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Breather solutions for a quasi-linear $(1 + 1)$-dimensional wave equation

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Abstract

We consider the $(1 + 1)$-dimensional quasi-linear wave equation $g(x)w_{tt} - w_{xx} + h(x)(w_t^3)_t = 0$ on $\mathbb{R} \times \mathbb{R}$ that arises in the study of localized electromagnetic waves modeled by Kerr-nonlinear Maxwell equations. We are interested in time-periodic, spatially localized solutions. Here $g \in L^\infty(\mathbb{R})$ is even with $g \not\equiv 0$ and $h(x) = \gamma \delta_0(x)$ with $\gamma \in \mathbb{R}\backslash\{0\}$ and δ_0 the delta-distribution supported in 0. We assume that 0 lies in a spectral gap of the operators $L_k = -\frac{d^2}{dx^2} - k^2\omega^2 g$ on $L^2(\mathbb{R})$ for all $k \in 2\mathbb{Z} + 1$ together with additional properties of the fundamental set of solutions of L_k. By expanding w into a Fourier series in time, we transfer the problem of finding a suitably defined weak solution to finding a minimizer of a functional on a sequence space. The solutions that we have found are exponentially localized in space. Moreover, we show that they can be well approximated by truncating the Fourier series in time. The guiding examples, where all assumptions are fulfilled, are explicitly given step potentials and periodic step potentials g. In these examples, we even find infinitely many distinct breathers.

1 | INTRODUCTION AND MAIN RESULTS

We study the $(1 + 1)$-dimensional quasi-linear wave equation

$$g(x)w_{tt} - w_{xx} + h(x)(w_t^3)_t = 0 \quad \text{for } (x, t) \in \mathbb{R} \times \mathbb{R}, \tag{1}$$

and we look for real-valued, time-periodic, and spatially localized solutions $w(x, t)$. Such solutions are also called breathers. At the end of this introduction, we give a motivation for this equation arising in the study of localized electromagnetic waves modeled by Kerr-nonlinear Maxwell equations. We also cite some relevant papers. To the best of our knowledge for (1) in its general form, no rigorous existence results are available. A first result is given in this paper by taking an extreme case where $h(x)$ is a spatial delta distribution at $x = 0$. Our basic assumption on the coefficients g and h is the following:

$$g \in L^\infty(\mathbb{R}) \text{ even, } g \not\equiv 0 \text{ and } h(x) = \gamma\delta_0(x) \text{ with } \gamma \neq 0, \tag{C0}$$

where δ_0 denotes the delta-distribution supported in 0. The special form of h enables us to reduce (1) to a linear wave equation with a nonlinear Neumann boundary condition (18). We then use in Section 2 a special series ansatz for the solution where each term directly solves the linear wave equation. To satisfy the nonlinear Neumann boundary condition, it remains to determine the free multiplicative coefficients in the series by minimizing a suitable energy functional.

We have two prototypical examples for the potential g: a step potential (Theorem 1) and a periodic step potential (Theorem 2). The general version is given in Theorem 3 below.

Theorem 1. *For $a, b, c > 0$, let*

$$g(x) := \begin{cases} -a, & \text{if } |x| > c, \\ b, & \text{if } |x| < c. \end{cases}$$

For every frequency ω such that $\sqrt{b}\omega c \frac{2}{\pi} \in \frac{2\mathbb{N}+1}{2\mathbb{N}+1}$ and $\gamma < 0$ there exist infinitely many nontrivial, real-valued, spatially localized and time-periodic weak solutions of (1) with period $T = \frac{2\pi}{\omega}$. For each solution w, there are constants $C, \rho > 0$ such that $|w(x, t)| \leq Ce^{-\rho|x|}$.

Theorem 2. *For $a, b > 0$, $a \neq b$ and $\Theta \in (0, 1)$, let*

$$g(x) := \begin{cases} a, & \text{if } |x| < \pi\Theta, \\ b, & \text{if } \pi\Theta < |x| < \pi, \end{cases}$$

and extend g as a 2π-periodic function to $\mathbb{R}$. Assume in addition

$$\sqrt{\frac{b}{a}} \frac{1 - \Theta}{\Theta} \in \frac{2\mathbb{N} + 1}{2\mathbb{N} + 1}. \tag{2}$$

For every frequency ω such that $4\sqrt{a}\theta\omega \in \frac{2\mathbb{N}+1}{2\mathbb{N}+1}$, there exist infinitely many nontrivial, real-valued, spatially localized and time-periodic weak solutions of (1) with period $T = \frac{2\pi}{\omega}$. For each solution w, there are constants $C, \rho > 0$ such that $|w(x,t)| \leq Ce^{-\rho|x|}$.

Weak solutions of (1) are understood in the following sense. We write $D := \mathbb{R} \times \mathbb{T}_T$ and denote by $\mathbb{T}_T$ the one-dimensional torus with period T.

Definition 1. Under the assumption (C0), a function $w \in H^1(\mathbb{R} \times \mathbb{T}_T)$ with $\partial_t w(0, \cdot) \in L^3(\mathbb{T}_T)$ is called a weak solution of (1) if for every $\psi \in C_c^\infty(\mathbb{R} \times \mathbb{T}_T)$

$$\int_D -g(x)\partial_t w \, \partial_t \psi + \partial_x w \, \partial_x \psi \mathrm{d}(x,t) - \gamma \int_0^T (\partial_t w(0,t))^3 \partial_t \psi(0,t) \mathrm{d}t = 0. \tag{3}$$

Theorem 1 and Theorem 2 are special cases of Theorem 3, which applies to much more general potentials g. In Section A.1 and Section A.2 of the Appendix, we will show that the special potentials g from these two theorems satisfy the conditions (C1) and (C2) of Theorem 3. The basic preparations and assumptions for Theorem 3 will be given next.

As we are looking for time-periodic solutions, it is appropriate to make the Fourier ansatz $w(x,t) = \sum_{k \in \mathbb{Z}_{odd}} w_k(x)e^{ik\omega t}$ with $\mathbb{Z}_{odd} := 2\mathbb{Z} + 1$. The reason for dropping even Fourier modes is that the 0-mode belongs to the kernel of the wave operator $L = g(x)\partial_t^2 - \partial_x^2$. The restriction to odd Fourier modes generates $T/2 = \pi/\omega$-antiperiodic functions w, and is therefore compatible with the structure of (1) and in particular the cubic nonlinearity. Notice the decomposition $(Lw)(x,t) = \sum_{k \in \mathbb{Z}_{odd}} L_k w_k(x)e^{ik\omega t}$ with self-adjoint operators

$$L_k = -\frac{d^2}{dx^2} - k^2\omega^2 g(x) : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}). \tag{4}$$

Clearly, $L_k = L_{-k}$ so that is suffices to study L_k for $k \in \mathbb{N}_{odd}$. Our first assumption is concerned with the spectrum $\sigma(L_k)$:

$$\forall k \in \mathbb{N}_{odd}, 0 \notin \sigma_{\mathrm{ess}}(L_k) \cup \sigma_{\mathrm{D}}(L_k), \tag{C1}$$

where by $\sigma_{\mathrm{D}}(L_k)$ we denote the spectrum of L_k with an extra Dirichlet condition at 0, that is, the spectrum of L_k restricted to $\{\varphi \in H^2(\mathbb{R}) \mid \varphi(0) = 0\}$. This is the same as the spectrum of L_k restricted to functions that are odd around $x = 0$.

Lemma 1. *Under the assumption (C0) and (C1), there exists for every $k \in \mathbb{N}_{odd}$ a function $\Phi_k \in H^2(0, \infty)$ with $L_k \Phi_k = 0$ on $(0, \infty)$ and $\Phi_k(0) = 1$.*

Proof. We have either that 0 is in the point spectrum (but not the Dirichlet spectrum) or that 0 is in the resolvent set of L_k. In the first case, there is an eigenfunction $\Phi_k \in H^2(\mathbb{R})$ with $L_k \Phi_k = 0$

and w.l.o.g. $\Phi_k(0) = 1$. In the second case $0 \in \rho(L_k)$ so that there exists a unique solution $\tilde{\Phi}_k$ of $L_k\tilde{\Phi}_k = 1_{[-2,-1]}$ on $\mathbb{R}$. Clearly, if restricted to $(0, \infty)$, the function $\tilde{\Phi}_k$ solves $L_k\tilde{\Phi}_k = 0$ on $(0, \infty)$. Moreover, $\tilde{\Phi}_k(0) \neq 0$ since otherwise we could reflect $\tilde{\Phi}_k|_{[0,\infty)}$ in an odd way to $\mathbb{R}$ and thus obtain an odd $H^2(\mathbb{R})$-eigenfunction of L_k that is excluded due to $0 \in \rho(L_k)$. Thus, a suitably rescaled version of $\tilde{\Phi}_k$ satisfies the claim of the lemma. ∎

Our second set of assumptions concerns the structure of the decaying fundamental solution according to Lemma 1.

$$\text{There exist } \rho, M > 0 \text{ such that for all } k \in \mathbb{N}_{odd} : |\Phi_k(x)| \leq Me^{-\rho x} \text{ on } [0, \infty). \qquad (C2)$$

We see that the existence of the decaying fundamental solution follows from (C0). Whether or not (C2) is fulfilled depends on the potential g in (1). In the examples in Theorems 1 and 2, this is the case.

Now we can formulate our third main theorem as a generalization of Theorem 1 and Theorem 2. The fact that the solutions which we find can be well approximated by truncation of the Fourier series in time is explained in Lemma 7 below. Moreover, a further extension yielding infinitely many different solutions is given in Theorem 7 in Section 4.

Theorem 3. *Assume (C0), (C1), and (C2) for a potential g and a frequency $\omega > 0$. Then (1) has a nontrivial, T-periodic weak solution w in the sense of Definition 1 with $T = \frac{2\pi}{\omega}$, provided that*

(i) $\gamma < 0$ and the sequence $(\Phi_k'(0))_{k \in \mathbb{N}_{odd}}$ has at least one positive element,
(ii) $\gamma > 0$ and the sequence $(\Phi_k'(0))_{k \in \mathbb{N}_{odd}}$ has at least one negative element.

Moreover, there is a constant $C > 0$ such that $|w(x, t)| \leq Ce^{-\rho|x|}$ for all $(x, t) \in \mathbb{R}^2$ with ρ as in (C2).

Remark 1. (a) It turns out that the above assumptions can be weakened as follows: it suffices to verify (C1) and (C2) and (i), (ii) for all integers $k \in r \cdot \mathbb{Z}_{odd}$ for some $r \in \mathbb{N}_{odd}$. We will prove this observation in Section 4.

(b) Our variational approach also works if we consider (1) with Dirichlet boundary conditions on a bounded interval $(-l, l)$ instead of the real line. There are many possible results. For illustration purposes, we just formulate the simplest one. For example, if we assume that $\frac{\omega l}{\pi} \in \frac{\mathbb{N}_{odd}}{4\mathbb{N}}$, then

$$w_{tt} - w_{xx} + \gamma\delta_0(x)(w_t^3)_t = 0 \text{ on } (-l, l) \times \mathbb{R} \text{ with } w(\pm l, t) = 0 \text{ for all } t \qquad (5)$$

has a nontrivial, real-valued time-periodic weak solution with period $T = \frac{2\pi}{\omega}$ both for $\gamma > 0$ and $\gamma < 0$. The operator $L_k = -\frac{d^2}{dx^2} - \omega^2 k^2$ is now a self-adjoint operator on $H^2(-l, l) \cap H_0^1(-l, l)$. The assumption $\frac{\omega l}{\pi} \in \frac{\mathbb{Z}_{odd}}{4\mathbb{Z}}$ guarantees (C1) for all $k \in \mathbb{Z}_{odd}$. The functions Φ_k are given by $\Phi_k(x) = \frac{\sin(\omega k(l-x))}{\sin(\omega kl)}$ so that $\Phi_k'(0) = -\omega k \cot(\omega kl)$. The assumption $\frac{\omega l}{\pi} \in \frac{\mathbb{N}_{odd}}{4\mathbb{N}}$ now guarantees that the sequence $\{\cot(\omega kl) \mid k \in \mathbb{Z}_{odd}\}$ is finite and does not contain 0 or $\pm\infty$. Moreover, $\frac{\omega l}{\pi} = \frac{2p-1}{4q}$ for $p, q \in \mathbb{N}$ yields $\Phi_k'(0)\Phi_{k+2q}'(0) < 0$, that is, we also have the required sign-change that allows for both signs of γ.

We observe that the growth of $(\Phi'_k(0))_{k \in \mathbb{Z}_{odd}}$ is connected to regularity properties of our solutions.

Theorem 4. *Assume (C0), (C1), and (C2) and additionally $\Phi'_k(0) = O(k)$. Then the weak solution w from Theorem 3 belongs to $H^{1+\nu}_{per}(\mathbb{T}_T, L^2(\mathbb{R})) \cap H^{\nu}_{per}(\mathbb{T}_T, H^1(\mathbb{R}))$ for any $\nu \in (0, \frac{1}{4})$.*

Here, for $\nu \in \mathbb{R}$, the fractional Sobolev spaces of time-periodic functions are defined by

$$H^{\nu}_{per}(\mathbb{T}_T, L^2(\mathbb{R})) := \left\{ u(x,t) = \sum_{k \in \mathbb{Z}} \hat{u}_k(x) e^{i\omega k t} \ \middle| \ \sum_{k \in \mathbb{Z}} \left(1 + |k|^2\right)^{\nu} \|\hat{u}_k\|^2_{L^2(\mathbb{R})} < \infty \right\}, \qquad (6)$$

$$H^{\nu}_{per}(\mathbb{T}_T, H^1(\mathbb{R})) := \left\{ u(x,t) = \sum_{k \in \mathbb{Z}} \hat{u}_k(x) e^{i\omega k t} \ \middle| \ \sum_{k \in \mathbb{Z}} \left(1 + |k|^2\right)^{\nu} \|\hat{u}_k\|^2_{H^1(\mathbb{R})} < \infty \right\}. \qquad (7)$$

We shortly motivate (1) and give some references to the literature. Consider Maxwell's equations in the absence of charges and currents

$$\nabla \cdot \mathbf{D} = 0, \qquad \nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \qquad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}(\mathbf{E}), \qquad (8)$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{H} = \partial_t \mathbf{D}, \qquad \mathbf{B} = \mu_0 \mathbf{H}. \qquad (9)$$

We assume that the dependence of the polarization $\mathbf{P}$ on the electric field $\mathbf{E}$ is instantaneous and it is the sum of a linear and a cubic term given by $\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi_1(\mathbf{x})\mathbf{E} + \varepsilon_0 \chi_3(\mathbf{x})|\mathbf{E}|^2\mathbf{E}$ with $\mathbf{x} \in \mathbb{R}^3$, cf. Ref. 1, Section 2.3 (for simplicity, more general cases where instead of a factor multiplying $|\mathbf{E}|^2\mathbf{E}$ one can take χ_3 as an $\mathbf{x}$-dependent tensor of type (1,3) are not considered here). Here ε_0, μ_0 are constants such that $c^2 = (\varepsilon_0\mu_0)^{-1}$ with c being the speed of light in vacuum and χ_1, χ_3 are given material functions. By direct calculations, one obtains the quasi-linear curl-curl-equation

$$0 = \nabla \times \nabla \times \mathbf{E} + \partial_t^2 \left(V(\mathbf{x})\mathbf{E} + \Gamma(\mathbf{x})|\mathbf{E}|^2\mathbf{E}\right), \qquad (10)$$

where $V(\mathbf{x}) = \mu_0\varepsilon_0(1 + \chi_1(\mathbf{x}))$ and $\Gamma(\mathbf{x}) = \mu_0\varepsilon_0\chi_3(\mathbf{x})$. Once (10) is solved for the electric field $\mathbf{E}$, the magnetic induction $\mathbf{B}$ is obtained by time integration from $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$ and it will satisfy $\nabla \cdot \mathbf{B} = 0$, provided that it does so at time $t = 0$. By construction, the magnetic field $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$ satisfies $\nabla \times \mathbf{H} = \partial_t \mathbf{D}$. To complete the full set of nonlinear Maxwell's equations, one only needs to check Gauss's law $\nabla \cdot \mathbf{D} = 0$ in the absence of external charges. This will follow directly from the constitutive equation $\mathbf{D} = \varepsilon_0(1 + \chi_1(\mathbf{x}))\mathbf{E} + \varepsilon_0\chi_3(\mathbf{x})|\mathbf{E}|^2\mathbf{E}$ and the two different specific forms of $\mathbf{E}$ given next:

$$\mathbf{E}(\mathbf{x},t) = (0, u(x_1 - \kappa t, x_3), 0)^T \qquad \text{polarized wave traveling in } x_1\text{-direction}, \qquad (11)$$

$$\mathbf{E}(\mathbf{x},t) = (0, u(x_1, t), 0)^T \qquad \text{polarized standing wave.} \qquad (12)$$

In the first case, $\mathbf{E}$ is a polarized wave independent of x_2 traveling with speed κ in the x_1 direction and with profile u. If additionally $V(\mathbf{x}) = V(x_3)$ and $\Gamma(\mathbf{x}) = \Gamma(x_3)$, then the quasi-linear curl-curl-

equation (10) turns into the following equation for $u = u(\tau, x_3)$ with the moving coordinate $\tau = x_1 - \kappa t$:

$$-u_{x_3 x_3} + (\kappa^2 V(x_3) - 1)u_{\tau\tau} + \kappa^2 \Gamma(x_3)(u^3)_{\tau\tau} = 0. \tag{13}$$

Setting $u = w_\tau$ and integrating once w.r.t. τ, we obtain (1).

In the second case, $\mathbf{E}$ is a polarized standing wave that is independent of x_2, x_3. If we assume furthermore that $V(\mathbf{x}) = V(x_1)$ and $\Gamma(\mathbf{x}) = \Gamma(x_1)$, then this time the quasi-linear curl-curl-equation (10) for $u = w_t$ turns (after one time integration) directly into (1).

In the literature, (10) has mostly been studied by considering time-harmonic waves $\mathbf{E}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x})e^{i\kappa t}$. This reduces the problem to the stationary elliptic equation

$$0 = \nabla \times \nabla \times \mathbf{U} - \kappa^2 (V(\mathbf{x})\mathbf{U} + \Gamma(\mathbf{x})|\mathbf{U}|^2 \mathbf{U}) \text{ in } \mathbb{R}^3. \tag{14}$$

Here $\mathbf{E}$ is no longer real-valued. This may be justified by extending the ansatz to $\mathbf{E}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x})e^{i\kappa t} + c.c.$ and by either neglecting higher harmonics generated from the cubic nonlinearity or by assuming the time-averaged constitutive relation $\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi_1(\mathbf{x})\mathbf{E} + \varepsilon_0 \chi_3(\mathbf{x})\frac{1}{T}\int_0^T |\mathbf{E}|^2 \, dt \mathbf{E}$ with $T = 2\pi/\kappa$, cf. Refs. [2], [3]. For results on (14), we refer to Refs. [4] and [5] and in particular to the survey.[6] Time-harmonic traveling waves have been found in a series of papers.[2,7,8] The number of results for monochromatic standing polarized wave profiles $U(\mathbf{x}) = (0, u(x_1), 0)$ with u satisfying $0 = -u'' - \kappa^2(V(x_1)u + \Gamma(x_1)|u|^2 u)$ on $\mathbb{R}$ is too large to cite, so we restrict ourselves to Cazenave's book.[9]

Our approach differs substantially from the approaches by monochromatic waves described above. Our ansatz $w(x, t) = \sum_{k \in \mathbb{Z}_{odd}} w_k(x)e^{ik\omega t}$ with $\mathbb{Z}_{odd} := 2\mathbb{Z} + 1$ is automatically polychromatic because it couples all integer multiples of the frequency ω. A similar polychromatic approach is considered in Ref. [10]. The authors seek spatially localized traveling wave solutions of the 1+1-dimensional quasi-linear Maxwell model, where in the direction of propagation, χ_1 is a periodic arrangement of delta functions. Based on a multiple scale approximation ansatz, the field profile is expanded into infinitely many modes that are time-periodic in both the fast and slow time variables. As the periodicities in the fast and slow time variables differ, the field becomes quasi-periodic in time. To a certain extent, the authors of Ref. [10] analytically deal with the resulting system for these infinitely many coupled modes through bifurcation methods, with a rigorous existence proof still missing. However, numerical results from Ref. [10] indicate that spatially localized traveling waves could exist.

With our case of allowing χ_1 to be a bounded function but taking χ_3 to be a delta function at $x = 0$, we consider an extreme case. On the other hand, our existence results (possibly for the first time) rigorously establish localized solutions of the full nonlinear Maxwell problem (10) without making the assumption of either neglecting higher harmonics or of assuming a time-averaged nonlinear constitutive law.

The existence of localized breathers of the quasi-linear problem (1) with bounded coefficients g, h remains generally open. We can, however, provide specific functions g, h for which (1) has a breather-type solution that decays to 0 as $|x| \to \infty$. Let

$$b(x) := (1 + x^2)^{-1/2}, \quad h(x) := \frac{1 - 2x^2}{1 + x^2}, \quad g(x) := \frac{2 + x^4}{(1 + x^2)^2} \tag{15}$$

and consider a time-periodic solution a of the ordinary differential equation

$$-a'' - (a'^3)' = a \tag{16}$$

with minimal prescribed period $T \in (0, 2\pi)$. Then $w(x, t) := a(t)b(x)$ satisfies (1). Note that h is sign-changing and w is not exponentially localized. We found this solution by inserting the ansatz for w with separated variables into (1). We then defined $b(x) := (1 + x^2)^{-1/2}$ and set $g(x) := -b''(x)/b(x)$ and $h(x) := -b''(x)/b(x)^3$. The remaining equation for a then turned out to be the above one.

The paper is structured as follows: In Section 2, we develop the variational setting and give the proof of Theorem 3. The proof of the additional regularity results of Theorem 4 is given in Section 3. In Section 4, we give the proof of Theorem 7 on the existence of infinitely many different breathers. In Section 5, we show that our breathers can be well approximated by truncation of the Fourier series in time. Finally, in the Appendix, we give details on the background and proof of Theorem 1 (Section A.1) and Theorem 2 (Section A.2) as well as a technical detail on a particular embedding of Hölder spaces into Sobolev spaces (Section A.3).

2 | VARIATIONAL APPROACH AND PROOF OF THEOREM 3

The main result of our paper is Theorem 3 that will be proved in this section. It is a consequence of Lemma 5 and Theorem 5 below.

Formally, (1) is the Euler–Lagrange equation of the functional

$$I(w) := \int_D -\frac{1}{2}g(x)|\partial_t w|^2 + \frac{1}{2}|\partial_x w|^2 \mathrm{d}(x, t) - \frac{1}{4}\gamma \int_0^T |\partial_t w(0, t)|^4 \mathrm{d}t \tag{17}$$

defined on a suitable space of T-periodic functions. Instead of directly searching for a critical point of this functional, we first rewrite the problem into a nonlinear Neumann boundary value problem under the assumption that w is even in x. In this case, (1) amounts to the following linear wave equation on the half-axis with nonlinear Neumann boundary conditions:

$$\begin{cases} g(x)w_{tt} - w_{xx} = 0 & \text{for } (x, t) \in (0, \infty) \times \mathbb{R}, \\ 2w_x(0_+, t) = \gamma\big(w_t(0, t)^3\big)_t & \text{for } t \in \mathbb{R}, \end{cases} \tag{18}$$

where solutions $w \in H^1([0, \infty) \times \mathbb{T}_T)$ with $\partial_t w(0, \cdot) \in L^3(\mathbb{T}_T)$ of (18) are understood in the sense that

$$2\int_{D_+} -g(x)\partial_t w \, \partial_t \psi + \partial_x w \, \partial_x \psi \mathrm{d}(x, t) - \gamma \int_0^T (\partial_t w(0, t))^3 \partial_t \psi(0, t)\mathrm{d}t = 0 \tag{19}$$

for all $\psi \in C_c^\infty([0, \infty) \times \mathbb{T}_T)$ with $D_+ = (0, \infty) \times \mathbb{T}_T$. It is clear that evenly extended solutions w of (19) also satisfy (3). To see this, note that every $\psi \in C_c^\infty(\mathbb{R} \times \mathbb{T}_T)$ can be split into an even and an odd part $\psi = \psi_e + \psi_o$ both belonging to $C_c^\infty(\mathbb{R} \times \mathbb{T}_T)$. Testing with ψ_o in (3) produces zeroes in all spatial integrals due to the evenness of w and also in the temporal integral since $\psi_o(0, \cdot) \equiv 0$ due to oddness. Testing with ψ_e in (3) produces twice the spatial integrals appearing in (19). In the

following, we concentrate on finding solutions of (18) for the linear wave equation with nonlinear Neumann boundary conditions.

Motivated by the linear wave equation in (18), we make the ansatz that

$$w(x,t) = \sum_{k \in \mathbb{Z}_{odd}} \frac{\hat{\alpha}_k}{k} \Phi_k(|x|) e_k(t), \tag{20}$$

where $e_k(t) := \frac{1}{\sqrt{T}} e^{i\omega k t}$ denotes the $L^2(\mathbb{T}_T)$-orthonormal Fourier base of $\mathbb{T}_T$, and where Φ_k are the decaying fundamental solutions Φ_k of L_k (cf. Lemma 1). Such a function w will always solve the linear wave equation in (18) and we will determine real sequences $\hat{\alpha} = (\hat{\alpha}_k)_{k \in \mathbb{Z}_{odd}}$ such that the nonlinear Neumann condition is satisfied as well. The additional factor $\frac{1}{k}$ is only for convenience, because ∂_t generates a multiplicative factor $i\omega k$.

The convolution between two sequences $\hat{z}, \hat{y} \in \mathbb{R}^{\mathbb{Z}}$ is defined pointwise (whenever it converges) by $(\hat{z} * \hat{y})_k := \sum_{l \in \mathbb{Z}} \hat{z}_l \hat{y}_{k-l}$.

To obtain real-valued functions w by the ansatz (20), we require the sequence $\hat{\alpha}$ to be real and odd in k, that is, $\hat{\alpha}_k \in \mathbb{R}$ and $\hat{\alpha}_k = -\hat{\alpha}_{-k}$. As (20) already solves the wave equation in (18), it remains to find $\hat{\alpha}$ such that

$$2w_x(0_+, t) = 2 \sum_{k \in \mathbb{Z}_{odd}} \frac{\hat{\alpha}_k}{k} \Phi'_k(0) e_k(t) = \frac{1}{T} \sum_{k \in \mathbb{Z}_{odd}} \gamma \omega^4 k (\hat{\alpha} * \hat{\alpha} * \hat{\alpha})_k e_k(t) = \gamma (w_t(0,t)^3)_t, \tag{21}$$

where we have used $\Phi_k(0) = 1$. As the above identity needs to hold for all $t \in \mathbb{R}$, we find

$$(\hat{\alpha} * \hat{\alpha} * \hat{\alpha})_k = \frac{2T \Phi'_k(0)}{\gamma \omega^4 k^2} \hat{\alpha}_k \quad \text{for all } k \in \mathbb{Z}_{odd}. \tag{22}$$

This will be accomplished by searching for critical points $\hat{\alpha}$ of the functional

$$J(\hat{z}) := \frac{1}{4} (\hat{z} * \hat{z} * \hat{z} * \hat{z})_0 + \frac{T}{\gamma \omega^4} \sum_k \frac{\Phi'_k(0)}{k^2} \hat{z}_k^2, \tag{23}$$

defined on a suitable Banach space of real sequences $\hat{z}$ with $\hat{z}_k = -\hat{z}_{-k}$. Indeed, computing (formally) the Fréchet derivative of J at $\hat{\alpha}$, we find

$$J'(\hat{\alpha})[\hat{y}] = (\hat{\alpha} * \hat{\alpha} * \hat{\alpha} * \hat{y})_0 + \frac{2T}{\gamma \omega^4} \sum_k \frac{\Phi'_k(0)}{k^2} \hat{\alpha}_k \hat{y}_k. \tag{24}$$

Let us indicate how (24) amounts to (22). For fixed $k_0 \in \mathbb{Z}_{odd}$, we define the test sequence $\hat{y} := (\delta_{k,k_0} - \delta_{k,-k_0})_{k \in \mathbb{Z}_{odd}}$ that has exactly two nonvanishing entries at k_0 and at $-k_0$. Thus, $\hat{y}$ belongs to the same space of odd, real sequences as $\hat{\alpha}$ and can therefore be used as a test sequence in $J'(\hat{\alpha})[\hat{y}] = 0$. After a short calculation using $\hat{\alpha}_k = -\hat{\alpha}_{-k}$, $\Phi'_k = \Phi'_{-k}$, we obtain (22) for k_0.

It turns out that a real Banach space of real-valued sequences that is suitable for J can be given by

$$\mathcal{D}(J) := \left\{ \hat{z} \in \mathbb{R}^{\mathbb{Z}_{odd}} \mid |||\hat{z}||| < \infty, \ \hat{z}_k = -\hat{z}_{-k} \right\} \quad \text{where } |||\hat{z}||| := \|\hat{z} * \hat{z}\|_{l^2}^{\frac{1}{2}}. \tag{25}$$

The relation between the function I defined in (17) and the new functional J is formally given by

$$I\left(\sum_{k\in\mathbb{Z}_{odd}}\frac{\hat{z}_k}{k}\Phi_k(|x|)e_k(t)\right) = -\frac{\gamma\omega^4}{T}J(\hat{z}). \tag{26}$$

Lemma 2. *The space $(\mathcal{D}(J),|||\cdot|||)$ is a separable, reflexive, real Banach space and isometrically embedded into the real Banach space $L^4(\mathbb{T}_T,i\mathbb{R})$ of purely imaginary-valued measurable functions. Moreover, for $\hat{u},\hat{v},\hat{w},\hat{z}\in\mathcal{D}(J)$, we have*

$$(\hat{u}*\hat{u}*\hat{u}*\hat{u})_0 = |||\hat{u}|||^4, \tag{27}$$

$$|(\hat{u}*\hat{v}*\hat{w}*\hat{z})_0| \le |||\hat{u}|||\,|||\hat{v}|||\,|||\hat{w}|||\,|||\hat{z}|||, \tag{28}$$

$$\|\hat{z}\|_{l^2} \le |||\hat{z}|||. \tag{29}$$

Proof. We first recall the correspondence between real-valued sequences $\hat{z}\in l^2$ with $\hat{z}_k = -\hat{z}_{-k}$ and purely imaginary-valued functions $z\in L^2(\mathbb{T}_T,i\mathbb{R})$ by setting

$$\hat{z}_k := \langle z,e_k\rangle_{L^2(\mathbb{T}_T)} \quad \text{and} \quad z(t) := \sum_{k\in\mathbb{Z}}\hat{z}_k e_k(t). \tag{30}$$

Parseval's identity provides the isomorphism $\|z\|_{L^2(\mathbb{T}_T)} = \|\hat{z}\|_{l^2}$. The following identity

$$T\|z\|_{L^4(\mathbb{T}_T)}^4 = T\int_0^T z(t)^4\,dt = (\hat{z}*\hat{z}*\hat{z}*\hat{z})_0 = \|\hat{z}*\hat{z}\|_{l^2}^2 = |||\hat{z}|||^4 \tag{31}$$

shows that $|||\cdot|||$ is indeed a norm on $\mathcal{D}(J)$ and it provides the isometric embedding of $\mathcal{D}(J)$ into a subspace of $L^4(\mathbb{T}_T,i\mathbb{R})$. By Parseval's equality and Hölder's inequality, we see that

$$\|\hat{z}\|_{l^2} = \|z\|_{L^2(\mathbb{T}_T)} \le T^{\frac{1}{4}}\|z\|_{L^4(\mathbb{T}_T)} = |||\hat{z}||| \tag{32}$$

so that $\mathcal{D}(J)$ is indeed a subspace of l^2. Finally, for any $\hat{u},\hat{v},\hat{w},\hat{z}\in\mathcal{D}(J)$, we see that

$$|(\hat{u}*\hat{v}*\hat{w}*\hat{z})_0| = T\left|\int_0^T u(t)v(t)w(t)z(t)\,dt\right| \le T\|u\|_{L^4}\|v\|_{L^4}\|w\|_{L^4}\|z\|_{L^4}$$

$$= |||\hat{u}|||\,|||\hat{v}|||\,|||\hat{w}|||\,|||\hat{z}|||. \tag{33}$$

This finishes the proof of the lemma. ∎

For $\frac{T}{2}$-antiperiodic functions $\psi : D\to\mathbb{R}$ of the space-time variable $(x,t)\in D$, we use the notation

$$\psi(x,t) = \sum_{k\in\mathbb{Z}_{odd}}\hat{\psi}_k(x)e_k(t) = \sum_{k\in\mathbb{Z}_{odd}}\frac{1}{k}\Psi_k(x)e_k(t) \tag{34}$$

with $\frac{1}{k}\Psi_k(x) = \hat{\psi}_k(x) := <\psi(x,\cdot), e_k>_{L^2(\mathbb{T}_T)}$. The Parseval identity and the definition of $|||\cdot|||$ immediately lead to the following lemma.

Lemma 3. *For $\psi : D \to \mathbb{R}$ as in (34) the following holds:*

(i) $\|\psi_x\|^2_{L^2(D)} = \sum_k \frac{1}{k^2}\|\Psi'_k\|^2_{L^2(\mathbb{R})}$,

(ii) $\|\psi_t\|^2_{L^2(D)} = \omega^2 \sum_k \|\Psi_k\|^2_{L^2(\mathbb{R})}$,

(iii) $T\|\psi_t(0,\cdot)\|^4_{L^4(\mathbb{T}_T)} = \omega^4 |||\hat{y}|||^4$ *where* $\hat{y}_k = \Psi_k(0)$ *for* $k \in \mathbb{Z}_{odd}$.

The next result gives some estimates on the growth of norms of Φ_k. It serves as a preparation for the proof of regularity properties for functions w as in (20) stated in Lemma 5.

Lemma 4. *Assume (C0), (C1), and (C2). Then*

$$\|\Phi_k\|_{L^2(0,\infty)} = O(1), \quad \|\Phi'_k\|_{L^2(0,\infty)} = O(k), \quad \|\Phi'_k\|_{L^\infty(0,\infty)} = O(k^{\frac{3}{2}}). \tag{35}$$

In particular, $|\Phi'_k(0)| = O(k^{\frac{3}{2}})$.

Proof. The first part of (35) is a direct consequence of (C2).

We multiply $L_k\Phi_k = 0$ first with Φ_k and then with Φ'_k, integrate both times from $a \geq 0$ to ∞ and, respectively, get

$$\int_a^\infty -\omega^2 k^2 g(x)\Phi_k(x)^2 + \Phi'_k(x)^2\,dx = -\Phi_k(a)\Phi'_k(a), \tag{36}$$

$$\int_a^\infty -2\omega^2 k^2 g(x)\Phi_k(x)\Phi'_k(x)\,dx = -\Phi'_k(a)^2. \tag{37}$$

Applying the Cauchy–Schwarz inequality to (37) and using the first part of (35), we find

$$\|\Phi'_k\|^2_{L^\infty(0,\infty)} \leq O(k^2)\|\Phi'_k\|_{L^2(0,\infty)} \tag{38}$$

and from (36) and (38), we get

$$\|\Phi'_k\|^2_{L^2(0,\infty)} \leq O(k^2) + \|\Phi_k\|_{L^\infty(0,\infty)}\|\Phi'_k\|_{L^\infty(0,\infty)} \tag{39}$$

$$\leq O(k^2) + \|\Phi_k\|_{L^\infty(0,\infty)}O(k)\|\Phi'_k\|^{\frac{1}{2}}_{L^2(0,\infty)}. \tag{40}$$

The L^∞-assumption in (C2) leads to

$$\|\Phi'_k\|^2_{L^2(0,\infty)} \leq O(k^2) + O(k)\|\Phi'_k\|^{\frac{1}{2}}_{L^2(0,\infty)} \leq O(k^2) + C_\varepsilon O(k^{\frac{4}{3}}) + \varepsilon\|\Phi'_k\|^2_{L^2(0,\infty)}, \tag{41}$$

where we have used Young's inequality with exponents 4/3 and 4. Taking, for example, $\epsilon = \frac{1}{2}$ this implies the second inequality in (35). Inserting this into (38), we obtain the third inequality in (35). ∎

Lemma 5. *Assume (C0), (C1), and (C2). For $\hat{\alpha} \in \mathcal{D}(J)$ and $w : D \to \mathbb{R}$ as in (20), we have $w_x, w_t \in L^2(D)$, $w_t(0, \cdot) \in L^4(\mathbb{T}_T)$ and there are values $C > 0$ and $\rho > 0$ such that $|w(x, t)| \le Ce^{-\rho|x|}$.*

Remark 2. The lemma does not require $\hat{\alpha}$ to be a critical point of J. The smoothness and decay of w as in (20) is simply a consequence of $\hat{\alpha} \in \mathcal{D}(J)$ and (C2).

Proof. We use the characterization from Lemma 3 as well as (29) from Lemma 2. Let us begin with the estimate for $\|\partial_t w\|_{L^2(D)}$. By Lemma 4, we have $\sup_k \|\Phi_k\|_{L^2(0,\infty)} < \infty$ so that

$$\|\partial_t w\|_{L^2(D)}^2 = 2\omega^2 \sum_k \hat{\alpha}_k^2 \|\Phi_k\|_{L^2(0,\infty)}^2 \le 2\omega^2 \Big(\sup_k \|\Phi_k\|_{L^2(0,\infty)} \Big)^2 \|\hat{\alpha}\|_{l^2}^2 \qquad (42)$$

$$\le 2\omega^2 \Big(\sup_k \|\Phi_k\|_{L^2(0,\infty)} \Big)^2 |||\hat{\alpha}|||^2 < \infty, \qquad (43)$$

which finishes our first goal. Next, we estimate $\|\partial_x w\|_{L^2(D)}$. Here, we use again Lemma 4 to find

$$\|\partial_x w\|_{L^2(D)}^2 = 2 \sum_k \frac{\hat{\alpha}_k^2}{k^2} \|\Phi_k'\|_{L^2(0,\infty)}^2 \le C\|\hat{\alpha}\|_{l^2}^2 \le C|||\hat{\alpha}|||^2 < \infty, \qquad (44)$$

which finishes our second goal. Next, we show that $w_t(0, \cdot) \in L^4(\mathbb{T}_T)$. Using $\Phi_k(0) = 1$, we observe that

$$T\|w_t(0, \cdot)\|_{L^4(\mathbb{T}_T)}^4 = T \int_0^T \Big(\sum_{k \in \mathbb{Z}_{odd}} i\omega \hat{\alpha}_k \Phi_k(0) e_k(t) \Big)^4 dt \qquad (45)$$

$$= \omega^4 |||\hat{\alpha}|||^4 < \infty. \qquad (46)$$

Finally, we show the uniform-in-time exponential decay of w. By construction w is even in x, hence we only consider $x > 0$. By (C2), we see that

$$|w(x, t)| \le \sum_k \frac{|\hat{\alpha}_k|}{|k|} |\Phi_k(x)| = \sum_k \frac{|\hat{\alpha}_k|}{|k|} Ce^{-\rho x} \le \|\hat{\alpha}\|_{l^2} \Big(\sum_k \frac{1}{k^2} \Big)^{1/2} Ce^{-\rho x} \le \tilde{C}e^{-\rho x}, \qquad (47)$$

which finishes the proof of the lemma. ∎

In the following result, we will show that minimizers of J on $\mathcal{D}(J)$ exist, are solutions of (22), and indeed correspond to weak solutions of (1).

Theorem 5. *Assume (C0), (C1), and (C2). Then the functional J is well defined on its domain $\mathcal{D}(J)$, Fréchet-differentiable, bounded from below, and attains its negative minimum provided*

(i) $\gamma < 0$ and the sequence $(\Phi'_k(0))_{k \in \mathbb{N}_{odd}}$ has at least one positive element, or

(ii) $\gamma > 0$ and the sequence $(\Phi'_k(0))_{k \in \mathbb{N}_{odd}}$ has at least one negative element.

For every critical point $\hat{\alpha} \in \mathcal{D}(J)$, the corresponding function $w(x,t) := \sum_{k \in \mathbb{Z}_{odd}} \frac{\hat{\alpha}_k}{k} \Phi_k(|x|) e_k(t)$ is a nontrivial weak solution of (1).

Proof. For the existence of a minimizer, we refer to Ref. [11, Theorem 1.2, Chapter I] where a coercive, sequentially weakly lower semicontinuous functional on a reflexive Banach space is shown to have a minimizer. Note that $J(\hat{z}) = \frac{1}{4} |||\hat{z}|||^4 + J_1(\hat{z})$, where $J_1(\hat{z}) := \sum_k a_k \hat{z}_k^2$ with $a_k := \frac{T\Phi'_k(0)}{\gamma \omega^4 k^2}$. By Lemma 4, the sequence $(a_k)_k$ is converging to 0 as $|k| \to \infty$, so, in particular, it is bounded. Due to (29), one finds that J is well defined and continuous on $\mathcal{D}(J)$, and moreover, that for $\hat{z} \in \mathcal{D}(J)$

$$J(\hat{z}) \geq \frac{1}{4} |||\hat{z}|||^4 - \sup_k |a_k| \sum_k \hat{z}_k^2 \geq \frac{1}{4} |||\hat{z}|||^4 - \sup_k |a_k| |||\hat{z}|||^2. \tag{48}$$

This implies that J is coercive and bounded from below. The weak lower semicontinuity of J follows from the convexity and continuity of the map $\hat{z} \mapsto |||\hat{z}|||^4$ and the weak continuity of J_1. To see the latter take an arbitrary $\epsilon > 0$. Then there is $k_0 \in \mathbb{N}$ such that $|a_k| \leq \epsilon$ for $|k| > k_0$ and this implies the inequality

$$|J_1(\hat{z}) - J_1(\hat{y})| \leq \sup_k |a_k| \sum_{|k| \leq k_0} |\hat{z}_k^2 - \hat{y}_k^2| + \epsilon(\|\hat{z}\|_{l^2}^2 + \|\hat{y}\|_{l^2}^2) \quad \forall \hat{z}, \hat{y} \in \mathcal{D}(J). \tag{49}$$

Since $(\mathcal{D}(J), |||\cdot|||)$ continuously embeds into l^2 any weakly convergent sequence in $(\mathcal{D}(J), |||\cdot|||)$ also weakly converges in l^2 and in particular pointwise. This pointwise convergence together with the boundedness of the sequence and (49) yields the weak continuity of J_1 and thus the weak lower semicontinuity of J. As a consequence (cf. Theorem 1.2 in Ref. 11), we get the existence of a minimizer.

To check that the minimizer is nontrivial, it suffices to verify that J attains negative values. Here we distinguish between cases (i) and (ii) in the assumptions of the theorem. In case (i) when $\gamma < 0$, we find an index k_0 such that $\Phi'_{k_0}(0) > 0$. In case (ii) when $\gamma > 0$, we choose k_0 such that $\Phi'_{k_0}(0) < 0$. In both cases, we obtain that $\Phi'_{k_0}(0)/\gamma < 0$. If we set $\hat{y} := (\delta_{k,k_0} - \delta_{k,-k_0})_{k \in \mathbb{Z}_{odd}}$, then $\hat{y}$ has exactly two nonvanishing entries, namely, $+1$ at k_0 and -1 at $-k_0$. Hence, $\hat{y} \in \mathcal{D}(J)$. Using the property $\Phi'_{k_0} = \Phi'_{-k_0}$, we find for $t \in \mathbb{R}$

$$J(t\hat{y}) = t^4 \frac{1}{4} |||\hat{y}|||^4 + 2t^2 \frac{T\Phi'_{k_0}(0)}{\gamma \omega^4 k_0^2}, \tag{50}$$

which is negative by the choice of k_0, provided that $t > 0$ is sufficiently small. Thus, $\inf_{\mathcal{D}(J)} J < 0$ and every minimizer $\hat{\alpha}$ is nontrivial.

Next, we show for every critical point $\hat{\alpha}$ of J that $w(x,t) := \sum_{k \in \mathbb{Z}_{odd}} \frac{\hat{\alpha}_k}{k} \Phi_k(|x|) e_k(t)$ is a weak solution of (1). The regularity properties $w \in H^1(\mathbb{R} \times \mathbb{T}_T)$, $\partial_t w(0,\cdot) \in L^4(\mathbb{T}_T)$ and the exponential decay have already been shown in Lemma 5. The proof that $J \in C^1(\mathcal{D}(J), \mathbb{R})$ and that its Fréchet-derivative is given by (24) is straightforward from the definition (cf. Ref. [12, Chapter 1]). We skip it

and just give two comments. First, as $|||\hat{y}|||^4$ is a fourfold convolution and convolution is commutative, its derivative is four times a triple convolution that explains the first term of J'. Second, as the second term of J is a quadratic form, its derivative will be twice the corresponding (symmetric) bilinear form. We will show that (3) holds for any ψ as in (34) with even functions $\Psi_k \in H^1(\mathbb{R})$, $\Psi_k = -\Psi_{-k}$ such that $\psi_x, \psi_t \in L^2(D)$ and $\psi_t(0, \cdot) \in L^4(\mathbb{T}_T)$ as described in Lemma 3. We begin by deriving expressions and estimates for the functionals

$$H_1(\psi) := \int_D g(x) w_t \psi_t \, d(x,t), \quad H_2(\psi) := \int_D w_x \psi_x \, d(x,t), \quad H_3(\psi) := \int_0^T w_t(0,t)^3 \psi_t(0,t) \, dt. \tag{51}$$

In a first step, we assume that the sum in (34) is finite to justify the exchange of summation and integration in the following. Then, starting with H_1, we find

$$H_1(\psi) = -\omega^2 \int_D g(x) \sum_{k,l} \hat{\alpha}_k \Phi_k(|x|) \Psi_l(|x|) e_k(t) e_l(t) d(x,t) \tag{52}$$

$$= -2\omega^2 \sum_k \hat{\alpha}_k \int_0^\infty g(x) \Phi_k(x) \Psi_{-k}(x) dx \tag{53}$$

$$= 2\omega^2 \sum_k \hat{\alpha}_k \int_0^\infty g(x) \Phi_k(x) \Psi_k(x) dx, \tag{54}$$

$$|H_1(\psi)| \leq 2\omega^2 \|g\|_{L^\infty(\mathbb{R})} \left(\sum_k \hat{\alpha}_k^2 \|\Phi_k\|_{L^2(0,\infty)}^2 \right)^{\frac{1}{2}} \left(\sum_k \|\Psi_k\|_{L^2(0,\infty)}^2 \right)^{\frac{1}{2}} = \|g\|_{L^\infty(\mathbb{R})} \|w_t\|_{L^2(D)} \|\psi_t\|_{L^2(D)}, \tag{55}$$

and similarly for H_2, we find

$$H_2(\psi) = \int_D \sum_{k,l} \frac{\hat{\alpha}_k}{k} \Phi_k'(|x|) \frac{1}{l} \Psi_l'(|x|) e_k(t) e_l(t) d(x,t) \tag{56}$$

$$= 2 \sum_k \frac{\hat{\alpha}_k}{-k^2} \int_0^\infty \Phi_k'(x) \Psi_{-k}'(x) dx \tag{57}$$

$$= 2 \sum_k \frac{\hat{\alpha}_k}{k^2} \int_0^\infty \Phi_k'(x) \Psi_k'(x) dx \tag{58}$$

$$= 2\omega^2 \sum_k \hat{\alpha}_k \int_0^\infty g(x) \Phi_k(x) \Psi_k(x) dx - 2 \sum_k \frac{\hat{\alpha}_k}{k^2} \Phi_k'(0) \Psi_k(0), \tag{59}$$

where in the last equality we used $L_k \Phi_k = 0$ on $(0,\infty)$ and integration by parts. Thus we obtain

$$|H_2(\psi)| \leq 2 \left(\sum_k \frac{\hat{\alpha}_k^2}{k^2} \|\Phi_k'\|_{L^2(0,\infty)}^2 \right)^{\frac{1}{2}} \left(\sum_k \frac{1}{k^2} \|\Psi_k'\|_{L^2(0,\infty)}^2 \right)^{\frac{1}{2}} = \|w_x\|_{L^2(D)} \|\psi_x\|_{L^2(D)}.$$

Moreover, considering H_3 and setting $\hat{y}_k := \Psi_k(0)$ for $k \in \mathbb{Z}_{odd}$ one sees

$$H_3(\psi) = \omega^4 \int_0^T \left(\sum_k \hat{\alpha}_k e_k(t) \right)^3 \left(\sum_l \Psi_l(0) e_l(t) \right) \mathrm{d}t \tag{60}$$

$$= \frac{\omega^4}{T} (\hat{\alpha} * \hat{\alpha} * \hat{\alpha} * \hat{y})_0, \tag{61}$$

$$|H_3(\psi)| \leq \frac{\omega^4}{T} |||\hat{\alpha}|||^3 |||\hat{y}||| = \|w_t(0, \cdot)\|_{L^4(\mathbb{T}_T)}^3 \|\psi_t(0, \cdot)\|_{L^4(\mathbb{T}_T)}. \tag{62}$$

Note that the estimates on H_1, H_2, H_3 only depend on the $L^2(D)$-norm of w_t, w_x, ψ_t, ψ_x and the $L^4(\mathbb{T}_T)$-norm of $w_t(0, \cdot), \psi_t(0, \cdot)$. As finite sums are dense in the Banach-space Z of functions ψ of the form (34) with $\|\psi_t\|_{L^2(D)}, \|\psi_x\|_{L^2(D)}, \|\psi_t(0, \cdot)\|_{L^4(\mathbb{T}_T)} < \infty$, we see that H_1, H_2, and H_3 are bounded linear functionals on Z. For such ψ, we use the above formulas for H_1, H_2, H_3 and compute the linear combination

$$-H_1(\psi) + H_2(\psi) - \gamma H_3(\psi) = -2 \sum_k \frac{\hat{\alpha}_k}{k^2} \Phi_k'(0) \Psi_k(0) - \frac{\gamma \omega^4}{T} (\hat{\alpha} * \hat{\alpha} * \hat{\alpha} * \hat{y})_0 = 0 \tag{63}$$

due to the Euler–Lagrange equation for the functional J, that is, the vanishing of $J'(\hat{\alpha})[\hat{y}]$ in (24) for all $\hat{y} \in \mathcal{D}(J)$. The last equality implies that w is a weak solution of (1). ∎

3 | FURTHER REGULARITY

Here we prove Theorem 4. We observe first that in the example of a periodic step potential in Theorem 2, we find that not only $\Phi_k'(0) = O(k^{\frac{3}{2}})$ holds (as Lemma 4 shows) but even $\Phi_k'(0) = O(k)$ is satisfied. It is exactly this weaker growth that we can exploit to prove additional smoothness of the solutions of (1). We begin by defining for $\nu > 0$ the Banach space of sequences

$$h^\nu := \left\{ \hat{z} \in l^2 \text{ s.t. } \|\hat{z}\|_{h^\nu}^2 := \sum_k (1 + k^2)^\nu |\hat{z}_k|^2 < \infty \right\}. \tag{64}$$

Moreover, we use the isometric isomorphism between h^ν and

$$H^\nu(\mathbb{T}_T) = \left\{ z(t) = \sum_k \hat{z}_k e_k(t) \text{ s.t. } \hat{z} \in h^\nu \right\} \tag{65}$$

by setting $\|z\|_{H^\nu} := \|\hat{z}\|_{h^\nu}$. We also use the Morrey embedding $H^{1+\nu}(\mathbb{T}_T) \hookrightarrow C^{0,\frac{1}{2}+\nu}(\mathbb{T}_T)$ for $\nu \in (0, 1/2)$ and the following embedding: $C^{0,\nu}(\mathbb{T}_T) \hookrightarrow H^{\tilde{\nu}}(\mathbb{T}_T)$ for $0 < \tilde{\nu} < \nu \leq 1$, cf. Lemma A1 in the Appendix. The latter embedding means that $\hat{z} \in h^{\tilde{\nu}}$ provided $z \in C^{0,\nu}(\mathbb{T}_T)$ and $0 < \tilde{\nu} < \nu \leq 1$.

Theorem 6. *Assume (C0), (C1), and (C2) and in addition $\Phi_k'(0) = O(k)$. For every $\hat{\alpha} \in \mathcal{D}(J)$ with $J'(\hat{\alpha}) = 0$, we have $\hat{\alpha} \in h^\nu$ for every $\nu \in (0, 1/4)$.*

Proof. Let $\hat{\alpha} \in \mathcal{D}(J)$ with $J'(\hat{\alpha}) = 0$. Recall from (22) that

$$(\hat{\alpha} * \hat{\alpha} * \hat{\alpha})_k = \hat{\eta}_k \hat{\alpha}_k \quad \text{where} \quad \hat{\eta}_k := \frac{2T\Phi'_k(0)}{\gamma \omega^4 k^2} \text{ for } k \in \mathbb{Z}_{odd} \tag{66}$$

so that $|\hat{\eta}_k| \le C/k$. If we define the convolution of two T-periodic functions $f, g \in L^2(\mathbb{T}_T)$ on the torus $\mathbb{T}_T$ as

$$(f * g)(t) := \frac{1}{\sqrt{T}} \int_0^T f(s)g(t-s)\mathrm{d}s, \tag{67}$$

and if we set

$$\alpha(t) := \sum_k \hat{\alpha}_k e_k(t), \quad \eta(t) := \sum_k \hat{\eta}_k e_k(t), \tag{68}$$

then the equation

$$\alpha^3 = \alpha * \eta \tag{69}$$

for the T-periodic function $\alpha \in L^4(\mathbb{T}_T)$ is equivalent to Equation (66) for the sequence $\hat{\alpha} \in \mathcal{D}(J)$. We will analyze (69) with a bootstrap argument.

Step 1: We show that $\alpha \in C^{0,\frac{1}{6}}(\mathbb{T}_T)$. The right-hand side of (69) is an $H^1(\mathbb{T}_T)$-function since

$$\|\alpha * \eta\|^2_{H^1(\mathbb{T}_T)} = \|\hat{\alpha}\hat{\eta}\|^2_{h^1} \le \sum_{k \in \mathbb{Z}_{odd}} (1 + k^2)\hat{\alpha}_k^2 \frac{C^2}{k^2} \le 2C^2 \|\hat{\alpha}\|^2_{l^2} < \infty. \tag{70}$$

Therefore, using (69), we see that $\alpha^3 \in H^1(\mathbb{T}_T)$ and by the Morrey embedding that $\alpha^3 \in C^{0,\frac{1}{2}}(\mathbb{T}_T)$. As the inverse of the mapping $x \mapsto x^3$ is given by $x \mapsto |x|^{-\frac{2}{3}}x$, which is a $C^{0,\frac{1}{3}}(\mathbb{R})$-function, we obtain $\alpha \in C^{0,\frac{1}{6}}(\mathbb{T}_T)$.

Step 2: We fix $q \in (0, 1)$ and show that if $\alpha \in C^{0,\nu_n}(\mathbb{T}_T)$ for some $\nu_n \in (0, 1/2)$ solves (69), then $\alpha \in C^{0,\nu_{n+1}}(\mathbb{T}_T)$ with $\nu_{n+1} = \frac{q\nu_n}{3} + \frac{1}{6}$. For the proof, we iterate the process from Step 1 and we start with $\alpha \in C^{0,\nu_n}(\mathbb{T}_T)$. Then, according to Lemma A1 of the Appendix, $\alpha \in H^{q\nu_n}(\mathbb{T}_T)$ and hence $\hat{\alpha} \in h^{q\nu_n}$. Then as before the convolution of α with η generates one more weak derivative, namely,

$$\|\alpha * \eta\|^2_{H^{1+q\nu_n}(\mathbb{T}_T)} = \|\hat{\alpha}\hat{\eta}\|^2_{h^{1+q\nu_n}} \le \sum_k (1 + k^2)^{1+q\nu_n} \hat{\alpha}_k^2 \frac{C^2}{k^2} \le C^2 \|\hat{\alpha}\|_{h^{q\nu_n}} < \infty. \tag{71}$$

Hence, by (69), we conclude $\alpha^3 \in H^{1+q\nu_n}(\mathbb{T}_T)$ and by the Morrey embedding $\alpha^3 \in C^{0,\frac{1}{2}+q\nu_n}(\mathbb{T}_T)$ provided $q\nu_n \in (0, 1/2)$. As in Step 1, this implies $\alpha \in C^{0,\nu_{n+1}}(\mathbb{T}_T)$ with $\nu_{n+1} = \frac{1}{6} + \frac{q\nu_n}{3}$.

Starting with $\nu_1 = 1/6$ from Step 1, we see by Step 2 that $\nu_n \nearrow \frac{1}{2(3-q)}$. As $q \in (0, 1)$ can be chosen arbitrarily close to 1, this finishes the proof. ∎

With this preparation, the proof of Theorem 4 is now immediate.

Proof of Theorem 4. Let $w(x,t) = \sum_{k\in\mathbb{Z}_{odd}} \frac{\hat{\alpha}_k}{k}\Phi_k(|x|)e_k(t)$ with $\hat{\alpha} \in \mathcal{D}(J)$ such that $J'(\hat{\alpha}) = 0$. Recall from assumption (C2) that $C := \sup_k \|\Phi_k\|^2_{L^2(0,\infty)} < \infty$. Likewise, from Lemma 4, we have $\|\Phi'_k\|^2_{L^2(0,\infty)} \leq \tilde{C}k^2$ for all $k \in \mathbb{Z}_{odd}$ and some $\tilde{C} > 0$. Therefore, using Theorem 6, we find for all $\nu < \frac{1}{4}$

$$\|\partial_t^{1+\nu}w\|^2_{L^2(D)} = 2\omega^{2+2\nu}\sum_k \hat{\alpha}_k^2 |k|^{2\nu}\|\Phi_k\|^2_{L^2(0,\infty)} \leq 2\omega^{2+2\nu}C\|\hat{\alpha}\|^2_{h^\nu} < \infty \tag{72}$$

and likewise

$$\|\partial_t^\nu w_x\|^2_{L^2(D)} = 2\omega^{2\nu}\sum_k \hat{\alpha}_k^2 |k|^{2\nu-2}\|\Phi'_k\|^2_{L^2(0,\infty)} \leq 2\omega^{2\nu}\tilde{C}\|\hat{\alpha}\|^2_{h^\nu} < \infty. \tag{73}$$

This establishes the claim. ■

4 | EXISTENCE OF INFINITELY MANY BREATHERS

In this section, we extend Theorem 3 by the following multiplicity result.

Theorem 7. *Assume (C0), (C1), and (C2). Then (1) has infinitely many nontrivial, T-periodic weak solutions w in the sense of Definition 1 with $T = \frac{2\pi}{\omega}$, provided that*

(i) $\gamma < 0$ and there exists an integer $l_- \in \mathbb{N}_{odd}$ such that for infinitely many $j \in \mathbb{N}$, the sequence $(\Phi'_{m\cdot l_-^j}(0))_{m\in\mathbb{N}_{odd}}$ has at least one positive element,
(ii) $\gamma > 0$ and there exists an integer $l_+ \in \mathbb{N}_{odd}$ such that for infinitely many $j \in \mathbb{N}$, the sequence $(\Phi'_{m\cdot l_+^j}(0))_{m\in\mathbb{N}_{odd}}$ has at least one negative element.

Remark 3. In the above theorem, conditions (C1) and (C2) can be weakened: instead of requiring them for all $k \in \mathbb{N}_{odd}$, it suffices to require them for $k \in l_-^j\mathbb{N}_{odd}$, $k \in l_+^j\mathbb{N}_{odd}$, respectively. We prove this observation together with the one in Remark 1 at the end of this section.

We start with an investigation about the types of symmetries that are compatible with our equation. The Euler–Lagrange equation (22) for critical points $\hat{\alpha} \in \mathcal{D}(J)$ of J takes the form $(\hat{\alpha} * \hat{\alpha} * \hat{\alpha})_k = \hat{\eta}_k\hat{\alpha}_k$ with $\hat{\eta}_k := \frac{2T\Phi'_k(0)}{\gamma\omega^4 k^2}$ for $k \in \mathbb{Z}_{odd}$. Next, we describe subspaces of $\mathcal{D}(J)$ that are invariant under triple convolution and pointwise multiplication with $(\hat{\eta}_k)_{k\in\mathbb{Z}_{odd}}$. It turns out that these subspaces are made up of sequences $\hat{z}$ where only the rth entry modulus $2r$ is occupied.

Definition 2. For $r \in \mathbb{N}_{odd}, p \in \mathbb{N}_{even}$ with $r < p$, let

$$\mathcal{D}(J)_{r,p} = \{\hat{z} \in \mathcal{D}(J) : \forall k \in \mathbb{Z}, k \neq r \bmod p : \hat{z}_k = 0\}. \tag{74}$$

Lemma 6. *For $r \in \mathbb{N}_{odd}, p \in \mathbb{N}_{even}$ with $r < p$ and $p \neq 2r$, we have $\mathcal{D}(J)_{r,p} = \{0\}$.*

Proof. Let $\hat{z} \in \mathcal{D}(J)_{r,p}$. For all $k \notin r + p\mathbb{Z}$, we have $\hat{z}_k = 0$ by definition of $\mathcal{D}(J)_{r,p}$. Let therefore $k = r + pl_1$ for some $l_1 \in \mathbb{Z}$. Then $-k = -r - pl_1 \notin r + p\mathbb{Z}$ because otherwise $2r = -p(l_1 + l_2) = p|l_1 + l_2|$ for some $l_2 \in \mathbb{Z}$. Since by assumption $p > r$, we get $|l_1 + l_2| < 2$. But clearly $|l_1 + l_2| \notin \{0, 1\}$ since $r \neq 0$ and $p \neq 2r$ by assumption. By this contradiction, we have shown $-k \notin r + p\mathbb{Z}$ so that necessarily $0 = \hat{z}_{-k} = -\hat{z}_k$. This shows $\hat{z} = 0$. ∎

In the following, we continue by only considering $\mathcal{D}_r := \mathcal{D}(J)_{r,2r}$ for $r \in \mathbb{N}_{odd}$.

Proposition 1. *Let $r \in \mathbb{N}_{odd}$.*

(i) *The elements $\hat{z} \in \mathcal{D}_r$ are exactly those elements of $\mathcal{D}(J)$ that generate $\frac{T}{2r}$-antiperiodic functions*
   $\sum_{k \in \mathbb{Z}_{odd}} \frac{\hat{z}_k}{k} \Phi_k(x) e_k(t).$

(ii) *If $\hat{z} \in \mathcal{D}_r$, then $(\hat{z} * \hat{z} * \hat{z})_k = 0$ for all $k \notin r + 2r\mathbb{Z}$.*

Proof.

(i) An element $\hat{z} \in \mathcal{D}(J)$ generates a $\frac{T}{2r}$-antiperiodic function $z(x, t) = \sum_{k \in \mathbb{Z}_{odd}} \frac{\hat{z}_k}{k} \Phi_k(x) e_k(t)$ if and only if $z(x, t + \frac{T}{2r}) = -z(x, t)$. Comparing the Fourier coefficients, we see that this is the case if for all $k \in \mathbb{Z}_{odd}$ we have $\hat{z}_k \left(\exp(\frac{i\omega kT}{2r}) + 1 \right) = 0$, that is, either $k \in r + 2r\mathbb{Z}$ or $\hat{z}_k = 0$. This is exactly the condition that $\hat{z} \in \mathcal{D}_r$.

(ii) Let $\hat{z} \in \mathcal{D}_r$ and assume that there is $k \in \mathbb{Z}$ such that $0 \neq (\hat{z} * \hat{z} * \hat{z})_k = \sum_{l,m} \hat{z}_l \hat{z}_{m-l} \hat{z}_{k-m}$. So, there is $l_0, m_0 \in \mathbb{Z}_{odd}$ such that $\hat{z}_{l_0}, \hat{z}_{m_0-l_0}, \hat{z}_{k-m_0} \neq 0$, which means by the definition of $\mathcal{D}_r$ that $l_0, m_0 - l_0, k - m_0 \in r + 2r\mathbb{Z}$. Thus, $k = l_0 + m_0 - l_0 + k - m_0 \in 3r + 2r\mathbb{Z} = r + 2r\mathbb{Z}$.

∎

Proof of Theorem 7. We give the proof in case (i); for case (ii) the proof only needs a trivial modification. Let $r = l^j$ where j is an index such that the sequence $(\Phi'_{k \cdot l^j}(0))_{k \in \mathbb{N}_{odd}}$ has a positive element (we have changed the notation from l_- to l for the sake of readability). As $\mathcal{D}_r$ is a closed subspace of $\mathcal{D}(J)$, we have as before in Theorem 5 the existence of a minimizer $\hat{\alpha}^{(r)} \in \mathcal{D}_r$, that is, $J(\hat{\alpha}^{(r)}) = \min_{\mathcal{D}_r} J < 0$. Moreover, $\hat{\alpha}^{(r)}$ satisfies the restricted Euler–Lagrange equation

$$0 = J'(\hat{\alpha}^{(r)})[\hat{x}] = (\hat{\alpha}^{(r)} * \hat{\alpha}^{(r)} * \hat{\alpha}^{(r)} * \hat{x})_0 + \frac{2T}{\gamma \omega^4} \sum_k \frac{\Phi'_k(0)}{k^2} \hat{\alpha}_k^{(r)} \hat{x}_k \qquad \forall \hat{x} \in \mathcal{D}_r. \qquad (75)$$

We need to show that (75) holds for every $\hat{z} \in \mathcal{D}(J)$. If for an arbitrary $\hat{z} \in \mathcal{D}(J)$, we define $\hat{x}_k := \hat{z}_k$ for $k \in r + 2r\mathbb{Z}$ and $\hat{x}_k := 0$ else then $\hat{x} \in \mathcal{D}_r$. If we furthermore define $\hat{y} := \hat{z} - \hat{x}$, then $\hat{y}_k = 0$ for all $k \in r + 2r\mathbb{Z}$. This implies, in particular, that

$$\sum_k \frac{\Phi'_k(0)}{k^2} \hat{\alpha}_k^{(r)} \hat{y}_k = 0 \qquad (76)$$

and by using (ii) of Proposition 1 also

$$(\hat{\alpha} * \hat{\alpha} * \hat{\alpha} * \hat{y})_0 = \sum_k \left(\hat{\alpha}^{(r)} * \hat{\alpha}^{(r)} * \hat{\alpha}^{(r)} \right)_k \hat{y}_{-k} = 0. \tag{77}$$

This implies $J'(\hat{\alpha}^{(r)})[\hat{y}] = 0$ and since by (75) also $J'(\hat{\alpha}^{(r)})[\hat{x}] = 0$, we have succeeded in proving that $J'(\hat{\alpha}^{(r)}) = 0$.

It remains to show the multiplicity result. For this purpose, we only consider $r = l^{j_m}$ for $j_m \to \infty$ as $m \to \infty$ where j_m is an index such that the sequence $(\Phi'_{l^{j_m}k}(0))_{k \in \mathbb{N}_{odd}}$ has a positive element. First, we observe that $\mathcal{D}_{l^{j_m}} \supsetneq \mathcal{D}_{l^{j_{m+1}}}$. Assume for contradiction that the set $\{\hat{\alpha}^{(l^{j_m})}\}$ is finite. Then we have a subsequence $(j_{m_n})_{n \in \mathbb{N}}$ such that $\hat{\alpha} = \hat{\alpha}^{(l^{j_{m_n}})}$ is constant. But then

$$\hat{\alpha} \in \bigcap_{n \in \mathbb{N}} \mathcal{D}_{l^{j_{m_n}}} = \bigcap_{j \in \mathbb{N}} \mathcal{D}_{l^j} = \{0\}. \tag{78}$$

This contradiction shows the existence of infinitely many distinct critical points of the function J and finishes the proof of the theorem. ∎

Proof of Remark 1 *and Remark* 3. The proof of Theorem 7 works on the basis that it suffices to minimize the functional J on $\mathcal{D}_r$. In this way, a $\frac{T}{2r}$-antiperiodic breather is obtained. For $\hat{z} \in \mathcal{D}_r$, only the entries $\hat{z}_k$ with $k \in r\mathbb{Z}_{odd}$ are nontrivial, whereas all other entries vanish. Therefore, (C1) and (C2) and the values of $\Phi'_k(0)$ are only relevant for $k \in r\mathbb{Z}_{odd}$. In the special case of Remark 3, we take $r = l^j_{\pm}$. ∎

5 | APPROXIMATION BY FINITELY MANY HARMONICS

Here we give some analytical results on finite-dimensional approximation of the breathers obtained in Theorem 3. The finite-dimensional approximation is obtained by cutting off the ansatz (20) and only considering harmonics of order $|k| \leq N$. Here a summand in the series (20) of the form $\Phi_k(|x|)e_k(t)$ is a called a harmonic because it satisfies the linear wave equation in (18). We will prove that J restricted to spaces $\mathcal{D}(J^{(N)})$ of cutoff ansatz functions still attains its minimum and that the sequence of the corresponding minimizers converges up to a subsequence to a minimizer of J on $\mathcal{D}(J)$.

Definition 3. Let $N \in \mathbb{N}_{odd}$. Define

$$J^{(N)} := J|_{\mathcal{D}(J^{(N)})}, \qquad \mathcal{D}(J^{(N)}) := \left\{ \hat{z} \in \mathcal{D}(J) \,\big|\, \forall |k| > N : \hat{z}_k = 0 \right\}. \tag{79}$$

Lemma 7. *Under the assumptions of Theorem 3, the following holds:*

(i) *For every $N \in \mathbb{N}_{odd}$ sufficiently large, there exists $\hat{\alpha}^{(N)} \in \mathcal{D}(J^{(N)})$ such that $J(\hat{\alpha}^{(N)}) = \inf J^{(N)} < 0$ and $\lim_{N \to \infty} J(\hat{\alpha}^{(N)}) = \inf J$.*

(ii) *There is $\hat{\alpha} \in \mathcal{D}(J)$ such that up to a subsequence (again denoted by $(\hat{\alpha}^{(N)})_N$), we have*

$$\hat{\alpha}^{(N)} \to \hat{\alpha} \qquad in \ \mathcal{D}(J) \tag{80}$$

and $J(\hat{\alpha}) = \inf J$.

Remark 4. The Euler–Lagrange equation for $\hat{\alpha}^{(N)}$ reads:

$$0 = J'\big(\hat{\alpha}^{(N)}\big)[\hat{y}] = \big(\hat{\alpha}^{(N)} * \hat{\alpha}^{(N)} * \hat{\alpha}^{(N)} * \hat{y}\big)_0 + \frac{2T}{\gamma\omega^4} \sum_k \frac{\Phi'_k(0)}{k^2} \hat{\alpha}_k^{(N)} \hat{y}_k \qquad \forall \hat{y} \in \mathcal{D}(J^{(N)}). \tag{81}$$

This amounts to satisfying (3) in Definition 1 for functions $\psi(x,t) = \sum_{k \in \mathbb{Z}_{odd}, |k| \leq N} \hat{\psi}_k(x) e_k(t)$ with $\hat{\psi}_k \in H^1(\mathbb{R})$. Clearly, in general, $\hat{\alpha}^{(N)}$ is not a critical point of J.

Proof.

(i) We choose $N \in \mathbb{N}_{odd}$ so large, such that we have the assumed sign of the one element in $(\Phi'_k(0))_{|k| \leq N}$. The restriction of J to the $\frac{N+1}{2}$-dimensional space $\mathcal{D}(J^{(N)})$ preserves coercivity. The continuity of $J^{(N)}$ therefore guarantees the existence of a minimizer $\hat{\alpha}^{(N)} \in \mathcal{D}(J^{(N)})$. As before we see that $J(\hat{\alpha}^{(N)}) = \inf J^{(N)} < 0$, so in particular $\hat{\alpha}^{(N)} \neq 0$. Next, we observe that $\mathcal{D}(J^{(N)}) \subset \mathcal{D}(J)$, that is, $J(\hat{\alpha}^{(N)}) \geq \inf J = J(\hat{\beta})$ for a minimizer $\hat{\beta} \in \mathcal{D}(J)$ of J. Let us define $\hat{\beta}_k^{(N)} = \hat{\beta}_k$ for $|k| \leq N$ and $\hat{\beta}_k^{(N)} = 0$. As the Fourier series $\beta(t) = \sum_k \hat{\beta}_k e_k(t)$ converges in $L^4(\mathbb{T})$, cf. Theorem 4.1.8 in Ref. 13, we see that $\hat{\beta}^{(N)} \to \hat{\beta}$ in $\mathcal{D}(J)$. By the minimality of $\hat{\alpha}^{(N)} \in \mathcal{D}(J^{(N)})$ and continuity of J, we conclude

$$\inf_{\mathcal{D}(J)} J \leq J(\hat{\alpha}^{(N)}) \leq J(\hat{\beta}^{(N)}) \longrightarrow J(\hat{\beta}) = \inf_{\mathcal{D}(J)} J. \tag{82}$$

Hence $\lim_{N \to \infty} J(\hat{\alpha}^{(N)}) = \inf J$ as claimed.

(ii) Since $\mathcal{D}(J^{(N)}) \subset \mathcal{D}(J^{(N+1)}) \subset \mathcal{D}(J)$, we see that $J(\hat{\alpha}^{(N)}) \geq J(\hat{\alpha}^{(N+1)}) \geq \inf J$ so that in particular the sequence $(J(\hat{\alpha}^{(N)}))_N$ is bounded. By coercivity of J, we conclude that $(\hat{\alpha}^{(N)})_N$ is bounded in $\mathcal{D}(J)$ so that there is $\hat{\alpha} \in \mathcal{D}(J)$ and a subsequence (again denoted by $(\hat{\alpha}^{(N)})_N$) such that

$$\hat{\alpha}^{(N)} \rightharpoonup \hat{\alpha} \qquad in \ \mathcal{D}(J). \tag{83}$$

By part (i) and weak lower semicontinuity of J, we obtain

$$\inf J = \lim_{N \to \infty} J(\hat{\alpha}^{(N)}) \geq J(\hat{\alpha}), \tag{84}$$

that is, $\hat{\alpha}$ is a minimizer of J. Recall that $J(\cdot) = \frac{1}{4}|||\cdot|||^4 + J_1(\cdot)$ where J_1 is weakly continuous, cf. proof of Theorem 5. Therefore, since $\hat{\alpha}^{(N)} \rightharpoonup \hat{\alpha}$ and $J(\hat{\alpha}^{(N)}) \to J(\hat{\alpha})$, we see that $|||\hat{\alpha}^{(N)}||| \to |||\hat{\alpha}|||$

as $N \to \infty$. As $\mathcal{D}(J)$ is strictly uniformly convex, we obtain the norm-convergence of $(\hat{\alpha}^{(N)})_N$ to $\hat{\alpha}$. ∎

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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APPENDIX A

A.1 | Details on exponentially decreasing fundamental solutions for step potentials

Here we consider a second-order ordinary differential operator

$$L_k := -\frac{d^2}{dx^2} - k^2\omega^2 g(x) \tag{A1}$$

with g as in Theorem 1. Clearly, L_k is a self-adjoint operator on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. Moreover, $\sigma_{ess}(L_k) = [k^2\omega^2 a, \infty)$. By the assumption on ω, we have

$$\sqrt{b}\omega c\frac{2}{\pi} = \frac{p}{q} \quad \text{with } p, q \in \mathbb{N}_{odd}. \tag{A2}$$

Hence, with $k \in q\mathbb{N}_{odd}$, $k\sqrt{b}\omega c$ is an odd multiple of $\pi/2$. In the following, we see that 0 is not an eigenvalue of L_k for $k \in q\mathbb{N}_{odd}$ so that (C1) as in Remark 1 is fulfilled. A potential eigenfunction ϕ_k for the eigenvalue 0 would have to look like

$$\phi_k(x) = \begin{cases} -A\sin(k\omega\sqrt{b}c)e^{k\omega\sqrt{a}(x+c)}, & x < -c, \\ A\sin(k\omega\sqrt{b}x) + B\cos(k\omega\sqrt{b}x), & -c < x < c, \\ A\sin(k\omega\sqrt{b}c)e^{-k\omega\sqrt{a}(x-c)}, & c < x, \end{cases} \tag{A3}$$

with $A, B \in \mathbb{R}$ to be determined. Note that we have used $\cos(k\omega\sqrt{b}c) = 0$. The C^1-matching of ϕ_k at $x = \pm c$ (which is due to the embedding $H^2(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$) leads to the two equations

$$-Bk\omega\sqrt{b}\sin(k\omega\sqrt{b}c) = -Ak\omega\sqrt{a}\sin(k\omega\sqrt{b}c), \tag{A4}$$

$$Bk\omega\sqrt{b}\sin(k\omega\sqrt{b}c) = -Ak\omega\sqrt{a}\sin(k\omega\sqrt{b}c), \tag{A5}$$

and since $\sin(k\omega\sqrt{b}c) = \pm1$, this implies $A = B = 0$ so that there is no eigenvalue 0 of L_k. Next we need to find the fundamental solution ϕ_k of L_k that decays to zero at $+\infty$ and is normalized by $\phi_k(0) = 1$. Here we can use the same ansatz as in (A3) and just ignore the part of ϕ_k on $(-\infty, 0)$. Now the normalization $\phi_k(0) = 1$ leads to $B = 1$ and the C^1-matching at $x = c$ leads to $A = \sqrt{\frac{b}{a}}B = \sqrt{\frac{b}{a}}$ so that the decaying fundamental solution is completely determined. We find

that

$$|\phi_k(x)| \leq \begin{cases} A + B, & 0 \leq x \leq c \\ A, & c < x \leq 2c \\ Ae^{-\frac{1}{2}k\omega\sqrt{a}x}, & x > 2c \end{cases} \tag{A6}$$

so that $|\phi_k(x)| \leq (A + B)e^{-\rho_k x} \leq Me^{-\rho x}$ on $[0, \infty)$ with $\rho_k = \frac{1}{2}k\omega\sqrt{a}$, $\rho = \frac{1}{2}\omega\sqrt{a}$ and $M = A + B$. This shows that (C2) also holds. Finally, since $\phi_k'(0) = \frac{bk\omega}{\sqrt{a}} > 0$, the existence of infinitely many breathers can only be shown for $\gamma < 0$. At the same time, due to $|\phi_k'(0)| = O(k)$, Theorem 4 applies.

A.2 | Details on Bloch modes for periodic step potentials

Here we consider a second-order periodic ordinary differential operator

$$L := -\frac{d^2}{dx^2} + V(x) \tag{A7}$$

with $V \in L^\infty(\mathbb{R})$ which we assume to be even and 2π-periodic. Moreover, we assume that 0 does not belong to the spectrum of $L : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$. We first describe what Bloch modes are and why they exist. Later we show that the specific periodic potential from Theorem 2 meets the assumptions of Theorem 3 , that is, we verify conditions (C1) and (C2).

A function $\Phi \in C^1(\mathbb{R})$ that is twice almost everywhere differentiable such that

$$L\Phi = 0 \quad \text{a.e. in } \mathbb{R}, \qquad \Phi(\cdot + 2\pi) = \rho\Phi(\cdot), \tag{A8}$$

where $\rho \in (-1, 1) \setminus \{0\}$ is called the (exponentially decreasing for $x \to +\infty$) Bloch mode of L and ρ is called the Floquet multiplier. The existence of Φ is guaranteed by the assumption that $0 \notin \sigma(L)$. This is essentially Hill's theorem, cf. [14]. Note that $\Psi(x) := \Phi(-x)$ is a second Bloch mode of L, which is exponentially increasing for $x \to +\infty$. The functions Φ and Ψ form a fundamental system of solutions for the operator L on $\mathbb{R}$. Next we explain how Φ is constructed, why it can be taken real-valued and why it does not vanish at $x = 0$ so that we can assume w.l.o.g. $\Phi(0) = 1$.

According to Ref. [14, Theorem 1.1.1], there are linearly independent functions $\Psi_1, \Psi_2 : \mathbb{R} \to \mathbb{C}$ and Floquet-multipliers $\rho_1, \rho_2 \in \mathbb{C}$ such that $L\Psi_j = 0$ a.e. on $\mathbb{R}$ and $\Psi_j(\cdot + 2\pi) = \rho_j\Psi_j(\cdot)$ for $j = 1, 2$. We define ϕ_j, $j = 1, 2$ as the solutions to the initial value problems

$$\begin{cases} L\phi_1 = 0, \\ \phi_1(0) = 1, \quad \phi_1'(0) = 0, \end{cases} \quad \text{and} \quad \begin{cases} L\phi_2 = 0, \\ \phi_2(0) = 0, \quad \phi_2'(0) = 1, \end{cases} \tag{A9}$$

and consider the Wronskian

$$W(x) := \begin{pmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{pmatrix} \tag{A10}$$

and the monodromy matrix

$$A := W(2\pi) = \begin{pmatrix} \phi_1(2\pi) & \phi_2(2\pi) \\ \phi_1'(2\pi) & \phi_2'(2\pi) \end{pmatrix}. \tag{A11}$$

Then $\det A = 1$ is the Wronskian determinant of the fundamental system ϕ_1, ϕ_2 and the Floquet multipliers $\rho_{1,2} = \frac{1}{2}(\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4})$ are the eigenvalues of A with corresponding eigenvectors $v_1 = (v_{1,1}, v_{1,2}) \in \mathbb{C}^2$ and $v_2 = (v_{2,1}, v_{2,2}) \in \mathbb{C}^2$. Thus, $\Psi_j(x) = v_{j,1}\phi_1(x) + v_{j,2}\phi_2(x)$. By Hill's theorem (see Ref. 14), we know that

$$0 \in \sigma(L) \Leftrightarrow |\text{tr}(A)| \leq 2. \tag{A12}$$

Due to the assumption that $0 \notin \sigma(L)$, we see that ρ_1, ρ_2 are real with $\rho_1, \rho_2 \in \mathbb{R} \setminus \{-1, 0, 1\}$ and $\rho_1\rho_2 = 1$, that is, one of the two Floquet multipliers has modulus smaller than 1 and the other one has modulus bigger than 1. W.l.o.g. we assume $0 < |\rho_2| < 1 < |\rho_1|$. Furthermore, since ρ_1, ρ_2 are real and A has real entries, we can choose v_1, v_2 to be real and so Ψ_1, Ψ_2 are both real-valued. As a result, we have found a real-valued Bloch mode $\Psi_2(x)$ that is exponentially decreasing as $x \to +\infty$ due to $|\rho_2| < 1$. Let us finally verify that $\Psi_2(0) \neq 0$ so that we may assume by rescaling that $\Psi_2(0) = 1$. Assume for contradiction that $\Psi_2(0) = 0$. As the potential $V(x)$ is even in x, this implies that Ψ_2 is odd and hence (due to the exponential decay at $+\infty$) in $L^2(\mathbb{R})$. But this contradicts that $0 \notin \sigma(L)$.

Now we explain how the precise choice of the data $a, b > 0, \Theta \in (0, 1)$ and ω for the step-potential g in Theorem 2 allows to fulfill the conditions (C1) and (C2). Let us define

$$\tilde{g}(x) := \begin{cases} a, & x \in [0, 2\Theta\pi), \\ b, & x \in (2\Theta\pi, 2\pi), \end{cases} \tag{A13}$$

and extend $\tilde{g}$ as a 2π-periodic function to $\mathbb{R}$. Then $\tilde{g}(x) = g(x - \Theta\pi)$, and the corresponding exponentially decaying Bloch modes $\tilde{\phi}_k$ and ϕ_k are similarly related by $\tilde{\phi}_k(x) = \phi_k(x - \Theta\pi)$. For the computation of the exponentially decaying Bloch modes, it is, however, more convenient to use the definition $\tilde{g}$ instead of g.

Now we will calculate the monodromy matrix A_k from (A11) for the operator L_k. For a constant value $c > 0$, the solution of the initial value problem

$$-\phi''(x) - k^2\omega^2 c\phi(x) = 0, \quad \phi(x_0) = \alpha, \quad \phi'(x_0) = \beta \tag{A14}$$

is given by

$$\begin{pmatrix} \phi(x) \\ \phi'(x) \end{pmatrix} = T_k(x - x_0, c) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{A15}$$

with the propagation matrix

$$T_k(s, c) := \begin{pmatrix} \cos(k\omega\sqrt{c}s) & \frac{1}{k\omega\sqrt{c}}\sin(k\omega\sqrt{c}s) \\ -k\omega\sqrt{c}\sin(k\omega\sqrt{c}s) & \cos(k\omega\sqrt{c}s) \end{pmatrix}. \tag{A16}$$

Therefore, we can write the Wronskian as follows:

$$W_k(x) = \begin{cases} T_k(x, a) & x \in [0, 2\Theta\pi] \\ T_k(x - 2\Theta\pi, b)T_k(2\Theta\pi, a) & x \in [2\Theta\pi, 2\pi] \end{cases} \tag{A17}$$

and the monodromy matrix as

$$A_k = W_k(2\pi) = T_k(2\pi(1-\Theta), b)T_k(2\Theta\pi, a). \tag{A18}$$

To get the exact form of A_k, let us use the notation

$$l := \sqrt{\frac{b}{a}}\frac{1-\Theta}{\Theta}, \qquad m := 2\sqrt{a}\Theta\omega. \tag{A19}$$

Hence

$$A_k = \sin(kml\pi)\sin(km\pi)$$

$$\cdot \begin{pmatrix} \cot(kml\pi)\cot(km\pi) - \sqrt{\frac{a}{b}} & \frac{1}{k\omega\sqrt{a}}\cot(kml\pi) + \frac{1}{k\omega\sqrt{b}}\cot(km\pi) \\ -k\omega\sqrt{b}\cot(km\pi) - k\omega\sqrt{a}\cot(kml\pi) & -\sqrt{\frac{b}{a}} + \cot(kml\pi)\cot(km\pi) \end{pmatrix} \tag{A20}$$

and

$$\operatorname{tr}(A_k) = 2\cos(kml\pi)\cos(km\pi) - \left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right)\sin(kml\pi)\sin(km\pi). \tag{A21}$$

To verify (C1), we aim for $|\operatorname{tr}(A_k)| > 2$. However, instead of showing $|\operatorname{tr}(A_k)| > 2$ for all $k \in \mathbb{Z}_{odd}$, we may restrict to $k \in r \cdot \mathbb{Z}_{odd}$ for fixed $r \in \mathbb{N}_{odd}$ according to Remark 1. Next we will choose $r \in \mathbb{Z}_{odd}$. Due to the assumptions from Theorem 2, we have

$$l = \frac{\tilde{p}}{\tilde{q}}, \quad 2m = \frac{p}{q} \in \frac{\mathbb{N}_{odd}}{\mathbb{N}_{odd}}. \tag{A22}$$

Therefore, by setting[1] $r = \tilde{q}q$, we obtain $\cos(km\pi) = \cos(kml\pi) = 0$ and $\sin(km\pi), \sin(kml\pi) \in \{\pm 1\}$ for all $k \in r \cdot \mathbb{Z}_{odd}$. Together with $a \neq b$ this implies $|\operatorname{tr}(A_k)| = \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} > 2$ so that (C1) holds and A_k takes the simple diagonal form

$$A_k = \begin{pmatrix} -\sqrt{\frac{a}{b}}\sin(kml\pi)\sin(km\pi) & 0 \\ 0 & -\sqrt{\frac{b}{a}}\sin(kml\pi)\sin(km\pi) \end{pmatrix}. \tag{A23}$$

In the following, we assume w.l.o.g. $0 < a < b$, that is, the Floquet exponent with modulus less than 1 is $\rho_k = -\sqrt{\frac{a}{b}}\sin(kml\pi)\sin(km\pi)$. Note that $|\rho_k| = \sqrt{a/b}$ is independent of k. Furthermore, the Bloch mode $\tilde{\phi}_k$ that is decaying to 0 at $+\infty$ and normalized by $\tilde{\phi}_k(\Theta\pi) = 1$ is deduced

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[1] Instead of $r = \tilde{q}q$, we may have chosen any odd multiple of $\tilde{q}q$, for example, $r = (\tilde{q}q)^j$ for any $j \in \mathbb{N}$. This is important for the applicability of Theorem 7 to obtain infinitely many breathers.

from the upper left element of the Wronskian, that is,

$$\tilde{\phi}_k(x) = \frac{1}{\cos(k\omega\sqrt{a}\Theta\pi)} \begin{cases} \cos(k\omega\sqrt{a}x), & x \in (0, 2\Theta\pi), \\ \cos(k\omega\sqrt{b}(x - 2\Theta\pi))\cos(k\omega\sqrt{a}2\Theta\pi) \\ -\sqrt{\frac{a}{b}}\sin(k\omega\sqrt{b}(x - 2\Theta\pi))\sin(k\omega\sqrt{a}2\Theta\pi), & x \in (2\Theta\pi, 2\pi), \end{cases} \tag{A24}$$

and on shifted intervals of lengths 2π, one has $\tilde{\phi}_k(x + 2m\pi) = \rho_k^m \tilde{\phi}_k(x)$. Notice that by (A22), the expression $k\omega\sqrt{a}\Theta\pi = k\frac{p}{q}\frac{\pi}{4}$ is an odd multiple of $\pi/4$ since $k \in q\tilde{q}\mathbb{Z}_{odd}$ and hence $|\cos(k\omega\sqrt{a}\Theta\pi)| = 1/\sqrt{2}$. Therefore, $\|\phi_k\|_{L^\infty(0,\infty)} = \|\tilde{\phi}_k\|_{L^\infty(\Theta\pi,\infty)} \leq \|\tilde{\phi}_k\|_{L^\infty(0,2\pi)} \leq \sqrt{2}(1 + \sqrt{a/b})$. Thus, we have shown that $|\phi_k(x)| \leq Me^{-\rho x}$ for $x \in [0, \infty)$ with $M > 0$ and $\rho = \frac{1}{4\pi}(\ln b - \ln a) > 0$. Finally, let us compute

$$\phi_k'(0) = \tilde{\phi}_k'(\Theta\pi) = -k\omega\sqrt{a}\tan(k\omega\sqrt{a}\Theta\pi) \in \{\pm k\omega\sqrt{a}\}. \tag{A25}$$

This shows that $|\phi_k'(0)| = O(k)$ holds which allows to apply Theorem 4. It also shows that the estimate $|\phi_k'(0)| = O(k^{\frac{3}{2}})$ from Lemma 4 can be improved in special cases. To see that $\phi_k'(0)$ is alternating in k, observe that moving from $k \in r\mathbb{Z}_{odd}$ to $k + 2r \in r\mathbb{Z}_{odd}$ the argument of tan changes by $2r\omega\sqrt{a}\Theta\pi$ that is an odd multiple of $\pi/2$. Since $\tan(x + \mathbb{Z}_{odd}\frac{\pi}{2}) = -1/\tan(x)$, we see that the sequence $\phi_k'(0)$ is alternating for $k \in r\mathbb{Z}_{odd}$. This shows, in particular, that for any $j \in \mathbb{N}$, the sequence $(\phi_{hrj}'(0))_{h \in \mathbb{N}_{odd}}$ contains infinitely many positive and negative elements, and hence, Theorem 7 for the existence of infinitely many breathers is applicable. This concludes the proof Theorem 2 because we have shown that the potential g satisfies the assumptions (C1) and (C2) from Theorem 3.

A.3 | Embedding of Hölder-spaces into Sobolev-spaces

For $0 < s < 1$, recall the definition of the Slobodeckij-seminorm for a function $z : \mathbb{T}_T \to \mathbb{R}$

$$[z]_s := \left(\int_{\mathbb{T}_T} \int_{\mathbb{T}_T} \frac{|z(t) - z(\tau)|^2}{|t - \tau|^{1+2s}} dt d\tau \right)^{1/2}. \tag{A26}$$

Lemma A1. *For $0 < \tilde{\nu} < \nu < 1$, the embedding $C^{0,\nu}(\mathbb{T}_T) \hookrightarrow H^{\tilde{\nu}}(\mathbb{T}_T)$ is continuous.*

Proof. Let $z(t) = \sum_k \hat{z}_k e_k(t)$ be a function in $C^{0,\nu}(\mathbb{T}_T)$. We need to show the finiteness of the spectral norm $\|z\|_{H^{\tilde{\nu}}}$. For this, we use the equivalence of the spectral norm $\|\cdot\|_{H^{\tilde{\nu}}}$ with the Slobodeckij norm, cf. Lemma A2. Therefore, it suffices to check the estimate

$$\int_{\mathbb{T}_T} \int_{\mathbb{T}_T} \frac{|z(t) - z(\tau)|^2}{|t - \tau|^{1+2\tilde{\nu}}} dt d\tau \leq \|z\|_{C^\nu(\mathbb{T}_T)}^2 \int_{\mathbb{T}_T} \int_{\mathbb{T}_T} |t - \tau|^{-1+2(\nu-\tilde{\nu})} dt d\tau \leq C(\nu, \tilde{\nu}) \|z\|_{C^\nu(\mathbb{T}_T)}^2, \tag{A27}$$

where the double integral is finite due to $\nu > \tilde{\nu}$. ∎

Lemma A2. *For functions $z \in H^s(\mathbb{T}_T)$, $0 < s < 1$, the spectral norm $\|z\|_{H^s} = (\sum_k (1 + k^2)^s |\hat{z}_k|^2)^{1/2}$ and the Slobodeckij norm $\|\|z\|\|_{H^s} := (\|z\|_{L^2(\mathbb{T}_T)}^2 + [z]_s^2)^{1/2}$ are equivalent.*

Proof. The Solobodeckij space and the spectrally defined fractional Sobolev space are both Hilbert spaces. Hence, by the open mapping theorem, if suffices to verify the estimate $|||z|||_{H^s} \leq C\|z\|_{H^s}$. By direct computation, we get

$$\int_{\mathbb{T}_T} \int_{\mathbb{T}_T} \frac{|z(t) - z(\tau)|^2}{|t - \tau|^{1+2s}} dt d\tau = \int_0^T \int_{-\tau}^{T-\tau} \frac{|z(x + \tau) - z(\tau)|^2}{|x|^{1+2s}} dx d\tau \tag{A28}$$

$$= \int_0^T \left(\int_0^{T-\tau} \frac{|z(x + \tau) - z(\tau)|^2}{x^{1+2s}} dx + \int_{T-\tau}^T \frac{|z(x + \tau) - z(\tau)|^2}{(T - x)^{1+2s}} dx \right) d\tau \tag{A29}$$

$$= \int_0^T \int_0^T \frac{|z(x + \tau) - z(\tau)|^2}{g(x, \tau)^{1+2s}} dx d\tau \tag{A30}$$

with

$$g(x, \tau) = \begin{cases} x & \text{if } 0 \leq x \leq T - \tau, \\ T - x & \text{if } T - \tau \leq x \leq T. \end{cases} \tag{A31}$$

Since $g(x, \tau) \geq \text{dist}(x, \partial \mathbb{T}_T)$ and due to Parseval's identity, we find

$$\int_{\mathbb{T}_T} \int_{\mathbb{T}_T} \frac{|z(t) - z(\tau)|^2}{|t - \tau|^{1+2s}} dt d\tau \leq \int_{\mathbb{T}_T} \frac{\|\widehat{z(\cdot + x)} - \hat{z}\|_{l^2}^2}{\text{dist}(x, \partial \mathbb{T}_T)^{1+2s}} dx \tag{A32}$$

$$= \int_{\mathbb{T}_T} \sum_k \frac{|\exp(ik\omega x) - 1|^2 |\hat{z}_k|^2}{\text{dist}(x, \partial \mathbb{T}_T)^{1+2s}} dx \tag{A33}$$

$$= 4 \int_0^{T/2} \sum_k \frac{1 - \cos(k\omega x)}{x^{1+2s}} |\hat{z}_k|^2 dx \tag{A34}$$

$$\leq 4\tilde{C} \sum_k k^{2s} |\hat{z}_k|^2 \tag{A35}$$

with $\tilde{C} = \int_0^\infty \frac{1 - \cos(\omega\xi)}{\xi^{1+2s}} d\xi$. This finishes the proof. ∎