Stability and convergence of the Ritz map in the maximum norm for nonconforming finite elements *

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Abstract. In this report, we consider the Poisson problem on a domain with regular boundary and discretize it with isoparametric finite elements of order $k \geq 1$. We study a (generalized) Ritz map and show stability and convergence of optimal order k in $W^{1,\infty}$.

Key words. nonconforming space discretization, isoparametric finite elements, Ritz map, maximum norm estimates, weighted norms.

1. Introduction

In the present paper we study the spatial discretization of the elliptic problem

$$\begin{aligned} -\Delta u(x) &= f(x), && x \in \Omega\,, \\ u(x) &= 0, && x \in \Gamma = \partial\Omega\,, \end{aligned}$$

on a smooth domain Ω with isoparametric finite elements. Since this is a nonconforming method, we define a (generalized) Ritz map and prove stability and convergence estimates in the $W^{1,\infty}$-norm. For conforming discretizations, such estimates are well known for many years now. In fact, the first quasi-optimal error bounds in the maximum norm in the conforming case were already given in the seventies by Natterer [13] and Scott [22]. Many extension and refinements have been achieved in the following years, see, e.g., [2, 7, 10, 14–17, 19–21, 23].

However, none of these papers provides stability and convergence estimates of the Ritz map in the nonconforming case. More recently in the context of nonconforming space discretization, maximum norm error bounds for linear finite elements applied to an inhomogeneous Neumann problem were derived in [9]. For evolving surface finite element methods, similar estimates are considered in [11].

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For the stability result, we closely follow the approach in [3, Ch. 8]. First, a regularized δ-function is introduced in order to move from the pointwise property to a variational setting and the stability is reduced to an estimate in $W^{1,1}$. Inserting appropriate weight functions, this is estimated by weighted H^1-norms. In order to cover the nonconformity of the finite element space, the additional terms stemming from the boundary perturbation have to be bounded carefully.

This strategy is adapted for the convergence result. Bounding certain additional geometric errors, the estimate is again reduced to the same $W^{1,1}$-estimates which are already established in the stability analysis.

The paper is organized as follows: In Section 2, we present the analytical framework and the space discretization by isoparametric Lagrange finite elements. After providing some properties of the discretized objects, we state our main results on the stability and convergence of the Ritz map. The proof of the stability is presented in Section 3 and the convergence rate is shown in Section 4. Some results on the elliptic regularity are postponed to Appendix A.

Notation

In the rest of the paper we use the notation

$$a \lesssim b$$

if there is a constant $C > 0$ independent of the spatial parameter h such that $a \leq Cb$. Further, for $\phi \in W^{j,p}(\Omega)$ we denote by $\nabla_j \phi$ the tensor of j-th order derivatives of ϕ. If it is clear from the context, we write L^p instead of $L^p(\Omega)$ or $L^p(\Omega_h)$.

2. General Setting

For a convex domain $\Omega \subset \mathbb{R}^N$, $N \in \{2, 3\}$, with boundary $\partial\Omega \in C^{s,1}$, $s \in \mathbb{N}$, we study for $f \in L^2(\Omega)$ the variational problem

$$(u \mid \psi)_{H_0^1(\Omega)} = (f \mid \psi)_{L^2(\Omega)}, \qquad \forall \psi \in H_0^1(\Omega), \tag{2.1}$$

and denote in the following $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. Due to the unique solvability of (2.1), we define the corresponding solution operator $S \colon H \to V$ by $S \colon f \mapsto u$. For the analysis, we heavily rely on the following elliptic regularity result [6, Thm. 2.4.2.5].

Theorem 2.1 (Elliptic regularity). *Let $\partial\Omega \in C^{1,1}$, then for all $1 < p < \infty$ there is a constant $C_p > 0$ such that for all $\varphi \in L^p(\Omega)$ it holds*

$$\|S\varphi\|_{W^{2,p}} \leq C_p \|\varphi\|_{L^p}.$$

Space discretization

We study the nonconforming space discretization of (2.1) based on isoparametric finite elements. For further details on this approach, we refer to [5]. In particular, we introduce a shape-regular and quasi-uniform mesh $\mathcal{T}_h$, consisting of isoparametric elements of degree $k \in \mathbb{N}$. We assume that the boundary $\partial\Omega$ is of class $C^{k+1,1}$. The computational domain Ω_h is given by

$$\Omega_h = \bigcup_{K \in \mathcal{T}_h} K \approx \Omega,$$

where the subscript h denotes the maximal diameter of all elements $K \in \mathcal{T}_h$. Based on the transformations F_K mapping the reference element $\widehat{K}$ to $K \in \mathcal{T}_h$, we introduce the isoparametric finite element space of degree k

$$W_h = \{\varphi \in C_0(\overline{\Omega}) \mid \varphi|_K = \widehat{\varphi} \circ (F_K)^{-1} \text{ with } \widehat{\varphi} \in \mathcal{P}^k(\widehat{K}) \text{ for all } K \in \mathcal{T}_h\} \subset V.$$

Here, $\mathcal{P}^k(\widehat{K})$ consists of all polynomials on $\widehat{K}$ of degree at most k. The discrete approximation spaces are given by

$$H_h = \big(W_h, (\cdot \mid \cdot)_{L^2(\Omega_h)}\big), \qquad V_h = \big(W_h, (\cdot \mid \cdot)_{H_0^1(\Omega_h)}\big), \qquad X_h = V_h \times H_h.$$

Following the detailed construction in [5, Sec. 5], we introduce the lift operator $\mathcal{L}_h \colon H_h \to H$. In particular, for $p \in [1, \infty]$ there are constants $c_p, C_p > 0$ with

$$c_p \left\|\varphi_h\right\|_{L^p(\Omega_h)} \le \left\|\mathcal{L}_h\varphi_h\right\|_{L^p(\Omega)} \le C_p \left\|\varphi_h\right\|_{L^p(\Omega_h)}, \qquad \varphi_h \in L^p(\Omega_h), \tag{2.2a}$$

$$c_p \left\|\varphi_h\right\|_{W^{1,p}(\Omega_h)} \le \left\|\mathcal{L}_h\varphi_h\right\|_{W^{1,p}(\Omega)} \le C_p \left\|\varphi_h\right\|_{W^{1,p}(\Omega_h)}, \qquad \varphi_h \in W^{1,p}(\Omega_h), \tag{2.2b}$$

cf. [5, Prop. 5.8]. Further, we denote the nodal interpolation operator by $I_h \colon C_0(\Omega) \to V_h$. As shown in [5, Thm. 5.9], we have for $m \in \{0, 1\}$, $1 \le p \le \infty$, and $1 \le \ell \le k$ the estimates

$$\left\|(\mathrm{Id} - \mathcal{L}_h I_h)\varphi\right\|_{W^{m,p}(\Omega)} \lesssim h^{\ell+1-m} \left\|\varphi\right\|_{W^{\ell+1,p}(\Omega)}, \qquad \varphi \in W^{\ell+1,p}(\Omega). \tag{2.3}$$

Further, $\ell = 0$ is allowed for $N < p \le \infty$.

We define the adjoint lift operator $\mathcal{L}_h^{V*} \colon V \to V_h$ by

$$\big(\mathcal{L}_h^{V*}\varphi \mid \psi_h\big)_{V_h} = (\varphi \mid \mathcal{L}_h\psi_h)_V, \qquad \varphi \in V,\ \psi_h \in V_h. \tag{2.4}$$

From [8, Thm. 5.3] and [5, Lem. 8.24], we obtain for $0 \le \ell \le k$

$$\left\|(\mathrm{Id} - \mathcal{L}_h\mathcal{L}_h^{V*})\varphi\right\|_{V_h} \lesssim h^{\ell} \left\|\varphi\right\|_{H^{\ell+1}(\Omega)}, \qquad \varphi \in H^{\ell+1}(\Omega).$$

We will also employ the inverse estimate, cf. [3, Thm. 4.5.11] or [12, Lem. 5.6],

$$\left\|\varphi_h\right\|_{L^\infty} \le Ch^{-N/p} \left\|\varphi_h\right\|_{L^p}$$

for $1 \le p < \infty$ and $C > 0$ independent of h.

Definition 2.2. Consider the adjoint lift $\mathcal{L}_h^{V*}$ given by (2.4). We define the generalized Ritz map by

$$\mathcal{L}_h \mathcal{L}_h^{V*} \colon V \to V\,. \tag{2.5}$$

We note that in the conforming case, this is simply the Ritz projection. However, the generalized Ritz map does not satisfy an orthogonality condition, but only an estimate of the form

$$\left(u - \mathcal{L}_h \mathcal{L}_h^{V*} u \mid \mathcal{L}_h \varphi_h\right)_V \lesssim h^k \left\|\mathcal{L}_h^{V*} u\right\|_{V_h} \|\varphi_h\|_{V_h}\,, \qquad u \in V, \varphi_h \in V_h.$$

This fact induces several additional error terms in the maximum norm error analysis which require a detailed inspection. We are now in the position to state our main results.

Theorem 2.3. *Let $\partial\Omega \in C^{k+1,1}$. Then the generalized Ritz map defined in (2.5) is stable in $W^{1,\infty}(\Omega)$, i.e.,*

$$\left\|\mathcal{L}_h \mathcal{L}_h^{V*} \varphi\right\|_{W^{1,\infty}(\Omega)} \lesssim \|\varphi\|_{W^{1,\infty}(\Omega)}\,, \qquad \varphi \in W^{1,\infty}(\Omega).$$

The proof is given in Section 3. We note that by (2.2) it is sufficient to show

$$\left\|\mathcal{L}_h^{V*} \varphi\right\|_{W^{1,\infty}(\Omega_h)} \lesssim \|\varphi\|_{W^{1,\infty}(\Omega)}\,, \qquad \varphi \in W^{1,\infty}(\Omega).$$

Our second main result is concerned with the approximation properties.

Theorem 2.4. *Let $k \geq 1$ and $\partial\Omega \in C^{k+1,1}$. Then, it holds for all $\varphi \in W^{k+1,\infty}(\Omega)$*

$$\left\|(\mathrm{Id} - \mathcal{L}_h \mathcal{L}_h^{V*})\varphi\right\|_{W^{1,\infty}(\Omega)} \leq C h^k \|\varphi\|_{W^{k+1,\infty}(\Omega)}\,,$$

where C is independent of h.

2.1. Properties of weighted norms

The main technical tool are weighted norms. To this end, we introduce the family $\{\sigma_z\}_{z \in \Omega}$ of weight functions with

$$\sigma_z \colon \Omega \to \mathbb{R}, \qquad \sigma_z(x) = \left(|x - z|^2 + \gamma^2 h^2\right)^{\frac{1}{2}}. \tag{2.6}$$

The parameter $\gamma > 0$ is fixed below. We first establish certain properties of the weight functions.

Lemma 2.5. *Consider the weights defined in (2.6).*

(a) For $\lambda \in \mathbb{R}$, there are constants $C > 0$ independent of $x, z \in \Omega$ and h such that the following bounds hold:

$$\max_{K \in \mathcal{T}_h} \left(\sup_{x \in K} \sigma_z^\lambda(x) / \inf_{x \in K} \sigma_z^\lambda(x)\right) \leq C\,,$$

$$\left\|\sigma_z^\lambda\right\|_{L^\infty} \leq C \max\{1, (\gamma h)^\lambda\}\,,$$

$$\left|D_x^\beta \sigma_z^\lambda(x)\right| \leq C \sigma_z^{\lambda - |\beta|}(x)\,, \quad x \in \Omega_h\,.$$

(b) If $\alpha > N$, then $\sigma_z^{-\alpha} \in L^1(\Omega)$ and

$$\int_\Omega \sigma_z^{-\alpha}(x)\,\mathrm{d}x \le C \max\{1, \tfrac{1}{\alpha-N}\}(\gamma h)^{-\alpha+N}\,. \tag{2.8}$$

Further, we use slight extensions of the estimates in [5, Lem. 8.24] in order to treat the errors stemming from nonconformity.

Lemma 2.6. *Let $\varphi_h, \psi_h \in V_h$.*

(a) The errors in the bilinear forms are estimated for any $\alpha \in \mathbb{R}$

$$\Big|(\mathcal{L}_h\varphi_h \mid \mathcal{L}_h\psi_h)_H - (\varphi_h \mid \psi_h)_{H_h}\Big| \le Ch^k\Big(\int_\Omega \sigma_z^\alpha\,|\mathcal{L}_h\varphi_h|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega \sigma_z^{-\alpha}\,|\mathcal{L}_h\psi_h|^2\,\mathrm{d}x\Big)^{1/2},$$

$$\Big|(\mathcal{L}_h\varphi_h \mid \mathcal{L}_h\psi_h)_V - (\varphi_h \mid \psi_h)_{V_h}\Big| \le Ch^k\Big(\int_\Omega \sigma_z^\alpha\,|\nabla\mathcal{L}_h\varphi_h|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega \sigma_z^{-\alpha}\,|\nabla\mathcal{L}_h\psi_h|^2\,\mathrm{d}x\Big)^{1/2},$$

$$\Big|(\mathcal{L}_h\varphi_h \mid \mathcal{L}_h\psi_h)_H - (\varphi_h \mid \psi_h)_{H_h}\Big| \le Ch^{k+1/2}\Big(\int_\Omega \sigma_z^\alpha\,|\mathcal{L}_h\varphi_h|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega \sigma_z^{-\alpha}\,|\nabla\mathcal{L}_h\psi_h|^2\,\mathrm{d}x\Big)^{1/2}.$$

with $C > 0$ independent of h and α.

(b) For any $p \in [1,\infty]$ the bilinear forms are estimated by

$$\Big|(\mathcal{L}_h\varphi_h \mid \mathcal{L}_h\psi_h)_H - (\varphi_h \mid \psi_h)_{H_h}\Big| \le Ch^k\,\|\mathcal{L}_h\varphi_h\|_{L^p(\Omega)}\,\|\mathcal{L}_h\psi_h\|_{L^{p'}(\Omega)}\,,$$

$$\Big|(\mathcal{L}_h\varphi_h \mid \mathcal{L}_h\psi_h)_V - (\varphi_h \mid \psi_h)_{V_h}\Big| \le Ch^k\,\|\mathcal{L}_h\nabla\varphi_h\|_{L^p(\Omega)}\,\|\mathcal{L}_h\nabla\psi_h\|_{L^{p'}(\Omega)}\,.$$

with $C > 0$ independent of h.

As the final property, we state a weighted inverse inequality, which is a straightforward generalization of [3, Thm. 4.5.11].

Lemma 2.7. *Let $\varphi_h \in V_h$. Then, for $j \ge 1$ it holds*

$$\int\limits_{\Omega_h} \sigma_z^\lambda|\nabla_j\varphi_h|^2\,\mathrm{d}x \lesssim h^{-2j}\int\limits_{\Omega_h} \sigma_z^\lambda|\varphi_h|^2\,\mathrm{d}x\,,$$

where the derivatives are considered elements-wise.

3. Stability of the adjoint lift operator, Theorem 2.3

In this section, we prove Theorem 2.3, i.e., the stability of the adjoint lift operator $\mathcal{L}_h^{V*}$ in $W^{1,\infty}$. To this end, we extend the results of [3, Ch. 8] for conforming space discretizations to domain approximations with isoparametric finite elements, cf. [5]. We emphasize that we follow the lines of [3] and add certain modifications due to the nonconformity, but give a rather complete proof for the sake of readability.

3.1. Reduction to weighted norm estimates

Let $z \in K^z$ with $K^z \in \mathcal{T}_h$. There exists $\delta^z \in C_0^\infty(K^z)$, see [18] for a construction, with zero extension to a function on Ω_h, such that

$$(\delta^z \mid \varphi_h)_{H_h} = \varphi_h(z), \qquad\qquad \varphi_h \in H_h,$$

and

$$\|\partial^\alpha \delta^z\|_{L^\infty} \lesssim h^{-N-|\alpha|}, \qquad\qquad \alpha \in \mathbb{N}^N. \tag{3.1}$$

Here, we use the notation $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_N^{\alpha_N}$ and $|\alpha| = \sum_{i=1}^N \alpha_i$. We further introduce the solutions $g_h^z \in V_h$ and $g^z \in V$ of the elliptic variational problems

$$\begin{aligned} (g_h^z \mid \varphi_h)_{V_h} &= (-\partial_i \delta^z \mid \varphi_h)_{H_h}, \quad \varphi_h \in V_h, \\ (g^z \mid \varphi)_V &= (-\partial_i \mathcal{L}_h \delta^z \mid \varphi)_H, \quad \varphi \in V. \end{aligned} \tag{3.2}$$

Using integration by parts as well as the definition (2.4) of the adjoint lift operator, this implies for $1 \le i \le N$

$$\begin{aligned} \partial_i\big(\mathcal{L}_h^{V*} u\big)(z) &= \big(\partial_i\big(\mathcal{L}_h^{V*} u\big) \mid \delta^z\big)_{H_h} \\ &= \big(\mathcal{L}_h^{V*} u \mid -\partial_i \delta^z\big)_{H_h} \\ &= \big(\mathcal{L}_h^{V*} u \mid g_h^z\big)_{V_h} \\ &= (u \mid \mathcal{L}_h g_h^z)_V \\ &= (u \mid g^z)_V + (u \mid \mathcal{L}_h g_h^z - g^z)_V \\ &= \big(u \mid -\partial_i\big(\mathcal{L}_h \delta^z\big)\big)_H + (u \mid \mathcal{L}_h g_h^z - g^z)_V \\ &= (\partial_i u \mid \mathcal{L}_h \delta^z)_H + (u \mid \mathcal{L}_h g_h^z - g^z)_V \,. \end{aligned} \tag{3.3}$$

Hence, Hölder's inequality yields

$$\big|\partial_i\big(\mathcal{L}_h^{V*} u\big)(z)\big| \lesssim \big(\|\mathcal{L}_h \delta^z\|_{L^1} + \|\mathcal{L}_h g_h^z - g^z\|_{W^{1,1}}\big) \|u\|_{W^{1,\infty}} \,.$$

Due to the stability (2.2) of $\mathcal{L}_h$, we have

$$\|\mathcal{L}_h \delta^z\|_{L^1} \lesssim \int\limits_{K^z} |\delta^z| \, \mathrm{d}x \lesssim h^N h^{-N} \le C. \tag{3.4}$$

Since we provide a bound on $\|\mathcal{L}_h g_h^z - g^z\|_{W^{1,1}}$ in Lemma 3.1, the stability estimate in Theorem 2.3 follows with the Poincaré inequality. Hence, it remains to prove the following estimate.

Lemma 3.1. *Let $g_h^z \in V_h$ and $g^z \in V$ be defined by (3.2). Then, there is a constant $C > 0$ such that*

$$\|\mathcal{L}_h g_h^z - g^z\|_{W^{1,1}} \le C.$$

with C independent of h and z.

In order to move from L^1 to L^2, we introduce a weight function and obtain the following upper bound by a weighted L^2-norms.

Lemma 3.2. *Let*

$$M_h := \sup_{z\in\Omega}\Big(\int_\Omega \sigma_z^{N+\lambda}\,|\nabla(g^z-\mathcal{L}_h g_h^z)|^2\,\mathrm{d}x\Big)^{1/2}.$$

Then, for $\lambda\in(0,1)$ it holds

$$\|\mathcal{L}_h g_h^z - g^z\|_{W^{1,1}} \le C M_h \lambda^{-1/2}(\gamma h)^{-\lambda/2},$$

with a constant $C>0$ independent of γ,λ,h.

Proof. By the Hölder inequality we have

$$\big\|\nabla\big(\mathcal{L}_h g_h^z - g^z\big)\big\|_{L^1} \le M_h\Big(\int_\Omega \sigma_z^{-N-\lambda}\,\mathrm{d}x\Big)^{1/2} \le C M_h\lambda^{-1/2}(\gamma h)^{-\lambda/2}$$

where we used (2.8) with $\alpha = N+\lambda$ for the last inequality. The application of the Poincaré inequality yields the assertion. □

From this, we see that it is sufficient to prove the following proposition from which Lemma 3.1 directly follows.

Proposition 3.3. *There is a $\lambda>0$ and $\gamma>1$ such that for all $0<h<h_0$ it holds*

$$M_h^2 = \sup_{z\in\Omega}\int_\Omega \sigma_z^{N+\lambda}\,|\nabla(g^z-\mathcal{L}_h g_h^z)|^2\,\mathrm{d}x \le Ch^\lambda\,, \tag{3.5}$$

with a constant $C>0$ independent of h.

Before we proof Proposition 3.3, we state the following estimate on weighted norms of δ_z. Later, they give the desired convergence rate h^λ.

Lemma 3.4. *For all $\mu>0$, the following bounds holds*

$$\int_\Omega \sigma_z^{N+\mu}\,|\nabla\mathcal{L}_h\delta^z|^2\,\mathrm{d}x \le Ch^{\mu-2}, \qquad \int_\Omega \sigma_z^{N+\mu}\,|\mathcal{L}_h\delta^z|^2\,\mathrm{d}x \le Ch^{\mu}\,,$$

with a constant $C>0$ independent of h.

Proof. By the shape-regularity and the definition of the weight in (2.6), we obtain

$$\big\|\sigma_z^{N+\mu}\big\|_{L^\infty(K^z)} \lesssim h^{N+\mu}\,,$$

and use $\delta^z\in C_0^\infty(K^z)$ together with (3.1) to bound

$$\int_\Omega \sigma_z^{N+\mu}\,|\nabla\mathcal{L}_h\delta^z|^2\,\mathrm{d}x \lesssim h^N h^{N+\mu}h^{-2(N+1)} \lesssim h^{\mu-2},$$
$$\int_\Omega \sigma_z^{N+\mu}\,|\mathcal{L}_h\delta^z|^2\,\mathrm{d}x \lesssim h^N h^{N+\mu}h^{-2N} \lesssim h^{\mu}.$$

□

3.2. Proof of Proposition 3.3

In the following, we present an extension of [3, Prop. 8.3.1]. In this step, the weighted H^1-norm in (3.5) is replaced a weighted L^2-norm and some additional error terms. We point out that in the conforming case the differences in the scalar product simply vanishes.

Proposition 3.5. *Let $g^z \in V$ and $g_h^z \in V_h$ be the solutions of (3.2) and define the errors $e = g^z - \mathcal{L}_h g_h^z$ and $\widehat{e} = (\mathrm{Id} - \mathcal{L}_h I_h) g^z$. Then*

$$\begin{aligned}\int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 \,\mathrm{d}x \lesssim \int_\Omega \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x + \int_\Omega \sigma_z^{N+\lambda-2} |\widehat{e}|^2 \,\mathrm{d}x + \int_\Omega \sigma_z^{N+\lambda} |\nabla \widehat{e}|^2 \,\mathrm{d}x \\ + \left|(\partial_i \mathcal{L}_h \delta^z \mid \mathcal{L}_h I_h \psi)_H - (\partial_i \delta^z \mid I_h \psi)_{H_h}\right| \\ + \left|(g_h^z \mid I_h \psi)_{V_h} - (\mathcal{L}_h g_h^z \mid \mathcal{L}_h I_h \psi)_V\right|\end{aligned}$$

with $\psi = \sigma_z^{N+\lambda} \mathcal{L}_h (I_h g^z - g_h^z)$.

Proof. Let $\widetilde{e} = I_h g^z - g_h^z$, then we have $\psi = \sigma_z^{N+\lambda} \mathcal{L}_h \widetilde{e}$. We note that it holds

$$\mathcal{L}_h \widetilde{e} = e - \widehat{e} \tag{3.6}$$

and compute

$$\begin{aligned}\int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 \,\mathrm{d}x &= \left(e \mid \sigma_z^{N+\lambda} e\right)_V - \int_\Omega \nabla e \cdot (\nabla \sigma_z^{N+\lambda}) e \,\mathrm{d}x \\ &= \left(e \mid \sigma_z^{N+\lambda} \widehat{e}\right)_V + (e \mid \psi)_V - \int_\Omega \nabla e \cdot (\nabla \sigma_z^{N+\lambda}) e \,\mathrm{d}x.\end{aligned}$$

Along the lines of the proof of [3, Prop. 8.3.1], we show

$$\begin{aligned}\int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 \,\mathrm{d}x \lesssim \int_\Omega \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x \\ + \int_\Omega \sigma_z^{N+\lambda-2} |\widehat{e}|^2 \,\mathrm{d}x + \int_\Omega \sigma_z^{N+\lambda} |\nabla \widehat{e}|^2 \,\mathrm{d}x + |(e \mid \psi)_V| \,.\end{aligned}$$

Hence, we turn to the term

$$(e \mid \psi)_V = (e \mid \psi - \mathcal{L}_h I_h \psi)_V + (g^z - \mathcal{L}_h g_h^z \mid \mathcal{L}_h I_h \psi)_V \tag{3.7}$$

and note that in the conforming case the last term vanishes by orthogonality. However, for the first term Lemma 3.6 below shows that for any $a > 0$ it holds

$$\begin{aligned}(e \mid \psi - \mathcal{L}_h I_h \psi)_V &\le a \int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 \,\mathrm{d}x + a^{-1} \int_\Omega \sigma_z^{-N-\lambda} |\nabla(\psi - \mathcal{L}_h I_h \psi)|^2 \,\mathrm{d}x. \\ &\lesssim a \int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 \,\mathrm{d}x + a^{-1} \int_\Omega \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x + a^{-1} \int_\Omega \sigma_z^{N+\lambda-2} |\widehat{e}|^2 \,\mathrm{d}x\end{aligned}$$

and absorption leaves the right terms. For the second term in (3.7) it remains to expand

$$\begin{aligned}(g^z - \mathcal{L}_h g_h^z \mid \mathcal{L}_h I_h \psi)_V &= (g^z \mid \mathcal{L}_h I_h \psi)_V - (\mathcal{L}_h g_h^z \mid \mathcal{L}_h I_h \psi)_V + (g_h^z \mid I_h \psi)_{V_h} - (g_h^z \mid I_h \psi)_{V_h} \\ &= (-\partial_i \mathcal{L}_h \delta^z \mid \mathcal{L}_h I_h \psi)_H + (\partial_i \delta^z \mid I_h \psi)_{H_h} \\ &\quad + (g_h^z \mid I_h \psi)_{V_h} - (\mathcal{L}_h g_h^z \mid \mathcal{L}_h I_h \psi)_V \,,\end{aligned}$$

where we used (3.2) in the second inequality, and the claim follows. □

We state the next lemma which was already used above, since we need it several more times in the following computations. It can be found as an auxiliary result in the proof of [3, Prop. 8.3.1].

Lemma 3.6. *Let $\psi = \sigma_z^{N+\lambda}\mathcal{L}_h\widetilde{e}$. Then, it holds the estimate*

$$\begin{aligned}\int_\Omega \sigma_z^{-N-\lambda}\big(|\nabla(\psi - \mathcal{L}_h I_h\psi)|^2 + |\psi - \mathcal{L}_h I_h \psi|^2\big)\,\mathrm{d}x \\ \lesssim \int_\Omega \sigma_z^{N+\lambda-2}\,|e|^2\,\mathrm{d}x + \int_\Omega \sigma_z^{N+\lambda-2}\,|\widehat{e}|^2\,\mathrm{d}x\,.\end{aligned}$$

The following two lemmas are devoted to control the defects stemming from the non-conformity. For the sake of presentation, we bound the two errors in two separate lemmas. We begin with the difference in the energy scalar product.

Lemma 3.7. *For any $a > 0$, there is a constant $C_a > 0$ such that*

$$\begin{aligned}\Big|(g_h^z \mid I_h\psi)_{V_h} - (\mathcal{L}_h g_h^z \mid \mathcal{L}_h I_h\psi)_V\Big| &\lesssim (a + a^{-1}h^2)\int_\Omega \sigma_z^{N+\lambda}\,|\nabla e|^2\,\mathrm{d}x + a^{-1}h^\lambda \\ &\quad + a\int_\Omega \sigma_z^{N+\lambda-2}\,|e|^2\,\mathrm{d}x \\ &\quad + a\int_\Omega \sigma_z^{N+\lambda}\,|\nabla\widehat{e}|^2\,\mathrm{d}x + a\int_\Omega \sigma_z^{N+\lambda-2}\,|\widehat{e}|^2\,\mathrm{d}x\,.\end{aligned}$$

Proof. From Lemma 2.6 we have with $\alpha = N + \lambda$ and Young

$$\begin{aligned}&\Big|(g_h^z \mid I_h\psi)_{V_h} - (\mathcal{L}_h g_h^z \mid \mathcal{L}_h I_h\psi)_V\Big| \\ &\le Ch\Big(\int_\Omega \sigma_z^{N+\lambda}\,|\nabla\mathcal{L}_h g_h^z|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega \sigma_z^{-N-\lambda}\,|\nabla\mathcal{L}_h I_h\psi|^2\,\mathrm{d}x\Big)^{1/2} \\ &\le a^{-1}h^2\int_\Omega \sigma_z^{N+\lambda}\,|\nabla\mathcal{L}_h g_h^z|^2\,\mathrm{d}x + a\int_\Omega \sigma_z^{-N-\lambda}\,|\nabla\mathcal{L}_h I_h\psi|^2\,\mathrm{d}x \\ &= \Delta_1 + \Delta_2\,.\end{aligned}$$

We recall $\mathcal{L}_h g_h^z = g^z - e$, and estimate

$$\Delta_1 \le a^{-1}h^2\int_\Omega \sigma_z^{N+\lambda}\,|\nabla e|^2\,\mathrm{d}x + a^{-1}h^2\int_\Omega \sigma_z^{N+\lambda}\,|\nabla g^z|^2\,\mathrm{d}x\,.$$

Using Lemma 2.5 and the estimate [3, eq. (8.4.3)] with the subsequent calculations with right hand side $\partial_i\mathcal{L}_h\delta^z$, we obtain

$$\begin{aligned}a^{-1}h^2\int_\Omega \sigma_z^{N+\lambda}\,|\nabla g^z|^2\,\mathrm{d}x &\lesssim a^{-1}h^2\int_\Omega \sigma_z^{N+\lambda-2}\,|\nabla g^z|^2\,\mathrm{d}x \\ &\lesssim a^{-1}h^2\int_\Omega \sigma_z^{N+\lambda}\,|\nabla\mathcal{L}_h\delta^z|^2\,\mathrm{d}x + a^{-1}h^2(\gamma h)^{-2}\int_\Omega \sigma_z^{N+\lambda}\,|\mathcal{L}_h\delta^z|^2\,\mathrm{d}x \\ &\lesssim C_a h^\lambda\,,\end{aligned}$$

where we used Lemma 3.4 in the last line. For the second term we expand

$$
\begin{aligned}
\Delta_2 &\lesssim a \int_\Omega \sigma_z^{-N-\lambda} |\nabla \psi|^2 \,\mathrm{d}x + a \int_\Omega \sigma_z^{-N-\lambda} |\nabla(\psi - \mathcal{L}_h I_h \psi)|^2 \,\mathrm{d}x \\
&\lesssim a \int_\Omega \sigma_z^{N+\lambda} |\nabla \mathcal{L}_h \widetilde{e}|^2 \,\mathrm{d}x + a \int_\Omega \sigma_z^{N+\lambda-2} |\mathcal{L}_h \widetilde{e}|^2 \,\mathrm{d}x \\
&\quad + a \int_\Omega \sigma_z^{-N-\lambda} |\nabla(\psi - \mathcal{L}_h I_h \psi)|^2 \,\mathrm{d}x \qquad (3.8) \\
&\lesssim a \int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 \,\mathrm{d}x + a \int_\Omega \sigma_z^{N+\lambda} |\nabla \widehat{e}|^2 \,\mathrm{d}x + a \int_\Omega \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x \\
&\quad + a \int_\Omega \sigma_z^{N+\lambda-2} |\widehat{e}|^2 \,\mathrm{d}x \,,
\end{aligned}
$$

where we used the definition of $\psi = \sigma_z^{N+\lambda} \mathcal{L}_h \widetilde{e}$, the representation (3.6), and Lemma 3.6. □

By similar techniques, we derive the second bound.

Lemma 3.8. *For any $a > 0$, there is a constant $C_a > 0$ such that*

$$
\begin{aligned}
\left| (\partial_i \mathcal{L}_h \delta_z \mid \mathcal{L}_h I_h \psi)_H - (\partial_i \delta_z \mid I_h \psi)_{H_h} \right| &\lesssim a \int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 \,\mathrm{d}x + C_a h^\lambda \\
&\quad + a \int_\Omega \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x \\
&\quad + a \int_\Omega \sigma_z^{N+\lambda} |\nabla \widehat{e}|^2 \,\mathrm{d}x + a \int_\Omega \sigma_z^{N+\lambda-2} |\widehat{e}|^2 \,\mathrm{d}x.
\end{aligned}
$$

Proof. We employ Lemmas 2.6 and 3.4 to conclude

$$
\begin{aligned}
&\left| (\partial_i \mathcal{L}_h \delta_z \mid \mathcal{L}_h I_h \psi)_H - (\partial_i \delta_z \mid I_h \psi)_{H_h} \right| \\
&\le C_a h^2 \int_\Omega \sigma_z^{N+\lambda} |\partial_i \mathcal{L}_h \delta_z|^2 \,\mathrm{d}x + a \int_\Omega \sigma_z^{-N-\lambda} |\nabla \mathcal{L}_h I_h \psi|^2 \,\mathrm{d}x \\
&\lesssim C_a h^\lambda + a \int_\Omega \sigma_z^{-N-\lambda} |\nabla \mathcal{L}_h I_h \psi|^2 \,\mathrm{d}x \,,
\end{aligned}
$$

and the claim follows as in (3.8). □

If we combinde the bounds from Proposition 3.5, Lemma 3.7 and Lemma 3.8, we have shown, for a, h sufficiently small, that it holds

$$
\begin{aligned}
\int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 \,\mathrm{d}x &\lesssim \int_\Omega \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x + h^\lambda \\
&\quad + \int_\Omega \sigma_z^{N+\lambda-2} |\widehat{e}|^2 \,\mathrm{d}x + \int_\Omega \sigma_z^{N+\lambda} |\nabla \widehat{e}|^2 \,\mathrm{d}x \,.
\end{aligned}
$$

Hence, it remains two absorb the weighted L^2-norm of e and to obtain a factor h^λ for the $\widehat{e}$ terms. This is done in the following two propositions. The first one estimates the interpolation error, which we state from [3] for completeness.

Proposition 3.9. *For $\widehat{e} = (\mathrm{Id} - \mathcal{L}_h I_h) g^z$, there is some constant $C > 0$ independent of h and λ s.t.*

$$\int_\Omega \sigma_z^{N+\lambda-2} |\widehat{e}|^2 \,\mathrm{d}x + \int_\Omega \sigma_z^{N+\lambda} |\nabla \widehat{e}|^2 \,\mathrm{d}x \le C h^\lambda.$$

Proof. Using the interpolation estimate, one obtains with the Hessian ∇_2

$$\int_\Omega \sigma_z^{N+\lambda-2} |\widehat{e}|^2 \,\mathrm{d}x + \int_\Omega \sigma_z^{N+\lambda} |\nabla \widehat{e}|^2 \,\mathrm{d}x \lesssim h^2 \int_\Omega \sigma_z^{N+\lambda} |\nabla_2 g^z|^2 \,\mathrm{d}x\,.$$

The application of [3, Lem. 8.3.11] and Lemma 3.4 then yields the result. □

The proof is closed once we have shown the following bound which extends the result of [3, Prop. 8.3.5] again due to the lack of orthogonality.

Proposition 3.10. *For any $\varepsilon > 0$, there is $\gamma_0 > 1$ such that*

$$\int_\Omega \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x \le \varepsilon \int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 \,\mathrm{d}x + C_\epsilon h^\lambda$$

for all $\gamma \ge \gamma_0$.

Proof. We define $v \in V$ as the solution of

$$(v \mid \phi)_V = \left(\sigma_z^{N+\lambda-2} e \mid \phi\right)_H \quad \forall \phi \in V\,,$$

and obtain

$$\int_\Omega \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x = (e \mid v)_V = (e \mid v - \mathcal{L}_h I_h v)_V + (e \mid \mathcal{L}_h I_h v)_V\,.$$

Note again, that in the conforming case the second term vanishes. The first term is estimated as in the proof of [3, Prop. 8.3.5] by

$$\begin{aligned}(e \mid v - \mathcal{L}_h I_h v)_V \le{}& \epsilon \int_\Omega \sigma_z^{N+\lambda} \left(|\nabla e|^2 + |e|^2\right) \mathrm{d}x \\ &+ \frac{C}{\lambda \epsilon \gamma^2} \int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 + \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x\,.\end{aligned} \tag{3.9}$$

Turning to the second term, using (3.2) we obtain

$$\begin{aligned}(e \mid \mathcal{L}_h I_h v)_V &= (g^z - \mathcal{L}_h g_h^z \mid \mathcal{L}_h I_h v)_V \\ &= (g^z \mid \mathcal{L}_h I_h v)_V - (\mathcal{L}_h g_h^z \mid \mathcal{L}_h I_h v)_V \\ &= (-\partial_i \mathcal{L}_h \delta^z \mid \mathcal{L}_h I_h v)_H - (g_h^z \mid I_h v)_{V_h} + (g_h^z \mid I_h v)_{V_h} - (\mathcal{L}_h g_h^z \mid \mathcal{L}_h I_h v)_V \\ &= (-\partial_i \mathcal{L}_h \delta^z \mid \mathcal{L}_h I_h v)_H + (\partial_i \delta^z \mid I_h v)_{H_h} + (g_h^z \mid I_h v)_{V_h} - (\mathcal{L}_h g_h^z \mid \mathcal{L}_h I_h v)_V \\ &= \Delta_H + \Delta_V\,.\end{aligned}$$

The two terms are estimated separately in the following.

(1) We apply Lemma 2.6 with $k = 1$ to obtain

$$\begin{aligned}
\Delta_H &\leq C h^{3/2} \Big(\int_\Omega \sigma_z^{N+\lambda} \, |\nabla \mathcal{L}_h \delta^z|^2 \, \mathrm{d}x \Big)^{1/2} \Big(\int_\Omega \sigma_z^{-N-\lambda} \, |\nabla \mathcal{L}_h I_h v|^2 \, \mathrm{d}x \Big)^{1/2} \\
&\leq h^2 \int_\Omega \sigma_z^{N+\lambda} \, |\nabla \mathcal{L}_h \delta^z|^2 \, \mathrm{d}x + h \int_\Omega \sigma_z^{-N-\lambda} \, |\nabla \mathcal{L}_h I_h v|^2 \, \mathrm{d}x \\
&\leq h^\lambda + h \int_\Omega \sigma_z^{-N-\lambda} \, |\nabla (v - \mathcal{L}_h I_h v)|^2 \, \mathrm{d}x + h \int_\Omega \sigma_z^{-N-\lambda} \, |\nabla v|^2 \, \mathrm{d}x \,,
\end{aligned}$$

where we used Lemma 3.4 in the last step. For the second term, we derive analogous to (3.9)

$$\int_\Omega \sigma_z^{-N-\lambda} \, |\nabla (v - \mathcal{L}_h I_h v)|^2 \, \mathrm{d}x \leq \frac{C}{\lambda \gamma^2} \int_\Omega \sigma_z^{N+\lambda} \, |\nabla e|^2 + \sigma_z^{N+\lambda-2} \, |e|^2 \, \mathrm{d}x \,. \tag{3.10}$$

Finally, we employ Lemma A.1

$$\begin{aligned}
h \int_\Omega \sigma_z^{-N-\lambda} \, |\nabla v|^2 \, \mathrm{d}x &\lesssim h (\gamma h)^{-1} \int_\Omega \sigma_z^{4-N-\lambda} \left| \nabla (\sigma_z^{N+\lambda-2} e) \right|^2 \mathrm{d}x \\
&\lesssim \gamma^{-1} \int_\Omega \sigma_z^{N+\lambda} \, |\nabla e|^2 + \sigma_z^{N+\lambda-2} \, |e|^2 \, \mathrm{d}x \,,
\end{aligned}$$

and collect this to derive

$$\Delta_H \lesssim h^\lambda + \big(h (\lambda \gamma^2)^{-1} + \gamma^{-1} \big) \int_\Omega \sigma_z^{N+\lambda} \, |\nabla e|^2 + \sigma_z^{N+\lambda-2} \, |e|^2 \, \mathrm{d}x \,. \tag{3.11}$$

(2) We employ Lemma 2.6 and obtain with $k = 1$

$$\begin{aligned}
\Delta_V &\leq C h \Big(\int_\Omega \sigma_z^{N+\lambda-1} \, |\nabla \mathcal{L}_h g_h^z|^2 \, \mathrm{d}x \Big)^{1/2} \Big(\int_\Omega \sigma_z^{-N-\lambda+1} \, |\nabla \mathcal{L}_h I_h v|^2 \, \mathrm{d}x \Big)^{1/2} \\
&\leq a h \int_\Omega \sigma_z^{N+\lambda-1} \, |\nabla \mathcal{L}_h g_h^z|^2 \, \mathrm{d}x + a^{-1} h \int_\Omega \sigma_z^{-N-\lambda+1} \, |\nabla \mathcal{L}_h I_h v|^2 \, \mathrm{d}x \,.
\end{aligned}$$

For the first term we obtain as in Lemma 3.7 using $h \leq \sigma_z(x)$

$$\begin{aligned}
a h \int_\Omega \sigma_z^{N+\lambda-1} \, |\nabla \mathcal{L}_h g_h^z|^2 \, \mathrm{d}x &\leq a h \int_\Omega \sigma_z^{N+\lambda-1} \, |\nabla e|^2 \, \mathrm{d}x + a h \int_\Omega \sigma_z^{N+\lambda-1} \, |\nabla g^z|^2 \, \mathrm{d}x \\
&\leq a \int_\Omega \sigma_z^{N+\lambda} \, |\nabla e|^2 \, \mathrm{d}x + a h \int_\Omega \sigma_z^{N+\lambda-1} \, |\nabla g^z|^2 \, \mathrm{d}x \,.
\end{aligned}$$

With Lemma A.3, $\alpha = 1$ and $f = \mathcal{L}_h \delta^z$ we obtain

$$\begin{aligned}
a h \int_\Omega \sigma_z^{N+\lambda-1} \, |\nabla g^z|^2 \, \mathrm{d}x &\leq a h \int_\Omega \sigma_z^{N+\lambda+1} \, |\nabla \mathcal{L}_h \delta^z|^2 \, \mathrm{d}x + a h (\gamma h)^{-1} \int_\Omega \sigma_z^{N+\lambda} \, |\mathcal{L}_h \delta^z|^2 \, \mathrm{d}x \\
&\lesssim h^\lambda \,,
\end{aligned}$$

where we used Lemma 3.4 in the last step. Further, we expand

$$\tfrac{1}{a}h\int_\Omega \sigma_z^{-N-\lambda+1}\,|\nabla\mathcal{L}_hI_hv|^2\,\mathrm{d}x \le \tfrac{1}{a}h\int_\Omega \sigma_z^{-N-\lambda+1}\,|\nabla(v-\mathcal{L}_hI_hv)|^2\,\mathrm{d}x + \tfrac{1}{a}h\int_\Omega \sigma_z^{-N-\lambda+1}\,|\nabla v|^2\,\mathrm{d}x\,,$$

and the first is treated by an interpolation estimate as in (3.10)

$$\tfrac{1}{a}h\int_\Omega \sigma_z^{-N-\lambda+1}\,|\nabla(v-\mathcal{L}_hI_hv)|^2\,\mathrm{d}x \lesssim \frac{h}{a\lambda\gamma^2}\int_\Omega \sigma_z^{N+\lambda}\,|\nabla e|^2 + \sigma_z^{N+\lambda-2}\,|e|^2\,\mathrm{d}x\,.$$

So it remains to bound by Lemma A.2

$$\begin{aligned}\tfrac{1}{a}h\int_\Omega \sigma_z^{-N-\lambda+1}\,|\nabla v|^2\,\mathrm{d}x &\le Ch(a\lambda)^{-1}(\gamma h)^{-1}\int_\Omega \sigma_z^{4-N-\lambda}\,|\nabla f|^2\,\mathrm{d}x\\ &\le C(a\lambda\gamma)^{-1}\int_\Omega \sigma_z^{N+\lambda}\,|\nabla e|^2 + \sigma_z^{N+\lambda-2}\,|e|^2\,\mathrm{d}x\,,\end{aligned}$$

and collecting the above estimates gives

$$\begin{aligned}\Delta_V &\lesssim h^\lambda + a\int_\Omega \sigma_z^{N+\lambda}\,|\nabla e|^2\,\mathrm{d}x\\ &\quad + \Big(\frac{h}{a\lambda\gamma^2}+\frac{1}{a\lambda\gamma}\Big)\int_\Omega \sigma_z^{N+\lambda}\,|\nabla e|^2 + \sigma_z^{N+\lambda-2}\,|e|^2\,\mathrm{d}x\,.\end{aligned} \tag{3.12}$$

We close the proof using (3.9), (3.11), and (3.12) and absorb the right-hand side for ϵ and λ fixed by first choosing some $a > 0$ sufficiently small and then some sufficiently large $\gamma = \gamma(\epsilon, \lambda, a)$. □

4. Convergence of the adjoint lift operator, Theorem 2.4

In the following section, we give the proof of Theorem 2.4. We follow the approach of the stability analysis and reduce the estimate to functions on the finite element space, in order to employ the properties of δ^z. We first estimate by (2.3)

$$\begin{aligned}\big\|u-\mathcal{L}_h\mathcal{L}_h^{V*}u\big\|_{W^{1,\infty}(\Omega)} &\lesssim \|u-\mathcal{L}_hI_hu\|_{W^{1,\infty}(\Omega)} + \big\|I_hu-\mathcal{L}_h^{V*}u\big\|_{W^{1,\infty}(\Omega_h)}\\ &\lesssim h^k\,\|u\|_{W^{k+1,\infty}(\Omega)} + \big\|I_hu-\mathcal{L}_h^{V*}u\big\|_{W^{1,\infty}(\Omega_h)}\end{aligned}$$

We employ (3.3) and derive

$$\begin{aligned}\partial_i\big(I_hu-\mathcal{L}_h^{V*}u\big)(z) &= (\partial_iI_hu\mid\delta^z)_{H_h} - \big(\partial_i\mathcal{L}_h^{V*}u\mid\delta^z\big)_{H_h}\\ &= (\partial_iI_hu\mid\delta^z)_{H_h} - (\partial_iu\mid\mathcal{L}_h\delta^z)_H - (u\mid\mathcal{L}_hg_h^z-g^z)_V\\ &= \big(\partial_i\big(\mathcal{L}_hI_hu-u\big)\mid\mathcal{L}_h\delta^z\big)_{H_h} + (\mathcal{L}_hI_hu-u\mid\mathcal{L}_hg_h^z-g^z)_V + \widetilde{\Delta}_1 - \widetilde{\Delta}_2\,,\end{aligned}$$

with defects

$$
\begin{aligned}
\widetilde{\Delta}_1 &= (\partial_i I_h u \mid \delta^z)_{H_h} - (\partial_i \mathcal{L}_h I_h u \mid \mathcal{L}_h \delta^z)_{H_h} \\
\widetilde{\Delta}_2 &= (\mathcal{L}_h I_h u \mid \mathcal{L}_h g_h^z - g^z)_V \ .
\end{aligned}
$$

We note that both terms vanish in the conforming case. Again, we apply the interpolation estimate (2.3) and the Hölder inequality to derive

$$
\begin{aligned}
\left\| \partial_i \big(I_h u - \mathcal{L}_h^{V*} u \big) \right\|_{L^\infty(\Omega_h)} &\leq \left\| \big(\mathcal{L}_h I_h u - u \big) \right\|_{W^{1,\infty}(\Omega)} \left\| \mathcal{L}_h \delta^z \right\|_{L^1(\Omega)} \\
&\quad + \left\| \big(\mathcal{L}_h I_h u - u \big) \right\|_{W^{1,\infty}(\Omega)} \left\| \mathcal{L}_h g_h^z - g^z \right\|_{W^{1,1}(\Omega)} \\
&\quad + |\widetilde{\Delta}_1| + |\widetilde{\Delta}_2| \\
&\lesssim h^k \left\| u \right\|_{W^{k+1,\infty}(\Omega)} + |\widetilde{\Delta}_1| + |\widetilde{\Delta}_2| \,,
\end{aligned}
$$

where we used (3.4) and Lemma 3.1 in the last step. Thus, Theorem 2.4 follows once we have employed the Poincaré inequality and shown that

$$
|\widetilde{\Delta}_1| + |\widetilde{\Delta}_2| \lesssim h^k \left\| u \right\|_{W^{k+1,\infty}(\Omega)} \ .
$$

This inequality is proved in the following series of lemmas.

Lemma 4.1. *There is a constant $C > 0$ such that*

$$
|\widetilde{\Delta}_1| \leq C h^k \left\| u \right\|_{W^{1,\infty}(\Omega)} \ ,
$$

with C independent of h.

Proof. We obtain by Lemma 2.6 and (3.4)

$$
|\widetilde{\Delta}_1| \leq h^k \left\| \partial_i \mathcal{L}_h I_h u \right\|_{L^\infty} \left\| \mathcal{L}_h \delta^z \right\|_{L^1} \lesssim h^k \left\| u \right\|_{W^{1,\infty}}
$$

where we used the stability of the lift (2.2) and the interpolation (2.3) in the last step. $\square$

In the next lemma, we decompose the remaining defect even further into two differences of bilinear forms.

Lemma 4.2. *The defect $\widetilde{\Delta}_2$ can be represented by*

$$
\widetilde{\Delta}_2 = \widetilde{\Delta}_H + \widetilde{\Delta}_V
$$

where $\widetilde{\Delta}_H$ and $\widetilde{\Delta}_V$ are given by

$$
\begin{aligned}
\widetilde{\Delta}_H &= (\mathcal{L}_h I_h u \mid \partial_i \mathcal{L}_h \delta^z)_H - (I_h u \mid \partial_i \delta^z)_{H_h} \ , \\
\widetilde{\Delta}_V &= (\mathcal{L}_h I_h u \mid \mathcal{L}_h g_h^z)_V - (I_h u \mid g_h^z)_{V_h} \ .
\end{aligned}
$$

Proof. Using the definitions of g^z and g_h^z in (3.2), we derive

$$\begin{aligned}\widetilde{\Delta}_2 &= (\mathcal{L}_h I_h u \mid \mathcal{L}_h g_h^z - g^z)_V \\ &= (\mathcal{L}_h I_h u \mid \mathcal{L}_h g_h^z)_V - (\mathcal{L}_h I_h u \mid g^z)_V \\ &= (I_h u \mid g_h^z)_{V_h} + \widetilde{\Delta}_V + (\mathcal{L}_h I_h u \mid \partial_i \mathcal{L}_h \delta^z)_H \\ &= -(I_h u \mid \partial_i \delta^z)_{H_h} + \widetilde{\Delta}_V + (\mathcal{L}_h I_h u \mid \partial_i \mathcal{L}_h \delta^z)_H \\ &= \widetilde{\Delta}_H + \widetilde{\Delta}_V \,. \end{aligned}$$ □

The final bounds are derived in the next lemma.

Lemma 4.3. *There is a constant $C > 0$ such that*

$$|\widetilde{\Delta}_H| + |\widetilde{\Delta}_V| \leq Ch^k \, \|u\|_{W^{1,\infty}(\Omega)}$$

with C independent of h.

Proof. We consider the two terms separately.

(a) Using integration by parts, Lemma 2.6, and (3.4) we obtain

$$\begin{aligned}|\widetilde{\Delta}_H| &= |\, (\partial_i \mathcal{L}_h I_h u \mid \mathcal{L}_h \delta^z)_H - (\partial_i I_h u \mid \delta^z)_{H_h} \,| \\ &\lesssim h^k \, \|\mathcal{L}_h I_h u\|_{W^{1,\infty}} \, \|\mathcal{L}_h \delta^z\|_{L^1} \\ &\lesssim h^k \, \|u\|_{W^{1,\infty}} \end{aligned}$$

where we used the stability of the lift (2.2) and the interpolation (2.3) in the last step.

(b) For the second term, we introduce the following band around Γ defined by $U_\delta := \{x \in \Omega \mid \operatorname{dist}(x, \Gamma) < \delta\} \subset \Omega$. For h sufficiently small, there is a constant $c_\Gamma > 0$ such that all boundary elements are contained in the band $U_{c_\Gamma h}$. As in the proof of [5, Lem. 8.24] we obtain

$$\begin{aligned}|\widetilde{\Delta}_V| &= |\, (\mathcal{L}_h I_h u \mid \mathcal{L}_h g_h^z)_V - (I_h u \mid g_h^z)_{V_h} \,| \\ &\lesssim h^k \, \|\mathcal{L}_h I_h u\|_{W^{1,\infty}(\Omega_h)} \, \|\mathcal{L}_h g_h^z\|_{W^{1,1}(U_{c_\Gamma h})} \\ &\lesssim h^k \, \|u\|_{W^{1,\infty}(\Omega)} \Big(\|\mathcal{L}_h g_h^z - g^z\|_{W^{1,1}(\Omega)} + \|g^z\|_{W^{1,1}(U_{c_\Gamma h})} \Big) \\ &\lesssim h^k \, \|u\|_{W^{1,\infty}(\Omega)} \,, \end{aligned}$$

where we used Lemmas 3.1 and A.4 in the last inequality. □

References

[1] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, Second, Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003. MR2424078

[2] N. Yu. Bakaev, V. Thomée, and L. B. Wahlbin, *Maximum-norm estimates for resolvents of elliptic finite element operators*, Math. Comp. **72** (2003), no. 244, 1597–1610. MR1986795

[3] S. C. Brenner and L. R. Scott, *The mathematical theory of finite element methods*, Third, Texts in Applied Mathematics, vol. 15, Springer, New York, 2008. MR2373954 (2008m:65001)

[4] C. M. Elliott and T. Ranner, *Finite element analysis for a coupled bulk-surface partial differential equation*, IMA J. Numer. Anal. **33** (2013), no. 2, 377–402. MR3047936

[5] ———, *A unified theory for continuous-in-time evolving finite element space approximations to partial differential equations in evolving domains*, IMA J. Numer. Anal. **41** (2021), no. 3, 1696–1845. MR4286249

[6] P. Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics, vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985. MR775683

[7] J. Guzmán, D. Leykekhman, J. Rossmann, and A. H. Schatz, *Hölder estimates for Green's functions on convex polyhedral domains and their applications to finite element methods*, Numer. Math. **112** (2009), no. 2, 221–243. MR2495783

[8] D. Hipp, M. Hochbruck, and C. Stohrer, *Unified error analysis for nonconforming space discretizations of wave-type equations*, IMA J. Numer. Anal. **39** (2019), no. 3, 1206–1245. MR3984056

[9] T. Kashiwabara and T. Kemmochi, *Pointwise error estimates of linear finite element method for Neumann boundary value problems in a smooth domain*, Numer. Math. **144** (2020), no. 3, 553–584. MR4071825

[10] N. Kopteva, *Logarithm cannot be removed in maximum norm error estimates for linear finite elements in 3D*, Math. Comp. **88** (2019), no. 318, 1527–1532. MR3925475

[11] B. Kovács and C. A. Power Guerra, *Maximum norm stability and error estimates for the evolving surface finite element method*, Numer. Methods Partial Differential Equations **34** (2018), no. 2, 518–554. MR3765711

[12] B. Maier, *Error analysis for space and time discretizations of quasilinear wave-type equations*, Ph.D. Thesis, 2020. https://doi.org/10.5445/IR/1000120935.

[13] F. Natterer, *Über die punktweise Konvergenz finiter Elemente*, Numer. Math. **25** (1975/76), no. 1, 67–77. MR474884

[14] J. A. Nitsche, *L_∞-convergence of finite element approximation*, Journées "Éléments Finis" (Rennes, 1975), 1975, pp. 18. MR568857

[15] ———, *L_∞-convergence of finite element approximations*, Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975), 1977, pp. 261–274. Lecture Notes in Math., Vol. 606. MR0488848

[16] R. Rannacher, *Zur L^∞-Konvergenz linearer finiter Elemente beim Dirichlet-Problem*, Math. Z. **149** (1976), no. 1, 69–77. MR488859

[17] R. Rannacher and R. Scott, *Some optimal error estimates for piecewise linear finite element approximations*, Math. Comp. **38** (1982), no. 158, 437–445. MR645661

[18] A. H. Schatz, I. H. Sloan, and L. B. Wahlbin, *Superconvergence in finite element methods and meshes that are locally symmetric with respect to a point*, SIAM J. Numer. Anal. **33** (1996), no. 2, 505–521. MR1388486

[19] A. H. Schatz and L. B. Wahlbin, *Interior maximum norm estimates for finite element methods*, Math. Comp. **31** (1977), no. 138, 414–442. MR431753

[20] ———, *On the quasi-optimality in L_∞ of the $\dot{H}^1$-projection into finite element spaces*, Math. Comp. **38** (1982), no. 157, 1–22. MR637283

[21] ———, *Interior maximum-norm estimates for finite element methods. II*, Math. Comp. **64** (1995), no. 211, 907–928. MR1297478

[22] R. Scott, *Optimal L^∞ estimates for the finite element method on irregular meshes*, Math. Comp. **30** (1976), no. 136, 681–697. MR436617

[23] L. B. Wahlbin, *Maximum norm error estimates in the finite element method with isoparametric quadratic elements and numerical integration*, RAIRO Anal. Numér. **12** (1978), no. 2, 173–202, v. MR502070

A. Appendix

In this section, we collect the regularity results used in the above analysis. These are taken from [3, Chap. 8] and stated here in a slightly more general version.

The first result is an extension of [3, Lem. 8.3.7], where the Hessian is replaced by the gradient which allows to obtain a factor h^{-1} instead of h^{-2}.

Lemma A.1. *Let $v \in V$ be the solution of*

$$(v \mid \phi)_V = (f \mid \phi)_H\,, \quad \forall \phi \in V\,.$$

Then, we have

$$\int_\Omega \sigma_z^{-N-\lambda} |\nabla v|^2 \,\mathrm{d}x \le C\lambda^{-1}(\gamma h)^{-1} \int_\Omega \sigma_z^{4-N-\lambda} |\nabla f|^2 \,\mathrm{d}x.$$

Proof. In the proof of [3, Lem. 8.3.7], one first estimates by Hölder's inequality

$$\int_\Omega \sigma_z^{-N-\lambda} |\nabla v|^2 \,\mathrm{d}x \lesssim (\gamma h)^{-\lambda-N/p} \|\nabla v\|_{L^{2p}}^2\,.$$

Once, we have shown that for any $p, s > 1$

$$\|\nabla v\|_{L^{2p}} \lesssim \|\nabla f\|_{L^1} \lesssim \|\nabla f\|_{L^s}\,, \tag{A.1}$$

we conclude with $s = \frac{2pN}{N+3p} := \frac{2}{q} \in (1,2)$

$$\begin{aligned}
\|\nabla f\|_{L^s}^s &= \int_\Omega |\nabla f|^{2/q} \,\mathrm{d}x \\
&= \int_\Omega \sigma_z^{-\frac{4-N-\lambda}{q}} \sigma_z^{\frac{4-N-\lambda}{q}} |\nabla f|^{2/q} \,\mathrm{d}x \\
&\le \Big(\int_\Omega \sigma_z^{-(4-N-\lambda)\frac{q'}{q}} \,\mathrm{d}x\Big)^{1/q'} \Big(\int_\Omega \sigma_z^{4-N-\lambda} |\nabla f|^2 \,\mathrm{d}x\Big)^{1/q}
\end{aligned}$$

and hence

$$\|\nabla f\|_{L^s}^2 = \int_\Omega |\nabla f|^{2/q} \,\mathrm{d}x \le \Big(\int_\Omega \sigma_z^{-(4-N-\lambda)\frac{q'}{q}} \,\mathrm{d}x\Big)^{q/q'} \int_\Omega \sigma_z^{4-N-\lambda} |\nabla f|^2 \,\mathrm{d}x.$$

With $\frac{q}{q'} = q-1$ we have

$$\begin{aligned}
\Big(\int_\Omega \sigma_z^{-(4-N-\lambda)\frac{q'}{q}} \,\mathrm{d}x\Big)^{q/q'} &= \Big(\int_\Omega \sigma_z^{-(4-N-\lambda)(q-1)} \,\mathrm{d}x\Big)^{q-1} \\
&\le C(\gamma h)^{-(4-N-\lambda)+N(q-1)} \\
&= C(\gamma h)^{-1+\lambda+\frac{N}{p}}
\end{aligned}$$

since

$$-(4-N-\lambda)+N(q-1) = -4+N+\lambda+N(\frac{N+3p}{pN}-1) = -1+\lambda+\frac{N}{p} < 0$$

for $p > \frac{N}{1-\lambda}$ and hence the claim follows.

It remains to prove (A.1). We employ Theorem 2.1 and [1, Thm. 4.12] to obtain

$$\|\nabla v\|_{L^{2p}} \lesssim \|v\|_{W^{2,N/2}} \lesssim \|f\|_{L^{3/2}} \lesssim \|f\|_{W^{1,1}} \leq \|\nabla f\|_{L^1}$$

where we use Case B ($mp = N$) for the first inequality, Case C ($m = p = 1$) for the third, and the Poincaré inequality for the last. $\square$

The next lemma is a straight forward extension of [3, Lem 8.3.7], where the case $\alpha = 2$ is derived.

Lemma A.2. *Let $v \in V$ be the solution of*

$$(v \mid \phi)_V = (f \mid \phi)_H\,, \quad \forall \phi \in V\,.$$

Then for $0 < \alpha \leq 2$, we have

$$\int_\Omega \sigma_z^{-N-\lambda+2-\alpha}\Big(|\nabla v|^2 + |\nabla_2 v|^2\Big)\,\mathrm{d}x \leq C\lambda^{-1}(\gamma h)^{-\alpha}\int_\Omega \sigma_z^{4-N-\lambda}\,|\nabla f|^2\,\mathrm{d}x.$$

Proof. In order to adapt the proof, it is sufficient to guarantee the existence of a $p \in (1,\infty)$ such that the conditions

$$p > \frac{N}{2-\lambda}$$

and

$$(-N-\lambda+2-\alpha)p' + N < 0 \quad \iff \quad \frac{N}{p} > 2-\alpha-\lambda$$

are both satisfied. For $2 \leq \alpha+\lambda$, the latter condition is empty. In the other cases, it is equivalent to

$$p < \frac{N}{2-\lambda-\alpha}\,,$$

and since $\alpha > 0$, such a p can be found. $\square$

The following lemma builds upon the estimates in [3, Lem. 8.3.11]. In the proof the result is shown for $\alpha = 0$.

Lemma A.3. *Let $v \in V$ be the solution of*

$$(\phi \mid v)_V = (\nu\cdot\nabla f \mid \phi)_H\,, \quad \forall \phi \in V\,.$$

Then for $0 \leq \alpha < 2-\lambda$, we have

$$\int_\Omega \sigma_z^{N+\lambda-2+\alpha}\,|\nabla v|^2\,\mathrm{d}x \leq C\int_\Omega \sigma_z^{N+\lambda+\alpha}\,|\nabla f|^2\,\mathrm{d}x + (\gamma h)^{-2+\alpha}\int_\Omega \sigma_z^{N+\lambda}\,|f|^2\,\mathrm{d}x.$$

Proof. We compute

$$\begin{aligned}\int_\Omega \sigma_z^{N+\lambda-2+\alpha} |\nabla v|^2 &= \Big(v \mid \sigma_z^{N+\lambda-2+\alpha} v\Big)_V - \int_\Omega \nabla v \nabla(\sigma_z^{N+\lambda-2+\alpha}) v \,\mathrm{d}x \\ &\leq \Big|\Big(\nu\cdot\nabla f \mid \sigma_z^{N+\lambda-2+\alpha} v\Big)_H\Big| + a \int_\Omega \sigma_z^{N+\lambda-2+\alpha} |\nabla v|^2 + \tfrac{1}{a}\int_\Omega \sigma_z^{N+\lambda-4+\alpha} |v|^2 \\ &\lesssim \int_\Omega \sigma_z^{N+\lambda+\alpha} |\nabla f|^2 + a \int_\Omega \sigma_z^{N+\lambda-2+\alpha} |\nabla v|^2 + \tfrac{1}{a}\int_\Omega \sigma_z^{N+\lambda-4+\alpha} |v|^2\end{aligned}$$

and by absorption, it only remains to bound the last term. We claim

$$\int_\Omega \sigma_z^{N+\lambda-4+\alpha} |v|^2 \leq C(\gamma h)^{-2+\alpha} \int_\Omega \sigma_z^{N+\lambda} |f|^2 \,\mathrm{d}x$$

which can be adapted from the proof of [3, Lem. 8.3.11], if one can find $r > 1$ with

$$r < \frac{2N}{2N-2+\lambda+\alpha},$$

which is possible since $\alpha + \lambda < 2$. □

The last lemma exploits the fact, that the solution g^z of the regularized δ-function only has to be bounded on a narrow strip around the boundary of Ω.

Lemma A.4. *There is a constant $C > 0$ such that*

$$\|g^z\|_{W^{1,1}(U_{c_\Gamma h})} \leq C$$

with C independent of h.

Proof. The key tool is the generalized version of the narrow band inequality shown in [4, Lem. 4.10]. We recall $U_\delta = \{x \in \Omega \mid \operatorname{dist}(x,\Gamma) < \delta\}$. Then for any $1 \leq p < \infty$, there is a constant $C_p > 0$ such that for any $\varphi \in W^{1,p}(\Omega)$ it holds

$$\|\varphi\|_{L^p(U_\delta)} \leq C_p\, \delta^{1/p}\, \|\varphi\|_{W^{1,p}(\Omega)}\,. \tag{A.2}$$

We apply (A.2) with $p = 1$ and $\delta = c_\Gamma h$ and obtain

$$\|g^z\|_{W^{1,1}(\mathcal{L}_h[J_h\neq 1])} \lesssim h\, \|\nabla_2 g^z\|_{L^1(\Omega)}\,.$$

Finally, we deduce by (2.8) and the elliptic regularity shown in [3, eq. (8.3.10)] the bound

$$\|\nabla_2 g^z\|^2_{L^1(\Omega)} \lesssim h^{-\lambda} \int\limits_\Omega \sigma_z^{N+\lambda} |\nabla_2 g^z| \,\mathrm{d}x \lesssim h^{-\lambda} h^{\lambda-2} \lesssim h^{-2}$$

and, taking the square roots, the assertion follows. □