Variational techniques for breathers in nonlinear wave equations

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der KIT-Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

DISSERTATION

von

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Tag der mündlichen Prüfung: 23. Juni 2021

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Acknowledgement

This thesis would not have been possible without the support of many people. The following list may be incomplete, length is not correlated with gratitude.

First of all I thank my supervisor Prof. Dr. Wolfgang Reichel for the endless support, the trust in the thesis and the great patience with the development of my ideas. Moreover I thank my co-supervisor Prof. Dr. Michael Plum. Our discussions may have been lower in numbers but not in depth. Especially I thank the CRC 1173 for funding my position and giving me the opportunity to research ins such a great environment. Furthermore I say "thank you" to my working group for always being open for discussions, helping me with references (especially Gabriele Brüll) and making my office an enjoyable place.

As a side note I thank the coffee for fueling my body and brain through countless hours of work.

Last but not least I thank my family and especially my wife Alexandra for always supporting me in all non-mathematical aspects.

Contents

Introduction

In this thesis we investigate the quasilinear wave equation

$$g(x)w_{tt} - w_{xx} + h(x)(w_t^3)_t = 0 \quad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

and the semilinear wave equation

$$V(x)u_{tt} - \Delta u = f(x,t,u), \qquad \text{on } (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.1}$$

where Δ denotes the Laplacian acting only on the variable x. Most of the time we refer to x as *space* and to t as *time*. We are specially interested in spatially localized and time-periodic solutions, so-called *breathers*.

Both equations typically arise in the study of localized electromagnetic waves modeled by Kerr-nonlinear Maxwell equations. Consider Maxwell's equations in the absence of charges and currents

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 0, & \nabla \times \mathbf{E} &= -\partial_t \mathbf{B}, & \mathbf{D} &= \varepsilon_0 \mathbf{E} + \mathbf{P}(\mathbf{E}), \\ \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{H} &= \partial_t \mathbf{D}, & \mathbf{B} &= \mu_0 \mathbf{H}. \end{aligned}$$

Assuming that $\mathbf{P}(\mathbf{E}) = \varepsilon_0\chi_1(\mathbf{x})\mathbf{E} + \varepsilon_0\chi_3(\mathbf{x})|\mathbf{E}|^2\mathbf{E}$ and $\mathbf{E}$ is either a standing or traveling polarized wave we can obtain (1.1). More details of this derivation are carried out in Section 1.1. As seen in [BCBLS11], equation (2.1) can be considered as an approximation of (1.1).

Chapter 1 is devoted to equation (1.1). Up to some additional explanations this Chapter 1 is available as the preprint [KR21]. This preprint is joint work with Wolfgang Reichel and at the moment of the submission of this thesis, the preprint is still in the review process. We consider the $(1+1)$-dimensional quasilinear wave equation (1.1). Here $g \in L^\infty(\mathbb{R})$ is even with $g \not\equiv 0$ and $h(x) = \gamma\,\delta_0(x)$ with $\gamma \in \mathbb{R}\backslash\{0\}$ and δ_0 the delta-distribution supported in 0. We assume that 0 lies in a spectral gap of the operators $L_k = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - k^2\omega^2 g(x)$ on $L^2(\mathbb{R})$ for all $k \in 2\mathbb{Z}+1$ together with additional properties of the fundamental set of solutions of L_k, see Section 1.1 for details. By expanding w into a Fourier series in time we transfer the problem of finding a suitably defined weak solution to finding a minimizer of a functional on a sequence space, see Section 1.2. By Lemma 1.11 the solutions that we have found are exponentially localized in space. Moreover, we show in Section 1.5 that they can be well approximated by truncating the Fourier series in time. The guiding examples, where all assumptions are fulfilled, are:

Theorem (See 1.1)**.** *For $a, b, c > 0$ let*

$$g(x) := \begin{cases} -a, & \text{if } |x| > c, \\ b, & \text{if } |x| < c. \end{cases}$$

For every frequency ω such that $\sqrt{b}\omega c\frac{2}{\pi} \in \frac{2\mathbb{N}+1}{2\mathbb{N}+1}$ and $\gamma < 0$ there exist infinitely many nontrivial, real-valued, spatially localized and time-periodic weak solutions of (1.1) with period $T = \frac{2\pi}{\omega}$. For each solution w there are constants $C, \rho > 0$ such that $|w(x,t)| \leq C\mathrm{e}^{-\rho|x|}$.

Theorem (See 1.2)**.** *For $a, b > 0$, $a \neq b$ and $\Theta \in (0,1)$ let*

$$g(x) := \begin{cases} a, & \text{if } |x| < \pi\Theta, \\ b, & \text{if } \pi\Theta < |x| < \pi \end{cases}$$

and extend g as a 2π-periodic function to $\mathbb{R}$. Assume in addition

$$\sqrt{\frac{b}{a}}\frac{1-\Theta}{\Theta} \in \frac{2\mathbb{N}+1}{2\mathbb{N}+1}.$$

For every frequency ω such that $4\sqrt{a}\theta\omega \in \frac{2\mathbb{N}+1}{2\mathbb{N}+1}$ there exist infinitely many nontrivial, real-valued, spatially localized and time-periodic weak solutions of (1.1) with period $T = \frac{2\pi}{\omega}$. For each solution w there are constants $C, \rho > 0$ such that $|w(x,t)| \leq C\mathrm{e}^{-\rho|x|}$.

In these examples we even find infinitely many distinct breathers as seen in Section 1.4.

Chapter 2 and Chapter 3 are devoted to (2.1), where we consider new examples for sign-changing potentials $V(x)$ and superlinear right hand sides. In fact, in Chapter 2 we prove the following existence results:

Theorem (See 2.21)**.** *Assume $\alpha, \beta > 0$, $p > 1$ and set $\omega := \frac{2\pi}{T}$. Let $\Gamma \in L^\infty(\mathbb{R})$ with $\inf_{\tilde{\Omega}} \Gamma > 0$ for all compact $\tilde{\Omega} \subset \Omega$ and $\lim_{|x|\to\infty} \Gamma(x) = 0$ and $V(x) := \beta\delta_0(x) - \alpha$. Then there exists a nontrivial weak solution u of the equation*

$$V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u, \qquad (x,t) \in \mathbb{R} \times \mathbb{T}_T, \tag{2.2}$$

with minimal energy among all $\frac{T}{2}$-anti-periodic, weak solutions.

Theorem (See 2.38)**.** *Assume $\alpha, \gamma, r > 0$, $p \in (1,3)$ and set $\omega := \frac{\pi}{2r\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$, $\beta := \alpha + \gamma$. Let $\Gamma \in L^\infty(\mathbb{R})$ such that $\inf_{\tilde{\Omega}} \Gamma > 0$ for all compact $\tilde{\Omega} \subset \Omega$ and $\lim_{|x|\to\infty} \Gamma(x) = 0$ and $V(x) := -\alpha + \beta\,\mathbb{1}_{[-r,r]}(x)$. Then there exists a nontrivial weak solution u of the equation*

$$V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u, \qquad (x,t) \in \mathbb{R} \times \mathbb{T}_T, \tag{2.2}$$

with minimal energy among all $\frac{T}{2}$-anti-periodic, weak solutions.

Theorem (See 2.49)**.** *Assume $\alpha, \gamma, R > 0$, $p \in (1,2)$ and set $\omega := \frac{\pi}{2R\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$, $\beta := \alpha + \gamma$. Let $\Gamma \in L^\infty(0,\infty)$ $\inf_{\tilde{\Omega}} \Gamma > 0$ for all compact $\tilde{\Omega} \subset \Omega$ and $V(x) := -\alpha + \beta\,\mathbb{1}_{B_R(0)}(x)$. Then there exists a nontrivial weak solution u of the equation*

$$V(x)u_{tt} - \Delta u = \Gamma(|x|)|u|^{p-1}u, \qquad (x,t) \in \mathbb{R}^2 \times \mathbb{T}_T. \tag{2.3}$$

with minimal energy among all $\frac{T}{2}$-anti-periodic, weak solutions.

Observe that in the first two results we assumed decay of Γ, in the higher dimensional case we do not need this assumption due to radial symmetry. In Chapter 3 we consider nondecaying potentials Γ, use a more complicated technique and prove the following existence results:

Theorem (See 3.45)**.** *Assume* $\alpha, \beta > 0$, $p \in (1,3)$ *and* $\omega := \frac{2\pi}{T} < \frac{2\sqrt{\alpha}}{\beta}$. *Let* $\Gamma \in L^\infty(\mathbb{R})$ *be periodic with* $\inf \Gamma > 0$ *and* $V(x) := \beta\delta_0(x) - \alpha$. *Then there exists a nontrivial weak solution* u *of the equation*

$$V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u, \qquad (x,t) \in \mathbb{R} \times \mathbb{T}_T, \tag{2.2}$$

with minimal energy among all $\frac{T}{2}$*-anti-periodic, weak solutions.*

Theorem (See 3.49)**.** *Assume* $\alpha, \gamma, r > 0$, $p \in (1,3)$ *and set* $\omega := \frac{\pi}{2r\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$, $\beta := \alpha + \gamma$. *Let* $\Gamma \in L^\infty(\mathbb{R})$ *be periodic, even,* $\inf \Gamma > 0$ *and let* $V(x) := -\alpha + \beta\,\mathbb{1}_{[-r,r]}(x)$. *Then there exists a nontrivial weak solution* u *of the equation*

$$V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u, \qquad (x,t) \in \mathbb{R} \times \mathbb{T}_T, \tag{2.2}$$

with minimal energy among all $\frac{T}{2}$*-anti-periodic, even in* t *and spatially odd weak solutions.*

We develop uniform methods to deduce the above examples. Formally, weak, time-periodic solutions of (2.1) with time-period T correspond to critical points of the indefinite energy functional

$$I\,(u) = \frac{1}{2}\int_{\mathbb{R}^N \times (0,T)} -V(x)|u_t|^2 + |u_x|^2 \,\mathrm{d}(x,t) - \frac{1}{p+1}\int_{\mathbb{R}^N \times (0,T)} F(x,t,u)\,\mathrm{d}(x,t)\,,$$

with $F(x,t,u) = \int_0^u f(x,t,s)\,\mathrm{d}s$. This is the starting point of our analysis.

In Chapter 2 we focus on the difficulties arising from sign-changing $V(x)$. Then a domain of the operator $L = V(x)\partial_t^2 - \Delta$ such that L is self-adjoint is hard to characterize. Furthermore the formal bilinear form $b_L\,(u,v) = \int_{\mathbb{R}^N \times \mathbb{T}_T} -V(x)u_t\,v_t + u_x\,v_x\,\mathrm{d}(x,t)$ is neither bounded from above nor below, i.e., we can not use Friedrich's extension theorem, cf. [RS10], to recover a self-adjoint operator L from b_L. We use the ideas of [HR19] and generalize their approach to overcome this difficulty. We will obtain a toolbox, applicable to our examples mentioned above. The toolbox roughly reads as follows:

1. Formally given an operator $L = V(x)\partial_t^2 - \Delta$ on $(x,t) \in \mathbb{R}^N \times \mathbb{T}_T$, we decompose it formally into the self-adjoint $L_k = -\Delta - \omega^2 k^2 V(x)$ using Fourier series with $T > 0$ as time-period and $\omega = \frac{2\pi}{T}$.
2. Then we calculate the spectra $\sigma(L_k)$ and verify a growing spectral gap around 0 in the sense that there are $c, K > 0$ and $a > 0$ such that for all $k > K$ we have $(-c|k|^a, c|k|^a) \subset \sigma(L_k)^C$. Here we verify in addition that if $0 \in \sigma(L_k)$, then it is an eigenvalue of at most finite multiplicity.
3. We then use Section 2.2.2 as a black-box to construct a sequence space $\mathcal{H}$, a bilinear form $b_{\mathcal{L}}$ and a self-adjoint operator $\mathcal{L} := \bigoplus_k L_k$ such that $b_{\mathcal{L}}$ is the corresponding closed and hermitian bilinear form to $\mathcal{L}$. In Section 2.2.3 we additionally equip $\mathcal{H}$ with a special scalar product, which is used in the other sections.

4. We apply the results of Section 2.2.3 to calculate $p^* = p^*(a, N, V)$ as in Theorem 2.16 such that $S\colon \mathcal{H} \hookrightarrow L^{p+1}(\mathbb{R}^N \times \mathbb{T}_T)$, $(S\hat{u}) := \sum_k \hat{u}_k(x)e_k(t)$ is continuous and locally compact.

5. Finally we use either Theorem 2.12 or Theorem 2.14 to obtain a ground state of $\mathcal{I}$ and hence a weak solution to (2.1).

The step 2. will be done completely in the example sections. The steps 3. and 4. are quite technical and are carried out in Section 2.2. Here we do not work with concrete examples but we generalize the technique of [HR19] to the above mentioned toolbox. With the Hilbert space $\mathcal{H}$ and an embedding $S\colon \mathcal{H} \hookrightarrow L^{p+1}(\mathbb{R}^N \times \mathbb{T}_T)$ at hand, we reformulate and slightly adjust the work of [SW10] to prove the existence of a ground state of (2.1), if the right hand side f satisfies some often used assumptions. This is step 5. The core of [SW10] is a minimization procedure on the Nehari-manifold. Here we need a compactness result for minimizing sequences, which either results form decay of f or from cylindrical symmetry of the equation (2.1). We will apply this procedure in three examples mentioned above. In those three examples more general right hand sides as stated in Theorem 2.12 and Theorem 2.14 are admissible, even a time-dependent right hand side like the function $\Gamma(x) \cdot \frac{1}{2+\sin(t)} \cdot |u|^{p-1}u$ is possible.

In Chapter 3 we focus on $N = 1$ and $f(x, t, u) = \Gamma(x)|u|^{p-1}u$ with non-decaying Γ in equation (2.1), since this case is not covered in Chapter 2. We use two dual variational approaches, inspired by [Fre13] and [DPR11] using the method of the Nehari Manifold and [Str08] using a constrained minimization approach, and again give a toolbox to guarantee the existence results: First check 1. to 4. as in Chapter 2. Then:

5'. We check that $\mathcal{L}\colon \mathcal{H} \to \mathcal{H}^*$ is invertible.

6. Furthermore we identify an elliptic operator $\widetilde{\mathcal{L}}$ such that formally $\mathcal{L} = \widetilde{\mathcal{L}} + \mathcal{W}$ and $\mathcal{W}$ is localized in space.

7.a) Finally we check Assumption 3.14, an a-priori estimate for ground state energies, and Assumption 3.15, a transfer of Palais-Smale sequences.

7.b) Finally we check Assumption 3.21, an a-priori estimate for ground state energies, and Assumption 3.22, an improvement for special weakly convergent sequences.

8. Now we apply the abstract existence result Theorem 3.16 in the case 7.a) and Theorem 3.24 in the case 7.b).

We give some details on the ideas of this line of actions. Using the invertibility of $\mathcal{L}$ and the special form of the right hand side we transform (2.1) into

$$Kv = |v|^{q-1}v \qquad \text{on } \ (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{3.3}$$

the so-called dual equation. Here we formally write $K = \Gamma^{\frac{1}{p+1}} L^{-1} \Gamma^{\frac{1}{p+1}}$ and $q = \frac{1}{p}$. Using the sublinear growth of the right had side of (3.3) we construct a bounded Palais-Smale in Lemma 3.13 and prove further properties of such sequences in Section 3.1.1. In Section 3.1.2 we consider

$$\tilde{L}u = \Gamma(x)|u|^{p-1}u \qquad \text{on } \ (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{3.2}$$

the so called equation at infinity (since the difference of L and $\tilde{L}$ is compact). In our examples equation (3.2) will be the elliptic, semilinear equation

$$-\alpha u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u \qquad \text{on} \quad (x,t) \in \mathbb{R} \times \mathbb{R}.$$

Considering only $\frac{T}{2}$-anti-periodic functions, this equation has a ground state. We identify two assumptions (depending on which approach) linking (3.3) and (3.2) as sufficient conditions for an abstract compactness result, yielding the abstract existence theorem of ground states of equation (3.3), i.e., Theorem 3.16 or Theorem 3.24. In Section 3.1.3.1 we give some abstract method on how to verify Assumption 3.14 or Assumption 3.21 and Assumption 3.15 or Assumption 3.22. In Section 3.1.3.1 we also compare the ground state levels resulting from the different approaches and obtain a one-to-one correspondence of the ground state energies and the ground states. Hence the approaches yield the same result but can use different structural advantages. We also give some comparison arguments for different symmetry classes. Then in Section 3.2 we apply the developed techniques. First we analyze the ground state of (3.2), which will be the same in the two examples. Further ideas to the dual method can be found at the beginning of Section 2.1.

Despite using many known ideas from previous works, this thesis contains several significant new contributions. To the best of the author's knowledge we are the first using minimization techniques to find solutions for the quasilinear wave equation (1.1) with $h(x) = \gamma\delta_0(x)$ as in Chapter 1. Regarding Chapter 2 we give a rather general toolbox to construct a domain for $L = V(x)\partial_t^2 - \Delta$, such that L is self-adjoint, the corresponding bilinear form is closed and its domain embeds continuously and locally compact into L^{p+1} when considering $\frac{T}{2}$-anti-periodic functions. Here we additionally construct the norm in a way, such that it fits perfectly into the setting of [SW10]. This toolbox is not restricted to specific potentials $V(x)$, but only needs spectral information of the operators L_k and the basic structure that $V(x)$ is in front of ∂_t^2. Looking into Chapter 3 we use a more advanced notation, in the sense that we do not work directly on functions but on multiple Hilbert spaces. Dual variational techniques are used to prove the existence of weak solution e.g. for nonlinear Helmholtz systems, cf. [MS17], or for nonlinear Schrödinger equations, cf. [Fre13]. Here one usually considers Sobolev and Lebesgue spaces, whereas we consider some Hilbert space $\mathcal{H}$ such that there is a continuous and locally compact embedding $S\colon \mathcal{H} \to L^{p+1}$, such that we can reuse the constructions of Chapter 2. The idea of deriving a compactness result by comparing with an equation at infinity was used for example for direct variational techniques for positive, self-adjoint operators $L\colon H^2 \to L^2$ in [DPR11] and for dual variational techniques with indefinite operator, self-adjoint operators $L\colon H^2 \to L^2$ in [Fre13]. We generalize their ideas into an abstract setting arguing on Hilbert spaces and identifying core steps for an abstract existence result. Checking the conditions for the abstract result in the examples is nontrivial and requires many technical calculations and generalizations of known ideas.

Our techniques naturally generate breather solutions of equation (1.1) and (2.1). Such breathers are a purely nonlinear phenomenon and can not be observed in dispersive media considering only linear equations. Possible applications could be localized optical pulses in nonlinear optics with a breather profile for a traveling wave, or an optical storage device with a standing breather. One of the first breathers discovered was the sine-Gordon-breather [AKNS73] and due to non-persistence results like [BMW94] and [Den93] one could get the impression they are a rare phenomenon. On the other hand we want to mention recent works like the variational techniques of [HR19], the bifurcation methods of

[BCBLS11] or the smart ansatz choice and ODE methods of [PR16]. Here new examples of breathers in different settings have been found. This field of mathematics is still in the development and we hope that this thesis brings us one step closer to a more unified understanding.

For future work the author wants to give an outlook, on how the ideas of this theses could be continued. One interesting question is "Is the critical exponents p^* for the growth of the right hand side optimal?" He would guess not, as the example in Section 2.3.1 indicates. Of course this instantly implies the question on how to improve the exponent. Another question is "Can we find breather solutions of semilinear wave equations as (2.1) with $L = V(x)(-\mathrm{i}\partial_t)^\mu + A(-\mathrm{i}\partial_x)$ for $\mu \in \mathbb{N}$ and a positive function A?" He would guess yes, since all abstract techniques aim for the position of the potential V in L and only rely on spectral information of L_k. Maybe such nonlinear Schrödinger equations, i.e., $\mu = 1$ and $A(x) = x^2$ could also be a field of new examples for breathers. Last but not least we can always ask for sharpness of our assumptions and seek for counter examples if one or more assumptions are not satisfied.

Preliminaries and Notation

We shortly give some preliminaries and notations. Many are self explanatory with context or are defined in the text. Nevertheless we collect some of the most important here.

$C^k(\Omega)$	the space of $k \in \mathbb{N}_0$ times continuously differentialble functions equipped with $\|u\|_{C^k(\Omega)} = \|u\|_{L^\infty(\Omega)} + \sum_{\|\alpha\|=N} \left\| \frac{\partial^\alpha u}{\partial x^\alpha} \right\|_{L^\infty(\Omega)}$.
$L^p(\Omega)$	the Lebesgue-spaces of an open set $\Omega \subset \mathbb{R}^N$, equipped with $\|u\|^p_{L^p(\Omega)} = \int_\Omega \|u\|^p \,\mathrm{d}x$ if $p < \infty$ and $\|u\|_{L^\infty(\Omega)} = \text{ess sup}\,\|u\|$ if $p = \infty$.
$W^{k,p}(\Omega)$	the Sobolev-spaces of an open set Ω, with k weak derivatives, all L^p-integrable, equipped with $\|u\|^p_{W^{k,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + \sum_{\|\alpha\|=N} \left\| \frac{\partial^\alpha u}{\partial x^\alpha} \right\|^p_{L^p(\Omega)}$
$H^k(\Omega)$	$:= W^{k,2}(\Omega)$ if $k \in \mathbb{N}_0$.
$H^s(\Omega)$	the Sobolev-spaces on $\mathbb{R}^N$, with $s > 0$ weak, fractional derivatives, equipped with $\|u\|_{H^k(\mathbb{R}^N)} = \left\| (1 + \|\cdot\|^s)^{\frac{1}{2}} \mathcal{F}u \right\|_{L^2(\mathbb{R}^N)}$.
$\mathcal{F}$	the Fourier transform on $\mathbb{R}^N$ defined by $(\mathcal{F}u)(\xi) = \frac{1}{\sqrt{2\pi}^N} \int_{\mathbb{R}^N} u(x) \mathrm{e}^{\mathrm{i}x\cdot\xi} \,\mathrm{d}x$.
$l^p(\mathbb{Z}, X)$	a sequence space equipped with $\|\hat{u}\|^p_{l^p(\mathbb{Z},X)} = \sum_k \|\hat{u}_k\|^p_X$.
$h^s(\mathbb{Z}, X)$	a sequence space equipped with $\|\hat{u}\|^2_{h^s(\mathbb{Z},X)} = \sum_k \|k\|^{2s} \|\hat{u}_k\|^2_X$.
$\bigoplus_{k \in K} X_k$	the Cartesian product of the sets X_k.
X^*	the dual of a Banach space X,
A^*	$: Y^* \to X^*$ the adjoint of an operator $A\colon X \to Y$.
$\langle \cdot, \cdot \rangle_{X \times X^*}$	the dual pairing induced by $\langle x, y \rangle_{X \times X^*} = y(x)$.
$\sigma(A)$	the spectrum of a linear operator A.
$\rho(A)$	the resolvent of a linear operator A.
$\mathbb{T}^N_T$	the Cartesian product of tori with periods $T = (T_1, \dots, T_N)$. If $N = 1$, we drop the superscr
$\\|\cdot\\|_{\mathcal{D}(A)}$	the graph norm corresponding to an operator $A\colon \mathcal{D}(A) \to X$ defined by $\|u\|_{\mathcal{D}(A)} := \|u\|_X + \|Au\|_X$.
$\mathbb{Z}_{odd}$	$:= 2\mathbb{Z} + 1$, i.e., odd integers.
$B_R(x)$	the open ball with radius R centered at x.

$\mathrm{o}\,(f)$	Landau notation: $g \in \mathrm{o}\,(f)$ as $x \to x_0$ iff $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = 0$.
$\mathrm{O}\,(f)$	Landau notation: $g \in \mathrm{O}\,(f)$ as $x \to x_0$ iff $\limsup\limits_{x\to x_0} \left\lvert \dfrac{f(x)}{g(x)} \right\rvert < \infty$.
p'	$:= \dfrac{p}{p-1}$ if $p \in [1, \infty]$, the conjugate Hölder exponent.

If we write $L^p(\Omega, B)$, then we specify the range as B and analogous for all other function spaces. If we put an index onto a function space, then we mean:

loc	locally integrable
per	periodic
ap	anti-periodic in time
odd	odd in space
c	compactly supported

For an operator $A\colon X \to X^*$ we define the corresponding bilinear form as $b_A\,(x, y) := \langle Ax, y\rangle_{X\times X^*}$. If we do not explicitly say that b_A is defined as the corresponding bilinear form, then it can be any bilinear form and the index A just indicates a formal motivation but has no further meaning. Observe that if $X = L^{p+1}(\Omega, \mathbb{R})$, then $\langle f, g\rangle_{X\times X^*} = \int_\Omega f\, g\, \mathrm{d}x$.

Considering a nonlinear functional $I\colon X \to \mathbb{R}$, often called energy functional, we call a critical point of I a *bound state*, and a critical point with minimal energy among all critical points a *ground state*.

An *embedding* is always an injective map.

We say an equation is *compatible* with a symmetry, if each object in the equation is stable under the symmetry. E.g. the function $\Gamma(x)|u|^{p-1}u$ is compatible with $\frac{T}{2}$-anti-periodicity, since if u is $\frac{T}{2}$-anti-periodic, then $\Gamma(x)|u|^{p-1}u$ is anti-periodic again. The same holds for $V(x)u_{tt} - u_{xx}$.

1 Breather Solutions for a Quasilinear $(1+1)$-dimensional Wave Equation

We follow the strategy mentioned in the Introduction.

1.1 Introduction and Main Results

We study the $(1+1)$-dimensional quasilinear wave equation

$$g(x)w_{tt} - w_{xx} + h(x)(w_t^3)_t = 0 \quad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

and we look for real-valued, time-periodic and spatially localized solutions $w(x,t)$. At the end of this introduction we give a motivation for this equation arising in the study of localized electromagnetic waves modeled by Kerr-nonlinear Maxwell equations. We also cite some relevant papers. To the best of our knowledge for (1.1) in its general form no rigorous existence results are available. A first result is given in this paper by taking an extreme case where $h(x)$ is a spatial delta distribution at $x = 0$. Our basic assumption on the coefficients g and h is the following:

$$g \in L^\infty(\mathbb{R}) \text{ even, } g \not\equiv 0 \text{ and } h(x) = \gamma\delta_0(x) \text{ with } \gamma \neq 0 \tag{C0}$$

where δ_0 denotes the delta-distribution supported in 0. We have two prototypical examples for the potential g: a step potential (Theorem 1.1) and a periodic step potential (Theorem 1.2). The general version is given in Theorem 1.5 below.

Theorem 1.1. *For $a, b, c > 0$ let*

$$g(x) := \begin{cases} -a, & \text{if } |x| > c, \\ b, & \text{if } |x| < c. \end{cases}$$

For every frequency ω such that $\sqrt{b}\omega c \frac{2}{\pi} \in \frac{2\mathbb{N}+1}{2\mathbb{N}+1}$ and $\gamma < 0$ there exist infinitely many nontrivial, real-valued, spatially localized and time-periodic weak solutions of (1.1) *with period $T = \frac{2\pi}{\omega}$. For each solution w there are constants $C, \rho > 0$ such that $|w(x,t)| \leq C\mathrm{e}^{-\rho|x|}$.*

Theorem 1.2. *For $a, b > 0$, $a \neq b$ and $\Theta \in (0,1)$ let*

$$g(x) := \begin{cases} a, & \text{if } |x| < \pi\Theta, \\ b, & \text{if } \pi\Theta < |x| < \pi \end{cases}$$

and extend g as a 2π-periodic function to $\mathbb{R}$. Assume in addition

$$\sqrt{\frac{b}{a}}\,\frac{1-\Theta}{\Theta} \in \frac{2\mathbb{N}+1}{2\mathbb{N}+1}.$$

For every frequency ω such that $4\sqrt{a}\theta\omega \in \frac{2\mathbb{N}+1}{2\mathbb{N}+1}$ there exist infinitely many nontrivial, real-valued, spatially localized and time-periodic weak solutions of (1.1) *with period $T = \frac{2\pi}{\omega}$. For each solution w there are constants $C, \rho > 0$ such that $|w(x,t)| \le C\mathrm{e}^{-\rho|x|}$.*

Weak solutions of (1.1) are understood in the following sense. We write $D := \mathbb{R} \times \mathbb{T}_T$ and denote by $\mathbb{T}_T$ the one-dimensional torus with period T.

Definition 1.3. *Under the assumption* (C0) *a function $w \in H^1(\mathbb{R} \times \mathbb{T}_T)$ with $\partial_t w(0,\cdot) \in L^3(\mathbb{T}_T)$ is called a weak solution of* (1.1) *if for every $\psi \in C_c^\infty(\mathbb{R} \times \mathbb{T}_T)$*

$$\int_D -g(x)\partial_t w\,\partial_t\psi + \partial_x w\,\partial_x\psi\,\mathrm{d}(x,t) - \gamma\int_0^T (\partial_t w(0,t))^3\partial_t\psi(0,t)\,\mathrm{d}t = 0. \tag{1.2}$$

Theorem 1.1 and Theorem 1.2 are special cases of Theorem 1.5, which applies to much more general potentials g. In Section 1.6.1 and Section 1.6.2 of the Appendix we will show that the special potentials g from these two theorems satisfy the conditions (C1) and (C2) of Theorem 1.5. The basic preparations and assumptions for Theorem 1.5 will be given next.

Since we are looking for time-periodic solutions, it is appropriate to make the Fourier ansatz $w(x,t) = \sum_{k\in\mathbb{Z}_{odd}} w_k(x)\mathrm{e}^{ik\omega t}$ with $\mathbb{Z}_{odd} := 2\mathbb{Z}+1$. The reason for dropping even Fourier modes is that the 0-mode belongs to the kernel of the wave operator $L = g(x)\partial_t^2 - \partial_x^2$. The restriction to odd Fourier modes generates $T/2 = \pi/\omega$-antiperiodic functions w, is therefore compatible with the structure of (1.1) and in particular the cubic nonlinearity. Notice the decomposition $(Lw)(x,t) = \sum_{k\in\mathbb{Z}_{odd}} L_k w_k(x)\mathrm{e}^{ik\omega t}$ with self-adjoint operators

$$L_k = -\frac{d^2}{dx^2} - k^2\omega^2 g(x) : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Clearly $L_k = L_{-k}$ so that is suffices to study L_k for $k \in \mathbb{N}_{odd}$. Our first assumption is concerned with the spectrum $\sigma(L_k)$:

$$\forall\, k \in \mathbb{N}_{odd}, 0 \notin \sigma_{\mathrm{ess}}(L_k) \cup \sigma_{\mathrm{D}}(L_k), \tag{C1}$$

where by $\sigma_{\mathrm{D}}(L_k)$ we denote the spectrum of L_k with an extra Dirichlet condition at 0, i.e., the spectrum of L_k restricted to $\{\varphi \in H^2(\mathbb{R}) \mid \varphi(0) = 0\}$. This is the same as the spectrum of L_k restricted to functions which are odd around $x = 0$.

Lemma 1.4. *Under the assumption* (C0) *and* (C1) *there exists for every $k \in \mathbb{N}_{odd}$ a function $\Phi_k \in H^2(0,\infty)$ with $L_k\Phi_k = 0$ on $(0,\infty)$ and $\Phi_k(0) = 1$.*

Proof. We have either that 0 is in the point spectrum (but not the Dirichlet spectrum) or that 0 is in the resolvent set of L_k. In the first case there is an eigenfunction $\Phi_k \in H^2(\mathbb{R})$ with $L_k\Phi_k = 0$ and w.l.o.g. $\Phi_k(0) = 1$. In the second case $0 \in \rho(L_k)$ so that there exists a unique solution $\tilde{\Phi}_k$ of $L_k\tilde{\Phi}_k = 1_{[-2,-1]}$ on $\mathbb{R}$. Clearly, if restricted to $(0,\infty)$, the function $\tilde{\Phi}_k$ solves $L_k\tilde{\Phi}_k =$ on $(0,\infty)$. Moreover, $\tilde{\Phi}_k(0) \neq 0$ since otherwise $\tilde{\Phi}_k$ would be an odd eigenfunction of L_k which is excluded due to $0 \in \rho(L_k)$. Thus a suitably rescaled version of $\tilde{\Phi}_k$ satisfies the claim of the lemma. □

Our second set of assumptions concerns the structure of the decaying fundamental solution according to Lemma 1.4.

$$\text{There exist } \rho, M > 0 \text{ such that for all } k \in \mathbb{N}_{odd}\colon |\Phi_k(x)| \leq Me^{-\rho x} \text{ on } [0,\infty). \tag{C2}$$

Now we can formulate our third main theorem as a generalization of Theorem 1.1 and Theorem 1.2. The fact that the solutions which we find, can be well approximated by truncation of the Fourier series in time, is explained in Lemma 1.21 below. Moreover, a further extension yielding infinitely many different solutions is given in Theorem 1.15 in Section 1.4.

Theorem 1.5. *Assume* (C0), (C1) *and* (C2) *for a potential g and a frequency $\omega > 0$. Then* (1.1) *has a nontrivial, T-periodic weak solution w in the sense of Definition 1.3 with $T = \frac{2\pi}{\omega}$ provided*

(i) $\gamma < 0$ and the sequence $(\Phi_k'(0))_{k\in\mathbb{N}_{odd}}$ has at least one positive element,

(ii) $\gamma > 0$ and the sequence $(\Phi_k'(0))_{k\in\mathbb{N}_{odd}}$ has at least one negative element.

Moreover, there is a constant $C > 0$ such that $|w(x,t)| \leq C\mathrm{e}^{-\rho|x|}$ for all $(x,t) \in \mathbb{R}^2$ with ρ as in (C2).

Remark 1.6. *(a) It turns out that the above assumptions can be weakened as follows: it suffices to verify* (C1) *and* (C2) *and (i), (ii) for all integers $k \in r \cdot \mathbb{Z}_{odd}$ for some $r \in \mathbb{N}_{odd}$. We will prove this observation in Section 1.4.*

(b) Our variational approach also works if we consider (1.1) *with Dirichlet boundary conditions on a bounded interval $(-l,l)$ instead of the real line. There are many possible results. For illustration purposes we just formulate the simplest one. E.g., if we assume that $\frac{\omega l}{\pi} \in \frac{\mathbb{Z}_{odd}}{4\mathbb{Z}}$ then*

$$w_{tt} - w_{xx} + \gamma\delta_0(x)(w_t^3)_t = 0 \text{ on } (-l,l)\times\mathbb{R} \text{ with } w(\pm l,t) = 0 \text{ for all } t$$

has a nontrivial, real-valued time-periodic weak solution with period $T = \frac{2\pi}{\omega}$ both for $\gamma > 0$ and $\gamma < 0$. The operator $L_k = -\frac{d^2}{dx^2} - \omega^2 k^2$ is now a self-adjoint operator on $H^2(-l,l) \cap H_0^1(-l,l)$. The assumption $\frac{\omega l}{\pi} \in \frac{\mathbb{Z}_{odd}}{4\mathbb{Z}}$ guarantees (C1) *for all $k \in \mathbb{Z}_{odd}$. The functions Φ_k are given by $\Phi_k(x) = \frac{\sin(\omega k(l-x))}{\sin(\omega k k)}$ so that $\Phi_k'(0) = -\omega k\cot(\omega kl)$. The assumption $\frac{\omega l}{\pi} \in \frac{\mathbb{Z}_{odd}}{4\mathbb{Z}}$ now guarantees that the sequence $\{\cot(\omega kl) \mid k \in \mathbb{Z}_{odd}\}$ is finite and does not contain 0 or $\pm\infty$. Moreover $\frac{\omega l}{\pi} = \frac{2p+1}{4q}$ yields $\Phi_k'(0)\Phi_{k+2q}'(0) < 0$, i.e., we also have the required sign-change which allows for both signs of γ.*

We observe that the growth of $(\Phi_k'(0))_{k\in\mathbb{Z}_{odd}}$ is connected to regularity properties of our solutions.

Theorem 1.7. *Assume* (C0), (C1) *and* (C2) *and additionally $\Phi_k'(0) = O(k)$. Then the weak solution w from Theorem 1.5 belongs to $H^{1+\nu}_{per}(\mathbb{T}_T, L^2(\mathbb{R})) \cap H^{\nu}_{per}(\mathbb{T}_T, H^1(\mathbb{R}))$ for any $\nu \in (0, \frac{1}{4})$.*

Here, for $\nu \in \mathbb{R}$ the fractional Sobolev spaces of time-periodic functions are defined by

$$H^{\nu}_{per}(\mathbb{T}_T, L^2(\mathbb{R})) := \left\{ u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}_k(x) \mathrm{e}^{\mathrm{i}\omega k t} \;\middle|\; \sum_{k\in\mathbb{Z}} \left(1+|k|^2\right)^{\nu} \|\hat{u}_k\|^2_{L^2(\mathbb{R})} < \infty \right\},$$
$$H^{\nu}_{per}(\mathbb{T}_T, H^1(\mathbb{R})) := \left\{ u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}_k(x) \mathrm{e}^{\mathrm{i}\omega k t} \;\middle|\; \sum_{k\in\mathbb{Z}} \left(1+|k|^2\right)^{\nu} \|\hat{u}_k\|^2_{H^1(\mathbb{R})} < \infty \right\}.$$

We shortly motivate (1.1) and give some references to the literature. Consider Maxwell's equations in the absence of charges and currents

$$\begin{aligned} &\nabla \cdot \mathbf{D} = 0, && \nabla \times \mathbf{E} = -\partial_t \mathbf{B}, && \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}(\mathbf{E}),\\ &\nabla \cdot \mathbf{B} = 0, && \nabla \times \mathbf{H} = \partial_t \mathbf{D}, && \mathbf{B} = \mu_0 \mathbf{H}. \end{aligned}$$

We assume that the dependence of the polarization $\mathbf{P}$ on the electric field $\mathbf{E}$ is instantaneous and it is the sum of a linear and a cubic term given by $\mathbf{P}(\mathbf{E}) = \varepsilon_0\chi_1(\mathbf{x})\mathbf{E} + \varepsilon_0\chi_3(\mathbf{x})|\mathbf{E}|^2\mathbf{E}$, cf. [Agr19], Section 2.3 (for simplicity, more general cases where instead of a factor multiplying $|\mathbf{E}|^2\mathbf{E}$ one can take χ_3 as an $\mathbf{x}$-dependent tensor of type $(1,3)$ are not considered here). Here ε_0, μ_0 are constants such that $c^2 = (\varepsilon_0\mu_0)^{-1}$ with c being the speed of light in vacuum and χ_1, χ_3 are given material functions. By direct calculations one obtains the quasilinear curl-curl-equation

$$0 = \nabla \times \nabla \times \mathbf{E} + \partial_t^2 \left(V(\mathbf{x})\mathbf{E} + \Gamma(\mathbf{x})|\mathbf{E}|^2\mathbf{E} \right), \tag{1.3}$$

where $V(\mathbf{x}) = \mu_0\varepsilon_0 (1 + \chi_1(\mathbf{x}))$ and $\Gamma(\mathbf{x}) = \mu_0\varepsilon_0\chi_3(\mathbf{x})$. Once (1.3) is solved for the electric field $\mathbf{E}$, the magnetic induction $\mathbf{B}$ is obtained by time-integration from $\nabla\times\mathbf{E} = -\partial_t\mathbf{B}$ and it will satisfy $\nabla\cdot\mathbf{B} = 0$ provided it does so at time $t = 0$. By construction, the magnetic field $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$ satisfies $\nabla \times \mathbf{H} = \partial_t\mathbf{D}$. In order to complete the full set of nonlinear Maxwell's equations one only needs to check Gauss's law $\nabla\cdot\mathbf{D} = 0$ in the absence of external charges. This will follow directly from the constitutive equation $\mathbf{D} = \varepsilon_0(1+\chi_1(\mathbf{x}))\mathbf{E}+\varepsilon_0\chi_3(\mathbf{x})|\mathbf{E}|^2\mathbf{E}$ and the two different specific forms of $\mathbf{E}$ given next:

$\mathbf{E}(\mathbf{x},t) = (0, u(x_1 - \kappa t, x_3), 0)^T$ — polarized wave traveling in x_1-direction

$\mathbf{E}(\mathbf{x},t) = (0, u(x_1, t), 0)^T$ — polarized standing wave

In the first case $\mathbf{E}$ is a polarized wave independent of x_2 traveling with speed κ in the x_1 direction and with profile u. If additionally $V(\mathbf{x}) = V(x_3)$ and $\Gamma(\mathbf{x}) = \Gamma(x_3)$ then the quasilinear curl-curl-equation (1.3) turns into the following equation for $u = u(\tau, x_3)$ with the moving coordinate $\tau = x_1 - \kappa t$:

$$-u_{x_3x_3} + (\kappa^2 V(x_3) - 1)u_{\tau\tau} + \kappa^2\Gamma(x_3)(u^3)_{\tau\tau} = 0.$$

Setting $u = w_\tau$ and integrating once w.r.t. τ we obtain (1.1).

In the second case $\mathbf{E}$ is a polarized standing wave which is independent of x_2, x_3. If we assume furthermore that $V(\mathbf{x}) = V(x_1)$ and $\Gamma(\mathbf{x}) = \Gamma(x_1)$ then this time the quasilinear curl-curl-equation (1.3) for $u = w_t$ turns (after one time-integration) directly into (1.1).

In the literature, (1.3) has mostly been studied by considering time-harmonic waves $\mathbf{E}(\mathbf{x},t)=\mathbf{U}(\mathbf{x})e^{i\kappa t}$. This reduces the problem to the stationary elliptic equation

$$0=\nabla\times\nabla\times\mathbf{U}-\kappa^2\left(V(\mathbf{x})\mathbf{U}+\Gamma(\mathbf{x})|\mathbf{U}|^2\mathbf{U}\right)\text{ in }\mathbb{R}^3. \tag{1.4}$$

Here case $\mathbf{E}$ is no longer real-valued. This may be justified by extending the ansatz to $\mathbf{E}(\mathbf{x},t)=\mathbf{U}(\mathbf{x})e^{i\kappa t}+c.c.$ and by either neglecting higher harmonics generated from the cubic nonlinearity or by assuming the time-averaged constitutive relation $\mathbf{P}(\mathbf{E})=\varepsilon_0\chi_1(\mathbf{x})\mathbf{E}+\varepsilon_0\chi_3(\mathbf{x})\frac{1}{T}\int_0^T|\mathbf{E}|^2\,dt\mathbf{E}$ with $T=2\pi/\kappa$, cf. [Stu93], [SMK03]. For results on (1.4) we refer to [BDPR16], [Med15] and in particular to the survey [BM17]. Time-harmonic traveling waves have been found in a series of papers [Stu90, Stu93, SZ10]. The number of results for monochromatic standing polarized wave profiles $U(\mathbf{x})=(0,u(x_1),0)$ with u satisfying $0=-u''-\kappa^2\left(V(x_1)u+\Gamma(x_1)|u|^2u\right)$ on $\mathbb{R}$ is too large to cite so we restrict ourselves to Cazenave's book [Caz03].

Our approach differs substantially from the approaches by monochromatic waves described above. Our ansatz $w(x,t)=\sum_{k\in\mathbb{Z}_{odd}}w_k(x)\mathrm{e}^{ik\omega t}$ with $\mathbb{Z}_{odd}:=2\mathbb{Z}+1$ is automatically polychromatic since it couples all integer multiples of the frequency ω. A similar polychromatic approach is considered in [PSW12]. The authors seek spatially localized traveling wave solutions of the 1+1-dimensional quasilinear Maxwell model, where in the direction of propagation χ_1 is a periodic arrangement of delta functions. Based on a multiple scale approximation ansatz, the field profile is expanded into infinitely many modes which are time-periodic in both the fast and slow time variables. Since the periodicities in the fast and slow time variables differ, the field becomes quasiperiodic in time. To a certain extent the authors of [PSW12] analytically deal with the resulting system for these infinitely many coupled modes through bifurcation methods, with a rigorous existence proof still missing. However, numerical results from [PSW12] indicate that spatially localized traveling waves could exist.

With our case of allowing χ_1 to be a bounded function but taking χ_3 to be a delta function at $x=0$ we consider an extreme case. On the other hand our existence results (possibly for the first time) rigorously establish localized solutions of the full nonlinear Maxwell problem (1.3) without making the assumption of either neglecting higher harmonics or of assuming a time-averaged nonlinear constitutive law.

The existence of localized breathers of the quasilinear problem (1.1) with bounded coefficients g,h remains generally open. We can, however, provide specific functions g, h for which (1.1) has a breather-type solution that decays to 0 as $|x|\to\infty$. Let

$$b(x):=(1+x^2)^{-1/2},\quad h(x):=\frac{1-2x^2}{1+x^2},\quad g(x):=\frac{2+x^4}{(1+x^2)^2}$$

and consider a time-periodic solution a of the ODE

$$-a''-(a'^3)'=a$$

with minimal prescribed period $T\in(0,2\pi)$. Then $w(x,t):=a(t)b(x)$ satisfies (1.1). Note that h is sign-changing and w is not exponentially localized. We found this solution by inserting the ansatz for w with separated variables into (1.1). We then defined $b(x):=(1+x^2)^{-1/2}$ and set $g(x):=-b''(x)/b(x)$ and $h(x):=-b''(x)/b(x)^3$. The remaining equation for a then turned out to be the above one.

The paper is structured as follows: In Section 1.2 we develop the variational setting and give the proof of Theorem 1.5. The proof of the additional regularity results of Theorem 1.7 is given in Section 1.3. In Section 1.4 we give the proof of Theorem 1.15 on the existence of infinitely many different breathers. In Section 1.5 we show that our breathers can be well approximated by truncation of the Fourier series in time. Finally, in the Appendix we give details on the background and proof of Theorem 1.1 (Section 1.6.1) and Theorem 1.2 (Section 1.6.2) as well as a technical detail on a particular embedding of Hölder spaces into Sobolev spaces (Section 1.6.3).

1.2 Variational Approach and Proof of Theorem 1.5

The main result of our paper is Theorem 1.5 which will be proved in this section. It is a consequence of Lemma 1.11 and Theorem 1.13 below.

Formally (1.1) is the Euler-Lagrange-equation of the functional

$$I(w) := \int_D -\frac{1}{2}g(x)|\partial_t w|^2 + \frac{1}{2}|\partial_x w|^2 \,\mathrm{d}(x,t) - \frac{1}{4}\gamma \int_0^T |\partial_t w(0,t)|^4 \,\mathrm{d}t \tag{1.5}$$

defined on a suitable space of T-periodic functions. Instead of directly searching for a critical point of this functional we first rewrite the problem into a nonlinear Neumann boundary value problem under the assumption that w is even in x. In this case (1.1) amounts to the following linear wave equation on the half-axis with nonlinear Neumann boundary conditions:

$$\begin{cases} g(x)w_{tt} - w_{xx} = 0 & \text{for } (x,t) \in (0,\infty)\times\mathbb{R}, \\ 2w_x(0_+,t) = \gamma\left(w_t(0,t)^3\right)_t & \text{for } t \in \mathbb{R} \end{cases} \tag{1.6}$$

where solutions $w \in H^1([0,\infty)\times\mathbb{T}_T)$ with $\partial_t w(0,\cdot) \in L^3(\mathbb{T}_T)$ of (1.6) are understood in the sense that

$$2\int_{D_+} -g(x)\partial_t w\,\partial_t\psi + \partial_x w\,\partial_x\psi \,\mathrm{d}(x,t) - \gamma\int_0^T (\partial_t w(0,t))^3\partial_t\psi(0,t)\,\mathrm{d}t = 0 \tag{1.7}$$

for all $\psi \in C_c^\infty([0,\infty)\times\mathbb{T}_T)$ with $D_+ = (0,\infty)\times\mathbb{T}_T$. It is clear that evenly extended solutions w of (1.7) also satisfy (1.2). To see this note that every $\psi \in C_c^\infty(\mathbb{R}\times\mathbb{T}_T)$ can be split into an even and an odd part $\psi = \psi_e + \psi_o$ both belonging to $C_c^\infty(\mathbb{R}\times\mathbb{T}_T)$. Testing with ψ_o in (1.2) produces zeroes in all spatial integrals due to the evenness of w and also in the temporal integral since $\psi_o(0,\cdot) \equiv 0$ due to oddness. Testing with ψ_e in (1.2) produces twice the spatial integrals appearing in (1.7). In the following we concentrate on finding solutions of (1.6) for the linear wave equation with nonlinear Neumann boundary conditions.

Motivated by the linear wave equation in (1.6) we make the ansatz that

$$w(x,t) = \sum_{k\in\mathbb{Z}_{odd}} \frac{\hat{\alpha}_k}{k}\Phi_k(|x|)e_k(t), \tag{1.8}$$

where $e_k(t) := \frac{1}{\sqrt{T}}\mathrm{e}^{\mathrm{i}\omega kt}$ denotes the $L^2(\mathbb{T}_T)$-orthonormal Fourier base of $\mathbb{T}_T$, and where Φ_k are the decaying fundamental solutions Φ_k of L_k, cf. Lemma 1.4. Such a function w

will always solve the linear wave equation in (1.6) and we will determine real sequences $\hat{\alpha} = (\hat{\alpha}_k)_{k\in\mathbb{Z}_{odd}}$ such that the nonlinear Neumann condition is satisfied as well. The additional factor $\frac{1}{k}$ is only for convenience, since ∂_t generates a multiplicative factor $\mathrm{i}\omega k$.

The convolution between two sequences $\hat{z}, \hat{y} \in \mathbb{R}^{\mathbb{Z}}$ is defined pointwise (whenever it converges) by $(\hat{z} * \hat{y})_k := \sum_{l\in\mathbb{Z}} \hat{z}_l \hat{y}_{k-l}$.

In order to obtain real-valued functions w by the ansatz (1.8) we require the sequence $\hat{\alpha}$ to be real and odd in k, i.e., $\hat{\alpha}_k \in \mathbb{R}$ and $\hat{\alpha}_k = -\hat{\alpha}_{-k}$. Since (1.8) already solves the wave equation in (1.6), it remains to find $\hat{\alpha}$ such that

$$2w_x(0_+,t) = 2\sum_{k\in\mathbb{Z}_{odd}} \frac{\hat{\alpha}_k}{k}\Phi_k'(0)e_k(t) \overset{!}{=} \frac{1}{T}\sum_{k\in\mathbb{Z}_{odd}} \gamma\omega^4 k(\hat{\alpha} * \hat{\alpha} * \hat{\alpha})_k e_k(t) = \gamma(w_t(0,t)^3)_t,$$

where we have used $\Phi_k(0) = 1$. As the above identity needs to hold for all $t \in \mathbb{R}$ we find

$$(\hat{\alpha} * \hat{\alpha} * \hat{\alpha})_k = \frac{2T\Phi_k'(0)}{\gamma\omega^4 k^2}\hat{\alpha}_k \quad \text{for all } k \in \mathbb{Z}_{odd}. \tag{1.9}$$

This will be accomplished by searching for critical points $\hat{\alpha}$ of the functional

$$J(\hat{z}) := \frac{1}{4}(\hat{z} * \hat{z} * \hat{z} * \hat{z})_0 + \frac{T}{\gamma\omega^4}\sum_k \frac{\Phi_k'(0)}{k^2}\hat{z}_k^2.$$

defined on a suitable Banach space of real sequences $\hat{z}$ with $\hat{z}_k = -\hat{z}_{-k}$. Indeed, computing (formally) the Fréchet derivative of J at $\hat{\alpha}$ we find

$$J'(\hat{\alpha})[\hat{y}] = (\hat{\alpha} * \hat{\alpha} * \hat{\alpha} * \hat{y})_0 + \frac{2T}{\gamma\omega^4}\sum_k \frac{\Phi_k'(0)}{k^2}\hat{\alpha}_k\hat{y}_k. \tag{1.10}$$

Let us indicate how (1.10) amounts to (1.9). For fixed $k_0 \in \mathbb{Z}_{odd}$ we define the test sequence $\hat{y} := (\delta_{k,k_0} - \delta_{k,-k_0})_{k\in\mathbb{Z}_{odd}}$ which has exactly two non-vanishing entries at k_0 and at $-k_0$. Thus, $\hat{y}$ belongs to the same space of odd, real sequences as $\hat{\alpha}$ and can therefore be used as a test sequence in $J'(\hat{\alpha})[\hat{y}] = 0$. After a short calculation using $\hat{\alpha}_k = -\hat{\alpha}_{-k}$, $\Phi_k' = \Phi_{-k}'$ we obtain (1.9) for k_0.

It turns out that a real Banach space of real-valued sequences which is suitable for J can be given by

$$\mathcal{D}(J) := \left\{\hat{z} \in \mathbb{R}^{\mathbb{Z}_{odd}} \;\middle|\; |||\hat{z}||| < \infty,\ \hat{z}_k = -\hat{z}_{-k}\right\} \text{ where } |||\hat{z}||| := \|\hat{z} * \hat{z}\|_{l^2}^{\frac{1}{2}}.$$

The relation between the function I defined in (1.5) and the new functional J is formally given by

$$I\left(\sum_{k\in\mathbb{Z}_{odd}} \frac{\hat{z}_k}{k}\Phi_k(|x|)e_k(t)\right) = -\frac{\gamma\omega^4}{T}J(\hat{z}).$$

Lemma 1.8. *The space $(\mathcal{D}(J), |||\cdot|||)$ is a separable, reflexive, real Banach space and isometrically embedded into the real Banach space $L^4(\mathbb{T}_T, \mathrm{i}\mathbb{R})$ of purely imaginary-valued measurable functions. Moreover for $\hat{u}, \hat{v}, \hat{w}, \hat{z} \in \mathcal{D}(J)$ we have*

$$(\hat{u} * \hat{u} * \hat{u} * \hat{u})_0 = |||\hat{u}|||^4, \tag{1.11}$$

$$|(\hat{u} * \hat{v} * \hat{w} * \hat{z})_0| \le |||\hat{u}|||\,|||\hat{v}|||\,|||\hat{w}|||\,|||\hat{z}|||, \tag{1.12}$$

$$\|\hat{z}\|_{l^2} \le |||\hat{z}|||. \tag{1.13}$$

Proof. We first recall the correspondence between real-valued sequences $\hat{z} \in l_2$ with $\hat{z}_k = -\hat{z}_{-k}$ and purely imaginary-valued functions $z \in L^2(\mathbb{T}_T, \mathrm{i}\mathbb{R})$ by setting

$$\hat{z}_k \coloneqq \langle z, e_k \rangle_{L^2(\mathbb{T}_T)} \text{ and } z(t) \coloneqq \sum_{k \in \mathbb{Z}} \hat{z}_k e_k(t)$$

Parseval's identity provides the isomorphism $\|z\|_{L^2(\mathbb{T}_T)} = \|\hat{z}\|_{l^2}$. The following identity

$$T\|z\|^4_{L^4(\mathbb{T}_T)} = T \int_0^T z(t)^4\,dt = (\hat{z} * \hat{z} * \hat{z} * \hat{z})_0 = \|\hat{z} * \hat{z}\|^2_{l^2} = |||\hat{z}|||^4$$

shows that $|||\cdot|||$ is indeed a norm on $\mathcal{D}(J)$ and its provides the isometric embedding of $\mathcal{D}(J)$ into a subspace of $L^4(\mathbb{T}_T, \mathrm{i}\mathbb{R})$. By Parseval's equality and Hölder's inequality we see that

$$\|\hat{z}\|_{l^2} = \|z\|_{L^2(\mathbb{T}_T)} \le T^{\frac{1}{4}}\|z\|_{L^4(\mathbb{T}_T)} = |||\hat{z}|||$$

so that $\mathcal{D}(J)$ is indeed a subspace of l^2. Finally, for any $\hat{u}, \hat{v}, \hat{w}, \hat{z} \in \mathcal{D}(J)$ we see that

$$|(\hat{u} * \hat{v} * \hat{w} * \hat{z})_0| = T\left|\int_0^T u(t)v(t)w(t)z(t)\,dt\right| \le T\|u\|_{L^4}\|v\|_{L^4}\|w\|_{L^4}\|z\|_{L^4} = |||\hat{u}|||\,|||\hat{v}|||\,|||\hat{w}|||\,|||\hat{z}|||.$$

This finishes the proof of the lemma. □

For $\frac{T}{2}$-anti-periodic functions $\psi\colon D \to \mathbb{R}$ of the space-time variable $(x,t) \in D$ we use the notation

$$\psi(x,t) = \sum_{k \in \mathbb{Z}_{odd}} \hat{\psi}_k(x) e_k(t) = \sum_{k \in \mathbb{Z}_{odd}} \frac{1}{k}\Psi_k(x) e_k(t) \tag{1.14}$$

with $\frac{1}{k}\Psi_k(x) = \hat{\psi}_k(x) \coloneqq \langle \psi(x,\cdot), e_k \rangle_{L^2(\mathbb{T}_T)}$. The Parseval identity and the definition of $|||\cdot|||$ immediately lead to the following lemma.

Lemma 1.9. *For $\psi : D \to \mathbb{R}$ as in (1.14) the following holds:*

(i) $\|\psi_x\|^2_{L^2(D)} = \sum_k \frac{1}{k^2}\|\Psi'_k\|^2_{L^2(\mathbb{R})}$,

(ii) $\|\psi_t\|^2_{L^2(D)} = \omega^2 \sum_k \|\Psi_k\|^2_{L^2(\mathbb{R})}$,

(iii) $T\|\psi_t(0,\cdot)\|^4_{L^4(\mathbb{T}_T)} = \omega^4 |||\hat{y}|||^4$ *where* $\hat{y}_k = \Psi_k(0)$ *for* $k \in \mathbb{Z}_{odd}$.

The next result give some estimates on the growth of norms of Φ_k. It serves as a preparation for the proof of regularity properties for functions w as in (1.8) stated in Lemma 1.11.

Lemma 1.10. *Assume* (C0), (C1) *and* (C2). *Then*

$$\|\Phi_k\|_{L^2(0,\infty)} = O(1), \quad \|\Phi_k'\|_{L^2(0,\infty)} = O(k), \quad \|\Phi_k'\|_{L^\infty(0,\infty)} = O(k^{\frac{3}{2}}). \tag{1.15}$$

In particular $|\Phi_k'(0)| = O(k^{\frac{3}{2}})$.

Proof. The first part of (1.15) is a direct consequence of (C2).

Multiplying $L_k\Phi_k = 0$ with Φ_k, Φ_k' and integrating from $a \geq 0$ to ∞ we get

$$\int_a^\infty -\omega^2 k^2 g(x)\Phi_k(x)^2 + \Phi_k'(x)^2\,dx = -\Phi_k(a)\Phi_k'(a), \tag{1.16}$$

$$\int_a^\infty -2\omega^2 k^2 g(x)\Phi_k(x)\Phi_k'(x)\,dx = -\Phi_k'(a)^2, \tag{1.17}$$

respectively. Applying the Cauchy-Schwarz inequality to (1.17) and using the first part of (1.15) we find

$$\|\Phi_k'\|^2_{L^\infty(0,\infty)} \leq O(k^2)\|\Phi_k'\|_{L^2(0,\infty)} \tag{1.18}$$

and from (1.16), (1.18) we get

$$\begin{aligned}\|\Phi_k'\|^2_{L^2(0,\infty)} &\leq O(k^2) + \|\Phi_k\|_{L^\infty(0,\infty)}\|\Phi_k'\|_{L^\infty(0,\infty)}\\ &\leq O(k^2) + \|\Phi_k\|_{L^\infty(0,\infty)}O(k)\|\Phi_k'\|^{\frac{1}{2}}_{L^2(0,\infty)}.\end{aligned}$$

The L^∞-assumption in (C2) leads to

$$\|\Phi_k'\|^2_{L^2(0,\infty)} \leq O(k^2) + O(k)\|\Phi_k'\|^{\frac{1}{2}}_{L^2(0,\infty)} \leq O(k^2) + C_\epsilon O(k^{\frac{4}{3}}) + \epsilon\|\Phi_k'\|^2_{L^2(0,\infty)},$$

where we have used Young's inequality with exponents 4/3 and 4. This implies the second inequality in (1.15). Inserting this into (1.18) we obtain the third inequality in (1.15). □

Lemma 1.11. *Assume* (C0), (C1) *and* (C2). *For* $\hat\alpha \in \mathcal{D}(J)$ *and* $w : D \to \mathbb{R}$ *as in* (1.8) *we have* $w_x, w_t \in L^2(D)$, $w_t(0,\cdot) \in L^4(\mathbb{T}_T)$ *and there are values* $C > 0$ *and* $\rho > 0$ *such that* $|w(x,t)| \leq C\mathrm{e}^{-\rho|x|}$.

Remark 1.12. *The lemma does not require* $\hat\alpha$ *to be a critical point of* J. *The smoothness and decay of* w *as in* (1.8) *is simply a consequence of* $\hat\alpha \in \mathcal{D}(J)$ *and* (C2).

Proof. We use the characterization from Lemma 1.9. Let us begin with the estimate for $\|\partial_t w\|_{L^2(D)}$. By Lemma 1.10 we have $\sup_k \|\Phi_k\|_{L^2(0,\infty)} < \infty$ so that

$$\begin{aligned}\|\partial_t w\|^2_{L^2(D)} &= 2\omega^2 \sum_k \hat\alpha_k^2 \|\Phi_k\|^2_{L^2(0,\infty)} \leq 2\omega^2\Big(\sup_k \|\Phi_k\|_{L^2(0,\infty)}\Big)^2 \|\hat\alpha\|^2_{l^2}\\ &\leq 2\omega^2\Big(\sup_k \|\Phi_k\|_{L^2(0,\infty)}\Big)^2 |||\hat\alpha|||^2 < \infty,\end{aligned}$$

which finishes our first goal. Next we estimate $\|\partial_x w\|_{L^2(D)}$. Here we use again Lemma 1.10 to find

$$\|\partial_x w\|^2_{L^2(D)} = 2\sum_k \frac{\hat\alpha_k^2}{k^2}\|\Phi_k'\|^2_{L^2(0,\infty)} \leq C\|\hat\alpha\|^2_{l^2} \leq C|||\hat\alpha|||^2 < \infty$$

which finishes our second goal. Next we show that $w_t(0,\cdot) \in L^4(\mathbb{T}_T)$. Using $\Phi_k(0)=1$ we observe that

$$T\|w_t(0,\cdot)\|^4_{L^4(\mathbb{T}_T)} = T\int_0^T \Big(\sum_{k\in\mathbb{Z}_{odd}} \mathrm{i}\omega\hat{\alpha}_k\Phi_k(0)e_k(t)\Big)^4 \,\mathrm{d}t$$
$$= \omega^4 |||\hat{\alpha}|||^4 < \infty.$$

Finally we show the uniform-in-time exponential decay of w. By construction w is even in x, hence we only consider $x>0$. By (C2) we see that

$$|w(x,t)| \le \sum_k \frac{|\hat{\alpha}_k|}{|k|}|\Phi_k(x)| = \sum_k \frac{|\hat{\alpha}_k|}{|k|} Ce^{-\alpha x} \le \|\hat{\alpha}\|_{l^2}\left(\sum_k \frac{1}{k^2}\right)^{1/2} Ce^{-\alpha x} \le \tilde{C}e^{-\alpha x}$$

which finishes the proof of the lemma. □

In the following result we will show that minimizers of J on $\mathcal{D}(J)$ exist, are solutions of (1.9) and indeed correspond to weak solutions of (1.1).

Theorem 1.13. *Assume* (C0), (C1) *and* (C2). *Then the functional J is well defined on its domain $\mathcal{D}(J)$, Fréchet-differentiable, bounded from below and attains its negative minimum provided*

(i) $\gamma < 0$ and the sequence $(\Phi_k'(0))_{k\in\mathbb{N}_{odd}}$ has at least one positive element, or

(ii) $\gamma > 0$ and the sequence $(\Phi_k'(0))_{k\in\mathbb{N}_{odd}}$ has at least one negative element.

For every critical point $\hat{\alpha}\in\mathcal{D}(J)$ the corresponding function $w(x,t) := \sum_{k\in\mathbb{Z}_{odd}} \frac{\hat{\alpha}_k}{k}\Phi_k(|x|)e_k(t)$ is a nontrivial weak solution of (1.1).

Proof. Note that $J(\hat{z}) = \frac{1}{4}|||\hat{z}|||^4 + J_1(\hat{z})$, where $J_1(\hat{z}) = \sum_k a_k\hat{z}_k^2$ with $a_k = \frac{T\Phi_k'(0)}{\gamma\omega^4k^2}$. By Lemma 1.10 the sequence $(a_k)_k$ is converging to 0 as $|k|\to\infty$, so in particular it is bounded. Due to (1.13) one finds that J is well defined and continuous on $\mathcal{D}(J)$, and moreover, that for $\hat{z}\in\mathcal{D}(J)$

$$J(\hat{z}) \ge \frac{1}{4}|||\hat{z}|||^4 - \sup_k |a_k| \sum_k \hat{z}_k^2 \ge \frac{1}{4}|||\hat{z}|||^4 - \sup_k |a_k| |||\hat{z}|||^2.$$

This implies that J is coercive and bounded from below. The weak lower semi-continuity of J follows from the convexity and continuity of the map $\hat{z}\mapsto|||\hat{z}|||^4$ and the weak continuity of J_1. To see the latter take an arbitrary $\epsilon>0$. Then there is $k_0\in\mathbb{N}$ such that $|a_k|\le\epsilon$ for $|k|>k_0$ and this implies the inequality

$$|J_1(\hat{z}) - J_1(\hat{y})| \le \sup_k |a_k| \sum_{|k|\le k_0} |\hat{z}_k^2 - \hat{y_k}^2| + \epsilon(\|\hat{z}\|^2_{l^2} + \|\hat{y}\|^2_{l^2}) \quad \forall\, \hat{z},\hat{y}\in\mathcal{D}(J). \tag{1.19}$$

Since $(\mathcal{D}(J), |||\cdot|||)$ continuously embeds into l^2 any weakly convergent sequence in $(\mathcal{D}(J), |||\cdot|||)$ also weakly converges in l^2 and in particular pointwise. This pointwise convergence together with the boundedness of the sequence and (1.19) yields the weak continuity of J_1 and thus the weak lower semi-continuity of J. As a consequence, cf. Theorem 1.2 in [Str08], we get the existence of a minimizer.

In order to check that the minimizer is nontrivial is suffices to verify that J attains negative values. Here we distinguish between case (i) and (ii) in the assumptions of the theorem. In case (i) when $\gamma < 0$ we find an index k_0 such that $\Phi'_{k_0}(0) > 0$. In case (ii) when $\gamma > 0$ we choose k_0 such that $\Phi'_{k_0}(0) < 0$. In both cases we obtain that $\Phi'_{k_0}(0)/\gamma < 0$. If we set $\hat{y} := (\delta_{k,k_0} - \delta_{k,-k_0})_{k\in\mathbb{Z}_{odd}}$ then $\hat{y}$ has exactly two non-vanishing entries, namely $+1$ at k_0 and -1 at $-k_0$. Hence $\hat{y} \in \mathcal{D}(J)$. Using the property $\Phi'_{k_0} = \Phi'_{-k_0}$ we find for $t \in \mathbb{R}$

$$J(t\hat{y}) = t^4 \frac{1}{4} |||\hat{y}|||^4 + 2t^2 \frac{T\Phi'_{k_0}(0)}{\gamma\omega^4 k_0^2}$$

which is negative by the choice of k_0 provided $t > 0$ is sufficiently small. Thus, $\inf_{\mathcal{D}(J)} J < 0$ and every minimizer $\hat{\alpha}$ is nontrivial.

Next we show for every critical point $\hat{\alpha}$ of J that $w(x,t) := \sum_{k\in\mathbb{Z}_{odd}} \frac{\hat{\alpha}_k}{k} \Phi_k(|x|) e_k(t)$ is a weak solution of (1.1). The regularity properties $w \in H^1(\mathbb{R}\times\mathbb{T}_T)$, $\partial_t w(0,\cdot) \in L^4(\mathbb{T}_T)$ and the exponential decay have already been shown in Lemma 1.11. We skip the standard proof that $J \in C^1(\mathcal{D}(J),\mathbb{R})$ and that its Fréchet-derivative is given by (1.10). We will show that (1.2) holds for any ψ as in (1.14) with even functions $\Psi_k \in H^1(\mathbb{R})$, $\Psi_k = -\Psi_{-k}$ such that $\psi_x, \psi_t \in L^2(D)$ and $\psi(0,\cdot) \in L^4(\mathbb{T}_T)$ as described in Lemma 1.9. We begin by deriving expressions and estimates for the functionals

$$H_1(\psi) = \int_D g(x) w_t \psi_t \, d(x,t), \quad H_2(\psi) = \int_D w_x \psi_x \, d(x,t), \quad H_3(\psi) = \int_0^T w_t(0,t)^3 \psi_t(0,t)\, dt.$$

In a first step we assume that the sum in (1.14) is finite in order to justify the exchange of summation and integration in the following. Then, starting with H_1 we find

$$\begin{aligned}
H_1(\psi) &= -\omega^2 \int_D g(x) \sum_{k,l} \hat{\alpha}_k \Phi_k(|x|) \Psi_l(|x|) e_k(t) e_l(t)\, \mathrm{d}(x,t) \\
&= -2\omega^2 \sum_k \hat{\alpha}_k \int_0^\infty g(x) \Phi_k(x) \Psi_{-k}(x)\, \mathrm{d}x \\
&= 2\omega^2 \sum_k \hat{\alpha}_k \int_0^\infty g(x) \Phi_k(x) \Psi_k(x)\, \mathrm{d}x\,,
\end{aligned}$$

$$|H_1(\psi)| \le 2\omega^2 \|g\|_{L^\infty(\mathbb{R})} \Big(\sum_k \hat{\alpha}_k^2 \|\Phi_k\|_{L^2(0,\infty)}^2\Big)^{\frac{1}{2}} \Big(\sum_k \|\Psi_k\|_{L^2(0,\infty)}^2\Big)^{\frac{1}{2}} = \|g\|_{L^\infty(\mathbb{R})} \|w_t\|_{L^2(D)} \|\psi_t\|_{L^2(D)}$$

and similarly for H_2 we find using (1.24)

$$\begin{aligned}
H_2(\psi) &= \int_D \sum_{k,l} \frac{\hat{\alpha}_k}{k} \Phi'_k(|x|) \frac{1}{l} \Psi'_l(|x|) e_k(t) e_l(t)\, \mathrm{d}(x,t) \\
&= 2 \sum_k \frac{\hat{\alpha}_k}{-k^2} \int_0^\infty \Phi'_k(x) \Psi'_{-k}(x)\, \mathrm{d}x \\
&= 2 \sum_k \frac{\hat{\alpha}_k}{k^2} \int_0^\infty \Phi'_k(x) \Psi'_k(x)\, \mathrm{d}x \\
&= 2\omega^2 \sum_k \hat{\alpha}_k \int_0^\infty g(x) \Phi_k(x) \Psi_k(x)\, \mathrm{d}x - 2 \sum_k \frac{\hat{\alpha}_k}{k^2} \Phi'_k(0) \Psi_k(0),
\end{aligned}$$

$$|H_2(\psi)| \le 2 \Big(\sum_k \frac{\hat{\alpha}_k^2}{k^2} \|\Phi'_k\|_{L^2(0,\infty)}^2\Big)^{\frac{1}{2}} \Big(\sum_k \frac{1}{k^2} \|\Psi'_k\|_{L^2(0,\infty)}^2\Big)^{\frac{1}{2}} = \|w_x\|_{L^2(D)} \|\psi_x\|_{L^2(D)}.$$

Moreover, considering H_3 and setting $\hat{y}_k := \Psi_k(0)$ for $k \in \mathbb{Z}_{odd}$ one sees

$$\begin{aligned} H_3(\psi) &= \omega^4 \int_0^T \Big(\sum_k \hat{\alpha}_k e_k(t)\Big)^3 \Big(\sum_l \Psi_l(0) e_l(t)\Big)\,\mathrm{d}t \\ &= \frac{\omega^4}{T}(\hat{\alpha} * \hat{\alpha} * \hat{\alpha} * \hat{y})_0, \\ |H_3(\psi)| &\le \frac{\omega^4}{T} |||\hat{\alpha}|||^3 |||\hat{y}||| = \|w_t(0,\cdot)\|^3_{L^4(\mathbb{T}_T)} \|\psi_t(0,\cdot)\|_{L^4(\mathbb{T}_T)}. \end{aligned}$$

Hence H_1, H_2 and H_3 are bounded linear functionals of the variable ψ as in (1.14) with $\psi_x, \psi_t \in L^2(D)$ and $\psi_t(0,\cdot) \in L^4(\mathbb{T}_T)$. For such ψ we use the above formulae for H_1, H_2, H_3 and compute the linear combination

$$-H_1(\psi) + H_2(\psi) - \gamma H_3(\psi) = -2\sum_k \frac{\hat{\alpha}_k}{k^2}\Phi_k'(0)\Psi_k(0) - \frac{\gamma\omega^4}{T}(\hat{\alpha} * \hat{\alpha} * \hat{\alpha} * \hat{y})_0 = 0$$

due to the Euler-Lagrange equation for the functional J, i.e., the vanishing of $J'(\hat{\alpha})[\hat{y}]$ in (1.10) for all $\hat{y} \in \mathcal{D}(J)$. The last equality means that w is a weak solution of (1.1). □

1.3 Further Regularity

Here we prove Theorem 1.7. We observe first that in the example of a periodic step-potential in Theorem 1.2 we find that not only $\Phi_k'(0) = O(k^{\frac{3}{2}})$ holds (as Lemma 1.10 shows) but even $\Phi_k'(0) = O(k)$ is satisfied. It is exactly this weaker growth that we can exploit in order to prove additional smoothness of the solutions of (1.1). We begin by defining for $\nu > 0$ the Banach space of sequences

$$h^\nu := \Big\{\hat{z} \in l^2 \text{ s.t. } \|\hat{z}\|^2_{h^\nu} := \sum_k (1+k^2)^\nu |\hat{z}_k|^2 < \infty\Big\}.$$

Moreover, we use the isometric isomorphism between h^ν and

$$H^\nu(\mathbb{T}_T) = \Big\{z(t) = \sum_k \hat{z}_k e_k(t) \text{ s.t. } \hat{z} \in h^\nu\Big\}$$

by setting $\|z\|_{H^\nu} := \|\hat{z}\|_{h^\nu}$. We also use the Morrey embedding $H^{1+\nu}(\mathbb{T}_T) \to C^{0,\frac{1}{2}+\nu}(\mathbb{T}_T)$ for $\nu \in (0, 1/2)$ and the following embedding: $C^{0,\nu}(\mathbb{T}_T) \to H^{\tilde{\nu}}(\mathbb{T}_T)$ for $0 < \tilde{\nu} < \nu \le 1$, cf. Lemma 1.23 in the Appendix. The latter embedding means that $\hat{z} \in h^{\tilde{\nu}}$ provided $z \in C^{0,\nu}(\mathbb{T}_T)$ and $0 < \tilde{\nu} < \nu \le 1$.

Theorem 1.14. *Assume* (C0), (C1), (C2) *and in addition* $\Phi_k'(0) = O(k)$. *For every* $\hat{\alpha} \in \mathcal{D}(J)$ *with* $J'(\hat{\alpha}) = 0$ *we have* $\hat{\alpha} \in h^\nu$ *for every* $\nu \in (0, 1/4)$.

Proof. Let $\hat{\alpha} \in \mathcal{D}(J)$ with $J'(\hat{\alpha}) = 0$. Recall from (1.9) that

$$(\hat{\alpha} * \hat{\alpha} * \hat{\alpha})_k = \hat{\eta}_k \hat{\alpha}_k \quad \text{where} \quad \hat{\eta}_k := \frac{2T\Phi_k'(0)}{\gamma\omega^4 k^2} \text{ for } k \in \mathbb{Z}_{odd} \tag{1.20}$$

so that $|\hat{\eta}_k| \leq C/k$. If we define the convolution of two T-periodic functions $f, g \in L^2(\mathbb{T}_T)$ on the torus $\mathbb{T}_T$ as

$$(f * g)(t) := \frac{1}{\sqrt{T}} \int_0^T f(s)g(t-s)\,\mathrm{d}s$$

and if we set

$$\alpha(t) := \sum_k \hat{\alpha}_k e_k(t), \quad \eta(t) := \sum_k \hat{\eta}_k e_k(t)$$

then the equation

$$\alpha^3 = \alpha * \eta \tag{1.21}$$

for the T-periodic function $\alpha \in L^4(\mathbb{T}_T)$ is equivalent to the equation (1.20) for the sequence $\hat{\alpha} \in \mathcal{D}(J)$. We will analyze (1.21) with a bootstrap argument.

Step 1: We show that $\alpha \in C^{0,\frac{1}{6}}(\mathbb{T}_T)$. The right hand side of (1.21) is an $H^1(\mathbb{T}_T)$-function since

$$\|\alpha * \eta\|^2_{H^1(\mathbb{T}_T)} = \|\hat{\alpha}\hat{\eta}\|^2_{h^1} \leq \sum_{k \in \mathbb{Z}_{odd}} (1+k^2)\hat{\alpha}_k^2 \frac{C^2}{k^2} \leq 2C^2 \|\hat{\alpha}\|^2_{l^2} < \infty.$$

Therefore, using (1.21) we see that $\alpha^3 \in H^1(\mathbb{T}_T)$ and by the Morrey embedding that $\alpha^3 \in C^{0,\frac{1}{2}}(\mathbb{T}_T)$. Since the inverse of the mapping $x \mapsto x^3$ is given by $x \mapsto |x|^{-\frac{2}{3}}x$, which is a $C^{0,\frac{1}{3}}(\mathbb{R})$-function, we obtain $\alpha \in C^{0,\frac{1}{6}}(\mathbb{T}_T)$.

Step 2: We fix $q \in (0,1)$ and show that if $\alpha \in C^{0,\nu_n}(\mathbb{T}_T)$ for some $\nu_n \in (0,1/2)$ solves (1.21) then $\alpha \in C^{0,\nu_{n+1}}(\mathbb{T}_T)$ with $\nu_{n+1} = \frac{q\nu_n}{3} + \frac{1}{6}$. For the proof we iterate the process from Step 1 and we start with $\alpha \in C^{0,\nu_n}(\mathbb{T}_T)$. Then, according to Lemma 1.23 of the Appendix, $\alpha \in H^{q\nu_n}(\mathbb{T}_T)$ and hence $\hat{\alpha} \in h^{q\nu_n}$. Then as before the convolution of α with η generates one more weak derivative, namely

$$\|\alpha * \eta\|^2_{H^{1+q\nu_n}(\mathbb{T}_T)} = \|\hat{\alpha}\hat{\eta}\|^2_{h^{1+q\nu_n}} \leq \sum_k (1+k^2)^{1+q\nu_n} \hat{\alpha}_k^2 \frac{C^2}{k^2} \leq C^2 \|\hat{\alpha}\|_{h^{q\nu_n}} < \infty.$$

Hence by (1.21) we conclude $\alpha^3 \in H^{1+q\nu_n}(\mathbb{T}_T)$ and by the Morrey embedding $\alpha^3 \in C^{0,\frac{1}{2}+q\nu_n}(\mathbb{T}_T)$ provided $q\nu_n \in (0,1/2)$. As in Step 1 this implies $\alpha \in C^{0,\nu_{n+1}}(\mathbb{T}_T)$ with $\nu_{n+1} = \frac{1}{6} + \frac{q\nu_n}{3}$.

Starting with $\nu_1 = 1/6$ from Step 1 we see by Step 2 that $\nu_n \nearrow \frac{1}{2(3-q)}$. Since $q \in (0,1)$ can be chosen arbitrarily close to 1 this finishes the proof. □

With this preparation the proof of Theorem 1.7 is now immediate.

Proof of Theorem 1.7. Let $w(x,t) = \sum_{k \in \mathbb{Z}_{odd}} \frac{\hat{\alpha}_k}{k} \Phi_k(|x|)e_k(t)$ with $\hat{\alpha} \in \mathcal{D}(J)$ such that $J'(\hat{\alpha}) = 0$. Recall from assumption (C2) that $C := \sup_k \|\Phi_k\|^2_{L^2(0,\infty)} < \infty$. Likewise, from Lemma 1.10 we have $\|\Phi_k'\|^2_{L^2(0,\infty)} \leq \tilde{C}k^2$ for all $k \in \mathbb{Z}_{odd}$ and some $\tilde{C} > 0$. Therefore, using Theorem 1.14 we find for all $\nu < \frac{1}{4}$

$$\left\|\partial_t^{1+\nu} w\right\|^2_{L^2(D)} = 2\omega^{2+2\nu} \sum_k \hat{\alpha}_k^2 |k|^{2\nu} \|\Phi_k\|^2_{L^2(0,\infty)} \leq 2\omega^{2+2\nu} C \|\hat{\alpha}\|^2_{h^\nu} < \infty$$

and likewise

$$\|\partial_t^\nu w_x\|_{L^2(D)}^2 = 2\omega^{2\nu} \sum_k \hat{\alpha}_k^2 |k|^{2\nu-2} \|\Phi_k'\|_{L^2(0,\infty)}^2 \le 2\omega^{2\nu} \tilde{C} \|\hat{\alpha}\|_{h^\nu}^2 < \infty.$$

This establishes the claim. □

1.4 Existence of Infinitely Many Breathers

In this section we extend Theorem 1.5 by the following multiplicity result.

Theorem 1.15. *Assume* (C0), (C1) *and* (C2). *Then* (1.1) *has infinitely many nontrivial, T-periodic weak solution w in the sense of Definition 1.3 with $T = \frac{2\pi}{\omega}$ provided*

(i) $\gamma < 0$ *and there exists an integer $l_- \in \mathbb{N}_{odd}$ such that for infinitely many $j \in \mathbb{N}$ the sequence* $\left(\Phi'_{m\cdot l_-^j}(0)\right)_{m\in\mathbb{N}_{odd}}$ *has at least one positive element,*

(ii) $\gamma > 0$ *and there exists an integer $l_+ \in \mathbb{N}_{odd}$ such that for infinitely many $j \in \mathbb{N}$ the sequence* $\left(\Phi'_{m\cdot l_+^j}(0)\right)_{m\in\mathbb{N}_{odd}}$ *has at least one negative element.*

Remark 1.16. *In the above Theorem, conditions* (C1) *and* (C2) *can be weakened: instead of requiring them for all $k \in \mathbb{N}_{odd}$ it suffices to require them for $k \in l_-^j\mathbb{N}_{odd}$, $k \in l_+^j\mathbb{N}_{odd}$ respectively. We prove this observation together with the one in Remark 1.6 at the end of this section.*

We start with an investigation about the types of symmetries which are compatible with our equation. The Euler-Lagrange equation (1.9) for critical points $\hat{\alpha} \in \mathcal{D}(J)$ of J takes the form $(\hat{\alpha} * \hat{\alpha} * \hat{\alpha})_k = \hat{\eta}_k \hat{\alpha}_k$ with $\hat{\eta}_k := \frac{2T\Phi_k'(0)}{\gamma\omega^4 k^2}$ for $k \in \mathbb{Z}_{odd}$. Next we describe subspaces of $\mathcal{D}(J)$ which are invariant under triple convolution and pointwise multiplication with $(\hat{\eta}_k)_{k\in\mathbb{Z}_{odd}}$. It turns out that these subspaces are made of sequences $\hat{z}$ where only the r^{th} entry modulus $2r$ is occupied.

Definition 1.17. *For $r \in \mathbb{N}_{odd}, p \in \mathbb{N}_{even}$ with $r < p$ let*

$$\mathcal{D}(J)_{r,p} = \{\hat{z} \in \mathcal{D}(J) : \forall\, k \in \mathbb{Z}, k \neq r \bmod p\colon \hat{z}_k = 0\}.$$

Lemma 1.18. *For $r \in \mathbb{N}_{odd}, p \in \mathbb{N}_{even}$ with $r < p$ and $p \neq 2r$ we have $\mathcal{D}(J)_{r,p} = \{0\}$.*

Proof. Let $\hat{z} \in \mathcal{D}(J)_{r,p}$. For all $k \notin r + p\mathbb{Z}$ we have $\hat{z}_k = 0$ by definition of $\mathcal{D}(J)_{r,p}$. Let therefore $k = r + pl_1$ for some $l_1 \in \mathbb{Z}$. Then $-k = -r - pl_1 \notin r + p\mathbb{Z}$ because otherwise $2r = -p(l_1+l_2) = p|l_1+l_2|$ for some $l_2 \in \mathbb{Z}$. Since by assumption $p > r$ we get $|l_1+l_2| < 2$. But clearly $|l_1 + l_2| \notin \{0, 1\}$ since $r \neq 0$ and $p \neq 2r$ by assumption. By this contradiction we have shown $-k \notin r + p\mathbb{Z}$ so that necessarily $0 = \hat{z}_{-k} = -\hat{z}_k$. This shows $\hat{z} = 0$. □

In the following we continue by only considering $\mathcal{D}_r := \mathcal{D}(J)_{r,2r}$ for $r \in \mathbb{N}_{odd}$.

Proposition 1.19. *Let $r \in \mathbb{N}_{odd}$.*

(i) The elements $\hat z \in \mathcal{D}_r$ are exactly those elements of $\mathcal{D}(J)$ which generate $\frac{T}{2r}$-antiperiodic functions $\sum_{k\in\mathbb{Z}_{odd}} \frac{\hat z_k}{k}\Phi_k(x)e_k(t)$.

(ii) If $\hat z \in \mathcal{D}_r$ then $(\hat z * \hat z * \hat z)_k = 0$ for all $k \notin r + 2r\mathbb{Z}$.

Proof. (i) An element $\hat z \in \mathcal{D}(J)$ generates a $\frac{T}{2r}$-antiperiodic function $z(x,t) = \sum_{k\in\mathbb{Z}_{odd}} \frac{\hat z_k}{k}\Phi_k(x)e_k(t)$ if and only if $z(x, t + \frac{T}{2r}) = -z(x,t)$. Comparing the Fourier coefficients we see that this is the case if for all $k \in \mathbb{Z}_{odd}$ we have $\hat z_k\big(\exp(\frac{\mathrm{i}\omega kT}{2r}) + 1\big) = 0$, i.e., either $k \in r + 2r\mathbb{Z}$ or $\hat z_k = 0$. This is exactly the condition that $\hat z \in \mathcal{D}_r$.
(ii) Let $\hat z \in \mathcal{D}_r$ and assume that there is $k \in \mathbb{Z}$ such that $0 \neq (\hat z * \hat z * \hat z)_k = \sum_{l,m} \hat z_l \hat z_{m-l}\hat z_{k-m}$. So there is $l_0, m_0 \in \mathbb{Z}_{odd}$ such that $\hat z_{l_0}, \hat z_{m_0-l_0}, \hat z_{k-m_0} \neq 0$ which means by the definition of $\mathcal{D}_r$ that $l_0, m_0-l_0, k-m_0 \in r+2r\mathbb{Z}$. Thus $k = l_0+m_0-l_0+k-m_0 \in 3r+2r\mathbb{Z} = r+2r\mathbb{Z}$. □

Proof of Theorem 1.15. We give the proof in case (i); for case (ii) the proof only needs a trivial modification. Let $r = l^j$ where j is an index such that the sequence $\left(\Phi'_{k\cdot l^j}(0)\right)_{k\in\mathbb{N}_{odd}}$ has a positive element (we have changed the notation from l_- to l for the sake of readability). Since $\mathcal{D}_r$ is a closed subspace of $\mathcal{D}(J)$ we have as before in Theorem 1.13 the existence of a minimizer $\hat\alpha^{(r)} \in \mathcal{D}_r$, i.e., $J(\hat\alpha^{(r)}) = \min_{\mathcal{D}_r} J < 0$. Moreover, $\hat\alpha^{(r)}$ satisfies the restricted Euler-Lagrange-equation

$$0 = J'\left(\hat\alpha^{(r)}\right)[\hat x] = \left(\hat\alpha^{(r)} * \hat\alpha^{(r)} * \hat\alpha^{(r)} * \hat x\right)_0 + \frac{2T}{\gamma\omega^4}\sum_k \frac{\Phi'_k(0)}{k^2}\hat\alpha^{(r)}_k \hat x_k \qquad \forall \hat x \in \mathcal{D}_r. \quad (1.22)$$

We need to show that (1.22) holds for every $\hat z \in \mathcal{D}(J)$. If for an arbitrary $\hat z \in \mathcal{D}(J)$ we define $\hat x_k := \hat z_k$ for $k \in r + 2r\mathbb{Z}$ and $\hat x_k := 0$ else then $\hat x \in \mathcal{D}_r$. If we furthermore define $\hat y := \hat z - \hat x$ then $\hat y_k = 0$ for all $k \in r + 2r\mathbb{Z}$. This implies in particular that

$$\sum_k \frac{\Phi'_k(0)}{k^2}\hat\alpha^{(r)}_k \hat y_k = 0$$

and by using (ii) of Proposition 1.19 also

$$(\hat\alpha * \hat\alpha * \hat\alpha * \hat y)_0 = \sum_k \left(\hat\alpha^{(r)} * \hat\alpha^{(r)} * \hat\alpha^{(r)}\right)_k \hat y_{-k} = 0.$$

This implies $J'(\hat\alpha^{(r)})[\hat y] = 0$ and since by (1.22) also $J'(\hat\alpha^{(r)})[\hat x] = 0$ we have succeeded in proving that $J'(\hat\alpha^{(r)}) = 0$.

It remains to show the multiplicity result. For this purpose we only consider $r = l^{j_m}$ for $j_m \to \infty$ as $m \to \infty$ where j_m is an index such that the sequence $\left(\Phi'_{l^{j_m}k}(0)\right)_{k\in\mathbb{N}_{odd}}$ has a positive element. First we observe that $\mathcal{D}_{l^{j_m}} \supsetneq \mathcal{D}_{l^{j_{m+1}}}$. Assume for contradiction that the set $\{\hat\alpha^{(l^{j_m})}\}$ is finite. Then we have a subsequence $(j_{m_n})_{n\in\mathbb{N}}$ such that $\hat\alpha = \hat\alpha^{(l^{j_{m_n}})}$ is constant. But then

$$\hat\alpha \in \bigcap_{n\in\mathbb{N}} \mathcal{D}_{l^{j_{m_n}}} = \bigcap_{j\in\mathbb{N}} \mathcal{D}_{l^j} = \{0\}.$$

This contradiction shows the existence of infinitely many distinct critical points of the function J and finishes the proof of the theorem. □

Proof of Remark 1.6 and Remark 1.16. The proof of Theorem 1.15 works on the basis that it suffices to minimize the functional J on $\mathcal{D}_r$. In this way a $\frac{T}{2r}$-antiperiodic breather is obtained. For $\hat{z} \in \mathcal{D}_r$ only the entries $\hat{z}_k$ with $k \in r\mathbb{Z}_{odd}$ are nontrivial while all other entries vanish. Therefore, (C1) and (C2) and the values of $\Phi_k'(0)$ are only relevant for $k \in r\mathbb{Z}_{odd}$. In the special case of Remark 1.16 we take $r = l_\pm^j$. □

1.5 Approximation by Finitely Many Harmonics

Here we give some analytical results on finite dimensional approximation of the breathers obtained in Theorem 1.5. The finite dimensional approximation is obtained by cutting-off the ansatz (1.8) and only considering harmonics of order $|k| \leq N$. Here a summand in the series (1.8) of the form $\Phi_k(|x|)e_k(t)$ is a called a harmonic since it satisfies the linear wave equation in (1.6). We will prove that J restricted to spaces $\mathcal{D}(J^{(N)})$ of cut-off ansatz functions still attains its minimum and that the sequence of the corresponding minimizers converges up to a subsequence to a minimizer of J on $\mathcal{D}(J)$.

Definition 1.20. *Let $N \in \mathbb{N}_{odd}$. Define*

$$J^{(N)} := J|_{\mathcal{D}(J^{(N)})}, \qquad \mathcal{D}(J^{(N)}) := \left\{\hat{z} \in \mathcal{D}(J) \;\middle|\; \forall\, |k| > N\colon \hat{z}_k = 0\right\}$$

Lemma 1.21. *Under the assumptions of Theorem 1.5 the following holds:*

(i) For every $N \in \mathbb{N}_{odd}$ sufficiently large there exists $\hat{\alpha}^{(N)} \in \mathcal{D}(J^{(N)})$ such that $J(\hat{\alpha}^{(N)}) = \inf J^{(N)} < 0$ and $\lim_{N\to\infty} J(\hat{\alpha}^{(N)}) = \inf J$.

(ii) There is $\hat{\alpha} \in \mathcal{D}(J)$ such that up to a subsequence (again denoted by $(\hat{\alpha}^{(N)})_N$) we have

$$\hat{\alpha}^{(N)} \to \hat{\alpha} \qquad \text{in } \mathcal{D}(J)$$

and $J(\hat{\alpha}) = \inf J$.

Remark 1.22. *The Euler-Lagrange-equation for $\hat{\alpha}^{(N)}$ reads:*

$$0 = J'\left(\hat{\alpha}^{(N)}\right)[\hat{y}] = \left(\hat{\alpha}^{(N)} * \hat{\alpha}^{(N)} * \hat{\alpha}^{(N)} * \hat{y}\right)_0 + \frac{2T}{\gamma\omega^4}\sum_k \frac{\Phi_k'(0)}{k^2}\hat{\alpha}_k^{(N)}\hat{y}_k \qquad \forall\, \hat{y} \in \mathcal{D}(J^{(N)}).$$

This amounts to satisfying (1.2) in Definition 1.3 for functions $\psi(x,t) = \sum_{k\in\mathbb{Z}_{odd},|k|\leq N}\hat{\psi}_k(x)e_k(t)$ with $\hat{\psi}_k \in H^1(\mathbb{R})$. Clearly, in general $\hat{\alpha}^{(N)}$ is not a critical point of J.

Proof. (i) We choose $N \in \mathbb{N}_{odd}$ so large, such that we have the assumed sign of the the one element in $(\Phi_k'(0))_{|k|\leq N}$. The restriction of J to the $\frac{N+1}{2}$-dimensional space $\mathcal{D}(J^{(N)})$ preserves coercivity. The continuity of $J^{(N)}$ therefore guarantees the existence of a minimizer $\hat{\alpha}^{(N)} \in \mathcal{D}(J^{(N)})$. As before we see that $J(\hat{\alpha}^{(N)}) = \inf J^{(N)} < 0$, so in particular $\hat{\alpha}^{(N)} \neq 0$. Next we observe that $\mathcal{D}(J^{(N)}) \subset \mathcal{D}(J)$, i.e., $J(\hat{\alpha}^{(N)}) \geq \inf J = J(\hat{\beta})$ for a minimizer $\hat{\beta} \in \mathcal{D}(J)$ of J. Let us define $\hat{\beta}_k^{(N)} = \hat{\beta}_k$ for $|k| \leq N$ and $\hat{\beta}_k^{(N)} = 0$. Since the Fourier-series $\beta(t) = \sum_k \hat{\beta}_k e_k(t)$ converges in $L^4(\mathbb{T})$, cf. Theorem 4.1.8 in [Gra08], we see that $\hat{\beta}^{(N)} \to \hat{\beta}$ in $\mathcal{D}(J)$. By the minimality of $\hat{\alpha}^{(N)} \in \mathcal{D}(J^{(N)})$ and continuity of J we conclude

$$\inf_{\mathcal{D}(J)} J \le J(\hat{\alpha}^{(N)}) \le J(\hat{\beta}^{(N)}) \longrightarrow J(\hat{\beta}) = \inf_{\mathcal{D}(J)} J.$$

Hence $\lim_{N\to\infty} J(\hat{\alpha}^{(N)}) = \inf J$ as claimed.

(ii) Since $\mathcal{D}(J^{(N)}) \subset \mathcal{D}(J^{(N+1)}) \subset \mathcal{D}(J)$ we see that $J(\hat{\alpha}^{(N)}) \ge J(\hat{\alpha}^{(N+1)}) \ge \inf J$ so that in particular the sequence $(J(\hat{\alpha}^{(N)}))_N$ is bounded. By coercivity of J we conclude that $(\hat{\alpha}^{(N)})_N$ is bounded in $\mathcal{D}(J)$ so that there is $\hat{\alpha} \in \mathcal{D}(J)$ and a subsequence (again denoted by $(\hat{\alpha}^{(N)})_N$) such that

$$\hat{\alpha}^{(N)} \rightharpoonup \hat{\alpha} \qquad \text{in } \mathcal{D}(J).$$

By part (i) and weak lower semi-continuity of J we obtain

$$\inf J = \lim_{N\to\infty} J(\hat{\alpha}^{(N)}) \ge J(\hat{\alpha}),$$

i.e., $\hat{\alpha}$ is a minimizer of J. Recall that $J(\cdot) = \frac{1}{4}|||\cdot|||^4 + J_1(\cdot)$ where J_1 is weakly continuous, cf. proof of Theorem 1.13. Therefore, since $\hat{\alpha}^{(N)} \rightharpoonup \hat{\alpha}$ and $J(\hat{\alpha}^{(N)}) \to J(\hat{\alpha})$ we see that $|||\hat{\alpha}^{(N)}||| \to |||\hat{\alpha}|||$ as $N \to \infty$. Since $\mathcal{D}(J)$ is strictly uniformly convex, we obtain the norm-convergence of $(\hat{\alpha}^{(N)})_N$ to $\hat{\alpha}$. ☐

1.6 Appendix

1.6.1 Details on exponentially decreasing fundamental solutions for step potentials

Here we consider a second-order ordinary differential operator

$$L_k := -\frac{d^2}{dx^2} - k^2\omega^2 g(x)$$

with g as in Theorem 1.1. Clearly, L_k is a self-adjoint operator on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. Moreover, $\sigma_{ess}(L_k) = [k^2\omega^2 a, \infty)$. By the assumption on ω we have

$$\sqrt{b}\omega c \frac{2}{\pi} = \frac{p}{q} \text{ with } p, q \in \mathbb{N}_{odd}.$$

Hence, with $k \in q\mathbb{N}_{odd}$, $k\sqrt{b}\omega c$ is an odd multiple of $\pi/2$. In the following we shall see that 0 is not an eigenvalue of L_k for $k \in q\mathbb{N}_{odd}$ so that (C1) as in Remark 1.16 is fulfilled. A potential eigenfunction ϕ_k for the eigenvalue 0 would have to look like

$$\phi_k(x) = \begin{cases} -A\sin\left(k\omega\sqrt{b}c\right)e^{k\omega\sqrt{a}(x+c)}, & x < -c, \\ A\sin\left(k\omega\sqrt{b}x\right) + B\cos\left(k\omega\sqrt{b}x\right), & -c < x < c, \\ A\sin\left(k\omega\sqrt{b}c\right)e^{-k\omega\sqrt{a}(x-c)}, & c < x. \end{cases} \tag{1.23}$$

with $A, B \in \mathbb{R}$ to be determined. Note that we have used $\cos\left(k\omega\sqrt{b}c\right) = 0$. The C^1-matching of ϕ_k at $x = \pm c$ lead to the two equations

$$-Bk\omega\sqrt{b}\sin\left(k\omega\sqrt{b}c\right) = -Ak\omega\sqrt{a}\sin\left(k\omega\sqrt{b}c\right),$$
$$Bk\omega\sqrt{b}\sin\left(k\omega\sqrt{b}c\right) = -Ak\omega\sqrt{a}\sin\left(k\omega\sqrt{b}c\right)$$

and since $\sin\left(k\omega\sqrt{b}c\right) = \pm 1$ this implies $A = B = 0$ so that there is no eigenvalue 0 of L_k. Next we need to find the fundamental solution ϕ_k of L_k that decays to zero at $+\infty$ and is normalized by $\phi_k(0) = 1$. Here we can use the same ansatz as in (1.23) and just ignore the part of ϕ_k on $(-\infty, 0)$. Now the normalization $\phi_k(0) = 1$ leads to $B = 1$ and the C^1-matching at $x = c$ leads to $A = \sqrt{\frac{b}{a}}B = \sqrt{\frac{b}{a}}$ so that the decaying fundamental solution is completely determined. We find that

$$|\phi_k(x)| \leq \begin{cases} A + B, & 0 \leq x \leq c \\ A, & c < x \leq 2c \\ Ae^{-\frac{1}{2}k\omega\sqrt{a}x}, & x > 2c \end{cases}$$

so that $|\phi_k(x)| \leq (A+B)e^{-\rho_k x} \leq Me^{-\rho x}$ on $[0,\infty)$ with $\rho_k = \frac{1}{2}k\omega\sqrt{a}$, $\rho = \frac{1}{2}\omega\sqrt{a}$ and $M = A + B$. This shows that also (C2) holds. Finally, since $\phi_k'(0) = \frac{bk\omega}{\sqrt{a}} > 0$ the existence of infinitely many breathers can only be shown for $\gamma < 0$. At the same time, due to $|\phi_k(0)| = O(k)$, Theorem 1.7 applies.

1.6.2 Details on Bloch Modes for periodic step potentials

Here we consider a second-order periodic ordinary differential operator

$$L := -\frac{d^2}{dx^2} + V(x)$$

with $V \in L^\infty(\mathbb{R})$ which we assume to be even and 2π-periodic. Moreover, we assume that 0 does not belong to the spectrum of $L : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$. We first describe what Bloch modes are and why they exists. Later we show that this is the situation which occurs in Theorem 1.5 and we verify conditions (C1) and (C2).

A function $\Phi \in C^1(\mathbb{R})$ which is twice almost everywhere differentiable such that

$$L\Phi = 0 \quad \text{a.e. in } \mathbb{R}, \qquad \Phi(\cdot + 2\pi) = \rho\Phi(\cdot). \tag{1.24}$$

with $\rho \in (-1,1) \setminus \{0\}$ is called the (exponentially decreasing for $x \to \infty$) Bloch mode of L and ρ is called the Floquet multiplier. The existence of Φ is guaranteed by the assumption that $0 \notin \sigma(L)$. This is essentially Hill's theorem, cf. [Eas73]. Note that $\Psi(x) := \Phi(-x)$ is a second Bloch mode of L, which is exponentially increasing for $x \to \infty$. The functions Φ and Ψ form a fundamental system of solutions for operator L on $\mathbb{R}$. Next we explain how Φ is constructed, why it can be taken real-valued and why it does not vanish at $x = 0$ so that we can assume w.l.o.g $\Phi(0) = 1$.

According to [Eas73], Theorem 1.1.1 there are linearly independent functions $\Psi_1, \Psi_2 \colon \mathbb{R} \to \mathbb{C}$ and Floquet-multipliers $\rho_1, \rho_2 \in \mathbb{C}$ such that $L\Psi_j = 0$ a.e. on $\mathbb{R}$ and $\Psi_j(x + 2\pi) = \rho_j \Psi_j(x)$ for $j = 1, 2$. We define ϕ_j, $j = 1, 2$ as the solutions to the initial value problems

$$\begin{cases} L\phi_1 = 0, \\ \phi_1(0) = 1, \quad \phi_1'(0) = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} L\phi_2 = 0, \\ \phi_2(0) = 0, \quad \phi_2'(0) = 1 \end{cases}$$

and consider the Wronskian

$$W(x) := \begin{pmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{pmatrix} \tag{1.25}$$

and the monodromy matrix

$$A := W(2\pi) = \begin{pmatrix} \phi_1(2\pi) & \phi_2(2\pi) \\ \phi_1'(2\pi) & \phi_2'(2\pi) \end{pmatrix}. \tag{1.26}$$

Then $\det A = 1$ is the Wronskian determinant of the fundamental system ϕ_1, ϕ_2 and the Floquet multipliers $\rho_{1,2} = \frac{1}{2}\left(\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4}\right)$ are the eigenvalues of A with corresponding eigenvectors $v_1 = (v_{1,1}, v_{1,2}) \in \mathbb{C}^2$ and $v_2 = (v_{2,1}, v_{2,2}) \in \mathbb{C}^2$. Thus, $\Psi_j(x) = v_{j,1}\phi_1(x) + v_{j,2}\phi_2(x)$. By Hill's theorem (see [Eas73]) we know that

$$0 \in \sigma(L) \qquad \Leftrightarrow \qquad |\operatorname{tr}(A)| \le 2.$$

Due to the assumption that $0 \notin \sigma(L)$ we see that ρ_1, ρ_2 are real with $\rho_1, \rho_2 \in \mathbb{R} \setminus \{-1, 0, 1\}$ and $\rho_1\rho_2 = 1$, i.e., one of the two Floquet multipliers has modulus smaller then one and other one has modulus bigger than one. W.l.o.g. we assume $0 < |\rho_2| < 1 < |\rho_1|$. Furthermore, since ρ_1, ρ_2 are real and A has real entries we can choose v_1, v_2 to be real and so Ψ_1, Ψ_2 are both real valued. As a result we have found a real-valued Bloch mode $\Psi_2(x)$ which is exponentially decreasing as $x \to \infty$ due to $|\rho_2| < 1$. Let us finally verify that $\Psi_2(0) \neq 0$ so that we may assume by rescaling that $\Psi_2(0) = 1$. Assume for contradiction that $\Psi_2(0) = 0$. Since the potential $V(x)$ is even in x this implies that Ψ_2 is odd and hence (due to the exponential decay at $+\infty$) in $L^2(\mathbb{R})$. But this contradicts that $0 \notin \sigma(L)$.

Now we explain how the precise choice of the data $a, b > 0, \Theta \in (0, 1)$ and ω for the step-potential g in Theorem 1.2 allows to fulfill the conditions (C1) and (C2). Let us define

$$\tilde{g}(x) := \begin{cases} a, & x \in [0, 2\Theta\pi), \\ b, & x \in (2\Theta\pi, 2\pi). \end{cases}$$

and extend $\tilde{g}$ as a 2π-periodic function to $\mathbb{R}$. Then $\tilde{g}(x) = g(x - \Theta\pi)$, and the corresponding exponentially decaying Bloch modes $\tilde{\phi}_k$ and ϕ_k are similarly related by $\tilde{\phi}_k(x) = \phi_k(x - \Theta\pi)$. For the computation of the exponentially decaying Bloch modes, it is, however, more convenient to use the definition $\tilde{g}$ instead of g.

Now we will calculate the monodromy matrix A_k from (1.26) for the operator L_k. For a constant value $c > 0$ the solution of the initial value problem

$$-\phi''(x) - k^2\omega^2 c\phi(x) = 0, \quad \phi(x_0) = \alpha, \quad \phi'(x_0) = \beta$$

is given by

$$\begin{pmatrix} \phi(x) \\ \phi'(x) \end{pmatrix} = T_k(x - x_0, c) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

with the propagation matrix

$$T_k(s, c) := \begin{pmatrix} \cos(k\omega\sqrt{c}s) & \frac{1}{k\omega\sqrt{c}} \sin(k\omega\sqrt{c}s) \\ -k\omega\sqrt{c}\sin(k\omega\sqrt{c}s) & \cos(k\omega\sqrt{c}s) \end{pmatrix}.$$

Therefore we can write the Wronskian as follows

$$W_k(x) = \begin{cases} T_k(x, a) & x \in [0, 2\Theta\pi] \\ T_k(x - 2\Theta\pi, b)T_k(2\Theta\pi, a) & x \in [2\Theta\pi, 2\pi] \end{cases}$$

and the monodromy matrix as

$$A_k = W_k(2\pi) = T_k(2\pi(1 - \Theta), b)T_k(2\Theta\pi, a).$$

To get the exact form of A_k let us use the notation

$$l := \sqrt{\frac{b}{a}} \frac{1 - \Theta}{\Theta}, \qquad m := 2\sqrt{a}\Theta\omega.$$

Hence

$$\begin{aligned} A_k =& \sin(kml\pi)\sin(km\pi) \\ &\cdot \begin{pmatrix} \cot(kml\pi)\cot(km\pi) - \sqrt{\frac{a}{b}} & \frac{1}{k\omega\sqrt{a}}\cot(kml\pi) + \frac{1}{k\omega\sqrt{b}}\cot(km\pi) \\ -k\omega\sqrt{b}\cot(km\pi) - k\omega\sqrt{a}\cot(kml\pi) & -\sqrt{\frac{b}{a}} + \cot(kml\pi)\cot(km\pi) \end{pmatrix} \end{aligned}$$

and

$$\operatorname{tr}(A_k) = 2\cos(kml\pi)\cos(km\pi) - \left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right)\sin(kml\pi)\sin(km\pi).$$

In order to verify (C1) we aim for $|\operatorname{tr}(A_k)| > 2$. However, instead of showing $|\operatorname{tr}(A_k)| > 2$ for all $k \in \mathbb{Z}_{odd}$ we may restrict to $k \in r \cdot \mathbb{Z}_{odd}$ for fixed $r \in \mathbb{N}_{odd}$ according to Remark 1.6. Next we will choose $r \in \mathbb{Z}_{odd}$. Due to the assumptions from Theorem 1.2 we have

$$l = \frac{\tilde{p}}{\tilde{q}}, \quad 2m = \frac{p}{q} \in \frac{\mathbb{N}_{odd}}{\mathbb{N}_{odd}}. \tag{1.27}$$

Therefore, by setting $r = \tilde{q}q$[1] we obtain $\cos(km\pi) = \cos(kml\pi) = 0$ and $\sin(km\pi), \sin(kml\pi) \in \{\pm 1\}$ for all $k \in r \cdot \mathbb{Z}_{odd}$. Together with $a \neq b$ this implies $|\operatorname{tr}(A_k)| = \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} > 2$ so that (C1) holds and A_k takes the simple diagonal form

$$A_k = \begin{pmatrix} -\sqrt{\frac{a}{b}}\sin(kml\pi)\sin(km\pi) & \\ 0 & -\sqrt{\frac{b}{a}}\sin(kml\pi)\sin(km\pi). \end{pmatrix}$$

[1]Instead of $r = \tilde{q}q$ we may have chosen any odd multiple of $\tilde{q}q$, e.g. $r = (\tilde{q}q)^j$ for any $j \in \mathbb{N}$. This is important for the applicability of Theorem 1.15 to obtain infinitely many breathers.

In the following we assume w.l.o.g $0 < a < b$, i.e., the Floquet exponent with modulus less than 1 is $\rho_k = -\sqrt{\frac{a}{b}}\sin(kml\pi)\sin(km\pi)$. Note that $|\rho_k| = \sqrt{a/b}$ is independent of k. Furthermore the Bloch mode $\tilde{\phi}_k$ that is decaying to 0 at $+\infty$ and normalized by $\tilde{\phi}_k(\Theta\pi) = 1$ is deduced from the upper left element of the Wronskian, i.e.,

$$\tilde{\phi}_k(x) = \frac{1}{\cos(k\omega\sqrt{a}\Theta\pi)}\begin{cases} \cos(k\omega\sqrt{a}x), & x \in (0, 2\Theta\pi), \\ \cos\Big(k\omega\sqrt{b}(x-2\Theta\pi)\Big)\cos(k\omega\sqrt{a}2\Theta\pi) & \\ -\sqrt{\frac{a}{b}}\sin\Big(k\omega\sqrt{b}(x-2\Theta\pi)\Big)\sin(k\omega\sqrt{a}2\Theta\pi), & x \in (2\Theta\pi, 2\pi) \end{cases}$$

and on shifted intervals of lengths 2π one has $\tilde{\phi}_k(x+2m\pi) = \rho_k^m\tilde{\phi}_k(x)$. Notice that by (1.27) the expression $k\omega\sqrt{a}\Theta\pi = k\frac{p}{q}\frac{\pi}{4}$ is an odd multiple of $\pi/4$ since $k \in q\tilde{q}\mathbb{Z}_{odd}$ and hence $|\cos(k\omega\sqrt{a}\Theta\pi)| = 1/\sqrt{2}$. Therefore $\|\phi_k\|_{L^\infty(0,\infty)} = \|\tilde{\phi}_k\|_{L^\infty(\Theta\pi,\infty)} \leq \|\tilde{\phi}_k\|_{L^\infty(0,2\pi)} \leq \sqrt{2}(1+\sqrt{a/b})$. Thus we have shown that $|\phi_k(x)| \leq Me^{-\rho x}$ for $x \in [0,\infty)$ with $M > 0$ and $\rho = \frac{1}{4\pi}(\ln b - \ln a) > 0$. Finally, let us compute

$$\phi_k'(0) = \tilde{\phi}_k'(\Theta\pi) = -k\omega\sqrt{a}\tan\big(k\omega\sqrt{a}\Theta\pi\big) \in \{\pm k\omega\sqrt{a}\}.$$

This shows that $|\phi_k'(0)| = O(k)$ holds which allows to apply Theorem 1.7. It also shows that the estimate $|\phi_k(0)| = O(k^{\frac{3}{2}})$ from Lemma 1.10 can be improved in special cases. To see that $\phi_k'(0)$ is alternating in k, observe that moving from $k \in r\mathbb{Z}_{odd}$ to $k + 2r \in r\mathbb{Z}_{odd}$ the argument of tan changes by $2r\omega\sqrt{a}\Theta\pi$ which is an odd multiple of $\pi/2$. Since $\tan\big(x + \mathbb{Z}_{odd}\frac{\pi}{2}\big) = -1/\tan(x)$ we see that the sequence $\phi_k'(0)$ is alternating for $k \in r\mathbb{Z}_{odd}$. This shows in particular that for any $j \in \mathbb{N}$ the sequence $(\phi_{hr^j}'(0))_{h\in\mathbb{N}_{odd}}$ contains infinitely many positive and negative elements, and hence Theorem 1.15 for the existence of infinitely many breathers is applicable. This concludes the proof Theorem 1.2 since we have shown that the potential g satisfies the assumptions (C1) and (C2) from Theorem 1.5.

1.6.3 Embedding of Hölder-spaces into Sobolev-spaces

Lemma 1.23. *For $0 < \tilde{\nu} < \nu < 1$ there is the continuous embedding $C^{0,\nu}(\mathbb{T}_T) \to H^{\tilde{\nu}}(\mathbb{T}_T)$.*

Proof. Let $z(t) = \sum_k \hat{z}_k e_k(t)$ be a function in $C^{0,\nu}(\mathbb{T}_T)$. We need to show the finiteness of the spectral norm $\|z\|_{H^{\tilde{\nu}}}$. For this we use the equivalence of the spectral norm $\|\cdot\|_{H^{\tilde{\nu}}}$ with the Slobodeckij norm, cf. Lemma 1.24. Therefore it suffices to check the estimate

$$\int_{\mathbb{T}_T}\int_{\mathbb{T}_T}\frac{|z(t)-z(\tau)|^2}{|t-\tau|^{1+2\tilde{\nu}}}\,\mathrm{d}t\,\mathrm{d}\tau \leq \|z\|_{C^\nu(\mathbb{T}_T)}^2\int_{\mathbb{T}_T}\int_{\mathbb{T}_T}|t-\tau|^{-1+2(\nu-\tilde{\nu})}\,\mathrm{d}t\,\mathrm{d}\tau \leq C(\nu,\tilde{\nu})\|z\|_{C^\nu(\mathbb{T}_T)}^2$$

where the double integral is finite due to $\nu > \tilde{\nu}$. □

For $0 < s < 1$ recall the definition of the Slobodeckij-seminorm for a function $z : \mathbb{T}_T \to \mathbb{R}$

$$[z]_s := \left(\int_{\mathbb{T}_T}\int_{\mathbb{T}_T}\frac{|z(t)-z(\tau)|^2}{|t-\tau|^{1+2s}}\,\mathrm{d}t\,\mathrm{d}\tau\right)^{1/2}.$$

Lemma 1.24. *For functions $z \in H^s(\mathbb{T}_T)$, $0 < s < 1$ the spectral norm $\|z\|_{H^s} = (\sum_k(1+k^2)^s|\hat{z}_k|^2)^{1/2}$ and the Solobodeckij norm $|||z|||_{H^s} := (\|z\|_{L^2(\mathbb{T}_T)}^2 + [z]_s^2)^{1/2}$ are equivalent.*

Proof. The Solobodeckij space and the spectrally defined fractional Sobolev space are both Hilbert spaces. Hence, by the open mapping theorem, if suffices to verify the estimate $|||z|||_{H^s} \leq C\|z\|_{H^s}$. By direct computation we get

$$\begin{aligned}
\int_{\mathbb{T}_T}\int_{\mathbb{T}_T} \frac{|z(t)-z(\tau)|^2}{|t-\tau|^{1+2s}}\,\mathrm{d}t\,\mathrm{d}\tau &= \int_0^T\int_{-\tau}^{T-\tau} \frac{|z(x+\tau)-z(\tau)|^2}{|x|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_0^T \left(\int_0^{T-\tau} \frac{|z(x+\tau)-z(\tau)|^2}{x^{1+2s}}\,\mathrm{d}x + \int_{T-\tau}^T \frac{|z(x+\tau)-z(\tau)|^2}{(T-x)^{1+2s}}\,\mathrm{d}x\right)\mathrm{d}\tau \\
&= \int_0^T\int_0^T \frac{|z(x+\tau)-z(\tau)|^2}{g(x,\tau)^{1+2s}}\,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}$$

with

$$g(x,\tau) = \begin{cases} x & \text{if } 0 \leq x \leq T-\tau, \\ T-x & \text{if } T-\tau \leq x \leq T. \end{cases}$$

Since $g(x,\tau) \geq \operatorname{dist}(x,\partial\mathbb{T}_T)$ and due to Parseval's identity we find

$$\begin{aligned}
\int_{\mathbb{T}_T}\int_{\mathbb{T}_T} \frac{|z(t)-z(\tau)|^2}{|t-\tau|^{1+2s}}\,\mathrm{d}t\,\mathrm{d}\tau &\leq \int_{\mathbb{T}_T} \frac{\|\widehat{z(\cdot+x)}-\hat{z}\|_{l^2}^2}{\operatorname{dist}(x,\partial\mathbb{T}_T)^{1+2s}}\,\mathrm{d}x \\
&= \int_{\mathbb{T}_T}\sum_k \frac{|\exp(\mathrm{i}k\omega x)-1|^2|\hat{z}_k|^2}{\operatorname{dist}(x,\partial\mathbb{T}_T)^{1+2s}}\,\mathrm{d}x \\
&= 4\int_0^{T/2}\sum_k \frac{1-\cos(k\omega x)}{x^{1+2s}}|\hat{z}_k|^2\,\mathrm{d}x \\
&\leq 4\tilde{C}\sum_k k^{2s}|\hat{z}_k|^2
\end{aligned}$$

with $\tilde{C} = \int_0^\infty \frac{1-\cos(\omega\xi)}{\xi^{1+2s}}\,\mathrm{d}\xi$. This finishes the proof. □

2 Some Direct Methods

2.1 Abstract results on ground states

In many applications semilinear equations of the form

$$Lu = f(x,t,u), \qquad \text{on} \quad (x,t) \in \mathbb{R}^N \times \mathbb{T}_T, \tag{2.1}$$

are of interest. Often L is a closed, densely defined linear operator on a Hilbert space and f is superlinear in u. We give new examples in the case

$$L = V(x)\partial_t^2 - \Delta \qquad \text{and} \qquad f(x,t,u) = \Gamma(x)|u|^{p-1}u$$

for potentials V with negative background strength but also attaining positive values, and bounded potentials Γ. Formally weak solutions of (2.1) correspond to critical points of the energy functional

$$I(u) = \frac{1}{2}b_L(u,u) - \int_\Omega F(x,t,u)\,\mathrm{d}x \qquad \text{with} \quad F(x,t,u) := \int_0^x f(x,t,s)\,\mathrm{d}s\,.$$

We call a weak solution of (2.1) a *bound state*. We call a bound state with minimal energy among all bound states a *ground state*. The above situation can arise in wave guides. The strategy for our abstract existence results is a rearrangement of the work [SW10] by A. Szulkin and T. Weth. On the one hand we simplify their general Hilbert space results to results on "almost function spaces", on the other hand we generalize their examples with fully periodic structure to a theorem with different kinds of symmetries. By "almost function spaces" we mean a Hilbert space which embeds into a space of functions. To highlight this, we write $\mathcal{H}$ for the abstract Hilbert space and $\hat{u}$ for an element in this space. After applying S on $\hat{u}$, we obtain a function. One can think of a sequence of Fourier coefficients $\hat{u}$ and S as the Fourier reconstruction operator $S\hat{u} = \sum_k \hat{u}_k \mathrm{e}^{\mathrm{i}\omega k t}$. In this abstract part we do not explicitly use that b_L corresponds to an operator, but this is implicitly used in the examples to construct the Hilbert space $\mathcal{H}$ with the desired properties.

2.1.1 Existence of a critical point

In this subsection we construct in a relatively general setting a critical point as candidate for a ground state. The proof that this candidate is non-trivial will be done in the next section since different settings require different strategies. Our assumptions throughout this section are

(A1) Let $\mathcal{H}$ be a real Hilbert space, $\langle\cdot,\cdot\rangle$ the scalar product on $\mathcal{H}$, $\Omega \subset \mathbb{R}^N$ be open, $p > 1$. Assume the embedding $S\colon \mathcal{H} \hookrightarrow L^{p+1}(\Omega,\mathbb{R})$ is continuous and $S\colon \mathcal{H} \hookrightarrow L^{p+1}_{loc}(\Omega,\mathbb{R})$ is compact. Furthermore assume that there is an orthogonal decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^0 \oplus \mathcal{H}^-$ with $\mathcal{H}^+ \neq \{0\}$ and $\dim(\mathcal{H}^0) < \infty$.

(A2) Let $b_{\mathcal{L}}: \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a continuous, hermitian bilinear form. Assume that $b_{\mathcal{L}}$ is positive definite on $\mathcal{H}^+$, $b_{\mathcal{L}}|_{\mathcal{H}^0 \times \mathcal{H}^0} \equiv 0$, negative definite on $\mathcal{H}^-$ and

$$b_{\mathcal{L}}(\hat{u}, \hat{u}) = \left\|\hat{u}^+\right\|^2 - \left\|\hat{u}^-\right\|^2.$$

where $\hat{u}^{\pm}$ is the projection on $\mathcal{H}^{\pm}$.

(A3) Let $f: \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory-function, and let $F(x, u) := \int_0^x f(x, s)\, \mathrm{d}s$. Assume

a) there is $C > 0$ such that $\forall\, (x, u) \in \Omega \times \mathbb{R}\colon\ |f(x, u)| \le C|u|^p$.

b) $f(x, u) = \mathrm{o}\,(u)$ uniformly in $x \in \Omega$ as $u \to 0$.

c) $u \mapsto f(x, u)/|u|$ is strictly increasing on $\mathbb{R} \setminus \{0\}$.

d) For all compact $\tilde{\Omega} \subset \Omega$ we have $F(x, u)/u^2 \to \infty$ uniformly in $x \in \tilde{\Omega}$ as $|u| \to \infty$.

Our guiding example $f(x, u) = \Gamma(x)|u|^{p-1}u$ with $\Gamma \in L^\infty(\Omega, \mathbb{R})$ and $\inf_{\tilde{\Omega}} \Gamma > 0$ for all compact $\tilde{\Omega} \subset \Omega$ satisfies $(A3)$. If $S: \mathcal{H} \hookrightarrow L^{p_1+1}(\Omega, \mathbb{R}) \cap L^{p_2+1}(\Omega, \mathbb{R})$ for $1 \le p_1 < p_2 < \infty$, then $(A3)$ part $a)$ can be modified to $|f(x, u)| \le C(|u|^{p_1} + |u|^{p_2})$. If Ω is bounded, then $(A3)$ part $a)$ can be modified to $|f(x, u)| \le C(1 + |u|^p)$. For simplicity we only prove the above version. We can also switch the sign of f in $(A3)$ by switching the roles of $\mathcal{H}^+$ and $\mathcal{H}^-$, if both are nontrivial. For convenience we only prove the above version. Note that $\mathcal{H}^0 = \{0\}$ and/or $\mathcal{H}^- = \{0\}$ is allowed.

Proposition 2.1. *Assume* $(A3)$. *Then* $\forall\, u \in \mathbb{R} \setminus \{0\}\colon\ \frac{1}{2} f(x, u)u > F(x, u) > 0$.

The proof of this observation is lengthy but elementary. We omit it here.

Definition 2.2. *Assume* $(A1)$, $(A2)$ *and* $(A3)$. *Let*

$$\mathcal{I}\colon \mathcal{H} \to \mathbb{R}, \quad \mathcal{I}(\hat{u}) := \frac{1}{2} b_{\mathcal{L}}(\hat{u}, \hat{u}) - \int_\Omega F(x, S\hat{u})\, \mathrm{d}x\,.$$

Proposition 2.3. *Assume* $(A1)$, $(A2)$ *and* $(A3)$. *Then:*

(i) $\mathcal{I} \in C^1(\mathcal{H}, \mathbb{R})$ *with*

$$\forall\, \hat{u}, \hat{v} \in \mathcal{H}\colon \mathcal{I}'(\hat{u})[\hat{v}] = b_{\mathcal{L}}(\hat{u}, \hat{v}) - \int_\Omega f(x, S\hat{u})\, S\hat{v}\, \mathrm{d}x$$

(ii) $\mathcal{I}'(\hat{u}) = \mathrm{o}\,(\|\hat{u}\|)$ *as* $\hat{u} \to 0$.

(iii) Let $V \subset \mathcal{H} \setminus \{0\}$ *be weakly closed. Then* $\lim_{r\to\infty} \frac{1}{r^2} \mathcal{I}(r\hat{u}) = \infty$ *uniformly for* $\hat{u} \in V$.

Proof. The proof for (i) is straightforward, we omit it. Part (ii) directly follows from $(A3)$ part b). Part (iii) can be found in the proof of Theorem 16 in [SW10], where we have to modify one argument. Recall the constructions leading to the term $\frac{1}{s_n^2} I(s_n u_n)$ and the weak limit $u \in E \setminus \{0\}$. Since u is nontrivial we can find a compact subset $\tilde{\Omega} \subset \Omega$ such that $\|u\|_{L^2(\tilde{\Omega})} > 0$. Then we conclude similar as before

$$\liminf_{n\to\infty} \frac{I(s_n u_n)}{s_n^2} \ge \liminf_{n\to\infty} \int_{\tilde{\Omega}} \frac{F(x, s_n u_n)}{(s_n u_n)^2} u_n^2\, \mathrm{d}x \ge \liminf_{n\to\infty} \inf_{x\in\tilde{\Omega}} \frac{F(x, s_n u_n)}{(s_n u_n)^2} \cdot \|u\|_{L^2(\tilde{\Omega})} = \infty$$

by Fatou's Lemma and $(A3)$. □

Definition 2.4. *Assume (A1), (A2) and (A3). Let*

$$\mathcal{N} := \left\{ \hat{u} \in \mathcal{H} \setminus \mathcal{H}^0 \;\middle|\; \mathcal{I}'(\hat{u})[\hat{u}] = 0, \;\; \forall\, \hat{v} \in \mathcal{H}^- : \mathcal{I}'(\hat{u})[\hat{v}] = 0 \right\}, \qquad c := \inf_{\mathcal{N}} \mathcal{I}.$$

Then the set $\mathcal{N}$ contains all nontrivial critical points of $\mathcal{I}$ and $\mathcal{N} \subset \mathcal{H} \setminus (\mathcal{H}^0 \oplus \mathcal{H}^-)$. Furthermore

$$\forall\, \hat{u} \in \mathcal{H} \colon \mathcal{I}(\hat{u}) = \int_\Omega \frac{1}{2} f(x, S\hat{u}) S\hat{u} - F(x, S\hat{u})\, \mathrm{d}x + \frac{1}{2}\mathcal{I}'(\hat{u})[\hat{u}],$$

hence $\mathcal{I}$ is positive on $\mathcal{N}$, i.e., $c \geq 0$. In the following we write $B^+ := \{\hat{u} \in \mathcal{H}^+ \mid \|\hat{u}\| = 1\}$.

The set $\mathcal{N}$ is often called the Nehari or Nehari-Pankov manifold, see e.g. [Pan04]. The value c is often called the ground state level. In our setting $\mathcal{N}$ is not necessary a C^1-manifold. We will obtain our candidate for the ground state by minimizing $\mathcal{I} \circ m \colon B^+ \to \mathbb{R}$ and extracting a weak limit. The projection m is constructed in Lemma 2.5 This minimization is strongly related to minimizing $\mathcal{I}$ on $\mathcal{N}$, but the latter may have not enough smoothness in order to obtain a critical point (since $\mathcal{N}$ may not be a C^1-manifold). Our procedure will be done in several steps, where most of the time we refer to [SW10] for the proofs. We start with observations on the projection m onto $\mathcal{N}$.

Lemma 2.5 (See [SW10])**.** *Assume (A1), (A2) and (A3). Then:*

(i) Let $\hat{w} \in \mathcal{H} \setminus (\mathcal{H}^0 \oplus \mathcal{H}^-)$ be arbitrary. Then the map $\mathbb{R}_{>0} \times (\mathcal{H}^0 \oplus \mathcal{H}^-) \to \mathbb{R}$, $(s, \hat{v}) \mapsto I(s\,\hat{w}^+ + \hat{v})$ has exactly one critical point $\tilde{m}(\hat{w})$. Furthermore this critical point is a strict maximum and $\tilde{m}(\hat{w}) \in \mathcal{N}$.

(ii) The map $\tilde{m} \colon \mathcal{H} \setminus (\mathcal{H}^0 \oplus \mathcal{H}^-) \to \mathcal{N}$ is continuous. Furthermore the map $m := \tilde{m}|_{B^+} \colon B^+ \to \mathcal{N}$ is a homemorphism.

Part (i) of this Lemma is a reformulation of Proposition 39 in [SW10]. Observe that $\tilde{m}(\hat{w}) = \tilde{m}(\hat{w}^+) = \tilde{m}(\frac{1}{\|\hat{w}^+\|}\hat{w}^+)$. Part (ii) of this Lemma is Proposition 31 in [SW10]. We do not give a proof here. We now define our auxiliary functional Ψ.

Definition 2.6. *Assume (A1), (A2) and (A3). Define $\tilde{\Psi} \colon \mathcal{H}^+ \setminus \{0\} \to \mathbb{R}$, $\tilde{\Psi}(\hat{w}) := \mathcal{I}(\tilde{m}(\hat{w}))$ and $\Psi := \tilde{\Psi}|_{B^+}$.*

Note that we use Ψ since $\mathcal{N}$ may not be a C^1-manifold but S^+ is. The map $\tilde{m}$ may not be C^1, but the composition $\tilde{\Psi} = \mathcal{I} \circ \tilde{m}$ is C^1 by the next Proposition.

Proposition 2.7 (See [SW10])**.** *Assume (A1), (A2) and (A3). Then:*

(i) $\tilde{\Psi} \in C^1(\mathcal{H}^+ \setminus \{0\}, \mathbb{R})$ and $\forall\, \hat{w}, \hat{v} \in \mathcal{H}^+ \setminus \{0\} \colon \; \tilde{\Psi}'(\hat{w})[\hat{v}] = \frac{\|\tilde{m}(\hat{w})^+\|}{\|\hat{w}\|} I'(\tilde{m}(\hat{w}))[\hat{v}]$.

(ii) $\Psi \in C^1(B^+, \mathbb{R})$ and $\forall\, \hat{w}, \hat{v} \in \mathcal{H}^+ \setminus \{0\} \colon \; \Psi'(\hat{w})[\hat{v}] = \|m(\hat{w})^+\| I'(m(\hat{w}))[\hat{v}]$.

(iii) If $(\hat{w}_n)_n$ is a Palais-Smale sequence for Ψ, then $(m(\hat{w}_n))_n$ is a Palais-Smale sequence for $\mathcal{I}$. If $(\hat{u}_n)_n \subset \mathcal{N}$ is a bounded Palais-Smale sequence for $\mathcal{I}$, then $(m^{-1}(\hat{u}_n))_n$ is a Palais-Smale sequence for Ψ.

(iv) $\hat{w}$ is a critical point of Ψ if and only if $m(\hat{w})$ is a nontrivial critical point of $\mathcal{I}$. Moreover $\inf_{B^+} \Psi = \inf_{\mathcal{N}} \mathcal{I}$.

This proposition is a combination of Proposition 32 and Corollary 33 in [SW10]. We finish our preparations with the following Lemma.

Lemma 2.8 (See [SW10]). *Assume* $(A1)$, $(A2)$ *and* $(A3)$. *Then:*

(i) $\exists\, \delta > 0\ \forall\, \hat{w} \in \mathcal{H} \setminus (\mathcal{H}^0 \oplus \mathcal{H}^-)\colon\ \|\tilde{m}(\hat{w})^+\| \geq \delta$.

(ii) For each compact set $K \subset \mathcal{H} \setminus (\mathcal{H}^0 \oplus \mathcal{H}^-)$ *there is some constant* C_K *such that* $\forall\, \hat{w} \in K\colon\ \|\tilde{m}(\hat{w})\| \leq C_K$.

This lemma follows from Proposition 2.3, but the proof is not straightforward. The details can be found in the proof for Theorem 35 in [SW10]. Next we observe some compactness of the nonlinearity. This lemma is not explicitly stated in [SW10], hence we give a proof for it.

Lemma 2.9. *Assume* $(A1)$ *and* $(A3)$. *Let* $v \in L^{p+1}(\Omega, \mathbb{R})$ *be fixed. Then the following map is weakly continuous:*

$$\mathcal{H} \ni \hat{u} \mapsto \int_\Omega f(x, S\hat{u}) v \, \mathrm{d}x \,.$$

Proof. Observe that the map $L^{p+1}(\Omega) \ni u \mapsto f(\cdot, u(\cdot)) \in L^{(p+1)'}(\Omega)$ is continuous and Lipschitz-continuous on $L^{p+1}(\Omega)$-bounded sets since f is a Caratheodory-function and satisfies the growth restriction $(A3)$ part a). The proof of this observation is lengthy, and not very insightful, hence we refer to [Str08] for details. Analogously for any compact set $\widetilde{\Omega} \subset \Omega$ the map $L^{p+1}(\widetilde{\Omega}) \ni u \mapsto f(\cdot, u(\cdot)) \in L^{(p+1)'}(\widetilde{\Omega})$ is continuous. Let $\hat{u}_n \rightharpoonup \hat{u}$ in $\mathcal{H}$ and let $\varepsilon > 0$ be arbitrary. Then by $(A1)$ we know $\sup_n \|f(\cdot, S\hat{u}_n) - f(\cdot, S\hat{u})\|_{L^{(p+1)'}(\Omega)} < \infty$ and for any compact $\widetilde{\Omega} \subset \Omega$ we have $S\hat{u}_n \to S\hat{u}$ in $L^{p+1}(\widetilde{\Omega})$. We choose a compact set $\widetilde{\Omega} \subset \Omega$ so large, that $\sup_n \|f(\cdot, S\hat{u}_n) - f(\cdot, S\hat{u})\|_{L^{(p+1)'}(\Omega)} \cdot \|v\|_{L^{p+1}(\Omega \setminus \widetilde{\Omega})} < \varepsilon$. Then we calculate

$$\begin{aligned}
&\left| \int_\Omega f(x, S\hat{u}_n) v \, \mathrm{d}x - \int_\Omega f(x, u) v \, \mathrm{d}x \right| \leq \int_\Omega |f(x, S\hat{u}_n) - f(x, S\hat{u})||v| \, \mathrm{d}x \\
&\quad = \int_{\widetilde{\Omega}} |f(x, S\hat{u}_n) - f(x, S\hat{u})||v| \, \mathrm{d}x + \int_{\Omega \setminus \widetilde{\Omega}} |f(x, S\hat{u}_n) - f(x, S\hat{u})||v| \, \mathrm{d}x \\
&\quad \leq \|f(\cdot, S\hat{u}_n) - f(\cdot, S\hat{u})\|_{L^{(p+1)'}(\widetilde{\Omega})} \|v\|_{L^{p+1}(\widetilde{\Omega})} + \varepsilon \ \longrightarrow\ \varepsilon .
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the claim is proven. □

We have gathered all technical preparations for the main result of this subsection: We will show how to find a good candidate for the ground state.

Theorem 2.10. *Assume* $(A1)$, $(A2)$, $(A3)$. *Then:*

(i) There is a sequence $(\hat{w}_n)_n \subset B^+$ *such that* $\Psi(\hat{w}_n) \to \inf_{B^+} \Psi$ *and* $\Psi'(\hat{w}_n) \to 0$.

(ii) Setting $\hat{u}_n := m(\hat{w}_n)$ *we have* $(\hat{u}_n)_n \subset \mathcal{N}$, $\mathcal{I}(\hat{u}_n) \to \inf_{\mathcal{N}} I$ *and* $\mathcal{I}'(\hat{u}_n) \to 0$. *Furthermore* $\inf_n \|\hat{u}_n\| > 0$ *and there is some* $\hat{u} \in \mathcal{H}$ *such that* $\hat{u}_n \rightharpoonup \hat{u}$ *in* $\mathcal{H}$ *up to a subsequence.*

(iii) Let $(\hat{u}_n)_n \subset \mathcal{H}$, $\hat{u} \in \mathcal{H}$ *such that* $\hat{u}_n \rightharpoonup \hat{u}$, *and* $\mathcal{I}'(\hat{u}_n) \to 0$. *Then* $\mathcal{I}'(\hat{u}) = 0$.

Proof. (i) Take a minimizing sequence $(\hat{w}_n)_n \subset B^+$ for Ψ. Since $\Psi \in C^1(B^+, \mathbb{R})$ and B^+ is a C^1-manifold, we can use Ekeland's variational principle to assume w.l.o.g. $\Psi'(\hat{w}_n) \to 0$.

(ii) $(\hat{u}_n)_n \subset \mathcal{N}$, $\mathcal{I}(\hat{u}_n) \to \inf_{\mathcal{N}} \mathcal{I}$ and $\mathcal{I}'(\hat{u}_n) \to 0$ are clear by construction of m and Proposition 2.7. The fact that $\inf_n \|\hat{u}_n\| > 0$ follows by Lemma 2.8. Following the lines of the proof for Proposition 36 in [SW10], we see that $(\hat{u}_n)_n$ is bounded in $\mathcal{H}$. Hence there is some $\hat{u} \in \mathcal{H}$ and some subsequence (again denoted by $(\hat{u}_n)_n$), such that $\hat{u}_n \rightharpoonup \hat{u}$ in $\mathcal{H}$.

(iii) Let $\hat{\varphi} \in \mathcal{H}$ be arbitrary. By continuity of $\mathcal{I}'$, $b_{\mathcal{L}}$ and Lemma 2.9 we see

$$\begin{aligned}\mathcal{I}'(\hat{u})[\hat{\varphi}] &= b_{\mathcal{L}}(\hat{u}, \hat{\varphi}) - \int_\Omega f(x, S\hat{u}) S\varphi \,\mathrm{d}x \\ &= \lim_{n\to\infty} \left(b_{\mathcal{L}}(\hat{u}_n, \hat{\varphi}) - \int_\Omega f(x, S\hat{u}_n) S\hat{\varphi} \,\mathrm{d}x \right) = \lim_{n\to\infty} \mathcal{I}'(\hat{u}_n)[\hat{\varphi}] = 0.\end{aligned}$$

Hence $\mathcal{I}'(\hat{u}) = 0$.

□

Theorem 2.10 can be read in the following way: Part (i) guarantees the existence of a minimizing Palais-Smale-sequence for Ψ on B^+. Part (ii) translates it into a weakly convergent Palais-Smale sequence for $\mathcal{I}$ on $\mathcal{N}$, which is in addition bounded away from zero. Part (iii) says that the weak limit of any weakly convergent Palais-Smale-sequence is a critical point. We have not claimed yet, that the limits in (ii) or (iii) are non-trivial. This is the goal of the following subsection, where we provide two different approaches.

2.1.2 Non-Triviality

In this section we provide two settings where we can guarantee the existence of a non-trivial ground state.

2.1.2.1 Case 1: compact non-linearity

In this section we assume that we have a compact non-linearity.

(B) The derivative of the map $\mathcal{H} \ni \hat{u} \mapsto \int_\Omega F(x, S\hat{u}) \,\mathrm{d}x \in \mathbb{R}$ is weakly continuous.

We give two examples.

Proposition 2.11. *Assume* $(A1)$, $(A3)$ *and one of the following conditions:*

(i) $\Omega \subset \mathbb{R}^N$ *is bounded.*

(ii) $\Omega \subset \mathbb{R}^N$ *is unbounded and for each fixed* $\hat{u} \in \mathcal{H}$, $\varepsilon > 0$ *and* $\delta > 0$ *there is some compact* $\widetilde{\Omega} \subset \Omega$ *such that*

$$\sup\left\{ \left\| f\left(x, S(\hat{u} + \hat{h})\right) \right\|_{L^{(p+1)'}(\Omega \setminus \widetilde{\Omega})} \mid \hat{h} \in \mathcal{H}, \left\| S\hat{h} \right\|_{L^{p+1}(\Omega\setminus\widetilde{\Omega})} < \delta \right\} < \varepsilon,$$

$$\sup\left\{ \left\| f\left(x, S(\hat{u} + \hat{h})\right) - f(x, S\hat{u}) \right\|_{L^{(p+1)'}(\Omega \setminus \widetilde{\Omega})} \mid \hat{h} \in \mathcal{H}, \left\| S\hat{h} \right\|_{L^{p+1}(\Omega\setminus\widetilde{\Omega})} < \delta \right\} < \varepsilon.$$

Then assumption (B) *is true.*

An example for the case (ii) is $f(x,u) = \Gamma(x)|u|^{p-1}u$ with $p > 1$ and $\Gamma \in L^\infty(\Omega, \mathbb{R})$, $\lim_{|x|\to\infty} \Gamma(x) = 0$.

Proof. In case of (i) the claim directly follows from assumption $(A1)$. For case (ii) we do a calculation similar to Lemma 2.9. Let $\hat{u}_n \rightharpoonup \hat{u}$ in $\mathcal{H}$, $\varepsilon > 0$ be arbitrary and define $\delta := \sup_n \|S\hat{u}_n - S\hat{u}\|_{L^{p+1}(\Omega)}$. Next we choose some compact $\widetilde{\Omega} \subset \Omega$ so large, such that

$$\sup\left\{\left\|f\left(x, S(\hat{u}+\hat{h})\right) - f(x, S\hat{u})\right\|_{L^{(p+1)'}(\Omega\setminus\widetilde{\Omega})} \mid \hat{h} \in \mathcal{H}, \left\|S\hat{h}\right\|_{L^{p+1}(\Omega\setminus\widetilde{\Omega})} < \delta\right\} \cdot\|S\|_{\mathcal{H}\to L^{p+1}(\Omega)} \overset{!}{<} \varepsilon.$$

Then we calculate for any $\hat{v} \in \mathcal{H}$

$$\begin{aligned}
&\left|\int_\Omega f(x, S\hat{u}_n)S\hat{v}\,\mathrm{d}x - \int_\Omega f(x, S\hat{u})S\hat{v}\,\mathrm{d}x\right| \\
&\quad\le \int_{\widetilde{\Omega}} |f(x, S\hat{u}_n) - f(x, S\hat{u})| \cdot |S\hat{v}|\,\mathrm{d}x + \int_{\Omega\setminus\widetilde{\Omega}} |f(x, S\hat{u}_n) - f(x, S\hat{u})| \cdot |S\hat{v}|\,\mathrm{d}x \\
&\quad\le \|f(x, S\hat{u}_n) - f(x, S\hat{u})\|_{L^{(p+1)'}(\widetilde{\Omega})} \cdot \|S\hat{v}\|_{L^{p+1}(\widetilde{\Omega})} \\
&\qquad + \|f(x, S\hat{u}_n) - f(x, S\hat{u})\|_{L^{(p+1)'}(\Omega\setminus\widetilde{\Omega})} \cdot \|S\hat{v}\|_{L^{p+1}(\Omega\setminus\widetilde{\Omega})} \\
&\quad\le \|f(x, S\hat{u}_n) - f(x, S\hat{u})\|_{L^{(p+1)'}(\widetilde{\Omega})} \cdot \|S\|_{\mathcal{H}\to L^{p+1}(\Omega)}\|\hat{v}\| + \varepsilon \cdot \|\hat{v}\|,
\end{aligned}$$

hence,

$$\begin{aligned}
&\sup_{\|\hat{v}\|=1}\left|\int_\Omega f(x, S\hat{u}_n)S\hat{v}\,\mathrm{d}x - \int_\Omega f(x, S\hat{u})S\hat{v}\,\mathrm{d}x\right| \\
&\qquad\le \|f(x, S\hat{u}_n) - f(x, S\hat{u})\|_{L^{(p+1)'}(\widetilde{\Omega})}\|S\|_{\mathcal{H}\to L^{p+1}(\Omega)} + \varepsilon \to \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the derivative is weakly continuous. □

Theorem 2.12. *Assume* $(A1)$, $(A2)$, $(A3)$ *and* (B). *Then the ground state* c *for equation* (2.1) *is attained, i.e., there is some* $\hat{u}^* \in \mathcal{H} \setminus \{0\}$ *s.t.* $\mathcal{I}'(\hat{u}^*) = 0$ *and* $\mathcal{I}(\hat{u}^*) = c$.

Proof. By Theorem 2.10 part (i) there is some sequence $(\hat{u}_n)_n \subset \mathcal{N}$ and $\hat{u}^* \in \mathcal{H}$ such that $\hat{u}_n \rightharpoonup \hat{u}^*$ in $\mathcal{H}$, $\mathcal{I}(\hat{u}_n) \to c$ and $\mathcal{I}'(\hat{u}_n) \to 0$. We use (B) and follow the lines of the proof of Proposition 36 in [SW10]. Then we obtain that $\mathcal{I}$ satisfies the Palais-Smale condition. Hence $(\hat{u}_n)_n$ has a convergent subsequence (again denoted by $(\hat{u}_n)_n$), i.e., $\hat{u}_n \to \hat{u}$ in $\mathcal{H}$. Since $\inf_n \|\hat{u}_n\| > 0$, the limit $\hat{u}$ is not 0. Continuity of $\mathcal{I}$ and $\mathcal{I}'$ yield: $\hat{u}$ is a non-trivial ground state of $\mathcal{I}$. □

2.1.2.2 Case 2: Cylindrical symmetry

In this subsubsection we assume to work in a cylindrically symmetric setting. The first components are radially symmetric directions (either none or at least 2 but not 1), the middle components denote unbounded directions, where we assume translation invariance of the functional on a grid and the last components denote periodic directions. In our examples the last components will refer to time, when we look for breather solutions.

$(C0)$ Let $N_{rad}, N_{trans}, N_{per} \in \mathbb{N}_0$, $N_{rad} \neq 1$, $N := N_{rad} + N_{trans} + N_{per} \geq 1$. Define $O(N_{rad}) := \{U \in \mathbb{R}^{N_{rad} \times N_{rad}} \mid UU^T = \mathrm{Id}_{N_{rad}}\}$. Let $\zeta_1, \ldots, \zeta_{N_{trans}} \in \mathbb{R}_{>0}$, $\zeta := \mathrm{diag}(\zeta_1, \ldots, \zeta_{N_{trans}}) \in \mathbb{R}^{N_{trans} \times N_{trans}}$. Let $\mathbb{T}_T^{N_{per}}$ denote the Cartesian product of N_{per} toruses with periods $T = (T_1, \ldots, T_{N_{per}}) \in \mathbb{R}_{>0}^{N_{per}}$. Let $\Omega := \mathbb{R}^{N_{rad}} \times \mathbb{R}^{N_{trans}} \times \mathbb{T}_T^{N_{per}}$

$(C1)$ Let $1 \leq p_* < p < p^* < \infty$ and assume that $S\colon \mathcal{H} \hookrightarrow L^{p_*+1}(\Omega, \mathbb{R}) \cap L^{p^*+1}(\Omega, \mathbb{R})$ is continuous. Furthermore there is a sequence of balls $B_j \subset \mathbb{R}^{N_{rad} \times N_{trans}}$, $j \in \mathbb{N}$, such that $\bigcup_j B_j = \mathbb{R}^{N_{rad} \times N_{trans}}$, each point of $\mathbb{R}^{N_{rad} \times N_{trans}}$ is contained in at most N^* balls and there some $C > 0$ such that $\sum_j \|S\hat{u}\|^{p_*+1}_{L^{p^*+1}(B_j \times \mathbb{T}_T^{N_{per}})} \leq C\|\hat{u}\|^{p_*+1}$ for $\hat{u} \in \mathcal{H}$.

$(C2)$ For any $\hat{u} \in \mathcal{H}, k \in \mathbb{Z}^{N_{trans}}, U \in O(N_{rad})$ assume

$$(S\hat{u})(U\cdot, \cdot + \zeta k, \cdot) \in \mathrm{Range}(S) \qquad \text{and} \qquad \left\|S^{-1}\left((S\hat{u})(U\cdot, \cdot + \zeta k, \cdot)\right)\right\| = \|\hat{u}\|,$$

i.e., $\mathcal{H}$ respects the ζ-cylindrical symmetry of Ω.

$(C3)$ For any $\hat{u} \in \mathcal{H}, k \in \mathbb{Z}^{N_{trans}}, U \in O(N_{rad})$ assume

$$\mathcal{I}\left(S^{-1}\left((S\hat{u})(U\cdot, \cdot + \zeta k, \cdot)\right)\right) = \mathcal{I}(\hat{u}),$$

i.e., $\mathcal{I}$ respects the ζ-cylindrical symmetry of Ω.

In our examples we will have the case $p_* = 1$ and $p^* < 2^* - 1$ where 2^* is the critical Sobolev exponent. One key observation is the following variant of P.L. Lions concentration compactness lemma (1984) to obtain a compactness argument. It will tell us, that the mass of our Palais-Smale sequence not just vanishes, but can be found in balls of uniform radii whose centers are possibly converging to infinity. For this we need the technical assumption $(C1)$. If the nonlinearity f in $(A3)$ is cylindrically symmetric in x and $(C2)$ holds, then $(C3)$ is obviously true.

Lemma 2.13. *Assume $(A1)$, $(A3)$, $(C0)$ and $(C1)$. Let $r > 0$, $q \in [p_*, p^*)$ and $(\hat{u}_n)_n \subset \mathcal{H}$ be bounded in $\mathcal{H}$. Assume*

$$M_n := \sup\left\{\|S\hat{u}_n\|_{L^{q+1}(B_r(x,y) \times \mathbb{T}_T^{N_{per}})} \;\middle|\; (x,y) \in \mathbb{R}^{N_{rad}+N_{trans}}\right\} \to 0, \quad \textit{as } n \to \infty.$$

Then $S\hat{u}_n \to 0$ in $L^{t+1}(\Omega)$ for any $t \in (p_*, p^*)$.

The following proof is based on [Wil96] with a generalization inspired by the appendix of [HR19].

Proof. Let $\hat{u} \in \mathcal{H}$. Choose some $s \in (q, p^*)$, which will later be specified later. Defining $\lambda := \frac{(s+1)-(q+1)}{(p^*+1)-(q+1)} \frac{p^*+1}{s+1}$ we see $\frac{1}{s+1} = \frac{1-\lambda}{q+1} + \frac{\lambda}{p^*+1}$. By a corollary of Hölder's inequality we see for any $(x,y) \in \mathbb{R}^{N_{rad}+N_{trans}}$

$$\|S\hat{u}\|_{L^{s+1}(B_r(x,y) \times \mathbb{T}_T^{N_{per}})} \leq \|S\hat{u}\|^{1-\lambda}_{L^{q+1}(B_r(x,y) \times \mathbb{T}_T^{N_{per}})} \|S\hat{u}\|^{\lambda}_{L^{p^*+1}(B_r(x,y) \times \mathbb{T}_T^{N_{per}})}.$$

Observe that $\lambda \to 0 < \frac{p_*+1}{q+1}$ as $s \to q$ and $\lambda \to 1 > \frac{p_*+1}{p^*+1}$ as $s \to p^*$. Hence there is some $s \in (q, p^*)$ such that in addition $\lambda = \frac{p_*+1}{s+1}$. We calculate

$$\int_{B_r(x,y)\times\mathbb{T}_T^{N_{per}}} |S\hat{u}|^{s+1}\,\mathrm{d}(x,y,z) \le \|S\hat{u}\|^{(1-\lambda)(s+1)}_{L^{q+1}(B_r(x,y)\times\mathbb{T}_T^{N_{per}})} \|S\hat{u}\|^{p_*+1}_{L^{p^*+1}(B_r(x,y)\times\mathbb{T}_T^{N_{per}})}.$$

Summing over balls (for further details on this step, see in the proof of Lemma 2.20) and applying assumption $(C1)$ we obtain

$$\int_\Omega |S\hat{u}_n|^{s+1}\,\mathrm{d}(x,y,z) \le M_n^{(1-\lambda)(s+1)} \cdot C\|\hat{u}_n\|^{p_*+1}.$$

Hence $S\hat{u}_n \to 0$ in $L^{s+1}(\Omega)$. If $t \in (p_*, s)$, there is $\mu \in (0,1)$ such that $\frac{1}{t+1} = \frac{1-\mu}{p_*+1} + \frac{\mu}{s+1}$. Again by Hölder's inequality we see

$$\|S\hat{u}_n\|_{L^{t+1}(\Omega)} \le \|S\hat{u}_n\|^{1-\mu}_{L^{p_*+1}(\Omega)} \|S\hat{u}_n\|^{\mu}_{L^{s+1}(\Omega)} \to 0.$$

Analogously if $t \in [s, p^*)$, there is $\mu \in [0,1)$ such that $\frac{1}{t+1} = \frac{1-\mu}{s+1} + \frac{\mu}{p^*+1}$, i.e.,

$$\|S\hat{u}\|_{L^{t+1}(\Omega)} \le \|S\hat{u}_n\|^{1-\mu}_{L^{s+1}(\Omega)} \|S\hat{u}_n\|^{\mu}_{L^{p^*+1}(\Omega)} \to 0.$$

□

Observe that the previous lemma does not need any of our symmetry assumptions. We are now ready to prove the existence of a ground state.

Theorem 2.14. *Assume* $(A1)$, $(A2)$, $(A3)$ *and* $(C0)$, $(C1)$, $(C2)$, $(C3)$. *Then the ground state level* $c = \inf_{\mathcal{N}} \mathcal{I}$ *is attained, i.e., there is some* $\hat{u}^* \in \mathcal{H} \setminus \{0\}$ *s.t.* $\mathcal{I}'(\hat{u}^*) = 0$ *and* $\mathcal{I}(\hat{u}^*) = c$.

Proof. By Theorem 2.10 part (ii) there is some sequence $(\hat{u}_n)_n \subset \mathcal{N}$ such that $\mathcal{I}(\hat{u}_n) \to c$ and $\mathcal{I}'(\hat{u}_n) \to 0$. As in the proof of Theorem 20 in [SW10] we see $S\hat{u}_n \not\to 0$ in $L^{p+1}(\Omega)$. Setting $r = 1$ we obtain by Lemma 2.13 the existence of some constant $\delta > 0$ and points $(x_n, y_n) \in \mathbb{R}^{N_{rad}+N_{trans}}$ such that

$$\forall\, n \in \mathbb{N}: \quad \|S\hat{u}_n\|_{L^{p+1}(B_1(x_n,y_n)\times\mathbb{T}_T^{N_{per}})} \ge \delta.$$

If $N_{rad} = 0$, then we do consider no x-direction. Otherwise we have $N_{rad} \ge 2$. For $r > 2$ we write $A_r := B_{r+2}(0) \setminus B_{r-2}(0)$, i.e., A_r is an annulus centered at 0 with width 4. By rotational symmetry we can observe that there is a constant $c = c(N_{rad}) > 0$ such that for $r > 4$ we have

$$\begin{aligned}\|S\hat{u}_n\|^{p+1}_{L^{p+1}(\Omega)} &\ge \|S\hat{u}_n\|^{p+1}_{L^{p+1}(A_r\times\mathbb{R}^{N_{trans}}\times\mathbb{T}_T^{N_{per}})} \ge r\cdot c\cdot \|S\hat{u}_n\|^{p+1}_{L^{p+1}(B_1(r)\times\mathbb{R}^{N_{trans}}\times\mathbb{T}_T^{N_{per}})} \\ &\ge r\cdot c\cdot \|S\hat{u}_n\|^{p+1}_{L^{p+1}(B_1(r,y_n)\times\mathbb{T}_T^{N_{per}})}.\end{aligned}$$

Since the left hand side is bounded and the right hand side grows linear in r, we see that x_n can not be unbounded, i.e., $\rho_x := \sup_n |x_n| < \infty$. Let $\tilde{y}_n \in \{\zeta k \mid k \in \mathbb{Z}^{N_{trans}}\}$ such that

$$|y_n - \tilde{y}_n| = \min\{|y_n - \zeta k| \;:\; k \in \mathbb{Z}^{N_{trans}}\}.$$

Define $\rho_y := \sup_n |y_n - \tilde{y}_n| \leq \sqrt{N_{trans}} \max_j \zeta_j < \infty$, $\rho_z := \max_j T_j < \infty$ and write $\rho := 1 + \max\{\rho_x, \rho_y, \rho_z\}$. This number will be the radius of a ball around the origin where we do not lose L^{p+1}-mass. We define our new sequence $\hat{v}_n := S^{-1}\left((S\hat{u}_n)(U\cdot, \cdot + Sk, \cdot)\right)$. Note that possibly $\hat{v}_n \notin \mathcal{N}$, but we will prove that its limit will be the desired ground state. Since by assumption $(A2)$ and $(C3)$ $\mathcal{I}$ and $\|\cdot\|$ are invariant under such shifts, we still have $\mathcal{I}(\hat{v}_n) \to c$, $\mathcal{I}'(\hat{v}_n) \to 0$ and $\|\hat{v}_n\| = \|\hat{u}_n\|$. Hence $(\hat{v}_n)_n$ is a minimizing Palais-Smale sequence for $\mathcal{I}$ and is bounded in $\mathcal{H}$. Thus there is some $\hat{v}^* \in H$ and some subsequence (again subscripted by n) such that $\hat{v}_n \rightharpoonup \hat{v}^*$ in $\mathcal{H}$. Applying Theorem 2.10 part (iii) we see $\mathcal{I}'(\hat{v}^*) = 0$. It remains to show $\hat{v}^* \neq 0$ and $\mathcal{I}(\hat{v}^*) = c$. Observe that by our construction

$$\begin{aligned}\delta &\leq \|S\hat{u}_n\|_{L^{p+1}(B_1(x_n,y_n)\times\mathbb{T}_T^{N_{per}})} = \|S\hat{v}_n\|_{L^{p+1}(B_1(x_n,y_n-\tilde{y}_n)\times\mathbb{T}_T^{N_{per}})} \\ &\leq \|S\hat{v}_n\|_{L^{p+1}(B_\rho(0,0)\times\mathbb{T}_T^{N_{per}})}.\end{aligned}$$

Using the local compactness in assumption $(A1)$ we see $\hat{v}^* \neq 0$. Hence $\hat{v}^* \in \mathcal{N}$ and $\mathcal{I}(\hat{v}^*) \geq c$. Observe that the local compactness in assumption $(A1)$ also implies $S\hat{v}_n \to S\hat{v}^*$ pointwise almost everywhere on Ω after taking a suitable subsequence (again subscripted by n). By Fatou's Lemma and Proposition 2.1 we see

$$\begin{aligned}c &= \liminf_n \mathcal{I}(\hat{v}_n) = \liminf_n \int_\Omega \frac{1}{2} f(x, S\hat{v}_n) S\hat{v}_n - F(x, S\hat{v}_n)\,\mathrm{d}(x,y,z) \\ &\geq \int_\Omega \frac{1}{2} f(x, S\hat{v}^*) S\hat{v}^* - F(x, S\hat{v}^*)\,\mathrm{d}(x,y,z) = \mathcal{I}(S\hat{v}^*),\end{aligned}$$

i.e., $\mathcal{I}(S\hat{v}^*) = c$. □

2.2 Abstract spectral tools

In this section we give a toolbox to construct the Hilbert space $\mathcal{H}$ suitable to an operator L of the form

$$L = V(x)\partial_t^2 - \Delta, \qquad (x,t) \in \Omega \times \mathbb{T}_T$$

for some potential V independent of the variable t and Δ denoting the Laplacian acting only on the variables x. Shortly recall, we refer to the variables $x \in \Omega$ as *space* and refer to the periodic variable $t \in \mathbb{T}_T$ as *time*, where $\mathbb{T}_T$ denotes the one-dimensional torus of period T. Since we consider sign-changing potentials V, the bilinear form

$$b_L(u,v) = \int_{\Omega\times\mathbb{T}_T} -V(x) u_t \overline{v}_t + u_x \overline{v}_x \,\mathrm{d}(x,t)$$

formally corresponding to the operator L is neither bounded from above or below, hence we cannot use Friedrich's extension (see e.g. [RS10]) to construct L and its domain as a self-adjoint operator from the bilinear form, as mentioned in the Introduction. One key in our strategy is to decompose L by Fourier decomposition in time. This is formally done by the calculation

$$\begin{aligned}(Lu)(x,t) &= \left(V(x)\partial_t^2 - \Delta\right) \sum_k \hat{u}_k(x) e_k(t) \\ &= \sum_k \left(-\Delta \hat{u}_k(x) - k^2\omega^2 V(x)\hat{u}_k(x)\right) e_k(t),\end{aligned}$$

where

$$e_k(t) := \frac{1}{\sqrt{T}} e^{i\omega k t}, \qquad \omega := \frac{2\pi}{T}, \qquad \hat{u}_k(x) := \langle u(x,\cdot), e_k \rangle_{L^2(\mathbb{T}_T)},$$

i.e., $(e_k)_k$ denotes the $L^2(\mathbb{T}_T)$-orthonormal Fourier base on $\mathbb{T}_T$. The advantage of this decomposition is, that now we can analyze the operators $L_k = -\Delta - k^2\omega^2 V(x)$. Observe that $V(x)$ is not in front of derivatives and for suitable V the operator L_k is self-adjoint on $L^2(\Omega)$. On the other hand, we have to deal with countably many such operators and have to find a proper domain for the new bilinear form corresponding to $\mathcal{L} = \bigoplus_k L_k$. In Section 2.2 we use the self-adjointness of the operators L_k to construct a norm suitable for assumption $(A2)$. The embedding results in Section 2.2.3 will be the crucial ingredients to verify assumption $(A1)$. This whole section is inspired by the techniques in [HR19], but we formulate the procedure in a more general setting. Later we apply these results to the examples in Section 2.3. There we will consider the $L^2(\mathbb{R}^N)$-self-adjoint operators $L_k := -\Delta - k^2\omega^2 V(x)$ with the corresponding hermitian sesquilinear forms $b_{L_k}(\hat{u}_k, \hat{v}_k) = \int_{\mathbb{R}^N} \nabla \hat{u}_k \overline{\nabla \hat{v}_k} - k^2\omega^2 V(x)\hat{u}_k \overline{\hat{v}_k} \, dx$ for $\hat{u}_k, \hat{v}_k \in H^1(\mathbb{R}^N)$.

2.2.1 Decomposition of a Hilbert space by a self-adjoint operator

We first consider only one operator and list some known results on self-adjoint operators including deeper results on functional calculus. We omit many calculations and auxiliary constructions, for reference, one can find details and further results in [RS10]. Let $(X, \langle\cdot,\cdot\rangle_X)$ be a complex (or real) Hilbert space, $(H, \langle\cdot,\cdot\rangle_H)$ be a complex (or real) Hilbert space and a subspace of X, $A\colon \mathcal{D}(A) \to X$ be self-adjoint on X and $b_A\colon H \times H \to \mathbb{C}$ be a closed, hermitian sesquilinear form such that

$$\forall\, u \in \mathcal{D}(A), v \in H\colon \quad b_A(u,v) = \langle Au, v\rangle_X .$$

Please note that: For given A and X, functional calculus for self-adjoint operators uniquely defines b_A and H. For given X and a lower bounded, hermitian, closed sesquilinear form b_A, Friedrich's extension theorem uniquely defines H and A.
In the following we assume in addition: If $0 \in \sigma(A)$, then it is an eigenvalue of finite multiplicity and isolated from the rest of $\sigma(A)$, i.e., there is some $\varepsilon > 0$ such that $(-\varepsilon,\varepsilon) \cap \sigma(A) \subset \{0\}$. Using the functional calculus for self-adjoint operators, we obtain: For each $\lambda \in \mathbb{R}$ there exists a projection $P^\lambda\colon X \to X$ which is uniquely determined by

$$\forall\, u, v \in H\colon \quad b_A(u,v) = \int_{\mathbb{R}} \lambda \, d\left\langle P^\lambda u, v\right\rangle_X .$$

Note that by our additional assumption on the spectrum, P^0 has finite dimensional range, i.e., it is compact. Using these projections we define the positive and negative projectors $P^\pm, P^\pm\colon H \to H^\pm$ by

$$\begin{aligned} P^+ u := u^+ &:= \int_\varepsilon^\infty 1 \, d\left\langle P^\lambda u, \cdot\right\rangle_X & \qquad H^+ &:= P^+ H, \\ P^- u := u^- &:= \int_{-\infty}^{-\varepsilon} 1 \, d\left\langle P^\lambda u, \cdot\right\rangle_X & H^- &:= P^- H, \\ u^0 &:= P^0 u & H^0 &:= P^0 H. \end{aligned}$$

where $\varepsilon > 0$ is chosen as above. Last we define the sesquilinear-form corresponding to $|A|$:

$$b_{|A|}\colon H \times H \to \mathbb{C}, \quad b_{|A|}(u,v) := \int_{\mathbb{R}} |\lambda| \,\mathrm{d}\left\langle P^{\lambda} u, v \right\rangle_X .$$

This sesquilinear form is a scalar product on $H^+ \oplus H^-$ which is equivalent to $\langle \cdot,\cdot \rangle_H$. The sesquilinear form

$$\langle u,v \rangle_{|A|} := b_{|A|}(u,v) + \left\langle u^0, v^0 \right\rangle_H$$

is an equivalent scalar product to $\langle \cdot,\cdot \rangle_H$ on $H = H^+ \oplus H^0 \oplus H^-$. By construction we have

$$\begin{aligned} \forall\, u,v \in H\colon \quad & b_A\left(u^+,v^+\right) = b_{|A|}\left(u^+,v^+\right), \qquad b_A\left(u^-,v^-\right) = -b_{|A|}\left(u^-,v^-\right), \\ & b_A(u,u) = b_{|A|}\left(u^+,u^+\right) - b_{|A|}\left(u^-,u^-\right) = \left\langle u^+,u^+ \right\rangle_{|A|} - \left\langle u^-,u^- \right\rangle_{|A|} . \end{aligned}$$

Furthermore we can calculate for any $u,v \in H$

$$\begin{aligned} b_{|A|}(u,v) &= b_A\left(u^+,v^+\right) - b_A\left(u^-,v^-\right) \\ &= b_A\left(u-u^-, v-v^-\right) - b_A\left(u^-,v^-\right) \\ &= b_A(u,v) - b_A\left(u,v^-\right) - b_A\left(u^-,v\right), \\ b_{|A|}(u,u) &= b_A(u,u) - 2\,\mathrm{Re}\left(b_A\left(u,u^-\right)\right) . \end{aligned}$$

2.2.2 Abstract construction of a sequence space

We now consider at most countably many operators. Let $K \subset \mathbb{Z}$ and for all $k \in K$ let X_k be a Hilbert space, $H_k \subset X_k$ be a subspace and L_k be a self-adjoint operator on X_k such that: if $0 \in \sigma(L_k)$, then it is an eigenvalue of finite multiplicity and isolated from the rest of $\sigma(L_k)$. As in section 2.2.1: $\langle \cdot,\cdot \rangle_{|L_k|} := b_{|L_k|}(\cdot,\cdot) + \left\langle P_k^0 \cdot, P_k^0 \cdot \right\rangle$ is a scalar product equivalent to the scalar product on H_k. We now define the composite Hilbert spaces

$$\begin{aligned} \mathcal{H} &:= l^2\left(K, \bigoplus_{k\in K} (H_k, \langle \cdot,\cdot \rangle_{|L_k|})\right) := \left\{ \hat{u} = (\hat{u}_k)_k \in \bigoplus_{k\in K} H_k \;\middle|\; \sum_{k\in K} \langle \hat{u}_k, \hat{u}_k \rangle_{|L_k|} < \infty \right\}, \\ \mathcal{X} &:= l^2\left(K, \bigoplus_{k\in K} (X_k, \langle \cdot,\cdot \rangle_{X_k})\right) := \left\{ \hat{u} = (\hat{u}_k)_k \in \bigoplus_{k\in K} X_k \;\middle|\; \sum_{k\in K} \|\hat{u}_k\|_{X_k}^2 < \infty \right\}, \\ & \langle \hat{u}, \hat{v} \rangle_{\mathcal{H}} := \sum_{k\in K} \langle \hat{u}_k, \hat{v}_k \rangle_{|L_k|}, \quad \|\hat{u}\|_{\mathcal{H}}^2 := \sum_{k\in K} \langle \hat{u}_k, \hat{u}_k \rangle_{|L_k|}, \\ & \langle \hat{u}, \hat{v} \rangle_{\mathcal{X}} := \sum_{k\in K} \langle \hat{u}_k, \hat{v}_k \rangle_{X_k}, \quad \|\hat{u}\|_{\mathcal{X}}^2 := \sum_{k\in K} \|\hat{u}_k\|_{X_k}^2. \end{aligned}$$

Then the composite sesquilinear form

$$b_{\mathcal{L}}\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \quad b_{\mathcal{L}}(\hat{u},\hat{v}) := \sum_{k\in K} b_{L_k}(\hat{u}_k, \hat{v}_k)$$

is well defined and hermitian. Moreover the operator

$$\mathcal{D}(\mathcal{L}) := \bigoplus_{k\in K} \mathcal{D}(L_k), \qquad \mathcal{L}\hat{u} := (L_k \hat{u}_k)_k$$

is self-adjoint on $\mathcal{X}$, since each L_k is self-adjoint on X_k, and we have

$$\forall\, \hat{u} \in \mathcal{D}(\mathcal{L}), \hat{v} \in \mathcal{H}: \quad b_{\mathcal{L}}\left(\hat{u}, \hat{v}\right) = \langle \mathcal{L}\hat{u}, \hat{v}\rangle_{\mathcal{X}}\,.$$

Furthermore we have the projectors

$$\begin{aligned} P^{\pm}\colon \mathcal{H} \to \mathcal{H}^{\pm}, \quad P^{\pm}\hat{u} &:= \hat{u}^{\pm} := \left(P_k^{\pm}\hat{u}_k\right)_k, & \mathcal{H}^{\pm} &:= P^{\pm}\mathcal{H},\\ P^{0}\colon \mathcal{H} \to \mathcal{H}^{0}, \quad P^{0}\hat{u} &:= \hat{u}^{0} := \left(P_k^{0}\hat{u}_k\right)_k, & \mathcal{H}^{0} &:= P^{0}\mathcal{H}, \end{aligned}$$

i.e., we have an orthogonal decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^0 \oplus \mathcal{H}^-$ and $b_{\mathcal{L}}$ is positive definite on $\mathcal{H}^+$, $b_{\mathcal{L}}$ is negative definite on $\mathcal{H}^-$ and $b_{\mathcal{L}}|_{\mathcal{H}^0\times\mathcal{H}^0} \equiv 0$. Furthermore

$$\forall\, \hat{u} \in \mathcal{H}: \quad b_{\mathcal{L}}\left(\hat{u}, \hat{u}\right) = b_{\mathcal{L}}\left(\hat{u}^+, \hat{u}^+\right) - b_{\mathcal{L}}\left(\hat{u}^-, \hat{u}^-\right) = \left\|\hat{u}^+\right\|_{\mathcal{H}}^2 - \left\|\hat{u}^-\right\|_{\mathcal{H}}^2.$$

We observe that the sequence space $\mathcal{H}$ as domain for the sesquilinear form of a decomposed operator $\mathcal{L} = \bigoplus_k L_k$ with a structure suitable to assumption $(A2)$ can always be constructed if for any $k \in K$ we have: if $0 \in \sigma(L_k)$, then it is an eigenvale of finite multiplicity and isolated from the rest of $\sigma(L_k)$.

We recapitulate what we have achieved up to now and how we will continue: Formally given an operator $L = V(x)\partial_t^2 - \Delta$ for functions on $(x,t) \in \Omega \times \mathbb{T}_T$, we can formally decompose L by Fourier series in time into $L_k = -\Delta - k^2\omega^2 V(x)$. For these operators we can rigorously write down the sesquilinear forms $b_{L_k}\left(\hat{u}_k, \hat{v}_k\right) = \int_{\mathbb{R}^N} \nabla\hat{u}_k\overline{\nabla\hat{v}_k} - k^2\omega^2 V(x)\hat{u}_k\overline{\hat{v}_k}\,\mathrm{d}x$ and a corresponding domain. For suitable potentials V, these sesquilinear forms are closed, hermitian and semi-bounded, and hence L_k is self-adjoint. Using functional calculus, we construct new scalar products $\langle\cdot,\cdot\rangle_{|L_k|}$. Now we can write down the sequence space $\mathcal{H}$ such that the sesquilinear form $b_{\mathcal{L}}$ is well defined and hermitian and we can even write down the self-adjoint operator $\mathcal{L}$. For sufficiently regular functions u, v we can expect $b_{\mathcal{L}}\left(\hat{u}, \hat{v}\right) = b_L\left(u, v\right)$, where $u(x,t) = \sum_k \hat{u}_k(x)e_k(t)$ and for v respectively. For the existence of ground states in the examples, we do not use the operator $\mathcal{L}$ and we only use the sesquilinear form $b_{\mathcal{L}}$ and the Hilbert space $\mathcal{H}$. We did not claim, how the space $\mathcal{H}$ exactly looks like, except the fact that it is a subset of $\bigoplus_{k\in K} H_k$. In general this is a very hard task and only in the example Section 2.3.1 we can characterize $\mathcal{H}$ precisely as a function space H. In the following section we will prove an embedding of $\mathcal{H}$ into $L^{p+1}(\Omega \times \mathbb{T}_T)$, which yields sufficient knowledge to apply our abstract results of Section 2.1 in the other examples in Secion 2.3.2 and Section 2.3.3.

2.2.3 Embeddings by spectral information for wave-like operators

As mentioned in the Introduction this section is again inspired by [HR19]. We do not consider specific examples but give the more general toolbox mentioned in the beginning. Observe that here we mostly work on complex spaces. In the application later we will apply the results on real spaces.
We start with an inequality concerning the question: "Having $(-c|k|^a, c|k|^a) \in \sigma(L_k)^C$ for $L_k = -\Delta - \omega^2k^2V(x)$, how can we estimate $\langle\cdot,\cdot\rangle_{|L_k|}$ from below using $\|\nabla\hat{u}\|_{L^2(\Omega)}$?" We keep track of constants and in addition we keep different possible structures for V in mind, since $V \in L^\infty(\Omega, \mathbb{R})$ and $V(x) = -\alpha + \beta\delta_0(x)$ behave differently.

Theorem 2.15. *Let $(X, \langle\cdot,\cdot\rangle_X)$ be a Hilbert space, $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space and a subspace of X, $A\colon \mathcal{D}(A) \to X$ be self-adjoint on X and $b_A\colon H \times H \to \mathbb{C}$ be a closed, hermitian bilinear form such that*

$$\forall\, u \in \mathcal{D}(A), v \in H\colon \quad b_A\,(u,v) = \langle Au, v\rangle_X\,.$$

Furthermore let $0 \notin \sigma(A)$ and define $\rho := \operatorname{dist}(0, \sigma(A)) > 0$. Assume in addition $b_A = b_D + b_V$ for hermitian sesquilinear forms b_D, b_V such that $H \subset \mathcal{D}(b_D) \cap \mathcal{D}(b_V)$. Then:

(i) If there is a constant $C_V > 0$ such that

$$\forall\, \varepsilon > 0, u \in H\colon \quad |b_V\,(u,u)| \le C_V \cdot \left(\varepsilon b_D\,(u,u) + \frac{1}{4\varepsilon}\|u\|_X^2\right),$$

then

$$\forall\, u \in H\colon \quad b_{|A|}\,(u,u) \ge \frac{\frac{\rho}{C_V^2}}{1 + 2\frac{\rho}{C_V^2}} \cdot \left(b_D\left(u^+,u^+\right) + b_D\left(u^-,u^-\right)\right).$$

(ii) If there is some $C_V > 0$ such that

$$\forall\, u \in H\colon \quad |b_V\,(u,u)| \le C_V\|u\|_X^2,$$

then

$$\forall\, u \in H\colon \quad b_{|A|}\,(u,u) \ge \frac{1}{1 + \frac{C_V}{\rho}}\left(b_D\left(u^+,u^+\right) + b_D\left(u^-,u^-\right)\right).$$

Case (i) refers to δ-potentials, since $|u(0)|^2 = \|\delta_0(x)u\|_{L^2(\mathbb{R})}^2 \le \varepsilon\|\nabla u\|_{L^2(\mathbb{R})}^2 + \frac{1}{4\varepsilon}\|u\|_{L^2(\mathbb{R})}^2$ (e.g. c.f. [HR19]), and case (ii) refers to bounded potentials V, since $\|V(x)u\|_{L^2(\mathbb{R})}^2 \le \|V\|_{L^\infty(\mathbb{R})}\|u\|_{L^2(\mathbb{R})}^2$ for bounded functions V.

Proof. For a self-adjoint and semi-bounded operator $B\colon \mathcal{D}(V) \to X$ we have by [RS10]

$$\inf_{u\in\mathcal{D}(b_V)} \frac{b_V\,(u,u)}{\|u\|_X^2} = \inf\sigma(C).$$

We apply this to the positive operators A^+ and $-A^-$ defined as in Section 2.2.1. Hence

$$\inf_{u\in H^+\setminus\{0\}} \frac{b_A\,(u^+,u^+)}{\|u\|_X^2} = \inf_{u\in H^+\setminus\{0\}} \frac{b_{A^+}\,(u,u)}{\|u\|_X^2} \ge \rho,$$
$$\inf_{u\in H^-\setminus\{0\}} -\frac{b_A\,(u^-,u^-)}{\|u\|_X^2} = \inf_{u\in H^-\setminus\{0\}} \frac{b_{-A^-}\,(u,u)}{\|u\|_X^2} \ge \rho.$$

Observe that $H^0 = \{0\}$, since $0 \notin \sigma(A)$. We now directly obtain for all $u \in H$

$$b_{|A|}\,(u,u) = b_A\left(u^+,u^+\right) - b_A\left(u^-,u^-\right) \ge \rho\cdot\left(\left\|u^+\right\|_X^2 + \left\|u^-\right\|_X^2\right) = \rho\|u\|_X^2.$$

(i) We start considering only u^+. Let $\mu \in (0,1)$ be arbitrary. If

$$b_D\left(u^+,u^+\right) + \frac{1}{1-\mu} b_V\left(u^+,u^+\right) \geq 0,$$

then

$$b_A\left(u^+,u^+\right) = b_D\left(u^+,u^+\right) + b_V\left(u^+,u^+\right) \geq \mu b_D\left(u^+,u^+\right).$$

If

$$b_D\left(u^+,u^+\right) + \frac{1}{1-\mu} b_V\left(u^+,u^+\right) \leq 0,$$

then choosing $\varepsilon := \frac{1-\mu}{2C_V}$ we see

$$\begin{aligned} b_D\left(u^+,u^+\right) &\leq -\frac{1}{1-\mu} b_V\left(u^+,u^+\right) \leq \frac{C_V}{1-\mu}\left(\varepsilon b_D\left(u^+,u^+\right) + \frac{1}{4\varepsilon}\left\|u^+\right\|_X^2\right) \\ \Rightarrow \quad b_D\left(u^+,u^+\right) &\leq \frac{C_V}{1-\mu} \cdot \frac{4C_V}{4(1-\mu)}\left\|u^+\right\|_X^2 = \left(\frac{C_V}{1-\mu}\right)^2 \left\|u^+\right\|_X^2. \end{aligned}$$

If $b_D\left(u^+,u^+\right) > 0$ we conclude

$$\frac{b_{|A|}\left(u^+,u^+\right)}{b_D\left(u^+,u^+\right)} = \frac{b_A\left(u^+,u^+\right)}{\left\|u^+\right\|_X^2} \cdot \frac{\left\|u^+\right\|_X^2}{b_D\left(u^+,u^+\right)} \geq \rho \cdot \left(\frac{1-\mu}{C_V}\right)^2,$$

and hence

$$\forall\, \mu \in (0,1), u \in H\colon \quad b_{|A|}\left(u^+,u^+\right) \geq \min\left\{\mu, \frac{\rho}{C_V^2}(1-\mu)^2\right\} b_D\left(u^+,u^+\right).$$

Obviously this is also true if $b_D\left(u^+,u^+\right) \leq 0$. Maximizing in $\mu \in (0,1)$ we obtain

$$\forall\, u \in H\colon \quad b_{|A|}\left(u^+,u^+\right) \geq \frac{2\frac{\rho}{C_V^2}}{1 + 2\frac{\rho}{C_V^2} + \sqrt{1 + 4\frac{\rho}{C_V^2}}} b_D\left(u^+,u^+\right).$$

We next consider u^-. By construction of A^- we have

$$b_D\left(u^-,u^-\right) + b_V\left(u^-,u^-\right) = b_A\left(u^-,u^-\right) \leq -\rho\left\|u^-\right\|_X^2.$$

Setting $\varepsilon := \frac{1}{2C_V}$ we obtain

$$\begin{aligned} b_D\left(u^-,u^-\right) &\leq -b_V\left(u^-,u^-\right) - \rho\left\|u^-\right\|_X^2 \\ &\leq C_V \cdot \left(\varepsilon b_D\left(u^-,u^-\right) + \frac{1}{4\varepsilon}\left\|u^-\right\|_X^2\right) + \rho\left\|u^-\right\|_X^2 \\ \Rightarrow \quad b_D\left(u^-,u^-\right) &\leq \left(C_V^2 + 2\rho\right)\left\|u^-\right\|_X^2 \end{aligned}$$

If $b_D\left(u^-,u^-\right) > 0$ we see as before

$$\frac{b_{|A|}\left(u^-,u^-\right)}{b_D\left(u^-,u^-\right)} = -\frac{b_A\left(u^-,u^-\right)}{\left\|u^-\right\|_X^2} \cdot \frac{\left\|u^-\right\|_X^2}{b_D\left(u^-,u^-\right)} \geq \rho \cdot \frac{1}{C_V^2 + 2\rho} = \frac{\frac{\rho}{C_V^2}}{1 + 2\frac{\rho}{C_V^2}},$$

and hence

$$\forall\, u \in H\colon \quad b_{|A|}\left(u^-, u^-\right) \geq \frac{\frac{\rho}{C_V^2}}{1 + 2\frac{\rho}{C_V^2}} b_D\left(u^-, u^-\right).$$

Obviously this is again true if $b_D\left(u^-, u^-\right) \leq 0$. Putting all together we obtain for any $u \in H$:

$$\begin{aligned} b_{|A|}(u, u) &= b_{|A|}\left(u^+, u^+\right) + b_{|A|}\left(u^-, u^-\right) \\ &\geq \frac{2\frac{\rho}{C_V^2}}{1 + 2\frac{\rho}{C_V^2} + \sqrt{1 + 4\frac{\rho}{C_V^2}}} b_D\left(u^+, u^+\right) + \frac{\frac{\rho}{C_V^2}}{1 + 2\frac{\rho}{C_V^2}} b_D\left(u^-, u^-\right) \\ &\geq \frac{\frac{\rho}{C_V^2}}{1 + 2\frac{\rho}{C_V^2}} \left(b_D\left(u^+, u^+\right) + b_D\left(u^-, u^-\right)\right), \end{aligned}$$

since

$$\frac{2r}{1 + 2r + \sqrt{1 + 4r}} \geq \frac{r}{1 + 2r} \qquad \text{for } \; r > 0.$$

(ii) We follow the same strategy as before. Consider u^+ and $\mu \in (0, 1)$. If

$$b_D\left(u^+, u^+\right) + \frac{1}{1 - \mu} b_V\left(u^+, u^+\right) \geq 0,$$

then

$$b_{|A|}\left(u^+, u^+\right) = b_D\left(u^+, u^+\right) + b_V\left(u^+, u^+\right) \geq \mu b_D\left(u^+, u^+\right).$$

If

$$b_D\left(u^+, u^+\right) + \frac{1}{1 - \mu} b_V\left(u^+, u^+\right) \leq 0,$$

then

$$b_D\left(u^+, u^+\right) \leq -\frac{1}{1 - \mu} b_V\left(u^+, u^+\right) \leq \frac{C_V}{1 - \mu} \|u^+\|_X^2.$$

If $b_D\left(u^+, u^+\right) > 0$ we conclude

$$\frac{b_{|A|}\left(u^+, u^+\right)}{b_D\left(u^+, u^+\right)} = \frac{b_A\left(u^+, u^+\right)}{\|u^+\|_X^2} \cdot \frac{\|u^+\|_X^2}{b_D\left(u^+, u^+\right)} \geq \rho \cdot \frac{1 - \mu}{C_V},$$

and hence

$$\forall\, \mu \in (0, 1), u \in H\colon \quad b_{|A|}\left(u^+, u^+\right) \geq \min\left\{\mu, \frac{\rho}{C_V}(1 - \mu)\right\} b_D\left(u^+, u^+\right).$$

Obviously this is also true if $b_D\left(u^+, u^+\right) \leq 0$. Maximizing in $\mu \in (0, 1)$ we obtain

$$\forall\, u \in H\colon \quad b_A\left(u^+, u^+\right) \geq \frac{1}{1 + \frac{C_V}{\rho}} b_D\left(u^+, u^+\right).$$

We next consider u^-. By construction of A^- we have

$$b_D\left(u^-, u^-\right) + b_V\left(u^-, u^-\right) = b_A\left(u^-, u^-\right) \leq -\rho \left\|u^-\right\|_X^2 \leq 0.$$

Thus

$$b_D\left(u^-, u^-\right) \leq C_V \cdot \left\|u^-\right\|_X^2.$$

If $b_D\left(u^-, u^-\right) > 0$ we see as before

$$\frac{b_{|A|}\left(u^-, u^-\right)}{b_D\left(u^-, u^-\right)} = \frac{b_{-A}\left(u^-, u^-\right)}{\|u^-\|_X^2} \cdot \frac{\|u^-\|_X^2}{b_D\left(u^-, u^-\right)} \geq \rho \cdot \frac{1}{C_V},$$

and hence

$$\forall\, u \in H\colon \quad b_{|A|}\left(u^-, u^-\right) \geq \frac{1}{0 + \frac{C_V}{\rho}} b_D\left(u^-, u^-\right) \geq \frac{1}{1 + \frac{C_V}{\rho}} b_D\left(u^-, u^-\right).$$

Obviously this is again true if $b_D\left(u^-, u^-\right) \leq 0$. Putting all together we obtain for any $u \in H$

$$b_{|A|}(u, u) = b_{|A|}\left(u^+, u^+\right) + b_{|A|}\left(u^-, u^-\right) \geq \frac{1}{1 + \frac{C_V}{\rho}} \left(b_D\left(u^+, u^+\right) + b_D\left(u^-, u^-\right)\right),$$

i.e., we have proven the claim. □

The next theorem embeds a sequence space into L^p-spaces for p less than some critical exponent p^* using the Fourier reconstruction operator. Here we obtain an explicit formula for p^*. Moreover the embedding is locally compact.

Theorem 2.16. *Let $n \in \mathbb{N}$, $a \in (0, 2)$, $b > 0$ such that $a + b \geq 2$. Define*

$$\hat{H} := \left\{ \hat{u} \in \left(H^1(\mathbb{R}^n)\right)^{\mathbb{Z}\setminus\{0\}} \;\middle|\; \|\hat{u}\|_{\hat{H}} < \infty \right\},$$
$$\|\hat{u}\|_{\hat{H}}^2 := \sum_{k \in \mathbb{Z}\setminus\{0\}} |k|^a \|\hat{u}_k\|_{L^2(\mathbb{R}^n)}^2 + |k|^{-b} \|\nabla \hat{u}_k\|_{L^2(\mathbb{R}^n)}^2.$$

Furthermore let $p^* := \frac{2n(a+b)+4}{n(a+b)+2-2a}$. Define the Fourier reconstruction operator

$$\left(S\hat{u}\right)(x, t) := \sum_{k \in \mathbb{Z}\setminus\{0\}} \hat{u}_k(x) e_k(t) \qquad \textit{with} \;\; e_k(t) := \frac{1}{\sqrt{T}} \mathrm{e}^{\mathrm{i}\omega k t}, \quad \omega := \frac{2\pi}{T}.$$

Then for $p \in [2, p^*)$: $S\colon \hat{H} \hookrightarrow L^p(\mathbb{R}^n \times \mathbb{T}_T)$ is continuous and $S\colon \hat{H} \hookrightarrow L^p(\tilde{\Omega} \times \mathbb{T}_T)$ is compact for any compact set $\tilde{\Omega} \subset \mathbb{R}^n$.

Proof. We write the continuous Fourier transform on $\mathbb{R}^n$ as follows

$$(\mathcal{F}u)(\xi) := \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} u(x) \mathrm{e}^{-\mathrm{i}\langle x, \xi \rangle}\, \mathrm{d}x \qquad \text{for} \;\; u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),$$

continuously extend $\mathcal{F}$ to $\mathcal{F}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and we define

$$\mathcal{F}\hat{u} := \left(\mathcal{F}\hat{u}_k\right)_k \qquad \text{for} \;\; \hat{u} \in \left(L^2(\mathbb{R}^n)\right)^{\mathbb{Z}}$$

$$\|\hat{u}\|_{L^p(\mathbb{R}^n\times\mathbb{Z})} := \begin{cases} \left(\sum_{k\in\mathbb{Z}} \|\hat{u}_k\|^p_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{p}}, & p\in[1,\infty), \\ \sup_k \|\hat{u}_k\|_{L^\infty(\mathbb{R}^n)}, & p=\infty. \end{cases}$$

Often one writes $\|\hat{u}\|_{L^p(\mathbb{R}^n\times\mathbb{Z})}$ as $\|\hat{u}\|_{L^p_k(\mathbb{Z})L^p_x(\mathbb{R}^n)}$. We continue in two steps.
Step 1: A general interpolation argument.
By Parseval's and Plancherel's identity we know

$$\|S\hat{u}\|_{L^2(\mathbb{R}^n\times\mathbb{T}_T)} = \|\hat{u}\|_{L^2(\mathbb{R}^n\times\mathbb{Z})} = \|\mathcal{F}\hat{u}\|_{L^2(\mathbb{R}^n\times\mathbb{Z})} \qquad \text{for } \hat{u}\in\left(L^2(\mathbb{R}^n)\right)^{\mathbb{Z}}.$$

By a direct calculation we see: if $\mathcal{F}\hat{u}\in l^1\left(\mathbb{Z},L^1(\mathbb{R}^n)\right)$, then

$$\begin{aligned}\|S\hat{u}\|_{L^\infty(\mathbb{R}^n\times\mathbb{T}_T)} &= \left\|\sum_k \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\mathcal{F}\hat{u}_k(x)\mathrm{e}^{\mathrm{i}\langle x,\cdot\rangle}\,\mathrm{d}x\right\|_{L^\infty(\mathbb{R}^n)} \\ &\le \sum_k \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}|\mathcal{F}\hat{u}_k(x)|\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}^n}\|\mathcal{F}\hat{u}\|_{L^1(\mathbb{R}^n\times\mathbb{Z})} \le \|\mathcal{F}\hat{u}\|_{L^1(\mathbb{R}^n\times\mathbb{Z})}.\end{aligned}$$

Using the Riesz-Thorin-interpolation and the Hölder conjugate $p' = \frac{p}{p-1}$ we see

$$\forall\, p\in[2,\infty]:\quad \|S\hat{u}\|_{L^p(\mathbb{R}^n\times\mathbb{T}_T)} \le \|\mathcal{F}\hat{u}\|_{L^{p'}(\mathbb{R}^n\times\mathbb{Z})}.$$

Step 2: Estimating $L^{p'}$-norms of the Fourier transform.
We next bound $\|\mathcal{F}\hat{u}\|_{L^{p'}(\mathbb{R}^n\times\mathbb{Z})}$ using the $\|\cdot\|_{\hat{H}}$-norm. We start with an elementary but technical estimate. Let $\rho := a+b$. Then $\rho\ge 2 > a$. Using Young's inequality for products we calculate for $\xi\in\mathbb{R}^n$ and $k\in\mathbb{Z}\setminus\{0\}$:

$$\begin{aligned}|\xi|^{\frac{2a}{\rho}} &= \frac{1}{|k|^{\rho-a}}|\xi|^{\frac{2a}{\rho}}|k|^{\rho-a} \le \frac{1}{|k|^{\rho-a}}\left(\frac{1}{\frac{\rho}{a}}|\xi|^{\frac{2a}{\rho}\cdot\frac{\rho}{a}} + \frac{1}{\frac{\rho}{\rho-a}}|k|^{(\rho-a)\frac{\rho}{\rho-a}}\right) \\ &= \frac{a}{\rho}\frac{1}{|k|^{\rho-a}}|\xi|^2 + \frac{\rho-a}{\rho}|k|^a \le |k|^{-b}|\xi|^2 + |k|^a.\end{aligned}$$

Using Hölder's inequality we now calculate for $\hat{u}\in\hat{H}$ and $p'\in(1,2)$:

$$\begin{aligned}\|\mathcal{F}\hat{u}\|^{p'}_{L^{p'}(\mathbb{R}^n\times\mathbb{Z})} &= \sum_k\int_{\mathbb{R}^n}|\mathcal{F}\hat{u}_k|^{p'}\,\mathrm{d}\xi = \sum_k\int_{\mathbb{R}^n}|\mathcal{F}\hat{u}_k|^{p'}\cdot\frac{\left(|\xi|^{\frac{2a}{\rho}}+|k|^a\right)^{\frac{p'}{2}}}{\left(|\xi|^{\frac{2a}{\rho}}+|k|^a\right)^{\frac{p'}{2}}}\,\mathrm{d}\xi \\ &\le \left(\sum_k\int_{\mathbb{R}^n}|\mathcal{F}\hat{u}_k|^2\cdot\left(|\xi|^{\frac{2a}{\rho}}+|k|^a\right)\mathrm{d}\xi\right)^{\frac{p'}{2}}\cdot\left(\sum_k\int_{\mathbb{R}^n}\left(|\xi|^{\frac{2a}{\rho}}+|k|^a\right)^{-\frac{p'}{2}\cdot\frac{\frac{2}{p'}}{\frac{2}{p'}-1}}\mathrm{d}\xi\right)^{\frac{\frac{2}{p'}-1}{\frac{2}{p'}}} \\ &\le \left(\sum_k\int_{\mathbb{R}^n}|\mathcal{F}\hat{u}_k|^2\cdot 2\left(|\xi|^2|k|^{-b}+|k|^a\right)\mathrm{d}\xi\right)^{\frac{p'}{2}} \\ &\quad\cdot\left(\sum_k\int_{\mathbb{R}^n}|k|^{-\frac{ap'}{2-p'}}\left(1+|\xi|^{\frac{2a}{\rho}}|k|^{-a}\right)^{-\frac{p'}{2-p'}}\mathrm{d}\xi\right)^{1-\frac{p'}{2}}\end{aligned}$$

$$= 2^{\frac{p'}{2}} \cdot \|\hat{u}\|_{\hat{H}}^{p'} \cdot \left(\sum_k |k|^{\frac{n\rho}{2} - \frac{ap'}{2-p'}} \int_{\mathbb{R}^n} \left(1 + |\eta|^{\frac{2a}{\rho}}\right)^{-\frac{p'}{2-p'}} \mathrm{d}\eta \right)^{1-\frac{p'}{2}}.$$

We want the integral over η and the sum over k to converge. This is true according to the following:

$$\sum_k |k|^{\frac{n\rho}{2} - \frac{ap'}{2-p'}} < \infty \quad \Leftrightarrow \quad \frac{n\rho}{2} - \frac{ap'}{2-p'} < -1 \quad \Leftrightarrow \quad p' > \frac{2n\rho + 4}{2a + 2 + n\rho},$$

$$\int_{\mathbb{R}^n} \left(1 + |\eta|^{\frac{2a}{\rho}}\right)^{-\frac{p'}{2-p'}} \mathrm{d}\eta < \infty \quad \Leftrightarrow \quad \frac{2a}{\rho} \frac{p'}{2-p'} > n \quad \Leftrightarrow \quad p' > \frac{2n\rho}{2a + n\rho}.$$

Observe that $\frac{2n\rho+4}{2a+2+n\rho} > \frac{2n\rho}{2a+n\rho}$, i.e., the convergence of the sum implies the convergence of the integral. Moreover:

$$p' > \frac{2n\rho + 4}{2a + 2 + n\rho} \quad \Leftrightarrow \quad p < \frac{2n\rho + 4}{n\rho + 2 - 2a} =: p^*.$$

Combining both steps we see: For $p \in [2, p^*)$ the Fourier reconstruction operator $S\colon \mathcal{H} \hookrightarrow L^p(\mathbb{R}^n \times \mathbb{T}_T)$ is continuous. Last we prove the local compactness of S. Observe first that $p^* < 2\frac{n+1}{n-1}$ and $2\frac{n+1}{n-1}$ is the critical Sobolev exponent for $H^1(\mathbb{R}^n \times \mathbb{T}_T) \hookrightarrow L^{2\frac{n+1}{n-1}}(\mathbb{R}^n \times \mathbb{T}_T)$. Now let $\tilde{\Omega} \subset \mathbb{R}^n$ be compact. For fixed $K \in \mathbb{N}$ we define the map $(S^{(K)}\hat{u})(x,t) := \sum_{|k| \le K} \hat{u}_k(x) e_k(t)$. Since $S^{(K)}$ only sees finitely many Fourier coefficients $\hat{u}_k \in H^1(\tilde{\Omega})$, we obtain $S^{(K)}\hat{u} \in H^1(\tilde{\Omega} \times \mathbb{T}_T)$, and hence $S^{(K)}\colon \mathcal{H} \hookrightarrow L^p(\tilde{\Omega} \times \mathbb{T}_T)$ is compact since $p < p^* < 2\frac{n+1}{n-1}$. By an analogous calculation as above we see that

$$\left\| S^{(K)} - S \right\|_{\mathcal{H} \hookrightarrow L^p(\tilde{\Omega} \times \mathbb{T}_T)}^{p'} \le 2^{\frac{p'}{2}} \cdot \left(\sum_{|k| > K} |k|^{\frac{n\rho}{2} - \frac{ap'}{2-p'}} \int_{\mathbb{R}^n} \left(1 + |\eta|^{\frac{2a}{\rho}}\right)^{-\frac{p'}{2-p'}} \mathrm{d}\eta \right)^{1-\frac{p'}{2}} \to 0,$$

as $K \to \infty$. Here we used the absolute convergence of the sum. Hence $S\colon \mathcal{H} \hookrightarrow L^p(\tilde{\Omega} \times \mathbb{T}_T)$ is the limit of compact operators and therefore compact itself. □

We combine the last two theorems into two results, as in step 4. of the toolbox mentioned in the Introduction. The first result focuses on step potentials, the second result focuses on δ-potentials.

Theorem 2.17. *Let $V \in L^\infty(\mathbb{R}^n)$. Then $L_k := -\Delta - k^2\omega^2 V(x)\colon H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are self-adjoint operators on $L^2(\mathbb{R}^n)$ with $\mathcal{D}(b_{L_k}) = H^1(\mathbb{R}^n)$ for $k \in \mathbb{Z}_{odd}$. Assume:*

(i) There are $N \in \mathbb{N}_0$, $c > 0$ such that for all $|k| > N$ we have $(-c \cdot |k|, c \cdot |k|) \subset \rho(L_k)$.

(ii) If $|k| \le N$ and $0 \in \sigma(L_k)$, then 0 is an eigenvalue of finite multiplicity and isolated from the rest of $\sigma(L_k)$.

Let $\mathcal{H} \subset \bigoplus_{k \in \mathbb{Z}_{odd}} H^1(\mathbb{R}^n)$ with $\|\hat{u}\|_{\mathcal{H}}^2 := \sum_{k \in \mathbb{Z}_{odd}} \langle \hat{u}, \hat{u} \rangle_{|L_k|}$ as in Section 2.2.1. Then the Fourier reconstruction operator $S\colon \mathcal{H} \hookrightarrow L^{p+1}(\mathbb{R}^n \times \mathbb{T}_T)$ is continuous and locally compact for $p \in [1, 1 + \frac{2}{n})$.

Proof. For self-adjointness of L_k and the domain of the sesquilinear form we cite [RS10]. We will absorb constants independent of k into c, i.e., c may change from line to line but stays positive and bounded away from zero. By Section 2.2.2 the scalar products $\langle\cdot,\cdot\rangle_{|L_k|} := b_{|L_k|}(\cdot,\cdot) + \langle P_k^0\cdot, P_k^0\cdot\rangle$ are equivalent to the standard $H^1(\mathbb{R}^n)$ scalar product. Since V is bounded, we have for any $\hat{u}_k \in H^1(\mathbb{R}^n)$:

$$\left|b_{-k^2\omega^2 V(x)}(\hat{u}_k,\hat{u}_k)\right| = \left|-k^2\omega^2\int_{\mathbb{R}^n} V(x)|\hat{u}_k|^2\,\mathrm{d}x\right| \le k^2\omega^2\|V\|_{L^\infty(\mathbb{R}^n)}\|\hat{u}_k\|^2_{L^2(\mathbb{R}^n)}.$$

For $|k| > N$ we know $P_k^0 \equiv 0$ and we can use our spectral gap assumption to apply Theorem 2.15 part (ii). We obtain for $|k| > N$:

$$\forall\,\hat{u}_k \in H^1(\mathbb{R}^n)\colon\quad b_{|L_k|}(\hat{u}_k,\hat{u}_k) \ge \frac{1}{1+\frac{k^2\omega^2\|V\|_{L^\infty(\mathbb{R}^n)}}{c\cdot|k|}}\cdot\left(b_{-\Delta}\left(\hat{u}_k^+,\hat{u}_k^+\right) + b_{-\Delta}\left(\hat{u}_k^-,\hat{u}_k^-\right)\right),$$

Observe that $b_{-\Delta}(v,v) = \|\nabla v\|^2_{L^2(\mathbb{R}^n)}$ and in general ∇ and $P_k^\pm$ do not commute. Using triangle inequality we can estimate

$$\begin{aligned} b_{-\Delta}\left(\hat{u}_k^+,\hat{u}_k^+\right) + b_{-\Delta}\left(\hat{u}_k^-,\hat{u}_k^-\right) &= \left\|\nabla\left(\hat{u}_k^+\right)\right\|^2_{L^2(\mathbb{R}^n)} + \left\|\nabla\left(\hat{u}_k^-\right)\right\|^2_{L^2(\mathbb{R}^n)} \\ &\ge \frac{1}{2}\left\|\nabla\left(\hat{u}_k^+\right) + \nabla\left(\hat{u}_k^-\right)\right\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{2}\|\nabla\hat{u}_k\|^2_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Absorbing all constants independent of k into c we obtain:

$$\exists\,c>0\colon\forall\,|k|>N,\hat{u}_k\in H^1(\mathbb{R}^n)\colon\quad b_{|L_k|}(\hat{u}_k,\hat{u}_k) \ge c\cdot|k|^{-1}\|\nabla\hat{u}_k\|^2_{L^2(\mathbb{R}^n)}.$$

By the spectral gap assumption we know:

$$\forall\,|k|>N,\hat{u}_k\in H^1(\mathbb{R}^n)\colon\quad b_{|L_k|}(\hat{u}_k,\hat{u}_k) \ge c\cdot|k|^1\|\hat{u}_k\|^2_{L^2(\mathbb{R}^n)}.$$

Combining both estimates we see: $\exists\,c>0\colon\forall\,|k|>N,\hat{u}_k\in H^1(\mathbb{R}^n)$:

$$\langle\hat{u}_k,\hat{u}_k\rangle_{|L_k|} = b_{|L_k|}(\hat{u}_k,\hat{u}_k) \ge c\cdot\left(|k|^{-1}\|\nabla\hat{u}_k\|^2_{L^2(\mathbb{R}^n)} + |k|^1\|\hat{u}_k\|^2_{L^2(\mathbb{R}^n)}\right).$$

For $|k| \le N$ we use the fact that the $\langle\cdot,\cdot\rangle_{|L_k|}$ scalar product dominates the standard $H^1(\mathbb{R}^n)$ scalar product, i.e., there are constants $\mu_k > 0$ such that:

$$\langle\hat{u}_k,\hat{u}_k\rangle_{|L_k|} \ge \mu_k\cdot\left(\|\nabla\hat{u}_k\|^2_{L^2(\mathbb{R}^n)} + \|\hat{u}_k\|^2_{L^2(\mathbb{R}^n)}\right).$$

Possibly shrinking c such that $0 < c \overset{!}{\le} \min_{|k|\le N}\frac{\mu_k}{|k|}$, we obtain:

$$\exists\,c>0\colon\forall\,\hat{u}_k\in H^1(\mathbb{R}^n)\colon\quad \langle\hat{u}_k,\hat{u}_k\rangle_{|L_k|} \ge c\cdot\left(|k|^{-1}\|\nabla\hat{u}_k\|^2_{L^2(\mathbb{R}^n)} + |k|^1\|\hat{u}_k\|^2_{L^2(\mathbb{R}^n)}\right).$$

Hence, the $\|\cdot\|_{\mathcal{H}}$-norm dominates the $\|\cdot\|_{\hat{H}}$-norm with $a := b := 1$ and $\hat{H}$ as in Theorem 2.16. Using this theorem we calculate:

$$p^* := \frac{2n(a+b)+4}{n(a+b)+2-2a} - 1 = 1 + \frac{2}{n}.$$

□

Theorem 2.18. *Let* $V(x) = -\alpha + \sum_{l\in\mathbb{N}} \beta_l \delta_{x_l}(x)$ *with* $\alpha > 0$, $\sup_{l\in\mathbb{N}} |\beta_l| < \infty$ *and* $(x_l)_l \subset \mathbb{R}$ *has no accumulation point. Then* $L_k := -\Delta - k^2\omega^2 V(x)\colon \mathcal{D}(L_k) \to L^2(\mathbb{R})$ *are self-adjoint operators on* $L^2(\mathbb{R}^n)$ *with* $\mathcal{D}(b_{L_k}) = H^1(\mathbb{R}^n)$ *for* $k \in \mathbb{Z}_{odd}$. *Assume:*

(i) There are $N \in \mathbb{N}_0$, $c > 0$ *such that for all* $|k| > N$ *we have* $(-c\cdot k^2, c\cdot k^2) \subset \rho(L_k)$.

(ii) If $|k| \le N$ *and* $0 \in \sigma(L_k)$, *then* 0 *is an eigenvalue of finite multiplicity and isolated from the rest of* $\sigma(L_k)$.

Let $\mathcal{H} \subset \bigoplus_{k\in\mathbb{Z}_{odd}} H^1(\mathbb{R}^n)$ *with* $\|\hat{u}\|_{\mathcal{H}}^2 := \sum_{k\in\mathbb{Z}_{odd}} \langle \hat{u}, \hat{u}\rangle_{|L_k|}$ *as in Section 2.2.1. Then the Fourier reconstruction operator* $S\colon \mathcal{H} \hookrightarrow L^{p+1}(\mathbb{R}\times\mathbb{T}_T)$ *is continuous and locally compact for* $p \in [1, 5)$.

Proof. We cite [HR19] to observe that there is $C > 0$ such that

$$\forall\, \varepsilon > 0, w \in H^1(\mathbb{R})\colon \quad |b_V(w,w)| \le C\cdot\left(\varepsilon\|w'\|_{L^2(\mathbb{R})}^2 + \frac{1}{4\varepsilon}\|w\|_{L^2(\mathbb{R})}^2\right).$$

Using the same proof as for Theorem 2.17 with the obvious changes of using Theorem 2.15 part (i) and adjusting the growth of the spectral gap, we obtain:

$$\exists\, c > 0\colon \forall\, \hat{u}_k \in H^1(\mathbb{R})\colon \quad \langle \hat{u}_k, \hat{u}_k\rangle_{|L_k|} \ge c\cdot\left(|k|^{-2}\|\nabla\hat{u}_k\|_{L^2(\mathbb{R})}^2 + |k|^2\|\hat{u}_k\|_{L^2(\mathbb{R})}^2\right),$$

and hence

$$p^* := \frac{2n(a+b)+4}{n(a+b)+2-2a} - 1 = 5.$$

□

At this point we want to remark, that the authors in [HR19] guarantee a linear growing spectral gap in the case of periodically distributed δ-potentials and their calculations are optimal in the growth of the spectral gap. Hence Theorem 2.18 is not exactly the same as [HR19] and the quadratic growth of the spectral gap improves the range of applicable exponents p. We sketch another example with two δ's for the application Theorem 2.18. We omit many straightforward calculation, since we do not use this example in the rest of this work, but it could be treated exactly as in Section 2.3.

Remark 2.19. *Let* $\alpha, \beta, T, r > 0$ *and set* $\omega := \frac{2\pi}{T}$. *Let* $V(x) = -\alpha + \beta\delta_{-r}(x) + \beta\delta_{-r}(x)$ *and* $L_k := -\Delta - k^2\omega^2 V(x)\colon \mathcal{D}(L_k) \to L^2(\mathbb{R})$. *Argue as Section 2.3.1.1 for the exact characterization of* $\mathcal{D}(L_k)$. *Then* $\sigma(L_k) = \{\lambda_{k,1}, \lambda_{k,2}\} \cup [\alpha k^2\omega^2, \infty)$ *and* $\lambda_{k,1}, \lambda_{k,2} \in (-\infty, \alpha k^2\omega^2)$ *are the unique solutions of*

$$\begin{aligned}
\beta k^2\omega^2 &\overset{!}{=} \frac{\sqrt{\alpha k^2\omega^2 - \lambda_{k,1}}\cdot\exp\big(\sqrt{\alpha k^2\omega^2 - \lambda_{k,1}}\,r\big)}{\sinh\big(\sqrt{\alpha k^2\omega^2 - \lambda_{k,1}}\,r\big)} \\
&= \sqrt{\alpha}|k|\omega\cdot\sqrt{1 - \frac{\lambda_{k,1}}{\alpha k^2\omega^2}}\cdot\frac{\exp\left(\sqrt{\alpha}|k|\omega\cdot\sqrt{1-\frac{\lambda_{k,1}}{\alpha k^2\omega^2}}\,r\right)}{\sinh\left(\sqrt{\alpha}|k|\omega\cdot\sqrt{1-\frac{\lambda_{k,1}}{\alpha k^2\omega^2}}\,r\right)}, \\
\beta k^2\omega^2 &\overset{!}{=} \frac{\sqrt{\alpha k^2\omega^2 - \lambda_{k,2}}\cdot\exp\big(\sqrt{\alpha k^2\omega^2 - \lambda_{k,2}}\,r\big)}{\cosh\big(\sqrt{\alpha k^2\omega^2 - \lambda_{k,2}}\,r\big)}
\end{aligned}$$

$$= \sqrt{\alpha}|k|\omega \cdot \sqrt{1 - \frac{\lambda_{k,2}}{\alpha k^2 \omega^2}} \cdot \frac{\exp\left(\sqrt{\alpha}|k|\omega \cdot \sqrt{1 - \frac{\lambda_{k,2}}{\alpha k^2\omega^2}}\, r\right)}{\cosh\left(\sqrt{\alpha}|k|\omega \cdot \sqrt{1 - \frac{\lambda_{k,2}}{\alpha k^2\omega^2}}\, r\right)}.$$

Furthermore $\lambda_{k,j}$ is a simple eigenvalue. We omit the calculations for this statement. Observe that the left hand side grows quadratic in $|k|$ and the right hand side grows linear in $|k|$ provided $|\lambda_{k,j}| < \frac{1}{2}\alpha k^2\omega^2$. Hence there are $N \in \mathbb{N}_0$, $c > 0$ such that for all $|k| > N$ we have $(-\frac{1}{2}\alpha k^2\omega^2, \frac{1}{2}\alpha k^2\omega^2) \subset \rho(L_k)$. Applying Theorem 2.18 we see: The Fourier reconstruction operator $S\colon \mathcal{H} \hookrightarrow L^{p+1}(\mathbb{R} \times \mathbb{T}_T)$ is continuous and locally compact for $p \in [1, 5)$.

We do not expect that the Theorems 2.17 and 2.18 are optimal in the sense that the exponent p^* is maximal. In fact we will prove in Section 2.3.1, i.e., $n = 1$ and $V(x) = \beta\delta_0(x) - \alpha$, that the range of S is a strict subset of $H^1(\mathbb{R} \times \mathbb{T}_T)$ and hence $\mathcal{H}$ embeds into $L^{p+1}(\mathbb{R} \times \mathbb{T}_T)$ for any $p \in [1, \infty)$ via S.

Before starting with the example section, we prove another technical lemma concerning some kind of local Sobolev inequality combined with Minkowski's inequality for integrals.

Lemma 2.20. *In the setting of Section 2.2.2 let $K \subset \mathbb{Z} \setminus \{0\}$, $X_k := L^2(\mathbb{R}^N)$ and $H_k := H^1(\mathbb{R}^N)$. Assume (A1), (A2) and there are $1 < p < p^* < \frac{N+3}{N-1}$ if $N \geq 2$ and $1 < p < p^* < \infty$ if $N = 1$ such that $S\colon \mathcal{H} \hookrightarrow L^{p_*+1}(\mathbb{R}^N \times \mathbb{T}_T) \cap L^{p^*+1}(\mathbb{R}^N \times \mathbb{T}_T)$ is continuous. Assume further there are $a, b, c, K_0 > 0$ such that:*

$$\forall\, |k| > K_0, \hat{u}_k\colon \qquad \langle \hat{u}_k, \hat{u}_k\rangle_{|L_k|} \geq c \cdot \left(|k|^a \|\hat{u}_k\|^2_{L^2(\mathbb{R}^N)} + |k|^{-b}\|\nabla \hat{u}_k\|^2_{L^2(\mathbb{R}^N)}\right),$$

and in addition $S\colon \hat{H} \to L^{p^*+1}(\mathbb{R}^N \times \mathbb{T}_T)$ is continuous with $\hat{H}$ as in Theorem 2.16. Set $R := N + 1$. Then there is some $C > 0$ such that

$$\sum_{j\in\mathbb{Z}^N} \|S\hat{u}\|^2_{L^{p^*+1}(B_R(j)\times\mathbb{T}_T)} \leq C\|\hat{u}\|^2,$$

i.e., assumption (C1) with $p_* = 1$, $B_j = B_R(j)$ and $N^* = (4N + 5)^N$ is true.

Proof. For simplicity and better readability we prove the claim for $K_0 = 0$. If $K_0 > 0$, we can make adjustments for $|k| \leq N$ exactly as in the proof of Theorem 2.17. Clearly $\bigcup_{j\in\mathbb{Z}^N} B_R(j) = \mathbb{R}^N$ since $R > \sqrt{N}$. Moreover each ball overlaps with at most $N^* := (4N+5)^N$ other balls (this number N^* is far from being optimal but an upper bound). In the following we absorb any constant independent of k and j into $C > 0$, i.e., the constant C may change from line to line. Citing [HKT08] we see there are continuation operators $E_j\colon H^1(B_R(j)) \to H^1(\mathbb{R}^N)$ with $E_j(u)|_{B_R(j)} \equiv u$ and $\|E_j(u)\|_{L^2(\mathbb{R}^N)} \leq C\|u\|_{L^2(B_R(j))}$, $\|\nabla E_j(u)\|_{L^2(\mathbb{R}^N)} \leq C\|\nabla u\|_{L^2(B_R(j))}$ with $C > 0$ independent j. Using continuity of $S\colon \hat{H} \to L^{p^*+1}(\mathbb{R}^N \times \mathbb{T}_T)$ we see

$$\begin{aligned}
\|\hat{u}_k\|^2_{L^{p^*+1}(B_R(j))} &= \|E_j(\hat{u}_k)\|^2_{L^{p^*+1}(B_R(j))} \leq C\|E_j(\hat{u}_k)(x) \cdot e_k(t)\|^2_{L^{p^*+1}(\mathbb{R}^N\times\mathbb{T}_T)} \\
&\leq C\left(|k|^a\|E_j(\hat{u}_k)(x)\cdot e_k(t)\|^2_{L^2(\mathbb{R}^N\times\mathbb{T}_T)} + |k|^{-b}\|\nabla(E_j(\hat{u}_k)(x))\cdot e_k(t)\|^2_{L^2(\mathbb{R}^N\times\mathbb{T}_T)}\right) \\
&\leq C\left(|k|^a\|\hat{u}_k\|^2_{L^2(B_R(j))} + |k|^{-b}\|\nabla\hat{u}_k\|^2_{L^2(B_R(j))}\right)
\end{aligned}$$

Hence using Minkowski's integral inequality, cf. [HLP+52], we obtain

$$\begin{aligned}\sum_{j\in\mathbb{Z}^N} \|S\hat{u}\|^2_{L^{p^*+1}(B_R(j)\times\mathbb{T}_T)} &= \sum_{j\in\mathbb{Z}^N} \Big\| \sum_{k\in K} \hat{u}_k(x)e_k(t)\Big\|^2_{L^{p^*+1}(B_R(j)\times\mathbb{T}_T)} \\ &\le \sum_{j\in\mathbb{Z}^N}\sum_{k\in K} \|\hat{u}_k\|^2_{L^{p^*+1}(B_R(j))} \\ &= C\sum_{j\in\mathbb{Z}^N}\sum_{k\in K}\Big(|k|^a\|\hat{u}_k\|^2_{L^2(B_R(j))} + |k|^{-b}\|\nabla\hat{u}_k\|^2_{L^2(B_R(j))}\Big).\end{aligned}$$

We next sum over all balls using the following idea: We decompose the union $\bigcup_{j\in\mathbb{Z}^N} B_R(j) = \mathbb{R}^N$ into sets $(D_m)_{m\in\mathbb{N}}$ consisting of all nonempty intersections of balls $B_R(j)$ and let M an index family be such that $(D_m)_{m\in M}$ are disjoint and $\bigcup_{m\in M} D_m = \mathbb{R}^N$. We observe that for each $m\in\mathbb{N}$ we need at most N^* indices $m_{l_1},\dots,m_{l_L}\in M$ such that $D_m = \bigcup_{j=1}^{L} D_{m_{l_j}}$. Hence for any $u\in L^2(\mathbb{R}^N)$ we have

$$\sum_{j\in\mathbb{Z}^N}\|u\|^2_{L^2(B_R(j))} = \sum_{m\in\mathbb{N}}\|u\|^2_{L^2(D_j)} \le N^*\sum_{m\in M}\|u\|^2_{L^2(D_j)} = \|u\|^2_{L^2(\mathbb{R}^N)}.$$

Absorbing N^* into C we now obtain the claim by calculating

$$\sum_{j\in\mathbb{Z}^N}\|S\hat{u}\|^2_{L^{p^*+1}(B_R(j)\times\mathbb{T}_T)} \le C\sum_{k\in K}\Big(|k|^a\|\hat{u}_k\|^2_{L^2(\mathbb{R}^N)} + |k|^{-b}\|\nabla\hat{u}_k\|^2_{L^2(\mathbb{R}^N)}\Big) \le C\|\hat{u}\|^2.$$

□

2.3 Examples

This section is one of our our main contributions to new knowledge about semilinear wave equations like (2.1) applying direct variational calculus. We prove the existence of a ground state for

$$V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u, \qquad (x,t)\in\mathbb{R}\times\mathbb{T}_T, \tag{2.2}$$

$$V(x)u_{tt} - \Delta u = \Gamma(|x|)|u|^{p-1}u, \qquad (x,t)\in\mathbb{R}^N\times\mathbb{T}_T. \tag{2.3}$$

More precisely in Section 2.3.1 we analyze (2.2) with $V(x) = -\alpha + \beta\delta_0(x)$ and $\Gamma(x)\to 0$ as $|x|\to\infty$. In Section 2.3.2 we analyze (2.2) with $V(x) = -\alpha + \beta\,\mathbb{1}_{[-r,r]}(|x|)$ and and $\Gamma(x)\to 0$ as $|x|\to\infty$. In Section 2.3.3 we analyze (2.3) with $V(x) = -\alpha + \beta\,\mathbb{1}_{B_R(0)}(|x|)$, $0<\Gamma\in L^\infty$ and $N=2$. Equation (2.2) with $0<\Gamma\in L^\infty$ periodic and either $V(x) = -\alpha + \beta\delta_0(x)$ or $V(x) = -\alpha + \beta\,\mathbb{1}_{[-r,r]}(|x|)$ is treated in Chapter 3 and uses a different approach.

Each example will follow the same structure: First we state all assumptions, then we state the main theorem of the example. The rest of the section is devoted to the proof of the theorem, following the toolbox presented in the Introduction. For this we first analyze the spectra of L_k, then we apply Section 2.2.2 to construct $\mathcal{H}$, then we use Section 2.2.3 to obtain the embedding $S\colon\mathcal{H}\hookrightarrow L^{p+1}(\mathbb{R}^n\times\mathbb{T}_T)$ and finally we apply our abstract results of Section 2.1 to obtain the desired ground state. The final proofs will be rather short since we will have invested much work into the preparations in Section 2.1 and Section 2.2.2. For the sequence space $\mathcal{H}$ we will only consider odd k to keep 0 out of the spectrum for L_k (note that $L_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$). This will result in $\frac{T}{2}$-anti-periodic functions, which is compatible with the right hand side of our examples.

2.3.1 δ-potentials

In this section we invest the extreme case where V has negative background strength and one infinitesimally located and infinitely strong positive value at zero. In fact we assume

(H_δ) Let $\alpha, \beta, T > 0$. Define $V(x) := \beta\delta_0(x) - \alpha$.

We analyze the operator $L = V(x)\partial_t^2 - \partial_x^2$ for $\frac{T}{2}$-anti-periodic functions. The potential V is strictly negative for $x \neq 0$ and formally positive at $x = 0$ by the Dirac-delta-distribution $\delta_0(x)$ located at zero with strength β. Hence the operator L is elliptic everywhere, except at $x = 0$, where it is formally hyperbolic. The main result of this section is the following.

Theorem 2.21. *Assume (H_δ), $p > 1$ and set $\omega := \frac{2\pi}{T}$. Let $\Gamma \in L^\infty(\mathbb{R})$ with $\Gamma(x) > 0$ a.e. and $\lim_{|x|\to\infty} \Gamma(x) = 0$ and $V(x) := \beta\delta_0(x) - \alpha$. Then there exists a nontrivial weak solution u of the equation*

$$V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u, \qquad (x,t) \in \mathbb{R} \times \mathbb{T}_T, \tag{2.2}$$

with minimal energy among all $\frac{T}{2}$-anti-periodic, weak solutions.

Definition 2.22. *Assume (H_δ), $p > 1$. Let*

$$\|u\|_Y^2 := \|u\|_{H^1(\mathbb{R}\times\mathbb{T}_T)}^2 + \|u_t(0,\cdot)\|_{L^2(\mathbb{T}_T)}^2, \qquad Y := \overline{C^1(\mathbb{R}\times\mathbb{T}_T)}^{\|u\|_Y},$$
$$B_L\colon Y \times Y \to \mathbb{C}, \quad B_L(u,v) := \int_{\mathbb{R}\times\mathbb{T}_T} \alpha u_t\,\overline{v_t} + u_x\,\overline{v_x}\,\mathrm{d}(x,t) - \beta\int_{\mathbb{T}_T} u_t(0,\cdot)\,\overline{v_t}(0,\cdot)\,\mathrm{d}t\,.$$

Then $u \in Y$ is called a weak solution of the equation (2.2)*, if*

$$\forall\,\varphi \in Y\colon \qquad B_L(u,\varphi) = \int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(x)|u|^{p-1}\,\varphi\,\mathrm{d}(x,t)\,.$$

The rest of this section is dedicated to the proof of Theorem 2.21. In Section 2.3.1.1 we analyze L_k, construct $\mathcal{H}$ and using (H_δ) we are able to characterize $\mathcal{H}$ precisely as a function space H and we will obtain a strictly stronger embedding result than Theorem 2.18. In Section 2.3.1.2 we even give an explicit domain for L such that $L\colon \mathcal{D}(L) \to L^2(\mathbb{R}\times\mathbb{T}_T)$ is self-adjoint and if in addition $\frac{2\sqrt{\alpha}}{\beta\omega} \notin \mathbb{Z}_{odd}$ we can construct a rather explicit inverse L^{-1}. These additional results are not necessary to prove Theorem 2.21, but they will shorten the proof and will be used again in Chapter 3.

2.3.1.1 Analysis of $\mathcal{L}$

Definition 2.23. *For $k \in \mathbb{Z}_{odd}$ define*

$$L_k u := -\hat{u}_k'' + \alpha k^2\omega^2\hat{u}_k - \beta k^2\omega^2\delta_0(x)\,\hat{u}_k,$$
$$\mathcal{D}(L_k) := \big\{\hat{u}_k \in H^1(\mathbb{R}) \;\big|\; \hat{u}_k|_{(0,\infty)} \in H^2(0,\infty),\ \hat{u}_k|_{(-\infty,0)} \in H^2(-\infty,0),$$
$$\hat{u}_k'(0_+) - \hat{u}_k'(0_-) = -\beta k^2\omega^2\hat{u}(0)\big\}$$

with the corresponding sesquilinear forms

$$b_{L_k}\colon H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{C}, \quad b_{L_k}(\hat{u}_k,\hat{v}_k) := \int_{\mathbb{R}} \hat{u}_k'\overline{\hat{v}_k'} + \alpha k^2\omega^2\hat{u}_k\overline{\hat{v}_k}\,\mathrm{d}x - \beta k^2\omega^2\hat{u}_k(0)\overline{\hat{v}_k(0)}.$$

Furthermore define the notations

$$\alpha_k^2 := \alpha k^2\omega^2, \qquad \beta_k^2 := \frac{1}{2}\beta k^2\omega^2, \qquad \varphi_k(x) := \beta_k \exp^{-\beta_k^2|x|}, \qquad \lambda_k := \alpha_k^2 - \beta_k^4.$$

Observe in particular the factor $\frac{1}{2}$ in β_k^2. It simplifies notations but is necessary to remember to keep track of constants.

Proposition 2.24. *Assume* (H_δ)*. Then*

(i) b_{L_k} *is closed, hermitian and lower bounded.* L_k *is well-defined and self-adjoint.*

(ii) $\forall\, \hat{u}_k \in \mathcal{D}(L_k)$, $\hat{v}_k \in H^1(\mathbb{R})$: $\quad b_{L_k}(\hat{u}_k, \hat{v}_k) = \langle L_k\hat{u}_k, \hat{v}_k\rangle_{L^2(\mathbb{R})}$.

(iii) $\sigma(L_k) = \{\lambda_k\} \cup [\alpha_k^2, \infty)$*, where* λ_k *is a simple eigenvalue with eigenfunction* φ_k*. At most finitely many* λ_k *are positive and* $\lambda_k \searrow -\infty$ *as* $k \to \infty$*.*

Proof. For (i) we refer to [CS94]. The parts (ii) and (iii) follow by a lengthy but straightforward calculation. □

Apply Section 2.2.1, with L_k as above, $X_k = L^2(\mathbb{R})$ and $H_k = H^1(\mathbb{R})$. Note that $0 \in \sigma(L_k)$ if and only if $|k| = \frac{\sqrt{\alpha}}{\beta\omega}$. Hence there are at most two $k_0 \in \mathbb{Z}_{odd}$, such that $0 \in \sigma(L_{k_0})$ and in this case 0 is a simple eigenvalue for this L_{k_0}. In all other cases we have $0 \notin \sigma(L_k)$. We observe that the projections P_k^- and P_k^0 can be explicitly calculated: $\forall\, \hat{u}_k \in H^1(\mathbb{R})$:

$$\hat{u}_k^- = \begin{cases} \langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\, \varphi_k, & \lambda_k < 0, \\ 0, & \lambda_k > 0, \end{cases} \qquad \hat{u}_k^0 = \begin{cases} \langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\, \varphi_k, & \lambda_k = 0 \\ 0, & \lambda_k \neq 0 \end{cases}.$$

For the sesquilinear form $b_{|L_k|}$ we calculate in the case $\lambda_k < 0$:

$$\begin{aligned} b_{|L_k|}(\hat{u}_k, \hat{v}_k) &= b_{L_k}(\hat{u}_k, \hat{v}_k) - b_{L_k}\left(\hat{u}_k, \hat{v}_k^-\right) - b_{L_k}\left(\hat{u}_k^-, \hat{v}_k\right) \\ &= b_{L_k}(\hat{u}_k, \hat{v}_k) - \overline{\langle \hat{v}_k, \varphi_k\rangle_{L^2(\mathbb{R})}}\, b_{L_k}(\hat{u}_k, \varphi_k) - \langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\, b_{L_k}(\varphi_k, \hat{v}_k) \\ &= b_{L_k}(\hat{u}_k, \hat{v}_k) - \overline{\langle \hat{v}_k, \varphi_k\rangle_{L^2(\mathbb{R})}}\, \lambda_k \langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})} - \langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\, \overline{\lambda_k \langle \hat{v}_k, \varphi_k\rangle_{L^2(\mathbb{R})}} \\ &= b_{L_k}(\hat{u}_k, \hat{v}_k) + 2\cdot(-\lambda_k)\cdot \langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\, \overline{\langle \hat{v}_k, \varphi_k\rangle_{L^2(\mathbb{R})}} \\ &= \left\langle \hat{u}_k', \hat{v}_k'\right\rangle_{L^2(\mathbb{R})} + \alpha_k^2 \langle \hat{u}_k, \hat{v}_k\rangle_{L^2(\mathbb{R})} - 2\beta_k^2 \hat{u}_k(0)\overline{\hat{v}_k(0)} \\ &\quad + 2\cdot\left(\beta_k^4 - \alpha_k^2\right)\cdot \langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\, \overline{\langle \hat{v}_k, \varphi_k\rangle_{L^2(\mathbb{R})}}. \end{aligned}$$

In the case $\lambda_k > 0$ we simply have:

$$b_{|L_k|}(\hat{u}_k, \hat{v}_k) = b_{L_k}(\hat{u}_k, \hat{v}_k).$$

And in the case $\lambda_k = 0$ (which occurs at most twice) we have:

$$b_{|L_k|}(\hat{u}_k, \hat{v}_k) = b_{L_k}\left(\hat{u}_k^+, \hat{v}_k^+\right).$$

Next we define the composite Hilbert space. We assume an additional conjugation symmetry to obtain real-valued functions $S\hat{u}$.

Definition 2.25. *Assume* (H_δ). *Define*

$$\mathcal{H} := \left\{ \hat{u} \in \left(H^1(\mathbb{R})\right)^{\mathbb{Z}_{odd}} \;\middle|\; \sum_{k\in\mathbb{Z}_{odd}} \langle \hat{u}_k, \hat{u}_k\rangle_{|L_k|} < \infty, \;\; \overline{\hat{u}_k} = \hat{u}_{-k} \right\},$$

$$\mathcal{X} := \left\{ \hat{u} \in \left(L^2(\mathbb{R})\right)^{\mathbb{Z}_{odd}} \;\middle|\; \sum_{k\in\mathbb{Z}_{odd}} \|\hat{u}_k\|^2_{L^2(\mathbb{R})} < \infty, \;\; \overline{\hat{u}_k} = \hat{u}_{-k} \right\},$$

and apply all other constructions as in Section 2.2.2.

As seen in Section 2.2.2 $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a Hilbert space and $\mathcal{L}$ is self-adjoint on $\mathcal{X}$. We continue with the analysis the range of the Fourier-reconstruction operator S defined by

$$(S\hat{u})(x,t) := \sum_{k\in\mathbb{Z}_{odd}} \hat{u}_k(x) e_k(t), \qquad \hat{u} \in \mathcal{H}.$$

By Theorem 2.18: The map $S\colon \mathcal{H} \to L^{p+1}(\mathbb{R}\times\mathbb{T}_T, \mathbb{R})$ is continuous and locally compact for $p \in [1,5)$.

Remark 2.26. *Arguing similar as in Section 2.3.2, we could now prove Theorem 2.21 under the additional assumption* $p < 5$ *and with another concept of "solution", analogous to Definition 2.47.*

Investing more work, which will later also be used in Chapter 3, we can get rid of these additional assumptions. For this we use a completely different technique, which relies on many explicit calculations. Our first step is the following theorem.

Theorem 2.27. *Assume* (H_δ). *Let*

$$\|u\|_H^2 := \|u\|^2_{H^1(\mathbb{R}\times\mathbb{T}_T)} + \|u_t(0,\cdot)\|^2_{L^2(\mathbb{T}_T)}, \qquad H := \overline{C^1_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})}^{\|u\|_H}.$$

Then $\operatorname{Range}(S) = H$ *and* $S\colon \mathcal{H} \to H$ *is a continuous and continuously invertible.*

The next pages are devoted to the preparations and proof of Theorem 2.27. By this theorem we will directly obtain: The map $S\colon \mathcal{H} \to L^{p+1}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})$ is continuous and locally compact for any $p \in [1,\infty)$. The key to characterizing $\operatorname{Range}(S)$ is the following relation between the projection operators and the form of the potential $V(x) = -\alpha + \beta\delta_0(x)$ in the sesquilinear form with just the right exponents in the constants.

Lemma 2.28. *Assume* (H_δ). *Let* $\hat{u}_k \in H^1(\mathbb{R})$ *and* $\lambda_k < 0$. *Then:*

(i) $\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})} = \frac{2}{\beta_k}\hat{u}_k(0) + \frac{1}{\beta_k^2}\langle \hat{u}_k', \operatorname{sign}(x)\,\varphi_k\rangle_{L^2(\mathbb{R})}$,

(ii) $2\beta_k^2|\hat{u}_k(0)|^2 \le \frac{3}{4}\|\hat{u}_k'\|^2_{L^2(\mathbb{R})} + \frac{3}{2}\beta_k^4\left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right|^2$,

$2\beta_k^4\left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right|^2 \le 9\beta_k^2|\hat{u}_k(0)|^2 + 18\|\hat{u}_k'\|^2_{L^2(\mathbb{R})}$,

(iii) $b_{|L_k|}(\hat{u},\varphi_k) = \left|\alpha_k^2 - \beta_k^4\right| \langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}$.

Proof. Integration by parts and the embedding of $H^1(\mathbb{R})$ into continuous and bounded functions yields

$$\begin{aligned}
\langle \hat{u}_k, \varphi_k \rangle_{L^2(\mathbb{R})} &= \int_{-\infty}^{0} \hat{u}_k(x)\beta_k \mathrm{e}^{\beta_k^2 x}\,\mathrm{d}x + \int_{0}^{\infty} \hat{u}_k(x)\beta_k \mathrm{e}^{-\beta_k^2 x}\,\mathrm{d}x \\
&= \left[\hat{u}_k(x)\frac{1}{\beta_k}\mathrm{e}^{\beta_k^2 x}\right]_{x=-\infty}^{x=0} - \int_{-\infty}^{0} \hat{u}_k'(x)\frac{1}{\beta_k}\mathrm{e}^{\beta_k^2 x}\,\mathrm{d}x \\
&\quad + \left[\hat{u}_k(x)\frac{-1}{\beta_k}\mathrm{e}^{-\beta_k^2 x}\right]_{x=0}^{x=\infty} - \int_{0}^{\infty} \hat{u}_k'(x)\frac{-1}{\beta_k}\mathrm{e}^{-\beta_k^2 x}\,\mathrm{d}x \\
&= \frac{2}{\beta_k}\hat{u}_k(0) + \frac{1}{\beta_k^2}\left\langle \hat{u}_k', \operatorname{sign}(x)\,\varphi_k \right\rangle_{L^2(\mathbb{R})}.
\end{aligned}$$

Hence, (i) is proven. Using Young's inequality $|ab| \le \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$ with $\varepsilon = 1$ and $\|\varphi_k\|_{L^2(\mathbb{R})} = 1$ we obtain

$$\begin{aligned}
2\beta_k^2|\hat{u}_k(0)|^2 &= 2\beta_k^2\left|\frac{\beta_k}{2}\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})} - \frac{1}{2\beta_k}\left\langle \hat{u}_k', \operatorname{sign}(x)\,\varphi_k\right\rangle_{L^2(\mathbb{R})}\right|^2 \\
&\le 2\beta_k^2\left(\frac{\beta_k}{2}\left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right| + \frac{1}{2\beta_k}\left|\left\langle \hat{u}_k', \operatorname{sign}(x)\,\varphi_k\right\rangle_{L^2(\mathbb{R})}\right|\right)^2 \\
&\le 2\beta_k^2\left(\frac{\beta_k}{2}\left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right| + \frac{1}{2\beta_k}\left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}\right)^2 \\
&= \frac{1}{2}\beta_k^4\left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right|^2 + \beta_k^2\left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right|\left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})} + \frac{1}{2}\left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2 \\
&\le \frac{3}{2}\beta_k^4\left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right|^2 + \frac{3}{4}\left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2, \\
2\beta_k^4\left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right|^2 &= 2\beta_k^4\left|\frac{2}{\beta_k}\hat{u}_k(0) + \frac{1}{\beta_k^2}\left\langle \hat{u}_k', \operatorname{sign}(x)\,\varphi_k\right\rangle_{L^2(\mathbb{R})}\right|^2 \\
&\le 2\beta_k^4\left(\frac{2}{\beta_k}|\hat{u}_k(0)| + \frac{1}{\beta_k^2}\left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}\right)^2 \\
&= 8\beta_k^2|\hat{u}_k(0)|^2 + 8\beta_k|\hat{u}_k(0)|\left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})} + 2\left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2 \\
&\le 9\beta_k^2|\hat{u}_k(0)|^2 + 18\left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2,
\end{aligned}$$

which proofs (ii). Part (iii) is a consequence of φ_k being an eigenfunction of L_k with eigenvalue $\lambda_k = \beta_k^4 - \alpha_k^2$, i.e. $L_k\varphi_k = (\beta_k^4 - \alpha_k^2)\varphi_k$ and $\|\varphi_k\|_{L^2(\mathbb{R})} = 1$. We omit the straightforward calculation here. □

Part (i) describes the balance between the projection $\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}$, evaluation of the $\delta_0(x)$-potential $\hat{u}_k(0)$ and derivative $\hat{u}_k'$. Part (ii) exploits the factors β_k. Observe the factor $\frac{3}{4} < 1$ in (ii), since this will be crucial in the following calculations. We next give an estimate on $\|\cdot\|_{\mathcal{H}}$.

Corollary 2.29. *Assume* (H_δ)*. Then:*

(i) $\exists\, c > 0\ \forall\, \hat{u} \in \mathcal{H}$: $\|\hat{u}\|_{\mathcal{H}}^2 \ge c \cdot \sum_{k\in\mathbb{Z}_{odd}} \|\hat{u}_k'\|_{L^2(\mathbb{R})}^2 + \alpha_k^2\|\hat{u}_k\|_{L^2(\mathbb{R})}^2 + \beta_k^2|\hat{u}_k(0)|^2$.

(ii) $\exists\, C > 0\ \forall\, \hat{u} \in \mathcal{H}$: $\|\hat{u}\|_{\mathcal{H}}^2 \le C \cdot \sum_{k\in\mathbb{Z}_{odd}} \|\hat{u}_k'\|_{L^2(\mathbb{R})}^2 + \alpha_k^2\|\hat{u}_k\|_{L^2(\mathbb{R})}^2 + \beta_k^2|\hat{u}_k(0)|^2$.

Proof. We start the proof with some preparations. Let $\hat{u} \in \mathcal{H}$. We have

$$\|\hat{u}\|_{\mathcal{H}}^2 = \sum_{k\in\mathbb{Z}_{odd}} \langle \hat{u}_k, \hat{u}_k\rangle_{|L_k|} = \sum_{k\in\mathbb{Z}_{odd}} b_{L_k}\left(\hat{u}_k, \hat{u}_k\right) - 2\,\mathrm{Re}\left(b_{L_k}\left(\hat{u}_k, \hat{u}_k^-\right)\right) + \left\langle \hat{u}_k^0, \hat{u}_k^0\right\rangle_{H^1(\mathbb{R})}.$$

Furthermore for $\lambda_k < 0$ we have

$$\begin{aligned} 2\,\mathrm{Re}(b_{L_k}\left(\hat{u}_k, \hat{u}_k^-\right)) &= 2\,\mathrm{Re}\left(\overline{\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}}\, b_{L_k}\left(\hat{u}_k, \varphi_k\right)\right) \\ &= 2\,\mathrm{Re}\left(\overline{\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}}\, \langle \hat{u}_k, \lambda_k\varphi_k\rangle_{L^2(\mathbb{R})}\right) = 2\lambda_k\left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right|^2, \\ \left\langle \hat{u}_k^0, \hat{u}_k^0\right\rangle_{H^1(\mathbb{R})} &= 0, \end{aligned}$$

for $\lambda_k = 0$ we have

$$\begin{aligned} 2\,\mathrm{Re}(b_{L_k}\left(\hat{u}_k, \hat{u}_k^-\right)) &= 0, \\ \left\langle \hat{u}_k^0, \hat{u}_k^0\right\rangle_{H^1(\mathbb{R})} &= \left\|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\,\varphi_k\right\|_{H^1(\mathbb{R})}^2 = \left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right|^2 \cdot (1+\beta_k^4). \end{aligned}$$

and for $\lambda_k > 0$ we have

$$2\,\mathrm{Re}(b_{L_k}\left(\hat{u}_k, \hat{u}_k^-\right)) = 0, \qquad \left\langle \hat{u}_k^0, \hat{u}_k^0\right\rangle_{H^1(\mathbb{R})} = 0.$$

Recall $\beta_k^2 = \frac{1}{2}\beta k^2\omega^2$ and $\alpha_k^2 = \alpha k^2\omega^2$. With these preparations we prove (i) and (ii).

(i) We choose $K \geq \frac{4\sqrt{2\alpha}}{\beta\omega}$ and consider first $|k| > K$. Then $\frac{1}{2}\beta_k^4 - 2\alpha_k^2 \geq \frac{1}{4}\beta_k^4 > 0$, $\lambda_k = \alpha_k^2 - \beta_k^4 < 0$ and using Lemma 2.28 part (ii) and our preparations we obtain

$$\begin{aligned} &b_{L_k}\left(\hat{u}_k, \hat{u}_k\right) - 2\,\mathrm{Re}(b_{L_k}\left(\hat{u}_k, \hat{u}_k^-\right)) + \left\langle \hat{u}_k^0, \hat{u}_k^0\right\rangle_{H^1(\mathbb{R})} \\ &= \left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2 + \alpha_k^2\|\hat{u}_k\|_{L^2(\mathbb{R})}^2 - 2\beta_k^2|\hat{u}_k(0)|^2 + 2(\beta_k^4 - \alpha_k^2)\left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right|^2 \\ &\geq \frac{1}{4}\left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2 + \alpha_k^2\|\hat{u}_k\|_{L^2(\mathbb{R})}^2 + \left(\frac{1}{2}\beta_k^4 - 2\alpha_k^2\right)\left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right|^2 \\ &\geq \frac{1}{4}\left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2 + \alpha_k^2\|\hat{u}_k\|_{L^2(\mathbb{R})}^2 + \frac{1}{4}\beta_k^4\left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right|^2. \end{aligned}$$

Next we consider $|k| \leq K$. By Section 2.2.2 there are constants $c_k > 0$ such that $c_k\|\hat{u}_k\|_{H^1(\mathbb{R})}^2 \leq \langle \hat{u}_k, \hat{u}_k\rangle_{|L_k|}$. This yields

$$\begin{aligned} \langle \hat{u}_k, \hat{u}_k\rangle_{|L_k|} &\geq c_k\|\hat{u}_k\|_{H^1(\mathbb{R})}^2 \geq \frac{c_k}{2}\|\hat{u}_k\|_{L^2(\mathbb{R})}^2 + \frac{c_k}{2}\left(\|\hat{u}_k\|_{L^2(\mathbb{R})}^2 + \left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2\right) \\ &\geq \frac{c_k}{2\beta_k^4}\cdot\beta_k^4\left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right|^2 + \frac{c_k}{2}\|\hat{u}_k\|_{L^2(\mathbb{R})}^2 + \frac{c_k}{2\alpha_k^2}\alpha_k^2\left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Choosing $C_1 := \min_{|k|\leq K}\{\frac{1}{4}, \frac{c_k}{2\beta_k^4}, \frac{c_k}{2}, \frac{c_k}{2\alpha_k^2}\}$ we combine the above calculations and obtain

$$\|\hat{u}\|_{\mathcal{H}}^2 \geq C_1 \cdot \sum_{k\in\mathbb{Z}_{odd}} \left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2 + \alpha_k^2\|\hat{u}_k\|_{L^2(\mathbb{R})}^2 + \beta_k^4\left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right|^2.$$

Using again Lemma 2.28 part (ii) we see for any $k \in \mathbb{Z}_{odd}$

$$\beta_k^4\left|\langle \hat{u}_k, \varphi_k\rangle_{L^2(\mathbb{R})}\right|^2 \geq \frac{4}{3}\beta_k^2|\hat{u}_k(0)|^2 - \frac{1}{2}\left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}.$$

Hence,

$$\|\hat{u}\|_{\mathcal{H}}^2 \geq C_1 \cdot \sum_{k\in\mathbb{Z}_{odd}} \frac{1}{2}\left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2 + \alpha_k^2\|\hat{u}_k\|_{L^2(\mathbb{R})}^2 + \frac{4}{3}\beta_k^2|\hat{u}_k(0)|^2,$$

i.e., choosing $c := \frac{1}{2}C_1$, we obtain claim (i).

(ii) Using our preparations and Lemma 2.28 part (ii) we calculate

$$\begin{aligned}
\|\hat{u}\|_{\mathcal{H}}^2 &= \sum_{k\in\mathbb{Z}_{odd}} \left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2 + \alpha_k^2\|\hat{u}_k\|_{L^2(\mathbb{R})}^2 - 2\beta_k^2|\hat{u}_k(0)|^2 \\
&\qquad + \sum_{\lambda_k<0} 2(\beta_k^4-\alpha_k^2)\left|\langle \hat{u}_k,\varphi_k\rangle_{L^2(\mathbb{R})}\right|^2 + \sum_{\lambda_k=0}\left|\langle \hat{u}_k,\varphi_k\rangle_{L^2(\mathbb{R})}\right|^2\cdot(1+\beta_k^4) \\
&\leq \sum_{k\in\mathbb{Z}_{odd}} \left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2 + \alpha_k^2\|\hat{u}_k\|_{L^2(\mathbb{R})}^2 + (1+3\beta_k^4)\left|\langle \hat{u}_k,\varphi_k\rangle_{L^2(\mathbb{R})}\right|^2 \\
&\leq \sum_{k\in\mathbb{Z}_{odd}} \left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2 + \alpha_k^2\|\hat{u}_k\|_{L^2(\mathbb{R})}^2 + \frac{1+3\beta_k^4}{2\beta_k^4}\left(9\beta_k^2|\hat{u}_k(0)|^2 + 18\left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2\right) \\
&\leq \sum_{k\in\mathbb{Z}_{odd}} \left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2 + \alpha_k^2\|\hat{u}_k\|_{L^2(\mathbb{R})}^2 + \frac{1+3\left(\frac{1}{2}\beta\omega^2\right)^2}{2\left(\frac{1}{2}\beta\omega^2\right)^2}\left(9\beta_k^2|\hat{u}_k(0)|^2 + 18\left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2\right),
\end{aligned}$$

i.e., claim (ii) with $C := 19\cdot\left(\frac{3}{2}+\frac{2}{\beta^2\omega^4}\right)$ is proven. □

With this corollary we are now ready to proof Theorem 2.27.

Proof of Theorem 2.27. Observe that $\mathcal{H} \subset l^2\left(\mathbb{Z}, L^2(\mathbb{R})\right)$, $H^1(\mathbb{R}\times\mathbb{T}_T) \subset L^2(\mathbb{R}\times\mathbb{T}_T)$ and $S\colon l^2\left(\mathbb{Z}, L^2(\mathbb{R})\right) \to L^2(\mathbb{R}\times\mathbb{T}_T)$ is an isometric isomorphism. If $u \in C^1_{ap}(\mathbb{R}\times\mathbb{T}_T)\cap H^1(\mathbb{R}\times\mathbb{T}_T)$, then $u_t(0,\cdot)$ is continuous and we can evaluate it pointwise using Fourier series by $u_t(0,\cdot) = \sum_k ik\omega\hat{u}_k(0)e_k(t)$. Moreover writing $\hat{u} := S^{-1}u$ we see

$$\begin{aligned}
\|u\|_H^2 &= \|u_x\|_{L^2(\mathbb{R}\times\mathbb{T}_T)}^2 + \|u_t\|_{L^2(\mathbb{R}\times\mathbb{T}_T)}^2 + \|u_t(0,\cdot)\|_{L^2(\mathbb{T}_T)}^2 \\
&= \sum_{k\in\mathbb{Z}_{odd}} \left\|\hat{u}_k'\right\|_{L^2(\mathbb{R})}^2 + \sum_{k\in\mathbb{Z}_{odd}} k^2\omega^2\|\hat{u}_k\|_{L^2(\mathbb{R})}^2 + \sum_{k\in\mathbb{Z}_{odd}} k^2\omega^2|\hat{u}_k(0)|^2 \\
&\begin{cases} \leq \max\left\{1,\frac{1}{\alpha},\frac{1}{\beta}\right\}\cdot\frac{1}{c}\cdot\|\hat{u}\|_{\mathcal{H}}^2, \\ \geq \min\left\{1,\frac{1}{\alpha},\frac{1}{\beta}\right\}\cdot\frac{1}{C}\cdot\|\hat{u}\|_{\mathcal{H}}^2, \end{cases}
\end{aligned}$$

with $c, C > 0$ as in Corollary 2.29. The facts that $C^1_{ap}(\mathbb{R}\times\mathbb{T}_T)\cap H^1(\mathbb{R}\times\mathbb{T}_T)$ is dense in $L^2(\mathbb{R}\times\mathbb{T}_T)$ and $S\colon l^2\left(\mathbb{Z}, L^2(\mathbb{R})\right) \to L^2(\mathbb{R}\times\mathbb{T}_T)$ is an isometric isomorphism, yield that $\mathcal{Y} := S^{-1}(C^1_{ap}(\mathbb{R}\times\mathbb{T}_T)\cap H^1(\mathbb{R}\times\mathbb{T}_T))$ is dense in $\mathcal{H}$. Hence $S\colon (\mathcal{Y}, \|\cdot\|_{\mathcal{H}}) \to (C^1_{ap}(\mathbb{R}\times\mathbb{T}_T)\cap H^1(\mathbb{R}\times\mathbb{T}_T), \|\cdot\|_H)$ is continuous and continuously invertible and its extension $S\colon \mathcal{H}\to H$ is also continuous and continuously invertible. □

Since $\mathcal{H}$ and H are isomorphic via S, we write $\hat{u} := S^{-1}u$ if $u \in H$ and $u := S\hat{u}$ if $\hat{u}\in\mathcal{H}$ in the following.

Remark 2.30. *For arbitrary* $u \in H^1(\mathbb{R}\times\mathbb{T}_T)$*, the object* $u_t(0,\cdot)$ *might be not well defined. We give a definition in the sense of traces. Consider* $\operatorname{tr}\colon C^1_{ap}(\mathbb{R}\times\mathbb{T}_T)\to C(\mathbb{T}_T), \operatorname{tr}(u)(t) := u_t(0,\cdot)$*. Obviously the map* $\operatorname{tr}\colon (C^1_{ap}(\mathbb{R}\times\mathbb{T}_T), \|\cdot\|_H) \to (C(\mathbb{T}_T), \|\cdot\|_{L^2(\mathbb{T}_T)})$ *is continuous and continuously extends to* $\operatorname{tr}\colon (H,\|\cdot\|_H)\to(L^2(\mathbb{T}_T),\|\cdot\|_{L^2(\mathbb{T}_T)})$*. Hence, if* $u\in H$ *then* $u_t(0,\cdot)\in L^2(\mathbb{T}_T)$ *is well defined in this trace sense.*

Remark 2.31. *Arguing similar as in Section 2.3.2, we could now prove Theorem 2.21 with another concept of "solution", analogous to Definition 2.47.*

We invest some more work, which will later also be used in Chapter 3, and characterize $\mathcal{L}$ and $b_{\mathcal{L}}$, which act on sequences $\hat{u}$, with L and b_L, which act on functions.

2.3.1.2 Analysis of L

The aim of this section is to write down a self-adjoint operator $L\colon \mathcal{D}(L)\to L^2(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})$ and a closed, symmetric bilinear form $b_L\colon H\times H\to\mathbb{R}$ such that $S(\mathcal{D}(\mathcal{L}))=\mathcal{D}(L)$, $L\circ S=\mathcal{L}$ on $\mathcal{D}(\mathcal{L})$ and $b_L\circ S=b_{\mathcal{L}}$ on $\mathcal{H}$. One calculation is especially long and technical, therefore we put it into Section 2.3.1.4. At the end of this section we will observe some regularity of $\mathcal{D}(L)$.

Definition 2.32. *Define*

$$\begin{aligned}\mathcal{D}(L) := \big\{v\in H^1_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R}) \;\big|\; & v|_{(-\infty,0)\times\mathbb{T}_T}\in H^2((-\infty,0)\times\mathbb{T}_T),\\ & v|_{(0,\infty)\times\mathbb{T}_T}\in H^2((0,\infty)\times\mathbb{T}_T),\; v_{tt}(0,\cdot)\in L^2(\mathbb{T}_T),\\ & v_x(0_+,\cdot)-v_x(0_-,\cdot)=\beta\, v_{tt}(0,\cdot)\big\}\\ L\colon \mathcal{D}(L)\to L^2_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R}),\quad & Lu := -\alpha u_{tt}-u_{xx}+\beta\delta_0(x)u_{tt},\\ b_L\colon H\times H\to\mathbb{R},\quad b_L(u,v) := & \int_{\mathbb{R}\times\mathbb{T}_T}\alpha u_t v_t + u_x v_x \,\mathrm{d}(x,t) - \int_{\mathbb{T}_T}\beta u_t(0,\cdot)v_t(0,\cdot)\,\mathrm{d}t\,.\end{aligned}$$

Writing $v_{tt}(0,\cdot)\in L^2(\mathbb{T}_T)$ *we assume* $\mathcal{D}(L)\subset\overline{C^2_{ap}(\mathbb{R}\times\mathbb{T}_T)}^{\|v\|^2_L}$ *with* $\|v\|^2_L := \|v\|^2_{H^1(\mathbb{R}\times\mathbb{T}_T)} + \|v\|^2_{H^2((-\infty,0)\times\mathbb{T}_T)} + \|v\|^2_{H^2((0,\infty)\times\mathbb{T}_T)} + \|v_{tt}(0,\cdot)\|^2_{L^2(\mathbb{T}_T)}$.

Note that $\|\cdot\|^2_L$ is not the graph-norm $\|v\|^2_{\mathcal{D}(L)} := \|v\|^2_{L^2(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})}+\|Lv\|^2_{L^2(\mathbb{R}\times\mathbb{T}_T)}$.

Theorem 2.33. *Assume* (H_δ)*. Then* L *is self-adjoint on* $L^2_{ap}(\mathbb{R}\times\mathbb{T}_T)$*,* b_L *is closed and symmetric and* $S(\mathcal{D}(\mathcal{L}))=\mathcal{D}(L)$*,* $L\circ S=\mathcal{L}$ *on* $\mathcal{D}(\mathcal{L})$ *and* $b_L\circ S=b_{\mathcal{L}}$ *on* $\mathcal{H}$*. If* $\frac{2\sqrt{\alpha}}{\beta\omega}\notin\mathbb{N}_{odd}$*, then* L *is invertible.*

Proof. Symmetry of b_L and $b_L\circ S=b_{\mathcal{L}}$ on $\mathcal{H}$ are clear by construction. Since $b_{\mathcal{L}}$ is closed and $S\colon\mathcal{H}\to H$ is an isomorphism, b_L is closed. Observe that the set of functions $\{v\in C^\infty_c(\mathbb{R}\times\mathbb{T}_T,\mathbb{R}) \mid v(0,\cdot)=v_t(0,\cdot)\equiv 0\}$ is dense in $L^2(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})$ and hence in $\mathcal{D}(L)$. Hence L is densely defined. Using partial integration we calculate for $u\in\mathcal{D}(L), v\in H$ that $b_L(u,v)=\langle Lu,v\rangle_{L^2(\mathbb{R}\times\mathbb{T}_T)}$. Since b_L is symmetric, L is symmetric. Let $u\in\mathcal{D}(L)\cap C^2(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})$ and $v\in C^1_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})$. As in the proof of Theorem 2.27 we see

$$\langle Lu,v\rangle_{L^2(\mathbb{R}\times\mathbb{T}_T)} = b_L(u,v) = b_{\mathcal{L}}(\hat{u},\hat{v}) = \langle\mathcal{L}\hat{u},\hat{v}\rangle_{l^2(\mathbb{Z}_{odd},L^2(\mathbb{R}))}\,.$$

Using the density of $\mathcal{D}(L) \cap C^2(\mathbb{R} \times \mathbb{T}_T, \mathbb{R}) \subset \mathcal{D}(L) \subset L^2(\mathbb{R} \times \mathbb{T}_T, \mathbb{R})$, $C^1_{ap}(\mathbb{R} \times \mathbb{T}_T, \mathbb{R}) \subset H \subset L^2(\mathbb{R} \times \mathbb{T}_T, \mathbb{R})$, $S^{-1}(\mathcal{D}(L) \cap C^2(\mathbb{R} \times \mathbb{T}_T, \mathbb{R})) \subset \mathcal{D}(\mathcal{L}) \subset l^2\left(\mathbb{Z}_{odd}, L^2(\mathbb{R}, \mathbb{R})\right)$, and $S^{-1}(C^1_{ap}(\mathbb{R} \times \mathbb{T}_T, \mathbb{R})) \subset H \subset l^2\left(\mathbb{Z}_{odd}, L^2(\mathbb{R}, \mathbb{R})\right)$ and the fact that the map $S\colon \mathcal{H} \to H$, $S\colon L^2(\mathbb{R} \times \mathbb{T}_T) \to l^2\left(\mathbb{Z}_{odd}, L^2(\mathbb{R})\right)$ are isomorphisms, we obtain $S(\mathcal{D}(\mathcal{L})) = \mathcal{D}(L)$, $L \circ S = \mathcal{L}$. Hence $\mathcal{D}(L)$ is closed and therefore L is self-adjoint. It remains to prove invertibility of L, if $\frac{2\sqrt{\alpha}}{\beta\omega} \notin \mathbb{N}_{odd}$. This will be carried out in Section 2.3.1.4. □

Lemma 2.34. *Assume* (H_δ). *Then the embedding* $\mathcal{D}(L) \hookrightarrow H^\mu(\mathbb{R} \times \mathbb{T}_T, \mathbb{R})$ *is continuous for any* $\mu < \frac{3}{2}$.

Proof. Let $u \in \mathcal{D}(L)$ be arbitrary and define $f := Lu$. Then as in step 6 of the proof of Theorem 2.33 (see appendix) we obtain

$$u(x,t) = \sum_k \frac{\beta_k^2}{\alpha_k} \hat{u}_k(0) \mathrm{e}^{-\alpha_k |x|} e_k(t) + \sum_k \mathcal{F}^{-1}\left(\frac{1}{\alpha_k^2 + (\cdot)^2} \mathcal{F}\hat{f}_k\right)(x) e_k(t).$$

As in the proof of Theorem 2.33:

$$\left\| \sum_k \mathcal{F}^{-1}\left(\frac{1}{\alpha_k^2 + (\cdot)^2} \mathcal{F}\hat{f}_k\right)(x) e_k(t) \right\|_{H^2(\mathbb{R} \times \mathbb{T}_T)} \leq c(\alpha, \omega) \|f\|_{L^2(\mathbb{R} \times \mathbb{T}_T)} \leq c(\alpha, \omega) \|u\|_{\mathcal{D}(L)},$$

where $\|u\|_{\mathcal{D}(L)} := \|u\|_{L^2(\mathbb{R} \times \mathbb{T}_T)} + \|Lu\|_{L^2(\mathbb{R} \times \mathbb{T}_T)}$ denotes the graph norm of L. Moreover for $\mu \in [0, \frac{3}{2})$ we calculate

$$\begin{aligned}
&\left\| \sum_k \frac{\beta_k^2}{\alpha_k} \hat{u}_k(0) \mathrm{e}^{-\alpha_k |x|} e_k(t) \right\|^2_{H^\mu(\mathbb{R} \times \mathbb{T}_T)} \\
&= \sum_k \left\| \frac{\beta_k^2}{\alpha_k} \hat{u}_k(0) \mathrm{e}^{-\alpha_k |\cdot|} \right\|^2_{H^\mu(\mathbb{R})} + |k\omega|^{2\mu} \left\| \frac{\beta_k^2}{\alpha_k} \hat{u}_k(0) \mathrm{e}^{-\alpha_k |\cdot|} \right\|^2_{L^2(\mathbb{R})} \\
&= \sum_k \beta_k^4 |\hat{u}_k(0)|^2 \frac{2}{\pi} \left\| (1 + |\cdot|^\mu) \frac{1}{\alpha_k^2 + (\cdot)^2} \right\|^2_{L^2(\mathbb{R})} + |k\omega|^{2\mu} \frac{\beta_k^4}{\alpha_k^2} |\hat{u}_k(0)|^2 \cdot \frac{1}{\alpha_k} \\
&\leq \sum_k \left(\frac{\omega^4}{2\pi} \left\| \frac{1 + |\cdot|^\mu}{\alpha\omega^2 + (\cdot)^2} \right\|^2_{L^2(\mathbb{R})} + \omega^{2\mu+1} \frac{\beta^2}{\sqrt{\alpha}^3} \right) k^4 |\hat{u}_k(0)|^2 \\
&\leq c(\alpha, \beta, \omega, \mu) \|u\|^2_{\mathcal{D}(L)}.
\end{aligned}$$

Hence

$$\|u\|_{H^\mu(\mathbb{R} \times \mathbb{T}_T)} \leq c(\alpha, \beta, \omega, \mu) \|u\|_{\mathcal{D}(L)}.$$

□

Remark 2.35. *Observe that for any* $k \in \mathbb{Z}_{odd}$ *the function* $\Phi_k(x,t) := \varphi_k(x) e_k(t)$ *is in* $\mathcal{D}(L)$ *but not in* $H^{\frac{3}{2}}(\mathbb{R} \times \mathbb{T}_T, \mathbb{R})$. *Hence the embedding in Lemma 2.34 is optimal in the sense that* $\mathcal{D}(L) \not\hookrightarrow H^\mu(\mathbb{R} \times \mathbb{T}_T, \mathbb{R})$ *for all* $\mu \geq \frac{3}{2}$.

2.3.1.3 Proof of Theorem 2.21

Proof of Theorem 2.21. Define the functional

$$I\colon H \to \mathbb{R}, \quad I\left(u\right) := \frac{1}{2} b_L\left(u,u\right) - \frac{1}{p+1}\int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(x)|u|^{p+1}\,\mathrm{d}(x,t)\,.$$

By our analysis in Section 2.1 this functional is well-defined and C^1. Furthermore by Theorem 2.45 and Sobolev's embedding theorems, cf. [Ada75], the assumptions $(A1)$ and $(A2)$ are satisfied. Clearly $(A3)$ is also satisfied. By Proposition 2.11 the assumption (B) is satisfied, such that the existence of a ground state u of I in H follows from Theorem 2.12. Until now we only considered $\frac{T}{2}$-anti-periodic functions, it remains to prove that our ground state u satisfies the weak equation for any test-function $\varphi \in Y$ as claimed in Definition 2.22. Let $v = \sum_{k\in\mathbb{Z}} \hat{v} e_k \in Y$ be arbitrary. Set $v^{odd} := \sum_{k\in\mathbb{Z}_{odd}} \hat{v}_k e_k$ and $v^{even} := \sum_{k\in 2\mathbb{Z}} \hat{v}_k e_k$. Then $v^{odd} \in H$ and hence $b_L\left(u, v^{odd}\right) - \int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(x)|u|^{p-1}u\,v^{odd}\,\mathrm{d}(x,t) = 0$. Using symmetry in time we see $b_L\left(u, v^{even}\right) = b_{\mathcal{L}}\left(\hat{u}, \hat{v}^{even}\right) = 0$, since for even k we have $\hat{u}_k = 0$ and for odd k we have $\hat{v}_k^{even} = 0$. Furthermore $\Gamma(x)|u|^{p-1}u$ is $\frac{T}{2}$-anti-periodic and in particular $L^2(\mathbb{R}\times\mathbb{T}_T)$-orthogonal to v^{even}, and therefore $\int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(x)|u|^{p-1}u\,v^{even}\,\mathrm{d}(x,t) = 0$. Hence $b_L\left(u,v\right) - \int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(x)|u|^{p-1}u\,v\,\mathrm{d}(x,t) = 0$. Since $v \in Y$ was arbitrary, $u \in H$ is a weak solution of $Lu = \Gamma(x)|u|^{p-1}u$ in the sense of Definition 2.22. □

2.3.1.4 Proof of Theorem 2.33

For the remaining part of the proof of Theorem 2.33 we use convolutions on $\mathbb{R}$:

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x-y)\,\mathrm{d}y\,.$$

Citing e.g. [Gra08] we see that for $f, g \in L^2(\mathbb{R})$, $h \in H^m(\mathbb{R})$ with $m \in \mathbb{N}_0$ we have

$$\mathcal{F}\left(\frac{\mathrm{d}^m}{\mathrm{d}x^m}h\right)(\xi) = (-i\xi)^m \mathcal{F}h(\xi), \quad \|h\|_{H^m(\mathbb{R})} \cong \|(1+|\cdot|^m)\mathcal{F}h\|_{L^2(\mathbb{R})},$$
$$\mathcal{F}(f\cdot g) = \mathcal{F}f * \mathcal{F}g, \quad \mathcal{F}^{-1}(f\cdot g) = \mathcal{F}^{-1}f * \mathcal{F}^{-1}g.$$

Furthermore, for $a > 0$, $\mu < \frac{3}{2}$ a straightforward calculation yields

$$\mathcal{F}^{-1}\left(\frac{1}{a^2+(\cdot)^2}\right) = \sqrt{\frac{\pi}{2}}\frac{1}{a}\mathrm{e}^{-a|\cdot|} \in H^\mu(\mathbb{R}). \tag{2.4}$$

We start with a formal calculation for L_k, which motivates the starting point of the proof of Theorem 2.33. Let $\hat{u}_k \in \mathcal{D}(L_k)$ and $\hat{f}_k \in L^2(\mathbb{R})$. Then formally

$$\begin{aligned}
L_k\hat{u}_k = \hat{f}_k \quad &\Leftrightarrow \quad \left(-\partial_x^2 + \alpha_k^2\right)\hat{u}_k - 2\beta_k^2\delta_0(x)\hat{u}_k = \hat{f}_k \\
&\Leftrightarrow \quad \left(\xi^2 + \alpha_k^2\right)\mathcal{F}\hat{u}_k - 2\beta_k^2\mathcal{F}\left(\delta_0(x)\hat{u}_k\right) = \mathcal{F}\hat{f}_k \\
&\Leftrightarrow \quad \mathcal{F}\hat{u}_k - 2\beta_k^2\frac{1}{\xi^2+\alpha_k^2}\mathcal{F}\left(\delta_0(x)\hat{u}_k\right) = \frac{1}{\xi^2+\alpha_k^2}\mathcal{F}\hat{f}_k \\
&\Leftrightarrow \quad \hat{u}_k - 2\beta_k^2\mathcal{F}^{-1}\left(\frac{1}{\alpha_k^2+(\cdot)^2}\mathcal{F}\left(\delta_0(x)\hat{u}_k\right)\right) = \mathcal{F}^{-1}\left(\frac{1}{\alpha_k^2+(\cdot)^2}\right) * \hat{f}_k
\end{aligned}$$

$$
\begin{aligned}
&2\beta_k^2\mathcal{F}^{-1}\left(\frac{1}{\alpha_k^2+(\cdot)^2}\mathcal{F}\left(\delta_0(\cdot)\hat{u}_k\right)\right)(x)\\
&=2\beta_k^2\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{1}{\alpha_k^2+\xi^2}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\delta_0(y)\hat{u}_k(y)\mathrm{e}^{-\mathrm{i}y\xi}\,\mathrm{d}y\right)\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi\\
&=\beta_k^2\sqrt{\frac{2}{\pi}}\hat{u}_k(0)\cdot\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{1}{\alpha_k^2+\xi^2}\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi\\
&=\frac{\beta_k^2}{\alpha_k}\hat{u}_k(0)\mathrm{e}^{-\alpha_k|x|}.
\end{aligned}
$$

Definition 2.36. *For $\hat{u}_k\in C_c^\infty(\mathbb{R})$ define*

$$
K_k\hat{u}_k(x):=\frac{\beta_k^2}{\alpha_k}\hat{u}_k(0)\mathrm{e}^{-\alpha_k|x|}.
$$

Furthermore, define

$$
R_k\colon L^2(\mathbb{R})\to H^2(\mathbb{R}),\quad R_k\hat{f}_k:=\mathcal{F}^{-1}\left(\frac{1}{\alpha_k^2+(\cdot)^2}\right)*\hat{f}_k.
$$

Proposition 2.37. *1. The map K_k continuously extends to $K_k\colon H^\nu(\mathbb{R})\to H^\mu(\mathbb{R})$ for any $\nu>\frac{1}{2}$ and $\mu<\frac{3}{2}$. In particular, $K_k\colon H^\mu(\mathbb{R})\to H^\mu(\mathbb{R})$ is compact for any $\mu\in(\frac{1}{2},\frac{3}{2})$.*

2. The map R_k is continuous with $\|R_k\|_{L^2\to H^2}\le\max\left\{\frac{1}{\alpha_k^2},1\right\}$ and bijective.

Proof. 1. Using Sobolev's embedding, cf. [Ada75], we see that $H^\nu(\mathbb{R})\ni\hat{u}_k\mapsto\hat{u}_k(0)\in\mathbb{C}$ is continuous for $\nu>\frac{1}{2}$. A straightforward calculation yields for any $a>0$ that $\mathrm{e}^{-a|x|}\in H^\mu(\mathbb{R})$ for all $\mu<\frac{3}{2}$ but $\mathrm{e}^{-a|x|}\notin H^{\frac{3}{2}}(\mathbb{R})$. Hence K_k continuously extends to $K_k\colon H^\nu(\mathbb{R})\to H^\mu(\mathbb{R})$ for any $\nu>\frac{1}{2}$ and $\mu<\frac{3}{2}$. Observe that $\dim(\mathrm{Range}(K_k))=1$, hence $K_k\colon H^\mu(\mathbb{R})\to H^\mu(\mathbb{R})$ is compact for any $\mu\in(\frac{1}{2},\frac{3}{2})$.

2. We calculate using Fourier transform:

$$
\begin{aligned}
\left\|R_k\hat{f}_k\right\|_{H^2(\mathbb{R})}^2&=\left\|\frac{1}{\alpha_k^2+(\cdot)^2}\mathcal{F}\hat{f}_k\right\|_{L^2(\mathbb{R})}^2+\left\|\frac{(\cdot)^2}{\alpha_k^2+(\cdot)^2}\mathcal{F}\hat{f}_k\right\|_{L^2(\mathbb{R})}^2\\
&=\left\|\frac{\sqrt{1+(\cdot)^4}}{\alpha_k^2+(\cdot)^2}\mathcal{F}\hat{f}_k\right\|_{L^2(\mathbb{R})}^2.
\end{aligned}
$$

A straightforward maximization yields that $\frac{1+(\cdot)^4}{(\alpha_k^2+(\cdot)^2)^2}\le\max\left\{\frac{1}{\alpha_k^4},1\right\}$. Hence the claim follows.

□

Proof of Theorem 2.33. Our strategy is as follows: Using $\frac{2\sqrt{\alpha}}{\beta\omega}\notin\mathbb{N}_{odd}$ we construct a weak inverse of L_k, i.e., given $\hat{f}_k\in L^2(\mathbb{R})$ we find $\hat{u}_k\in H^1(\mathbb{R})$ such that for all $\hat{v}_k\in H^1(\mathbb{R})$ we have $b_{L_k}(\hat{u}_k,\hat{v}_k)=\left\langle\hat{f}_k,\hat{v}_k\right\rangle_{L^2(\mathbb{R})}$. This will yield $\hat{u}\in\mathcal{H}$ and this $\hat{u}$ is a weak solution to $\mathcal{L}\hat{u}=\hat{f}$. Applying S we find $u\in H$ is a weak solution to $Lu=f$. Using regularity

theory and some explicit calculations we verify $u \in \mathcal{D}(L)$ and hence we found a solution to $Lu = f$. Since $L \circ S = \mathcal{L}$ on the level of bilinear forms and they are uniquely determined by their bilinear forms, they are equal. Step 1: Solutions to $\hat{u}_k - K_k\hat{u}_k = R_k\hat{f}_k$.
We prove the following statement:

$$\begin{aligned} \forall\, \hat{f}_k \in L^2(\mathbb{R})\, \exists!\, \hat{u}_k \in H^1(\mathbb{R})\colon \quad & \hat{u}_k - K_k\hat{u}_k = R_k\hat{f}_k. \\ \text{Moreover: } \forall\, \mu < \frac{3}{2}\colon \quad & \hat{u}_k \in H^\mu(\mathbb{R}). \end{aligned} \tag{2.5}$$

We will use a corollary of Fredholm's alternative, cf. [Rud06]. By Remark 2.37 we only have to check injectivity of $\mathrm{Id} - K_k\colon H^1(\mathbb{R}) \to H^1(\mathbb{R})$ for existence and uniqueness of $\hat{u}_k$. Let $\hat{v}_k \in \mathrm{Ker}(\mathrm{Id} - K_k)$. Then

$$0 = \hat{v}_k - K_k\hat{v}_k = \hat{v}_k - \frac{\beta_k^2}{\alpha_k}\hat{v}_k(0)\mathrm{e}^{-\alpha_k|\cdot|}.$$

Assume $\hat{v}_k(0) \neq 0$. Then $\alpha_k^2 = \beta_k^4$ which contradicts $\frac{2\sqrt{\alpha}}{\beta\omega} \notin \mathbb{N}_{odd}$. Hence $\hat{v}_k(0) = 0$ and therefore $\hat{v}_k \equiv 0$, i.e., $\mathrm{Id} - K_k\colon H^1(\mathbb{R}) \to H^1(\mathbb{R})$ is injective. By Fredholm's alternative there is a unique $\hat{u}_k \in H^1(\mathbb{R})$ such that $\hat{u}_k - K_k\hat{u}_k = R_k\hat{f}_k$. Using this equation, $R_k\hat{f}_k \in H^2(\mathbb{R})$ and $K_k\hat{u}_k \in H^\mu(\mathbb{R})$, we obtain $\hat{u}_k \in H^\mu(\mathbb{R})$ for any $\mu < \frac{3}{2}$.

Step 2: $\hat{u}_k$ is a weak solution to $L_k\hat{u}_k = \hat{f}_k$.
By step 1 and (2.4) we have

$$\begin{aligned} & \hat{u}_k(x) - \frac{\beta_k^2}{\alpha_k}\hat{u}_k(0)\mathrm{e}^{-\alpha_k|x|} = \mathcal{F}^{-1}\left(\frac{1}{\alpha_k^2 + (\cdot)^2} \cdot \mathcal{F}\hat{f}_k\right)(x), && \forall\, x \in \mathbb{R}, \\ \Leftrightarrow \quad & \mathcal{F}\hat{u}_k(\xi) - \beta_k^2\hat{u}_k(0)\sqrt{\frac{2}{\pi}} \cdot \frac{1}{\alpha_k^2 + \xi^2} = \frac{1}{\alpha_k^2 + \xi^2} \cdot \mathcal{F}\hat{f}_k(\xi), && \text{for a.e. } \xi \in \mathbb{R}, \\ \Leftrightarrow \quad & \left(\xi^2 + \alpha_k^2\right)\mathcal{F}\hat{u}_k(\xi) - 2\beta_k^2\hat{u}_k(0) \cdot \frac{1}{\sqrt{2\pi}} = \mathcal{F}\hat{f}_k(\xi), && \text{for a.e. } \xi \in \mathbb{R}. \end{aligned}$$

For $\psi \in H^1(\mathbb{R})$ we obtain

$$\begin{aligned} b_{L_k}(\hat{u}_k, \psi) &= \int_{\mathbb{R}} \hat{u}_k'\overline{\psi'} + \alpha_k^2\hat{u}_k\overline{\psi}\,\mathrm{d}x - 2\beta_k^2\hat{u}_k(0)\overline{\psi(0)} \\ &= \int_{\mathbb{R}} -\mathrm{i}\xi\mathcal{F}\hat{u}_k\overline{(-i\xi\mathcal{F}\psi)} + \alpha_k^2\mathcal{F}\hat{u}_k\overline{\mathcal{F}\psi}\,\mathrm{d}\xi - 2\beta_k^2\hat{u}_k(0)\overline{\psi(0)} \\ &= \int_{\mathbb{R}} \left(\xi^2 + \alpha_k^2\right)\mathcal{F}\hat{u}_k\overline{\mathcal{F}\psi}\,\mathrm{d}\xi - 2\beta_k^2\hat{u}_k(0)\overline{\psi(0)} \\ &= \int_{\mathbb{R}} \left(\mathcal{F}\hat{f}_k + 2\beta_k^2\hat{u}_k(0) \cdot \frac{1}{\sqrt{2\pi}}\right)\overline{\mathcal{F}\psi}\,\mathrm{d}\xi - 2\beta_k^2\hat{u}_k(0)\overline{\psi(0)} \\ &= \int_{\mathbb{R}} \mathcal{F}\hat{f}_k\overline{\mathcal{F}\psi}\,\mathrm{d}\xi + 2\beta_k^2\hat{u}_k(0) \cdot \overline{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}\psi \cdot \mathrm{e}^{\mathrm{i}\xi\cdot 0}\,\mathrm{d}\xi} - 2\beta_k^2\hat{u}_k(0)\overline{\psi(0)} \\ &= \int_{\mathbb{R}} \mathcal{F}\hat{f}_k\overline{\mathcal{F}\psi}\,\mathrm{d}\xi + 2\beta_k^2\hat{u}_k(0) \cdot \overline{\psi(0)} - 2\beta_k^2\hat{u}_k(0)\overline{\psi(0)} \\ &= \left\langle \hat{f}_k, \psi \right\rangle_{L^2(\mathbb{R})}. \end{aligned}$$

Here we used that $\mathcal{F}\psi \in L^1(\mathbb{R})$. Hence, $\hat{u}_k$ is a weak solution to $L_k\hat{u}_k = \hat{f}_k$.

Step 3: $\hat{u}_k \in \mathcal{D}(L_k)$.

We already know by step 1 that $\hat{u}_k \in H^\mu(\mathbb{R})$ for any $\mu < \frac{3}{2}$. This step will prove that this regularity is in some sense optimal. Observe that if we take all test-functions $\psi \in H^1(\mathbb{R})$ with support in $(-\infty, 0)$ and $(0, \infty)$, we obtain that $\hat{u}_k$ is a weak solution to $-\hat{u}_k'' + \alpha_k^2 \hat{u}_k = \hat{f}_k$ on $(-\infty, 0)$ and $(0, \infty)$. Elliptic regularity theory directly yields $\hat{u}_k \in H^2_{loc}((-\infty, 0))$ and $\hat{u}_k \in H^2_{loc}((0, \infty))$. $\hat{f}_k \in L^2(\mathbb{R})$ yields $\hat{u}_k \in H^2((-\infty, 0))$ and $\hat{u}_k \in H^2((0, \infty))$. Hence the traces $\hat{u}_k'(0_-)$ and $\hat{u}_k'(0_+)$ exist in the classical sense. Now define for $j \in \mathbb{N}$

$$\psi_j(x) := \begin{cases} 0, & x \le -\frac{2}{j}, \\ j\left(x + \frac{2}{j}\right), & -\frac{2}{j} < x \le -\frac{1}{j}, \\ 1, & -\frac{1}{j} < x \le \frac{1}{j}, \\ -j\left(x - \frac{2}{j}\right), & \frac{1}{j} < x \le \frac{2}{j}, \\ 0, & \frac{2}{j} < x. \end{cases}$$

Since $\psi_j \to 0$ in $L^2(\mathbb{R})$, $\psi_j \in H^1(\mathbb{R})$ and $\psi_j(0) = 1$ we obtain by step 2

$$\begin{aligned} \mathrm{o}(1) = \left\langle \hat{f}_k, \psi_j \right\rangle_{L^2(\mathbb{R})} &= b_{L_k}(\hat{u}_k, \psi_j) = \int_{\mathbb{R}} \hat{u}_k' \overline{\psi_j'} + \alpha_k^2 \hat{u}_k \overline{\psi_j} \,\mathrm{d}x - 2\beta_k^2 \hat{u}_k(0) \overline{\psi_j(0)} \\ &= \int_{-\frac{2}{j}}^{-\frac{1}{j}} \hat{u}_k' \cdot j \,\mathrm{d}x + \int_{\frac{1}{j}}^{\frac{2}{j}} \hat{u}_k' \cdot (-j) \,\mathrm{d}x - 2\beta_k^2 \hat{u}_k(0) + \mathrm{o}(1) \\ &= j \cdot \left(\hat{u}_k\left(-\frac{1}{j}\right) - \hat{u}_k\left(-\frac{2}{j}\right)\right) - j \cdot \left(\hat{u}_k\left(\frac{2}{j}\right) - \hat{u}_k\left(\frac{1}{j}\right)\right) - 2\beta_k^2 \hat{u}_k(0) + \mathrm{o}(1) \\ &\longrightarrow \hat{u}_k'(0_-) - \hat{u}_k'(0_+) - 2\beta_k^2 \hat{u}_k(0). \end{aligned}$$

Hence $\hat{u}_k \in \mathcal{D}(L_k)$.

Step 4: $\hat{u} \in \mathcal{H}$

Since $\lambda_k = -\beta_k^4 + \alpha_k^2 \neq 0$ and $\varphi_k(x) = \beta_k \mathrm{e}^{-\beta_k^2 |x|}$ are an eigen-pair of L_k and $\hat{u}_k$ is a weak solution to $L_k \hat{u}_k = \hat{f}_k$ we obtain for $\lambda_k < 0$ using Lemma 2.28:

$$\begin{aligned} \min\left\{\alpha_k^2, |\lambda_k|\right\} \|\hat{u}_k\|_{L^2(\mathbb{R})}^2 &\le b_{|L_k|}(\hat{u}_k, \hat{u}_k) = b_{L_k}(\hat{u}_k, \hat{u}_k) + 2(-\lambda_k) \left| \langle \hat{u}_k, \varphi_k \rangle_{L^2(\mathbb{R})} \right|^2 \\ &= b_{L_k}(\hat{u}_k, \hat{u}_k) - 2 b_{L_k}(\hat{u}_k, \varphi_k) \overline{\langle \hat{u}_k, \varphi_k \rangle_{L^2(\mathbb{R})}} \\ &= \left\langle \hat{f}_k, \hat{u}_k \right\rangle_{L^2(\mathbb{R})} - 2 \left\langle \hat{f}_k, \varphi_k \right\rangle_{L^2(\mathbb{R})} \overline{\langle \hat{u}_k, \varphi_k \rangle_{L^2(\mathbb{R})}} \\ &\le \left\| \hat{f}_k \right\|_{L^2(\mathbb{R})} \|\hat{u}_k\|_{L^2(\mathbb{R})} + 2 \left\| \hat{f}_k \right\|_{L^2(\mathbb{R})} \|\varphi_k\|_{L^2(\mathbb{R})} \|\hat{u}_k\|_{L^2(\mathbb{R})} \|\varphi_k\|_{L^2(\mathbb{R})} \\ &= 3 \left\| \hat{f}_k \right\|_{L^2(\mathbb{R})} \|\hat{u}_k\|_{L^2(\mathbb{R})}. \end{aligned}$$

If $\lambda_k \ge 0$ we obtain $\min\{\alpha_k^2, |\lambda_k|\} \|\hat{u}_k\|_{L^2(\mathbb{R})}^2 \le \left\| \hat{f}_k \right\|_{L^2(\mathbb{R})} \|\hat{u}_k\|_{L^2(\mathbb{R})}$ in a similar way. Observe that there is a constant $c > 0$ such that $\min\{\alpha_k^2, |\lambda_k|\} \ge c$ uniform in $k \in \mathbb{Z}_{odd}$. Hence $\|\hat{u}_k\|_{L^2(\mathbb{R})} \le \frac{1}{c} \left\| \hat{f}_k \right\|_{L^2(\mathbb{R})}$. Since $f \in L^2(\mathbb{R} \times \mathbb{T}_T)$ we obtain $u = \sum_k \hat{u}_k e_k \in L^2(\mathbb{R} \times \mathbb{T}_T)$ with $\|u\|_{L^2(\mathbb{R} \times \mathbb{T}_T)} \le \frac{1}{c} \|f\|_{L^2(\mathbb{R} \times \mathbb{T}_T)}$. The above calculation also yields

$$\|\hat{u}\|_{\mathcal{H}}^2 = \sum_k b_{|L_k|}(\hat{u}_k, \hat{u}_k) \le 3 \sum_k \left\| \hat{f}_k \right\|_{L^2(\mathbb{R})} \|\hat{u}_k\|_{L^2(\mathbb{R})} \le 3 \|f\|_{L^2(\mathbb{R} \times \mathbb{T}_T)} \|u\|_{L^2(\mathbb{R} \times \mathbb{T}_T)} < \infty,$$

i.e., $\hat{u} \in \mathcal{H}$.

Step 5: u is a weak solution to $Lu = f$.
By step 3 and 4 we have for all $v \in H$

$$b_L(u,v) = \sum_k b_{L_k}(\hat{u}_k, \hat{v}_k) = \sum_k \left\langle \hat{f}_k, \hat{v}_k \right\rangle_{L^2(\mathbb{R})} = \langle f, v\rangle_{L^2(\mathbb{R}\times\mathbb{T}_T)},$$

i.e., u is a weak solution to $Lu = f$.

Step 6: $u \in \mathcal{D}(L)$.
As in step 3, observe that u is a weak solution of $-\alpha\, u_{tt} - u_{xx} = f$ on $(-\infty,0)\times\mathbb{T}_T$ and $(0,\infty)\times\mathbb{T}_T$. Elliptic regularity theory yields $u \in H^2_{loc}((-\infty,0)\times\mathbb{T}_T)$ and $u \in H^2_{loc}((0,\infty)\times\mathbb{T}_T)$. Since the $f \in L^2(\mathbb{R}\times\mathbb{T}_T)$, we obtain $u \in H^2((-\infty,0)\times\mathbb{T}_T)$ and $u \in H^2((0,\infty)\times\mathbb{T}_T)$. In particular the traces $u_x(0_-,\cdot)$ and $u_x(0_+,\cdot)$ exist as $L^2(\mathbb{T}_T)$-functions.
Next we show that the evaluation $u_{tt}(0,\cdot)$ is a well-defined $L^2(\mathbb{T}_T)$-function. Using step 1 and 4 we have

$$\begin{aligned} u(x,t) &= \sum_k \hat{u}_k(x)e_k(t) = \sum_k \left(K_k\hat{u}_k + R_k\hat{f}_k\right)(x)e_k(t) \\ &= \sum_k \left(\frac{\beta_k^2}{\alpha_k}\hat{u}_k(0)\mathrm{e}^{-\alpha_k|x|} + \mathcal{F}^{-1}\left(\frac{1}{\alpha_k^2+(\cdot)^2}\mathcal{F}\hat{f}_k\right)(x)\right)e_k(t). \end{aligned}$$

We now justify that we can split the sum into two sums. Observe that

$$\begin{aligned} &\left\|\sum_k \mathcal{F}^{-1}\left(\frac{1}{\alpha_k^2+(\cdot)^2}\mathcal{F}\hat{f}_k\right)(x)e_k(t)\right\|^2_{H^2(\mathbb{R}\times\mathbb{T}_T)} \\ &= \sum_k \left\|\mathcal{F}^{-1}\left(\frac{1}{\alpha_k^2+(\cdot)^2}\mathcal{F}\hat{f}_k\right)\right\|^2_{H^2(\mathbb{R})} + k^4\omega^4\left\|\mathcal{F}^{-1}\left(\frac{1}{\alpha_k^2+(\cdot)^2}\mathcal{F}\hat{f}_k\right)\right\|^2_{L^2(\mathbb{R})} \\ &= \sum_k \left\|\frac{1+(\cdot)^2}{\alpha_k^2+(\cdot)^2}\mathcal{F}\hat{f}_k\right\|^2_{L^2(\mathbb{R})} + \left\|\frac{k^2\omega^2}{\alpha_k^2+(\cdot)^2}\mathcal{F}\hat{f}_k\right\|^2_{L^2(\mathbb{R})} \\ &\leq \sum_k \left(1+\frac{1}{\alpha^2\omega^4}\right)\left\|\mathcal{F}\hat{f}_k\right\|^2_{L^2(\mathbb{R})} + \frac{1}{\alpha^2}\left\|\mathcal{F}\hat{f}_k\right\|^2_{L^2(\mathbb{R})} \\ &= \left(1+\frac{1}{\alpha^2\omega^4}+\frac{1}{\alpha^2}\right)\|f\|^2_{L^2(\mathbb{R}\times\mathbb{T}_T)} < \infty. \end{aligned}$$

Hence, writing $\mathbb{R}_+ := (0,\infty)$ and $\mathbb{R}_- := (-\infty,0)$, we see

$$\begin{aligned} \infty &> \left\|\sum_k \hat{u}_k(x)e_k(t) - \sum_k \mathcal{F}^{-1}\left(\frac{1}{\alpha_k^2+(\cdot)^2}\mathcal{F}\hat{f}_k\right)(x)e_k(t)\right\|^2_{H^2(\mathbb{R}_+\times\mathbb{T}_T)} \\ &= \left\|\sum_k \frac{\beta_k^2}{\alpha_k}\hat{u}_k(0)\mathrm{e}^{-\alpha_k|x|}e_k(t)\right\|^2_{H^2(\mathbb{R}_+\times\mathbb{T}_T)} \\ &= \sum_k \left\|\frac{\beta_k^2}{\alpha_k}\hat{u}_k(0)\mathrm{e}^{-\alpha_k|\cdot|}\right\|^2_{H^2(\mathbb{R}_+)} + k^4\omega^4\left\|\frac{\beta_k^2}{\alpha_k}\hat{u}_k(0)\mathrm{e}^{-\alpha_k|\cdot|}\right\|^2_{L^2(\mathbb{R}_+)} \\ &= \sum_k \frac{\beta_k^4}{\alpha_k^2}|\hat{u}_k(0)|^2\cdot\frac{1}{2}\left(\frac{1}{\alpha_k}+\alpha_k+\alpha_k^3\right) + k^4\omega^4\frac{\beta_k^4}{\alpha_k^2}|\hat{u}_k(0)|^2\cdot\frac{1}{2\alpha_k} \end{aligned}$$

$$\begin{cases} \leq C \cdot \sum_k k^5 |\hat{u}_k(0)|^2, \\ \geq \frac{1}{C} \cdot \sum_k k^5 |\hat{u}_k(0)|^2, \end{cases}$$

for some constant $C = C(\alpha, \beta, \omega) > 0$. The same calculation yields

$$\begin{aligned}
\infty &> \left\| \sum_k \hat{u}_k(x) e_k(t) - \sum_k \mathcal{F}^{-1} \left(\frac{1}{\alpha_k^2 + (\cdot)^2} \mathcal{F} \hat{f}_k \right)(x) e_k(t) \right\|^2_{H^2(\mathbb{R}_- \times \mathbb{T}_T)} \\
&= \left\| \sum_k \frac{\beta_k^2}{\alpha_k} \hat{u}_k(0) \mathrm{e}^{-\alpha_k |x|} e_k(t) \right\|^2_{H^2(\mathbb{R}_- \times \mathbb{T}_T)} \\
&= \sum_k \left\| \frac{\beta_k^2}{\alpha_k} \hat{u}_k(0) \mathrm{e}^{-\alpha_k |\cdot|} \right\|^2_{H^2(\mathbb{R}_-)} + k^4 \omega^4 \left\| \frac{\beta_k^2}{\alpha_k} \hat{u}_k(0) \mathrm{e}^{-\alpha_k |\cdot|} \right\|^2_{L^2(\mathbb{R}_-)} \\
&= \sum_k \frac{\beta_k^4}{\alpha_k^2} |\hat{u}_k(0)|^2 \cdot \frac{1}{2} \left(\frac{1}{\alpha_k} + \alpha_k + \alpha_k^3 \right) + k^4 \omega^4 \frac{\beta_k^4}{\alpha_k^2} |\hat{u}_k(0)|^2 \cdot \frac{1}{2\alpha_k} \\
&\begin{cases} \leq C \cdot \sum_k k^5 |\hat{u}_k(0)|^2, \\ \geq \frac{1}{C} \cdot \sum_k k^5 |\hat{u}_k(0)|^2. \end{cases}
\end{aligned}$$

Therefore

$$\|u_{tt}(0, \cdot)\|^2_{L^2(\mathbb{T}_T)} = \left\| \sum_k \hat{u}_k(0) \cdot (i\omega k)^2 e_k \right\|^2_{L^2(\mathbb{T}_T)} = \sum_k k^4 \omega^4 |\hat{u}_k(0)|^2 < \infty,$$

i.e., $u_{tt}(0, \cdot) \in L^2(\mathbb{T}_T)$. It remains to show, $u_x(0_+) - u_x(0_-) = \beta u_{tt}(0, \cdot) \in L^2(\mathbb{T}_T)$. Using $\hat{u}_k \in \mathcal{D}(L_k)$ we easily see

$$\begin{aligned}
u_x(0_+, \cdot) - u_x(0_-, \cdot) &= \sum_k \hat{u}_k'(0_+, \cdot) e_k - \hat{u}_k'(0_-, \cdot) e_k = \sum_k -\beta k^2 \omega^2 \hat{u}_k(0, \cdot) e_k \\
&= \sum_k \beta \hat{u}_k(0) \cdot \partial_t^2 e_k = \beta u_{tt}(0, \cdot).
\end{aligned}$$

Hence $u \in \mathcal{D}(L)$.

Step 7: Uniqueness of u.
Let $u, v \in \mathcal{D}(L)$ solve $Lu = f = Lv$. Then for any k we have $\hat{u}_k - K_k \hat{u}_k = R_k \hat{f}_k = \hat{v}_k - K_k \hat{v}_k$. The uniqueness in (2.5) yields $\hat{u}_k = \hat{v}_k$ for all k, i.e. $u = v$. □

2.3.2 Step-potentials in 1 space dimension

In this section we investigate a one dimensional step-potential V with negative background strength and a positive step symmetric around $x = 0$. In fact we assume

(H_S) Let $\alpha, \gamma, r > 0$. Define $\beta := \alpha + \gamma$ and for $x \in \mathbb{R}$ we set

$$V(x) := -\alpha + \beta \, \mathbb{1}_{[-r,r]}(x) = \begin{cases} \gamma, & |x| \leq r, \\ -\alpha, & |x| > r. \end{cases}$$

Observe that $-\alpha$ is the background strength, β is the height of the step relative to the background and γ measures how positive the step is. We analyze the operator $L = V(x)\partial_t^2 - \partial_x^2$ for $\frac{T}{2}$-anti-periodic functions. The potential V is strictly negative for $|x| > r$ and strictly positive for $|x| < r$. Hence this operator is elliptic for $|x| > r$ and hyperbolic for $|x| < r$. The main result in this section is the following.

Theorem 2.38. *Assume* (H_S)*,* $p \in (1,3)$ *and set* $\omega := \frac{\pi}{2r\sqrt{\gamma}}$*,* $T := \frac{\omega}{2\pi}$*. Let* $\Gamma \in L^\infty(\mathbb{R})$ *such that* $\Gamma(x) > 0$ *a.e. and* $\lim_{|x|\to\infty} \Gamma(x) = 0$ *and* $V(x) := -\alpha + \beta\, \mathbb{1}_{[-r,r]}(x)$*. Then there exists a nontrivial weak solution* u *of the equation*

$$V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u, \qquad (x,t) \in \mathbb{R} \times \mathbb{T}_T, \tag{2.2}$$

with minimal energy among all $\frac{T}{2}$*-anti-periodic, weak solutions.*

The term *weak solution* is defined in Definition 2.47, when we have all tools at hand to write down every object rigorously. In Proposition 2.48 we see: A weak solution is always a *very weak solution* in the following sense.

Definition 2.39. *Assume* (H_S)*,* $p > 1$ *and* $\Gamma \in L^\infty(\mathbb{R})$*.* $u \in L^{p+1}(\mathbb{R} \times \mathbb{T}_T)$ *is called a very weak solution of the equation* (2.2)*, if*

$$\forall\, \varphi \in C_c^2(\mathbb{R} \times \mathbb{T}_T)\colon \qquad \int_{\mathbb{R}\times\mathbb{T}_T} V(x)u\varphi_{tt} - u\varphi_{xx}\,\mathrm{d}(x,t) = \int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(x)|u|^{p-1}u\,\varphi\,\mathrm{d}(x,t)\,.$$

The rest of this section is dedicated to the proof of Theorem 2.38 following the strategy announced in the beginning of Section 2.3.

2.3.2.1 Spectral analysis of L_k

Definition 2.40. *Assume* (H_S) *and set* $\omega := \frac{\pi}{2r\sqrt{\gamma}}$*,* $T := \frac{\omega}{2\pi}$*. For* $k \in \mathbb{Z}_{odd}$ *define* $\mathcal{D}(L_k) := H^2(\mathbb{R})$*,*

$$L_k u := -\,\hat{u}_k'' - k^2\omega^2 \cdot V(x)\hat{u}_k$$

with the corresponding bilinear forms

$$b_{L_k}\colon H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{C}, \quad b_{L_k}(\hat{u}_k, \hat{v}_k) := \int_{\mathbb{R}} \hat{u}_k'\overline{\hat{v}_k'} - k^2\omega^2 V(x)\hat{u}_k\overline{\hat{v}_k}\,\mathrm{d}x\,.$$

Furthermore define the notations $\alpha_k^2 := \alpha k^2\omega^2$*,* $\beta_k^2 := \beta k^2\omega^2$*,* $\gamma_k^2 := \gamma k^2\omega^2$ *and for* $\lambda \in (-\gamma_k^2, \alpha_k^2)$ *write*

$$\varphi_{k,\lambda}^{odd}(x) := \begin{cases} -\sin\Big(\sqrt{\gamma_k^2+\lambda}\,r\Big)\mathrm{e}^{\sqrt{\alpha_k^2-\lambda}(x+r)}, & x < -r, \\ \sin\Big(\sqrt{\gamma_k^2+\lambda}\,x\Big), & |x| < r, \\ \sin\Big(\sqrt{\gamma_k^2+\lambda}\,r\Big)\mathrm{e}^{-\sqrt{\alpha_k^2-\lambda}(x-r)}, & x > r, \end{cases},$$

$$\varphi_{k,\lambda}^{even}(x) := \begin{cases} \cos\Big(\sqrt{\gamma_k^2+\lambda}\,r\Big)\mathrm{e}^{\sqrt{\alpha_k^2-\lambda}(x+r)}, & x < -r, \\ \cos\Big(\sqrt{\gamma_k^2+\lambda}\,x\Big), & |x| < r, \\ \cos\Big(\sqrt{\gamma_k^2+\lambda}\,r\Big)\mathrm{e}^{-\sqrt{\alpha_k^2-\lambda}(x-r)}, & x > r, \end{cases}$$

$$EV_k^{odd}(\lambda) :=\sqrt{\alpha_k^2-\lambda}\sin\Big(\sqrt{\gamma_k^2+\lambda}\,r\Big)+\sqrt{\gamma_k^2+\lambda}\cos\Big(\sqrt{\gamma_k^2+\lambda}\,r\Big),$$
$$EV_k^{even}(\lambda) :=\sqrt{\gamma_k^2+\lambda}\sin\Big(\sqrt{\gamma_k^2+\lambda}\,r\Big)-\sqrt{\alpha_k^2-\lambda}\cos\Big(\sqrt{\gamma_k^2+\lambda}\,r\Big).$$

Observe that in this section we do not have an additional factor $\frac{1}{2}$ in β_k^2.

Proposition 2.41. *Assume (H_S) and set $\omega := \frac{\pi}{2r\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. Then*

(i) L_k is self-adjoint on $L^2(\mathbb{R},\mathbb{R})$ and for $\hat{u}_k \in \mathcal{D}(L_k)$, $\hat{v}_k \in H^1(\mathbb{R},\mathbb{R})$ we have $b_{L_k}(\hat{u}_k,\hat{v}_k) = \langle L_k\hat{u}_k,\hat{v}_k\rangle_{L^2(\mathbb{R})}$.

(ii) There are $J_k \in \mathbb{N}$, $\lambda_{k,j} \in (-\gamma_k^2,\alpha_k^2)$ such that $\sigma(L_k) = \{\lambda_{k,j} \mid j = 1,\dots,J_k\} \cup [\alpha_k^2,\infty)$ and $0 < \inf_k \frac{1}{|k|}J_k$, $\sup_k \frac{1}{|k|}J_k < \infty$. Furthermore for $\lambda \in (-\gamma_k^2,\alpha_k^2)$ we have

$$\lambda \in \{\lambda_{k,j} \mid j = 1,\dots,J_k\} \qquad \Leftrightarrow \qquad EV_k^{odd}(\lambda)\cdot EV_k^{even}(\lambda) = 0.$$

The values $\lambda_{k,j}$ are simple eigenvalues. If $EV_k^{odd}(\lambda_{k,j}) = 0$, then the corresponding eigenfunction is $\varphi^{odd}_{k,\lambda_{k,j}}$ and if $EV_k^{even}(\lambda_{k,j}) = 0$, then the corresponding eigenfunction is $\varphi^{even}_{k,\lambda_{k,j}}$.

We want to remark that this proposition is true for any $\alpha,\gamma,k,\omega,r > 0$.

Proof. We only sketch the proof and cite [RS10] for details. Observe that $L_k = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\alpha_k^2-\gamma_k^2\,\mathbb{1}_{[-r,r]}(x)$. Writing $\tilde{L}_k := -\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\alpha_k^2\colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$, $B_k := -\gamma_k^2\,\mathbb{1}_{[-r,r]}(x)$ we see $L_k = \tilde{L}_k + B_k$ is a compact perturbation of the self-adjoint operator $\tilde{L}_k$ and hence self-adjoint with $\sigma_{\mathrm{ess}}(L_k) = \sigma_{\mathrm{ess}}(\tilde{L}_k) = [\alpha_k^2,\infty)$. A long but straightforward calculation yields: The point spectrum of L_k contains exactly all $\lambda \in (-\gamma_k^2,\alpha_k^2)$ such that $EV_k^{odd}(\lambda)\cdot EV_k^{even}(\lambda) = 0$ and the corresponding eigenfunctions are as claimed. We omit the calculation for this. Observe that if $\lambda_{k,j}$ is an eigenvalue of L_k then $\sin\Big(\sqrt{\gamma_k^2+\lambda_{k,j}}\,r\Big) \neq 0 \neq \cos\Big(\sqrt{\gamma_k^2+\lambda_{k,j}}\,r\Big)$ and either $EV_k^{odd}(\lambda_{k,j}) = 0$ or $EV_k^{even}(\lambda_{k,j}) = 0$ but not both are 0 and furthermore

$$EV_k^{odd}(\lambda_{k,j}) = 0 \qquad \Leftrightarrow \qquad -\sqrt{\frac{\gamma_k^2+\lambda_{k,j}}{\alpha_k^2-\lambda_{k,j}}} = \tan\Big(\sqrt{\gamma_k^2+\lambda_{k,j}}\,r\Big),$$
$$EV_k^{even}(\lambda_{k,j}) = 0 \qquad \Leftrightarrow \qquad \sqrt{\frac{\alpha_k^2-\lambda_{k,j}}{\gamma_k^2+\lambda_{k,j}}} = \tan\Big(\sqrt{\gamma_k^2+\lambda_{k,j}}\,r\Big).$$

The left hand sides are both monotone and the right hand side is periodic in $\lambda_{k,j}$. Observe that $\sqrt{\gamma_k^2+\lambda_{k,j}}\,r \in (0,\sqrt{\alpha+\gamma}\omega|k|r) = (0,\sqrt{\frac{\alpha}{\gamma}+1}\frac{\pi}{2}|k|)$, i.e., the function tan goes trough at least $\left\lfloor \frac{1}{2}\sqrt{\frac{\alpha}{\gamma}+1}\right\rfloor |k|$ and at most $\left\lceil \sqrt{\frac{1}{2}\frac{\alpha}{\gamma}+1}\right\rceil |k|$ periods (we used the Gaussian floor and ceil notation here). Hence the number of eigenvalues $\lambda_{k,j}$ of L_k grows linear in $|k|$ and the proposition is proven. □

Theorem 2.42. *Assume (H_S) and set $\omega := \frac{\pi}{2r\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. Then there is a constant $c > 0$ (independent of $k \in \mathbb{Z}_{odd}$) such that $(-c\cdot|k|, c\cdot|k|) \subset \rho(L_k)$.*

Proof. By assumption: $\sqrt{\gamma}\omega r = \frac{\pi}{2}$ and hence

$$\forall\, k \in \mathbb{Z}_{odd}: \qquad \cos(\gamma_k r) = \cos(\sqrt{\gamma} k\omega r) = 0, \quad \sin(\gamma_k r) = \sin(\sqrt{\gamma} k\omega r) = (-1)^{\frac{k-1}{2}}.$$

Therefore

$$\begin{aligned} EV_k^{odd}(0) \cdot EV_k^{even}(0) &= (\alpha_k \sin(\gamma_k r) + \gamma_k \cos(\gamma_k r)) \cdot (\gamma_k \sin(\gamma_k r) - \alpha_k \cos(\gamma_k r)) \\ &= \sqrt{\alpha}\sqrt{\gamma} k^2 \omega^2 = k^2 \cdot \sqrt{\frac{\alpha}{\gamma}} \left(\frac{\pi}{2r}\right)^2 > 0, \end{aligned}$$

and hence $0 \notin \sigma(L_k)$. Next we bound EV_k^{odd} and EV_k^{even} away from zero for $\lambda \in (-c \cdot |k|, c \cdot |k|)$. Since both are of very similar structure, namely

$$\begin{aligned} &\sqrt{\mu\, k^2\omega^2 \mp \lambda} \sin\left(\sqrt{\gamma_k^2 + \lambda\, r}\right) \pm \sqrt{\nu\, k^2\omega^2 \pm \lambda} \cos\left(\sqrt{\gamma_k^2 + \lambda\, r}\right), \\ &= \begin{cases} EV_k^{odd}(\lambda) & \text{with } \mu = \alpha,\ \nu = \gamma \text{ and choosing the upper signs,} \\ EV_k^{even}(\lambda) & \text{with } \mu = \gamma,\ \nu = \alpha \text{ and choosing the lower signs,} \end{cases} \end{aligned}$$

we will do this in one calculation. Here we also use the global estimates $|\cos(x)| \le \left|x - k\frac{\pi}{2}\right|$ and $|\sin(x)| \ge 1 - \frac{1}{2}\left(x - k\frac{\pi}{2}\right)^2$ for any $x \in \mathbb{R}, k \in \mathbb{Z}_{odd}$. Since we want to prove linear growth of the gap in k, we extract the factor $\frac{|\lambda|}{|k|}$ and will bound all other $|k|$ by 1 from below. Now let $\frac{|\lambda|}{|k|} \le \frac{\min\{\alpha,\gamma\}}{\gamma}\frac{\pi^2}{8r}$, $\mu, \nu \in \{\alpha, \gamma\}$, use $\sqrt{\gamma}\omega r = \frac{\pi}{2}$ and calculate

$$\begin{aligned} &\frac{r}{|k|\frac{\pi}{2}} \cdot \left| \sqrt{\mu\, k^2\omega^2 \pm \lambda} \sin\left(\sqrt{\gamma_k^2 + \lambda\, r}\right) \pm \sqrt{\nu\, k^2\omega^2 \pm \lambda} \cos\left(\sqrt{\gamma_k^2 + \lambda\, r}\right) \right| \\ &\ge \sqrt{\frac{\mu}{\gamma} - \frac{|\lambda|}{|k|}\frac{4r^2}{|k|\pi^2}} \cdot \left| \sin\left(\sqrt{\left(k\frac{\pi}{2}\right)^2 + \lambda\, r^2}\right) \right| \\ &\quad - \sqrt{\frac{\nu}{\gamma} + \frac{|\lambda|}{|k|}\frac{4r^2}{|k|\pi^2}} \cdot \left| \cos\left(\sqrt{\left(k\frac{\pi}{2}\right)^2 + \lambda\, r^2}\right) \right| \\ &\ge \sqrt{\frac{\mu}{\gamma} - \frac{|\lambda|}{|k|}\frac{4r^2}{|k|\pi^2}} \cdot \left(1 - \frac{1}{2}\left(\sqrt{\left(k\frac{\pi}{2}\right)^2 + \lambda\, r^2} - k\frac{\pi}{2}\right)^2\right) \\ &\quad - \sqrt{\frac{\nu}{\gamma} + \frac{|\lambda|}{|k|}\frac{4r^2}{|k|\pi^2}} \cdot \left| \sqrt{\left(k\frac{\pi}{2}\right)^2 + \lambda r^2} - k\frac{\pi}{2} \right| \\ &= \sqrt{\frac{\mu}{\gamma} - \frac{|\lambda|}{|k|}\frac{4r^2}{|k|\pi^2}} \cdot \left(1 - \frac{1}{2}\left(\frac{\lambda\, r^2}{\sqrt{\left(k\frac{\pi}{2}\right)^2 + \lambda\, r^2} + k\frac{\pi}{2}}\right)^2\right) \\ &\quad - \sqrt{\frac{\nu}{\gamma} + \frac{|\lambda|}{|k|}\frac{4r^2}{|k|\pi^2}} \cdot \left| \frac{\lambda\, r^2}{\sqrt{\left(k\frac{\pi}{2}\right)^2 + \lambda\, r^2} + k\frac{\pi}{2}} \right| \\ &\ge \sqrt{\frac{\mu}{\gamma} - \frac{|\lambda|}{|k|}\frac{4r^2}{|k|\pi^2}} - \frac{1}{2}\sqrt{\frac{\mu}{\gamma} - \frac{|\lambda|}{|k|}\frac{4r^2}{|k|\pi^2}} \left(\frac{\frac{\lambda}{k}\frac{2r^2}{\pi}}{\sqrt{1 - \frac{|\lambda|}{|k|}\frac{4r^2}{|k|\pi^2}} + 1}\right)^2 \\ &\quad - \sqrt{\frac{\nu}{\gamma} + \frac{|\lambda|}{|k|}\frac{4r^2}{|k|\pi^2}} \frac{\frac{|\lambda|}{|k|}\frac{2r^2}{\pi}}{\sqrt{1 - \frac{|\lambda|}{|k|}\frac{4r^2}{|k|\pi^2}} + 1} \end{aligned}$$

$$\geq \sqrt{\frac{\mu}{\gamma} - \frac{|\lambda|}{|k|}\frac{4r^2}{|k|\pi^2}} - \frac{1}{2}\sqrt{\frac{\mu}{\gamma} + \frac{|\lambda|}{|k|}\frac{4r^2}{|k|\pi^2}}\left(\frac{\lambda}{k}\right)^2\frac{4r^2}{\pi^2}r^2 - \sqrt{\frac{\nu}{\gamma} + \frac{|\lambda|}{|k|}\frac{4r^2}{|k|\pi^2}} \cdot \left|\frac{\lambda}{k}\right|\frac{2r^2}{\pi}$$
$$\geq \sqrt{\frac{\mu}{\gamma} - \frac{|\lambda|}{|k|}\frac{4r^2}{\pi^2}} - \frac{1}{2}\sqrt{\frac{\mu}{\gamma} + \frac{|\lambda|}{|k|}\frac{4r^2}{\pi^2}}\left(\frac{\lambda}{k}\right)^2\frac{4r^2}{\pi^2}r^2 - \sqrt{\frac{\nu}{\gamma} + \frac{|\lambda|}{|k|}\frac{4r^2}{\pi^2}} \cdot \left|\frac{\lambda}{k}\right|\frac{2r^2}{\pi}.$$

Observe that

$$\lim_{x\to 0}\sqrt{\frac{\mu}{\gamma} - x\cdot\frac{4r^2}{\pi^2}} - \frac{1}{2}\sqrt{\frac{\mu}{\gamma} + x\cdot\frac{4r^2}{\pi^2}}\cdot x^2\cdot\frac{4r^2}{\pi^2}r^2 - \sqrt{\frac{\nu}{\gamma} + x\cdot\frac{4r^2}{\pi^2}}\cdot x\cdot\frac{2r^2}{\pi} = \sqrt{\frac{\mu}{\gamma}} > 0$$

Hence, there is some $c > 0$ such that if $|\lambda| \leq c|k|$ then

$$\left|EV_k^{odd}(\lambda)\right|, |EV_k^{even}(\lambda)| \geq \sqrt{\frac{\min\{\alpha,\gamma\}}{\gamma}}\frac{\pi}{4r}|k|,$$

i.e., $\lambda \notin \sigma(L_k)$. □

Remark 2.43. *We note that the previous calculation is optimal in the exponent of $|k|$. Observe that*

$$\sqrt{\gamma_k^2 + \lambda}\,r - k\frac{\pi}{2} = \frac{\frac{2r^2}{\pi}}{\sqrt{1 - \frac{|\lambda|}{|k|}\frac{4r^2}{|k|\pi^2}} + 1}\cdot\frac{\lambda}{k}.$$

Hence, if $\lambda \in (-c\cdot|k|^s, c\cdot|k|^s)$ for $s > 1$, then sin *and* cos *will have large enough arguments to create zeros in the product.*

We now use this spectral information to define our sequence space $\mathcal{H}$. Observe that we assume symmetry in the coefficients such that the reconstructed function $S\hat{u}$ will be real valued.

Definition 2.44. *Assume (H_S) and set $\omega := \frac{\pi}{2r\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. Define*

$$\mathcal{H} := \left\{\hat{u} \in \left(H^1(\mathbb{R})\right)^{\mathbb{Z}_{odd}} \;\middle|\; \sum_{k\in\mathbb{Z}_{odd}}\langle\hat{u}_k,\hat{u}_k\rangle_{|L_k|} < \infty, \quad \overline{\hat{u}_k} = \hat{u}_{-k}\right\},$$
$$\mathcal{X} := \left\{\hat{u} \in \left(L^2(\mathbb{R})\right)^{\mathbb{Z}_{odd}} \;\middle|\; \sum_{k\in\mathbb{Z}_{odd}}\|\hat{u}_k\|^2_{L^2(\mathbb{R})} < \infty, \quad \overline{\hat{u}_k} = \hat{u}_{-k}\right\},$$

and apply all other constructions as in Section 2.2.2.

As seen in Section 2.2.2 $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a Hilbert space and $\mathcal{L}$ is self-adjoint on $\mathcal{X}$. Applying Theorem 2.17 we obtain the following corollary

Corollary 2.45. *Assume (H_S) and set $\omega := \frac{\pi}{2r\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. Then the map $S\colon \mathcal{H} \to L^{p+1}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})$ with*

$$\left(S\hat{u}\right)(x,t) := \sum_{k\in\mathbb{Z}_{odd}}\hat{u}_k(x)e_k(t), \qquad \hat{u}\in\mathcal{H}.$$

is continuous and locally compact for $p \in [1,3)$.

Lemma 2.46. *Assume* (H_S), *set* $\omega := \frac{\pi}{2r\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. *Then*

$$h^1\left(\mathbb{Z}_{odd}, L^2(\mathbb{R})\right) \cap l^2\left(\mathbb{Z}_{odd}, H^1(\mathbb{R})\right) \cap \{\overline{\hat{u}_k} = \hat{u}_{-k}\} \subset \mathcal{H}.$$

Proof. Let $\hat{u}_k \in H^1(\mathbb{R})$. As in Section 2.3.1.1 we use the fact that $\sigma(L_k) \cap (-\infty, 0) = \{\lambda_{k,j} \mid j = 1, \dots, J_k,\ \lambda_{k,j} < 0\}$ from Proposition 2.41 to see that

$$\hat{u}_k^- = \sum_{\{\lambda_{k,j}<0\}} \langle \hat{u}_k, \varphi_{k,j} \rangle_{L^2(\mathbb{R})} \varphi_{k,j},$$

where $\varphi_{k,j}$ is the eigenfunction to $\lambda_{k,j}$ of L_k normalized to $\|\varphi_{k,j}\|_{L^2(\mathbb{R})} = 1$. Using $L_k\varphi_{k,j} = \lambda_{k,j}\varphi_{k,j}$ we see by partial integration

$$\left\langle \varphi'_{k,j}, \varphi'_{k,l} \right\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} -\varphi''_{k,j}\varphi_{k,l}\,\mathrm{d}x = \left(\lambda_{k,j} - \alpha_k^2\right)\delta_{j,l} + \beta_k^2 \int_{-r}^{r} \varphi_{k,j}\varphi_{k,l}\,\mathrm{d}x\,.$$

Combining both we obtain

$$\begin{aligned}
\left\|(\hat{u}_k^-)'\right\|^2_{L^2(\mathbb{R})} &= \left\| \sum_{\{\lambda_{k,j}<0\}} \langle \hat{u}_k, \varphi_{k,j} \rangle_{L^2(\mathbb{R})} \varphi'_{k,j} \right\|^2_{L^2(\mathbb{R})} \\
&= \sum_{\{\lambda_{k,j}<0\}} \sum_{\{\lambda_{k,l}<0\}} \langle \hat{u}_k, \varphi_{k,l} \rangle_{L^2(\mathbb{R})} \overline{\langle \hat{u}_k, \varphi_{k,j} \rangle}_{L^2(\mathbb{R})} \left\langle \varphi'_{k,j}, \varphi'_{k,j} \right\rangle_{L^2(\mathbb{R})} \\
&= \sum_{\{\lambda_{k,j}<0\}} \left| \langle \hat{u}_k, \varphi_{k,l} \rangle_{L^2(\mathbb{R})} \right|^2 \left(\lambda_{k,j} - \alpha_k^2\right) + \beta_k^2 \int_{-r}^{r} \left|\hat{u}_k^-\right|^2 \mathrm{d}x \\
&\leq \beta_k^2 \left\|\hat{u}_k^-\right\|^2_{L^2(\mathbb{R})} \leq \beta_k^2 \|\hat{u}_k\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

where we used the inequality $\lambda_{k,j} < \alpha_k^2$ and the equality $\|\hat{u}_k\|^2_{L^2(\mathbb{R})} = \left\|\hat{u}_k^+\right\|^2_{L^2(\mathbb{R})} + \left\|\hat{u}_k^-\right\|^2_{L^2(\mathbb{R})}$. Next we calculate

$$\left\|(\hat{u}_k^+)'\right\|^2_{L^2(\mathbb{R})} \leq 2\left\|\hat{u}'_k\right\|^2_{L^2(\mathbb{R})} + 2\left\|(\hat{u}_k^-)'\right\|^2_{L^2(\mathbb{R})} \leq 2\left\|\hat{u}'_k\right\|^2_{L^2(\mathbb{R})} + 2\beta_k^2 \|\hat{u}_k\|^2_{L^2(\mathbb{R})}.$$

We combine these estimates and obtain for $\hat{u} \in h^1\left(\mathbb{Z}_{odd}, L^2(\mathbb{R})\right) \cap l^2\left(\mathbb{Z}_{odd}, H^1(\mathbb{R})\right)$

$$\begin{aligned}
\|\hat{u}\|^2_{\mathcal{H}} &= \sum_{k\in\mathbb{Z}_{odd}} \langle \hat{u}_k, \hat{u}_k \rangle_{|L_k|} = \sum_{k\in\mathbb{Z}_{odd}} b_{L_k}\left(\hat{u}_k^+, \hat{u}_k^+\right) - b_{L_k}\left(\hat{u}_k^-, \hat{u}_k^-\right) \\
&\leq \sum_{k\in\mathbb{Z}_{odd}} \left\|(\hat{u}_k^+)'\right\|^2_{L^2(\mathbb{R})} + (\alpha_k^2 + \gamma_k^2)\left\|\hat{u}_k^+\right\|^2_{L^2(\mathbb{R})} + \left\|(\hat{u}_k^-)'\right\|^2_{L^2(\mathbb{R})} + (\alpha_k^2 + \gamma_k^2)\left\|\hat{u}_k^-\right\|^2_{L^2(\mathbb{R})} \\
&\leq \sum_{k\in\mathbb{Z}_{odd}} 2\left\|\hat{u}'_k\right\|^2_{L^2(\mathbb{R})} + 4(\alpha k^2\omega^2 + \gamma k^2\omega^2)\|\hat{u}_k\|^2_{L^2(\mathbb{R})} \\
&\leq \max\{1, (\alpha+\gamma)\omega^2\}\|\hat{u}\|^2_{h^1(\mathbb{Z}_{odd}, L^2(\mathbb{R}))\cap l^2(\mathbb{Z}_{odd}, H^1(\mathbb{R}))},
\end{aligned}$$

where $\|\hat{u}\|^2_{h^1(\mathbb{Z}_{odd}, L^2(\mathbb{R}))\cap l^2(\mathbb{Z}_{odd}, H^1(\mathbb{R}))} := \sum_{k\in\mathbb{Z}_{odd}} \|\hat{u}'_k\|^2_{L^2(\mathbb{R})} + k^2\|\hat{u}_k\|^2_{L^2(\mathbb{R})}$. □

2.3.2.2 Proof of Theorem 2.38

Having Definition 2.44 and Corollary 2.45 we can now define the term *weak solution*.

Definition 2.47. *Assume (H_S), $p \in (1,3)$ and set $\omega := \frac{\pi}{2r\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. A function $u := S\hat{u}$ for $\hat{u} \in \mathcal{H}$ is called a weak solution of the equation (2.2) if*

$$\forall\, \hat{\varphi} \in \mathcal{H}\colon \qquad b_{\mathcal{L}}\left(\hat{u}, \hat{\varphi}\right) = \int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(x)|u|^{p-1}\, S\hat{\varphi}\, \mathrm{d}(x,t)\,.$$

Proposition 2.48. *Assume (H_S), $p \in (1,3)$ and set $\omega := \frac{\pi}{2r\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. Let $u := S\hat{u}$ for $\hat{u} \in \mathcal{H}$ be a weak solution of the equation (2.2) in the sense of Definition 2.47. Then u is a very weak solution of the equation (2.2) in the sense of Definition 2.39.*

Proof. Let $\varphi \in C_c^2(\mathbb{R}\times\mathbb{T}_T)$. Set $\varphi^{odd} := \sum_{k\in\mathbb{Z}_{odd}} \hat{\varphi}_k e_k$ and $\varphi^{even} := \sum_{k\in 2\mathbb{Z}} \hat{\varphi}_k e_k$. Using symmetry in time we see

$$\int_{\mathbb{R}\times\mathbb{T}_T} V(x)u\varphi_{tt}^{even} - u\varphi_{xx}^{even}\, \mathrm{d}(x,t) = 0, \qquad \int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(x)|u|^{p-1}u\,\varphi^{even}\, \mathrm{d}(x,t) = 0.$$

Moreover $\hat{\varphi}^{odd} := S^{-1}\varphi^{odd} \in h^1\left(\mathbb{Z}_{odd}, L^2(\mathbb{R})\right) \cap l^2\left(\mathbb{Z}_{odd}, H^1(\mathbb{R})\right) \subset \mathcal{H}$ by Proposition 2.46. Using the solution property of u we see

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(x)|u|^{p-1}u\,\varphi^{odd}\, \mathrm{d}(x,t) &= b_{\mathcal{L}}\left(\hat{u}, \hat{\varphi}^{odd}\right) = \sum_{k\in\mathbb{Z}_{odd}} b_{L_k}\left(\hat{u}_k, \hat{\varphi}_k\right) \\ &= \sum_{k\in\mathbb{Z}_{odd}} \int_{\mathbb{R}} -\hat{u}_k(\varphi_k^{odd})'' - k^2\omega^2 V(x)u\varphi\, \mathrm{d}x = \int_{\mathbb{R}\times\mathbb{T}_T} V(x)u\varphi_{tt}^{odd} - u\varphi_{xx}^{odd}\, \mathrm{d}(x,t)\,.\end{aligned}$$

Combining both calculations with $\varphi = \varphi^{odd} + \varphi^{even}$ yields

$$\int_{\mathbb{R}\times\mathbb{T}_T} V(x)u\varphi_{tt} - u\varphi_{xx}\, \mathrm{d}(x,t) - \int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(x)|u|^{p-1}u\,\varphi\, \mathrm{d}(x,t) = 0.$$

Since $\varphi \in C_c^2(\mathbb{R}\times\mathbb{T}_T)$ was arbitrary, the claim is proven. □

With all these preparations we can prove Theorem 2.38 rather quickly.

Proof of Theorem 2.38. We consider

$$\mathcal{I}\colon \mathcal{H} \to \mathbb{R}, \;\; \mathcal{I}\left(\hat{u}\right) := \frac{1}{2} b_{\mathcal{L}}\left(\hat{u}, \hat{u}\right) - \frac{1}{p+1}\int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(x)|S\hat{u}|^{p+1}\, \mathrm{d}(x,t)\,.$$

By Corollary 2.45 we have $(A1)$. Hence $\mathcal{I}$ is well defined. Condition $(A2)$ for $b_{\mathcal{L}}$ directly follows from the constructions in Section 2.2. By our assumptions condition $(A3)$ is also fulfilled. Applying Theorem 2.12, we obtain a ground state $\hat{u}$ of $\mathcal{I}$ in $\mathcal{H}$ and hence a weak solution. □

2.3.3 Radially symmetric step-potentials in 2 space dimensions

In this section we investigate a 2 dimensional, radially symmetric step-potential V with negative background strength and a positive step on a disk centered at $x=0$. In fact we assume

(H_R) Let $\alpha, \gamma, R > 0$. Define $\beta := \alpha + \gamma$ and for $x \in \mathbb{R}^2$ we set

$$V(x) := -\alpha + \beta\, \mathbb{1}_{B_R(0)}(x) = \begin{cases} \gamma, & |x| \le R, \\ -\alpha, & |x| > R. \end{cases}$$

Observe that $-\alpha$ is the background strength, β is the height of the step relative to the background and γ measures how positive the step is. We analyze the operator $L = V(x)\partial_t^2 - \Delta$ for radially symmetric and $\frac{T}{2}$-anti-periodic functions. The potential V is strictly negative for $|x| > R$ and strictly positive for $|x| < R$. Hence this operator is elliptic for $|x| > R$ and hyperbolic for $|x| < R$. The main result in this section is the following. Note that we do not assume any decay or periodicity on Γ.

Theorem 2.49. *Assume (H_R), $p \in (1,2)$ and set $\omega := \frac{\pi}{2R\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. Let $\Gamma \in L^\infty(0,\infty)$ be positive a.e. and $V(x) := -\alpha + \beta\, \mathbb{1}_{B_R(0)}(x)$. Then there exists a nontrivial weak solution u of the equation*

$$V(x)u_{tt} - \Delta u = \Gamma(|x|)|u|^{p-1}u, \qquad (x,t) \in \mathbb{R}^2 \times \mathbb{T}_T. \tag{2.3}$$

with minimal energy among all $\frac{T}{2}$-anti-periodic, weak solutions.

The term *weak solution* is defined in Definition 2.58, when we have all tools at hand to write down every object rigorously. In Proposition 2.59 we see: A weak solution is always a *very weak solution* in the following sense.

Definition 2.50. *Assume (H_R), $p > 1$ and $\Gamma \in L^\infty(\mathbb{R})$. $u \in L^{p+1}(\mathbb{R}^2 \times \mathbb{T}_T)$ is called a very weak solution of the equation (2.3), if*

$$\forall \varphi \in C_c^2(\mathbb{R}^2 \times \mathbb{T}_T): \qquad \int_{\mathbb{R}^2 \times \mathbb{T}_T} V(x)u\varphi_{tt} - u\,\Delta\varphi \,\mathrm{d}(x,t) = \int_{\mathbb{R}^2 \times \mathbb{T}_T} \Gamma(|x|)|u|^{p-1}u\,\varphi \,\mathrm{d}(x,t)\,.$$

The rest of this section is dedicated to the proof of Theorem 2.49 following the strategy announced in the beginning of Section 2.3.

2.3.3.1 Spectral analysis of L_k

Definition 2.51. *Assume (H_R) and set $\omega := \frac{\pi}{2R\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. For $k \in \mathbb{Z}_{odd}$ define $\mathcal{D}(L_k) := H^2(\mathbb{R}^2, \mathbb{R})$,*

$$L_k u := -\,\hat{u}_k'' - k^2\omega^2 \cdot V(x)\hat{u}_k$$

with the corresponding sesquilinear forms

$$b_{L_k} \colon H^1_{rad}(\mathbb{R}^2) \times H^1_{rad}(\mathbb{R}^2) \to \mathbb{C}, \quad b_{L_k}(\hat{u}_k, \hat{v}_k) := \int_{\mathbb{R}^2} \hat{u}_k' \overline{\hat{v}_k'} - \alpha k^2\omega^2 V(x)\hat{u}_k\overline{\hat{v}_k}\,\mathrm{d}x\,.$$

Furthermore define the notations $\alpha_k^2 := \alpha k^2\omega^2$, $\beta_k^2 := \beta k^2\omega^2$ *and* $\gamma_k^2 := \gamma k^2\omega^2$. *Let*

$$J_0(r) := \frac{1}{\pi}\int_0^\pi \cos(r\sin(\tau))\,\mathrm{d}\tau\,, \qquad K_0(r) := \int_0^\infty \mathrm{e}^{-r\cosh(\tau)}\,\mathrm{d}\tau \qquad \textit{for } r>0,$$

i.e., J_0 *is a solution of Bessel's differential equation* $r^2 f''(r) + rf'(r) + r^2 f(r) = 0$ *and* K_0 *is a solution of the modified Bessel differential equation* $r^2 f''(r) + rf'(r) - r^2 f(r) = 0$. J_0 *is often called the Bessel-function of the first kind,* K_0 *is often called the modified Bessel-function of the first kind. Next define the auxiliary functions*

$$j(r) := \frac{J_0(r)}{r\cdot J_0'(r)}, \qquad \kappa(r) := \frac{K_0(r)}{r\cdot K_0'(r)} \qquad \textit{for } r>0.$$

Last we define the functions

$$\varphi_{k,\lambda}(x) := \begin{cases} K_0\left(\sqrt{\alpha_k^2-\lambda}\cdot R\right)\cdot J_0\left(\sqrt{\gamma_k^2+\lambda}\cdot |x|\right), & |x|<R,\\ J_0\left(\sqrt{\gamma_k^2+\lambda}\cdot R\right)\cdot K_0\left(\sqrt{\alpha_k^2-\lambda}\cdot |x|\right), & |x|>R.\end{cases}$$

Proposition 2.52. *Assume* (H_R) *and set* $\omega := \frac{\pi}{2R\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. *Then*

(i) L_k *is self-adjoint on* $L^2(\mathbb{R}^2)$ *and for* $\hat{u}_k \in \mathcal{D}(L_k)$, $\hat{v}_k \in H^1(\mathbb{R}^2)$ *we have* $b_{L_k}(\hat{u}_k, \hat{v}_k) = \langle L_k\hat{u}_k, \hat{v}_k\rangle_{L^2(\mathbb{R}^2)}$.

(ii) There are $J_k \in \mathbb{N}_0, \lambda_{k,j} \in (-\gamma_k^2, \alpha_k^2)$ *such that* $\sigma(L_k) = \{\lambda_{k,j} \mid j=1,\ldots,J_k\} \cup [\alpha_k^2,\infty)$ *and* $\sup_k \frac{1}{|k|} J_k < \infty$. *Furthermore for* $\lambda \in (-\gamma_k^2, \alpha_k^2)$ *we have*

$$\lambda \in \{\lambda_{k,j} \mid j=1,\ldots,J_k\} \qquad \Leftrightarrow \qquad j\left(\sqrt{\gamma_k^2+\lambda_{k,j}}\cdot R\right) = \kappa\left(\sqrt{\alpha_k^2-\lambda_{k,j}}\cdot R\right).$$

The values $\lambda_{k,j}$ *are simple eigenvalues.*

Proof. We argue exactly as in Proposition 2.41 for part (*i*). Part (*ii*) are straightforward calculations, we only proof the at most linear growth of J_k. The key is the analysis of the behavior of j and κ at ∞. All cited asymptotics can be fund in [ASR88]. The following formulas for $r\to\infty$ are true:

$$J_0(r) = \sqrt{\frac{2}{\pi r}}\cdot\left(\cos\left(r-\frac{\pi}{4}\right) + O\left(|r|^{-1}\right)\right), \quad K_0(r) = \sqrt{\frac{2}{\pi r}}\mathrm{e}^{-r}\cdot\left(1-\frac{1}{8r}+O\left(|r|^{-2}\right)\right).$$

Using the recursion formula and asymptotic for the Bessel function of first kind J_1 we obtain

$$J_0'(r) = -J_1(r) = -\sqrt{\frac{2}{\pi r}}\cdot\left(\sin\left(r-\frac{\pi}{4}\right)+O\left(|r|^{-1}\right)\right), \qquad \text{as } r\to\infty.$$

Hence we obtain for $r\to\infty$

$$j(r) = \frac{J_0(r)}{r\cdot J_0'(r))} = -\frac{1}{r}\cdot\frac{\cos(r-\frac{\pi}{4}) + O\left(|r|^{-1}\right)}{\sin(r-\frac{\pi}{4})+O\left(|r|^{-1}\right)} = -\frac{1}{r}\cdot\cot\left(r-\frac{\pi}{4}\right)+O\left(|r|^{-2}\right).$$

Using the recursion formula and asymptotics for the modified Bessel function of first kind K_1 we obtain similarly

$$K_0'(r) = -K_1(r) = -\sqrt{\frac{2}{\pi r}}\mathrm{e}^{-r} \cdot \left(1 - \frac{3}{8r} + O\left(|r|^{-2}\right)\right), \qquad \text{as } r \to \infty.$$

Hence we obtain for $r \to \infty$

$$\kappa(r) = \frac{K_0(r)}{r \cdot K_0'(r)} = -\frac{1}{r} \cdot \frac{\sqrt{\frac{2}{\pi r}}\mathrm{e}^{-r} \cdot \left(1 - \frac{1}{8r} + O\left(|r|^{-2}\right)\right)}{\sqrt{\frac{2}{\pi r}}\mathrm{e}^{-r} \cdot \left(1 - \frac{3}{8r} + O\left(|r|^{-2}\right)\right)} = -\frac{1}{r} + O\left(|r|^{-2}\right) < 0.$$

Putting this together, for big r we see j will be oscillating between $-\infty$ and ∞ like $-\frac{1}{r} \cdot \cot\big(r - \frac{\pi}{4}\big)$ and κ will be strictly negative. Hence approximately in intervals of the form $(n\pi + \frac{\pi}{4}, n\pi + \frac{3\pi}{4})$ for $n \in \mathbb{N}$ the function j is negative and it is strictly monotonically decreasing from 0 to $-\infty$. Inserting now the arguments $\sqrt{\gamma_k^2 + \lambda_{k,j}} \cdot R$ and $\sqrt{\alpha_k^2 - \lambda_{k,j}} \cdot R$ into j and κ, we obtain

$$\lambda \in \{\lambda_{k,j} \mid j = 1, \ldots, J_k\} \qquad \Leftrightarrow \qquad j\left(\sqrt{\gamma_k^2 + \lambda} \cdot R\right) = \kappa\left(\sqrt{\alpha_k^2 - \lambda} \cdot R\right).$$

Arguing with the asymptotics and sign changes as before, we obtain the at most linear growth of J_k. For more details we refer to the next theorem. □

Note that we neither claim any lower bound on J_k nor do we claim that 0 is not in the spectrum of any L_k.

Theorem 2.53. *Assume* (H_R) *and set* $\omega := \frac{\pi}{2R\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. *Then there is* $K \in \mathbb{N}$ *and a constant* $c > 0$ *(independent of* $k \in \mathbb{Z}_{odd}$*) such that* $(-c \cdot |k|, c \cdot |k|) \subset \rho(L_k)$.

Proof. We continue the arguments of Proposition 2.52. For any $\varepsilon > 0$ there is some $r_0 > 0$ such that for any $n \in \mathbb{N}$ we have:

$$r \in \left(n\pi - \frac{\pi}{4} + \varepsilon, n\pi + \frac{\pi}{4} - \varepsilon\right) \cap (r_0, \infty), \quad \tilde{r} > r_0 \quad \Longrightarrow \quad j(r) > 0 > \kappa(\tilde{r}).$$

Choose $\varepsilon := \frac{\pi}{8}$ and let r_0 be as above. We want to construct $c > 0$ in such a way, that $\lambda \in (-c \cdot |k|, c \cdot |k|) \subset (-\gamma_k^2, \alpha_k^2)$ implies $\left|\sqrt{\gamma_k^2 + \lambda} \cdot R - k\pi\right| < \frac{\pi}{8}$ and hence $j\left(\sqrt{\gamma_k^2 + \lambda}R\right) > 0$, i.e., λ is not an eigenvalue of L_k. By assumption (H_R): $\sqrt{r}\omega R = \pi$. Hence

$$\sqrt{\gamma_k^2 + \lambda} \cdot R - k\pi = \sqrt{k^2\pi^2 + \lambda R^2} - k\pi = \frac{\lambda R^2}{\sqrt{k^2\pi^2 + \lambda R^2} + k\pi} = \frac{\frac{\lambda R^2}{k\pi}}{\sqrt{1 + \frac{1}{k\pi} \cdot \frac{\lambda R^2}{k\pi}} + 1}.$$

We define $c := \frac{\pi^2}{8R^2}$ and $K := \frac{r_0}{\pi} + 1$. Then if $|\lambda| < c \cdot |k|$ we have

$$\left|\sqrt{\gamma_k^2 + \lambda} \cdot R - k\pi\right| < \frac{c \cdot \frac{R^2}{\pi}}{\sqrt{1 - \frac{1}{k\pi} \cdot c \cdot \frac{R^2}{\pi}} + 1} = \frac{\frac{\pi}{8}}{\sqrt{1 - \frac{1}{8k}} + 1} < \frac{\pi}{8},$$

i.e., if in addition $k \geq K$ we have

$$j\left(\sqrt{\gamma_k^2 + \lambda}\right) > 0 > \kappa\left(\sqrt{\alpha_k^2 - \lambda}\right),$$

and hence λ is in the resolvent of L_k. □

Remark 2.54. *With a similar argument as in Remark 2.43 we see that for any* $s > 1$ *there are no* $c > 0$ *and* $K \in \mathbb{N}$ *such that for all* $k \in \mathbb{Z}_{odd}$ *with* $|k| \geq K$ *the whole interval* $(-c \cdot |k|^s, c \cdot |k|^s)$ *is in the resolvent of* L_k.

We now use this spectral information to define our sequence space $\mathcal{H}$. Observe that we assume symmetry in the coefficients such that the reconstructed function will be real valued.

Definition 2.55. *Assume* (H_R) *and set* $\omega := \frac{\pi}{2R\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. *Define*

$$\mathcal{H} := \left\{ \hat{u} \in \left(H^1_{rad}(\mathbb{R}^2)\right)^{\mathbb{Z}_{odd}} \;\middle|\; \sum_{k\in\mathbb{Z}_{odd}} \langle \hat{u}_k, \hat{u}_k \rangle_{|L_k|} < \infty, \quad \overline{\hat{u}_k} = \hat{u}_{-k} \right\},$$

$$\mathcal{X} := \left\{ \hat{u} \in \left(L^2_{rad}(\mathbb{R}^2)\right)^{\mathbb{Z}_{odd}} \;\middle|\; \sum_{k\in\mathbb{Z}_{odd}} \|\hat{u}_k\|^2_{L^2(\mathbb{R}^2)} < \infty, \quad \overline{\hat{u}_k} = \hat{u}_{-k} \right\},$$

and apply all other constructions as in Section 2.2.2.

As seen in Section 2.2.2 $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is a Hilbert space and $\mathcal{L}$ is self-adjoint on $\mathcal{X}$. Applying Theorem 2.17 we obtain the following corollary

Corollary 2.56. *Assume* (H_R) *and set* $\omega := \frac{\pi}{2R\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. *Then the map* $S\colon \mathcal{H} \to L^{p+1}(\mathbb{R}^2 \times \mathbb{T}_T, \mathbb{R})$ *with*

$$(S\hat{u})(x,t) := \sum_{k\in\mathbb{Z}_{odd}} \hat{u}_k(x) e_k(t), \qquad \hat{u} \in \mathcal{H}.$$

is continuous and locally compact for $p \in [1, 2)$.

Lemma 2.57. *Assume* (H_R), *set* $\omega := \frac{\pi}{2R\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. *Then*

$$h^1\left(\mathbb{Z}_{odd}, L^2(\mathbb{R}^2)\right) \cap l^2\left(\mathbb{Z}_{odd}, H^1(\mathbb{R}^2)\right) \cap \{\overline{\hat{u}_k} = \hat{u}_{-k}\} \subset \mathcal{H}.$$

Proof. Change $\mathbb{R}$ to $\mathbb{R}^2$ in the proof of Lemma 2.46. Possibly there are 0 eigenvalues but there are most finitely many. Hence we can argue as in the proof of Theorem 2.17 with equivalence of scalar products. □

2.3.3.2 Proof of Theorem 2.49

Having Definition 2.55 and Corollary 2.56 we can now define the term *weak solution.*

Definition 2.58. *Assume* (H_R), $p \in (1,2)$ *and set* $\omega := \frac{\pi}{2R\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. *A function* $u := S\hat{u}$ *for* $\hat{u} \in \mathcal{H}$ *is called a weak solution of the equation* (2.3) *if*

$$\forall\, \hat{\varphi} \in \mathcal{H}\colon \qquad b_{\mathcal{L}}(\hat{u}, \hat{\varphi}) = \int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(x)|u|^{p-1}\, S\hat{\varphi}\, \mathrm{d}(x,t)\,.$$

Proposition 2.59. *Assume* (H_R), $p \in (1,2)$ *and set* $\omega := \frac{\pi}{2R\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$. *Let* $u := S\hat{u}$ *for* $\hat{u} \in \mathcal{H}$ *be a weak solution of the equation* (2.3) *in the sense of Definition 2.58. Then* u *is a very weak solution of the equation* (2.3) *in the sense of Definition 2.50.*

Proof. Change $\mathbb{R}$ to $\mathbb{R}^2$ in the proof of Proposition 2.59. □

With all these preparations we can prove Theorem 2.49 rather quickly.

Proof of Theorem 2.49. We consider

$$\mathcal{I}\colon \mathcal{H} \to \mathbb{R}, \quad \mathcal{I}(\hat{u}) := \frac{1}{2} b_{\mathcal{L}}(\hat{u}, \hat{u}) - \frac{1}{p+1} \int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(x) |S\hat{u}|^{p+1} \, \mathrm{d}(x,t) \, .$$

By Corollary 2.56 we have $(A1)$. Hence $\mathcal{I}$ is well defined. Condition $(A2)$ for $b_{\mathcal{L}}$ directly follows from the constructions in Section 2.2. By our assumptions condition $(A3)$ is also fulfilled. Clearly the symmetry conditions $(C0)$, $(C2)$ and $(C3)$ are fulfilled since V and $\Gamma(|\cdot|)$ are radially symmetric. The assumption $(C1)$ is checked in Lemma 2.20. Applying Theorem 2.12, we obtain a ground state $\hat{u}$ of $\mathcal{I}$ in $\mathcal{H}$. □

3 A Dual Method

In this part we consider the $1+1$ dimensional semilinear wave equation

$$Lu = \Gamma(x)|u|^{p-1}u, \qquad (x,t) \in \mathbb{R} \times \mathbb{T}_T, \tag{3.1}$$

i.e., we consider (2.2) with a more specific right hand side. As in Chapter 2 the operator L is closed and linear but here we assume in addition invertibility. Our guiding example is $L = V(x)\partial_t^2 - \partial_x^2$, which justifies the term "wave" in the equation. For the right hand side we assume $\Gamma \in L^\infty(\mathbb{R})$ to be periodic and positive and $p > 1$. This case was not covered in Chapter 2. We will prove the existence of a ground state of (3.1) with $V(x) = -\alpha + \beta\delta_0(x)$ in Section 3.2.2 and with $V(x) = -\alpha + \beta\,\mathbb{1}_{[-r,r]}(x)$ Section 3.2.3. Observe that (3.1) is not translation invariant in space, which causes the main obstacle of our analysis. We will obtain a compactness result by comparing (3.1) with an "equation at infinity"

$$\tilde{L}u = \Gamma(x)|u|^{p-1}u, \qquad (x,t) \in \mathbb{R} \times \mathbb{T}_T. \tag{3.2}$$

We assume that (3.2) has a ground state, in our examples in Section 3.2 the operator $\tilde{L} = -\alpha\partial_t^2 - \partial_x^2$ is strictly elliptic considering only $\frac{T}{2}$-anti-periodic functions, i.e., this assumption is satisfied. Moreover $L - \tilde{L}$ will be compactly supported in space, what motivates the name "equation at infinity". Comparing the generalized Nehari manifold of (3.1) (see Section 2.1) and (3.2) and their corresponding ground state levels is possible but is not detailed enough to the best of the author's knowledge. For this reason we consider the dual formulations of the equations

$$Kv = |v|^{q-1}v, \qquad (x,t) \in \mathbb{R} \times \mathbb{T}_T, \tag{3.3}$$

$$\tilde{K}v = |v|^{q-1}v, \qquad (x,t) \in \mathbb{R} \times \mathbb{T}_T, \tag{3.4}$$

with formally $K = \Gamma^{\frac{1}{p+1}} L^{-1} \Gamma^{\frac{1}{p+1}}$, $\tilde{K} = \Gamma^{\frac{1}{p+1}} \tilde{L}^{-1} \Gamma^{\frac{1}{p+1}}$ and $q = \frac{1}{p} < 1$. The corresponding energy functionals read formally

$$J(u) = \frac{1}{q+1} \int_{\mathbb{R}\times\mathbb{T}_T} |v|^{q+1} \, \mathrm{d}(x,t) - \frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}_T} v\, Kv \, \mathrm{d}(x,t)\,,$$

$$\tilde{J}(u) = \frac{1}{q+1} \int_{\mathbb{R}\times\mathbb{T}_T} |v|^{q+1} \, \mathrm{d}(x,t) - \frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}_T} v\, \tilde{K}v \, \mathrm{d}(x,t)\,.$$

Weak solutions of (3.3) and (3.4) correspond to weak solutions of (3.1) and (3.2), and vice versa and analogously there is a one-to-one correspondence for ground states. We will provide two minimization methods to obtain a ground state of (3.4), both fueled by an a-priory energy estimate.

First we give an approach by the Nehari manifold, later used in the example with $V(x) = -\alpha + \beta\delta_0(x)$ in Section 3.2.2. Note that due to the sublinear growth on the right hand sides we do not need to consider the generalized Nehari manifolds. In addition the analysis of a generalized Nehari Manifold in the dual problem would need a whole new construction, since it is not clear how it will look like. We will see that Assumption 3.14 and Assumption 3.15, which link the equations (3.3) and (3.4), are sufficient for some compactness argument and hence we will prove the existence of a ground state of (3.3) in Theorem 3.16. Assumption 3.14 describes an a-priory estimate between the ground state levels and Assumption 3.15 translates a Palais-Smale sequence for J into a Palais-Smale sequence for $\tilde{J}$ without changing the energy level provided there is some loss of compactness. The second assumption mainly results from the fact that $L - \tilde{L}$ will be compact in space.

Second we give an approach by constrained minimization, later used in the example with $V(x) = -\alpha + \beta\,\mathbb{1}_{[-r,r]}(x)$ Section 3.2.3. We will see that Assumption 3.21 and Assumption 3.22, which link the equations (3.3) and (3.4), are sufficient for some compactness argument and hence we will prove the existence of a ground state of (3.3) in Theorem 3.24. Assumption 3.21 again describes an a-priori estimate between ground state levels and Assumption 3.22 improves the convergence of special weakly convergent sequences. The second assumption again mainly results from the fact that $L - \tilde{L}$ will be compact in space.

In this chapter we were mainly inspired by [Fre13] and [DPR11] for the approach of the Nehari Manifold and by [Str08] for the constrained minimization approach. All authors used dual variational techniques to construct good minimizing sequences and their compactness argument results from an a-priory energy estimate. We want to empathize at this point, that most of the core ideas in the abstract parts are not completely new, but we rearranged the arguments and generalized the techniques to make them applicable to new wave-like operators as introduced above. We give the rather general toolbox to apply these techniques to a larger class of examples, now including indefinite variational functionals as in Section 3.2, as mentioned in the Introduction. Our main contribution is the new and abstract Hilbert space notation not explicitly using Sobolev spaces and handling the possibly different domains of L and $\tilde{L}$. This, however, manifests in many technical problems.

Last in this chapter, namely in Section 3.1.3.1, we compare the different energy levels and realize a one to one correspondence between the ground state levels and the ground states of the different approaches. Furthermore we give some relation between different energy levels when restricting to specific symmetries. This will later be applied in the examples. We do not expect these last results to be completely new, but to the best of the authors knowledge, we did not find a suitable reference. Hence we prove them in this work.

3.1 Abstract results on dual ground states

In this abstract section we carry out the abstract procedure advertised above and prove the abstract existence theorems for ground states of equation (3.1), i.e., Theorem 3.16 and Theorem 3.24. First we provide general tools on equation (3.1) and general insights on Palais-Smale sequence in Section 3.1.1. Then we introduce the comparison with the "equation at infinity". Afterwards we split into two paths. First we take an approach using the Nehari manifold. Second we take an approach using a constrained minimization. Both times, one core idea is an a-priori estimate for minimal energy levels. Thereafter we provide some tool how to compare and link the energy levels of dual problems. Here we also consider the case that one energy level is an infimum and not attained, which yields additional technical challenges. We apply these techniques later in Section 3.2 to the announced examples.

3.1.1 The dual problem and Palais-Smale sequences

We aim for the $1+1$ dimensional equation (2.2), but we can do most constructions and proofs on an arbitrary open set Ω. The results Theorem 3.16 and Theorem 3.24 will be done in a $N_{trans} + N_{per}$ dimensional setting. Our assumptions are

$(A0)$ Let $N_{trans}, N_{per} \in \mathbb{N}_0$, $N := N_{trans} + N_{per} \geq 1$. Let $\mathbb{T}_T^{N_{per}}$ denote the N_{per}-dimensional torus with periods $T = (T_1, \dots, T_{N_{per}}) \in \mathbb{R}_{>0}^{N_{per}}$. Let $\Omega := \mathbb{R}^{N_{trans}} \times \mathbb{T}_T^{N_{per}}$.

$(A1)$ Let $\mathcal{H}$ be a real Hilbert space, $\langle \cdot, \cdot \rangle$ the scalar product on $\mathcal{H}$, $\Omega \subset \mathbb{R}^N$ be open, $p > 1$. Assume the embedding $S\colon \mathcal{H} \hookrightarrow L^{p+1}(\Omega, \mathbb{R})$ is continuous and $S\colon \mathcal{H} \hookrightarrow L^{p+1}_{loc}(\Omega, \mathbb{R})$ is compact. Furthermore assume that there is an orthogonal decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ with $\mathcal{H}^+ \neq \{0\}$.

$(A2)$ Let $b_{\mathcal{L}}\colon \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a continuous, symmetric bilinear form. Assume that $b_{\mathcal{L}}$ is positive definite on $\mathcal{H}^+$, negative definite on $\mathcal{H}^-$ such that

$$b_{\mathcal{L}}(\hat{u}, \hat{u}) = \left\|\hat{u}^+\right\|^2 - \left\|\hat{u}^-\right\|^2.$$

where $\hat{u}^{\pm}$ is the projection on $\mathcal{H}^{\pm}$. Let in addition $\mathcal{L}\colon \mathcal{H} \to \mathcal{H}^*$ be a continuous and continuously invertible linear operator satisfying

$$\forall\, \hat{u}, \hat{v} \in \mathcal{H}\colon b_{\mathcal{L}}(\hat{u}, \hat{v}) = \langle \mathcal{L}\hat{u}, \hat{v} \rangle_{\mathcal{H}^* \times \mathcal{H}}.$$

$(A3)$ Let $\Gamma \in L^\infty(\Omega)$ with $\inf \Gamma > 0$ and there is some $\zeta = (\zeta_1, \dots, \zeta_{N_{trans}}, 0, \dots, 0) \in \mathbb{R}_+^{N_{trans}} \times \{0\}^{N_{trans}}$ such that $\Gamma(\cdot + \operatorname{diag}(\zeta)k) = \Gamma(\cdot)$ for any $k \in \mathbb{Z}^{N_{trans}} \times \{0\}^{N_{trans}}$, i.e., Γ is periodic with periods ζ.

$(C1)$ Let $1 \leq p_* < p < p^* < \infty$ and assume that $S\colon \mathcal{H} \hookrightarrow L^{p_*+1}(\Omega, \mathbb{R}) \cap L^{p^*+1}(\Omega, \mathbb{R})$ is continuous. Furthermore there is a sequence of balls $B_j \subset \mathbb{R}^{N_{trans}}$, $j \in \mathbb{N}$, such that $\bigcup_j B_j = \mathbb{R}^{N_{trans}}$, at most N^* balls intersect at each point and there some $C > 0$ such that $\sum_j \|S\hat{u}\|^{p_*+1}_{L^{p^*+1}(B_j \times \mathbb{T}_T^{N_{per}})} \leq C\|u\|^{p_*+1}$.

Assumption $(A0)$ makes our domain of interest Ω a strip. We could also include additional bounded directions in this abstract section, but for the sake of simplicity we omit this here. The assumptions $(A1)$ and $(A2)$ are similar as in Chapter 2 but more restrictive here. Assumption $(A3)$ fixes the geometry of the right hand side. We can switch the sign of Γ in $(A3)$ by switching the roles of $\mathcal{H}^+$ and H^- if both are non-empty. In our examples $\mathcal{L}$ and $b_{\mathcal{L}}$ define each other uniquely. By assumption $(C1)$ we have Lemma 2.13, a variant of the often named "P.L. Lions concentration compactness lemma".

Remark 3.1. *The adjoint S^* of S is a continuous map $S^*\colon L^{\frac{1}{p}+1}(\Omega, \mathbb{R}) \to \mathcal{H}^*$.*

Definition 3.2. *Assume (A1), (A2) and (A3). Let* $q := \frac{1}{p}$. *Define the operators*

$$\begin{aligned} &\mathcal{S}\colon \mathcal{H} \to L^{p+1}(\Omega,\mathbb{R}), && (\mathcal{S}\hat{u})(x) := \Gamma(x)^{\frac{1}{p+1}} \cdot (S\hat{u})(x), \\ &\mathcal{K}\colon L^{q+1}(\Omega,\mathbb{R}) \to \mathcal{H}, && \mathcal{K}v := \mathcal{L}^{-1} \circ \mathcal{S}^* v, \\ &K\colon L^{q+1}(\Omega,\mathbb{R}) \to L^{p+1}(\Omega,\mathbb{R}), && Kv := \mathcal{S} \circ \mathcal{L}^{-1} \circ \mathcal{S}^* v = \mathcal{S} \circ \mathcal{K}v. \end{aligned}$$

We define further the functionals

$$\begin{aligned} &\mathcal{I}\colon \mathcal{H} \to \mathbb{R}, && \mathcal{I}(\hat{u}) := \frac{1}{2} b_{\mathcal{L}}(\hat{u},\hat{u}) - \frac{1}{p+1}\int_\Omega \Gamma(x)|S\hat{u}|^{p+1}\,\mathrm{d}x \\ & && \phantom{\mathcal{I}(\hat{u})} = \frac{1}{2} b_{\mathcal{L}}(\hat{u},\hat{u}) - \frac{1}{p+1}\int_\Omega |\mathcal{S}\hat{u}|^{p+1}\,\mathrm{d}x\,, \\ &J\colon L^{q+1}(\Omega,\mathbb{R}) \to \mathbb{R}, && J(v) := \frac{1}{q+1}\int_\Omega |v|^{q+1}\,\mathrm{d}x - \frac{1}{2} b_{\mathcal{K}}(v,v), \end{aligned}$$

where

$$b_{\mathcal{K}}\colon L^{q+1}(\Omega,\mathbb{R}) \times L^{q+1}(\Omega,\mathbb{R}) \to \mathbb{C}, \quad b_{\mathcal{K}}(v,w) := \langle \mathcal{K}v, \mathcal{S}^* w\rangle_{\mathcal{H}\times\mathcal{H}^*}\,.$$

The operators $\mathcal{K}$ and K are inspired by so-called Birmann-Schwinger Kernels, cf. [Sim82]. The operator $\mathcal{S}$ helps us to simplify notation, since it absorbs Γ.

Remark 3.3. *Observe that by positivity of* Γ *and injectivity of* S *we obtain injectivity of* $\mathcal{S}$. *Using the duality* $L^{q+1}(\Omega)^* \cong L^{p+1}(\Omega)$ *we have for* $v, w \in L^{q+1}(\Omega,\mathbb{R})$:

$$b_{\mathcal{K}}(v,w) = \langle \mathcal{S}\circ\mathcal{K}v, w\rangle_{L^{p+1}(\Omega)\times L^{q+1}(\Omega)} = \int_\Omega Kv\cdot w\,\mathrm{d}x = \langle Kv, w\rangle_{L^{p+1}(\Omega)\times L^{q+1}(\Omega)}\,.$$

Proposition 3.4. *Assume (A1), (A2) and (A3). Then:*

(i) The operators $\mathcal{S}$, K *and* $\mathcal{K}$ *are continuous.*

(ii) $\mathcal{I} \in C^1(\mathcal{H},\mathbb{R})$, $J \in C^1(L^{q+1}(\Omega,\mathbb{R}),\mathbb{R})$ *with*

$$\begin{aligned} &\forall\, \hat{u},\hat{z} \in \mathcal{H}\colon && \mathcal{I}'(\hat{u})[\hat{z}] = b_{\mathcal{L}}(\hat{u},\hat{z}) - \int_\Omega \Gamma(x)|S\hat{u}|^{p-1}S\hat{u}\,S\hat{z}\,\mathrm{d}x \\ & && \phantom{\mathcal{I}'(\hat{u})[\hat{z}]} = b_{\mathcal{L}}(\hat{u},\hat{z}) - \int_\Omega |\mathcal{S}\hat{u}|^{p-1}\mathcal{S}\hat{u}\,\mathcal{S}\hat{z}\,\mathrm{d}x \\ &\forall\, v,w \in L^{q+1}(\Omega)\colon && J'(v)[w] = \int_\Omega |v|^{q-1}v\,w\,\mathrm{d}x - b_{\mathcal{K}}(v,w) \end{aligned}$$

(iii) $b_{\mathcal{K}}$ *is symmetric, i.e.,* $\forall\, v,w \in L^{q+1}(\Omega,\mathbb{R})\colon \quad b_{\mathcal{K}}(v,w) = b_{\mathcal{K}}(w,v)$.

(iv) For $\hat{u} \in \mathcal{H}$ *we have* $\mathcal{I}'(\hat{u}) = \mathcal{L}\hat{u} - \mathcal{S}^*\left(|\mathcal{S}\hat{u}|^{p-1}\mathcal{S}\hat{u}\right)$ *in* $\mathcal{H}^*$.
For $v \in L^{q+1}(\Omega,\mathbb{R})$ *we can identify* $J'(v) = |v|^{q-1}v - Kv$ *in* $L^{p+1}(\Omega,\mathbb{R})$.

(v) If $\hat{u} \in \mathcal{H}$ *is a critical point of* $\mathcal{I}$, *then* $v := |\mathcal{S}\hat{u}|^{p-1}\mathcal{S}\hat{u}$ *is a critical point of* J.
If $v \in L^{q+1}(\Omega,\mathbb{R})$ *is a critical point of* J, *then* $\hat{u} := \mathcal{S}^{-1}(|v|^{q-1}v)$ *is a critical point of* $\mathcal{I}$.

(vi) *Define* $c_0 \coloneqq \inf\{\mathcal{I}(\hat{u}) \mid \mathcal{I}'(\hat{u}) = 0\}$ *and* $m_0 \coloneqq \inf\{J(v) \mid J'(v) = 0\}$*. Then we have* $\left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} c_0 = \left(\frac{1}{q+1} - \frac{1}{2}\right)^{-1} m_0$*. Furthermore if* $\hat{u} \in \mathcal{H}$ *is a ground state of* $\mathcal{I}$*, then* $v \coloneqq |\mathcal{S}\hat{u}|^{p-1}\mathcal{S}\hat{u}$ *is a ground state of* J *and if* $v \in L^{q+1}(\Omega)$ *is a ground state of* J*, then* $\hat{u} \coloneqq \mathcal{S}^{-1}(|v|^{q-1}v)$ *is a ground state of* I*.*

Proof. (i) The statement follows by continuity S, S^*, $\mathcal{L}^{-1}$ and boundedness of Γ.

(ii) This proof is a straightforward calculation. For details see e.g. [Str08].

(iii) This part directly follows from the fact that $\mathcal{L}$ is symmetric.

(iv) Let $\hat{u}, \hat{z} \in \mathcal{H}$. Using $L^{q+1}(\Omega)^* \cong L^{p+1}(\Omega)$ we calculate

$$\begin{aligned} \mathcal{I}'(\hat{u})[\hat{z}] &= \langle \mathcal{L}\hat{u}, \hat{z}\rangle_{\mathcal{H}^*\times\mathcal{H}} - \left\langle |\mathcal{S}\hat{u}|^{p-1}\mathcal{S}\hat{u}, \mathcal{S}\hat{z}\right\rangle_{L^{q+1}(\Omega)^*\times L^{p+1}(\Omega)} \\ &= \left\langle \mathcal{L}\hat{u} - \mathcal{S}^*\left(|\mathcal{S}\hat{u}|^{p-1}\mathcal{S}\hat{u}\right), \hat{z}\right\rangle_{\mathcal{H}^*\times\mathcal{H}}. \end{aligned}$$

Analogous we see for $v, w \in L^{q+1}(\Omega, \mathbb{R})$

$$\begin{aligned} J'(v)[w] &= \left\langle |v|^{q-1}v, w\right\rangle_{L^{p+1}(\Omega)\times L^{q+1}(\Omega)} - \langle Kv, w\rangle_{L^{q+1}(\Omega)^*\times L^{q+1}(\Omega)^*} \\ &= \left\langle |v|^{q-1}v - Kv, w\right\rangle_{L^{p+1}(\Omega)\times L^{q+1}(\Omega)}. \end{aligned}$$

Using $L^{q+1}(\Omega)^* \cong L^{p+1}(\Omega)$ again, the claim follows.

(v) We start with the first statement. Let $\hat{u} \in \mathcal{H}$ be a critical point of $\mathcal{I}$ and set $v \coloneqq |\mathcal{S}\hat{u}|^{p-1}\mathcal{S}\hat{u}$. By $(A1)$ and $(A2)$we find $v \in L^{q+1}(\Omega, \mathbb{R})$. Clearly $|v|^{q-1}v = \mathcal{S}\hat{u}$. Since $\mathcal{I}'(\hat{u}) = 0$, we see using part (iv)

$$\mathcal{L}\hat{u} = \mathcal{S}^*\left(|\mathcal{S}\hat{u}|^{p-1}\mathcal{S}\hat{u}\right) = \mathcal{S}^*v \qquad \text{in } \mathcal{H}^*.$$

Hence we obtain for any $w \in L^{q+1}(\Omega)$

$$J'(v)[w] = \int_\Omega |v|^{q-1}v\, w\, \mathrm{d}x - b_{\mathcal{K}}(v, w) = \int_\Omega \mathcal{S}\hat{u}\, w\, \mathrm{d}x - \int_\Omega \mathcal{S} \circ \underbrace{\mathcal{L}^{-1} \circ \mathcal{S}^* v}_{=\hat{u}} \cdot w\, \mathrm{d}x = 0,$$

i.e., v is a critical point of J.
Next we show the second statement. Let $v \in L^{q+1}(\Omega, \mathbb{R})$ be a critical point of J. Then $|v|^{q-1}v = Kv = \mathcal{S}\mathcal{K}v$ in $L^{q+1}(\Omega)^* \simeq L^{p+1}(\Omega)$. Observe that $|v|^{q-1}v$ is in the range of $\mathcal{S}$, i.e., $\hat{u} \coloneqq \mathcal{S}^{-1}(|v|^{q-1}v)$ is well defined by injectivity of $\mathcal{S} \colon \mathcal{H} \to L^{q+1}(\Omega, \mathbb{R})$. Hence $\hat{u} \in \mathcal{H}$ and clearly $v = |\mathcal{S}\hat{u}|^{p-1}\mathcal{S}\hat{u}$. Moreover $\hat{u} = \mathcal{L}^{-1}\mathcal{S}^*v$ and we obtain for any $\hat{z} \in \mathcal{H}$

$$\mathcal{I}'(\hat{u})[\hat{z}] = b_{\mathcal{L}}(\hat{u}, \hat{z}) - \int_\Omega |\mathcal{S}\hat{u}|^{p-1}\mathcal{S}\hat{u}\, \mathcal{S}\hat{z}\, \mathrm{d}x = \langle \mathcal{S}^*v, \hat{z}\rangle_{\mathcal{H}^*\times\mathcal{H}} - \int_\Omega v\, \mathcal{S}\hat{z}\, \mathrm{d}x = 0,$$

i.e., $\hat{u}$ is a critical point of I.

(vi) In part (v) we have seen, that $\Phi\colon \{\hat{u} \in \mathcal{H} \setminus \{0\} \mid \mathcal{I}'(\hat{u}) = 0\} \to \{v \in L^{q+1}(\Omega, \mathbb{R}) \setminus \{0\} \mid J'(v) = 0\}$, $\Phi(\hat{u}) := |\mathcal{S}\hat{u}|^{p-1}\mathcal{S}\hat{u}$ is a bijection with inverse $\Phi^{-1}(v) := \mathcal{S}^{-1}(|v|^{q-1}v)$. If both sets of critical points are empty, then both infima are $-\infty$ and hence the claim is true. Now assume one set (and hence both sets) of critical points is nonempty. Let $\hat{u} \in \{\hat{u} \in \mathcal{H} \setminus \{0\} \mid \mathcal{I}'(\hat{u}) = 0\}$. We then calculate:

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} c_0 &\le \left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} \mathcal{I}(\hat{u}) = \int_\Omega |\mathcal{S}\hat{u}|^{p+1}\,\mathrm{d}x \\ &= \int_\Omega |\Phi(\hat{u})|^{q+1}\,\mathrm{d}x = \left(\frac{1}{q+1} - \frac{1}{2}\right)^{-1} J(\Phi(\hat{u})) \ge \left(\frac{1}{q+1} - \frac{1}{2}\right)^{-1} m_0.\end{aligned}$$

Taking the infimum in $\hat{u}$ yields $\left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} c_0 \ge \left(\frac{1}{q+1} - \frac{1}{2}\right)^{-1} m_0$. By an analogous calculation we obtain $\left(\frac{1}{q+1} - \frac{1}{2}\right)^{-1} m_0 \ge \left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} c_0$. Hence we have equality. Now let $\hat{u} \in \{\mathcal{H} \setminus \{0\} \mid \mathcal{I}'(\hat{u}) = 0\}$ be a ground state for $\mathcal{I}$, i.e., $\mathcal{I}(\hat{u}) = c_0$. Then $\Phi(\hat{u}) \in \{v \in L^{q+1}(\Omega, \mathbb{R}) \setminus \{0\} \mid J'(v) = 0\}$ and

$$\begin{aligned}\left(\frac{1}{q+1} - \frac{1}{2}\right)^{-1} m_0 &\le \left(\frac{1}{q+1} - \frac{1}{2}\right)^{-1} J(\Phi(\hat{u})) = \left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} \mathcal{I}(\hat{u}) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} c_0 = \left(\frac{1}{q+1} - \frac{1}{2}\right)^{-1} m_0.\end{aligned}$$

Hence $\Phi(\hat{u})$ is a ground state of J. The other direction is done exactly analogously.

□

In the next section we will construct special Palais-Smale sequences and by the structure of J we get boundedness for free using the following proposition.

Definition 3.5. *Let X be a Banach space and $E \in C^1(X, \mathbb{R})$. Then $(x_n)_n \subset X$ is called a Palais-Smale sequence for E, if $(E(x_n))_n$ is bounded and $E'(x_n) \to 0$ in X^*.*

Proposition 3.6. *Assume (A1), (A2) and (A3). If $(v^{(n)})_n \in L^{q+1}(\Omega, \mathbb{R})$ is a Palais-Smale sequence for J, then $(v^{(n)})_n$ is bounded in $L^{q+1}(\Omega)$.*

Proof. Assume there is a subsequence, w.l.o.g. again superscripted with n, such that $\|v^{(n)}\|_{L^{q+1}(\Omega)} \nearrow \infty$ in $L^{q+1}(\Omega)$. Since $q > 0$ we then obtain

$$\begin{aligned}\mathrm{o}(1) &= \left\|v^{(n)}\right\|_{L^{q+1}(\Omega)}^{-1} J\left(v^{(n)}\right) - \frac{1}{2} J'\left(v^{(n)}\right)\left[\left\|v^{(n)}\right\|_{L^{q+1}(\Omega)}^{-1} v^{(n)}\right] \\ &= \left(\frac{1}{q+1} - \frac{1}{2}\right) \cdot \left\|v^{(n)}\right\|_{L^{q+1}(\Omega)}^{q} \to \infty,\end{aligned}$$

a contradiction. □

Since Palais-Smale sequences are bounded, we find a weak limit up to choosing some suitable subsequence. Next we show that any weak limit v^* of a Palais-Smale-sequence is a critical point of J and J is weakly lower semi-continuous on such sequences. Observe that we will not yet prove that $v^* \neq 0$. This is done in Section 3.1.2 and needs additional assumptions.

Lemma 3.7. *Let* $(v^{(n)})_n \subset L^{q+1}(\Omega, \mathbb{R})$ *and* $v^* \in L^{q+1}(\Omega, \mathbb{R})$ *such that* $v^{(n)} \rightharpoonup v^*$ *and* $J'\left(v^{(n)}\right) \to 0$*. Then* $J'(v^*) = 0$ *and* $\liminf_n J\left(v^{(n)}\right) \geq J(v^*)$*.*

Proof. We write $\hat{u}^{(n)} := \mathcal{K}v^{(n)}$ as before and define $\hat{u}^* := \mathcal{K}v^*$. By continuity of $\mathcal{K}$ we have $\hat{u}^{(n)} \rightharpoonup \hat{u}^*$ in $\mathcal{H}$. By $(A1)$ we know $\mathcal{S}\hat{u}^{(n)} \rightharpoonup \mathcal{S}\hat{u}^*$ in $L^{p+1}(\Omega)$ and $\mathcal{S}\hat{u}^{(n)} \to \mathcal{S}\hat{u}^*$ $L^{p+1}_{loc}(\Omega)$. Hence using Proposition 3.4 part (iv)

$$\left|v^{(n)}\right|^{q-1} v^{(n)} = J'\left(v^{(n)}\right) + Kv^{(n)} = \mathrm{o}(1) + \mathcal{S}\hat{u}^{(n)} \begin{cases} \rightharpoonup \mathcal{S}\hat{u}^*, & \text{in } L^{p+1}(\Omega), \\ \to \mathcal{S}\hat{u}^*, & \text{in } L^{p+1}_{loc}(\Omega), \end{cases}$$

where we consider the above equalities via the duality $L^{q+1}(\Omega)^* \cong L^{p+1}(\Omega)$. Hence

$$\forall \varphi \in C_c^\infty(\Omega, \mathbb{R}): \qquad \int_\Omega \left|v^{(n)}\right|^{q-1} v^{(n)} \varphi \,\mathrm{d}x \to \int_\Omega \mathcal{S}\hat{u}^* \varphi \,\mathrm{d}x\,.$$

On the other hand we calculate by weak convergence for any $\varphi \in C_c^\infty(\Omega, \mathbb{R})$:

$$\begin{aligned} b_\mathcal{K}\left(v^{(n)}, \varphi\right) &= \left\langle \mathcal{K}v^{(n)}, \mathcal{S}^*\varphi \right\rangle_{\mathcal{H}\times\mathcal{H}^*} = \left\langle \hat{u}^{(n)}, \mathcal{S}^*\varphi \right\rangle_{\mathcal{H}\times\mathcal{H}^*} \\ &\to \langle \hat{u}^*, \mathcal{S}^*\varphi \rangle_{\mathcal{H}\times\mathcal{H}^*} = \langle \mathcal{K}v^*, \mathcal{S}^*\varphi \rangle_{\mathcal{H}\times\mathcal{H}^*} = b_\mathcal{K}(v^*, \varphi). \end{aligned}$$

Hence:

$$\begin{aligned} \forall \varphi \in C_c^\infty(\Omega): \quad 0 &= \lim_{n\to\infty} J'\left(v^{(n)}\right)[\varphi] = \lim_{n\to\infty} \int_\Omega \left|v^{(n)}\right|^{q-1} v^{(n)} \varphi \,\mathrm{d}x - b_\mathcal{K}\left(v^{(n)}, \varphi\right) \\ &= \int_\Omega |v^*|^{q-1} v^* \varphi \,\mathrm{d}x - b_\mathcal{K}(v^*, \varphi) = J'(v^*)[\varphi]. \end{aligned}$$

By density of $C_c^\infty(\Omega, \mathbb{R})$ in $L^{p+1}(\Omega, \mathbb{R}) \simeq L^{q+1}(\Omega, \mathbb{R})^*$ we obtain $J'(v^*) = 0$. It remains to proof the lower semi-continuity ot J on weakly convergent Palais-Smale sequences. Since the p^{th} odd power as a map $L^{p+1}(\tilde{\Omega}, \mathbb{R}) \to L^{q+1}(\tilde{\Omega}, \mathbb{R})$ is Lipschitz-continuous on L^{p+1}-bounded sets for any open set $\tilde{\Omega} \subset \mathbb{R}^n$ (e.g. cf. [Str08]), we obtain

$$v^{(n)} = \left|\mathcal{S}\hat{u}^{(n)}\right|^{p-1} \mathcal{S}\hat{u}^{(n)} + \mathrm{o}(1) \to |\mathcal{S}\hat{u}^*|^{p-1}\mathcal{S}\hat{u}^*, \qquad \text{in } L^{q+1}_{loc}(\Omega).$$

Observe first that the convergence in $L^{q+1}_{loc}(\Omega)$ implies pointwise almost everywhere convergence up to a subsequence. Observe second that the convergence in $L^{q+1}_{loc}(\Omega)$ and boundedness in $L^{q+1}(\Omega)$ imply weak convergence in $L^{q+1}(\Omega)$. Hence $v^* = |\mathcal{S}\hat{u}^*|^{p-1}\mathcal{S}\hat{u}^*$ and therefore $\mathcal{S}\hat{u}^* = |v^*|^{q-1}v^*$. Using $J'\left(v^{(n)}\right) \to 0$, boundedness of $(v^{(n)})_n$ in $L^{q+1}(\Omega)$, $v^{(n)} \to v^*$ almost everywhere and Fatous's Lemma we calculate:

$$\begin{aligned} \liminf_n J\left(v^{(n)}\right) &= \liminf_n J\left(v^{(n)}\right) - \frac{1}{2}J'\left(v^{(n)}\right)\left[v^{(n)}\right] = \liminf_n \left(\frac{1}{q+1} - \frac{1}{2}\right) \int_\Omega \left|v^{(n)}\right|^{q+1} \mathrm{d}x \\ &\geq \left(\frac{1}{q+1} - \frac{1}{2}\right) \int_\Omega |v^*|^{q+1} \,\mathrm{d}x = J(v^*) - \frac{1}{2}J'(v^*)[v^*] = J(v^*), \end{aligned}$$

i.e., the claim is proven. □

3.1.2 Compactness by comparison with the equation at infinity

In this section we provide two criteria to ensure that the weak limit v^* of two special Palais-Smale-sequences is not 0. This will be done considering an additional "equation at infinity", indicated by tilde notation.

Definition 3.8. *Assume* $(A0)$, $(A1)$, $(A2)$, $(A3)$ *and* $(C1)$ *as in Section 2.1. We refer to this setting as the "equation of interest". Assume additionally* $(A0)$, $(A1)$, $(A2)$, $(A3)$ *and* $(C1)$ *are true for the Hilbert space* $\widetilde{\mathcal{H}}$*, the embedding* $\tilde{S}\colon \widetilde{\mathcal{H}} \to L^{p+1}(\Omega,\mathbb{R})$*, the operator* $\widetilde{\mathcal{L}}$*, the bilinear form* $b_{\widetilde{\mathcal{L}}}$ *but with the same* Ω*,* $p>1$*,* ζ *and* Γ *as the "equation of interest", i.e., without tilde notation. We refer to the setting with tilde notation as the "equation at infinity".*

3.1.2.1 A Nehari approach

In this approach we will need some additional symmetry in the "equation at infinity".

Definition 3.9. *Assume* $(A0)$, $(A1)$, $(A2)$, $(A3)$ *and* $(C1)$ *as in Section 2.1. Assume further we have an "equation at infinity" as in Definition 3.8. We say* $\widetilde{\mathcal{H}}$ *and* $\tilde{J}$ *respect* ζ*-translation symmetry of* Ω *as in* $(C2)$ *and* $(C3)$ *in Section 2.1.2.2 with* $N_{rad}=0$ *if for any* $\hat{u}\in\widetilde{\mathcal{H}}$*,* $v\in L^{q+1}(\Omega,\mathbb{R})$ *and* $k\in\mathbb{Z}^{N_{trans}}$ *we have* $(\tilde{S}\hat{u})(\cdot+\zeta k)\in \operatorname{Range}(\tilde{S})$*,*

$$\left\|\tilde{S}^{-1}\left((\tilde{S}\hat{u})(\cdot+\zeta k)\right)\right\| = \|\hat{u}\|_{\widetilde{\mathcal{H}}}, \qquad \text{and} \qquad \tilde{J}\left(v(\cdot+\zeta k)\right)=\tilde{J}\left(v\right).$$

We now present the method of the Nehari manifold in the dual problem.

Definition 3.10. *Assume* $(A1)$, $(A2)$ *and* $(A3)$*. Let*

$$\mathcal{M}^{(N)} := \left\{v\in L^{q+1}(\Omega,\mathbb{R})\setminus\{0\} \;\middle|\; J'\left(v\right)[v]=0\right\}, \qquad m^{(N)} := \inf_{\mathcal{M}^{(N)}} J.$$

Remark 3.11. *The set* $\mathcal{M}^{(N)}$ *is called Nehari manifold. It contains all critical points of* J *and* $\mathcal{M}^{(N)}\subset\{v\in L^{q+1}(\Omega,\mathbb{R}) \mid b_{\mathcal{K}}\left(v,v\right)>0\}$*. Furthermore*

$$\begin{aligned}\forall\, v\in L^{q+1}(\Omega,\mathbb{R})\colon\quad J'\left(v\right)[v] &= \int_\Omega |v|^{q+1}\,\mathrm{d}x - b_{\mathcal{K}}\left(v,v\right)\\ J\left(v\right) &= \left(\frac{1}{q+1}-\frac{1}{2}\right)\int_\Omega |v|^{q+1}\,\mathrm{d}x + \frac{1}{2}J'\left(v\right)[v],\end{aligned}$$

hence J *is positive on* $\mathcal{M}^{(N)}$*.*

Proposition 3.12. *Assume* $(A1)$, $(A2)$ *and* $(A3)$*. If* $\mathcal{M}^{(N)}\neq\emptyset$*, then* $m^{(N)}>0$*.*

Proof. Let $v\in\mathcal{M}^{(N)}$. Then we calculate:

$$\|v\|^{q+1}_{L^{q+1}(\Omega)} = b_{\mathcal{K}}\left(v,v\right) = \langle Kv,v\rangle_{L^{p+1}(\Omega)\times L^{q+1}(\Omega)} \le \|K\|_{L^{q+1}(\Omega,\mathbb{R})\to L^{p+1}(\Omega,\mathbb{R})}\|v\|^2_{L^{q+1}(\Omega)}.$$

Since $q+1<2$ we see $\|v\|_{L^{q+1}(\Omega)}$ is bounded away from zero on $\mathcal{M}^{(N)}$ and by Remark 3.11 the claim follows. □

The previous proposition is one main point why we consider the Nehari manifold of the dual problem. By the sub-quadratic growth of $\|v\|_{L^{q+1}(\Omega)}^{q+1}$ in J due to $q < 1$ we easily obtained that minimizing sequences of J on $\mathcal{M}^{(N)}$ are bounded and bounded away from 0 in $L^{q+1}(\Omega, \mathbb{R})$ and the infimum $m^{(N)}$ is also bounded away from 0. Observe that possibly $\mathcal{M}^{(N)} = \emptyset$, if $\mathcal{S}^*$ maps $L^{q+1}(\Omega, \mathbb{R})$ into the parts where $\mathcal{L}^{-1}$ is negative. We will use Section 3.1.3.1 to verify $\mathcal{M}^{(N)} \neq \emptyset$ in the examples.

The next lemma makes use of assumption $(C1)$ to apply our version of "P.L. Lion's concentration compactness lemma", i.e., Lemma 2.13 in Chapter 2.

Lemma 3.13. *Assume* $(A0)$, $(A1)$, $(A2)$, $(A3)$ *and* $(C1)$. *If* $\mathcal{M}^{(N)} \neq \emptyset$, *then there is* $(v^{(n)})_n \in \mathcal{M}^{(N)}$, $\xi^{(n)} \in \Omega$ *and* $\delta, r > 0$ *such that* $J\left(v^{(n)}\right) \to m$, $J'\left(v^{(n)}\right) \to 0$ *and writing* $\hat{u}^{(n)} := \mathcal{K}v^{(n)}$ *we have in addition*

$$\forall\, n \in \mathbb{N}: \quad \int_{B_r(\xi^{(n)}) \cap \Omega} \left| S\hat{u}^{(n)} \right|^{p+1} \mathrm{d}x \geq \delta.$$

We will prove the stronger result that $r > 0$ can be chosen arbitrary.

Proof. Let $(v^{(n)})_n \in \mathcal{M}^{(N)}$ such that $J\left(v^{(n)}\right) \to m$. By Ekeland's variational principle, cf. [Eke74], we can assume w.l.o.g. $J'\left(v^{(n)}\right) \to 0$. Set $\hat{u}^{(n)} := \mathcal{K}v^{(n)}$. Assume to the contrary

$$\limsup_n \sup_{B_r(y) \subset \Omega} \int_{B_r(y)} \left| S\hat{u}^{(n)} \right|^{p+1} \mathrm{d}x = 0.$$

By Lemma 2.13 we conclude $S\hat{u}^{(n)} \to 0$ in $L^{p+1}(\Omega)$. But then

$$\begin{aligned} 0 < \inf_n \left\| v^{(n)} \right\|_{L^{q+1}(\Omega)}^{q+1} &= b_{\mathcal{K}}(v, v) = \left\langle \mathcal{K}v, S^*\left(\Gamma(x)^{\frac{1}{p+1}} v\right) \right\rangle_{\mathcal{H} \times \mathcal{H}^*} \\ = \left\langle S\hat{u}^{(n)}, \Gamma(x)^{\frac{1}{p+1}} v^{(n)} \right\rangle_{L^{p+1}(\Omega) \times L^{q+1}(\Omega)} &\leq \left\| S\hat{u}^{(n)} \right\|_{L^{q+1}(\Omega)} \cdot \|\Gamma\|_{L^\infty(\mathbb{R})}^{\frac{1}{p+1}} \left\| v^{(n)} \right\|_{L^{q+1}(\Omega)} \to 0, \end{aligned}$$

a contradiction. The claim now follows, possibly choosing a suitable subsequence. □

Provided $\mathcal{M}^{(N)} \neq \emptyset$, we have constructed a special minimizing sequence $(v^{(n)})_n$. It is a Palais-Smale-sequence for J and we can guarantee, that the L^{p+1}-norm of $\hat{u}^{(n)} := \mathcal{K}v^{(n)}$ in balls with uniform radii and centers possibly diverging to infinity is bounded from below.

The next assumptions link the "equation of interest" (with no tilde notation) and the "equation at infinity" (with tilde notation).

Assumption 3.14. *Assume the setting as in Definition 3.8. Assume* $m^{(N)} < \tilde{m}^{(N)}$.

Assumption 3.15. *Assume the setting as in Definition 3.8. Assume there is some compact set* $\tilde{\Omega} \subset \Omega$ *and some* $s \in (p_*, p^*)$ *such that: If* $(v^{(n)})_n$ *is a Palais-Smale sequence for* J *to the level* $m^{(N)}$ *and* $\left\| \tilde{S}\tilde{\mathcal{K}}v^{(n)} \right\|_{L^{s+1}(\tilde{\Omega})} \to 0$, *then* $(v^{(n)})_n$ *is a Palais-Smale sequence for* $\tilde{J}$ *to the level* $m^{(N)}$, *i.e.,* $\tilde{J}\left(v^{(n)}\right) \to m$ *and* $\tilde{J}'\left(v^{(n)}\right) \to 0$.

These assumption are quite abstract at this point, but they will become more clear in the examples. In Section 3.1.3.1 we provide some abstract tools to verify these statements. The identification of these core steps and the proofs that these assumptions are true in our examples, is one of our main new contribution in this chapter. Assumption 3.14 is an a-priory estimate of ground state levels. Assumption 3.15 tells us, that if there is some "bad" convergence in L^{s+1} on a special set $\tilde{\Omega}$ (sometimes called "loss of compactness"), then our Palais-Smale sequence for J is in addition a Palais-Smale sequence for $\tilde{J}$ without changing the energy level $m^{(N)}$. We are now ready to proof our main theorem.

Theorem 3.16. *Assume the setting as in Definition 3.8 and 3.9. Assume further Assumption 3.14 and Assumption 3.15 are true. Then J has a ground state.*

Proof. We observe that by the strict inequality in Assumption 3.14, we have $m^{(N)} < \infty$ and hence $\mathcal{M}^{(N)} \neq \emptyset$. By Lemma 3.13 there is a sequence $(v^{(n)})_n \subset \mathcal{M}^{(N)}$, a sequence $\xi^{(n)} \in \Omega$ and some $\delta, r > 0$ such that $J\left(v^{(n)}\right) \to m^{(N)}$, $J'\left(v^{(n)}\right) \to 0$. Let $\tilde{\Omega} \subset \Omega$ be compact and $s \in (p_*, p^*)$ as in Assumption 3.15. We will prove that $\|\tilde{S}\tilde{\mathcal{K}}v^{(n)}\|_{L^{s+1}(\tilde{\Omega})} \nrightarrow 0$. Assume the contrary. Then by Assumption 3.15 we obtain that $(v^{(n)})_n$ is a Palais-Smale sequence for $\tilde{J}$ to the level $m^{(N)}$. Arguing as in the proof of Lemma 3.13 to $\tilde{J}$, there is a subsequence of $(v^{(n)})_n$, again superscripted by n, a sequence $\xi^{(n)} \in \Omega$ and some $\delta, r > 0$ such that

$$\forall\, n \in \mathbb{N}: \qquad \int_{B_r(\xi^{(n)})} \left|\tilde{S}\tilde{\mathcal{K}}v^{(n)}\right|^{p+1} \mathrm{d}x \geq \delta.$$

Case 1: $(\xi^{(n)})_n \in \Omega$ is bounded.
By Proposition 3.6 the sequence $(v^{(n)})_n$ is bounded in $L^{q+1}(\Omega)$. Hence we find some $v^* \in L^{q+1}(\Omega)$ and some subsequence, again superscripted by n, such that $v^{(n)} \rightharpoonup v^*$ in $L^{q+1}(\Omega)$. We see by continuity of $\tilde{\mathcal{K}}$ that $\tilde{\mathcal{K}}v^{(n)} \rightharpoonup \tilde{\mathcal{K}}v^*$ in $\tilde{\mathcal{H}}$. Since $\tilde{S}$ is locally compact and $(\xi^{(n)})_n \in \Omega$ is bounded, we obtain $\tilde{S}\tilde{\mathcal{K}}v^{(n)} \to \tilde{S}\tilde{\mathcal{K}}v^*$ in $L^{p+1}(B_R(0) \cap \Omega)$ for $R := r + \sup_n \left|\xi^{(n)}\right|$. Using $\|\tilde{S}\tilde{\mathcal{K}}v^{(n)}\|_{L^{p+1}(B_R(0)\cap\Omega)} \geq \delta$ we obtain $\tilde{S}\tilde{\mathcal{K}}v^* \neq 0$. Injectivity of $\tilde{S}$ and $\mathcal{K}$ yields $v^* \neq 0$. As in Lemma 3.7 we see that v^* is a critical point of $\tilde{J}$ with $m^{(N)} = \liminf_n \tilde{J}\left(v^{(n)}\right) \geq \tilde{J}\left(v^*\right)$. Combined with $v^* \neq 0$ we get the reverse inequality $\tilde{J}\left(v^*\right) \geq m^{(N)}$, hence v^* is a nontrivial critical point of $\tilde{J}$ with energy level $m^{(N)}$. This contradicts Assumption 3.14.
Case 2: $(\xi^{(n)})_n \in \Omega$ is unbounded.
For each $\xi^{(n)}$ choose a closest grid point $\eta^{(n)} \in \operatorname{diag}(\zeta)(\mathbb{Z}^{N_{trans}} \times \{0\}^{N_{per}})$ with ζ as in $(A3)$. Define $R := r + \rho$, with $\rho := \sup_n \left|\xi^{(n)} - \eta^{(n)}\right|$. Clearly $\rho \leq \sqrt{N_{trans}} \max_{j=1,\ldots,N_{trans}} \zeta_j$. We define the translations $w^{(n)} := v^{(n)}(\cdot + \eta^{(n)})$ and define $\hat{z}^{(n)} := \tilde{\mathcal{K}}w^{(n)}$. Observe that by the choice of $\eta^{(n)}$ we have $\hat{z}^{(n)} = (\tilde{\mathcal{K}}v^{(n)})(\cdot + \eta^{(n)}) = \tilde{\mathcal{K}}(v^{(n)}(\cdot + \eta^{(n)}))$, i.e., $\tilde{\mathcal{K}}$ commutes with translations and $\Gamma(\cdot) = \Gamma(\cdot + \eta^{(n)})$. Since $\tilde{J}$ is ζ-translation invariant, $(w^{(n)})_n$ is a Palais-Smale sequence for $\tilde{J}$ to the level $m^{(N)}$. Obviously $(w^{(n)})_n$ is bounded in $L^{q+1}(\Omega)$ and hence we find some $w^* \in L^{q+1}(\Omega, \mathbb{R})$ such that, up to taking a subsequence again superscripted by n, we have $w^{(n)} \rightharpoonup w^*$ in $L^{q+1}(\Omega)$. Observe that for $n \in \mathbb{N}$ we have

$$\delta \leq \int_{B_R(\eta^{(n)})} \left|\tilde{S}\tilde{\mathcal{K}}v^{(n)}\right|^{p+1} \mathrm{d}x = \int_{B_R(0)} \left|\tilde{S}\tilde{\mathcal{K}}, w^{(n)}\right|^{p+1} \mathrm{d}x = \int_{B_R(0)} \left|\tilde{S}\hat{z}^{(n)}\right|^{p+1} \mathrm{d}x\,,$$

i.e., $(S\hat{z}^{(n)})_n$ is bounded away from 0 in $L^{p+1}(B_R(0))$. By continuity of $\tilde{\mathcal{K}}$ and local compactness of $\tilde{S}$ we obtain $w^* \neq 0$. As in case 1 we see that w^* is a nontrivial dual

ground state of $\tilde{J}$ with energy level $m^{(N)}$. This contradicts again Assumption 3.14.
Conclusion: Since $\|\tilde{S}\tilde{\mathcal{K}}v^{(n)}\|_{L^{s+1}(\tilde{\Omega})} \nrightarrow 0$, we can choose a subsequence of $(v^{(n)})_n$, again superscripted by n, such that $\inf_n \|\tilde{S}\tilde{\mathcal{K}}v^{(n)}\|_{L^{s+1}(\tilde{\Omega})} > 0$. Arguing exactly as before we find another a subsequence of $(v^{(n)})_n$, again superscripted by n, and $v^* \in L^{q+1}(\Omega, \mathbb{R}) \setminus \{0\}$ such that $v^{(n)} \rightharpoonup v^*$ in $L^{q+1}(\Omega)$. Applying Lemma 3.7 to J we see v^* is a critical point of J with $m = \liminf_n J\left(v^{(n)}\right) \geq J\left(v^*\right)$. Combined with $v^* \neq 0$ we get the reverse inequality $J\left(v^*\right) \geq m$, hence v^* is a nontrivial dual ground state of J. □

We finish this section with an observation that reduces the comparison of the ground state levels $m^{(N)}$ and $\tilde{m}^{(N)}$ of two dual functionals J and $\tilde{J}$ to a comparison of the corresponding bilinear forms $b_{\tilde{\mathcal{K}}}$ and $b_{\mathcal{K}}$ for the ground state of the "equation at infinity".

Lemma 3.17. *Assume (A1), (A2) and (A3) for the "equation of interest" and the "equation at infinity". Assume in addition that $\tilde{J}$ has a ground state $v \in L^{q+1}(\Omega, \mathbb{R})$. Then: $b_{\tilde{\mathcal{K}}}(v, v) < b_{\mathcal{K}}(v, v)$ implies $m^{(N)} < \tilde{m}^{(N)}$.*

Proof. Since $v \in \widetilde{\mathcal{M}}^{(N)}$ we know $b_{\tilde{\mathcal{K}}}(v, v) = \|v\|^{q+1}_{L^{q+1}(\Omega)} > 0$ and hence $b_{\mathcal{K}}(v, v) > 0$. We define

$$\tau := \left(\frac{\|v\|^{q+1}_{L^{q+1}(\Omega)}}{b_{\mathcal{K}}(v, v)}\right)^{\frac{1}{1-q}} < \left(\frac{\|v\|^{q+1}_{L^{q+1}(\Omega)}}{b_{\tilde{\mathcal{K}}}(v, v)}\right)^{\frac{1}{1-q}} = 1.$$

By the polynomial structure of J we see that $tv \in \mathcal{M}^{(N)}$ for $t > 0$ if and only if $t = \tau$. Hence: $b_{\mathcal{K}}(\tau v, \tau v) = \|\tau v\|^{q+1}_{L^{q+1}(\Omega)}$. Now we calculate:

$$\begin{aligned} J(\tau v) &= \left(\frac{1}{1+q} - \frac{1}{2}\right) \|\tau v\|^{q+1}_{L^{q+1}(\Omega)} = \tau^{q+1} \cdot \left(\frac{1}{1+q} - \frac{1}{2}\right) \|v\|^{q+1}_{L^{q+1}(\Omega)} \\ &= \tau^{q+1} \cdot \tilde{J}(v) < \tilde{J}(v) = \tilde{m}^{(N)}. \end{aligned}$$

Hence $m^{(N)} < \tilde{m}^{(N)}$. □

3.1.2.2 A constrained minimization approach

In the abstract part of this minimization approach we do not see an additional symmetry assumption as Definition 3.9, but we will use such symmetry in the application to check all assumptions. We split the functional J in its quadratic nonlinear part J_0 and its part J_1. Then we minimize J_0 under the constrained $J_1 \equiv 1$.

Definition 3.18. *Assume (A1), (A2) and (A3). Let*

$$J_0, J_1 \colon \mathcal{D}(J) \to \mathbb{R}, \qquad J_0(v) := \frac{1}{q+1} \int_\Omega |v|^{q+1} \, \mathrm{d}x\,, \quad J_1(v) := \frac{1}{2} b_{\mathcal{K}}(v, v)$$

$$\mathcal{M}^{(L)} := \left\{ v \in \mathcal{D}(J) \;\middle|\; J_1(v) = 1 \right\}, \qquad m^{(L)} := \inf_{\mathcal{M}^{(L)}} J_0.$$

Remark 3.19. *Since J_0 and J_1 are C^1, we expect a Lagrange multiplier $\lambda \in \mathbb{R}$ for a minimizer v^* such that $J_1'(v^*) = \lambda J_0'(v^*)$, if such a minimizer exists.*

Proposition 3.20. *Assume* $(A1)$*,* $(A2)$ *and* $(A3)$*. If* $\mathcal{M}^{(L)} \neq \emptyset$*, then* $m^{(L)} > 0$*.*

Proof. Assume $(A1)$, $(A2)$ and $(A3)$. Let $v \in \mathcal{M}^{(L)}$. By continuity of K we calculate

$$1 = b_{\mathcal{K}}(v,v) \leq \|K\|_{L^{q+1}(\Omega)\to L^{p+1}(\Omega)} \|v\|^2_{L^{q+1}(\Omega)} = \|K\|_{L^{q+1}(\Omega)\to L^{p+1}(\Omega)} \left((q+1)J_0(v)\right)^{\frac{2}{q+1}}.$$

□

Observe that possibly $\mathcal{M}^{(L)} = \emptyset$, if $\mathcal{S}^*$ maps $L^{q+1}(\Omega,\mathbb{R})$ into the parts where $\mathcal{L}^{-1}$ is negative. We will use Section 3.1.3.1 to verify $\mathcal{M}^{(L)} \neq \emptyset$ in the examples. The next assumptions link the "equation of interest" (with no tilde notation) and the "equation at infinity" (with tilde notation).

Assumption 3.21. *Assume the setting as in Definition 3.8. Assume* $m^{(L)} < \tilde{m}^{(L)}$*.*

Assumption 3.22. *Assume the setting as in Definition 3.8. Assume that if* $w^{(n)} \rightharpoonup 0$ *in* $L^{q+1}(\Omega)$*, then* $b_{\mathcal{K}}\left(w^{(n)}, w^{(n)}\right) - b_{\tilde{\mathcal{K}}}\left(w^{(n)}, w^{(n)}\right) = \mathrm{o}(1)$*.*

Assumption 3.21 also implies that $m^{(L)} \neq \infty$, even if $\tilde{m}^{(L)} = \infty$, and hence the set $\mathcal{M}^{(L)}$ is nonempty.

We shortly state a technical lemma which instantly follows from the often called Brezis-Lieb-Lemma.

Lemma 3.23. *Let* $f_n, f \in L^{q+1}(\Omega)$ *be measurable,* $(f_n)_n$ *be uniformly bounded in* $L^{q+1}(\Omega)$ *and* $f_n \to f$ *almost everywhere. Then*

$$\lim_{n\to\infty}\left(\int_\Omega |f_n|^{q+1}\,\mathrm{d}x - \int_\Omega |f - f_n|^{q+1}\,\mathrm{d}x\right) = \int_\Omega |f|^{q+1}\,\mathrm{d}x\,.$$

A proof can be found in [Wil96].

Theorem 3.24. *Assume the setting as in Definition 3.8. Assume further Assumption 3.21 and Assumption 3.22 are true. Then* $m^{(L)}$ *is attained.*

Proof. We split the proof into several steps.

Step 1: Approximate Lagrangian multipliers.
Let $(v^{(n)})_n \subset \mathcal{M}^{(L)}$ be a minimizing sequence for J_0, i.e., $v^{(n)} \in \mathcal{D}(J)$ with $J_1(v^{(n)}) = 1$ and $J_0(v^{(n)}) \to m^{(L)}$. Clearly $(v^{(n)})_n$ is bounded in $L^{q+1}(\Omega)$, hence we find some $v^* \in \mathcal{D}(J)$ and some subsequence (again superscripted with n) such that $v^{(n)} \rightharpoonup v^*$ in $L^{q+1}(\Omega)$. Using Ekelands variational principle, cf. [Eke74], we argue exactly as in [Fre13], proof of Proposition 2.3.12 step 1 and the first half of step 2, to find a new, bounded, minimizing sequence $(v^{(n)})_n \subset \mathcal{M}^{(L)}$ and $\lambda_n \in \mathbb{R}$ with $\partial_{T_n} J_0(v^{(n)}) \to 0$ and $\partial_{S_n} J_0(v^{(n)}) = -\lambda_n \partial_{S_n} J_1(v^{(n)})$, where T_n and S_n denote the tangential and normal space in $v^{(n)}$ on $\mathcal{M}^{(L)}$. Observe that in [Fre13] the functional J was minimized over the Nehari manifold but all arguments are also valid in our setting when adjusting the functional and minimization set. Using additionally $\partial_{T_n} J_1(v^{(n)}) = 0$ we find

$$\mathrm{o}(1) = J_0'(v^{(n)}) - \lambda_n J_1'(v^{(n)}) = \left|v^{(n)}\right|^{q-1} v^{(n)} - \lambda_n K v^{(n)}, \qquad \text{in } L^{p+1}(\Omega).$$

We test this equation with $v^{(n)}$ and use the polynomial structures of J_0 and J_1. Hence we calculate

$$\begin{aligned}\mathrm{o}\,(1) &= J_0'(v^{(n)})[v^{(n)}] - \lambda_n J_1'(v^{(n)})[v^{(n)}] \\ &= (q+1)J_0(v^{(n)}) - 2\lambda_n J_1(v^{(n)}) = (q+1)m^{(L)} - 2\lambda_n + \mathrm{o}\,(1),\end{aligned}$$

i.e., $\lambda_n \to \frac{q+1}{2}m^{(L)} > 0$. Hence we obtain $J_0'(v^{(n)}) - \frac{q+1}{2}m^{(L)}J_1'(v^{(n)}) = \mathrm{o}\,(1)$.

Step 2: Construct a Palais-Smale Sequence and improve convergence.
We use again the polynomial structures of J_0 and J_1 and set $\tau := (\frac{q+1}{2}m^{(L)})^{\frac{1}{q-1}}$. Then $J_0'(\tau v^{(n)}) - J_1'(\tau v^{(n)}) = \mathrm{o}\,(1)$ and $J(\tau v^{(n)})$ is bounded, i.e., $(\tau v^{(n)})_n$ is a Palais-Smale sequence for J. Since $(v^{(n)})_n$ is bounded in $L^{q+1}(\Omega)$, we find $v^* \in L^{q+1}(\Omega)$ and a subsequence (again superscripted with n) such that $v^{(n)} \rightharpoonup v^*$. We now apply Lemma 3.7 for $(\tau v^{(n)})_n$ and obtain $0 = J'(\tau v^*) = \tau^q |v^*|^{q-1}v^* - \tau K v^*$. Hence, by the choice of τ, we conclude

$$0 = |v^*|^{q-1}v^* - \frac{q+1}{2}m^{(L)}Kv^*.$$

We observe that $\frac{q+1}{2}m^{(L)}b_{\mathcal{K}}(v^*, v^*) = \int_\Omega |v^*|^{q+1}\,\mathrm{d}x \geq 0$.

Step 3: Apply Brezis-Lieb Lemma.
Using the boundedness of $(v^{(n)})_n$ in $L^{q+1}(\Omega)$ and pointwise convergence $v^{(n)} \to v^*$ we obtain

$$m^{(L)} = J_0(v^{(n)}) + \mathrm{o}\,(1) = J_0(v^*) + J_0(v^{(n)} - v^*) + \mathrm{o}\,(1).$$

We will analyze the error term $J_0(v^{(n)} - v^*)$ in more detail later.

Step 4: Compare to the equation at infinity.
Motivated by the previous step we calculate

$$\begin{aligned}1 = J_1(v^{(n)}) &= \frac{1}{2}b_{\mathcal{K}}\left(v^{(n)}, v^{(n)}\right) \\ &= \frac{1}{2}b_{\mathcal{K}}\left(v^{(n)} - v^*, v^{(n)} - v^*\right) + b_{\mathcal{K}}\left(v^{(n)}, v^*\right) - \frac{1}{2}b_{\mathcal{K}}(v^*, v^*).\end{aligned}$$

Observe that by weak convergence we have $\lim_n b_{\mathcal{K}}\left(v^{(n)}, v^*\right) = b_{\mathcal{K}}(v^*, v^*)$. Using Assumption 3.22 we obtain

$$1 = \frac{1}{2}b_{\mathcal{K}}(v^*, v^*) + \frac{1}{2}b_{\tilde{\mathcal{K}}}\left(v^{(n)} - v^*, v^{(n)} - v^*\right) + \mathrm{o}\,(1).$$

Step 5: Scaling behavior and concavity.
We set $A := \frac{1}{2}b_{\mathcal{K}}(v^*, v^*)$ and $B_n := \frac{1}{2}b_{\tilde{\mathcal{K}}}\left(v^{(n)} - v^*, v^{(n)} - v^*\right)$. Clearly $A, B_n \geq 0$, $A + B_n \to 1$ and $B_n \to 1 - A =: B$. If $A > 0$, then $A^{-\frac{1}{2}}v^* \in \mathcal{M}^{(L)}$ and hence $m^{(L)} \leq J_0(A^{-\frac{1}{2}}v^*) = A^{-\frac{q+1}{2}}J_0(v^*)$. Therefore we obtain $A^{\frac{q+1}{2}}m^{(L)} \leq J_0(v^*)$. This equation is obviously true if $A = 0$. Similarly, if $B_n > 0$ then $B_n^{-\frac{1}{2}}(v^{(n)} - v^*) \in \widetilde{\mathcal{M}}^{(L)}$ and hence $\tilde{m}^{(L)} \leq J_0(B_n^{-\frac{1}{2}}(v^{(n)} - v^*)) = B_n^{-\frac{q+1}{2}}J_0(v^{(n)} - v^*)$. Therefore we obtain $B_n^{\frac{q+1}{2}}\tilde{m}^{(L)} \leq J_0(v^{(n)}v^*)$. This equation is obviously true if $B_n = 0$. We combine these results and see that

$$m^{(L)} = J_0(v^*) + J_0(v^{(n)} - v^*) + \mathrm{o}\,(1) \geq A^{\frac{q+1}{2}}m^{(L)} + B_n^{\frac{q+1}{2}}\tilde{m}^{(L)} + \mathrm{o}\,(1)$$

$$\to A^{\frac{q+1}{2}} m^{(L)} + B^{\frac{q+1}{2}} \tilde{m}^{(L)} \geq A^{\frac{q+1}{2}} m^{(L)} + B^{\frac{q+1}{2}} m^{(L)}.$$

Hence $A^{\frac{q+1}{2}} + B^{\frac{q+1}{2}} = 1$. The strict concavity of $x \mapsto x^{\frac{q+1}{2}}$ now yields either $A = 1$, $B = 0$ or $A = 0$, $B = 1$. The latter is impossible, since this would yield $m^{(L)} \geq \tilde{m}^{(L)}$ as seen in the calculation above, a contradiction to Assumption 3.21. Hence $A = 1$, $B = 0$.

Step 6: Conclusion.
Invertibility of $\tilde{K}$ and $B_n = \frac{1}{2} b_{\tilde{\mathcal{K}}} \left(v^{(n)} - v^*, v^{(n)} - v^* \right) \to 0$ now yields $v^{(n)} \to v^*$ in $L^{q+1}(\Omega)$. Therefore we see that $m^{(L)} = J_0(v^*) + J_0(v^{(n)} - v^*) + \mathrm{o}\,(1) = J_0(v^*)$ and $v^* \in \mathcal{M}^{(L)}$ since $A = 1$. Hence we found a minimizer. Utilizing $|v^*|^{q-1} v^* - \frac{q+1}{2} m^{(L)} K v^* = 0$ and $\tau = (\frac{q+1}{2} m^{(L)})^{\frac{1}{q-1}}$ as in step 3 we see that τv^* is critical point of J. □

3.1.3 Further abstract tools

3.1.3.1 Comparison of sesquilinear forms in the dual setting

This subsection is fueled by an idea of Ambrosetti and Struwe aiming to the question: How can we compare the bilinear forms of two dual problems? This knowledge helps us to compare ground state levels and to construct tools to prove the a-priori energy estimate in Assumption 3.14. In addition, and this is our new use for Lemma 3.25, we will exploit it to have a tool to prove Assumption 3.15 in our examples.

This and Lemma 3.17 are the main ideas to prove the a-priori energy estimates in Assumption 3.14 and Assumption 3.21. Moreover in our examples $\mathcal{W} = \mathcal{L} - \widetilde{\mathcal{L}}$ will have "compact support in space" (this term explains itself in the examples), which makes it way easier to calculate pairings with $\mathcal{W}$. Lemma 3.25 will also be used to proof Assumption 3.15, i.e., to transfer a Palais-Smale sequence for J to the level $m^{(N)}$ to a Palais-Smale sequence for $\tilde{J}$ without changing the energy level $m^{(N)}$ and will also be used to proof Assumption 3.22, i.e., to calculate a difference of bilinear forms. Here we assume in addition that $\mathcal{W}$ is seeing nothing outside a set $\widetilde{\Omega}$.

The next lemma contains the idea how to rewrite die difference of the bilinear forms for dual functionals and inspired by [AS86]. We recall: Up to now the Hilbert spaces $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ and the embeddings S and $\tilde{S}$ could have been completely distinct, but from now on we assume more structure. In our examples, both, $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, will be either subspaces of the function space $L^2(\mathbb{R} \times \mathbb{T}_T)$ or subspaces of the sequence space $l^2\left(\mathbb{Z}_{odd}, L^2(\mathbb{R})\right)$, and both, S and $\tilde{S}$, will be either the Sobolev-embedding for functions or the Fourier reconstruction operator which reconstructs a function out of a sequence.

Lemma 3.25. *Assume* (A1), (A2) *and* (A3) *for the "equation of interest" and the "equation at infinity". Let* $v, w \in L^{q+1}(\Omega, \mathbb{R})$, *set* $\hat{u} := \tilde{\mathcal{K}} v$, $\hat{z} := \tilde{\mathcal{K}} w \in \widetilde{\mathcal{H}}$ *and assume in addition*

(i) $\mathcal{S}^* v = \tilde{\mathcal{S}}^* v$ *and* $S^* w = \tilde{S}^* w$.

(ii) $\hat{u}, \hat{z} \in \mathcal{H}$.

(iii) $\mathcal{W}\hat{u} := \mathcal{L}\hat{u} - \widetilde{\mathcal{L}}\hat{u} \in \mathcal{H}^*$ *and* $\mathcal{W}\hat{z} := \mathcal{L}\hat{z} - \widetilde{\mathcal{L}}\hat{z} \in \mathcal{H}^*$.

Define $\hat{\varphi} = -\mathcal{L}^{-1}\mathcal{W}\hat{z} \in \mathcal{H}$. *Then* $b_{\mathcal{K}}(v, w) - b_{\tilde{\mathcal{K}}}(v, w) = \langle -\mathcal{W}\hat{u}, \hat{\varphi} \rangle_{\mathcal{H}^* \times \mathcal{H}} + \langle -\mathcal{W}\hat{u}, \hat{z} \rangle_{\mathcal{H}^* \times \mathcal{H}}$.

We can read Lemma 3.25 as follows: Taking some possibly special functions $v, w \in L^{q+1}(\Omega, \mathbb{R})$ (e.g. a ground state for "equation at infinity"), check (i), (ii) and (iii). Assumption (i) is always satisfied, if S and $\tilde{S}$ have the same formula but one has a larger domain, as in our examples. Assumption (ii) checks, whether v and w are mapped into $\mathcal{H}$ by $\tilde{\mathcal{K}}$ and not just $\widetilde{\mathcal{H}}$. This means that $\tilde{\mathcal{K}}$ should be regularizing in the sense that $\mathrm{Range}(\tilde{\mathcal{K}})$ is a strict subset of $\widetilde{\mathcal{H}}$. Writing $\hat{u} := \tilde{\mathcal{K}}v, \hat{z} := \tilde{\mathcal{K}}w$ we now have $\hat{u}, \hat{z} \in \widetilde{\mathcal{H}} \cap \mathcal{H}$. Assumption (iii) checks if the differences $\mathcal{W}\hat{u} := \mathcal{L}\hat{u} - \widetilde{\mathcal{L}}\hat{u}$ and $\mathcal{W}\hat{z} := \mathcal{L}\hat{z} - \widetilde{\mathcal{L}}\hat{z}$ are in $\mathcal{H}^*$. If the assumptions (i), (ii) and (iii) are true, then we can define $\hat{\psi} = -\mathcal{L}^{-1}\mathcal{W}\hat{u}$ and $\hat{\varphi} = -\mathcal{L}^{-1}\mathcal{W}\hat{z}$ and do the calculation of the lemma. Note that $\hat{\psi}$ is only needed in the calculation and does not appear in the statement.

Proof of Lemma 3.25. Assumption (ii) and (iii) yield that $\hat{\varphi}$ and $\hat{\psi}$ are well defined. We calculate on $\mathcal{H}^*$:

$$\begin{aligned}
\mathcal{L}\hat{\psi} &= -\mathcal{W}\hat{u} = \widetilde{\mathcal{L}}\hat{u} - \mathcal{L}\hat{u} = \tilde{\mathcal{S}}^*v - \mathcal{L}\hat{u} \quad &&\Leftrightarrow \quad \tilde{\mathcal{S}}^*v = \mathcal{L}(\hat{\psi} + \hat{u}),\\
\mathcal{L}\hat{\varphi} &= -\mathcal{W}\hat{z} = \widetilde{\mathcal{L}}\hat{z} - \mathcal{L}\hat{z} = \tilde{\mathcal{S}}^*w - \mathcal{L}\hat{z} \quad &&\Leftrightarrow \quad \tilde{\mathcal{S}}^*w = \mathcal{L}(\hat{\varphi} + \hat{z}).
\end{aligned}$$

By (i) we see $S^*v = \tilde{S}^*v = \mathcal{L}(\hat{\psi} + \hat{u}) \in \mathcal{H}^*$ and $S^*w = \tilde{S}^*w = \mathcal{L}(\hat{\varphi} + \hat{z}) \in \mathcal{H}^*$. We observe by (i) and $\hat{u} \in \mathcal{H} \cap \widetilde{\mathcal{H}}$ that

$$\langle \hat{u}, \mathcal{S}^*w \rangle_{\widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}}^*} = \langle \mathcal{S}\hat{u}, w \rangle_{L^{p+1}(\Omega) \times L^{q+1}(\Omega)} = \langle \hat{u}, \mathcal{S}^*w \rangle_{\mathcal{H} \times \mathcal{H}^*}.$$

Moreover we have

$$\begin{aligned}
\left\langle \hat{\psi}, \mathcal{S}^*w \right\rangle_{\mathcal{H} \times \mathcal{H}^*} &= \left\langle \mathcal{S}\hat{\psi}, w \right\rangle_{L^{p+1}(\Omega) \times L^{q+1}(\Omega)} = \int_\Omega \mathcal{S}\hat{\psi} \cdot w \,\mathrm{d}x\\
&= \int_\Omega w \cdot \mathcal{S}\hat{\psi} \,\mathrm{d}x = \left\langle w, \mathcal{S}\hat{\psi} \right\rangle_{L^{q+1}(\Omega) \times L^{p+1}(\Omega)} = \left\langle \mathcal{S}^*w, \hat{\psi} \right\rangle_{\mathcal{H}^* \times \mathcal{H}}
\end{aligned}$$

Hence using all from above we calculate

$$\begin{aligned}
b_{\mathcal{K}}(v, w) - b_{\tilde{\mathcal{K}}}(v, w) &= \langle \mathcal{L}^{-1}\mathcal{S}^*v, \mathcal{S}^*w \rangle_{\mathcal{H} \times \mathcal{H}^*} - \left\langle \widetilde{\mathcal{L}}^{-1}\tilde{\mathcal{S}}^*v, \tilde{\mathcal{S}}^*w \right\rangle_{\widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}}^*}\\
&= \left\langle \mathcal{L}^{-1}\tilde{\mathcal{S}}^*v, \mathcal{S}^*w \right\rangle_{\mathcal{H} \times \mathcal{H}^*} - \left\langle \widetilde{\mathcal{L}}^{-1}\tilde{\mathcal{S}}^*v, \mathcal{S}^*w \right\rangle_{\widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}}^*}\\
&= \left\langle \hat{\psi} + \hat{u}, \mathcal{S}^*w \right\rangle_{\mathcal{H} \times \mathcal{H}^*} - \langle \hat{u}, \mathcal{S}^*w \rangle_{\widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}}^*}\\
&= \left\langle \hat{\psi}, \mathcal{S}^*w \right\rangle_{\mathcal{H} \times \mathcal{H}^*} + \langle \hat{u}, \mathcal{S}^*w \rangle_{\mathcal{H} \times \mathcal{H}^*} - \langle \hat{u}, \mathcal{S}^*w \rangle_{\mathcal{H} \times \mathcal{H}^*}\\
&= \left\langle \mathcal{S}^*w, \hat{\psi} \right\rangle_{\mathcal{H}^* \times \mathcal{H}} = \left\langle \mathcal{L}(\hat{\varphi} + \hat{z}), \hat{\psi} \right\rangle_{\mathcal{H}^* \times \mathcal{H}}\\
&= \left\langle \mathcal{L}\hat{\psi}, \hat{\varphi} \right\rangle_{\mathcal{H}^* \times \mathcal{H}} + \left\langle \mathcal{L}\hat{\psi}, \hat{z} \right\rangle_{\mathcal{H}^* \times \mathcal{H}} = \langle -\mathcal{W}\hat{u}, \hat{\varphi} \rangle_{\mathcal{H}^* \times \mathcal{H}} + \langle -\mathcal{W}\hat{u}, \hat{z} \rangle_{\mathcal{H}^* \times \mathcal{H}}.
\end{aligned}$$

□

Combining Lemma 3.17 and Lemma 3.25 we have the following strategy for proving Assumption 3.14: Take a ground state v for the "equation at infinity", define $\hat{u} := \tilde{\mathcal{K}}v$ and check the assumptions of Lemma 3.25 for $v = w$. Using Lemma 3.17 we see

$$b_{\mathcal{K}}(v, v) - b_{\tilde{\mathcal{K}}}(v, v) = \left\langle -\mathcal{W}\hat{u}, \hat{\psi} \right\rangle_{\mathcal{H}^* \times \mathcal{H}} + \langle -\mathcal{W}\hat{u}, \hat{u} \rangle_{\mathcal{H}^* \times \mathcal{H}} \overset{!}{>} 0 \qquad \Rightarrow \qquad m < \tilde{m}.$$

In our examples $-\mathcal{W}$ will be a positive operator, i.e., we can hope that the right hand side of the first equation is positive.

We will also use Lemma 3.25 in the following lemma, where (iv) indicates a criterion for Assumption 3.15.

This and Lemma 3.17 are the main ideas to prove the a-priori energy estimates in Assumption 3.14 and Assumption 3.21. Moreover in our examples $\mathcal{W}$ will have "compact support in space" (this term explains itself in the examples), which makes it way easier to calculate pairings with $\mathcal{W}$.

Lemma 3.26. *Assume* (A1), (A2) *and* (A3) *for the "equation of interest" and the "equation at infinity". Let $v^{(n)} \in L^{q+1}(\Omega, \mathbb{R})$ be a Palais-Smale sequence for J to the level $m^{(N)}$ with $\tilde{S}\tilde{\mathcal{K}}v^{(n)} \to 0$ in $L^{s+1}(\widetilde{\Omega})$. Assume in addition*

(i) $S^* = \tilde{S}^*$ pointwise on $L^{q+1}(\Omega, \mathbb{R})$.

(ii) $\tilde{\mathcal{K}}\colon L^{q+1}(\Omega, \mathbb{R}) \to \mathcal{H}$ is continuous.

(iii) $\mathcal{W} \circ \tilde{\mathcal{K}}\colon L^{q+1}(\Omega, \mathbb{R}) \to \mathcal{H}^*$ with $\mathcal{W} \circ \tilde{\mathcal{K}}v := \mathcal{L}\tilde{\mathcal{K}}v - \widetilde{\mathcal{L}}\tilde{\mathcal{K}}v$ is compact.

(iv) If $\tilde{S}\tilde{\mathcal{K}}v = \tilde{S}\tilde{\mathcal{K}}w$ on $\widetilde{\Omega}$, then $\mathcal{W}\tilde{\mathcal{K}}v = \mathcal{W}\tilde{\mathcal{K}}w$ in $\mathcal{H}^*$.

Then there is a subsequence, again superscripted with n, such that $(v^{(n)})_n$ is a Palais-Smale sequence for $\tilde{J}$ to the level $m^{(N)}$. In other words: Assumption 3.15 with $\widetilde{\Omega} \subset \Omega$ and $s \in (p_*, p^*)$ as in 4. is true.

We can read Lemma 3.26 as follows: We first check the assumptions (i), (ii) and (iii). Then Lemma 3.25 is applicable for any $v \in L^{q+1}(\Omega, \mathbb{R})$. Next check assumption (iv). This is formally fulfilled, if $\mathcal{L} - \widetilde{\mathcal{L}}$ is formally supported in $\tilde{\Omega}$. Then Assumption 3.15 is true.

Proof of Lemma 3.26. Let $v^{(n)} \in L^{q+1}(\Omega, \mathbb{R})$ be a Palais-Smale sequence for J to the level $m^{(N)}$, i.e., $J\left(v^{(n)}\right) \to M$ and $J'\left(v^{(n)}\right) \to 0$ in $L^{q+1}(\Omega, \mathbb{R})$. By assumption (i), (ii) and (iii) we can apply Lemma 3.25 for each $v^{(n)}$. Let $\hat{u}^{(n)} := \tilde{\mathcal{K}}v^{(n)}$ and $\hat{\psi}^{(n)} := -\mathcal{L}^{-1}\mathcal{W}\hat{u}^{(n)}$. Then applying Lemma 3.25 with $w = v = v^{(n)}$ we obtain

$$\begin{aligned}\tilde{J}\left(v^{(n)}\right) - J\left(v^{(n)}\right) &= \frac{1}{2}\left(b_{\mathcal{K}}\left(v^{(n)}, v^{(n)}\right) - b_{\tilde{\mathcal{K}}}\left(v^{(n)}, v^{(n)}\right)\right) \\ &= \frac{1}{2}\left(\left\langle -\mathcal{W}\hat{u}^{(n)}, \hat{\psi}^{(n)}\right\rangle_{\mathcal{H}^*\times\mathcal{H}} + \left\langle -\mathcal{W}\hat{u}^{(n)}, \hat{u}^{(n)}\right\rangle_{\mathcal{H}^*\times\mathcal{H}}\right).\end{aligned}$$

For $w \in L^{q+1}(\Omega)$ define $\hat{z} := \tilde{\mathcal{K}}w \in \widetilde{\mathcal{H}}$ and $\mathcal{L}\hat{\varphi} = -\mathcal{W}\hat{z}$ on $\mathcal{H}^*$. Applying Lemma 3.25 we obtain

$$\begin{aligned}\tilde{J}'\left(v^{(n)}\right)[w] - J'\left(v^{(n)}\right)[w] &= b_{\mathcal{K}}\left(v^{(n)}, w\right) - b_{\tilde{\mathcal{K}}}\left(v^{(n)}, w\right) \\ &= \left\langle -\mathcal{W}\hat{u}^{(n)}, \hat{\varphi}\right\rangle_{\mathcal{H}^*\times\mathcal{H}} + \left\langle -\mathcal{W}\hat{u}^{(n)}, \hat{z}\right\rangle_{\mathcal{H}^*\times\mathcal{H}}.\end{aligned}$$

Obviously $\mathcal{W}\hat{u}^{(n)} \to 0$ in $\mathcal{H}^*$ is sufficient to prove the claim. By Proposition 3.6 the sequence $(v^{(n)})_n$ is bounded in $L^{q+1}(\Omega, \mathbb{R})$, hence $v^{(n)} \rightharpoonup v^*$ in $L^{q+1}(\Omega, \mathbb{R})$ up to taking a subsequence. Compactness in (iii) yields $\mathcal{W}\hat{u}^{(n)} \to \mathcal{W}\tilde{\mathcal{K}}v^*$ in $\mathcal{H}^*$, possibly after taking another subsequence. Looking back into the proof of Lemma 3.7 we see that (up to taking another a subsequence) we have $v^{(n)} \to v^*$ a.e. in Ω. Using $\tilde{S}\tilde{\mathcal{K}}v^{(n)} \to 0$ in $L^{s+1}(\widetilde{\Omega})$ and continuity of $\tilde{\mathcal{K}}$ and local compactness of $\tilde{S}$ we obtain $\tilde{S}\tilde{\mathcal{K}}v^* = 0$ in $\widetilde{\Omega}$. Hence $\mathcal{W}\tilde{\mathcal{K}}v^* = 0$ and the proof is done. □

3.1.3.2 Comparison of Nehari and constrained minimization energy level

Last in this section we connect the energy levels of the Nehari and constrained minimization approach.

Lemma 3.27. *Assume (A1), (A2) and (A3). Then:*

1. $m^{(L)} = 2(1+q)^{-\frac{1+q}{2}}(1-q)^{-\frac{1-q}{2}} {m^{(N)}}^{\frac{1-q}{2}}$.
2. *The value* $m^{(L)}$ *is attained if and only if* $m^{(N)}$ *is attained.*
3. *In the case* $m^{(L)}, m^{(N)} < \infty$ *set* $\tau := (\frac{q+1}{2} m^{(L)})^{\frac{1}{1-q}}$. *If* $v_L \in \mathcal{M}^{(L)}$ *with* $m^{(L)} = J_0(v_L)$, *then* $\tau v_L \in \mathcal{M}^{(N)}$ *with* $m^{(N)} = J(\tau v_L)$. *If* $v_N \in \mathcal{M}^{(N)}$ *with* $m^{(N)} = J(v_N)$, *then* $\frac{1}{\tau} v_N \in \mathcal{M}^{(L)}$ *with* $m^{(L)} = J_0(\frac{1}{\tau} v_N)$.

Proof. If $\mathcal{D}(J) \cap \{b_{\mathcal{K}}(v,v) > 0\} = \emptyset$, then $\mathcal{M}^{(N)} = \mathcal{M}^{(L)} = \emptyset$ and both sides are $+\infty$, i.e., the claim holds true. Now let $\mathcal{D}(J) \cap \{b_{\mathcal{K}}(v,v) > 0\} \neq \emptyset$, then $\mathcal{M}^{(N)}$ and $\mathcal{M}^{(L)}$ are both nonempty. For $v \in \mathcal{D}(J)$ with $b_{\mathcal{K}}(v,v) > 0$ we define

$$t_N(v) := \left(\frac{\int_\Omega |v|^{q+1}\,\mathrm{d}x}{b_{\mathcal{K}}(v,v)} \right)^{\frac{1}{1-q}} = \left(\frac{(q+1)J_0(v)}{2J_1(v)} \right)^{\frac{1}{1-q}}, \qquad t_L(v) := J_1(v)^{-\frac{1}{2}}.$$

Then: $tv \in \mathcal{M}^{(N)}$ if and only if $t = t_N(v)$, and $tv \in \mathcal{M}^{(L)}$ if and only if $t = t_L(v)$. With this preparation we start calculating. Let $v \in \mathcal{M}^{(N)}$. We observe that $(1+q)J_0(v) = 2J_1(v)$ and $J(v) = (1 - \frac{1+q}{2})J_0(v) = \frac{1-q}{2} J_0(v)$. Then we calculate

$$\begin{aligned} m^{(L)} &\le J_0(t_L(v)v) = t_L(v)^{1+q} J_0(v) = \left(\frac{1+q}{2} J_0(v) \right)^{-\frac{1+q}{2}} J_0(v) \\ &= \left(\frac{2}{1+q} \right)^{\frac{1+q}{2}} \left(\frac{2}{1-q} J(v) \right)^{1-\frac{1+q}{2}} = 2 \left(\frac{1}{1+q} \right)^{\frac{1+q}{2}} \left(\frac{1}{1-q} \right)^{\frac{1-q}{2}} J(v)^{\frac{1-q}{2}}. \end{aligned}$$

Taking the infimum over v on the right hand side we obtain $m^{(L)} \le 2(1+q)^{-\frac{1+q}{2}}(1-q)^{-\frac{1-q}{2}} {m^{(N)}}^{\frac{1-q}{2}}$. Now let $v \in \mathcal{M}^{(L)}$. We observe that $J_1(v) = 1$ and $t_N(v) = \left(\frac{(q+1)}{2} J_0(v) \right)^{\frac{1}{1-q}}$. Then we calculate

$$\begin{aligned} m^{(N)} &\le J(t_N(v)v) = \frac{1-q}{2} J_0(t_N(v)v) = \frac{1-q}{2} \left(\frac{(q+1)}{2} J_0(v) \right)^{\frac{1+q}{1-q}} J_0(v) \\ &= \left(\frac{1}{2} \right)^{\frac{2}{1-q}} (1+q)^{\frac{1+q}{1-q}} (1-q) J_0(v)^{\frac{2}{1-q}}. \end{aligned}$$

Taking the infimum over v on the right hand side we obtain after a short calculation $m^{(L)} \ge 2(1+q)^{-\frac{1+q}{2}}(1-q)^{-\frac{1-q}{2}} {m^{(N)}}^{\frac{1-q}{2}}$. Hence the claim 1 is proven. Claim 2 instantly follows from the calculations above. We only sketch the proof of claim 3, the left out calculations are lengthy but straight forward. If $v_L \in \mathcal{M}^{(L)}$ with $m^{(L)} = J_0(v_L)$, then by the Lagrangian multiplier rule have $J_0'(v_L) = \tau^{1-q} J_1'(v_L)$, where we obtain the exact value of the Lagrangian multiplier by testing the equation with v_L as in step 1 in the proof of Theorem 3.24. Using the scaling of J_0 and J_1 we obtain $J'(\tau v_L) = 0$ and therefore $\tau v_L \in \mathcal{M}^{(N)}$. Using additionally $\tau = t_N(v_L)$, $t_L(v_L) = 1$, and $m^{(L)} = 2(1+q)^{-\frac{1+q}{2}}(1-q)^{-\frac{1-q}{2}} {m^{(N)}}^{\frac{1-q}{2}}$ we see that $m^{(N)} = J(\tau v_L)$. The reverse claim is proven analogously. □

We have shown that due to the polynomial structure of our dual functional the presented minimization procedures are equivalent. For more general functionals this might not always be true.

3.2 Examples for "wave-guides"

We will apply our previously developed technique on dual variational functionals to semilinear wave equations. Especially the case of one spatial direction and time-periodicity with a non-translation-invariant wave operator L and a periodic in space right hand side is of interest, since it was not covered by the techniques in Chapter 2. Observe that Γ could have been t depended here. In our examples we have $L = \tilde{L} - W$ with $\tilde{L} = -\alpha\partial_t^2 - \partial_x^2$. Hence we first analyze ground states of our "equation at infinity" $\tilde{L}u = \Gamma(x)|u|^{p-1}u$ on $(x,t) \in \mathbb{R} \times \mathbb{T}_T$.

3.2.1 The ground state of the "equation at infinity"

In this section we consider the $1+1$ dimensional, elliptic, semilinear equation

$$-\alpha u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u, \qquad (x,t) \in \mathbb{R} \times \mathbb{T}_T, \tag{3.5}$$

with $\Gamma \in L^\infty_{per}(\mathbb{R})$ and $\inf \Gamma > 0$. As before $\mathbb{T}_T$ denotes the 1-dimensional torus with period $T > 0$. We consider $\frac{T}{2}$-anti-periodic functions on the cylinder $\mathbb{R} \times \mathbb{T}_T$, since (3.5) is compatible with this symmetry and it makes $\tilde{L}$ an elliptic operator. Then one can find a ground state by well known techniques, e.g. cf. [SW10]. Since we work on function spaces, we write non-calligraphic letters, as in Chapter 2.

Definition 3.28. *We define for $p \in (1,\infty)$, $k \in \mathbb{N}$, $\alpha > 0$ and $\Gamma \in L^\infty_{per}(\mathbb{R})$:*

$$\begin{aligned}
\tilde{L}&\colon H^2_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R}) \to L^2_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R}), \quad \tilde{L}u := -\alpha u_{tt} - u_{xx},\\
b_{\tilde{L}}&\colon H^1_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R}) \times H^1_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R}) \to \mathbb{R}, \quad b_{\tilde{L}}(u,v) := \int_{\mathbb{R}\times\mathbb{T}_T} \alpha u_t v_t + u_x v_x \,\mathrm{d}(x,t),\\
\tilde{I}&\colon H^1_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R}) \to \mathbb{R}, \quad \tilde{I}(u) := \frac{1}{2} b_{\tilde{L}}(u,u) - \frac{1}{p+1}\int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(x)|u|^{p+1}\,\mathrm{d}(x,t).
\end{aligned}$$

Proposition 3.29. *Let $p \in (1,\infty)$, $\alpha > 0$ and $\Gamma \in L^\infty_{per}(\mathbb{R})$ such that $\inf\Gamma > 0$. Then:*

(i) $\forall\, u \in H^2_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R}), v \in H^1_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})\colon \quad b_{\tilde{L}}(u,v) = \left\langle \tilde{L}u, v\right\rangle_{L^2(\mathbb{R}\times\mathbb{T}_T)}$.

(ii) $\tilde{L}$ is self-adjoint with spectrum $\sigma(\tilde{L}) = [\alpha\omega^2,\infty)$ with $\omega := \frac{2\pi}{T}$. Hence $\tilde{L}$ is continuously invertible.

(iii) $\tilde{I} \in C^1(H^1_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R}))$ with $\tilde{I}'(u)[v] = b_{\tilde{L}}(u,v) - \int_{\mathbb{R}\times\mathbb{T}_T}\Gamma(x)|u|^{p-1}u\,v\,\mathrm{d}(x,t)$.

(iv) $\tilde{I}$ has a ground state, i.e., there is $u \in H^1_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})\setminus\{0\}$ such that $0 < \tilde{I}(u) = \inf\{\tilde{I}(v) \mid \tilde{I}'(v) = 0\}$. Furthermore $u \in H^2_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})$.

Proof. (i), (ii) and (iii) are clear since $\tilde{L}$ and $\tilde{I}$ fit into our abstract setting. The existence of the ground state in (iv) can be found in [SW10]. The additional regularity of the ground state can be done by standard elliptic regularity theory, e.g. as in [GT15]. ☐

For the sake

3.2.1.1 Decay at infinity

We will have a closer look onto the ground state and analyze its behavior at infinity. Our main result in this section will be the following:

Theorem 3.30. *Let $u \in H^2_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})\setminus\{0\}$ be a solution of (3.5). Then for $s \in \{0,1,2\}$, $l \in (0,1)$ there are $\varepsilon > 0$ and complex numbers $\mathcal{U}^{(s)}_{\pm 1}, \sqcap^{(l)}_{\pm 1} \in \mathbb{C}$ such that as $|x| \to \infty$ we have uniformly in t:*

$$\frac{\partial^s}{\partial t^s}u(x,t) = \left(\mathcal{U}^{(s)}_1\cdot \mathrm{e}^{\mathrm{i}\omega t} + \mathcal{U}^{(s)}_{-1}\cdot \mathrm{e}^{-\mathrm{i}\omega t}\right)\cdot \mathrm{e}^{-\sqrt{\alpha}\omega|x|} + \mathrm{O}\left(\mathrm{e}^{-(\sqrt{\alpha}\omega+\varepsilon)|x|}\right),$$
$$\sum_{k\in\mathbb{Z}_{odd}} |\omega k|^{2+l}\hat{u}_k(x)e_k(t) = \left(\sqcap^{(l)}_1\cdot \mathrm{e}^{\mathrm{i}\omega t} + \sqcap^{(l)}_{-1}\cdot \mathrm{e}^{-\mathrm{i}\omega t}\right)\cdot \mathrm{e}^{-\sqrt{\alpha}\omega|x|} + \mathrm{O}\left(\mathrm{e}^{-(\sqrt{\alpha}\omega+\varepsilon)|x|}\right).$$

The first equality tells us the exact behavior of a solution u, u_t and u_{tt} at infinity. Note that the first frequencies $k = \pm 1$ dominate and the correction terms are decaying strictly faster and uniformly in time. The second equality tells us, that we also know the behavior of $\left|\frac{\partial}{\partial t}\right|^{2+l} u$ at infinity. We do not claim that u_{ttt} exists. The rest of this section is denoted to the proof of Theorem 3.30. We start with giving a formula for $\tilde{L}^{-1}$.

Lemma 3.31. *Let $\alpha, s > 0$. Then for $u \in L^2_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})$:*

$$\tilde{L}^{-1}u(x,t) = \int_{\mathbb{R}\times\mathbb{T}_T} \tilde{G}(y-x,s-t)u(y,s)\,\mathrm{d}(y,s)\,,$$
$$\tilde{G}(x,t) := -\frac{1}{4\pi\sqrt{\alpha}}\ln\Big(2\mathrm{e}^{-\sqrt{\alpha}\omega|x|}\cdot\big(\cosh\big(\sqrt{\alpha}\omega|x|\big)-\cos(\omega t)\big)\Big),\quad (x,t)\notin\{0\}\times T\mathbb{Z}.$$

The proof for this lemma is technical and not very insightful. Therefore we shift the details into the appendix Section 3.2.4.1. Later we use more than one integrable time derivative of $\tilde{G}$, but the second time derivative of $\tilde{G}$ is not integrable. Therefore we look at fractional derivatives.

Proposition 3.32. *Let $\alpha > 0$. Then $\tilde{G}$ is smooth outside $\{0\}\times T\mathbb{Z}$ and the following asymptotics are true:*

$$\begin{aligned}\tilde{G}(x,t) &= \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{2\sqrt{\alpha T}|k|\omega}\mathrm{e}^{-\sqrt{\alpha}|k|\omega|x|}e_k(t)\\ &= \begin{cases} -\frac{1}{4\pi\sqrt{\alpha}}\ln\big(\omega^2t^2+\alpha\omega^2x^2\big)+\mathrm{o}\,(1), & |x|,|t|\to 0,\\ \frac{1}{2\pi\sqrt{\alpha}}\mathrm{e}^{-\sqrt{\alpha}\omega|x|}\cos(\omega t)+\mathrm{O}\left(\mathrm{e}^{-2\sqrt{\alpha}\omega|x|}\right), & |x|\to\infty, t\in\mathbb{T}_T,\end{cases}\end{aligned}$$

$$\partial_t\tilde{G}(x,t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\mathrm{i}\cdot\operatorname{sign}(k)}{2\sqrt{\alpha T}} \mathrm{e}^{-\sqrt{\alpha}|k|\omega|x|} e_k(t) = -\frac{\omega}{4\pi\sqrt{\alpha}} \frac{\sin(\omega t)}{\cosh(\sqrt{\alpha}\omega|x|)-\cos(\omega t)},$$
$$= \begin{cases} -\frac{1}{2\pi\sqrt{\alpha}}\frac{t}{\alpha x^2+t^2} + \mathrm{o}(1), & |x|,|t|\to 0,\\ -\frac{\omega}{2\pi\sqrt{\alpha}}\mathrm{e}^{-\sqrt{\alpha}\omega|x|}\sin(\omega t) + \mathrm{O}\left(\mathrm{e}^{-2\sqrt{\alpha}\omega|x|}\right), & |x|\to\infty, t\in\mathbb{T}_T,\end{cases}$$
$$\tilde{G}^{(s)}(x,t) := \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|\omega k|^{s-1}}{2\sqrt{\alpha T}\omega} \mathrm{e}^{-\sqrt{\alpha}|k|\omega|x|} e_k(t)$$
$$= \begin{cases} \frac{\Gamma(s)\omega^{s-1}}{2\pi\sqrt{\alpha}}\left((\sqrt{\alpha}\omega|x|+\mathrm{i}\omega t)^{-s} + (\sqrt{\alpha}\omega|x|-\mathrm{i}\omega t)^{-s}\right) + \mathrm{o}(1), & |x|,|t|\to 0,\\ -\frac{\omega^{s-1}}{2\pi\sqrt{\alpha}}\mathrm{e}^{-\sqrt{\alpha}\omega|x|}\cos(\omega t) + \mathrm{O}\left(\mathrm{e}^{-2\sqrt{\alpha}\omega|x|}\right), & |x|\to\infty, t\in\mathbb{T}_T.\end{cases}$$
$$\partial_t^2\tilde{G}(x,t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{-|k|\omega}{2\sqrt{\alpha T}} \mathrm{e}^{-\sqrt{\alpha}|k|\omega|x|} e_k(t) = -\frac{\omega^2}{4\pi\sqrt{\alpha}} \frac{\cos(\omega t)\cosh(\sqrt{\alpha}\omega|x|)-1}{\left(\cosh(\sqrt{\alpha}\omega|x|)-\cos(\omega t)\right)^2}$$
$$= \begin{cases} -\frac{1}{2\pi\sqrt{\alpha}}\frac{\alpha x^2-t^2}{(\alpha x^2+t^2)^2} + \mathrm{O}(1), & |x|,|t|\to 0,\\ -\frac{\omega^2}{2\pi\sqrt{\alpha}}\mathrm{e}^{-\sqrt{\alpha}\omega|x|}\cos(\omega t) + \mathrm{O}\left(\mathrm{e}^{-2\sqrt{\alpha}\omega|x|}\right), & |x|\to\infty, t\in\mathbb{T}_T.\end{cases}$$

Hence $\tilde{G}, \partial_t\tilde{G}, \tilde{G}^{(s)} \in L^1(\mathbb{R}\times\mathbb{T}_T)$ *for* $s\in(0,2)$ *but* $\partial_t^2\tilde{G}\notin L^1_{loc}(\mathbb{R}\times\mathbb{T}_T)$.

Observe that $\tilde{G}^{(s)}$ is something like a s^{th} fractional time derivative of $\tilde{G}$.

Proof. Clearly $\tilde{G}$ is smooth outside $x=0$. The closed forms of the derivatives follow by the usual differentiation rules. The Fourier series representations now follow by smoothness and the usual differentiation rules. The behaviors for small arguments of $\tilde{G}$, $\partial_t\tilde{G}$ and $\partial_t^2\tilde{G}$ follow by inserting $\mathrm{e}^z = 1+\mathrm{O}(z)$, $\cos(z) = 1-\frac{1}{2}z^2+\mathrm{O}(z^4)$, $\sin(z) = z+\mathrm{O}(z^3)$ and $\cosh(z) = 1+\frac{1}{2}z^2+\mathrm{O}(z^4)$ for $|z|\to 0$ into the closed representations. We sketch these calculations, i.e. for $0<|x|,|t|\to 0$ we have

$$-4\pi\sqrt{\alpha}\tilde{G}(x,t) = \ln\Big(2\mathrm{e}^{-\sqrt{\alpha}\omega|x|}\cdot\big(\cosh(\sqrt{\alpha}\omega|x|)-\cos(\omega t)\big)\Big)$$
$$= \ln\Bigg(2\cdot(1+\mathrm{O}(|x|))\cdot\bigg(\Big(1+\frac{1}{2}\alpha\omega^2x^2+\mathrm{O}(x^4)\Big)-\Big(1-\frac{1}{2}\omega^2t^2+\mathrm{O}(t^4)\Big)\bigg)\Bigg)$$
$$= \ln(1+\mathrm{O}(|x|)) + \ln\big(\omega^2t^2+\alpha\omega^2x^2+\mathrm{O}(x^4+t^4)\big)$$
$$= \ln\big(\omega^2t^2+\alpha\omega^2x^2\big)+\mathrm{O}(|x|+|t|),$$
$$-4\pi\sqrt{\alpha}\tilde{G}_t(x,t) = \frac{\omega\sin(\omega t)}{\cosh(\sqrt{\alpha}\omega|x|)-\cos(\omega t)}$$
$$= \frac{\omega^2 t+\mathrm{O}\left(|t|^3\right)}{\left(1+\frac{1}{2}\alpha\omega^2x^2+\mathrm{O}(x^4)\right)-\left(1-\frac{1}{2}\omega^2t^2+\mathrm{O}(t^4)\right)} = \frac{2t}{\alpha x^2+t^2}+\mathrm{O}(|x|+|t|),$$
$$-4\pi\sqrt{\alpha}\tilde{G}_{tt}(x,t) = \omega^2\frac{\cos(\omega t)\cosh(\sqrt{\alpha}\omega|x|)-1}{\left(\cosh(\sqrt{\alpha}\omega|x|)-\cos(\omega t)\right)^2}$$
$$= \omega^2\frac{\left(1-\frac{1}{2}\omega^2t^2+\mathrm{O}(t^4)\right)\left(1+\frac{1}{2}\alpha\omega^2x^2+\mathrm{O}(x^4)\right)-1}{\left(\left(1+\frac{1}{2}\alpha\omega^2x^2+\mathrm{O}(x^4)\right)-\left(1-\frac{1}{2}\omega^2t^2+\mathrm{O}(t^4)\right)\right)^2}$$
$$= 2\frac{\alpha x^2-t^2+\mathrm{O}\left(x^4+t^4\right)}{\left(t^2+\alpha x^2+\mathrm{O}(x^4+t^4)\right)^2} = 2\frac{\alpha x^2-t^2}{\left(t^2+\alpha x^2\right)^2}+\mathrm{O}(1).$$

For the behavior of $\tilde{G}^{(s)}$ for small arguments, we cite the following fact about the so-called *Polylogarithm* in [Woo92]:

$$\begin{aligned} &\operatorname{Li}_\sigma(x) := \sum_{k=1}^{\infty} \frac{1}{k^\sigma} x^k, \quad x \in \mathbb{C}, |x| < 1, \operatorname{Re}(\sigma) < 1 \\ \Rightarrow \quad &\operatorname{Li}_\sigma(\mathrm{e}^\mu) - \Gamma(1-\sigma) \cdot (-\mu)^{\sigma-1} \to 0 \quad \text{as } \mu \to 0 \text{ with } \operatorname{Re}(\mu) < 0. \end{aligned}$$

Here we choose $(-\mu)^{-s} > 0$ if $\mu \in (-\infty, 0)$ and extend continuous into the slit plane $\mathbb{C} \setminus (0, \infty)$. Using this result with $\sigma = 1 - s$ and $\mu = -\sqrt{\alpha}\omega|x| \pm \mathrm{i}\omega t$ we see

$$\begin{aligned} 2\sqrt{\alpha T}\omega^{2-s}\tilde{G}^{(s)}(x,t) &= \sum_{k=1}^{\infty} \frac{1}{k^{1-s}} \mathrm{e}^{(-\sqrt{\alpha}\omega|x|+\mathrm{i}\omega t)k} + \sum_{k=1}^{\infty} \frac{1}{k^{1-s}} \mathrm{e}^{(-\sqrt{\alpha}\omega|x|-\mathrm{i}\omega t)k} \\ &= \Gamma(s) \cdot (\sqrt{\alpha}\omega|x| - \mathrm{i}\omega t)^{-s} + \Gamma(s) \cdot (\sqrt{\alpha}\omega|x| + \mathrm{i}\omega t)^{-s} + \mathrm{o}(1) \end{aligned}$$

as $0 < |x|, |t| \to 0$. We obtain the behavior for $|x| > 1$ easily by inserting the sequence representations and calculating:

$$\begin{aligned} &\sum_{k=1}^{\infty} k^n \mathrm{e}^{(-\sqrt{\alpha}\omega|x|\pm\mathrm{i}\omega t)k} \\ &\quad = \mathrm{e}^{-\sqrt{\alpha}\omega|x|} \cdot (\cos(\omega t) \pm \mathrm{i}\sin(\omega t)) + \mathrm{e}^{-2\sqrt{\alpha}\omega|x|} \sum_{k=0}^{\infty} (k+2)^n \mathrm{e}^{-\sqrt{\alpha}\omega|x|k} \mathrm{e}^{\pm\mathrm{i}\omega(k+2)t} \\ &\quad = \mathrm{e}^{-\sqrt{\alpha}\omega|x|} \cdot (\cos(\omega t) \pm \mathrm{i}\sin(\omega t)) + \mathrm{O}\left(\mathrm{e}^{-2\sqrt{\alpha}\omega|x|}\right), \end{aligned}$$

as $1 < |x| \to \infty$ for $n \in \mathbb{Z} \setminus \{0\}$. Note that $\mathrm{O}\left(\mathrm{e}^{-2\sqrt{\alpha}\omega|x|}\right)$ may be t-dependent but is uniformly bounded. Hence as $|x| \to \infty$ we obtain

$$\begin{aligned} \tilde{G}(x,t) &= \frac{1}{2\sqrt{\alpha T}\omega} \sum_{k=1}^{\infty} \frac{1}{k} \mathrm{e}^{(-\sqrt{\alpha}\omega|x|+\mathrm{i}\omega t)k} + \frac{1}{2\sqrt{\alpha T}\omega} \sum_{k=1}^{\infty} \frac{1}{k} \mathrm{e}^{(-\sqrt{\alpha}\omega|x|-\mathrm{i}\omega t)k} \\ &= \frac{1}{2\pi\sqrt{\alpha}} \mathrm{e}^{-\sqrt{\alpha}\omega|x|} \cos(\omega t) + \mathrm{O}\left(\mathrm{e}^{-2\sqrt{\alpha}\omega|x|}\right) \\ \tilde{G}_t(x,t) &= \frac{\mathrm{i}}{2\sqrt{\alpha T}} \sum_{k=1}^{\infty} \mathrm{e}^{(-\sqrt{\alpha}\omega|x|+\mathrm{i}\omega t)k} - \frac{\mathrm{i}}{2\sqrt{\alpha T}} \sum_{k=1}^{\infty} \mathrm{e}^{(-\sqrt{\alpha}\omega|x|-\mathrm{i}\omega t)k} \\ &= -\frac{\omega}{2\pi\sqrt{\alpha}} \mathrm{e}^{-\sqrt{\alpha}\omega|x|} \sin(\omega t) + \mathrm{O}\left(\mathrm{e}^{-2\sqrt{\alpha}\omega|x|}\right), \\ \tilde{G}_{tt}(x,t) &= \frac{-\omega}{2\sqrt{\alpha T}} \sum_{k=1}^{\infty} k\mathrm{e}^{(-\sqrt{\alpha}\omega|x|+\mathrm{i}\omega t)k} + \frac{-\omega}{2\sqrt{\alpha T}} \sum_{k=1}^{\infty} k\mathrm{e}^{(-\sqrt{\alpha}\omega|x|-\mathrm{i}\omega t)k} \\ &= \frac{-\omega^2}{2\pi\sqrt{\alpha}} \mathrm{e}^{-\sqrt{\alpha}\omega|x|} \cos(\omega t) + \mathrm{O}\left(\mathrm{e}^{-2\sqrt{\alpha}\omega|x|}\right). \end{aligned}$$

The claimed integrability at the singularities follow by $\ln(|\cdot|), |\cdot|^a \in L^1_{loc}(\mathbb{R}^2, \mathbb{R})$ for $a > -2$ and $|\cdot|^{-2} \notin L^1_{loc}(\mathbb{R}^2, \mathbb{R})$. □

The next Lemma is a tool to "read off" the behavior at infinity of a function with a convolution representation $u = g * f$. Observe that we will only need knowledge of the convolution-kernel g at infinity, i.e., the limiting function g_∞, fast decay of f, namely $b > a + \varepsilon$, and global integrability of g. Note that we do not need integrability of g_∞, since f is decaying fast enough. We will also keep track of the constants.

Lemma 3.33. *Let $\varepsilon, a, b, R > 0$ and suppose $b > a + \varepsilon$. Assume that*

$$u(x,t) = \int_{\mathbb{R}\times\mathbb{T}_T} f(y,s)g(x-y,t-s)\,\mathrm{d}(y,s)\,, \qquad (x,t) \in \mathbb{R}\times\mathbb{T}_T,$$

with functions $f, g, g_\infty \colon \mathbb{R}\times\mathbb{T}_T \to \mathbb{C}$ such that:

(i) $g \in L^1(\mathbb{R}\times\mathbb{T}_T)$ *and* $\forall (x,t) \in \mathbb{R}\times\mathbb{T}_T\colon\ |g_\infty(x,t)| \le C\mathrm{e}^{-ax}$,

(ii) $\forall\, x > R, t \in \mathbb{T}_T\colon\ |g(x,t) - g_\infty(x,t)| \le C\mathrm{e}^{-(a+\varepsilon)x}$,

(iii) $\forall (x,t) \in \mathbb{R}\times\mathbb{T}_T\colon\ |f(x,t)| \le C\mathrm{e}^{-b|x|}$.

Then there is $\tilde{C} > 0$ such that:

$$\left|u(x,t) - \int_{\mathbb{R}\times\mathbb{T}_T} f(y,s)g_\infty(x-y,s-t)\,\mathrm{d}(y,s)\right| \le \tilde{C}\mathrm{e}^{-(a+\varepsilon)x}, \qquad \forall\, x > R, t \in \mathbb{T}_T.$$

A similar result for asymptotics with $x < -R$ can be proven analogously.

Proof. Observe that $x - y > R$ is equivalent to $y < x - R$. We then calculate for $x > R$:

$$\begin{aligned}
&\left|u(x,t) - \int_{\mathbb{R}\times\mathbb{T}_T} f(y,s)g_\infty(x-y,s-t)\,\mathrm{d}(y,s)\right| \\
&\quad\le \int_{\mathbb{R}\times\mathbb{T}_T} |f(y,s)|\cdot|g(x-y,s-t) - g_\infty(x-y,s-t)|\,\mathrm{d}(y,s) \\
&\quad\le \int_{\mathbb{T}_T}\int_{x-R}^{\infty} C\mathrm{e}^{-by}\cdot\left(|g(x-y,s-t)| + C\mathrm{e}^{-a(x-y)}\right)\mathrm{d}y\,\mathrm{d}s \\
&\qquad + \int_{\mathbb{T}_T}\int_{-\infty}^{x-R} C\mathrm{e}^{-b|y|}\cdot C\mathrm{e}^{-(a+\varepsilon)(x-y)}\,\mathrm{d}y\,\mathrm{d}s \\
&\quad\le C\mathrm{e}^{-b(R-x)}\int_{\mathbb{R}\times\mathbb{T}_T}|g(x-y,s-t)|\,\mathrm{d}(y,s) + C^2T\mathrm{e}^{-ax}\int_{x-R}^{\infty}\mathrm{e}^{-(b-a)y}\,\mathrm{d}y \\
&\qquad + C^2T\mathrm{e}^{-(a+\varepsilon)x}\int_{-\infty}^{0}\mathrm{e}^{(b+a+\varepsilon)y}\,\mathrm{d}y + C^2T\mathrm{e}^{-(a+\varepsilon)x}\int_0^{x-R}\mathrm{e}^{-(b-a-\varepsilon)y}\,\mathrm{d}y \\
&\quad= C\|g\|_{L^1(\mathbb{R}\times\mathbb{T}_T)}\mathrm{e}^{bR}\cdot\mathrm{e}^{-bx} + \frac{C^2T}{b-a}\mathrm{e}^{(b-a)R}\cdot\mathrm{e}^{-bx} \\
&\qquad + \frac{C^2T}{b+a+\varepsilon}\mathrm{e}^{-(a+\varepsilon)x} + \frac{C^2T}{b-a-\varepsilon}\mathrm{e}^{-(a+\varepsilon)x}\left(1 - \mathrm{e}^{-(b-a-\varepsilon)(x-R)}\right).
\end{aligned}$$

Hence the constant $\tilde{C} := C\|g\|_{L^1(\mathbb{R}\times\mathbb{T}_T)}\mathrm{e}^{bR} + \frac{C^2}{b-a}\mathrm{e}^{(b-a)R} + \frac{2bC^2T}{b^2-(a+\varepsilon)^2} + \frac{C^2T}{b-a-\varepsilon}\mathrm{e}^{(b-a-\varepsilon)R}$ does the job. □

To apply the previous lemma, we must prove that u is exponentially decaying with the rate $\mathrm{e}^{-\sqrt{\alpha}\omega|x|}$. Then the right hand side $\Gamma(x)|u|^{p-1}u$ will decay strictly faster since $p > 1$.

Lemma 3.34. *Let $u \in H^2_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})\setminus\{0\}$ be a solution of* (3.5)*. Then*

$$\forall\,\mu < \sqrt{\alpha}\omega\ \exists\, C > 0\ \forall\,(x,t) \in \mathbb{T}_T\colon \qquad |u(x,t)| \le C\mathrm{e}^{-\mu|x|}.$$

Proof. Since u is a solution of (3.5) and we have an explicit formula for $\tilde{L}^{-1}$, we obtain

$$u(x,t) = \int_{\mathbb{R}\times\mathbb{T}_T} \tilde{G}(x-y, t-\tau) f(u(y,\tau))\,\mathrm{d}(y,\tau)\,, \qquad \text{with } f(u) := |u|^{p-1}u.$$

Following the lines of [BL97] chapter 3 until Theorem 3.1.4, we observe that the additional integration over $\mathbb{T}_T$ does not change the proof and we see: the function u is decaying exponentially with $\mathrm{e}^{\mu|x|}u(x,t) \in L^1(\mathbb{R}\times\mathbb{T}_T)$ for any $\mu < \sqrt{\alpha}\omega$. ☐

We now have all tools at hand to proof the main result for this section.

Proof of Theorem 3.30. We prove the result for $x \to \infty$, the proof for $x \to -\infty$ is done analogously. By Lemma 3.34 u decays exponentially with $\mathrm{e}^{\mu|x|}u(x,t) \in L^1(\mathbb{R}\times\mathbb{T}_T)$ for any $\mu < \sqrt{\alpha}\omega$. Since $p > 1$, the right hand side of equation (3.2) also decays exponentially and with higher rate $p\theta\sqrt{\alpha}\omega$ for any $\theta < 1$. Let $\varepsilon \in (0, \min\{(p-1)\sqrt{\alpha}\omega, 2\sqrt{\alpha}\omega\})$ be fixed. Using the formula for $\tilde{L}^{-1}$ from Lemma 3.31, the integrability and decay of $\tilde{G}$ from Proposition 3.32 and the asymptotics from Lemma 3.33 we obtain

$$\begin{aligned}
u(x,t) &= \int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(y)|u(y,s)|^{p-1}u(y,s)\tilde{G}(x-y,t-s)\,\mathrm{d}(y,s) \\
&= \int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(y)|u(y,s)|^{p-1}u(y,s)\cdot\frac{1}{2\pi\sqrt{\alpha}}\cos(\omega(t-s))\mathrm{e}^{-\sqrt{\alpha}\omega(x-y)}\,\mathrm{d}(y,s) \\
&\quad + \mathrm{O}\left(\mathrm{e}^{-(\sqrt{\alpha}\omega+\varepsilon)x}\right) \\
&= \frac{\sqrt{T}}{4\pi\sqrt{\alpha}}\int_{\mathbb{R}}\left(\widehat{\Gamma|u|^{p-1}u}\right)_1(y)\mathrm{e}^{\sqrt{\alpha}\omega y}\,\mathrm{d}y\cdot e_1(t)\mathrm{e}^{-\sqrt{\alpha}\omega x} \\
&\quad + \frac{\sqrt{T}}{4\pi\sqrt{\alpha}}\int_{\mathbb{R}}\left(\widehat{\Gamma|u|^{p-1}u}\right)_{-1}(y)\mathrm{e}^{\sqrt{\alpha}\omega y}\,\mathrm{d}y\cdot e_{-1}(t)\mathrm{e}^{-\sqrt{\alpha}\omega x} \\
&\quad + \mathrm{O}\left(\mathrm{e}^{-(\sqrt{\alpha}\omega+\varepsilon)x}\right), \qquad \text{as } x\to\infty.
\end{aligned}$$

Here we also used the fact that $\cos(\omega t) = \frac{\sqrt{T}}{2}(e_1(t)+e_{-1}(t))$. To obtain the behavior of u_t at infinity we put the time derivative onto $\tilde{G}$ in the convolution. Hence

$$\begin{aligned}
\frac{\partial}{\partial t}u(x,t) &= \int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(y)|u(y,s)|^{p-1}u(y,s)\frac{\partial}{\partial t}\tilde{G}(x-y,t-s)\,\mathrm{d}(y,s) \\
&= \int_{\mathbb{R}\times\mathbb{T}_T} \Gamma(y)|u(y,s)|^{p-1}u(y,s)\cdot\frac{-\omega}{2\pi\sqrt{\alpha}}\sin(\omega(t-s))\mathrm{e}^{-\sqrt{\alpha}\omega(x-y)}\,\mathrm{d}(y,s) \\
&\quad + \mathrm{O}\left(\mathrm{e}^{-(\sqrt{\alpha}\omega+\varepsilon)x}\right) \\
&= \frac{-\omega\sqrt{T}}{4\pi\mathrm{i}\sqrt{\alpha}}\int_{\mathbb{R}}\left(\widehat{\Gamma|u|^{p-1}u}\right)_1(y)\mathrm{e}^{\sqrt{\alpha}\omega y}\,\mathrm{d}y\cdot e_1(t)\mathrm{e}^{-\sqrt{\alpha}\omega x} \\
&\quad + \frac{\omega\sqrt{T}}{4\pi\mathrm{i}\sqrt{\alpha}}\int_{\mathbb{R}}\left(\widehat{\Gamma|u|^{p-1}u}\right)_{-1}(y)\mathrm{e}^{\sqrt{\alpha}\omega y}\,\mathrm{d}y\cdot e_{-1}(t)\mathrm{e}^{-\sqrt{\alpha}\omega x} \\
&\quad + \mathrm{O}\left(\mathrm{e}^{-(\sqrt{\alpha}\omega+\varepsilon)x}\right), \qquad \text{as } x\to\infty.
\end{aligned}$$

Observe that $|u_t(x,t)| \le C\mathrm{e}^{-\sqrt{\alpha}\omega|x|}$ for some $C > 0$. Hence there is some $C > 0$ such that $\forall\,(x,t)\in\mathbb{T}_T$:

$$\left|\frac{\partial}{\partial t}\left(\Gamma(x)|u(x,t)|^{p-1}u(x,t)\right)\right| = p\cdot\Gamma(x)\cdot\left||u(x,t)|^{p-1}u_t(x,t)\right| \le p\|\Gamma\|_\infty C^p\mathrm{e}^{-p\sqrt{\alpha}\omega|x|}.$$

Here we also used $p > 1$ to estimate $|u(x,t)|^{p-1}$. By the same arguments as before we can now put one time derivative on $\tilde{G}$ and one time derivative on $\Gamma(x)|u|^{p-1}u$ to obtain:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}u(x,t) &= \int_{\mathbb{R}\times\mathbb{T}_T} \frac{\partial}{\partial s}\left(\Gamma(y)|u(y,s)|^{p-1}u(y,s)\right)\frac{\partial}{\partial t}\tilde{G}(x-y,t-s)\,\mathrm{d}(y,s)\\
&= \int_{\mathbb{R}\times\mathbb{T}_T} p\Gamma(y)|u(y,s)|^{p-1}u_t(y,s)\cdot\frac{-\omega}{2\pi\sqrt{\alpha}}\sin(\omega(t-s))\mathrm{e}^{-\sqrt{\alpha}\omega(x-y)}\,\mathrm{d}(y,s)\\
&\quad + \mathrm{O}\left(\mathrm{e}^{-(\sqrt{\alpha}\omega+\varepsilon)x}\right)\\
&= \frac{-p\omega\sqrt{T}}{4\pi\mathrm{i}\sqrt{\alpha}}\int_{\mathbb{R}}\left(\widehat{\Gamma|u|^{p-1}u_t}\right)_1(y)\mathrm{e}^{\sqrt{\alpha}\omega y}\,\mathrm{d}y\cdot e_1(t)\mathrm{e}^{-\sqrt{\alpha}\omega x}\\
&\quad + \frac{p\omega\sqrt{T}}{4\pi\mathrm{i}\sqrt{\alpha}}\int_{\mathbb{R}}\left(\widehat{\Gamma|u|^{p-1}u_t}\right)_{-1}(y)\mathrm{e}^{\sqrt{\alpha}\omega y}\,\mathrm{d}y\cdot e_{-1}(t)\mathrm{e}^{-\sqrt{\alpha}\omega x}\\
&\quad + \mathrm{O}\left(\mathrm{e}^{-(\sqrt{\alpha}\omega+\varepsilon)x}\right), \qquad \text{as } x\to\infty.
\end{aligned}$$

Hence we have the claimed result for u_{tt}. We can not repeat this trick again with the same arguments, since $\tilde{G}_{tt} \notin L^1(\mathbb{R}\times\mathbb{T}_T)$ and we only assumed $p > 1$ but not $p > 2$. But we can gain something similar to a fractional $(2+l)$-th time derivative with $l \in (0,1)$. The core idea is that $\tilde{G}_t$ has integrable "fractional derivatives" up to order less than one. We calculate:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}_{odd}} |\omega k|^{2+l}\hat{u}_k(x)e_k(t) &= \sum_{k\in\mathbb{Z}_{odd}} |\omega k|^{l}\widehat{(u_{tt})}_k(x)e_k(t)\\
&= \int_{\mathbb{R}\times\mathbb{T}_T} \frac{\partial}{\partial s}\left(\Gamma(y)|u(y,s)|^{p-1}u(y,s)\right)\cdot\tilde{G}^{(1+l)}(x-y,t-s)\,\mathrm{d}(y,s)\\
&= \int_{\mathbb{R}\times\mathbb{T}_T} p\Gamma(y)|u(y,s)|^{p-1}u_t(y,s)\cdot\frac{-\omega^l}{2\pi\sqrt{\alpha}}\cos(\omega(t-s))\mathrm{e}^{-\sqrt{\alpha}\omega(x-y)}\,\mathrm{d}(y,s)\\
&\quad + \mathrm{O}\left(\mathrm{e}^{-(\sqrt{\alpha}\omega+\varepsilon)x}\right)\\
&= \frac{-p\sqrt{T}\omega^l}{4\pi\sqrt{\alpha}}\int_{\mathbb{R}}\left(\widehat{\Gamma|u|^{p-1}u_t}\right)_1(y)\mathrm{e}^{\sqrt{\alpha}\omega y}\,\mathrm{d}y\cdot e_1(t)\mathrm{e}^{-\sqrt{\alpha}\omega x}\\
&\quad + \frac{-p\sqrt{T}\omega^l}{4\pi\sqrt{\alpha}}\int_{\mathbb{R}}\left(\widehat{\Gamma|u|^{p-1}u_t}\right)_{-1}(y)\mathrm{e}^{\sqrt{\alpha}\omega y}\,\mathrm{d}y\cdot e_{-1}(t)\mathrm{e}^{-\sqrt{\alpha}\omega x}\\
&\quad + \mathrm{O}\left(\mathrm{e}^{-(\sqrt{\alpha}\omega+\varepsilon)x}\right).
\end{aligned}$$

□

3.2.1.2 Additional restrictions and symmetries

We will later use different energy levels concerning additional restrictions like Dirichlet boundary conditions and symmetries like oddness in space. If we restrict to Dirichlet boundary conditions in time (which can be extended to periodic boundary conditions), we obtain even ground states in time and space. We cite the following existence result.

Proposition 3.35. *Let $p \in (1,\infty)$, $\alpha > 0$ and $\Gamma \in L^\infty_{per}(\mathbb{R})$ with $\inf\Gamma > 0$. Then (3.5) has a ground state u in the class*

$$\widetilde{\mathcal{H}}_{odd} := \left\{u \in H^1_{ap}(\mathbb{R}\times\mathbb{T}_T) \;\middle|\; u\left(x, \pm\frac{T}{4}\right) \equiv 0\right\}.$$

Furthermore $u \in H^2_{ap}(\mathbb{R} \times \mathbb{T}_T)$. *Moreover there is a ground state* u*, which is positive on* $\mathbb{R} \times (-\frac{T}{4}, \frac{T}{4})$ *and even in* x *and* t *around* 0.

Proof. As in Proposition 3.29 we refer to [SW10] and [GT15] for existence of a positive solution and the regularity claim. The symmetry claim can be proven by a moving plane method as in [CCW98]. □

Proposition 3.36. *Let* $p \in (1, \infty)$*,* $\alpha > 0$ *and* $\Gamma \in L^\infty_{per}(\mathbb{R})$ *with* $\inf \Gamma > 0$*. Then* (3.5) *has no ground state solution* u *in the class*

$$\left\{ u \in H^1_{ap}(\mathbb{R} \times \mathbb{T}_T) \;\middle|\; u\left(x, \pm\frac{T}{4}\right) \equiv 0,\ u \text{ is odd in } x \right\}.$$

Proof. Assume the contrary. Without loss of generality we assume u to be non-negative on $(0, \infty) \times (-\frac{T}{4}, \frac{T}{4})$ by taking $|u|$ instead of u. Then by usual regularity theory as in as in [GT15] we obtain a $H^2(\mathbb{R} \times \mathbb{T}_T)$ solution. Ellipticity and the strong minimum principle yields positivity of u. As in [CCW98], we start the moving plane method in x at 0 and prove that u starts to increase. Since there is no right hand boundary, u will increase forever, contradicting integrability. □

We have seen, that sometimes the ground state energy is not attained. Motivated by the symmetry in Proposition 3.35 we will consider $\frac{T}{2}$-antiperiodic and even functions in t. Later we compare spatially even and odd functions. As notation we indicate the symmetry by two indices, the first index indicates symmetry in space and the second index indicates symmetry in time additionally to the $\frac{T}{2}$-anti-periodicity.

Definition 3.37. *We define*

$$\begin{aligned}
\mathcal{D}(\tilde{J})_{\cdot,e} &:= \left\{ v \in L^{q+1}_{ap}(\mathbb{R} \times \mathbb{T}_T) \mid v \text{ is even in } t \text{ around } 0 \right\}, \\
\mathcal{D}(\tilde{J})_{e,e} &:= \left\{ v \in \mathcal{D}(\tilde{J})_{\cdot,e} \mid v \text{ is even in } x \text{ around } 0 \right\}, \\
\mathcal{D}(\tilde{J})_{o,e} &:= \left\{ v \in \mathcal{D}(\tilde{J})_{\cdot,e} \mid v \text{ is odd in } x \text{ around } 0 \right\}.
\end{aligned}$$

Observe that "$\frac{T}{2}$-anti-periodic and even" implies "odd around $\pm\frac{T}{4}$". In the next lemma we will see how we can calculate the different minimal energy levels by the constrained minimization approach.

Lemma 3.38. *Let* $p \in (1, \infty)$*,* $\alpha > 0$ *and* $\Gamma \in L^\infty_{per}(\mathbb{R})$ *with* $\inf \Gamma > 0$*. Define:*

$$\begin{aligned}
\widetilde{\mathcal{M}}^{(L)}_{\cdot,e} &:= \left\{ v \in \mathcal{D}(\tilde{J})_{\cdot,e} \mid \tilde{J}_1(v) = 1 \right\}, & \tilde{m}^{(L)}_{\cdot,e} &:= \inf_{\widetilde{\mathcal{M}}^{(L)}_{\cdot,e}} J_0, \\
\widetilde{\mathcal{M}}^{(L)}_{e,e} &:= \left\{ v \in \mathcal{D}(\tilde{J})_{e,e} \mid \tilde{J}_1(v) = 1 \right\}, & \tilde{m}^{(L)}_{e,e} &:= \inf_{\widetilde{\mathcal{M}}^{(L)}_{e,e}} J_0, \\
\widetilde{\mathcal{M}}^{(L)}_{o,e} &:= \left\{ v \in \mathcal{D}(\tilde{J})_{o,e} \mid \tilde{J}_1(v) = 1 \right\}, & \tilde{m}^{(L)}_{o,e} &:= \inf_{\widetilde{\mathcal{M}}^{(L)}_{o,e}} J_0.
\end{aligned}$$

Then: $\tilde{m}^{(L)}_{\cdot,e} = \tilde{m}^{(L)}_{e,e} = 2^{-\frac{1-q}{2}} \tilde{m}^{(L)}_{o,e}$.

Proof. The first equality is clear due to the symmetry of the ground state in Proposition 3.35. We divide the proof of the second equality into several steps. W.l.o.g. let Γ be 1-periodic.

Step 1: Scaling.
Let $v^{(n)} \in \mathcal{D}(J)$ and assume $\tilde{J}_1(v^{(n)}) \to \tau > 0$ and $\limsup_n \tilde{J}_0(v^{(n)} \leq \mu$. Setting $t_n := \tilde{J}_1(v^{(n)})^{-\frac{1}{2}}$ we then calculate

$$\limsup_n \tilde{J}_0(t_n v^{(n)}) = \limsup_n t_n^{q+1} \tilde{J}_0(v^{(n)}) \leq \tau^{-\frac{q+1}{2}} \mu.$$

Step 2: $\tilde{m}_{o,e}^{(L)} \leq 2^{\frac{1-q}{2}} \tilde{m}_{\cdot,e}^{(L)}$
Let v^* be a ground state as in Proposition 3.35. We set $v^{(n)}(x,t) := v^*(x-n,t) - v^*(x+n,t)$, then clearly $v^{(n)} \in \mathcal{D}(\tilde{J})_{o,e}$. Using the translation invariance of Γ and the operator $\widetilde{\mathcal{L}}$ and the asymptotics of the ground state v^* we calculate

$$\begin{aligned}
\tilde{J}_1(v^{(n)}) &= \frac{1}{2} \int_\Omega v^*(\cdot - n, \cdot) \tilde{K} v^*(\cdot - n, \cdot) + v^*(\cdot + n, \cdot) \tilde{K} v^*(\cdot + n, \cdot) \\
&\qquad - v^*(\cdot - n, \cdot) \tilde{K} v^*(\cdot + n, \cdot) - v^*(\cdot + n, \cdot) \tilde{K} v^*(\cdot - n, \cdot) \, \mathrm{d}x \\
&= 2\tilde{J}_1(v^*) - \lambda \int_\Omega v^*(\cdot + n, \cdot) |v^*(\cdot - n, \cdot)|^{q-1} v^*(\cdot - n, \cdot) \, \mathrm{d}x \\
&= 2 - \mathrm{O}\left(\mathrm{e}^{-2\sqrt{\alpha}\omega n}\right).
\end{aligned}$$

Furthermore we use $|a-b|^{q+1} \leq |a|^{q+1} + |b|^{q+1} + (q+1)|a|^q|b|$. We shift the proof of this estimate into Proposition 3.40. Then we see

$$\begin{aligned}
\tilde{J}_0(v^{(n)}) &\leq \frac{1}{q+1} \int_\Omega |v^*(\cdot - \zeta k_n)|^{q+1} + |v^*(\cdot - \zeta k_n)|^{q+1} + (q+1)|v^*(\cdot + \zeta k_n)||v^*(\cdot - \zeta k_n)|^q \, \mathrm{d}x \\
&= 2\tilde{m}_{\cdot,e}^{(L)} + \mathrm{O}\left(\mathrm{e}^{-2\sqrt{\alpha}\omega n}\right).
\end{aligned}$$

We use step 1, set $t_n := \tilde{J}_1(v^{(n)})^{-\frac{1}{2}}$ and obtain

$$\tilde{m}_{o,e}^{(L)} \leq \limsup_n \tilde{J}_0(t_n v^{(n)}) \leq 2^{-\frac{q+1}{2}} \cdot 2\tilde{m}_{\cdot,e}^{(L)} = 2^{\frac{1-q}{2}} \tilde{m}_{\cdot,e}^{(L)}.$$

Step 3: $\tilde{m}_{\cdot,e}^{(L)} \leq 2^{-\frac{1-q}{2}} \tilde{m}_{o,e}^{(L)}$
Let $w^{(n)} \in \tilde{m}_{o,e}^{(L)}$ such that $\tilde{J}_0(w^{(n)}) \to \tilde{m}_{o,e}^{(L)}$. Set $v^{(n)} := \mathbb{1}_{\{x \geq 0\}} \cdot w^{(n)}$, then clearly $v^{(n)} \in \mathcal{D}(\tilde{J})_{\cdot,e}$. Using oddness of $w^{(n)}$ and the formula $\tilde{K}v = \tilde{G} * v$ we now calculate

$$\begin{aligned}
\tilde{J}_1(w^{(n)}) &= \frac{1}{2} \int_{\mathbb{R}_+ \times \mathbb{T}_T} w^{(n)} \tilde{K} w^{(n)} \, \mathrm{d}x \\
&= \frac{1}{2} \int_{\mathbb{R}_+ \times \mathbb{T}_T} w^{(n)}(x) \int_{\mathbb{R}_+ \times \mathbb{T}_T} \tilde{G}(x-y) w^{(n)}(y) \, \mathrm{d}y \, \mathrm{d}x \\
&\quad + \frac{1}{2} \int_{\mathbb{R}_+ \times \mathbb{T}_T} w^{(n)}(x) \int_{\mathbb{R}_- \times \mathbb{T}_T} \tilde{G}(x-y) w^{(n)}(y) \, \mathrm{d}y \, \mathrm{d}x \\
&\quad + \frac{1}{2} \int_{\mathbb{R}_- \times \mathbb{T}_T} w^{(n)}(x) \int_{\mathbb{R}_+ \times \mathbb{T}_T} \tilde{G}(x-y) w^{(n)}(y) \, \mathrm{d}y \, \mathrm{d}x \\
&\quad + \frac{1}{2} \int_{\mathbb{R}_- \times \mathbb{T}_T} w^{(n)}(x) \int_{\mathbb{R}_- \times \mathbb{T}_T} \tilde{G}(x-y) w^{(n)}(y) \, \mathrm{d}y \, \mathrm{d}x
\end{aligned}$$

$$= \int_{\mathbb{R}_+\times\mathbb{T}_T} w^{(n)}(x) \int_{\mathbb{R}_+\times\mathbb{T}_T} \tilde{G}(x-y) w^{(n)}(y)\,\mathrm{d}y\,\mathrm{d}x - \int_{\mathbb{R}_+\times\mathbb{T}_T} w^{(n)}(x) \int_{\mathbb{R}_+\times\mathbb{T}_T} \tilde{G}(x+y) w^{(n)}(y)\,\mathrm{d}y\,\mathrm{d}x\,.$$

Hence

$$\begin{aligned}\tilde{J}_1(v^{(n)}) &= \frac{1}{2}\int_{\mathbb{R}_+\times\mathbb{T}_T} w^{(n)}(x) \int_{\mathbb{R}_+\times\mathbb{T}_T} \tilde{G}(x-y) w^{(n)}(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{1}{2}\tilde{J}_1(w^{(n)}) + \frac{1}{2}\int_{\mathbb{R}_+\times\mathbb{T}_T} w^{(n)}(x) \int_{\mathbb{R}_+\times\mathbb{T}_T} \tilde{G}(x+y) w^{(n)}(y)\,\mathrm{d}y\,\mathrm{d}x\,.\end{aligned}$$

Next we will estimate the correction term using the asymptotics of $\tilde{G}$. We observe first that for any $s > 1$ and $M > 0$ we have $\left\|w^{(n)}\right\|_{L^s((0,M)\times\mathbb{T}_T)} \to 0$. If this were false, then $w^{(n)}$ would have a weakly convergent subsequence with a nontrivial limit and this limit would be a minimizer, contradicting that $\tilde{m}^{(L)}_{o,e}$ is not attained. We now calculate

$$\begin{aligned}&\left|\int_{\mathbb{R}_+\times\mathbb{T}_T} w^{(n)}(x) \int_{\mathbb{R}_+\times\mathbb{T}_T} \tilde{G}(x+y) w^{(n)}(y)\,\mathrm{d}y\,\mathrm{d}x\right| \\ &\le \int_{(0,M)\times\mathbb{T}_T} \left|w^{(n)}(x)\right| \int_{(0,M)\times\mathbb{T}_T} \left|\tilde{G}(x+y)\right|\left|w^{(n)}(y)\right|\mathrm{d}y\,\mathrm{d}x \\ &\quad + \int_{(0,M)\times\mathbb{T}_T} \left|w^{(n)}(x)\right| \int_{(M,\infty)\times\mathbb{T}_T} C\mathrm{e}^{-\sqrt{\alpha}\omega(x+y)}\left|w^{(n)}(y)\right|\mathrm{d}y\,\mathrm{d}x \\ &\quad + \int_{(M,\infty)\times\mathbb{T}_T} \left|w^{(n)}(x)\right| \int_{(0,M)\times\mathbb{T}_T} C\mathrm{e}^{-\sqrt{\alpha}\omega(x+y)}\left|w^{(n)}(y)\right|\mathrm{d}y\,\mathrm{d}x \\ &\quad + \int_{(M,\infty)\times\mathbb{T}_T} \left|w^{(n)}(x)\right| \int_{(M,\infty)\times\mathbb{T}_T} C\mathrm{e}^{-\sqrt{\alpha}\omega(x+y)}\left|w^{(n)}(y)\right|\mathrm{d}y\,\mathrm{d}x \\ &\le \int_{(0,M)\times\mathbb{T}_T} \left|w^{(n)}(x)\right| \int_{(0,M)\times\mathbb{T}_T} \left|\tilde{G}(x+y)\right|\left|w^{(n)}(y)\right|\mathrm{d}y\,\mathrm{d}x \\ &\quad + C(T,\alpha,\omega,s)\left\|v^{(n)}\right\|_{L^s((0,M)\times\mathbb{T}_T)} \cdot \left\|v^{(n)}\right\|_{L^s((M,\infty)\times\mathbb{T}_T)} \cdot M^{s'}\mathrm{e}^{-\sqrt{\alpha}\omega M} \\ &\quad + C(T,\alpha,\omega,s)\left\|v^{(n)}\right\|^2_{L^s((M,\infty)\times\mathbb{T}_T)} \cdot \mathrm{e}^{-\sqrt{\alpha}\omega M}.\end{aligned}$$

Hence, choosing $s = q+1$, we obtain

$$\begin{aligned}&\limsup_n \left|\int_{\mathbb{R}_+\times\mathbb{T}_T} w^{(n)}(x) \int_{\mathbb{R}_+\times\mathbb{T}_T} \tilde{G}(x+y) w^{(n)}(y)\,\mathrm{d}y\,\mathrm{d}x\right| \\ &\quad \le C(\Omega,\alpha,\omega,s)(\tilde{m}^{(L)}_{\cdot,e})^2 \cdot (1+M^{s'})\mathrm{e}^{-\sqrt{\alpha}\omega M} \to 0 \qquad \text{as } M\to\infty.\end{aligned}$$

We now conclude $\tilde{J}_1(v^{(n)}) = \frac{1}{2} + \mathrm{o}(1)$, where we have inserted $\tilde{J}_1(w^{(n)}) = 1$. Using again the oddness of $w^{(n)}$ we calculate

$$\tilde{J}_0(v^{(n)}) = \frac{1}{q+1}\int_{\mathbb{R}_+\times\mathbb{T}_T} \left|v^{(n)}\right|^{q+1}\mathrm{d}x = \frac{1}{2(q+1)}\int_{\mathbb{R}\times\mathbb{T}_T} \left|w^{(n)}\right|^{q+1}\mathrm{d}x \to \frac{1}{2}\tilde{m}^{(L)}_{o,e}.$$

We use step 1, set $t_n := \tilde{J}_1(v^{(n)})^{-\frac{1}{2}}$ and obtain

$$\tilde{m}^{(L)}_{\cdot,e} \le \lim_n \tilde{J}_0(t_n v^{(n)}) = 2^{\frac{q+1}{2}} \cdot \frac{1}{2}\tilde{m}^{(L)}_{o,e} = 2^{-\frac{1-q}{2}}\tilde{m}^{(L)}_{o,e}.$$

□

Remark 3.39. *Observe that the previous lemma can be proven in more generality. The essential assumptions were the existence of an exponentially decaying and even ground state, the fact that a cut-off as in step 3 keeps us in* $\mathcal{D}(\tilde{J})_{\cdot,e}$ *and a representation* $\tilde{K}v = \tilde{G}*v$ *with an exponentially decaying kernel. For the sake of simplicity we do not go into details.*

We next prove the technical inequality in the proof of Lemma 3.38. We want to remark that this estimate is good, if one of the parameters is very small but bad if both are almost equal.

Proposition 3.40. *If* $a, b \in \mathbb{R}$, $q \in (0,1)$, *then:* $|a-b|^{q+1} \le |a|^{q+1} + |b|^{q+1} + (q+1)|a|^q|b|$.

Proof. We start with a simplified version and regain complexity.

Step 1: $\forall\, h > 0\colon\ (1+h)^{q+1} \le 1 + h^{q+1} + (q+1)h$.
Obviously the left and right hand side are convex C^1-functions and equal at $h = 0$. Hence it suffices to prove that the derivatives are ordered. We calculate:

$$(q+1)(1+h)^q \le (q+1)h^q + (q+1) \qquad \Leftrightarrow \qquad (1+h)^{q+1} \le 1 + h^q.$$

The latter inequality holds true due to concavity of $h \mapsto h^q$.

Step 2: $\forall\, h \in \mathbb{R}\colon\ |1+h|^{q+1} \le 1 + |h|^{q+1} + (q+1)|h|$.
Observe that $|1+h| \le 1 + |h|$ and use step 1.

Step 3: $\forall\, a, b \in \mathbb{R}\colon\ |a-b|^{q+1} \le |a|^{q+1} + |b|^{q+1} + (q+1)|a|^q|b|$.
If $a = 0$, then the inequality is obviously true. If $a \neq 0$, then we set $h = -\frac{b}{a}$ in step 3 and multiply the equation by $|a|^{q+1}$. □

Next we give a short result on scaling in the time period T.

Lemma 3.41. *Let* $p \in (1,\infty)$, $\alpha > 0$ *and* $\Gamma \in L^\infty_{per}(\mathbb{R})$ *with* $\inf\Gamma > 0$. *Let* $z \in H^1_{ap}(\mathbb{R}\times\mathbb{T}_T)$ *be the ground state of* (3.5) *in the class* $\widetilde{\mathcal{H}}_{odd}$ *with* $T = 1$ *as in Proposition 3.35. Then* $u(x,t) = T^{-\frac{2}{p-1}} z(\frac{1}{T}x, \frac{1}{T}t)$ *is the ground state of* (3.5) *in the class* $\widetilde{\mathcal{H}}_{odd}$ *with* $T > 0$ *as in Proposition 3.35.*

Proof. Clearly $u \in \widetilde{\mathcal{H}}_{odd}$, u is positive on $\mathbb{R}\times(-\frac{T}{4}, \frac{T}{4})$ and even around 0. We calculate:

$$\begin{aligned}\left(\tilde{L}u\right)(x,t) &= T^{-\frac{2}{p-1}-2}\left(\tilde{L}z\right)\left(\frac{1}{T}x, \frac{1}{T}t\right) = T^{-\frac{2}{p-1}-2}\left(z\left(\frac{1}{T}x, \frac{1}{T}t\right)\right)^p \\ &= T^{-\frac{2}{p-1}-2+\frac{2p}{p-1}}(u(x,t))^p = (u(x,t))^p.\end{aligned}$$

The claim now follows by uniqueness. □

Using this scaling result we can refine the behavior at infinity by extracting the parameter r in $T = 4r\sqrt{\gamma}$.

Corollary 3.42. *Let $p \in (1,\infty)$, $\alpha > 0$ and $\Gamma \in L^\infty_{per}(\mathbb{R})$ with $\inf \Gamma > 0$. Let $u \in H^1_{ap}(\mathbb{R} \times \mathbb{T}_T)$ be the ground state of (3.5) in the class $\widetilde{\mathcal{H}}_{odd}$ with $T = 4r\sqrt{\gamma}$ as in Proposition 3.35. Then for $s \in \{0,1,2\}$, $l \in (0,1)$ there are $\varepsilon > 0$ and complex numbers $\mathcal{U}^{(s)}_{\pm 1}, \sqcap^{(l)}_{\pm 1} \in \mathbb{C}$ such that as $|x| \to \infty$ we have uniformly in t:*

$$\frac{\partial^s}{\partial t^s} u(x,t) = \left(\mathcal{U}^{(s)}_1 \cdot \mathrm{e}^{\mathrm{i}\omega t} + \mathcal{U}^{(s)}_{-1} \cdot \mathrm{e}^{-\mathrm{i}\omega t}\right) \cdot \mathrm{e}^{-\sqrt{\alpha}\omega|x|} + \mathrm{O}\left(\mathrm{e}^{-(\sqrt{\alpha}\omega+\varepsilon)|x|}\right),$$

$$\sum_{k\in\mathbb{Z}_{odd}} |\omega k|^{2+l} \hat{u}_k(x) e_k(t) = \left(\sqcap^{(l)}_1 \cdot \mathrm{e}^{\mathrm{i}\omega t} + \sqcap^{(l)}_{-1} \cdot \mathrm{e}^{-\mathrm{i}\omega t}\right) \cdot \mathrm{e}^{-\sqrt{\alpha}\omega|x|} + \mathrm{O}\left(\mathrm{e}^{-(\sqrt{\alpha}\omega+\varepsilon)|x|}\right).$$

Moreover there is $C(\alpha,\gamma,p,\Gamma,l)$ such that $\sqcap^{(l)}_\pm = C(\alpha,\gamma,p,\Gamma,l) \cdot r^{-2-l-\frac{2}{p-1}}$.

Proof. The asymptotics have already been proven in Theorem 3.30. It remains to proof the scaling of $\sqcap^{(l)}_\pm$. Consider z as in Lemma 3.41. Then there are $\zeta^{(l)}_1, \zeta^{(l)}_{-1} \in \mathbb{C}$ independent of T with

$$z(x,t) = \sum_{k\in\mathbb{Z}_{odd}} \hat{z}_k(x) \mathrm{e}^{2\pi\mathrm{i}kt} = \left(\zeta^{(l)}_1 \cdot \mathrm{e}^{2\pi\mathrm{i}t} + \zeta^{(l)}_{-1} \cdot \mathrm{e}^{-2\pi\mathrm{i}t}\right) \cdot \mathrm{e}^{-2\pi\sqrt{\alpha}|x|} + \mathrm{O}\left(\mathrm{e}^{-(2\pi\sqrt{\alpha}+\varepsilon)|x|}\right),$$

$$\sum_{k\in\mathbb{Z}_{odd}} |2\pi k|^{2+l} \hat{z}_k(x) \mathrm{e}^{2\pi\mathrm{i}kt} = (2\pi)^{2+l} \left(\zeta^{(l)}_1 \cdot \mathrm{e}^{2\pi\mathrm{i}t} + \zeta^{(l)}_{-1} \cdot \mathrm{e}^{-2\pi\mathrm{i}t}\right) \cdot \mathrm{e}^{-2\pi\sqrt{\alpha}|x|} + \mathrm{O}\left(\mathrm{e}^{-(2\pi\sqrt{\alpha}+\varepsilon)|x|}\right).$$

Using $u(x,t) = T^{-\frac{2}{p-1}} z(\frac{1}{T}x, \frac{1}{T}t)$ and the time scaling $t \rightsquigarrow \frac{1}{T}t$, which leads to $\left|\frac{\partial}{\partial t}\right|^s \rightsquigarrow \frac{1}{T^s}\left|\frac{\partial}{\partial t}\right|^s$, we obtain

$$\sum_{k\in\mathbb{Z}_{odd}} |\omega k|^{2+l} \hat{u}_k(x) \mathrm{e}^{2\pi\mathrm{i}kt} = T^{-\frac{2}{p-1}} \left(\frac{2\pi}{T}\right)^{2+l} \left(\zeta^{(l)}_1 \cdot \mathrm{e}^{\mathrm{i}\frac{2\pi}{T}t} + \zeta^{(l)}_{-1} \cdot \mathrm{e}^{-\mathrm{i}\frac{2\pi}{T}t}\right) \cdot \mathrm{e}^{-\sqrt{\alpha}\omega|x|} + \mathrm{O}\left(\mathrm{e}^{-(\sqrt{\alpha}\omega+\varepsilon)|x|}\right).$$

Last we insert $T = \frac{2\pi}{\omega} = 4r\sqrt{\gamma}$ and see

$$\sqcap^{(l)}_\pm = C(\alpha,\gamma,p,\Gamma) \cdot r^{-\frac{2}{p-1}} \cdot \omega^{2+l} = C(\alpha,\gamma,p,\Gamma) r^{-\frac{2}{p-1}} \cdot \left(\frac{\pi}{2r\sqrt{\gamma}}\right)^{2+l} = C(\alpha,\gamma,p,\Gamma,l) \cdot r^{-2-l-\frac{2}{p-1}}.$$

□

With these preparations we will later be able to proof the a-priori energy estimate in Section 3.2.3 quite fast.

3.2.1.3 Further remarks

Before we jump into the concrete examples, we develop some notation and results.

Definition 3.43. *Let $\alpha > 0$ and $p \in (1,\infty)$. Define*

$$\widetilde{\mathcal{H}} := l^1\left(\mathbb{Z}_{odd}, H^1(\mathbb{R})\right) \cap h^1\left(\mathbb{Z}_{odd}, L^2(\mathbb{R})\right) \cap \left\{\overline{\hat{u}_k} = \hat{u}_{-k}\right\}$$

$(\tilde{S}\hat{u})(x,t) := \sum_k \hat{u}_k(x)e_k(t)$ whenever it converges, $(\tilde{S}^{-1}u)_k := \langle u(x,\cdot), e_k\rangle_{L^2(\mathbb{T}_T)}$ whenever it exists, $\tilde{L}_k := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \alpha k^2\omega^2 \colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$, $\widetilde{\mathcal{L}} := \bigoplus_{k\in\mathbb{Z}_{odd}} L_k$ and $\tilde{K} := \tilde{L}^{-1}$. We refer to this construction as "equation at infinity" formulated on sequence spaces. Let further $\mathcal{H} = l^2(\mathbb{Z}_{odd}, H_k) \subset l^2\left(\mathbb{Z}_{odd}, L^2(\mathbb{R})\right)$, $(S\hat{u})(x,t) := \sum_k \hat{u}_k(x)e_k(t)$ $b_{\mathcal{L}}$ and Γ satisfy (A1), (A2) and (A3) and recall the constructions and results from Section 2.1.

Lemma 3.44. *Assume we are in the setting of previous definition. Then*

(i) $\tilde{S}\hat{u} = S\hat{u}$ whenever it converges. We no longer distinguish between S and $\tilde{S}$ if we know that either $S\hat{u}$ or $\tilde{S}\hat{u}$ exists.

(ii) $S\colon l^1\left(\mathbb{Z}, H^1(\mathbb{R})\right) \cap h^1\left(\mathbb{Z}, L^2(\mathbb{R})\right) \to H^1(\mathbb{R}\times\mathbb{T}_T)$ and $S\colon l^2\left(\mathbb{Z}, L^2(\mathbb{R})\right) \to L^2(\mathbb{R}\times\mathbb{T}_T)$ are isometries and $S^{-1}\colon L^{q+1}(\mathbb{R}\times\mathbb{T}_T) \to l^{q+1}\left(\mathbb{Z}, L^{p+1}(\mathbb{R})\right)$ is continuous.

(iii) Consider $S\colon \mathcal{H} \to L^{p+1}_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})$ and its adjoint $S^*\colon L^{q+1}_{ap}(\mathbb{R}\times\mathbb{T}_T) \to \mathcal{H}^*$. Then for $v \in L^2_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})$ we can represent $S^*v \in \mathcal{H}^*$ by an element of the space $l^{q+1}\left(\mathbb{Z}_{odd}, L^{p+1}_{ap}(\mathbb{R})\right)$ and this representative is $S^*v = S^{-1}v$.

(iv) For any $v \in L^{q+1}_{ap}(\mathbb{R}\times\mathbb{T}_T)$ we have $\tilde{S}\tilde{\mathcal{K}}v = \tilde{K}v$.

Proof. (i) This statement is clear.

(ii) E.g., cf [Gra08] for the isomety claims. We only prove $S^{-1}\colon L^{q+1}(\mathbb{R}\times\mathbb{T}_T) \to l^{q+1}\left(\mathbb{Z}, L^{p+1}(\mathbb{R})\right)$. We calculate:

$$\left\|(S^{-1}u)_k\right\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} \left|\int_{\mathbb{T}_T} u(x,t)\mathrm{e}^{-\mathrm{i}\omega kt}\,\mathrm{d}t\right| \mathrm{d}x \le T\|u\|_{L^1(\mathbb{R}\times\mathbb{T}_T)},$$

i.e., $\left\|S^{-1}\right\|_{L^1(\mathbb{R}\times\mathbb{T}_T)\to l^\infty(L^1(\mathbb{R}))} \le T$ is continuous. Using $\left\|S^{-1}\right\|_{L^2(\mathbb{R}\times\mathbb{T}_T)\to l^2(L^2(\mathbb{R}))} = 1$, we see by Riesz-Thorin interpolation that $\left\|S^{-1}\right\|_{L^{q+1}(\mathbb{R}\times\mathbb{T}_T)\to l^{q+1}(L^{p+1}(\mathbb{R}))} \le 1+T$.

(iii) Let $v \in L^{q+1}_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R}) \cap L^2_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})$. Then $S^*v \in \mathcal{H}^* = l^2(\mathbb{Z}_{odd}, H_k^*)$ since $\mathcal{H} = l^2(\mathbb{Z}_{odd}, H_k) \subset l^2\left(\mathbb{Z}_{odd}, L^2_{ap}(\mathbb{R})\right)$. Hence we see for any $\hat{z} \in \mathcal{H}$

$$\begin{aligned}\langle S^*v, \hat{z}\rangle_{\mathcal{H}^*\times\mathcal{H}} &= \langle v, S\hat{z}\rangle_{L^{q+1}(\mathbb{R}\times\mathbb{T}_T)\times L^{p+1}(\mathbb{R}\times\mathbb{T}_T)} = \int_{\mathbb{R}\times\mathbb{T}_T} v\cdot S\hat{z}\,\mathrm{d}(x,t)\\ &= \langle v, S\hat{z}\rangle_{L^2(\mathbb{R}\times\mathbb{T}_T)\times L^2(\mathbb{R}\times\mathbb{T}_T)} = \left\langle S^{-1}v, \hat{z}\right\rangle_{l^2(\mathbb{Z}_{odd},L^2(\mathbb{R}))\times l^2(\mathbb{Z}_{odd},L^2(\mathbb{R}))}.\end{aligned}$$

Here we used that the adjoint of an isometry is its inverse. The claim now follows by density.

(iv) Let $v \in L^{q+1}_{ap}(\mathbb{R}\times\mathbb{T}_T,\mathbb{R})$. Observe that we have $\tilde{K}v = (-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \alpha k^2\omega^2)^{-1}v \in W^{2,q+1}_{ap}(\mathbb{R}\times\mathbb{T}_T) \subset H^1_{ap}(\mathbb{R}\times\mathbb{T}_T)$. Moreover we have $(S^*v)_k = \hat{v}_k \in L^{p+1}(\mathbb{R})$. Looking into the proof of Lemma 3.31 we have $(S\tilde{K}v)_k = \tilde{G}_k * \hat{v}_k$. But this is exactly $\tilde{G}_k * \hat{v}_k = \tilde{L}_k^{-1}\hat{v}_k$. Using part (*ii*) we have $S\tilde{K}v = \widetilde{\mathcal{L}}^{-1}S^{-1}v$. Since the right hand side is in $l^2\left(\mathbb{Z}_{odd}, L^2_{ap}(\mathbb{R})\right)$, we can use that S is an isometry and obtain the claim.

□

3.2.2 δ-potentials

This section is an application of Theorem 3.16. Our main assumption throughout this example will be:

(H_δ) Let $\alpha, \beta, T > 0$. Define $V(x) := \beta\delta_0(x) - \alpha$.

We will prove the following theorem:

Theorem 3.45. *Assume* (H_δ)*,* $p \in (1,3)$ *and* $\omega := \frac{2\pi}{T} < \frac{2\sqrt{\alpha}}{\beta}$. *Let* $\Gamma \in L^\infty(\mathbb{R})$ *be periodic with* $\inf \Gamma > 0$ *and* $V(x) := \beta\delta_0(x) - \alpha$. *Then there exists a nontrivial weak solution* u *of the equation*

$$V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u, \qquad (x,t) \in \mathbb{R} \times \mathbb{T}_T, \tag{2.2}$$

with minimal energy among all $\frac{T}{2}$*-anti-periodic, weak solutions.*

See Definition 2.22 in Chapter 2 for the term *weak solution* in the case of (H_δ). Observe that compared to Theorem 2.21 in Chapter 2 the structure of the right hand side of Theorem 3.45 is less general and we also use the additional assumption $\omega < \frac{2\sqrt{\alpha}}{\beta}$. A short calculation using Proposition 2.24 shows: the assumption $\omega < \frac{2\sqrt{\alpha}}{\beta}$ is equivalent to positivity of the eigenvalue λ_1 of L_1. We will apply Theorem 3.16 to prove the existence of a ground state.

Assume (H_δ), $p \in (1,3)$ and $\omega := \frac{2\pi}{T}$ and recall the constructions and results for $\mathcal{H}$, $\mathcal{L}$, H, L etc. of Section 2.3.1.1 Section 2.3.1.2. We mostly work on function spaces indicated with non-calligraphic letters. Observe in addition that by Lemma 2.20, assumption ($C1$) is true. If we write calligraphic letters in between, we mean the corresponding objects on sequence spaces, which are precisely determined by Fourier series. Additionally recall the "equation at infinity" and its ground state, analyzed in Section 3.2.1. We will verify Assumption 3.14 and Assumption 3.15 and start with checking the assumptions of Lemma 3.26.

Lemma 3.46. *Assume* (H_δ)*,* $p \in (1,3)$ *and* $\omega := \frac{2\pi}{T} < \frac{2\sqrt{\alpha}}{\beta}$. *Let* $\Gamma \in L^\infty(\mathbb{R})$ *be periodic with* $\inf \Gamma > 0$. *Recall the notation and results as in Section 3.2.1. Then:*

(i) $\mathcal{S}^* = \tilde{\mathcal{S}}^*$ *pointwise on* $L^{q+1}(\Omega, \mathbb{R})$.

(ii) $\tilde{K} \colon L^{q+1}(\Omega, \mathbb{R}) \to H$ *is continuous.*

(iii) $W \circ \tilde{K} \colon L^{q+1}(\Omega, \mathbb{R}) \to H^*$ *with* $W \circ \tilde{K}v := L\tilde{K}v - \tilde{L}\tilde{K}v$ *is compact.*

(iv) If $\tilde{K}v = \tilde{K}w$ *on* $(-\varepsilon, \varepsilon) \times \mathbb{T}_T$ *for any* $\varepsilon > 0$*, then* $W\tilde{K}v = W\tilde{K}w$ *in* H^*.

Proof. (i) Since S and $\tilde{S}$ are the same map with just different domains and both ranges are subsets of $L^{p+1}(\Omega, \mathbb{R})$, this is clear.

(ii) By Lemma 3.44 we have $\tilde{K} \colon L^{q+1}(\mathbb{R} \times \mathbb{T}_T) \to W^{2,q+1}(\mathbb{R} \times \mathbb{T}_T)$. Using Sobolev's embedding we see $W^{2,q+1}(\mathbb{R} \times \mathbb{T}_T) \to H^1(\mathbb{R} \times \mathbb{T}_T)$ is continuous, e.g. cf [Ada75], and using standard trace theory we see $W^{2,q+1}(\mathbb{R} \times \mathbb{T}_T) \to H^1(\{0\} \times \mathbb{T}_T)$ is continuous since $q = \frac{1}{p} > \frac{1}{3}$, e.g. cf. [DNPV12]. Since $H = H^1(\mathbb{R} \times \mathbb{T}_T) \cap H^1(\{0\} \times \mathbb{T}_T)$, cf. Theorem 2.27, we obtain $\tilde{K} \colon L^{q+1}(\mathbb{R} \times \mathbb{T}_T, \mathbb{R}) \to W^{2,q+1}(\mathbb{R} \times \mathbb{T}_T, \mathbb{R}) \to H$ is compact as claimed.

(iii) Let $v \in L^{q+1}(\mathbb{R} \times \mathbb{T}_T, \mathbb{R})$, $z \in H$ be arbitrary and write $u := \tilde{K}v$. Then

$$\left|\left\langle W \circ \tilde{K}v, z \right\rangle_{H^* \times H}\right| = \left|\langle Lu, z\rangle_{H^*\times H} - \left\langle \tilde{L}u, z\right\rangle_{H^*\times H}\right| = \left|-\beta \int_{\mathbb{T}_T} u_t(0,\cdot) z_t(0,\cdot)\,\mathrm{d}t\right|$$
$$\leq \beta \|u_t(0,\cdot)\|_{L^2(\mathbb{T}_T)} \|z_t(0,\cdot)\|_{L^2(\mathbb{T}_T)} \leq \beta \|u\|_H \|z\|_H$$
$$\leq \beta \|\tilde{K}\|_{L^{q+1}(\Omega)\to H} \|v\|_{L^{q+1}(\Omega)} \|z\|_H,$$

i.e., $W \circ \tilde{K} := L\tilde{K} - \tilde{L}\tilde{K} \colon L^{q+1}(\Omega, \mathbb{R}) \to H^*$ is continuous. We obtain compactness of $W \circ \tilde{K}$ by observing compactness of the embedding $W^{2,q+1}(\mathbb{R} \times \mathbb{T}_T, \mathbb{R}) \to H^1(\{0\} \times \mathbb{T}_T)$ since $q = \frac{1}{p} > \frac{1}{3}$, e.g. cf. [DNPV12].

(iv) Looking again into the calculation in *(iii)* we observe that W only sees the time-derivative evaluated at $x = 0$. If two functions coincide in an open set around the line $\{0\} \times \mathbb{T}_T$, then clearly their time-derivatives are equal.

□

Corollary 3.47. *Assume (H_δ), $p \in (1, 3]$ and $\omega := \frac{2\pi}{T} < \frac{2\sqrt{\alpha}}{\beta}$. Let $\Gamma \in L^\infty(\mathbb{R})$ be periodic with $\inf \Gamma > 0$ a.e. Recall the notation and results as in Section 3.2.1. Then Assumption 3.15 holds true.*

Proof. Lemma 3.46 checks the assumptions for Lemma 3.26, hence the claim follows. □

Lemma 3.48. *Assume (H_δ), $p \in (1, 3)$ and $\omega := \frac{2\pi}{T} < \frac{2\sqrt{\alpha}}{\beta}$. Let $\Gamma \in L^\infty(\mathbb{R})$ be periodic with $\inf \Gamma > 0$. Recall the notation and results as in Section 3.2.1. Then $m < \tilde{m}$, i.e., the Assumption 3.14 is true.*

Proof. W.l.o.g. we assume $\Gamma \in L^\infty(\mathbb{R})$ is 1-periodic. Let $n \in \mathbb{N}$ be arbitrary, $u \in H^2_{ap}(\mathbb{R} \times \mathbb{T}_T) \setminus \{0\}$ be the ground state of $\tilde{I}$ (by Proposition 3.29) and define

$$v^{(n)}(x, t) := |(\mathcal{S}u)(x + n, t)|^{p-1} \mathcal{S}u(x + n, t)$$

i.e., $v^{(n)}$ is a ground state for $\tilde{J}$ by Proposition 3.4 and 1-periodicity of $\tilde{J}$. We follow the sketch mentioned after the proof of Lemma 3.25. We aim to prove $b_{\tilde{K}}\left(v^{(n)}, v^{(n)}\right) < b_K\left(v^{(n)}, v^{(n)}\right)$ for some $n \in \mathbb{N}$ big enough. Using Lemma 3.46, we can define $\psi^{(n)} = -L^{-1}Wu(\cdot + n) \in H^*$. We calculate using Fourier Series:

$$L\psi^{(n)} = -Wu(\cdot + n) \quad \Leftrightarrow \quad -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\hat{\psi}_k^{(n)} + \alpha_k^2 \hat{\psi}_k^{(n)} - 2\beta_k^2 \delta_0(x) \hat{\psi}_k^{(n)} = 2\beta_k^2 \delta_0(x) \hat{u}_k(\cdot + n).$$

Hence we see $\hat{\psi}_k^{(n)}(x) = \hat{\psi}_k^{(n)}(0)\mathrm{e}^{-\alpha_k|x|}$ and $\alpha_k \hat{\psi}_k^{(n)}(0) - \beta_k^2 \hat{\psi}_k^{(n)}(0) = \beta_k^2 \hat{u}_k(n)$. This yields

$$\psi^{(n)}(x, t) = \sum_{k \in \mathbb{Z}_{odd}} \frac{\beta_k^2}{\alpha_k - \beta_k^2} \hat{u}_k(n) \mathrm{e}^{-\alpha_k|x|} e_k(t).$$

Inserting the above formula for $\psi^{(n)}$ into Lemma 3.25 we can calculate explicitly

$$b_K\left(v^{(n)}, v^{(n)}\right) - b_{\tilde{K}}\left(v^{(n)}, v^{(n)}\right) = \int_{\mathbb{T}_T} \beta u_t(n, t) \psi_t(0, t)\,\mathrm{d}t + \int_{\mathbb{T}_T} \beta |u_t(n, t)|^2\,\mathrm{d}t$$

$$
\begin{aligned}
&= \beta \sum_{k\in\mathbb{Z}_{odd}} k^2\omega^2 \hat{u}_k(n)\hat{\psi}_k(0) + k^2\omega^2|\hat{u}_k(n)|^2 = \sum_{k\in\mathbb{Z}_{odd}} 2\beta_k^2 \left(\frac{\beta_k^2}{\alpha_k-\beta_k^2}+1\right)|\hat{u}_k(n)|^2 \\
&= 2\sum_{k\in\mathbb{Z}_{odd}} \frac{\alpha_k\beta_k^2}{\alpha_k-\beta_k^2}|\hat{u}_k(n)|^2 \\
&= 2\frac{\alpha_1\beta_1^2}{\alpha_1-\beta_1^2}\left(|\hat{u}_1(n)|^2+|\hat{u}_{-1}(n)|^2\right) + 2\sum_{|k|>1}\frac{\alpha_k\beta_k^2}{\alpha_k-\beta_k^2}|\hat{u}_k(n)|^2 \\
&= 2\frac{\alpha_1\beta_1^2}{\alpha_1-\beta_1^2}\cdot\frac{1}{\omega^4}\left(\left|\widehat{(u_{tt})}_1(n)\right|^2+\left|\widehat{(u_{tt})}_{-1}(n)\right|^2\right) + 2\sum_{|k|>1}\frac{\alpha_k\beta_k^2}{\alpha_k-\beta_k^2}\cdot\frac{1}{\omega^4k^4}\left|\widehat{(u_{tt})}_k(n)\right|^2 \\
&= 2\frac{\alpha_1\beta_1^2}{\alpha_1-\beta_1^2}\cdot\frac{1}{\omega^4}\left(\left|\mathcal{U}_1^{(2)}\right|^2+\left|\mathcal{U}_{-1}^{(2)}\right|^2\right)\cdot \mathrm{e}^{-2\sqrt{\alpha}\omega|n|} \\
&\qquad + 2\sum_{|k|>1}\frac{\alpha_k\beta_k^2}{\alpha_k-\beta_k^2}\cdot\frac{1}{\omega^4k^4}\cdot \mathrm{O}\left(\mathrm{e}^{-2(\sqrt{\alpha}\omega+\varepsilon)|n|}\right) \\
&\geq \left(2\frac{\alpha_1\beta_1^2}{\alpha_1-\beta_1^2}\cdot\frac{1}{\omega^4}\left(\left|\mathcal{U}_1^{(2)}\right|^2+\left|\mathcal{U}_{-1}^{(2)}\right|^2\right) - \mathrm{O}\left(\mathrm{e}^{-2\varepsilon|n|}\right)\cdot\sum_{|k|>1}\frac{1}{\omega^4|k|^3}\right)\cdot \mathrm{e}^{-2\sqrt{\alpha}\omega|n|} \\
&> 0 \qquad \text{for } n\in\mathbb{N} \text{ sufficiently large.}
\end{aligned}
$$

Here we used the approximation result for the ground state of $\tilde{I}$ in Theorem 3.30, the convergence of the sum $\sum_{k\in\mathbb{Z}}\frac{1}{|k|^3}$ and $\alpha_1 > \beta_1^2$. The latter is equivalent to $\omega < \frac{2\sqrt{\alpha}}{\beta}$. Observe that the terms absorbed in $\mathrm{O}(\cdot)$ are uniformly bounded in k. Having this strict inequality we obtain the claim by applying Lemma 3.17. □

The proof of Theorem 3.45 is now straightforward.

Proof of Theorem 3.45. Observe that by the constructions as in Section 2.3.1, Corollary 3.47 and Lemma 3.48 we have checked the assumptions of Theorem 3.16 and obtain a ground state of I. Arguing with symmetries exactly as in Theorem 2.21 we obtain that the ground state is a weak solution. □

3.2.3 Step-Potentials

This section is an application of Theorem 3.24. Our main assumption throughout this example will be:

(H_S) Let $\alpha,\gamma,r>0$. Define $\beta := \alpha+\gamma$ and for $x\in\mathbb{R}$ we set

$$V(x) := -\alpha + \beta\,\mathbb{1}_{[-r,r]}(x) = \begin{cases}\gamma, & |x|\leq r,\\ -\alpha, & |x|>r.\end{cases}$$

We will prove the following theorem:

Theorem 3.49. *Assume* (H_S), $p \in (1,3)$ *and set* $\omega := \frac{\pi}{2r\sqrt{\gamma}}$, $T := \frac{\omega}{2\pi}$, $\beta := \alpha + \gamma$. *Let* $\Gamma \in L^\infty(\mathbb{R})$ *be periodic, even,* $\inf \Gamma > 0$ *and let* $V(x) := -\alpha + \beta \mathbb{1}_{[-r,r]}(x)$. *Then there is some* $r_0 = r_0(\alpha, \gamma, p, \Gamma) > 0$ *such that for* $r > r_0$ *exists a nontrivial weak solution* u *of the equation*

$$V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u, \qquad (x,t) \in \mathbb{R} \times \mathbb{T}_T, \tag{2.2}$$

with minimal energy among all $\frac{T}{2}$*-anti-periodic, even in* t *and spatially odd weak solutions.*

See Definition 2.47 in Chapter 2 for the term *weak solution* in the case of (H_S). Observe that compared to Theorem 2.38 in Chapter 2 the structure of the right hand side of Theorem 3.49 is less general and we also consider only spatially odd functions when comparing energies. A short calculation using Proposition 2.41 will show later in Proposition 3.54: restricting to spatially odd functions yields that L_1 has at least one eigenvalue and all eigenvalues of L_1 are positive. This is necessary for the structure of our strategy. We will apply Theorem 3.24 to prove the existence of a ground state.

Assume (H_S), $p \in (1,3)$ and $\omega := \frac{2\pi}{T}$. Let $\Gamma \in L^\infty(\mathbb{R})$ be periodic, even, $\inf \Gamma > 0$ and recall the constructions and results for $\mathcal{H}$, $\mathcal{L}$ etc. of Section 2.3.2.1 and restrict every function to *odd in* x *around* 0. Often we highlight the oddness in space by the index "odd" but for the sake of readability we drop the index at calligraphic letters. Observe in addition that by Lemma 2.20, assumption $(C1)$ is true. We work on the abstract sequence spaces indicated with calligraphic letters. Additionally recall the "equation at infinity" and its ground state when considering even and odd functions and the non-existence of a ground state only considering spatially odd functions, analyzed in Section 3.2.1. Also recall the calculations comparing different energy levels of Section 3.2.1 when dealing with spatial symmetry, since these will be important in this example. We will verify Assumption 3.21 and Assumption 3.22 and start with checking the assumptions of Lemma 3.26.

Lemma 3.50. *Assume* (H_S), $p \in (1,3)$ *and* $\omega := \frac{2\pi}{T}$. *Let* $\Gamma \in L^\infty(\mathbb{R})$ *be periodic, even,* $\inf \Gamma > 0$. *Recall the notation and results as in Section 3.2.1. Then:*

(i) $S^* = \tilde{S}^*$ *pointwise on* $L^{q+1}_{ap,odd}(\mathbb{R} \times \mathbb{T}_T, \mathbb{R})$.

(ii) $\tilde{\mathcal{K}}\colon L^{q+1}_{ap,odd}(\mathbb{R} \times \mathbb{T}_T) \to \mathcal{H}$ *is continuous.*

(iii) $\mathcal{W} \circ \tilde{\mathcal{K}}\colon L^{q+1}_{ap,odd}(\mathbb{R} \times \mathbb{T}_T, \mathbb{R}) \to \mathcal{H}^*$ *with* $\mathcal{W} \circ \tilde{\mathcal{K}}v := \mathcal{L}\tilde{\mathcal{K}}v - \widetilde{\mathcal{L}}\tilde{\mathcal{K}}v$ *is compact.*

(iv) If $\tilde{S}\tilde{\mathcal{K}}v = \tilde{S}\tilde{\mathcal{K}}w$ *on* $(-r-\varepsilon, r+\varepsilon) \times \mathbb{T}_T$ *for any* $\varepsilon > 0$, *then* $\mathcal{W}\tilde{\mathcal{K}}v = \mathcal{W}\tilde{\mathcal{K}}w$ *in* $\mathcal{H}^*$.

Proof. (i) Since S and $\tilde{S}$ are the same map with just different domains and both ranges are subsets of $L^{p+1}_{ap,odd}(\mathbb{R} \times \mathbb{T}_T, \mathbb{R})$, this is clear.

(ii) By Lemma 3.44 we have $\tilde{K}\colon L^{q+1}(\mathbb{R} \times \mathbb{T}_T) \to W^{2,q+1}(\mathbb{R} \times \mathbb{T}_T)$. Using Sobolev's embedding we see $W^{2,q+1}(\mathbb{R} \times \mathbb{T}_T) \to H^1(\mathbb{R} \times \mathbb{T}_T)$ is continuous, cf. [Ada75]. Using Lemma 3.44 we obtain continuity of $\tilde{\mathcal{K}}\colon L^{q+1}_{ap,odd}(\mathbb{R} \times \mathbb{T}_T) \to \widetilde{\mathcal{H}}$ with

$$\widetilde{\mathcal{H}} = h^1\left(\mathbb{Z}_{odd}, L^2_{odd}(\mathbb{R})\right) \cap l^2\left(\mathbb{Z}_{odd}, H^1_{odd}(\mathbb{R})\right) \cap \left\{\overline{\hat{u}_k} = \hat{u}_{-k}\right\}.$$

Arguing exactly as in Lemma 2.46 we see $\widetilde{\mathcal{H}}$ embeds continuously into $\mathcal{H}$, i.e., $\tilde{\mathcal{K}}\colon L^{q+1}_{ap,odd}(\Omega) \to \mathcal{H}$ is continuous.

(*iii*) Let $v \in L^{q+1}_{ap,odd}(\mathbb{R} \times \mathbb{T}_T, \mathbb{R})$, $\hat{z} \in \mathcal{H}$ be arbitrary and write $\hat{u} := \tilde{\mathcal{K}} v$. First, using $p < 3$ we observe that $W^{2,q+1}(\mathbb{R} \times \mathbb{T}_T) \to H^{\frac{3}{2}}((-r,r) \times \mathbb{T}_T)$ is compact, cf. [DNPV12]. Second we observe that $\mathcal{H} \subset h^{\frac{1}{2}}\left(\mathbb{Z}_{odd}, L^2(\mathbb{R})\right)$ since $\langle \hat{z}_k, \hat{z}_k \rangle_{|L_k|} \geq c|k| \|\hat{z}_k\|^2_{L^2(\mathbb{R})}$ as seen in the proof of Theorem 2.45. Hence

$$
\begin{aligned}
\left| \left\langle \mathcal{W} \circ \tilde{\mathcal{K}} v, \hat{z} \right\rangle_{\mathcal{H}^* \times \mathcal{H}} \right| &= \left| \langle \mathcal{L} \hat{u}, \hat{z} \rangle_{H^* \times H} - \left\langle \widetilde{\mathcal{L}} \hat{u}, \hat{z} \right\rangle_{\mathcal{H}^* \times \mathcal{H}} \right| \\
= \left| \beta \sum_{k \in \mathbb{Z}_{odd}} \int_{(-r,r)} -k^2 \omega^2 \hat{u}_k \hat{z}_k \, \mathrm{d}x \right| &\leq \beta \sum_{k \in \mathbb{Z}_{odd}} k^{\frac{3}{2}} \omega^{\frac{3}{2}} \left\| S \tilde{\mathcal{K}} v \right\|_{L^2(-r,r)} k^{\frac{1}{2}} \omega^{\frac{1}{2}} \|\hat{z}_k\|_{L^2(\mathbb{R})} \\
\leq \beta \left\| S \tilde{\mathcal{K}} v \right\|_{H^{\frac{3}{2}}((-r,r) \times \mathbb{T}_T)} \|\hat{z}\|_{h^{\frac{1}{2}}(\mathbb{Z}_{odd}, L^2(\mathbb{R}))} &\leq \beta \left\| S \tilde{\mathcal{K}} v \right\|_{H^{\frac{3}{2}}((-r,r) \times \mathbb{T}_T)} \|\hat{z}\|_{\mathcal{H}},
\end{aligned}
$$

i.e., $\mathcal{W} \circ \tilde{\mathcal{K}}$ is compact.

(*iv*) Looking again into the calculation in (*iii*) we observe that W only sees the time-derivative evaluated on the strip $(-r,r) \times \mathbb{T}_T$. If two functions coincide on a set containing that strip, then clearly their time-derivatives are equal.

□

Corollary 3.51. *Assume* (H_δ), $p \in (1,3)$ *and* $\omega := \frac{2\pi}{T}$. *Let* $\Gamma \in L^\infty(\mathbb{R})$ *be periodic, even,* $\inf \Gamma > 0$. *Recall the notation and results as in Section 3.2.1. Then Assumption 3.22 holds true.*

Proof. Lemma 3.50 checks the assumptions for Lemma 3.26, hence we argue similarly as in the proof there. We set $\hat{z}^{(n)} := \tilde{\mathcal{K}} w^{(n)}$ and $\hat{\varphi}^{(n)} := -\mathcal{L}^{-1} \mathcal{W} \hat{z}^{(n)}$. Then applying Lemma 3.25 we obtain

$$
b_{\mathcal{K}}\left(w^{(n)}, w^{(n)}\right) - b_{\tilde{\mathcal{K}}}\left(w^{(n)}, w^{(n)}\right) = \left\langle -\mathcal{W} \hat{z}^{(n)}, \hat{\varphi}^{(n)} \right\rangle_{\mathcal{H}^* \times \mathcal{H}} + \left\langle -\mathcal{W} \hat{z}^{(n)}, \hat{z}^{(n)} \right\rangle_{\mathcal{H}^* \times \mathcal{H}}.
$$

Using weak convergence $w^{(n)} \rightharpoonup 0$ and compactness of $\mathcal{W} \circ \tilde{\mathcal{K}} \colon L^{q+1}_{ap}(\mathbb{R} \times \mathbb{T}_T) \to \mathcal{H}^*$ we obtain the claim. □

Lemma 3.52. *Assume* (H_δ), $p \in (1,3)$, $\omega := \frac{\pi}{2r\sqrt{\gamma}}$ *and* $T := \frac{2\pi}{\omega}$. *Let* $\Gamma \in L^\infty(\mathbb{R})$ *be periodic, even,* $\inf \Gamma > 0$. *Recall the notation and results as in Section 3.2.1, but restricted to odd functions. Then there is some* $r_0 = r_0(\alpha, \gamma, p, \Gamma)$ *such that for* $r > r_0$ *we have* $m^{(L)} < \tilde{m}^{(L)}$, *i.e., the Assumption 3.14 is true.*

Proof. Since the scaling in r is crucial in this proof, every generic constant c is independent on r. We utilize the results in Section 3.2.1. We recall the definitions

$$
\begin{aligned}
\mathcal{D}(\tilde{J})_{o,e} &:= \left\{ v \in L^{q+1}_{ap}(\mathbb{R} \times \mathbb{T}_T) \mid v \text{ is odd in } x \text{ around } 0 \text{ and even in } t \text{ around } 0 \right\}, \\
\mathcal{M}^{(L)}_{o,e} &:= \left\{ v \in \mathcal{D}(\tilde{J})_{o,e} \mid J_1(v) = 1 \right\}, \qquad m^{(L)}_{o,e} := \inf_{\mathcal{M}^{(L)}_{o,e}} J_0,
\end{aligned}
$$

By Lemma 3.38 we know $m^{(L)} = m^{(L)}_{\cdot,e} = 2^{-\frac{1-q}{2}} m^{(L)}_{o,e}$ and analogously with tilde. Hence is suffices to compare $m^{(L)}_{o,e}$ and $\tilde{m}^{(L)}_{o,e}$. We already know that $\tilde{m}^{(L)}_{o,e}$ is not attained but $v^{(n)}(x,t) := v^*(x-n,t) - v^*(x+n,t)$ is an infimizing sequence, where v^* is a ground state as in Proposition 3.35. We refine the calculations in the proof of Lemma 3.38. Let $C^* > 0$ be the smallest constant such that $v^*(x,t) \le C_* \mathrm{e}^{-p\sqrt{\alpha}\omega|x|}$. The existence of such a constant results from Corollary 3.42, and its scaling in r reads $C_*(r) = C(\alpha, p, \Gamma) \cdot r^{-\frac{2p}{p-1}}$ as seen in Lemma 3.41 when inserting $v^* = |u^*|^{p-1}u^*$. A direct calculation verifies for $a, n > 0$, $p > 1$:

$$\int_{\mathbb{R}} \mathrm{e}^{-ap|x\mp n|}\mathrm{e}^{-a|x\pm n|} = \frac{2}{a(p^2-1)}\left(p\mathrm{e}^{-2an} - \mathrm{e}^{-2apn}\right).$$

Hence

$$\begin{aligned}
\tilde{J}_0(v^{(n)}) &\le \frac{1}{q+1}\int_\Omega |v^*(\cdot+n)|^{q+1} + |v^*(\cdot-n)|^{q+1} + (q+1)|v^*(\cdot+n)||v^*(\cdot-n)|^q \,\mathrm{d}(x,t) \\
&\le 2\tilde{m}_{\cdot,e}^{(L)} + \frac{(p+1)TC_*^{q+1}}{\sqrt{\alpha}\omega(p^2-1)}\mathrm{e}^{-2\sqrt{\alpha}\omega n} = 2\tilde{m}_{\cdot,e}^{(L)} + C(\alpha,p,\Gamma)\cdot r^{-\frac{4}{p-1}}\mathrm{e}^{-2\sqrt{\alpha}\omega n}, \\
\tilde{J}_1(v^{(n)}) &= 2\tilde{J}_1(v^*) - 2\int_\Omega v^*(\cdot+n,\cdot)|v^*(\cdot-n,\cdot)|^{q-1}v^*(\cdot-n,\cdot)\,\mathrm{d}(x,t) \\
&\ge 2 - \frac{4pTC_*^{q+1}}{\sqrt{\alpha}\omega(p^2-1)}\mathrm{e}^{-2\sqrt{\alpha}\omega n} = 2 - C(\alpha,p,\Gamma)\cdot r^{-\frac{4}{p-1}}\mathrm{e}^{-2\sqrt{\alpha}\omega n},
\end{aligned}$$

where we plugged in $T = 4r\sqrt{\gamma}$, $\omega = \frac{\pi}{2r\sqrt{\gamma}}$ and $C_* = C(\alpha,p,\Gamma)\, r^{-\frac{2p}{p-1}}$ and calculated $2 - \frac{2p}{p-1}(q+1) = -\frac{4}{p-1}$. We next want to calculate the difference $J_1(v^{(n)}) - \tilde{J}_1(v^{(n)}) = \frac{1}{2}b_K\left(v^{(n)}, v^{(n)}\right) - \frac{1}{2}b_{\tilde{\mathcal{K}}}\left(v^{(n)}, v^{(n)}\right)$. This is done in the upcoming lengthy and technical calculations. Define $\hat{u}^{(n)} := \tilde{\mathcal{K}}v^{(n)}$. Using Lemma 3.46, we can define $\hat{\psi}^{(n)} = -\mathcal{L}^{-1}\mathcal{W}\hat{u}^{(n)} \in \mathcal{H}^*$. We do not have an explicit formula for $\hat{\psi}^{(n)}$ as in Section 3.2.2, which results in additional technical calculations. We use the decomposition of $\mathcal{L}$ into L_k and apply the spectral decomposition of the self-adjoint operators L_k as in Section 2.2.1. Then

$$\begin{aligned}
b_K\left(v^{(n)}, v^{(n)}\right) - b_{\tilde{\mathcal{K}}}\left(v^{(n)}, v^{(n)}\right) &= \sum_{k\in\mathbb{Z}_{odd}} \int_{\mathbb{R}} \hat{\psi}_k^{(n)} L_k \hat{\psi}_k^{(n)}\,\mathrm{d}x + \beta k^2\omega^2 \int_{\mathbb{R}} \mathbb{1}_{[-r,r]}(x)|\hat{u}_k^{(n)}|^2\,\mathrm{d}x \\
= \sum_{k\in\mathbb{Z}_{odd}} \int_{\mathbb{R}} \hat{\psi}_k^{(n)} L_k \hat{\psi}_k^{(n)}\,\mathrm{d}x &+ \frac{1}{\beta k^2\omega^2}\int_{\mathbb{R}} \left|-\beta k^2\omega^2 \mathbb{1}_{[-r,r]}(x)\hat{u}_k^{(n)}\right|^2\mathrm{d}x \\
= \sum_{k\in\mathbb{Z}_{odd}} \int_{\mathbb{R}} \hat{\psi}_k^{(n)} L_k \hat{\psi}_k^{(n)}\,\mathrm{d}x &+ \int_{\mathbb{R}} \frac{1}{\beta k^2\omega^2} L_k\hat{\psi}_k^{(n)} L_k \hat{\psi}_k^{(n)}\,\mathrm{d}x \\
= \sum_{k\in\mathbb{Z}_{odd}} \int_{\mathbb{R}} \lambda + \frac{1}{\beta k^2\omega^2}\lambda^2 \,\mathrm{d}\left\langle \hat{\psi}_k^{(n)}, P_k^\lambda \hat{\psi}_k^{(n)}\right\rangle &
\end{aligned}$$

In particular the projection P_k^λ can be evaluated explicitly if $\lambda = \lambda_{k,j}$, i.e., the j-th eigenvalue of L_k. Using the corresponding eigenfunction $\hat{\varphi}_{k,j}$ we see

$$\forall\, \hat{z}_k \in L^2(\mathbb{R})\colon \qquad \left\langle \hat{z}_k, P_k^{\lambda_{k,j}}\hat{z}_k\right\rangle_{L^2(\mathbb{R})} = \left\langle \hat{z}_k, \langle \hat{z}_k, \hat{\varphi}_{k,j}\rangle_{L^2(\mathbb{R})}\hat{\varphi}_k\right\rangle_{L^2(\mathbb{R})} = \left|\langle \hat{z}_k, \hat{\varphi}_k\rangle_{L^2(\mathbb{R})}\right|^2.$$

Hence, estimating integral over the continuous spectrum from below by 0 we obtain

$$b_K\left(v^{(n)}, v^{(n)}\right) - b_{\tilde{\mathcal{K}}}\left(v^{(n)}, v^{(n)}\right) \ge \sum_{k\in\mathbb{Z}_{odd}}\sum_j \left(\lambda_{k,j} + \frac{1}{\beta k^2\omega^2}\lambda_{k,j}^2\right)\left|\left\langle \hat{\psi}_k^{(n)}, \varphi_{k,j}\right\rangle_{L^2(\mathbb{R})}\right|^2$$

$$
\begin{aligned}
&\geq \sum_{k\in\mathbb{Z}_{odd}} \sum_j \left(\frac{1}{\lambda_{k,j}} + \frac{1}{\beta k^2\omega^2}\right) \left|\left\langle \hat{\psi}_k^{(n)}, L_k\varphi_{k,j}\right\rangle_{L^2(\mathbb{R})}\right|^2 \\
&= \sum_{k\in\mathbb{Z}_{odd}} \sum_j \left(\frac{1}{\lambda_{k,j}} + \frac{1}{\beta k^2\omega^2}\right) \left|\left\langle \beta k^2\omega^2 \mathbb{1}_{[-r,r]}(x)\hat{u}_k^{(n)}, \varphi_{k,j}\right\rangle_{L^2(\mathbb{R})}\right|^2 \\
&= \sum_{k\in\mathbb{Z}_{odd}} \sum_j \left(\frac{\beta^2 k^4\omega^4}{\lambda_{k,j}} + \beta k^2\omega^2\right) \left(\int_{-r}^r \hat{u}_k^{(n)}\varphi_{k,j}\,\mathrm{d}x\right)^2.
\end{aligned}
$$

We define for $l \in (\frac{1}{2}, 1)$ the asymptotics

$$
z(x,t) := \sum_{k\in\mathbb{Z}_{odd}} (\omega|k|)^l \widehat{(u_{tt})}_k(x) e_k(t)
$$

and obtain using Theorem 3.30:

$$
\begin{aligned}
z(x,t) &= \left(\sqcap_1^{(l)} \cdot \mathrm{e}^{\mathrm{i}\omega t} + \sqcap_{-1}^{(l)} \cdot \mathrm{e}^{-\mathrm{i}\omega t}\right) \cdot \mathrm{e}^{-\sqrt{\alpha}\omega|x|} + E(x,t), \\
\text{with} \quad E(x,t) &= \mathrm{O}\left(\mathrm{e}^{-(\sqrt{\alpha}\omega+\varepsilon)|x|}\right) \text{for} \quad |x| \to \infty.
\end{aligned}
$$

Now we insert the asymptotics, use orthogonality of $\varphi_{1,j}$, use $\varphi_{1,j} = \varphi_{-1,j}$ and $\lambda_{1,j} = \lambda_{-1,j}$ estimate $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$ and use $|x+n| = x+n$ for $x \in [-r,r], n > r$. Hence

$$
\begin{aligned}
&b_K\left(v^{(n)}, v^{(n)}\right) - b_{\tilde{\mathcal{K}}}\left(v^{(n)}, v^{(n)}\right) \\
&\geq \sum_{k\in\mathbb{Z}_{odd}} \sum_j \left(\frac{\beta^2}{\omega^{2l}} \frac{1}{|k|^{2l}\cdot\lambda_{k,j}} + \frac{\beta}{\omega^{2+2l}} \frac{1}{|k|^{2+2l}}\right) \left(\int_{-r}^r \hat{z}_k(\cdot+n)\varphi_{k,j}\,\mathrm{d}x\right)^2 \\
&\geq \frac{1}{2\omega^{2l}} \sum_j \left(\frac{\beta^2}{\lambda_{1,j}} + \frac{\beta}{\omega^2}\right) \mathrm{e}^{-2\sqrt{\alpha}\omega n} \cdot \left(\left|\sqcap_1^{(l)}\right|^2 + \left|\sqcap_{-1}^{(l)}\right|^2\right) \left(\int_{-r}^r \mathrm{e}^{-\sqrt{\alpha}\omega x}\varphi_{1,j}(x)\,\mathrm{d}x\right)^2 \\
&\quad - \frac{1}{\omega^{2l}} \sum_j \left(\frac{\beta^2}{\lambda_{1,j}} + \frac{\beta}{\omega^2}\right) \left\|\hat{E}_1(\cdot+n)\right\|^2_{L^2(-r,r)} \|\varphi_{1,j}\|^2_{L^2(\mathbb{R})} \\
&\quad - \sum_{|k|\geq 3} \sum_j \left(\frac{\beta^2}{\omega^{2l}} \frac{1}{|k|^{2l}\cdot c|k|} + \frac{\beta}{\omega^{2+2l}} \frac{1}{|k|^{2+2l}}\right) \left\|\hat{E}_k(\cdot+n)\right\|^2_{L^2(-r,r)} \|\varphi_{k,j}\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Here we estimated $|\lambda_{k,j}| \geq c|k|$ for some $c > 0$ by Theorem 2.42. Next we insert $\|\varphi_{k,j}\|_{L^2(\mathbb{R})}$, observe that the t-uniform bound of E carries on into a k-uniform bound of $\hat{E}_k$ and use $\sigma(L_k) = \{\lambda_{k,j} \mid j = 1,\ldots,J_k\} \cup [\alpha k^2\omega^2, \infty)$ with $0 < J_k \leq C|k|$ for some $C > 0$ and $|\lambda_{k,j}| \geq c|k|$ as in Proposition 2.41. This yields

$$
\begin{aligned}
&b_K\left(v^{(n)}, v^{(n)}\right) - b_{\tilde{\mathcal{K}}}\left(v^{(n)}, v^{(n)}\right) \\
&\geq \mathrm{e}^{-2\sqrt{\alpha}\omega n} \cdot \frac{1}{2\omega^{2l}} \sum_j \left(\frac{\beta^2}{\lambda_{1,j}} + \frac{\beta}{\omega^2}\right) \left(\left|\sqcap_1^{(l)}\right|^2 + \left|\sqcap_{-1}^{(l)}\right|^2\right) \left(\int_{-r}^r \mathrm{e}^{-\sqrt{\alpha}\omega x}\varphi_{1,j}(x)\,\mathrm{d}x\right)^2 \\
&\quad - C(\alpha,\gamma,p,r)\mathrm{e}^{-2(\sqrt{\alpha}\omega+\varepsilon)n} - C(\alpha,\gamma,p,r,l) \sum_{|k|\geq 3} \left(\frac{1}{|k|^{2l}} + \frac{1}{|k|^{1+2l}}\right) \cdot \mathrm{e}^{-2(\sqrt{\alpha}\omega+\varepsilon)n}
\end{aligned}
$$

We recall that the operator $L_1\colon H^2_{odd}(\mathbb{R}) \to L^2_{odd}(\mathbb{R})$ has at least one eigenvalue and all eigenvalues are positive by spatial oddness. We prove this in the subsequent Proposition 3.54. Moreover, applying again the subsequent Proposition 3.54, we obtain

$$\left(\frac{\beta^2}{\lambda_{1,j}} + \frac{\beta}{\omega^2}\right)\left(\int_{-r}^{r} \mathrm{e}^{-\sqrt{\alpha}\omega x}\varphi_{1,j}(x)\,\mathrm{d}x\right)^2 = C(\alpha,\gamma,j)\cdot r^5.$$

We give an explicit formula of the left hand term in Proposition 3.54. But we do not state it here to not interrupt the flow of arguments calculating $J_1(v^{(n)}) - \tilde{J}_1(v^{(n)})$. Using in addition the scaling formula for $\sqcap_\pm$ in Theorem 3.42 we obtain

$$\begin{aligned}
&b_K\left(v^{(n)}, v^{(n)}\right) - b_{\tilde{K}}\left(v^{(n)}, v^{(n)}\right)\\
&\geq C(\alpha,\gamma,p,\gamma,l)\cdot r^{2l}\cdot r^5\cdot r^{-4-2l-\frac{4}{p-1}}\cdot \mathrm{e}^{-2\sqrt{\alpha}\omega n} - C(\alpha,\gamma,p,r,l)\mathrm{e}^{-2(\sqrt{\alpha}\omega+\varepsilon)n}\\
&= C(\alpha,\gamma,p,\gamma,l)\cdot r^{1-\frac{4}{p-1}}\cdot \mathrm{e}^{-2\sqrt{\alpha}\omega n} - C(\alpha,\gamma,p,r,l)\mathrm{e}^{-2(\sqrt{\alpha}\omega+\varepsilon)n}.
\end{aligned}$$

Hence we conclude by this long and technical calculation the fact that

$$\begin{aligned}
J_1(v^{(n)}) &\geq \tilde{J}_1(v^{(n)}) + C(\alpha,\gamma,p,\Gamma,l)\cdot r^{1-\frac{4}{p-1}}\cdot \mathrm{e}^{-2\sqrt{\alpha}\omega n} - C(\alpha,\gamma,p,r,l)\mathrm{e}^{-2(\sqrt{\alpha}\omega+\varepsilon)n}\\
&\geq 2 + \left(C(\alpha,\gamma,p,\Gamma,l)\cdot r - C(\alpha,p,\Gamma)\right) r^{-\frac{4}{p-1}}\mathrm{e}^{-2\sqrt{\alpha}\omega n} - C(\alpha,\gamma,p,r,l)\mathrm{e}^{-2(\sqrt{\alpha}\omega+\varepsilon)n},
\end{aligned}$$

where we inserted the estimate on $\tilde{J}_1(v^{(n)})$. We observe that the factor $C(\alpha,\gamma,p,\Gamma,l)\cdot r - C(\alpha,p,\Gamma)$ will be positive for $r > r_0 = r_0(\alpha,\gamma,p,\Gamma,l)$ and this choice is independent of n. With these preparations we finally compare the desired energy levels. Recall step 1 and 2 in the proof of Lemma 3.38 and insert the above results. We conclude

$$\begin{aligned}
m^{(L)}_{o,e} \xleftarrow{\infty\leftarrow n} J_0(J_1(v^{(n)})^{-\frac{1}{2}}v^{(n)}) &= \frac{J_0(v^{(n)})}{J_1(v^{(n)})^{\frac{q+1}{2}}}\\
&\leq \frac{2\tilde{m}^{(L)} + C\cdot r^{-\frac{4}{p-1}}\mathrm{e}^{-2\sqrt{\alpha}\omega n}}{\left(2 + (C\cdot r - C)\,r^{-\frac{4}{p-1}}\mathrm{e}^{-2\sqrt{\alpha}\omega n} - C(r)\mathrm{e}^{-2(\sqrt{\alpha}\omega+\varepsilon)n}\right)^{\frac{q+1}{2}}}\\
&\xrightarrow{n\to\infty} 2^{\frac{1-q}{2}}\tilde{m}^{(L)} = \tilde{m}^{(L)}_{o,e}
\end{aligned}$$

For the sake of readability we dropped all dependencies of the different constants C except for the dependence on r in the constant in front of $\mathrm{e}^{-2(\sqrt{\alpha}\omega+\varepsilon)n}$. We improve this estimate by a variant of Taylor's approximation. Observe that for $a_0, a_1, b_1 \in \mathbb{R}$, $b_0 \in \mathbb{R}\setminus\{0\}$, $s > 0$ and $\mu \in (0,1)$ we find $\nu > 0$ such that

$$\frac{a_0 + a_1 x}{(b_0 + b_1 x + b_{1+\mu}x^{1+\mu})^s} = \frac{a_0}{b_0^s} + \frac{a_1 b_0 - a_0 b_1 s}{b_0^{s+1}}\cdot x + \mathrm{O}\left(x^{1+\nu}\right), \qquad \text{as } x \searrow 0.$$

Hence, choosing $r > r_0 = r_0(\alpha,\gamma,p,\Gamma,l)$ sufficiently big, we see that for $n \to \infty$ the fraction

$$\begin{aligned}
&\frac{2\tilde{m}^{(L)} + C\cdot r^{-\frac{4}{p-1}}\mathrm{e}^{-2\sqrt{\alpha}\omega n}}{\left(2 + (C\cdot r - C)\,r^{-\frac{4}{p-1}}\mathrm{e}^{-2\sqrt{\alpha}\omega n} - C(r)\mathrm{e}^{-2(\sqrt{\alpha}\omega+\varepsilon)n}\right)^{\frac{q+1}{2}}}\\
&\quad= \frac{2\tilde{m}^{(L)}}{2^{\frac{q+1}{2}}} + \frac{2\cdot C - 2\tilde{m}^{(L)}\cdot(C\cdot r - C)\cdot\frac{q+1}{2}}{2^{\frac{q+1}{2}+1}}\cdot r^{-\frac{4}{p-1}}\cdot \mathrm{e}^{-2\sqrt{\alpha}\omega n} + \mathrm{O}\left(\mathrm{e}^{-2\sqrt{\alpha}\omega\cdot(1+\nu)n}\right)
\end{aligned}$$

converges to $\tilde{m}^{(L)}_{o,e}$ strictly from below. We shortly remark, that the constant in front of $\mathrm{e}^{-2(\sqrt{\alpha}\omega+\varepsilon)n}$ may also be big, but since we first fix r and let $n \to \infty$ afterwards, this does not interrupt our argumentation. We conclude $m^{(L)}_{o,e} < \tilde{m}^{(L)}_{o,e}$ and with the argumentation at the beginning of the proof we have proven the claim $m^{(L)} < \tilde{m}^{(L)}$ if $r > r_0$. □

Remark 3.53. *The fact that we only have an infimizing sequence at hand and not a minimizer gives rise to additional technical difficulties, resulting in the additional assumption $r > r_0$ when comparing Theorem 3.49 and Theorem 3.45.*

Proposition 3.54. *Assume (H_S) and $\omega := \frac{2\pi}{T}$. Then the operator $L_1 \colon H^2_{odd}(\mathbb{R}) \to L^2_{odd}(\mathbb{R})$ has at least one eigenvalue and all eigenvalues are positive. Moreover, writing $\sigma(L_1) = \{\lambda_{1,j} \mid j = 0, 1, \ldots, J_1\}$ with $\lambda_{1,0} < \lambda_{1,1} < \ldots < \lambda_{1,J_1}$ we have for j even*

$$\left(\frac{\beta^2}{\lambda_{1,j}} + \frac{\beta}{\omega^2}\right)\left(\int_{-r}^{r} \mathrm{e}^{-\sqrt{\alpha}\omega x}\varphi_{1,j}(x)\,\mathrm{d}x\right)^2 = \frac{(\alpha+\gamma)\gamma}{\left(\frac{\pi}{2r}\right)^4\mu_j} \cdot \frac{4r\cos^2\left(\frac{\pi}{2}\sqrt{1+\mu_j}\right)}{\left(\frac{\alpha}{\gamma}+1+\mu_j\right)} \cdot \frac{\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}}{1+\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}} \cdot \left(\sqrt{\frac{\alpha}{\gamma}}\sinh\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\right) + \sqrt{\frac{\alpha}{\gamma}-\mu_j}\cosh\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\right)\right)^2,$$

and for j odd

$$\left(\frac{\beta^2}{\lambda_{1,j}} + \frac{\beta}{\omega^2}\right)\left(\int_{-r}^{r} \mathrm{e}^{-\sqrt{\alpha}\omega x}\varphi_{1,j}(x)\,\mathrm{d}x\right)^2 = \frac{(\alpha+\gamma)\gamma}{\left(\frac{\pi}{2r}\right)^4\mu_j} \cdot \frac{4r\sin^2\left(\frac{\pi}{2}\sqrt{1+\mu_j}\right)}{\left(\frac{\alpha}{\gamma}+1+\mu_j\right)} \cdot \frac{\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}}{1+\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}} \cdot \left(\sqrt{\frac{\alpha}{\gamma}-\mu_j}\sinh\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\right) + \sqrt{\frac{\alpha}{\gamma}}\cosh\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\right)\right)^2,$$

where $\mu_j := \lambda_{1,j}\frac{4r^2}{\pi^2}$ is independent of r.

Since the proof of this proposition is long, technical and interrupts the flow of arguments, we shifted it into Section 3.2.4.2. The proof of Theorem 3.49 is now straightforward, since we have already dealt with all technical difficulties and outsourced them into the previous results.

Proof of Theorem 3.49. Observe that by the constructions as in Section 2.3.2, Corollary 3.51 and Lemma 3.52 we have checked the assumptions of Theorem 3.24 and obtain a ground state of J. Using Proposition 3.4 we obtain a ground state for $\mathcal{I}$. We argue with symmetries analogously as in Theorem 2.38 and observe that oddness in space can be treated as $\frac{T}{2}$-anti-periodicity in time. Hence we obtain that the ground state is a weak solution. □

3.2.4 Proof of Lemma 3.31 and Proposition 3.54

3.2.4.1 Proof of Lemma 3.31

To prove this lemma we start by decomposing $\tilde{L}$ by Fourier series in time and then prove an explicit formula for the inverse of $\tilde{L}$.

Definition 3.55. *Let $\alpha > 0$ and $\omega := \frac{2\pi}{T}$. We define for $k \in \mathbb{Z}$*

$$\tilde{L}_k \colon H^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad \tilde{L}_k \hat{u}_k := -\hat{u}_k'' + \alpha k^2 \omega^2 \hat{u}_k,$$
$$b_{\tilde{L}_k} \colon H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{C}, \quad b_{\tilde{L}_k}(\hat{u}_k, \hat{v}_k) := \int_{\mathbb{R}} \hat{u}_k' \overline{\hat{v}_k'} + \alpha k^2 \omega^2 \hat{u}_k \overline{\hat{v}_k} \, \mathrm{d}x\,.$$

Proposition 3.56. *Let $\alpha > 0$ and $k \in \mathbb{Z} \setminus \{0\}$. Then:*

1. $\forall\, \hat{u}_k \in H^2(\mathbb{R}), \hat{v}_k \in H^1(\mathbb{R})\colon \quad b_{\tilde{L}_k}(\hat{u}_k, \hat{v}_k) = \left\langle \tilde{L}_k \hat{u}_k, \hat{v}_k \right\rangle_{L^2(\mathbb{R})}$.
2. *$\tilde{L}_k$ is self-adjoint with spectrum $\sigma(\tilde{L}_k) = [\alpha k^2 \omega^2, \infty)$ and $\tilde{L}_k$ is continuously invertible.*
3. $L^2_{ap}(\mathbb{R} \times \mathbb{T}_T) = \{u(x,t) = \sum_{k \in \mathbb{Z}_{odd}} \hat{u}_k(x) e_k(t) \mid \sum_{k \in \mathbb{Z}_{odd}} \|\hat{u}_k\|_{L^2(\mathbb{R})} < \infty\}$
4. $\forall\, u \in H^2_{ap}(\mathbb{R} \times \mathbb{T}_T)\colon \quad (\tilde{L}u)(x,t) = \sum_{k \in \mathbb{Z}_{odd}} (\tilde{L}_k \hat{u}_k)(x) e_k(t)$,
   $\forall\, u \in H^1_{ap}(\mathbb{R} \times \mathbb{T}_T), v \in H^1_{ap}(\mathbb{R} \times \mathbb{T}_T)\colon \quad b_{\tilde{L}}(u, v) = \sum_{k \in \mathbb{Z}_{odd}} b_{\tilde{L}_k}(\hat{u}_k, \hat{v}_k)$.

Proof of Lemma 3.31. We now invert each $\tilde{L}_k$ separately using the Fourier transform

$$\mathcal{F}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad \mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \mathrm{e}^{-\mathrm{i}x\xi} \, \mathrm{d}x\,,$$
$$\mathcal{F}^{-1}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad \mathcal{F}^{-1}f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\xi) \mathrm{e}^{\mathrm{i}x\xi} \, \mathrm{d}\xi\,.$$

Then $\mathcal{F}(f')(\xi) = -\mathrm{i}\xi \cdot (\mathcal{F}f)(\xi)$. Recall that for $a > 0$ we have

$$\mathcal{F}^{-1}\left(\frac{1}{a^2 + |\cdot|^2}\right) = \sqrt{\frac{\pi}{2}} \frac{1}{a} \mathrm{e}^{-a|\cdot|}.$$

We will also use convolutions in space and define for $r, s, t \in [1, \infty]$ with $\frac{1}{s} + \frac{1}{t} = 1 + \frac{1}{r}$:

$$*\colon L^s(\mathbb{R}) \times L^t(\mathbb{R}) \to L^r(\mathbb{R}), \quad (f * g)(x) = \int_{\mathbb{R}} f(y) g(x - y) \, \mathrm{d}y\,.$$

We do not use convolutions of sequences, hence there should be no confusion. Then $\sqrt{2\pi}\mathcal{F}^{-1}(fg) = \mathcal{F}^{-1}f * \mathcal{F}^{-1}g$. We now calculate for $u \in H^2_{ap}(\mathbb{R} \times \mathbb{T}_T)$, $f \in L^2_{ap}(\mathbb{R} \times \mathbb{T}_T)$:

$$
\begin{aligned}
& \tilde{L}u = -\alpha u_{tt} - u_{xx} = f, && (x,t) \in \mathbb{R} \times \mathbb{T}_T, \\
\Leftrightarrow \quad & -\hat{u}_k'' + \alpha k^2 \omega^2 \hat{u}_k = \hat{f}_k, && x \in \mathbb{R}, k \in \mathbb{Z}_{odd}, \\
\Leftrightarrow \quad & |\xi|^2 \mathcal{F}\hat{u}_k + \alpha k^2 \omega^2 \mathcal{F}\hat{u}_k = \mathcal{F}\hat{f}_k, && \xi \in \mathbb{R}, k \in \mathbb{Z}_{odd}, \\
\Leftrightarrow \quad & \mathcal{F}\hat{u}_k = \frac{1}{\alpha k^2 \omega^2 + |\xi|^2} \mathcal{F}\hat{f}_k, && \xi \in \mathbb{R}, k \in \mathbb{Z}_{odd},
\end{aligned}
$$

$$\Leftrightarrow \qquad \hat{u}_k = \frac{1}{2\sqrt{\alpha}|k|\omega} \mathrm{e}^{-\sqrt{\alpha}|k|\omega|\cdot|} * \hat{f}_k, \qquad x \in \mathbb{R}, k \in \mathbb{Z}_{odd},$$

$$\Leftrightarrow \qquad u(x,t) = \sum_{k \in \mathbb{Z}_{odd}} \left(\frac{1}{2\sqrt{\alpha}|k|\omega} \mathrm{e}^{-\sqrt{\alpha}|k|\omega|\cdot|} * \hat{f}_k \right)(x) e_k(t), \quad (x,t) \in \mathbb{R} \times \mathbb{T}_T.$$

We shortly justify that this formula generates the correct regularity, i.e., inserting $f \in L^2(\mathbb{R} \times \mathbb{T}_T)$ yields $u \in H^2(\mathbb{R} \times \mathbb{T}_T)$. Observe that for $a > 0$ we have $\left\| \mathrm{e}^{-a|\cdot|} \right\|_{L^1(\mathbb{R})} = \frac{2}{a}$ and by Young's convolution theorem we conclude

$$\begin{aligned}
&\left\| \frac{\partial^2}{\partial x^2} \sum_{k \in \mathbb{Z}_{odd}} \left(\frac{1}{2\sqrt{\alpha}|k|\omega} \mathrm{e}^{-\sqrt{\alpha}|k|\omega|\cdot|} * \hat{f}_k \right)(x) e_k(t) \right\|^2_{L^2(\mathbb{R} \times \mathbb{T}_T)} \\
&= \sum_{k \in \mathbb{Z}_{odd}} \left\| \frac{\partial^2}{\partial x^2} \left(\frac{1}{2\sqrt{\alpha}|k|\omega} \mathrm{e}^{-\sqrt{\alpha}|k|\omega|\cdot|} * \hat{f}_k \right) \right\|^2_{L^2(\mathbb{R})} = \sum_{k \in \mathbb{Z}_{odd}} \left\| \frac{\pi}{2} \frac{|\xi|^2}{\alpha k^2 \omega^2 + |\xi|^2} \mathcal{F} \hat{f}_k \right\|^2_{L^2(\mathbb{R})} \\
&\leq \sum_{k \in \mathbb{Z}_{odd}} \frac{\pi^2}{4} \left\| \hat{f}_k \right\|^2_{L^2(\mathbb{R})} = \frac{\pi^2}{4} \cdot \|f\|^2_{L^2(\mathbb{R} \times \mathbb{T}_T)}, \\
&\left\| \frac{\partial^2}{\partial t^2} \sum_{k \in \mathbb{Z}_{odd}} \left(\frac{1}{2\sqrt{\alpha}|k|\omega} \mathrm{e}^{-\sqrt{\alpha}|k|\omega|\cdot|} * \hat{f}_k \right)(x) e_k(t) \right\|^2_{L^2(\mathbb{R} \times \mathbb{T}_T)} \\
&= \sum_{k \in \mathbb{Z}_{odd}} \left\| \frac{|k|\omega}{2\sqrt{\alpha}} \mathrm{e}^{-\sqrt{\alpha}|k|\omega|\cdot|} * \hat{f}_k \right\|^2_{L^2(\mathbb{R})} \leq \sum_{k \in \mathbb{Z}_{odd}} \left\| \frac{|k|\omega}{2\sqrt{\alpha}} \mathrm{e}^{-\sqrt{\alpha}|k|\omega|\cdot|} \right\|^2_{L^1(\mathbb{R})} \left\| \hat{f}_k \right\|^2_{L^2(\mathbb{R})} \\
&= \sum_{k \in \mathbb{Z}_{odd}} \frac{1}{\alpha^2} \left\| \hat{f}_k \right\|^2_{L^2(\mathbb{R})} = \frac{1}{\alpha^2} \|f\|^2_{L^2(\mathbb{R} \times \mathbb{T}_T)}.
\end{aligned}$$

Hence the formula is a map $L^2(\mathbb{R} \times \mathbb{T}_T) \to H^2(\mathbb{R} \times \mathbb{T}_T)$ and we have:

$$\tilde{L}^{-1} u(x,t) := \sum_{k \in \mathbb{Z}_{odd}} \left(\frac{1}{2\sqrt{\alpha}|k|\omega} \mathrm{e}^{-\sqrt{\alpha}|k|\omega|\cdot|} * \hat{u}_k \right)(x) e_k(t)$$

Using convolutions in space and time we see for $u, v \in L^2(\mathbb{R} \times \mathbb{T}_T)$:

$$\begin{aligned}
&\int_{\mathbb{R} \times \mathbb{T}_T} v(x-y, t-s) u(y,s) \,\mathrm{d}(y,s) = \int_{\mathbb{R} \times \mathbb{T}_T} \sum_k \hat{v}_k(x-y) e_k(t-s) \cdot \sum_l \hat{u}_l(y) e_l(s) \,\mathrm{d}(y,s) \\
&= \int_{\mathbb{R}} \sum_l \sum_k \int_{\mathbb{T}_T} e_k(-s) e_l(s) \,\mathrm{d}s \, \sqrt{T} e_k(t) \hat{v}_k(x-y) \hat{u}_l(y) \,\mathrm{d}y = \sqrt{T} \sum_k (\hat{v}_k * \hat{u}_k)(x) e_k(t).
\end{aligned}$$

Here we used $\int_{\mathbb{T}_T} e_k(s) e_l(-s) \,\mathrm{d}s = 1$ if $k = l$ and $\int_{\mathbb{T}_T} e_k(s) e_l(-s) \,\mathrm{d}s = 0$ if $k \neq l$. Hence we define $\tilde{G}_k(x) := \frac{1}{2\sqrt{\alpha T}|k|\omega} \mathrm{e}^{-\sqrt{\alpha}|k|\omega|x|}$ for $k \neq 0$ and $\tilde{G}(x,t) := \sum_{k \in \mathbb{Z} \setminus \{0\}} \tilde{G}_k(x) e_k(t)$. We now obtain the claimed integral formula. It remains to prove the closed form of $\tilde{G}$. For this we use $\sum_{n=1}^{\infty} \frac{-1}{n} z^n = \ln(1-z)$ for $|z| < 1$ and calculate for $x \neq 0$:

$$\begin{aligned}
\tilde{G}(x,t) &= \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{2\sqrt{\alpha T}|k|\omega} \mathrm{e}^{-\sqrt{\alpha}|k|\omega|x|} e_k(t) \\
&= \frac{1}{2\sqrt{\alpha T}\omega} \frac{1}{\sqrt{T}} \sum_{k=1}^{\infty} \frac{1}{k} \mathrm{e}^{(-\sqrt{\alpha}\omega|x| + \mathrm{i}\omega t)k} + \frac{1}{k} \mathrm{e}^{(-\sqrt{\alpha}\omega|x| - \mathrm{i}\omega t)k}
\end{aligned}$$

$$= -\frac{1}{4\pi\sqrt{\alpha}} \cdot \left(\ln\left(1 - \mathrm{e}^{-\sqrt{\alpha}\omega|x|+\mathrm{i}\omega t}\right) + \ln\left(1 - \mathrm{e}^{-\sqrt{\alpha}\omega|x|-\mathrm{i}\omega t}\right)\right)$$
$$= -\frac{1}{4\pi\sqrt{\alpha}} \cdot \ln\left(1 - 2\mathrm{e}^{-\sqrt{\alpha}\omega|x|}\cos(\omega t) + \mathrm{e}^{-2\sqrt{\alpha}\omega|x|}\right).$$

□

We can observe that the inversion formula extends naturally to the inverse to the operator $-\alpha\partial_t^2 - \partial_x^2$ as a map $H^2(\mathbb{R}\times\mathbb{T}_T)\cap\{\hat{u}_0\equiv 0\} \to L^2(\mathbb{R}\times\mathbb{T}_T)\cap\{\hat{u}_0\equiv 0\}$, but we do not need this in the rest of our considerations.

3.2.4.2 Proof of Proposition 3.54

Recall the notation in Section 2.3.2. We write $\lambda\frac{4r^2}{\pi^2} = \mu$ to simplify the calculations. Then $\lambda \in (-\gamma\omega^2, \alpha\omega^2)$ is equivalent to $\mu \in (-1, \frac{\alpha}{\gamma})$. Observe that

$$EV_1^{odd}(\lambda) = 0 \quad\Leftrightarrow\quad \sqrt{\frac{\alpha}{\gamma} - \mu}\sin\left(\frac{\pi}{2}\sqrt{1+\mu}\right) + \sqrt{1+\mu}\cos\left(\frac{\pi}{2}\sqrt{1+\mu}\right) = 0$$
$$\Leftrightarrow\quad -\sqrt{\frac{1+\mu}{\frac{\alpha}{\gamma}-\mu}} = \tan\left(\frac{\pi}{2}\sqrt{1+\mu}\right) \tag{3.6}$$
$$EV_1^{even}(\lambda) = 0 \quad\Leftrightarrow\quad \sqrt{1+\mu}\sin\left(\frac{\pi}{2}\sqrt{1+\mu}\right) - \sqrt{\frac{\alpha}{\gamma} - \mu}\cos\left(\frac{\pi}{2}\sqrt{1+\mu}\right) = 0$$
$$\Leftrightarrow\quad \sqrt{\frac{\frac{\alpha}{\gamma}-\mu}{1+\mu}} = \tan\left(\frac{\pi}{2}\sqrt{1+\mu}\right). \tag{3.7}$$

The zeros of these equations only depend on the fraction $\frac{\alpha}{\gamma}$. Note (3.6) has corresponding odd eigenfunctions and (3.7) has corresponding even eigenfunctions. Observe that (3.7) has exactly one solution on $(-1,0)$, i.e., $L_1\colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$ has exactly one negative eigenvalue and the corresponding eigenfunction is even. In addition (3.6) has at least one solution and there is a smallest solution, i.e, $L_1\colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$ has at least one positive eigenvalue and the corresponding eigenfunction is odd. If $\frac{\alpha}{\gamma}$ is sufficiently large, there can be more positive eigenvalues, they will all be simple and the corresponding eigenfunctions alternate between being odd and even. Hence we can write the point spectrum of $L_1\colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$ as $\{\lambda_{1,j} \mid j = 0,\dots,J\}$ for some $J \geq 1$ and we have $\lambda_{1,0} < 0 < \lambda_{1,1} < \ldots < \lambda_{1,J}$. Note that we startet counting at $j = 0$, such that even indices refer to even eigenfunctions. We now calculate the claimed formulas. Writing $\mu_j \coloneqq \lambda_{1,j}\frac{4r^2}{\pi^2}$, we know μ_j solves (3.7) if j is even and μ_j solves (3.6) if j is odd. The eigenfunction corresponding to $\lambda_{1,j}$ is

$$\varphi_{1,j} = c_j \cdot \begin{cases} \cos\left(\frac{\pi}{2}\sqrt{1+\mu_j}\right)\exp\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}\,\frac{x+r}{r}\right), & x < -r, \\ \cos\left(\frac{\pi}{2}\sqrt{1+\mu_j}\,\frac{x}{r}\right), & -r < x < r, \\ \cos\left(\frac{\pi}{2}\sqrt{1+\mu_j}\right)\exp\left(-\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}\,\frac{x+r}{r}\right), & r < x, \end{cases} \qquad \text{for } j \text{ even,}$$

$$\varphi_{1,j} = c_j \cdot \begin{cases} -\sin\left(\frac{\pi}{2}\sqrt{1+\mu_j}\right)\exp\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}\,\frac{x+r}{r}\right), & x < -r, \\ \sin\left(\frac{\pi}{2}\sqrt{1+\mu_j}\,\frac{x}{r}\right), & -r < x < r, \\ \sin\left(\frac{\pi}{2}\sqrt{1+\mu_j}\right)\exp\left(-\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}\,\frac{x+r}{r}\right), & r < x, \end{cases} \qquad \text{for } j \text{ odd.}$$

The normalization constants c_j are such that $\|\varphi_j\|_{L^2(\mathbb{R})} = 1$, i.e., using (3.7) for j even we calculate

$$
\begin{aligned}
\frac{1}{c_j^2} &= \int_{-\infty}^{-r} \cos^2\Big(\frac{\pi}{2}\sqrt{1+\mu_j}\Big) \exp\Big(\pi\sqrt{\frac{\alpha}{\gamma}-\mu_j}\,\frac{x+r}{r}\Big)\,\mathrm{d}x + \int_{-r}^{r} \cos^2\Big(\frac{\pi}{2}\sqrt{1+\mu_j}\,\frac{x}{r}\Big)\,\mathrm{d}x \\
&\quad + \int_{r}^{\infty} \cos^2\Big(\frac{\pi}{2}\sqrt{1+\mu_j}\Big) \exp\Big(-\pi\sqrt{\frac{\alpha}{\gamma}-\mu_j}\,\frac{x+r}{r}\Big)\,\mathrm{d}x \\
&= r\frac{\cos^2\big(\frac{\pi}{2}\sqrt{1+\mu_j}\big)}{\pi\sqrt{\frac{\alpha}{\gamma}-\mu_j}} + r + r\frac{\cos\big(\frac{\pi}{2}\sqrt{1+\mu_j}\big)\sin\big(\frac{\pi}{2}\sqrt{1+\mu_j}\big)}{\frac{\pi}{2}\sqrt{1+\mu_j}} + r\frac{\cos^2\big(\frac{\pi}{2}\sqrt{1+\mu_j}\big)}{\pi\sqrt{\frac{\alpha}{\gamma}-\mu_j}} \\
&= r\frac{\cos^2\big(\frac{\pi}{2}\sqrt{1+\mu_j}\big)}{\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}} + r + r\frac{\sin^2\big(\frac{\pi}{2}\sqrt{1+\mu_j}\big)}{\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}} = r\frac{1+\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}}{\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}},
\end{aligned}
$$

and using (3.6) for j odd we calculate analogously

$$
\begin{aligned}
\frac{1}{c_j^2} &= \int_{-\infty}^{-r} \sin^2\Big(\frac{\pi}{2}\sqrt{1+\mu_j}\Big) \exp\Big(\pi\sqrt{\frac{\alpha}{\gamma}-\mu_j}\,\frac{x+r}{r}\Big)\,\mathrm{d}x + \int_{-r}^{r} \sin^2\Big(\frac{\pi}{2}\sqrt{1+\mu_j}\,\frac{x}{r}\Big)\,\mathrm{d}x \\
&\quad + \int_{r}^{\infty} \sin^2\Big(\frac{\pi}{2}\sqrt{1+\mu_j}\Big) \exp\Big(-\pi\sqrt{\frac{\alpha}{\gamma}-\mu_j}\,\frac{x+r}{r}\Big)\,\mathrm{d}x \\
&= r\frac{\sin^2\big(\frac{\pi}{2}\sqrt{1+\mu_j}\big)}{\pi\sqrt{\frac{\alpha}{\gamma}-\mu_j}} + r - r\frac{\cos\big(\frac{\pi}{2}\sqrt{1+\mu_j}\big)\sin\big(\frac{\pi}{2}\sqrt{1+\mu_j}\big)}{\frac{\pi}{2}\sqrt{1+\mu_j}} + r\frac{\sin^2\big(\frac{\pi}{2}\sqrt{1+\mu_j}\big)}{\pi\sqrt{\frac{\alpha}{\gamma}-\mu_j}} \\
&= r\frac{\sin^2\big(\frac{\pi}{2}\sqrt{1+\mu_j}\big)}{\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}} + r + r\frac{\cos^2\big(\frac{\pi}{2}\sqrt{1+\mu_j}\big)}{\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}} = r\frac{1+\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}}{\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}}.
\end{aligned}
$$

We next calculate the integrals $\int_{-r}^{r} \mathrm{e}^{-\sqrt{\alpha}\omega x}\varphi_{1,j}(x)\,\mathrm{d}x$. We start with j even using (3.7):

$$
\begin{aligned}
\int_{-r}^{r} \mathrm{e}^{-\sqrt{\alpha}\omega x}\varphi_{1,j}(x)\,\mathrm{d}x &= c_j\int_{-r}^{r} \mathrm{e}^{-\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\frac{x}{r}} \cos\Big(\frac{\pi}{2}\sqrt{1+\mu_j}\,\frac{x}{r}\Big)\,\mathrm{d}x \\
&= \frac{2rc_j}{\frac{\pi^2}{4}\left(\frac{\alpha}{\gamma}+1+\mu_j\right)} \cdot \Big(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\cos\Big(\frac{\pi}{2}\sqrt{1+\mu_j}\Big)\sinh\Big(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\Big) \\
&\qquad + \frac{\pi}{2}\sqrt{1+\mu_j}\sin\Big(\frac{\pi}{2}\sqrt{1+\mu_j}\Big)\cosh\Big(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\Big)\Big) \\
&= \frac{2rc_j\cos\big(\frac{\pi}{2}\sqrt{1+\mu_j}\big)}{\frac{\pi}{2}\left(\frac{\alpha}{\gamma}+1+\mu_j\right)} \cdot \Big(\sqrt{\frac{\alpha}{\gamma}}\sinh\Big(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\Big) + \sqrt{\frac{\alpha}{\gamma}-\mu_j}\cosh\Big(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\Big)\Big),
\end{aligned}
$$

and continue with j odd using (3.6):

$$
\begin{aligned}
\int_{-r}^{r} \mathrm{e}^{-\sqrt{\alpha}\omega x}\varphi_{1,j}(x)\,\mathrm{d}x &= c_j\int_{-r}^{r} \mathrm{e}^{-\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\frac{x}{r}} \sin\Big(\frac{\pi}{2}\sqrt{1+\mu_j}\,\frac{x}{r}\Big)\,\mathrm{d}x \\
&= \frac{2rc_j}{\frac{\pi^2}{4}\left(\frac{\alpha}{\gamma}+1+\mu_j\right)} \cdot \Big(\frac{\pi}{2}\sqrt{1+\mu_j}\cos\Big(\frac{\pi}{2}\sqrt{1+\mu_j}\Big)\sinh\Big(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\Big) \\
&\qquad - \frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\sin\Big(\frac{\pi}{2}\sqrt{1+\mu_j}\Big)\cosh\Big(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\Big)\Big)
\end{aligned}
$$

$$= -\frac{2rc_j \sin\left(\frac{\pi}{2}\sqrt{1+\mu_j}\right)}{\frac{\pi}{2}\left(\frac{\alpha}{\gamma}+1+\mu_j\right)} \cdot \left(\sqrt{\frac{\alpha}{\gamma}-\mu_j}\sinh\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\right) + \sqrt{\frac{\alpha}{\gamma}}\cosh\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\right)\right).$$

Hence we obtain for j even

$$\begin{aligned}
&\left(\frac{\beta^2}{\lambda_{1,j}}+\frac{\beta}{\omega^2}\right)\left(\int_{-r}^{r} \mathrm{e}^{-\sqrt{\alpha}\omega x}\varphi_{1,j}(x)\,\mathrm{d}x\right)^2 \\
&\quad = \frac{(\alpha+\gamma)\gamma}{\left(\frac{\pi}{2r}\right)^2\mu_j}\left(\frac{\alpha}{\gamma}+1+\mu_j\right)\cdot\frac{4r^2c_j^2\cos^2\left(\frac{\pi}{2}\sqrt{1+\mu_j}\right)}{\left(\frac{\pi}{2r}\right)^2\left(\frac{\alpha}{\gamma}+1+\mu_j\right)^2} \\
&\qquad\qquad \cdot\left(\sqrt{\frac{\alpha}{\gamma}}\sinh\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\right)+\sqrt{\frac{\alpha}{\gamma}-\mu_j}\cosh\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\right)\right)^2 \\
&\quad = \frac{(\alpha+\gamma)\gamma}{\left(\frac{\pi}{2r}\right)^4\mu_j}\cdot\frac{4r\cos^2\left(\frac{\pi}{2}\sqrt{1+\mu_j}\right)}{\left(\frac{\alpha}{\gamma}+1+\mu_j\right)}\cdot\frac{\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}}{1+\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}} \\
&\qquad\qquad \cdot\left(\sqrt{\frac{\alpha}{\gamma}}\sinh\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\right)+\sqrt{\frac{\alpha}{\gamma}-\mu_j}\cosh\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\right)\right)^2,
\end{aligned}$$

and for j odd analogously

$$\begin{aligned}
&\left(\frac{\beta^2}{\lambda_{1,j}}+\frac{\beta}{\omega^2}\right)\left(\int_{-r}^{r} \mathrm{e}^{-\sqrt{\alpha}\omega x}\varphi_{1,j}(x)\,\mathrm{d}x\right)^2 \\
&\quad = \frac{(\alpha+\gamma)\gamma}{\left(\frac{\pi}{2r}\right)^2\mu_j}\left(\frac{\alpha}{\gamma}+1+\mu_j\right)\cdot\frac{4r^2c_j^2\sin^2\left(\frac{\pi}{2}\sqrt{1+\mu_j}\right)}{\left(\frac{\pi}{2r}\right)^2\left(\frac{\alpha}{\gamma}+1+\mu_j\right)^2} \\
&\qquad\qquad \cdot\left(\sqrt{\frac{\alpha}{\gamma}-\mu_j}\sinh\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\right)+\sqrt{\frac{\alpha}{\gamma}}\cosh\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\right)\right)^2 \\
&\quad = \frac{(\alpha+\gamma)\gamma}{\left(\frac{\pi}{2r}\right)^4\mu_j}\cdot\frac{4r\sin^2\left(\frac{\pi}{2}\sqrt{1+\mu_j}\right)}{\left(\frac{\alpha}{\gamma}+1+\mu_j\right)}\cdot\frac{\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}}{1+\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}-\mu_j}} \\
&\qquad\qquad \cdot\left(\sqrt{\frac{\alpha}{\gamma}-\mu_j}\sinh\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\right)+\sqrt{\frac{\alpha}{\gamma}}\cosh\left(\frac{\pi}{2}\sqrt{\frac{\alpha}{\gamma}}\right)\right)^2.
\end{aligned}$$

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Declaration

Hiermit versichere ich, dass ich diese Arbeit selbständig verfasst und keine anderen, als die angegebenen Quellen und Hilfsmittel benutzt, die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht und die Satzung des Karlsruher Instituts für Technologie zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet habe.

Karlsruhe, 25. Mai 2021