Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 54-9 (2021) 615–619

Conservation of Generalized Momentum Maps in the Optimal Control of Constrained Mechanical Systems

Peter Betsch, Simeon Schneider

Karlsruhe Institute of Technology (KIT), 76131 Karlsruhe, Germany (e-mail: peter.betsch@kit.edu, simeon.schneider@kit.edu).

Abstract: We show that the optimal control of constrained mechanical systems satisfies an optimal control version of Noether's theorem. In particular, the symmetry of the uncontrolled mechanical system subject to algebraic constraints is handed down to the optimal control problem. We also show that the corresponding generalized momentum map on the level of the optimal control problem can be preserved under discretization.

Keywords: Optimal control, symmetry, constrained mechanical systems, differential-algebraic equations, structure-preservation.

1. INTRODUCTION

In the present work we deal with the optimal control of discrete mechanical systems subject to holonomic constraints. Constrained mechanical systems are particularly important for the description of multibody system dynamics (Bauchau (2011)). The motion of such systems is governed by differential-algebraic equations (DAEs). The DAEs are typically in index-3 Hessenberg form (Ascher and Petzold (1998); Kunkel and Mehrmann (2006)). For some applications minimal coordinates can be found such that the holonomic constraints can be eliminated by applying size-reduction approaches, see e.g. Leyendecker et al. (2010). Alternatively, coordinate partitioning techniques might be applied to satisfy the algebraic constraints. However, to prevent singularities, coordinate switching can be necessary which, on the other hand, is highly inconvenient in the solution of optimal control boundary value problems. In contrast to that, using redundant coordinates associated with the underlying DAEs facilitates a general and singularity-free description of the state equations of constrained mechanical systems.

2. OPTIMAL CONTROL OF CONSTRAINED MECHANICAL SYSTEMS

We focus on the GGL-stabilized (Gear et al. (1985)) index-2 variant of the state equations given by

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u}) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}) \end{aligned} \tag{1}$$

Here, the state variables are comprised of redundant coordinates $\boldsymbol{q} \in \mathbb{R}^n$ and momenta $\boldsymbol{p} \in \mathbb{R}^n$. Accordingly, $\boldsymbol{x} = (\boldsymbol{q},\boldsymbol{p}) \in P$, where $P = \mathbb{R}^n \times \mathbb{R}^n$. The algebraic constraints $(1)_2$ contain both the holonomic constraints

$$\alpha_i(\boldsymbol{q}) = 0\,, \quad i = 1,\ldots,m \tag{2}$$

and constraints on momentum level, $\beta_i(\boldsymbol{q},\boldsymbol{p}) = 0$, $i = 1,\ldots,m$, resulting from the time derivative of (2). Accordingly, $\boldsymbol{g} : P \mapsto \mathrm{R}^{2m}$ is the algebraic constraint function. The constraints $(1)_2$ are enforced by means of Lagrange multipliers $\boldsymbol{y} = \mathbb{R}^{2m}$. Moreover, $\boldsymbol{u} \in \mathbb{R}^{n_u}$ contains the control inputs. The right-hand side of $(1)_1$ can be written as

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u}) = \boldsymbol{J}\nabla H(\boldsymbol{x}) + \sum_{i=1}^{2m} y_i \boldsymbol{h}_i(\boldsymbol{x}) + \boldsymbol{F}(\boldsymbol{x},\boldsymbol{u}) \tag{3}$$

Here, $\boldsymbol{J} \in \mathbb{R}^{2n\times 2n}$ represents the canonical symplectic matrix, $H : P \mapsto \mathbb{R}$ contains kinetic plus potential energies of the unconstrained mechanical system, $\boldsymbol{h}_i : P \mapsto \mathbb{R}^{2n}$ originates from the constraints $(1)_2$ and contributes to the constraint forces, and $\boldsymbol{F} : P \times \mathbb{R}^{n_u} \mapsto \mathbb{R}^{2n}$ accounts for the actuating forces. The optimal control problem seeks to minimize the cost functional

$$\int_0^T l(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u})\,dt \tag{4}$$

subject to the state equations (1), which need to be satisfied throughout the time interval $[0,T]$. In the last equation, $l : P \times \mathbb{R}^{2m} \times \mathbb{R}^{n_u} \mapsto \mathbb{R}$ is the cost density function. The necessary conditions of optimality are well-known (Roubícek and Valásek (2002); Gerdts (2012)) and can be formulated by introducing the Hamiltonian of the optimal control problem

$$\widetilde{H}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{\lambda}) := \boldsymbol{\psi}\cdot\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u}) + \boldsymbol{\lambda}\cdot\widetilde{\boldsymbol{g}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u}) - l(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u}) \tag{5}$$

where

$$\widetilde{\boldsymbol{g}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u}) := D\boldsymbol{g}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u}) \tag{6}$$

The necessary optimality conditions are comprised of the adjoint DAEs

$$\begin{aligned} \dot{\boldsymbol{\psi}} &= -\partial_x \widetilde{H}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{\lambda}) \\ \boldsymbol{0} &= \partial_y \widetilde{H}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{\lambda}) \end{aligned} \tag{7}$$

along with the optimality condition

$$\boldsymbol{0} = \partial_u \widetilde{H}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{\lambda}) \tag{8}$$

For simplicity of exhibition, the endpoint conditions and possible restrictions on $\boldsymbol{u}$ and $\boldsymbol{x}$ are not addressed herein.

10.1016/j.ifacol.2021.06.123

3. CONSERVATION OF GENERALIZED MOMENTUM MAPS

There exists an optimal control version of Noether's theorem. In particular, for state equations in the form of ODEs this has been shown in Djukić (1973); van der Schaft (1987); Torres (2002). We show that these results can be extended to the present case. Accordingly, if the optimal control problem has symmetry, an associated generalized momentum map is conserved along the solution of the optimal control problem.

Suppose that the underlying *uncontrolled* mechanical system has symmetry. Specifically, assume that $H : P \mapsto \mathbb{R}$ and $g_i : P \mapsto \mathbb{R}$ (i=1,. . . , 2m) are invariant under the action of a group G on P given by a smooth mapping $\Phi : G \times P \mapsto P$. Accordingly, the symmetry of the underlying mechanical system is characterized by the invariance properties

$$H \circ \Phi_g = H \quad \text{and} \quad \boldsymbol{g} \circ \Phi_g = \boldsymbol{g} \tag{9}$$

for $\Phi_g : P \mapsto P$ and any $g \in G$. We focus on the standard actions of a matrix Lie group on $\mathbb{R}^{2n}$. In particular, we consider the one-parameter subgroup $\varphi_\xi(s) = \exp(s\xi)$ of G associated with $\xi \in \mathfrak{g}$, the Lie algebra of G. The infinitesimal generator associated to ξ at $\boldsymbol{x} \in P$ is given by

$$\xi_P(\boldsymbol{x}) = \frac{d}{ds}\bigg|_{s=0} \Phi(\varphi_\xi(s), \boldsymbol{x}) \tag{10}$$

The symmetry of the uncontrolled mechanical system is a prerequisite for the symmetry of the mechanical control system. Specifically, the mechanical control system has symmetry, if the following properties hold:

$$\begin{aligned} \boldsymbol{f}(\Phi_{\varphi_\xi(s)}(\boldsymbol{x}), \boldsymbol{y}^s, \boldsymbol{u}^s) &= D\Phi_{\varphi_\xi(s)}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) \\ \widetilde{\boldsymbol{g}}(\Phi_{\varphi_\xi(s)}(\boldsymbol{x}), \boldsymbol{y}^s, \boldsymbol{u}^s) &= \widetilde{\boldsymbol{g}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) \\ l(\Phi_{\varphi_\xi(s)}(\boldsymbol{x}), \boldsymbol{y}^s, \boldsymbol{u}^s) &= l(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) \end{aligned} \tag{11}$$

Here, $\boldsymbol{y}^s$ and $\boldsymbol{u}^s$ denote one-parameter families of multipliers and controls such that $\boldsymbol{y}^0 = \boldsymbol{y}$ and $\boldsymbol{u}^0 = \boldsymbol{u}$.

Proposition 1. If the mechanical control system satisfies the symmetry conditions (11), generalized momentum maps of the form

$$J_\xi = \boldsymbol{\psi} \cdot \xi_P(\boldsymbol{x}) \tag{12}$$

are preserved along the solution of the optimal control problem.

Proof. Symmetry property $(11)_3$ leads to

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0} l(\Phi_{\varphi_\xi(s)}(\boldsymbol{x}), \boldsymbol{y}^s, \boldsymbol{u}^s) &= \partial_x l(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) \cdot \xi_P(\boldsymbol{x}) \\ &+ \partial_y l(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) \cdot \frac{d}{ds}\bigg|_{s=0} \boldsymbol{y}^s \\ &+ \partial_u l(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) \cdot \frac{d}{ds}\bigg|_{s=0} \boldsymbol{u}^s \\ &= 0 \end{aligned}$$

Inserting from (7) and (8) into the last equation yields

$$\begin{aligned} 0 &= \dot{\boldsymbol{\psi}} \cdot \xi_P(\boldsymbol{x}) \\ &+\boldsymbol{\psi} \cdot \left[\partial_x \boldsymbol{f} \cdot \xi_P(\boldsymbol{x}) + \partial_y \boldsymbol{f} \cdot \frac{d}{ds}\boldsymbol{y}^s + \partial_u \boldsymbol{f} \cdot \frac{d}{ds}\boldsymbol{u}^s\right]_{s=0} \\ &+\boldsymbol{\lambda} \cdot \left[\partial_x \widetilde{\boldsymbol{g}} \cdot \xi_P(\boldsymbol{x}) + \partial_y \widetilde{\boldsymbol{g}} \cdot \frac{d}{ds}\boldsymbol{y}^s + \partial_u \widetilde{\boldsymbol{g}} \cdot \frac{d}{ds}\boldsymbol{u}^s\right]_{s=0} \end{aligned}$$

Taking into account the infinitesimal version of $(11)_1$,

$$\partial_x \boldsymbol{f} \cdot \xi_P(\boldsymbol{x}) + \left[\partial_y \boldsymbol{f} \cdot \frac{d}{ds}\boldsymbol{y}^s + \partial_u \boldsymbol{f} \cdot \frac{d}{ds}\boldsymbol{u}^s\right]_{s=0} = D\xi_P(\boldsymbol{x})\dot{\boldsymbol{x}}$$

and $(11)_2$,

$$\partial_x \widetilde{\boldsymbol{g}} \cdot \xi_P(\boldsymbol{x}) + \left[\partial_y \widetilde{\boldsymbol{g}} \cdot \frac{d}{ds}\boldsymbol{y}^s + \partial_u \widetilde{\boldsymbol{g}} \cdot \frac{d}{ds}\boldsymbol{u}^s\right]_{s=0} = \boldsymbol{0}$$

one obtains

$$\dot{\boldsymbol{\psi}} \cdot \xi_P(\boldsymbol{x}) + \boldsymbol{\psi} \cdot D\xi_P(\boldsymbol{x})\dot{\boldsymbol{x}} = 0$$

or

$$\frac{d}{dt}\left(\boldsymbol{\psi} \cdot \xi_P(\boldsymbol{x})\right) = 0$$

Consequently, the generalized momentum map (12) is a conserved quantity.

4. STRUCTURE-PRESERVING SCHEME

We show that the direct transcription method recently proposed in Martens and Gerdts (2020) is capable to preserve generalized momentum maps of the form (12). A similar result has been found previously for optimal control problems in which the state equations assume the form of ODEs, see Betsch and Becker (2017).

The direct approach devised by Martens and Gerdts (2020) is based on the application of the implicit Euler method for the time discretization of the state equations (1):

$$\begin{aligned} \boldsymbol{x}_{n+1} - \boldsymbol{x}_n &= h\boldsymbol{f}(\boldsymbol{x}_{n+1}, \boldsymbol{y}_{n+1}, \boldsymbol{u}_{n+1}) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}_{n+1}) \end{aligned} \tag{13}$$

Accordingly, the time interval $[0, T]$ is partitioned into N time steps of length $h = t_{n+1} - t_n$, $n = 0, \ldots, N-1$. The resulting discrete versions of (7) and (8) assume the form

$$\begin{aligned} \boldsymbol{\psi}_{n+1} - \boldsymbol{\psi}_n &= -h\partial_x \widetilde{H}_d(\boldsymbol{x}_{n+1}, \boldsymbol{y}_{n+1}, \boldsymbol{u}_{n+1}, \boldsymbol{\psi}_n, \boldsymbol{\lambda}_n) \\ \boldsymbol{0} &= \partial_y \widetilde{H}_d(\boldsymbol{x}_{n+1}, \boldsymbol{y}_{n+1}, \boldsymbol{u}_{n+1}, \boldsymbol{\psi}_n, \boldsymbol{\lambda}_n) \\ \boldsymbol{0} &= \partial_u \widetilde{H}_d(\boldsymbol{x}_{n+1}, \boldsymbol{y}_{n+1}, \boldsymbol{u}_{n+1}, \boldsymbol{\psi}_n, \boldsymbol{\lambda}_n) \end{aligned} \tag{14}$$

In the discrete necessary conditions of optimality (14), the discrete optimal control Hamiltonian is defined by

$$\widetilde{H}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{\psi}, \boldsymbol{\lambda}) := \boldsymbol{\psi}\cdot\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) + \boldsymbol{\lambda}\cdot\widetilde{\boldsymbol{g}}_d(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) - l(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u})$$

In the last equation, $\widetilde{\boldsymbol{g}}_d(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u})$ is given by

$$\widetilde{\boldsymbol{g}}_d(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) = \frac{1}{h}\left(\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x} - h\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}))\right) \tag{15}$$

Note that the imposition of $\widetilde{\boldsymbol{g}}_d(\boldsymbol{x}_{n+1}, \boldsymbol{y}_{n+1}, \boldsymbol{u}_{n+1}) = \boldsymbol{0}$, $n = 0, \ldots, N-1$ is equivalent to the constraint enforcement through $(13)_2$, provided that consistent initial conditions for $\boldsymbol{x}_0 \in P$ are used. That is, $\boldsymbol{g}(\boldsymbol{x}_0) = 0$ needs be insured.

Proposition 2. If the mechanical control system satisfies the symmetry conditions (11), the above scheme is capable to conserve generalized momentum maps of the form (12) in the sense that

$$J_\xi(\boldsymbol{x}_{n+1}, \boldsymbol{\psi}_{n+1}) = J_\xi(\boldsymbol{x}_n, \boldsymbol{\psi}_n) \tag{16}$$

for $n = 0, \ldots, N-1$.

Proof. Similar to the corresponding proof of the continuous case we start with the infinitesimal version of symmetry property $(11)_3$ given by

$$\frac{d}{ds}\bigg|_{s=0} l(\Phi_{\varphi_\xi(s)}(\boldsymbol{x}_{n+1}), \boldsymbol{y}^s_{n+1}, \boldsymbol{u}^s_{n+1}) = 0$$

Continuing along the lines of the continuous proof, substitute from (14) into the last equation and subsequently take into account the infinitesimal versions of $(11)_{1,2}$ to arrive at

$$(\boldsymbol{\psi}_{n+1} - \boldsymbol{\psi}_n) \cdot \xi_P(\boldsymbol{x}_{n+1}) + \boldsymbol{\psi}_n \cdot (\xi_P(\boldsymbol{x}_{n+1}) - \xi_P(\boldsymbol{x}_n)) = 0$$

The last equation can be recast in the form

$$\boldsymbol{\psi}_{n+1} \cdot \xi_P(\boldsymbol{x}_{n+1}) - \boldsymbol{\psi}_n \cdot \xi_P(\boldsymbol{x}_n) = 0$$

which corroborates the discrete conservation property (16).

5. NUMERICAL EXAMPLE

In the numerical example we consider the pendulum depicted in Fig. 1.

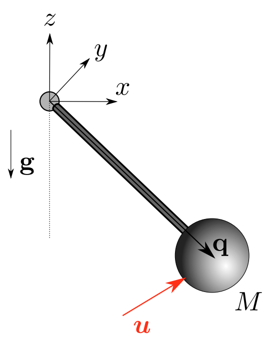

Fig. 1. Pendulum

To describe the configuration of the pendulum, we choose redundant Cartesian coordinates $\boldsymbol{q} \in \mathbb{R}^3$ subject to $m = 1$ holonomic constraint (2), where the constraint function is given by

$$\alpha(\boldsymbol{q}) = \frac{1}{2}(\boldsymbol{q} \cdot \boldsymbol{q} - L) \tag{17}$$

The linear momentum is given by $\boldsymbol{p} = M\dot{\boldsymbol{q}}$, leading to the state vector $\boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{p}) \in P$, where $P = \mathbb{R}^3 \times \mathbb{R}^3$. In the above equations, L and M denote the length of the pendulum and the mass, respectively. The total energy of the pendulum $H : P \mapsto \mathbb{R}$ assumes the form

$$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2M}\boldsymbol{p} \cdot \boldsymbol{p} + M\mathrm{g}z \tag{18}$$

where g denotes the gravitational acceleration. The constraint on momentum level follows from $\frac{d}{dt}\alpha(\boldsymbol{q}) = 0$ and is given by

$$\beta(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{M}\boldsymbol{q} \cdot \boldsymbol{p} \tag{19}$$

Concerning the contribution to the constraint forces in (3), the holonomic constraint (17) gives rise to

$$\boldsymbol{h}_1(\boldsymbol{q}, \boldsymbol{p}) = \begin{bmatrix} \boldsymbol{0} \\ \nabla\alpha(\boldsymbol{q}) \end{bmatrix} \tag{20}$$

while the GGL stabilization technique yields

$$\boldsymbol{h}_2(\boldsymbol{q}, \boldsymbol{p}) = \begin{bmatrix} \nabla\alpha(\boldsymbol{q}) \\ \boldsymbol{0} \end{bmatrix} \tag{21}$$

Furthermore, we choose

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}) = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{u} \end{bmatrix} \tag{22}$$

where $\boldsymbol{u} \in \mathbb{R}^3$ is a force vector acting on mass M (Fig. 1). To minimize the control effort, we choose $l(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) = \boldsymbol{u} \cdot \boldsymbol{u}$.

Consider the one-parameter subgroup $\varphi_\xi(s)$ of SO(3) associated with $\xi \in \mathrm{so}(3)$, which in the present example can be identified with vector $\boldsymbol{\xi} \in \mathbb{R}^3$ being co-linear to base vector $\boldsymbol{e}_z$ (Fig. 1). Now,

$$\Phi(\varphi_\xi(s), \boldsymbol{x}) = (\boldsymbol{R}_\xi(s)\boldsymbol{q}, \boldsymbol{R}_\xi(s)\boldsymbol{p}) \tag{23}$$

where $\boldsymbol{R}_\xi(s) \in \mathrm{SO}(3)$ is a rotation matrix which describes rotations about the z-axis. It can be easily verified that both H and g_i $(i = 1, 2)$ are rotationally invariant in the sense that $H \circ \Phi_{\varphi_\xi(s)} = H$ and $g_i \circ \Phi_{\varphi_\xi(s)} = g_i$, respectively. The infinitesimal generator (10) assumes the form

$$\xi_P(\boldsymbol{x}) = (\boldsymbol{\xi} \times \boldsymbol{q}, \boldsymbol{\xi} \times \boldsymbol{p}) \tag{24}$$

Choosing $\boldsymbol{y}^s = \boldsymbol{y}$ and $\boldsymbol{u}^s = \boldsymbol{R}_\xi(s)\boldsymbol{u}$, it can be shown in a straightforward way that the symmetry conditions in (11) are satisfied. The generalized momentum map (12) of the optimal control problem at hand is obtained as

$$J_\xi = \boldsymbol{\xi} \cdot (\boldsymbol{q} \times \boldsymbol{\psi}_q + \boldsymbol{p} \times \boldsymbol{\psi}_p) \tag{25}$$

where the adjoint variables are partitioned as $\boldsymbol{\psi} = (\boldsymbol{\psi}_q, \boldsymbol{\psi}_p)$.

In the numerical example we choose $T = 1.5$, $M = 1$, $L = 5$. The time discretization is based on $N = 150$ steps. We consider a maneuver of the pendulum with initial conditions

$$\boldsymbol{q}(0) = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{p}(0) = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$

and end conditions

$$\boldsymbol{q}(T) = \begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix}, \qquad \boldsymbol{p}(T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The computed time-evolution of the state variables $\boldsymbol{x}(t)$ and the adjoint variables $\boldsymbol{\psi}(t)$ is depicted in Figs. 2 and 3, respectively. Using the present scheme, the generalized momentum map of the optimal control problem is preserved and the constraints are fulfilled (up to numerical round-off), as can be observed from Figs. 4 and 5, respectively. Similarly, the condition $\boldsymbol{u} \cdot \boldsymbol{q} = 0$ is satisfied. Furthermore, Figs. 6 and 7 depict the control inputs and the Lagrange multiplier y_1 versus time, respectively. Eventually, to illustrate the resulting motion of the pendulum, Fig. 8 shows a sequence of subsequent snapshots in time of the pendulum, wherein the arrows indicate both magnitude and direction of the control forces acting on the mass.

6. CONCLUSION

We have dealt with generalized momentum maps in the optimal control of constrained mechanical systems. In particular, we have shown that when the optimal control problem has symmetry, the corresponding generalized momentum map is a conserved quantity. This result can be viewed as generalization to constrained systems of previous work on symmetries in optimal control by Djukić (1973); van der Schaft (1987); Torres (2002) and Betsch and Becker (2017). Typically, the symmetry of the optimal control problem is inherited from the symmetry of the underlying uncontrolled mechanical system. Reliable numerical methods for the optimal control of nonlinear mechanical systems subject to holonomic constraints are still in their infancy. The direct transcription approach adopted herein has been recently developed by Martens and Gerdts (2020) and is based on the GGL-stabilized index-2 DAEs. We have shown that this scheme is capable to conserve the generalized momentum map on the level of the optimal control problem and thus inherits the conservation property from the continuous problem.

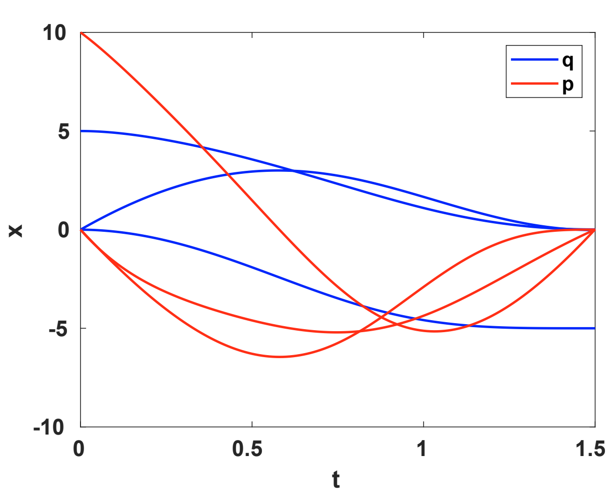

Fig. 2. Evolution of the state variables

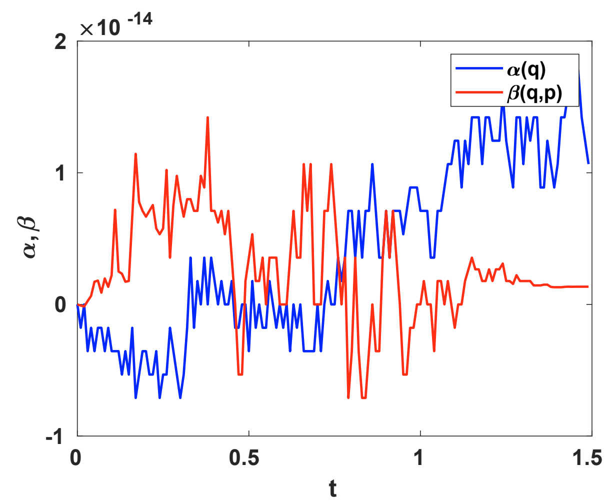

Fig. 5. Fulfillment of the constraints $\alpha(\mathbf{q})$ and $\beta(\mathbf{q}, \mathbf{p})$

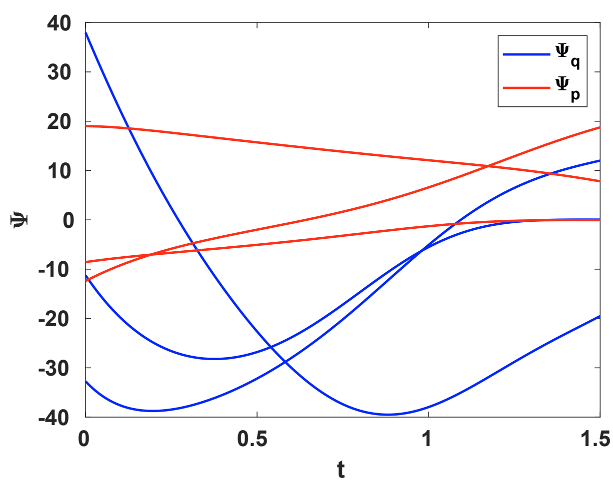

Fig. 3. Evolution of the adjoint variables

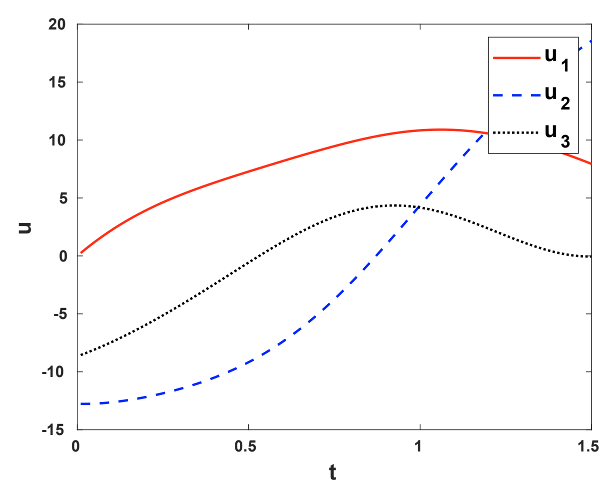

Fig. 6. Evolution of the controls $\boldsymbol{u}$

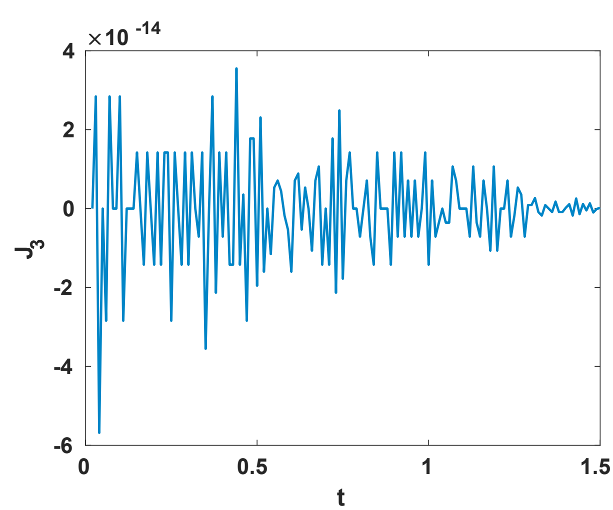

Fig. 4. Conservation of the generalized momentum map (25)

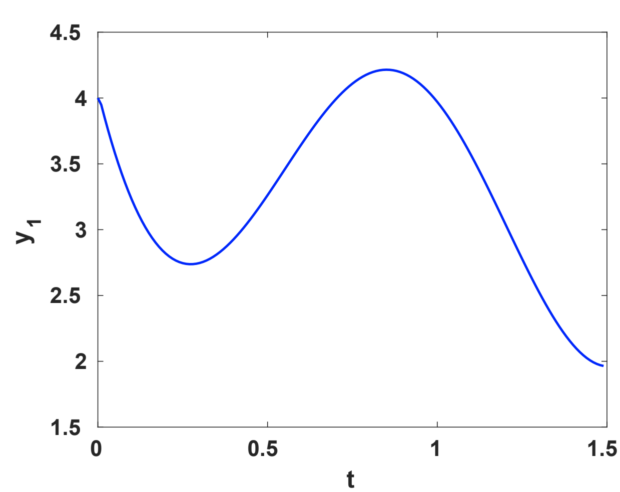

Fig. 7. Evolution of the Lagrangian multiplier y_1

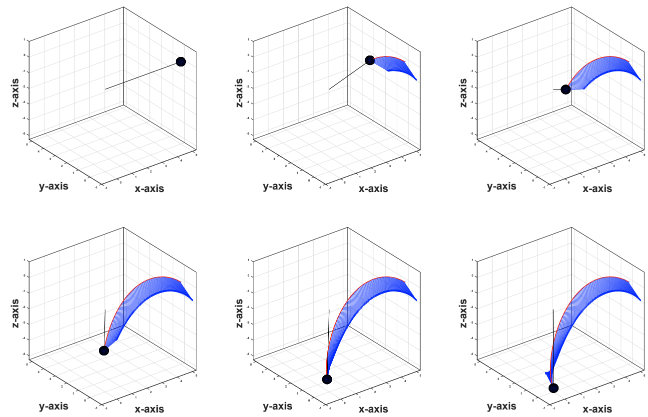

Fig. 8. Snapshots of the motion at times $t \in \{0,\ 0.3,\ 0.6,\ 0.9,\ 1.2,\ 1.5\}$

ACKNOWLEDGEMENTS

This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 442997215. This support is gratefully acknowledged.

REFERENCES

Ascher, U. and Petzold, L. (1998). *Computer methods for ordinary differential equations and differential-algebraic equations*. SIAM.

Bauchau, O. (2011). *Flexible Multibody Dynamics*, volume 176 of *Solid Mechanics and its Applications*. Springer-Verlag.

Betsch, P. and Becker, C. (2017). Conservation of generalized momentum maps in mechanical optimal control problems with symmetry. *Int. J. Numer. Meth. Engng*, 111(2), 144–175.

Djukić, D. (1973). Noether's theorem for optimum control systems. *Int. J. Control*, 18(3), 667–672.

Gear, C., Gupta, G., and Leimkuhler, B. (1985). Automatic integration of the Euler-Lagrange equations with constraints. *J. Comp. Appl. Math.*, 12, 77–90.

Gerdts, M. (2012). *Optimal Control of ODEs and DAEs*. De Gruyter, Berlin/Boston.

Kunkel, P. and Mehrmann, V. (2006). *Differential-Algebraic Equations*. European Mathematical Society.

Leyendecker, S., Ober-Blöbaum, S., Marsden, J., and Ortiz, M. (2010). Discrete mechanics and optimal control for constrained systems. *Optim. Control Appl. Meth.*, 31(6), 505–528.

Martens, B. and Gerdts, M. (2020). Convergence analysis for approximations of optimal control problems subject to higher index differential-algebraic equations and mixed control-state constraints. *SIAM J. Control Optim.*, 58(1), 1–33.

Roubícek, T. and Valásek, M. (2002). Optimal control of causal differential–algebraic systems. *J. Math. Anal. Appl.*, 269(2), 616–641.

Torres, D. (2002). On the Noether theorem for optimal control. *European Journal of Control*, 8(1), 56–63.

van der Schaft, A. (1987). Symmetries in optimal control. *SIAM J. Control and Optimization*, 25(2), 245–259.