{-2,2}-1024S_{-3,1}+2048S_{-2,1,1}\Big)S_1$$

$$+\Big(-128S_3-512S_{-2,1}\Big)S_1^2+\Big(-\frac{8344}{27}+384S_3+1536S_{-2,1}\Big)S_2+\frac{3112}{9}S_3-\frac{880}{3}S_4$$

$$+64S_5+\Big(\frac{16P_2}{(-1+N)N(1+N)(2+N)}+\frac{32(-241+134N+134N^2)S_1}{9(-1+N)(2+N)}-\frac{352}{3}S_2$$

$$-64S_3-1536S_{2,1}+128S_{-2,1}-192\zeta_3\Big)S_{-2}+\Big(48-192S_1\Big)S_{-2}^2+\Big(256S_1^2-768S_2$$

$$-320S_{-2}+\frac{32(-107+67N+67N^2)}{9(-1+N)(2+N)}-\frac{352}{3}S_1\Big)S_{-3}+\Big(-\frac{208}{3}+320S_1\Big)S_{-4}$$

$$-704S_{-5}-384S_{2,3}-768S_{2,-3}+\frac{704}{3}S_{3,1}+384S_{4,1}$$

$$-\frac{64(-107+67N+67N^2)S_{-2,1}}{9(-1+N)(2+N)}-\frac{352}{3}S_{-2,2}$$

$$+1088S_{-2,3}-448S_{-4,1}+1536[S_{2,1,-2}+S_{-2,2,1}+S_{-3,1,1}]-768S_{3,1,1}$$

$$+\frac{1408}{3}S_{-2,1,1}+512S_{-2,1,-2}-3072S_{-2,1,1,1}-\frac{24(-6+5N+5N^2)\zeta_3}{(-1+N)(2+N)}\Big]\Big\}$$

$$+C_F^2\Big\{T_FN_F\Big[92+\Big(-\frac{8(-8+55N+55N^2)}{3N(1+N)}+\frac{1280}{9}S_2-\frac{512}{3}S_3-\frac{512}{3}S_{-2,1}$$

$$+128\zeta_3\Big)S_1-\frac{80}{3}S_2-\frac{128}{3}S_2^2+\frac{1856}{9}S_3-\frac{512}{3}S_4+\Big(\frac{2560}{9}S_1-\frac{256}{3}S_2\Big)S_{-2}+\Big(\frac{1280}{9}$$

$$-\frac{256}{3}S_1\Big)S_{-3}-\frac{256}{3}S_{-4}+\frac{256}{3}S_{3,1}-\frac{2560}{9}S_{-2,1}-\frac{256}{3}S_{-2,2}+\frac{1024}{3}S_{-2,1,1}-96\zeta_3\Big]$$

$$+C_A\Big[-\frac{151}{2}+\Big(-\frac{8(-206+211N+211N^2)}{3(-1+N)N(1+N)(2+N)}-\frac{4288}{9}S_2+\frac{1984}{3}S_3+320S_4$$

$$-1024S_{3,1}+\frac{1984}{3}S_{-2,1}+3712S_{-2,2}+3840S_{-3,1}-7168S_{-2,1,1}\Big)S_1+\Big(256S_3$$

$$+1792 S_{-2,1} \bigg) S_1^2 + \left(\frac{604}{3} - 832 S_3 - 5248 S_{-2,1} \right) S_2 + \frac{352}{3} S_2^2 - \frac{6160}{9} S_3 + \frac{2416}{3} S_4$$

$$+ \left(-\frac{48 P_2}{(-1+N)N(1+N)(2+N)} + \left(-\frac{64 P_7}{9(-1+N)N(1+N)(2+N)} - 256 S_2 \right) S_1 \right.$$

$$+ \frac{992}{3} S_2 + 64 S_3 + 5376 S_{2,1} - 384 S_{-2,1} + 576 \zeta_3 \bigg) S_{-2} + \left(-96 + 512 S_1 \right) S_{-2}^2$$

$$+ \left(-\frac{32\left(-187+134N+134N^2\right)}{9(-1+N)(2+N)} + \frac{992}{3} S_1 - 1152 S_1^2 + 2624 S_2 + 960 S_{-2} \right) S_{-3}$$

$$+ \left(\frac{560}{3} - 1472 S_1 \right) S_{-4} + 2304 S_{-5} + 768 S_{2,3} + 2688 S_{2,-3} - \frac{1856}{3} S_{3,1} - 768 S_{4,1}$$

$$+ \frac{64\left(-187+134N+134N^2\right) S_{-2,1}}{9(-1+N)(2+N)} + \frac{992}{3} S_{-2,2} - 3648 S_{-2,3} + 1728 S_{-4,1}$$

$$-5376 [S_{2,1,-2} + S_{-2,2,1} + S_{-3,1,1}] + 1536 [S_{3,1,1} - S_{-2,1,-2}] - \frac{3968}{3} S_{-2,1,1}$$

$$+ 10752 S_{-2,1,1,1} + \frac{72\left(-6+5N+5N^2\right)\zeta_3}{(-1+N)(2+N)} \bigg] \bigg\}$$

$$+ C_F^3 \Bigg\{ -29 + \left(\frac{384\left(-1+N+N^2\right)}{(-1+N)N(1+N)(2+N)} + 128 S_2^2 - 384 S_3 + 128 S_4 + 512 S_{3,1} \right.$$

$$- 384 S_{-2,1} - 3328 S_{-2,2} - 3584 S_{-3,1} + 6144 S_{-2,1,1} \bigg) S_1 - 256 S_{-2}^2 S_1 + \bigg(12 + 512 S_3$$

$$+ 4352 S_{-2,1} \bigg) S_2 - 96 S_2^2 + 104 S_3 - 480 S_4 + \left(\frac{32 P_2}{(-1+N)N(1+N)(2+N)} \right.$$

$$+ \left(\frac{384}{(-1+N)N(1+N)(2+N)} + 512 S_2 \right) S_1 - 192 S_2 + 128 S_3 - 4608 S_{2,1} + 256 S_{-2,1}$$

$$- 384 \zeta_3 \bigg) S_{-2} + \left(\frac{192}{(-1+N)(2+N)} - 192 S_1 + 1280 S_1^2 - 2176 S_2 - 640 S_{-2} \right) S_{-3}$$

$$+ \left(-96 + 1664 S_1 \right) S_{-4} - 1792 S_{-5} + 384 [-S_{2,3} + S_{3,1} + S_{4,1}] - 2304 S_{2,-3}$$

$$- \frac{384 S_{-2,1}}{(-1+N)(2+N)} - 1536 S_1^2 S_{-2,1} - 192 S_{-2,2} + 2944 S_{-2,3} - 1664 S_{-4,1}$$

$$+ 4608 S_{2,1,-2} - 768 S_{3,1,1} + 768 S_{-2,1,1} + 1024 S_{-2,1,-2} + 4608 [S_{-2,2,1} + S_{-3,1,1}]$$

$$- 9216 S_{-2,1,1,1} - \frac{48\left(-6+5N+5N^2\right)\zeta_3}{(-1+N)(2+N)} \bigg\} \Bigg\}, \tag{49}$$

$$\gamma_{\text{NS}}^{(2),\text{tr},-} = \frac{1}{2}\left[1 - (-1)^N\right] \Bigg\{ C_F \Bigg\{ T_F^2 N_F^2 \left[\frac{8\left(-8+17N+17N^2\right)}{9N(1+N)} - \frac{128}{27} S_1 - \frac{640}{27} S_2 \right.$$

$$+\frac{128}{9}S_3\Bigg] + C_A T_F N_F\Bigg[-\frac{16(-8+49N+90N^2+45N^3)}{9N(1+N)^2}$$

$$+\left(-\frac{16(-27+209N+209N^2)}{27N(1+N)}+64S_3+\frac{256}{3}S_{-2,1}-128\zeta_3\right)S_1$$

$$+\frac{5344}{27}S_2-\frac{448}{3}S_3+\frac{320}{3}S_4+\left(-\frac{1280}{9}S_1+\frac{128}{3}S_2\right)S_{-2}+\left(-\frac{640}{9}+\frac{128}{3}S_1\right)S_{-3}$$

$$+\frac{128}{3}S_{-4}-\frac{256}{3}S_{3,1}+\frac{1280}{9}S_{-2,1}+\frac{128}{3}S_{-2,2}-\frac{512}{3}S_{-2,1,1}$$

$$+96\zeta_3\Bigg] + C_A^2\Bigg[\frac{P_{23}}{18(-1+N)N(1+N)^2(2+N)}+\left(\frac{4(12+245N+245N^2)}{3N(1+N)}-176S_3\right.$$

$$\left.-256S_4+512S_{3,1}-\frac{704}{3}S_{-2,1}-1024S_{-2,2}-1024S_{-3,1}+2048S_{-2,1,1}\right)S_1$$

$$+\left(-128S_3-512S_{-2,1}\right)S_1^2+\left(-\frac{8344}{27}+384S_3+1536S_{-2,1}\right)S_2+\frac{3112}{9}S_3-\frac{880}{3}S_4$$

$$+64S_5+\left(\frac{16(-5+3N+3N^2)}{(-1+N)(2+N)}+\frac{32(81+134N+134N^2)}{9N(1+N)}S_1-\frac{352}{3}S_2-64S_3\right.$$

$$\left.-1536S_{2,1}+128S_{-2,1}-192\zeta_3\right)S_{-2}+\left(48-192S_1\right)S_{-2}^2+\left(\frac{32(81+67N+67N^2)}{9N(1+N)}\right.$$

$$\left.-\frac{352}{3}S_1+256S_1^2-768S_2-320S_{-2}\right)S_{-3}+\left(-\frac{208}{3}+320S_1\right)S_{-4}-704S_{-5}$$

$$-384S_{2,3}-768S_{2,-3}+\frac{704}{3}S_{3,1}+384S_{4,1}-\frac{64(81+67N+67N^2)S_{-2,1}}{9N(1+N)}-\frac{352}{3}S_{-2,2}$$

$$+1088S_{-2,3}-448S_{-4,1}+1536S_{2,1,-2}-768S_{3,1,1}+\frac{1408}{3}S_{-2,1,1}+512S_{-2,1,-2}$$

$$+1536[S_{-2,2,1}+S_{-3,1,1}]-3072S_{-2,1,1,1}-\frac{24(12+5N+5N^2)\zeta_3}{N(1+N)}\Bigg]\Bigg\}$$

$$+C_F^2\Bigg\{ T_F N_F\Bigg[\frac{4(112+415N+414N^2+207N^3)}{9N(1+N)^2}+\left(-\frac{8(8+55N+55N^2)}{3N(1+N)}\right.$$

$$\left.+\frac{1280}{9}S_2-\frac{512}{3}S_3-\frac{512}{3}S_{-2,1}+128\zeta_3\right)S_1-\frac{80}{3}S_2-\frac{128}{3}S_2^2+\frac{1856}{9}S_3-\frac{512}{3}S_4$$

$$+\left(\frac{2560}{9}S_1-\frac{256}{3}S_2\right)S_{-2}+\left(\frac{1280}{9}-\frac{256}{3}S_1\right)S_{-3}-\frac{256}{3}S_{-4}+\frac{256}{3}S_{3,1}-\frac{2560}{9}S_{-2,1}$$

$$-\frac{256}{3}S_{-2,2}+\frac{1024}{3}S_{-2,1,1}-96\zeta_3\Bigg] + C_A\Bigg[\frac{P_{20}}{18(-1+N)N(1+N)^2(2+N)}$$

$$
+\left(-\frac{8P_6}{3(-1+N)N^2(1+N)^2(2+N)}-\frac{4288}{9}S_2+\frac{1984}{3}S_3+320S_4-1024S_{3,1}\right.
$$

$$
\left.+\frac{1984}{3}S_{-2,1}+3712S_{-2,2}+3840S_{-3,1}-7168S_{-2,1,1}\right)S_1+\left(256S_3+1792S_{-2,1}\right)S_1^2
$$

$$
+\left(\frac{4(-24+151N+151N^2)}{3N(1+N)}-832S_3-5248S_{-2,1}\right)S_2+\frac{352}{3}S_2^2-\frac{6160}{9}S_3
$$

$$
+\frac{2416}{3}S_4+\left(-\frac{48(-5+3N+3N^2)}{(-1+N)(2+N)}+\left(-\frac{64P_8}{9(-1+N)N(1+N)(2+N)}-256S_2\right)S_1\right.
$$

$$
\left.+\frac{992}{3}S_2+64S_3+5376S_{2,1}-384S_{-2,1}+576\zeta_3\right)S_{-2}+\left(-96+512S_1\right)S_{-2}^2
$$

$$
+\left(-\frac{32(243+134N+134N^2)}{9N(1+N)}+\frac{992}{3}S_1-1152S_1^2+2624S_2+960S_{-2}\right)S_{-3}
$$

$$
+\left(\frac{560}{3}-1472S_1\right)S_{-4}+2304S_{-5}+768S_{2,3}+2688S_{2,-3}-\frac{1856}{3}S_{3,1}-768S_{4,1}
$$

$$
+\frac{64(243+134N+134N^2)}{9N(1+N)}S_{-2,1}+\frac{992}{3}S_{-2,2}-3648S_{-2,3}+1728S_{-4,1}-5376S_{2,1,-2}
$$

$$
-1536[S_{-2,1,-2}-S_{3,1,1}]-5376[S_{-2,2,1}+S_{-3,1,1}]-\frac{3968}{3}S_{-2,1,1}+10752S_{-2,1,1,1}
$$

$$
\left.\left.+\frac{72(12+5N+5N^2)\zeta_3}{N(1+N)}\right]\right\}+C_F^3\left\{\frac{P_{21}}{(-1+N)N(1+N)^2(2+N)}\right.
$$

$$
+\left(-\frac{32P_1}{(-1+N)N^2(1+N)^2(2+N)}+128S_2^2-384S_3+128S_4+512S_{3,1}-384S_{-2,1}\right.
$$

$$
\left.-3328S_{-2,2}-3584S_{-3,1}+6144S_{-2,1,1}\right)S_1-256S_{-2}^2S_1+\left(\frac{4(16+3N+3N^2)}{N(1+N)}\right.
$$

$$
\left.+512S_3+4352S_{-2,1}\right)S_2-96S_2^2+104S_3-480S_4+\left(\frac{32(-5+3N+3N^2)}{(-1+N)(2+N)}\right.
$$

$$
+\left(\frac{128(-9+4N+4N^2)}{(-1+N)N(1+N)(2+N)}+512S_2\right)S_1-192S_2+128S_3-4608S_{2,1}
$$

$$
\left.+256S_{-2,1}-384\zeta_3\right)S_{-2}+\left(\frac{576}{N(1+N)}-192S_1+1280S_1^2-2176S_2-640S_{-2}\right)S_{-3}
$$

$$
+\left(-96+1664S_1\right)S_{-4}-1792S_{-5}-384S_{2,3}-2304S_{2,-3}+384S_{3,1}+384S_{4,1}
$$

$$
-\frac{1152}{N(1+N)}S_{-2,1}-1536S_1^2S_{-2,1}-192S_{-2,2}+2944S_{-2,3}-1664S_{-4,1}+4608S_{2,1,-2}
$$

$$
-768S_{3,1,1}+768S_{-2,1,1}+1024S_{-2,1,-2}+4608S_{-2,2,1}+4608S_{-3,1,1}-9216S_{-2,1,1,1}
$$

$$-\frac{48\left(12+5N+5N^2\right)\zeta_3}{N(1+N)}\Bigg\}\Bigg\}. \tag{50}$$

We have calculated the transversity anomalous dimensions for the first time directly and without any assumptions.

The polynomials in Eqs. (47)–(50) read

$$P_1 = N^4 - 2N^3 - 3N^2 + 8N + 4, \tag{51}$$

$$P_2 = 3N^4 + 6N^3 - 8N^2 - 11N - 2, \tag{52}$$

$$P_3 = 5N^4 + 10N^3 + N^2 - 4N - 4, \tag{53}$$

$$P_4 = 13N^4 + 26N^3 + 13N^2 - 16N - 20, \tag{54}$$

$$P_5 = 15N^4 + 30N^3 + 79N^2 + 16N - 24, \tag{55}$$

$$P_6 = 17N^4 + 58N^3 - 5N^2 - 94N - 24, \tag{56}$$

$$P_7 = 134N^4 + 268N^3 - 107N^2 - 241N + 27, \tag{57}$$

$$P_8 = 134N^4 + 268N^3 - 17N^2 - 151N - 243, \tag{58}$$

$$P_9 = 134N^4 + 268N^3 + 89N^2 - 81N - 72, \tag{59}$$

$$P_{10} = 134N^4 + 268N^3 + 116N^2 - 18N - 27, \tag{60}$$

$$P_{11} = 134N^4 + 268N^3 + 137N^2 + 3N + 27, \tag{61}$$

$$P_{12} = 134N^4 + 268N^3 + 203N^2 + 69N + 27, \tag{62}$$

$$P_{13} = 165N^4 + 330N^3 + 165N^2 + 256N + 80, \tag{63}$$

$$P_{14} = 245N^4 + 490N^3 - 117N^2 - 362N - 112, \tag{64}$$

$$P_{15} = 268N^4 + 536N^3 + 301N^2 - 3N + 90, \tag{65}$$

$$P_{16} = 268N^4 + 536N^3 + 487N^2 + 183N + 126, \tag{66}$$

$$P_{17} = 385N^4 + 770N^3 + 427N^2 + 6N - 126, \tag{67}$$

$$P_{18} = 389N^4 + 778N^3 + 398N^2 + 9N - 81, \tag{68}$$

$$P_{19} = 453N^4 + 906N^3 + 1325N^2 + 344N - 264, \tag{69}$$

$$P_{20} = -1359N^5 - 4077N^4 - 11887N^3 - 14003N^2 + 14494N + 13376, \tag{70}$$

$$P_{21} = -29N^5 - 87N^4 + 227N^3 + 503N^2 - 230N - 256, \tag{71}$$

$$P_{22} = 81N^5 + 243N^4 - 337N^3 - 893N^2 - 526N - 60, \tag{72}$$

$$P_{23} = 1657N^5 + 4971N^4 + 4801N^3 + 261N^2 - 6938N - 3600, \tag{73}$$

$$P_{24} = 3N^6 + 9N^5 + 9N^4 + 9N^3 + 2N^2 + 4N + 2, \tag{74}$$

$$P_{25} = 3N^6 + 9N^5 + 9N^4 + 51N^3 + 76N^2 + 60N + 16, \tag{75}$$

$$P_{26} = 22N^6 + 66N^5 + 95N^4 + 88N^3 + 197N^2 + 160N + 52, \tag{76}$$

$$P_{27} = 27N^6 + 81N^5 - 209N^4 - 487N^3 - 272N^2 - 48N - 9, \tag{77}$$

$$P_{28} = 51N^6 + 153N^5 + 57N^4 + 35N^3 + 96N^2 + 16N - 24, \tag{78}$$

$$P_{29} = 135N^6 - 31N^5 - 601N^4 - 569N^3 + 487N^2 + 621N + 216, \tag{79}$$

$$P_{30} = 209N^6 + 627N^5 + 627N^4 + 209N^3 - 108N^2 - 108N - 54, \tag{80}$$

$$P_{31} = 270N^6 + 810N^5 - 427N^4 - 936N^3 + 269N^2 + 238N - 132, \tag{81}$$

$$P_{32} = 4971N^6 + 14913N^5 - 24035N^4 - 41453N^3 - 452N^2 + 7024N - 2904, \tag{82}$$

$$P_{33} = -1359N^8 - 5436N^7 - 8274N^6 - 13452N^5 - 15103N^4 - 12528N^3$$
$$-4120N^2 + 2560N + 1584, \tag{83}$$

$$P_{34} = 207N^8 + 828N^7 + 1491N^6 + 1779N^5 + 1210N^4 + 453N^3 - 8N^2$$
$$-160N - 72, \tag{84}$$

$$P_{35} = 245N^8 + 980N^7 + 1542N^6 + 1196N^5 + 395N^4 - 60N^3 + 156N^2$$
$$+222N + 90, \tag{85}$$

$$P_{36} = -29N^{10} - 145N^9 - 226N^8 - 46N^7 - N^6 - 469N^5 - 976N^4 - 940N^3$$
$$-576N^2 - 208N - 32, \tag{86}$$

$$P_{37} = 5N^4 + 10N^3 + 9N^2 + 4N + 4, \tag{87}$$

$$P_{38} = 13N^4 + 26N^3 + 13N^2 - 16N - 20, \tag{88}$$

$$P_{39} = 134N^4 + 268N^3 + 188N^2 + 54N + 45, \tag{89}$$

$$P_{40} = 134N^4 + 268N^3 + 215N^2 + 45N + 54, \tag{90}$$

$$P_{41} = 134N^4 + 268N^3 + 245N^2 + 111N + 135, \tag{91}$$

$$P_{42} = 134N^4 + 268N^3 + 311N^2 + 177N + 135, \tag{92}$$

$$P_{43} = 268N^4 + 536N^3 + 625N^2 + 321N + 414, \tag{93}$$

$$P_{44} = 268N^4 + 536N^3 + 811N^2 + 507N + 450, \tag{94}$$

$$P_{45} = 81N^5 + 162N^4 - 391N^3 - 286N^2 + 156N + 72, \tag{95}$$

$$P_{46} = 3N^6 + 9N^5 + 9N^4 + 9N^3 + 6N^2 + 8N + 2, \tag{96}$$

$$P_{47} = 3N^6 + 9N^5 + 9N^4 + 51N^3 + 12N^2 - 4N - 16, \tag{97}$$

$$P_{48} = 22N^6 + 66N^5 + 95N^4 - 40N^3 - 115N^2 - 120N - 44, \tag{98}$$

$$P_{49} = 27N^6 + 81N^5 - 155N^4 - 379N^3 - 92N^2 + 78N + 27, \tag{99}$$

$$P_{50} = 135N^6 - 31N^5 - 481N^4 - 617N^3 - 395N^2 - 309N - 144, \tag{100}$$

$$P_{51} = 165N^6 + 495N^5 + 495N^4 + 421N^3 + 144N^2 - 112N - 96, \tag{101}$$

$$P_{52} = 209N^6 + 627N^5 + 627N^4 + 209N^3 + 36N^2 + 36N + 18, \tag{102}$$

$$P_{53} = 453N^6 + 1359N^5 + 2231N^4 + 1669N^3 + 368N^2 + 24N + 144, \tag{103}$$

$$P_{54} = 207N^8 + 828N^7 + 1443N^6 + 1635N^5 + 90N^4 - 779N^3 - 632N^2 + 120, \tag{104}$$

$$P_{55} = 245N^8 + 980N^7 + 1542N^6 + 1196N^5 + 475N^4 + 100N^3 + 36N^2 + 22N - 6, \tag{105}$$

$$P_{56} = 270N^8 + 1080N^7 + 347N^6 - 1471N^5 - 1507N^4 - 417N^3 - 362N^2$$
$$-12N + 144, \tag{106}$$

$$P_{57} = -1359N^{10} - 6795N^9 - 15246N^8 - 27870N^7 - 5163N^6 + 40241N^5 + 34648N^4$$
$$-12280N^3 - 32592N^2 - 17616N - 3456, \tag{107}$$

$$P_{58} = -29N^{10} - 145N^9 - 130N^8 + 338N^7 + 383N^6 + 107N^5 + 464N^4 + 1748N^3$$
$$+1600N^2 + 752N + 160, \tag{108}$$

$$P_{59} = 4971N^{10} + 24855N^9 + 11770N^8 - 70578N^7 - 147665N^6 - 144917N^5$$
$$- 85692N^4 - 18992N^3 + 22824N^2 + 15840N + 2592. \tag{109}$$

Finally, we turn to the non–singlet anomalous dimension $\gamma_{\mathrm{NS}}^{(2),\mathrm{s}}$ for which we obtain

$$\gamma_{\mathrm{NS}}^{(2),\mathrm{s}} = 4N_F \frac{d_{abc}d^{abc}}{N_C} \Bigg[\frac{2P_{60}}{(-1+N)N^5(1+N)^5(2+N)}$$

$$- \frac{P_{61}}{(-1+N)N^4(1+N)^4(2+N)} S_1 + \Bigg(-\frac{2P_{62}}{(-1+N)N^3(1+N)^3(2+N)}$$

$$- \frac{4(2+N+N^2)^2}{(-1+N)N^2(1+N)^2(2+N)} S_1 \Bigg) S_{-2}$$

$$- \frac{(2+N+N^2)}{N^2(1+N)^2} [S_3 - 2S_{-3} + 4S_{-2,1}] \Bigg], \tag{110}$$

with $d_{abc}d^{abc}/N_C = 40/9$ for $N_C = 3$ in QCD and the polynomials

$$P_{60} = N^8 + 4N^7 + 13N^6 + 25N^5 + 57N^4 + 77N^3 + 55N^2 + 20N + 4, \tag{111}$$
$$P_{61} = 3N^8 + 12N^7 + 16N^6 + 6N^5 + 30N^4 + 64N^3 + 73N^2 + 40N + 12, \tag{112}$$
$$P_{62} = N^6 + 3N^5 - 8N^4 - 21N^3 - 23N^2 - 12N - 4. \tag{113}$$

Note that $\gamma_{\mathrm{NS}}^{(2),\mathrm{s}}$ has no pole at $N = 1$, but vanishes.

The splitting functions in z space are given by

$$P_{\mathrm{NS}}^{(2),+,\delta} = -\Bigg\{ \delta(1-z) \Bigg\{ C_F^2 \Bigg[C_A \Bigg[-\frac{151}{2} + \left(\frac{820}{3} - 32\zeta_3 \right) \zeta_2 + \frac{1976}{15} \zeta_2^2 - \frac{1688}{3} \zeta_3$$

$$- 240\zeta_5 \Bigg] + T_F N_F \Bigg[92 - \frac{80\zeta_2}{3} - \frac{928\zeta_2^2}{15} + \frac{544\zeta_3}{3} \Bigg] \Bigg] + C_F \Bigg[C_A T_F N_F \Bigg[-80 + \frac{5344\zeta_2}{27}$$

$$- \frac{16\zeta_2^2}{5} - \frac{800\zeta_3}{9} \Bigg] + T_F^2 N_F^2 \Bigg[\frac{136}{9} - \frac{640\zeta_2}{27} + \frac{128\zeta_3}{9} \Bigg] + C_A^2 \Bigg[\frac{1657}{18} - \frac{8992\zeta_2}{27} + 4\zeta_2^2$$

$$+ \frac{3104\zeta_3}{9} - 80\zeta_5 \Bigg] \Bigg] + C_F^3 \Bigg[-29 + \left(-36 + 64\zeta_3 \right) \zeta_2 - \frac{576}{5}\zeta_2^2 - 136\zeta_3 + 480\zeta_5 \Bigg] \Bigg\} \Bigg\}, \tag{114}$$

$$P_{\mathrm{NS}}^{(2),+,\mathrm{plu}} = -\Bigg\{ \frac{1}{1-z} \Bigg\{ C_F \Bigg[T_F^2 N_F^2 \frac{128}{27} + C_A^2 \left(-\frac{980}{3} + \frac{2144\zeta_2}{9} - \frac{352\zeta_2^2}{5} - \frac{176\zeta_3}{3} \right)$$

$$+ C_A T_F N_F \left(\frac{3344}{27} - \frac{640\zeta_2}{9} + \frac{448\zeta_3}{3} \right) \Bigg] + C_F^2 T_F N_F \Bigg[\frac{440}{3} - 128\zeta_3 \Bigg] \Bigg\} \Bigg\}, \tag{115}$$

$$P_{\mathrm{NS}}^{(2),+,\mathrm{reg}} = -\Bigg\{ C_F \Bigg\{ T_F^2 N_F^2 \Bigg[\frac{256(-1+z)}{9} - 128/27 + \frac{64(11 - 12z + 11z^2)}{27(-1+z)} \mathrm{H}_0$$

$$+\frac{32(1+z^2)}{9(-1+z)}\mathrm{H}_0^2\Bigg] + C_A T_F N_F \Bigg[-\left(\frac{3344}{27} - \frac{640}{9}\zeta_2 + \frac{448}{3}\zeta_3\right)$$

$$+\left(-\frac{16(317-408z+425z^2)}{27(-1+z)} + \frac{256(4+3z+4z^2)}{9(1+z)}\mathrm{H}_{-1} + \frac{32(3+z)(1+z^2)\zeta_2}{3(-1+z)(1+z)}\right)\mathrm{H}_0$$

$$+\frac{9896(1-z)}{27} + \left(-\frac{16(11+28z+11z^2+34z^3)}{9(-1+z)(1+z)} + \frac{32(1+z^2)}{3(1+z)}\mathrm{H}_{-1}\right)\mathrm{H}_0^2$$

$$-\frac{64z(1+z^2)}{9(-1+z)(1+z)}\mathrm{H}_0^3 + \left(\frac{16(1+z^2)}{(-1+z)}\mathrm{H}_0^2 + 64(1-z)\right)\mathrm{H}_1 + \left(32(1+z) + \frac{32(1+z^2)}{(1-z)}\mathrm{H}_0\right.$$

$$\left.-\frac{128(1+z^2)}{3(1+z)}\mathrm{H}_{-1}\right)\mathrm{H}_{0,1} - \frac{256(4+3z+4z^2)}{9(1+z)}\mathrm{H}_{0,-1} + \frac{128z(1+z^2)}{3(-1+z)(1+z)}\mathrm{H}_{0,0,1}$$

$$-\frac{64(1+z^2)}{3(1+z)}\mathrm{H}_{0,0,-1} + \frac{32(-9-6z+23z^2)}{9(1+z)}\zeta_2 + \frac{128(1+z^2)}{3(1+z)}\mathrm{H}_{0,1,-1}$$

$$+\frac{128(1+z^2)}{3(1+z)}\mathrm{H}_{0,-1,1} + \frac{128(1+z^2)}{3(1+z)}\mathrm{H}_{-1}\zeta_2 - \frac{64(3-z-4z^2+6z^3)}{3(-1+z)(1+z)}\zeta_3\Bigg]$$

$$+C_A^2\Bigg[-\left(-\frac{980}{3} + \frac{2144}{9}\zeta_2 - \frac{352}{5}\zeta_2^2 - \frac{176}{3}\zeta_3\right) + \frac{19474(-1+z)}{27}$$

$$+\left(\frac{4(1967-1933z-205z^2+4343z^3)}{27(-1+z)(1+z)} + \left(-\frac{32(91+48z+91z^2)}{9(1+z)}\right.\right.$$

$$\left.\left.-\frac{352(1+z^2)}{(1+z)}\zeta_2\right)\mathrm{H}_{-1} - 64(1+z)\mathrm{H}_{-1}^2 + \frac{32z(7+9z^2)}{(-1+z)(1+z)}\zeta_3\right.$$

$$\left.+\frac{8(-69-95z+3z^2+37z^3)}{3(-1+z)(1+z)}\zeta_2\right)\mathrm{H}_0 + \left(\frac{2(121+969z+121z^2+345z^3)}{9(-1+z)(1+z)}\right.$$

$$\left.+\frac{8(19+60z+19z^2)}{3(1+z)}\mathrm{H}_{-1} - \frac{64(1+z^2)}{(1+z)}\mathrm{H}_{-1}^2 + \frac{4(15+8z+17z^2)}{(1+z)}\zeta_2\right)\mathrm{H}_0^2$$

$$+\left(\frac{16z(23+8z^2)}{9(-1+z)(1+z)} + \frac{16(1+z^2)}{(1+z)}\mathrm{H}_{-1}\right)\mathrm{H}_0^3 - \frac{2z(3+5z^2)}{3(-1+z)(1+z)}\mathrm{H}_0^4$$

$$+\left(352(-1+z) - \frac{4(19-16z+19z^2)}{(-1+z)}\mathrm{H}_0^2 - \frac{16(1+z^2)}{(-1+z)}\mathrm{H}_0^3 + 64(-1+z)\zeta_2\right.$$

$$\left.-\frac{32(1+z^2)}{(-1+z)}\mathrm{H}_0\zeta_2 + \frac{288(1+z^2)}{(-1+z)}\zeta_3\right)\mathrm{H}_1 - \frac{32(1+z^2)}{(-1+z)}\mathrm{H}_0^2\mathrm{H}_1^2$$

$$+\left(-\frac{32(35+48z+35z^2)}{3(1+z)}\zeta_2 - \frac{384(1+z^2)}{(1+z)}\zeta_3\right)\mathrm{H}_{-1} + \left(\frac{8(19-16z+19z^2)}{(-1+z)}\mathrm{H}_0\right.$$

$$\left.-224(1+z) + \frac{4(3+5z^2)}{(-1+z)}\mathrm{H}_0^2 + \frac{128(1+z^2)}{(-1+z)}\mathrm{H}_0\mathrm{H}_1 + \frac{32(29+36z+29z^2)}{3(1+z)}\mathrm{H}_{-1}\right.$$

$$-\frac{256(1+z^2)}{(1+z)}\mathrm{H}_{-1}^2 + \frac{192(1+z^2)}{(-1+z)}\mathrm{H}_{0,-1} - \frac{16(-5+3z-7z^2+z^3)\zeta_2}{(-1+z)(1+z)}\Bigg)\mathrm{H}_{0,1}$$

$$-\frac{64(1+z^2)}{(-1+z)}\mathrm{H}_{0,1}^2 + \Bigg(\frac{32(91+48z+91z^2)}{9(1+z)} + \Bigg(-\frac{32(-2+z+4z^2)}{(-1+z)}$$

$$+\frac{256(1+z^2)}{(1+z)}\mathrm{H}_{-1}\Bigg)\mathrm{H}_0 + \frac{8(-3-z-5z^2+z^3)}{(-1+z)(1+z)}\mathrm{H}_0^2 + \frac{192(1+z^2)}{(-1+z)}\mathrm{H}_0\mathrm{H}_1$$

$$+128(1+z)\mathrm{H}_{-1} + \frac{16(7+5z^2)}{(1+z)}\zeta_2\Bigg)\mathrm{H}_{0,-1} - \frac{16(3+2z+3z^2)}{(-1+z)}\mathrm{H}_{0,-1}^2$$

$$+\Bigg(-\frac{32(-3-4z+3z^2+26z^3)}{3(-1+z)(1+z)} + \frac{16(3+z^2)}{(-1+z)}\mathrm{H}_0 + \frac{256(1+z^2)}{(1+z)}\mathrm{H}_{-1}\Bigg)\mathrm{H}_{0,0,1}$$

$$+\Bigg(\frac{16(-5+29z+19z^2+29z^3)}{3(-1+z)(1+z)} - \frac{16(-13+z-15z^2+3z^3)}{(-1+z)(1+z)}\mathrm{H}_0$$

$$-\frac{384(1+z^2)}{(-1+z)}\mathrm{H}_1 - \frac{256(1+z^2)}{(1+z)}\mathrm{H}_{-1}\Bigg)\mathrm{H}_{0,0,-1} - \frac{128(1+z^2)}{(-1+z)}\mathrm{H}_0\mathrm{H}_{0,1,1}$$

$$+\Bigg(-\frac{32(29+36z+29z^2)}{3(1+z)} - \frac{192(1+z^2)}{(-1+z)}\mathrm{H}_0 + \frac{512(1+z^2)}{(1+z)}\mathrm{H}_{-1}\Bigg)[\mathrm{H}_{0,1,-1} + \mathrm{H}_{0,-1,1}]$$

$$+\Bigg(-128(1+z) - \frac{32(-11+3z-13z^2+5z^3)}{(-1+z)(1+z)}\mathrm{H}_0\Bigg)\mathrm{H}_{0,-1,-1} + \frac{128z(1+z^2)}{(1-z)(1+z)}$$

$$\times\mathrm{H}_{0,0,0,1} - \frac{384(1+z^2)}{(-1+z)(1+z)}\mathrm{H}_{0,0,0,-1} + \frac{128(1+z^2)}{(-1+z)}[\mathrm{H}_{0,0,1,1} - \mathrm{H}_{0,-1,0,1}]$$

$$-\frac{256(1+z^2)}{(1+z)}\big[\mathrm{H}_{0,0,1,-1} + \mathrm{H}_{0,0,-1,1} - \mathrm{H}_{0,0,-1,-1}\big] - \frac{512(1+z^2)}{(1+z)}[\mathrm{H}_{0,1,-1,-1}$$

$$+\mathrm{H}_{0,-1,1,-1} + \mathrm{H}_{0,-1,-1,1}] - \frac{32(-63-102z+28z^2)}{9(1+z)}\zeta_2 + \frac{256(1+z^2)}{(1+z)}\mathrm{H}_{-1}^2\zeta_2$$

$$+\frac{4(81+63z-57z^2+121z^3)}{5(-1+z)(1+z)}\zeta_2^2 + \frac{16(-27-56z-11z^2+84z^3)}{3(-1+z)(1+z)}\zeta_3\Bigg]\Bigg\}$$

$$+C_F^2\Bigg\{T_F N_F\Bigg[-\Bigg(\frac{440}{3} - 128\zeta_3 - \frac{2048}{3}(1-z)\mathrm{H}_1\Bigg) + \frac{1300(1-z)}{9}$$

$$+\Bigg(-\frac{512(4+3z+4z^2)}{9(1+z)}\mathrm{H}_{-1} + \frac{128z(1+z^2)}{3(-1+z)(1+z)}\zeta_2$$

$$+\frac{16(-22-8z+45z^2)}{9(-1+z)}\Bigg)\mathrm{H}_0 + \Bigg(\frac{32(-8+24z-3z^2+5z^3)}{9(-1+z)(1+z)} - \frac{64(1+z^2)}{3(1+z)}\mathrm{H}_{-1}\Bigg)$$

$$\times\mathrm{H}_0^2 - \frac{16(3-5z+9z^2+z^3)}{9(-1+z)(1+z)}\mathrm{H}_0^3 + \Bigg(256(-1+z)3 - \frac{256(4-3z+4z^2)}{9(-1+z)}\mathrm{H}_0$$

$$
-\frac{128(1+z^2)}{3(-1+z)}\mathrm{H}_0^2\Bigg)\mathrm{H}_1 + \left(-\frac{128(1+z)}{3} + \frac{128(1+z^2)}{3(-1+z)}\mathrm{H}_0 + \frac{256(1+z^2)}{3(1+z)}\mathrm{H}_{-1}\right)\mathrm{H}_{0,1}
$$

$$
+\frac{512(4+3z+4z^2)}{9(1+z)}\mathrm{H}_{0,-1} + \frac{128(1+z^2)}{3(1+z)}\mathrm{H}_{0,0,-1} - \frac{256(1+z^2)}{3(1+z)}[\mathrm{H}_{0,1,-1}
$$

$$
+\mathrm{H}_{0,-1,1}] - \frac{256z(1+z^2)}{3(-1+z)(1+z)}\mathrm{H}_{0,0,1} - \frac{256(1+z^2)}{3(1+z)}\mathrm{H}_{-1}\zeta_2 - \frac{640(1+z^2)}{9(1+z)}\zeta_2
$$

$$
+\frac{128(1+z-2z^2+4z^3)}{3(-1+z)(1+z)}\zeta_3\Bigg] + C_A\left[\left(-\frac{4(-185-1797z+314z^2+2574z^3)}{9(-1+z)(1+z)}\right.\right.
$$

$$
+\left(\frac{32(173+78z+173z^2)}{9(1+z)} + \frac{1344(1+z^2)}{(1+z)}\zeta_2\right)\mathrm{H}_{-1} + 224(1+z)\mathrm{H}_{-1}^2
$$

$$
-\frac{16(-81-179z+27z^2+151z^3)}{3(-1+z)(1+z)}\zeta_2 - \frac{32(-3+21z-9z^2+25z^3)}{(-1+z)(1+z)}\zeta_3\Bigg)\mathrm{H}_0
$$

$$
+\frac{1064(-1+z)}{9} + \left(\frac{8(118-705z+33z^2+320z^3)}{9(-1+z)(1+z)} - \frac{8(107+276z+107z^2)}{3(1+z)}\mathrm{H}_{-1}\right.
$$

$$
+\frac{288(1+z^2)}{(1+z)}\mathrm{H}_{-1}^2 - \frac{32(-6+6z-3z^2+11z^3)}{(-1+z)(1+z)}\zeta_2\Bigg)\mathrm{H}_0^2
$$

$$
+\left(\frac{4(33-343z+99z^2+119z^3)}{9(-1+z)(1+z)} - \frac{272(1+z^2)}{3(1+z)}\mathrm{H}_{-1}\right)\mathrm{H}_0^3 + \frac{8z(3+7z^2)}{3(-1+z)(1+z)}
$$

$$
\times\mathrm{H}_0^4 + \left(-\frac{3344(-1+z)}{3} + \left(\frac{32(109-84z+109z^2)}{9(-1+z)} - \frac{64(1+z^2)}{(-1+z)}\zeta_2\right)\mathrm{H}_0\right.
$$

$$
+\frac{16(43-24z+43z^2)}{3(-1+z)}\mathrm{H}_0^2 + \frac{176(1+z^2)}{3(-1+z)}\mathrm{H}_0^3 - 224(-1+z)\zeta_2 - \frac{768(1+z^2)}{(-1+z)}\zeta_3\Bigg)
$$

$$
\times\mathrm{H}_1 + \frac{64(1+z^2)}{(-1+z)}\mathrm{H}_0^2\mathrm{H}_1^2 + \left(\frac{1984(2+3z+2z^2)}{3(1+z)}\zeta_2 + \frac{1344(1+z^2)}{(1+z)}\zeta_3\right)\mathrm{H}_{-1}
$$

$$
+\left(\frac{2032(1+z)}{3} + \left(-\frac{256(4-3z+4z^2)}{3(-1+z)} - \frac{64(1+z^2)}{(1+z)}\mathrm{H}_{-1}\right)\mathrm{H}_0 - \frac{64(1+2z^2)}{(-1+z)}\mathrm{H}_0^2\right.
$$

$$
-\frac{256(1+z^2)}{(-1+z)}\mathrm{H}_0\mathrm{H}_1 - \frac{32(103+144z+103z^2)}{3(1+z)}\mathrm{H}_{-1} + \frac{896(1+z^2)}{(1+z)}\mathrm{H}_{-1}^2
$$

$$
-\frac{512(1+z^2)}{(-1+z)}\mathrm{H}_{0,-1} + \frac{64(3-z+3z^2)}{(1+z)}\zeta_2\Bigg)\mathrm{H}_{0,1} + \frac{128(1+z^2)}{(-1+z)}\mathrm{H}_{0,1}^2
$$

$$
+\left(-\frac{32(173+78z+173z^2)}{9(1+z)} + \left(\frac{16(-29+14z+27z^2)}{(-1+z)} - \frac{1024(1+z^2)}{(1+z)}\mathrm{H}_{-1}\right)\right.
$$

$$
\times\mathrm{H}_0 - \frac{32(-3-2z-6z^2+z^3)}{(-1+z)(1+z)}\mathrm{H}_0^2 - \frac{512(1+z^2)}{(-1+z)}\mathrm{H}_0\mathrm{H}_1 - 448(1+z)\mathrm{H}_{-1}
$$

$$
-\frac{64\big(-4+7z-3z^2+6z^3\big)}{(-1+z)(1+z)}\zeta_2\bigg)\mathrm{H}_{0,-1}+\Bigg(\frac{16\big(-39-61z+21z^2+167z^3\big)}{3(-1+z)(1+z)}
$$

$$
+\frac{32\big(-3-z+5z^2+7z^3\big)}{(-1+z)(1+z)}\mathrm{H}_0-\frac{896\big(1+z^2\big)}{(1+z)}\mathrm{H}_{-1}\Bigg)\mathrm{H}_{0,0,1}+\frac{64\big(3+z+3z^2\big)}{(-1+z)}\mathrm{H}_{0,-1}^2
$$

$$
+\Bigg(-\frac{16\big(-67+79z+77z^2+55z^3\big)}{3(-1+z)(1+z)}+\frac{32\big(-29+5z-33z^2+13z^3\big)}{(-1+z)(1+z)}\mathrm{H}_0
$$

$$
+\frac{1024\big(1+z^2\big)}{(-1+z)}\mathrm{H}_1+\frac{896\big(1+z^2\big)}{(1+z)}\mathrm{H}_{-1}\Bigg)\mathrm{H}_{0,0,-1}+\frac{256\big(1+z^2\big)}{(-1+z)}\mathrm{H}_0\mathrm{H}_{0,1,1}
$$

$$
+\Bigg(\frac{32\big(103+144z+103z^2\big)}{3(1+z)}+\frac{64(7+9z)\big(1+z^2\big)}{(-1+z)(1+z)}\mathrm{H}_0-\frac{1792\big(1+z^2\big)}{(1+z)}\mathrm{H}_{-1}\Bigg)
$$

$$
\times[\mathrm{H}_{0,1,-1}+\mathrm{H}_{0,-1,1}]+\Bigg(448(1+z)+\frac{128\big(-11+4z-12z^2+5z^3\big)}{(-1+z)(1+z)}\mathrm{H}_0\Bigg)\mathrm{H}_{0,-1,-1}
$$

$$
+\frac{64\big(-1+6z-4z^2+3z^3\big)}{(-1+z)(1+z)}\mathrm{H}_{0,0,0,1}-\frac{64\big(-26+5z-23z^2+8z^3\big)}{(-1+z)(1+z)}\mathrm{H}_{0,0,0,-1}
$$

$$
-\frac{256\big(1+z^2\big)}{(-1+z)}\mathrm{H}_{0,0,1,1}+\frac{896\big(1+z^2\big)}{(1+z)}[\mathrm{H}_{0,0,1,-1}+\mathrm{H}_{0,0,-1,1}-\mathrm{H}_{0,0,-1,-1}]
$$

$$
+\frac{384\big(1+z^2\big)}{(-1+z)}\mathrm{H}_{0,-1,0,1}+\frac{1792\big(1+z^2\big)}{(1+z)}[\mathrm{H}_{0,1,-1,-1}+\mathrm{H}_{0,-1,1,-1}+\mathrm{H}_{0,-1,-1,1}]
$$

$$
-\frac{16(163+684z+253z^2)}{9(1+z)}\zeta_2-\frac{896\big(1+z^2\big)}{(1+z)}\zeta_2\mathrm{H}_{-1}^2-\frac{16\big(67+67z-3z^2+7z^3\big)}{5(-1+z)(1+z)}\zeta_2^2
$$

$$
-\frac{16\big(-101-209z-2z^2+292z^3\big)}{3(-1+z)(1+z)}\zeta_3\Bigg]\Bigg\}
$$

$$
+C_F^3\Bigg\{-124(-1+z)+\Bigg(\frac{4\big(-15-259z-12z^2+280z^3\big)}{(-1+z)(1+z)}+\Bigg(64(1+z)
$$

$$
-\frac{1280\big(1+z^2\big)\zeta_2}{(1+z)}\Bigg)\mathrm{H}_{-1}+\frac{64(1+2z)\big(-6-2z+7z^2\big)}{(-1+z)(1+z)}\zeta_2-192(1+z)\mathrm{H}_{-1}^2
$$

$$
+\frac{64\big(-3+7z-9z^2+7z^3\big)}{(-1+z)(1+z)}\zeta_3\Bigg)\mathrm{H}_0+\Bigg(-\frac{4\big(-2-100z+89z^2\big)}{(-1+z)}
$$

$$
+\frac{16\big(23+52z+23z^2\big)}{(1+z)}\mathrm{H}_{-1}-\frac{320\big(1+z^2\big)}{(1+z)}\mathrm{H}_{-1}^2+\frac{16\big(13+15z^2\big)}{(1+z)}\zeta_2\Bigg)\mathrm{H}_0^2
$$

$$
+\Bigg(-\frac{8\big(-3+46z+31z^2\big)}{3(1+z)}+\frac{352\big(1+z^2\big)}{3(1+z)}\mathrm{H}_{-1}\Bigg)\mathrm{H}_0^3-\frac{4\big(-1+5z-7z^2+11z^3\big)}{3(-1+z)(1+z)}
$$

$$
\times\mathrm{H}_0^4+\Bigg(1120(-1+z)-32(-1+z)\mathrm{H}_0-\frac{96\big(1+z^2\big)}{(-1+z)}\mathrm{H}_0^2-\frac{64\big(1+z^2\big)}{3(-1+z)}\mathrm{H}_0^3
$$

$$+192(-1+z)\zeta_2+\frac{384(1+z^2)}{(-1+z)}\zeta_3\Big)H_1+\Big(-\frac{384(3+5z+3z^2)}{(1+z)}\zeta_2$$

$$-\frac{1152(1+z^2)}{(1+z)}\zeta_3\Big)H_{-1}+\Big(-648(1+z)+\Big(\frac{16(9-8z+11z^2)}{(-1+z)}$$

$$+\frac{128(1+z^2)}{(1+z)}H_{-1}\Big)H_0+\frac{48(1+3z^2)}{(-1+z)}H_0^2+\frac{192(5+8z+5z^2)}{(1+z)}H_{-1}$$

$$+\frac{128(1+z^2)}{(-1+z)}H_0H_1-\frac{768(1+z^2)}{(1+z)}H_{-1}^2+\frac{256(1+z^2)}{(-1+z)}H_{0,-1}-\frac{192(1+z^2)}{(1+z)}\zeta_2\Big)H_{0,1}$$

$$-\frac{64(1+z^2)}{(-1+z)}H_{0,1}^2+\Big(-64(1+z)+\Big(-32(21+11z)+\frac{1024(1+z^2)}{(1+z)}H_{-1}\Big)H_0$$

$$+\frac{32(-3-3z-7z^2+z^3)}{(-1+z)(1+z)}H_0^2+\frac{256(1+z^2)}{(-1+z)}H_0H_1+384(1+z)H_{-1}$$

$$+\frac{64(-1+7z)(1+z^2)}{(-1+z)(1+z)}\zeta_2\Big)H_{0,-1}-\frac{192(1+z^2)}{(-1+z)}H_{0,-1}^2+\Big(-\frac{96(3+8z+7z^2)}{(1+z)}$$

$$-\frac{64(-1+z+3z^2+5z^3)}{(-1+z)(1+z)}H_0+\frac{768(1+z^2)}{(1+z)}H_{-1}\Big)H_{0,0,1}+\Big(\frac{32(19+12z-z^2)}{(1+z)}$$

$$-\frac{128(-2+z)(4+z+5z^2)}{(-1+z)(1+z)}H_0-\frac{512(1+z^2)}{(-1+z)}H_1-\frac{768(1+z^2)}{(1+z)}H_{-1}\Big)H_{0,0,-1}$$

$$-\frac{128(1+z^2)}{(-1+z)}H_0H_{0,1,1}+\Big(-\frac{192(5+8z+5z^2)}{(1+z)}-\frac{128(1+3z)(1+z^2)}{(-1+z)(1+z)}H_0$$

$$+\frac{1536(1+z^2)}{(1+z)}H_{-1}\Big)H_{0,1,-1}+\Big(-\frac{192(5+8z+5z^2)}{(1+z)}-\frac{128(1+3z)(1+z^2)}{(-1+z)(1+z)}H_0$$

$$+\frac{1536(1+z^2)}{(1+z)}H_{-1}\Big)H_{0,-1,1}+\Big(-384(1+z)-\frac{128(-11+5z)(1+z^2)}{(-1+z)(1+z)}H_0\Big)$$

$$\times H_{0,-1,-1}+\frac{128(-14+5z-11z^2+8z^3)}{(-1+z)(1+z)}H_{0,0,0,-1}-\frac{64(1-4z+z^2)}{(1+z)}H_{0,0,0,1}$$

$$+\frac{128(1+z^2)}{(-1+z)}H_{0,0,1,1}-z\frac{768(1+z^2)}{(1+z)z}[H_{0,0,1,-1}+H_{0,0,-1,1}-H_{0,0,-1,-1}]$$

$$-\frac{1536(1+z^2)}{(1+z)}[H_{0,1,-1,-1}+H_{0,-1,1,-1}+H_{0,-1,-1,1}]-\frac{256(1+z^2)}{(-1+z)}H_{0,-1,0,1}$$

$$+8(77+93z)\zeta_2+\frac{768(1+z^2)}{(1+z)}H_{-1}^2\zeta_2-\frac{16(-81-11z-7z^2+63z^3)}{5(-1+z)(1+z)}\zeta_2^2$$

$$+\frac{96(2+3z)(2+5z)}{(1+z)}\zeta_3\Big\}\Big\}. \tag{116}$$

For transversity the contributions to the splitting function $P_2^{\mathrm{NS},+,\mathrm{tr}}$ read

$$
P_{\mathrm{NS}}^{(2),+,\mathrm{tr},\delta} = -\Bigg\{ C_F^2 \Bigg[C_A \Bigg[-\frac{151}{2} + \frac{820}{3}\zeta_2 + \frac{1976}{15}\zeta_2^2 + \left(-\frac{1688}{3} - 32\zeta_2 \right)\zeta_3 - 240\zeta_5 \Bigg]
$$

$$
+ T_F N_F \Bigg[92 - \frac{80\zeta_2}{3} - \frac{928\zeta_2^2}{15} + \frac{544\zeta_3}{3} \Bigg] \Bigg] + C_F \Bigg[C_A T_F N_F \Bigg[-80 + \frac{5344\zeta_2}{27}
$$

$$
- \frac{16\zeta_2^2}{5} - \frac{800\zeta_3}{9} \Bigg] + T_F^2 N_F^2 \Bigg[\frac{136}{9} - \frac{640\zeta_2}{27} + \frac{128\zeta_3}{9} \Bigg] + C_A^2 \Bigg[\frac{1657}{18} - \frac{8992\zeta_2}{27}
$$

$$
+ 4\zeta_2^2 + \frac{3104\zeta_3}{9} - 80\zeta_5 \Bigg] \Bigg] + C_F^3 \Bigg\{ -29 - 36\zeta_2 - \frac{576}{5}\zeta_2^2 + \left(-136 + 64\zeta_2 \right)\zeta_3
$$

$$
+ 480\zeta_5 \Bigg\} \Bigg\} \delta(1-z), \tag{117}
$$

$$
P_{\mathrm{NS}}^{(2),+,\mathrm{tr},\mathrm{plu}} = -\frac{1}{1-z} \Bigg\{ C_F \Bigg[T_F^2 N_F^2 \frac{128}{27} + C_A^2 \Bigg[-\frac{4}{45}\left(3675 - 2680\zeta_2 + 792\zeta_2^2 \right)
$$

$$
- \frac{176\zeta_3}{3} \Bigg] + C_A T_F N_F \Bigg[-\frac{16}{27}\left(-209 + 120\zeta_2 \right) + \frac{448\zeta_3}{3} \Bigg] \Bigg]
$$

$$
+ \frac{8}{3} C_F^2 T_F N_F \Bigg[55 - 48\zeta_3 \Bigg] \Bigg\}, \tag{118}
$$

$$
P_{\mathrm{NS}}^{(2),+,\mathrm{tr},\mathrm{reg}} = -\Bigg\{ C_F^2 \Bigg\{ T_F N_F \Bigg[-\frac{440}{3} - \left(\frac{16z(-9+4z)}{3(-1+z)} - \frac{2560z}{9(1+z)}\mathrm{H}_{-1} \right.
$$

$$
\left. + \frac{256z\zeta_2}{3(1-z)(1+z)} \right)\mathrm{H}_0 + \left(-\frac{32z(-29+11z)}{9(-1+z)(1+z)} + \frac{128z}{3(1+z)}\mathrm{H}_{-1} \right)\mathrm{H}_0^2
$$

$$
+ \frac{64z(-1+3z)}{9(1-z)(1+z)}\mathrm{H}_0^3 + \left(\frac{64}{3}(1-z) + \frac{1280z}{9(1-z)}\mathrm{H}_0 + \frac{256z}{3(1-z)}\mathrm{H}_0^2 \right)\mathrm{H}_1 + \left(\frac{256z}{3(-1+z)}\mathrm{H}_0 \right.
$$

$$
\left. - \frac{512z}{3(1+z)}\mathrm{H}_{-1} \right)\mathrm{H}_{0,1} - \frac{2560z}{9(1+z)}\mathrm{H}_{0,-1} - \frac{512z}{3(-1+z)(1+z)}\mathrm{H}_{0,0,1}
$$

$$
- \frac{256z}{3(1+z)}\mathrm{H}_{0,0,-1} + \frac{512z}{3(1+z)}\mathrm{H}_{0,1,-1} + \frac{512z}{3(1+z)}\mathrm{H}_{0,-1,1} + \frac{1280z\zeta_2}{9(1+z)}
$$

$$
+ \frac{512z\zeta_2}{3(1+z)}\mathrm{H}_{-1} + \frac{128(3+z)(-1+2z)\zeta_3}{3(-1+z)(1+z)} \Bigg] + C_A \Bigg[\left(\frac{4z\left(-537 - 43z + 278z^2 \right)}{3(-1+z)(1+z)} \right.
$$

$$
+ \left(\frac{16\left(27 - 482z + 27z^2 \right)}{9(1+z)} - \frac{2688z\zeta_2}{1+z} \right)\mathrm{H}_{-1} - \frac{48(1+z)\left(1+z^2 \right)}{z}\mathrm{H}_{-1}^2
$$

$$
- \frac{32z\left(4 - 36z - 18z^2 + 9z^3 \right)\zeta_2}{3(-1+z)(1+z)} + \frac{64z(-23+6z)\zeta_3}{(-1+z)(1+z)} \Bigg)\mathrm{H}_0
$$

$$
+\left(\frac{8z\left(-412+151z+27z^2\right)}{9(-1+z)(1+z)}+\frac{16\left(9-13z+9z^2\right)}{3(1+z)}H_{-1}-\frac{576z}{1+z}H_{-1}^2\right.
$$

$$
\left.+\frac{32z(-17+9z)\zeta_2}{(-1+z)(1+z)}\right)H_0^2+\left(-\frac{16z\left(47-33z+9z^2\right)}{9(-1+z)(1+z)}+\frac{544z}{3(1+z)}H_{-1}\right)H_0^3
$$

$$
+\frac{80z}{3(-1+z)(1+z)}H_0^4+\left(\frac{1112}{3}(-1+z)+\left(\frac{4288z}{9(-1+z)}-\frac{128z\zeta_2}{-1+z}\right)H_0\right.
$$

$$
\left.-\frac{992z}{3(1-z)}H_0^2+\frac{352z}{3(-1+z)}H_0^3+\frac{48(1-z)\left(1+z^2\right)\zeta_2}{z}+\frac{1536z\zeta_3}{1-z}\right)H_1
$$

$$
-\frac{128z}{1-z}H_0^2H_1^2+\left(\frac{16\left(9-18z-178z^2-18z^3+9z^4\right)\zeta_2}{3z(1+z)}-\frac{2688z\zeta_3}{1+z}\right)H_{-1}
$$

$$
+\left(96(1+z)+\left(-\frac{1280z}{3(-1+z)}+\frac{128z}{1+z}H_{-1}\right)H_0+\frac{192z}{1-z}H_0^2+\frac{512z}{1-z}H_0H_1\right.
$$

$$
\left.-\frac{32\left(9-80z^2+9z^4\right)}{3z(1+z)}H_{-1}-\frac{1792z}{1+z}H_{-1}^2+\frac{1024z}{1-z}H_{0,-1}-\frac{448z\zeta_2}{1+z}\right)H_{0,1}
$$

$$
-\frac{256z}{1-z}H_{0,1}^2+\left(-\frac{16\left(27-482z+27z^2\right)}{9(1+z)}+\left(\frac{96\left(1+z-z^2+z^3\right)}{-1+z}+\frac{2048z}{1+z}H_{-1}\right)\right.
$$

$$
\times H_0-\frac{32z(1+9z)}{(1-z)(1+z)}H_0^2+\frac{1024z}{1-z}H_0H_1+\frac{96(1+z)\left(1+z^2\right)}{z}H_{-1}
$$

$$
\left.+\frac{64z(13-7z)\zeta_2}{(1-z)(1+z)}\right)H_{0,-1}-\frac{448z}{1-z}H_{0,-1}^2+\left(\frac{32z\left(62-18z-9z^2+9z^3\right)}{3(-1+z)(1+z)}\right.
$$

$$
\left.+\frac{64z(3+z)}{(-1+z)(1+z)}H_0+\frac{1792z}{1+z}H_{-1}\right)H_{0,0,1}+\left(\frac{32\left(9+58z-22z^2+9z^3+18z^4\right)}{3(1-z)(1+z)}\right.
$$

$$
\left.+\frac{64z(-9+31z)}{(1-z)(1+z)}H_0-\frac{2048z}{1-z}H_1-\frac{1792z}{1+z}H_{-1}\right)H_{0,0,-1}-\frac{512z}{1-z}H_0H_{0,1,1}
$$

$$
+\left(\frac{32\left(9-80z^2+9z^4\right)}{3z(1+z)}-\frac{128z(9+7z)}{(1-z)(1+z)}H_0+\frac{3584z}{1+z}H_{-1}\right)H_{0,1,-1}
$$

$$
+\left(\frac{32\left(9-80z^2+9z^4\right)}{3z(1+z)}-\frac{128z(9+7z)}{(1-z)(1+z)}H_0+\frac{3584z}{1+z}H_{-1}\right)H_{0,-1,1}
$$

$$
+\left(-\frac{96(1+z)\left(1+z^2\right)}{z}+\frac{128z(-9+23z)}{(1-z)(1+z)}H_0\right)H_{0,-1,-1}+\frac{64z(-9+5z)}{(1-z)(1+z)}H_{0,0,0,1}
$$

$$
-\frac{64z(-13+49z)}{(1-z)(1+z)}H_{0,0,0,-1}+\frac{512z}{1-z}H_{0,0,1,1}-\frac{1792z}{1+z}\left(H_{0,0,1,-1}+H_{0,0,-1,1}\right.
$$

$$
\left.-H_{0,0,-1,-1}\right)-\frac{3584z}{1+z}\left(H_{0,1,-1,-1}+H_{0,-1,1,-1}+H_{0,-1,-1,1}\right)-\frac{768z}{1-z}H_{0,-1,0,1}
$$

$$
-\frac{16\left(54+349z+27z^2\right)\zeta_2}{9(1+z)}+\frac{1792z\zeta_2}{1+z}\mathrm{H}_{-1}^2+\frac{32z(37+32z)\zeta_2^2}{5(1-z)(1+z)}
$$

$$
+\frac{16z\left(128-112z-45z^2+9z^3\right)\zeta_3}{3(1-z)(1+z)}\Bigg]\Bigg\}
$$

$$
+C_F\Bigg\{T_F^2N_F^2\Bigg[\frac{64}{27}(-5+3z)-\frac{640z}{27(1-z)}\mathrm{H}_0-\frac{64z}{9(1-z)}\mathrm{H}_0^2\Bigg]
$$

$$
+C_AT_FN_F\Bigg[-\frac{176}{27}(13+6z)+\left(\frac{16z(-343+9z)}{27(-1+z)}-\frac{64z(1+3z)\zeta_2}{3(1-z)(1+z)}\right.
$$

$$
-\frac{1280z}{9(1+z)}\mathrm{H}_{-1}\Bigg)\mathrm{H}_0+\left(\frac{32z(31+11z)}{9(1-z)(1+z)}-\frac{64z}{3(1+z)}\mathrm{H}_{-1}\right)\mathrm{H}_0^2+\frac{128z}{9(1-z)(1+z)}
$$

$$
\times\mathrm{H}_0^3-\left(\frac{16}{3}(1-z)+\frac{32z}{1-z}\mathrm{H}_0^2\right)\mathrm{H}_1+\left(\frac{64z}{1-z}\mathrm{H}_0+\frac{256z}{3(1+z)}\mathrm{H}_{-1}\right)\mathrm{H}_{0,1}
$$

$$
+\frac{1280z}{9(1+z)}\mathrm{H}_{0,-1}-\frac{256z}{3(1-z)(1+z)}\mathrm{H}_{0,0,1}+\frac{128z}{3(1+z)}\mathrm{H}_{0,0,-1}-\frac{256z}{3(1+z)}[\mathrm{H}_{0,1,-1}
$$

$$
+\mathrm{H}_{0,-1,1}]+\frac{640\zeta_2}{9(1+z)}-\frac{256z\zeta_2}{3(1+z)}\mathrm{H}_{-1}+\frac{64\left(-7+5z+6z^2\right)\zeta_3}{3(1-z)(1+z)}\Bigg]
$$

$$
+C_A^2\Bigg[\frac{4}{9}(614+121z)+\left(\frac{8z\left(-1565-881z+360z^2\right)}{27(1-z)(1+z)}\right.
$$

$$
+\left(-\frac{16\left(9-250z+9z^2\right)}{9(1+z)}+\frac{704z\zeta_2}{1+z}\right)\mathrm{H}_{-1}+\frac{16(1+z)\left(1-z+z^2\right)}{z}\mathrm{H}_{-1}^2
$$

$$
-\frac{16z\left(-29-39z+6z^3\right)\zeta_2}{3(1-z)(1+z)}+\frac{512z\zeta_3}{(-1+z)(1+z)}\Bigg)\mathrm{H}_0+\left(\frac{176z}{3(1+z)}\mathrm{H}_{-1}\right.
$$

$$
+\frac{128z}{1+z}\mathrm{H}_{-1}^2+\frac{4z\left(-675-121z+18z^2\right)}{9(1-z)(1+z)}-\frac{96z\zeta_2}{1+z}\Bigg)\mathrm{H}_0^2+\left(-\frac{496z}{9(1-z)(1+z)}\right.
$$

$$
-\frac{32z}{1+z}\mathrm{H}_{-1}\Bigg)\mathrm{H}_0^3+\frac{16z}{3(1-z)(1+z)}\mathrm{H}_0^4+\left(\frac{320}{3}(1-z)+\frac{88z}{1-z}\mathrm{H}_0^2+\frac{32z}{1-z}\mathrm{H}_0^3\right.
$$

$$
+\frac{16(-1+z)\left(1+z+z^2\right)\zeta_2}{z}+\frac{64z\zeta_2}{1-z}\mathrm{H}_0-\frac{576z\zeta_3}{1-z}\Bigg)\mathrm{H}_1+\frac{64z}{1-z}\mathrm{H}_0^2\mathrm{H}_1^2+\left(\frac{768z\zeta_3}{1+z}\right.
$$

$$
-\frac{16\left(3+3z-44z^2+3z^3+3z^4\right)\zeta_2}{3z(1+z)}\Bigg)\mathrm{H}_{-1}+\left(-32(1+z)-\frac{176z}{1-z}\mathrm{H}_0-\frac{32z}{1-z}\mathrm{H}_0^2\right.
$$

$$
-\frac{256z}{1-z}\mathrm{H}_0\mathrm{H}_1+\frac{32\left(3+3z-22z^2+3z^3+3z^4\right)}{3z(1+z)}\mathrm{H}_{-1}+\frac{512z}{1+z}\mathrm{H}_{-1}^2-\frac{384z}{1-z}\mathrm{H}_{0,-1}
$$

$$
+\frac{64z(1-3z)\zeta_2}{(1-z)(1+z)}\Bigg)\mathrm{H}_{0,1}+\frac{128z}{1-z}\mathrm{H}_{0,1}^2+\left(\frac{16\left(9-250z+9z^2\right)}{9(1+z)}+\left(-\frac{512z}{1+z}\mathrm{H}_{-1}\right.\right.
$$

$$+ \frac{32z(3 - z + z^2)}{1 - z}\bigg)H_0 + \frac{64z^2}{(1 - z)(1 + z)}H_0^2 - \frac{384z}{1 - z}H_0H_1$$

$$- \frac{32(1 + z)(1 - z + z^2)}{z}H_{-1} - \frac{192z\zeta_2}{1 + z}\bigg)H_{0,-1} + \frac{128z}{1 - z}H_{0,-1}^2$$

$$+ \left(\frac{32z(22 - 3z + 3z^3)}{3(1 - z)(1 + z)} - \frac{64z}{1 - z}H_0 - \frac{512z}{1 + z}H_{-1}\right)H_{0,0,1} + \left(-\frac{32z(29 + z + 6z^3)}{3(1 - z)(1 + z)}\right.$$

$$+ \frac{64z(1 - 7z)}{(1 - z)(1 + z)}H_0 + \frac{768z}{1 - z}H_1 + \frac{512z}{1 + z}H_{-1}\bigg)H_{0,0,-1} + \frac{256z}{1 - z}H_0H_{0,1,1}$$

$$+ \left(-\frac{32(3 + 3z - 22z^2 + 3z^3 + 3z^4)}{3z(1 + z)} + \frac{384z}{1 - z}H_0 - \frac{1024z}{1 + z}H_{-1}\right)H_{0,1,-1}$$

$$+ \left(-\frac{32(3 + 3z - 22z^2 + 3z^3 + 3z^4)}{3z(1 + z)} + \frac{384z}{1 - z}H_0 - \frac{1024z}{1 + z}H_{-1}\right)H_{0,-1,1}$$

$$+ \left(\frac{32(1 + z)(1 - z + z^2)}{z} + \frac{256z(1 - 3z)}{(1 - z)(1 + z)}H_0\right)H_{0,-1,-1} + \frac{256z}{(1 - z)(1 + z)}H_{0,0,0,1}$$

$$+ \frac{768z^2}{(1 - z)(1 + z)}H_{0,0,0,-1} - \frac{256z}{1 - z}H_{0,0,1,1} + \frac{512z}{1 + z}H_{0,0,1,-1} + \frac{512z}{1 + z}H_{0,0,-1,1}$$

$$- \frac{512z}{1 + z}H_{0,0,-1,-1} + \frac{1024z}{1 + z}H_{0,1,-1,-1} + \frac{256z}{1 - z}H_{0,-1,0,1} + \frac{1024z}{1 + z}H_{0,-1,1,-1}$$

$$+ \frac{1024z}{1 + z}H_{0,-1,-1,1} + \frac{16(-116 + 27z + 9z^2)\zeta_2}{9(1 + z)} - \frac{512z\zeta_2}{1 + z}H_{-1}^2$$

$$- \frac{32(-11 + 23z + 14z^2)\zeta_2^2}{5(1 - z)(1 + z)} - \frac{16(-11 + 28z - 30z^2 + 3z^4)\zeta_3}{3(1 - z)(1 + z)}\bigg]\bigg\}$$

$$+ C_F^3\left\{\left(\frac{4z(-85 + 27z + 64z^2)}{(1 - z)(1 + z)} + \left(-32(1 + z) + \frac{2560z\zeta_2}{1 + z}\right)H_{-1} + \frac{32(1 + z)^3}{z}\right.\right.$$

$$\times H_{-1}^2 - \frac{64z(6 - 4z - 6z^2 + z^3)\zeta_2}{(1 - z)(1 + z)} - \frac{128z(7 - 6z)\zeta_3}{(1 - z)(1 + z)}\bigg)H_0 + \left(-\frac{4z(-17 + 4z)}{-1 + z}\right.$$

$$- \frac{96(1 + z + z^2)}{1 + z}H_{-1} + \frac{640z}{1 + z}H_{-1}^2 - \frac{448z\zeta_2}{1 + z}\bigg)H_0^2 + \left(\frac{16z(-1 + 2z)}{1 + z}\right.$$

$$- \frac{704z}{3(1 + z)}H_{-1}\bigg)H_0^3 + \frac{32(2 - z)z}{3(1 - z)(1 + z)}H_0^4 + \left(-256(-1 + z) + \frac{192z}{1 - z}H_0^2\right.$$

$$+ \frac{128z}{3(1 - z)}H_0^3 - \frac{32(1 - z)^3\zeta_2}{z} - \frac{768z\zeta_3}{1 - z}\bigg)H_1 + \left(\frac{2304z\zeta_3}{1 + z}\right.$$

$$- \frac{32(1 - 8z - 30z^2 - 8z^3 + z^4)\zeta_2}{z(1 + z)}\bigg)H_{-1} + \left(-64(1 + z) - \left(\frac{192z}{1 - z}\right.\right.$$

$$+ \frac{256z}{1+z} \mathrm{H}_{-1}\Bigg) \mathrm{H}_0 - \frac{192z}{1-z} \mathrm{H}_0^2 - \frac{256z}{1-z} \mathrm{H}_0 \mathrm{H}_1 + \frac{64\left(1 - 2z - 12z^2 - 2z^3 + z^4\right)}{z(1+z)} \mathrm{H}_{-1}$$

$$+ \frac{1536z}{1+z} \mathrm{H}_{-1}^2 - \frac{512z}{1-z} \mathrm{H}_{0,-1} + \frac{384z\zeta_2}{1+z}\Bigg) \mathrm{H}_{0,1} + \frac{128z}{1-z} \mathrm{H}_{0,1}^2 + \Bigg(32(1+z)$$

$$+ \Bigg(-64\left(-3 + z^2\right) - \frac{2048z}{1+z} \mathrm{H}_{-1}\Bigg) \mathrm{H}_0 + \frac{64z(1+5z)}{(1-z)(1+z)} \mathrm{H}_0^2 - \frac{512z}{1-z} \mathrm{H}_0 \mathrm{H}_1$$

$$- \frac{64(1+z)^3}{z} \mathrm{H}_{-1} - \frac{128(7-z)z\zeta_2}{(1-z)(1+z)}\Bigg) \mathrm{H}_{0,-1} + \frac{384z}{1-z} \mathrm{H}_{0,-1}^2 + \Bigg(\frac{128z(3+z)}{(1-z)(1+z)} \mathrm{H}_0$$

$$- \frac{64z\left(-6 - 2z + z^2\right)}{1+z} - \frac{1536z}{1+z} \mathrm{H}_{-1}\Bigg) \mathrm{H}_{0,0,1} + \Bigg(\frac{64\left(-3 - 3z + 5z^2 + 2z^3\right)}{1+z}$$

$$- \frac{128z(-7+17z)}{(1-z)(1+z)} \mathrm{H}_0 + \frac{1024z}{1-z} \mathrm{H}_1 + \frac{1536z}{1+z} \mathrm{H}_{-1}\Bigg) \mathrm{H}_{0,0,-1} + \frac{256z}{1-z} \mathrm{H}_0 \mathrm{H}_{0,1,1}$$

$$+ \Bigg(-\frac{64\left(1 - 2z - 12z^2 - 2z^3 + z^4\right)}{z(1+z)} + \frac{256z(3+z)}{(1-z)(1+z)} \mathrm{H}_0 - \frac{3072z}{1+z} \mathrm{H}_{-1}\Bigg) \mathrm{H}_{0,1,-1}$$

$$+ \Bigg(-\frac{64\left(1 - 2z - 12z^2 - 2z^3 + z^4\right)}{z(1+z)} + \frac{256z(3+z)}{(1-z)(1+z)} \mathrm{H}_0 - \frac{3072z}{1+z} \mathrm{H}_{-1}\Bigg) \mathrm{H}_{0,-1,1}$$

$$+ \Bigg(\frac{64(1+z)^3}{z} + \frac{256z(5-11z)}{(1-z)(1+z)} \mathrm{H}_0 \Bigg) \mathrm{H}_{0,-1,-1} + \frac{384z}{1+z} \mathrm{H}_{0,0,0,1} - \frac{128z(13-25z)}{(1-z)(1+z)}$$

$$\times \mathrm{H}_{0,0,0,-1} - \frac{256z}{1-z} \mathrm{H}_{0,0,1,1} + \frac{1536z}{1+z} [\mathrm{H}_{0,0,1,-1} + \mathrm{H}_{0,0,-1,1} - \mathrm{H}_{0,0,-1,-1}]$$

$$+ \frac{3072z}{1+z} [\mathrm{H}_{0,1,-1,-1} + \mathrm{H}_{0,-1,1,-1} + \mathrm{H}_{0,-1,-1,1}] + \frac{512z}{1-z} \mathrm{H}_{0,-1,0,1} + 32(2+z)\zeta_2$$

$$- \frac{1536z\zeta_2}{1+z} \mathrm{H}_{-1}^2 + \frac{64z(13-22z)\zeta_2^2}{5(1-z)(1+z)} + \frac{32z\left(-24 - 14z + z^2\right)\zeta_3}{1+z}\Bigg\}\Bigg\}. \tag{119}$$

In the following we use the subsidiary functions

$$p_{qq} = \frac{1+z^2}{1+z}, \tag{120}$$

$$p_{qq}^{\mathrm{tr}} = \frac{z}{1+z}. \tag{121}$$

The difference terms between the $+$ and $-$-type splitting functions are given by

$$P_{\mathrm{NS}}^{(2),+}(z) - P_{\mathrm{NS}}^{(2),-}(z) =$$

$$- \Bigg\{ \Bigg(C_F - \frac{C_A}{2} \Bigg) \Bigg\{ C_F T_F N_F \Bigg[-\frac{3904}{9}(-1+z) + p_{qq} \Bigg(-\frac{128}{3} \mathrm{H}_{-1} \mathrm{H}_0^2 + \frac{128}{9} \mathrm{H}_0^3$$

$$+ \frac{512}{3} \mathrm{H}_{-1} \mathrm{H}_{0,1} - \frac{256}{3} \mathrm{H}_{0,0,1} + \frac{256}{3} \mathrm{H}_{0,0,-1} - \frac{512}{3} \mathrm{H}_{0,1,-1} - \frac{512}{3} \mathrm{H}_{0,-1,1} - \frac{1280}{9}\zeta_2$$

$$+\frac{128}{3}\mathrm{H}_0\zeta_2 - \frac{512}{3}\mathrm{H}_{-1}\zeta_2 + 128\zeta_3\bigg) + \frac{2624}{9}(1+z)\mathrm{H}_0$$

$$-\frac{1024\big(4+3z+4z^2\big)}{9(1+z)}\mathrm{H}_{-1}\mathrm{H}_0 + \frac{64\big(19+18z+19z^2\big)}{9(1+z)}\mathrm{H}_0^2$$

$$+\frac{512}{3}(-1+z)\mathrm{H}_1 - \frac{256}{3}(1+z)\mathrm{H}_{0,1} + \frac{1024\big(4+3z+4z^2\big)}{9(1+z)}\mathrm{H}_{0,-1}\bigg] + C_F^2\bigg[\frac{7424}{9}(1-z)$$

$$+p_{qq}\bigg(\big(-128\mathrm{H}_{-1}^2 + 64\zeta_2\big)\mathrm{H}_0^2 + \frac{320}{3}\mathrm{H}_{-1}\mathrm{H}_0^3 + \big(256\mathrm{H}_{-1}\mathrm{H}_0 + 512\mathrm{H}_{-1}^2 + 128\zeta_2\big)\mathrm{H}_{0,1}$$

$$+\big(-128\mathrm{H}_0 - 512\mathrm{H}_{-1}\big)\mathrm{H}_{0,0,1} + 512\mathrm{H}_{-1}\mathrm{H}_{0,0,-1} + \big(-256\mathrm{H}_0 - 1024\mathrm{H}_{-1}\big)\mathrm{H}_{0,1,-1}$$

$$+\big(-256\mathrm{H}_0 - 1024\mathrm{H}_{-1}\big)\mathrm{H}_{0,-1,1} + 128\mathrm{H}_{0,0,0,1} + 896\mathrm{H}_{0,0,0,-1} + 512[\mathrm{H}_{0,0,1,-1}$$

$$+\mathrm{H}_{0,0,-1,1} - \mathrm{H}_{0,0,-1,-1}] + 1024[\mathrm{H}_{0,1,-1,-1} + \mathrm{H}_{0,-1,1,-1} + \mathrm{H}_{0,-1,-1,1}] + 256\mathrm{H}_{-1}\mathrm{H}_0\zeta_2$$

$$-512\mathrm{H}_{-1}^2\zeta_2 + 768\mathrm{H}_{-1}\zeta_3\bigg) + \bigg(\frac{32\big(-103-8z+41z^2\big)}{9(1+z)} + \frac{512\big(41+15z+41z^2\big)}{9(1+z)}\mathrm{H}_{-1}$$

$$+\frac{64\big(-11-12z+4z^2\big)}{3(1+z)}\zeta_2 - \frac{128\big(2+z^2\big)}{1+z}\zeta_3\bigg)\mathrm{H}_0 + \bigg(-\frac{32\big(188+135z+215z^2\big)}{9(1+z)}$$

$$+\frac{32\big(25+24z+25z^2\big)}{3(1+z)}\mathrm{H}_{-1}\bigg)\mathrm{H}_0^2 - \frac{160\big(2+6z+11z^2\big)}{9(1+z)}\mathrm{H}_0^3 - \frac{32z^2}{3(1+z)}\mathrm{H}_0^4$$

$$+\bigg(-\frac{3200}{3}(-1+z) + 256(-1+z)\mathrm{H}_0 - 128(-1+z)\zeta_2\bigg)\mathrm{H}_1$$

$$+\bigg(\frac{1600(1+z)}{3} - 128(1+z)\mathrm{H}_0 - \frac{128\big(25+24z+25z^2\big)}{3(1+z)}\mathrm{H}_{-1}\bigg)\mathrm{H}_{0,1}$$

$$+\bigg(-\frac{512\big(41+15z+41z^2\big)}{9(1+z)} + 704(-1+z)\mathrm{H}_0 + 64(-1+z)\mathrm{H}_0^2$$

$$-\frac{128\big(3+z^2\big)}{1+z}\zeta_2\bigg)\mathrm{H}_{0,-1} + 128(1-z)\mathrm{H}_{0,-1}^2 + \frac{64\big(19+24z+31z^2\big)}{3(1+z)}\mathrm{H}_{0,0,1}$$

$$+\bigg(-\frac{64\big(-41+24z+91z^2\big)}{3(1+z)} - \frac{128\big(3+5z^2\big)}{1+z}\mathrm{H}_0\bigg)\mathrm{H}_{0,0,-1}$$

$$+\frac{128\big(25+24z+25z^2\big)}{3(1+z)}[\mathrm{H}_{0,1,-1} + \mathrm{H}_{0,-1,1}] - 256(1-z)\mathrm{H}_0\mathrm{H}_{0,-1,-1}$$

$$+\frac{128\big(49-45z+40z^2\big)}{9(1+z)}\zeta_2 + \frac{128\big(25+24z+25z^2\big)}{3(1+z)}\mathrm{H}_{-1}\zeta_2$$

$$-\frac{64\big(7+8z^2\big)}{1+z}\zeta_2^2 - \frac{64\big(24+12z+z^2\big)}{1+z}\zeta_3\bigg]\bigg\} + \bigg(C_F - \frac{C_A}{2}\bigg)^2 C_F\bigg\{\frac{11744}{9}(-1+z)$$

$$
+p_{qq}\bigg(-512H_{-1}^2 H_0^2 + 128H_{-1}H_0^3 + \big(-2048H_{-1}^2 - 512\zeta_2\big)H_{0,1} + 2048H_{-1}H_0 H_{0,-1}
$$

$$
+2048H_{-1}H_{0,0,1} - 2048H_{-1}H_{0,0,-1} + 4096H_{-1}[H_{0,1,-1} + H_{0,-1,1}] - 512H_{0,0,0,1}
$$

$$
+1536H_{0,0,0,-1} - 2048[H_{0,0,1,-1} + H_{0,0,-1,1} - H_{0,0,-1,-1}] - 4096[H_{0,1,-1,-1}
$$

$$
+H_{0,-1,1,-1} + H_{0,-1,-1,1}] - 2816H_{-1}H_0\zeta_2 + 2048H_{-1}^2\zeta_2 - 3072H_{-1}\zeta_3\bigg)
$$

$$
+\bigg(\frac{32\big(-41 + 386z + 535z^2\big)}{9(1+z)} - \frac{128\big(173 + 78z + 173z^2\big)}{9(1+z)}H_{-1}
$$

$$
+\frac{64\big(47 + 132z + 89z^2\big)}{3(1+z)}\zeta_2 + \frac{128\big(7 + 9z^2\big)}{1+z}\zeta_3\bigg)H_0
$$

$$
+\bigg(\frac{128\big(38 + 29z^2\big)}{9(1+z)} + \frac{64\big(13 + 48z + 13z^2\big)}{3(1+z)}H_{-1}
$$

$$
+\frac{32\big(11 + 13z^2\big)}{1+z}\zeta_2\bigg)H_0^2 + \frac{64\big(5 - 18z + 8z^2\big)H_0^3}{9(1+z)} - \frac{8\big(3 + 5z^2\big)H_0^4}{3(1+z)} + \bigg(\frac{8960}{3}(-1 + z)
$$

$$
+512(-1 + z)\zeta_2\bigg)H_1 + \bigg(-\frac{4480}{3}(1+z) + \frac{256\big(35 + 48z + 35z^2\big)}{3(1+z)}H_{-1}\bigg)H_{0,1}
$$

$$
+\bigg(\frac{128\big(173 + 78z + 173z^2\big)}{9(1+z)} - 128(3 + 13z)H_0 + \frac{64\big(1 + 3z^2\big)}{1+z}H_0^2 + \frac{128\big(7 + 5z^2\big)}{1+z}\zeta_2\bigg)
$$

$$
\times H_{0,-1} + 128(-1 + z)H_{0,-1}^2 - \frac{128\big(23 + 48z + 47z^2\big)}{3(1+z)}H_{0,0,1} + \bigg(\frac{128\big(5 + 48z + 65z^2\big)}{3(1+z)}
$$

$$
-\frac{128\big(7 + 9z^2\big)}{1+z}H_0\bigg)H_{0,0,-1} - \frac{256\big(35 + 48z + 35z^2\big)}{3(1+z)}[H_{0,1,-1} + H_{0,-1,1}]
$$

$$
-\frac{256\big(7 + 9z^2\big)}{1+z}H_0 H_{0,-1,-1} + \frac{128\big(23 + 171z + 14z^2\big)}{9(1+z)}\zeta_2
$$

$$
-\frac{256\big(35 + 48z + 35z^2\big)}{3(1+z)}H_{-1}\zeta_2 + \frac{32\big(7 + 9z^2\big)}{1+z}\zeta_2^2 + \frac{128\big(18 + 24z + 17z^2\big)}{1+z}\zeta_3\bigg\}\bigg\} \quad (122)
$$

and

$$
P_{\mathrm{NS}}^{(2),+,\mathrm{tr}}(z) - P_{\mathrm{NS}}^{(2),-,\mathrm{tr}}(z) =
$$

$$
-\bigg\{\bigg(C_F - \frac{C_A}{2}\bigg)\bigg\{C_F T_F N_F\bigg[\frac{448}{9}(-1 + z) + p_{qq}^{\mathrm{tr}}\bigg(-\frac{1280}{9}H_0^2 - \frac{256}{9}H_0^3 + \bigg(\frac{5120}{9}H_0
$$

$$
+\frac{256}{3}H_0^2 - \frac{1024}{3}H_{0,1} + \frac{1024}{3}\zeta_2\bigg)H_{-1} - \frac{5120}{9}H_{0,-1} + \frac{512}{3}[H_{0,0,1} - H_{0,0,-1}]
$$

$$
+\frac{1024}{3}[H_{0,1,-1} + H_{0,-1,1}] + \frac{2560}{9}\zeta_2 - \frac{256}{3}H_0\zeta_2 - 256\zeta_3\bigg) - \frac{128}{3}(-1 + z)H_1\bigg]
$$

$$+ C_F^2 \Bigg[-\frac{4480}{9}(-1+z) + p_{qq}^{\mathrm{tr}} \Bigg(\Bigg(256 H_{0,0,1} + 1024 H_{0,0,-1} + 512[H_{0,1,-1} + H_{0,-1,1}]$$

$$-\frac{64}{3}\big(-4-8z+z^2\big)\zeta_2 + 384\zeta_3 \Bigg) H_0 - \frac{32}{9}\big(-44-8z+z^2\big)H_0^3 + \frac{32}{3}H_0^4$$

$$+ \Bigg(\Bigg(-512 H_{0,1} - 512\zeta_2 \Bigg) H_0 - \frac{640}{3}H_0^3 + 1024[H_{0,0,1} - H_{0,0,-1}] + 2048[H_{0,1,-1}$$

$$+ H_{0,-1,1}] - \frac{3328}{3}\zeta_2 - 1536\zeta_3 \Bigg) H_{-1} + \Bigg(256 H_0^2 - 1024 H_{0,1} + 1024\zeta_2 \Bigg) H_{-1}^2$$

$$+ \frac{128}{3}\big(-22-8z+z^2\big)H_{0,0,1} - 256 H_{0,0,0,1} - 1792 H_{0,0,0,-1} - 1024[H_{0,0,1,-1}$$

$$+ H_{0,0,-1,1} - H_{0,0,-1,-1}] - 2048[H_{0,1,-1,-1} + H_{0,-1,1,-1} + H_{0,-1,-1,1}] - 128 H_0^2 \zeta_2$$

$$- 256 H_{0,1}\zeta_2 + 512 H_{0,-1}\zeta_2 + 960\zeta_2^2 + \frac{64}{3}\big(30-8z+z^2\big)\zeta_3 \Bigg)$$

$$+ \Bigg(-\frac{32\big(1-14z+3z^2\big)}{3(1+z)} + 64(1-z)H_1 + \frac{64\big(-1+9z-27z^2+3z^3\big)}{3z}$$

$$\times H_{0,-1} \Bigg) H_0 - \frac{128(1+z)\big(1-10z+z^2\big)}{3z}H_{-1}^2 H_0 - \frac{16\big(6-497z+42z^2+9z^3\big)}{9(1+z)}$$

$$\times H_0^2 + \Bigg(\frac{352}{3}(-1+z) - \frac{64(-1+z)\big(1-8z+z^2\big)\zeta_2}{3z} \Bigg) H_1$$

$$+ \Bigg(\frac{32\big(9+24z-1042z^2+24z^3+9z^4\big)}{9z(1+z)}H_0 + \frac{32\big(1-8z-44z^2-8z^3+z^4\big)}{3z(1+z)}[H_0^2$$

$$-4H_{0,1}] + \frac{256(1+z)\big(1-10z+z^2\big)}{3z}H_{0,-1} \Bigg) H_{-1} + \frac{128}{3}(1+z)H_{0,1}$$

$$- \frac{64\big(-1+8z-80z^2-56z^3+7z^4\big)}{3z(1+z)}H_{0,0,-1}$$

$$- \frac{32\big(9+24z-1042z^2+24z^3+9z^4\big)}{9z(1+z)}H_{0,-1} + \frac{128\big(1-8z-44z^2-8z^3+z^4\big)}{3z(1+z)}$$

$$\times[H_{0,1,-1} + H_{0,-1,1}] + \frac{32\big(-24-545z+24z^2+9z^3\big)}{9(1+z)}\zeta_2$$

$$- \frac{256(1+z)\big(1-10z+z^2\big)}{3z}H_{0,-1,-1} \Bigg] \Bigg\}$$

$$+ \Bigg(C_F - \frac{C_A}{2} \Bigg)^2 C_F \Bigg\{ \Bigg\{ \frac{5728}{9}(-1+z) + p_{qq}^{\mathrm{tr}} \Bigg(\Bigg(2048 H_{0,0,-1} + 4096 H_{0,-1,-1}$$

$$+ \frac{128}{3}\big(-29-24z+3z^2\big)\zeta_2 - 2048\zeta_3 \Bigg) H_0 + \Bigg(\frac{32}{9}\big(-277-6z+3z^2\big)$$

$$-256H_{0,-1}-768\zeta_2\Big)H_0^2-\frac{1984}{9}H_0^3+\frac{64}{3}H_0^4+\Bigg(\Big(-4096H_{0,-1}+5632\zeta_2\Big)H_0$$

$$+\frac{1408}{3}H_0^2-256H_0^3-4096[H_{0,0,1}-H_{0,0,-1}]-8192[H_{0,1,-1}+H_{0,-1,1}]$$

$$+6144\zeta_3\Big)H_{-1}+\Big(1024H_0^2+4096H_{0,1}-4096\zeta_2\Big)H_{-1}^2-\frac{128}{3}\big(-49-24z$$

$$+3z^2)H_{0,0,1}+\frac{256}{3}\big(-38-24z+3z^2\big)H_{0,0,-1}+1024H_{0,0,0,1}-3072H_{0,0,0,-1}$$

$$+4096[H_{0,0,1,-1}+H_{0,0,-1,1}-H_{0,0,-1,-1}]+8192[H_{0,1,-1,-1}+H_{0,-1,1,-1}$$

$$+H_{0,-1,-1,1}]+1024H_{0,1}\zeta_2-1536H_{0,-1}\zeta_2-512\zeta_2^2+64\big(-31-8z+z^2\big)\zeta_3\Bigg)$$

$$+\Bigg(\frac{64\big(1-22z-5z^2\big)}{3(1+z)}+128(9-z)zH_{0,-1}\Bigg)H_0+\frac{64\big(1+z\big)\big(1-10z+z^2\big)}{z}$$

$$\times H_{-1}^2H_0+\Bigg(-\frac{1088}{3}(-1+z)+\frac{64(-1+z)\big(1-8z+z^2\big)}{z}\zeta_2\Bigg)H_1$$

$$+\Bigg(-\frac{64\big(3+12z-518z^2+12z^3+3z^4\big)}{9z(1+z)}H_0+\frac{128\big(3-24z-98z^2-24z^3+3z^4\big)}{3z(1+z)}$$

$$\times H_{0,1}-\frac{128(1+z)\big(1-10z+z^2\big)}{z}H_{0,-1}-\frac{64\big(3-24z-142z^2-24z^3+3z^4\big)}{3z(1+z)}\zeta_2\Bigg)$$

$$\times H_{-1}-128(1+z)H_{0,1}+\frac{64\big(3+12z-518z^2+12z^3+3z^4\big)}{9z(1+z)}H_{0,-1}$$

$$-\frac{128\big(3-24z-98z^2-24z^3+3z^4\big)}{3z(1+z)}[H_{0,1,-1}+H_{0,-1,1}]$$

$$+\frac{128(1+z)\big(1-10z+z^2\big)}{z}H_{0,-1,-1}-\frac{64\big(-18-295z-6z^2+3z^3\big)}{9(1+z)}\zeta_2\Bigg\}\Bigg\}\Bigg\}\qquad(123)$$

in the vector and transversity cases. Here we used the shorthand notation $H_{\vec{a}}(z)\equiv H_{\vec{a}}$. 21 harmonic polylogarithms of up to weight $w=4$ are contributing. 18 harmonic polylogarithms of up to weight $w=4$ are contributing to the difference terms.

The difference terms $P_{\mathrm{NS}}^{(2),+}(z)-P_{\mathrm{NS}}^{(2),-}(z)$ and $P_{\mathrm{NS}}^{(2),+,\mathrm{tr}}(z)-P_{\mathrm{NS}}^{(2),-,\mathrm{tr}}(z)$ do not contain soft contributions. Their expansion around $z=1$ is given by

$$P_{\mathrm{NS}}^{(2),+}(z)-P_{\mathrm{NS}}^{(2),-}(z)=\frac{16}{3}(1-z)\Big(C_F-\frac{C_A}{2}\Big)C_F(11C_A-4T_FN_F)+O((1-z)^2),$$

$$(124)$$

$$P_{\mathrm{NS}}^{(2),+,\mathrm{tr}}(z)-P_{\mathrm{NS}}^{(2),-,\mathrm{tr}}(z)=-(1-z)\Bigg\{\Big(C_F-\frac{C_A}{2}\Big)\Bigg[C_F^2\Big(-\frac{8}{9}\big(-404-36\zeta_2\big)$$

$$-\frac{608}{3}H_1\Big)+\frac{64}{9}C_FT_FN_F\big(-7+6H_1\big)\Bigg]+\Big(C_F-\frac{C_A}{2}\Big)^2C_F\Bigg[-\frac{8}{9}\big(116+144\zeta_2\big)$$

$$-\frac{8}{9}\big(-696+144\zeta_2\big)H_1\bigg]\bigg\} + O((1-z)^2). \tag{125}$$

In N space the leading term for $N \to \infty$ is $\propto \ln(N)/N^2$.

Finally, the splitting function $P_{\mathrm{NS}}^{(2),s}$ reads

$$
\begin{aligned}
P_{\mathrm{NS}}^{(2),s}(z) = -2N_F \frac{d_{abc}d_{abc}}{N_C} \bigg\{ &-\frac{400}{3}(1-z) + \bigg(-\frac{2}{3}(100+9z) \\
&+\frac{4(1+z)\big(4-7z+4z^2\big)}{3z}H_{-1}^2 + \frac{52}{3}(1+z)H_{-1} + 16\zeta_3\bigg)H_0 + \bigg(\frac{1}{3}(-38-29z) \\
&-\frac{2(1+z)\big(8-5z+8z^2\big)}{3z}H_{-1}\bigg)H_0^2 + \frac{2}{9}(3+8z^2)H_0^3 - \frac{1}{3}H_0^4 \\
&+\bigg(\frac{182}{3}(-1+z) + 3(1-z)H_0^2\bigg)H_1 + \bigg(-\frac{82}{3}(1+z) - 6(1-z)H_0 + 2(1+z)H_0^2 \\
&-\frac{16(1+z)\big(2+z+2z^2\big)}{3z}H_{-1}\bigg)H_{0,1} + \bigg(-\frac{52}{3}(1+z) \\
&+\frac{16\big(2+3z^2\big)}{3z}H_0 + 4(1-z)H_0^2 - \frac{8(1+z)\big(4-7z+4z^2\big)}{3z}H_{-1}\bigg)H_{0,-1} + 8(1-z) \\
&\times H_{0,-1}^2 + \bigg(\frac{4}{3}(3+9z+8z^2) - 8(1+z)H_0\bigg)H_{0,0,1} + \bigg(\frac{4\big(-8+3z-21z^2+8z^3\big)}{3z} \\
&-8(1-z)H_0\bigg)H_{0,0,-1} + \frac{16(1+z)\big(2+z+2z^2\big)}{3z}[H_{0,1,-1} + H_{0,-1,1}] \\
&+\bigg(\frac{8(1+z)\big(4-7z+4z^2\big)}{3z} - 16(1-z)H_0\bigg)H_{0,-1,-1} + \bigg(\frac{2}{3}(67+41z) \\
&-\frac{2}{3}(3+27z+32z^2)H_0 + 2(5+3z)H_0^2 - \frac{4(-1+z)\big(4+7z+4z^2\big)}{3z}H_1 \\
&+\frac{4(1+z)\big(4-z+4z^2\big)}{z}H_{-1} + 8(1+z)H_{0,1} - 8(1-z)H_{0,-1}\bigg)\zeta_2 - 2(3+5z)\zeta_2^2 \\
&-\frac{16}{3}(3+5z^2)\zeta_3\bigg\}. \tag{126}
\end{aligned}
$$

5. Comparison to the literature

We confirm the results for the non–singlet case for $\gamma_{\mathrm{NS}}^{(2),+}$, $\gamma_{\mathrm{NS}}^{(2),-}$ and $\gamma_{\mathrm{NS}}^{(2),s}$ in Ref. [4] where the on–shell forward Compton amplitude has been used for the calculation. The contributions $\propto T_F$ have already been calculated independently as a by–product of the massive on–shell operator matrix elements in Ref. [10]. We also agree with the fixed moments, which were calculated in

Refs. [40,83–86] and the prediction of the leading N_F terms for $P_{\mathrm{NS}}^{(2),+}$ and $P_{\mathrm{NS}}^{(2),-}$ computed in [87].

Furthermore, we derive the small z limit of the splitting functions, given by

$$
P_{\mathrm{NS}}^{(2),+} \simeq -\left\{ -\frac{4}{3}C_F^3 \mathrm{H}_0^4 + \left[8C_F^3 + C_F^2\left(-\frac{44C_A}{3} + \frac{16T_F N_F}{3} \right) \right]\mathrm{H}_0^3 \right.
$$
$$
+ \left[C_F\left[\frac{176C_A T_F N_F}{9} - \frac{32}{9}T_F^2 N_F^2 + C_A^2\left(-\frac{242}{9} + 60\zeta_2 \right) \right] + C_F^2\left[\frac{256 T_F N_F}{9} \right.\right.
$$
$$
\left.\left. + C_A\left(-\frac{944}{9} - 192\zeta_2 \right) \right] + C_F^3\left(-8 + 208\zeta_2 \right) \right]\mathrm{H}_0^2 + \left[C_F\left(-\frac{704}{27}T_F^2 N_F^2 \right.\right.
$$
$$
+ C_A T_F N_F\left(\frac{5072}{27} - 32\zeta_2 \right) + C_A^2\left(-\frac{7868}{27} + 184\zeta_2 \right) \bigg) + C_F^2\left(\frac{352 T_F N_F}{9} \right.
$$
$$
\left. + C_A\left(-\frac{740}{9} - 432\zeta_2 - 96\zeta_3 \right) \right) + C_F^3\left(60 + 384\zeta_2 + 192\zeta_3 \right) \bigg]\mathrm{H}_0
$$
$$
+ C_F^2\left[C_A\left(-\frac{1064}{9} - \frac{2608\zeta_2}{9} + \frac{1072\zeta_2^2}{5} \right.\right.
$$
$$
\left.\left. -\frac{1616\zeta_3}{3} \right) + T_F N_F\left(\frac{1300}{9} - \frac{640\zeta_2}{9} - \frac{128\zeta_3}{3} \right) \right] + C_F\left[-\frac{256}{9}T_F^2 N_F^2 \right.
$$
$$
\left. + C_A T_F N_F\left(\frac{9896}{27} - 32\zeta_2 + 64\zeta_3 \right) + C_A^2\left(-\frac{19474}{27} + 224\zeta_2 - \frac{324\zeta_2^2}{5} \right.\right.
$$
$$
\left.\left.\left. + 144\zeta_3 \right) \right] + C_F^3\left(124 + 616\zeta_2 - \frac{1296\zeta_2^2}{5} + 384\zeta_3 \right) \right\} + O(z), \tag{127}
$$

$$
P_{\mathrm{NS}}^{(2),+} - P_{\mathrm{NS}}^{(2),-} \simeq 2C_F(C_A - 2C_F)\left\{ (C_A - 2C_F)\mathrm{H}_0^4 + \left(-\frac{40C_A}{9} + \frac{32T_F N_F}{9} \right)\mathrm{H}_0^3 \right.
$$
$$
+ \left(\frac{304 T_F N_F}{9} + 8C_F\left(-4 + 13\zeta_2 \right) - \frac{4}{9}C_A\left(152 + 99\zeta_2 \right) \right)\mathrm{H}_0^2
$$
$$
+ \left(\frac{16}{9}T_F N_F\left(41 + 6\zeta_2 \right) + 32C_F\left(-4 + 6\zeta_2 + 5\zeta_3 \right) - \frac{4}{9}C_A\left(-41 + 282\zeta_2 \right.\right.
$$
$$
\left.\left. + 252\zeta_3 \right) \right)\mathrm{H}_0 - 8C_F\left(15 - 32\zeta_2 + 7\zeta_2^2 - 24\zeta_3 \right) - \frac{16}{9}T_F N_F\left(-61 + 20\zeta_2 \right.
$$
$$
\left.\left. -18\zeta_3 \right) - \frac{4}{9}C_A\left(-367 + 92\zeta_2 + 648\zeta_3 + 63\zeta_2^2 \right) \right\} + O(z), \tag{128}
$$

$$
P_{\mathrm{NS}}^{(2),s} \simeq 2N_F\frac{d_{abc}d_{abc}}{N_C}\left[\frac{1}{3}\mathrm{H}_0^4 - \frac{2}{3}\mathrm{H}_0^3 + \left(18 - 10\zeta_2 \right)\mathrm{H}_0^2 + \left(56 + 2\zeta_2 - 16\zeta_3 \right)\mathrm{H}_0 \right.
$$
$$
\left. + 144 - 66\zeta_2 + 6\zeta_2^2 + 16\zeta_3 \right] + O(z). \tag{129}
$$

The leading small z terms for $P_{\text{NS}}^{(2),+}$ and $P_{\text{NS}}^{(2),-}$ agree with the prediction in Ref. [88] after correcting some misprints there [89], see also [90]. Numerically the leading contributions are not dominant but they are significantly reduced by sub–leading corrections, cf. [89]. For $N_C = 3$ and $N_F = 3$ one obtains

$$P_{\text{NS}}^{(2),+} \simeq 3.16049 \, \text{H}_0^4 + 45.0370 \, \text{H}_0^3 + 407.565 \, \text{H}_0^2 + 1684.87 \, \text{H}_0 + 3469.02, \tag{130}$$

$$P_{\text{NS}}^{(2),-} \simeq 2.86420 \, \text{H}_0^4 + 52.1481 \, \text{H}_0^3 + 570.854 \, \text{H}_0^2 + 1973.93 \, \text{H}_0 + 3769.92. \tag{131}$$

There are no predictions from genuine small z calculations for sub–leading terms. Also the small z behavior of $P_{\text{NS}}^{(2),s}$ has not been predicted.

The splitting functions in the case of transversity do not contain logarithmically enhanced terms in the small z region to three–loop order, but approach the following constants

$$\lim_{z \to 0} P_{\text{NS}}^{(2),\text{tr},+}(z) = \frac{1}{9} C_F [484 C_A^2 - 352 C_A T_F N_F + 64 T_F^2 N_F^2], \tag{132}$$

$$\lim_{z \to 0} P_{\text{NS}}^{(2),\text{tr},-}(z) = -C_F \left[100 C_A^2 + \frac{128}{9} C_A T_F N_F - \frac{64}{9} T_F^2 N_F^2 \right]$$

$$+ C_F^2 \left[C_A \left(\frac{3344}{9} + 32\zeta_2 \right) - T_F N_F \frac{448}{9} \right] - C_F^3 64[2 + \zeta_2]. \tag{133}$$

For transversity we agree with the moments $\propto T_F$ calculated in [41] and the corresponding complete N and z–space expressions given in [10]. In [91] the moments 1–8 of the transversity anomalous dimension have been computed, to which we agree,[6] as well as to the result given in the attachment to [92]. There the anomalous dimensions have been obtained from 15 moments, under certain special assumptions on their mathematical structure.[7] We also agree to the 16th moment of the transversity anomalous dimension calculated in [93].

6. Conclusions

We have calculated the three–loop non–singlet anomalous dimensions $\gamma_{\text{NS}}^{(2),+}$, $\gamma_{\text{NS}}^{(2),-}$, $\gamma_{\text{NS}}^{(2),\text{tr},+}$, $\gamma_{\text{NS}}^{(2),\text{tr},-}$ and $\gamma_{\text{NS}}^{(2),s}$ in Quantum Chromodynamics for unpolarized and polarized deep–inelastic scattering. The method used in this first complete recalculation of the former results in [4,92] has been the traditional one, cf. [16,18], of massless off shell operator matrix elements, unlike the on–shell Compton amplitude at virtuality Q^2 in [4]. The present method requests to obtain the anomalous dimensions in a gauge–dependent framework. We confirm results given in the literature, also on partial results both in the unpolarized and polarized case. The former three–loop calculations have been performed using gauge–invariant quantities. For the non–singlet anomalous dimensions a finite renormalization can be avoided in the polarized case, due to a known Ward identity and all the results are obtained in the $\overline{\text{MS}}$ scheme directly. The present calculation has been performed fully automatically in all its parts using a chain of dedicated codes from diagram generation to the final results. The three–loop anomalous dimensions have a comparatively simple mathematical structure, since they can be expressed in harmonic sums

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[6] In the last term of the 1st moment a factor N_F^2 is missing.

[7] There are the following sign errors in Eqs. (A.1)–(A.15) of [92]: the signs of the $C_F^2 T_F N_F \zeta_3$ terms in (A.1), $C_A C_F T_F N_F \zeta_3$ in (A.2), $C_F^2 T_F N_F \zeta_3$ in (A.3), $C_F^2 T_F N_F \zeta_3$ in (A.4), and $C_F^2 T_F N_F \zeta_3$ (A.15) have to be changed.

only [74,75]. We remark that also the three–loop unpolarized and polarized singlet anomalous dimensions $(\Delta)\gamma_{\mathrm{PS}}^{(2)}$ and $(\Delta)\gamma_{\mathrm{qg}}^{(2)}$ have been recalculated in complete form using the framework of massive on–shell OMEs in [12–14]. The flavor non–singlet anomalous dimensions play a particular role in the associated scheme–invariant evolution equations for non–singlet structure functions [94], allowing for a direct measurement of the strong coupling constant.

The expressions of the anomalous dimensions and splitting functions have also been given in computer-readable form as an attachment to this paper.

CRediT authorship contribution statement

All authors have very essentially contributed to the different part of the present work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Relation between the Larin and the $\overline{\mathrm{MS}}$ scheme

The known Ward identity in the non–singlet case allows to derive the transformation between the Larin scheme and the $\overline{\mathrm{MS}}$ scheme directly. We calculated the anomalous dimension $\Delta\gamma_2^{\mathrm{NS},-}$ in both schemes and obtain the following transformation relations at three–loop order

$$\Delta\gamma_2^{\mathrm{NS},-,\overline{\mathrm{MS}}} = \Delta\gamma_2^{\mathrm{NS},-,\mathrm{L}} - 2\beta_0 \left[z_{qq}^{(1)2} - 2z_{qq}^{(2),\mathrm{NS}} \right] + 2\beta_1 z_{qq}^{(1)}, \tag{134}$$

$$z_{qq}^{(1)} = -\frac{8C_F}{N(N+1)}, \tag{135}$$

$$z_{qq}^{(2),\mathrm{NS}} = C_F^2 \left[\frac{8P_{63}}{N^3(N+1)^2} + \frac{16(1+2N)}{N^2(N+1)^2} S_1 + \frac{16}{N(N+1)}[S_2 + 2S_{-2}] \right]$$

$$+ C_F C_A \left[-\frac{P_{64}}{9N^3(N+1)^3} - \frac{16}{N(N+1)} S_{-2} \right] + C_F T_F N_F \frac{5N^2 - N - 3}{9N^2(N+1)^2}, \tag{136}$$

with

$$P_{63} = 2N^4 + N^3 + 8N^2 + 5N + 2, \tag{137}$$

$$P_{64} = 103N^4 + 140N^3 + 58N^2 + 21N + 36. \tag{138}$$

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