Logarithmic contributions to the polarized $O(\alpha_s^3)$ asymptotic massive Wilson coefficients and operator matrix elements in deeply inelastic scattering

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We compute the logarithmic contributions to the polarized massive Wilson coefficients for deep-inelastic scattering in the asymptotic region $Q^2 \gg m^2$ to three-loop order in the fixed-flavor number scheme and present the corresponding expressions for the polarized massive operator matrix elements needed in the variable flavor number scheme. The calculation is performed in the Larin scheme. For the massive operator matrix elements $A_{qq,Q}^{(3),\mathrm{PS}}$ and $A_{qg,Q}^{(3),\mathrm{S}}$ the complete results are presented. The expressions are given in Mellin-N space and in momentum fraction z space.

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I. INTRODUCTION

At leading twist the scaling violations of deeply–inelastic structure functions obtain corrections from the massless scale evolution of the parton densities, the massive Wilson coefficients, and target-mass corrections, Ref. [1], provided that the electroweak radiative corrections are properly removed [2–4]. At low scales of the virtuality $Q^2 = -q^2$, all kind of other dynamical contributions, such as dynamical higher twist effects, particular small x effects, with x the Bjorken variable, and particular hadronic contributions due to vector meson dominance etc. are present. For these reasons one usually applies respective cuts of the kind $Q^2 > 10\ \mathrm{GeV}^2, W^2 = Q^2(1-x)/x > 15\ \mathrm{GeV}^2$ [5] to eliminate these effects in order to obtain clean twist-two data for which a dedicated QCD analysis is going to be performed. The target mass effects can be accounted for analytically [1,6–8]. Furthermore, for a precision measurement of the strong coupling constant $\alpha_s(M_Z^2) = g_s^2/(4\pi)$, one should use targets with no (or only soft) nuclear binding, such as the case for p and d targets. In the latter case nuclear corrections still have to be performed.

Under these circumstances the largest source of scaling violations next to the massless higher order corrections are implied by the heavy quark corrections, in the form of single and two-mass corrections. Both in the unpolarized and polarized cases the scaling violations due to the heavy flavor Wilson coefficients turn out to be different from those of the massless corrections in very wide kinematic regions covered by the present experiments. Therefore, one can not model these effects by just adding more quasi-massless flavors in massless higher order corrections. In future analysis of the polarized structure functions[1] at the EIC operating at high luminosity [10] these corrections are important for precision measurements of the strong coupling constant [11] and the charm quark mass, Ref. [12].

In the polarized case, the leading order corrections were derived in [13,14], the next-to-leading order (NLO) asymptotic corrections in [15–18], the complete NLO corrections in analytic form for the nonsinglet and pure singlet corrections in [15,19,20] and numerically in [21]. At three-loop order, the massive operator matrix elements (OMEs) for the nonsinglet $A_{qq,Q}^{(3),\mathrm{NS}}$, pure singlet $A_{Qq}^{(3),\mathrm{PS}}$, and the OME $A_{gq,Q}^{(3)}$ have been calculated in [22–24]. Furthermore, two-mass corrections to different OMEs at three-loop order were calculated in [24–27]. The three-loop heavy flavor nonsinglet contributions at leading twist to the structure functions $g_1(x,Q^2)$ and $g_2(x,Q^2)$ were computed in [28]. Likewise, in the unpolarized case the leading order corrections were obtained in [29–34] at NLO in numerical form in [35] and in analytic form in the nonsinglet and pure

[1]For a review on the theory and phenomenology of polarized deep-inelastic structure functions see Ref. [9].

singlet cases in [19,36,37] and in the asymptotic case in [19,36,38–41]. At three-loop order a series of moments has been computed in [42] for all massive OMEs. A series of three-loop massive OMEs has been calculated in Refs. [22,43–47] and first analytic results for $A_{Qg}^{(3)}$ have been given in [48]. Two-mass three-loop corrections were calculated in Refs. [25,49,50].

In the present paper we calculate the three-loop logarithmic corrections to the polarized massive OMEs A_{ij} both in Mellin N and momentum fraction z-space in the single mass case. These OMEs appear as building blocks of the variable flavor number scheme (VFNS) at three-loop order, through which the matching of parton distribution functions at N_F and N_F+1 massless heavy flavors can be obtained [38]. Because of the close values of the heavy quark masses m_c and m_b, two-mass decoupling has been considered and implemented in the VFNS in [18,25,51]. In the present paper we will consider the heavy quark masses in the on-shell scheme.[2] The OMEs determine also the polarized massive Wilson coefficients in the region of large virtualities $Q^2 \gg m_Q^2$, with m_Q the corresponding heavy quark mass $m_Q = m_c(m_b)$ of charm and bottom quarks, which we derive for the structure function $g_1(x,Q^2)$. It turns out that all functions can either be expressed in terms of harmonic sums [52,53] or harmonic polylogarithms [54]. The analytic continuation to z-space in Mellin space evolution have been studied in [55,56] in the case of harmonic sums. The expressions for the different OMEs and massive Wilson coefficients turn out to be rather long both in N and z space. As they are at present, they can be used to study the corrections at the logarithmic level and the expressions provide the frame for the final results.

The paper is organized as follows. In Sec. II, we summarize the basic formalism. It widely follows the representation in the unpolarized case [57] and resumes the representations of the structure functions and the relations for the single mass variable flavor number scheme. We then present the complete polarized massive operator matrix elements (OMEs) $A_{qq,Q}^{(3),\mathrm{PS}}$ and $A_{qg,Q}^{(3),\mathrm{S}}$ beyond the logarithmic terms in Sec. III. In Sec. IV the polarized OME $A_{Qg}^{(3),\mathrm{S}}$ is given in N space, followed by the expressions for the OMEs $A_{gg,Q}^{(3),\mathrm{S}}$ in Sec. V. The OMEs $A_{qq,Q}^{(3),\mathrm{NS}}, A_{Qq}^{(3),\mathrm{PS}}$ and $A_{gq,Q}^{(3),\mathrm{S}}$ have already been published before in Refs. [22–24] in complete form. In Secs. VI–VIII we present the polarized three-loop Wilson coefficients $L_q^{\mathrm{PS}}, L_g^{\mathrm{S}}, H_{Qq}^{\mathrm{PS}}$ and H_{Qg}^{S}. The logarithmic contributions can be represented in terms of harmonic sums and a large number of polynomials. Section IX contains the conclusions. In Appendix A the corresponding expressions for the OMEs in momentum fraction z space are presented, and in Appendix B the corresponding expressions for the polarized Wilson coefficients are presented. Here, the yet unknown constant parts of the polarized massive OMEs $a_{Qg}^{(3)}$ and $a_{gg,Q}^{(3)}$ and the yet missing three-loop polarized massless Wilson coefficients, are left as symbols. In Refs. [22,28] the flavor nonsinglet contributions were presented in the $\overline{\mathsf{MS}}$ scheme to three-loop order. For a consistent treatment, we present the transformation for the massive OME $A_{qq,Q}^{\mathrm{NS}}$ and the asymptotic massive Wilson coefficient L_q^{NS} in the Larin scheme in Appendix C.

[2]The transformation to the $\overline{\mathsf{MS}}$-scheme is straightforward.

II. THE FORMALISM

The explicit expressions showing the principal structure of the different massive OMEs have been derived in Refs. [42,57], as well as for the asymptotic massive Wilson coefficients. The different polarized massive OMEs obey the following expansion in the strong coupling constant

$$A_{ij}^{(k)} = \delta_{ij} + \sum_{l=1}^{\infty} a_s^l A_{ij}^{(l),k}, \tag{1}$$

with $a_s = \alpha_s/(4\pi)$, the indices $i,j = q,g$ label the partonic channels, and k denotes the different OMEs. Here, and in the following, we use the shorthand notations [42]

$$\hat{f}(N_F) = f(N_F+1) - f(N_F) \tag{2}$$

$$\tilde{f}(N_F) = \frac{1}{N_F} f(N_F). \tag{3}$$

The heavy flavor Wilson coefficients, accounting for the single mass contributions to the structure function $g_1(x,Q^2)$, are given by

$$\begin{aligned} L_{q,(1)}^{\mathsf{NS,Q}}(N_F+1) = {}& a_s^2[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1) + \hat{C}_{q,(1)}^{(2),\mathsf{NS}}(N_F)] \\ & + a_s^3[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1) \\ & + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1) C_{q,(1)}^{(1),\mathsf{NS}}(N_F+1) \\ & + \hat{C}_{q,(1)}^{(3),\mathsf{NS}}(N_F)], \end{aligned} \tag{4}$$

$$\begin{aligned} L_{q,(1)}^{\mathsf{PS}}(N_F+1) = {}& a_s^3[A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1) \\ & + A_{gq,Q}^{(2)}(N_F+1) N_F \tilde{C}_{g,(1)}^{(1)}(N_F+1) \\ & + N_F \hat{\tilde{C}}_{q,(1)}^{(3),\mathsf{PS}}(N_F)], \end{aligned} \tag{5}$$

$$L^{\mathsf{S}}_{g,(1)}(N_F+1) = a_s^2 A^{(1)}_{gg,Q}(N_F+1) N_F \tilde{C}^{(1)}_{g,(2,L)}(N_F+1) + a_s^3 [A^{(3)}_{qg,Q}(N_F+1) + A^{(1)}_{gg,Q}(N_F+1) N_F \tilde{C}^{(2)}_{g,(2,L)}(N_F+1) + A^{(2)}_{gg,Q}(N_F+1) N_F \tilde{C}^{(1)}_{g,(2,L)}(N_F+1) + A^{(1)}_{Qg}(N_F+1) N_F \tilde{C}^{(2),\mathsf{PS}}_{q,(2,L)}(N_F+1) + N_F \hat{\tilde{C}}^{(3)}_{g,(2,L)}(N_F)], \tag{6}$$

$$H^{\mathsf{PS}}_{q,(1)}(N_F+1) = a_s^2 [A^{(2),\mathsf{PS}}_{Qq}(N_F+1) + \tilde{C}^{(2),\mathsf{PS}}_{q,(2,L)}(N_F+1)] + a_s^3 [A^{(3),\mathsf{PS}}_{Qq}(N_F+1) + \tilde{C}^{(3),\mathsf{PS}}_{q,(2,L)}(N_F+1) + A^{(2)}_{gq,Q}(N_F+1) \tilde{C}^{(1)}_{g,(2,L)}(N_F+1) + A^{(2),\mathsf{PS}}_{Qq}(N_F+1) C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+1)], \tag{7}$$

$$\begin{aligned} H^{\mathsf{S}}_{g,(1)}(N_F+1) = {} & a_s [A^{(1)}_{Qg}(N_F+1) + \tilde{C}^{(1)}_{g,(2,L)}(N_F+1)] \\ & + a_s^2 [A^{(2)}_{Qg}(N_F+1) + A^{(1)}_{Qg}(N_F+1) C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+1) + A^{(1)}_{gg,Q}(N_F+1) \tilde{C}^{(1)}_{g,(2,L)}(N_F+1) + \tilde{C}^{(2)}_{g,(2,L)}(N_F+1)] \\ & + a_s^3 [A^{(3)}_{Qg}(N_F+1) + A^{(2)}_{Qg}(N_F+1) C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+1) + A^{(2)}_{gg,Q}(N_F+1) \tilde{C}^{(1)}_{g,(2,L)}(N_F+1) \\ & + A^{(1)}_{Qg}(N_F+1) \{C^{(2),\mathsf{NS}}_{q,(2,L)}(N_F+1) + \tilde{C}^{(2),\mathsf{PS}}_{q,(2,L)}(N_F+1)\} \\ & + A^{(1)}_{gg,Q}(N_F+1) \tilde{C}^{(2)}_{g,(2,L)}(N_F+1) + \tilde{C}^{(3)}_{g,(2,L)}(N_F+1)], \end{aligned} \tag{8}$$

with $\beta_{0,Q} = -(4/3) T_F$. The QCD color factors are given by $C_A = N_c, C_F = (N_c^2-1)/(2N_c), T_F = 1/2$ for $SU(N_c)$ and $N_c = 3$. N_F denotes the number of massless flavors. Here the massive OMEs A_{ij} depend on m^2/μ^2 and the massless Wilson coefficients depend on Q^2/μ^2. Note that we extended the original notion in Refs. [38,42].

The argument (N_F+1) in the massive OMEs signals that these functions depend on N_F massless and one massive flavor, while the setting of N_F in the massless Wilson coefficients is a functional one. The massless Wilson coefficients are labeled by $C_i^{(l),k}$, where i refers to the parton species and l to the expansion order in the strong coupling constant. The massless Wilson coefficients are known to two-loop order [20,58–61] and in the non-singlet case to three-loop order [62].

In the present paper we express all relations in the Larin scheme [63,64], which is a consistent scheme also in the massive case. The anomalous dimensions were calculated to two-loop order [65,66] and to three-loop order in [67,68].

The twist-two contributions to the structure function $g_1(x,Q^2)$ in the single mass case for pure virtual photon exchange[3] are given by

$$\begin{aligned} \frac{1}{x} g_1(x,Q^2) = {} & \sum_{k=1}^{N_F} e_k^2 \Bigg\{ L^{\mathsf{NS}}_{q,(2,L)}\left(x, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes [\Delta f_k(x,\mu^2,N_F) + \Delta f_{\bar{k}}(x,\mu^2,N_F)] \\ & + \frac{1}{N_F} L^{\mathsf{PS}}_{q,(2,L)}\left(x, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes \Delta\Sigma(x,\mu^2,N_F) + \frac{1}{N_F} L^{\mathsf{S}}_{g,(2,L)}\left(x, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes \Delta G(x,\mu^2,N_F) \Bigg\} \\ & + e_Q^2 \left[H^{\mathsf{PS}}_{q,(2,L)}\left(x, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes \Delta\Sigma(x,\mu^2,N_F) + H^{\mathsf{S}}_{g,(2,L)}\left(x, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes \Delta G(x,\mu^2,N_F) \right]. \end{aligned} \tag{9}$$

Here, the polarized quark and antiquark densities are denoted by Δf_k and $\Delta f_{\bar{k}}$, ΔG denotes the polarized gluon density, and the polarized singlet distribution is given by

$$\Delta\Sigma = \sum_{k=1}^{N_F} [\Delta f_k + \Delta f_{\bar{k}}]. \tag{10}$$

The structure function $g_2(x,Q^2)$ at twist-two is obtained by the Wandzura-Wilczek relation [6,7,69–71]

[3]For the structure of electroweak gauge boson exchange see e.g., [6,69].

$$g_2(x,Q^2) = -g_1(x,Q^2) + \int_x^1 \frac{dz}{z} g_1(z,Q^2). \quad (11)$$

Besides the Wilson coefficients (4)–(8) the massive OMEs determine the relations of the matching of parton distributions in the VFNS at large enough scales μ^2, [38,42]. Here, the PDFs for N_F+1 massless quarks are related to the former N_F massless quarks process independently. The corresponding relations to three-loop order read, (see also [38])[4]

$$\Delta f_k(N_F+1,\mu^2) + \Delta f_{\bar{k}}(N_F+1,\mu^2) = A_{qq,Q}^{\mathsf{NS}}\left(N_F,\frac{\mu^2}{m^2}\right) \otimes [\Delta f_k(N_F,\mu^2) + \Delta f_{\bar{k}}(N_F,\mu^2)] + \tilde{A}_{qq,Q}^{\mathsf{PS}}\left(N_F,\frac{\mu^2}{m^2}\right) \otimes \Delta\Sigma(N_F,\mu^2) + \tilde{A}_{qg,Q}^{\mathsf{S}}\left(N_F,\frac{\mu^2}{m^2}\right) \otimes \Delta G(N_F,\mu^2) \quad (12)$$

$$\Delta f_{Q+\bar{Q}}(N_F+1,\mu^2) = A_{Qq}^{\mathsf{PS}}\left(N_F,\frac{\mu^2}{m^2}\right) \otimes \Delta\Sigma(N_F,\mu^2) + A_{Qg}^{\mathsf{S}}\left(N_F,\frac{\mu^2}{m^2}\right) \otimes \Delta G(N_F,\mu^2) \quad (13)$$

$$\Delta G(N_F+1,\mu^2) = A_{gq,Q}^{\mathsf{S}}\left(N_F,\frac{\mu^2}{m^2}\right) \otimes \Delta\Sigma(N_F,\mu^2) + A_{gg,Q}^{S}\left(N_F,\frac{\mu^2}{m^2}\right) \otimes \Delta G(N_F,\mu^2) \quad (14)$$

$$\Delta\Sigma(N_F+1,\mu^2) = \left[A_{qq,Q}^{\mathsf{NS}}\left(N_F,\frac{\mu^2}{m^2}\right) + N_F\tilde{A}_{qq,Q}^{\mathsf{PS}}\left(N_F,\frac{\mu^2}{m^2}\right) + A_{Qq}^{\mathsf{PS}}\left(N_F,\frac{\mu^2}{m^2}\right)\right] \otimes \Delta\Sigma(N_F,\mu^2) + \left[N_F\tilde{A}_{qg,Q}^{\mathsf{S}}\left(N_F,\frac{\mu^2}{m^2}\right) + A_{Qg}^{\mathsf{S}}\left(N_F,\frac{\mu^2}{m^2}\right)\right] \otimes \Delta G(N_F,\mu^2). \quad (15)$$

The N_F dependence of the OMEs is understood as functional and μ^2 denotes the matching scale, which for the heavy-to-light transitions are normally much larger than the mass scale m^2, [72]. The corresponding matching equations in the two-mass case are given in Ref. [25].

The results of the calculations presented in the subsequent sections have been obtained making mutual use of the packages `HarmonicSums.m` [52–54,56,73,74] and `Sigma.m` [75,76], which is based on algorithms in difference-ring theory [77].

In the subsequent expressions we abbreviate the following logarithms by

$$L_Q = \ln\left(\frac{Q^2}{\mu^2}\right) \quad \text{and} \quad L_M = \ln\left(\frac{m^2}{\mu^2}\right), \quad (16)$$

where we set the renormalization and factorization scales equal $\mu \equiv \mu_F = \mu_R$.

III. $A_{qq,Q}^{(3),\mathrm{PS}}$ AND $A_{qg,Q}^{(3),\mathrm{S}}$

In what follows, the OMEs $A_{qq,Q}^{(3),\mathrm{PS}}$ and $A_{qg,Q}^{(3),\mathrm{S}}$ are presented in complete form analytically, where the $O(\varepsilon^0)$ contribution to the unrenormalized OMEs, $a_{ij,Q}^{(3)}$, are given separately. Up to three-loop order the massive OMEs $A_{qq,Q}^{\mathrm{PS}}$ and $A_{qg,Q}^{\mathrm{S}}$ do not yet receive double mass contributions. The OMEs are expressed in terms of harmonic sums [52,53] $S_{\vec{a}}(N) \equiv S_{\vec{a}}$, which are defined by

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\mathrm{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \qquad S_\varnothing = 1, b, a_i \in \mathbb{Z}\backslash\{0\}. \quad (17)$$

[4]Here, we have corrected some typographical errors in (13)–(15) in [38], in accordance with the Appendix of Ref. [38].

The OME $A_{qq,Q}^{PS(3)}$ is given by

$$A_{qq,Q}^{PS(3)} = a_s^3\Big\{a_{qq,Q}^{PS(3)} + C_F N_F T_F^2\Big\{-\frac{32L_M^3(N-1)(2+N)}{9N^2(1+N)^2} + \frac{32(N-1)(2+N)(98+369N+408N^2+164N^3)}{81N^2(1+N)^5}$$
$$+ L_M^2\left[-\frac{32(2+N)(3+4N-3N^2+8N^3)}{9N^3(1+N)^3} + \frac{32(N-1)(2+N)}{3N^2(1+N)^2}S_1\right]$$
$$+ L_M\left[-\frac{32(2+N)P_1}{27N^4(1+N)^4} + \frac{64(2+N)(3+4N-3N^2+8N^3)}{9N^3(1+N)^3}S_1 - \frac{32(N-1)(2+N)}{3N^2(1+N)^2}S_1^2 - \frac{32(N-1)(2+N)}{3N^2(1+N)^2}S_2\right]$$
$$+ \left[-\frac{32(N-1)(2+N)(22+41N+28N^2)}{27N^2(1+N)^4} - \frac{16(N-1)(2+N)S_2}{3N^2(1+N)^2}\right]S_1$$
$$+ \frac{16(N-1)(2+N)(2+5N)}{9N^2(1+N)^3}(S_1^2+S_2) - \frac{16(N-1)(2+N)}{9N^2(1+N)^2}(S_1^3+2S_3)$$
$$+ \left[\frac{16(2+N)(3+2N-6N^2+13N^3)}{9N^3(1+N)^3} - \frac{32(N-1)(2+N)S_1}{3N^2(1+N)^2}\right]\zeta_2 + \frac{32(N-1)(2+N)}{9N^2(1+N)^2}\zeta_3\Big\}\Big\}, \tag{18}$$

with

$$P_1 = 86N^5 + 38N^4 + 40N^3 - 8N^2 - 15N - 9. \tag{19}$$

Here $\zeta_k, k \geq 2, k \in \mathbb{N}$ denotes the Riemann ζ function at integer arguments.

Likewise, the OME $A_{qg,Q}^{(3)}$ is obtained by

$$A_{qg,Q}^{(3)} = a_s^3\Big\{a_{qg,Q}^{(3)} + C_A N_F T_F^2\Big\{-\frac{8(N-1)P_8}{81N^5(1+N)^5} + L_M^3\left[-\frac{64(N-1)}{9N^2(1+N)^2} + \frac{32(N-1)}{9N(1+N)}S_1\right]$$
$$+ L_M^2\left[\frac{8P_4}{9N^3(1+N)^3} + \frac{32(1+5N^2)}{9N(1+N)^2}S_1 - \frac{16(N-1)}{3N(1+N)}(S_1^2+S_2) - \frac{32(N-1)}{3N(1+N)}S_{-2}\right]$$
$$+ L_M\left[\frac{16P_7}{27N^4(1+N)^4} + \left[\frac{16(-1+44N+67N^2+94N^3)}{27N(1+N)^3} - \frac{16(N-1)S_2}{3N(1+N)}\right]S_1\right.$$
$$- \frac{32(-2+5N^2)}{9N(1+N)^2}S_1^2 + \frac{16(N-1)}{9N(1+N)}S_1^3 - \frac{32(-2+6N+5N^2)}{9N(1+N)^2}S_2 + \frac{32(N-1)}{9N(1+N)}S_3$$
$$\left.- \frac{64(-2+5N)}{9N(1+N)}S_{-2} + \frac{64(N-1)}{3N(1+N)}S_{-3} + \frac{64(N-1)}{3N(1+N)}S_{2,1}\right]$$
$$- \frac{16(N-1)(283+584N+328N^2)}{81N(1+N)^3}S_1 - \frac{8(N-1)}{3N(1+N)^2}S_1^2 + \frac{8(N-1)(1+2N)}{3N(1+N)^2}S_2$$
$$+ \left[-\frac{8P_3}{9N^3(1+N)^3} - \frac{32(-2+5N^2)}{9N(1+N)^2}S_1 + \frac{8(N-1)S_1^2}{3N(1+N)}\right.$$
$$\left.+ \frac{8(N-1)S_2}{3N(1+N)} + \frac{16(N-1)S_{-2}}{3N(1+N)}\right]\zeta_2 + \left[\frac{64(N-1)}{9N^2(1+N)^2} - \frac{32(N-1)S_1}{9N(1+N)}\right]\zeta_3\Big\}$$
$$+ C_F N_F T_F^2\Big\{-\frac{(N-1)P_{10}}{81N^6(1+N)^6} + L_M^3\left[\frac{8(N-1)P_2}{9N^3(1+N)^3} - \frac{32(N-1)}{9N(1+N)}S_1\right]$$
$$+ L_M^2\left[\frac{4(N-1)P_6}{9N^4(1+N)^4} - \frac{32(N-1)(3+5N)}{9N^2(1+N)}S_1 + \frac{16(N-1)}{3N(1+N)}(S_1^2+S_2)\right]$$
$$+ L_M\left[\frac{4P_9}{27N^5(1+N)^5} + \left[-\frac{16(-24-52N+103N^2)}{27N^2(1+N)} - \frac{16(N-1)}{3N(1+N)}S_2\right]S_1\right.$$

$$+\frac{16(N-1)(3+10N)}{9N^2(1+N)}S_1^2-\frac{16(N-1)}{9N(1+N)}S_1^3+\frac{16(-3+5N+10N^2)}{9N^2(1+N)}S_2+\frac{64(N-1)}{9N(1+N)}S_3\Bigg]+\frac{5248(N-1)S_1}{81N(1+N)}$$
$$-\frac{896(N-1)S_2}{27N(1+N)}+\frac{160(N-1)S_3}{9N(1+N)}-\frac{32(N-1)}{3N(1+N)}S_4+\Bigg[-\frac{4(N-1)P_5}{9N^4(1+N)^4}+\frac{16(N-1)(3+10N)}{9N^2(1+N)}S_1$$
$$-\frac{8(N-1)S_1^2}{3N(1+N)}-\frac{8(N-1)S_2}{N(1+N)}\Bigg]\zeta_2+\Bigg[-\frac{8(N-1)P_2}{9N^3(1+N)^3}+\frac{32(N-1)S_1}{9N(1+N)}\Bigg]\zeta_3\Bigg\}\Bigg\}, \tag{20}$$

with the polynomials

$$P_2 = 3N^4+6N^3-N^2-4N+12, \tag{21}$$

$$P_3 = 6N^5+6N^4-67N^3+6N^2+43N-18, \tag{22}$$

$$P_4 = 9N^5+9N^4-79N^3+15N^2+22N-24, \tag{23}$$

$$P_5 = 18N^6+54N^5+5N^4-20N^3+95N^2-132N-108, \tag{24}$$

$$P_6 = 33N^6+99N^5+41N^4-11N^3+86N^2-216N-144, \tag{25}$$

$$P_7 = 99N^7+198N^6-410N^5-344N^4+128N^3-130N^2-39N+90, \tag{26}$$

$$P_8 = 255N^8+1020N^7-532N^6-4536N^5-4344N^4-1138N^3+3N^2+36N+108, \tag{27}$$

$$P_9 = 159N^9+477N^8-220N^7-710N^6+117N^5-1081N^4+2536N^3 + 1026N^2-1800N-1080, \tag{28}$$

$$P_{10} = 1551N^{10}+7755N^9+10982N^8+1910N^7+2427N^6+14975N^5 + 13952N^4-1488N^3-7488N^2-6912N-2592. \tag{29}$$

The $O(\varepsilon^0)$ parts of the unrenormalized massive OMEs, $a_{qq,Q}^{(3),\mathrm{PS}}$ and $a_{qg,Q}^{(3)}$ are given by

$$a_{qq,Q}^{(3),\mathrm{PS}} = C_F T_F^2 N_F\Bigg\{\frac{32(2+N)R_2}{243N^5(1+N)^5}+\Bigg(-\frac{64(2+N)R_1}{81N^4(1+N)^4}-\frac{128(N-1)(2+N)}{9N^2(1+N)^2}S_2\Bigg)S_1$$
$$+\frac{64(2+N)(3+2N-6N^2+13N^3)}{27N^3(1+N)^3}[S_1^2+S_2]-\frac{128(N-1)(2+N)}{27N^2(1+N)^2}[S_1^3+2S_3]$$
$$+\Bigg(\frac{16(2+N)(3+2N-6N^2+13N^3)}{9N^3(1+N)^3}-\frac{32(N-1)(2+N)}{3N^2(1+N)^2}S_1\Bigg)\zeta_2$$
$$-\frac{224(N-1)(2+N)}{9N^2(1+N)^2}\zeta_3\Bigg\} \tag{30}$$

$$
\begin{aligned}
a_{qg,Q}^{(3)} = {} & C_F T_F^2 N_F \Biggl\{ \frac{112(N-1)R_3}{9N^3(1+N)^3}\zeta_3 - \frac{8(N-1)R_6}{9N^4(1+N)^4}\zeta_2 - \frac{16R_8}{243N^6(1+N)^6} + \frac{(N-1)(3+10N)}{N^2(1+N)} \\
& \times \left(\frac{32}{81}S_1^3 + \frac{32}{27}S_1S_2 + \frac{32}{9}S_1\zeta_2\right) + \frac{N-1}{N(1+N)}\left(-\frac{8}{27}S_1^4 - \frac{16}{9}S_1^2S_2 - \frac{8}{9}S_2^2 - \frac{64}{27}S_1S_3\right. \\
& \left. - \frac{80}{9}S_4 - 16\left(\frac{1}{3}S_1^2 + S_2\right)\zeta_2 - \frac{448}{9}S_1\zeta_3\right) - \frac{32(420+2399N-555N^2-2507N^3)}{243N^2(1+N)^2}S_1 \\
& - \frac{16(-24-164N+215N^2)}{81N^2(1+N)}S_1^2 - \frac{16(-8-148N+221N^2)}{27N^2(1+N)}S_2 + \frac{64(-3+5N)(1+8N)}{81N^2(1+N)}S_3 \Biggr\} \\
& + C_A T_F^2 N_F \Biggl\{ -\frac{32R_4}{243N(1+N)^4}S_1 - \frac{16R_5}{9N^3(1+N)^3}\zeta_2 - \frac{32R_7}{243N^5(1+N)^5} \\
& + \frac{-2+5N^2}{N(1+N)^2}\left(-\frac{64}{81}S_1^3 + \frac{64}{27}S_1S_2 - \frac{128}{81}S_3 - \frac{256}{27}S_{2,1} - \frac{64}{9}S_1\zeta_2\right) \\
& + \frac{N-1}{N(1+N)}\left(\left(-\frac{320}{27}S_3 + \frac{128}{9}S_{2,1}\right)S_1 + \frac{8}{27}S_1^4 - \frac{16}{9}S_1^2S_2 + \frac{8}{9}S_2^2 + \frac{112}{9}S_4 + \frac{256}{9}S_{-4} + \frac{256}{9}S_{3,1} - \frac{128}{9}S_{2,1,1}\right. \\
& \left. + \left(\frac{16}{3}[S_2 + S_1^2] + \frac{32}{3}S_{-2}\right)\zeta_2 + \frac{448}{9}S_1\zeta_3\right) + \frac{16(-59-68N+125N^2+206N^3)}{81N(1+N)^3}S_1^2 \\
& - \frac{16(131+26N-509N^2-332N^3)}{81N(1+N)^3}S_2 - \frac{256}{9(1+N)^2}S_3 - \frac{64(7-11N)(10+11N)}{81N(1+N)^2}S_{-2} \\
& - \frac{256(-2+5N)}{27N(1+N)}S_{-3} - \frac{896(N-1)}{9N^2(1+N)^2}\zeta_3 \Biggr\}, \qquad (31)
\end{aligned}
$$

with

$$R_1 = 142N^5 + 64N^4 + 2N^3 - 52N^2 - 15N - 9, \qquad (32)$$

$$R_2 = 1648N^7 + 2444N^6 + 777N^5 - 1091N^4 - 106N^3 + 201N^2 + 72N + 27, \qquad (33)$$

$$R_3 = 3N^4 + 6N^3 - N^2 - 4N + 12, \qquad (34)$$

$$R_4 = 2228N^4 + 3601N^3 + 669N^2 - 2191N - 1055, \qquad (35)$$

$$R_5 = 6N^5 + 6N^4 - 67N^3 + 6N^2 + 43N - 18, \qquad (36)$$

$$R_6 = 18N^6 + 54N^5 + 5N^4 - 20N^3 + 95N^2 - 132N - 108, \qquad (37)$$

$$R_7 = 2040N^9 + 6120N^8 - 4816N^7 - 16208N^6 - 1776N^5 + 7268N^4 + 2443N^3 + 756N^2 - 225N - 810, \qquad (38)$$

$$
\begin{aligned}
R_8 = {} & 2322N^{11} + 9288N^{10} + 5975N^9 - 11499N^8 - 7124N^7 + 4346N^6 + 522N^5 \\
& - 20360N^4 - 12027N^3 + 5193N^2 + 11880N + 4860. \qquad (39)
\end{aligned}
$$

IV. $A_{Qg}^{(3),\mathrm{S}}$

The logarithmic contributions to the massive OME $A_{Qg}^{(3),\mathrm{S}}$ are given by

$$
\begin{aligned}
A_{Qg} =& -a_s \frac{4L_M(N-1)}{N(1+N)} T_F + a_s^2 \Bigg\{ -\frac{16L_M^2(N-1)T_F^2}{3N(1+N)} \\
&+ C_F T_F \Bigg\{ \frac{2P_{95}}{N^4(1+N)^4(2+N)} + L_M^2 \left[\frac{2(N-1)(2+3N+3N^2)}{N^2(1+N)^2} - \frac{8(N-1)S_1}{N(1+N)} \right] \\
&+ L_M \left[-\frac{4(N-1)P_{23}}{N^3(1+N)^3} + \frac{16(N-1)S_1}{N^2(1+N)} - \frac{8(N-1)S_1^2}{N(1+N)} + \frac{8(N-1)S_2}{N(1+N)} \right] \\
&+ \left[\frac{4(-36-22N-2N^2+N^3)}{N^2(1+N)(2+N)} - \frac{4(N-1)S_2}{N(1+N)} \right] S_1 + \frac{4(-2+3N+3N^2)}{N^2(1+N)(2+N)} S_1^2 \\
&- \frac{4(N-1)S_1^3}{3N(1+N)} + \frac{4P_{16}}{N^2(1+N)^2(2+N)} S_2 + \frac{16(N-1)S_3}{3N(1+N)} \Bigg\} \\
&+ C_A T_F \Bigg\{ \frac{4P_{92}}{N^4(1+N)^4(2+N)} + L_M^2 \left[-\frac{16(N-1)}{N^2(1+N)^2} + \frac{8(N-1)S_1}{N(1+N)} \right] \\
&+ L_M \left[-\frac{8P_{43}}{N^3(1+N)^3} - \frac{32S_1}{N(1+N)^2} + \frac{8(N-1)S_1^2}{N(1+N)} + \frac{8(N-1)S_2}{N(1+N)} + \frac{16(N-1)S_{-2}}{N(1+N)} \right] \\
&+ \left[-\frac{4P_{14}}{N(1+N)^3(2+N)} + \frac{12(N-1)S_2}{N(1+N)} \right] S_1 - \frac{4(5+4N+N^2)}{N(1+N)^2(2+N)} S_1^2 + \frac{4(N-1)S_1^3}{3N(1+N)} \\
&- \frac{4(-16+15N+24N^2+7N^3)}{N^2(1+N)^2(2+N)} S_2 + \frac{32(N-1)S_3}{3N(1+N)} + \left[\frac{16(N-1)}{N(1+N)^2} + \frac{16(N-1)S_1}{N(1+N)} \right] S_{-2} + \frac{8(N-1)S_{-3}}{N(1+N)} \\
&- \frac{16(N-1)S_{-2,1}}{N(1+N)} \Bigg\} \Bigg\} + a_s^3 \Bigg\{ a_{Qg}^{(3)} + C_A T_F^2 \Bigg\{ \frac{8P_{106}}{81N^5(1+N)^5(2+N)} + L_M^3 \left[-\frac{448(N-1)}{9N^2(1+N)^2} + \frac{224(N-1)S_1}{9N(1+N)} \right] \\
&+ L_M^2 \left[-\frac{8P_{53}}{9N^3(1+N)^3} + \frac{32(-23+5N^2)}{9N(1+N)^2} S_1 + \frac{16(N-1)S_1^2}{N(1+N)} + \frac{16(N-1)S_2}{N(1+N)} + \frac{32(N-1)S_{-2}}{N(1+N)} \right] \\
&+ L_M \Bigg[\frac{16P_{97}}{27N^4(1+N)^4(2+N)} + \left[\frac{16P_{39}}{27N(1+N)^3(2+N)} + \frac{80(N-1)S_2}{3N(1+N)} \right] S_1 - \frac{32(11+10N+13N^2+5N^3)}{9N(1+N)^2(2+N)} S_1^2 \\
&+ \frac{16(N-1)S_1^3}{3N(1+N)} - \frac{32P_{25}}{9N^2(1+N)^2(2+N)} S_2 + \frac{32(N-1)S_3}{N(1+N)} + \left[-\frac{64(4-3N+5N^2)}{9N(1+N)^2} + \frac{128(N-1)S_1}{3N(1+N)} \right] S_{-2} \\
&+ \frac{128(N-1)S_{-3}}{3N(1+N)} + \frac{64(N-1)S_{2,1}}{3N(1+N)} - \frac{128(N-1)S_{-2,1}}{3N(1+N)} \Bigg] \\
&+ \left[-\frac{16P_{60}}{81N(1+N)^4(2+N)} - \frac{32(-13+3N^2)}{3N(1+N)^2(2+N)} S_2 - \frac{640(N-1)S_3}{9N(1+N)} + \frac{256(N-1)S_{-2,1}}{3N(1+N)} \right] S_1 \\
&+ \left[\frac{8P_{18}}{3N(1+N)^3(2+N)} - \frac{80(N-1)S_2}{3N(1+N)} \right] S_1^2 + \frac{32(5+4N+N^2)}{9N(1+N)^2(2+N)} S_1^3 \\
&- \frac{8(N-1)S_1^4}{9N(1+N)} + \frac{8P_{66}}{3N^3(1+N)^3(2+N)} S_2 - \frac{8(N-1)S_2^2}{3N(1+N)} + \frac{256(-6+8N+7N^2+N^3)}{9N^2(1+N)^2(2+N)} S_3 - \frac{48(N-1)S_4}{N(1+N)} \\
&+ \left[\frac{128(3+N^2)}{3N(1+N)^3} - \frac{256(N-1)S_1}{3N(1+N)^2} - \frac{128(N-1)S_1^2}{3N(1+N)} - \frac{128(N-1)S_2}{3N(1+N)} \right] S_{-2} \\
&+ \left[-\frac{128(N-1)}{3N(1+N)^2} - \frac{128(N-1)S_1}{3N(1+N)} \right] S_{-3} - \frac{64(N-1)S_{-4}}{3N(1+N)} + \frac{64(N-1)S_{3,1}}{3N(1+N)} \\
&+ \frac{256(N-1)S_{-2,1}}{3N(1+N)^2} + \frac{128(N-1)S_{-2,2}}{3N(1+N)} + \frac{128(N-1)S_{-3,1}}{3N(1+N)} + \frac{64(N-1)S_{2,1,1}}{3N(1+N)} - \frac{256(N-1)S_{-2,1,1}}{3N(1+N)}
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{4P_{56}}{9N^3(1+N)^3}-\frac{160(-2+N)(2+N)S_1}{9N(1+N)^2}-\frac{40(N-1)S_1^2}{3N(1+N)}-\frac{40(N-1)S_2}{3N(1+N)}-\frac{80(N-1)S_{-2}}{3N(1+N)}\right]\zeta_2\\
&+\left[\frac{448(N-1)}{9N^2(1+N)^2}-\frac{224(N-1)S_1}{9N(1+N)}\right]\zeta_3\Bigg\}+C_AN_FT_F^2\Bigg\{\frac{16P_{102}}{3N^5(1+N)^5(2+N)}+L_M^3\left[-\frac{64(N-1)}{9N^2(1+N)^2}\right.\\
&\left.+\frac{32(N-1)S_1}{9N(1+N)}\right]+L_M^2\left[-\frac{8P_{48}}{9N^3(1+N)^3}-\frac{32(1+5N^2)}{9N(1+N)^2}S_1+\frac{16(N-1)S_1^2}{3N(1+N)}\right.\\
&\left.+\frac{16(N-1)S_2}{3N(1+N)}+\frac{32(N-1)S_{-2}}{3N(1+N)}\right]+L_M\left[\frac{8P_{98}}{27N^4(1+N)^4(2+N)}+\left[\frac{16P_{36}}{27N(1+N)^3(2+N)}+\frac{32(N-1)S_2}{3N(1+N)}\right]S_1\right.\\
&-\frac{16(7+8N+23N^2+10N^3)}{9N(1+N)^2(2+N)}S_1^2+\frac{32(N-1)S_1^3}{9N(1+N)}-\frac{16P_{29}}{9N^2(1+N)^2(2+N)}S_2+\frac{160(N-1)S_3}{9N(1+N)}\\
&\left.+\left[-\frac{64(1+5N^2)}{9N(1+N)^2}+\frac{64(N-1)S_1}{3N(1+N)}\right]S_{-2}+\frac{32(N-1)S_{-3}}{N(1+N)}+\frac{64(N-1)S_{2,1}}{3N(1+N)}-\frac{64(N-1)S_{-2,1}}{3N(1+N)}\right]\\
&+\left[-\frac{16P_{45}}{3N(1+N)^4(2+N)}-\frac{16(-13+3N^2)}{3N(1+N)^2(2+N)}S_2-\frac{320(N-1)S_3}{9N(1+N)}+\frac{128(N-1)S_{-2,1}}{3N(1+N)}\right]S_1\\
&+\left[\frac{8P_{13}}{3N(1+N)^3(2+N)}-\frac{40(N-1)S_2}{3N(1+N)}\right]S_1^2+\frac{16(5+4N+N^2)}{9N(1+N)^2(2+N)}S_1^3-\frac{4(N-1)S_1^4}{9N(1+N)}\\
&+\frac{8P_{62}}{3N^3(1+N)^3(2+N)}S_2-\frac{4(N-1)S_2^2}{3N(1+N)}+\frac{128(-6+8N+7N^2+N^3)}{9N^2(1+N)^2(2+N)}S_3-\frac{24(N-1)S_4}{N(1+N)}\\
&+\left[\frac{64(3+N^2)}{3N(1+N)^3}-\frac{128(N-1)S_1}{3N(1+N)^2}-\frac{64(N-1)S_1^2}{3N(1+N)}-\frac{64(N-1)S_2}{3N(1+N)}\right]S_{-2}+\left[-\frac{64(N-1)}{3N(1+N)^2}-\frac{64(N-1)S_1}{3N(1+N)}\right]S_{-3}\\
&-\frac{32(N-1)S_{-4}}{3N(1+N)}+\frac{32(N-1)S_{3,1}}{3N(1+N)}+\frac{128(N-1)S_{-2,1}}{3N(1+N)^2}+\frac{64(N-1)S_{-2,2}}{3N(1+N)}+\frac{64(N-1)S_{-3,1}}{3N(1+N)}+\frac{32(N-1)S_{2,1,1}}{3N(1+N)}\\
&-\frac{128(N-1)S_{-2,1,1}}{3N(1+N)}+\left[\frac{4P_{49}}{9N^3(1+N)^3}+\frac{16(7+5N^2)}{9N(1+N)^2}S_1-\frac{16(N-1)S_1^2}{3N(1+N)}-\frac{16(N-1)S_2}{3N(1+N)}-\frac{32(N-1)S_{-2}}{3N(1+N)}\right]\zeta_2\\
&+\left[\frac{64(N-1)}{9N^2(1+N)^2}-\frac{32(N-1)S_1}{9N(1+N)}\right]\zeta_3\Bigg\}+C_FN_FT_F^2\Bigg\{-\frac{4P_{111}}{3N^6(1+N)^6(2+N)}+L_M^3\left[\frac{8(N-1)P_{20}}{9N^3(1+N)^3}-\frac{32(N-1)S_1}{9N(1+N)}\right]\\
&+L_M^2\left[-\frac{4(N-1)P_{72}}{9N^4(1+N)^4}+\frac{32(N-1)(3+5N)}{9N^2(1+N)}S_1-\frac{16(N-1)S_1^2}{3N(1+N)}-\frac{16(N-1)S_2}{3N(1+N)}\right]\\
&+L_M\left[\frac{2P_{107}}{27N^5(1+N)^5(2+N)}+\left[-\frac{32(138+91N+58N^2+19N^3)}{27N^2(1+N)(2+N)}-\frac{32(N-1)S_2}{3N(1+N)}\right]S_1\right.\\
&\left.+\frac{32(-6-4N+11N^2+5N^3)}{9N^2(1+N)(2+N)}S_1^2-\frac{32(N-1)S_1^3}{9N(1+N)}+\frac{16P_{63}}{3N^3(1+N)^3(2+N)}S_2+\frac{224(N-1)S_3}{9N(1+N)}\right]\\
&+\left[\frac{16P_{17}}{3N^2(1+N)^2(2+N)}-\frac{16(-2+3N+3N^2)}{3N^2(1+N)(2+N)}S_2+\frac{32(N-1)S_3}{9N(1+N)}+\frac{64(N-1)S_{2,1}}{3N(1+N)}\right]S_1\\
&+\left[-\frac{8(-36-22N-6N^2+N^3)}{3N^2(1+N)(2+N)}+\frac{8(N-1)S_2}{3N(1+N)}\right]S_1^2-\frac{16(-2+3N+3N^2)}{9N^2(1+N)(2+N)}S_1^3+\frac{4(N-1)S_1^4}{9N(1+N)}\\
&-\frac{8P_{94}}{3N^4(1+N)^4(2+N)}S_2+\frac{4(N-1)S_2^2}{3N(1+N)}-\frac{16P_{65}}{9N^3(1+N)^3(2+N)}S_3-\frac{8(N-1)S_4}{N(1+N)}+\frac{64S_{2,1}}{3N^2}+\frac{64(N-1)S_{3,1}}{3N(1+N)}\\
&-\frac{128(N-1)S_{2,1,1}}{3N(1+N)}+\left[-\frac{2(N-1)P_{84}}{9N^4(1+N)^4}-\frac{16(N-1)(6+5N)}{9N^2(1+N)}S_1+\frac{16(N-1)S_1^2}{3N(1+N)}\right]\zeta_2\\
&+\left[-\frac{8(N-1)P_{20}}{9N^3(1+N)^3}+\frac{32(N-1)S_1}{9N(1+N)}\right]\zeta_3\Bigg\}+C_FT_F^2\Bigg\{-\frac{2P_{112}}{3N^6(1+N)^6(2+N)}+L_M^3\left[\frac{16(N-1)P_{27}}{9N^3(1+N)^3}-\frac{128(N-1)S_1}{9N(1+N)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+L_M^2\left[-\frac{4(N-1)P_{79}}{9N^4(1+N)^4}+\frac{32(N-1)(9+5N)}{9N^2(1+N)}S_1-\frac{16(N-1)S_1^2}{N(1+N)}+\frac{16(N-1)S_2}{3N(1+N)}\right]\\
&+L_M\left[\frac{2P_{109}}{27N^5(1+N)^5(2+N)}+\left[-\frac{16(600+380N+134N^2+29N^3)}{27N^2(1+N)(2+N)}-\frac{16(N-1)S_2}{N(1+N)}\right]S_1\right.\\
&\left.+\frac{16(-18+N+31N^2+10N^3)}{9N^2(1+N)(2+N)}S_1^2-\frac{16(N-1)S_1^3}{3N(1+N)}+\frac{16P_{19}}{3N^2(1+N)^2(2+N)}S_2+\frac{32(N-1)S_3}{N(1+N)}\right]\\
&+\left[\frac{32P_{17}}{3N^2(1+N)^2(2+N)}-\frac{32(-2+3N+3N^2)}{3N^2(1+N)(2+N)}S_2+\frac{64(N-1)S_3}{9N(1+N)}+\frac{128(N-1)S_{2,1}}{3N(1+N)}\right]S_1\\
&+\left[-\frac{16(-36-22N-6N^2+N^3)}{3N^2(1+N)(2+N)}+\frac{16(N-1)S_2}{3N(1+N)}\right]S_1^2-\frac{32(-2+3N+3N^2)}{9N^2(1+N)(2+N)}S_1^3+\frac{8(N-1)S_1^4}{9N(1+N)}\\
&-\frac{16P_{64}}{3N^3(1+N)^3(2+N)}S_2+\frac{8(N-1)S_2^2}{3N(1+N)}-\frac{32P_{22}}{9N^2(1+N)^2(2+N)}S_3-\frac{16(N-1)S_4}{N(1+N)}+\frac{128S_{2,1}}{3N^2}\\
&+\frac{128(N-1)S_{3,1}}{3N(1+N)}-\frac{256(N-1)S_{2,1,1}}{3N(1+N)}+\left[\frac{2(N-1)P_{82}}{9N^4(1+N)^4}-\frac{80(N-1)(3+N)S_1}{9N^2(1+N)}+\frac{40(N-1)S_1^2}{3N(1+N)}-\frac{8(N-1)S_2}{N(1+N)}\right]\zeta_2\\
&+\left[-\frac{16(N-1)P_{27}}{9N^3(1+N)^3}+\frac{128(N-1)S_1}{9N(1+N)}\right]\zeta_3\Bigg\}+C_AC_FT_F\Bigg\{\frac{P_{113}}{3N^6(1+N)^6(2+N)}+L_M^3\left[-\frac{2(N-1)(2+3N+3N^2)}{9N^3(1+N)^3}\right.\\
&\left.\times(-48+11N+11N^2)-\frac{8(N-1)(60+7N+7N^2)}{9N^2(1+N)^2}S_1+\frac{64(N-1)S_1^2}{3N(1+N)}\right]\\
&+L_M^2\left[-\frac{8S_1P_{52}}{9N^3(1+N)^3}+\frac{P_{61}}{9N^3(1+N)^3}+\frac{8(27-40N-12N^2+N^3)}{3N^2(1+N)^2}S_1^2+\frac{32(N-1)S_1^3}{N(1+N)}+\frac{8(N-1)(9+N+N^2)}{3N^2(1+N)^2}S_2\right.\\
&\left.-\frac{16(N-1)S_3}{N(1+N)}-\frac{24(N-1)S_{-2}}{N(1+N)}-\frac{16(N-1)S_{-3}}{N(1+N)}+\frac{32(N-1)S_{-2,1}}{N(1+N)}\right]+L_M\left[\frac{P_{100}}{54N^5(1+N)^5(2+N)}\right.\\
&+\left[\frac{8P_{90}}{27N^4(1+N)^3(2+N)}-\frac{8P_{34}}{3N^2(1+N)^2(2+N)}S_2+\frac{128(N-1)S_3}{N(1+N)}-\frac{192(N-1)S_{2,1}}{N(1+N)}-\frac{64(N-1)S_{-2,1}}{N(1+N)}\right]S_1\\
&+\left[-\frac{4P_{77}}{9N^3(1+N)^3(2+N)}+\frac{144(N-1)S_2}{N(1+N)}\right]S_1^2+\frac{8P_{30}}{9N^2(1+N)^2(2+N)}S_1^3+\frac{16(N-1)S_1^4}{N(1+N)}\\
&+\frac{4P_{71}}{3N^3(1+N)^3(2+N)}S_2-\frac{32(N-1)S_2^2}{N(1+N)}-\frac{8(N-1)(216+41N+77N^2)}{9N^2(1+N)^2}S_3\\
&+\left[-\frac{32P_{46}}{N^3(1+N)^3(2+N)}-\frac{16(N-1)(20+4N+N^2+3N^3)}{N^2(1+N)^2(2+N)}S_1+\frac{160(N-1)S_1^2}{N(1+N)}\right]S_{-2}+\frac{48(N-1)S_{-2}^2}{N(1+N)}\\
&+\left[\frac{8(N-1)(3+N)(-2+3N)}{N^2(1+N)^2}+\frac{32(N-1)S_1}{N(1+N)}\right]S_{-3}+\frac{80(N-1)S_{-4}}{N(1+N)}+\frac{48(N-1)(4+N+N^2)}{N^2(1+N)^2}S_{2,1}\\
&+\frac{32(N-1)S_{3,1}}{N(1+N)}+\frac{16(N-1)(6-N+3N^2)}{N^2(1+N)^2}S_{-2,1}+\frac{32(N-1)S_{-2,2}}{N(1+N)}+\frac{96(N-1)S_{2,1,1}}{N(1+N)}-\frac{128(N-1)S_{-2,1,1}}{N(1+N)}\\
&\left.+\left[-\frac{96(N-1)(-5+3N+3N^2)}{N^2(1+N)^2}-\frac{96(N-1)S_1}{N(1+N)}\right]\zeta_3\right]+\left[\frac{8P_{105}}{3N^5(1+N)^5(2+N)}+\frac{8P_{55}}{3N^2(1+N)^3(2+N)}S_2\right.\\
&-\frac{8(N-1)S_2^2}{N(1+N)}+\frac{8P_{40}}{9N^2(1+N)^2(2+N)}S_3-\frac{48(N-1)S_4}{N(1+N)}-\frac{16(36-11N+12N^2+11N^3)}{3N^2(1+N)^2}S_{2,1}-\frac{32(N-1)S_{3,1}}{N(1+N)}\\
&\left.-\frac{32(N-1)(2-N+3N^2)}{N^2(1+N)^2}S_{-2,1}+\frac{64(N-1)S_{-2,2}}{N(1+N)}+\frac{64(N-1)S_{-3,1}}{N(1+N)}+\frac{160(N-1)S_{2,1,1}}{N(1+N)}-\frac{128(N-1)S_{-2,1,1}}{N(1+N)}\right]S_1\\
&+\left[-\frac{4P_{89}}{3N^3(1+N)^4(2+N)}+\frac{4P_{35}}{3N^2(1+N)^2(2+N)}S_2-\frac{352(N-1)S_3}{3N(1+N)}-\frac{64(N-1)S_{2,1}}{N(1+N)}+\frac{128(N-1)S_{-2,1}}{N(1+N)}\right]S_1^2
\end{aligned}
$$

$$+\left[\frac{8P_{74}}{9N^3(1+N)^3(2+N)}-\frac{48(N-1)S_2}{N(1+N)}\right]S_1^3-\frac{2P_{12}}{9N^2(1+N)^2(2+N)}S_1^4-\frac{8(N-1)S_1^5}{3N(1+N)}-\frac{4P_{96}}{3N^4(1+N)^4(2+N)}S_2$$
$$-\frac{2(N-1)(-15+N+N^2)}{3N^2(1+N)^2}S_2^2+\frac{4P_{83}}{9N^3(1+N)^3(2+N)}S_3+\frac{4(N-1)(-3+19N+19N^2)}{N^2(1+N)^2}S_4$$
$$+\left[-\frac{16P_{47}}{N^2(1+N)^4}+\left[\frac{32P_{26}}{N^2(1+N)^3}-\frac{64(N-1)S_2}{N(1+N)}\right]S_1+\frac{16(N-1)^2(-2+3N)}{N^2(1+N)^2}S_1^2-\frac{64(N-1)S_1^3}{N(1+N)}\right.$$
$$\left.+\frac{16(N-1)(2+3N+3N^2)}{N^2(1+N)^2}S_2\right]S_{-2}+\left[\frac{16(N-1)(2+9N+15N^2+6N^3)}{N^2(1+N)^3}+\frac{16(N-1)(2-N+3N^2)}{N^2(1+N)^2}S_1\right.$$
$$\left.-\frac{64(N-1)S_1^2}{N(1+N)}\right]S_{-3}+\left[\frac{8(N-1)(2+3N+3N^2)}{N^2(1+N)^2}-\frac{32(N-1)S_1}{N(1+N)}\right]S_{-4}-\frac{16(-24+11N+11N^2)}{3N^3(1+N)}S_{2,1}$$
$$-\frac{8(N-1)(-42+31N+31N^2)}{3N^2(1+N)^2}S_{3,1}-\frac{32(N-1)(2+9N+15N^2+6N^3)}{N^2(1+N)^3}S_{-2,1}$$
$$-\frac{16(N-1)(2+3N+3N^2)}{N^2(1+N)^2}S_{-2,2}-\frac{16(N-1)(2+3N+3N^2)}{N^2(1+N)^2}S_{-3,1}+\frac{8(N-1)(-102+35N+35N^2)}{3N^2(1+N)^2}S_{2,1,1}$$
$$+\frac{32(N-1)(2+3N+3N^2)}{N^2(1+N)^2}S_{-2,1,1}+\left(\frac{192\ln(2)(N-1)}{N(1+N)}+\frac{4S_1P_{59}}{9N^3(1+N)^3}+\frac{P_{86}}{18N^4(1+N)^4}\right.$$
$$-\frac{8(27-40N-12N^2+N^3)}{3N^2(1+N)^2}S_1^2-\frac{32(N-1)S_1^3}{N(1+N)}+\frac{12(N-1)^2(2+N)}{N^2(1+N)^2}S_2+\frac{8(N-1)S_3}{N(1+N)}$$
$$\left.+\left[\frac{8(N-1)(1+3N+3N^2)}{N^2(1+N)^2}-\frac{16(N-1)S_1}{N(1+N)}\right]S_{-2}+\frac{8(N-1)S_{-3}}{N(1+N)}-\frac{16(N-1)S_{-2,1}}{N(1+N)}\right)\zeta_2$$
$$\left.+\left[-\frac{2(N-1)P_{42}}{9N^3(1+N)^3}+\frac{8(N-1)(60+7N+7N^2)}{9N^2(1+N)^2}S_1-\frac{64(N-1)S_1^2}{3N(1+N)}\right]\zeta_3\right\}$$
$$+C_F^2T_F\left\{\frac{P_{110}}{N^6(1+N)^6}+L_M^3\left[-\frac{2(N-1)(2+3N+3N^2)^2}{3N^3(1+N)^3}+\frac{16(N-1)(2+3N+3N^2)}{3N^2(1+N)^2}S_1-\frac{32(N-1)S_1^2}{3N(1+N)}\right]\right.$$
$$+L_M^2\left[\frac{(N-1)P_{73}}{N^4(1+N)^4}+\left[-\frac{8(N-1)P_{24}}{N^3(1+N)^3}+\frac{48(N-1)S_2}{N(1+N)}\right]S_1+\frac{4(N-1)(2+N)(5+3N)}{N^2(1+N)^2}S_1^2-\frac{16(N-1)S_1^3}{N(1+N)}\right.$$
$$-\frac{12(N-1)(2+3N+3N^2)}{N^2(1+N)^2}S_2+\frac{32(N-1)S_3}{N(1+N)}+\left[-\frac{32(N-1)}{N^2(1+N)^2}+\frac{64(N-1)S_1}{N(1+N)}\right]S_{-2}$$
$$\left.+\frac{32(N-1)S_{-3}}{N(1+N)}-\frac{64(N-1)S_{-2,1}}{N(1+N)}\right]+L_M\left[\frac{P_{101}}{2N^5(1+N)^5(2+N)}+\left[\frac{8P_{93}}{N^4(1+N)^4(2+N)}\right.\right.$$
$$\left.+\frac{16P_{15}}{N^2(1+N)^2(2+N)}S_2-\frac{96(N-1)S_3}{N(1+N)}+\frac{128(N-1)S_{2,1}}{N(1+N)}\right]S_1$$
$$+\left[-\frac{8P_{67}}{N^3(1+N)^3(2+N)}-\frac{16(N-1)S_2}{N(1+N)}\right]S_1^2+\frac{64(-1+N+N^2)}{N^2(1+N)(2+N)}S_1^3-\frac{8(N-1)S_1^4}{N(1+N)}$$
$$+\frac{4P_{70}}{N^3(1+N)^3(2+N)}S_2-\frac{24(N-1)S_2^2}{N(1+N)}-\frac{8(N-1)(-6+11N+3N^2)}{N^2(1+N)^2}S_3$$
$$-\frac{80(N-1)S_4}{N(1+N)}+\left[\frac{32P_{44}}{N^2(1+N)^3(2+N)}-\frac{128(N-1)S_1}{N(1+N)^2}-\frac{64(N-1)S_2}{N(1+N)}\right]S_{-2}$$
$$-\frac{32(N-1)S_{-2}^2}{N(1+N)}+\left[-\frac{64(N-1)^2}{N^2(1+N)^2}-\frac{128(N-1)S_1}{N(1+N)}\right]S_{-3}-\frac{160(N-1)S_{-4}}{N(1+N)}-\frac{64(N-1)S_{2,1}}{N^2(1+N)^2}+\frac{32(N-1)S_{3,1}}{N(1+N)}$$
$$\left.+\frac{128(N-1)S_{-2,1}}{N(1+N)^2}+\frac{64(N-1)S_{-2,2}}{N(1+N)}+\frac{128(N-1)S_{-3,1}}{N(1+N)}-\frac{96(N-1)S_{2,1,1}}{N(1+N)}+\frac{48(N-1)(-2+3N+3N^2)}{N^2(1+N)^2}\zeta_3\right]$$

$$
\begin{aligned}
&+\Big[-\frac{4P_{103}}{N^5(1+N)^5(2+N)}-\frac{4P_{69}}{N^3(1+N)^3(2+N)}S_2+\frac{4(N-1)S_2^2}{N(1+N)}-\frac{8P_{28}}{3N^2(1+N)^2(2+N)}S_3-\frac{24(N-1)S_4}{N(1+N)}\\
&-\frac{16(-6-9N-4N^2+3N^3)}{N^2(1+N)^2}S_{2,1}+\frac{64(N-1)S_{3,1}}{N(1+N)}-\frac{128(N-1)S_{2,1,1}}{N(1+N)}\Big]S_1+\Big[\frac{2P_{54}}{N^3(1+N)^2(2+N)}\\
&-\frac{2P_{21}}{N^2(1+N)^2(2+N)}S_2+\frac{32(N-1)S_3}{3N(1+N)}+\frac{64(N-1)S_{2,1}}{N(1+N)}\Big]S_1^2+\Big[-\frac{4P_{50}}{3N^3(1+N)^2(2+N)}+\frac{8(N-1)S_2}{N(1+N)}\Big]S_1^3\\
&+\frac{P_{11}}{3N^2(1+N)^2(2+N)}S_1^4+\frac{4(N-1)S_1^5}{3N(1+N)}+\frac{2P_{91}}{N^4(1+N)^4(2+N)}S_2-\frac{(N-1)(2+3N+3N^2)}{N^2(1+N)^2}S_2^2\\
&+\frac{4P_{76}}{3N^3(1+N)^3(2+N)}S_3+\frac{6(N-1)(2+3N+3N^2)}{N^2(1+N)^2}S_4-\frac{16(2+3N+3N^2)}{N^3(1+N)}S_{2,1}\\
&-\frac{16(N-1)(2+3N+3N^2)}{N^2(1+N)^2}S_{3,1}+\frac{32(N-1)(2+3N+3N^2)}{N^2(1+N)^2}S_{2,1,1}\\
&+\Big[-\frac{384\ln(2)(N-1)}{N(1+N)}+\frac{(N-1)P_{85}}{2N^4(1+N)^4}+\Big[-\frac{8(N-1)P_{33}}{N^3(1+N)^3}-\frac{32(N-1)S_2}{N(1+N)}\Big]S_1\\
&-\frac{4(N-1)(2+N)(5+3N)}{N^2(1+N)^2}S_1^2+\frac{16(N-1)S_1^3}{N(1+N)}+\frac{8(N-1)(2+3N+3N^2)}{N^2(1+N)^2}S_2-\frac{16(N-1)S_3}{N(1+N)}\\
&+\Big[\frac{16(N-1)}{N^2(1+N)^2}-\frac{32(N-1)S_1}{N(1+N)}\Big]S_{-2}-\frac{16(N-1)S_{-3}}{N(1+N)}+\frac{32(N-1)S_{-2,1}}{N(1+N)}\Big]\zeta_2\\
&+\Big[\frac{2(N-1)P_{41}}{3N^3(1+N)^3}-\frac{16(N-1)(2+3N+3N^2)}{3N^2(1+N)^2}S_1+\frac{32(N-1)S_1^2}{3N(1+N)}\Big]\zeta_3\Big\}\\
&+C_A^2T_F\Big\{-\frac{4(-24+11N+11N^2)P_{102}}{3N^6(1+N)^6(2+N)}+L_M^3\Big[\frac{16(N-1)(-24+11N+11N^2)}{9N^3(1+N)^3}\\
&-\frac{8(N-1)(-48+11N+11N^2)}{9N^2(1+N)^2}S_1-\frac{32(N-1)S_1^2}{3N(1+N)}\Big]+L_M^2\Big[\frac{16P_{87}}{9N^4(1+N)^4}+\Big[\frac{8P_{57}}{9N^3(1+N)^3}-\frac{48(N-1)S_2}{N(1+N)}\Big]S_1\\
&-\frac{4(24-83N+11N^3)}{3N^2(1+N)^2}S_1^2-\frac{16(N-1)S_1^3}{N(1+N)}-\frac{4(N-1)(-72+11N+11N^2)}{3N^2(1+N)^2}S_2-\frac{16(N-1)S_3}{N(1+N)}\\
&+\Big[-\frac{8(N-1)(-48+11N+11N^2)}{3N^2(1+N)^2}-\frac{64(N-1)S_1}{N(1+N)}\Big]S_{-2}-\frac{16(N-1)S_{-3}}{N(1+N)}+\frac{32(N-1)S_{-2,1}}{N(1+N)}\Big]\\
&+L_M\Big[-\frac{4P_{108}}{27N^5(1+N)^5(2+N)}+\Big[-\frac{8P_{99}}{27N^4(1+N)^4(2+N)}-\frac{8P_{32}}{3N^2(1+N)^2(2+N)}S_2-\frac{160(N-1)S_3}{N(1+N)}\\
&+\frac{64(N-1)S_{2,1}}{N(1+N)}+\frac{320(N-1)S_{-2,1}}{N(1+N)}\Big]S_1+\Big[\frac{4P_{80}}{9N^3(1+N)^3(2+N)}-\frac{128(N-1)S_2}{N(1+N)}\Big]S_1^2\\
&-\frac{8(-184-119N+4N^2+11N^3)}{9N(1+N)^2(2+N)}S_1^3-\frac{8(N-1)S_1^4}{N(1+N)}+\frac{4P_{81}}{9N^3(1+N)^3(2+N)}S_2-\frac{8(N-1)S_2^2}{N(1+N)}\\
&-\frac{8(441-568N+55N^3)}{9N^2(1+N)^2}S_3-\frac{16(N-1)S_4}{N(1+N)}+\Big[\frac{16P_{78}}{9N^3(1+N)^3(2+N)}-\frac{16(54-113N+24N^2+11N^3)}{3N^2(1+N)^2}S_1\\
&-\frac{160(N-1)S_1^2}{N(1+N)}-\frac{64(N-1)S_2}{N(1+N)}\Big]S_{-2}-\frac{48(N-1)S_{-2}^2}{N(1+N)}+\Big[-\frac{8(54-73N+11N^3)}{N^2(1+N)^2}-\frac{288(N-1)S_1}{N(1+N)}\Big]S_{-3}\\
&-\frac{176(N-1)S_{-4}}{N(1+N)}-\frac{16(N-1)(24+11N+11N^2)}{3N^2(1+N)^2}S_{2,1}-\frac{64(N-1)S_{3,1}}{N(1+N)}+\frac{16(66-101N+11N^3)}{3N^2(1+N)^2}S_{-2,1}\\
&+\frac{224(N-1)S_{-2,2}}{N(1+N)}+\frac{256(N-1)S_{-3,1}}{N(1+N)}-\frac{384(N-1)S_{-2,1,1}}{N(1+N)}+\Big[\frac{48(N-1)(-8+3N+3N^2)}{N^2(1+N)^2}+\frac{96(N-1)S_1}{N(1+N)}\Big]\zeta_3\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl[-\frac{4P_{104}}{3N^5(1+N)^5(2+N)}-\frac{4P_{68}}{3N^3(1+N)^3(2+N)}S_2+\frac{4(N-1)S_2^2}{N(1+N)}+\frac{16P_{37}}{9N^2(1+N)^2(2+N)}S_3+\frac{72(N-1)S_4}{N(1+N)}\\
&-\frac{32(N-1)S_{3,1}}{N(1+N)}-\frac{32(N-1)(-24+23N+11N^2)}{3N^2(1+N)^2}S_{-2,1}-\frac{64(N-1)S_{-2,2}}{N(1+N)}-\frac{64(N-1)S_{-3,1}}{N(1+N)}-\frac{32(N-1)S_{2,1,1}}{N(1+N)}\\
&+\frac{128(N-1)S_{-2,1,1}}{N(1+N)}\Biggr]S_1+\Biggl[\frac{2P_{75}}{3N^2(1+N)^4(2+N)}+\frac{2P_{38}}{3N^2(1+N)^2(2+N)}S_2+\frac{320(N-1)S_3}{3N(1+N)}-\frac{128(N-1)S_{-2,1}}{N(1+N)}\Biggr]S_1^2\\
&+\Biggl[-\frac{4P_{51}}{9N^2(1+N)^3(2+N)}+\frac{40(N-1)S_2}{N(1+N)}\Biggr]S_1^3+\frac{P_{31}}{9N^2(1+N)^2(2+N)}S_1^4+\frac{4(N-1)S_1^5}{3N(1+N)}\\
&-\frac{2(-24+11N+11N^2)P_{62}}{3N^4(1+N)^4(2+N)}S_2+\frac{(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2}S_2^2\\
&-\frac{32(-24+11N+11N^2)(-6+8N+7N^2+N^3)}{9N^3(1+N)^3(2+N)}S_3+\frac{6(N-1)(-24+11N+11N^2)}{N^2(1+N)^2}S_4\\
&+\Biggl[-\frac{16(3+N^2)(-24+11N+11N^2)}{3N^2(1+N)^4}+\Biggl[\frac{32(-3+N)(-8+15N+5N^2)}{3N^2(1+N)^3}+\frac{64(N-1)S_2}{N(1+N)}\Biggr]S_1\\
&+\frac{16(N-1)(-24+35N+11N^2)}{3N^2(1+N)^2}S_1^2+\frac{64(N-1)S_1^3}{N(1+N)}+\frac{16(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2}S_2\Biggr]S_{-2}\\
&+\Biggl[\frac{16(N-1)(-24+11N+11N^2)}{3N^2(1+N)^3}+\frac{16(N-1)(-24+23N+11N^2)}{3N^2(1+N)^2}S_1+\frac{64(N-1)S_1^2}{N(1+N)}\Biggr]S_{-3}\\
&+\Biggl[\frac{8(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2}+\frac{32(N-1)S_1}{N(1+N)}\Biggr]S_{-4}-\frac{8(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2}S_{3,1}\\
&-\frac{32(N-1)(-24+11N+11N^2)}{3N^2(1+N)^3}S_{-2,1}-\frac{16(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2}S_{-2,2}\\
&-\frac{16(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2}S_{-3,1}-\frac{8(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2}S_{2,1,1}\\
&+\frac{32(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2}S_{-2,1,1}+\Biggl[-\frac{2P_{88}}{9N^4(1+N)^4}+\Biggl[-\frac{4P_{58}}{9N^3(1+N)^3}+\frac{32(N-1)S_2}{N(1+N)}\Biggr]S_1\\
&+\frac{4(24-83N+11N^3)}{3N^2(1+N)^2}S_1^2+\frac{16(N-1)S_1^3}{N(1+N)}+\frac{4(N-1)(-48+11N+11N^2)}{3N^2(1+N)^2}S_2+\frac{8(N-1)S_3}{N(1+N)}\\
&+\Biggl[\frac{8(N-1)(-36+11N+11N^2)}{3N^2(1+N)^2}+\frac{48(N-1)S_1}{N(1+N)}\Biggr]S_{-2}+\frac{8(N-1)S_{-3}}{N(1+N)}-\frac{16(N-1)S_{-2,1}}{N(1+N)}\Biggr]\zeta_2\\
&+\Biggl[-\frac{16(N-1)(-24+11N+11N^2)}{9N^3(1+N)^3}+\frac{8(N-1)(-48+11N+11N^2)}{9N^2(1+N)^2}S_1+\frac{32(N-1)S_1^2}{3N(1+N)}\Biggr]\zeta_3\Biggr\}\\
&+T_F^3\Biggl[-\frac{64L_M^3(N-1)}{9N(1+N)}+\frac{64(N-1)\zeta_3}{9N(1+N)}\Biggr]\Biggr\}, \qquad (40)
\end{aligned}
$$

and the polynomials P_i read

$$P_{11}=-3N^4-54N^3-95N^2-12N+36, \qquad (41)$$

$$P_{12}=N^4-94N^3-256N^2-161N+78, \qquad (42)$$

$$P_{13}=N^4+2N^3-5N^2-12N+2, \qquad (43)$$

$$P_{14}=N^4+4N^3-N^2-10N+2, \qquad (44)$$

$$P_{15}=N^4+10N^3+27N^2+30N+4, \qquad (45)$$

$$P_{16}=N^4+17N^3+43N^2+33N+2, \qquad (46)$$

$$P_{17}=2N^4-4N^3-3N^2+20N+12, \qquad (47)$$

$$P_{18}=2N^4+3N^3-12N^2-23N+6, \qquad (48)$$

$$P_{19}=2N^4+39N^3+100N^2+73N+2, \qquad (49)$$

$$P_{20} = 3N^4 + 6N^3 - N^2 - 4N + 12, \quad (50)$$

$$P_{21} = 3N^4 + 30N^3 + 47N^2 + 4N - 20, \quad (51)$$

$$P_{22} = 3N^4 + 48N^3 + 123N^2 + 98N + 8, \quad (52)$$

$$P_{23} = 5N^4 + 10N^3 + 8N^2 + 7N + 2, \quad (53)$$

$$P_{24} = 5N^4 + 13N^3 + 14N^2 + 16N + 6, \quad (54)$$

$$P_{25} = 5N^4 + 37N^3 + 82N^2 + 41N - 48, \quad (55)$$

$$P_{26} = 6N^4 + 11N^3 - 6N^2 - N - 2, \quad (56)$$

$$P_{27} = 6N^4 + 12N^3 + 7N^2 + N + 6, \quad (57)$$

$$P_{28} = 9N^4 + 102N^3 + 245N^2 + 192N + 12, \quad (58)$$

$$P_{29} = 10N^4 + 53N^3 + 92N^2 + 37N - 48, \quad (59)$$

$$P_{30} = 11N^4 - 68N^3 - 263N^2 - 184N + 72, \quad (60)$$

$$P_{31} = 11N^4 - 26N^3 - 227N^2 - 286N + 48, \quad (61)$$

$$P_{32} = 11N^4 + 4N^3 - 239N^2 - 304N + 240, \quad (62)$$

$$P_{33} = 13N^4 + 23N^3 + 4N^2 - 14N - 5, \quad (63)$$

$$P_{34} = 13N^4 + 140N^3 + 365N^2 + 190N - 276, \quad (64)$$

$$P_{35} = 17N^4 + 34N^3 + 82N^2 + 161N - 78, \quad (65)$$

$$P_{36} = 29N^4 + 60N^3 + 149N^2 + 336N + 74, \quad (66)$$

$$P_{37} = 55N^4 + 86N^3 - 343N^2 - 422N + 384, \quad (67)$$

$$P_{38} = 55N^4 + 182N^3 - 175N^2 - 542N + 240, \quad (68)$$

$$P_{39} = 76N^4 + 183N^3 + 196N^2 + 267N - 38, \quad (69)$$

$$P_{40} = 97N^4 + 494N^3 + 1079N^2 + 898N - 408, \quad (70)$$

$$P_{41} = 153N^4 + 306N^3 + 165N^2 + 12N + 4, \quad (71)$$

$$P_{42} = 183N^4 + 366N^3 + 305N^2 + 122N + 96, \quad (72)$$

$$P_{43} = N^5 + N^4 - 4N^3 + 3N^2 - 7N - 2, \quad (73)$$

$$P_{44} = 2N^5 + 6N^4 + 3N^3 + 11N + 2, \quad (74)$$

$$P_{45} = 2N^5 + 10N^4 + 29N^3 + 64N^2 + 67N + 8, \quad (75)$$

$$P_{46} = 3N^5 + 8N^4 + 6N^3 + 10N^2 + 7N + 2, \quad (76)$$

$$P_{47} = 8N^5 + 7N^4 - 9N^3 + 7N^2 + 13N + 6, \quad (77)$$

$$P_{48} = 9N^5 + 9N^4 - 79N^3 + 15N^2 + 22N - 24, \quad (78)$$

$$P_{49} = 15N^5 + 15N^4 - 103N^3 + 33N^2 - 20N - 36, \quad (79)$$

$$P_{50} = 18N^5 - 15N^4 - 198N^3 - 381N^2 - 216N + 4, \quad (80)$$

$$P_{51} = 18N^5 + 47N^4 - 35N^3 - 141N^2 - 5N - 120, \quad (81)$$

$$P_{52} = 40N^5 + 73N^4 - 142N^3 - 163N^2 - 150N + 54, \quad (82)$$

$$P_{53} = 45N^5 + 45N^4 - 47N^3 + 27N^2 - 190N - 24, \quad (83)$$

$$P_{54} = 51N^5 + 89N^4 + 6N^3 - 66N^2 - 104N - 72, \quad (84)$$

$$P_{55} = 66N^5 + 336N^4 + 627N^3 + 415N^2 - 10N - 194, \quad (85)$$

$$P_{56} = 69N^5 + 69N^4 - 55N^3 + 51N^2 - 338N - 36, \quad (86)$$

$$P_{57} = 85N^5 + 85N^4 - 73N^3 + 197N^2 - 342N - 108, \quad (87)$$

$$P_{58} = 103N^5 + 103N^4 - 79N^3 + 317N^2 - 612N - 144, \quad (88)$$

$$P_{59} = 337N^5 + 403N^4 - 541N^3 - 583N^2 - 300N + 108, \quad (89)$$

$$P_{60} = 436N^5 + 1780N^4 + 2689N^3 + 2782N^2 + 2167N - 134, \quad (90)$$

$$P_{61} = 489N^5 + 489N^4 - 1187N^3 - 57N^2 - 742N - 144, \quad (91)$$

$$P_{62} = N^6 + 18N^5 + 63N^4 + 84N^3 + 30N^2 - 64N - 16, \quad (92)$$

$$P_{63} = N^6 + 23N^5 + 73N^4 + 85N^3 + 58N^2 + 24N - 24, \quad (93)$$

$$P_{64} = 3N^6 + 30N^5 + 107N^4 + 124N^3 + 48N^2 + 20N + 8, \quad (94)$$

$$P_{65} = 3N^6 + 51N^5 + 153N^4 + 185N^3 + 160N^2 + 80N - 72, \quad (95)$$

$$P_{66} = 4N^6 + 41N^5 + 126N^4 + 163N^3 + 58N^2 - 128N - 32, \quad (96)$$

$$P_{67} = 5N^6 + 26N^5 + 77N^4 + 168N^3 + 159N^2 + 19N - 22, \tag{97}$$

$$P_{68} = 6N^6 + 75N^5 + 345N^4 + 719N^3 + 323N^2 - 696N - 96, \tag{98}$$

$$P_{69} = 18N^6 + 87N^5 + 199N^4 + 185N^3 + 63N^2 + 44N + 20, \tag{99}$$

$$P_{70} = 23N^6 + 39N^5 - 89N^4 - 219N^3 - 172N^2 - 130N - 28, \tag{100}$$

$$P_{71} = 25N^6 - 118N^5 - 662N^4 - 500N^3 + 421N^2 + 186N - 264, \tag{101}$$

$$P_{72} = 33N^6 + 99N^5 + 41N^4 - 11N^3 + 86N^2 - 216N - 144, \tag{102}$$

$$P_{73} = 33N^6 + 99N^5 + 137N^4 + 157N^3 + 62N^2 + 8N - 16, \tag{103}$$

$$P_{74} = 36N^6 + 48N^5 - 297N^4 - 977N^3 - 976N^2 - 362N + 24, \tag{104}$$

$$P_{75} = 37N^6 + 207N^5 + 753N^4 + 1771N^3 + 1598N^2 - 118N + 48, \tag{105}$$

$$P_{76} = 57N^6 + 297N^5 + 567N^4 + 615N^3 + 468N^2 + 220N + 16, \tag{106}$$

$$P_{77} = 80N^6 + 201N^5 - 775N^4 - 3495N^3 - 4405N^2 - 2238N - 72, \tag{107}$$

$$P_{78} = 94N^6 + 282N^5 + 79N^4 + 42N^3 + 286N^2 - 585N - 18, \tag{108}$$

$$P_{79} = 129N^6 + 387N^5 + 509N^4 + 349N^3 + 50N^2 + 240N + 144, \tag{109}$$

$$P_{80} = 170N^6 + 543N^5 + 221N^4 - 15N^3 + 425N^2 - 864N - 288, \tag{110}$$

$$P_{81} = 170N^6 + 873N^5 + 1547N^4 + 951N^3 - 1717N^2 - 2976N + 864, \tag{111}$$

$$P_{82} = 243N^6 + 729N^5 + 923N^4 + 583N^3 + 14N^2 + 300N + 216, \tag{112}$$

$$P_{83} = 321N^6 + 1353N^5 + 1521N^4 - 713N^3 - 2842N^2 - 2216N + 96, \tag{113}$$

$$P_{84} = 333N^6 + 999N^5 + 1075N^4 + 389N^3 - 68N^2 + 384N + 216, \tag{114}$$

$$P_{85} = 633N^6 + 1899N^5 + 1967N^4 + 697N^3 - 4N^2 - 48N + 8, \tag{115}$$

$$P_{86} = -891N^7 - 1782N^6 - 3712N^5 - 3058N^4 + 6775N^3 + 7144N^2 + 276N - 144, \tag{116}$$

$$P_{87} = 9N^7 + 18N^6 - 124N^5 - 109N^4 + 199N^3 - 191N^2 + 138N + 72, \tag{117}$$

$$P_{88} = 69N^7 + 138N^6 - 667N^5 - 541N^4 + 952N^3 - 1277N^2 + 990N + 432, \tag{118}$$

$$P_{89} = 95N^7 + 378N^6 + 853N^5 + 1832N^4 + 2190N^3 + 364N^2 - 780N - 432, \tag{119}$$

$$P_{90} = 251N^7 + 1335N^6 + 1745N^5 + 243N^4 - 529N^3 - 4161N^2 - 5400N - 756, \tag{120}$$

$$P_{91} = N^8 + 427N^7 + 2161N^6 + 4081N^5 + 3554N^4 + 1404N^3 + 228N^2 + 64N + 16, \tag{121}$$

$$P_{92} = 2N^8 + 10N^7 + 22N^6 + 36N^5 + 29N^4 + 4N^3 + 33N^2 + 12N + 4, \tag{122}$$

$$P_{93} = 2N^8 + 29N^7 + 135N^6 + 297N^5 + 333N^4 + 204N^3 + 28N^2 - 44N - 24, \tag{123}$$

$$P_{94} = 3N^8 + 33N^7 + 149N^6 + 267N^5 + 196N^4 + 104N^3 + 64N^2 - 88N - 48, \tag{124}$$

$$P_{95} = 12N^8 + 52N^7 + 60N^6 - 25N^4 - 2N^3 + 3N^2 + 8N + 4, \tag{125}$$

$$P_{96} = 36N^8 + 348N^7 + 1210N^6 + 2229N^5 + 2168N^4 + 505N^3 - 424N^2 - 68N + 48, \tag{126}$$

$$P_{97} = 111N^8 + 480N^7 + 286N^6 - 468N^5 + 82N^4 + 246N^3 + 295N^2 + 228N + 252, \tag{127}$$

$$P_{98} = 201N^8 + 840N^7 + 565N^6 - 699N^5 - 344N^4 - 645N^3 - 314N^2 + 324N + 360, \tag{128}$$

$$P_{99} = 296N^8 + 1184N^7 + 2744N^6 + 5900N^5 + 4088N^4 + 476N^3 + 9477N^2 + 4725N + 702, \tag{129}$$

$$P_{100} = -7299N^{10} - 39375N^9 - 79900N^8 - 85198N^7 - 17323N^6 + 129917N^5 + 137090N^4 + 25904N^3 + 12072N^2 + 30672N + 8640, \tag{130}$$

$$P_{101} = -149N^{10} - 793N^9 - 1404N^8 - 1170N^7 - 1341N^6 - 1221N^5 + 1710N^4 + 2800N^3 + 2256N^2 + 368N - 32, \tag{131}$$

$$P_{102} = 4N^{10} + 22N^9 + 45N^8 + 36N^7 - 11N^6 - 15N^5 + 25N^4 - 41N^3 - 21N^2 - 16N - 4, \tag{132}$$

$$P_{103} = 10N^{10} + 62N^9 + 403N^8 + 1523N^7 + 2997N^6 + 3197N^5 + 1812N^4 + 478N^3 + 46N^2 + 24N + 8, \tag{133}$$

$$P_{104} = 26N^{10} + 132N^9 + 159N^8 - 351N^7 - 877N^6 + 531N^5 + 1820N^4 - 300N^3 - 252N^2 - 192N - 48, \tag{134}$$

$$P_{105} = 28N^{10} + 139N^9 + 444N^8 + 803N^7 + 451N^6 + 3N^5 + 490N^4 + 219N^3 + 51N^2 - 60N - 12, \tag{135}$$

$$P_{106} = 435N^{10} + 2391N^9 + 6946N^8 + 11512N^7 + 4822N^6 - 7016N^5 - 5369N^4 - 6743N^3 - 2406N^2 - 1764N - 216, \tag{136}$$

$$P_{107} = 531N^{10} + 2799N^9 + 4124N^8 + 446N^7 - 3445N^6 - 5245N^5 + 4358N^4 + 18128N^3 - 1968N^2 - 10800N - 4320, \tag{137}$$

$$P_{108} = 939N^{10} + 4893N^9 + 5386N^8 - 5198N^7 - 10400N^6 - 17636N^5 - 18137N^4 + 7177N^3 - 21672N^2 - 14112N - 4104, \tag{138}$$

$$P_{109} = 1773N^{10} + 9153N^9 + 14204N^8 + 2930N^7 - 9151N^6 - 8431N^5 - 250N^4 + 13772N^3 + 1920N^2 - 8928N - 4320, \tag{139}$$

$$P_{110} = -23N^{11} - 92N^{10} - 53N^9 + 322N^8 + 465N^7 - 348N^6 - 929N^5 - 384N^4 + 132N^3 + 102N^2 + 32N + 8, \tag{140}$$

$$P_{111} = 87N^{12} + 490N^{11} + 949N^{10} + 368N^9 - 1285N^8 - 2214N^7 - 1591N^6 - 126N^5 + 644N^4 - 86N^3 - 268N^2 - 184N - 48, \tag{141}$$

$$P_{112} = 385N^{12} + 2182N^{11} + 4181N^{10} + 1458N^9 - 5589N^8 - 8414N^7 - 5041N^6 - 1754N^5 - 760N^4 - 176N^3 + 152N^2 + 224N + 96, \tag{142}$$

$$P_{113} = 1623N^{12} + 9602N^{11} + 20093N^{10} + 15520N^9 - 3305N^8 - 13494N^7 - 5099N^6 + 9414N^5 + 10456N^4 + 5270N^3 + 1624N^2 + 40N - 96. \tag{143}$$

V. $A_{gg,Q}^{(3),\mathrm{S}}$

For the logarithmic contributions to the OME $A_{gg,Q}^{(3),\mathrm{S}}$ we obtain

$$
\begin{aligned}
A_{gg,Q} = {} & \frac{4a_s L_M T_F}{3} + a_s^2\Bigg\{\frac{16L_M^2 T_F^2}{9} + C_F T_F\left[\frac{4L_M P_{147}}{N^3(1+N)^3} + \frac{P_{166}}{N^4(1+N)^4} + \frac{4L_M^2(N-1)(2+N)}{N^2(1+N)^2}\right] \\
& + C_A T_F\Bigg\{\frac{2P_{155}}{27N^3(1+N)^3} + L_M\left[\frac{16P_{134}}{9N^2(1+N)^2} - \frac{80}{9}S_1\right] + L_M^2\left[\frac{16}{3N(1+N)} - \frac{8}{3}S_1\right] - \frac{4(47+56N)S_1}{27(1+N)}\Bigg\}\Bigg\} \\
& + a_s^3\Bigg\{\frac{64L_M^3}{27}T_F^3 + C_F T_F^2\Bigg\{\frac{2P_{177}}{9N^5(1+N)^5} + \frac{80L_M^3(N-1)(2+N)}{9N^2(1+N)^2} + L_M^2\left[\frac{8P_{153}}{9N^3(1+N)^3} + \frac{32(N-1)(2+N)S_1}{3N^2(1+N)^2}\right] \\
& + L_M\left[-\frac{8P_{173}}{27N^4(1+N)^4} + \frac{32(N-1)(2+N)(-6-8N+N^2)}{9N^3(1+N)^3}S_1 + \frac{16(N-1)(2+N)S_1^2}{3N^2(1+N)^2} - \frac{16(N-1)(2+N)S_2}{N^2(1+N)^2}\right] \\
& + \left[-\frac{8P_{154}}{9N^3(1+N)^3} - \frac{16(N-1)(2+N)S_1}{3N^2(1+N)^2}\right]\zeta_2 - \frac{80(N-1)(2+N)\zeta_3}{9N^2(1+N)^2}\Bigg\} \\
& + C_F N_F T_F^2\Bigg\{\frac{2P_{178}}{81N^5(1+N)^5} + \frac{64L_M^3(N-1)(2+N)}{9N^2(1+N)^2} + L_M\left[-\frac{4P_{171}}{9N^4(1+N)^4} - \frac{32(N-1)(2+N)(4+6N+N^2)}{3N^3(1+N)^3}S_1\right. \\
& \left. + \frac{16(N-1)(2+N)S_1^2}{N^2(1+N)^2} - \frac{80(N-1)(2+N)S_2}{3N^2(1+N)^2}\right] \\
& + \left[\frac{32(N-1)(2+N)(22+41N+28N^2)}{27N^2(1+N)^4} + \frac{16(N-1)(2+N)S_2}{3N^2(1+N)^2}\right]S_1 - \frac{16(N-1)(2+N)(2+5N)}{9N^2(1+N)^3}S_1^2 \\
& + \frac{16(N-1)(2+N)S_1^3}{9N^2(1+N)^2} - \frac{16(N-1)(2+N)(2+5N)}{9N^2(1+N)^3}S_2 + \frac{32(N-1)(2+N)S_3}{9N^2(1+N)^2} \\
& + \left[\frac{4P_{162}}{9N^3(1+N)^3} + \frac{16(N-1)(2+N)S_1}{3N^2(1+N)^2}\right]\zeta_2 - \frac{64(N-1)(2+N)\zeta_3}{9N^2(1+N)^2}\Bigg\} \\
& + C_A^2 T_F\Bigg\{-\frac{4P_{174}}{243N^4(1+N)^4} + L_M^3\left[-\frac{352}{27N(1+N)} + \frac{176}{27}S_1\right] \\
& + L_M^2\left[-\frac{2P_{149}}{9N^3(1+N)^3} + \left[-\frac{8P_{140}}{9N^2(1+N)^2} + \frac{64}{3}S_2\right]S_1 - \frac{128S_2}{3N(1+N)} + \frac{32}{3}S_3\right. \\
& \left. + \left[-\frac{128}{3N(1+N)} + \frac{64}{3}S_1\right]S_{-2} + \frac{32}{3}S_{-3} - \frac{64}{3}S_{-2,1}\right] \\
& + L_M\left[\frac{16S_2P_{133}}{9N^2(1+N)^2} + \frac{16S_{-3}P_{139}}{9N^2(1+N)^2} - \frac{32S_{-2,1}P_{139}}{9N^2(1+N)^2} + \frac{8S_3P_{142}}{9N^2(1+N)^2} + \frac{P_{183}}{81(N-1)N^5(1+N)^5(2+N)}\right. \\
& + \left[-\frac{4P_{176}}{81(N-1)N^4(1+N)^4(2+N)} + \frac{640}{9}S_2 - \frac{32}{3}S_3\right]S_1 + \left[-\frac{16P_{163}}{9(N-1)N^3(1+N)^3(2+N)}\right. \\
& \left.\left. + \frac{32P_{157}}{9(N-1)N^2(1+N)^2(2+N)}S_1\right]S_{-2} + \frac{32}{3}S_{-2}^2 + \left[\frac{64(-3+2N+2N^2)}{N^2(1+N)^2} - 64S_1\right]\zeta_3\right] \\
& - \frac{8(2339+4876N+2834N^2)}{243(1+N)^2}S_1 - \frac{44S_1^2}{9(1+N)} + \frac{44(1+2N)S_2}{9(1+N)} \\
& + \left[\frac{4P_{160}}{27N^3(1+N)^3} + \left(\frac{16(36+72N+N^2+2N^3+N^4)}{27N^2(1+N)^2} - \frac{32}{3}S_2\right)S_1\right. \\
& \left. + \frac{64S_2}{3N(1+N)} - \frac{16}{3}S_3 + \left[\frac{64}{3N(1+N)} - \frac{32}{3}S_1\right]S_{-2} - \frac{16}{3}S_{-3} + \frac{32}{3}S_{-2,1}\right]\zeta_2 + \left[\frac{352}{27N(1+N)} - \frac{176}{27}S_1\right]\zeta_3\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+C_A N_F T_F^2\Big\{\frac{16P_{172}}{243N^4(1+N)^4}+L_M\Big[-\frac{16S_1P_{145}}{81N^2(1+N)^2}-\frac{4P_{164}}{81N^3(1+N)^3}\Big]+L_M^3\Big[\frac{128}{27N(1+N)}-\frac{64}{27}S_1\Big]\\
&+\frac{32(283+584N+328N^2)}{243(1+N)^2}S_1+\frac{16S_1^2}{9(1+N)}-\frac{16(1+2N)S_2}{9(1+N)}+\Big[-\frac{4P_{137}}{27N^2(1+N)^2}+\frac{160}{27}S_1\Big]\zeta_2\\
&+\Big[-\frac{128}{27N(1+N)}+\frac{64}{27}S_1\Big]\zeta_3\Big\}+C_A T_F^2\Big\{-\frac{8P_{167}}{81N^4(1+N)^4}+L_M\Big[-\frac{8S_1P_{141}}{9N^2(1+N)^2}-\frac{2P_{159}}{27N^3(1+N)^3}\Big]\\
&+L_M^2\Big[\frac{8P_{143}}{27N^2(1+N)^2}-\frac{640}{27}S_1\Big]+L_M^3\Big[\frac{448}{27N(1+N)}-\frac{224}{27}S_1\Big]+\frac{16(283+584N+328N^2)}{81(1+N)^2}S_1\\
&+\frac{8S_1^2}{3(1+N)}-\frac{8(1+2N)S_2}{3(1+N)}+\Big[-\frac{4P_{144}}{27N^2(1+N)^2}+\frac{560}{27}S_1\Big]\zeta_2+\Big[-\frac{448}{27N(1+N)}+\frac{224}{27}S_1\Big]\zeta_3\Big\}\\
&+C_F^2 T_F\Big\{\frac{8S_3P_{136}}{3N^3(1+N)^3}+\frac{4S_2P_{151}}{N^4(1+N)^4}+\frac{P_{180}}{N^6(1+N)^6}\\
&+L_M^3\Big[-\frac{4(N-1)(2+N)(2+3N+3N^2)}{3N^3(1+N)^3}+\frac{16(N-1)(2+N)S_1}{3N^2(1+N)^2}\Big]\\
&+L_M^2\Big[-\frac{8(N-1)(2+N)(-2-3N+N^3+2N^4)}{N^4(1+N)^4}+\frac{8(N-1)^2(2+N)(2+3N)}{N^3(1+N)^3}S_1-\frac{16(N-1)(2+N)S_2}{N^2(1+N)^2}\Big]\\
&+L_M\Big[-\frac{4S_2P_{138}}{N^3(1+N)^3}-\frac{2P_{181}}{(N-1)N^5(1+N)^5(2+N)}+\Big[-\frac{8P_{152}}{N^4(1+N)^4}+\frac{24(N-1)(2+N)S_2}{N^2(1+N)^2}\Big]S_1\\
&+\frac{4(-6-13N+3N^3)}{N^2(1+N)^3}S_1^2-\frac{8(N-1)(2+N)S_1^3}{3N^2(1+N)^2}+\frac{16(14+5N+5N^2)}{3N^2(1+N)^2}S_3\\
&+\Big[-\frac{32(10+N+N^2)}{(N-1)N(1+N)(2+N)}+\frac{256S_1}{N^2(1+N)^2}\Big]S_{-2}+\frac{128S_{-3}}{N^2(1+N)^2}-\frac{32(N-1)(2+N)S_{2,1}}{N^2(1+N)^2}\\
&-\frac{256S_{-2,1}}{N^2(1+N)^2}-\frac{96(2+N+N^2)\zeta_3}{N^2(1+N)^2}\Big]+\Big[-\frac{8P_{132}}{N^3(1+N)^3}+\frac{8(-2+3N+3N^2)}{N^3(1+N)^2}S_2\\
&-\frac{16(N-1)(2+N)S_3}{3N^2(1+N)^2}-\frac{32(N-1)(2+N)S_{2,1}}{N^2(1+N)^2}\Big]S_1\\
&+\Big[\frac{4(-36-22N-6N^2+N^3)}{N^3(1+N)^2}-\frac{4(N-1)(2+N)S_2}{N^2(1+N)^2}\Big]S_1^2\\
&+\frac{8(-2+3N+3N^2)}{3N^3(1+N)^2}S_1^3-\frac{2(N-1)(2+N)S_1^4}{3N^2(1+N)^2}-\frac{2(N-1)(2+N)S_2^2}{N^2(1+N)^2}\\
&+\frac{12(N-1)(2+N)S_4}{N^2(1+N)^2}-\frac{32(2+N)S_{2,1}}{N^3(1+N)}-\frac{32(N-1)(2+N)S_{3,1}}{N^2(1+N)^2}\\
&+\frac{64(N-1)(2+N)S_{2,1,1}}{N^2(1+N)^2}+\Big[128\ln(2)-\frac{2P_{170}}{N^4(1+N)^4}\\
&-\frac{4(N-1)(2+N)(-4-3N+3N^2)}{N^3(1+N)^3}S_1-\frac{4(N-1)(2+N)S_1^2}{N^2(1+N)^2}\\
&+\frac{12(N-1)(2+N)S_2}{N^2(1+N)^2}\Big]\zeta_2+\Big[-\frac{4P_{158}}{3N^3(1+N)^3}-\frac{16(N-1)(2+N)S_1}{3N^2(1+N)^2}\Big]\zeta_3\Big\}\\
&+C_A C_F T_F\Big\{-\frac{4S_2P_{148}}{N^4(1+N)^4}+\frac{P_{179}}{18N^6(1+N)^6}\\
&+L_M^3\Big[-\frac{8(N-1)(2+N)(-12+11N+11N^2)}{9N^3(1+N)^3}-\frac{16(N-1)(2+N)S_1}{3N^2(1+N)^2}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+L_M^2\Biggl[-\frac{8S_1P_{150}}{3N^3(1+N)^3}-\frac{2P_{169}}{9N^4(1+N)^4}-\frac{16(N-1)(2+N)S_2}{N^2(1+N)^2}\\
&-\frac{32(N-1)(2+N)S_{-2}}{N^2(1+N)^2}\Biggr]+L_M\Biggl[\frac{4S_2P_{135}}{N^3(1+N)^3}+\frac{8P_{182}}{27(N-1)N^5(1+N)^5(2+N)}\\
&+\Biggl[-\frac{8P_{175}}{9(N-1)N^4(1+N)^4(2+N)}-\frac{40(N-1)(2+N)S_2}{N^2(1+N)^2}\Biggr]S_1\\
&-\frac{4(-12-16N+5N^2+11N^3)}{3N^3(1+N)^2}S_1^2+\frac{8(N-1)(2+N)S_1^3}{3N^2(1+N)^2}-\frac{16(26+5N+5N^2)}{3N^2(1+N)^2}S_3\\
&+\Biggl[-\frac{16P_{146}}{(N-1)N^2(1+N)^3(2+N)}+\frac{32P_{130}}{(N-1)N^2(1+N)^2(2+N)}S_1\Biggr]S_{-2}\\
&+\frac{16(-22+5N+5N^2)}{N^2(1+N)^2}S_{-3}+\frac{32(N-1)(2+N)S_{2,1}}{N^2(1+N)^2}-\frac{32(-14+N+N^2)S_{-2,1}}{N^2(1+N)^2}\\
&+\Biggl[-\frac{32(-3+N)(4+N)}{N^2(1+N)^2}+64S_1\Biggr]\zeta_3\Biggr]+\Biggl[-\frac{2P_{165}}{9N^2(1+N)^5}+\frac{8(-13+3N^2)}{N^2(1+N)^3}S_2\\
&+\frac{160(N-1)(2+N)S_3}{3N^2(1+N)^2}-\frac{64(N-1)(2+N)S_{-2,1}}{N^2(1+N)^2}\Biggr]S_1+\Biggl[-\frac{4P_{131}}{N^2(1+N)^4}\\
&+\frac{20(N-1)(2+N)S_2}{N^2(1+N)^2}\Biggr]S_1^2-\frac{8(5+4N+N^2)}{3N^2(1+N)^3}S_1^3+\frac{2(N-1)(2+N)S_1^4}{3N^2(1+N)^2}\\
&+\frac{2(N-1)(2+N)S_2^2}{N^2(1+N)^2}-\frac{64(-6+8N+7N^2+N^3)}{3N^3(1+N)^3}S_3+\frac{36(N-1)(2+N)S_4}{N^2(1+N)^2}\\
&+\Biggl[-\frac{32(2+N)(3+N^2)}{N^2(1+N)^4}+\frac{64(N-1)(2+N)S_1}{N^2(1+N)^3}+\frac{32(N-1)(2+N)S_1^2}{N^2(1+N)^2}\\
&+\frac{32(N-1)(2+N)S_2}{N^2(1+N)^2}\Biggr]S_{-2}+\Biggl[\frac{32(N-1)(2+N)}{N^2(1+N)^3}+\frac{32(N-1)(2+N)S_1}{N^2(1+N)^2}\Biggr]S_{-3}\\
&+\frac{16(N-1)(2+N)S_{-4}}{N^2(1+N)^2}-\frac{16(N-1)(2+N)S_{3,1}}{N^2(1+N)^2}-\frac{64(N-1)(2+N)S_{-2,1}}{N^2(1+N)^3}\\
&-\frac{32(N-1)(2+N)S_{-2,2}}{N^2(1+N)^2}-\frac{32(N-1)(2+N)S_{-3,1}}{N^2(1+N)^2}-\frac{16(N-1)(2+N)S_{2,1,1}}{N^2(1+N)^2}\\
&+\frac{64(N-1)(2+N)S_{-2,1,1}}{N^2(1+N)^2}+\Biggl[-64\ln(2)-\frac{4S_1P_{161}}{3N^3(1+N)^3}+\frac{4P_{168}}{9N^4(1+N)^4}\\
&+\frac{4(N-1)(2+N)S_1^2}{N^2(1+N)^2}+\frac{12(N-1)(2+N)S_2}{N^2(1+N)^2}+\frac{24(N-1)(2+N)S_{-2}}{N^2(1+N)^2}\Biggr]\zeta_2\\
&+\Biggl[\frac{8P_{156}}{9N^3(1+N)^3}+\frac{16(N-1)(2+N)S_1}{3N^2(1+N)^2}\Biggr]\zeta_3\Biggr\}-\frac{64}{27}T_F^3\zeta_3+a_{gg,Q}^{(3)}\Biggr\}.
\end{aligned}
\tag{144}
$$

The polynomials P_i read

$$P_{130}=N^4+2N^3-7N^2-8N+28, \tag{145}$$

$$P_{131}=N^4+2N^3-5N^2-12N+2, \tag{146}$$

$$P_{132}=2N^4-4N^3-3N^2+20N+12, \tag{147}$$

$$P_{133} = 3N^4 + 6N^3 - 89N^2 - 92N + 12, \tag{148}$$

$$P_{134} = 3N^4 + 6N^3 + 16N^2 + 13N - 3, \tag{149}$$

$$P_{135} = 3N^4 + 32N^3 + 65N^2 - 16N - 60, \tag{150}$$

$$P_{136} = 3N^4 + 48N^3 + 123N^2 + 98N + 8, \tag{151}$$

$$P_{137} = 9N^4 + 18N^3 + 113N^2 + 104N - 24, \tag{152}$$

$$P_{138} = 11N^4 + 36N^3 + 43N^2 + 46N + 8, \tag{153}$$

$$P_{139} = 20N^4 + 40N^3 + 11N^2 - 9N + 54, \tag{154}$$

$$P_{140} = 23N^4 + 46N^3 + 23N^2 + 96N + 48, \tag{155}$$

$$P_{141} = 40N^4 + 74N^3 + 25N^2 - 9N + 16, \tag{156}$$

$$P_{142} = 40N^4 + 80N^3 + 73N^2 + 33N + 54, \tag{157}$$

$$P_{143} = 63N^4 + 126N^3 + 271N^2 + 208N - 48, \tag{158}$$

$$P_{144} = 99N^4 + 198N^3 + 463N^2 + 364N - 84, \tag{159}$$

$$P_{145} = 136N^4 + 254N^3 + 37N^2 - 81N + 144, \tag{160}$$

$$P_{146} = 3N^5 + 7N^4 - 29N^3 - 51N^2 - 2N - 8, \tag{161}$$

$$P_{147} = N^6 + 3N^5 + 5N^4 + N^3 - 8N^2 + 2N + 4, \tag{162}$$

$$P_{148} = N^6 + 18N^5 + 63N^4 + 84N^3 + 30N^2 - 64N - 16, \tag{163}$$

$$P_{149} = 3N^6 + 9N^5 - 163N^4 - 341N^3 + 164N^2 - 432N - 192, \tag{164}$$

$$P_{150} = 3N^6 + 9N^5 + 20N^4 + 25N^3 - 11N^2 - 46N - 12, \tag{165}$$

$$P_{151} = 3N^6 + 30N^5 + 107N^4 + 124N^3 + 48N^2 + 20N + 8, \tag{166}$$

$$P_{152} = 6N^6 + 23N^5 - 14N^4 - 121N^3 - 114N^2 - 20N + 8, \tag{167}$$

$$P_{153} = 15N^6 + 45N^5 + 49N^4 - 13N^3 - 64N^2 + 40N + 48, \tag{168}$$

$$P_{154} = 15N^6 + 45N^5 + 56N^4 + N^3 - 68N^2 + 29N + 42, \tag{169}$$

$$P_{155} = 15N^6 + 45N^5 + 374N^4 + 601N^3 + 161N^2 - 24N + 36, \tag{170}$$

$$P_{156} = 18N^6 + 54N^5 + 65N^4 + 40N^3 - 23N^2 - 34N + 24, \tag{171}$$

$$P_{157} = 20N^6 + 60N^5 + 11N^4 - 78N^3 - 13N^2 + 36N - 108, \tag{172}$$

$$P_{158} = 24N^6 + 72N^5 + 69N^4 + 18N^3 + N^2 + 4N + 4, \tag{173}$$

$$P_{159} = 27N^6 + 81N^5 - 1247N^4 - 2341N^3 - 720N^2 + 32N - 240, \tag{174}$$

$$P_{160} = 27N^6 + 81N^5 + 148N^4 + 161N^3 + 253N^2 - 390N - 144, \tag{175}$$

$$P_{161} = 30N^6 + 90N^5 + 79N^4 + 8N^3 + 23N^2 + 70N + 12, \tag{176}$$

$$P_{162} = 63N^6 + 189N^5 + 157N^4 + 35N^3 + 80N^2 + 4N - 24, \tag{177}$$

$$P_{163} = 95N^6 + 285N^5 + 92N^4 - 291N^3 - 97N^2 + 96N - 36, \tag{178}$$

$$P_{164} = 297N^6 + 891N^5 - 461N^4 - 2119N^3 - 872N^2 - 96N - 432, \tag{179}$$

$$P_{165} = 233N^7 + 1093N^6 + 1970N^5 + 1538N^4 - 167N^3 - 2143N^2 - 2412N - 288, \tag{180}$$

$$P_{166} = -15N^8 - 60N^7 - 82N^6 - 44N^5 - 15N^4 - 4N^2 - 12N - 8, \tag{181}$$

$$P_{167} = 3N^8 + 12N^7 + 2080N^6 + 5568N^5 + 4602N^4 + 1138N^3 - 3N^2 - 36N - 108, \tag{182}$$

$$P_{168} = 15N^8 + 60N^7 + 242N^6 + 417N^5 + 344N^4 + 285N^3 + 185N^2 + 456N + 108, \tag{183}$$

$$P_{169} = 33N^8 + 132N^7 - 82N^6 - 840N^5 - 571N^4 + 564N^3 + 308N^2 + 984N + 288, \tag{184}$$

$$P_{170} = 40N^8 + 160N^7 + 205N^6 + 61N^5 + 18N^4 + 113N^3 + 75N^2 - 20N - 12, \tag{185}$$

$$P_{171} = 67N^8 + 268N^7 + 194N^6 - 508N^5 - 533N^4 + 480N^3 + 616N^2 + 344N + 144, \tag{186}$$

$$P_{172} = 126N^8 + 504N^7 - 1306N^6 - 5052N^5 - 4473N^4 - 1138N^3 + 3N^2 + 36N + 108, \tag{187}$$

$$P_{173} = 219N^8 + 876N^7 + 1142N^6 + 288N^5 - 217N^4 + 240N^3 + 410N^2 + 366N + 180, \tag{188}$$

$$P_{174} = 1386N^8 + 5544N^7 - 11270N^6 - 46284N^5 - 39915N^4 - 9422N^3 + 33N^2 + 396N + 1188, \tag{189}$$

$$P_{175} = 15N^{10} + 75N^9 + 2N^8 - 469N^7 - 506N^6 + 524N^5 + 781N^4 + 26N^3 - 1192N^2 - 1128N - 432, \tag{190}$$

$$P_{176} = 310N^{10} + 1748N^9 + 4811N^8 + 14192N^7 + 24974N^6 + 3194N^5 - 29393N^4 - 16866N^3 + 8694N^2 + 7128N + 1944, \tag{191}$$

$$P_{177} = 391N^{10} + 1955N^9 + 3622N^8 + 3046N^7 + 1595N^6 + 1327N^5 + 1152N^4 + 216N^3 - 288N^2 - 360N - 144, \tag{192}$$

$$P_{178} = 1593N^{10} + 7965N^9 + 11578N^8 + 1594N^7 - 1379N^6 + 12793N^5 + 17152N^4 + 4432N^3 - 1728N^2 - 2160N - 864, \tag{193}$$

$$P_{179} = -3135N^{12} - 18810N^{11} - 42713N^{10} - 44692N^9 - 22145N^8 - 9290N^7 - 8167N^6 - 4136N^5 - 960N^4 + 11232N^3 + 6720N^2 + 3360N + 576, \tag{194}$$

$$P_{180} = -39N^{12} - 234N^{11} - 521N^{10} - 492N^9 - 85N^8 - 42N^7 - 883N^6 - 1660N^5 - 1324N^4 - 492N^3 - 52N^2 + 48N + 16, \tag{195}$$

$$P_{181} = N^{12} + 6N^{11} - 3N^{10} - 58N^9 - 21N^8 + 222N^7 + 609N^6 + 1144N^5 + 1122N^4 + 142N^3 - 180N^2 + 40N + 48, \tag{196}$$

$$P_{182} = 276N^{12} + 1656N^{11} + 3334N^{10} + 869N^9 - 6591N^8 - 7395N^7 + 7452N^6 + 13479N^5 + 3167N^4 + 7303N^3 + 1110N^2 - 5004N - 2376, \quad (197)$$

$$P_{183} = 2493N^{12} + 14958N^{11} + 42317N^{10} + 75910N^9 + 45511N^8 - 60782N^7 - 29777N^6 + 17194N^5 - 130384N^4 - 115536N^3 + 25776N^2 + 24192N + 5184. \quad (198)$$

Next we turn to the polarized massive Wilson coefficients in the asymptotic region $Q^2 \gg m^2$ in the single mass case.

VI. THE POLARIZED WILSON COEFFICIENT $L_q^{\rm PS}$ AND $L_g^{\rm S}$

The massive Wilson coefficients (5) and (6) are related to the expansion coefficients of the massive OMEs and the massless Wilson coefficients. In presenting the Wilson coefficients for the structure function $g_1(x,Q^2)$ we leave the three-loop massive OMEs, calculated in the previous sections and Ref. [23] and the three-loop contribution to the polarized massless Wilson coefficients, symbolic. The latter depend also on the logarithmic terms L_Q (16). In the nonsinglet case the complete asymptotic Wilson coefficient to the structure function $g_1(x,Q^2)$ has already been calculated in Ref. [28].

For $L_q^{\rm PS}$ we obtain

$$\begin{aligned}
L_q^{PS} = {} & \frac{1}{2}[1-(-1)^N]\Bigg\{a_s^3\Bigg\{C_F N_F T_F^2\Bigg\{-\frac{32(N-1)^2(2+N)(22+41N+28N^2)}{27N^3(1+N)^4} \\
& + L_M\left[-\frac{64(N-1)^2(2+N)(2+5N)}{9N^3(1+N)^3} + (N-1)\left[-\frac{64(2+N)(3+2N+2N^2)}{9N^3(1+N)^3}S_1 + \frac{64(2+N)S_1^2}{3N^2(1+N)^2}\right]\right] \\
& + L_M^2\left[-\frac{32(N-1)^2(2+N)}{3N^3(1+N)^2} - \frac{32(N-1)(2+N)S_1}{3N^2(1+N)^2}\right] + (N-1)\Bigg[L_Q\left[\frac{32L_M^2(2+N)}{3N^2(1+N)^2} + \frac{32(2+N)(22+41N+28N^2)}{27N^2(1+N)^4}\right. \\
& \left. + L_M\left[\frac{64(2+N)(2+5N)}{9N^2(1+N)^3} - \frac{64(2+N)S_1}{3N^2(1+N)^2}\right] - \frac{32(2+N)(2+5N)}{9N^2(1+N)^3}S_1 + \frac{16(2+N)S_1^2}{3N^2(1+N)^2} + \frac{16(2+N)S_2}{3N^2(1+N)^2}\right] \\
& + \left[-\frac{32(2+N)(6+37N+35N^2+13N^3)}{27N^3(1+N)^4} - \frac{16(2+N)S_2}{3N^2(1+N)^2}\right]S_1 + \frac{16(2+N)(3+4N+7N^2)}{9N^3(1+N)^3}S_1^2 - \frac{16(2+N)S_1^3}{3N^2(1+N)^2}\Bigg] \\
& - \frac{16(N-1)^2(2+N)}{3N^3(1+N)^2}S_2\Bigg\} + A_{qq,Q}^{PS(3)} + N_F\hat{\hat{C}}_q^{PS(3)}(L_Q,N_F)\Bigg\}\Bigg\}. \quad (199)
\end{aligned}$$

$L_g^{\rm S}$ is given by

$$\begin{aligned}
L_g^S = {} & \frac{1}{2}[1-(-1)^N]\Bigg\{a_s^2 N_F T_F^2\Bigg\{\frac{16L_M L_Q(N-1)}{3N(1+N)} + L_M\left[-\frac{16(N-1)^2}{3N^2(1+N)} - \frac{16(N-1)S_1}{3N(1+N)}\right]\Bigg\} \\
& + a_s^3\Bigg\{N_F T_F^3\Bigg\{\frac{64L_M^2 L_Q(N-1)}{9N(1+N)} + L_M^2\left[-\frac{64(N-1)^2}{9N^2(1+N)} - \frac{64(N-1)S_1}{9N(1+N)}\right]\Bigg\} \\
& + C_A N_F T_F^2\Bigg\{-\frac{8(N-1)^2 Q_9}{27N^5(1+N)^4} + L_Q\Bigg[(N-1)\left[\frac{8Q_9}{27N^4(1+N)^4} + L_M^2\left[\frac{64}{3N^2(1+N)^2} - \frac{32S_1}{3N(1+N)}\right] - \frac{16(47+56N)S_1}{27N(1+N)^2}\right. \\
& \left. + L_M\left[\frac{32Q_5}{9N^3(1+N)^3} + (N-1)\left[\frac{32S_1^2}{3N(1+N)} - \frac{32S_2}{3N(1+N)} - \frac{64S_{-2}}{3N(1+N)}\right] - \frac{64(-9-2N+3N^2+2N^3)}{9N^2(1+N)^2}S_1\right]\right]\Bigg] \\
& + L_M^2\left[-\frac{64(N-1)^2}{3N^3(1+N)^2} + (N-1)\left[\frac{32(-3+N^2)S_1}{3N^2(1+N)^2} + \frac{32S_1^2}{3N(1+N)}\right]\right] + (N-1)\Bigg[\frac{8S_1Q_{10}}{27N^4(1+N)^4} \\
& + L_M L_Q^2\left[\frac{64}{3N^2(1+N)^2} - \frac{32S_1}{3N(1+N)}\right] + \frac{16(47+56N)S_1^2}{27N(1+N)^2}
\end{aligned}$$

$$
\begin{aligned}
&+L_M\Bigg[-\frac{16Q_{13}}{9(N-1)N^4(1+N)^4(2+N)^2}+(N-1)\Bigg[-\frac{16S_1^3}{9N(1+N)}-\frac{64S_{2,1}}{3N(1+N)}\Bigg]\\
&+\Bigg[-\frac{16Q_8}{9N^3(1+N)^3(2+N)}+\frac{80(N-1)S_2}{3N(1+N)}\Bigg]S_1+\frac{32(-9-10N+3N^2+7N^3)}{9N^2(1+N)^2}S_1^2\\
&+\frac{32(3-2N+N^2+N^3)}{3N^2(1+N)^2}S_2+\frac{32(2+5N+5N^2)}{9N(1+N)(2+N)}S_3\\
&+\Bigg[\frac{64Q_3}{3(N-1)N(1+N)^2(2+N)^2}+\frac{64(2+N+N^2)}{3N(1+N)(2+N)}S_1\Bigg]S_{-2}-\frac{64(-4+N+N^2)}{3N(1+N)(2+N)}S_{-3}\\
&-\frac{256S_{-2,1}}{3N(1+N)(2+N)}-\frac{32(2+N+N^2)}{N(1+N)(2+N)}\zeta_3\Bigg]\Bigg\}+C_FN_FT_F^2\Bigg\{\frac{4(N-1)^2Q_{12}}{N^6(1+N)^5}\\
&+(N-1)\Bigg[\frac{4S_1Q_{12}}{N^5(1+N)^5}+L_ML_Q^2\Bigg[\frac{8Q_2}{3N^3(1+N)^3}-\frac{32S_1}{3N(1+N)}\Bigg]\Bigg]\\
&+L_Q\Bigg[\frac{16L_M^2(N-1)^2(2+N)}{N^3(1+N)^3}+(N-1)\Bigg[-\frac{4Q_{12}}{N^5(1+N)^5}+L_M\Bigg[\frac{16S_1Q_1}{3N^3(1+N)^3}\\
&-\frac{16Q_6}{3N^4(1+N)^4}+\frac{64S_1^2}{3N(1+N)}-\frac{64S_2}{3N(1+N)}\Bigg]\Bigg]\Bigg]+L_M^2\Bigg[-\frac{16(N-1)^3(2+N)}{N^4(1+N)^3}\\
&-\frac{16(N-1)^2(2+N)}{N^3(1+N)^3}S_1\Bigg]+L_M\Bigg[\frac{8S_2Q_4}{3N^3(1+N)^3}-\frac{8Q_{14}}{3(N-1)N^5(1+N)^5(2+N)^2}\\
&+(N-1)\Bigg[-\frac{8(-4+2N+3N^2)(3+2N+3N^2)}{3N^3(1+N)^3}S_1^2-\frac{80S_1^3}{9N(1+N)}+\frac{64S_{2,1}}{3N(1+N)}\Bigg]\\
&+\Bigg[-\frac{32Q_{11}}{3N^4(1+N)^4(2+N)}+\frac{16(N-1)S_2}{N(1+N)}\Bigg]S_1-\frac{256(1+N+N^2)}{9N(1+N)(2+N)}S_3\\
&+\Bigg[\frac{64Q_7}{3(N-1)N^2(1+N)^2(2+N)^2}-\frac{512S_1}{3N(1+N)(2+N)}\Bigg]S_{-2}-\frac{256S_{-3}}{3N(1+N)(2+N)}\\
&+\frac{512S_{-2,1}}{3N(1+N)(2+N)}+\frac{64(2+N+N^2)}{N(1+N)(2+N)}\zeta_3\Bigg]\Bigg\}+A_{qg,Q}^{(3)}+N_F\hat{\tilde{C}}_g^{S(3)}(L_Q,N_F)\Bigg\}\Bigg\},
\end{aligned}
\tag{200}
$$

with the polynomials Q_i given by

$$Q_1=3N^4+2N^3-N^2-12, \tag{201}$$

$$Q_2=3N^4+6N^3-N^2-4N+12, \tag{202}$$

$$Q_3=N^5+3N^4-3N^3-9N^2-8N-8, \tag{203}$$

$$Q_4=9N^5-N^4-23N^3-15N^2-14N+12, \tag{204}$$

$$Q_5=9N^5+9N^4-4N^3+15N^2-41N-12, \tag{205}$$

$$Q_6=N^6-19N^4-22N^3+22N^2-34N-36, \tag{206}$$

$$Q_7=N^6+3N^5+13N^4+21N^3+22N^2+12N-24, \tag{207}$$

$$Q_8=13N^6+44N^5+71N^4+94N^3-90N^2-288N-72, \tag{208}$$

$$Q_9=15N^6+45N^5+374N^4+601N^3+161N^2-24N+36, \tag{209}$$

$$Q_{10}=97N^6+161N^5-392N^4-807N^3-255N^2+24N-36, \tag{210}$$

$$Q_{11} = N^8 + 3N^7 + 11N^6 + 20N^5 - 15N^4 - 17N^3 + 49N^2 - 20N - 36 \tag{211}$$

$$Q_{12} = 15N^8 + 60N^7 + 82N^6 + 44N^5 + 15N^4 + 4N^2 + 12N + 8, \tag{212}$$

$$Q_{13} = 24N^{10} + 102N^9 + 58N^8 - 210N^7 - 209N^6 + 23N^5 + 529N^4 + 1109N^3 + 234N^2 - 388N - 120, \tag{213}$$

$$Q_{14} = 8N^{12} + 50N^{11} + 122N^{10} + 98N^9 - 457N^8 - 1398N^7 - 1232N^6 - 634N^5 - 793N^4 + 388N^3 + 1128N^2 - 16N - 336. \tag{214}$$

VII. THE POLARIZED WILSON COEFFICIENTS $H_{Qq}^{(3),\mathrm{PS}}$

The next Wilson coefficient is $H_{Qq}^{(3),\mathrm{PS}}$. It is given by

$$\begin{aligned}
H_q^{PS} = {}& \frac{1}{2}[1-(-1)^N]\Biggl\{a_s^2 C_F T_F \Biggl\{\frac{8Q_{16}}{(N-1)N^4(1+N)^4(2+N)} + (2+N)\Biggl[\frac{8L_M(1+2N+N^3)}{N^3(1+N)^3} \\
&+ L_Q\Biggl[-\frac{8(2+N-N^2+2N^3)}{N^3(1+N)^3} - \frac{8(N-1)S_1}{N^2(1+N)^2}\Biggr] + (N-1)\Biggl[-\frac{4L_M^2}{N^2(1+N)^2} + \frac{4L_Q^2}{N^2(1+N)^2} \\
&+ \frac{4S_1^2}{N^2(1+N)^2} - \frac{12S_2}{N^2(1+N)^2}\Biggr] + \frac{8(2+N-N^2+2N^3)}{N^3(1+N)^3}S_1\Biggr] - \frac{64}{(N-1)N(1+N)(2+N)}S_{-2}\Biggr\} \\
&+ a_s^3\Biggl\{C_F^2 T_F (2+N)\Biggl\{\frac{8S_2Q_{15}}{N^4(1+N)^4} + \frac{4(-1-3N-4N^2+4N^3)(-2+5N+6N^2+9N^3)}{N^6(1+N)^5} \\
&+ L_Q\Biggl[-\frac{4(2+3N+3N^2)(-1-3N-4N^2+4N^3)}{N^5(1+N)^5} + L_M\Biggl[\frac{8(2+3N+3N^2)(1+2N+N^3)}{N^4(1+N)^4} - \frac{32(1+2N+N^3)}{N^3(1+N)^3}S_1\Biggr] \\
&+ (N-1)\Biggl[L_M^2\Biggl[-\frac{4(2+3N+3N^2)}{N^3(1+N)^3} + \frac{16S_1}{N^2(1+N)^2}\Biggr] - \frac{8(2+3N+3N^2)}{N^3(1+N)^3}S_2\Biggr] \\
&+ \Biggl[\frac{16(-1-3N-4N^2+4N^3)}{N^4(1+N)^4} + \frac{32(N-1)S_2}{N^2(1+N)^2}\Biggr]S_1\Biggr] + L_M\Biggl[-\frac{8(1+2N+N^3)(-2+5N+6N^2+9N^3)}{N^5(1+N)^4} \\
&+ \frac{8(-2+3N+3N^2)(1+2N+N^3)}{N^4(1+N)^4}S_1 + \frac{16(1+2N+N^3)}{N^3(1+N)^3}S_1^2 - \frac{16(1+2N+N^3)}{N^3(1+N)^3}S_2\Biggr] \\
&+ (N-1)\Biggl[L_M^2\Biggl[\frac{4(-2+5N+6N^2+9N^3)}{N^4(1+N)^3} - \frac{4(-2+3N+3N^2)}{N^3(1+N)^3}S_1 - \frac{8S_1^2}{N^2(1+N)^2} + \frac{8S_2}{N^2(1+N)^2}\Biggr] + \frac{16S_2^2}{N^2(1+N)^2}\Biggr] \\
&+ \Biggl[-\frac{4(-2+3N+3N^2)(-1-3N-4N^2+4N^3)}{N^5(1+N)^5} - \frac{8(N-1)(-2+3N+3N^2)}{N^3(1+N)^3}S_2\Biggr]S_1 \\
&+ \Biggl[-\frac{8(-1-3N-4N^2+4N^3)}{N^4(1+N)^4} - \frac{16(N-1)S_2}{N^2(1+N)^2}\Biggr]S_1^2\Biggr\} + C_F T_F^2 (2+N)\Biggl\{-\frac{32(N-1)^2(22+41N+28N^2)}{27N^3(1+N)^4} \\
&+ L_M\Biggl[-\frac{64(N-1)^2(2+5N)}{9N^3(1+N)^3} + (N-1)\Biggl[-\frac{64(3+2N+2N^2)}{9N^3(1+N)^3}S_1 + \frac{64S_1^2}{3N^2(1+N)^2}\Biggr]\Biggr] + L_M^2\Biggl[-\frac{32(N-1)^2}{3N^3(1+N)^2} \\
&- \frac{32(N-1)S_1}{3N^2(1+N)^2}\Biggr] + (N-1)\Biggl[L_Q\Biggl[\frac{32L_M^2}{3N^2(1+N)^2} + \frac{32(22+41N+28N^2)}{27N^2(1+N)^4} + L_M\Biggl[\frac{64(2+5N)}{9N^2(1+N)^3} - \frac{64S_1}{3N^2(1+N)^2}\Biggr] \\
&- \frac{32(2+5N)S_1}{9N^2(1+N)^3} + \frac{16S_1^2}{3N^2(1+N)^2} + \frac{16S_2}{3N^2(1+N)^2}\Biggr] + \Biggl[-\frac{32(6+37N+35N^2+13N^3)}{27N^3(1+N)^4} - \frac{16S_2}{3N^2(1+N)^2}\Biggr]S_1 \\
&+ \frac{16(3+4N+7N^2)}{9N^3(1+N)^3}S_1^2 - \frac{16S_1^3}{3N^2(1+N)^2}\Biggr] - \frac{16(N-1)^2S_2}{3N^3(1+N)^2}\Biggr\} + A_{Qq}^{PS(3)} + \tilde{C}_q^{PS(3)}(L_Q, N_F+1)\Biggr\}\Biggr\},
\end{aligned} \tag{215}$$

where

$$Q_{15} = 9N^5 + 6N^4 - 12N^2 - 8N + 1, \tag{216}$$

$$Q_{16} = 3N^8 + 10N^7 - N^6 - 22N^5 - 14N^4 - 18N^3 - 30N^2 + 8. \tag{217}$$

The polarized massive OME $A_{Qq}^{(3),\mathrm{PS}}$ has been calculated in Ref. [23].

VIII. THE POLARIZED WILSON COEFFICIENT $H_{Qg}^{(3),\mathrm{S}}$

Finally, $H_{Qg}^{(3),\mathrm{S}}$, (8), is obtained by

$$\begin{aligned}
H_g^S =& \frac{1}{2}[1-(-1)^N]\Biggl\{a_s T_F\Biggl\{-\frac{4(N-1)^2}{N^2(1+N)}+(N-1)\Biggl[-\frac{4L_M}{N(1+N)}+\frac{4L_Q}{N(1+N)}-\frac{4S_1}{N(1+N)}\Biggr]\Biggr\} \\
&+a_s^2\Biggl\{C_A T_F\Biggl\{-\frac{8Q_{78}}{(N-1)N^4(1+N)^4(2+N)^2}+L_M\Biggl[-\frac{8Q_{42}}{N^3(1+N)^3}+(N-1)\Biggl[\frac{8S_1^2}{N(1+N)}+\frac{8S_2}{N(1+N)}+\frac{16S_{-2}}{N(1+N)}\Biggr] \\
&-\frac{32S_1}{N(1+N)^2}\Biggr]+L_Q\Biggl[\frac{8(-2+N)Q_{19}}{N^3(1+N)^3}+(N-1)\Biggl[\frac{8S_1^2}{N(1+N)}-\frac{8S_2}{N(1+N)}-\frac{16S_{-2}}{N(1+N)}\Biggr] \\
&+\frac{16(3-N-N^2+N^3)}{N^2(1+N)^2}S_1\Biggr]+(N-1)\Biggl[L_Q^2\Biggl[\frac{16}{N^2(1+N)^2}-\frac{8S_1}{N(1+N)}\Biggr] \\
&+L_M^2\Biggl[-\frac{16}{N^2(1+N)^2}+\frac{8S_1}{N(1+N)}\Biggr]-\frac{16S_{2,1}}{N(1+N)}\Biggr]+\Biggl[-\frac{16Q_{55}}{N^3(1+N)^3(2+N)}+\frac{32(N-1)S_2}{N(1+N)}\Biggr]S_1 \\
&-\frac{4(12-N+N^2+2N^3)}{N^2(1+N)(2+N)}S_1^2+\frac{4Q_{21}}{N^2(1+N)^2(2+N)}S_2+\frac{8(-2+3N+3N^2)}{N(1+N)(2+N)}S_3 \\
&+\Biggl[\frac{16Q_{18}}{(N-1)N(1+N)(2+N)^2}+\frac{32S_1}{2+N}\Biggr]S_{-2}-\frac{8(-2+N)(3+N)}{N(1+N)(2+N)}S_{-3}-\frac{16(2+N+N^2)}{N(1+N)(2+N)}S_{-2,1}-\frac{24(2+N+N^2)}{N(1+N)(2+N)}\zeta_3\Biggr\} \\
&+C_F T_F\Biggl\{\frac{4Q_{79}}{(N-1)N^4(1+N)^4(2+N)^2}+(N-1)\Biggl[L_M^2\Biggl[\frac{2(2+3N+3N^2)}{N^2(1+N)^2}-\frac{8S_1}{N(1+N)}\Biggr] \\
&+L_Q^2\Biggl[\frac{2(2+3N+3N^2)}{N^2(1+N)^2}-\frac{8S_1}{N(1+N)}\Biggr]+L_Q\Biggl[-\frac{4Q_{27}}{N^3(1+N)^3}+L_M\Biggl[-\frac{4(2+3N+3N^2)}{N^2(1+N)^2}+\frac{16S_1}{N(1+N)}\Biggr] \\
&+\frac{4(-6-N+3N^2)}{N^2(1+N)^2}S_1+\frac{16S_1^2}{N(1+N)}-\frac{16S_2}{N(1+N)}\Biggr]+L_M\Biggl[\frac{4Q_{27}}{N^3(1+N)^3} \\
&-\frac{4(-6-N+3N^2)}{N^2(1+N)^2}S_1-\frac{16S_1^2}{N(1+N)}+\frac{16S_2}{N(1+N)}\Biggr]-\frac{8S_1^3}{N(1+N)}+\frac{16S_{2,1}}{N(1+N)}\Biggr] \\
&+\Biggl[\frac{4Q_{54}}{N^3(1+N)^3(2+N)}+\frac{8(N-1)S_2}{N(1+N)}\Biggr]S_1-\frac{2Q_{30}}{N^2(1+N)^2(2+N)}S_1^2+\frac{2Q_{34}}{N^2(1+N)^2(2+N)}S_2-\frac{16(2+N+N^2)}{N(1+N)(2+N)}S_3 \\
&+\Biggl[\frac{16(10+N+N^2)}{(N-1)(2+N)^2}-\frac{128S_1}{N(1+N)(2+N)}\Biggr]S_{-2}-\frac{64S_{-3}}{N(1+N)(2+N)}+\frac{128S_{-2,1}}{N(1+N)(2+N)}+\frac{48(2+N+N^2)}{N(1+N)(2+N)}\zeta_3\Biggr\} \\
&+T_F^2\Biggl\{(N-1)\Biggl[-\frac{16L_M^2}{3N(1+N)}+\frac{16L_ML_Q}{3N(1+N)}\Biggr]+L_M\Biggl[-\frac{16(N-1)^2}{3N^2(1+N)}-\frac{16(N-1)S_1}{3N(1+N)}\Biggr]\Biggr\}\Biggr\} \\
&+a_s^3\Biggl\{C_F N_F T_F^2\Biggl\{(N-1)\Biggl[L_M L_Q^2\Biggl[-\frac{8Q_{24}}{3N^3(1+N)^3}+\frac{32S_1}{3N(1+N)}\Biggr]+(L_ML_Q)\Biggl[-\frac{16S_1Q_{36}}{9N^3(1+N)^3}+\frac{8Q_{62}}{9N^4(1+N)^4} \\
&-\frac{32S_1^2}{3N(1+N)}+\frac{32S_2}{N(1+N)}\Biggr]\Biggr]+L_M\Biggl[-\frac{2Q_{84}}{27N^5(1+N)^5(2+N)}+(N-1)\Biggl[\frac{8S_1^2Q_{36}}{9N^3(1+N)^3}-\frac{8S_2Q_{39}}{9N^3(1+N)^3} \\
&+\Biggl[\frac{8Q_{64}}{27N^4(1+N)^4}-\frac{32S_2}{3N(1+N)}\Biggr]S_1+\frac{32S_1^3}{9N(1+N)}+\frac{352S_3}{9N(1+N)}-\frac{64S_{2,1}}{3N(1+N)}\Biggr]+\frac{256S_{-2}}{N^2(1+N)^2(2+N)}\Biggr]\Biggr\}
\end{aligned}$$

$$
\begin{aligned}
&+C_A T_F^2\Bigg\{-\frac{8(N-1)^2Q_{58}}{27N^5(1+N)^4}+L_Q\Bigg[(N-1)\Bigg[\frac{8Q_{58}}{27N^4(1+N)^4}+L_M^2\Bigg[\frac{64}{3N^2(1+N)^2}-\frac{32S_1}{3N(1+N)}\Bigg]-\frac{16(47+56N)S_1}{27N(1+N)^2}\Bigg]\\
&+L_M\Bigg[\frac{32Q_{48}}{9N^3(1+N)^3}+(N-1)\Bigg[\frac{32S_1^2}{3N(1+N)}-\frac{32S_2}{3N(1+N)}-\frac{64S_{-2}}{3N(1+N)}\Bigg]-\frac{64(-9-2N+3N^2+2N^3)}{9N^2(1+N)^2}S_1\Bigg]\Bigg]\\
&+L_M^2\Bigg[-\frac{64(N-1)^2}{3N^3(1+N)^2}+(N-1)\Bigg[\frac{32(-3+N^2)S_1}{3N^2(1+N)^2}+\frac{32S_1^2}{3N(1+N)}\Bigg]\Bigg]\\
&+(N-1)\Bigg[\frac{8S_1Q_{63}}{27N^4(1+N)^4}+L_M L_Q^2\Bigg[\frac{64}{3N^2(1+N)^2}-\frac{32S_1}{3N(1+N)}\Bigg]+\frac{16(47+56N)S_1^2}{27N(1+N)^2}\Bigg]\\
&+L_M\Bigg[-\frac{16Q_{82}}{9(N-1)N^4(1+N)^4(2+N)^2}+(N-1)\Bigg[-\frac{16S_1^3}{9N(1+N)}-\frac{64S_{2,1}}{3N(1+N)}\Bigg]\\
&+\Bigg[-\frac{16Q_{57}}{9N^3(1+N)^3(2+N)}+\frac{80(N-1)S_2}{3N(1+N)}\Bigg]S_1+\frac{32(-9-10N+3N^2+7N^3)}{9N^2(1+N)^2}S_1^2+\frac{32(3-2N+N^2+N^3)}{3N^2(1+N)^2}S_2\\
&+\frac{32(2+5N+5N^2)}{9N(1+N)(2+N)}S_3+\Bigg[\frac{64Q_{43}}{3(N-1)N(1+N)^2(2+N)^2}+\frac{64(2+N+N^2)}{3N(1+N)(2+N)}S_1\Bigg]S_{-2}-\frac{64(-4+N+N^2)}{3N(1+N)(2+N)}S_{-3}\\
&-\frac{256S_{-2,1}}{3N(1+N)(2+N)}-\frac{32(2+N+N^2)}{N(1+N)(2+N)}\zeta_3\Bigg]\Bigg\}\\
&+C_F T_F^2\Bigg\{\frac{4(N-1)^2Q_{76}}{N^6(1+N)^5}+(N-1)\Bigg[\frac{4S_1Q_{76}}{N^5(1+N)^5}+L_M L_Q^2\Bigg[\frac{8(2+3N+3N^2)}{3N^2(1+N)^2}-\frac{32S_1}{3N(1+N)}\Bigg]\\
&+L_Q\Bigg[-\frac{4Q_{76}}{N^5(1+N)^5}+L_M^2\Bigg[-\frac{16Q_{22}}{3N^3(1+N)^3}+\frac{64S_1}{3N(1+N)}\Bigg]+L_M\Bigg[-\frac{16Q_{53}}{3N^4(1+N)^4}+\frac{16(-6-N+3N^2)}{3N^2(1+N)^2}S_1\\
&+\frac{64S_1^2}{3N(1+N)}-\frac{64S_2}{3N(1+N)}\Bigg]\Bigg]+L_M^2\Bigg[-\frac{16S_1Q_{23}}{3N^3(1+N)^3}+\frac{16Q_{49}}{3N^4(1+N)^3}-\frac{32S_1^2}{3N(1+N)}+\frac{32S_2}{3N(1+N)}\Bigg]\Bigg]\\
&+L_M\Bigg[-\frac{8Q_{86}}{3(N-1)N^5(1+N)^4(2+N)^2}+(N-1)\Bigg[-\frac{8(-8+3N+9N^2)}{3N^2(1+N)^2}S_1^2-\frac{80S_1^3}{9N(1+N)}+\frac{64S_{2,1}}{3N(1+N)}\Bigg]\\
&+\Bigg[-\frac{32Q_{70}}{3N^4(1+N)^4(2+N)}+\frac{16(N-1)S_2}{N(1+N)}\Bigg]S_1+\frac{8(4-19N-10N^2+9N^3)}{3N^2(1+N)^2}S_2-\frac{256(1+N+N^2)}{9N(1+N)(2+N)}S_3\\
&+\Bigg[\frac{64(10+N+N^2)}{3(N-1)(2+N)^2}-\frac{512S_1}{3N(1+N)(2+N)}\Bigg]S_{-2}-\frac{256S_{-3}}{3N(1+N)(2+N)}+\frac{512S_{-2,1}}{3N(1+N)(2+N)}+\frac{64(2+N+N^2)}{N(1+N)(2+N)}\zeta_3\Bigg]\Bigg\}\\
&+C_A C_F T_F\Bigg\{-\frac{4(-2+5N+6N^2+9N^3)Q_{72}}{N^6(1+N)^5(2+N)}+L_M^2\Bigg[(N-1)\Bigg[\frac{16(-2+5N+6N^2+9N^3)}{N^4(1+N)^3}+\Bigg[-\frac{8Q_{32}}{N^3(1+N)^3}-\frac{16S_2}{N(1+N)}\Bigg]S_1\\
&+\frac{16S_1^3}{N(1+N)}+\frac{32S_2}{N^2(1+N)^2}\Bigg]+\frac{24(N-1)^2(2+N)}{N^2(1+N)^2}S_1^2\Bigg]+L_Q\Bigg[\frac{4(2+3N+3N^2)Q_{72}}{N^5(1+N)^5(2+N)}\\
&+L_M\Bigg[-\frac{2Q_{68}}{9N^4(1+N)^4}+(N-1)\Bigg[-\frac{32S_1^3}{N(1+N)}-\frac{16(-1+4N+4N^2)}{N^2(1+N)^2}S_2+\frac{32S_3}{N(1+N)}+\frac{48S_{-2}}{N(1+N)}\\
&+\frac{32S_{-3}}{N(1+N)}-\frac{64S_{-2,1}}{N(1+N)}\Bigg]+\Bigg[\frac{4Q_{51}}{9N^3(1+N)^3}-\frac{32(N-1)S_2}{N(1+N)}\Bigg]S_1+\frac{16(-3+17N+10N^3)}{3N^2(1+N)^2}S_1^2\Bigg]\\
&+(N-1)\Bigg[L_M^2\Bigg[-\frac{16(2+3N+3N^2)}{N^3(1+N)^3}+\frac{8(10+3N+3N^2)}{N^2(1+N)^2}S_1-\frac{32S_1^2}{N(1+N)}\Bigg]-\frac{16S_1^4}{3N(1+N)}+\frac{32(2+3N+3N^2)}{3N^2(1+N)^2}S_3\\
&+\Bigg[\frac{16(2+3N+3N^2)}{N^2(1+N)^3}+\frac{16(2-N+3N^2)}{N^2(1+N)^2}S_1-\frac{64S_1^2}{N(1+N)}\Bigg]S_{-2}+\Bigg[\frac{8(2+3N+3N^2)}{N^2(1+N)^2}-\frac{32S_1}{N(1+N)}\Bigg]S_{-3}\\
&-\frac{16(2+3N+3N^2)}{N^2(1+N)^2}S_{-2,1}\Bigg]+\Bigg[-\frac{4Q_{74}}{N^4(1+N)^4(2+N)}+(N-1)\Bigg[-\frac{128S_3}{3N(1+N)}+\frac{64S_{-2,1}}{N(1+N)}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4Q_{33}}{N^2(1+N)^2(2+N)}S_2\Bigg]S_1+\left[\frac{4Q_{46}}{N^2(1+N)^3(2+N)}-\frac{48(N-1)S_2}{N(1+N)}\right]S_1^2+\frac{4Q_{25}}{3N^2(1+N)^2(2+N)}S_1^3\\
&-\frac{4(2+3N+3N^2)(-16+15N+24N^2+7N^3)}{N^3(1+N)^3(2+N)}S_2\Bigg]+L_M\Bigg[\frac{2S_2Q_{52}}{9N^3(1+N)^3}+\frac{Q_{85}}{54N^5(1+N)^5(2+N)}\\
&+(N-1)\Bigg[\left[-\frac{2Q_{41}}{9N^3(1+N)^3}-\frac{16S_2}{N(1+N)}\right]S_1^2+\frac{16S_1^4}{N(1+N)}-\frac{8(-36+121N+121N^2)}{9N^2(1+N)^2}S_3+\frac{32S_4}{N(1+N)}\\
&+\frac{48S_{-2}^2}{N(1+N)}+\left[-\frac{32}{N^2(1+N)^2}+\frac{32S_1}{N(1+N)}\right]S_{-3}+\frac{80S_{-4}}{N(1+N)}+\frac{16(-6+11N+11N^2)}{3N^2(1+N)^2}S_{2,1}\\
&+\frac{96S_{3,1}}{N(1+N)}-\frac{32S_{-2,2}}{N(1+N)}-\frac{64S_{-3,1}}{N(1+N)}-\frac{96S_{2,1,1}}{N(1+N)}-\frac{24(1+3N)(2+3N)}{N^2(1+N)^2}\zeta_3\Bigg]\\
&+\left[-\frac{2Q_{69}}{27N^4(1+N)^4}+(N-1)\left[-\frac{96S_3}{N(1+N)}+\frac{64S_{2,1}}{N(1+N)}+\frac{64S_{-2,1}}{N(1+N)}+\frac{192\zeta_3}{N(1+N)}\right]+\frac{32(1+5N^2)}{3N(1+N)^2}S_2\right]S_1\\
&+\frac{16(9-53N+8N^3)}{9N^2(1+N)^2}S_1^3+\left[-\frac{16Q_{56}}{N^3(1+N)^3(2+N)}+(N-1)\left[\frac{48S_1}{N(1+N)}-\frac{32S_1^2}{N(1+N)}\right]\right]S_{-2}\Bigg]\\
&+(N-1)\Bigg[L_ML_Q^2\left[\frac{22(2+3N+3N^2)}{3N^2(1+N)^2}-\frac{88S_1}{3N(1+N)}\right]+\frac{8S_1^5}{3N(1+N)}-\frac{32(-2+5N+6N^2+9N^3)}{3N^3(1+N)^2}S_3\\
&+\Bigg[-\frac{16(-2+5N+6N^2+9N^3)}{N^3(1+N)^3}+\left[-\frac{16Q_{31}}{N^3(1+N)^3}-\frac{32S_2}{N(1+N)}\right]S_1+\frac{16(2+N)(-1+3N)}{N^2(1+N)^2}S_1^2\\
&+\frac{32S_1^3}{N(1+N)}-\frac{32S_2}{N(1+N)^2}\Bigg]S_{-2}+\Bigg[-\frac{8(-2+5N+6N^2+9N^3)}{N^3(1+N)^2}+\frac{8(-2+3N+3N^2)}{N^2(1+N)^2}S_1\\
&+\frac{16S_1^2}{N(1+N)}-\frac{16S_2}{N(1+N)}\Bigg]S_{-3}+\frac{16(-2+5N+6N^2+9N^3)}{N^3(1+N)^2}S_{-2,1}\Bigg]\\
&+\Bigg[\frac{4Q_{80}}{N^5(1+N)^5(2+N)}+(N-1)\left[-\frac{24S_2^2}{N(1+N)}+\frac{32(-2+3N+3N^2)}{3N^2(1+N)^2}S_3-\frac{16(-2+3N+3N^2)}{N^2(1+N)^2}S_{-2,1}\right]\\
&-\frac{4Q_{61}}{N^3(1+N)^3(2+N)}S_2\Bigg]S_1+\Bigg[\frac{4Q_{71}}{N^4(1+N)^4(2+N)}+(N-1)\left[\frac{64S_3}{3N(1+N)}-\frac{32S_{-2,1}}{N(1+N)}\right]\\
&+\frac{4Q_{29}}{N^2(1+N)^2(2+N)}S_2\Bigg]S_1^2+\left[-\frac{4Q_{59}}{3N^3(1+N)^3(2+N)}+\frac{64(N-1)S_2}{3N(1+N)}\right]S_1^3\\
&+\frac{4(4-38N-29N^2+3N^4)}{3N^2(1+N)^2(2+N)}S_1^4+\left[-\frac{4Q_{73}}{N^4(1+N)^4(2+N)}+(N-1)\left[-\frac{64S_3}{3N(1+N)}+\frac{32S_{-2,1}}{N(1+N)}\right]\right]S_2\\
&+\frac{8(-16+15N+24N^2+7N^3)}{N^2(1+N)^2(2+N)}S_2^2\Bigg\}\\
&+C_F^2T_F\Bigg\{-\frac{2(-2+5N+6N^2+9N^3)Q_{75}}{N^6(1+N)^5(2+N)}+L_M^2\Bigg[(N-1)\Bigg[-\frac{2(2+3N+3N^2)(-2+5N+6N^2+9N^3)}{N^4(1+N)^3}\\
&+\left[\frac{2Q_{38}}{N^3(1+N)^3}+\frac{16S_2}{N(1+N)}\right]S_1-\frac{16S_1^3}{N(1+N)}-\frac{4(2+3N+3N^2)}{N^2(1+N)^2}S_2\Bigg]-\frac{12(N-1)^2(2+N)}{N^2(1+N)^2}S_1^2\Bigg]\\
&+L_Q\Bigg[\frac{2(2+3N+3N^2)Q_{75}}{N^5(1+N)^5(2+N)}+(N-1)\Bigg[L_M^2\left[\frac{2(2+3N+3N^2)^2}{N^3(1+N)^3}-\frac{16(2+3N+3N^2)}{N^2(1+N)^2}S_1+\frac{32S_1^2}{N(1+N)}\right]\\
&+L_M\Bigg[\frac{2Q_{60}}{N^4(1+N)^4}+\left[-\frac{4Q_{35}}{N^3(1+N)^3}-\frac{128S_2}{N(1+N)}\right]S_1-\frac{64(2+N)S_1^2}{N^2(1+N)^2}+\frac{64S_1^3}{N(1+N)}+\frac{32(2+3N+3N^2)}{N^2(1+N)^2}S_2\\
&-\frac{64S_3}{N(1+N)}+\left[\frac{64}{N^2(1+N)^2}-\frac{128S_1}{N(1+N)}\right]S_{-2}-\frac{64S_{-3}}{N(1+N)}+\frac{128S_{-2,1}}{N(1+N)}\Bigg]+\frac{16S_1^4}{3N(1+N)}+\frac{16(2+3N+3N^2)}{3N^2(1+N)^2}S_3\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\left[-\frac{4Q_{77}}{N^4(1+N)^4(2+N)}-\frac{4Q_{28}}{N^2(1+N)^2(2+N)}S_2-\frac{64(N-1)S_3}{3N(1+N)}\right]S_1\\
&+\left[-\frac{4Q_{45}}{N^3(1+N)^2(2+N)}+\frac{16(N-1)S_2}{N(1+N)}\right]S_1^2-\frac{4Q_{26}}{3N^2(1+N)^2(2+N)}S_1^3+\frac{4(2+3N+3N^2)Q_{20}}{N^3(1+N)^3(2+N)}S_2\Bigg]\\
&+(N-1)\Bigg[L_M L_Q^2\left[-\frac{2(2+3N+3N^2)^2}{N^3(1+N)^3}+\frac{16(2+3N+3N^2)}{N^2(1+N)^2}S_1-\frac{32S_1^2}{N(1+N)}\right]\\
&-\frac{8S_1^5}{3N(1+N)}-\frac{16(-2+5N+6N^2+9N^3)}{3N^3(1+N)^2}S_3\Bigg]\\
&+L_M\Bigg[\frac{Q_{83}}{2N^5(1+N)^5(2+N)}+(N-1)\Bigg[-\frac{2S_2Q_{40}}{N^3(1+N)^3}+\left[\frac{2Q_{37}}{N^3(1+N)^3}+\frac{112S_2}{N(1+N)}\right]S_1^2\\
&-\frac{24S_1^4}{N(1+N)}-\frac{40S_2^2}{N(1+N)}+\frac{8(-2+9N+9N^2)}{N^2(1+N)^2}S_3-\frac{48S_4}{N(1+N)}-\frac{96S_{-2}^2}{N(1+N)}\\
&+\left[\frac{64}{N^2(1+N)^2}-\frac{64S_1}{N(1+N)}\right]S_{-3}-\frac{160S_{-4}}{N(1+N)}-\frac{16(-2+3N+3N^2)}{N^2(1+N)^2}S_{2,1}-\frac{160S_{3,1}}{N(1+N)}\\
&+\frac{64S_{-2,2}}{N(1+N)}+\frac{128S_{-3,1}}{N(1+N)}+\frac{96S_{2,1,1}}{N(1+N)}+\frac{288\zeta_3}{N(1+N)}\Bigg]+\Bigg[\frac{2Q_{66}}{N^4(1+N)^4}+(N-1)\Bigg[\frac{32(-4+2N+3N^2)}{N^2(1+N)^2}S_2\\
&+\frac{96S_3}{N(1+N)}-\frac{64S_{2,1}}{N(1+N)}-\frac{128S_{-2,1}}{N(1+N)}-\frac{192\zeta_3}{N(1+N)}\Bigg]\Bigg]S_1-\frac{16(N-1)^2(4+3N)}{N^2(1+N)^2}S_1^3\\
&+\left[\frac{32Q_{44}}{N^2(1+N)^3(2+N)}+(N-1)\left[-\frac{64S_1}{N^2(1+N)^2}+\frac{128S_1^2}{N(1+N)}-\frac{64S_2}{N(1+N)}\right]\right]S_{-2}\Bigg]\\
&+\left[\frac{2Q_{81}}{N^5(1+N)^5(2+N)}+(N-1)\left[\frac{8S_2^2}{N(1+N)}+\frac{16(-2+3N+3N^2)}{3N^2(1+N)^2}S_3\right]+\frac{8Q_{47}}{N^2(1+N)^3(2+N)}S_2\right]S_1\\
&+\left[\frac{4Q_{65}}{N^3(1+N)^4(2+N)}-\frac{4Q_{17}}{N^2(1+N)^2(2+N)}S_2+\frac{32(N-1)S_3}{3N(1+N)}\right]S_1^2+\left[\frac{4Q_{50}}{3N^3(1+N)^2(2+N)}-\frac{16(N-1)S_2}{3N(1+N)}\right]S_1^3\\
&-\frac{4(-8-5N+N^2)(-2+3N+3N^2)}{3N^2(1+N)^2(2+N)}S_1^4+\left[-\frac{4Q_{67}}{N^3(1+N)^4(2+N)}-\frac{32(N-1)S_3}{3N(1+N)}\right]S_2-\frac{8Q_{20}}{N^2(1+N)^2(2+N)}S_2^2\Bigg\}\\
&+T_F^3\left\{\frac{64L_M^2L_Q(N-1)}{9N(1+N)}+L_M^2\left[-\frac{64(N-1)^2}{9N^2(1+N)}-\frac{64(N-1)S_1}{9N(1+N)}\right]\right\}+A_{Qg}^{(3)}+\tilde{C}_g^{S(3)}(L_Q,N_F+1)\Bigg\}\Bigg\}, \qquad (218)
\end{aligned}
$$

with the polynomials Q_i

$$Q_{17}=N^4-22N^3-79N^2-72N-4, \qquad (219)$$

$$Q_{18}=N^4+3N^3-4N^2-8N-4, \qquad (220)$$

$$Q_{19}=N^4+3N^3-2N^2+3N+3, \qquad (221)$$

$$Q_{20}=N^4+17N^3+43N^2+33N+2, \qquad (222)$$

$$Q_{21}=2N^4-N^3-24N^2-17N+28, \qquad (223)$$

$$Q_{22}=3N^4+6N^3+2N^2-N+6, \qquad (224)$$

$$Q_{23}=3N^4+6N^3+4N^2+N-6, \qquad (225)$$

$$Q_{24}=3N^4+6N^3+11N^2+8N-12, \qquad (226)$$

$$Q_{25} = 3N^4 + 18N^3 + 47N^2 + 56N - 4, \tag{227}$$

$$Q_{26} = 3N^4 + 42N^3 + 71N^2 + 8N - 28, \tag{228}$$

$$Q_{27} = 4N^4 + 5N^3 + 3N^2 - 4N - 4, \tag{229}$$

$$Q_{28} = 7N^4 + 74N^3 + 171N^2 + 128N + 4, \tag{230}$$

$$Q_{29} = 9N^4 + 6N^3 - 55N^2 - 44N + 44, \tag{231}$$

$$Q_{30} = 9N^4 + 6N^3 - 35N^2 - 16N + 20, \tag{232}$$

$$Q_{31} = 9N^4 + 12N^3 + 8N^2 + 5N - 2, \tag{233}$$

$$Q_{32} = 9N^4 + 15N^3 + 17N^2 + 9N - 6, \tag{234}$$

$$Q_{33} = 9N^4 + 46N^3 + 93N^2 + 48N - 76, \tag{235}$$

$$Q_{34} = 11N^4 + 42N^3 + 47N^2 + 32N + 12, \tag{236}$$

$$Q_{35} = 25N^4 + 26N^3 - 3N^2 - 52N - 36, \tag{237}$$

$$Q_{36} = 29N^4 + 58N^3 + 5N^2 - 24N + 36, \tag{238}$$

$$Q_{37} = 43N^4 + 86N^3 + 107N^2 - 8N - 56, \tag{239}$$

$$Q_{38} = 45N^4 + 78N^3 + 53N^2 + 12N - 12, \tag{240}$$

$$Q_{39} = 85N^4 + 170N^3 + 61N^2 - 24N + 36, \tag{241}$$

$$Q_{40} = 111N^4 + 198N^3 + 135N^2 + 24N - 32, \tag{242}$$

$$Q_{41} = 763N^4 + 1418N^3 + 985N^2 + 978N + 72, \tag{243}$$

$$Q_{42} = N^5 + N^4 - 4N^3 + 3N^2 - 7N - 2, \tag{244}$$

$$Q_{43} = N^5 + 3N^4 - 3N^3 - 9N^2 - 8N - 8, \tag{245}$$

$$Q_{44} = 2N^5 + 6N^4 + N^3 - 6N^2 + 11N + 10, \tag{246}$$

$$Q_{45} = 4N^5 - 13N^4 - 114N^3 - 241N^2 - 144N + 4, \tag{247}$$

$$Q_{46} = 4N^5 + 13N^4 - 19N^3 - 69N^2 - 15N - 10, \tag{248}$$

$$Q_{47} = 5N^5 + 39N^4 + 118N^3 + 171N^2 + 93N + 2, \tag{249}$$

$$Q_{48} = 9N^5 + 9N^4 - 4N^3 + 15N^2 - 41N - 12, \tag{250}$$

$$Q_{49} = 9N^5 + 15N^4 + 8N^3 + 3N^2 + 7N - 6, \tag{251}$$

$$Q_{50} = 15N^5 + 36N^4 - 97N^3 - 366N^2 - 264N + 16, \tag{252}$$

$$Q_{51} = 439N^5 + 439N^4 - 937N^3 - 367N^2 - 582N - 144, \tag{253}$$

$$Q_{52} = 815N^5 + 923N^4 - 1349N^3 - 491N^2 - 258N - 216, \tag{254}$$

$$Q_{53} = N^6 - 7N^4 - 4N^3 + 16N^2 - 10N - 12, \tag{255}$$

$$Q_{54} = 2N^6 + 5N^5 - 22N^4 - 95N^3 - 114N^2 - 24N + 16, \tag{256}$$

$$Q_{55} = 2N^6 + 5N^5 - 3N^4 - 7N^3 + 2N^2 - 11N - 8, \tag{257}$$

$$Q_{56} = 11N^6 + 30N^5 + 9N^4 - 22N^3 - 10N^2 + 2N + 4, \tag{258}$$

$$Q_{57} = 13N^6 + 44N^5 + 71N^4 + 94N^3 - 90N^2 - 288N - 72, \tag{259}$$

$$Q_{58} = 15N^6 + 45N^5 + 374N^4 + 601N^3 + 161N^2 - 24N + 36, \tag{260}$$

$$Q_{59} = 15N^6 + 57N^5 + 47N^4 - N^3 + 12N^2 - 38N + 4, \tag{261}$$

$$Q_{60} = 21N^6 + 45N^5 + 55N^4 - 13N^3 - 12N^2 - 8N + 8, \tag{262}$$

$$Q_{61} = 25N^6 + 85N^5 + 119N^4 + 75N^3 - 118N^2 - 102N + 44, \tag{263}$$

$$Q_{62} = 57N^6 + 153N^5 + 233N^4 + 163N^3 - 70N^2 + 120N + 144, \tag{264}$$

$$Q_{63} = 97N^6 + 161N^5 - 392N^4 - 807N^3 - 255N^2 + 24N - 36, \tag{265}$$

$$Q_{64} = 247N^6 + 795N^5 + 555N^4 - 71N^3 + 210N^2 - 360N - 432, \tag{266}$$

$$Q_{65} = 15N^7 + 28N^6 - 116N^5 - 453N^4 - 575N^3 - 221N^2 + 114N + 88, \tag{267}$$

$$Q_{66} = 21N^7 - 52N^6 - 86N^5 - 60N^4 + 17N^3 + 176N^2 + 24N - 40, \tag{268}$$

$$Q_{67} = 21N^7 + 220N^6 + 713N^5 + 1132N^4 + 1010N^3 + 486N^2 + 38N - 52, \tag{269}$$

$$Q_{68} = 753N^7 + 1308N^6 - 44N^5 - 1118N^4 - 1549N^3 - 766N^2 - 600N - 288, \tag{270}$$

$$Q_{69} = 3479N^7 + 7444N^6 - 5160N^5 - 13414N^4 - 8111N^3 - 5478N^2 + 1368N + 864, \tag{271}$$

$$Q_{70} = N^8 + 3N^7 + 5N^6 + 5N^5 - 9N^4 - 8N^3 + 19N^2 - 8N - 12, \tag{272}$$

$$Q_{71} = N^8 + 14N^7 + 88N^6 + 229N^5 + 202N^4 + 47N^3 + 77N^2 + 14N + 8, \tag{273}$$

$$Q_{72} = 2N^8 + 10N^7 + 22N^6 + 36N^5 + 29N^4 + 4N^3 + 33N^2 + 12N + 4, \tag{274}$$

$$Q_{73} = 4N^8 - 43N^7 - 277N^6 - 500N^5 - 308N^4 + 25N^3 + 245N^2 + 102N - 24, \tag{275}$$

$$Q_{74} = 11N^8 + 55N^7 + 99N^6 + 119N^5 + 90N^4 + 2N^3 + 136N^2 + 48N + 16, \tag{276}$$

$$Q_{75} = 12N^8 + 52N^7 + 60N^6 - 25N^4 - 2N^3 + 3N^2 + 8N + 4, \tag{277}$$

$$Q_{76} = 15N^8 + 60N^7 + 82N^6 + 44N^5 + 15N^4 + 4N^2 + 12N + 8, \tag{278}$$

$$Q_{77} = 21N^8 + 101N^7 + 193N^6 + 321N^5 + 528N^4 + 550N^3 + 302N^2 + 88N + 8, \tag{279}$$

$$Q_{78} = N^{10} + 3N^9 - 15N^8 - 56N^7 - 8N^6 + 90N^5 + 60N^4 + 67N^3 + 86N^2 - 12N - 24, \tag{280}$$

$$Q_{79} = 5N^{10} + 23N^9 + 31N^8 - N^7 + 54N^6 + 268N^5 + 342N^4 + 98N^3 - 60N^2 - 8N + 16, \tag{281}$$

$$Q_{80} = 15N^{10} + 87N^9 + 154N^8 + 96N^7 + 18N^6 - 64N^5 + 47N^4 + 153N^3 - 22N^2 - 12N - 8, \tag{282}$$

$$Q_{81} = 18N^{10} + 162N^9 + 740N^8 + 2296N^7 + 4511N^6 + 5341N^5 + 3593N^4 + 1065N^3 - 130N^2 - 148N - 8, \tag{283}$$

$$Q_{82} = 24N^{10} + 102N^9 + 58N^8 - 210N^7 - 209N^6 + 23N^5 + 529N^4 + 1109N^3 + 234N^2 - 388N - 120, \tag{284}$$

$$Q_{83} = 29N^{10} + 229N^9 + 620N^8 + 1434N^7 + 2173N^6 + 505N^5 - 86N^4 + 712N^3 + 704N^2 - 16N - 160, \tag{285}$$

$$Q_{84} = 1371N^{10} + 6171N^9 + 10220N^8 + 5678N^7 - 9493N^6 - 17113N^5 - 9154N^4 - 10864N^3 - 10656N^2 + 1872N + 4320, \tag{286}$$

$$Q_{85} = 20283N^{10} + 92379N^9 + 127804N^8 + 11278N^7 - 154181N^6 - 222809N^5 - 170666N^4 - 111392N^3 - 62568N^2 + 5040N + 8640, \tag{287}$$

$$Q_{86} = 8N^{11} + 42N^{10} + 44N^9 - 102N^8 - 415N^7 - 539N^6 - 195N^5 - 241N^4 - 414N^3 + 268N^2 + 104N - 96. \tag{288}$$

IX. CONCLUSIONS

We have calculated the logarithmic contributions to the massive operator matrix elements A_{ij} and the massive Wilson coefficients in the asymptotic region in the polarized case to three-loop order, also referring to previous results in the literature. These quantities provide important corrections to the deep-inelastic scattering structure functions $g_1(x,Q^2)$ and $g_2(x,Q^2)$, as well as the matching functions in the polarized variable flavor number scheme to three-loop order. Here the OMEs $A_{qq,Q}^{(3),\mathrm{PS}}$ and $A_{qg,Q}^{(3),\mathrm{S}}$ are new quantities, which have not been calculated in complete form. At present, except for the massless flavor nonsinglet Wilson coefficients [62], the polarized massless three-loop Wilson coefficients are not yet known. Therefore, we have given them only symbolically in the corresponding expressions. Similarly, the calculation of the constant part of the unrenormalized polarized massive OMEs $A_{Qg}^{(3)}$ and $A_{gg,Q}^{(3)}$ is still underway. However, the logarithmic contributions are already available for use. These contributions, unlike the case for the nonlogarithmic parts, can all be represented in terms of harmonic sums in Mellin N space or by harmonic polylogarithms in momentum fraction z space. Finally, one has to know these expressions to three-loop order, because this is required by the experimental accuracy expected in future high-luminosity deep–inelastic scattering experiments. Likewise, the matrix elements for the polarized variable flavor number scheme are needed to describe the decoupling of heavy flavors at polarized hadron colliders, to give full account of these QCD corrections. Working in the Larin scheme provides an alternative to other schemes in matching the massive OMEs with the massless Wilson coefficients and polarized parton densities, which would also be described in this scheme. This implies a modification for their evolution, but maintains their universality and leads to the same predictions of all observables. The analytic expressions for the massive operator matrix elements and Wilson coefficients in the asymptotic regions are lengthy, as expected for single differential three-loop quantities (see also [62]). For this reason we also provide computer-readable files as attachments to this paper.

In addition to the single mass corrections, on which we have concentrated in the present paper, there are double mass corrections due to charm and bottom quarks of the polarized massive OMEs starting with three-loop order. They have been calculated for all OMEs except $A_{Qg}^{(3)}$ in Refs. [25–27], in complete analytic form and the calculation of $A_{Qg}^{(3)}$ maintaining the leading terms in the mass expansion m_c^2/m_b^2 is underway. We also presented the nonsinglet OME and the associated Wilson coefficient in the Larin scheme since it previously has been given in the $\overline{\mathrm{MS}}$ scheme only [22,28].

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APPENDIX A: THE z-SPACE EXPRESSIONS FOR THE MASSIVE OMEs

The massive OMEs in z space decompose into the three contributions A_δ, A_+ and $A_{\rm reg}$. Their convolutions with regular functions, such as parton distribution functions, are given by [78]

$$A_\delta(z) \otimes f(z) = f(1) \tag{A1}$$

$$A_+(z) \otimes f(z) = \int_z^1 \frac{dy}{y} A_+(y)\left[f\left(\frac{z}{y}\right) - f(z)\right] - f(z)\int_0^z dy A_+(y) \tag{A2}$$

$$A_{\rm reg}(z) \otimes f(z) = \int_z^1 \frac{dy}{y} A_{\rm reg}(y) f\left(\frac{z}{y}\right). \tag{A3}$$

The following terms can all be expressed by harmonic polylogarithms [54]. They correspond to words out of letters of the three-letter alphabet

$$\mathfrak{A} = \left\{f_0(x) = \frac{1}{x}, f_1(x) = \frac{1}{1-x}, f_{-1}(x) = \frac{1}{1+x}\right\} \tag{A4}$$

and perform algebraic reductions, Ref. [73]. The harmonic polylogarithms are given by

$$H_{b,\vec{a}}(x) = \int_0^x dy f_a(y) H_{\vec{a}}(y), \quad H_\emptyset = 1, \quad a, a_i \in \{0,1,-1\}. \tag{A5}$$

With the exception of $A_{gg,Q}^{(3)}$ and $A_{qq,Q}^{(3),\rm NS}$, all OMEs shown in the following have only regular contributions. For the OME $A_{qq,Q}^{(3),\rm PS}$ they read

$$\begin{aligned}
A_{qq,Q}^{PS}(z) = a_s^3\Bigg\{ C_F N_F T_F^2 \Bigg\{ & L_M^3\left[\frac{160}{9}(z-1) - \frac{64}{9}(1+z)H_0\right] \\
& + L_M^2\left[-\frac{32}{9}(z-1)(-1+15H_1) + (1+z)\left(-\frac{32}{3}H_0^2 + \frac{64}{3}H_{0,1} - \frac{64}{3}\zeta_2\right) - \frac{32}{9}(7+z)H_0\right] \\
& + L_M\Bigg[(1+z)\left(-\frac{176}{9}H_0^2 - \frac{32}{9}H_0^3 + \frac{128}{3}H_{0,0,1} - \frac{128}{3}H_{0,1,1}\right) \\
& - \frac{32}{27}(55+31z)H_0 + (z-1)\left(\frac{5312}{27} - \frac{64}{9}H_1 + \frac{160}{3}H_1^2\right) + \frac{64}{9}(7+z)H_{0,1} \\
& + \left[-\frac{64}{9}(7+z) - \frac{128}{3}(1+z)H_0\right]\zeta_2\Bigg] + (1+z)\left(-\frac{64}{3}H_{0,1,1,1} + \frac{448}{15}\zeta_2^2\right) \\
& + \frac{128}{81}(49+82z)H_0 + (z-1)\left(-\frac{10880}{81} + \frac{3712}{27}H_1 - \frac{64}{9}H_1^2 + \frac{80}{9}H_1^3\right) - \frac{128}{27}(11+14z)H_{0,1} \\
& + \frac{64}{9}(2+5z)H_{0,1,1} + \left[\frac{16}{27}(103+97z) + (1+z)\left(\frac{176}{9}H_0 + \frac{16}{3}H_0^2 - \frac{64}{3}H_{0,1}\right) + \frac{160}{3}(z-1)H_1\right]\zeta_2 \\
& + \left[-\frac{32}{9}(-1+15z) + \frac{64}{9}(1+z)H_0\right]\zeta_3\Bigg\} + a_{qq,Q}^{PS(3)}\Bigg\}.
\end{aligned} \tag{A6}$$

The contributions to $A_{qg,Q}^{(3)}$ are given by

$$\begin{aligned}
A_{qg,Q} = a_s^3\Bigg\{ C_F N_F T_F^2 \Bigg\{ & \frac{1}{27}(58501 - 59018z) + L_M^3\left[\frac{8}{3}(-41+42z) + (-1+2z)\left(\frac{16}{3}H_0^2 - \frac{32}{9}H_1\right) - \frac{32}{9}(13+19z)H_0\right] \\
& + L_M^2\Bigg[\frac{4}{3}(-285+296z) + (-1+2z)\left(\frac{32}{3}H_0^3 + \frac{16}{3}H_1^2 + \frac{32}{3}H_{0,1} - \frac{32}{3}\zeta_2\right) - \frac{8}{9}(425+2z)H_0 - 16(7+6z)H_0^2 \\
& - \frac{32}{9}(1+4z)H_1\Bigg] + L_M\Bigg[\frac{212}{9}(-69+70z) + (-1+2z)\left(\frac{20}{3}H_0^4 - \frac{16}{9}H_1^3 + \frac{32}{3}H_{0,0,1} - \frac{32}{3}H_{0,1,1} + \frac{32}{3}\zeta_3\right) \\
& + \left[-\frac{8}{27}(4103+1565z) + \frac{64}{3}(z-1)H_1 - \frac{32}{3}(-1+2z)H_{0,1}\right]H_0 + \left[\frac{4}{3}(-343+10z) + \frac{16}{3}(-1+2z)H_1\right]H_0^2
\end{aligned}$$

$$
\begin{aligned}
&-\frac{16}{9}(50+23z)H_0^3-\frac{16}{27}(-28+131z)H_1+\frac{32}{9}(-2+7z)H_1^2+\frac{64}{9}(1+4z)H_{0,1}-\frac{64}{9}(-2+7z)\zeta_2\Bigg]\\
&+(-1+2z)\left(-\frac{4}{15}H_0^5+\frac{5248}{81}H_1-\frac{896}{27}H_{0,1}+\frac{160}{9}H_{0,0,1}-\frac{32}{3}H_{0,0,0,1}+\frac{16}{3}\zeta_2^2\right)\\
&+\left[\frac{16}{81}(7333+4195z)+(-1+2z)\left(\frac{896}{27}H_1-\frac{160}{9}H_{0,1}+\frac{32}{3}H_{0,0,1}\right)\right]H_0\\
&+\left[\frac{8}{27}(1331+140z)+(-1+2z)\left(\frac{80}{9}H_1-\frac{16}{3}H_{0,1}\right)\right]H_0^2+\left[-\frac{16}{27}(-101+28z)+\frac{16}{9}(-1+2z)H_1\right]H_0^3\\
&+\frac{4}{9}(13+7z)H_0^4+\Bigg[-\frac{4}{3}(-299+305z)+(-1+2z)\left(-8H_0^3-\frac{8}{3}H_1^2-\frac{32}{3}H_{0,1}\right)\\
&+\left[\frac{4}{9}(757+178z)+\frac{16}{3}(-1+2z)H_1\right]H_0+\frac{8}{3}(34+25z)H_0^2+\frac{32}{9}(-2+7z)H_1\Bigg]\zeta_2\\
&+\left[-\frac{8}{3}(-41+42z)+(-1+2z)\left(-\frac{16}{3}H_0^2+\frac{32}{9}H_1\right)+\frac{32}{9}(13+19z)H_0\right]\zeta_3\Bigg\}\\
&+C_AN_FT_F^2\Bigg\{-\frac{8}{27}(-4028+4113z)+L_M^3\left[\frac{64}{3}(z-1)-\frac{64}{9}(1+z)H_0+\frac{32}{9}(-1+2z)H_1\right]\\
&+L_M^2\Bigg[\frac{8}{3}(-65+68z)+\left[-\frac{16}{9}(47+38z)-\frac{32}{3}(1+2z)H_{-1}\right]H_0-\frac{32}{3}H_0^2+\frac{32}{9}(1+4z)H_1\\
&-\frac{16}{3}(-1+2z)H_1^2+\frac{32}{3}(1+2z)H_{0,-1}-\frac{32}{3}\zeta_2\Bigg]+L_M\Bigg[\frac{16}{9}(-514+547z)\\
&+(-1+2z)\left(\frac{16}{9}H_1^3+\frac{32}{3}H_{0,0,1}+32H_{0,1,1}\right)+\Bigg[-\frac{32}{27}(463+211z)+(-1+2z)\left(\frac{16}{3}H_1^2-\frac{32}{3}H_{0,1}\right)\\
&-\frac{64}{9}(2+7z)H_{-1}+\frac{64}{3}(1+2z)H_{0,-1}\Bigg]H_0+\left[-\frac{8}{9}(133+72z)+\frac{16}{3}(-1+2z)H_1-\frac{32}{3}(1+2z)H_{-1}\right]H_0^2\\
&+\frac{16}{9}(-5+6z)H_0^3+\left[\frac{16}{27}(-1+95z)-\frac{64}{3}(-1+2z)H_{0,1}\right]H_1\\
&-\frac{32}{9}(-2+7z)H_1^2+\frac{64}{3}zH_{0,1}+\frac{64}{9}(2+7z)H_{0,-1}-\frac{64}{3}(1+2z)H_{0,0,-1}\\
&+\left[-\frac{64}{9}(2+3z)+\frac{32}{3}(-1+2z)H_1\right]\zeta_2+32\zeta_3\Bigg]+\left[\frac{8}{81}(5836+4771z)-\frac{16}{3}(-1+2z)H_1\right]H_0\\
&-\frac{4}{27}(-649+128z)H_0^2+\frac{16}{27}(17+2z)H_0^3-\frac{4}{9}(-1+2z)H_0^4-\frac{16}{81}(-283+611z)H_1\\
&-\frac{8}{3}(z-1)H_1^2-\frac{16}{9}(3+10z)H_{0,1}+\frac{32}{3}zH_{0,1,1}\\
&+\Bigg[-\frac{8}{9}(-231+205z)+\left[\frac{8}{9}(97+100z)+\frac{16}{3}(1+2z)H_{-1}\right]H_0-\frac{8}{3}(-3+2z)H_0^2\\
&-\frac{32}{9}(-2+7z)H_1+\frac{8}{3}(-1+2z)H_1^2-\frac{16}{3}(1+2z)H_{0,-1}\Bigg]\zeta_2+\frac{16}{3}\zeta_2^2\\
&+\left[-\frac{32}{3}(-2+3z)+\frac{64}{9}(1+z)H_0-\frac{32}{9}(-1+2z)H_1\right]\zeta_3\Bigg\}+a_{qg,Q}^{(3)}\Bigg\}
\end{aligned}
\tag{A7}
$$

and the functions $a_{qq,Q}^{(3)}(z)$ and $a_{qg,Q}^{(3)}(z)$ are

$$
\begin{aligned}
a_{qq,Q}^{(3),\mathrm{PS}}(z) = {} & C_F T_F^2 N_F \Biggl\{ (1-z)\Biggl(\frac{85504}{243} - \frac{25472}{81} H_1 + \frac{320}{27} H_1^2 - \frac{640}{27} H_1^3 - \frac{160}{3} H_1 \zeta_2\Biggr) \\
& + (1+z)\Biggl[\Biggl(\frac{54592}{243} - 64\zeta_3\Biggr) H_0 + \frac{2288}{81} H_0^2 + \frac{176}{81} H_0^3 + \frac{8}{27} H_0^4 - \frac{9152}{81} H_{0,1} \\
& - \frac{704}{27} H_{0,0,1} + \frac{1408}{27} H_{0,1,1} - \frac{128}{9} H_{0,0,0,1} + \frac{256}{9} H_{0,0,1,1} - \frac{512}{9} H_{0,1,1,1} \\
& + \Biggl(\frac{1232}{27} H_0 + \frac{112}{9} H_0^2 - \frac{64}{3} H_{0,1}\Biggr)\zeta_2 + \frac{704}{15}\zeta_2^2\Biggr] + \frac{16}{81}(617+527z)\zeta_2 + \frac{32}{27}(-127+83z)\zeta_3 \Biggr\} \qquad \text{(A8)}
\end{aligned}
$$

$$
\begin{aligned}
a_{qg,Q}^{(3)}(z) = {} & C_F T_F^2 N_F \Biggl\{ -\frac{16}{81}(-49511+50285z) - (1-2z)\Biggl(-\frac{8}{3}H_0^5 + \frac{32}{27}H_0^3 H_1 - \frac{8}{27}H_1^4 - \frac{32}{9}H_0^2 H_{0,1} + \frac{64}{9}H_0 H_{0,0,1} \\
& - \frac{64}{9}H_{0,0,0,1} - \frac{64}{9}H_{0,1,1,1}\Biggr) + \frac{16}{243}(107935+46408z)H_0 + \frac{8}{81}(21955+3853z)H_0^2 - \frac{8}{81}(-3997+1118z)H_0^3 \\
& + \frac{8}{27}(160+61z)H_0^4 + \Biggl(\frac{32}{243}(-1559+4066z) + \frac{896}{81}(-5+13z)H_0 + \frac{64}{27}(-2+7z)H_0^2\Biggr)H_1 \\
& - \frac{80}{81}(-28+71z)H_1^2 + \frac{64}{81}(-2+7z)H_1^3 + \Biggl(-\frac{1792}{81}(-5+13z) - \frac{128}{27}(-2+7z)H_0\Biggr)H_{0,1} \\
& + \frac{128}{27}(-2+7z)[H_{0,0,1} + H_{0,1,1}] + \Biggl(-\frac{8}{81}(-7513+6779z) - (1-2z)\Biggl(-16H_0^3 + \frac{32}{3}H_0 H_1 - \frac{16}{3}H_1^2 - \frac{64}{3}H_{0,1}\Biggr) \\
& + \frac{8}{9}(757+178z)H_0 + \frac{16}{3}(34+25z)H_0^2 + \frac{64}{9}(-2+7z)H_1\Biggr)\zeta_2 - \frac{608}{45}(1-2z)\zeta_2^2 \\
& + \Biggl(\frac{16}{27}(-2567+2590z) + (-1+2z)\Biggl(\frac{224}{3}H_0^2 - \frac{448}{9}H_1\Biggr) - \frac{448}{9}(13+19z)H_0\Biggr)\zeta_3\Biggr\} \\
& + C_A T_F^2 N_F \Biggl\{ -\frac{4352}{81}(-104+109z) - (1-2z)\Biggl(-\frac{32}{9}H_0^3 H_1 - \frac{16}{9}H_0^2 H_1^2 + \frac{32}{27}H_0 H_1^3 + \frac{8}{27}H_1^4 \\
& + \Biggl(\frac{32}{3}H_0^2 - \frac{64}{9}H_1^2\Biggr)H_{0,1} - \frac{64}{9}H_{0,1}^2 + \Biggl(-\frac{64}{3}H_0 + \frac{128}{9}H_1\Biggr)H_{0,0,1} + \Biggl(\frac{64}{9}H_0 + \frac{128}{9}H_1\Biggr)H_{0,1,1} + \frac{64}{3}H_{0,0,0,1} \\
& - \frac{64}{9}H_{0,0,1,1} - \frac{64}{9}H_{0,1,1,1}\Biggr) + (1+2z)\Biggl(\frac{128}{27}H_{-1}H_0^3 - \frac{128}{9}H_0^2 H_{0,-1} - \frac{256}{9}H_{0,0,0,-1} + \frac{256}{27}H_{0,0,-1}\Bigl(2+3H_0\Bigr)\Biggr) \\
& + \Biggl(\frac{32}{243}(23638+13471z) + \frac{64}{81}(70+191z)H_{-1}\Biggr)H_0 + \Biggl(\frac{16}{81}(3423+680z) + \frac{128}{27}(2+7z)H_{-1}\Biggr)H_0^2 \\
& + \frac{80}{81}(85+8z)H_0^3 - \frac{8}{27}(-15+26z)H_0^4 + \Biggl(-\frac{32}{243}(-1055+3283z) - \frac{32}{9}(-4+11z)H_0 - \frac{64}{27}(-2+7z)H_0^2\Biggr)H_1 \\
& + \Biggl(\frac{16}{81}(-59+265z) - \frac{64}{27}(-2+7z)H_0\Biggr)H_1^2 - \frac{64}{81}(-2+7z)H_1^3 + \Biggl(-\frac{128}{27}(3+26z) + \frac{64}{27}(-4+17z)H_0 \\
& + \frac{256}{27}(-2+7z)H_1\Biggr)H_{0,1} + \Biggl(-\frac{64}{81}(70+191z) - \frac{128}{27}(4+11z)H_0\Biggr)H_{0,-1} - \frac{64}{27}(-4+35z)H_{0,0,1} \\
& - \frac{128}{9}(-2+3z)H_{0,1,1} + \Biggl(-\frac{16}{81}(-2359+1311z) - (1-2z)\Biggl(-\frac{64}{9}H_0 H_1 + \frac{80}{9}H_1^2 + \frac{64}{9}H_{0,1}\Biggr) \\
& + (1+2z)\Biggl(\frac{32}{3}H_{-1}H_0 - \frac{32}{3}H_{0,-1}\Biggr) + \frac{16}{9}(97+120z)H_0 - \frac{16}{3}(-3+2z)H_0^2 - \frac{320}{27}(-2+7z)H_1\Biggr)\zeta_2 \\
& + \frac{1376}{45}\zeta_2^2 + \Biggl(\frac{128}{9}(-23+22z) - \frac{896}{9}(1+z)H_0 - \frac{320}{9}(1-2z)H_1\Biggr)\zeta_3\Biggr\}. \qquad \text{(A9)}
\end{aligned}
$$

The single mass contributions to the OME $A_{Qg}^{(3)}$ read

$$
\begin{aligned}
A_{Qg} = & -4a_s L_M(-1+2z)T_F + a_s^2\Biggl\{-\frac{16}{3}L_M^2(-1+2z)T_F^2 + C_F T_F\Biggl\{2(-11+23z) + L_M^2[6+(-1+2z)(-4H_0-8H_1)] \\
& + L_M[-4(2+3z)+(-1+2z)(-4H_0^2-8H_1^2+16\zeta_2)+[-4(-1+16z)-16(-1+2z)H_1]H_0-32(z-1)H_1] \\
& + (-1+2z)\left((5+2z+4H_1)H_0^2+\frac{2}{3}H_0^3-\frac{4}{3}H_1^3+24H_{0,0,1}-8H_{0,1,1}-8\zeta_3\right) \\
& + (-2(8+25z)+4(-11+8z+2z^2)H_1-16(-1+2z)H_{0,1})H_0-4(-16+15z)H_1-4(z-1)(3+z)H_1^2 \\
& - 4(-29+12z+4z^2)H_{0,1}+8(-9+2z+z^2)\zeta_2\Biggr\} \\
& + C_A T_F\Biggl\{4(-51+53z)+L_M^2[48(z-1)-16(1+z)H_0+8(-1+2z)H_1]+(1+z)(-16H_{0,-1}-64H_{0,0,1}) \\
& + (-1+2z)\left(\frac{4}{3}H_1^3+8H_{0,1,1}\right)+(1+2z)\left(-8H_{-1}^2H_0-\frac{4}{3}H_0^3+8H_{0,0,-1}-16H_{0,-1,-1}\right) \\
& + L_M[8(-12+11z)+(1+2z)(16H_{-1}H_0+8H_0^2-16H_{0,-1})-8(1+8z)H_0+32(z-1)H_1+8(-1+2z)H_1^2+16\zeta_2] \\
& + [-4(14+37z)-4(z-1)(25+z)H_1-4(-1+2z)H_1^2+32(1+z)H_{0,1}-8(1+2z)H_{0,-1}]H_0 \\
& - 2(3+z^2)H_0^2-4H_1+2(z-1)(5+z)H_1^2+[(1+2z)(4H_0^2+16H_{0,-1}-8\zeta_2)+16(1+z)H_0]H_{-1} \\
& + 4(-25+28z+2z^2)H_{0,1}-4(-4+4z+z^2)\zeta_2+8(7+8z)\zeta_3\Biggr\}\Biggr\} \\
& + a_s^3\Biggl\{T_F^3(-1+2z)\left(-\frac{64L_M^3}{9}+\frac{64\zeta_3}{9}\right)+C_A T_F^2\Biggl\{-\frac{8}{27}(-7758+7613z) \\
& + L_M^3\left[\frac{448}{3}(z-1)-\frac{448}{9}(1+z)H_0+\frac{224}{9}(-1+2z)H_1\right]+(1+z)\left(-\frac{128}{3}H_{0,0,-1}+\frac{256}{3}H_{0,-1,-1}\right) \\
& + (-1+2z)\left(-\frac{8}{9}H_1^4+\frac{64}{3}H_{0,1}^2+\frac{320}{3}H_{0,0,1,1}+\frac{64}{3}H_{0,1,1,1}\right)+(1+2z)\left(-\frac{128}{9}H_{-1}^3H_0-\frac{64}{3}H_{0,0,0,-1}+\frac{256}{3}H_{0,0,1,-1}\right. \\
& \left.+\frac{256}{3}H_{0,0,-1,1}-\frac{128}{3}H_{0,0,-1,-1}+\frac{128}{3}H_{0,-1,0,1}+\frac{256}{3}H_{0,-1,-1,-1}\right) \\
& + L_M^2\left[\frac{8}{3}(-151+136z)+(1+2z)(32H_{-1}H_0-32H_{0,-1})-\frac{16}{9}(59+134z)H_0+\frac{32}{3}(1+4z)H_0^2\right. \\
& \left.+\frac{32}{9}(-23+28z)H_1+16(-1+2z)H_1^2+32\zeta_2\right] \\
& + L_M\left[\frac{16}{9}(-812+849z)+(-1+2z)\left(\frac{16}{3}H_1^3+\frac{160}{3}H_{0,1,1}\right)+(1+2z)\left(-\frac{64}{3}H_{-1}^2H_0-\frac{128}{3}H_{0,-1,-1}\right)\right. \\
& + \left[-\frac{32}{27}(589+544z)-\frac{32}{3}(z-1)(25+z)H_1-\frac{16}{3}(-1+2z)H_1^2+32(3+2z)H_{0,1}\right]H_0 \\
& + \left[-\frac{8}{9}(151+72z+6z^2)+\frac{16}{3}(-1+2z)H_1\right]H_0^2+\frac{16}{9}(-7+2z)H_0^3 \\
& + \left[\frac{304}{27}(-1+5z)+(-1+2z)\left(-\frac{64H_{0,1}}{3}+\frac{32\zeta_2}{3}\right)\right]H_1+\frac{16}{9}(-11-2z+3z^2)H_1^2 \\
& + \left[(1+2z)\left(\frac{128H_{0,-1}}{3}-\frac{64\zeta_2}{3}\right)-\frac{64}{9}(-4+z)H_0\right]H_{-1}+\frac{32}{3}(-25+30z+2z^2)H_{0,1}+\frac{64}{9}(-4+z)H_{0,-1} \\
& \left.-\frac{32}{3}(17+14z)H_{0,0,1}-\frac{32}{9}(-8+18z+3z^2)\zeta_2+\frac{32}{3}(17+16z)\zeta_3\right]+\left[\frac{8}{81}(7672+14815z)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+(-1+2z)\left(\frac{64}{9}H_1^3-\frac{320}{3}H_{0,1,1}\right)+(1+2z)\left(\frac{64}{3}H_{0,0,-1}-\frac{128}{3}H_{0,1,-1}-\frac{128}{3}H_{0,-1,1}+\frac{128}{3}H_{0,-1,-1}\right)\\
&+\left[-\frac{16}{3}(63-66z+4z^2)+64(-1+2z)H_{0,1}\right]H_1+\frac{64}{3}(z-1)H_1^2+\frac{32}{3}(-27+8z+z^2)H_{0,1}\\
&+\frac{128}{3}(1+z)H_{0,-1}-\frac{128}{3}(3+2z)H_{0,0,1}+\frac{8}{9}(115+244z)\zeta_2+\frac{64}{9}(19+31z)\zeta_3\Big]H_0\\
&+\left[-\frac{4}{27}(-577+1244z+72z^2)-\frac{16}{3}(z-1)(25+z)H_1-\frac{16}{3}(-1+2z)H_1^2+\frac{128}{3}(1+z)H_{0,1}-\frac{32}{3}(1+2z)H_{0,-1}\right.\\
&\left.-\frac{8}{3}(3+14z)\zeta_2\right]H_0^2-\frac{16}{27}(-2+z)(4+3z)H_0^3-\frac{4}{9}(1+6z)H_0^4+\left[-\frac{16}{81}(-67+503z)\right.\\
&\left.+(-1+2z)\left(-\frac{320}{3}H_{0,0,1}-\frac{64}{3}H_{0,1,1}+\frac{160}{9}\zeta_3\right)-\frac{160}{9}(-4+5z)\zeta_2\right]H_1+\left[\frac{8}{3}(3-5z+4z^2)-\frac{40}{3}(-1+2z)\zeta_2\right]H_1^2\\
&-\frac{16}{9}(z-1)(5+z)H_1^3+\left[(1+z)\left(-\frac{64}{3}H_0^2-\frac{256}{3}H_{0,-1}+\frac{128}{3}\zeta_2\right)+(1+2z)\left(\frac{32}{9}H_0^3-\frac{256}{3}H_{0,0,1}+\frac{128}{3}H_{0,0,-1}\right.\right.\\
&\left.\left.-\frac{256}{3}H_{0,-1,-1}+64\zeta_3\right)+\left[-128(1+z)+(1+2z)\left(\frac{128}{3}H_{0,1}-\frac{128}{3}H_{0,-1}-\frac{80}{3}\zeta_2\right)\right]H_0\right]H_{-1}\\
&+\left[(1+2z)\left(\frac{32}{3}H_0^2+\frac{128}{3}H_{0,-1}-\frac{64}{3}\zeta_2\right)+\frac{128}{3}(1+z)H_0\right]H_{-1}^2+\frac{16}{9}(189-238z+24z^2)H_{0,1}\\
&+\left[128(1+z)+\frac{80}{3}(1+2z)\zeta_2\right]H_{0,-1}-\frac{32}{3}(-29-8z+z^2)H_{0,0,1}-\frac{32}{3}(-4+7z+z^2)H_{0,1,1}+128H_{0,0,0,1}\\
&\left.-\frac{4}{9}(-561+620z+48z^2)\zeta_2-\frac{432}{5}\zeta_2^2+\frac{32}{3}(14-43z+z^2)\zeta_3\right\}\\
&+C_AN_FT_F^2\left\{-\frac{32}{3}(-53+51z)+L_M^3\left[\frac{64}{3}(z-1)-\frac{64}{9}(1+z)H_0+\frac{32}{9}(-1+2z)H_1\right]\right.\\
&+(1+z)\left(-\frac{64}{3}H_{0,0,-1}+\frac{128}{3}H_{0,-1,-1}\right)+(-1+2z)\left(-\frac{4}{9}H_1^4+\frac{32}{3}H_{0,1}^2+\frac{160}{3}H_{0,0,1,1}+\frac{32}{3}H_{0,1,1,1}\right)\\
&+(1+2z)\left(-\frac{64}{9}H_{-1}^3H_0-\frac{4}{9}H_0^4-\frac{32}{3}H_{0,0,0,-1}+\frac{128}{3}H_{0,0,1,-1}+\frac{128}{3}H_{0,0,-1,1}-\frac{64}{3}H_{0,0,-1,-1}\right.\\
&\left.+\frac{64}{3}H_{0,-1,0,1}+\frac{128}{3}H_{0,-1,-1,-1}\right)+L_M^2\left[-\frac{8}{3}(-65+68z)+(1+2z)\left(\frac{32}{3}H_{-1}H_0-\frac{32}{3}H_{0,-1}\right)\right.\\
&\left.+\frac{16}{9}(47+38z)H_0+\frac{32}{3}H_0^2-\frac{32}{9}(1+4z)H_1+\frac{16}{3}(-1+2z)H_1^2+\frac{32}{3}\zeta_2\right]\\
&+L_M\left[\frac{8}{9}(-778+845z)+(-1+2z)\left(\frac{32}{9}H_1^3+\frac{128}{3}H_{0,1,1}\right)+(1+2z)\left(-\frac{32}{3}H_{-1}^2H_0-\frac{32}{3}H_{0,0,-1}-\frac{64}{3}H_{0,-1,-1}\right)\right.\\
&+\left[-\frac{8}{27}(1319+866z)-\frac{16}{3}(z-1)(25+z)H_1+\frac{32}{3}(5+2z)H_{0,1}+\frac{32}{3}(1+2z)H_{0,-1}\right]H_0\\
&+\left[-\frac{8}{9}(108+68z+3z^2)+\frac{16}{3}(-1+2z)H_1\right]H_0^2+\frac{16}{9}(-5+2z)H_0^3\\
&+\left[-\frac{16}{27}(-37+8z)+(-1+2z)\left(-\frac{64H_{0,1}}{3}+\frac{32\zeta_2}{3}\right)\right]H_1+\frac{8}{9}(-7-16z+3z^2)H_1^2\\
&+\left[(1+2z)\left(-\frac{16}{3}H_0^2+\frac{64}{3}H_{0,-1}-\frac{32}{3}\zeta_2\right)-\frac{64}{9}(-1+4z)H_0\right]H_{-1}+\frac{16}{3}(-25+30z+2z^2)H_{0,1}
\end{aligned}
$$

$$+\frac{64}{9}(-1+4z)H_{0,-1}-32(3+2z)H_{0,0,1}-\frac{16}{9}(-4+18z+3z^2)\zeta_2+\frac{64}{3}(5+4z)\zeta_3\Big]$$
$$+\Big[\frac{16}{3}(17+93z)+(-1+2z)\Big(\frac{32}{9}H_1^3-\frac{160}{3}H_{0,1,1}\Big)+(1+2z)\Big(\frac{32}{3}H_{0,0,-1}-\frac{64}{3}H_{0,1,-1}$$
$$-\frac{64}{3}H_{0,-1,1}+\frac{64}{3}H_{0,-1,-1}\Big)+\Big[-\frac{32}{3}(-16+z)(z-1)+32(-1+2z)H_{0,1}\Big]H_1$$
$$+\frac{32}{3}(z-1)H_1^2+\frac{16}{3}(-27+8z+z^2)H_{0,1}+\frac{64}{3}(1+z)H_{0,-1}-\frac{64}{3}(3+2z)H_{0,0,1}$$
$$-\frac{16}{9}(22+7z)\zeta_2+\frac{64}{9}(7+13z)\zeta_3\Big]H_0+\Big[-\frac{8}{3}(2+31z+2z^2)-\frac{8}{3}(z-1)(25+z)H_1$$
$$-\frac{8}{3}(-1+2z)H_1^2+\frac{64}{3}(1+z)H_{0,1}-\frac{16}{3}(1+2z)H_{0,-1}-\frac{8}{3}(3+2z)\zeta_2\Big]H_0^2$$
$$-\frac{8}{9}(3+z^2)H_0^3+\Big[\frac{32}{3}(-2+z)+(-1+2z)\Big(-\frac{160}{3}H_{0,0,1}-\frac{32}{3}H_{0,1,1}+\frac{160}{9}\zeta_3\Big)$$
$$-\frac{16}{9}(-7+2z)\zeta_2\Big]H_1+\Big[\frac{8}{3}(1-2z+2z^2)-\frac{16}{3}(-1+2z)\zeta_2\Big]H_1^2-\frac{8}{9}(z-1)(5+z)H_1^3$$
$$+\Big[(1+z)\Big(-\frac{32}{3}H_0^2-\frac{128}{3}H_{0,-1}+\frac{64}{3}\zeta_2\Big)+(1+2z)\Big(\frac{16}{9}H_0^3-\frac{128}{3}H_{0,0,1}+\frac{64}{3}H_{0,0,-1}$$
$$-\frac{128}{3}H_{0,-1,-1}+32\zeta_3\Big)+\Big[-64(1+z)+(1+2z)\Big(\frac{64}{3}H_{0,1}-\frac{64}{3}H_{0,-1}-\frac{32}{3}\zeta_2\Big)\Big]H_0\Big]H_{-1}$$
$$+\Big[(1+2z)\Big(\frac{16}{3}H_0^2+\frac{64}{3}H_{0,-1}-\frac{32}{3}\zeta_2\Big)+\frac{64}{3}(1+z)H_0\Big]H_{-1}^2+\frac{32}{3}(16-19z+2z^2)H_{0,1}$$
$$+\Big[64(1+z)+\frac{32}{3}(1+2z)\zeta_2\Big]H_{0,-1}-\frac{16}{3}(-29-8z+z^2)H_{0,0,1}-\frac{16}{3}(-4+8z+z^2)H_{0,1,1}$$
$$+64H_{0,0,0,1}-\frac{4}{3}(89-62z+8z^2)\zeta_2-\frac{608}{15}\zeta_2^2+\frac{16}{3}(4-32z+z^2)\zeta_3\Big\}$$
$$+C_FT_F^2\Big\{-\frac{22}{3}(-339+374z)+L_M^3\Big[\frac{16}{3}(-19+21z)+(-1+2z)\Big(\frac{16}{3}H_0^2-\frac{128}{9}H_1\Big)$$
$$-\frac{16}{9}(23+44z)H_0\Big]+L_M^2\Big[\frac{4}{3}(-287+244z)+(-1+2z)\Big(\frac{32}{3}H_0^3-16H_1^2-\frac{32}{3}H_{0,1}+32\zeta_2\Big)$$
$$+\Big[-\frac{8}{9}(433+94z)-\frac{64}{3}(-1+2z)H_1\Big]H_0-\frac{320}{3}(1+z)H_0^2-\frac{32}{9}(-13+8z)H_1\Big]$$
$$+L_M\Big[\frac{2}{9}(-7895+8486z)+(-1+2z)\Big(\frac{20}{3}H_0^4-\frac{16}{3}H_1^3+\frac{256}{3}H_{0,0,1}-32H_{0,1,1}-\frac{32}{3}\zeta_3\Big)$$
$$+\Big[-\frac{8}{27}(4285+1843z)+\frac{64}{9}(-22+20z+3z^2)H_1-64(-1+2z)H_{0,1}\Big]H_0$$
$$+\Big[\frac{4}{9}(-1069+128z+24z^2)+\frac{64}{3}(-1+2z)H_1\Big]H_0^2-\frac{8}{9}(103+40z)H_0^3$$
$$-\frac{16}{27}(-260+289z)H_1-\frac{32}{9}(-7-z+3z^2)H_1^2-\frac{64}{9}(-47+19z+6z^2)H_{0,1}$$
$$+\frac{64}{9}(-25-z+3z^2)\zeta_2\Big]+(-1+2z)\Big(-\frac{4}{15}H_0^5+\frac{8}{9}H_1^4-\frac{128}{3}H_{0,1}^2-\frac{64}{3}H_{0,0,0,1}$$
$$-\frac{320}{3}H_{0,1,1,1}-\frac{976}{15}\zeta_2^2\Big)+\Big[\frac{16}{3}(261+155z)+(-1+2z)\Big(\frac{128}{3}H_{0,0,1}+\frac{256}{3}H_{0,1,1}\Big)$$

$$+\left[\frac{32}{3}(2-5z+4z^2)+(-1+2z)\left(-\frac{128H_{0,1}}{3}+\frac{64\zeta_2}{3}\right)\right]H_1-\frac{64}{3}(-2+8z+z^2)H_{0,1}$$
$$+\frac{4}{9}(769+316z)\zeta_2-\frac{16}{9}(-47+4z)\zeta_3\Bigg]H_0+\Bigg[\frac{8}{3}(142-11z+8z^2)$$
$$+(-1+2z)\left(-\frac{64H_{0,1}}{3}-\frac{16\zeta_3}{3}\right)+\frac{16}{3}(-11+8z+2z^2)H_1+\frac{4}{3}(65+56z)\zeta_2\Bigg]H_0^2$$
$$+\left[\frac{8}{9}(64-19z+4z^2)+(-1+2z)\left(\frac{32H_1}{9}-8\zeta_2\right)\right]H_0^3+\frac{2}{9}(25+16z)H_0^4+\Bigg[\frac{32}{3}(-5+7z)$$
$$+(-1+2z)\left(\frac{256}{3}H_{0,0,1}+\frac{256}{3}H_{0,1,1}-\frac{640}{9}\zeta_3\right)+\frac{80}{9}(-5+4z)\zeta_2\Bigg]H_1$$
$$+\left[-\frac{16}{3}(16-19z+4z^2)+(-1+2z)\left(-\frac{64H_{0,1}}{3}+\frac{104\zeta_2}{3}\right)\right]H_1^2+\frac{32}{9}(z-1)(3+z)H_1^3$$
$$+\left[-\frac{32}{3}(8-31z+8z^2)+48(-1+2z)\zeta_2\right]H_{0,1}+\frac{32}{3}(3+24z+2z^2)H_{0,0,1}$$
$$+\frac{64}{3}(-9+2z+z^2)H_{0,1,1}+\frac{2}{3}(685-924z+64z^2)\zeta_2-\frac{16}{3}(-27+61z+4z^2)\zeta_3\Bigg\}$$
$$+C_FN_FT_F^2\Bigg\{-\frac{4}{3}(-827+914z)+L_M^3\Bigg[\frac{8}{3}(-41+42z)+(-1+2z)\left(\frac{16}{3}H_0^2-\frac{32}{9}H_1\right)$$
$$-\frac{32}{9}(13+19z)H_0\Bigg]+L_M^2\Bigg[-\frac{4}{3}(-285+296z)+(-1+2z)\Bigg(-\frac{32}{3}H_0^3-\frac{16}{3}H_1^2-\frac{32}{3}H_{0,1}$$
$$+\frac{32}{3}\zeta_2\Bigg)+\frac{8}{9}(425+2z)H_0+16(7+6z)H_0^2+\frac{32}{9}(1+4z)H_1\Bigg]+L_M\Bigg[\frac{2}{9}(-8933+9110z)$$
$$+(-1+2z)\left(\frac{20}{3}H_0^4-\frac{32}{9}H_1^3-\frac{64}{3}H_{0,1,1}+192H_{0,0,0,1}-\frac{384}{5}\zeta_2^2\right)+\Bigg[-\frac{8}{27}(3349+3562z)$$
$$+(-1+2z)\left(-64H_{0,0,1}-128\zeta_3\right)+\frac{16}{9}(323-322z+6z^2)H_1+\frac{32}{3}(31+28z)H_{0,1}\Bigg]H_0$$
$$+\left[\frac{8}{9}(-401+34z+6z^2)+16(-1+2z)H_1\right]H_0^2-16(5+3z)H_0^3-\frac{32}{27}(-58+77z)H_1$$
$$-\frac{16}{9}(-5-8z+3z^2)H_1^2-\frac{16}{9}(277-338z+12z^2)H_{0,1}-\frac{32}{3}(59+62z)H_{0,0,1}$$
$$+\frac{32}{9}(-23-8z+3z^2)\zeta_2+192(3+4z)\zeta_3\Bigg]+(-1+2z)\Bigg(\frac{4}{15}H_0^5+\frac{4}{9}H_1^4-\frac{64}{3}H_{0,1}^2$$
$$-\frac{160}{3}H_{0,1,1,1}-128H_{0,0,0,0,1}+128\zeta_5\Bigg)+\Bigg[\frac{8}{3}(-111+479z)+(-1+2z)\Bigg(\frac{128}{3}H_{0,1,1}$$
$$+128H_{0,0,0,1}\Bigg)+\left[\frac{16}{3}(-10+7z+4z^2)+(-1+2z)\left(-\frac{64H_{0,1}}{3}+\frac{16\zeta_2}{3}\right)\right]H_1$$
$$-\frac{32}{3}(46-34z+z^2)H_{0,1}-\frac{32}{3}(41+44z)H_{0,0,1}+\frac{8}{9}(-259+86z)\zeta_2+\frac{32}{9}(109+151z)\zeta_3\Bigg]H_0$$
$$+\Bigg[\frac{4}{3}(-116-437z+8z^2)+(-1+2z)\left(-32H_{0,0,1}-\frac{208\zeta_3}{3}\right)+\frac{8}{3}(115-118z+2z^2)H_1$$
$$+\frac{16}{3}(29+32z)H_{0,1}-\frac{232}{3}(1+z)\zeta_2\Bigg]H_0^2+\Bigg[\frac{4}{9}(-59+50z+4z^2)$$
$$+(-1+2z)\left(\frac{16H_1}{9}+8\zeta_2\right)\Bigg]H_0^3-\frac{4}{9}(8+11z)H_0^4+\Bigg[\frac{16}{3}(-5+7z)+(-1+2z)\Bigg(\frac{128}{3}H_{0,0,1}$$

$$
\begin{aligned}
&+\frac{128}{3}H_{0,1,1}-\frac{352}{9}\zeta_3\bigg)+\frac{16}{9}(-7+2z)\zeta_2\bigg]H_1+\bigg[-\frac{8}{3}(16-19z+4z^2)\\
&+(-1+2z)\bigg(-\frac{32H_{0,1}}{3}+16\zeta_2\bigg)\bigg]H_1^2+\frac{16}{9}(z-1)(3+z)H_1^3+\bigg[-\frac{16}{3}(-4-19z+8z^2)\\
&+\frac{80}{3}(-1+2z)\zeta_2\bigg]H_{0,1}+\frac{16}{3}(69-18z+2z^2)H_{0,0,1}+\frac{32}{3}(-9+2z+z^2)H_{0,1,1}\\
&+\frac{32}{3}(37+34z)H_{0,0,0,1}+\frac{2}{3}(-77-194z+32z^2)\zeta_2-\frac{64}{15}(29+50z)\zeta_2^2\\
&-\frac{8}{3}(-169+250z+4z^2)\zeta_3\bigg\}+C_F^2T_F\bigg\{451-474z+L_M^3\bigg[(-1+2z)\bigg(-6-\frac{4}{3}H_0^2-\frac{32}{3}H_1^2\\
&+\frac{32}{3}\zeta_2\bigg)+\bigg[-\frac{8}{3}(z-1)-\frac{32}{3}(-1+2z)H_1\bigg]H_0+16H_1\bigg]+(1+2z)(16H_{-1}^2H_0\zeta_2\\
&+16H_{0,0,-1}\zeta_2+32H_{0,-1,-1}\zeta_2)+L_M^2\bigg[-439+472z+(-1+2z)(-16H_1^3+48H_{0,0,1}\\
&+16H_{0,1,1})+(1+2z)\bigg(-32H_{-1}^2H_0-\frac{8}{3}H_0^3-32H_{0,0,-1}-64H_{0,-1,-1}\bigg)+\bigg[-2(101+138z)\\
&+(-1+2z)(-48H_1^2-32H_{0,1})-16(-7+4z)H_1\\
&+16(-1+6z)\zeta_2\bigg]H_0+\bigg[-4(11+10z)-16(-1+2z)H_1\bigg]H_0^2+\bigg[-8(-4+9z)\\
&+64(-1+2z)\zeta_2\bigg]H_1-4(-19+16z)H_1^2+\bigg[(1+2z)(16H_0^2+64H_{0,-1}-32\zeta_2)\\
&+96(1+z)H_0\bigg]H_{-1}-16(3+7z)H_{0,1}-96(1+z)H_{0,-1}+16(2+11z)\zeta_2+16(1+6z)\zeta_3\bigg]\\
&+L_M\bigg[\frac{1}{2}(3633-3782z)+(-1+2z)(-8H_1^4-32H_{0,1}^2+32H_{0,-1}^2-80H_{0,0,1,1}\\
&-32H_{0,1,1,1}-64H_{0,-1,0,1})+\bigg[2(209+350z)+(1+2z)(-96H_{0,-1}-96H_{0,0,-1})\\
&+(-1+2z)\bigg(-\frac{32}{3}H_1^3+160H_{0,1,1}-64H_{0,1,-1}-64H_{0,-1,1}\bigg)\\
&+\bigg[-12(-15+26z)+(-1+2z)(-96H_{0,1}+64H_{0,-1}+192\zeta_2)\bigg]H_1-8(-5+6z)H_1^2\\
&-8(-21+22z+4z^2)H_{0,1}-16(1+14z)H_{0,0,1}-256zH_{0,-1,-1}+32(-7+12z)\zeta_2\\
&+16(-11+10z)\zeta_3\bigg]H_0+\bigg[2(-21-220z+24z^2)+(-1+2z)(-24H_1^2+48\zeta_2)\\
&+4(19-28z+4z^2)H_1+8(3+2z)H_{0,1}+16(1+6z)H_{0,-1}\bigg]H_0^2+\bigg[\frac{4}{3}(-17-23z+4z^2)\\
&-\frac{32}{3}(-1+2z)H_1\bigg]H_0^3+\frac{1}{3}(-1-6z)H_0^4+\bigg[-8(-83+81z)+(-1+2z)(96H_{0,0,1}\\
&-128H_{0,0,-1}+96H_{0,1,1}+224\zeta_3)-192(z-1)H_{0,1}+16(-21+22z)\zeta_2\bigg]H_1\\
&+\bigg[-8(-39+44z)+(-1+2z)(-64H_{0,1}+96\zeta_2)\bigg]H_1^2-16(z-1)(5+z)H_1^3
\end{aligned}
$$

$$
\begin{aligned}
&+\bigg[(1+z)(64H_0^2-128H_{0,-1}+64\zeta_2)+\bigg[-32(2+3z+3z^2)+64(1+2z)H_{0,-1}\bigg]H_0\\
&+\frac{16}{3}(1+2z)H_0^3\bigg]H_{-1}+\bigg[64(1+z)H_0-32(1+2z)H_0^2\bigg]H_{-1}^2+\bigg[-4(-23+38z)\\
&-128z\zeta_2\bigg]H_{0,1}+\bigg[32(2+3z+3z^2)+64(-1+2z)H_{0,1}-128z\zeta_2\bigg]H_{0,-1}\\
&+8(-29+24z+4z^2)H_{0,0,1}+64(1+4z)H_{0,0,-1}-16(-19+3z+6z^2)H_{0,1,1}\\
&+128(1+z)H_{0,-1,-1}+64(-1+8z)H_{0,0,0,1}-32(-5+2z)H_{0,0,0,-1}-16(21-29z+6z^2)\zeta_2\\
&-\frac{8}{5}(-47+126z)\zeta_2^2+16(-62+37z+6z^2)\zeta_3\bigg]+(-1+2z)\bigg(\frac{1}{15}H_0^5+\frac{4}{3}H_1^5-112H_{0,0,0,0,1}\\
&+160H_{0,0,0,1,1}+96H_{0,0,1,0,1}-432H_{0,0,1,1,1}-224H_{0,1,0,1,1}-144H_{0,1,1,1,1}+64\zeta_5\bigg)\\
&+\bigg[-2(91+33z)+(-1+2z)\bigg(\frac{4}{3}H_1^4-32H_{0,1}^2+48H_{0,0,0,1}-160H_{0,1,1,1}\bigg)\\
&+\bigg[4(-214+215z+24z^2)+(-1+2z)\bigg(96H_{0,0,1}+192H_{0,1,1}-\frac{448}{3}\zeta_3\bigg)\\
&-32(-7+4z+z^2)H_{0,1}-4(-19+16z+8z^2)\zeta_2\bigg]H_1+\bigg[112(z-1)\\
&+(-1+2z)(-64H_{0,1}+72\zeta_2)\bigg]H_1^2+\frac{16}{3}(z-1)(3+z)H_1^3+\bigg[-8(3+2z)(-7+4z)\\
&+152(-1+2z)\zeta_2\bigg]H_{0,1}-8(-19-18z+4z^2)H_{0,0,1}+32(-21+2z+2z^2)H_{0,1,1}\\
&-2(-143+12z)\zeta_2-\frac{8}{5}(-37+94z)\zeta_2^2-\frac{8}{3}(-20+95z+24z^2)\zeta_3\bigg]H_0+\bigg[-128+263z\\
&+48z^2+(-1+2z)\bigg(8H_{0,0,1}+64H_{0,1,1}-\frac{68}{3}\zeta_3\bigg)+\bigg[2(-27+42z+16z^2)\\
&-48(-1+2z)H_{0,1}\bigg]H_1+4(-11+8z+2z^2)H_1^2-4(1+22z)H_{0,1}-2(-21-3z\\
&+8z^2)\zeta_2\bigg]H_0^2+\bigg[\frac{1}{3}(-39+50z+32z^2)+(-1+2z)\bigg(\frac{8}{3}H_1^2-8H_{0,1}\bigg)\\
&+\frac{4}{3}(-25+16z+4z^2)H_1-\frac{2}{3}(-5+2z)\zeta_2\bigg]H_0^3+\bigg[\frac{1}{3}(-2+5z+4z^2)+\frac{4}{3}(-1+2z)H_1\bigg]H_0^4\\
&+\bigg[-4(-159+169z)+(-1+2z)\bigg(-64H_{0,1}^2-\frac{1056}{5}\zeta_2^2\bigg)\\
&+\bigg[-32(z-1)+64(-1+2z)\zeta_2\bigg]H_{0,1}+64(-7+4z+z^2)H_{0,0,1}\\
&-96H_{0,1,1}-104(-3+4z)\zeta_2-32(-11+6z+2z^2)\zeta_3\bigg]H_1\\
&+\bigg[-2(56-131z+24z^2)+(-1+2z)\bigg(96H_{0,0,1}+64H_{0,1,1}-\frac{112}{3}\zeta_3\bigg)+24H_{0,1}\\
&-4(13-8z+4z^2)\zeta_2\bigg]H_1^2+\bigg[-\frac{8}{3}(44-47z+12z^2)+(-1+2z)\bigg(-\frac{64H_{0,1}}{3}+32\zeta_2\bigg)\bigg]H_1^3
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3}(-51+32z+16z^2)H_1^4+\Big[(1+2z)(-8H_0^2\zeta_2-32H_{0,-1}\zeta_2+16\zeta_2^2)-48(1+z)H_0\zeta_2\Big]H_{-1}\\
&+\Big[-4(-364+97z+48z^2)+(-1+2z)(-32H_{0,0,1}+96H_{0,1,1}+160\zeta_3)\\
&+4(73+74z)\zeta_2\Big]H_{0,1}-8(-13+8z+2z^2)H_{0,1}^2+\Big[4(-85-40z+16z^2)\\
&-200(-1+2z)\zeta_2\Big]H_{0,0,1}+\Big[-8(-1-85z+24z^2)-128(-1+2z)\zeta_2\Big]H_{0,1,1}\\
&+16(-17+2z+6z^2)H_{0,0,0,1}-32(-30-z+2z^2)H_{0,0,1,1}+8(-83+20z+12z^2)H_{0,1,1,1}\\
&+\Big[-384\ln(2)(-1+2z)+\frac{1}{2}(-1005-506z+192z^2)+\frac{16}{3}(-7+2z)\zeta_3\Big]\zeta_2\\
&+48(1+z)H_{0,-1}\zeta_2-\frac{8}{5}(-46+251z+42z^2)\zeta_2^2+2(141-154z+96z^2)\zeta_3\Big\}\\
&+C_A^2T_F\Big\{\frac{8}{3}(-6427+6405z)+(1+z)(-128H_{0,1,-1,-1}-128H_{0,-1,1,-1}-128H_{0,-1,-1,1})\\
&+(-7+10z)(-32H_{0,0,0,0,-1}+32H_{0,0,-1,0,1})+(-1+4z)(-64H_{0,0,-1,0,-1}-512H_{0,0,-1,-1,-1}\\
&-256H_{0,-1,0,-1,-1})+(1+2z)(-16H_{-1}^4H_0-128H_{0,0,1,1,-1}-128H_{0,0,1,-1,1}-192H_{0,0,1,-1,-1}\\
&-128H_{0,0,-1,1,1}-192H_{0,0,-1,1,-1}-192H_{0,0,-1,-1,1}-128H_{0,1,-1,-1,-1}-64H_{0,-1,0,1,1}\\
&-128H_{0,-1,0,1,-1}-128H_{0,-1,0,-1,1}-128H_{0,-1,1,-1,-1}-128H_{0,-1,-1,0,1}-128H_{0,-1,-1,1,-1}\\
&-128H_{0,-1,-1,-1,1}-384H_{0,-1,-1,-1,-1})+L_M^3\Big[-\frac{1328}{3}(z-1)+\Big[\frac{16}{9}(107+59z)\\
&-\frac{32}{3}(-1+2z)H_1\Big]H_0-\frac{32}{3}(-2+z)H_0^2-\frac{8}{9}(-155+166z)H_1-\frac{32}{3}(-1+2z)H_1^2\\
&+32(1+2z)H_{0,1}-\frac{128}{3}(1+z)\zeta_2\Big]+L_M^2\Big[-\frac{16}{3}(-301+298z)+(1+2z)(32H_{-1}^2H_0\\
&+16H_{0,0,1}+32H_{0,1,-1}+32H_{0,-1,1}+64H_{0,-1,-1})+\Big[\frac{368}{9}(1+28z)+128(z-1)H_1\\
&-48(1+2z)H_{0,1}+16(-5+14z)H_{0,-1}+32(-1+4z)\zeta_2\Big]H_0+\Big[-\frac{16}{3}(25+33z)\\
&+8(-1+2z)H_1\Big]H_0^2+\frac{64}{3}(z-1)H_0^3+\Big[-\frac{40}{9}(-115+98z)+32(-1+2z)\zeta_2\Big]H_1\\
&-\frac{4}{3}(-131+142z)H_1^2-16(-1+2z)H_1^3+\Big[(1+2z)(-24H_0^2-32H_{0,1}-64H_{0,-1}\\
&+64\zeta_2)-\frac{8}{3}(155+166z)H_0\Big]H_{-1}+16(9+4z)H_{0,1}+\frac{8}{3}(155+166z)H_{0,-1}\\
&-16(-13+22z)H_{0,0,-1}+64(1+z)H_{0,1,1}-\frac{8}{3}(161+72z)\zeta_2+16(-17+8z)\zeta_3\Big]\\
&+L_M\Big[-\frac{4}{9}(-33406+33719z)+(1+z)(-64H_{0,1,-1}-64H_{0,-1,1})+(-1+2z)(-8H_1^4
\end{aligned}
$$

$$+32H_{0,1,1,1}) + (1+2z)\biggl(-\frac{160}{3}H_{-1}^3 H_0 + 368H_{0,0,0,-1} - 32H_{0,0,1,-1} - 32H_{0,0,-1,1}$$
$$+160H_{0,0,-1,-1} + 64H_{0,1,1,-1} + 64H_{0,1,-1,1} + 64H_{0,-1,1,1} + 320H_{0,-1,-1,-1}\biggr)$$
$$+\biggl[\frac{8}{27}(13987+34141z) + (-1+2z)\biggl(\frac{80}{3}H_1^3 - 96H_{0,1,-1} - 96H_{0,-1,1}\biggr)$$
$$+\biggl[\frac{4}{3}(z-1)(2363+11z) + (-1+2z)(64H_{0,1} + 96H_{0,-1} + 160\zeta_2)\biggr]H_1 + 480(z-1)H_1^2$$
$$-\frac{8}{3}(574+181z+6z^2)H_{0,1} + \frac{8}{3}(-275+26z)H_{0,-1} - 64(5+7z)H_{0,0,1} - 64(4+3z)H_{0,0,-1}$$
$$-64(2+11z)H_{0,1,1} - 128(-1+5z)H_{0,-1,-1} + 8(-29+15z)\zeta_2 - 96(11+6z)\zeta_3\biggr]H_0$$
$$+\biggl[\frac{2}{9}(2817-3380z+141z^2) + \frac{4}{3}(62-79z+6z^2)H_1 + 16(-1+2z)H_1^2 + 4(-1+46z)H_{0,1}$$
$$-32(-2+z)H_{0,-1} - 4(11+2z)\zeta_2\biggr]H_0^2 + \biggl[\frac{4}{9}(235+254z+6z^2) - \frac{40}{3}(-1+2z)H_1\biggr]H_0^3$$
$$-\frac{2}{3}(-19+20z)H_0^4 + \biggl[-\frac{4}{27}(-2629+3221z) + (-1+2z)(-256H_{0,0,1} - 192H_{0,0,-1}$$
$$+544\zeta_3) - \frac{16}{3}(-61+50z)H_{0,1} - \frac{8}{3}(-17+28z)\zeta_2\biggr]H_1 + \biggl[-\frac{2}{9}(-1217+844z+33z^2)$$
$$+(-1+2z)(-32H_{0,1}+32\zeta_2)\biggr]H_1^2 - \frac{8}{9}(-92+94z+9z^2)H_1^3 + \biggl[(1+2z)\biggl(\frac{8}{3}H_0^3$$
$$+32H_{0,0,1} - 160H_{0,0,-1} - 64H_{0,1,1} - 320H_{0,-1,-1} + 192\zeta_3\biggr) + \biggl[-\frac{16}{9}(1097+1030z+27z^2)$$
$$+(1+2z)(64H_{0,-1}+64\zeta_2)\biggr]H_0 + \frac{4}{3}(101+112z)H_0^2 + 64(1+z)H_{0,1} - \frac{16}{3}(221+232z)H_{0,-1}$$
$$+\frac{8}{3}(197+208z)\zeta_2\biggr]H_{-1} + \biggl((1+2z)(8H_0^2 + 160H_{0,-1} - 80\zeta_2) + \frac{8}{3}(221+232z)H_0\biggr)H_{-1}^2$$
$$+\biggl[-\frac{4}{3}(-2009+2064z+22z^2) - 16(-13+18z)\zeta_2\biggr]H_{0,1} + 32(-1+8z)H_{0,1}^2$$
$$+\biggl[\frac{16}{9}(1097+1030z+27z^2) + 96(-1+2z)H_{0,1} - 16(-1+26z)\zeta_2\biggr]H_{0,-1}$$
$$+32(-3+8z)H_{0,-1}^2 + 8(333+132z+2z^2)H_{0,0,1} - \frac{8}{3}(-449+164z)H_{0,0,-1}$$
$$-\frac{16}{3}(-46+44z+9z^2)H_{0,1,1} + \frac{16}{3}(221+232z)H_{0,-1,-1} + 16(63+26z)H_{0,0,0,1}$$
$$+64(1+9z)H_{0,0,1,1} - 128(z-1)H_{0,-1,0,1} - \frac{4}{9}(3326+864z+75z^2)\zeta_2 - \frac{4}{5}(879+490z)\zeta_2^2$$
$$+\frac{16}{3}(-946+100z+9z^2)\zeta_3\biggr] + \biggl[-\frac{4}{3}(2971+8427z) + (1+2z)(-64H_{0,0,1,-1}$$
$$-64H_{0,0,-1,1} - 96H_{0,0,-1,-1} + 64H_{0,1,1,-1} + 64H_{0,1,-1,1} + 64H_{0,1,-1,-1} + 64H_{0,-1,1,1}$$
$$+64H_{0,-1,1,-1} + 64H_{0,-1,-1,1}) + \biggl[\frac{8}{3}(1388-1387z+11z^2) + (-1+2z)\biggl(32H_{0,0,1}$$
$$+224H_{0,1,1} - \frac{160}{3}\zeta_3\biggr) - 8(-141+150z+2z^2)H_{0,1} - 16(z-1)(27+z)\zeta_2\biggr]H_1$$

$$
\begin{aligned}
&+\Big[-\frac{8}{3}(-200+203z)+(-1+2z)(-96H_{0,1}-8\zeta_2)\Big]H_1^2+\frac{8}{9}(92-106z+3z^2)H_1^3\\
&-\frac{20}{3}(-1+2z)H_1^4+\Big[-\frac{4}{3}(-2301+412z+35z^2)+8(19+22z)\zeta_2\Big]H_{0,1}\\
&+48(3+4z)H_{0,1}^2+\Big[-\frac{16}{3}(71+131z)-24(-1+10z)\zeta_2\Big]H_{0,-1}-32(-1+4z)H_{0,-1}^2\\
&-\frac{16}{3}(-330-118z+3z^2)H_{0,0,1}-\frac{8}{3}(155+118z)H_{0,0,-1}+\frac{8}{3}(-577+782z\\
&+12z^2)H_{0,1,1}+\frac{16}{3}(83+94z)H_{0,1,-1}+\frac{16}{3}(83+94z)H_{0,-1,1}-\frac{16}{3}(119+106z)H_{0,-1,-1}\\
&-64(-7+4z)H_{0,0,0,1}+32(-5+8z)H_{0,0,0,-1}-32(7+18z)H_{0,0,1,1}-32(-7+2z)H_{0,1,1,1}\\
&+128(z-1)H_{0,-1,0,1}-768zH_{0,-1,-1,-1}-\frac{2}{9}(-667+5468z)\zeta_2-\frac{32}{5}(-31+19z)\zeta_2^2\\
&-\frac{16}{9}(344+695z+18z^2)\zeta_3\Big]H_0+\Big[\frac{2}{3}(-218+1781z+22z^2)+(1+2z)(32H_{0,1,-1}\\
&+32H_{0,-1,1})+\Big[\frac{2}{3}(z-1)(1559+35z)+(-1+2z)(-32H_{0,1}-4\zeta_2)\Big]H_1\\
&+\frac{2}{3}(-209+214z+6z^2)H_1^2-\frac{8}{3}(139+154z)H_{0,1}+\frac{4}{3}(107+94z)H_{0,-1}\\
&+32(-1+4z)H_{0,0,1}-48(-1+2z)H_{0,0,-1}-144H_{0,1,1}+16(-1+10z)H_{0,-1,-1}\\
&-\frac{2}{3}(-87-226z+12z^2)\zeta_2+\frac{16}{3}(-31+20z)\zeta_3\Big]H_0^2+\Big[\frac{2}{9}(201+324z+35z^2)\\
&+\frac{8}{3}(z-1)(25+z)H_1+\frac{8}{3}(-1+2z)H_1^2-\frac{64}{3}(1+z)H_{0,1}+\frac{16}{3}(-1+4z)H_{0,-1}\\
&-\frac{16}{3}(-2+5z)\zeta_2\Big]H_0^3+\frac{1}{9}(71+94z+6z^2)H_0^4-\frac{8}{15}(z-1)H_0^5+\Big[\frac{8}{3}(-752+739z)\\
&+(-1+2z)(-80H_{0,1}^2+64H_{0,0,0,1}-96H_{0,0,1,1}+128H_{0,1}\zeta_2-184\zeta_2^2)\\
&+\frac{8}{3}(-763+806z+12z^2)H_{0,0,1}+\frac{8}{3}(-83+94z)H_{0,1,1}+\frac{4}{9}(-2189+2122z)\zeta_2\\
&-\frac{8}{9}(-919+938z+36z^2)\zeta_3\Big]H_1+\Big[-\frac{2}{3}(167-226z+22z^2)+(-1+2z)\Big(176H_{0,0,1}\\
&+16H_{0,1,1}-\frac{208}{3}\zeta_3\Big)-\frac{4}{3}(101-118z+6z^2)\zeta_2\Big]H_1^2+\Big[-\frac{2}{9}(415-440z+61z^2)\\
&+\frac{32}{3}(-1+2z)\zeta_2\Big]H_1^3+\frac{1}{9}(-203+190z+24z^2)H_1^4+\frac{4}{3}(-1+2z)H_1^5\\
&+\Big[(1+z)\Big(128H_{0,1,-1}+128H_{0,-1,1}-\frac{2576}{3}\zeta_2\Big)+(1+2z)\Big(-\frac{4}{3}H_0^4-32H_{0,0,0,1}\\
&-160H_{0,0,0,-1}+128H_{0,0,1,1}+192H_{0,0,1,-1}+192H_{0,0,-1,1}+128H_{0,1,-1,-1}+128H_{0,-1,0,1}\\
&+128H_{0,-1,1,-1}+128H_{0,-1,-1,1}+384H_{0,-1,-1,-1}-\frac{376}{5}\zeta_2^2\Big)+\Big[1712(1+z)\\
&+(1+2z)(64H_{0,0,1}+96H_{0,0,-1}-64H_{0,1,1}-64H_{0,1,-1}-64H_{0,-1,1}+128H_{0,-1,-1}-64\zeta_3)\\
&-\frac{16}{3}(83+94z)H_{0,1}+\frac{16}{3}(95+106z)H_{0,-1}+\frac{8}{3}(71+82z)\zeta_2\Big]H_0
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{1144(1+z)}{3}+(1+2z)(-32H_{0,1}-16H_{0,-1}+36\zeta_2)\right]H_0^2-\frac{4}{9}(59+70z)H_0^3+\left[192(1+z)\right.\\
&\left.+96(1+2z)\zeta_2\right]H_{0,1}+\left[\frac{4000(1+z)}{3}-16(1+2z)\zeta_2\right]H_{0,-1}+\frac{32}{3}(89+100z)H_{0,0,1}\\
&\left.-\frac{16}{3}(71+82z)H_{0,0,-1}+\frac{32}{3}(107+118z)H_{0,-1,-1}-8(111+122z)\zeta_3\right]H_{-1}\\
&+\left[(1+2z)\left(-\frac{8}{3}H_0^3-96H_{0,0,1}-64H_{0,1,-1}-64H_{0,-1,1}-192H_{0,-1,-1}+160\zeta_3\right)\right.\\
&+\left[-\frac{2000}{3}(1+z)+(1+2z)(32H_{0,1}-64H_{0,-1}+40\zeta_2)\right]H_0-\frac{4}{3}(119+130z)H_0^2\\
&\left.-64(1+z)H_{0,1}-\frac{16}{3}(107+118z)H_{0,-1}+\frac{8}{3}(131+142z)\zeta_2\right]H_{-1}^2+\left[(1+2z)\left(\frac{64}{3}H_0^2\right.\right.\\
&\left.\left.+\frac{64}{3}H_{0,1}+64H_{0,-1}-\frac{160}{3}\zeta_2\right)+\frac{16}{9}(107+118z)H_0\right]H_{-1}^3+\left[-\frac{8}{3}(1442-779z+22z^2)\right.\\
&\left.-96(5+8z)H_{0,0,1}+160(-1+2z)H_{0,1,1}-32(17+2z)\zeta_2+160(1+2z)\zeta_3\right]H_{0,1}\\
&-\frac{8}{3}(-170+178z+3z^2)H_{0,1}^2+\left[-1712(1+z)+(-1+4z)(64H_{0,0,-1}+128H_{0,-1,-1})\right.\\
&\left.-32(-7+10z)H_{0,0,1}-\frac{8}{3}(119+106z)\zeta_2+64(-2+5z)\zeta_3\right]H_{0,-1}+64H_{0,-1}^2\\
&+\left[\frac{4}{3}(-3019-472z+35z^2)-8(23+14z)\zeta_2\right]H_{0,0,1}+\left[\frac{16}{3}(-1+119z)\right.\\
&\left.+24(-5+14z)\zeta_2\right]H_{0,0,-1}+\left[-\frac{4}{3}(824-1048z+61z^2)-16(-11+34z)\zeta_2\right]H_{0,1,1}\\
&+\left[-192(1+z)-96(1+2z)\zeta_2\right]H_{0,1,-1}+\left[-192(1+z)-96(1+2z)\zeta_2\right]H_{0,-1,1}\\
&+\left[-\frac{4000}{3}(1+z)-16(-5+14z)\zeta_2\right]H_{0,-1,-1}+16(-163+12z+3z^2)H_{0,0,0,1}\\
&+\frac{8}{3}(203+142z)H_{0,0,0,-1}-\frac{8}{3}(-517+926z+12z^2)H_{0,0,1,1}-\frac{32}{3}(89+100z)H_{0,0,1,-1}\\
&-\frac{32}{3}(89+100z)H_{0,0,-1,1}+\frac{16}{3}(71+82z)H_{0,0,-1,-1}+\frac{8}{3}(5+2z+18z^2)H_{0,1,1,1}\\
&-\frac{16}{3}(95+106z)H_{0,-1,0,1}-\frac{32}{3}(107+118z)H_{0,-1,-1,-1}+32(-27+2z)H_{0,0,0,0,1}\\
&+32(83+128z)H_{0,0,0,1,1}+128(-5+8z)H_{0,0,0,1,-1}+128(-5+8z)H_{0,0,0,-1,1}\\
&-32(-11+14z)H_{0,0,0,-1,-1}+32(29+44z)H_{0,0,1,0,1}+32(-7+10z)H_{0,0,1,0,-1}\\
&-256(-2+7z)H_{0,0,1,1,1}-32(-7+20z)H_{0,1,0,1,1}-32(1+4z)H_{0,1,1,1,1}\\
&+\left[\frac{2}{3}(344+4849z+44z^2)+\frac{8}{3}(139+166z)\zeta_3\right]\zeta_2-\frac{8}{15}(-3215-348z+63z^2)\zeta_2^2\\
&\left.+\frac{4}{3}(1156+2164z+61z^2)\zeta_3-544(-2+z)\zeta_5\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+C_AC_FT_F\biggl\{\frac{1}{3}(-1555+3178z)+(1+z)(128H_{0,1,-1,-1}+128H_{0,-1,1,-1}+128H_{0,-1,-1,1})\\
&+(1+2z)(16H_{-1}^4H_0-32H_{0,0,0,0,-1}+64H_{0,0,0,1,-1}+64H_{0,0,0,-1,1}\\
&-64H_{0,0,0,-1,-1}+32H_{0,0,1,0,-1}+128H_{0,0,1,1,-1}+128H_{0,0,1,-1,1}+192H_{0,0,1,-1,-1}\\
&+32H_{0,0,-1,0,1}+32H_{0,0,-1,0,-1}+128H_{0,0,-1,1,1}+192H_{0,0,-1,1,-1}\\
&+192H_{0,0,-1,-1,1}+256H_{0,0,-1,-1,-1}+128H_{0,1,-1,-1,-1}+64H_{0,-1,0,1,1}\\
&+128H_{0,-1,0,1,-1}+128H_{0,-1,0,-1,1}+128H_{0,-1,0,-1,-1}+128H_{0,-1,1,-1,-1}+128H_{0,-1,-1,0,1}\\
&+128H_{0,-1,-1,1,-1}+128H_{0,-1,-1,-1,1}+384H_{0,-1,-1,-1,-1})+L_M^3\biggl[\frac{2}{3}(-155+144z)\\
&+\biggl[\frac{4}{9}(-131+214z)+\frac{64}{3}(-1+2z)H_1\biggr]H_0-\frac{32}{3}(1+z)H_0^2+\frac{8}{9}(-173+166z)H_1\\
&+\frac{64}{3}(-1+2z)H_1^2-32(1+2z)H_{0,1}+\frac{32}{3}(5+2z)\zeta_2\biggr]+L_M^2\biggl[\frac{1}{3}(435-272z)\\
&+(-1+2z)\biggl(32H_1^3-\frac{128}{3}H_{0,1}\biggr)+(1+2z)\biggl[(-24H_0+8H_0^2+32H_{0,1}-32\zeta_2)H_{-1}+24H_{0,-1}\\
&+16H_{0,0,-1}-48H_{0,1,1}-32H_{0,1,-1}-32H_{0,-1,1}\biggr]+\biggl[\frac{2}{9}(155+3026z)+320(z-1)H_1\\
&+48(-1+2z)H_1^2-16(5+2z)H_{0,1}-16(1+2z)H_{0,-1}-16(-3+2z)\zeta_2\biggr]H_0+\biggl[-8(1+6z)\\
&+8(-1+2z)H_1\biggr]H_0^2+\biggl[\frac{8}{9}(-611+571z)-96(-1+2z)\zeta_2\biggr]H_1+\frac{8}{3}(-94+95z)H_1^2\\
&+64(2+z)H_{0,0,1}-\frac{8}{3}(-95+88z)\zeta_2-16(-1+8z)\zeta_3\biggr]+L_M\biggl[\frac{1}{18}(-57083+54650z)\\
&+(1+z)\biggl[-32(7+3z)H_{0,1,-1}-32(7+3z)H_{0,-1,1}\biggr]+(1+2z)\biggl(\frac{160}{3}H_{-1}^3H_0-96H_{0,0,1,-1}\\
&-96H_{0,0,-1,1}-32H_{0,0,-1,-1}-64H_{0,1,1,-1}-64H_{0,1,-1,1}-256H_{0,1,-1,-1}-64H_{0,-1,1,1}\\
&-256H_{0,-1,1,-1}-256H_{0,-1,-1,1}-320H_{0,-1,-1,-1}\biggr)+\biggl[\frac{2}{27}(-15607+7346z)\\
&+\biggl[-\frac{4}{9}(-1979+2068z+66z^2)+(-1+2z)(32H_{0,1}-96H_{0,-1}-256\zeta_2)\biggr]H_1\\
&-8(-43+39z)H_1^2-16(-1+2z)H_1^3+\frac{8}{3}(-65+34z+18z^2)H_{0,1}+8(-41+44z)H_{0,-1}\\
&+16(-7+30z)H_{0,0,1}-16(-1+22z)H_{0,0,-1}+48(5+6z)H_{0,1,1}+32(-5+2z)H_{0,1,-1}\\
&+32(-5+2z)H_{0,-1,1}-128(1+z)H_{0,-1,-1}+24(17+28z+8z^2)\zeta_2+16(45+32z)\zeta_3\biggr]H_0\\
&+\biggl[-\frac{2}{9}(68+3197z+390z^2)-4(-59+56z+6z^2)H_1-24(-1+2z)H_1^2\\
&-16(-5+4z)H_{0,1}-16(3+4z)H_{0,-1}-8(5+18z)\zeta_2\biggr]H_0^2+\biggl[-\frac{2}{3}(-39+46z+36z^2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{8}{3}(-1+2z)H_1\Bigg]H_0^3+\frac{2}{3}(5+8z)H_0^4+\Bigg[\frac{8}{27}(-8663+8914z)+(-1+2z)(160H_{0,0,1}\\
&+192H_{0,0,-1}-96H_{0,1,1}-672\zeta_3)+48(-11+10z)H_{0,1}+8(13-18z+6z^2)\zeta_2\Bigg]H_1\\
&+\Bigg[\frac{4}{9}(-1435+1322z+33z^2)+(-1+2z)(96H_{0,1}-128\zeta_2)\Bigg]H_1^2+\frac{8}{9}(-182+166z\\
&+27z^2)H_1^3+16(-1+2z)H_1^4+\Bigg[(1+2z)\Bigg(\frac{104}{3}H_0^3+96H_{0,0,1}+32H_{0,0,-1}+64H_{0,1,1}\\
&+256H_{0,1,-1}+256H_{0,-1,1}+320H_{0,-1,-1}-384\zeta_3\Bigg)+\Bigg[16(1+z)(1+9z)\\
&+(1+2z)(64H_{0,1}+128H_{0,-1}-352\zeta_2)\Bigg]H_0+4(41+44z+12z^2)H_0^2\\
&+32(1+z)(7+3z)H_{0,1}+48(11+12z+2z^2)H_{0,-1}-8(61+76z+18z^2)\zeta_2\Bigg]H_{-1}\\
&+\Bigg[(1+2z)(-72H_0^2-128H_{0,1}-160H_{0,-1}+208\zeta_2)-24(11+12z+2z^2)H_0\Bigg]H_{-1}^2\\
&+\Bigg[\frac{4}{9}(-4939+1742z+132z^2)+64(-4+3z)\zeta_2\Bigg]H_{0,1}-192zH_{0,1}^2+\Bigg[-16(1+z)(1+9z)\\
&-96(-1+2z)H_{0,1}+16(13+14z)\zeta_2\Bigg]H_{0,-1}-64zH_{0,-1}^2-\frac{16}{3}(100+76z+27z^2)H_{0,0,1}\\
&-8(-41+132z+12z^2)H_{0,0,-1}+\frac{8}{3}(-193+62z+54z^2)H_{0,1,1}\\
&-48(11+12z+2z^2)H_{0,-1,-1}-16(1+50z)H_{0,0,0,1}+32(1+32z)H_{0,0,0,-1}\\
&-32(3+10z)H_{0,0,1,1}+64(-3+4z)H_{0,-1,0,1}+\frac{8}{9}(1498+163z+129z^2)\zeta_2+\frac{24}{5}(25+38z)\zeta_2^2\\
&+16(221-94z+6z^2)\zeta_3\Bigg]+\Bigg[-\frac{2}{3}(-627+3853z)+(-1+2z)\Bigg(\frac{16}{3}H_1^4+288H_{0,1,1,1}\Bigg)\\
&+(1+2z)(16H_{0,-1}^2+16H_{0,0,0,-1}+64H_{0,0,1,-1}+64H_{0,0,-1,1}+96H_{0,0,-1,-1}-64H_{0,1,1,-1}\\
&-64H_{0,1,-1,1}-64H_{0,1,-1,-1}+32H_{0,-1,0,1}-64H_{0,-1,1,1}-64H_{0,-1,1,-1}-64H_{0,-1,-1,1}\\
&+192H_{0,-1,-1,-1})+\Bigg[-\frac{4}{3}(-872+851z+80z^2)+(-1+2z)\Bigg(-128H_{0,0,1}-416H_{0,1,1}\\
&+\frac{608}{3}\zeta_3\Bigg)+\frac{8}{3}(-433+404z+18z^2)H_{0,1}+\frac{4}{3}(-181+122z+36z^2)\zeta_2\Bigg]H_1\\
&+\Bigg[8(-16+11z)+(-1+2z)(160H_{0,1}-64\zeta_2)\Bigg]H_1^2-8z^2H_1^3+\Bigg[\frac{8}{3}(-43+406z+47z^2)\\
&-24(-3+14z)\zeta_2\Bigg]H_{0,1}-8(19+10z)H_{0,1}^2+\Bigg[-80(z-1)+48(1+2z)\zeta_2\Bigg]H_{0,-1}\\
&+\frac{16}{3}(-22-118z+9z^2)H_{0,0,1}+8(3+8z)H_{0,0,-1}-\frac{8}{3}(-791+712z+36z^2)H_{0,1,1}\\
&-48H_{0,1,-1}-48H_{0,-1,1}+16(3+4z)H_{0,-1,-1}+64(5+4z)H_{0,0,0,1}+32(7+18z)H_{0,0,1,1}\\
&-\frac{2}{9}(2590+5197z)\zeta_2-\frac{8}{5}(131+28z)\zeta_2^2+\frac{4}{9}(-631-58z+216z^2)\zeta_3\Bigg]H_0
\end{aligned}
$$

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\begin{aligned}
&+\Bigg[\frac{1}{3}(556+1519z-160z^2)+(1+2z)(-32H_{0,1,-1}-32H_{0,-1,1}-32H_{0,-1,-1})\\
&+\Bigg[-\frac{2}{3}(-433+340z+94z^2)+(-1+2z)(80H_{0,1}+4\zeta_2)\Bigg]H_1-2(-67+56z+6z^2)H_1^2\\
&+\frac{4}{3}(-109+266z)H_{0,1}+4H_{0,-1}-64(2+3z)H_{0,0,1}-8(-23+22z)H_{0,1,1}\\
&+\frac{1}{3}(83+218z+72z^2)\zeta_2+\frac{8}{3}(-5+22z)\zeta_3\Bigg]H_0^2+\Bigg[\frac{1}{9}(241-2020z-188z^2)\\
&-\frac{4}{9}(-299+286z+18z^2)H_1-\frac{16}{3}(-1+2z)H_1^2+\frac{32}{3}(2+5z)H_{0,1}-\frac{8}{3}(1+2z)H_{0,-1}\\
&+4(1+4z)\zeta_2\Bigg]H_0^3+\Bigg[\frac{1}{18}(83-70z-36z^2)-\frac{4}{3}(-1+2z)H_1\Bigg]H_0^4+\frac{2}{15}H_0^5\\
&+\Bigg[-\frac{32}{3}(-215+208z)+(-1+2z)\Bigg(144H_{0,1}^2-64H_{0,0,0,1}+96H_{0,0,1,1}+\frac{1976}{5}\zeta_2^2\Bigg)\\
&+\Bigg[128(z-1)-192(-1+2z)\zeta_2\Bigg]H_{0,1}-\frac{8}{3}(-857+808z+36z^2)H_{0,0,1}\\
&-\frac{8}{3}(-341+376z)H_{0,1,1}-\frac{4}{9}(-1649+1348z)\zeta_2+\frac{8}{9}(-1759+1718z+108z^2)\zeta_3\Bigg]H_1\\
&+\Bigg[\frac{4}{3}(709-844z+40z^2)+(-1+2z)\Bigg(-272H_{0,0,1}-80H_{0,1,1}+\frac{320}{3}\zeta_3\Bigg)\\
&+\frac{8}{3}(-83+94z)H_{0,1}+8(48-55z+3z^2)\zeta_2\Bigg]H_1^2+\Bigg[\frac{4}{9}(477-502z+97z^2)\\
&+(-1+2z)\Bigg(\frac{64H_{0,1}}{3}-\frac{128\zeta_2}{3}\Bigg)\Bigg]H_1^3-\frac{2}{9}(-178+143z+36z^2)H_1^4-\frac{8}{3}(-1+2z)H_1^5\\
&+\Bigg[(1+z)(-128H_{0,1,-1}-128H_{0,-1,1})+(1+2z)\Bigg(\frac{4}{3}H_0^4+32H_{0,0,0,1}+160H_{0,0,0,-1}\\
&-128H_{0,0,1,1}-192H_{0,0,1,-1}-192H_{0,0,-1,1}-128H_{0,1,-1,-1}-128H_{0,-1,0,1}-128H_{0,-1,1,-1}\\
&-128H_{0,-1,-1,1}-384H_{0,-1,-1,-1}+\frac{296}{5}\zeta_2^2\Bigg)+\Bigg[-16(11+19z)+(1+2z)(-64H_{0,0,1}\\
&-96H_{0,0,-1}+64H_{0,1,1}+64H_{0,1,-1}+64H_{0,-1,1}-128H_{0,-1,-1}+64\zeta_3)+48H_{0,1}\\
&-16(7+4z)H_{0,-1}+8(16+19z)\zeta_2\Bigg]H_0+\Bigg[-8(11+5z)+(1+2z)(32H_{0,1}+16H_{0,-1}\\
&-28\zeta_2)\Bigg]H_0^2-\frac{4}{3}(5+8z)H_0^3+\Bigg[-192(1+z)-96(1+2z)\zeta_2\Bigg]H_{0,1}+\Bigg[32(-5+z)\\
&+48(1+2z)\zeta_2\Bigg]H_{0,-1}-32(5+2z)H_{0,0,1}-16(1+4z)H_{0,0,-1}\\
&-32(11+8z)H_{0,-1,-1}+16(17+11z)\zeta_2+8(37+28z)\zeta_3\Bigg]H_{-1}+\Bigg[(1+2z)\Bigg(\frac{8}{3}H_0^3+96H_{0,0,1}\\
&+64H_{0,1,-1}+64H_{0,-1,1}+192H_{0,-1,-1}-160\zeta_3\Bigg)+\Bigg[-16(-5+z)+(1+2z)(-32H_{0,1}\\
&+64H_{0,-1}-56\zeta_2)\Bigg]H_0+12(5+4z)H_0^2+64(1+z)H_{0,1}+16(11+8z)H_{0,-1}\\
&-8(19+16z)\zeta_2\Bigg]H_{-1}^2+\Bigg[(1+2z)\Bigg(-\frac{64}{3}H_0^2-\frac{64}{3}H_{0,1}-64H_{0,-1}+\frac{160}{3}\zeta_2\Bigg)-\frac{16}{3}(11+8z)H_0\Bigg]H_{-1}^3
\end{aligned}
$$

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\begin{aligned}
&+\left[\frac{4}{3}(-272+421z+160z^2)+16(25+46z)H_{0,0,1}\right.\\
&\left.-32(-5+22z)H_{0,1,1}-\frac{4}{3}(-253+482z)\zeta_2-32(3+26z)\zeta_3\right]H_{0,1}\\
&+\frac{8}{3}(-247+296z+9z^2)H_{0,1}^2+(16(11+19z)+(1+2z)(-32H_{0,0,1}-32H_{0,0,-1}\\
&-64H_{0,-1,-1}-16\zeta_3)-8(12+11z)\zeta_2)H_{0,-1}+32H_{0,-1}^2+\left[-\frac{4}{3}(93+1146z+94z^2)\right.\\
&\left.+8(1+70z)\zeta_2\right]H_{0,0,1}+\left[16(1+15z)-40(1+2z)\zeta_2\right]H_{0,0,-1}+\left[\frac{8}{3}(774-577z+97z^2)\right.\\
&\left.+16(-13+62z)\zeta_2\right]H_{0,1,1}+\left[192(1+z)+96(1+2z)\zeta_2\right]H_{0,1,-1}+\left[192(1+z)\right.\\
&\left.+96(1+2z)\zeta_2\right]H_{0,-1,1}+\left[-32(-5+z)-16(1+2z)\zeta_2\right]H_{0,-1,-1}\\
&-\frac{8}{3}(-157-172z+54z^2)H_{0,0,0,1}-8(7+16z)H_{0,0,0,-1}+8(-187+142z+12z^2)H_{0,0,1,1}\\
&+32(5+2z)H_{0,0,1,-1}+32(5+2z)H_{0,0,-1,1}+16(1+4z)H_{0,0,-1,-1}\\
&-\frac{8}{3}(79-332z+54z^2)H_{0,1,1,1}+16(7+4z)H_{0,-1,0,1}+32(11+8z)H_{0,-1,-1,-1}\\
&-16(19+2z)H_{0,0,0,0,1}-48(49+86z)H_{0,0,0,1,1}-16(55+106z)H_{0,0,1,0,1}\\
&+64(-8+55z)H_{0,0,1,1,1}+16(-13+98z)H_{0,1,0,1,1}+16(-7+26z)H_{0,1,1,1,1}\\
&+\left[192\ln(2)(-1+2z)+\frac{1}{6}(-11243+8530z-640z^2)-\frac{4}{3}(151+178z)\zeta_3\right]\zeta_2\\
&\left.\left.+\frac{8}{15}(-1831+1325z+189z^2)\zeta_2^2-\frac{2}{3}(1289-2624z+388z^2)\zeta_3+8(7+22z)\zeta_5\right\}+a_{Qg}^{(3)}\right\}.
\end{aligned}
\tag{A10}
$$

Finally, the single mass contributions to the OME $A_{gg,Q}^{(3)}$ are given by

$$
\begin{aligned}
A_{gg,Q}^{(\delta)}=&\left\{a_s\frac{4L_M}{3}T_F+a_s^2\left[C_FT_F(-15+4L_M)+C_AT_F\frac{2}{9}(5+24L_M)+\frac{16L_M^2}{9}T_F^2\right]\right.\\
&+a_s^3\left\{\frac{64L_M^3}{27}T_F^3+C_AN_FT_F^2\left[\frac{224}{27}-\frac{44L_M}{3}-\frac{4\zeta_2}{3}\right]+C_FN_FT_F^2\left[\frac{118}{3}\right.\right.\\
&\left.-\frac{268L_M}{9}+28\zeta_2\right]+C_AT_F^2\left[-\frac{8}{27}-2L_M+\frac{56L_M^2}{3}-\frac{44}{3}\zeta_2\right]+C_FT_F^2\left[\frac{782}{9}\right.\\
&\left.-\frac{584L_M}{9}+\frac{40L_M^2}{3}-\frac{40}{3}\zeta_2\right]+C_A^2T_F\left[-\frac{616}{27}+L_M\left(\frac{277}{9}+\frac{16}{3}\zeta_2^2+\frac{160}{9}\zeta_3\right)\right.\\
&\left.+L_M^2\left(-\frac{2}{3}+\frac{16\zeta_3}{3}\right)+\left(4-\frac{8\zeta_3}{3}\right)\zeta_2\right]+C_F^2T_F[-39-2L_M+16[-5+8\ln(2)]\zeta_2\\
&-32\zeta_3]+C_AC_FT_F\left[-\frac{1045}{6}+\frac{736L_M}{9}-\frac{22L_M^2}{3}-\frac{4}{3}[-5+48\ln(2)]\zeta_2+16\zeta_3\right]\\
&\left.\left.-\frac{64}{27}T_F^3\zeta_3+a_{gg,Q}^{(3),\delta}\right\}\right\}\delta(1-z)
\end{aligned}
\tag{A11}
$$

$$
\begin{aligned}
A_{gg,Q}^{(+)} = & -a_s^2 \frac{8(28+30L_M+9L_M^2)}{27(z-1)} C_A T_F + a_s^3 \Biggl\{ C_A N_F T_F^2 \Biggl[\frac{1}{z-1} \Biggl[-\frac{2176 L_M}{81} - \frac{64 L_M^3}{27} \\
& - \frac{4}{243}(-2624 - 441\zeta_2 + 81 z\zeta_2 - 144\zeta_3) + \frac{32}{9} H_0 \Biggr] + \frac{4}{3}\zeta_2 \Biggr] + C_A T_F^2 \Biggl[\frac{1}{z-1} \Biggl[-\frac{320 L_M}{9} \\
& - \frac{640 L_M^2}{27} - \frac{224 L_M^3}{27} - \frac{4}{81}(-1312 - 717\zeta_2 + 297 z\zeta_2 - 168\zeta_3) + \frac{16}{3} H_0 \Biggr] + \frac{44}{3}\zeta_2 \Biggr] \\
& + C_A^2 T_F \Biggl\{ \frac{1}{z-1} \Biggl[\frac{176 L_M^3}{27} - \frac{4}{243}(5668 + 207\zeta_2 - 243 z\zeta_2 + 324\zeta_2^2 + 396\zeta_3 - 162\zeta_2\zeta_3 \\
& + 162 z\zeta_2\zeta_3) + \left(-\frac{88}{9} + \frac{32 H_1 \zeta_2}{3} \right) H_0 + \frac{8}{3} H_0^2 \zeta_2 \Biggr] + L_M \Biggl[\frac{1}{z-1} \Biggl[-\frac{16}{9} H_0^2 (10 + 3H_1) \\
& + \frac{8}{81}(-155 + 360\zeta_2 - 54\zeta_2^2 + 54 z\zeta_2^2 - 1044\zeta_3 + 180 z\zeta_3) + \Biggl(-\frac{16}{3} - \frac{640}{9} H_1 \\
& + \frac{32}{3} H_{0,1} - \frac{64}{3} H_{0,-1} \Biggr) H_0 - \frac{64}{3} H_{0,0,1} + \frac{128}{3} H_{0,0,-1} \Biggr] - \frac{16}{3}\zeta_2^2 - \frac{160}{9}\zeta_3 \Biggr] \\
& + L_M^2 \Biggl[\frac{1}{z-1} \Biggl[\frac{8}{9}(-23 + 12\zeta_2 - 6\zeta_3 + 6 z\zeta_3) - \frac{16}{3} H_0^2 - \frac{64}{3} H_0 H_1 \Biggr] - \frac{16}{3}\zeta_3 \Biggr] \\
& + \left(-4 + \frac{8\zeta_3}{3} \right) \zeta_2 \Biggr\} + C_A C_F T_F \Biggl\{ \frac{1}{z-1} \Biggl[-\frac{2}{9}(233 + 210\zeta_2 - 288 \ln(2)\zeta_2 \\
& - 30 z\zeta_2 + 288 \ln(2) z\zeta_2 + 72\zeta_3 - 72 z\zeta_3) + \frac{8}{3} L_M (-5 + 24\zeta_3) - 8 L_M^2 \Biggr] \\
& + \frac{4}{3}[-5 + 48\ln(2)]\zeta_2 - 16\zeta_3 \Biggr\} + C_F^2 T_F \Biggl\{ 16[-5\zeta_2 + 8\ln(2)\zeta_2 - 2\zeta_3] \\
& - 16[-5 + 8\ln(2)]\zeta_2 + 32\zeta_3 \Biggr\} + a_{gg,Q}^{(3),(+)} \Biggr\}
\end{aligned}
\tag{A12}
$$

$$
\begin{aligned}
A_{gg,Q}^{\rm reg} = & a_s^2 \Biggl\{ C_F T_F \Biggl\{ -56(z-1) + L_M^2[-20(z-1) + 8(1+z)H_0] + L_M[-40(z-1) \\
& - 8(-5+z)H_0 + 8(1+z)H_0^2] + 12(3+z)H_0 - 2(-5+z)H_0^2 + \frac{4}{3}(1+z)H_0^3 \Biggr\} \\
& + C_A T_F \Biggl\{ -\frac{8}{3} L_M^2 (-1+2z) - \frac{2}{27}(-337 + 449z) + L_M \Biggl[-\frac{16}{9}(-14 + 19z) + \frac{16}{3}(1+z)H_0 \Biggr] \\
& + \frac{4}{9}(22+z)H_0 + \frac{4}{3}(1+z)H_0^2 + \frac{4}{3} z H_1 \Biggr\} \Biggr\} + a_s^3 \Biggl\{ C_F T_F^2 \Biggl\{ (z-1)\left(\frac{776\zeta_2}{9} + \frac{80 H_1 \zeta_2}{3} \right) \\
& + L_M^3 \Biggl[-\frac{400}{9}(z-1) + \frac{160}{9}(1+z)H_0 \Biggr] + L_M^2 \Biggl[-\frac{32}{9}(z-1)(26 + 15H_1) + (1+z)\Biggl(\frac{64}{3} H_0^2 \\
& + \frac{64}{3} H_{0,1} - \frac{64}{3}\zeta_2 \Biggr) - \frac{64}{9}(-13 + 8z)H_0 \Biggr] + L_M \Biggl[-\frac{16}{9}(z-1)H_1(52 + 15H_1) + (1+z)\Biggl(\frac{80}{9} H_0^3 \\
& - \frac{128}{3} H_{0,0,1} + \frac{64}{3} H_{0,1,1} \Biggr) + \frac{32}{27}(335 - 335z + 18\zeta_3 + 18 z\zeta_3) + \Biggl[\frac{16}{27}(373 + 55z) \\
& + (1+z)\left(\frac{128 H_{0,1}}{3} - \frac{128\zeta_2}{3} \right) - \frac{320}{3}(z-1)H_1 \Biggr] H_0 - \frac{8}{9}(-59 + 55z)H_0^2 + \frac{64}{9}(-2 + 7z)H_{0,1} \\
& + \frac{64}{9}(-13 + 8z)\zeta_2 \Biggr] + (1+z)\left(-\frac{4}{3} H_0^4 - \frac{32}{3} H_{0,1}\zeta_2 + \frac{32}{3}\zeta_2^2 \right) + \frac{16}{9}(-216 + 216z - 25\zeta_3
\end{aligned}
$$

$$
\begin{aligned}
&+25z\zeta_3)+\left[-\frac{32}{9}(81+9z+5\zeta_3+5z\zeta_3)+\frac{8}{9}(-97+41z)\zeta_2\right]H_0+\left[-24(3+z)\right.\\
&\left.-\frac{56}{3}(1+z)\zeta_2\right]H_0^2+\frac{8}{3}(-5+z)H_0^3\Bigg\}+C_FN_FT_F^2\Bigg\{L_M^3\left[-\frac{320}{9}(z-1)+\frac{128}{9}(1+z)H_0\right]\\
&+(z-1)\left[\left(-\frac{3712}{27}-\frac{80\zeta_2}{3}\right)H_1+\frac{64}{9}H_1^2-\frac{80}{9}H_1^3\right]+L_M\left[-\frac{16}{3}(z-1)H_1(32+15H_1)\right.\\
&+(1+z)\left(\frac{32}{3}H_0^3-\frac{256}{3}H_{0,0,1}+64H_{0,1,1}\right)+\frac{64}{9}(89-89z+3\zeta_3+3z\zeta_3)+\left[\frac{32}{9}(85+19z)\right.\\
&\left.+(1+z)\left(\frac{256H_{0,1}}{3}-\frac{256\zeta_2}{3}\right)-\frac{640}{3}(z-1)H_1\right]H_0-\frac{16}{9}(-29+49z)H_0^2+\frac{64}{3}(-2+3z)H_{0,1}\\
&\left.+\frac{64}{3}(-8+7z)\zeta_2\right]+(1+z)\left(-\frac{8}{9}H_0^4+\frac{64}{3}H_{0,1,1,1}-\frac{96}{5}\zeta_2^2\right)+\frac{64}{81}(-494+494z-27\zeta_3\\
&+90z\zeta_3)+\left[-\frac{64}{81}(341+191z+18\zeta_3+18z\zeta_3)-\frac{16}{9}(19+7z)\zeta_2\right]H_0+\left[-16(3+z)\right.\\
&\left.-\frac{16}{3}(1+z)\zeta_2\right]H_0^2+\frac{16}{9}(-5+z)H_0^3+\left[\frac{128}{27}(11+14z)+\frac{32}{3}(1+z)\zeta_2\right]H_{0,1}\\
&-\frac{64}{9}(2+5z)H_{0,1,1}-\frac{80}{27}(29+11z)\zeta_2\Bigg\}+C_AN_FT_F^2\Bigg\{-\frac{64}{27}L_M^3(-1+2z)\\
&+L_M\left[-\frac{32}{81}(-329+397z)+(1+z)\left(\frac{32}{3}H_0^2+\frac{256}{9}H_{0,1}-\frac{256}{9}\zeta_2\right)-\frac{80}{27}(-20+13z)H_0\right.\\
&\left.-\frac{16}{9}(-41+39z)H_1\right]+\frac{32}{243}(-1147+1475z-18\zeta_3+36z\zeta_3)+\left[-\frac{16}{81}(295+31z)\right.\\
&\left.-\frac{32}{9}(1+z)\zeta_2\right]H_0-\frac{16}{27}(22+z)H_0^2-\frac{32}{27}(1+z)H_0^3-\frac{256}{27}zH_1+\frac{16}{9}zH_1^2\\
&+\frac{32}{27}(-14+19z)\zeta_2\Bigg\}+C_AT_F^2\Bigg\{-\frac{224}{27}L_M^3(-1+2z)+L_M^2\left[-\frac{128}{27}(-14+19z)\right.\\
&\left.+\frac{128}{9}(1+z)H_0\right]+L_M\left[-\frac{32}{9}(-37+47z)+(1+z)\left(\frac{80}{9}H_0^2+\frac{128}{9}H_{0,1}-\frac{128}{9}\zeta_2\right)\right.\\
&\left.-\frac{8}{27}(-188+61z)H_0-\frac{8}{9}(-41+35z)H_1\right]+\frac{16}{81}(-1147+1475z-42\zeta_3+84z\zeta_3)\\
&+\left[-\frac{8}{27}(295+31z)-\frac{112}{9}(1+z)\zeta_2\right]H_0-\frac{8}{9}(22+z)H_0^2-\frac{16}{9}(1+z)H_0^3-\frac{128}{9}zH_1+\frac{8}{3}zH_1^2\\
&+\frac{112}{27}(-14+19z)\zeta_2\Bigg\}+C_A^2T_F\Bigg\{\frac{176}{27}L_M^3(-1+2z)+L_M\left[(1+z)\left[\frac{16(2-35z+2z^2)H_{-1}^2H_0}{3z}\right.\right.\\
&\left.+\left(-\frac{1832}{9}+96\zeta_2\right)H_{0,1}+\frac{32(2-35z+2z^2)H_{0,-1,-1}}{3z}-192H_{0,0,0,1}+192H_{0,0,0,-1}\right]\\
&-\frac{4}{81}(39029-38719z-10512\zeta_3+6696z\zeta_3+1080z^2\zeta_3)+\left[-\frac{4}{27}(2870+7703z-1296\zeta_3)\right.\\
&-\frac{64}{9}(-19+29z)H_1-\frac{8}{3}(-37+17z)H_{0,1}+\frac{64(1+2z^2)H_{0,-1}}{3z}+96(1+z)H_{0,0,1}\\
&\left.+32(-5+z)H_{0,0,-1}+192(z-1)H_{0,-1,-1}-\frac{8(-59-2z+69z^2+16z^3)}{3(1+z)}\zeta_2\right]H_0
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[\frac{4(84+169z+81z^2+36z^3)}{9(1+z)}+\frac{4}{3}(-29+25z)H_1-24(1+z)H_{0,1}-48(z-1)H_{0,-1} \\
&+8(5+3z)\zeta_2\Bigg]H_0^2+\frac{8}{9}(1-12z+4z^2)H_0^3-\frac{4}{3}H_0^4+\Bigg[\frac{4}{9}(-1087+1065z) \\
&-\frac{16(z-1)(2+35z+2z^2)}{3z}\zeta_2\Bigg]H_1+\Bigg[(1+z)\Bigg[-\frac{8(4-31z+4z^2)H_0^2}{3z} \\
&-\frac{32(2-35z+2z^2)H_{0,-1}}{3z}+\frac{16(2-9z+2z^2)\zeta_2}{z}\Bigg]-\frac{16}{9z(1+z)}(18-233z-462z^2 \\
&-233z^3+18z^4)H_0-\frac{64(1+z)^3H_{0,1}}{3z}\Bigg]H_{-1}+\Bigg[\frac{16}{9z(1+z)}(18-233z-462z^2 \\
&-233z^3+18z^4)+96(z-1)\zeta_2\Bigg]H_{0,-1}-96(z-1)H_{0,-1}^2+\frac{16}{3}(-23+31z+4z^2)H_{0,0,1} \\
&+\frac{16(1+4z)(-4-11z+z^2)}{3z}H_{0,0,-1}+\frac{64(1+z)^3H_{0,1,-1}}{3z}+\frac{64(1+z)^3H_{0,-1,1}}{3z} \\
&-\frac{8(-539-960z-425z^2+36z^3)}{9(1+z)}\zeta_2-\frac{24}{5}(13+23z)\zeta_2^2-\frac{160}{9}\zeta_3\Bigg] \\
&+L_M^2\Bigg[\frac{8}{9}(-100+77z+6\zeta_3)+\Bigg[-\frac{32}{3}(-3+13z)-\frac{64}{3}(-1+2z)H_1\Bigg]H_0 \\
&+\frac{64(2+3z+2z^2)}{3(1+z)}H_{-1}H_0+\frac{16(3+4z)H_0^2}{3(1+z)}-\frac{64(2+3z+2z^2)}{3(1+z)}H_{0,-1}+\frac{32(1+2z)^2\zeta_2}{3(1+z)} \\
&-\frac{16}{3}\zeta_3\Bigg]-\frac{8}{243}(-11843+14677z-198\zeta_3+396z\zeta_3)+\Bigg[\frac{44}{81}(295+31z)+\frac{8}{9}(-7+89z)\zeta_2 \\
&+\frac{32}{3}(-1+2z)H_1\zeta_2\Bigg]H_0+\Bigg[\frac{44(22+z)}{27}-\frac{8(3+4z)\zeta_2}{3(1+z)}\Bigg]H_0^2+\frac{88}{27}(1+z)H_0^3+\frac{704}{27}zH_1 \\
&-\frac{44}{9}zH_1^2+\Bigg[-\frac{4}{27}(-563+559z+18\zeta_3)+\frac{8}{3}\zeta_3\Bigg]\zeta_2-\frac{32(2+3z+2z^2)}{3(1+z)}H_{-1}H_0\zeta_2 \\
&+\frac{32(2+3z+2z^2)}{3(1+z)}H_{0,-1}\zeta_2-\frac{16(1+2z)^2\zeta_2^2}{3(1+z)}\Bigg\}+C_AC_FT_F\Bigg\{L_M^3\Bigg[\frac{8}{9}(z-1)(223+30H_1) \\
&+(1+z)\Bigg(-\frac{32H_{0,1}}{3}+\frac{32\zeta_2}{3}\Bigg)-\frac{16}{9}(53+38z)H_0+\frac{16}{3}(-2+z)H_0^2\Bigg]+(1+z)\Bigg[-\frac{160}{3}H_{-1}^3H_0 \\
&+(224H_0+40H_0^2+160H_{0,-1}-80\zeta_2)H_{-1}^2+320H_{0,0,1,-1}+320H_{0,0,-1,1}-160H_{0,0,-1,-1} \\
&+160H_{0,-1,0,1}+320H_{0,-1,-1,-1}-864H_{0,0,0,1,1}-288H_{0,0,1,0,1}+128H_{0,0,1,1,1}+32H_{0,1,0,1,1} \\
&+32H_{0,1,1,1,1}\Bigg]+(z-1)\Bigg[\frac{4}{15}H_0^5+(-80-20\zeta_2)H_1^2-\frac{88}{3}H_1^3-\frac{10}{3}H_1^4+80H_{0,1}^2 \\
&+128H_{0,0,0,0,-1}-384H_{0,0,0,1,-1}-384H_{0,0,0,-1,1}+192H_{0,0,0,-1,-1}-128H_{0,0,1,0,-1} \\
&-128H_{0,0,-1,0,1}+64H_{0,0,-1,0,-1}+512H_{0,0,-1,-1,-1}+256H_{0,-1,0,-1,-1}\Bigg] \\
&+L_M^2\Bigg[(1+z)(160H_{-1}H_0-160H_{0,-1}-32H_{0,0,1})+(z-1)\Bigg(-\frac{32}{3}H_0^3+\frac{440}{3}H_1+128H_{0,0,-1}\Bigg) \\
&-\frac{8}{9}(862-853z-144\zeta_3+72z\zeta_3)+\Bigg[-\frac{8}{9}(187+571z)+(z-1)(-80H_1-64H_{0,-1}-32\zeta_2) \\
&+32(1+z)H_{0,1}\Bigg]H_0+\frac{56}{3}(1+4z)H_0^2-\frac{32}{3}(10+z)H_{0,1}+\frac{16}{3}(35+17z)\zeta_2\Bigg]
\end{aligned}
$$

$$+L_M\Bigg[(z-1)\Bigg[\Bigg[-\frac{7720}{9}+160H_{0,1}+\frac{32(2+47z+2z^2)\zeta_2}{3z}\Bigg]H_1+\frac{4}{3}H_1^2-\frac{40}{3}H_1^3+224H_{0,-1}^2\Bigg]$$
$$+(1+z)\Bigg[-\frac{16(4-79z+4z^2)H_{-1}^2H_0}{3z}-32H_{0,1}^2-\frac{32(4-79z+4z^2)H_{0,-1,-1}}{3z}-32H_{0,0,1,1}$$
$$+32H_{0,1,1,1}\Bigg]+\frac{8}{27}(3845-3890z+990\zeta_3+2016z\zeta_3+360z^2\zeta_3)+\Bigg[(z-1)\Bigg(-\frac{2960}{3}H_1$$
$$-120H_1^2-448H_{0,-1,-1}\Bigg)-\frac{8}{27}(-4205+2956z+216\zeta_3+324z\zeta_3)+\frac{16}{3}(38+107z)H_{0,1}$$
$$-\frac{16(8+9z+93z^2)H_{0,-1}}{3z}-192(1+2z)H_{0,0,1}+64(3+z)H_{0,0,-1}+96(1+z)H_{0,1,1}$$
$$+\frac{64}{3}(-11+13z+4z^2)\zeta_2\Bigg]H_0+\Bigg[-\frac{4}{9}(-1261+659z+108z^2)$$
$$+(z-1)(-232H_1+48H_{0,-1})+(1+z)(112H_{0,1}-96\zeta_2)\Bigg]H_0^2-\frac{16}{9}(-79-31z+4z^2)H_0^3$$
$$-\frac{4}{3}(-11+7z)H_0^4+\Bigg[(1+z)\Bigg[\frac{16(18-37z+18z^2)H_0}{3z}+\frac{8(8-11z+8z^2)H_0^2}{3z}$$
$$+\frac{32(4-79z+4z^2)H_{0,-1}}{3z}-\frac{16(4-21z+4z^2)\zeta_2}{z}\Bigg]+\frac{128(1+z)^3H_{0,1}}{3z}\Bigg]H_{-1}$$
$$+\Bigg[\frac{272}{9}(-19+44z)-256(1+z)\zeta_2\Bigg]H_{0,1}+\Bigg[-\frac{16(1+z)(18-37z+18z^2)}{3z}$$
$$-224(z-1)\zeta_2\Bigg]H_{0,-1}-\frac{16}{3}(131+179z+8z^2)H_{0,0,1}-\frac{16(-8-21z-189z^2+8z^3)}{3z}H_{0,0,-1}$$
$$-\frac{16}{3}(-19+29z)H_{0,1,1}-\frac{128(1+z)^3H_{0,1,-1}}{3z}-\frac{128(1+z)^3H_{0,-1,1}}{3z}+96(1+7z)H_{0,0,0,1}$$
$$-96(3+5z)H_{0,0,0,-1}+\frac{16}{9}(-289-193z+54z^2)\zeta_2+\frac{16}{5}(89+73z)\zeta_2^2\Bigg]$$
$$-\frac{2}{9}(-34613+34846z+3860\zeta_3+6472z\zeta_3+2520\zeta_5-1080z\zeta_5)+\Bigg[(1+z)(-48H_{0,1}^2-160H_{0,1,-1}$$
$$-160H_{0,-1,1}+128H_{0,0,1,1}-64H_{0,1,1,1})+(z-1)\Bigg[(1344+240H_{0,1}+40\zeta_2)H_1+112H_1^2$$
$$+\frac{80}{3}H_1^3+32H_{0,-1}^2-96H_{0,0,0,-1}-64H_{0,-1,0,1}+128H_{0,-1,-1,-1}\Bigg]+\frac{8}{9}(2676+5541z$$
$$+178\zeta_3+598z\zeta_3)+[224(-6+z)-16(1+z)\zeta_2]H_{0,1}+[32(1+8z)+48(z-1)\zeta_2]H_{0,-1}$$
$$-32(29+11z)H_{0,0,1}+16(13+5z)H_{0,0,-1}-16(-21+25z)H_{0,1,1}+32(9+5z)H_{0,-1,-1}$$
$$+64(-5+2z)H_{0,0,0,1}+\frac{4}{9}(127+745z)\zeta_2+\frac{16}{5}(-38+5z)\zeta_2^2\Bigg]H_0+\Bigg[(z-1)(-424H_1$$
$$-20H_1^2+32H_{0,0,-1}-32H_{0,-1,-1})-\frac{4}{3}(-192+441z-56\zeta_3+52z\zeta_3)+8(29+19z)H_{0,1}$$
$$-8(9+5z)H_{0,-1}-32(-2+z)H_{0,0,1}+16(1+z)H_{0,1,1}-\frac{8}{3}(-2+19z)\zeta_2\Bigg]H_0^2$$
$$+\Bigg[-\frac{4}{9}(-28+5z)+(z-1)\Bigg(-\frac{16}{3}H_{0,-1}+8\zeta_2\Bigg)\Bigg]H_0^3-\frac{2}{9}(1+4z)H_0^4$$
$$+\Bigg[(z-1)\Bigg(-400H_{0,0,1}-80H_{0,1,1}-\frac{460}{3}\zeta_2\Bigg)+\frac{8}{3}(81-75z-50\zeta_3+50z\zeta_3)\Bigg]H_1$$

$$+\Bigg[(1+z)\Bigg[(-672+160H_{0,1}-160H_{0,-1}-120\zeta_2)H_0-112H_0^2+\frac{40}{3}H_0^3-448H_{0,-1}$$
$$-320H_{0,0,1}+160H_{0,0,-1}-320H_{0,-1,-1}+224\zeta_2\Bigg]+240(\zeta_3+z\zeta_3)\Bigg]H_{-1}+\Bigg[-\frac{40}{3}(-96+93z$$
$$+4\zeta_3+4z\zeta_3)+160(1+z)H_{0,0,1}+\frac{8}{3}(32+5z)\zeta_2\Bigg]H_{0,1}+[-96(-7-7z-\zeta_3+z\zeta_3)$$
$$+(z-1)(128H_{0,0,1}-64H_{0,0,-1}-128H_{0,-1,-1})+8(23+15z)\zeta_2]H_{0,-1}-64H_{0,-1}^2$$
$$+[16(115+29z)+16(1+z)\zeta_2]H_{0,0,1}+[-32(-5+9z)-96(z-1)\zeta_2]H_{0,0,-1}$$
$$+[-8(-30+31z)+16(1+z)\zeta_2]H_{0,1,1}+[448(1+z)+64(z-1)\zeta_2]H_{0,-1,-1}$$
$$+48(29+3z)H_{0,0,0,1}-16(17+5z)H_{0,0,0,-1}+16(-17+28z)H_{0,0,1,1}+96zH_{0,1,1,1}$$
$$-128(-4+z)H_{0,0,0,0,1}-\frac{4}{9}(-487+2179z+258\zeta_3+222z\zeta_3)\zeta_2-\frac{8}{15}(1402+229z)\zeta_2^2$$
$$-16\zeta_3\Bigg\}+C_F^2T_F\Bigg\{L_M^3\Bigg[-\frac{4}{3}(z-1)(13+20H_1)+(1+z)\left(\frac{8}{3}H_0^2+\frac{32}{3}H_{0,1}-\frac{32}{3}\zeta_2\right)$$
$$-\frac{16}{3}(-2+3z)H_0\Bigg]+(z-1)\Bigg[(280-80H_{0,1}+100\zeta_2)H_1^2+24H_1^3+\frac{10}{3}H_1^4\Bigg]$$
$$+(1+z)\Bigg(-\frac{2}{15}H_0^5-32H_{0,0,1,1}+192H_{0,0,0,0,1}+768H_{0,0,0,1,1}+320H_{0,0,1,0,1}-416H_{0,0,1,1,1}$$
$$-192H_{0,1,0,1,1}-32H_{0,1,1,1,1}\Bigg)+L_M^2\Bigg[(1+z)\left(\frac{16}{3}H_0^3-32H_{0,0,1}\right)+8(25-25z+4\zeta_3+4z\zeta_3)$$
$$+[-8(-19+7z)+(1+z)(32H_{0,1}-32\zeta_2)-80(z-1)H_1]H_0-16(-3+2z)H_0^2$$
$$-120(z-1)H_1+16(1+5z)H_{0,1}-96\zeta_2\Bigg]+L_M\Bigg[(z-1)\Bigg[(-200-160H_{0,1}-256\zeta_2)H_1$$
$$-20H_1^2+\frac{40}{3}H_1^3-128H_{0,-1}^2\Bigg]+(1+z)\Bigg[-256H_{-1}^2H_0+2H_0^4+\Bigg[-\frac{32(2+3z+2z^2)H_0}{z}$$
$$+128H_0^2+512H_{0,-1}-256\zeta_2\Bigg]H_{-1}+32H_{0,1}^2-512H_{0,-1,-1}-416H_{0,0,0,1}+384H_{0,0,0,-1}$$
$$-32H_{0,1,1,1}\Bigg]+8(-31+31z-70\zeta_3+54z\zeta_3)+[(1+z)(288H_{0,0,1}-64H_{0,1,1})$$
$$+(z-1)(168H_1+80H_1^2+256H_{0,-1}+256H_{0,-1,-1})-4(-13+133z-16\zeta_3+48z\zeta_3)$$
$$-64(-3+4z)H_{0,1}-256H_{0,0,-1}-416\zeta_2]H_0+[2(69+73z+16z^2)$$
$$+(z-1)(128H_1-64H_{0,-1})+(1+z)(-64H_{0,1}-32\zeta_2)]H_0^2-\frac{8}{3}(-11+5z)H_0^3$$
$$+[-8(-67+41z)+128(1+z)\zeta_2]H_{0,1}+\Bigg[\frac{32(1+z)(2+3z+2z^2)}{z}+128(z-1)\zeta_2\Bigg]H_{0,-1}$$
$$+32(9+8z)H_{0,0,1}-256(-1+3z)H_{0,0,-1}+16(-7+13z)H_{0,1,1}-16(33-10z+4z^2)\zeta_2$$
$$-\frac{32}{5}(13+23z)\zeta_2^2\Bigg]-\frac{4}{3}(-660+660z-323\zeta_3+17z\zeta_3+120\zeta_5+120z\zeta_5)$$
$$+\Bigg[(z-1)[(168-160H_{0,1}+80\zeta_2)H_1+320H_{0,1,1}]+(1+z)\Bigg(32H_{0,1}^2-160H_{0,0,0,1}$$

$$
\begin{aligned}
&-128H_{0,0,1,1}+\frac{448}{5}\zeta_2^2\Bigg)-\frac{8}{3}(-51-282z-32\zeta_3+6z\zeta_3)+[-16(-17+27z)\\
&-32(1+z)\zeta_2]H_{0,1}+16(9+19z)H_{0,0,1}+2(83+255z)\zeta_2\Bigg]H_0+\Bigg[\frac{2}{3}(-51-201z+44\zeta_3\\
&+44z\zeta_3)+140(z-1)H_1-8(5+13z)H_{0,1}+48(1+z)H_{0,0,1}+4(-7+3z)\zeta_2\Bigg]H_0^2\\
&+\Bigg[\frac{2}{3}(1+83z)+(1+z)\Bigg(-\frac{16H_{0,1}}{3}-4\zeta_2\Bigg)+\frac{40}{3}(z-1)H_1\Bigg]H_0^3+\frac{2}{3}(-3+z)H_0^4\\
&+\Bigg[(z-1)(-32H_{0,1}+320H_{0,0,1}+320H_{0,1,1}+148\zeta_2)-\frac{8}{3}(27-27z-110\zeta_3+110z\zeta_3)\Bigg]H_1\\
&+\Bigg[(1+z)(-128H_{0,0,1}+64H_{0,1,1})+\frac{8}{3}(15+174z+44\zeta_3+44z\zeta_3)+8(-5+13z)\zeta_2\Bigg]H_{0,1}\\
&-32(-3+5z)H_{0,1}^2+[8(-45+43z)-32(1+z)\zeta_2]H_{0,0,1}\\
&+[8(-58+5z)-80(1+z)\zeta_2]H_{0,1,1}-16(7+23z)H_{0,0,0,1}-16(-23+27z)H_{0,1,1,1}\\
&+\frac{2}{3}(633-1389z+40\zeta_3+40z\zeta_3)\zeta_2-\frac{8}{5}(-163+23z)\zeta_2^2+32\zeta_3\Bigg\}+a_{gg,Q}^{(3),\mathrm{reg}}\Bigg\}. \qquad \text{(A13)}
\end{aligned}
$$

APPENDIX B: THE z-SPACE EXPRESSIONS FOR THE ASYMPTOTIC MASSIVE WILSON COEFFICIENTS

Here we list the z-space contributions to the massive Wilson coefficient $L_q^{\mathsf{PS}}, L_g^{\mathsf{S}}$ and H_g^{S}. For $L_q^{\mathsf{PS}}(z)$ we obtain

$$
\begin{aligned}
L_q^{PS}(z)=a_s^3C_FN_FT_F^2\Bigg\{&L_M^2\Bigg[\frac{32}{3}(z-1)(12+5H_1)+(1+z)\Bigg(-\frac{32}{3}H_0^2-\frac{64}{3}H_{0,1}+\frac{64}{3}\zeta_2\Bigg)\\
&+\frac{32}{3}(-7+z)H_0\Bigg]+(z-1)\Bigg[\frac{3296}{9}-\frac{32}{27}H_1(-125+12H_0)+\frac{16}{9}H_1^2(28+15H_0)+\frac{80}{3}H_1^3\Bigg]\\
&+L_Q\Bigg[(1+z)\Bigg(\frac{64}{3}H_{0,1,1}-\frac{64\zeta_3}{3}\Bigg)+L_M^2\Bigg[-\frac{160}{3}(z-1)+\frac{64}{3}(1+z)H_0\Bigg]\\
&+L_M\Bigg[\frac{64}{9}(z-1)(-4+15H_1)+(1+z)\Bigg(-\frac{128H_{0,1}}{3}+\frac{128\zeta_2}{3}\Bigg)+\frac{128}{9}(2+5z)H_0\Bigg]\\
&+\frac{128}{27}(11+14z)H_0+(z-1)\Bigg(-\frac{3712}{27}+\frac{128}{9}H_1-\frac{80}{3}H_1^2\Bigg)-\frac{64}{9}(2+5z)H_{0,1}\\
&+\frac{64}{9}(2+5z)\zeta_2\Bigg]+L_M\Bigg[(z-1)\Bigg[-\frac{64}{3}-\frac{64}{9}H_1(32+15H_0)-\frac{320}{3}H_1^2\Bigg]\\
&+(1+z)\Bigg[\frac{128}{9}H_{0,1}(1+3H_0)-\frac{128}{3}H_{0,0,1}+\frac{256}{3}H_{0,1,1}-\frac{128}{3}\zeta_3\Bigg]-\frac{512}{9}(1+2z)H_0\\
&-\frac{64}{9}(2+5z)H_0^2+\Bigg[\frac{64}{9}(-17+13z)-\frac{128}{3}(1+z)H_0\Bigg]\zeta_2\Bigg]+(1+z)\Bigg(\frac{64}{3}H_{0,0,1,1}-64H_{0,1,1,1}+\frac{32}{15}\zeta_2^2\Bigg)\\
&+\frac{128}{27}(-40+z)H_0-\frac{64}{27}(11+14z)H_0^2+\Bigg(\frac{128}{27}(-8+z)+\frac{64}{9}(2+5z)H_0\Bigg)H_{0,1}-\frac{64}{9}(2+5z)H_{0,0,1}\\
&+\Bigg[\frac{64}{9}(1+4z)-\frac{64}{3}(1+z)H_0\Bigg]H_{0,1,1}+\Bigg[\frac{128}{27}(5+2z)-\frac{64}{9}(2+5z)H_0-\frac{160}{3}(z-1)H_1+\frac{64}{3}(1+z)H_{0,1}\Bigg]\zeta_2\\
&+\Bigg[-\frac{32}{9}(-17+13z)+\frac{64}{3}(1+z)H_0\Bigg]\zeta_3+A_{qq,Q}^{PS(3)}(z)+N_F\hat{\hat{C}}_q^{PS(3)}(L_Q,N_F,z)\Bigg\}. \qquad \text{(B1)}
\end{aligned}
$$

For $L_g^{\mathsf{S}}(z)$ we obtain

$$
\begin{aligned}
L_g^S(z) &= a_s^2 N_F T_F^2 \Big\{ \frac{16}{3} L_M L_Q (2z-1) + L_M \Big[-\frac{16}{3}(-3+4z) + (2z-1)\Big(-\frac{16}{3}H_0 - \frac{16}{3}H_1\Big)\Big]\Big\} \\
&+ a_s^3 \Big\{ N_F T_F^3 \Big\{ \frac{64}{9} L_M^2 L_Q (2z-1) + L_M^2 \Big[-\frac{64}{9}(-3+4z) + (2z-1)\Big(-\frac{64}{9}H_0 - \frac{64}{9}H_1\Big)\Big]\Big\} \\
&+ C_A N_F T_F^2 \Big\{ \frac{8}{27}(-3943+3928z) + L_M L_Q^2 \Big[-64(z-1) + \frac{64}{3}(1+z)H_0 - \frac{32}{3}(2z-1)H_1 \Big] \\
&+ L_M^2 \Big[\frac{512}{3}(z-1) + (1+z)\Big(-\frac{32}{3}H_0^2 - \frac{64}{3}H_{0,1}\Big) + \Big[\frac{64}{3}(-4+z) + \frac{32}{3}(2z-1)H_1\Big]H_0 \\
&+ \frac{32}{3}(-9+10z)H_1 + \frac{32}{3}(2z-1)H_1^2 + 32\zeta_2 \Big] + L_Q \Big[-\frac{40}{9}(-108+107z) \\
&+ L_M^2 \Big[-64(z-1) + \frac{64}{3}(1+z)H_0 - \frac{32}{3}(2z-1)H_1 \Big] + L_M \Big[-\frac{32}{3}(-22+19z) \\
&+ \Big[\frac{32}{9}(5+98z) + \frac{64}{3}(2z-1)H_1\Big]H_0 - \frac{64}{3}(1+2z)H_{-1}H_0 - \frac{64}{3}(1+3z)H_0^2 \\
&+ \frac{128}{9}(-8+7z)H_1 + \frac{32}{3}(2z-1)H_1^2 - \frac{128}{3}(1+z)H_{0,1} + \frac{64}{3}(1+2z)H_{0,-1} + \frac{128}{3}\zeta_2 \Big] \\
&+ \frac{8}{27}(785+644z)H_0 + \frac{16}{9}(17+2z)H_0^2 - \frac{16}{9}(2z-1)H_0^3 - \frac{16}{27}(-47+103z)H_1 - \frac{32}{3}zH_{0,1} \\
&+ \frac{32}{3}z\zeta_2 \Big] + (2z-1)\Big(\frac{4}{9}H_0^4 + \frac{32}{3}H_{0,0,0,1} - \frac{64}{15}\zeta_2^2\Big) + L_M \Big[\frac{16}{9}(-325+301z) \\
&+ \Big[\frac{32}{9}(-58-42z+3z^2) - \frac{128}{9}(-8+7z)H_1 - 16(2z-1)H_1^2 - \frac{32}{3}(-3-6z+2z^2)H_{0,1} \\
&+ \frac{64}{3}(-1-2z+z^2)H_{0,-1}\Big]H_0 + \frac{32}{3}(1+2z+2z^2)H_{-1}^2 H_0 + \Big[\frac{8}{9}(23-124z+44z^2) \\
&+ \frac{16}{3}(3-6z+2z^2)H_1\Big]H_0^2 + \frac{16}{9}(5+14z)H_0^3 + \Big[\frac{16}{9}(-180+161z+6z^2) \\
&+ \frac{64}{3}(2z-1)H_{0,1}\Big]H_1 - \frac{32}{9}(-8+z)H_1^2 - \frac{16}{9}(2z-1)H_1^3 \\
&+ \Big[-\frac{64(2+6z+12z^2+11z^3)}{9z}H_0 - \frac{32}{3}(-1-2z+z^2)H_0^2 + \frac{64}{3}(1+2z)H_{0,1} \\
&- \frac{64}{3}(1+2z+2z^2)H_{0,-1}\Big]H_{-1} - \frac{32}{9}(41+62z)H_{0,1} + \frac{64(2+6z+12z^2+11z^3)}{9z}H_{0,-1} \\
&+ \frac{32}{3}(-3+2z+2z^2)H_{0,0,1} - \frac{64}{3}(-1-2z+z^2)H_{0,0,-1} - \frac{32}{3}(-7+2z)H_{0,1,1} \\
&- \frac{64}{3}(1+2z)H_{0,1,-1} - \frac{64}{3}(1+2z)H_{0,-1,1} + \frac{64}{3}(1+2z+2z^2)H_{0,-1,-1} \\
&+ \Big[-\frac{32}{9}(3-90z+22z^2) - \frac{64}{3}(3+4z)H_0 - \frac{64}{3}(z-1)^2 H_1 + \frac{32}{3}(-1-2z+2z^2)H_{-1}\Big]\zeta_2 \\
&- \frac{64}{3}(2+z+2z^2)\zeta_3 \Big] + \Big[\frac{8}{27}(-2405+413z) + \frac{16}{27}(-47+103z)H_1 + \frac{32}{3}zH_{0,1}\Big]H_0 \\
&- \frac{4}{27}(989+764z)H_0^2 + \frac{160}{27}(-2+z)H_0^3 + \frac{8}{27}(-1884+1981z)H_1 + \frac{16}{27}(-47+103z)H_1^2 \\
&- \frac{8}{27}(785+572z)H_{0,1} - \frac{32}{9}(17+5z)H_{0,0,1} + \frac{64}{3}zH_{0,1,1} + \Big[\frac{8}{9}(293+122z) - \frac{32}{9}(-17+z)H_0
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{16}{3}(2z-1)H_0^2\Big]\zeta_2+\Big[-\frac{32}{9}(-17+z)-\frac{32}{3}(2z-1)H_0\Big]\zeta_3\Big\}+C_FN_FT_F^2\Big\{4(-889+904z)\\
&+L_ML_Q^2\Big[8(-41+42z)+(2z-1)\Big(16H_0^2-\frac{32}{3}H_1\Big)-\frac{32}{3}(13+19z)H_0\Big]\\
&+L_Q\Big[-20(-67+70z)+L_M^2\Big[-336(z-1)+48(3+4z)H_0-16(2z-1)H_0^2\Big]\\
&+L_M\Big[-\frac{16}{3}(-472+473z)+(2z-1)\Big(-32H_0^3+\frac{64}{3}H_1^2-64H_{0,0,1}+64\zeta_3\Big)\\
&+\Big[\frac{32}{3}(156+25z)+\frac{128}{3}(2z-1)H_1\Big]H_0+\frac{16}{3}(73+52z)H_0^2-\frac{16}{3}(-109+106z)H_1\\
&+\frac{64}{3}(14+17z)H_{0,1}+\Big[-64(4+7z)+64(2z-1)H_0\Big]\zeta_2\Big]+32(23+17z)H_0\\
&-8(-23+9z)H_0^2+\frac{8}{3}(9+4z)H_0^3-\frac{4}{3}(2z-1)H_0^4\Big]+L_M^2\Big[928(z-1)\\
&+(2z-1)\Big(\frac{16}{3}H_0^3+32H_{0,0,1}-32\zeta_3\Big)-16(30+7z)H_0-8(11+4z)H_0^2\\
&+336(z-1)H_1-48(3+4z)H_{0,1}+\Big[48(3+4z)-32(2z-1)H_0\Big]\zeta_2\Big]\\
&+L_M\Big[\frac{8}{9}(-8533+8509z)+(1+z)^2\Big(-\frac{128}{3}H_{-1}^2H_0-\frac{256}{3}H_{0,-1,-1}\Big)+(2z-1)\Big(\frac{28}{3}H_0^4\\
&-\frac{80}{9}H_1^3+64H_{0,0,1,1}-\frac{32}{5}\zeta_2^2\Big)+\Big[-\frac{8}{9}(5606+209z+24z^2)+(2z-1)\Big(-\frac{64}{3}H_1^2\\
&+64H_{0,0,1}\Big)+48(-11+10z)H_1+\frac{32}{3}(-25-40z+4z^2)H_{0,1}\\
&-\frac{128}{3}(-2+4z+z^2)H_{0,-1}\Big]H_0+\Big[-\frac{4}{9}(3063+462z+32z^2)-\frac{16}{3}(-1+2z+4z^2)H_1\Big]H_0^2\\
&-\frac{16}{9}(104+17z)H_0^3+\Big[-\frac{32}{3}(229-230z+2z^2)-\frac{64}{3}(2z-1)H_{0,1}\Big]H_1+24(-11+10z)H_1^2\\
&+\Big[(1+z)^2\Big(\frac{64}{3}H_0^2+\frac{256}{3}H_{0,-1}\Big)+\frac{64(4+45z+48z^2+4z^3)}{9z}H_0\Big]H_{-1}-\frac{16}{3}(203+156z)H_{0,1}\\
&-\frac{64(4+45z+48z^2+4z^3)}{9z}H_{0,-1}-\frac{64}{3}(z-1)(-11+2z)H_{0,0,1}\\
&+\frac{128}{3}(-5+6z+z^2)H_{0,0,-1}-\frac{32}{3}(31+28z)H_{0,1,1}+\Big[\frac{32}{9}(543+99z+8z^2)\\
&+\frac{32}{3}(83+56z)H_0-96(2z-1)H_0^2+\frac{64}{3}(-1+2z+2z^2)H_1-\frac{128}{3}(1+z)^2H_{-1}\Big]\zeta_2\\
&+\Big[\frac{64}{3}(33-4z+4z^2)-128(2z-1)H_0\Big]\zeta_3\Big]+(2z-1)\Big(\frac{4}{15}H_0^5+32H_{0,0,0,0,1}-32\zeta_5\Big)\\
&+4(-519+10z)H_0-8(69+8z)H_0^2-\frac{8}{3}(32+3z)H_0^3+\frac{2}{3}(-11+4z)H_0^4+20(-67+70z)H_1\\
&-32(23+17z)H_{0,1}+16(-23+9z)H_{0,0,1}-16(9+4z)H_{0,0,0,1}+\Big[32(23+17z)\\
&-16(-23+9z)H_0+8(9+4z)H_0^2-\frac{16}{3}(2z-1)H_0^3\Big]\zeta_2+\Big[\frac{32}{5}(9+4z)-\frac{64}{5}(2z-1)H_0\Big]\zeta_2^2
\end{aligned}
$$

$$+\left[-16(-23+9z)+16(9+4z)H_0-16(2z-1)H_0^2\right]\zeta_3\Big\}+A_{qg,Q}^{(3)}+N_F\hat{\tilde{C}}_g^{S(3)}(L_Q,N_F,z)\Big\}. \tag{B2}$$

For $H_q^{\mathsf{PS}}(z)$ we obtain

$$\begin{aligned}
H_q^{\mathrm{PS}} = {} & a_s^2 C_F T_F\Big\{-\frac{4}{3}(z-1)(148+66H_1+15H_1^2)+L_Q[8(z-1)(11+5H_1)\\
& +(1+z)(-16H_0^2-16H_{0,1}+16\zeta_2)+32(-2+z)H_0]+L_M^2[20(z-1)-8(1+z)H_0]\\
& +L_Q^2[-20(z-1)+8(1+z)H_0]+L_M[8(z-1)-8(-1+3z)H_0+8(1+z)H_0^2]\\
& +(1+z)\left(\frac{16}{3}H_0^3-32H_{0,0,1}+16H_{0,1,1}+16\zeta_3\right)+\left[-\frac{256}{3}(-2+z)-80(z-1)H_1\right.\\
& \left.-\frac{32(1+z)^3H_{-1}}{3z}+32(1+z)H_{0,1}\right]H_0+\frac{8}{3}(21+2z^2)H_0^2+16(-1+3z)H_{0,1}\\
& +\frac{32(1+z)^3H_{0,-1}}{3z}+\left[-\frac{32}{3}(9-3z+z^2)-32(1+z)H_0\right]\zeta_2\Big\}\\
& +a_s^3\Big\{C_F T_F^2\Big\{L_M^2\left[\frac{32}{3}(z-1)(12+5H_1)+(1+z)\left(-\frac{32}{3}H_0^2-\frac{64}{3}H_{0,1}+\frac{64}{3}\zeta_2\right)\right.\\
& \left.+\frac{32}{3}(-7+z)H_0\right]+L_Q\left[(1+z)\left(\frac{64}{3}H_{0,1,1}-\frac{64\zeta_3}{3}\right)+L_M^2\left[-\frac{160}{3}(z-1)+\frac{64}{3}(1+z)H_0\right]\right.\\
& +L_M\left[\frac{64}{9}(z-1)(-4+15H_1)+(1+z)\left(-\frac{128H_{0,1}}{3}+\frac{128\zeta_2}{3}\right)+\frac{128}{9}(2+5z)H_0\right]\\
& +\frac{128}{27}(11+14z)H_0+(z-1)\left(-\frac{3712}{27}+\frac{128}{9}H_1-\frac{80}{3}H_1^2\right)-\frac{64}{9}(2+5z)H_{0,1}\\
& \left.+\frac{64}{9}(2+5z)\zeta_2\right]+L_M\left[(1+z)\left(\frac{128}{9}H_{0,1}-\frac{128}{3}H_{0,0,1}+\frac{256}{3}H_{0,1,1}-\frac{128}{3}\zeta_3\right)\right.\\
& +\left[-\frac{512}{9}(1+2z)-\frac{320}{3}(z-1)H_1+\frac{128}{3}(1+z)H_{0,1}\right]H_0-\frac{64}{9}(2+5z)H_0^2\\
& \left.+(z-1)\left(-\frac{64}{3}-\frac{2048}{9}H_1-\frac{320}{3}H_1^2\right)+\left[\frac{64}{9}(-17+13z)-\frac{128}{3}(1+z)H_0\right]\zeta_2\right]\\
& +(1+z)\left(\frac{64}{3}H_{0,0,1,1}-64H_{0,1,1,1}+\frac{32}{15}\zeta_2^2\right)+\left[\frac{128}{27}(-40+z)+\frac{16}{9}(z-1)H_1(-8+15H_1)\right.\\
& \left.+\frac{64}{9}(2+5z)H_{0,1}-\frac{64}{3}(1+z)H_{0,1,1}\right]H_0-\frac{64}{27}(11+14z)H_0^2+(z-1)\left(\frac{3296}{9}+\frac{4000}{27}H_1\right.\\
& \left.+\frac{448}{9}H_1^2+\frac{80}{3}H_1^3\right)+\frac{128}{27}(-8+z)H_{0,1}-\frac{64}{9}(2+5z)H_{0,0,1}+\frac{64}{9}(1+4z)H_{0,1,1}\\
& +\left[\frac{128}{27}(5+2z)-\frac{64}{9}(2+5z)H_0-\frac{160}{3}(z-1)H_1+\frac{64}{3}(1+z)H_{0,1}\right]\zeta_2\\
& +\left[-\frac{32}{9}(-17+13z)+\frac{64}{3}(1+z)H_0\right]\zeta_3\Big\}+C_F^2T_F\Big\{(z-1)\Big[2656+8H_1(103+40H_{0,0,1})\\
& +\frac{8}{3}H_0^4+144H_1^2-80H_{0,1}^2+160H_{0,1,1,1}\Big]+L_M^2\Big[4(z-1)(92+43H_1+10H_1^2)\\
& +(1+z)\left(-\frac{8}{3}H_0^3+48H_{0,0,1}-32H_{0,1,1}-16\zeta_3\right)+[-4(51+7z)+80(z-1)H_1
\end{aligned}$$

$$
\begin{aligned}
&-32(1+z)H_{0,1}]H_0+8(-6+5z)H_0^2-80zH_{0,1}+[80+16(1+z)H_0]\zeta_2\Big]\\
&+L_Q[L_M^2[-4(z-1)(13+20H_1)+(1+z)(8H_0^2+32H_{0,1}-32\zeta_2)-16(-2+3z)H_0]\\
&+(z-1)(-392-288H_1-160H_{0,1,1})+L_M\Big[-8(z-1)(5+4H_1)\\
&+(1+z)\left(-8H_0-\frac{16}{3}H_0^3-64H_{0,0,1}+64\zeta_3\right)+32zH_0^2+32(-1+3z)H_{0,1}\\
&+[-32(-1+3z)+64(1+z)H_0]\zeta_2\Big]+(1+z)\left(\frac{2}{3}H_0^4-64H_{0,0,0,1}+128H_{0,0,1,1}-\frac{256}{5}\zeta_2^2\right)\\
&+[-52(-3+z)+8(z-1)H_1(13+10H_1)+(1+z)(96H_{0,0,1}-64H_{0,1,1})\\
&-32(-3+2z)H_{0,1}]H_0+[6(3+19z)+80(z-1)H_1-32(1+z)H_{0,1}]H_0^2-\frac{16}{3}zH_0^3\\
&+8(19+17z)H_{0,1}-16(1+5z)H_{0,0,1}+[-48(1+5z)+(1+z)(-16H_0^2+64H_{0,1})\\
&+16(-1+3z)H_0-160(z-1)H_1]\zeta_2+[48(-3+5z)-96(1+z)H_0]\zeta_3\Big]\\
&+L_M\Big[(z-1)\left(312-\frac{64}{3}H_0^3+88H_1+16H_1^2\right)+(1+z)\left(\frac{4}{3}H_0^4-160H_{0,0,0,1}+64H_{0,0,1,1}\right.\\
&\left.+\frac{288}{5}\zeta_2^2\right)+[-8(-2+21z)+32(z-1)H_1-32(-1+3z)H_{0,1}+64(1+z)H_{0,0,1}]H_0\\
&+4(7+13z)H_0^2-8(-11+21z)H_{0,1}+32(1+7z)H_{0,0,1}-32(-1+3z)H_{0,1,1}\\
&+[8(-7+17z)-32(3+z)H_0-16(1+z)H_0^2]\zeta_2+[-64(1+2z)+32(1+z)H_0]\zeta_3]\\
&+(1+z)\left(-\frac{2}{15}H_0^5-80H_{0,0,0,0,1}+832H_{0,0,0,1,1}+256H_{0,0,1,0,1}-128H_{0,0,1,1,1}+80\zeta_5\right)\\
&+\Big[-4(263+195z)+(z-1)\Big[-32H_1(14+5H_{0,1})-172H_1^2-\frac{80}{3}H_1^3\Big]\\
&+(1+z)(32H_{0,1}^2+96H_{0,0,0,1}-224H_{0,0,1,1}+64H_{0,1,1,1})+16(1+10z)H_{0,1}\\
&-16(-1+17z)H_{0,0,1}+160(-2+3z)H_{0,1,1}\Big]H_0+[-6(26+15z)-4(z-1)H_1(43+10H_1)\\
&+(1+z)(-48H_{0,0,1}+32H_{0,1,1})+80zH_{0,1}]H_0^2+\Big[-2(5+23z)-\frac{80}{3}(z-1)H_1\\
&+\frac{32}{3}(1+z)H_{0,1}\Big]H_0^3+[4(-169+35z)-128(1+z)H_{0,0,1}]H_{0,1}-4(109+33z)H_{0,0,1}\\
&+8(-49+13z)H_{0,1,1}+32(-8+15z)H_{0,0,0,1}-144(-1+3z)H_{0,0,1,1}+\Big[4(57+77z)\\
&+8(z-1)H_1(43+10H_1)+(1+z)\left(\frac{8}{3}H_0^3+96H_{0,0,1}-64H_{0,1,1}-96\zeta_3\right)+[12(5+13z)\\
&+160(z-1)H_1-64(1+z)H_{0,1}]H_0+8(3+z)H_0^2-160zH_{0,1}\Big]\zeta_2+\Big[-\frac{8}{5}(-145+83z)\\
&+\frac{48}{5}(1+z)H_0\Big]\zeta_2^2+[4(207+7z)+(1+z)(24H_0^2+128H_{0,1})-32(-13+8z)H_0\\
&-320(z-1)H_1]\zeta_3\Big\}+A_{Qq}^{\mathrm{PS}(3)}+\tilde{C}^{\mathrm{PS}(3)}(L_Q,N_F+1)\Big\}. \qquad \text{(B3)}
\end{aligned}
$$

Finally, $H^{\mathsf{S}}(z)$ reads

$$
\begin{aligned}
H_g^S = {} & a_s T_F\{-4(-3+4z)+(2z-1)(-4L_M+4L_Q-4H_0-4H_1)\}\\
& + a_s^2\Bigg\{C_A T_F\Bigg\{-\frac{8}{3}(-101+104z)+L_Q^2[-48(z-1)\\
& + 16(1+z)H_0-8(2z-1)H_1]+L_M^2[48(z-1)-16(1+z)H_0+8(2z-1)H_1]\\
& + (1+2z)(-16H_{0,1,-1}-16H_{0,-1,1})+L_M[8(-12+11z)+(1+2z)(16H_{-1}H_0+8H_0^2\\
& - 16H_{0,-1})-8(1+8z)H_0+32(z-1)H_1+8(2z-1)H_1^2+16\zeta_2]+L_Q[8(-20+21z)\\
& + (1+2z)(-16H_{-1}H_0+16H_{0,-1})+[24(-5+4z)+16(2z-1)H_1]H_0-8(3+4z)H_0^2\\
& + 16(-7+8z)H_1+8(2z-1)H_1^2-32(1+z)H_{0,1}+32\zeta_2]+\Bigg[\frac{4}{3}(194-163z+6z^2)\\
& - 4(-53+56z+z^2)H_1-16(2z-1)H_1^2-8(-7-10z+2z^2)H_{0,1}\\
& + 8(-3-6z+2z^2)H_{0,-1}\Bigg]H_0+16z^2H_{-1}^2H_0+\Bigg[\frac{2}{3}(126-48z+41z^2)\\
& + 4(3-6z+2z^2)H_1\Bigg]H_0^2+\frac{8}{3}(3+4z)H_0^3+[4(43-53z+2z^2)+16(2z-1)H_{0,1}]H_1\\
& + 2(19-24z+z^2)H_1^2+\Bigg[-\frac{16(2+3z+9z^2+11z^3)}{3z}H_0-4(-3-6z+2z^2)H_0^2\\
& + 16(1+2z)H_{0,1}-32z^2H_{0,-1}\Bigg]H_{-1}+4(-19+28z+2z^2)H_{0,1}\\
& + \frac{16(2+3z+9z^2+11z^3)}{3z}H_{0,-1}+8(-9-10z+2z^2)H_{0,0,1}-8(-3-6z+2z^2)H_{0,0,-1}\\
& + 48H_{0,1,1}+32z^2H_{0,-1,-1}+\Bigg[-\frac{4}{3}(114-84z+47z^2)-32(2+z)H_0\\
& - 16(z-1)^2H_1+16(-1-2z+z^2)H_{-1}\Bigg]\zeta_2-8(-1-10z+4z^2)\zeta_3\Bigg\}\\
& + C_F T_F\Bigg\{-\frac{20}{3}(-20+17z)+L_M^2(6+(2z-1)(-4H_0-8H_1))\\
& + L_Q^2[6+(2z-1)(-4H_0-8H_1)]+L_M(-4(-17+13z)+(2z-1)(-8H_0^2-16H_1^2\\
& + 8H_{0,1}+24\zeta_2)+[-16(-3+2z)-32(2z-1)H_1]H_0-4(-17+20z)H_1)\\
& + L_Q[4(-17+13z)+L_M[-12+(2z-1)(8H_0+16H_1)]+(2z-1)(8H_0^2+16H_1^2\\
& - 8H_{0,1}-24\zeta_2)+[16(-3+2z)+32(2z-1)H_1]H_0+4(-17+20z)H_1]\\
& + (2z-1)\left(-\frac{8}{3}H_0^3-8H_1^3+24H_{0,1,1}\right)+(1+z)^2(-32H_{-1}^2H_0-64H_{0,-1,-1})\\
& + \Bigg[-\frac{8}{3}(-46+53z+6z^2)+8(8-14z+z^2)H_1-16(2z-1)H_1^2+32(z-1)^2H_{0,1}\\
& - 32(z-1)^2H_{0,-1}\Bigg]H_0+\Bigg[-\frac{4}{3}(-27+6z+23z^2)-16z^2H_1\Bigg]H_0^2+[-4(-47+41z+4z^2)\\
& - 16(2z-1)H_{0,1}]H_1-2(-33+40z+2z^2)H_1^2+\Bigg[(1+z)^2(16H_0^2+64H_{0,-1})\\
& + \frac{16(4+12z^2+13z^3)}{3z}H_0\Bigg]H_{-1}-16(-6+z^2)H_{0,1}-\frac{16(4+12z^2+13z^3)}{3z}H_{0,-1}\\
& - 32(z-1)^2H_{0,0,1}+32(1-6z+z^2)H_{0,0,-1}+\Bigg[\frac{8}{3}(-60+42z+29z^2)+32(2z-1)H_0
\end{aligned}
$$

$$
\begin{aligned}
&+16(-1+2z+2z^2)H_1-32(1+z)^2H_{-1}\Bigg]\zeta_2+8(1+14z+8z^2)\zeta_3\Bigg\}\\
&+T_F^2\Bigg\{(2z-1)\left(-\frac{16L_M^2}{3}+\frac{16L_ML_Q}{3}\right)+L_M\left[-\frac{16}{3}(-3+4z)+(2z-1)\left(-\frac{16}{3}H_0-\frac{16}{3}H_1\right)\right]\Bigg\}\Bigg\}\\
&+a_s^3\Bigg\{C_FN_FT_F^2\Bigg\{L_ML_Q^2\left[8(-43+42z)+(2z-1)\left(16H_0^2+\frac{32}{3}H_1\right)-\frac{32}{3}(14+17z)H_0\right]\\
&+(L_ML_Q)\Bigg[-\frac{8}{3}(-709+690z)+(2z-1)\left(-\frac{64}{3}H_0^3-\frac{32}{3}H_1^2-64H_{0,0,1}+64\zeta_3\right)\\
&+\left[-\frac{16}{9}(-673+26z)-\frac{128}{3}(2z-1)H_1\right]H_0+\frac{32}{3}(25+16z)H_0^2-\frac{16}{9}(-425+454z)H_1\\
&+64(4+7z)H_{0,1}+\left[-\frac{64}{3}(14+17z)+64(2z-1)H_0\right]\zeta_2\Bigg]+L_M\Bigg[\frac{2}{9}(-29415+28958z)\\
&+(1+z)[-64(-5+z)H_{-1}H_0+64(-5+z)H_{0,-1}]+(2z-1)\Bigg(\frac{20}{3}H_0^4+\frac{32}{9}H_1^3\\
&+256H_{0,0,-1}-64H_{0,0,0,1}+64H_{0,0,1,1}+\frac{96}{5}\zeta_2^2\Bigg)+\Bigg[-\frac{8}{27}(13571+1958z)\\
&+(2z-1)\left(\frac{32}{3}H_1^2-128H_{0,-1}+64H_{0,0,1}\right)+\frac{16}{3}(-151+170z)H_1-\frac{32}{3}(23+44z)H_{0,1}\Bigg]H_0\\
&+\left[8(-129+18z+4z^2)+\frac{80}{3}(2z-1)H_1\right]H_0^2-\frac{304}{9}(4+z)H_0^3+\Bigg[\frac{104}{27}(-523+542z)\\
&+\frac{64}{3}(2z-1)H_{0,1}\Bigg]H_1+\frac{8}{9}(-425+454z)H_1^2-\frac{704}{9}(5+11z)H_{0,1}+\frac{32}{3}(1+46z)H_{0,0,1}\\
&-\frac{64}{3}(11+23z)H_{0,1,1}+\Bigg[-\frac{16}{9}(-853+26z+36z^2)+(2z-1)\left(-64H_0^2-\frac{128}{3}H_1\right)\\
&+\frac{64}{3}(31+16z)H_0\Bigg]\zeta_2+\left[-\frac{64}{3}(-20+19z)-64(2z-1)H_0\right]\zeta_3\Bigg]\Bigg\}+C_AT_F^2\Bigg\{\frac{8}{27}(-3943\\
&+3928z)+L_ML_Q^2\left[-64(z-1)+\frac{64}{3}(1+z)H_0-\frac{32}{3}(2z-1)H_1\right]+L_M^2\Bigg[\frac{512}{3}(z-1)\\
&+(1+z)\left(-\frac{32}{3}H_0^2-\frac{64}{3}H_{0,1}\right)+\left[\frac{64}{3}(-4+z)+\frac{32}{3}(2z-1)H_1\right]H_0+\frac{32}{3}(-9+10z)H_1\\
&+\frac{32}{3}(2z-1)H_1^2+32\zeta_2\Bigg]+L_Q\Bigg[-\frac{40}{9}(-108+107z)+L_M^2\Bigg[-64(z-1)+\frac{64}{3}(1+z)H_0\\
&-\frac{32}{3}(2z-1)H_1\Bigg]+L_M\Bigg[-\frac{32}{3}(-22+19z)+(1+2z)\left(-\frac{64}{3}H_{-1}H_0+\frac{64}{3}H_{0,-1}\right)\\
&+\left[\frac{32}{9}(5+98z)+\frac{64}{3}(2z-1)H_1\right]H_0-\frac{64}{3}(1+3z)H_0^2+\frac{128}{9}(-8+7z)H_1+\frac{32}{3}(2z-1)H_1^2\\
&-\frac{128}{3}(1+z)H_{0,1}+\frac{128}{3}\zeta_2\Bigg]+\frac{8}{27}(785+644z)H_0+\frac{16}{9}(17+2z)H_0^2-\frac{16}{9}(2z-1)H_0^3-\frac{16}{27}(-47+103z)H_1\\
&-\frac{32}{3}zH_{0,1}+\frac{32}{3}z\zeta_2\Bigg]+(2z-1)\left(\frac{4}{9}H_0^4+\frac{32}{3}H_{0,0,0,1}-\frac{64}{15}\zeta_2^2\right)\\
&+L_M\Bigg[\frac{16}{9}(-325+301z)+(1+2z)\left(-\frac{64}{3}H_{0,1,-1}-\frac{64}{3}H_{0,-1,1}\right)+\Bigg[\frac{32}{9}(-58-42z+3z^2)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{128}{9}(-8+7z)H_1-16(2z-1)H_1^2-\frac{32}{3}(-3-6z+2z^2)H_{0,1} \\
& +\frac{64}{3}(-1-2z+z^2)H_{0,-1}\Bigg]H_0+\frac{32}{3}(1+2z+2z^2)H_{-1}^2H_0+\Bigg[\frac{8}{9}(23-124z+44z^2) \\
& +\frac{16}{3}(3-6z+2z^2)H_1\Bigg]H_0^2+\frac{16}{9}(5+14z)H_0^3+\Bigg[\frac{16}{9}(-180+161z+6z^2) \\
& +\frac{64}{3}(2z-1)H_{0,1}\Bigg]H_1-\frac{32}{9}(-8+z)H_1^2-\frac{16}{9}(2z-1)H_1^3+\Bigg[-\frac{64(2+6z+12z^2+11z^3)}{9z}H_0 \\
& -\frac{32}{3}(-1-2z+z^2)H_0^2+\frac{64}{3}(1+2z)H_{0,1}-\frac{64}{3}(1+2z+2z^2)H_{0,-1}\Bigg]H_{-1} \\
& -\frac{32}{9}(41+62z)H_{0,1}+\frac{64(2+6z+12z^2+11z^3)}{9z}H_{0,-1}+\frac{32}{3}(-3+2z+2z^2)H_{0,0,1} \\
& -\frac{64}{3}(-1-2z+z^2)H_{0,0,-1}-\frac{32}{3}(-7+2z)H_{0,1,1}+\frac{64}{3}(1+2z+2z^2)H_{0,-1,-1} \\
& +\Bigg[-\frac{32}{9}(3-90z+22z^2)-\frac{64}{3}(3+4z)H_0-\frac{64}{3}(z-1)^2H_1+\frac{32}{3}(-1-2z+2z^2)H_{-1}\Bigg]\zeta_2 \\
& -\frac{64}{3}(2+z+2z^2)\zeta_3\Bigg]+\Bigg[\frac{8}{27}(-2405+413z)+\frac{16}{27}(-47+103z)H_1+\frac{32}{3}zH_{0,1}\Bigg]H_0 \\
& -\frac{4}{27}(989+764z)H_0^2+\frac{160}{27}(-2+z)H_0^3+\frac{8}{27}(-1884+1981z)H_1+\frac{16}{27}(-47+103z)H_1^2 \\
& -\frac{8}{27}(785+572z)H_{0,1}-\frac{32}{9}(17+5z)H_{0,0,1}+\frac{64}{3}zH_{0,1,1}+\Bigg[\frac{8}{9}(293+122z)-\frac{32}{9}(-17+z)H_0 \\
& -\frac{16}{3}(2z-1)H_0^2\Bigg]\zeta_2+\Bigg[-\frac{32}{9}(-17+z)-\frac{32}{3}(2z-1)H_0\Bigg]\zeta_3\Bigg\}+C_FT_F^2\Bigg\{4(-889+904z) \\
& +L_ML_Q^2\Bigg[8+(2z-1)\Bigg(-\frac{16}{3}H_0-\frac{32}{3}H_1\Bigg)\Bigg]+L_Q\Bigg[-20(-67+70z)+L_M^2\Bigg[-16(-20+21z) \\
& +(2z-1)\Bigg(-16H_0^2+\frac{64}{3}H_1\Bigg)+\frac{80}{3}(5+8z)H_0\Bigg]+L_M\Bigg[-\frac{16}{3}(-136+137z) \\
& +(2z-1)\Bigg(-\frac{32}{3}H_0^3+\frac{64}{3}H_1^2-\frac{32}{3}H_{0,1}-32\zeta_2\Bigg)+\Bigg[\frac{32}{3}(51+25z)+\frac{128}{3}(2z-1)H_1\Bigg]H_0 \\
& +\frac{16}{3}(25+16z)H_0^2+\frac{16}{3}(-17+20z)H_1\Bigg]+32(23+17z)H_0-8(-23+9z)H_0^2 \\
& +\frac{8}{3}(9+4z)H_0^3-\frac{4}{3}(2z-1)H_0^4\Bigg]+L_M^2\Bigg[\frac{16}{3}(-155+164z)+(2z-1)\Bigg(\frac{16}{3}H_0^3-\frac{32}{3}H_1^2 \\
& +32H_{0,0,1}-32\zeta_3\Bigg)+\Bigg[-\frac{16}{3}(79+13z)-\frac{64}{3}(2z-1)H_1\Bigg]H_0-\frac{8}{3}(31+16z)H_0^2 \\
& +16(-18+17z)H_1-\frac{16}{3}(29+32z)H_{0,1}+\Bigg(\frac{80}{3}(5+8z)-32(2z-1)H_0\Bigg)\zeta_2\Bigg] \\
& +L_M\Bigg[\frac{8}{9}(-1549+1525z)+(1+z)^2\Bigg(-\frac{128}{3}H_{-1}^2H_0-\frac{256}{3}H_{0,-1,-1}\Bigg)+(2z-1)\Bigg(\frac{8}{3}H_0^4 \\
& -\frac{80}{9}H_1^3+\frac{128}{3}H_{0,1,1}+64H_{0,0,0,1}-\frac{128}{5}\zeta_2^2\Bigg)+\Bigg[-\frac{8}{9}(1394-547z+24z^2)-48(-3+4z)H_1 \\
& -\frac{64}{3}(2z-1)H_1^2+\frac{64}{3}(1-2z+2z^2)H_{0,1}-\frac{128}{3}(z-1)^2H_{0,-1}\Bigg]H_0
\end{aligned}
$$

$$
\begin{aligned}
&+\left[-\frac{4}{9}(885+732z+104z^2)-\frac{16}{3}(-1+2z+4z^2)H_1\right]H_0^2+\frac{8}{9}(-61+14z)H_0^3\\
&+\left[-\frac{32}{3}(61-62z+2z^2)-\frac{64}{3}(2z-1)H_{0,1}\right]H_1-24(-3+4z)H_1^2\\
&+\left[(1+z)^2\left(\frac{64}{3}H_0^2+\frac{256}{3}H_{0,-1}\right)+\frac{64(4+12z^2+13z^3)}{9z}H_0\right]H_{-1}-\frac{16}{3}(119+30z)H_{0,1}\\
&-\frac{64(4+12z^2+13z^3)}{9z}H_{0,-1}-\frac{64}{3}(14+5z+2z^2)H_{0,0,1}+\frac{128}{3}(1-6z+z^2)H_{0,0,-1}\\
&+\left[\frac{32}{9}(138+99z+26z^2)+\frac{32}{3}(23+20z)H_0-32(2z-1)H_0^2+\frac{64}{3}(-1+2z+2z^2)H_1\right.\\
&\left.-\frac{128}{3}(1+z)^2H_{-1}\right]\zeta_2+\left[\frac{32}{3}(27+28z+8z^2)-64(2z-1)H_0\right]\zeta_3\bigg]\\
&+(2z-1)\left(\frac{4}{15}H_0^5+32H_{0,0,0,0,1}-32\zeta_5\right)+4(-519+10z)H_0-8(69+8z)H_0^2\\
&-\frac{8}{3}(32+3z)H_0^3+\frac{2}{3}(-11+4z)H_0^4+20(-67+70z)H_1-32(23+17z)H_{0,1}\\
&+16(-23+9z)H_{0,0,1}-16(9+4z)H_{0,0,0,1}+\bigg[32(23+17z)-16(-23+9z)H_0\\
&+8(9+4z)H_0^2-\frac{16}{3}(2z-1)H_0^3\bigg]\zeta_2+\left[\frac{32}{5}(9+4z)-\frac{64}{5}(2z-1)H_0\right]\zeta_2^2\\
&+[-16(-23+9z)+16(9+4z)H_0-16(2z-1)H_0^2]\zeta_3\bigg\}+C_F^2T_F\bigg\{-2(101+7z)\\
&+L_ML_Q^2[(2z-1)(-18-4H_0^2-32H_1^2+32\zeta_2)+[-8(z-1)-32(2z-1)H_1]H_0+48H_1]\\
&+L_M^2\bigg[-2(-67+94z)+(2z-1)\left(-\frac{4}{3}H_0^3-16H_1^3+8H_{0,0,1}+16H_{0,1,1}+8\zeta_3\right)\\
&+[-2(-31+58z)-96(z-1)H_1-32(2z-1)H_1^2]H_0+[4(5+3z)-16(2z-1)H_1]H_0^2\\
&-2(-103+58z)H_1-12(-7+8z)H_1^2+8(3+13z)H_{0,1}+[-8(15+z)\\
&+(2z-1)(24H_0+64H_1)]\zeta_2\bigg]+L_Q\bigg[2(23+13z)+L_M^2[(2z-1)(18+4H_0^2+32H_1^2-32\zeta_2)\\
&+[8(z-1)+32(2z-1)H_1]H_0-48H_1]+(2z-1)\bigg(\frac{16}{3}H_1^4+16H_{0,1}^2+16H_{0,0,0,1}-80H_{0,0,1,1}\\
&+80H_{0,1,1,1}+\frac{184}{5}\zeta_2^2\bigg)+L_M[-2(-305+284z)+(2z-1)(64H_1^3-112H_{0,0,1}-64H_{0,1,1})\\
&+(1+2z)(64H_{-1}^2H_0+64H_{0,0,-1}+128H_{0,-1,-1})+[56(5+14z)+(2z-1)(160H_1^2+64H_{0,1})\\
&+32(-13+10z)H_1]H_0+[8(6+7z)+64(2z-1)H_1]H_0^2+\frac{8}{3}(1+6z)H_0^3\\
&+4(-119+94z)H_1+320(z-1)H_1^2+[(1+2z)(-32H_0^2-128H_{0,-1})-192(1+z)H_0]H_{-1}\\
&+16(3+z)H_{0,1}+192(1+z)H_{0,-1}+[-16(-11+21z)-16(-5+18z)H_0-256(2z-1)H_1\\
&+64(1+2z)H_{-1}]\zeta_2-16(1+14z)\zeta_3]+\bigg[2(28+37z)+(2z-1)\bigg(\frac{16}{3}H_1^3-48H_{0,0,1}\\
&+32H_{0,1,1}\bigg)+[4(-53+46z)+32(2z-1)H_{0,1}]H_1-8(-5+4z)H_1^2+8[-49+14z\\
&+4z^2]H_{0,1}]H_0+[15+14z+(2z-1)(-8H_1^2+24H_{0,1})-4(-25+16z+4z^2)H_1]H_0^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left[-\frac{4}{3}(-2+5z+4z^2)-\frac{16}{3}(2z-1)H_1\right]H_0^3+\frac{1}{3}(1-2z)H_0^4+[12(-10+3z)\\
&-64(2z-1)H_{0,0,1}]H_1+16(-17+16z)H_1^2+4(-13+8z+4z^2)H_1^3+4(55+46z)H_{0,1}\\
&-32(-15+4z+z^2)H_{0,0,1}+8(-87+28z+12z^2)H_{0,1,1}+[-8(1+46z)\\
&+(2z-1)(8H_0^2-16H_1^2-96H_{0,1})+(8(13+4z)+32(2z-1)H_1)H_0+16(-5+4z)H_1]\zeta_2\\
&+[-16(-21+10z+6z^2)+32(2z-1)H_0]\zeta_3\bigg]+L_M\bigg[\frac{1}{2}(-2757+2786z)\\
&+(1+z)(-192H_{0,1,-1}-192H_{0,-1,1}-192H_{0,-1,-1})+(2z-1)(-24H_1^4-16H_{0,1}^2\\
&-288H_{0,0,0,-1}-192H_{0,0,1,1}+224H_{0,1,1,1}-192H_{0,-1,0,1})+(1+2z)\bigg(\frac{128}{3}H_{-1}^3H_0+80H_{0,1,1}\\
&-64H_{0,0,1,-1}-64H_{0,0,-1,1}-64H_{0,0,-1,-1}-128H_{0,1,-1,-1}-128H_{0,-1,1,-1}-128H_{0,-1,-1,1}\\
&-256H_{0,-1,-1,-1}\bigg)+\bigg[-6(149+196z)+(2z-1)\bigg(-\frac{256}{3}H_1^3+224H_{0,1,1}-192H_{0,1,-1}\\
&-192H_{0,-1,1}\bigg)+[-4(-307+230z)+(2z-1)(-160H_{0,1}+192H_{0,-1})]H_1\\
&-8(-67+82z)H_1^2+8(25+18z)H_{0,1}-224(1+2z)H_{0,-1}-16(3+10z)H_{0,0,1}\\
&-32(5+2z)H_{0,0,-1}-256zH_{0,-1,-1}\bigg]H_0+(4(-64-125z+12z^2)-4(-67+76z)H_1\\
&-56(2z-1)H_1^2+8(3+2z)H_{0,1}+16(1+6z)H_{0,-1})H_0^2+\left[\frac{4}{3}(-27+z)-\frac{64}{3}(2z-1)H_1\right]H_0^3\\
&+\frac{1}{3}(-5-14z)H_0^4+[-2(-353+332z)+(2z-1)(224H_{0,0,1}-384H_{0,0,-1}-96H_{0,1,1})\\
&+48(-3+4z)H_{0,1}]H_1+[-2(-365+322z)+32(2z-1)H_{0,1}]H_1^2-16(-13+16z)H_1^3\\
&+[(1+z)(192H_0^2+192H_{0,1}+192H_{0,-1})+(1+2z)(16H_0^3+64H_{0,0,1}+64H_{0,0,-1}\\
&+128H_{0,1,-1}+128H_{0,-1,1}+256H_{0,-1,-1})+[-32(-12-11z+3z^2)\\
&+128(1+2z)H_{0,-1}]H_0]H_{-1}+[(1+2z)(-80H_0^2-64H_{0,1}-128H_{0,-1})-96(1+z)H_0]H_{-1}^2\\
&-12(49+6z)H_{0,1}+[32(-12-11z+3z^2)+192(2z-1)H_{0,1}]H_{0,-1}-64H_{0,-1}^2\\
&+8(-71+18z)H_{0,0,1}+64(1+8z)H_{0,0,-1}+16(-3+22z)H_{0,0,0,1}+[-32(8-31z+3z^2)\\
&+[16(-13+11z)+320(2z-1)H_1]H_0+8(-5+18z)H_0^2+64(-13+16z)H_1\\
&+160(2z-1)H_1^2+[-288(1+z)-64(1+2z)H_0]H_{-1}+128(1+2z)H_{-1}^2\\
&-64(-1+4z)H_{0,1}-32(1+6z)H_{0,-1}]\zeta_2+\frac{48}{5}(13+2z)\zeta_2^2+[16(-6+31z)\\
&+16(-3+26z)H_0+256(2z-1)H_1-224(1+2z)H_{-1}]\zeta_3]+(2z-1)\bigg(\frac{1}{15}H_0^5-\frac{8}{3}H_1^5\\
&+104H_{0,0,0,0,1}-656H_{0,0,0,1,1}-192H_{0,0,1,0,1}+32H_{0,0,1,1,1}-64H_{0,1,0,1,1}-128H_{0,1,1,1,1}-72\zeta_5\bigg)\\
&+\bigg[-2(159+625z)+(2z-1)\bigg(-\frac{16}{3}H_1^4-24H_{0,1}^2-80H_{0,0,0,1}+192H_{0,0,1,1}-144H_{0,1,1,1}\bigg)\\
&+(-4(-11-43z+18z^2)+8(-13+4z+4z^2)H_{0,1}+32(2z-1)H_{0,1,1})H_1\\
&+[-10(-19+18z)-16(2z-1)H_{0,1}]H_1^2-\frac{4}{3}(-23+20z+8z^2)H_1^3\\
&-16(-14+39z+3z^2)H_{0,1}+16(-30+43z+4z^2)H_{0,0,1}
\end{aligned}
$$

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\begin{aligned}
&-16(-61+31z+8z^2)H_{0,1,1}]H_0+[-103+287z-36z^2+(2z-1)(32H_{0,0,1}-40H_{0,1,1})\\
&+2(-51+22z+12z^2)H_1+2(-19+20z)H_1^2-8(z-1)(25+2z)H_{0,1}]H_0^2\\
&+\left[\frac{1}{3}(-53-6z+24z^2)+(2z-1)\left(\frac{8}{3}H_1^2-8H_{0,1}\right)+\frac{4}{3}(-31+28z+4z^2)H_1\right]H_0^3\\
&+\left[\frac{1}{3}(-6+z+4z^2)+\frac{4}{3}(2z-1)H_1\right]H_0^4+[-2(-593+575z)-16(2z-1)H_{0,1}^2\\
&-16(-13+4z+4z^2)H_{0,0,1}]H_1+[4(172-166z+9z^2)+32(2z-1)H_{0,0,1}]H_1^2\\
&-\frac{4}{3}(-181+148z+18z^2)H_1^3-4(-9+8z+2z^2)H_1^4+[2(226-423z+72z^2)\\
&+(2z-1)(96H_{0,0,1}+32H_{0,1,1})]H_{0,1}+4(-13+4z+4z^2)H_{0,1}^2+2(149+562z+24z^2)H_{0,0,1}\\
&-4(-187+104z+36z^2)H_{0,1,1}-8(-74+141z+16z^2)H_{0,0,0,1}\\
&+8(-155+100z+16z^2)H_{0,0,1,1}-8(-129+44z+16z^2)H_{0,1,1,1}\\
&+\left[-2(248-337z+36z^2)+(2z-1)\left(-\frac{4}{3}H_0^3+\frac{64}{3}H_1^3-96H_{0,0,1}+80H_{0,1,1}\right)\right.\\
&+[2(-271+18z)-8(-19+20z)H_1+80(2z-1)H_{0,1}]H_0+[-4(13+7z)\\
&-16(2z-1)H_1]H_0^2+[20(-19+18z)+32(2z-1)H_{0,1}]H_1+4(-23+20z+8z^2)H_1^2\\
&\left.+16(-48+27z+4z^2)H_{0,1}\right]\zeta_2+\left[\frac{16}{5}(25+74z+14z^2)-16(2z-1)H_0\right]\zeta_2^2\\
&+[2(-815-130z+72z^2)+(2z-1)(-4H_0^2+32H_1^2-32H_{0,1})\\
&\left.+[8(-39+33z+16z^2)+32(2z-1)H_1]H_0+32(-17+12z+4z^2)H_1]\zeta_3\right\}\\
&+C_AC_FT_F\left\{4(-1921+1903z)+L_ML_Q^2\left[22+(2z-1)\left(-\frac{44}{3}H_0-\frac{88}{3}H_1\right)\right]\right.\\
&+(1+z)(-176H_{0,1,-1}-176H_{0,-1,1}-64H_{0,1,1,-1}-64H_{0,1,-1,1}-64H_{0,-1,1,1}+512H_{0,0,1,0,1})\\
&+(2z-1)\left(\frac{8}{3}H_1^5-128H_{0,1,0,1,1}+32H_{0,1,1,1,1}\right)+(1+2z)\left(-8H_{-1}^4H_0-\frac{2}{15}H_0^5+64H_{0,0,0,0,-1}\right.\\
&-80H_{0,0,0,1,-1}-80H_{0,0,0,-1,1}+144H_{0,0,0,-1,-1}-48H_{0,0,1,0,-1}-32H_{0,0,1,1,-1}-32H_{0,0,1,-1,1}\\
&+32H_{0,0,1,-1,-1}-32H_{0,0,-1,0,1}+48H_{0,0,-1,0,-1}-32H_{0,0,-1,1,1}+32H_{0,0,-1,1,-1}+32H_{0,0,-1,-1,1}\\
&-352H_{0,0,-1,-1,-1}-64H_{0,1,1,-1,-1}-64H_{0,1,-1,1,-1}-64H_{0,1,-1,-1,1}-128H_{0,1,-1,-1,-1}\\
&-32H_{0,-1,0,1,1}+32H_{0,-1,0,1,-1}+32H_{0,-1,0,-1,1}-128H_{0,-1,0,-1,-1}-64H_{0,-1,1,1,-1}\\
&-64H_{0,-1,1,-1,1}-128H_{0,-1,1,-1,-1}+64H_{0,-1,-1,0,1}-64H_{0,-1,-1,1,1}-128H_{0,-1,-1,1,-1}\\
&\left.-128H_{0,-1,-1,-1,1}-192H_{0,-1,-1,-1,-1}\right)+L_M^2[992(z-1)+[32(-16+z)\\
&+24(-11+12z)H_1+32(2z-1)H_1^2-48(1+2z)H_{0,1}]H_0+[8(-13+12z)\\
&+16(2z-1)H_1]H_0^2-\frac{16}{3}(1+z)H_0^3+8(-73+64z)H_1+24(-7+8z)H_1^2+16(2z-1)H_1^3\\
&-320zH_{0,1}+16(5+8z)H_{0,0,1}-32(1+4z)H_{0,1,1}+[8(33+4z)+48H_0-64(2z-1)H_1]\zeta_2\\
&-16(1+4z)\zeta_3]+L_M\left[\frac{1}{18}(-15897+22658z)+(2z-1)(16H_1^4-16H_{0,-1}^2-224H_{0,1,1,1})\right.
\end{aligned}
$$

$$
\begin{aligned}
&+(1+2z)\Bigg[-\frac{32}{3}H_{-1}^3H_0+(-24H_0+24H_0^2+32H_{0,-1})H_{-1}^2-48H_{0,1,-1}\\
&-48H_{0,-1,1}-48H_{0,-1,-1}+96H_{0,0,1,-1}+96H_{0,0,-1,1}-32H_{0,0,-1,-1}-64H_{0,1,1,-1}\\
&-64H_{0,1,-1,1}-64H_{0,-1,1,1}+64H_{0,-1,-1,-1}\Bigg]+\Bigg[-\frac{2}{27}(-7111+4982z)\\
&+(2z-1)\left(\frac{112}{3}H_1^3-80H_{0,1,1}\right)+\left[-\frac{4}{3}(211+52z)+(2z-1)(64H_{0,1}-96H_{0,-1})\right]H_1\\
&+\frac{8}{3}(-73+71z)H_1^2+\frac{80}{3}(-5+z)H_{0,1}-48(-3+2z)H_{0,-1}+16(5+14z)H_{0,0,1}\\
&+16(7+6z)H_{0,0,-1}+32(-5+2z)H_{0,1,-1}+32(-5+2z)H_{0,-1,1}+32(1+6z)H_{0,-1,-1}\Bigg]H_0\\
&+\left[-2(-146-91z+12z^2)-\frac{20}{3}(-5+16z)H_1-16H_{0,1}-8(3+10z)H_{0,-1}\right]H_0^2\\
&+\left[-\frac{10}{9}(-59+70z)-8(2z-1)H_1\right]H_0^3+\frac{2}{3}(5+4z)H_0^4+\Bigg[\frac{2}{27}(-10201+6722z)\\
&+(2z-1)(-96H_{0,0,1}+192H_{0,0,-1}+96H_{0,1,1})-\frac{16}{3}(-29+40z)H_{0,1}\Bigg]H_1\\
&+\left[\frac{2}{9}(-2279+1516z)-32(2z-1)H_{0,1}\right]H_1^2+\frac{16}{9}(-71+79z)H_1^3+\Bigg[(1+2z)\Bigg(\frac{8}{3}H_0^3+48H_{0,1}\\
&+48H_{0,-1}-96H_{0,0,1}+32H_{0,0,-1}+64H_{0,1,1}-64H_{0,-1,-1}\Bigg)+[16(7-z+3z^2)\\
&+(1+2z)(64H_{0,1}-64H_{0,-1})]H_0-24H_0^2\Bigg]H_{-1}-\frac{8}{9}(-124+71z)H_{0,1}+[-16(7-z+3z^2)\\
&-96(2z-1)H_{0,1}]H_{0,-1}+\frac{8}{3}(133+40z)H_{0,0,1}+48(-5+4z)H_{0,0,-1}-\frac{8}{3}(85+76z)H_{0,1,1}\\
&-128(1+4z)H_{0,0,0,1}+16(-13+10z)H_{0,0,0,-1}+64(-1+6z)H_{0,0,1,1}+64(-1+4z)H_{0,-1,0,1}\\
&+\Bigg[\frac{4}{9}(637+298z+108z^2)+(1+2z)(-8H_{-1}(9+8H_0)-16H_0^2-16H_{-1}^2)\\
&+\left[\frac{16}{3}(-11+10z)-64(2z-1)H_1\right]H_0-\frac{8}{3}(-97+80z)H_1-64(2z-1)H_1^2\\
&+16(-3+10z)H_{0,1}+64(1+3z)H_{0,-1}\Bigg]\zeta_2+\frac{16}{5}(15+22z)\zeta_2^2+\Bigg[-\frac{16}{3}(-71+94z)+80H_0\\
&-112(2z-1)H_1+16(1+2z)H_{-1}\Bigg]\zeta_3\Bigg]+L_Q\Bigg[-4(-281+275z)+(1+z)(80H_{0,-1}\\
&+64H_{0,1,-1}+64H_{0,-1,1})+(2z-1)\left(-\frac{16}{3}H_1^4+16H_{0,1}^2-48H_{0,1,1,1}\right)+(1+2z)\Bigg(-\frac{64}{3}H_{-1}^3H_0\\
&+\frac{2}{3}H_0^4+16H_{0,-1}^2-16H_{0,0,0,-1}+192H_{0,0,1,1}+32H_{0,0,1,-1}+32H_{0,0,-1,1}+32H_{0,0,-1,-1}\\
&+64H_{0,1,-1,-1}+32H_{0,-1,0,1}+64H_{0,-1,1,-1}+64H_{0,-1,-1,1}+128H_{0,-1,-1,-1}\Bigg)\\
&+L_M^2[-144(z-1)+[-16(-5+8z)-32(2z-1)H_1]H_0+16(1+z)H_0^2\\
&-24(-9+8z)H_1-32(2z-1)H_1^2+48(1+2z)H_{0,1}-16(5+2z)\zeta_2]
\end{aligned}
$$

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\begin{aligned}
&+L_M\Bigg[\frac{2}{3}(-425+174z)+(2z-1)(-32H_1^3+32H_{0,1,1})+(1+2z)[(48H_0-16H_0^2\\
&-64H_{0,1})H_{-1}+56H_{0,1}-48H_{0,-1}-32H_{0,0,-1}+64H_{0,1,-1}+64H_{0,-1,1}]\\
&+\Bigg[-\frac{4}{9}(623+566z)+(2z-1)(-32H_1^2-32H_{0,1})+\frac{32}{3}(1+10z)H_1+32(1+2z)H_{0,-1}\Bigg]H_0\\
&+\Bigg[\frac{8}{3}(-23+58z)+16(2z-1)H_1\Bigg]H_0^2-\frac{32}{3}(1+z)H_0^3-\frac{4}{9}(-515+76z)H_1\\
&-\frac{16}{3}(-23+13z)H_1^2-32(3+2z)H_{0,0,1}+\Bigg[-\frac{8}{3}(7+82z)+(1+2z)(64H_0+64H_{-1})\\
&+64(2z-1)H_1\Bigg]\zeta_2+192z\zeta_3\Bigg]+\Bigg[-8(-49+24z)+[16(-15+16z)+32(2z-1)H_{0,1}]H_1\\
&+4(-43+40z)H_1^2+\frac{16}{3}(2z-1)H_1^3-8(-29+26z+2z^2)H_{0,1}+56H_{0,-1}+192(1+z)H_{0,0,1}\\
&-48(1+6z)H_{0,1,1}\Bigg]H_0+[2(29+148z)+8(z-1)(25+z)H_1+8(2z-1)H_1^2-64(1+z)H_{0,1}\\
&+8(1+2z)H_{0,-1}]H_0^2+\frac{8}{3}(1+z^2)H_0^3+[-44(-20+21z)-64(2z-1)H_{0,0,1}]H_1\\
&-8(-5+3z)H_1^2-4(-11+8z+2z^2)H_1^3+\Bigg[(1+2z)\Bigg(-\frac{16}{3}H_0^3-32H_{0,0,1}-32H_{0,0,-1}\\
&-64H_{0,1,-1}-64H_{0,-1,1}-128H_{0,-1,-1}\Bigg)+[-80(1+z)-32(1+2z)H_{0,-1}]H_0\\
&-4(11+8z)H_0^2-64(1+z)H_{0,1}-16(7+4z)H_{0,-1}\Bigg]H_{-1}+[(1+2z)(24H_0^2+32H_{0,1}\\
&+64H_{0,-1})+8(7+4z)H_0]H_{-1}^2+8(59+43z)H_{0,1}+16(-1+z+z^2)H_{0,0,1}\\
&+8(-3+8z)H_{0,0,-1}-16(-24+31z+3z^2)H_{0,1,1}+16(7+4z)H_{0,-1,-1}\\
&-32(5+4z)H_{0,0,0,1}+[-24(13+25z)+(1+2z)(-16H_0^2-64H_{-1}^2-48H_{0,-1})\\
&+16(-5+z)H_0-288(z-1)H_1+16(2z-1)H_1^2+[24(5+4z)+64(1+2z)H_0]H_{-1}\\
&+128(1+z)H_{0,1}]\zeta_2-\frac{144}{5}(3+2z)\zeta_2^2+[16(-28+28z+3z^2)-48(3+4z)H_0\\
&+48(2z-1)H_1+112(1+2z)H_{-1}]\zeta_3]+[-20(143+100z)+(1+z)(-64H_{0,1,-1}\\
&-64H_{0,-1,1})+(1+2z)(-8H_{0,-1}^2-32H_{0,0,0,-1}-32H_{0,0,1,-1}-32H_{0,0,-1,1}-64H_{0,0,-1,-1}\\
&-64H_{0,1,-1,-1}-32H_{0,-1,0,1}-64H_{0,-1,1,-1}-64H_{0,-1,-1,1}-96H_{0,-1,-1,-1})\\
&+[4(231-235z+9z^2)+(2z-1)(32H_{0,0,1}+64H_{0,1,1})-8(-59+60z+2z^2)H_{0,1}]H_1\\
&+[-4(-101+96z)-32(2z-1)H_{0,1}]H_1^2+\frac{4}{3}(29-36z+4z^2)H_1^3+\frac{4}{3}(2z-1)H_1^4\\
&+8(-6+47z+3z^2)H_{0,1}+24(3+2z)H_{0,1}^2-40(3+2z)H_{0,-1}-8(-7+96z+4z^2)H_{0,0,1}\\
&+8(-9+28z)H_{0,0,-1}+32(-29+49z+2z^2)H_{0,1,1}+8(-9+28z)H_{0,-1,-1}\\
&+32(7+6z)H_{0,0,0,1}-16(25+34z)H_{0,0,1,1}+16(7+10z)H_{0,1,1,1}]H_0\\
&+\Bigg[2(-223-174z+9z^2)+(1+2z)(8H_{0,0,-1}+24H_{0,-1,-1})+[-4(-105+104z+3z^2)\\
&-16(2z-1)H_{0,1}]H_1-2(-49+52z)H_1^2-\frac{8}{3}(2z-1)H_1^3+4(-5+50z+2z^2)H_{0,1}
\end{aligned}
$$

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\begin{aligned}
&-8(1+10z)H_{0,-1}-96(1+z)H_{0,0,1}+24(1+6z)H_{0,1,1}\Big]H_0^2+\Big[-2(17+56z+2z^2)\\
&-\frac{8}{3}(z-1)(25+z)H_1-\frac{8}{3}(2z-1)H_1^2+\frac{64}{3}(1+z)H_{0,1}-\frac{8}{3}(1+2z)H_{0,-1}\Big]H_0^3\\
&-\frac{2}{3}(5-4z+z^2)H_0^4+[4(-697+712z)-32(2z-1)H_{0,1}^2+16(-59+60z+2z^2)H_{0,0,1}]H_1\\
&+[-2(337-348z+9z^2)+64(2z-1)H_{0,0,1}]H_1^2+\frac{4}{3}(-88+64z+9z^2)H_1^3\\
&+4(-8+8z+z^2)H_1^4+\Big[(1+z)(176H_{0,1}+64H_{0,1,1})+(1+2z)\Big(\frac{4}{3}H_0^4-64H_{0,0,0,1}\\
&+32H_{0,0,1,1}-32H_{0,0,1,-1}-32H_{0,0,-1,1}+96H_{0,0,-1,-1}+64H_{0,1,1,-1}+64H_{0,1,-1,1}\\
&+128H_{0,1,-1,-1}-32H_{0,-1,0,1}+64H_{0,-1,1,1}+128H_{0,-1,1,-1}+128H_{0,-1,-1,1}+192H_{0,-1,-1,-1}\Big)\\
&+[64(1+z)(8+H_{0,1})+(1+2z)(32H_{0,0,1}+32H_{0,0,-1}+64H_{0,1,-1}+64H_{0,-1,1}\\
&+32H_{0,-1,-1})+24(3+4z)H_{0,-1}]H_0+4(41+32z)H_0^2+\frac{4}{3}(17+20z)H_0^3\\
&+48(10+7z)H_{0,-1}+8(1+4z)H_{0,0,1}+8(17+20z)H_{0,0,-1}+16(13+16z)H_{0,1,-1}\\
&+16(13+16z)H_{0,-1,1}+32(11+14z)H_{0,-1,-1}\Big]H_{-1}+[(1+2z)(-8H_0^3+16H_{0,0,1}\\
&-48H_{0,0,-1}-32H_{0,1,1}-64H_{0,1,-1}-64H_{0,-1,1}-96H_{0,-1,-1})+[-24(10+7z)\\
&+(1+2z)(-32H_{0,1}-16H_{0,-1})]H_0-2(35+44z)H_0^2-8(13+16z)H_{0,1}\\
&-16(11+14z)H_{0,-1}]H_{-1}^2+\Big[(1+2z)\Big(\frac{40}{3}H_0^2+\frac{64}{3}H_{0,1}+32H_{0,-1}\Big)+\frac{16}{3}(11+14z)H_0\Big]H_{-1}^3\\
&+[-4(437-86z+18z^2)-256(1+z)H_{0,0,1}+64(2z-1)H_{0,1,1}]H_{0,1}\\
&-4(-59+60z+2z^2)H_{0,1}^2+[-512(1+z)+(1+2z)(48H_{0,0,1}-16H_{0,0,-1}\\
&+64H_{0,-1,-1})]H_{0,-1}-160zH_{0,-1}^2-4(235+166z+6z^2)H_{0,0,1}-8(11+12z)H_{0,0,-1}\\
&+8(-157+52z+9z^2)H_{0,1,1}-48(10+7z)H_{0,-1,-1}+64(-8+22z+z^2)H_{0,0,0,1}\\
&-32(-4+11z)H_{0,0,0,-1}-16(-27+105z+4z^2)H_{0,0,1,1}-8(1+4z)H_{0,0,1,-1}\\
&-8(1+4z)H_{0,0,-1,1}-8(17+20z)H_{0,0,-1,-1}+8(-55+70z+8z^2)H_{0,1,1,1}\\
&-16(13+16z)H_{0,1,-1,-1}-8(5+4z)H_{0,-1,0,1}-16(13+16z)H_{0,-1,1,-1}\\
&-16(13+16z)H_{0,-1,-1,1}-32(11+14z)H_{0,-1,-1,-1}-16(17+10z)H_{0,0,0,0,1}\\
&+64(26+27z)H_{0,0,0,1,1}-16(-7+62z)H_{0,0,1,1,1}+\Big[4(334+149z+9z^2)\\
&+(1+2z)\Big(\frac{8}{3}H_0^3-\frac{112}{3}H_{-1}^3-16H_{0,0,-1}-32H_{0,-1,-1}\Big)+[4(101+186z)+288(z-1)H_1\\
&-16(2z-1)H_1^2-128(1+z)H_{0,1}+32(1+2z)H_{0,-1}]H_0+8(6+5z)H_0^2+[8(-71+75z)\\
&+64(2z-1)H_{0,1}]H_1-4(7-8z+4z^2)H_1^2-\frac{64}{3}(2z-1)H_1^3+[-8(52+43z)\\
&+(1+2z)(-32H_0^2-32H_{0,-1})-48(5+6z)H_0]H_{-1}+[48(4+5z)+80(1+2z)H_0]H_{-1}^2\\
&-16(1+28z+2z^2)H_{0,1}+8(13+32z)H_{0,-1}+192(1+z)H_{0,0,1}-16(1+22z)H_{0,1,1}\\
&-32(7+3z)\zeta_3\Big]\zeta_2+\Big[-\frac{4}{5}(-587+302z+28z^2)-\frac{32}{5}(-7+4z)H_0-48(2z-1)H_1
\end{aligned}
$$

$$+\frac{64}{5}(1+2z)H_{-1}\Big]\zeta_2^2+\Big[-4(-597-112z+18z^2)+(1+2z)(104H_{-1}^2+24H_{0,-1}) \\ +[-32(-27+19z+2z^2)-48(2z-1)H_1]H_0+48(1+z)H_0^2 \\ -4(-267+260z+16z^2)H_1-88(2z-1)H_1^2+[-12(29+36z)-80(1+2z)H_0]H_{-1} \\ +8(29+38z)H_{0,1}]\zeta_3-4(-73+46z)\zeta_5\Big\}+T_F^3\Big\{\frac{64}{9}L_M^2L_Q(2z-1)+L_M^2\Big[-\frac{64}{9}(-3+4z) \\ +(2z-1)\Big(-\frac{64}{9}H_0-\frac{64}{9}H_1\Big)\Big]\Big\}+A_{Qg}^{(3)}+\tilde{C}_g^{(3)}(N_F+1)\Big\}. \tag{B4}$$

APPENDIX C: THE TRANSFORMATION BETWEEN $\overline{\text{MS}}$ AND THE LARIN SCHEME IN THE NONSINGLET CASE

In Refs. [22,25] the massive OME $A_{qq,Q}^{(3),\text{NS}}$ has been given in the M̄S scheme. This also applies to the three-loop heavy flavor massive nonsinglet Wilson coefficient in [28], related to the associated three-loop massless Wilson coefficient [79]. As we present here all OMEs in the Larin scheme, we also provide $A_{qq,Q}^{(3),\text{NS}}$ in this scheme. We first decompose $A_{qq,Q}^{(3),\text{NS}}$ into its single and double mass pieces

$$A_{qq,Q}^{(3),\text{NS}}=A_{qq,Q}^{(3),\text{NS,s}}+\tilde{A}_{qq,Q}^{(3),\text{NS}}. \tag{C1}$$

The renormalized single mass OME [22] is given by

$$A_{qq,Q}^{(2),\text{NS,L}}=A_{qq,Q}^{(2),\text{NS,M}}+\frac{1}{2}[\hat{\gamma}_{qq}^{(1),\text{NS,L}}-\hat{\gamma}_{qq}^{(1),\text{NS,M}}]L+a_{qq,Q}^{(2),\text{NS,L}}-a_{qq,Q}^{(2),\text{NS,M}}, \tag{C2}$$

$$A_{qq,Q}^{(3),\text{NS,L,s}}=A_{qq,Q}^{(3),\text{NS,M,s}}+\frac{1}{2}[(\gamma_{qq}^{(1),\text{NS,L}}-\gamma_{qq}^{(1),\text{NS,M}})\beta_{0,Q}-(\hat{\gamma}_{qq}^{(1),\text{NS,L}}-\hat{\gamma}_{qq}^{(1),\text{NS,M}})(\beta_0+\beta_{0,Q})]L^2 \\ +\frac{1}{2}[\hat{\gamma}_{qq}^{(2),\text{NS,L}}-\hat{\gamma}_{qq}^{(2),\text{NS,M}}-4(a_{qq,Q}^{(2),\text{NS,L}}-a_{qq,Q}^{(2),\text{NS,M}})(\beta_0+\beta_{0,Q})]L+4(\bar{a}_{qq,Q}^{(2),\text{NS,L}}-\bar{a}_{qq,Q}^{(2),\text{NS,M}}) \\ -\frac{\beta_{0,Q}\zeta_2}{4}[\gamma_{qq}^{(1),\text{NS,L}}-\gamma_{qq}^{(1),\text{NS,M}}]+\delta m_1^{(0)}[\hat{\gamma}_{qq}^{(1),\text{NS,L}}-\hat{\gamma}_{qq}^{(1),\text{NS,M}}]+2\delta m_1^{(-1)}(a_{qq,Q}^{(2),\text{NS,L}}-a_{qq,Q}^{(2),\text{NS,M}}) \\ +a_{qq,Q}^{(3),\text{NS,L}}-a_{qq,Q}^{(3),\text{NS,M}}, \tag{C3}$$

where the formula applies structurally in N and z space. In z space products have to be understood as convolutions. The constant part of the unrenormalized OME $A_{qq,Q}^{(3),\text{NS,L}}$, $a_{qq,Q}^{(3),\text{NS,L}}$, is related to the corresponding expression in the M scheme by

$$a_{qq,Q}^{(3),\text{NS,L}}=a_{qq,Q}^{(3),\text{NS,M}}+C_F\Big\{T_F^2\Big[-\frac{128W_2}{81N^3(1+N)^3}-N_F\Big(\frac{128W_3}{81N^3(1+N)^3}+\frac{64}{9N(1+N)}\zeta_2\Big)-\frac{128}{9N(1+N)}\zeta_2\Big] \\ +C_AT_F\Big[\frac{32W_1}{9N^3(1+N)^3}S_1-\frac{16W_4}{81N^4(1+N)^4}+\frac{64}{3N(1+N)}S_3-\frac{128(-3+4N+10N^2)}{27N^2(1+N)^2}S_{-2} \\ +\frac{128}{9N(1+N)}S_{-3}+\frac{256}{9N(1+N)}S_{-2,1}+\frac{176\zeta_2}{9N(1+N)}-\frac{128\zeta_3}{3N(1+N)}\Big]\Big\} \\ +C_F^2T_F\Big[-\frac{128(12+17N-14N^3+3N^4)}{27N^3(1+N)^3}S_1+\frac{32W_5}{27N^4(1+N)^4}+\frac{256(-3-N+5N^2)}{27N^2(1+N)^2}S_2 \\ +\frac{256(-3+4N+10N^2)}{27N^2(1+N)^2}S_{-2}-\frac{512}{9N(1+N)}S_3-\frac{256}{9N(1+N)}S_{-3}-\frac{512}{9N(1+N)}S_{-2,1}+\frac{128}{3N(1+N)}\zeta_3\Big], \tag{C4}$$

with the polynomials

$$W_1 = 3N^4 + 6N^3 + 5N^2 + 2N + 2, \tag{C5}$$

$$W_2 = 29N^4 + 38N^3 + 17N^2 + 8N + 6, \tag{C6}$$

$$W_3 = 30N^4 + 50N^3 + 24N^2 + 4N + 3, \tag{C7}$$

$$W_4 = 13N^6 - 113N^5 - 107N^4 - 255N^3 - 136N^2 - 294N - 144, \tag{C8}$$

$$W_5 = 115N^6 + 198N^5 + 199N^4 + 28N^3 + 72N^2 + 16N - 12. \tag{C9}$$

Here the expansion coefficients of the QCD-β function [41,80] are

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F N_F \tag{C10}$$

$$\beta_1 = \frac{34}{3}C_A^2 - 4\left(\frac{5}{3}C_A + C_F\right)T_F N_F \tag{C11}$$

and the expansion parameters of the unrenormalized heavy quark mass [41,81] are

$$\delta m_1^{(-1)} = 6C_F \tag{C12}$$

$$\delta m_1^{(0)} = -4C_F. \tag{C13}$$

Furthermore, the transformation relation of the asymptotic massive Wilson coefficient is

$$L_{qq,Q}^{(2),\mathrm{NS,L}} = L_{qq,Q}^{(2),\mathrm{NS,M}} + A_{qq,Q}^{(2),\mathrm{NS,L}} - A_{qq,Q}^{(2),\mathrm{NS,M}} + \hat{C}_q^{(2),\mathrm{NS,L}} - \hat{C}_q^{(2),\mathrm{NS,M}} \tag{C14}$$

$$L_{qq,Q}^{(3),\mathrm{NS,L}} = L_{qq,Q}^{(3),\mathrm{NS,M}} + A_{qq,Q}^{(3),\mathrm{NS,L}} - A_{qq,Q}^{(3),\mathrm{NS,M}} + A_{qq,Q}^{(2),\mathrm{NS,L}} C_q^{(1),\mathrm{NS,L}} - A_{qq,Q}^{(2),\mathrm{NS,M}} C_q^{(1),\mathrm{NS,M}} + C_q^{(3),\mathrm{NS,L}} - C_q^{(3),\mathrm{NS,M}}, \tag{C15}$$

which structurally again applies for Mellin N and z space. Here, the following functions contribute

$$C_q^{(1),\mathrm{NS,L}} = C_q^{(1),\mathrm{NS,M}} - C_F \frac{8}{N(N+1)} \tag{C16}$$

$$\hat{C}_q^{(2),\mathrm{NS,L}} = \hat{C}_q^{(2),\mathrm{NS,M}} - C_F T_F \frac{16(3+N-5N^2)}{9N^2(1+N)^2} \tag{C17}$$

$$\gamma_{qq}^{(1),\mathrm{NS,L}} = \gamma_{qq}^{(1),\mathrm{NS,M}} - 2\beta_0 z_{qq}^{(1)} \tag{C18}$$

$$a_{qq,Q}^{(2),\mathrm{NS,L}} = a_{qq,Q}^{(2),\mathrm{NS,M}} - 16 C_F T_F \frac{5N^2 - N - 3}{9N^2(N+1)^2} \tag{C19}$$

$$\bar{a}_{qq,Q}^{(2),\mathrm{NS,L}} = \bar{a}_{qq,Q}^{(2),\mathrm{NS,M}} - C_F T_F \left[\frac{8(9+12N+10N^2+26N^3+28N^4)}{27N^3(N+1)^3} + \frac{8\zeta_2}{3N(N+1)}\right] \tag{C20}$$

$$\gamma_{qq}^{(2),\mathrm{NS,L}} = \gamma_{qq}^{(2),\mathrm{NS,M}} + 2\beta_0[(z_{qq}^{(1)})^2 - 2z_{qq}^{(2),\mathrm{NS}}] - 2\beta_1 z_{qq}^{(1)}. \tag{C21}$$

The functions $z_{ij}^{(k)}$ are given in Refs. [64,67,68].

$$z_{qq}^{(1)} = -\frac{8C_F}{N(N+1)}, \tag{C22}$$

$$z_{qq}^{(2),\mathrm{NS}} = C_F T_F N_F \frac{16(-3-N+5N^2)}{9N^2(1+N)^2} + C_A C_F \left\{ -\frac{4W_6}{9N^3(1+N)^3} - \frac{16}{N(1+N)} S_{-2} \right\} + C_F^2 \left\{ \frac{8(2+5N+8N^2+N^3+2N^4)}{N^3(1+N)^3} + \frac{16(1+2N)}{N^2(1+N)^2} S_1 + \frac{16}{N(1+N)} S_2 + \frac{32}{N(1+N)} S_{-2} \right\}, \tag{C23}$$

with

$$W_6 = 103N^4 + 140N^3 + 58N^2 + 21N + 36. \tag{C24}$$

The corresponding expressions in z space read

$$\begin{aligned}
a_{qq,Q}^{(3),\mathrm{NS,L}}(z) = {} & a_{qq,Q}^{(3),\mathrm{NS,M}}(z) + C_F \Bigg\{ T_F^2 \Bigg[N_F \left((1-z)\left(-\frac{1280}{27} - \frac{640}{81} H_0 - \frac{64}{27} H_0(z)^2 \right) - \frac{64(1-z)}{9}\zeta_2 \right) \\
& + (1-z)\left(-\frac{3712}{81} - \frac{1280}{81} H_0 - \frac{128}{27} H_0^2 \right) - \frac{128}{9}(1-z)\zeta_2 \Bigg] \\
& + C_A T_F \Bigg[(1-z)\left(-\frac{3664}{81} + \frac{352}{9} H_1 + \frac{32}{3} H_0^2 H_1 - \frac{64}{3} H_0 H_{0,1} \right) \\
& + (1+z)\Bigg(\frac{1280}{27} H_{-1} H_0 + \frac{64}{9} H_{-1} H_0^2 - \frac{1280}{27} H_{0,-1} - \frac{128}{9} H_{0,0,-1} + \frac{256}{9} H_{0,1,-1} \\
& + \frac{256}{9} H_{0,-1,1} - \frac{128}{9} H_{0,1}(-1+2H_{-1}) \Bigg) - \frac{32}{81}(200+7z) H_0 - \frac{16}{27}(47+53z) H_0^2 \\
& - \frac{128}{27} H_0^3 + \frac{256}{9} H_{0,0,1} + \Bigg(-\frac{16}{27}(-89+57z) - \frac{64}{9}(-1+3z) H_0 \\
& + \frac{256}{9}(1+z) H_{-1} \Bigg)\zeta_2 + \frac{128}{3}(-2+z)\zeta_3 \Bigg] \Bigg\} + C_F^2 T_F \Bigg[(1-z)\Bigg(\frac{4480}{27} \\
& - \frac{896}{9} H_1 - \frac{1280}{27} H_0 H_1 - \frac{256}{9} H_0^2 H_1 + \frac{256}{9} H_0 H_{0,1} \Bigg) + (1+z)\Bigg(-\frac{2560}{27} H_{-1} H_0 \\
& - \frac{128}{9} H_{-1} H_0^2 + \frac{2560}{27} H_{0,-1} + \frac{256}{9} H_{0,0,-1} - \frac{512}{9} H_{0,1,-1} - \frac{512}{9} H_{0,-1,1} \\
& + \frac{128}{9} H_{0,1}(-3+4H_{-1}) \Bigg) - \frac{32}{27}(-112+3z) H_0 + \frac{256}{27}(4+7z) H_0^2 \\
& + \frac{64}{27}(1+3z) H_0^3 - \frac{512}{9} H_{0,0,1} + \left(\frac{256}{9} H_0 + (1+z)\left(-\frac{128}{27} - \frac{512}{9} H_{-1} \right) \right)\zeta_2 - \frac{256}{9}(-4+z)\zeta_3 \Bigg]
\end{aligned} \tag{C25}$$

and

$$C_q^{(1),\mathrm{NS,L}} = C_q^{(2),\mathrm{NS,M}} - 8C_F(1-z), \tag{C26}$$

$$\hat{C}_q^{(2),\mathrm{NS,L}} = \hat{C}_q^{(2),\mathrm{NS,M}} + \frac{16}{9} C_F T_F (1-z)(5+3H_0), \tag{C27}$$

$$\gamma_{qq}^{(1),\mathrm{NS,L}} = \gamma_{qq}^{(1),\mathrm{NS,M}} + 16\beta_0 C_F (1-z), \tag{C28}$$

$$a_{qq,Q}^{(2),\mathrm{NS,L}} = a_{qq,Q}^{(2),\mathrm{NS,M}} - \frac{16}{9} C_F T_F (1-z)(5+3H_0), \tag{C29}$$

$$\bar{a}_{qq,Q}^{(2),\mathrm{NS,L}} = \bar{a}_{qq,Q}^{(2),\mathrm{NS,M}} - (1-z) C_F T_F \left(\frac{224}{27} + \frac{8}{3}\zeta_2 + \frac{40}{9} H_0 + \frac{4}{3} H_0^2 \right), \tag{C30}$$

$$\hat{\gamma}_{qq}^{(2),\mathrm{NS,L}} = \hat{\gamma}_{qq}^{(2),\mathrm{NS,M}} + C_A C_F T_F \left[-\frac{19904}{27} - \left(\frac{256}{9}(10-z) - \frac{256}{3}(1+z)H_{-1} \right) H_0 - \frac{128}{3}H_0^2 - \frac{256}{3}(1+z)H_{0,-1} + \frac{256}{3}\zeta_2 \right]$$
$$+ C_F^2 T_F \left[\frac{1472}{3} + \left(\frac{128}{3}(5+6z) - \frac{512}{3}(1+z)H_{-1} \right) H_0 + \frac{128}{3}(1+z)H_0^2 - \frac{256}{3}(1-z)H_0 H_1 + \frac{512}{3}(1+z)H_{0,-1} \right.$$
$$\left. - \frac{256}{3}(1+z)\zeta_2 \right] + C_F T_F^2 (1-2N_F)(1-z) \left[\frac{1280}{27} + \frac{256}{9} H_0 \right]. \tag{C31}$$

For the double mass contributions [25] we obtain

$$\tilde{A}_{qq,Q}^{(3),\mathrm{NS,L}} = \tilde{A}_{qq,Q}^{(3),\mathrm{NS,M}} + \frac{1}{2}\beta_{0,Q}(\hat{\gamma}_{qq}^{(1),\mathrm{NS,L}} - \hat{\gamma}_{qq}^{(1),\mathrm{NS,M}})(L_1^2 + L_2^2) + 4(a_{qq,Q}^{(2),\mathrm{NS,L}} - a_{qq,Q}^{(2),\mathrm{NS,M}})(L_1 + L_2)$$
$$+ 8\beta_{0,Q}(\bar{a}_{qq,Q}^{(2),\mathrm{NS,L}} - \bar{a}_{qq,Q}^{(2),\mathrm{NS,M}}) + \tilde{a}_{qq,Q}^{(3),\mathrm{NS,L}} - \tilde{a}_{qq,Q}^{(3),\mathrm{NS,M}}, \tag{C32}$$

with

$$\tilde{a}_{qq,Q}^{(3),\mathrm{NS,L}}(N) = \tilde{a}_{qq,Q}^{(3),\mathrm{NS,M}}(N) + C_F T_F^2 \left\{ -\frac{256}{9N(N+1)}(L_1^2 + L_1 L_2 + L_2^2 + \zeta_2) + \frac{256(3+N+5N^2)}{27N^2(N+1)^2}(L_1 + L_2) \right.$$
$$\left. - \frac{256(6+8N+17N^2+38N^3+29N^4)}{81N^3(N+1)^3} \right\}, \tag{C33}$$

$$\tilde{a}_{qq,Q}^{(3),\mathrm{NS,L}}(z) = \tilde{a}_{qq,Q}^{(3),\mathrm{NS,M}}(z) + C_F T_F^2 (1-z) \left\{ -\frac{256}{9}(L_1^2 + L_1 L_2 + L_2^2 + \zeta_2) \right.$$
$$\left. - \frac{256}{27}(5+3H_0)(L_1+L_2) - \frac{256}{27}H_0^2 - \frac{2560}{81}H_0 - \frac{7424}{81} \right\}. \tag{C34}$$

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