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Full-Order Observer Design for a Class of Nonlinear Port-Hamiltonian Systems

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Abstract: In this paper, we present a simple method to design a full-order observer for a class of nonlinear port-Hamiltonian systems (PHSs). We provide a sufficient condition for the observer to be globally exponentially convergent. This condition exploits the natural damping of the system. The observer and its design are illustrated by means of an academic example system. Numerical simulations verify the convergence of the reconstructions towards the unknown system variables.

Keywords: Port-Hamiltonian systems, observer design, nonlinear systems, state estimation, output estimation, system damping

1. INTRODUCTION

Port-Hamiltonian systems (PHSs) have been identified as a powerful framework for the treatment of complex physical systems. PHSs have first been introduced for real-valued, continuous-time, finite-dimensional systems, see, e.g., Maschke and van der Schaft (1992). Meanwhile, the port-Hamiltonian framework has been extended to complex-valued systems (see, e.g., Mehl et al. (2016)), discrete-time systems (see, e.g., Kotyczka and Lefèvre (2018)), and infinite-dimensional systems (see, e.g., Le Gorrec et al. (2005)).

The literature on PHSs contains numerous contributions for the design of controllers, see, e.g., Ortega et al. (2008) and van der Schaft (2017). In contrast, the design of observers for PHSs has received rather limited attention. The available methods are presented in the sequel. Thereby, we distinguish between observer designs for linear and nonlinear PHSs.

Linear PHSs are a special class of linear state-space systems. Hence, for the state reconstruction of such systems it is natural to approach with a standard Luenberger observer, see, e.g., Khalil et al. (2012). Cardoso Ribeiro (2016) and Kotyczka et al. (2019) show that the Luenberger observer is also a viable option if the linear model arises from the structure-preserving discretization of an infinite-dimensional PHS. Toledo et al. (2020) address the design of passive observers for infinite-dimensional boundary-controlled PHSs. A compensator for linear finite-dimensional PHSs based on a dual observer has been proposed by Kotyczka and Wang (2015). Atitallah et al. (2015) address the combined input-state reconstruction for linear PHSs. Likewise, Pfeifer et al. (2019) derive an interval input-state-output estimator for linear PHSs.

For nonlinear PHSs, there exist also several observer design methods. Regarding observers for nonlinear PHSs, we differentiate between two kinds of nonlinearities, viz. (a) nonlinearities in the interconnection structure and (b) nonlinearities in the storages. The former are characterized by state-dependent matrices of the PHSs; the latter are characterized by possibly non-quadratic Hamiltonians.

For PHSs with nonlinearities of type (a), Vincent et al. (2016) present two nonlinear, passivity-based observers: a proportional observer and a proportional observer with integral action. Another two notable methods which can be applied to a class of PHSs with nonlinearities of type (a) stem from Biedermann et al. (2018) and Biedermann and Meurer (2021): the former present a passivity-based observer design for a class of state-affine systems; the latter propose a dissipativity- and IDA-PBC-based observer design for nonlinear systems that can be decomposed into a time-varying state-affine part, a nonlinear feedback part, and a perturbation term.

For PHSs with nonlinearities of type (b), Yaghmaei and Yazdanpanah (2019b) propose an IDA-PBC-based observer design. The observer from Yaghmaei and Yazdanpanah (2019b) allows for a separation principle as known from linear systems theory (see Yaghmaei and Yazdanpanah (2019a)).

To the best of our knowledge, there are only two observer designs for PHSs with nonlinearities of both types, (a) and (b). Wang et al. (2005) develop adaptive and non-adaptive state observers. However, the observers are in general not asymptotically convergent as they only converge when the system reaches a steady state. Venkatraman and van der Schaft (2010) present a passivity-based, globally exponentially convergent observer for PHSs with nonlinearities of type (a) and (b). The proposed observer design is delicate

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as it requires the solution of a set of algebraic equations and partial differential equations (PDEs).

In this paper, we present a full-order observer with a simple design scheme for a class of real-valued, continuous-time, finite-dimensional PHSs with nonlinearities of both types, (a) and (b). We provide a sufficient condition for the observer to be globally exponentially convergent. This condition is mild as it makes use of the system damping.

Structure: The remainder of this paper is structured as follows. Section 2 formally outlines the problem under consideration. In Section 3, we propose an observer and provide a sufficient condition for its global exponential convergence. The results from Section 3 are discussed in Section 4. Hereafter, the observer and its design are illustrated for an academic example in Section 5. Section 6 summarizes the insights and concludes the paper.

Notation: Sets, groups, and spaces are written in blackboard bold. For the dimension of a vector space $\mathbb{X}$, we write $\dim(\mathbb{X})$. The symbol $\times$ denotes the Cartesian product. Vectors, matrices, and vector-valued functions are written in bold font. Let $\boldsymbol{A} \in \mathbb{R}^{n\times m}$ be a matrix with n rows and m columns. For the transposed of $\boldsymbol{A}$ we write $\boldsymbol{A}^\top$. Now let $n = m$. $\boldsymbol{A} \succ 0$ and $\boldsymbol{A} \succeq 0$ mean that $\boldsymbol{A}$ is positive-definite and positive semi-definite, respectively. With $\mathrm{diag}(\cdot)$ we denote a diagonal matrix; likewise, $\mathrm{blkdiag}(\cdot)$ is a block diagonal matrix of matrices. Now let $\boldsymbol{x} \in \mathbb{R}^n$ be a (column) vector. For the kernel of the linear map $\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}$ we write $\ker(\boldsymbol{A})$. Throughout this paper, the time-dependence "(t)" of vectors is omitted in the notation.

2. PROBLEM FORMULATION

Consider an explicit PHS of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{x}'\\ \boldsymbol{x}''\end{pmatrix} = (\boldsymbol{J}(\boldsymbol{x}') - \boldsymbol{R}(\boldsymbol{x}'))\frac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x}')\boldsymbol{u}, \quad (1a)$$

$$\boldsymbol{y} = \boldsymbol{G}^\top(\boldsymbol{x}')\frac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x}), \quad (1b)$$

with initial value $\boldsymbol{x}|_{t=0} = \boldsymbol{x}_0$, $\boldsymbol{x}' \in \mathbb{X}' \subset \mathbb{R}^{n_1}$, $\boldsymbol{x}'' \in \mathbb{X}'' \subset \mathbb{R}^{n-n_1}$, $\boldsymbol{u} \in \mathbb{U} \subset \mathbb{R}^p$, and $\boldsymbol{y} \in \mathbb{Y} \subset \mathbb{R}^p$, where $\mathbb{X}'$ and $\mathbb{X}''$ are closed and bounded and therewith compact. The overall state vector is defined as $\boldsymbol{x} := (\boldsymbol{x}'^\top\ \boldsymbol{x}''^\top)^\top \in \mathbb{X} = \mathbb{X}' \times \mathbb{X}''$, where $\mathbb{X}$ is then also compact. The matrices in (1) are of proper sizes, continuously differentiable in $\boldsymbol{x}'$, and satisfy $\boldsymbol{J}(\boldsymbol{x}') = -\boldsymbol{J}^\top(\boldsymbol{x}')$, $\boldsymbol{R}(\boldsymbol{x}') = \boldsymbol{R}^\top(\boldsymbol{x}') \succeq 0$. Let the Hamiltonian of (1) be of the form

$$H(\boldsymbol{x}) = \frac{1}{2}\left(\boldsymbol{x}'^\top\ \boldsymbol{x}''^\top\right)\begin{pmatrix}\boldsymbol{Q}' & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{Q}''\end{pmatrix}\begin{pmatrix}\boldsymbol{x}'\\ \boldsymbol{x}''\end{pmatrix} + \boldsymbol{N}(\boldsymbol{x}'), \quad (2)$$

where $\boldsymbol{Q} := \mathrm{blkdiag}(\boldsymbol{Q}', \boldsymbol{Q}'') = \boldsymbol{Q}^\top \succ 0$ and $\boldsymbol{N} : \mathbb{X}' \to \mathbb{R}$, $\boldsymbol{x}' \mapsto \boldsymbol{N}(\boldsymbol{x}')$. The function $\boldsymbol{N}$ may be any function that is positive semi-definite and twice continuously differentiable in $\boldsymbol{x}'$. Suppose $\boldsymbol{u}$ is known but $\boldsymbol{x}$ and $\boldsymbol{y}$ are unknown. Moreover, assume measurements $\boldsymbol{m} \in \mathbb{R}^q$ with $q \geq n_1$ of the form $\boldsymbol{m} = \boldsymbol{C}(\boldsymbol{x}')\boldsymbol{Q}\boldsymbol{x}$ where $\boldsymbol{C}(\boldsymbol{x}')$ depends continuously on $\boldsymbol{x}'$:

$$\begin{pmatrix}\boldsymbol{m}_1\\ \boldsymbol{m}_2\end{pmatrix} = \begin{pmatrix}\boldsymbol{Q}'^{-1} & \boldsymbol{0}\\ \boldsymbol{C}'(\boldsymbol{x}') & \boldsymbol{C}''(\boldsymbol{x}')\end{pmatrix}\begin{pmatrix}\boldsymbol{Q}' & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{Q}''\end{pmatrix}\begin{pmatrix}\boldsymbol{x}'\\ \boldsymbol{x}''\end{pmatrix}. \quad (3)$$

Note that we have $\boldsymbol{m}_1 = \boldsymbol{x}'$, i.e., $\boldsymbol{x}'$ is the measured part of the state vector $\boldsymbol{x}$.

The problem addressed in this paper reads: What is an asymptotic observer for (1) that produces reconstructions of $\boldsymbol{x}$ and $\boldsymbol{y}$ based on knowledge on $\boldsymbol{m}$? How can we design such an observer in a simple manner?

Remark 1. In (1), we consider a PHS without feedthrough. This is for a compact notation and without loss of generality. As the system inputs are known, it is straightforward to extend the problem and methods to systems with feedthrough (cf. Ludyk (1995, p. 7)). Nevertheless, the above setting excludes some systems of practical interest. Firstly, this is due to (1) and (3) where the matrices are dependent only on the measured part of the state vector. Secondly, we have the requirement $\boldsymbol{Q} \succ 0$ in (2). Despite these restrictions, the treated class of systems covers a considerable number of physical examples such as mechanical and electromechanical PHSs, see, e.g., Venkatraman and van der Schaft (2010) and Yaghmaei and Yazdanpanah (2019b, Eq. (23) and (27)). Moreover, note that the measurement equation (3) can also be written in the form[1]

$$\begin{pmatrix}\boldsymbol{m}_1\\ \boldsymbol{m}_2\end{pmatrix} = \begin{pmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \tilde{\boldsymbol{C}}'(\boldsymbol{x}') & \tilde{\boldsymbol{C}}''(\boldsymbol{x}')\end{pmatrix}\begin{pmatrix}\boldsymbol{x}'\\ \boldsymbol{x}''\end{pmatrix}. \quad (4)$$

In (4), we have $\boldsymbol{m}_1 = \boldsymbol{x}'$ and $\boldsymbol{m}_2 = \tilde{\boldsymbol{C}}(\boldsymbol{x}')\boldsymbol{x}$ which reveals the generality of this formulation.

3. NONLINEAR OBSERVER DESIGN

First, we provide three preliminary statements, viz. Lemma 2, Lemma 3, and Lemma 4. Afterwards, the proposed observer and its design are summarized in Theorem 6. Finally, in Corollary 7 and Corollary 8, the result from Theorem 6 are analyzed in more detail.

The state-output reconstruction problem described in the previous section involves three equations, viz. a dynamics equation (1a), an output equation (1b), and a measurement equation (3). Note that the measurement equation may be nonlinear in the states. In the following lemma, we show that the state-output reconstruction problem can be reduced to a state reconstruction problem which involves only two equations.

Lemma 2. Consider the situation in Section 2. Let $\hat{\boldsymbol{x}}$ be a reconstruction of $\boldsymbol{x}$ with $\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| \leq k_1 e^{-k_2 t}$ for $t \geq 0$ and some positive constants $k_1, k_2 \in \mathbb{R}_{>0}$. Then, we can calculate an output reconstruction

$$\hat{\boldsymbol{y}} = \boldsymbol{G}^\top(\hat{\boldsymbol{x}}')\frac{\partial H}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}}), \quad (5)$$

with $\|\boldsymbol{y} - \hat{\boldsymbol{y}}\| \leq k_3 e^{-k_2 t}$ for all $t \geq 0$ and some positive constant $k_3 \in \mathbb{R}_{>0}$.

Proof. Because $\mathbb{X}'$ and $\mathbb{X}$ are compact, $\boldsymbol{G}^\top$ and $\frac{\partial H}{\partial \boldsymbol{x}}$ are bounded in $\boldsymbol{x}'$ and $\boldsymbol{x}$, respectively, i.e., there exist constants $k_G, k_H \in \mathbb{R}_{>0}$ such that $\|\boldsymbol{G}^\top(\boldsymbol{x}')\| < k_G$ and $\|\frac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x})\| < k_H$ for all $\boldsymbol{x}' \in \mathbb{X}'$ and $\boldsymbol{x} \in \mathbb{X}$.

Since $\boldsymbol{G}^\top$ is continuously differentiable and $\mathbb{X}'$ is compact, $\boldsymbol{G}^\top$ is Lipschitz continuous on $\mathbb{X}'$ with constant $L_G = \sup_{\boldsymbol{x}'\in\mathbb{X}'}\|\frac{\partial \boldsymbol{G}}{\partial \boldsymbol{x}'}(\boldsymbol{x}')\|$ that is $\|\boldsymbol{G}^\top(\boldsymbol{x}_1') - \boldsymbol{G}^\top(\boldsymbol{x}_2')\| \leq L_G\|\boldsymbol{x}_1' - \boldsymbol{x}_2'\|$ for all $\boldsymbol{x}_1', \boldsymbol{x}_2' \in \mathbb{X}'$. Likewise $\frac{\partial H}{\partial \boldsymbol{x}}$ is Lipschitz continuous with a constant L_H on $\mathbb{X}$.

[1] To bring (4) to the form (3), we write $\boldsymbol{m} = \tilde{\boldsymbol{C}}(\boldsymbol{x}')\boldsymbol{Q}^{-1}\boldsymbol{Q}\boldsymbol{x} = \boldsymbol{C}(\boldsymbol{x}')\boldsymbol{Q}\boldsymbol{x}$ with $\boldsymbol{C}(\boldsymbol{x}') = \tilde{\boldsymbol{C}}(\boldsymbol{x}')\boldsymbol{Q}^{-1}$.

We now can conclude

$$
\begin{aligned}
\|\boldsymbol{y}-\hat{\boldsymbol{y}}\| &= \left\|\boldsymbol{G}^\top(\boldsymbol{x}')\frac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x}) - \boldsymbol{G}^\top(\hat{\boldsymbol{x}}')\frac{\partial H}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}})\right\| \\
&\leq \left\|\boldsymbol{G}^\top(\boldsymbol{x}')\frac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x}) - \boldsymbol{G}^\top(\boldsymbol{x}')\frac{\partial H}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}})\right\| + \\
&\quad \left\|\boldsymbol{G}^\top(\boldsymbol{x}')\frac{\partial H}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}}) - \boldsymbol{G}^\top(\hat{\boldsymbol{x}}')\frac{\partial H}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}})\right\| \\
&\leq \|\boldsymbol{G}^\top(\boldsymbol{x}')\|\left\|\frac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x}) - \frac{\partial H}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}})\right\| + \\
&\quad \|\boldsymbol{G}^\top(\boldsymbol{x}') - \boldsymbol{G}^\top(\hat{\boldsymbol{x}}')\|\left\|\frac{\partial H}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}})\right\| \\
&\leq k_G L_H\|\boldsymbol{x}-\hat{\boldsymbol{x}}\| + L_G\|\boldsymbol{x}'-\hat{\boldsymbol{x}}'\|k_H \\
&\leq (k_G L_H + L_G k_H)k_1 e^{-k_2 t},
\end{aligned}
\tag{6}
$$

where in the last step we used $\|\boldsymbol{x}'-\hat{\boldsymbol{x}}'\| \leq \|\boldsymbol{x}-\hat{\boldsymbol{x}}\|$ and $\|\boldsymbol{x}-\hat{\boldsymbol{x}}\| \leq k_1 e^{-k_2 t}$. □

Lemma 2 shows that an exponentially convergent reconstruction of the output can always be obtained from an exponentially convergent reconstruction of the state. Hence, the state-output reconstruction problem can be formulated as an ordinary state reconstruction problem that involves two equations, viz. (1a) and (3). This motivates to approach with a Luenberger-like observer consisting of an internal model of the system dynamics and a measurement error injection term. This is the approach we follow in the subsequent lemma.

Lemma 3. Consider a system with dynamics (1a) and measurements (3). Suppose there exists a matrix $\boldsymbol{L} \in \mathbb{R}^{n\times q}$ depending continuously on $\boldsymbol{x}'$ such that

$$
\boldsymbol{R}(\boldsymbol{x}') + \frac{1}{2}\boldsymbol{L}(\boldsymbol{x}')\boldsymbol{C}(\boldsymbol{x}') + \frac{1}{2}\boldsymbol{C}^\top(\boldsymbol{x}')\boldsymbol{L}^\top(\boldsymbol{x}') \succ 0, \tag{7}
$$

for all $\boldsymbol{x}' \in \mathbb{X}'$. Then, there exists a globally exponentially convergent state observer of the form

$$
\begin{aligned}
\dot{\hat{\boldsymbol{x}}} &= (\boldsymbol{J}(\boldsymbol{x}') - \boldsymbol{R}(\boldsymbol{x}'))\frac{\partial H}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}}) \\
&\quad + \boldsymbol{G}(\boldsymbol{x}')\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{x}')\left(\boldsymbol{m} - \boldsymbol{C}(\boldsymbol{x}')\boldsymbol{Q}\hat{\boldsymbol{x}}\right),
\end{aligned}
\tag{8}
$$

with initial value $\hat{\boldsymbol{x}}|_{t=0} = \hat{\boldsymbol{x}}_0$. The vectors $\hat{\boldsymbol{x}}' \in \mathbb{X}'$ and $\hat{\boldsymbol{x}}'' \in \mathbb{X}''$ of the splitting $\hat{\boldsymbol{x}} = (\hat{\boldsymbol{x}}'^\top\ \hat{\boldsymbol{x}}''^\top)^\top$ are mimicking the splitting of $\boldsymbol{x} = (\boldsymbol{x}'^\top\ \boldsymbol{x}''^\top)^\top$.

Proof. Let us define the reconstruction error as $\boldsymbol{\varepsilon} := \boldsymbol{x} - \hat{\boldsymbol{x}}$. With (1a), (2), (3), and (8), the error dynamics can be expressed as

$$
\dot{\boldsymbol{\varepsilon}} = (\boldsymbol{J}(\boldsymbol{x}') - \boldsymbol{R}(\boldsymbol{x}') - \boldsymbol{L}(\boldsymbol{x}')\boldsymbol{C}(\boldsymbol{x}'))\,\boldsymbol{Q}\boldsymbol{\varepsilon}, \tag{9}
$$

with initial value $\boldsymbol{\varepsilon}_0 = \boldsymbol{x}_0 - \hat{\boldsymbol{x}}_0$. Obviously, $\boldsymbol{\varepsilon} \equiv \boldsymbol{0}$ is an equilibrium of (9). Next, we analyze the stability of this equilibrium by using Lyapunov's direct method. Consider the Lyapunov candidate $V(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon}^\top\boldsymbol{Q}\boldsymbol{\varepsilon}$. As shown in Proposition 9 in the appendix, with this Lyapunov candidate for a system (9) we obtain $\dot{V}(\boldsymbol{\varepsilon}) = -\boldsymbol{\varepsilon}^\top\boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{Q}\boldsymbol{\varepsilon}$ where

$$
\boldsymbol{\Gamma} = \boldsymbol{R}(\boldsymbol{x}') + \frac{1}{2}\boldsymbol{L}(\boldsymbol{x}')\boldsymbol{C}(\boldsymbol{x}') + \frac{1}{2}\boldsymbol{C}^\top(\boldsymbol{x}')\boldsymbol{L}^\top(\boldsymbol{x}'). \tag{10}
$$

It is noteworthy that (10) is independent of the matrix $\boldsymbol{J}(\boldsymbol{x}')$. Now let (7) hold. We then have $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^\top \succ 0$ which is equivalent to $\boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{Q} = (\boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{Q})^\top \succ 0$. From this follows that $\dot{V}(\boldsymbol{\varepsilon})$ is negative-definite and thus $\boldsymbol{\varepsilon} \equiv \boldsymbol{0}$ an asymptotically stable equilibrium of (9). Moreover, as shown in Proposition 10 in the appendix, the positive definiteness of $\boldsymbol{Q}$ and $\boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{Q}$ implies the existence of positive constants $k_1, k_2, k_3 \in \mathbb{R}_{>0}$ such that $k_1\|\boldsymbol{\varepsilon}\|^2 \leq V(\boldsymbol{\varepsilon}) \leq k_2\|\boldsymbol{\varepsilon}\|^2$ and $\dot{V}(\boldsymbol{\varepsilon}) \leq -k_3\|\boldsymbol{\varepsilon}\|^2$ hold for all $\boldsymbol{x} \in \mathbb{X}$. Hence, $\boldsymbol{\varepsilon} \equiv \boldsymbol{0}$ is a globally exponentially stable equilibrium of (9) Khalil (2002). This implies (8) to be an exponentially convergent observer for the system consisting of (1) and (3). □

Equation (7) is a sufficient condition for the existence of an exponentially convergent observer of the form (8). Thus, the observer design problem is to find a matrix $\boldsymbol{L}(\boldsymbol{x}')$ such that (7) is fulfilled. In the sequel, we present an approach for finding a matrix $\boldsymbol{L}(\boldsymbol{x}')$ such that (7) is satisfied.

Recall that $\boldsymbol{R}(\boldsymbol{x}') \succeq 0$ for all $\boldsymbol{x}' \in \mathbb{X}'$. For (7) to hold, we need to find a matrix $\boldsymbol{L}(\boldsymbol{x}')$ which "moves the zero eigenvalues of $-\boldsymbol{R}(\boldsymbol{x}')$ to the left". In the following lemma, we propose a choice of $\boldsymbol{L}(\boldsymbol{x}')$ which has the best chances to accomplish this:

Lemma 4. Consider two matrices $\boldsymbol{R}(\boldsymbol{s}) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{C}(\boldsymbol{s}) \in \mathbb{R}^{q\times n}$ depending on some parameter $\boldsymbol{s} \in \mathbb{S}$. Let $\boldsymbol{R}(\boldsymbol{s}) = \boldsymbol{R}^\top(\boldsymbol{s}) \succeq 0$ for all $\boldsymbol{s} \in \mathbb{S}$. There exists a matrix $\boldsymbol{L}(\boldsymbol{s}) \in \mathbb{R}^{n\times q}$ which satisfies

$$
\boldsymbol{R}(\boldsymbol{s}) + \frac{1}{2}\boldsymbol{L}(\boldsymbol{s})\boldsymbol{C}(\boldsymbol{s}) + \frac{1}{2}\boldsymbol{C}^\top(\boldsymbol{s})\boldsymbol{L}^\top(\boldsymbol{s}) \succ 0, \quad \forall \boldsymbol{s} \in \mathbb{S}, \tag{11}
$$

if and only if (11) is satisfied for $\boldsymbol{L}(\boldsymbol{s}) = \boldsymbol{C}^\top(\boldsymbol{s})$.

Proof. We show that the following two statements are equivalent:

(*i*) $\forall \boldsymbol{s} \in \mathbb{S}\colon \exists \boldsymbol{L}(\boldsymbol{s}) \in \mathbb{R}^{n\times q}$ s.t.

$$
\boldsymbol{R}(\boldsymbol{s}) + \frac{1}{2}\boldsymbol{L}(\boldsymbol{s})\boldsymbol{C}(\boldsymbol{s}) + \frac{1}{2}\boldsymbol{C}^\top(\boldsymbol{s})\boldsymbol{L}^\top(\boldsymbol{s}) \succ 0, \tag{12a}
$$

(*ii*) $\forall \boldsymbol{s} \in \mathbb{S}\colon \boldsymbol{R}(\boldsymbol{s}) + \boldsymbol{C}^\top(\boldsymbol{s})\boldsymbol{C}(\boldsymbol{s}) \succ 0.$ (12b)

By setting $\boldsymbol{L}(\boldsymbol{s}) = \boldsymbol{C}^\top(\boldsymbol{s})$ it is easy to see that (*ii*) implies (*i*). We now show that (*i*) also implies (*ii*). To this end, we show the contraposition, i.e., that if $\boldsymbol{R}(\boldsymbol{s}) + \boldsymbol{C}^\top(\boldsymbol{s})\boldsymbol{C}(\boldsymbol{s})$ is not positive-definite, then the matrix in (*i*) is not positive-definite for all $\boldsymbol{L}(\boldsymbol{s})$.

Let (*ii*) be violated. The matrix $\boldsymbol{R}(\boldsymbol{s}) + \boldsymbol{C}^\top(\boldsymbol{s})\boldsymbol{C}(\boldsymbol{s})$ is positive semi-definite, i.e.,

$$
\boldsymbol{R}(\boldsymbol{s}) + \boldsymbol{C}^\top(\boldsymbol{s})\boldsymbol{C}(\boldsymbol{s}) \succeq 0, \quad \forall \boldsymbol{s} \in \mathbb{S}, \tag{13}
$$

as $\boldsymbol{R}(\boldsymbol{s}) \succeq 0$ and

$$
\boldsymbol{v}^\top\boldsymbol{C}^\top(\boldsymbol{s})\boldsymbol{C}(\boldsymbol{s})\boldsymbol{v} = \|\boldsymbol{C}(\boldsymbol{s})\boldsymbol{v}\| \geq 0, \quad \forall \boldsymbol{s} \in \mathbb{S}, \forall \boldsymbol{v} \in \mathbb{R}^n, \tag{14}
$$

i.e., $\boldsymbol{C}^\top(\boldsymbol{s})\boldsymbol{C}(\boldsymbol{s}) \succeq 0$ for all $\boldsymbol{s} \in \mathbb{S}$. From (13) and the violation of (*ii*) follows, that there exists a non-zero vector $\boldsymbol{v} \in \mathbb{R}^n$ and a value $\boldsymbol{s}_0 \in \mathbb{S}$ such that

$$
\boldsymbol{v}^\top\left(\boldsymbol{R}(\boldsymbol{s}_0) + \boldsymbol{C}^\top(\boldsymbol{s}_0)\boldsymbol{C}(\boldsymbol{s}_0)\right)\boldsymbol{v} = 0. \tag{15}
$$

For this $\boldsymbol{v}$ and $\boldsymbol{s}_0$ we have

$$
\begin{aligned}
&\boldsymbol{v}^\top\boldsymbol{R}(\boldsymbol{s}_0)\boldsymbol{v} + \boldsymbol{v}^\top\boldsymbol{C}^\top(\boldsymbol{s}_0)\boldsymbol{C}(\boldsymbol{s}_0)\boldsymbol{v} = 0, \\
\Leftrightarrow\quad &\boldsymbol{v}^\top\boldsymbol{R}(\boldsymbol{s}_0)\boldsymbol{v} = 0 \quad\wedge\quad \boldsymbol{v}^\top\boldsymbol{C}^\top(\boldsymbol{s}_0)\boldsymbol{C}(\boldsymbol{s}_0)\boldsymbol{v} = 0, \\
\Leftrightarrow\quad &\boldsymbol{v}^\top\boldsymbol{R}(\boldsymbol{s}_0)\boldsymbol{v} = 0 \quad\wedge\quad \boldsymbol{v} \in \ker\left(\boldsymbol{C}(\boldsymbol{s}_0)\right).
\end{aligned}
\tag{16}
$$

For the left hand side of (12a) we obtain

$$
\underbrace{\boldsymbol{v}^\top\boldsymbol{R}(\boldsymbol{s}_0)\boldsymbol{v}}_{=0} + \frac{1}{2}\boldsymbol{v}^\top\boldsymbol{L}(\boldsymbol{s}_0)\underbrace{\boldsymbol{C}(\boldsymbol{s}_0)\boldsymbol{v}}_{=\boldsymbol{0}} + \frac{1}{2}\underbrace{\boldsymbol{v}^\top\boldsymbol{C}^\top(\boldsymbol{s}_0)}_{=\boldsymbol{0}}\boldsymbol{L}^\top(\boldsymbol{s}_0)\boldsymbol{v}, \tag{17}
$$

i.e., zero. Hence, for $\boldsymbol{s}_0 \in \mathbb{S}$ and for all $\boldsymbol{L}(\boldsymbol{s})$ the matrix

$$
\boldsymbol{R}(\boldsymbol{s}_0) + \frac{1}{2}\boldsymbol{L}(\boldsymbol{s}_0)\boldsymbol{C}(\boldsymbol{s}_0) + \frac{1}{2}\boldsymbol{C}^\top(\boldsymbol{s}_0)\boldsymbol{L}^\top(\boldsymbol{s}_0) \tag{18}
$$

is not positive-definite. This is the contraposition of statement (*i*). □

Remark 5. Lemma 4 holds also for an observer gain $\boldsymbol{L}(\boldsymbol{s}) = \alpha \boldsymbol{C}^\top(\boldsymbol{s})$ where $\alpha \in \mathbb{R}_{>0}$. The parameter α can be used to increase ($\alpha > 1$) or decrease ($\alpha < 1$) the convergence rate of those observer states that are influenced by the error injection.

In Lemma 3, we propose a state observer for the PHS (1). Lemma 4 provides a simple design for such an observer. From Lemma 2 we know, that a state observer can be easily extended to a state-output observer. In the following theorem, we summarize these insights to formulate a globally exponentially convergent state-output observer for the PHS (1):

Theorem 6. Consider a nonlinear PHS (1) with Hamiltonian (2) and measurements (3). Let

$$\boldsymbol{R}(\boldsymbol{x}') + \boldsymbol{C}^\top(\boldsymbol{x}')\boldsymbol{C}(\boldsymbol{x}') \succ 0, \quad \forall \boldsymbol{x}' \in \mathbb{X}'. \tag{19}$$

hold. A globally exponentially convergent full-order state-output observer for the system is given by

$$\dot{\hat{\boldsymbol{x}}} = (\boldsymbol{J}(\boldsymbol{x}') - \boldsymbol{R}(\boldsymbol{x}'))\frac{\partial H}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}}) + \boldsymbol{G}(\boldsymbol{x}')\boldsymbol{u} + \boldsymbol{C}^\top(\boldsymbol{x}')\left(\boldsymbol{m} - \boldsymbol{C}(\boldsymbol{x}')\boldsymbol{Q}\hat{\boldsymbol{x}}\right), \tag{20a}$$

$$\hat{\boldsymbol{y}} = \boldsymbol{G}^\top(\boldsymbol{x}')\frac{\partial H}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}}), \tag{20b}$$

with initial value $\hat{\boldsymbol{x}}|_{t=0} = \hat{\boldsymbol{x}}_0$.

Proof. The proof follows directly from Lemma 2, Lemma 3, and Lemma 4. In the latter we substitute $\boldsymbol{s} \in \mathbb{S}$ with $\boldsymbol{x}' \in \mathbb{X}'$. □

It is important to note that the observer from Theorem 6 is directly obtained from the system model. In particular, there are no free observer parameters which makes its design very simple.

In the following two corollaries, we analyze the results obtained so far in more detail. First, we consider the case of linear measurements, i.e., the case where in (3) we have $\boldsymbol{C}(\boldsymbol{x}') = \boldsymbol{C} = \text{const}$.

Corollary 7. Given a system with dynamics (1a) and measurements (3) where the measurement matrix is a constant matrix $\boldsymbol{C}(\boldsymbol{x}') = \boldsymbol{C}$. The existence condition (7) for an observer of the form (8) is satisfied if and only if it is satisfied for the constant matrix $\boldsymbol{L} = \boldsymbol{C}^\top$.

Proof. The claim follows directly from Lemma 3 and Lemma 4 under $\boldsymbol{C}(\boldsymbol{x}') = \boldsymbol{C}$. □

The main point from Corollary 7 is as follows. Despite the fact that the matrix $\boldsymbol{R}(\boldsymbol{x}')$ is parametrized over $\boldsymbol{x}'$, a constant observer gain $\boldsymbol{L}$ is sufficient to evaluate if the existence condition (7) is solvable or not. In other words, for $\boldsymbol{C}(\boldsymbol{x}') = \boldsymbol{C} = \text{const.}$, there is no benefit in approaching with a parametrized observer gain $\boldsymbol{L}(\boldsymbol{x}')$. In this context, Corollary 7 reflects the idea behind Lemma 4. Loosely speaking, if the output error injection allows to access those parts of $-\boldsymbol{R}(\boldsymbol{x}')$ which corresponds to zero eigenvalues, we can shift them to the left. In the case of linear measurements, a constant observer gain which is independent of $\boldsymbol{x}'$ is sufficient towards this endeavor. On the other hand, if $\boldsymbol{R}(\boldsymbol{x}')$ is already positive-definite, the observer (8) is globally exponentially convergent without any error injection, see the following corollary.

Corollary 8. Consider a strictly passive PHS (1) with measurements (3), i.e., the case where $\boldsymbol{R}(\boldsymbol{x}') \succ 0$ for all $\boldsymbol{x}' \in \mathbb{X}'$. A globally exponentially convergent state observer for the system is given by (8) with $\boldsymbol{L} = \boldsymbol{0}$.

Proof. The statement follows from Lemma 3 under $\boldsymbol{R}(\boldsymbol{x}') \succ 0$ for all $\boldsymbol{x}' \in \mathbb{X}'$. □

4. DISCUSSION

Theorem 6 is the main theoretical result of this paper. The theorem provides a sufficient condition and a design scheme for a state-output observer applicable to a class of PHSs. This class of PHSs allows for nonlinearities in the interconnection structure as well as in the storages, i.e., PHSs with state-dependent matrices and a possibly non-quadratic Hamiltonian. Venkatraman and van der Schaft (2010) consider an almost identical class of systems. A limitation of this class of PHSs is the assumption that those states which are responsible for the state-dependence of the PHS matrices and which constitute the non-quadratic part of the Hamiltonian are measured. In order to apply our approach to a practical system, this assumption has to be satisfied, e.g., by an appropriate sensor placement in which these states are being measured.

The observer from Theorem 6 obviates a dedicated "design" as it can be derived directly from the system model. This is in contrast to the observer design from Venkatraman and van der Schaft (2010) which requires the closed-form solution of a set of PDEs and algebraic equations.

The existence condition (19) of the observer requires the error system to be sufficiently damped. For this we note that the damping consists of two parts, viz. the natural damping of the system and a virtual damping arising from the error injection. To ensure a fast convergence of all observer states, the error injection must access those states subject to no or weak natural dissipation. On the other hand, if the natural damping is sufficiently strong on all states (i.e., the system is strictly passive), we can completely omit the error injection in the observer (cf. Corollary 8). This damping interpretation is closely related to well-known insights for the control of PHSs, see, e.g., van der Schaft (2017, Sec. 7.1).

5. EXAMPLE

In this section, we illustrate the nonlinear observer from Theorem 6 by means of an academic example.

Consider the following PHS

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} d & de^{\kappa x_1} & 0 \\ de^{\kappa x_1} & de^{2\kappa x_1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \frac{\partial H}{\partial \boldsymbol{x}} + \begin{pmatrix} 0 & d \\ 1 & de^{\kappa x_1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{21a}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -d & -de^{\kappa x_1} & 0 \end{pmatrix} \frac{\partial H}{\partial \boldsymbol{x}} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{21b}$$

with $d > 0$, $\kappa > 0$ and the non-quadratic Hamiltonian

$$H(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{pmatrix} \boldsymbol{x} + \frac{1}{4}x_1^4, \tag{22}$$

where $q_1, q_2, q_3 > 0$. For the system, consider two measurements $m_1 = x_1$ and $m_2 = e^{\kappa x_1} x_3$. The corresponding measurement equation reads

$$\boldsymbol{m} = \underbrace{\begin{pmatrix} q_1^{-1} & 0 & 0 \\ 0 & 0 & q_3^{-1} e^{\kappa x_1} \end{pmatrix}}_{=\boldsymbol{C}(x_1)} \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{23}$$

Following the notation from Section 2, we have $\boldsymbol{x}' = x_1$ and $\boldsymbol{x}'' = (x_2 \; x_3)^\top$. For the observer, we have

$$\boldsymbol{R}(x_1) + \boldsymbol{C}^\top(x_1)\boldsymbol{C}(x_1) = \begin{pmatrix} d + q_1^{-2} & d e^{\kappa x_1} & 0 \\ d e^{\kappa x_1} & d e^{2\kappa x_1} & 0 \\ 0 & 0 & q_3^{-2} e^{2\kappa x_1} \end{pmatrix} \succ 0, \tag{24}$$

for all $\boldsymbol{x} \in \mathbb{X}$. Thus, the observer existence condition (19) is satisfied and a globally exponentially convergent state-output observer is given by (20).

We illustrate the results obtained from numerical simulation of the system (21) and the observer (20). The system parameters are chosen to $d = 1$, $q_1 = \frac{1}{2}$, $q_2 = \frac{1}{3}$, $q_3 = \frac{1}{4}$, and $\kappa = 0.1$. The initial values of the system and the observer are given by $\boldsymbol{x}_0 = (0 \;\; 0 \;\; 0)^\top$ and $\hat{\boldsymbol{x}}_0 = (1 \;\; 1 \;\; 1)^\top$, respectively. The input signals are specified to $u_1 = \sigma(t - 10\,\mathrm{s})$ and $u_2 = \sin(0.1\,\mathrm{s}^{-1}\, t)$, where $\sigma(\cdot)$ is the unit step function.

Figure 1 depicts the states x_i (solid, blue) and the reconstructions $\hat{x}_i$ (dashed, red) for $i = 1, 2, 3$. As can be seen, the state reconstructions reach the true states in less than ten seconds. The reconstructions of the system output are given in Figure 2. The figure shows that the reconstructed outputs also converge to the true outputs as described by Lemma 2.

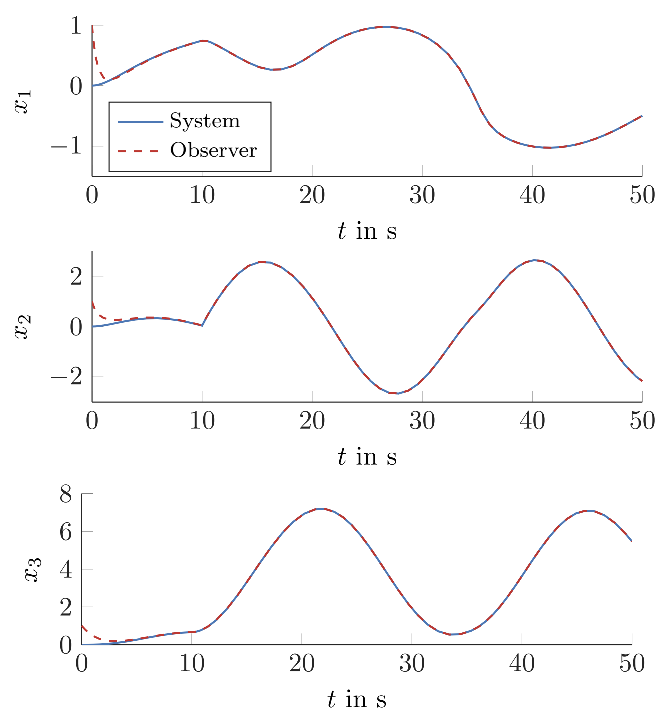

Fig. 1. States of the system (21) (solid, blue) and reconstructions from the observer (20) (dashed, red)

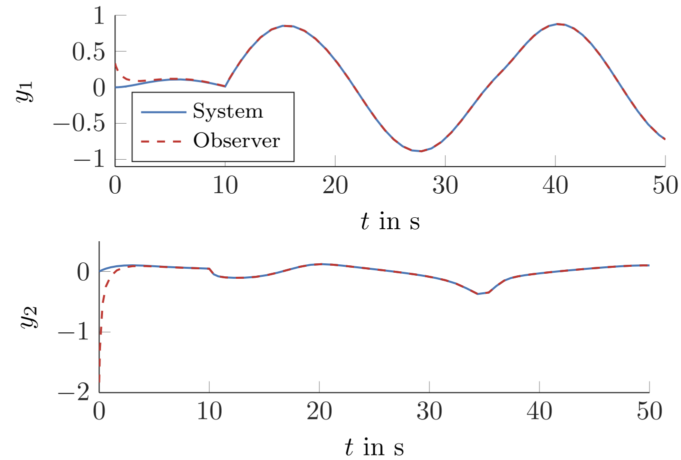

Fig. 2. Outputs of the system (21) (solid, blue) and reconstructions from the observer (20) (dashed, red)

6. CONCLUSION

In this paper, we presented a simple design scheme for a globally exponentially convergent full-order state-output observer for a class of PHSs with nonlinearities in both, the interconnection structure and storages. In contrast to existing approaches, our observer does not require the solution of PDEs but can directly be obtained from the system model (Theorem 6). This makes the approach simple and appealing for a practical application. For the observer, we provide a sufficient existence condition which exploits the natural damping contained in the system. Future work will focus on a robustification of our observer with respect to measurement noise and the extension of the approach to systems with unknown inputs.

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APPENDIX

In the proof of Lemma 3, we applied Lyapunov's direct method to prove $\mathbf{0}$ to be a globally exponentially stable equilibrium point of an error system. In this proof, we made use of the following two propositions:

Proposition 9. Consider the autonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{s})\boldsymbol{Q}\boldsymbol{x}, \tag{25}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{A}(\boldsymbol{s}) \in \mathbb{R}^{n\times n}$, and $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ with $\boldsymbol{Q} = \boldsymbol{Q}^\top \succ 0$ for some parameter $\boldsymbol{s} \in \mathbb{S}$. In order to analyze the stability of the equilibrium $\boldsymbol{x} \equiv \mathbf{0}$ suppose the Lyapunov function candidate $V(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top\boldsymbol{Q}\boldsymbol{x}$. The derivative of $V(\boldsymbol{x})$ with respect to time can be expressed as

$$\dot{V}(\boldsymbol{x}) = \boldsymbol{x}^\top\boldsymbol{Q}\left(\tfrac{1}{2}\left(\boldsymbol{A}(\boldsymbol{s}) + \boldsymbol{A}^\top(\boldsymbol{s})\right)\right)\boldsymbol{Q}\boldsymbol{x}. \tag{26}$$

Equation (26) depends only on the symmetric part of the matrix $\boldsymbol{A}(\boldsymbol{s})$, i.e., $\dot{V}(\boldsymbol{x})$ it is independent of the skew-symmetric part of $\boldsymbol{A}(\boldsymbol{s})$.

Proof. The derivative of $V(\boldsymbol{x})$ reads

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &= \frac{1}{2}\dot{\boldsymbol{x}}^\top\boldsymbol{Q}\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^\top\boldsymbol{Q}\dot{\boldsymbol{x}} \\ &\overset{(25)}{=} \frac{1}{2}\left(\boldsymbol{A}(\boldsymbol{s})\boldsymbol{Q}\boldsymbol{x}\right)^\top\boldsymbol{Q}\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^\top\boldsymbol{Q}\boldsymbol{A}(\boldsymbol{s})\boldsymbol{Q}\boldsymbol{x} \\ &= \boldsymbol{x}^\top\boldsymbol{Q}\left(\tfrac{1}{2}\left(\boldsymbol{A}(\boldsymbol{s}) + \boldsymbol{A}^\top(\boldsymbol{s})\right)\right)\boldsymbol{Q}\boldsymbol{x}. \end{aligned} \tag{27}$$

Proposition 10. Given a vector $\boldsymbol{x} \in \mathbb{R}^n$ and a family of symmetric, positive-definite matricies $\boldsymbol{D}(\boldsymbol{s}) \in \mathbb{R}^{n\times n}$ depending continuously on some parameter $\boldsymbol{s} \in \mathbb{S}$ with $\mathbb{S}$ compact. Then, there exist positive constants $k_1, k_2 \in \mathbb{R}_{>0}$ such that

$$k_1\|\boldsymbol{x}\|^2 \leq \boldsymbol{x}^\top\boldsymbol{D}(\boldsymbol{s})\boldsymbol{x} \leq k_2\|\boldsymbol{x}\|^2, \quad \forall \boldsymbol{s} \in \mathbb{S}, \forall \boldsymbol{x} \in \mathbb{R}^n. \tag{28}$$

Proof. We first show that, without loss of generality, $\boldsymbol{D}(\boldsymbol{s})$ can be assumed to be diagonal.

As $\boldsymbol{D}(\boldsymbol{s})$ is symmetric there exists a continuous family of orthogonal matrices $\boldsymbol{T}(\boldsymbol{s})$ such that

$$\boldsymbol{x}^\top\boldsymbol{D}(\boldsymbol{s})\boldsymbol{x} = \boldsymbol{x}^\top\boldsymbol{T}^\top(\boldsymbol{s})\underbrace{\boldsymbol{T}(\boldsymbol{s})\boldsymbol{D}(\boldsymbol{s})\boldsymbol{T}^\top(\boldsymbol{s})}_{=:\tilde{\boldsymbol{D}}(\boldsymbol{s})}\boldsymbol{T}(\boldsymbol{s})\boldsymbol{x}, \tag{29}$$

for all $\boldsymbol{s} \in \mathbb{S}$ and for all $\boldsymbol{x} \in \mathbb{R}^n$ where $\tilde{\boldsymbol{D}}(\boldsymbol{s})$ is a diagonal matrix with the (positive) eigenvalues of $\boldsymbol{D}(\boldsymbol{s})$ on its diagonal. By defining $\boldsymbol{y} \coloneqq \boldsymbol{T}(\boldsymbol{s})\boldsymbol{x}$ we may rewrite (28) as

$$k_1\|\boldsymbol{y}\|^2 \leq \boldsymbol{y}^\top\tilde{\boldsymbol{D}}(\boldsymbol{s})\boldsymbol{y} \leq k_2\|\boldsymbol{y}\|^2, \quad \forall \boldsymbol{s} \in \mathbb{S}, \forall \boldsymbol{y} \in \mathbb{R}^n. \tag{30}$$

In (30), we use that $\|\boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2$ for all $\boldsymbol{s} \in \mathbb{S}$ which follows from the invariance of the Euclidean norm under orthogonal transformations. Equation (30) shows that, without loss of generality, we may assume $\boldsymbol{D}(\boldsymbol{s})$ to be diagonal.

Now for the claim from the proposition. Let $\boldsymbol{D}(\boldsymbol{s})$ be a positive-definite and diagonal matrix for all $\boldsymbol{s} \in \mathbb{S}$. Recall that $\boldsymbol{D}(\boldsymbol{s})$ depends continuously on $\boldsymbol{s}$. Hence, the eigenvalues $\lambda_i(\boldsymbol{s})$ of $\boldsymbol{D}(\boldsymbol{s})$ are also continuous in $\boldsymbol{s}$ for $i = 1, \ldots, n$. From the positive definiteness of $\boldsymbol{D}(\boldsymbol{s})$ and the compactness of $\mathbb{S}$ we conclude that all eigenvalues $\lambda_i(\boldsymbol{s})$ are contained in a compact subset of $\mathbb{R}_{>0}$. Thus, there exist positive constants $k_1, k_2 \in \mathbb{R}_{>0}$ with $k_1 \leq \lambda_i(\boldsymbol{s}) \leq k_2$ for all $\boldsymbol{s} \in \mathbb{S}$ and $i = 1, \ldots, n$. Such constants then fulfill (28) as $k_1\boldsymbol{x}^\top\boldsymbol{I}\boldsymbol{x} \leq \boldsymbol{x}^\top\boldsymbol{D}(\boldsymbol{s})\boldsymbol{x} \leq k_2\boldsymbol{x}^\top\boldsymbol{I}\boldsymbol{x}$, for all $\boldsymbol{s} \in \mathbb{S}$, $\boldsymbol{x} \in \mathbb{R}^n$.