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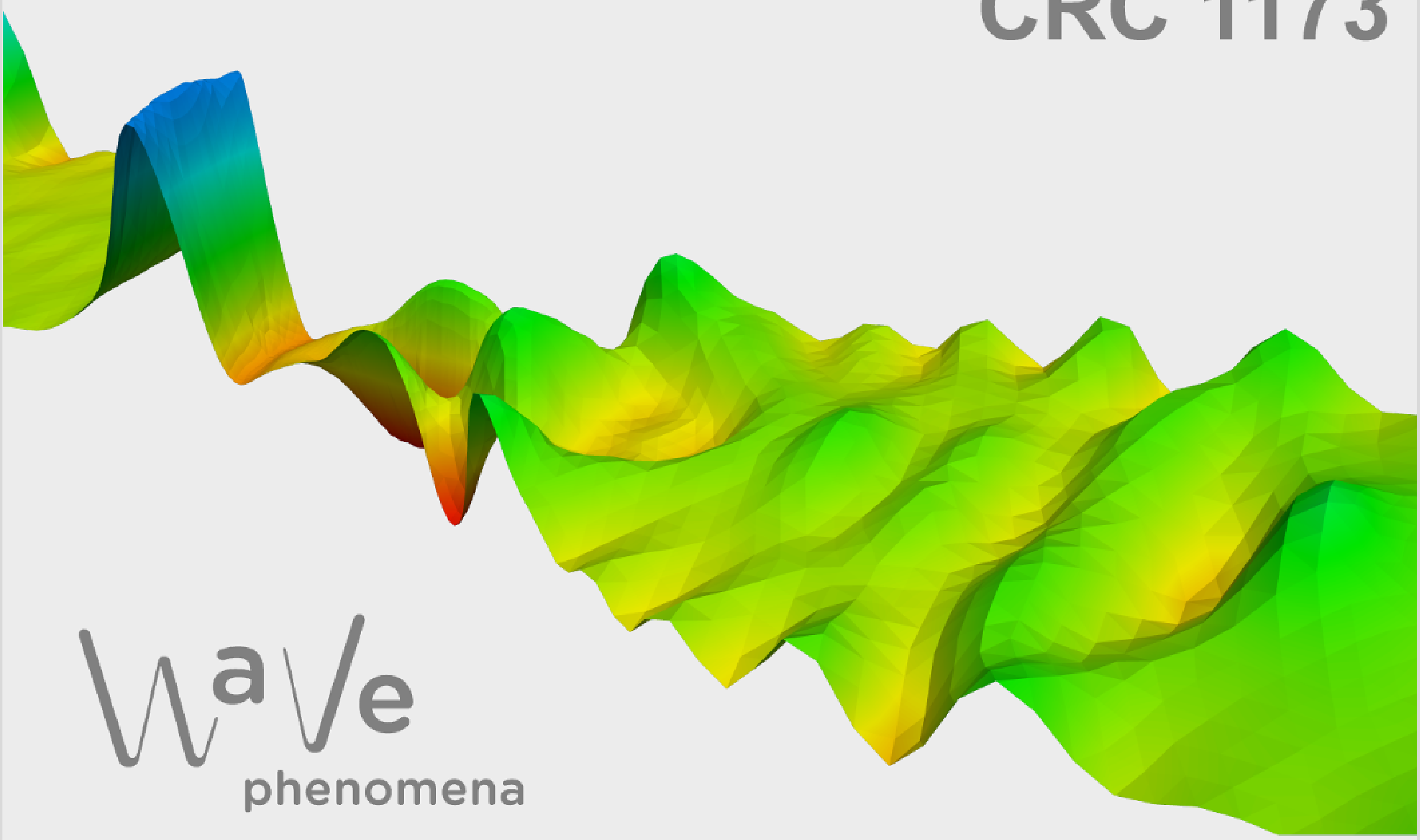
CRC Preprint 2021/44, November 2021



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ISSN 2365-662X

A unified error analysis for nonlinear wave-type equations with application to acoustic boundary conditions

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[Received on 17 November 2021]

In this work we present a unified error analysis for abstract space discretizations of nonlinear wave-type equations. This yields an error bound in terms of discretization and interpolation errors that can be applied to various equations and space discretizations fitting in the abstract setting. We use the unified error analysis to prove novel convergence rates for a non-conforming finite element space discretization of wave equations with nonlinear acoustic boundary conditions and illustrate the error bound by some numerical experiments.

Keywords: nonlinear evolution equations, monotone operators, non-conforming space discretization, acoustic boundary conditions, dynamic boundary conditions, nonlinear wave equations, a-priori error analysis, finite element method

1. Introduction

In this paper we consider the space discretization of wave equations with nonlinear acoustic boundary conditions. These boundary conditions are an effective model for a boundary that is subject to small oscillations in normal direction which are caused by a wave propagation in the interior of the domain.

Such boundary conditions were first mentioned in Beale & Rosencrans (1974). Since then, many papers studied their properties, wellposedness, and stability, and they are still in the focus of current research, cf. Beale (1976); Frota & Vicente (2018); Gal *et al.* (2003); Ma & Souza (2017); Vicente & Frota (2013) and references therein.

However, the only numerical paper we are aware of considering these boundary conditions is Hipp *et al.* (2019). In this paper a space discretization for wave equations with linear acoustic boundary conditions was derived and analyzed. In the present paper, we now consider the space discretization of nonlinear acoustic boundary conditions as proposed in Graber (2012); Graber & Said-Houari (2012); Wu (2012), and extend the results from Hipp *et al.* (2019) to this case.

Since acoustic boundary conditions include derivatives on the boundary, they are usually posed on domains with smooth boundary. Hence, the domain has to be approximated by the finite element method wich renders the space discretization non-conforming. This makes the error analysis much more involved since the exact and the numerical solution are not defined on the same domain. To tackle this difficulty, in Hipp (2017); Hipp *et al.* (2019) a unified error analysis for linear wave equations was introduced and extended in Hochbruck & Leibold

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(2020) to semilinear equations. The unified error analysis is an abstract framework in which wave equations as well as their spatial discretizations are considered as evolution equations in Hilbert spaces. In this framework, the error analysis is performed which gives an abstract error bound in terms of approximation properties of the space discretization method. This error bound can then be applied to all equations and space discretizations fitting into the abstract setting.

The aim of this paper is to extend the unified error analysis to nonlinear evolution equations with quasi-monotone operators and to use this theory to prove error bounds for a finite element discretization of the wave equation with nonlinear acoustic boundary conditions. This is a generalization of the results in the thesis Leibold (2021). A major difficulty lies in the discretization of the nonlinearities. This must be done in such a way that it preserves the quasi-monotonicity of the operator to ensure the stability of the numerical scheme.

We are not aware of any other results in this direction, neither of such a general error analysis for non-conforming space discretizations of nonlinear wave-type equations, nor of results conserning the discretization of wave equations with nonlinear acoustic boundary conditions. Nevertheless, we mention the following works going in the same direction. In Emmrich *et al.* (2015), a full discretization in an abstract framework similar to the one used in this paper was considered. But only a conforming space discretization was analyzed and no error bounds but only weak convergence of the discretization was shown. For quasilinear equations, a related framework was introduced in Maier (2020); Hochbruck & Maier (2021), covering quasilinear wave and Maxwell equations. However, the error analysis in this work relies on properties of quasilinear operators that cannot be used for nonlinear acoustic boundary conditions and in general for equations with maximal quasi-monotone operators.

This paper is structured as follows. In Section 2 we introduce the wave equation with nonlinear acoustic boundary conditions with a corresponding finite element space discretization and state our main result, an error bound of the spatial discretization. We then present in Section 3 the unified error analysis for nonlinear first-order evolution equations and use the results in Section 4 to analyze nonlinear second-order wave-type equations. Finally, in Section 5 we use the results of the unified error analysis to prove the space discretization error bound for the wave equations with nonlinear acoustic boundary conditions and illustrate it with some numerical experiments.

2. The wave equation with nonlinear acoustic boundary conditions

In this section we present the analytical framework for the wave equation with acoustic boundary conditions and a suitable finite element space discretization. Additionally, we present our main result, a space discretization error bound.

2.1 *Problem statement and analytical framework*

Let $\Omega \subset \mathbb{R}^n, n = 2,3,$ be a domain with C^2-boundary Γ and outer normal vector $\mathbf{n}$. We consider the acousitc wave equation with non-local reacting acoustic boundary conditions in the following form: Seek $u\colon [0,T]\times\Omega\to\mathbb{R}, \delta\colon [0,T]\times\Gamma\to\mathbb{R}$ satisfying

$$u_{tt}+k_\Omega u-c_\Omega\Delta u=f_\Omega, \qquad t\geqslant 0,\mathbf{x}\in\Omega, \tag{2.1a}$$
$$\mu\delta_{tt}+d\delta_t+k_\Gamma\delta+\rho u_t-c_\Gamma\Delta_\Gamma\delta=f_\Gamma, \qquad t\geqslant 0,\mathbf{x}\in\partial\Omega, \tag{2.1b}$$
$$\eta(\delta_t)=\partial_{\mathbf{n}}u+\theta(u_t), \qquad t\geqslant 0,\mathbf{x}\in\partial\Omega, \tag{2.1c}$$
$$u(0)=u^0,\quad u_t(0)=v^0,\quad \delta(0)=\delta^0,\quad \delta_t(0)=\vartheta^0. \tag{2.1d}$$

Here Δ_Γ denotes the Laplace–Beltrami operator an Γ.

Remark 2.1 It is possible to include nonlinear forcing terms $F_\Omega(\mathbf{x},u)$ and $F_\Gamma(\mathbf{x},\delta)$ at the right hand side of (2.1a) and (2.1b), respectively. This was considered in Leibold (2021) for the wave equation with kinetic boundary conditions and such terms can be treated similarly for the acoustic boundary conditions. We omit this here for the sake of a clearer presentation.

We make the following assumptions on the coefficients and nonlinearities in (2.1).

Assumption 2.2

a) The constants satisfy $c_\Omega,c_\Gamma,\mu>0,\quad k_\Omega,k_\Gamma\geqslant 0,\quad d,\rho\in\mathbb{R}$.

b) The function $\theta\in C(\mathbb{R};\mathbb{R})$ satisfies $\theta(0)=0$ and is strictly monotonically increasing with

$$(\theta(\xi_1)-\theta(\xi_2))\,(\xi_1-\xi_2)\geqslant\theta_0|\xi_1-\xi_2|^2, \qquad \xi_1,\xi_2\in\mathbb{R},$$

for some $\theta_0>0$. Further, there exist

$$1\leqslant\zeta\begin{cases}<\infty, & n=2,\\ \leqslant 3, & n=3,\end{cases} \tag{2.2}$$

and a constant $C>0$ such that for all $\xi\in\mathbb{R}$

$$|\theta(\xi)|\leqslant C(1+|\xi|^\zeta). \tag{2.3}$$

c) The function $\eta\colon\mathbb{R}\to\mathbb{R}$ is globally Lipschitz continuous. We then have that $\widetilde{\eta}$ defined via $\widetilde{\eta}(\xi)=\eta(\xi)-\frac{\rho}{c_\Omega}\xi$ is also Lipschitz continuous and denote the Lipschitz constant of $\widetilde{\eta}$ by L_η.

d) The inhomogeneities satisfy $f_\Omega\in W^{1,1}_{loc}([0,\infty);C(\overline{\Omega}))$ and $f_\Gamma\in W^{1,1}_{loc}([0,\infty);C(\Gamma))$.

Weak formulation To prove wellposedness and derive a finite element discretization, we now present a weak formulation of the wave equation with acoustic boundary conditions (2.1). We make use of the densely embedded Hilbert spaces

$$V=\mathbb{H}^1\hookrightarrow H=\mathbb{H}^0,$$

where

$$\mathbb{H}^0:=L^2(\Omega)\times L^2(\Gamma),\quad \mathbb{H}^k:=H^k(\Omega)\times H^k(\Gamma),\quad k\geqslant 1.$$

By multiplying (2.1a) and (2.1b) with test functions defined on Ω and Γ, respectively, applying integration by parts and inserting the nonlinear coupling (2.1c), we obtain the the weak formulation of (2.1): Seek $\vec{u} = [u,\delta]^\mathsf{T} : [0,T] \to V$ satisfying

$$\begin{aligned} m\big(\vec{u}'',\vec{\varphi}\big) + \langle \mathcal{D}(\vec{u}'),\vec{\varphi}\rangle_{V^*\times V} + a\big(\vec{u},\vec{\varphi}\big) &= m\big(\vec{f},\vec{\varphi}\big), \qquad \text{for } t \geqslant 0 \text{ and all } \vec{\varphi} \in V,\\ \vec{u}(0) = \vec{u}^0, \qquad u'(0) &= \vec{v}^0, \end{aligned} \tag{2.4}$$

where for $\vec{v} = [v,z]^\mathsf{T}, \vec{\varphi} = [\varphi,\psi]^\mathsf{T} \in V$ we have

$$m\big(\vec{v},\vec{\varphi}\big) = \int_\Omega v\varphi \,\mathrm{d}\mathbf{x} + \mu \int_\Gamma z\psi \,\mathrm{d}s, \tag{2.5a}$$

$$a\big(\vec{v},\vec{\varphi}\big) = c_\Omega \int_\Omega \nabla v \cdot \nabla\varphi \,\mathrm{d}\mathbf{x} + k_\Omega \int_\Omega v\varphi \,\mathrm{d}\mathbf{x} \tag{2.5b}$$

$$+ c_\Gamma \int_\Gamma \nabla_\Gamma z \cdot \nabla_\Gamma \psi \,\mathrm{d}\mathbf{x} + k_\Gamma \int_\Gamma z\psi \,\mathrm{d}s, \tag{2.5c}$$

$$\langle \mathcal{D}(\vec{v}),\vec{\varphi}\rangle_{V^*\times V} = \int_\Gamma c_\Omega \left(\theta(v) - \eta(z)\right)\varphi + \left(dz + \rho v\right)\psi \,\mathrm{d}s, \tag{2.5d}$$

$$\vec{f} = [f_\Omega, f_\Gamma]^\mathsf{T}. \tag{2.5e}$$

Note that m is an inner product on H and $\tilde{a} := a + m$ is an inner product on V.

Remark 2.3 Assumption 2.2 ensures that (2.1) is globally wellposed, we comment on this in Sections 4.1 and 5.

2.2 *Finite element space discretization*

For the space discretization of (2.1) we consider the bulk-surface finite element from Elliott & Ranner (2013) which was also used in Hipp *et al.* (2019) to discretize the wave equation with linear acoustic boundary conditions. We give a brief introduction of the finite element spaces and refer to Elliott & Ranner (2013) for further details on the bulk-surface finite element method.

The bulk-surface finite element method Let $\Gamma \in C^{p+1}$ for some $p \geqslant 1$ and let $\mathcal{T}_h^\Omega$ be a consistent and quasi-uniform mesh consisting of isoparametric elements K of degree p which discretizes Ω . By h we denote the maximal mesh width of $\mathcal{T}_h^\Omega$. The discretized domain is then given by

$$\Omega_h = \bigcup_{K\in\mathcal{T}_h} K$$

and its boundary by $\Gamma_h = \partial\Omega_h$. The bulk and the surface finite element space of order p are then defined by

$$\begin{aligned} V_{h,p}^\Omega &:= \big\{ v_h \in C(\Omega_h) \mid v_h\big|_K = \widehat{v}_h \circ (F_{K^e})^{-1} \text{ with } \widehat{v}_h \in \mathbb{P}_p(\widehat{K}) \text{ for all } K \in \mathcal{T}_h \big\},\\ V_{h,p}^\Gamma &:= \big\{ \vartheta_h \in C(\Gamma_h) \mid \vartheta_h = v_h\big|_{\Gamma_h} \text{ with } v_h \in V_{h,p}^\Omega \big\}, \end{aligned}$$

respectively. Here, $\widehat{K}$ denotes the reference triangle with corresponding polynomial space $\mathbb{P}_p(\widehat{K})$ of order p, and F_{K^e} is the transformation from $\widehat{K}$ to K. Note that by construction we have $v_h\big|_{\Gamma_h} \in V_{h,p}^{\Gamma}$ for all $v_h \in V_{h,p}^{\Omega}$.

As approximation space for V we set $V_h = V_{h,p}^{\Omega} \times V_{h,p}^{\Gamma}$. Note that, since Ω_h is only an approximation of Ω, we have $V_h \nsubseteq V$, i.e., the discretization is non-conforming. Hence, to relate functions in V_h with functions in V, in Elliott & Ranner (2013), for $\vec{v}_h = [v_h, \vartheta_h]^\mathsf{T} \in V_h$ a lifted version

$$\vec{v}_h^{\ell} = \left[v_h^{\ell}, \vartheta_h^{\ell}\right]^\mathsf{T} \in C(\Omega) \times C(\Gamma) \subset V \tag{2.6}$$

was constructed. By $I_{h,\Omega}\colon C(\overline{\Omega}) \to V_{h,p}^{\Omega}$ and $I_{h,\Gamma}\colon C(\Gamma) \to V_{h,p}^{\Gamma}$ we denote the order p nodal interpolation operators in Ω and on Γ, respectively, and set for $\vec{v} = [v, \vartheta]^\mathsf{T} \in V$

$$I_h \vec{v} = \left[I_{h,\Omega} v, I_{h,\Gamma} \vartheta\right]^\mathsf{T} \in V_h.$$

THE SPATIALLY DISCRETIZED EQUATION We now state the finite element discretization of (2.1). For this, let

$$\sum_{\Gamma_h} \cdot \Delta s\colon C(\Gamma_h) \to \mathbb{R}$$

be an elementwise defined quadrature formula that approximates the integral $\int_{\Gamma_h} \cdot \,\mathrm{d}s$. We require that the quadrature formula has positive weights and is of order greater than $2p$, s.t. polynomials up to degree $2p$ are integrated exactly and we have for all $z_h, \psi_h \in V_{h,p}^{\Gamma}$

$$\int_{\Gamma_h} z_h \psi_h \,\mathrm{d}s = \sum_{\Gamma_h} z_h \psi_h \Delta s. \tag{2.7}$$

For $\vec{v}_h = [v_h, z_h]^\mathsf{T}, \vec{\varphi}_h = [\varphi_h, \psi_h]^\mathsf{T} \in V_h$ we define

$$m_h\big(\vec{v}_h, \vec{\varphi}_h\big) = \int_{\Omega_h} v_h \varphi_h \,\mathrm{d}\mathbf{x} + \int_{\Gamma_h} z_h \psi_h \,\mathrm{d}s, \tag{2.8a}$$

$$a_h\big(\vec{v}_h, \vec{\varphi}_h\big) = c_\Omega \int_{\Omega_h} \nabla v_h \cdot \nabla \varphi_h \,\mathrm{d}\mathbf{x} + k_\Omega \int_{\Omega_h} v_h \varphi_h \,\mathrm{d}\mathbf{x} \tag{2.8b}$$

$$+ c_\Gamma \int_{\Gamma_h} \nabla_\Gamma z_h \cdot \nabla_\Gamma \psi_h \,\mathrm{d}s + k_\Gamma \int_{\Gamma_h} z_h \psi_h \,\mathrm{d}s, \tag{2.8c}$$

$$m_h\big(\mathcal{D}(\vec{v}_h), \vec{\varphi}_h\big) = \sum_{\Gamma_h} c_\Omega \left(\theta(v_h) - \eta(z_h)\right) \varphi_h + \left(d z_h + \rho v_h\right) \psi_h \Delta s, \tag{2.8d}$$

$$m_h\big(\vec{f}_h, \vec{\varphi}_h\big) = \int_{\Omega_h} I_{h,\Omega} f_\Omega \varphi_h \,\mathrm{d}\mathbf{x} + \int_{\Gamma} I_{h,\Gamma} f_\Gamma \psi_h \,\mathrm{d}s. \tag{2.8e}$$

Then, the spatial discretization of (2.1) is given by: Seek $\vec{u}_h\colon [0,T] \to V_h$ s.t.

$$\begin{aligned} m_h\big(\vec{u}_h'', \vec{\varphi}_h\big) + m_h\big(\mathcal{D}_h(\vec{u}_h'), \vec{\varphi}_h\big) + a_h\big(\vec{u}_h, \vec{\varphi}_h\big) &= m_h\big(\vec{f}_h, \vec{\varphi}_h\big), \qquad \text{for } t \geqslant 0, \vec{\varphi}_h \in V_h, \\ \vec{u}_h(0) = \vec{u}_h^0, \qquad \vec{u}_h'(0) &= \vec{v}_h^0. \end{aligned}$$

REMARK 2.4 The use of the quadrature formulas instead of the interpolation in the definition of the discretized nonlinearity $\mathcal{D}_h$ is required to prove that $\mathcal{D}_h$ is quasi-monotone, cf. Lemma 5.3.

To prove an error bound of the discretization we pose the following assumptions on the exact solution and the data:

ASSUMPTION 2.5

a) Let $T>0$. For the inhomogeneities and the nonlinearities in (2.1) we assume the additional regularity

$$f_\Omega \in L^\infty\big([0,T];H^{\max\{2,p\}}(\Omega)\big), \quad f_\Gamma \in L^\infty\big([0,T];H^{\max\{2,p\}}(\Gamma)\big), \tag{2.9a}$$
$$\theta,\eta \in C^p(\mathbb{R};\mathbb{R}). \tag{2.9b}$$

Furthermore, we assume that the strong solution u of (2.1) satisfies on $[0,T]$

$$\begin{aligned} u &\in L^\infty\big([0,T];H^{p+1}(\Omega)\big), & u' &\in L^\infty\big([0,T];W^{p+1,\infty}(\Omega)\big),\\ u'' &\in L^\infty\big([0,T];H^{\max\{2,p\}}(\Omega)\big), & &\\ \delta &\in L^\infty\big([0,T];H^{p+1}(\Gamma)\big), & \delta' &\in L^\infty\big([0,T];H^{p+1}(\Gamma)\cap W^{p,\infty}(\Gamma)\big),\\ \delta'' &\in L^\infty\big([0,T];H^{\max\{2,p\}}(\Gamma)\big). & & \end{aligned}$$

b) Let the discrete initial values satisfy

$$\|\vec{u}_h^0 - I_h\vec{u}^0\|_{\mathbb{H}^1} + \|\vec{v}_h^0 - I_h\vec{v}^0\|_{\mathbb{H}^0} \leqslant C_{\mathrm{iv}} h^p$$

with a constant C_{iv} independent of h.

As main theorem, we state the following error bound for the finite element discretization of the wave equation with nonlinear acoustic boundary conditions.

THEOREM 2.6. *Let Assumption 2.2 be satisfied and $\vec{u}=[u,\delta]^\intercal$ be the solution of (2.1) on $[0,T]$. Further, let Assumption 2.5 be satisfied and let $\vec{u}_h=[u_h,\delta_h]^\intercal$ be the spatial approximation of $\vec{u}$, obtained with the bulk-surface finite element method of order p. Then, the error bound*

$$\|\vec{u}_h^\ell - \vec{u}\|_{\mathbb{H}^1} + \|(\vec{u}_h')^\ell(t) - \vec{u}'(t)\|_{\mathbb{H}^0} \leqslant C\mathrm{e}^{(\frac{1}{2}+c')t}(1+t)h^p$$

holds true with a constant C independent of h.

In the next two sections we will now present a general theory for the error analysis of non-conforming space discretizations which we then use to proof Theorem 2.6 in Section 5.

3. Abstract space discretizations of first-order evolution equations with monotone operators

In this section we present the unified error analysis for abstract space discretizations of first-order evolution equations with maximal monotone operators. This generalizes the results from Hipp *et al.* (2019) and Hochbruck & Leibold (2020) for linear and semilinear equations, respectively. The results of this section are part of the dissertation Leibold (2021).

We first present the continuous equation and the corresponding abstract space discretization, before we prove an error bound.

3.1 *Analytical setting*

Let X be a Hilbert space with scalar product p in which we consider the evolution equation

$$x'(t) + \mathcal{S}(x(t)) = g(t), \qquad t \geqslant 0, \tag{3.1a}$$

$$x(0) = x^0 \in D(\mathcal{S}). \tag{3.1b}$$

In the following, we omit the t arguments in evolution equations. We pose the following classical assumptions to ensure that (3.1) is wellposed.

ASSUMPTION 3.1

a) The nonlinear operator $\mathcal{S} \colon D(\mathcal{S}) \to X$ is quasi-monotone and maximal, i.e., there is a $c_{\mathrm{qm}} > 0$ s.t.

$$p\big(\mathcal{S}(y) - \mathcal{S}(z), y - z\big) \geqslant -c_{\mathrm{qm}} \|y - z\|_X^2 \qquad \text{for all } y, z \in D(\mathcal{S}),$$

and there exists some $\lambda > c_{\mathrm{qm}}$ s.t. $\mathrm{range}(\lambda + \mathcal{S}) = X$. Furthermore, $D(\mathcal{S})$ is dense in X.

b) The inhomogeneity satisfies $g \in W^{1,1}_{loc}([0,\infty); X)$.

The following wellposedness result can, e.g., be found in (Showalter, 1997, Corollary IV.4.1).

THEOREM 3.2. *Let Assumption 3.1 hold true. Then, the evolution equation* (3.1) *is globally wellposed, i.e.,* (3.1) *has a unique strong solution* $x \in C([0,\infty); X)$ *which satisfies* $x(t) \in D(\mathcal{S})$ *for all* $t \in [0,\infty)$, $x(0) = x^0$, *and* (3.1a) *is satisfied for almost all* $t \in [0,\infty)$.

We further state the following stability result which is essential for the latter error analysis.

THEOREM 3.3. *Let Assumption 3.1 be satisfied and for* $T > 0$ *and* $i = 1,2$ *let* x_i *be the strong solutions of*

$$x_i' + \mathcal{S}(x_i) = g_i, \qquad t \in [0,T],$$
$$x_i(0) = x_i^0$$

with $g_i \in W^{1,1}([0,T]; X)$. *Then for all* $t \in [0,T]$

$$\|x_1(t) - x_2(t)\|_X \leqslant \mathrm{e}^{c_{\mathrm{qm}} t} \left(\|x_1^0 - x_2^0\|_X + \int_0^t \|g_1(s) - g_2(s)\|_X \,\mathrm{d}s \right).$$

Proof. The result can be derived with energy estimates similar to (Showalter, 1997, Theorem IV.4.1A). □

3.2 *Abstract space discretization*

We now present an abstract space discretization of the evolution equation (3.1). Let $(X_h)_h$ be a family of finite dimensional vector spaces with scalar products p_h, where h is a discretization parameter, e.g., the maximal mesh width of a finite element discretization. For all $X_h \in (X_h)_h$ we seek an approximations $x_h \in X_h$ to the solution x of (3.1). Therefore, let $\mathcal{S}_h$ and g_h be approximations of $\mathcal{S}$ and g, respectively, which satisfy the following assumptions similar to Assumption 3.1.

ASSUMPTION 3.4

a) The nonlinear operator $\mathcal{S}_h : X_h \to X_h$ is quasi-monotone, i.e., there is a $\widehat{c}_{\mathrm{qm}} > 0$ independent of h s.t.

$$p_h\big(\mathcal{S}_h(y_h) - \mathcal{S}_h(z_h), y_h - z_h\big) \geqslant -\widehat{c}_{\mathrm{qm}} \|y_h - z_h\|_{X_h}^2 \qquad \text{for all } y_h, z_h \in X_h. \tag{3.2}$$

b) The inhomogeneity satisfies $g_h \in W^{1,1}_{loc}([0;\infty); X_h)$.

The discretized evolution equation is then given by

$$x_h' + \mathcal{S}_h(x_h) = g_h, \qquad t \geqslant 0, \tag{3.3a}$$

$$x_h(0) = x_h^0. \tag{3.3b}$$

Since these assumptions are similar to the continuous case, we obtain by Theorem 3.2 that (3.3) is globally wellposed.

In the following we introduce a framework for the error analysis of the abstract space discretization that is similar to the linear case presented in Hipp *et al.* (2019). To cover non-conforming space discretizations where $X_h \not\subseteq X$, as they appear in Section 2, we make the following assumptions to relate the discrete and the continuous problem.

ASSUMPTION 3.5

a) There exists a *lift operator* $\mathcal{L}_h \in \mathcal{L}(X_h, X)$ which satisfies

$$\|\mathcal{L}_h y_h\|_X \leqslant \widehat{C}_X \|y_h\|_{X_h} \qquad \text{for all } y_h \in X_h \tag{3.4}$$

for some constant $\widehat{C}_X > 0$ independent of h. The adjoint of the lift operator $\mathcal{L}_h^* \in \mathcal{L}(X, X_h)$ is defined via

$$p_h\big(\mathcal{L}_h^* y, y_h\big) = p\big(y, \mathcal{L}_h y_h\big), \qquad \text{for all } y \in X, y_h \in X_h.$$

b) Let $Z \hookrightarrow X$ be a densely embedded subspace of X on which a *reference operator* $J_h \in \mathcal{L}(Z; X_h)$ is defined which satisfies

$$\|J_h\|_{X_h \leftarrow Z} \leqslant \widehat{C}_{J_h}$$

for some constant $\widehat{C}_{J_h} > 0$ independent of h.

The reference operator should satisfy $\mathcal{L}_h J_h z \approx z$ for all $z \in Z$ in a suitable sense and could, e.g., be an interpolation or a projection operator.

The space discretization error bound is given in terms of the following terms:

DEFINITION 3.6 (Remainder and error terms)

a) The *remainder of the nonlinear monotone operator* is given by

$$R_h : D(\mathcal{S}) \cap Z \to X_h, \qquad R_h(z) := \mathcal{L}_h^* \mathcal{S}(z) - \mathcal{S}_h\left(J_h z\right). \tag{3.5}$$

b) We define the error term

$$\begin{aligned} E_h(t) = &\left\| x_h^0 - J_h x^0 \right\|_{X_h} + t\|(\mathcal{L}_h^* - J_h)x'\|_{L^\infty([0,t];X_h)} \\ &+ t\|R_h(x)\|_{L^\infty([0,t];X_h)} + t\|\mathcal{L}_h^* g - g_h\|_{L^\infty([0,t];X_h)}. \end{aligned} \tag{3.6}$$

We now can state and prove an error bound of the abstract space discretization, cf. (Leibold, 2021, Thm. 2.10).

THEOREM 3.7. *Let Assumptions 3.1, 3.4, and 3.5 be satisfied and x be the strong solution of* (3.1) *on* $[0,T]$ *with* $x, x' \in L^\infty([0,T];Z)$. *Furthermore, let* x_h *be the solution of* (3.3) *on* $[0,T]$. *Then, for all* $t \in [0,T]$ *the lifted discrete solution satisfies the error bound*

$$\|\mathcal{L}_h x_h(t) - x(t)\|_X \leqslant \widehat{C}_X \mathrm{e}^{\widehat{c}_{\mathrm{qm}} t} E_h(t) + \|(\mathrm{I} - \mathcal{L}_h J_h)x(t)\|_X. \tag{3.7}$$

Proof. We split the error via $\mathcal{L}_h x_h(t) - x(t) = \mathcal{L}_h e_h + (\mathcal{L}_h J_h - \mathrm{I})x(t)$, where

$$e_h(t) = x_h(t) - J_h x(t) \in X_h$$

is the discrete error. The full error can thus be bounded by

$$\|\mathcal{L}_h x_h(t) - x(t)\|_X \leqslant \widehat{C}_X \|e_h\|_{X_h} + \|(\mathcal{L}_h J_h - \mathrm{I})x(t)\|_X \tag{3.8}$$

and we further investigate the discrete error. By applying the adjoint lift to (3.1a) we obtain

$$\mathcal{L}_h^* x' + \mathcal{L}_h^* \mathcal{S}(x) = \mathcal{L}_h^* g.$$

Adding $J_h x', \mathcal{S}_h(J_h x)$, and g_h on both sides yields

$$J_h x' + \mathcal{S}_h(J_h x) = g_h + \Delta_h \tag{3.9}$$

where

$$\Delta_h = (J_h - \mathcal{L}_h^*)\, x' + \mathcal{S}_h(J_h x) - \mathcal{L}_h^* \mathcal{S}(x) + \mathcal{L}_h^* g - g_h. \tag{3.10}$$

Under Assumption 3.4, the stability estimate from Theorem 3.3 holds also true in the discrete case with $\widehat{c}_{\mathrm{qm}}$ instead of c_{qm}. Hence, we obtain by Theorem 3.3 applied to (3.3) and (3.9) the following bound for the discrete error

$$\begin{aligned} \|e_h(t)\|_{X_h} &\leqslant \mathrm{e}^{\widehat{c}_{\mathrm{qm}} t} \left(\|x_h^0 - J_h x^0\|_{X_h} + \int_0^t \|\Delta_h(s)\|_{X_h} \,\mathrm{d}s \right) \\ &\leqslant \mathrm{e}^{\widehat{c}_{\mathrm{qm}} t} \left(\|x_h^0 - J_h x^0\|_{X_h} + t\|\Delta_h\|_{L^\infty([0,T];X_h)} \right) \\ &\leqslant \mathrm{e}^{\widehat{c}_{\mathrm{qm}} t} E_h(t), \end{aligned} \tag{3.11}$$

where we used (3.10) and (3.5). Together with (3.8), we finally obtain (3.7). □

In the following section we will use this result to derive error bounds for second-order nonlinear wave-type equations.

4. Abstract space discretizations of second-order evolution equations with nonlinear damping

In this section we apply the theory of Section 3 to second-order evolution equations. As in the previous section, we first introduce the continuous problem and then present and analyze the abstract space discretization. This is a generalization of the linear unified error analysis introduced in Hipp *et al.* (2019) and also an extension of the framework considered in the dissertation Leibold (2021) which does not cover the acoustic boundary conditions with nonlinear coupling from Section 2, cf. Remark 4.2 and Section 5.

4.1 *Analytical setting*

Let V, H be Hilbert spaces es and let V be densely embedded in H. We consider the following variational equation, which is typical for a weak formulation of a second-order partial differential equation. Seek $u \in C^2([0,T];H) \cap C^1([0,T];V)$ with

$$\begin{aligned} m\big(u'', \varphi\big) + \langle \mathcal{D}(u'), \varphi \rangle_{V^* \times V} + a\big(u, \varphi\big) &= m\big(f, \varphi\big), \qquad \text{for } t \geqslant 0 \text{ and all } \varphi \in V, \\ u(0) = u^0, \qquad u'(0) &= v^0, \end{aligned} \tag{4.1}$$

To ensure the wellposedness of (4.1) we pose the following assumptions.

Assumption 4.1

a) The bilinear form $m \colon H \times H \to \mathbb{R}$ is a scalar product on H with induced norm $\|\cdot\|_m$. In the following, we equip H with m.

b) The bilinear form $a \colon V \times V \to \mathbb{R}$ is symmetric and there exists a constant $c_G \geqslant 0$ s.t.

$$\tilde{a} := a + c_G m$$

is a scalar product on V with induced norm $\|\cdot\|_{\tilde{a}}$. From now on, we equip V with $\tilde{a}$.

c) The nonlinearity $\mathcal{D} \in C(V; V^*)$ satisfies $\mathcal{D}(0) = 0$ and is quasi-monotone, i.e., there is a constant $\beta_{\mathrm{qm}} \geqslant 0$ s.t.

$$\langle \mathcal{D}(v) - \mathcal{D}(w), v - w \rangle_{V^* \times V} \geqslant -\beta_{\mathrm{qm}} \|v - w\|_m^2 \qquad \text{for all } v, w \in V.$$

d) The inhomogeneity satisfies $f \in W^{1,1}_{loc}([0,\infty); H)$.

We denote by $C_{H,V}$ the embedding constant of V into H, i.e.,

$$\|v\|_m \leqslant C_{H,V} \|v\|_{\tilde{a}} \qquad \text{for all } v \in V. \tag{4.2}$$

Formulation as evolution equation We identify H with its dual space H^* to obtain the Gelfand triple

$$V \hookrightarrow H \cong H^* \hookrightarrow V^* \tag{4.3}$$

with dense embeddings. To reformulate (4.1) as an evolution equation, we define the operator $\mathcal{A} \in \mathcal{L}(V, V^*)$ associated to a via

$$\langle \mathcal{A}v, w \rangle_{V^* \times V} := a\big(v, w\big) \qquad \text{for all } v, w \in V. \tag{4.4}$$

Then, we can rewrite (4.1) equivalently as an evolution equation in V^*: Seek $u \in C^2([0,T];H) \cap C^1([0,T];V)$ satisfying

$$\begin{aligned} u'' + \mathcal{D}(u') + \mathcal{A}u &= f, \qquad t \geqslant 0, \\ u(0) = u^0, \quad u'(0) &= v^0. \end{aligned} \tag{4.5}$$

Note that (4.5) implicitly contains the condition

$$\mathcal{D}(u') + \mathcal{A}u \in H$$

due to $u'', f \in H$.

REMARK 4.2 In Leibold (2021), the stricter assumption $\mathcal{D} \in C(V;H)$ was posed. However, this does not cover the acoustic boundary conditions with nonlinear coupling (2.1c) as we will see in Section 5, cf. Remark 5.2.

FIRST-ORDER FORMULATION We rewrite (4.5) into an first-order formulation in the framework of Section 3.1. For this let $u' = v$ and we define

$$x = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathcal{S}(x) = \begin{bmatrix} -v \\ \mathcal{A}u + \mathcal{D}(v) \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ f \end{bmatrix}, \quad x^0 = \begin{bmatrix} u^0 \\ v^0 \end{bmatrix} \tag{4.6}$$

with

$$X = V \times H, \qquad D(\mathcal{S}) = \Big\{ [u,v]^\intercal \in V \times H \mid \mathcal{A}u + \mathcal{D}(v) \in H \Big\}. \tag{4.7}$$

Then, (4.5) is equivalent to the first-order evolution equation (3.1).

In the following we show that the assumptions of Section 3.1 are satisfied. The subsequent lemma is a slight extension of (Leibold, 2021, Lemma 2.14).

LEMMA 4.3. *The nonlinear operator $\mathcal{S}$ is maximal and quasi-monotone with constant*

$$c_{\mathrm{qm}} = \frac{1}{2} c_G C_{H,V} + \beta_{\mathrm{qm}}$$

and $D(\mathcal{S})$ is dense in X.

Proof. We start by proving the quasi-monotonicity. For $x_1 = [u_1, v_1]^\intercal, x_2 = [u_2, v_2]^\intercal \in D(\mathcal{S})$ we calculate by using Assumption 4.1, (4.3), and the definitions of $\mathcal{S}$ and $\mathcal{A}$

$$\begin{aligned}
&p\big(\mathcal{S}(x_1) - \mathcal{S}(x_2), x_1 - x_2\big) \\
&= -\tilde{a}\big(v_1 - v_2, u_1 - u_2\big) + m\big(\mathcal{A}u_1 + \mathcal{D}(v_1) - \mathcal{A}u_2 - \mathcal{D}(v_2), v_1 - v_2\big) \\
&= -\tilde{a}\big(v_1 - v_2, u_1 - u_2\big) + a\big(u_1 - u_2, v_1 - v_2\big) + \langle \mathcal{D}(v_1) - \mathcal{D}(v_2), v_1 - v_2 \rangle_{V^* \times V} \\
&\geqslant -c_G m\big(v_1 - v_2, u_1 - u_2\big) - \beta_{\mathrm{qm}} \|v_1 - v_2\|_m^2 \\
&\geqslant -c_G \|v_1 - v_2\|_m \|u_1 - u_2\|_m - \beta_{\mathrm{qm}} \|v_1 - v_2\|_m^2 \\
&\geqslant -c_G C_{H,V} \|u_1 - u_2\|_{\tilde{a}} \|v_1 - v_2\|_m - \beta_{\mathrm{qm}} \|v_1 - v_2\|_m^2 \\
&\geqslant -\frac{1}{2} c_G C_{H,V} \left(\|u_1 - u_2\|_{\tilde{a}}^2 + \|v_1 - v_2\|_m^2 \right) - \beta_{\mathrm{qm}} \|v_1 - v_2\|_m^2 \\
&\geqslant -\left(\frac{1}{2} c_G C_{H,V} + \beta_{\mathrm{qm}} \right) \|x_1 - x_2\|_X^2 .
\end{aligned}$$

In the next step we prove the maximality and proceed similar as in the proof of (Vitillaro, 2017, Theorem 4.1). We have to show that there exists a $\lambda > 0$ such that for every $h = [h_1, h_2]^\intercal \in X = V \times H$ there exists a solution $x = [v, w]^\intercal \in D(\mathcal{S})$ of the stationary problem $(\lambda + \mathcal{S})x = h$ or equivalently

$$\lambda v - w = h_1, \tag{4.8a}$$
$$\lambda w + \mathcal{A}v + \mathcal{D}(w) = h_2. \tag{4.8b}$$

By solving (4.8a) for v and plugging it into (4.8b) we obtain

$$\lambda w + \frac{1}{\lambda}\mathcal{A}w + \mathcal{D}(w) = h_2 - \frac{1}{\lambda}\mathcal{A}h_1 := \widetilde{h} \in V^*. \tag{4.9}$$

We thus investigate the operator $T = \lambda + \frac{1}{\lambda}\mathcal{A} + \mathcal{D} \in C(V; V^*)$ which can be decomposed via $T = T_1 + T_2$ with

$$T_1 = \frac{1}{\lambda}\left(\frac{\lambda^2}{2} + \mathcal{A}\right), \qquad T_2 = \frac{\lambda}{2} + \mathcal{D}.$$

For

$$\lambda > \max\{c_{\mathrm{qm}}, \sqrt{2c_G}, 2\beta_{\mathrm{qm}}\}$$

we then have that T is monotone as the sum of monotone operators. Further, we have for all $v \in V$

$$\begin{aligned}\langle T(v), v\rangle_{V^*\times V} &= \langle T_1(v), v\rangle_{V^*\times V} + \langle T_2(v), v\rangle_{V^*\times V}\\ &\geqslant \frac{1}{\lambda}\|v\|_{\tilde{a}}^2 + \langle T_2(v) - T_2(0), v - 0\rangle_{V^*\times V}\\ &\geqslant \frac{1}{\lambda}\|v\|_{\tilde{a}}^2,\end{aligned}$$

where we used that T_1 is coercive due to the choice of λ, and T_2 is monotone with $T_2(0) = 0$. Thus, T is coercive, i.e.

$$\frac{\langle T(v), v\rangle_{V^*\times V}}{\|v\|_{\tilde{a}}} \to \infty \quad \text{for } \|v\|_{\tilde{a}} \to \infty.$$

We apply (Barbu, 2010, Corollary 2.3) stating that continuous, monotone, and coercive operators from a reflexive Banach space to its dual space are surjective. This yields the existence of a solution $v \in V$ of (4.9) and thus also of a solution $x = [v, w]^\intercal \in V \times H$ of (4.8). We further obtain by (4.8b) $x \in D(\mathcal{S})$ since

$$\mathcal{A}v + \mathcal{D}(w) = h_2 - \lambda w \in H.$$

The density of $D(\mathcal{S})$ in X follows from the maximality and the quasi-monotonicity of $\mathcal{S}$ and $\mathcal{S}(0) = 0$, cf (Showalter, 1997, Prop. I.4.2). □

COROLLARY 4.4. *Assumption 4.1 implies that the first-order formulation of* (4.5) *satisfies Assumption 3.1.*

Proof. By Lemma 4.3 we have that Assumption 3.1 a) is satisfied. Assumption 3.1 b) is directly implied by Assumption 4.1 d). □

By Theorem 3.2 we then directly obtain the wellposedness of (4.1).

COROLLARY 4.5. *Let* (4.1) *hold true and let* $\left[u^0, v^0\right]^\intercal \in D(\mathcal{S})$*. Then,* (4.1) *is globally wellposed, i.e., there exists a unique strong solution* $[u, v]^\intercal \in C([0, \infty); X)$*.*

4.2 *Space discretization*

We consider a family $(V_h)_h$ of finite dimensional vector spaces related to a discretization parameter h and the following discretized version of (4.1) in $V_h \in (V_h)_h$. Seek $u_h \in C^2([0,T];V_h)$ with

$$\begin{aligned} m_h\big(u_h'',\varphi_h\big)+m_h\big(\mathcal{D}_h(u_h'),\varphi_h\big)+a_h\big(u_h,\varphi_h\big)&=m_h\big(f_h,\varphi_h\big), \quad \text{for all } \varphi_h \in V_h, t \geqslant 0, \\ u_h(0)=u_h^0, \qquad u_h'(0)&=v_h^0. \end{aligned} \tag{4.10}$$

Here, $m_h, a_h, \mathcal{D}_h$, and f_h are approximations of the corresponding continuous counterparts.

We pose the following assumptions similar to Assumption 4.1.

ASSUMPTION 4.6 All constants in the following statements are independent of h.

a) The bilinear form m_h is a scalar product on V_h. We denote V_h equipped with this scalar product m_h by H_h and the induced norm by $\|\cdot\|_{m_h}$.

b) The bilinear form $a_h \colon V_h \times V_h \to \mathbb{R}$ is symmetric and there exists a constant $\widehat{c}_G \geqslant 0$ s.t.

$$\tilde{a}_h \coloneqq a_h + \widehat{c}_G m_h$$

is a scalar product on V_h with induced norm $\|\cdot\|_{\tilde{a}_h}$. In the following, we equip V_h with $\tilde{a}_h$.

c) The nonlinearity $\mathcal{D}_h \in C(V_h;H_h)$ satisfies $\mathcal{D}(0)=0$ and is continuous and quasi-monotone with constant $\widehat{\beta}_{\mathrm{qm}}$.

d) The inhomogeneity satisfies $f_h \in W^{1,1}_{loc}([0,\infty);H_h)$.

e) There exists a constant $\widehat{C}_{H,V} > 0$ s.t.

$$\|v_h\|_{m_h} \leqslant \widehat{C}_{H,V}\|v_h\|_{\tilde{a}_h} \qquad \text{for all } v_h \in V_h. \tag{4.11}$$

The operator $\mathcal{A}_h, \in \mathcal{L}(V_h;V_h)$ related to a_h is defined via

$$m_h\big(\mathcal{A}_h v_h, w_h\big) \coloneqq a_h\big(v_h, w_h\big) \quad \text{for all } v_h, w_h \in V_h.$$

We then can reformulate (4.10) as an evolution equation in V_h:

$$\begin{aligned} u_h''+\mathcal{D}_h(u_h')+\mathcal{A}_h u_h &= f_h, \qquad t \geqslant 0, \\ u_h(0)=u_h^0, \qquad u_h'(0)&=v_h^0. \end{aligned} \tag{4.12}$$

Analogously to the continuous equation we rewrite (4.12) in a first-order formulation and therefore define $X_h = V_h \times H_h$. With

$$x_h = \begin{bmatrix} u_h \\ v_h \end{bmatrix}, \quad \mathcal{S}_h(x_h) = \begin{bmatrix} -v_h \\ \mathcal{A}_h u_h + \mathcal{D}_h(v_h) \end{bmatrix}, \quad g_h = \begin{bmatrix} 0 \\ f_h \end{bmatrix}, \quad x_h^0 = \begin{bmatrix} u_h^0 \\ v_h^0 \end{bmatrix}, \tag{4.13}$$

(4.12) is then of the form (3.3).

COROLLARY 4.7. *Assumption 4.6 implies that the first-order formulation of* (4.12) *satisfies Assumption 3.4. Furthermore,* (3.2) *holds true with* $\widehat{c}_{\mathrm{qm}} = \frac{1}{2}\widehat{c}_G \widehat{C}_{H,V} + \widehat{\beta}_{\mathrm{qm}}$.

Proof. Since the setting in the discrete case from Assumption 4.6 is similar to the continuous one from Assumption 4.1 with constants independent of h, the proof of Lemma 4.3 transfers directly to the discrete case. □

Similar to the first-order case, we require the existence of suitable operators to relate continuous and discrete functions of the abstract non-conforming space discretization.

ASSUMPTION 4.8

a) There exists a lift operator $\mathcal{L}_h^V \in \mathcal{L}(V_h; V)$ satisfying

$$\|\mathcal{L}_h^V v_h\|_m \leqslant \widehat{C}_H \|v_h\|_{m_h}, \qquad \|\mathcal{L}_h^V v_h\|_{\tilde{a}} \leqslant \widehat{C}_V \|v_h\|_{\tilde{a}_h}, \tag{4.14}$$

for all $v_h \in V_h$ with constants $\widehat{C}_H, \widehat{C}_V > 0$ independent of h.

b) There exists an interpolation operator $I_h \in \mathcal{L}(Z^V; V_h)$, defined on a dense subspace Z^V of V, which satisfies

$$\|I_h\|_{H_h \leftarrow Z^V} \leqslant \widehat{C}_{I_h} \tag{4.15}$$

with a constant $\widehat{C}_{I_h} > 0$ independent of h.

To apply the results of Section 3.2, we now define the first-order reference and lift operator.

DEFINITION 4.9

a) The *adjoint lift operators* $\mathcal{L}_h^{V*} \colon V \to V_h$ and $\mathcal{L}_h^{H*} \colon H \to H_h$ w.r.t. the scalar products of V and H are defined via

$$\begin{aligned} m_h\big(\mathcal{L}_h^{H*} v, w_h\big) &:= m\big(v, \mathcal{L}_h^V w_h\big) \quad \text{for all } v \in H, w_h \in H_h, \\ \tilde{a}_h\big(\mathcal{L}_h^{V*} v, w_h\big) &:= \tilde{a}\big(v, \mathcal{L}_h^V w_h\big) \quad \text{for all } v \in V, w_h \in V_h. \end{aligned} \tag{4.16}$$

b) We define the *first-order lift operator* $\mathcal{L}_h \colon X_h \to X$ by

$$\mathcal{L}_h \begin{bmatrix} v_h \\ w_h \end{bmatrix} := \begin{bmatrix} \mathcal{L}_h^V v_h \\ \mathcal{L}_h^V w_h \end{bmatrix}.$$

c) We define the *first-order reference operator* $J_h \colon Z \to X_h$ by

$$J_h \begin{bmatrix} v \\ w \end{bmatrix} := \begin{bmatrix} \mathcal{L}_h^{V*} v \\ I_h w \end{bmatrix} \tag{4.17}$$

on $Z = V \times Z^V \overset{d}{\hookrightarrow} X$.

LEMMA 4.10. *The first-order lift and reference operators from Definition 4.9 satisfy Assumption 3.5 with* $\widehat{C}_X = \max\{\widehat{C}_V, \widehat{C}_H\}$ *and* $\widehat{C}_{J_h} = \max\{\widehat{C}_V, \widehat{C}_{I_h}\}$.

Proof. This is a direct consequence of Assumption 4.8. □

In the following we now bound the first-order remainder term which is for $z = [v,w]^\intercal \in Z$ given by

$$R_h(z) = \mathcal{L}_h^* \mathcal{S}(z) - \mathcal{S}_h J_h(z) = \begin{bmatrix} -(\mathcal{L}_h^{V*} - I_h)w \\ \mathcal{L}_h^{H*}(\mathcal{A}v + \mathcal{D}(w)) - \big(\mathcal{A}_h \mathcal{L}_h^{V*} v + \mathcal{D}_h(I_h w)\big) \end{bmatrix}. \tag{4.18}$$

To do so, we use the following error terms in the scalar products, which are for $v_h, w_h \in V_h$ defined via

$$\begin{aligned} \Delta m(v_h, w_h) &:= m(\mathcal{L}_h^V v_h, \mathcal{L}_h^V w_h) - m_h(v_h, w_h), \\ \Delta \tilde{a}(v_h, w_h) &:= \tilde{a}(\mathcal{L}_h^V v_h, \mathcal{L}_h^V w_h) - \tilde{a}_h(v_h, w_h). \end{aligned} \tag{4.19}$$

We obtain the following bound for the remainder term, cf. (Leibold, 2021, Lem. 2.23)

LEMMA 4.11. *Let Assumption 4.1 and 4.6 be satisfied. Then, for $z = [v,w]^\intercal \in D(\mathcal{S}) \cap Z$, the remainder of the monotone operator can be bounded by*

$$\begin{aligned} \|R_h(z)\|_{X_h} \leqslant C\Big(&\max_{\|\varphi_h\|_{\tilde{a}_h}=1} \big|\Delta\tilde{a}(I_h w, \varphi_h)\big| + \max_{\|\varphi_h\|_{\tilde{a}_h}=1} \big|\Delta\tilde{a}(I_h v, \varphi_h)\big| \\ &+ \max_{\|\psi_h\|_{m_h}=1} \big|\Delta m(I_h v, \psi_h)\big| + \|(\mathrm{I} - \mathcal{L}_h^V I_h)v\|_{\tilde{a}} + \|\Big(\mathrm{I} - \mathcal{L}_h^V I_h\Big) w\|_{\tilde{a}} \\ &+ \max_{\|\psi_h\|_{m_h}=1} \big|\langle \mathcal{D}(w), \mathcal{L}_h^V \psi_h\rangle_{V^* \times V} - m_h\big(\mathcal{D}_h(I_h w), \psi_h\big)\big|\Big), \end{aligned} \tag{4.20}$$

i.e., against errors in the scalar products, interpolation errors, and the discretization error of the nonlinear operator.

Proof. The proof works similar to the proof of (Hipp *et al.*, 2019, Lemma 4.7) and relies on the identity

$$\|R_h(z)\|_{X_h} = \max_{\|y_h\|_{X_h}=1} p_h\big(R_h(z), y_h\big).$$

Thus, let $y_h = [\varphi_h, \psi_h]^\intercal \in X_h$ with $\|y_h\|_{X_h} = 1$. By (4.18) we obtain

$$\begin{aligned} &p_h\big(R_h(z), y_h\big) \\ &= -\tilde{a}_h\big((\mathcal{L}_h^{V*} - I_h)w, \varphi_h\big) + m_h\big(\mathcal{L}_h^{H*}(Av + \mathcal{D}(w)) - \big(\mathcal{A}_h \mathcal{L}_h^{V*} v + \mathcal{D}_h(I_h w)\big), \psi_h\big) \\ &= -\Big(\tilde{a}\big(w, \mathcal{L}_h^V \varphi_h\big) - \tilde{a}_h\big(I_h w, \varphi_h\big)\Big) + \Big(a\big(v, \mathcal{L}_h^V \psi_h\big) - a_h\big(\mathcal{L}_h^{V*} v, \psi_h\big)\Big) \\ &\quad + \langle \mathcal{D}(w), \mathcal{L}_h^V \psi_h\rangle_{V^* \times V} - m_h\big(\mathcal{D}_h(I_h w), \psi_h\big), \end{aligned} \tag{4.21}$$

and we bound the first two summands separately. To bound the first one, we use (4.19), (4.14), and $\|\varphi_h\|_{\tilde{a}_h} \leqslant 1$ to obtain

$$\begin{aligned} \tilde{a}\big(w, \mathcal{L}_h^V \varphi_h\big) - \tilde{a}_h\big(I_h w, \varphi_h\big) &= \tilde{a}\big(w, \mathcal{L}_h^V \varphi_h\big) - \tilde{a}\big(\mathcal{L}_h^V I_h w, \mathcal{L}_h^V \varphi_h\big) + \Delta\tilde{a}\big(I_h w, \varphi_h\big) \\ &\leqslant \|\Big(\mathrm{I} - \mathcal{L}_h^V I_h\Big) w\|_{\tilde{a}} \|\mathcal{L}_h^V \varphi_h\|_{\tilde{a}} + \big|\Delta\tilde{a}\big(I_h w, \varphi_h\big)\big| \\ &\leqslant \widehat{C}_V \|\Big(\mathrm{I} - \mathcal{L}_h^V I_h\Big) w\|_{\tilde{a}} + \max_{\|\varphi_h\|_{\tilde{a}_h}=1} \big|\Delta\tilde{a}\big(I_h w, \varphi_h\big)\big|. \end{aligned} \tag{4.22}$$

By using the definitions of $\tilde{a}, \tilde{a}_h$, $\|\psi_h\|_{m_h} \leqslant 1$ and (4.19), (4.2), (4.14), (4.11), we bound the second summand in (4.21) via

$$\begin{aligned}
&a\big(v, \mathcal{L}_h^V \psi_h\big) - a_h\big(\mathcal{L}_h^{V*} v, \psi_h\big) \\
&= \tilde{a}\big(v, \mathcal{L}_h^V \psi_h\big) - \tilde{a}_h\big(\mathcal{L}_h^{V*} v, \psi_h\big) - \Big(c_G m\big(v, \mathcal{L}_h^V \psi_h\big) - \widehat{c}_G m_h\big(\mathcal{L}_h^{V*} v, \psi_h\big)\Big) \\
&\leqslant \max\{c_G, \widehat{c}_G\} \big| m\big(v, \mathcal{L}_h^V \psi_h\big) - m_h\big(\mathcal{L}_h^{V*} v, \psi_h\big) \big| \\
&\leqslant \max\{c_G, \widehat{c}_G\} \Big(\big| m\big((\mathrm{I} - \mathcal{L}_h^V I_h) v, \mathcal{L}_h^V \psi_h\big) \big| + \big| \Delta m\big(I_h v, \psi_h\big) \big| \\
&\qquad\qquad + m_h\big((I_h - \mathcal{L}_h^{V*}) v, \psi_h\big) \Big) \\
&\leqslant \max\{c_G, \widehat{c}_G\} \Big(\widehat{C}_H C_{H,V} \|(\mathrm{I} - \mathcal{L}_h^V I_h) v\|_{\tilde{a}} + \max_{\|\psi_h\|_{m_h} = 1} \big| \Delta m\big(I_h v, \psi_h\big) \big| \\
&\qquad\qquad + \widehat{C}_{H,V} \|(I_h - \mathcal{L}_h^{V*}) v\|_{\tilde{a}_h} \Big).
\end{aligned}$$

Similar to (4.22), we further estimate

$$\begin{aligned}
\|(I_h - \mathcal{L}_h^{V*}) v\|_{\tilde{a}_h} &= \max_{\|\varphi_h\|_{\tilde{a}_h} = 1} \tilde{a}_h\big((I_h - \mathcal{L}_h^{V*}) v, \varphi_h\big) \\
&= \max_{\|\varphi_h\|_{\tilde{a}_h} = 1} \tilde{a}_h\big(I_h v, \varphi_h\big) - \tilde{a}\big(v, \mathcal{L}_h^V \varphi_h\big) \\
&\leqslant \widehat{C}_V \Big\| \Big(\mathrm{I} - \mathcal{L}_h^V I_h\Big) v \Big\|_{\tilde{a}} + \max_{\|\varphi_h\|_{\tilde{a}_h} = 1} \big| \Delta \tilde{a}\big(I_h v, \varphi_h\big) \big|.
\end{aligned}$$

We finally obtain the assertion by collecting all terms. □

We are now in the position to prove the following error bound for the abstract second-order equations, cf. (Leibold, 2021, Thm. 2.24).

THEOREM 4.12. *Let Assumptions 4.1, 4.6, and 4.8 be satisfied and u be the strong solution of (4.5) on $[0,T]$ with $u, u', u'' \in L^\infty([0,T]; Z^V)$. Further, let u_h be the semidiscrete solution of (4.12) on $[0,T]$. Then, for all $t \in [0,T]$, the lifted semidiscrete solution satisfies the error bound*

$$\|\mathcal{L}_h^V u_h(t) - u(t)\|_{\tilde{a}} + \|\mathcal{L}_h^V u_h'(t) - u'(t)\|_m \leqslant C \mathrm{e}^{\widehat{c}_{\mathrm{qm}} t} (1 + t) \sum_{i=1}^{4} E_{h,i} \tag{4.23}$$

with a constant C that is independent of h and t. The other constants are given by

$$\widehat{c}_{\mathrm{qm}} = \frac{1}{2} \widehat{c}_G \widehat{C}_{H,V} + \widehat{\beta}_{\mathrm{qm}},$$

and the abstract space discretization errors

$$
\begin{aligned}
E_{h,1} =& \|u_h^0 - \mathcal{L}_h^{V*}u^0\|_{\tilde{a}_h} + \|v_h^0 - I_h v^0\|_{m_h} + \|\mathcal{L}_h^{H*}f - f_h\|_{L^\infty([0,T];H)}, \\
E_{h,2} =& \Big\| \max_{\|\psi_h\|_{m_h}=1} |\langle \mathcal{D}(u'), \mathcal{L}_h^V \psi_h\rangle_{V^*\times V} - m_h\big(\mathcal{D}_h(I_h u'), \psi_h\big)| \Big\|_{L^\infty(0,T)}, \\
E_{h,3} =& \|(\mathrm{I} - \mathcal{L}_h^V I_h)u\|_{L^\infty([0,T];V)} + \|(\mathrm{I} - \mathcal{L}_h^V I_h)u'\|_{L^\infty([0,T];V)} \\
&+ \|(\mathrm{I} - \mathcal{L}_h^V I_h)u''\|_{L^\infty([0,T];H)}, \\
E_{h,4} =& \Big\| \max_{\|\varphi_h\|_{\tilde{a}_h}=1} \Delta\tilde{a}\big(I_h u, \varphi_h\big) \Big\|_{L^\infty(0,T)} + \Big\| \max_{\|\psi_h\|_{m_h}=1} \Delta m\big(I_h u, \psi_h\big) \Big\|_{L^\infty(0,T)} \\
&+ \Big\| \max_{\|\varphi_h\|_{\tilde{a}_h}=1} \Delta\tilde{a}\big(I_h u', \varphi_h\big) \Big\|_{L^\infty(0,T)} + \Big\| \max_{\|\psi_h\|_{m_h}=1} \Delta m\big(I_h u'', \psi_h\big) \Big\|_{L^\infty(0,T)}.
\end{aligned}
\tag{4.24}
$$

Proof. By Corollaries 4.5, 4.7, and Lemma 4.10, we have that the first-order formulations of (4.5) and (4.12) satisfy all assumptions of Theorem 3.7.

By applying Theorem 3.7 and employing the error bound (3.7), we obtain

$$
\begin{aligned}
&\|\mathcal{L}_h^V u_h(t) - u(t)\|_{\tilde{a}} + \|\mathcal{L}_h^V u_h'(t) - u'(t)\|_m \\
&\leqslant 2\left(\|\mathcal{L}_h^V u_h(t) - u(t)\|_{\tilde{a}}^2 + \|\mathcal{L}_h^V u_h'(t) - u'(t)\|_m^2 \right)^{\frac{1}{2}} = 2\|\mathcal{L}_h x_h(t) - x(t)\|_X \\
&\leqslant 2\widehat{C}_X \mathrm{e}^{\left(\widehat{L}_{T,M_h} + \widehat{c}_{\mathrm{qm}}\right)t} E_h(t) + 2\|(\mathrm{I} - \mathcal{L}_h J_h)x(t)\|_X
\end{aligned}
$$

with

$$
\begin{aligned}
E_h(t) = & \big\|x_h^0 - J_h x^0\big\|_{X_h} + t\big\|(\mathcal{L}_h^* - J_h)x'\big\|_{L^\infty([0,T];X_h)} \\
&+ t\|R_h(x)\|_{L^\infty([0,T];X_h)} + t\|\mathcal{L}_h^* g - g_h\|_{L^\infty([0,T];X_h)}.
\end{aligned}
$$

In the remaining proof, we bound the different terms against $E_{h,i}, i = 1, \dots, 4$. For the remainder term we apply the bound (4.20) and obtain for all $t \in [0,T]$

$$
\|R_h(x(t))\|_{X_h} \leqslant C(E_{h,2} + E_{h,3} + E_{h,4}),
$$

By the definitions of J_h and $\mathcal{L}_h^*$ we further have for the discretization errors of the initial values and the inhomogeneity

$$
\big\|x_h^0 - J_h x^0\big\|_{X_h} + \|\mathcal{L}_h^* g - g_h\|_{L^\infty([0,T];X_h)} \leqslant C E_{h,1}.
$$

The reference error can be decomposed for all $t \in [0,T]$ via

$$
\begin{aligned}
\|(\mathrm{I} - \mathcal{L}_h J_h)x(t)\|_X &\leqslant \|(\mathrm{I} - \mathcal{L}_h^V \mathcal{L}_h^{V*})u(t)\|_{\tilde{a}} + \|(\mathrm{I} - \mathcal{L}_h^V I_h)u'(t)\|_m \\
&\leqslant \|(\mathrm{I} - \mathcal{L}_h^V \mathcal{L}_h^{V*})u(t)\|_{\tilde{a}} + E_{h,3},
\end{aligned}
$$

where we have similar to (4.22)

$$
\begin{aligned}
\|(\mathrm{I}-\mathcal{L}_h^V\mathcal{L}_h^{V*})u\|_{\tilde a} &\leqslant \|(\mathrm{I}-\mathcal{L}_h^V I_h)u\|_{\tilde a}+\|\mathcal{L}_h^V(I_h-\mathcal{L}_h^{V*})u\|_{\tilde a}\\
&\leqslant CE_{h,3}+\widehat{C}_V\max_{\|\varphi_h\|_{\tilde a_h}=1}\Big(\tilde a_h\big(I_hu,\varphi_h\big)-\tilde a\big(u,\mathcal{L}_h^V\varphi_h\big)\Big)\\
&\leqslant CE_{h,3}+\widehat{C}_V^2\|\Big(\mathrm{I}-\mathcal{L}_h^V I_h\Big)u\|_{\tilde a}+\widehat{C}_V\max_{\|\varphi_h\|_{\tilde a_h}=1}\big|\Delta\tilde a\big(I_hu,\varphi_h\big)\big|\\
&\leqslant C(E_{h,3}+E_{h,4}).
\end{aligned}
$$

In the same way, we finally bound

$$
\begin{aligned}
\|(\mathcal{L}_h^*-J_h)x'\|_{X_h} &\leqslant \|(\mathcal{L}_h^{H*}-I_h)u''\|_{m_h}\\
&\leqslant \widehat{C}_H\|\Big(\mathrm{I}-\mathcal{L}_h^V I_h\Big)u''\|_m+\max_{\|\psi_h\|_{m_h}=1}\big|\Delta m\big(I_hu'',\psi_h\big)\big|\\
&\leqslant C(E_{h,3}+E_{h,4}).
\end{aligned}
$$

□

Having this abstract theory at hand, we can now return to the wave equation with nonlinear acoustic boundary conditions from Section 2 and give the proof of Theorem 2.6 in the next section.

5. Numerical analysis of wave equations with nonlinear acoustic boundary conditions

In this section we will use the unified error analysis for second-order equations from Section 4 to prove the error bound from Theorem 2.6. We start by verifying that all assumptions are satisfied.

LEMMA 5.1. *Let Assumption 2.2 be satisfied. Then, with the definitions in* (2.5)*, Assumption 4.1 is satisfied with* $\beta_{\mathrm{qm}}=\frac{1}{\mu}\Big(\frac{L_\eta^2}{4c_\Omega\theta_0}-d\Big), c_G=1,$ *and* $C_{H,V}=1$.

Proof. We clearly have that m is a scalar product on H and that $\tilde a \coloneqq a+m$ is a scalar product on V. Further, Assumption 2.2 d) implies directly Assumption 4.1 d).

Thus it remains to prove Assumption 4.1 c). By Assumption 2.2 a), b), c) and (2.5a), (2.5d) we obtain

$$
\begin{aligned}
&\langle \mathcal{D}(\vec{v}_1) - \mathcal{D}(\vec{v}_2), \vec{v}_1 - \vec{v}_2 \rangle_{V^* \times V} \\
&= \int_\Gamma c_\Omega \left(\theta(v_1) - \theta(v_2) - (\eta(z_1) - \eta(z_2))\right)(v_1 - v_2) \\
&\quad + (d(z_1 - z_2) + \rho(v_1 - v_2))(z_1 - z_2)\,\mathrm{d}s \\
&\geqslant \int_\Gamma c_\Omega \theta_0 |v_1 - v_2|^2 + d|z_1 - z_2|^2 + c_\Omega \left(\frac{\rho}{c_\Omega}(z_1 - z_2) - (\eta(z_1) - \eta(z_2))\right)(v_1 - v_2)\,\mathrm{d}s \\
&\geqslant \int_\Gamma c_\Omega \theta_0 |v_1 - v_2|^2 + d|z_1 - z_2|^2 - L_\eta |(z_1 - z_2)(v_1 - v_2)|\,\mathrm{d}s \\
&\geqslant \int_\Gamma c_\Omega \theta_0 |v_1 - v_2|^2 + d|z_1 - z_2|^2 - c_\Omega \theta_0 |v_1 - v_2|^2 - \frac{L_\eta^2}{4 c_\Omega \theta_0} |z_1 - z_2|^2\,\mathrm{d}s \\
&\geqslant \left(d - \frac{L_\eta^2}{4 c_\Omega \theta_0}\right) \|z_1 - z_2\|_{L^2(\Gamma)}^2 \\
&\geqslant -\beta_{\mathrm{qm}} \|\vec{v}_1 - \vec{v}_2\|_m^2.
\end{aligned}
$$

This proves the the quasi-monotonicity of $\mathcal{D}$.

In the next step we show $\mathcal{D} \in C(V; V^*)$. We emphasize that the trace inequality

$$
v \mapsto v\big|_\Gamma \in C(H^1(\Omega); L^q(\Gamma)) \tag{5.1}
$$

holds true for $q = \zeta + 1$ with ζ from the growth condition (2.3), cf. (Adams & Fournier, 2003, Thm. 5.36). For $\vec{v}_1 = [v_1, z_1]^\intercal, \vec{v}_2 = [v_2, z_2]^\intercal, \vec{\varphi} = [\varphi, \psi]^\intercal \in V$ with $\|\vec{\varphi}\|_{\tilde{a}} = 1$ this yields together with the Hölder and the Minkowski inequalities and the global Lipschitz continuity of η

$$
\begin{aligned}
&|\langle \mathcal{D}(\vec{v}_2) - \mathcal{D}(\vec{v}_2), \vec{\varphi} \rangle_{V^* \times V}| \\
&= \Big| \int_\Gamma c_\Omega \left(\theta(v_1) - \theta(v_2) - \big(\eta(z_1) - \eta(z_2)\big)\right)\varphi + (d(z_1 - z_2) + \rho(v_1 - v_2))\psi\,\mathrm{d}s \Big| \\
&\leqslant c_\Omega \Big(\|\theta(v_1) - \theta(v_2)\|_{L^{\frac{q}{q-1}}(\Gamma)} + \|\eta(z_1) - \eta(z_2)\|_{L^{\frac{q}{q-1}}(\Gamma)} \Big) \|\varphi\|_{L^q(\Gamma)} \\
&\quad + \Big(d\|z_1 - z_2\|_{L^2(\Gamma)} + \rho\|v_1 - v_2\|_{L^2(\Gamma)} \Big) \|\psi\|_{L^2(\Gamma)} \\
&\leqslant c_\Omega \Big(\|\theta(v_1) - \theta(v_2)\|_{L^{\frac{q}{q-1}}(\Gamma)} + C\|z_1 - z_2\|_{L^2(\Gamma)} \Big) \max\{1, \frac{1}{c_\Omega}\} \|\vec{\varphi}\|_{\tilde{a}} \\
&\quad + \Big(d\|z_1 - z_2\|_{L^2(\Gamma)} + \rho C\|v_1 - v_2\|_{H^1(\Omega)} \Big) \frac{1}{\mu} \|\vec{\varphi}\|_{\tilde{a}} \\
&\leqslant \max\{c_\Omega, 1\} \Big(\|\theta(v_1) - \theta(v_2)\|_{L^{\frac{q}{q-1}}(\Gamma)} + C\|z_1 - z_2\|_{L^2(\Gamma)} \Big) \\
&\quad + \frac{1}{\mu} \Big(d\|z_1 - z_2\|_{L^2(\Gamma)} + \rho C\|v_1 - v_2\|_{H^1(\Omega)} \Big).
\end{aligned} \tag{5.2}
$$

We hence obtain

$$\begin{aligned}
&\|\mathcal{D}(\vec{v}_2)-\mathcal{D}(\vec{v}_2)\|_{V^*}\\
&=\sup_{\|\vec{\varphi}\|_{\tilde{a}}=1}|\langle\mathcal{D}(\vec{v}_2)-\mathcal{D}(\vec{v}_2),\vec{\varphi}\rangle_{V^*\times V}|\\
&\leqslant C\left(\|\theta(v_1)-\theta(v_2)\|_{L^{\frac{q}{q-1}}(\Gamma)}+\|z_1-z_2\|_{L^2(\Gamma)}+\|v_1-v_2\|_{H^1(\Omega)}\right).
\end{aligned}$$

By the trace inequality (5.1), the growth condition (2.3), the relation $\zeta=q-1$, and (Goldberg *et al.*, 1992, Theorem 4.2) we further have $v\mapsto\theta(v)\in C(H^1(\Omega);L^{\frac{q}{q-1}}(\Gamma))$. This yields

$$\|\mathcal{D}(\vec{v}_2)-\mathcal{D}(\vec{v}_2)\|_{V^*}\to 0 \quad \text{for } \|\vec{v}_1-\vec{v}_2\|_{\tilde{a}}\to 0$$

which proves $\mathcal{D}\in C(V,V^*)$. □

REMARK 5.2 It is not possible to prove the stronger condition $\mathcal{D}\in C(V,H)$. This is due to the fact that the calculation (5.2) strongly relies on $\varphi\in H^1(\Omega)$ and is not possible for a test function $\varphi\in L^2(\Omega)$.

Lemma 5.1 ensures that the weak formulation (2.4) of (2.1) fits in the setting of Section 4.1 and, hence, is locally wellposed by Corollary 4.5.

We now prove, that the bulk-surface finite element space discretization from Section 2.2 fits into the abstract setting of Section 4.2.

LEMMA 5.3. *Let Assumption 2.2 hold true. Then, the bulk-surface finite element space discretization of* (2.1) *satisfies Assumption 4.6 with* $\widehat{\beta}_{\mathrm{qm}}=\frac{1}{\mu}\left(\frac{L_\eta^2}{4c_\Omega\theta_0}-d\right)$, $\widehat{c}_G=1$, *and* $\widehat{C}_{H,V}=1$.

Proof. Since a_h and m_h are defined as in continuous case, Assumption 4.6 a) and b) are satisfied. Assumption 4.6 d) follows from Assumption 2.2 d) and the continuity of the interpolation operator.

It remains to prove Assumption 4.6 c).

$$m_h\big(\mathcal{D}(\vec{v}_h),\vec{\varphi}_h\big)=\sum_{\Gamma_h}c_\Omega\left(\theta(v_h)-\eta(z_h)\right)\varphi_h+\left(dz_h+\rho v_h\right)\psi_h\,\mathrm{d}s.$$

To prove the quasi-monotonicity, we proceed analogously to the proof in the continuous case from Lemma 5.1 and obtain

$$\begin{aligned}
m_h\big(\mathcal{D}(\vec{v}_h^1)-\mathcal{D}(\vec{v}_h^2),\vec{v}_h^1-\vec{v}_h^2\big)&=\sum_{\Gamma_h}c_\Omega\left(\theta(v_h^1)-\theta(v_h^2)-(\eta(z_h^1)-\eta(z_h^2))\right)(v_h^1-v_h^2)\\
&\quad+\left(d(z_h^1-z_h^2)+\rho(v_h^1-v_h^2)\right)(z_h^1-z_h^2)\,\mathrm{d}s\\
&\geqslant\left(d-\frac{L_\eta^2}{4c_\Omega\theta_0}\right)\sum_{\Gamma_h}|z_h^1-z_h^2|^2\,\mathrm{d}s\\
&=\left(d-\frac{L_\eta^2}{4c_\Omega\theta_0}\right)\|z_h^1-z_h^2\|_{L^2(\Gamma_h)}^2\\
&\geqslant\widehat{\beta}_{\mathrm{qm}}\|\vec{v}_h^1-\vec{v}_h^2\|_{m_h},
\end{aligned}$$

where we used that the quadrature formula has positive weights and satisfies (2.7).

Finally, $\mathcal{D}_h$ is continuous, since V_h is a finite dimensional space and, thus, convergence in V_h implies uniform pointwise convergence and especially convergence in all quadrature nodes. □

To prove an error bound for the semidiscretization, we apply the theory of Section 4.2 and therefore have to specify the operators from Assumption 4.8.

Definition 5.4

a) The lift operator $\mathcal{L}_h^V \in \mathcal{L}(V_h;V)$ is defined via

$$\mathcal{L}_h^V [v_h, z_h]^\intercal := \left[v_h^\ell, z_h^\ell\right]^\intercal \qquad \text{for all } [v_h, z_h]^\intercal \in V_h$$

with v_h^ℓ from (2.6).

b) We set $Z^V := H^2(\Omega) \times H^2(\Gamma) \subset C(\overline{\Omega}) \times C(\Gamma)$.

c) We define the interpolation operator via $I_h [v_h, z_h]^\intercal := \left[I_{h,\Omega} v_h, I_{h,\Gamma} z_h\right]^\intercal$.

Our error analysis relies on the following properties of the lift and the interpolation operators. First of all, there exist element-wise norm equivalences related to the lift, which were shown in (Elliott & Ranner, 2020, Lemmas 5.3 and 7.3).

Lemma 5.5. *There exists $C_{\Omega,\Omega_h} > c_{\Omega,\Omega_h} > 0$, $C_{\Gamma,\Gamma_h} > c_{\Gamma,\Gamma_h} > 0$ independent of h s.t. for all $v_h \in V_{h,p}^\Omega$, $\vartheta_h \in V_{h,p}^\Gamma$, $k = 0,1,\dots,p+1$, and $K_\Omega \in \mathcal{T}_h$, $K_\Gamma \in \mathcal{T}_h^\Gamma$ we have*

$$\begin{aligned} c_{\Omega,\Omega_h} \|v_h\|_{H^k(K_\Omega)} &\leqslant \|v_h^\ell\|_{H^k(K_\Omega^\ell)} \leqslant C_{\Omega,\Omega_h} \|v_h\|_{H^k(K_\Omega)}, \\ c_{\Gamma,\Gamma_h} \|\vartheta_h\|_{H^k(K_\Gamma)} &\leqslant \|\vartheta_h^\ell\|_{H^k(K_\Gamma^\ell)} \leqslant C_{\Gamma,\Gamma_h} \|\vartheta_h\|_{H^k(K_\Gamma)}, \end{aligned} \tag{5.3}$$

where $K_\Omega^\ell = G_h(K_\Omega), K_\Gamma^\ell = G_h(K_\Gamma)$. By construction, the lift additionally preserves the L^∞ norm, i.e.,

$$\begin{aligned} \|v_h^\ell\|_{L^\infty(K_\Omega^\ell)} &= \|v_h\|_{L^\infty(K_\Omega)}, \\ \|\vartheta_h^\ell\|_{L^\infty(K_\Gamma^\ell)} &= \|\vartheta_h\|_{L^\infty(K_\Gamma)}. \end{aligned}$$

Further, we have the following bounds of the geometric errors stemming from the domain approximation (cf. (Elliott & Ranner, 2013, proof of Lemma 6.2)).

Lemma 5.6. *For $u_h, \varphi_h \in V_{h,p}^\Omega$ and $\vartheta_h, \psi_h \in V_{h,p}^\Gamma$, the following bounds hold true:*

$$\left| \int_\Omega u_h^\ell \varphi_h^\ell \,\mathrm{d}\mathbf{x} - \int_{\Omega_h} u_h \varphi_h \,\mathrm{d}\mathbf{x} \right| \leqslant Ch^p \|u_h\|_{L^2(\Omega_h)} \|\varphi_h\|_{L^2(\Omega_h)}, \tag{5.4a}$$

$$\left| \int_\Omega \nabla u_h^\ell \nabla \varphi_h^\ell \,\mathrm{d}\mathbf{x} - \int_{\Omega_h} \nabla u_h \nabla \varphi_h \,\mathrm{d}\mathbf{x} \right| \leqslant Ch^p \|\nabla u_h\|_{L^2(\Omega_h)} \|\nabla \varphi_h\|_{L^2(\Omega_h)}, \tag{5.4b}$$

$$\left| \int_\Gamma \vartheta_h^\ell \psi_h^\ell \,\mathrm{d}s - \int_{\Gamma_h} \vartheta_h \psi_h \,\mathrm{d}s \right| \leqslant Ch^{p+1} \|\vartheta_h\|_{L^2(\Gamma_h)} \|\psi_h\|_{L^2(\Gamma_h)}, \tag{5.4c}$$

$$\left| \int_\Gamma \nabla_\Gamma \vartheta_h^\ell \nabla_\Gamma \psi_h^\ell \,\mathrm{d}s - \int_{\Gamma_h} \nabla_{\Gamma_h} \vartheta_h \nabla_{\Gamma_h} \psi_h \,\mathrm{d}s \right| \leqslant Ch^{p+1} \|\nabla_{\Gamma_h} \vartheta_h\|_{L^2(\Gamma_h)} \|\nabla_{\Gamma_h} \psi_h\|_{L^2(\Gamma_h)}. \tag{5.4d}$$

The nodal interpolation satisfy the following error bounds, which follow from (Elliott & Ranner, 2020, Theorem 4.28, Theorem 5.9) for the bulk and (Elliott & Ranner, 2020, Theorem 6.24, Theorem 7.10) for the surface interpolation, respectively.

LEMMA 5.7. *Let* $1 \leqslant k \leqslant p$.

a) Globally, the interpolation operators satisfy for all $v \in H^{k+1}(\Omega)$, *and* $\vartheta \in H^{k+1}(\Gamma)$ *the error bounds*

$$\|v-(I_{h,\Omega}v)^\ell\|_{L^2(\Omega)}+h\|v-(I_{h,\Omega}v)^\ell\|_{H^1(\Omega)} \leqslant Ch^{k+1}\|v\|_{H^{k+1}(\Omega)}, \tag{5.5a}$$
$$\|\vartheta-(I_{h,\Gamma}\vartheta)^\ell\|_{L^2(\Gamma)}+h\|\vartheta-(I_{h,\Gamma}\vartheta)^\ell\|_{H^1(\Gamma)} \leqslant Ch^{k+1}\|\vartheta\|_{H^{k+1}(\Gamma)}, \tag{5.5b}$$

with a constant C *independent of* h.

b) Locally, on each element $K_\Omega \in \mathcal{T}_h^\Omega$, $K_\Gamma \in \mathcal{T}_h^\Gamma$, *the interpolation operators satisfy for all* $0 \leqslant r \leqslant k$ *and all* $v \in H^{k+1}(K_\Omega^\ell), \vartheta \in H^{k+1}(K_\Gamma^\ell)$, *the error bounds*

$$\|v-(I_{h,\Omega}v)^\ell\|_{H^r(K_\Omega^\ell)} \leqslant Ch^{k+1-r}\|v\|_{H^{k+1}(K_\Omega^\ell)}, \tag{5.6a}$$
$$\|\vartheta-(I_{h,\Gamma}\vartheta)^\ell\|_{H^r(K_\Gamma^\ell)} \leqslant Ch^{k+1-r}\|\vartheta\|_{H^{k+1}(K_\Gamma^\ell)}, \tag{5.6b}$$

with a constant C *independent of* h.

c) Locally, on each element $K_\Omega \in \mathcal{T}_h^\Omega$, $K_\Gamma \in \mathcal{T}_h^\Gamma$, *and for every* $v_h \in H^{k+1}(K_\Omega), \vartheta_h \in H^{k+1}(K_\Gamma)$, *the* L^∞ *error bounds*

$$\|v_h - I_{h,\Omega}v_h^\ell\|_{L^\infty(K_\Omega)} \leqslant Ch^{k+1}\|v_h\|_{W^{k+1,\infty}(K_\Omega)}, \tag{5.7a}$$
$$\|\vartheta_h - I_{h,\Gamma}\vartheta_h^\ell\|_{L^\infty(K_\Gamma)} \leqslant Ch^{k+1}\|\vartheta_h\|_{W^{k+1,\infty}(K_\Gamma)} \tag{5.7b}$$

hold true with a constant C *independent of* h.

The following lemma is a direct consequence of Lemma 5.6 and Lemma 5.7.

LEMMA 5.8. *The operators defined in Definition 5.4 satisfy Assumption 4.8 with*

$$\widehat{C}_V = \max\{C_{\Omega,\Omega_h}, C_{\Gamma,\Gamma_h}\},$$

where C_{Ω,Ω_h} *and* C_{Γ,Γ_h} *are given in* (5.3).

We are now in the position to prove the error bound of the space discretization.

Proof of Theorem 2.6. We apply Theorem 4.12. By Lemmas 5.1, 5.3, and 5.8 we have that all assumptions are satisfied and we have to bound the space discretization error terms $E_{h,i}$ in (4.24).

The terms $E_{h,1}, E_{h,3}$ and $E_{h,4}$ also appeared in the linear case and were bounded under Assumption 2.5 in (Hipp *et al.*, 2019, Proof of Thm 5.3) by order h^p.

It thus remains to bound $E_{h,2}$. For $t \in [0,T]$ and $\vec{v} = \vec{u}'(t) \in V_h$ we calculate

$$\begin{aligned}
&\max_{\|\vec{\varphi}_h\|_{m_h}=1} |\langle \mathcal{D}(\vec{v}), \mathcal{L}_h^V \vec{\varphi}_h\rangle_{V^*\times V} - m_h\big(\mathcal{D}_h(I_h\vec{v}), \vec{\varphi}_h\big)| \\
&\leqslant \max_{\|\vec{\varphi}_h\|_{m_h}=1} \Big| c_\Omega \Big(\int_\Gamma \theta(v)\varphi_h^\ell \,\mathrm{d}s - \sum_{\Gamma_h} \theta(I_{h,\Omega} v)\varphi_h \,\mathrm{d}s \Big) \Big| \\
&\quad + \Big| c_\Omega \Big(\int_\Gamma \eta(z)\varphi_h^\ell \,\mathrm{d}s - \sum_{\Gamma_h} \eta(I_{h,\Gamma} z)\varphi_h \,\mathrm{d}s \Big) \Big| \\
&\quad + \Big| \rho \Big(\int_\Gamma v\psi_h^\ell \,\mathrm{d}s - \sum_{\Gamma_h} I_{h,\Omega} v\psi_h \,\mathrm{d}s \Big) \Big| + \Big| d \Big(\int_\Gamma z\psi_h^\ell \,\mathrm{d}s - \sum_{\Gamma_h} I_{h,\Gamma} z\psi_h \,\mathrm{d}s \Big) \Big|.
\end{aligned} \tag{5.8}$$

We bound the different terms separately. Let $\|\vec{\varphi}_h\|_{m_h} = \|[\varphi_h, \psi_h]^\mathsf{T}\|_{m_h} = 1$. We then have

$$\begin{aligned}
\Big| \int_\Gamma \theta(v)\varphi_h^\ell \,\mathrm{d}s - \sum_{\Gamma_h} \theta(I_{h,\Omega} v)\varphi_h \,\mathrm{d}s \Big| &\leqslant \Big| \int_\Gamma \theta(v)\varphi_h^\ell \,\mathrm{d}s - \int_\Gamma (I_{h,\Gamma}\theta(v))^\ell \varphi_h^\ell \,\mathrm{d}s \Big| \\
&\quad + \Big| \int_\Gamma (I_{h,\Gamma}\theta(v))^\ell \varphi_h^\ell \,\mathrm{d}s - \int_{\Gamma_h} (I_{h,\Gamma}\theta(v))\varphi_h \,\mathrm{d}s \Big| \\
&\quad + \Big| \int_{\Gamma_h} (I_{h,\Gamma}\theta(v))\varphi_h \,\mathrm{d}s - \sum_{\Gamma_h} \theta(I_{h,\Omega} v)\varphi_h \,\mathrm{d}s \Big|.
\end{aligned} \tag{5.9}$$

For the first summand on the right hand side of (5.9) we have by the continuity of the lift operator and the interpolation error (5.5b)

$$\begin{aligned}
\Big| \int_\Gamma \theta(v)\varphi_h^\ell \,\mathrm{d}s - \int_\Gamma (I_{h,\Gamma}\theta(v))^\ell \varphi_h^\ell \,\mathrm{d}s \Big| &\leqslant \|\theta(v) - (I_{h,\Gamma}\theta(v))^\ell\|_{L^2(\Gamma)} \|\varphi_h^\ell\|_{L^2(\Gamma)} \\
&\leqslant Ch^p \|\theta(v)\|_{H^{\max\{2,p\}}(\Gamma_h)}.
\end{aligned}$$

The second summand can be bounded using the geometric error estimate (5.4c) by

$$\begin{aligned}
\Big| \int_\Gamma (I_{h,\Gamma}\theta(v))^\ell \varphi_h^\ell \,\mathrm{d}s - \int_{\Gamma_h} (I_{h,\Gamma}\theta(v))\varphi_h \,\mathrm{d}s \Big| &\leqslant Ch^{p+1} \|I_{h,\Gamma}\theta(v)\|_{L^2(\Gamma_h)} \|\varphi_h\|_{L^2(\Gamma_h)} \\
&\leqslant Ch^p \|\theta(v)\|_{H^2(\Gamma_h)}.
\end{aligned}$$

To bound the third summand on the right hand side od (5.9) we use that for the nodal interpolation we have $I_{h,\Gamma}\theta(v) = I_{h,\Gamma}\theta((I_{h,\Gamma} v)^\ell) \in V_{h,p}^\Gamma$ and that the order of quadrature formula is greater than $2p$ to obtain

$$
\begin{aligned}
&\Big|\int_{\Gamma_h} (I_{h,\Gamma}\theta(v))\varphi_h \,\mathrm{d}s - \sum_{\Gamma_h} \theta(I_{h,\Gamma}v)\varphi_h \,\mathrm{d}s\Big| \\
&\qquad = \Big|\sum_{\Gamma_h} \big(I_{h,\Gamma}\theta((I_{h,\Gamma}v)^\ell)\big)\varphi_h \,\mathrm{d}s - \sum_{\Gamma_h} \theta(I_{h,\Gamma}v)\varphi_h \,\mathrm{d}s\Big| \\
&\qquad \leqslant \Big(\sum_{\Gamma_h} \big(I_{h,\Gamma}\theta((I_{h,\Gamma}v)^\ell) - \theta(I_{h,\Gamma}v)\big)^2 \,\mathrm{d}s\Big)^{\frac{1}{2}} \Big(\sum_{\Gamma_h} \varphi_h^2 \,\mathrm{d}s\Big)^{\frac{1}{2}} \\
&\qquad \leqslant \sigma(\Gamma_h)\|I_{h,\Gamma}\theta((I_{h,\Gamma}v)^\ell) - \theta(I_{h,\Gamma}v)\|_{L^\infty(\Gamma_h)}\|\varphi_h\|_{L^2(\Gamma_h)} \\
&\qquad \leqslant Ch^p \sum_{F\in\mathcal{T}_h^\Gamma} \|\theta(I_{h,\Gamma}v)\|_{W^{p,\infty}(F)},
\end{aligned}
$$

where we denote by $\sigma(\Gamma_h)$ the measure of Γ_h and we used the L^∞ interpolation error bound (5.7b). This term is bounded since $v \in W^{p+1,\infty}(\Omega)$, $v \mapsto v\big|_\Gamma \in C(W^{p+1,\infty}(\Omega); W^{p,\infty}(\Gamma))$, $I_{h,\Gamma} \in C(W^{p,\infty}(\Gamma); W^{p,\infty}(\Gamma_h))$, and $\theta \in C^p$. In total we obtain in (5.9)

$$
\Big|\int_\Gamma \theta(v)\varphi_h^\ell \,\mathrm{d}s - \sum_{\Gamma_h} \theta(I_{h,\Omega}v)\varphi_h \,\mathrm{d}s\Big| \leqslant Ch^p
$$

and similarly

$$
\Big|\int_\Gamma \eta(z)\varphi_h^\ell \,\mathrm{d}s - \sum_{\Gamma_h} \eta(I_{h,\Gamma}z)\varphi_h \,\mathrm{d}s\Big| \leqslant Ch^p.
$$

To bound the third term in (5.8), we make use of the classical inverse estimate $\|v_h\|_{H^1(\Omega_h)} \leqslant Ch^{-1}\|v_h\|_{L^2(\Omega_h)}$ (cf. (Brenner & Scott, 2008, Lem. 4.5.3)), (5.5b), and the trace inequality $\|v\|_{H^{p+1}(\Gamma)} \leqslant C\|v\|_{H^{p+2}(\Omega)}$ to obtain

$$
\begin{aligned}
&\Big|\Big(\int_\Gamma v\psi_h^\ell \,\mathrm{d}s - \sum_{\Gamma_h} I_{h,\Omega}v\psi_h \,\mathrm{d}s\Big)\Big| \\
&= \Big|\Big(\int_\Gamma v\psi_h^\ell \,\mathrm{d}s - \int_{\Gamma_h} I_{h,\Omega}v\psi_h \,\mathrm{d}s\Big)\Big| \\
&\leqslant \Big|\Big(\int_\Gamma v\psi_h^\ell \,\mathrm{d}s - \int_\Gamma (I_{h,\Omega}v)^\ell\psi_h^\ell \,\mathrm{d}s\Big)\Big| + \Big|\Big(\int_\Gamma (I_{h,\Omega}v)^\ell\psi_h^\ell \,\mathrm{d}s - \int_{\Gamma_h} I_{h,\Omega}v\psi_h \,\mathrm{d}s\Big)\Big| \\
&\leqslant \|v - (I_{h,\Omega}v)^\ell\|_{L^2(\Gamma)}\|\psi_h^\ell\|_{L^2(\Gamma)} + Ch^{p+1}\|I_{h,\Omega}v\|_{L^2(\Gamma_h)}\|\psi_h\|_{L^2(\Gamma_h)} \\
&\leqslant Ch^{p+1}\|v\|_{H^{p+1}(\Gamma)}\|\psi_h\|_{H^1(\Omega_h)} + Ch^{p+1}\|I_{h,\Omega}v\|_{H^1(\Omega_h)}\|\psi_h\|_{H^1(\Omega_h)} \\
&\leqslant Ch^{p+1}\|v\|_{H^{p+2}(\Omega)}h^{-1}\|\psi_h\|_{L^2(\Omega_h)} + Ch^{p+1}\|v\|_{H^2(\Omega_h)}h^{-1}\|\psi_h\|_{L^2(\Omega_h)} \\
&\leqslant Ch^p.
\end{aligned}
$$

Similarly, we have

$$
\Big|\int_\Gamma z\psi_h^\ell \,\mathrm{d}s - \sum_{\Gamma_h} I_{h,\Gamma}z\psi_h \,\mathrm{d}s\Big| \leqslant Ch^p,
$$

and thus in total $E_{h,2} \leqslant Ch^p$. Theorem 4.12 gives then the desired result. $\square$

6. Numerical experiment

In this section we illustrate Theorem 2.6 with a numerical experiment.

Let $\Omega = B_1(0) \subset \mathbb{R}^2$ be the unit disc and in (2.1) we set

$$
\begin{aligned}
c_\Omega = c_\Gamma = \rho = 1, \qquad k_\Omega = k_\Gamma = d = 0,\\
\theta(\xi) = \xi^3, \qquad \eta(\xi) = -64\pi^3\cos^3(\xi),\\
f_\Omega(\mathbf{x}) = \sin(2\pi t)\big(-16\pi^2 r^3 + 24\pi^2 r^2 - 144 r^2 + 96 r\big),\\
f_\Gamma(\mathbf{x}) = 2\pi - 4\pi\cos(2\pi t),\\
u^0(\mathbf{x}) = 0, \quad v^0(\mathbf{x}) = 2\pi(4r^3 - 6r^2), \quad \delta^0(\mathbf{x}) = 0, \quad \vartheta^0 = 0
\end{aligned}
$$

where $r = r(\mathbf{x}) = \mathbf{x}_1^2 + \mathbf{x}_2^2$. Then, Assumption 2.2 is satisfied and the exact solution of (2.1) is given by

$$
u(t,\mathbf{x}) = \sin(2\pi t)\big(4r^3 - 6r^2\big), \quad \delta(t,\mathbf{x}) = \pi t^2.
$$

We implemented the experiments in the `C++` finite element library `deal.ii`, cf. Arndt *et al.* (2021b,a). The code which was used for the numerical is available at `https://doi.org/10.5445/IR/1000139898`. For the time integration we use the implicit midpoint rule with sufficiently small time step size ($\approx 10^{-3}$), such that the time integration error is negligible, and solve the arising nonlinear systems with the simplified newton method. For the spatial discretization we use the bulk-surface finite element method of order $p = 1$ and $p = 2$.

We consider the error

$$
\begin{aligned}
\mathbf{E}(t) \coloneqq \|u_h(t) - u(t)\big|_{\Omega_h}\|_{H^1(\Omega_h)} + \|u_h'(t) - u'(t)\big|_{\Omega_h}\|_{L^2(\Omega_h)}\\
+ \|\delta_h(t) - I_{h,\Gamma}\delta(t)\|_{H^1(\Gamma_h)} + \|\delta_h'(t) - I_{h,\Gamma}\delta'(t)\|_{L^2(\Gamma_h)}
\end{aligned}
\tag{6.1}
$$

instead of the error from Theorem 2.6 since the computation of the lift is quite laborious. We evaluated the integrals with a quadrature rule of degree $2p$, so that the quadrature error is negligible. The restriction of u to Ω_h is possible for this example since we have $\Omega_h \subset \Omega$.

In Figure 1 the error $\mathbf{E}(t)$ is plotted against the maximal mesh width h. We observe that the error converges with error p as predicted by Theorem 2.6 which indicates that our proven convergence rates are optimal.

Acknowledgements I thank Benjamin Dörich and Marlis Hochbruck for the careful reading of this manuscript and helpful discussions.

Funding Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

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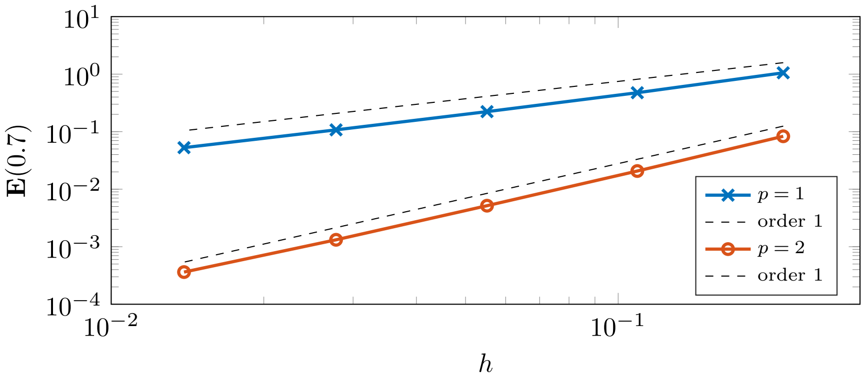

FIG. 1. Error from (6.1) at $t = 0.7$ for the test example.

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